CALABI-YAU SUBMANIFOLDS OF JOYCE MANIFOLDS OF THE FIRST KIND

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A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Mathematics

2011

ABSTRACT

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Akbulut and Salur suggested the study of Calabi-Yau submanifolds of G_2 manifolds that come from a certain process. The author, in a joint paper with Akbulut and Salur, applied this process to a Joyce manifold, more specifically, to the Joyce manifold $J(1/2, 0, 0, 1/2, 1/2)$, and obtained a pair of Borcea-Voisin 3-folds with Hodge numbers $h^{1,1} = h^{2,1} = 19$.

In this thesis, we first list all possible Joyce manifolds of the first kind. Then we describe the Calabi-Yau submanifolds of these manifolds that come from the process, we mentioned above, using the coordinate directions. This way we obtain two different Borcea-Voisin manifolds, as well as $\mathbb{T}^2 \times K3$ and $\mathbb{T}^6$.

ACKNOWLEDGMENT

I would like to thank my supervisor Dr. Selman Akbulut for his excellent guidance, patience and constant encouragement. Many thanks to my committee members Dr. Ronald Finthushel, Dr. Rajesh Kulkarni, Dr. Benjamin Schmidt and Dr. Xiaodong Wang. I would also like to thank Dr. Sema Salur for her advices and fruitful discussions.

Finally, special thanks to my family, my wife Melike and my mother Fatma, for their constant support and encouragement.

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Chapter 1

Introduction

In this thesis we study Calabi-Yau submanifolds of a family of G_2 manifolds constructed by Joyce in [J1, J2]. The method we use to find these Calabi-Yau 3-folds is given by Akbulut and Salur in [AS2]. We first find a (locally) non-vanishing vector field on the G_2 manifold, then the submanifold normal to this vector field has a Calabi-Yau structure induced from the G_2 structure. Akbulut and Salur defines 2 such submanifolds to be a mirror pair if they come from the same G_2 structure. They call this as a mirror duality inside a G_2 manifold. This is an interesting concept as we will see in our examples. As we will note later in the text, in some cases the concept of mirror duality and mirror symmetry coincides, in other words, some mirror pairs are actually mirror symmetric.

Another motivation comes from string theory. As far as the author understands, in string theory electrons and quarks are considered to be 1-dimensional strings rather than 0-dimensional objects. Physicists study the motion of these strings. In different theories, they consider the space time to have not only 4 dimensions (space plus time) but as many as 26 (bosonic case), 10 (superstring theory) or 11 (M-theory) dimensions. To model the extra 6 dimensions in superstring theory they use Calabi-Yau manifolds, and to model the extra 7 dimensions in M-theory they use G_2 manifolds.

Therefore, considering Calabi-Yau manifolds as submanifolds of G_2 manifolds might have interesting meanings from the point of view of a physicist. Again, these are only interpretations of the author who does not know physics.

The outline of this thesis is as follows: In the second chapter we give basic definitions and some well known facts on holonomy and Calabi-Yau manifolds. Also we introduce a famous type of Calabi-Yau manifold constructed by Borcea and Voisin. This 3-folds will appear in our examples on the last chapter. In chapter 3 we give the definition and examples of G_2 manifolds, and we explain the construction of Akbulut and Salur. In the last chapter, we give our results. We consider Joyce manifold of the first kind and study their Calabi-Yau submanifolds obtained by the method given by Akbulut and Salur.

Chapter 2

Background Material

In this chapter we will recall basic definitions and some examples. In the first section we introduce the notion of holonomy for Riemannian manifolds and give a classifying theorem of Berger. For further reading on holonomy groups we refer the reader to [Bes],[J3] and [P]. In the second section we define Calabi-Yau manifolds. In the third section we define orbifolds and give some facts on their resolutions. Finally, on the last section we give an important example to Calabi-Yau manifolds, namely, Borcea-Voisin 3-folds.

2.1 Holonomy Groups

Let (M, g) be a Riemannian manifold of dimension n and let ∇ be the Levi-Civita connection. Suppose that $\gamma : [0,1] \longrightarrow M$ is a smooth curve with $\gamma(0) = x$ and $\gamma(1) = y$, where $x, y \in M$. Then for each $u \in T_xM$ there exists a unique section s of $\gamma^*(TM)$ satisfying $\nabla_{\dot{\gamma}(t)} s(t) = 0$ for each $t \in [0,1]$, with $s(0) = u$. Define $P_\gamma : T_xM \longrightarrow T_yM$ by $P_\gamma(u) = s(1)$. P_γ is a well-defined linear map, called the *parallel transport map.* One can generalize this definition to the case when γ is continuous and piecewise smooth by requiring s to be continuous, and differentiable whenever γ is differentiable.

Definition 2.1.1. The *holonomy group* $Hol_x(g)$ based at x is defined to be

$$Hol_x(g) = \left\{P_\gamma : \gamma \textit{ is a loop based at } x\right\} \subseteq GL(n)$$

If M is connected, the holonomy group is independent of the base point because if $x, y \in M$ can be connected by a piecewise smooth curve γ in M then $P_\gamma Hol_x(g) P_\gamma^{-1} \cong Hol_y(g)$. Therefore in this case, we can drop the subscripts x and write the holonomy group as $Hol(g)$. Under certain assumptions on M and g, Berger [Ber] gave a list of all possible holonomy groups:

Theorem 2.1.2. (Berger) Suppose that (M, g) is a simply-connected Riemannian manifold of dimension n, and that g is irreducible and non-symmetric, then exactly one of the following seven cases holds.

i) $Hol(g) = SO(n)$,

ii) $n = 2m$ with $m \geq 2$, and $Hol(g) = U(m)$ in $SO(2m)$,

iii) $n = 2m$ with $m \geq 2$, and $Hol(g) = SU(m)$ in $SO(2m)$,

iv) $n = 4m$ with $m \geq 2$, and $Hol(g) = Sp(m)$ in $SO(4m)$,

v) $n = 4m$ with $m \geq 2$, and $Hol(g) = Sp(m)Sp(1)$ in $SO(4m)$,

vi) $n = 7$ and $Hol(g) = G_2$ in $SO(7)$,

vii) $n = 8$ and $Hol(g) = Spin(7)$ in $SO(8)$.

2.2 Calabi-Yau Manifolds

There are several inequivalent definitions of Calabi-Yau manifolds in use in the literature. We will use the following as our definition which is equivalent to saying that

$Hol(g) \subseteq SU(n)$, which is also equivalent to saying that the canonical bundle of the manifold is trivial (see [J3]).

Definition 2.2.1. A compact n dimensional Kähler manifold (N, J, ω) is called a *Calabi-Yau manifold* if N has a holomorphic n-form Ω that vanishes nowhere.

Example 2.2.2. In dimension one the only examples are tori, T^2. In dimension two all Calabi-Yau manifolds are either T^4 or $K3$ surfaces (compact, complex surface with $h^{1,0} = 0$ and trivial canonical bundle).

Example 2.2.3. Let N be a hypersurface of degree $n+1$ in $\mathbb{CP}^n$, so $N = \{[z_0, z_1, ..., z_n] \in \mathbb{CP}^n : f(z_0, z_1, ..., z_n) = 0\}$. One can show that (for example [J3],section 6.7) N is a Calabi-Yau manifold of complex dimension $n-1$. This is perhaps the simplest known method of finding Calabi-Yau manifolds. But all nonsingular hypersurfaces in $\mathbb{CP}^n$ of degree $n+1$ are diffeomorphic, and thus, this method provides only one smooth manifold in each dimension.

Some of the well-known properties of Calabi-Yau manifolds are given by the following two propositions. For proofs, we refer the reader to [J3].

Proposition 2.2.4. *If (N, J, g) is a compact Kähler manifold of dimension n and $Hol(g)$ is $SU(n)$ or $Sp(n/2)$, then g is Ricci flat and irreducible and N has finite fundamental group.*

Proposition 2.2.5. *Let (N, J, g) be a Calabi-Yau manifold of dimension n with $Hol(g) = SU(n)$ and let $h^{p,q}$ be its Hodge numbers. Then $h^{0,0} = h^{n,0} = 1$ and $h^{p,0} = 0$ for $p \neq 0, n$.*

Therefore, for Calabi-Yau manifolds of complex dimension 3 with holonomy $SU(3)$, the Hodge diamond is given by:

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{2,1} & & h^{2,1} & & 1 \\
 & 0 & & h^{1,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

2.3 Orbifolds

In this section we will give definition and some facts on orbifolds. For further reading refer to Satake [Sat], who calls them V-manifolds, and Joyce [J3].

Definition 2.3.1. An *orbifold* is a singular real manifold X of dimension n where singularities are locally isomorphic to quotient singularities $\mathbb{R}^n/G$ for finite subgroups $G \subset GL(n)$, such that if $1 \neq \gamma \in G$ then the subspace V_γ of $\mathbb{R}^n$ fixed by γ has $dim\ V_\gamma \leq n-2$.

Definition 2.3.2. For each singular point $x \in X$ in an orbifold X, there is a finite group $G_x \subset GL(n)$, unique up to conjugation, such that an open neighborhood of $x \in X$ is homeomorphic to an open neighborhood of $0 \in \mathbb{R}^n/G_x$. We call x an *orbifold point* of X and G_x the *orbifold group* of x.

Example 2.3.3. If M is a manifold and G is a finite group that acts smoothly on M, with non-identity fixed point sets of codimension at least two, then M/G is an orbifold.

The following proposition, taken from [J3], describes the singular set of M/G.

Proposition 2.3.4. *Let M be a smooth manifold and G be a finite group acting smoothly and faithfully on M preserving orientation. Then M/G is an orbifold. For* *each $x \in M$ define the stabilizer subgroup of x to be $Stab(x) = \{\gamma \in G : \gamma \cdot x = x\}$. If $Stab(x) = \{1\}$ then xG is a non-singular point of M/G. If $Stab(x) \neq \{1\}$ then xG is a singular point of M/G and has orbifold group $Stab(x)$. Thus the singular set of M/G is*

$$S = \{xG \in M/G : x \in M \text{ and } \gamma \cdot x = x \text{ for some } \gamma \in G\}$$

Definition 2.3.5. A *complex orbifold* is a singular complex manifold of dimension n whose singularities are locally isomorphic to $\mathbb{C}^n/G$, where G is a finite subgroup of $GL(n, \mathbb{C})$. The *orbifold points* and *orbifold groups* are defined as above.

Recall that a metric g on a complex manifold (M, J) is called a *Hermitian metric* if $g(u, v) = g(Ju, Jv)$ for all vectors u, v on M. The corresponding *Hermitian form* ω is the 2-form defined by $\omega(u, v) = g(Ju, v)$. This form ω is called a *Kähler form* if it is closed ($d\omega = 0$), and in this case g is called a *Kähler metric* and M is called a *Kähler manifold*.

Definition 2.3.6. We say that g is a *Kähler metric* on a complex orbifold (X, J), if g is Kähler in the usual sense on the non-singular part of X, and wherever X is locally isomorphic to $\mathbb{C}^n/G$, we can identify g with the quotient of a $G-$invariant Kähler metric defined near $0 \in \mathbb{C}^n$. In this case (X, J, g) is called a *Kähler orbifold*.

Many definitions and results about manifolds can be generalized to orbifolds, such as the definition of Kähler metrics above. In particular, the ideas of smooth k-forms, (p,q)-forms makes sense, De Rham and Dolbeault cohomology are well defined and have nearly all their usual properties. If all the orbifold groups of X lie in $SL(n, \mathbb{C})$, then the canonical bundle K_X (n^{th} exterior power of the cotangent bundle) is a genuine line bundle over X. The singularities of orbifolds may be resolved to obtain non-singular manifolds. To understand these resolutions we need to understand them locally first.

Definition 2.3.7. A *resolution* (X, π) of $\mathbb{C}^n/G$ is a nonsingular complex manifold X of dimension n with a proper biholomorphic map $\pi : X \to \mathbb{C}^n/G$ that induces a biholomorphism between dense open sets. We call X a *crepant resolution* if the canonical bundles are isomorphic, $K_X \cong \pi^*(K_{\mathbb{C}^n/G})$.

Each singularity $\mathbb{C}^2/G$ for G a finite subgroup of $SU(2)$ admits a unique crepant resolution ([M]). For $G \subset SL(n, \mathbb{C})$ any finite subgroup, $\mathbb{C}^3/G$ admits a crepant resolution ([R]). For $n \geq 4$, $\mathbb{C}^4/G$ may or may not admit a crepant resolution.

There is a conjecture from [IR], usually called the *McKay Correspondence*, which aims to describe the topology and geometry of crepant resolutions (X, π) of $\mathbb{C}^n/G$ in terms of the group G. We need the following definition to state the conjecture, and after the conjecture we will give the cases that have been proved.

Definition 2.3.8. Let $G \subset SL(n, \mathbb{C})$ be a finite subgroup. Then each $\gamma \in G$ has n eigenvalues $e^{2\pi i a_1}, ..., e^{2\pi i a_n}$, where $a_1, ..., a_n \in [0, 1)$ are uniquely defined up to order. Define the *age* of γ to be $age(\gamma) = a_1 + ... + a_n$. Since $det(\gamma) = 1 = e^{2\pi i \ age(\gamma)}$, $age(\gamma)$ is an integer between 0 and $n - 1$.

Conjecture 2.3.9. Let G be a finite subgroup of $SL(n, \mathbb{C})$, and (X, π) a crepant resolution of $\mathbb{C}^n/G$. Then there exists a basis of $H^*(X, \mathbb{Q})$ consisting of algebraic cycles in $1 - 1$ correspondence with conjugacy classes of G, such that conjugacy classes with *age* k correspond to basis elements of $H^{2k}(X, \mathbb{Q})$. In particular, $b^{2k}(X)$ is the number of conjugacy classes of G with *age* k, and $b^{2k+1}(X) = 0$, so the Euler characteristic $\chi(X)$ is the number of conjugacy classes in G.

The case $n = 2$ is already known to be true by McKay [M]. Ito and Reid [IR] proved that the conjecture is true for $n = 3$, Batyrev and Dais [BD] proved for arbitrary n when G is abelian, using toric geometry, and also gave their own proof for $n = 3$ case.

2.4 Borcea-Voisin 3-folds

In this section we will give an important example to Calabi-Yau 3-folds, constructed by Borcea [Bo] and Voisin [V].

The proofs of the following Lemma and Theorem can be found in [V].

Lemma 2.4.1. *If S is a $K3$ surface with and involution j such that j induces a non-trivial automorphism on $H^{2,0}(S)$, then the fixed points of j have several possibilities:*

i) no fixed points,

ii) a finite number of rational curves and at most one curve with genus > 0, or

iii) two elliptic curves.

Theorem 2.4.2. Let E be an elliptic curve $\mathbb{C}/\Lambda$ and i the involution induced by the involution on $\mathbb{C}$, $z \mapsto -z$. Let S be a $K3$ surface with involution j inducing a non-trivial automorphism on $H^{2,0}(S)$, and let $k(e,s) := (i(e), j(s))$ be the product automorphism on $E \times S$. Then $X = (\widetilde{E \times S})/k$, the minimal resolution of the orbifold $(E \times S)/k$, is a Calabi-Yau manifold.

Voisin gives a formula for the Hodge numbers of X of this theorem: Let n be the number of fixed curves of j on S as in the above Lemma (n is possibly 0), and let n' be the total genus of these n fixed curves. Then the Hodge numbers of X are given by

$$h^{1,1} = 11 + 5n - n' \textit{ and } h^{2,1} = 11 + 5n' - n. \tag{2.4.1}$$

The importance of this construction is that Nikulin's classification [N] implies that if (S, j) is a $K3$ surface with an involution that has n fixed curves with total genus n', then there exists a complementary pair (S', j') with n' fixed curves with total genus n. If we let X' to be the manifold obtained by Borcea-Voisin construction on S', then

X' will have Hodge numbers $h^{1,1} = 11 + 5n' - n$ and $h^{2,1} = 11 + 5n - n'$. Therefore manifolds constructed using this method always come in mirror pairs.

Chapter 3

G_2 Manifolds

3.1 Holonomy Group G_2

For the proofs of the facts listed in this section we refer to [AS2], [Br1], [J3].

Definition 3.1.1. Let $(x_1, ..., x_7)$ be coordinates on $\mathbb{R}^7$. Write $dx^{ij...l}$ for the exterior form $dx^i \wedge dx^j \wedge ... \wedge dx^l$ on $\mathbb{R}^7$. Define a 3-form φ_0 on $\mathbb{R}^7$ by

$$\varphi_0 = dx^{123} + dx^{145} + dx^{167} + dx^{246} - dx^{257} - dx^{347} - dx^{356} \tag{3.1.1}$$

The subgroup of $GL(7)$ preserving φ_0 is the exceptional Lie group G_2.

Theorem 3.1.2. The subgroup $G_2 \subseteq GL(7)$ is compact, connected, simply-connected and of dimension 14. Moreover, G_2 acts irreducibly on $\mathbb{R}^7$ and transitively on the spaces of lines in $\mathbb{R}^7$ and 2-planes in $\mathbb{R}^7$. Finally, G_2 is isomorphic to the group of algebra automorphisms of octanions.

Definition 3.1.3. A smooth 7-manifold M has a G_2 structure if there is a 3-form $\varphi \in \Omega^3(M)$ such that at each $x \in M$ the pair $(T_xM, \varphi(x))$ is isomorphic to $(T_0\mathbb{R}^7, \varphi_0)$. We call (M, φ) a manifold with G_2 structure.

A G_2 structure φ on M gives an orientation $\mu \in \Omega^7(M)$ on M, and μ determines a metric $g_\varphi =<,>$ on M by:

$$< u, v >= (u \lrcorner \varphi \wedge v \lrcorner \varphi \wedge \varphi)/6\mu \tag{3.1.2}$$

Definition 3.1.4. A manifold with G_2 structure (M, φ) is called a G_2 manifold if $Hol(g_\varphi) \subseteq G_2$.

Equivalent definitions can be given by the following proposition which follows from [Sal](section 11.5).

Proposition 3.1.5. *Let (M, φ) be a 7-manifold with a G_2 structure. Then the following are equivalent:*

i) $Hol(g_\varphi) \subseteq G_2$

ii) $\nabla\varphi = 0$, *where* ∇ *is the Levi-Civita connection of* g_φ

iii) $d\varphi = d^*\varphi = 0$

iv) $d\varphi = d(*_\varphi \varphi) = 0$

Example 3.1.6. Let (N, ω, Ω) be a Calabi-Yau 3-fold, then $N \times S^1$ has holonomy $SU(3) \subset G_2$. In this case $\varphi = Re\Omega + \omega \wedge dt$. Similarly, $N \times \mathbb{R}$ is a non-compact G_2 manifold.

Example 3.1.7. Let Y be a Riemannian 3-manifold with constant sectional curvature $+1$. Bryant and Salamon [BS] gave an explicit metric on the spinor bundle $S \cong Y \times \mathbb{R}^4$.

Example 3.1.8. Joyce [J1, J2] gave the first examples of compact 7-manifolds with holonomy G_2. We will give details to this construction in section 3.4.

3.2 Akbulut-Salur Construction

In this section we will describe the method of finding Calabi-Yau submanifolds of G_2 manifolds given by Akbulut and Salur [AS2]. Let (M, φ) be a 7-manifold with a G_2 structure, and let $g_\varphi =<,>$ be the corresponding metric given by equation (3.1.2).

Definition 3.2.1. Define a cross product structure $\times$ on the tangent bundle of M as follows

$$\varphi(u, v, w) =< u \times v, w > \tag{3.2.1}$$

We can also view cross product as a tangent bundle valued 2-form $\psi \in \Omega^2(M, TM)$ defined by $\psi(u, v) = u \times v$

Definition 3.2.2. Define the tangent bundle valued 3-form $\chi \in \Omega^3(M, TM)$ by

$$< \chi(u, v, w), z >= *\varphi(u, v, w, z) \tag{3.2.2}$$

Definition 3.2.3. Let ξ be a nonvanishing vector field of M. We can define a symplectic ω_ξ and a complex structure J_ξ on the 6-plane bundle $V_\xi = \xi^\perp$ by

$$\omega_\xi =< \psi, \xi >= \xi \lrcorner \varphi \ \ \textit{and} \ \ J_\xi(X) = X \times \xi \tag{3.2.3}$$

And define a complex valued (3,0) form $\Omega_\xi = Re\Omega_\xi + i\ Im\Omega_\xi$ by

$$Re\Omega_\xi = \varphi|_{V_\xi} \ \ \textit{and} \ \ Im\Omega_\xi =< \chi, \xi >= \xi \lrcorner * \varphi \tag{3.2.4}$$

Example 3.2.4. (example of section 3.1 in [AS2]) Consider $M = T^7$ as G_2 manifold with calibration 3-form $\varphi = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}$ where $\{e_1, ..., e_7\}$ is the basis of TM. If we choose $\xi = e_7$, then $V_\xi =< e_1, ..., e_6 >$, the symplectic form is $\omega_\xi = e^{16} - e^{25} - e^{34}$, the complex structure J_ξ is $e_1 \mapsto -e_6$, $e_2 \mapsto e_5$,

$e_3 \mapsto e_4$, and the complex valued (3,0) form is $\Omega_\xi = (e^1 + i\, e^6) \wedge (e^2 - i\, e^5) \wedge (e^3 - i\, e^4)$; note that this is just $\Omega_\xi = (e^1 - i\ J_\xi(e^1)) \wedge (e^2 - i\ J_\xi(e^2)) \wedge (e^3 - i\ J_\xi(e^3))$.

Example 3.2.5. (continued from previous example) If we choose $\xi' = e_3$, then $V_{\xi'} =< e_1, ..., \hat{e}_3, ..., e_7 >$, the symplectic form is $\omega_{\xi'} = e^{12} - e^{47} - e^{56}$, the complex structure $J_{\xi'}$ is $e_1 \mapsto -e_2$, $e_4 \mapsto e_7$, $e_5 \mapsto e_6$, and $\Omega_{\xi'} = (e^1 + i\, e^2) \wedge (e^4 - i\, e^7) \wedge (e^5 - i\, e^6)$ which is just $\Omega_{\xi'} = (e^1 - i\ J_{\xi'}(e^1)) \wedge (e^2 - i\ J_{\xi'}(e^2)) \wedge (e^3 - i\ J_{\xi'}(e^3))$.

Now let $\xi \in \Omega^0(M, TM)$ be a non-vanishing unit vector field, which gives a codimension one distribution $V_\xi = \xi^\perp$ on M, which is equipped with the structures $(V_\xi, \omega_\xi, \Omega_\xi, J_\xi)$ as in the definitions above. Let $\xi^\sharp$ be the dual 1-form of ξ. Let $e_{\xi^\sharp}$ and $\xi \lrcorner$ denote the exterior and interior product operations respectively. Since $e_{\xi^\sharp} \circ \xi \lrcorner + \xi \lrcorner \circ e_{\xi^\sharp} = id$ we have

$$\varphi = e_{\xi^\sharp} \circ \xi \lrcorner (\varphi) + \xi \lrcorner \circ e_{\xi^\sharp} (\varphi) = \omega_\xi \wedge \xi^\sharp + Re\Omega_\xi \tag{3.2.5}$$

Recall that the condition that the distribution V_ξ be integrable (the involutive condition which implies $\xi^\perp$ comes from a foliation) is given by

$$d\xi^\sharp \wedge \xi^\sharp = 0 \tag{3.2.6}$$

By Thomas [T], even if V_ξ is not integrable, it is homotopic to a foliation. Let X_ξ be a page of this foliation, and for simplicity assume that this 6-manifold is smooth. Now we will give the following two Lemmas with proofs from [AS2].

Lemma 3.2.6. *J_ξ is compatible with ω_ξ, and it is metric invariant.*

Proof. Let $u, v \in V_\xi$, then

$$\begin{aligned}\omega_\xi\left(J_\xi(u), v\right) &= \omega_\xi(u \times \xi, v) = \varphi(u \times \xi, v, \xi) \\ &= -\varphi(\xi, \xi \times u, v) = -<\xi \times (\xi \times u), v> \\ &= -<-|\xi|^2 u + <\xi, u> \xi, v> \\ &= |\xi|^2 <u, v> - <\xi, u><\xi, v> \\ &= <u, v>\end{aligned}$$

The first line comes from the definitions of ω_ξ and J_ξ, the second line comes from the definition of cross product, and the identity $\xi \times (\xi \times u) = -|\xi|^2 u + <\xi, u> \xi$ for the third line is from [Br2]. Hence $<J_\xi(u), J_\xi(v)> = -\omega_\xi\left(u, J_\xi(v)\right) = <u, v>$ $\square$

Lemma 3.2.7. Ω_ξ *is a nonvanishing (3,0) form.*

Proof.

$$\begin{aligned}(-i/2)\Omega_\xi \wedge \bar{\Omega}_\xi = Im\Omega_\xi \wedge Re\Omega_\xi &= (\xi \lrcorner * \varphi) \wedge \left(\xi \lrcorner (\xi^\sharp \wedge \varphi)\right) \\ &= -\xi \lrcorner \left((\xi \lrcorner * \varphi) \wedge (\xi^\sharp \wedge \varphi)\right) \\ &= \xi \lrcorner \left(*(\xi^\sharp \wedge \varphi) \wedge (\xi^\sharp \wedge \varphi)\right) \\ &= \left|\xi^\sharp \wedge \varphi\right|^2 \xi \lrcorner vol(M) \\ &= 4\left|\xi^\sharp\right|^2 (*\xi^\sharp) = 4\ vol(X_\xi)\end{aligned}$$

Third line is from the identity $*(\xi \lrcorner \alpha) = (-1)^{k+1}(\xi^\sharp \wedge *\alpha)$ for $\alpha \in \Omega^k(M)$. And the identity $|\varphi \wedge \beta|^2 = 4|\beta|^2$ for $\beta \in \Omega^1(M)$ is from [Br2]. $\square$

The following observations is again from [AS2]. One can show that $*Re\Omega_\xi = -Im\Omega_\xi \wedge \xi^\sharp$ and $*Im\Omega_\xi = Re\Omega_\xi \wedge \xi^\sharp$. And if $\star$ is the star operator of X_ξ, then $\star Re\Omega_\xi = Im\Omega_\xi$. Note that ω_ξ is a symplectic structure on X_ξ whenever $d\varphi = 0$ and $L_\xi(\varphi)|_{V_\xi} = 0$, where L is the Lie derivative. This comes from $\omega_\xi = \xi \lrcorner \varphi$ and $d\omega_\xi = L_\xi(\varphi) - \xi \lrcorner d\varphi = L_\xi(\varphi)$. We also have $d^*\varphi = 0 \Rightarrow d^\star\omega_\xi = 0$, since $*\varphi =$

$\star\omega_\xi - Im\Omega_\xi \wedge \xi^\sharp$, and hence $d(\star\omega_\xi) = d(*\varphi|_{X_\xi}) = 0$. Also $d\varphi = 0 \Rightarrow d(Re\Omega_\xi) = d(\varphi|_{X_\xi}) = 0$. Furthermore, if $d^*\varphi = 0$ and $L_\xi(*\varphi)|_{V_\xi} = 0$ then $d(Im\Omega_\xi) = 0$, which follows from $Im\Omega_\xi = \xi \lrcorner (*\varphi)$. Also, J_ξ is integrable when $d\Omega = 0$ ([H]). Using the following definition, all these observations sum up to the theorem below.

Definition 3.2.8. (X^6, ω, Ω, J) is called an almost Calabi-Yau manifold, if X is a Riemannian manifold with a non-degenerate 2-form ω which is co-closed, and J is a metric invariant almost complex structure which is compatible with ω, and Ω is a non-vanishing (3,0) form with $Re\Omega$ closed. Furthermore, when ω and $Im\Omega$ are closed, we call this a Calabi-Yau manifold.

Theorem 3.2.9. Let (M, φ) be a G_2 manifold, and ξ be a unit vector field which comes from a codimension one foliation on M, then $(X_\xi, \omega_\xi, \Omega_\xi, J_\xi)$ is an almost Calabi-Yau manifold with $\varphi|_{X_\xi} = Re\Omega_\xi$ and $*\varphi|_{X_\xi} = \star\omega_\xi$. Furthermore, if $L_\xi(\varphi)|_{X_\xi} = 0$ then $d\omega_\xi = 0$, and if $L_\xi(*\varphi)|_{X_\xi} = 0$ then J_ξ is integrable; when both of these conditions are satisfied, $(X_\xi, \omega_\xi, \Omega_\xi, J_\xi)$ is a Calabi-Yau manifold.

3.3 MirrorDuality

In this section, we will define the concept of *mirror duality* of Calabi-Yau manifolds inside a G_2 manifold, introduced by Akbulut and Salur in [AS1, AS2].

Definition 3.3.1. Let (M, φ) be a manifold with a G_2 structure. A 4-dimensional submanifold $X \subset M$ is called a *co-associative* if $\varphi|X = 0$. A 3-dimensional submanifold $Y \subset M$ is called an *associative* if $\varphi|Y = vol(Y)$.

By a theorem of Thomas, all orientable 7-manifolds admit non-vanishing 2-frame fields [T]. Using this, one obtains an additional structure on the tangent bundle of G_2 manifolds [AS1].

Lemma 3.3.2. *A non-vanishing oriented 2-plane field* Λ *on a manifold with* G_2 *structure* (M, φ) *induces a splitting of* $TM = \mathbf{E} \oplus \mathbf{V}$*, where* $\mathbf{E}$ *is a bundle of associative 3-planes, and* $\mathbf{V} = \mathbf{E}^{\perp}$ *is a bundle of coassociative 4-planes. The unit sections* ξ *of the bundle* $\mathbf{E} \to M$ *give complex structures* J_ξ *on* $\mathbf{V}$*.*

Proof. Let $\Lambda = span\,\{u, v\}$ be the 2-plane spanned by the basis vectors of an orthonormal 2-frame $\{u, v\}$ in M. Then we define $\mathbf{E} = span\,\{u, v, u \times v\}$, and $\mathbf{V} = \mathbf{E}^{\perp}$. We can define the complex structure on $\mathbf{V}$ by $J_\xi(x) = x \times \xi$. □

Note that, this complex structure J_ξ extends naturally to a complex structure on V_ξ as in section 3.2.

Definition 3.3.3. Two Calabi-Yau manifolds are *mirror pairs* of each other, if their complex structures are induced from the same calibration 3-form in a G_2 manifold. Furthermore, we call them *strong mirror pairs* if their normal vector fields ξ and ξ' are homotopic to each other through non-vanishing vector fields.

Example 3.3.4. If we reconsider the examples 3.2.4 and 3.2.5 with the notions of this section, we have $\Lambda = span\,\{e_1, e_2\}$ and $\mathbf{E} = span\,\{e_1, e_2, e_3 = e_1 \times e_2\}$, $\mathbf{V} = span\,\{e_4, e_5, e_6, e_7\}$. For $\xi = e_3$ and $\xi' = e_7$ we obtained two different complex structures on $\mathbb{T}^6$, which are by definition mirror pairs of each other. In this case the notion of mirror duality in a G_2 manifold and the famous notion of *mirror symmetry* ($h^{1,1}(V_\xi) = h^{2,1}(V_{\xi'})$ and $h^{1,1}(V_{\xi'}) = h^{2,1}(V_\xi)$) coincides.

Example 3.3.5. In [AES], we considered a Joyce manifold of the first kind, more specifically $J(1/2, 0, 0, 1/2, 1/2)$, and we found a pair of Borcea-Voisin 3-folds with Hodge numbers $h^{1,1} = h^{2,1} = 19$ as mirror pairs, which are again mirror symmetric. This example will be considered in a more general setting in chapter 4 in which we will see that mirror pairs are not always mirror symmetric.

3.4 Joyce Manifolds

Joyce's construction of 7-manifolds with holonomy G_2 is similar to the Kummer construction for K3 surfaces, and he calls his construction as the *generalized Kummer construction*. For more details reader is referred to [J1, J2, J3]. He starts with a flat Riemannian 7-torus $\mathbb{T}^7$, divide it by the action of a finite group, Γ, of automorphisms of $\mathbb{T}^7$ (not an arbitrary group but one that gives nice singularities), then resolves the singularities to obtain the 7-manifold M. Then he shows the existence of a G_2 metric on this manifold. Let's give a little more detail on this construction.

Definition 3.4.1. Let $\mathbb{T}^n$ be an n-torus with a flat Riemannian metric. Let Γ be a finite group of isometries of $\mathbb{T}^n$. Let S be the singular set of $\mathbb{T}^n/\Gamma$. Let M be a compact, smooth n-manifold and $\Phi : M \to \mathbb{T}^n/\Gamma$ be a surjective continuous map that is smooth except at S. The quadraple $(\mathbb{T}^n, \Gamma, M, \Phi)$ is called a *generalized Kummer construction* if it has the following properties:

i) Φ is injective on $\Phi^{-1}(M - S)$,

ii) $\Phi^{-1}(S)$ is a finite union of compact submanifolds of M, and

iii) for each $s \in S$, $\Phi^{-1}(s)$ is a connected, simply-connected, finite union of compact submanifolds of M.

To avoid "bad" types of singularities where two or more singular submanifolds of S intersect, we put a condition on Γ:

Condition*. Let Γ be a finite group of orientation preserving isometries of $\mathbb{T}^n$. Suppose that whenever γ_1, γ_2 are non-identity elements of Γ that have fixed points in $\mathbb{T}^n$, then either $\gamma_1\gamma_2$ has no fixed points in $\mathbb{T}^n$ or $\gamma_1 = \gamma_2^k$ for some $k \in \mathbb{Z}$.

This condition on Γ guarantees the following (Lemma 2.1.3 of [J2]):

Lemma 3.4.2. *S is a disjoint union of connected components, and each connected component has a neighborhood isometric to a neighborhood of the singular set in $(\mathbb{T}^{n-2l} \times (\mathbb{R}^{2l}/\mathbb{Z}_p))/F$. Here l is a positive integer with $2l \leq n$, $\mathbb{Z}_2$ is a nontrivial cyclic subgroup of $SO(2l)$ acting freely on $\mathbb{R}^{2l} - \{0\}$, and F is a finite group of isometries of $\mathbb{T}^{n-2l} \times (\mathbb{R}^{2l}/\mathbb{Z}_p)$ that acts freely on $\mathbb{T}^{n-2l}$.*

Definition 3.4.3. Let G be a finite subgroup of $SO(n)$ that acts freely on $\mathbb{R}^n - \{0\}$ (so n is even, otherwise G has fixed points). An ALE (*asymptotically locally Euclidean*) space X is a complete Riemannian manifold with one end modeled on the end of $\mathbb{R}^n/G$, such that the metric g on X is asymptotic to the Euclidean metric h on $\mathbb{R}^n/G$ in the sense that,

There exists a surjective continuous map $\phi : X \to \mathbb{R}^n/G$ that is smooth in the appropriate sense, such that $\phi^{-1}(0)$ is a connected, simply-connected, finite union of compact submanifolds of X and ϕ induces a diffeomorphism from $X - \phi^{-1}(0)$ to $(\mathbb{R}^n - \{0\})/G$ such that

$\phi_*(g) - h = O(r^{-4})$, $\partial\phi_*(g) = O(r^{-5})$, $\partial^2\phi_*(g) = O(r^{-6})$ for large r,

where r is the distance from the origin in $\mathbb{R}^n/G$ and ∂ is the flat connection on $\mathbb{R}^n/G$.

To construct M, Joyce uses $\mathbb{T}^{n-2l} \times X$ to desingularize the singular component modeled on $\mathbb{T}^{n-2l} \times (\mathbb{R}^{2l}/\mathbb{Z}_p)$ where X is an ALE space for the group $\mathbb{Z}_p \subset SO(2l)$. In his paper, he considers only the cases $n = 7$ and $l = 2$ or 3. And he proves the following (Theorem 2.2.2 of [J2]):

Theorem 3.4.4. Let $\widehat{\varphi}$ be a flat G_2 structure on $\mathbb{T}^7$, and let Γ be a finite group of diffeomorphisms of $\mathbb{T}^7$ preserving $\widehat{\varphi}$. Let $S_1, ..., S_k$ be the connected components of the singular set S of $\mathbb{T}^7/\Gamma$. Suppose that for each $j = 1, ..., k$ either

(i) S_j has a neighborhood isometric to a neighborhood of the singular set of $(\mathbb{T}^3 \times \mathbb{C}^2/G_j)/F_j$, where $\mathbb{T}^3$ is a flat Riemannian torus, G_j a finite subgroup of $SU(2)$, and F_j is a group of isometries of $\mathbb{T}^3 \times \mathbb{C}^2/G_j$ acting freely on $\mathbb{T}^3$. There is an ALE space

X_j with holonomy $SU(2)$ asymptotic to $\mathbb{C}^2/G_j$, and an action of F_j on X_j such that $(\mathbb{T}^3 \times X_j)/F_j$ is asymptotic to $(\mathbb{T}^3 \times \mathbb{C}^2/G_j)/F_j$, or

(ii) S_j has a neighborhood isometric to a neighborhood of the singular set of $(\mathbb{S}^1 \times \mathbb{C}^3/G_j)/F_j$, where G_j a finite subgroup of $SU(3)$ acting freely except at 0, and F_j is a group of isometries of $\mathbb{S}^1 \times \mathbb{C}^3/G_j$ acting freely on $\mathbb{S}^1$. There is an ALE space X_j with holonomy $SU(3)$ asymptotic to $\mathbb{C}^3/G_j$, and an action of F_j on X_j such that $(\mathbb{S}^1 \times X_j)/F_j$ is asymptotic to $(\mathbb{S}^1 \times \mathbb{C}^3/G_j)/F_j$.

Then there exists a compact 7-manifold M constructed from $\mathbb{T}^7/\Gamma$ and $X_1, ..., X_k$, a positive constant θ, and a family $\{\varphi_t : t \in (0, \theta]\}$ of smooth, closed sections of $\Lambda^3_+ M$. Let g_t be the metric on M associated to φ_t. There exists a family $\{\psi_t : t \in (0, \theta]\}$ of smooth 3-forms on M with $d^*\psi_t = d^*\varphi_t$, where d^* is defined using g_t. There exist positive constants $D_1, ..., D_5$ independent of t, such that the following five conditions hold for each $t \in (0, \theta]$, where all norms are calculated using g_t:

i) $\|\psi_t\|_2 \leq D_1 t^4$ and $\|\psi_t\|_{C^2} \leq D_1 t^4$

ii) the injectivity radius $\delta(g_t)$ satisfies $\delta(g_t) \geq D_2 t$

iii) the Riemannian curvature $R(g_t)$ of g_t satisfies $\|R(g_t)\|_{C^0} \leq D_3 t^{-2}$

iv) the volume $vol(M)$ satisfies $vol(M) \geq D_4$ and

v) the diameter $diam(M)$ satisfies $diam(M) \leq D_5$

Then Joyce shows that on this manifold M, for small t one can deform φ_t (by adding $d\eta_t$ where η_t is a small 2-form) to obtain a family of torsion free G_2 structures $\tilde{\varphi}_t$, and he proves the following (Theorem 2.2.3 of [J2]):

Theorem 3.4.5. *M of the previous theorem admits a smooth family of torsion free G_2 structures of dimension $b^3(M)$.*

Chapter 4

Calabi-Yau Submanifolds of Joyce Manifolds

4.1 Joyce Manifolds of First Kind: $J(b_1, b_2, c_1, c_2, c_3)$

In this section we will consider a special family of Joyce manifolds, which is usually referred as Joyce manifolds of the first kind. Let $(x_1, ..., x_7)$ be coordinates on $\mathbb{T}^7 = \mathbb{R}^7/\mathbb{Z}^7$ where $x_i \in \mathbb{R}/\mathbb{Z}$. Define a section $\widehat{\varphi}$ of $\Lambda^3_+\mathbb{T}^7$ by:

$$\begin{aligned}\widehat{\varphi} \;&=\; dx^1 \wedge dx^2 \wedge dx^3 + dx^1 \wedge dx^4 \wedge dx^5 + dx^1 \wedge dx^6 \wedge dx^7 + dx^2 \wedge dx^4 \wedge dx^6 \\ &\quad -dx^2 \wedge dx^5 \wedge dx^7 - dx^3 \wedge dx^4 \wedge dx^7 - dx^3 \wedge dx^5 \wedge dx^6\end{aligned}$$

Let $\Gamma =< \alpha, \beta, \gamma >\cong \mathbb{Z}_2^3$ defined by:

	$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{x_4}$	$\mathbf{x_5}$	$\mathbf{x_6}$	$\mathbf{x_7}$
α	x_1	x_2	x_3	$-x_4$	$-x_5$	$-x_6$	$-x_7$
β	x_1	$-x_2$	$-x_3$	x_4	x_5	$b_1 - x_6$	$b_2 - x_7$
γ	$-x_1$	x_2	$c_1 - x_3$	x_4	$c_2 - x_5$	x_6	$c_3 - x_7$

Table 4.1: The action of generators of Γ on $\mathbb{T}^7$

where $b_i, c_i \in \{0, 1/2\}$.

Clearly α, β, γ preserves $\widehat{\varphi}$ and $\alpha^2 = \beta^2 = \gamma^2 = 1$, $\alpha\beta = \beta\alpha$, $\alpha\gamma = \gamma\alpha$, $\beta\gamma = \gamma\beta$. For any choice of b_i's and c_i's, each of α, β, γ fixes 16 copies of $\mathbb{T}^3$:

$$
\begin{aligned}
Fix(\alpha) &= \{(x_1, ..., x_7) \in \mathbb{T}^7 : \quad x_4, x_5, x_6, x_7 \in \{0, 1/2\}\} \\
Fix(\beta) &= \{(x_1, ..., x_7) \in \mathbb{T}^7 : \quad x_2, x_3 \in \{0, 1/2\}, \quad x_6 \in \{b_1/2, (1+b_1)/2\}, \\
&\quad x_7 \in \{b_2/2, (1+b_2)/2\}\} \\
Fix(\gamma) &= \{(x_1, ..., x_7) \in \mathbb{T}^7 : \quad x_1 \in \{0, 1/2\}, \quad x_3 \in \{c_1/2, (1+c_1)/2\}, \\
&\quad x_5 \in \{c_2/2, (1+c_2)/2\}, \quad x_7 \in \{c_3/2, (1+c_3)/2\}\}
\end{aligned}
$$

We need to put extra conditions on b_i's and c_i's so that Γ satisfies condition *. In other words, we don't want $\alpha\beta$, $\beta\gamma$, $\gamma\alpha$, $\alpha\beta\gamma$ to have any fixed points.

There are only 13 possible 5-tuples $(b_1, b_2, c_1, c_2, c_3)$ satisfying this condition as explained below.

	$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{x_4}$	$\mathbf{x_5}$	$\mathbf{x_6}$	$\mathbf{x_7}$
$\alpha\beta$	x_1	$-x_2$	$-x_3$	$-x_4$	$-x_5$	$b_1 + x_6$	$b_2 + x_7$
$\beta\gamma$	$-x_1$	$-x_2$	$c_1 + x_3$	x_4	$c_2 - x_5$	$b_1 - x_6$	$b_2 + c_3 + x_7$
$\gamma\alpha$	$-x_1$	x_2	$c_1 - x_3$	$-x_4$	$c_2 + x_5$	$-x_6$	$c_3 + x_7$
$\alpha\beta\gamma$	$-x_1$	$-x_2$	$c_1 + x_3$	$-x_4$	$c_2 + x_5$	$b_1 + x_6$	$b_2 - x_7$

Table 4.2: The action of mixed terms of Γ

i) If $b_1 = b_2 = 0$ then $\alpha\beta$ will have fixed points,

ii) If $c_1 = 0$ and $b_2 = c_3$ then $\beta\gamma$ will have fixed points,

iii) If $c_2 = c_3 = 0$ then $\gamma\alpha$ will have fixed points,

iv) If $c_1 = c_2 = b_1 = 0$ then $\alpha\beta\gamma$ will have fixed points.

We need to choose b_i's and c_i's so that they don't satisfy any of the above conditions i),...,iv). Condition iv) is unnecessary since if $c_1 = c_2 = b_1 = 0$ then $b_2 = 1/2$ from condition i) and hence $c_3 = 0$ from condition ii) which is impossible as long as we take care of condition iii). Now we will find all possible 5-tuples:

$$
\begin{aligned}
b_1 = b_2 = 1/2 \quad &\Longrightarrow \quad c_1 = 1/2 \textit{ or } c_3 = 0 \ (\textit{by ii}) \\
&\Longrightarrow \quad (c_1, c_2, c_3) = (1/2, 1/2, 0) \textit{ or } (1/2, 0, 1/2) \\
&\qquad\quad \textit{or } (1/2, 1/2, 1/2) \textit{ or } (0, 1/2, 0) \ (\textit{by iii}) \\
b_1 = 0, b_2 = 1/2 \quad &\Longrightarrow \quad c_1 = 1/2 \textit{ or } c_3 = 0 \ (\textit{by ii}) \\
&\Longrightarrow \quad (c_1, c_2, c_3) = (1/2, 1/2, 0) \textit{ or } (1/2, 0, 1/2) \\
&\qquad\quad \textit{or } (1/2, 1/2, 1/2) \textit{ or } (0, 1/2, 0) \ (\textit{by iii}) \\
b_1 = 1/2, b_2 = 0 \quad &\Longrightarrow \quad c_1 = 1/2 \textit{ or } c_3 = 1/2 \ (\textit{by ii}) \\
&\Longrightarrow \quad (c_1, c_2, c_3) = (1/2, 1/2, 0) \textit{ or } (1/2, 0, 1/2) \\
&\qquad\quad \textit{or } (1/2, 1/2, 1/2) \textit{ or } (0, 0, 1/2) \textit{ or } (0, 1/2, 1/2) \ (\textit{by iii})
\end{aligned}
$$

Therefore the only posibilities are:

$$
\begin{array}{lll}
(1/2, 1/2, 1/2, 1/2, 0), & (1/2, 1/2, 1/2, 0, 1/2), & (1/2, 1/2, 1/2, 1/2, 1/2) \\
(1/2, 1/2, 0, 1/2, 0), & (0, 1/2, 1/2, 1/2, 0), & (0, 1/2, 1/2, 0, 1/2) \\
(0, 1/2, 1/2, 1/2, 1/2), & (0, 1/2, 0, 1/2, 0), & (1/2, 0, 1/2, 1/2, 0) \\
(1/2, 0, 1/2, 0, 1/2), & (1/2, 0, 1/2, 1/2, 1/2), & (1/2, 0, 0, 0, 1/2) \\
(1/2, 0, 0, 1/2, 1/2) & &
\end{array}
$$

For the 13 cases that satisfy condition *, let's see how (the neighborhood of) the singular set looks like. Since Γ is abelian, α will preserve (setwise) $Fix(\beta)$ and $Fix(\gamma)$.

$$
\begin{aligned}
x \in Fix(\beta) \;&\Longrightarrow\; \beta\alpha(x) = \alpha\beta(x) = \alpha(x) \;\Longrightarrow\; \alpha(x) \in Fix(\beta) \\
x \in Fix(\gamma) \;&\Longrightarrow\; \gamma\alpha(x) = \alpha\gamma(x) = \alpha(x) \;\Longrightarrow\; \alpha(x) \in Fix(\gamma)
\end{aligned}
$$

Similarly β will preserve $Fix(\alpha)$ and $Fix(\gamma)$, and γ will preserve $Fix(\alpha)$ and $Fix(\beta)$.

Consider the 16 $\mathbb{T}^3$ fixed by α. Their neighborhoods in $\mathbb{T}^7/ < \alpha >$ look like $\mathbb{T}^3 \times \mathbb{C}^2/\{\pm 1\}$. As b_1 and b_2 are not 0 at the same time, the action of β on at least one of x_6 or x_7 is $x \leftrightarrow 1/2 - x$. And the fixed $\mathbb{T}^3$'s of α have $x_6, x_7 \in \{0, 1/2\}$. Hence β pairs 8 of these $\mathbb{T}^3$'s with the remaining 8. As c_2 and c_3 are not 0 at the same time, the action of γ on at least one of x_5 or x_7 is $x \leftrightarrow 1/2 - x$. And the fixed $\mathbb{T}^3$'s of α have $x_5, x_7 \in \{0, 1/2\}$. Hence γ pairs 8 of these $\mathbb{T}^3$'s with the remaining 8. So the contribution to the singular set of $\mathbb{T}^7/\Gamma$ coming from $Fix(\alpha)$ (with neighborhoods) is either 4 copies of $\mathbb{T}^3 \times \mathbb{C}^2/\{\pm 1\}$ or 8 copies of $(\mathbb{T}^3 \times \mathbb{C}^2/\{\pm 1\})/ < \beta\gamma >$. The latter is the case $b_1 = 0$, $b_2 = 1/2$, $c_2 = 0$, $c_3 = 1/2$, when β and γ both changes only x_7 on $Fix(\alpha)$. Note that in this case since $b_2 = c_3$ we must have $c_1 = 1/2$.

Similar observations can be made on $Fix(\beta)$ and $Fix(\gamma)$. Consider the neighborhoods $\mathbb{T}^3 \times \mathbb{C}^2/\{\pm 1\}$ of the 16 fixed $\mathbb{T}^3$'s in $\mathbb{T}^7/ < \beta >$, the $\mathbb{C}^2$ part coming from coordinates x_2, x_3, x_6, x_7. The action of α on x_6 and x_7 is $x \leftrightarrow -x$. Since b_1 and b_2 are not 0 at the same time, α will change at least one of x_6 or x_7 on $Fix(\beta)$. The action of γ on x_3 is $x \leftrightarrow c_1 - x$ and on x_7 it is $x \leftrightarrow c_3 - x$. So γ changes x_3 on $Fix(\beta)$ if and only if $c_1 = 1/2$ and changes x_7 on $Fix(\beta)$ if and only if $b_2 + c_3 = 1/2$. Hence the contribution from $Fix(\beta)$ is either 4 copies of $\mathbb{T}^3 \times \mathbb{C}^2/\{\pm 1\}$ or 8 copies of $(\mathbb{T}^3 \times \mathbb{C}^2/\{\pm 1\})/ < \gamma\alpha >$. The latter is the case $b_1 = 0$, $b_2 = 1/2$, $c_1 = 0$, $c_3 = 0$, when α and γ both changes only x_7 on $Fix(\beta)$. Note that in this case since $c_3 = 0$ we must have $c_2 = 1/2$.

Finally, consider the neighborhoods $\mathbb{T}^3 \times \mathbb{C}^2/\{\pm 1\}$ of the 16 fixed $\mathbb{T}^3$'s in $\mathbb{T}^7/ < \gamma >$, the $\mathbb{C}^2$ part coming from coordinates x_1, x_3, x_5, x_7. The action of α on x_5 and x_7 is $x \leftrightarrow -x$. Since c_2 and c_3 are not 0 at the same time, α will change at least one of x_5 or x_7 on $Fix(\gamma)$. The action of β on x_3 is $x \leftrightarrow -x$ and on x_7 it is $x \leftrightarrow b_2 - x$. So β changes x_3 on $Fix(\gamma)$ if and only if $c_1 = 1/2$ and changes x_7 on $Fix(\gamma)$ if and only if $b_2 + c_3 = 1/2$. Hence the contribution from $Fix(\gamma)$ is either 4 copies of $\mathbb{T}^3 \times \mathbb{C}^2/\{\pm 1\}$ or 8 copies of $(\mathbb{T}^3 \times \mathbb{C}^2/\{\pm 1\})/ < \gamma\alpha >$. The latter is the case $c_2 = 0$, $c_3 = 1/2$, $c_1 = 0$, $b_2 = 0$, when α and β both changes only x_7 on $Fix(\gamma)$. Note that in this case since $b_2 = 0$ we must have $b_1 = 1/2$.

Therefore, the contributions to singular set (with neighborhoods) for each case is given by:

case	**from** $Fix(\alpha)$	**from** $Fix(\beta)$	**from** $Fix(\gamma)$
$(0, 1/2, 0, 1/2, 0)$	$4\mathbb{T}^3 \times \mathbb{C}^2/\pm$	$8(\mathbb{T}^3 \times \mathbb{C}^2/\pm)/\mathbb{Z}_2$	$4\mathbb{T}^3 \times \mathbb{C}^2/\pm$
$(0, 1/2, 1/2, 0, 1/2)$	$8(\mathbb{T}^3 \times \mathbb{C}^2/\pm)/\mathbb{Z}_2$	$4\mathbb{T}^3 \times \mathbb{C}^2/\pm$	$4\mathbb{T}^3 \times \mathbb{C}^2/\pm$
$(1/2, 0, 0, 0, 1/2)$	$4\mathbb{T}^3 \times \mathbb{C}^2/\pm$	$4\mathbb{T}^3 \times \mathbb{C}^2/\pm$	$8(\mathbb{T}^3 \times \mathbb{C}^2/\pm)/\mathbb{Z}_2$
other 10	$4\mathbb{T}^3 \times \mathbb{C}^2/\pm$	$4\mathbb{T}^3 \times \mathbb{C}^2/\pm$	$4\mathbb{T}^3 \times \mathbb{C}^2/\pm$

Table 4.3: Fixed point sets for all possible 5-tuples

To obtain the smooth 7-manifold underlying the Joyce manifold, Joyce uses *Eguchi-Hanson space* (we will call it X) which is a complete hyperkähler metric on $T^*\mathbb{C}P^1$. So he replaces the neighborhoods (of the singular sets) of the form $\mathbb{T}^3 \times \mathbb{C}^2/\{\pm 1\}$ by $\mathbb{T}^3 \times X$ and $(\mathbb{T}^3 \times \mathbb{C}^2/\{\pm 1\})/\mathbb{Z}_2$ by $(\mathbb{T}^3 \times X)/\mathbb{Z}_2$. The former resolution is the unique crepant resolution, yet the latter can be made in topologically two different ways: depending on the induced action of $\mathbb{Z}_2$ on $[\mathbb{C}P^1] \in H^2(T^*\mathbb{C}P^1)$, which can be chosen as either id or $-id$.

4.2 Calabi-Yau Submanifolds

In this section we describe Calabi-Yau submanifolds of Joyce manifolds of the first kind that are obtained by the construction of Akbulut-Salur that we mentioned in the previous chapter, by choosing ξ to be the direction corresponding to each coordinate of $\mathbb{T}^7$ separately.

In the following diagrams, for the left diagram $\mathbb{T}^7 = S^1 \times \mathbb{T}^6$ where S^1 is the base

$\mathbb{T}^6$ is the fiber, empty circles are fixed $\mathbb{T}^3$'s of α, β or γ, filled circles are $\mathbb{T}^2$'s (parts of fixed $\mathbb{T}^3$'s on fibers). The right hand diagram is $\mathbb{T}^7/\Gamma$ as a fibration over the interval $[0, 1/2]$ or $[0, 1/4]$ depending on what we obtain as S^1/Γ (image of the base of left hand diagram) with generic fiber shown. In each case, the singular set of this generic fiber is a number of copies of $\mathbb{T}^2 \times \mathbb{C}^2/\{\pm 1\}$ which will be resolved by replacing it with $\mathbb{T}^2 \times X$ where X is the Eguchi Hanson space. Fixed sets in each graph from up to down comes from α, β and γ in this order. Numbers near the fixed sets gives the number of fixed $\mathbb{T}^3$'s or $\mathbb{T}^2$'s.

4.2.1 x_1 direction

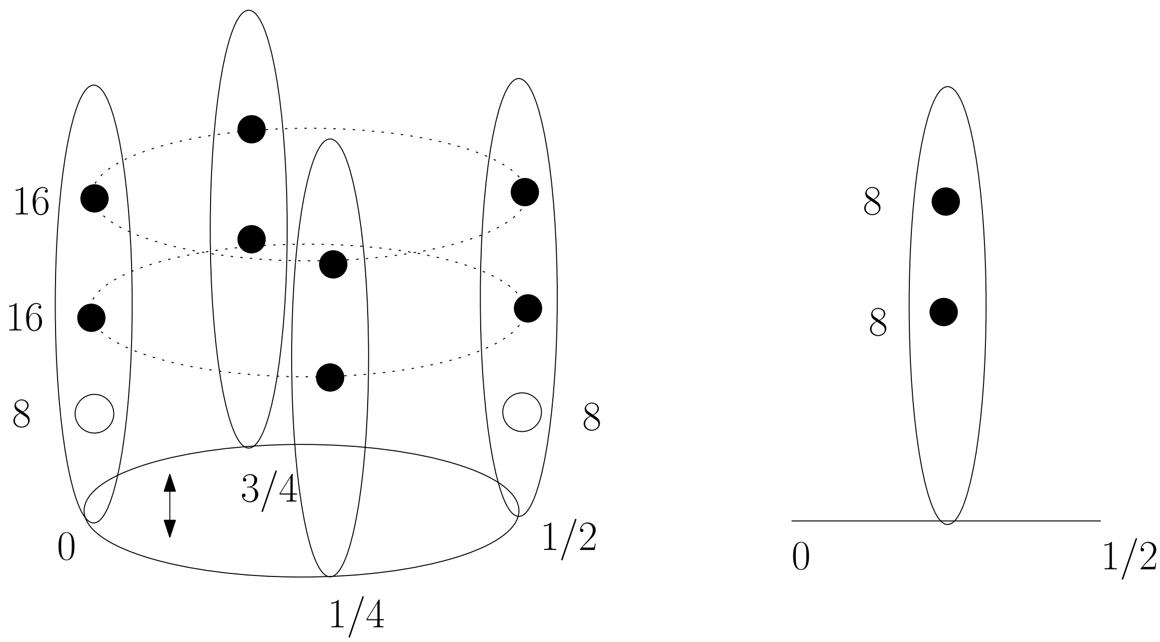

Figure 4.1: Base S^1 has x_1 coordinate

In this case (figure 4.1), the base is S^1 corresponding to x_1 coordinate. Fixed $\mathbb{T}^3$'s of α and β have x_1 component, therefore they are drawn as dotted circles on the left hand graph where these circles meet each fiber at a $\mathbb{T}^2$. Fixed $\mathbb{T}^3$'s of γ lies on the two fibers $x_1 = 0$ and $x_1 = 1/2$. The action of γ on x_1 is $x_1 \leftrightarrow -x_1$ so it will pair the fibers as shown in the graph, fixing the fibers over $x_1 = 0$ and $x_1 = 1/2$. Therefore, after taking the quotient we obtain a fibration over the interval $[0, 1/2]$ (right hand diagram in figure 4.1) with fibers over $(0, 1/2)$ equal to $\mathbb{T}^6/ < \alpha, \beta >$, whose fixed set is now 16 copies of $\mathbb{T}^2$, 8 from α and 8 from β, since α pairs the fixed set of β and β pairs the fixed set of α.

	$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{x_4}$	$\mathbf{x_5}$	$\mathbf{x_6}$	$\mathbf{x_7}$
α	x_2	x_3	$-x_4$	$-x_5$	$-x_6$	$-x_7$
β	$-x_2$	$-x_3$	x_4	x_5	$b_1 - x_6$	$b_2 - x_7$

Table 4.4: The action of $< \alpha, \beta >$ on $\mathbb{T}^6$

Let N be the corresponding resolution of $\mathbb{T}^6/ < \alpha, \beta >$ in M. We can think of the resolution that gives N in two steps. First, resolve $\mathbb{T}^6/ < \alpha >= \mathbb{T}^2 \times \mathbb{T}^4/\mathbb{Z}_2$, where $\mathbb{Z}_2$ action on $\mathbb{T}^4$ (with coordinates x_4, x_5, x_6, x_7) is $\{\pm 1\}$.

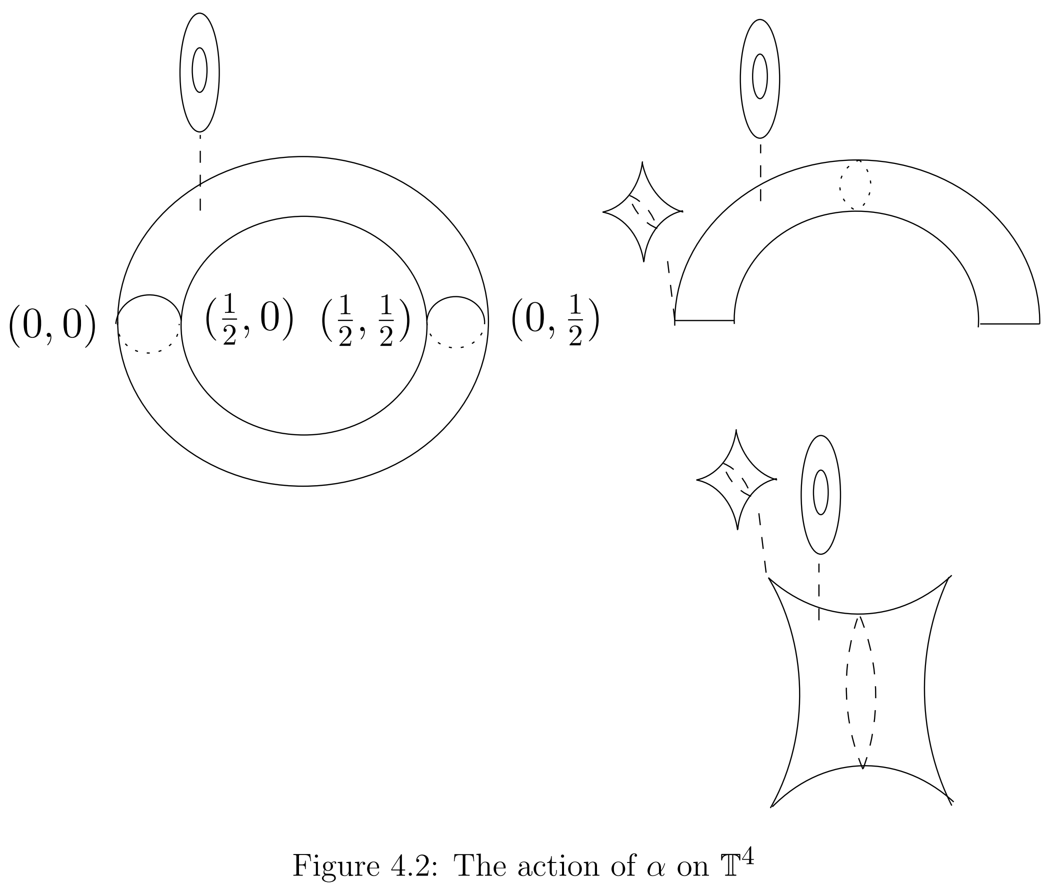

Figure 4.2: The action of α on $\mathbb{T}^4$

Figure 4.2 explains the $\mathbb{Z}_2$ action of α on $\mathbb{T}^4$. We think $\mathbb{T}^4$ as $\mathbb{T}^2 \times \mathbb{T}^2$, base $\mathbb{T}^2$

corresponding to the coordinates x_6, x_7 and fiber $\mathbb{T}^2$ corresponding to the coordinates x_4, x_5 (as in the first diagram of figure 4.2). α will fold the base making it a pillow (sphere with 4 singular points), fibers over non-singular points will be $\mathbb{T}^2$'s and fibers over singular points will be $\mathbb{T}^2/\mathbb{Z}_2$, which are again pillows (as in the diagrams on the right in figure 4.2). 16 fixed points (4 corners of pillow fibers above 4 corners of the base pillow) of this action on $\mathbb{T}^4$ will be replaced by Eguchi-Hanson spaces X (blow up of $\mathbb{C}^2$ at the origin), which gives a $K3$ surface. Hence after resolving $\mathbb{T}^6/<\alpha>$ we obtain $\mathbb{T}^2 \times K3$.

The action of β can be extended trivially on $\mathbb{T}^2 \times K3$. β will act by $\{\pm 1\}$ on $\mathbb{T}^2$ and by a holomorphic involution that acts by -1 on its holomorphic two form lifted from $dz_1 \wedge dz_2$ on $\mathbb{T}^4$ ($z_1 = x_4 + ix_5,\ z_2 = x_6 + ix_7$). The fixed set of the action of β on $K3$ consists of two copies of $\mathbb{T}^2$'s. These tori, in the last diagram of figure 4.2, are two regular fibers at two points on one of the belt circles (dotted circle on the base pillow). For example, if $(b_1, b_2) = (1/2, 1/2)$ then the fibers are at the points that are the images of $(x_6, x_7) = (1/4, 1/4), (1/4, 3/4)$.

Therefore, N is a Borcea-Voisin 3-fold as we described in section 2.4. Fixed points of the involution on $K3$ is two copies of $\mathbb{T}^2$'s as in part iii) of lemma 2.4.1, so the formula 2.4.1 with $n = n' = 2$ gives the Hodge numbers of N as $h^{1,1} = h^{2,1} = 19$.

4.2.2 x_2 direction

In this case (figure 4.3), the base is S^1 corresponding to x_2 coordinate. The generic fiber of $\mathbb{T}^7/\Gamma$ is $\mathbb{T}^6/<\alpha, \gamma>$ with the following action:

	$\mathbf{x_1}$	$\mathbf{x_3}$	$\mathbf{x_4}$	$\mathbf{x_5}$	$\mathbf{x_6}$	$\mathbf{x_7}$
α	x_1	x_3	$-x_4$	$-x_5$	$-x_6$	$-x_7$
γ	$-x_1$	$c_1 - x_3$	x_4	$c_2 - x_5$	x_6	$c_3 - x_7$

Table 4.5: The action of $<\alpha, \gamma>$ on $\mathbb{T}^6$

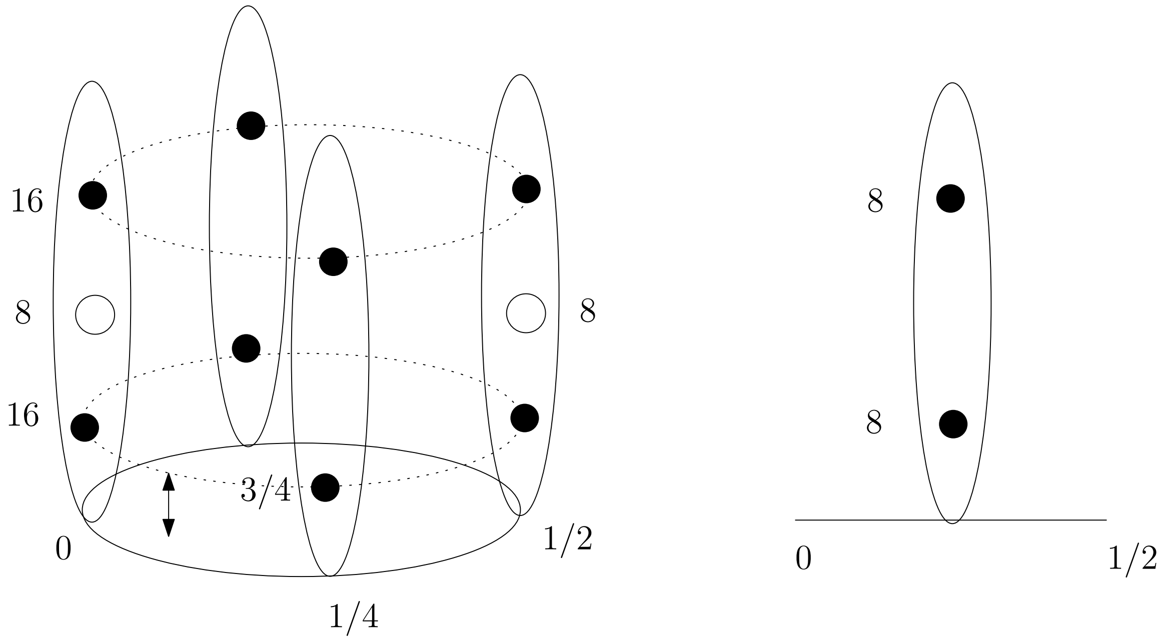

Figure 4.3: Base S^1 has x_2 coordinate

Note that, all of the analysis of the previous section can be done here, by changing the roles of β and γ. And we obtain the same manifold N, a Borcea-Voisin 3-fold with Hodge numbers $h^{1,1} = h^{2,1} = 19$.

4.2.3 x_3 direction

We will divide this case into two subcases depending on the choice of the number c_1, because it will decide the position of the fixed $\mathbb{T}^3$'s of γ.

If $c_1 = 0$

In this case (figure 4.4), the base is S^1 corresponding to x_3 coordinate. Both β and γ changes x_3 coordinate but $\beta\gamma$ fixes it. Therefore, the generic fiber is $\mathbb{T}^6/ < \alpha, \beta\gamma >$, with fixed point set being 8 copies of $\mathbb{T}^2$'s coming from the fixed points of α. The action is given by the following table (recall that from section 4.1 if $c_1 = 0$ then $b_2 \neq c_3$):

Let N be the resulting resolution of $\mathbb{T}^6/ < \alpha, \beta\gamma >$. As in section 4.2.1, we will

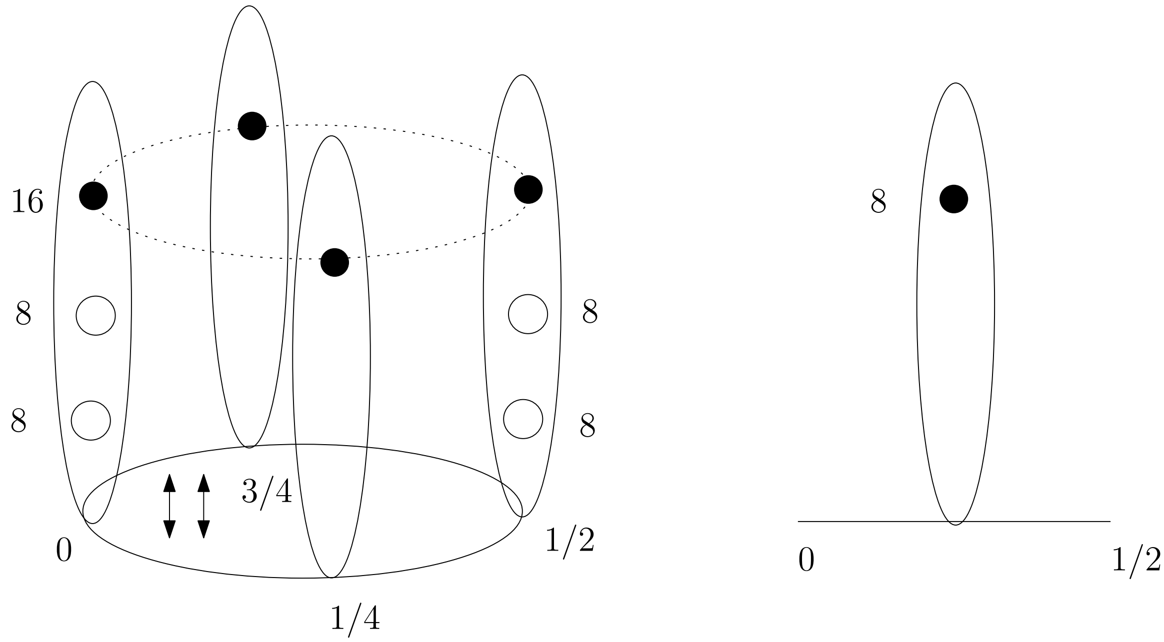

Figure 4.4: Base S^1 has x_3 coordinate, $c_1 = 0$

	$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_4}$	$\mathbf{x_5}$	$\mathbf{x_6}$	$\mathbf{x_7}$
α	x_1	x_2	$-x_4$	$-x_5$	$-x_6$	$-x_7$
$\beta\gamma$	$-x_1$	$-x_2$	x_4	$c_2 - x_5$	$b_1 - x_6$	$1/2 + x_7$

Table 4.6: The action of $< \alpha, \beta\gamma >$ on $\mathbb{T}^6$

consider this resolution in two steps. First resolve $\mathbb{T}^2 \times \mathbb{T}^4 / < \alpha >$ as before to obtain $\mathbb{T}^2 \times K3$. The action of $\beta\gamma$ on $\mathbb{T}^2$ is $\{\pm 1\}$ and on $K3$ (after we lift the action to $K3$) is an involution which acts by -1 on its holomorphic two form lifted from $dz_1 \wedge dz_2$ on $\mathbb{T}^4$ (this time $z_1 = x_5 + ix_6,\ z_2 = x_4 + ix_7$). But the difference in this case is, the action of $\beta\gamma$ on $K3$ has no fixed points, since $\beta\gamma(x_7) = 1/2 + x_7$.

Therefore, N is a Borcea-Voisin 3-fold with Hodge numbers $h^{1,1} = h^{2,1} = 11$, which are obtained from the formula 2.4.1 with $n = n' = 0$.

If $c_1 = 1/2$

In this case (figure 4.5), fixed $\mathbb{T}^3$'s of β will be in fibers at $x_3 = 0, 1/2$ and fixed $\mathbb{T}^3$'s of γ will be in fibers at $x_3 = 1/4, 3/4$. β will fold the base S^1 fixing the points 0

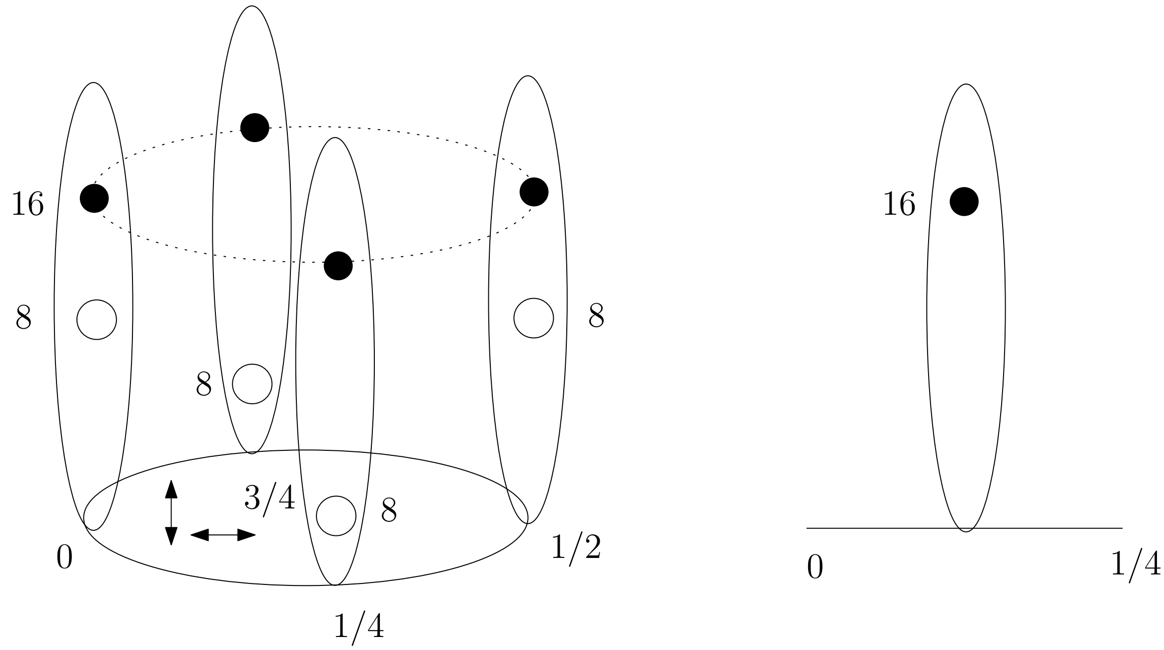

Figure 4.5: Base S^1 has x_3 coordinate, $c_1 = 1/2$

and 1/2, while γ will fold the base in the perpendicular direction, fixing 1/4 and 3/4. The resulting orbifold $\mathbb{T}^7/\Gamma$ is then will be a fibration over the interval [0,1/4] with generic fiber $\mathbb{T}^6/<\alpha>$, which is $\mathbb{T}^2 \times \mathbb{T}^4/\{\pm 1\}$. The resolution N of $\mathbb{T}^6/<\alpha>$ is therefore, $N = \mathbb{T}^2 \times K3$. Using the Hodge diamonds of $\mathbb{T}^2$ and $K3$:

$$\begin{array}{ccc} & 1 & \\ 1 & & 1 \\ & 1 & \end{array} \quad and \quad \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

and using the Künneth formula, we obtain the Hodge numbers of N as $h^{1,1} = h^{2,1} = 21$.

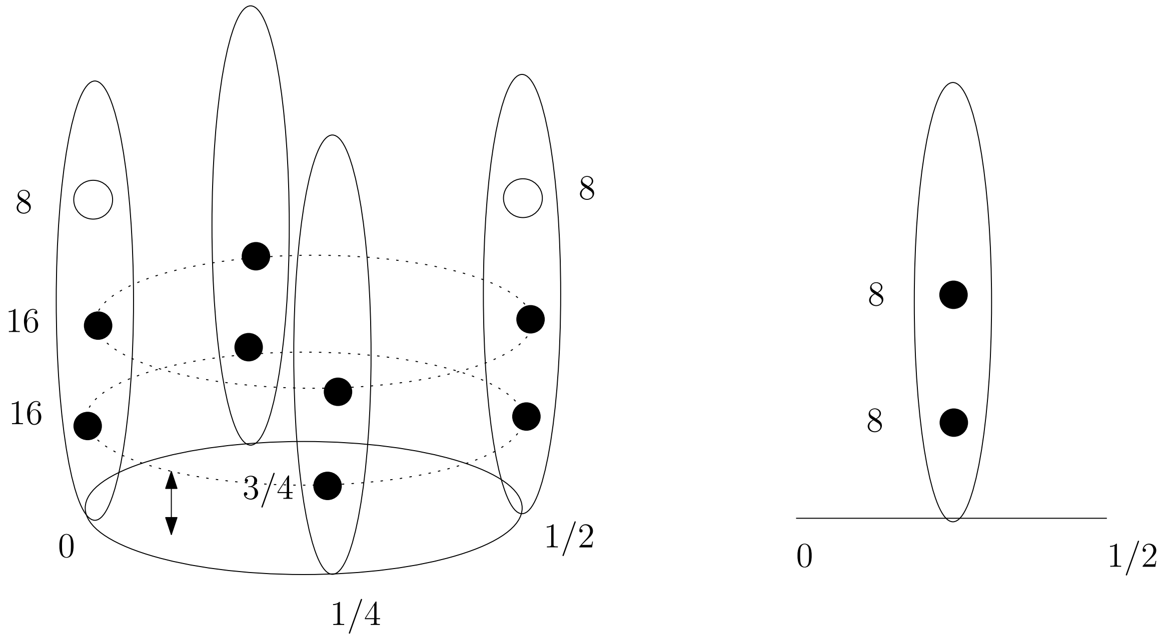

Figure 4.6: Base S^1 has x_4 coordinate

4.2.4 x_4 direction

In this case (figure 4.6), the base is S^1 corresponding to x_4 coordinate. This case is similar to x_1 direction. The generic fiber is $\mathbb{T}^6/<\beta,\gamma>$, with fixed point set being 8 copies of $\mathbb{T}^2$'s coming from the fixed points of β and 8 copies coming from the fixed points of γ. The resolution N of $\mathbb{T}^6/<\beta,\gamma>$ is again a Borcea-Voisin 3-fold with Hodge numbers $h^{1,1}=h^{2,1}=19$.

4.2.5 x_5 direction

We have two subcases depending on the choice of c_2.

If $c_2 = 0$

In this case (figure 4.7), generic fiber of $\mathbb{T}^7/\Gamma$ is $\mathbb{T}^6/<\beta,\alpha\gamma>$ similar to the $c_1=0$ case of the x_3 direction. $\mathbb{T}^6/<\beta>=\mathbb{T}^2\times\mathbb{T}^4/\{\pm1\}$ where $\mathbb{T}^4$ has coordinates x_2,x_3,x_6,x_7. After we obtain $\mathbb{T}^2\times K3$, the action of $\alpha\gamma$ on $K3$ will have no fixed

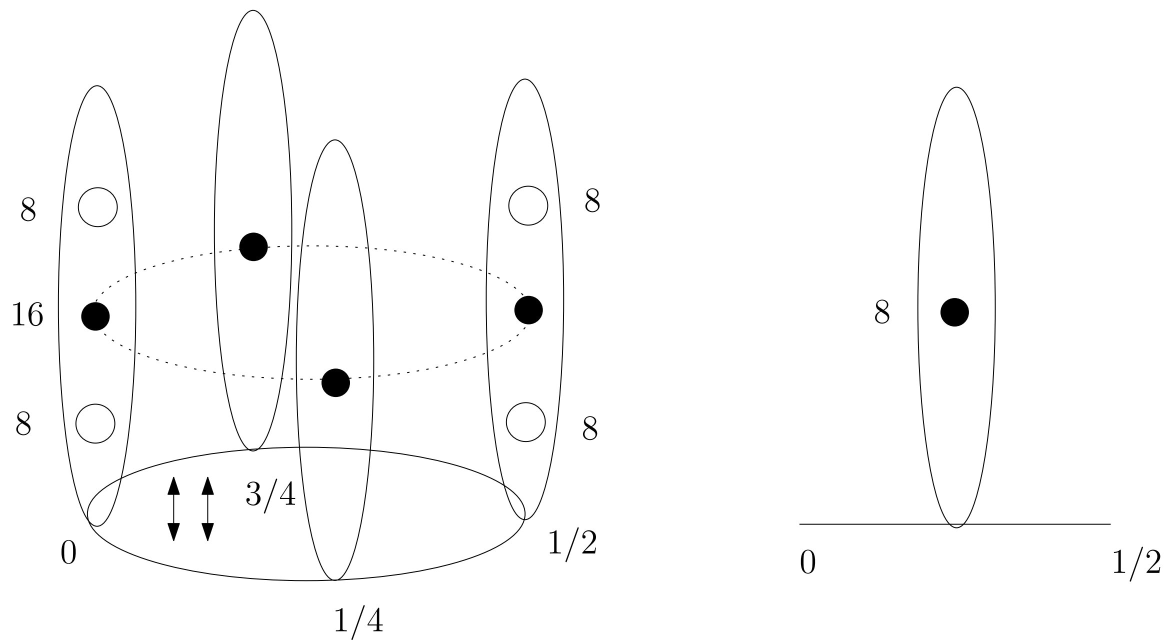

Figure 4.7: Base S^1 has x_5 coordinate, $c_2 = 0$

points, because from section 4.1; c_2 and c_3 can not be 0 at the same time, therefore $\alpha\gamma$ will change x_7 coordinate. If we call N to be the resolution of $\mathbb{T}^6/ < \beta, \alpha\gamma >$, then N is a Borcea-Voisin 3-fold with Hodge numbers $h^{1,1} = h^{2,1} = 11$.

If $c_2 = 1/2$

This case (figure 4.8) is similar to $c_1 = 1/2$ case of x_3 direction. Generic fiber before resolution is $\mathbb{T}^6/ < \beta >= \mathbb{T}^2 \times \mathbb{T}^4/\{\pm 1\}$. And after resolution we obtain $N = \mathbb{T}^2 \times K3$ with Hodge numbers $h^{1,1} = h^{2,1} = 21$.

4.2.6 x_6 direction

We have two subcases depending on the choice of b_1.

If $b_1 = 0$

The generic fiber before resolution is $\mathbb{T}^6/ < \gamma, \alpha\beta >$ (figure 4.9). Similar to $c_1 = 0$ case of x_3 direction, we first divide by the action of γ and resolve the resulting orbifold

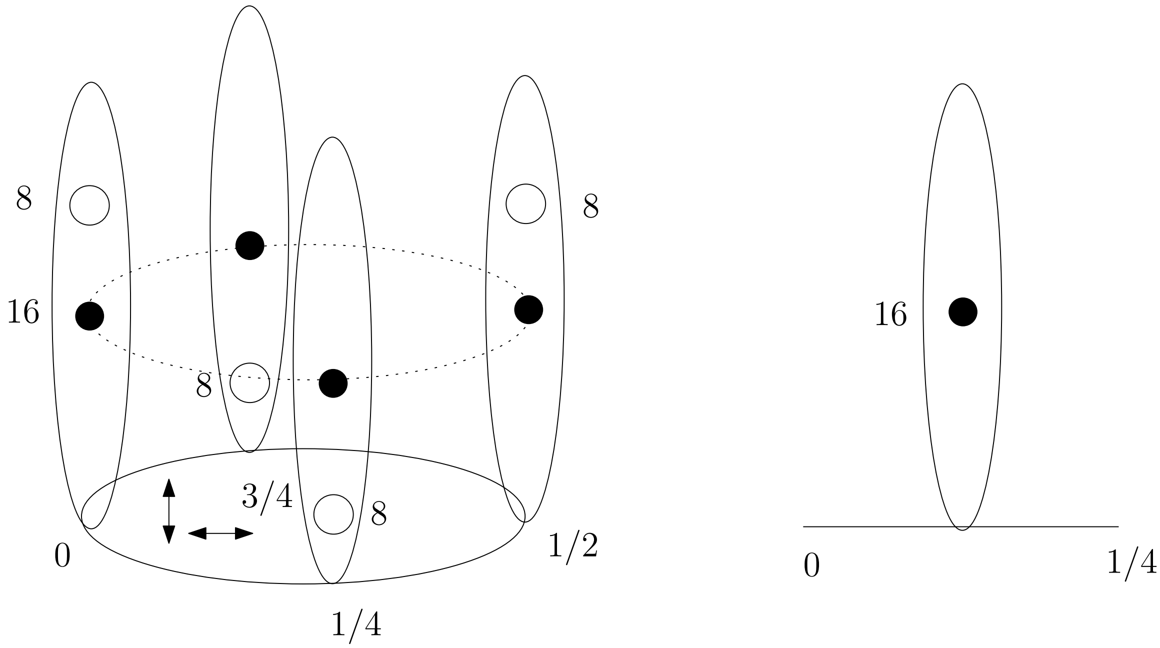

Figure 4.8: Base S^1 has x_5 coordinate, $c_2 = 1/2$

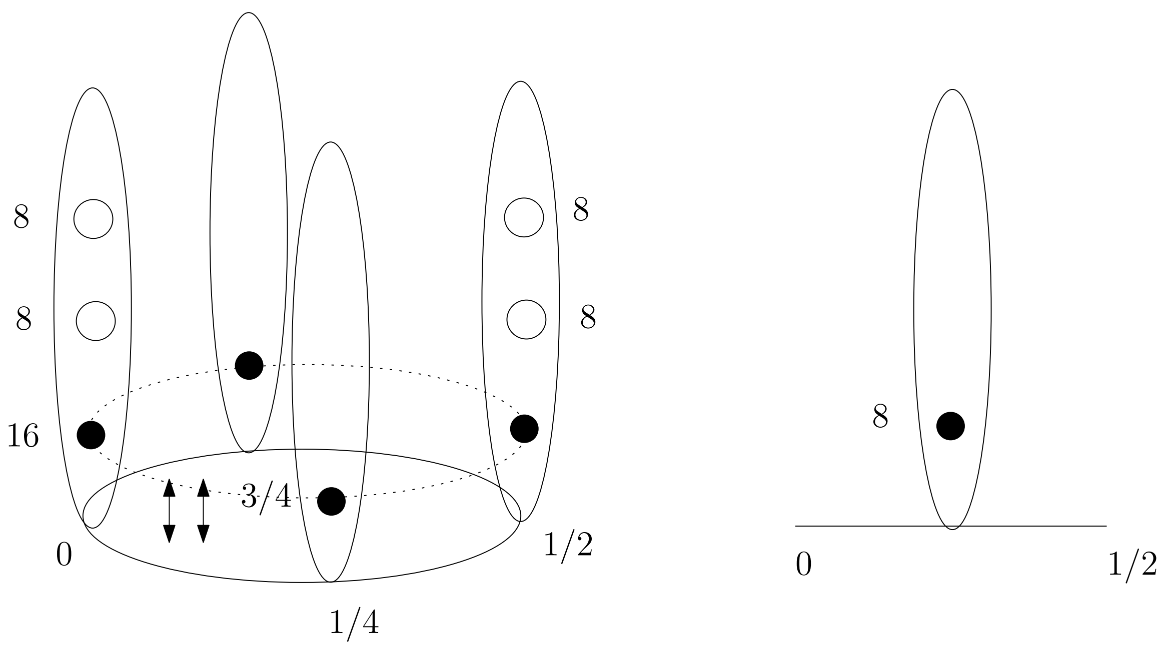

Figure 4.9: Base S^1 has x_6 coordinate, $b_1 = 0$

to obtain $\mathbb{T}^2 \times K3$. The action of $\alpha\beta$ on $K3$ has no fixed points since $\alpha\beta(x_7) = 1/2 + x_7$ (b_1 and b_2 can not be 0 at the same time). Therefore we obtain a Borcea-Voisin 3-fold N with Hodge numbers $h^{1,1} = h^{2,1} = 11$.

If $b_1 = 1/2$

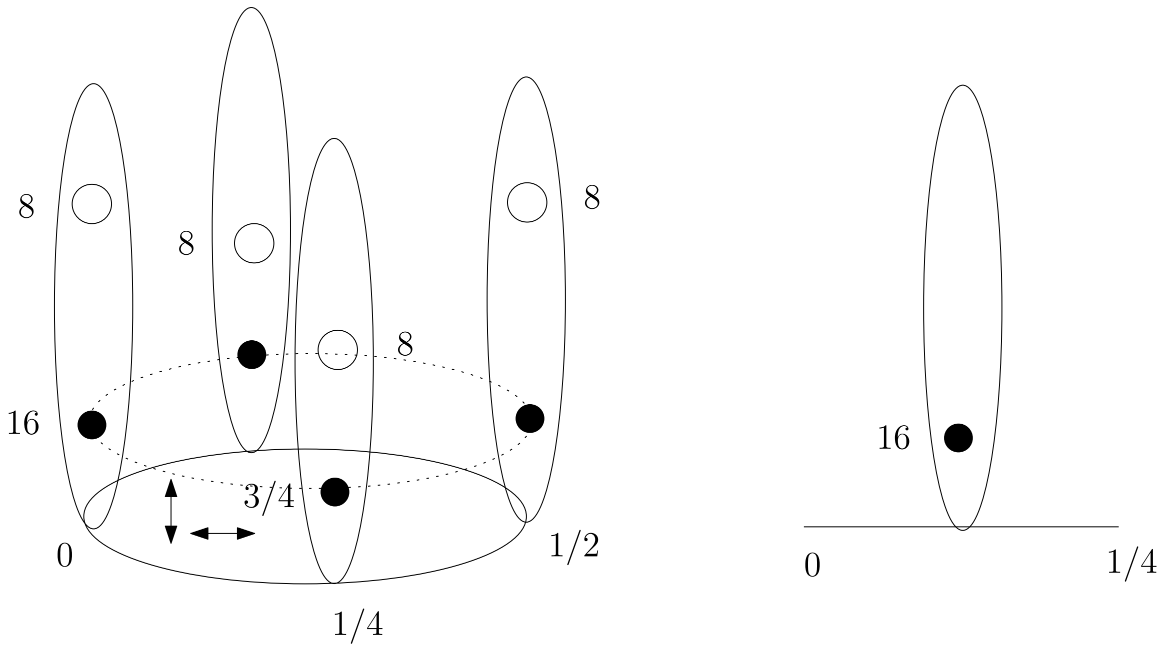

Figure 4.10: Base S^1 has x_6 coordinate, $b_1 = 1/2$

In this case (figure 4.10) generic fiber before resolution is $\mathbb{T}^6/<\gamma>$. Similar to the case $c_1 = 1/2$ in of x_3 direction, we obtain $N = \mathbb{T}^2 \times K3$ with Hodge numbers $h^{1,1} = h^{2,1} = 21$.

4.2.7 x_7 direction

We have two cases: either $(b_2, c_3) = (0, 1/2)$ or $(b_2, c_3) = (1/2, 0)$. In both cases (see figures 4.11 and 4.12) The fixed $\mathbb{T}^3$'s of α, β and γ lie on the fibers at $x_7 = 0, 1/4, 1/2, 3/4$. Therefore the generic fiber of $\mathbb{T}^7/\Gamma$ will be $\mathbb{T}^6/<\alpha\beta>= \mathbb{T}^6$ and $\mathbb{T}^6/<\alpha\gamma>= \mathbb{T}^6$ respectively. So $N = \mathbb{T}^6$ and has Hodge numbers $h^{1,1} = h^{2,1} = 9$.

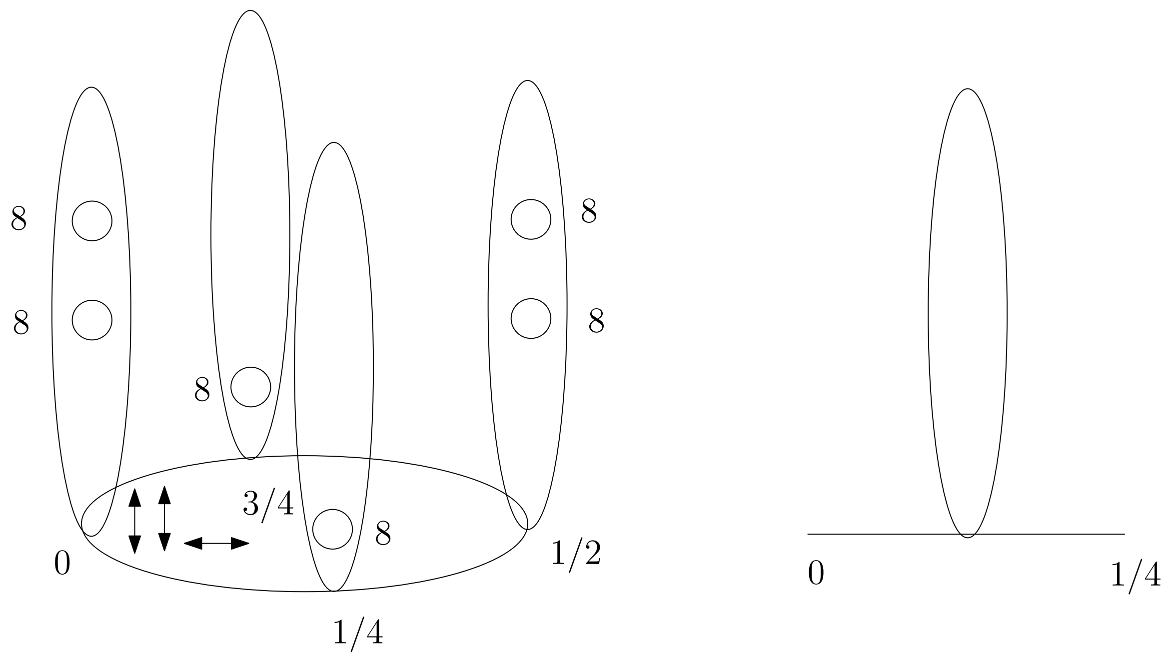

Figure 4.11: Base S^1 has x_7 coordinate, $b_2 = 0$

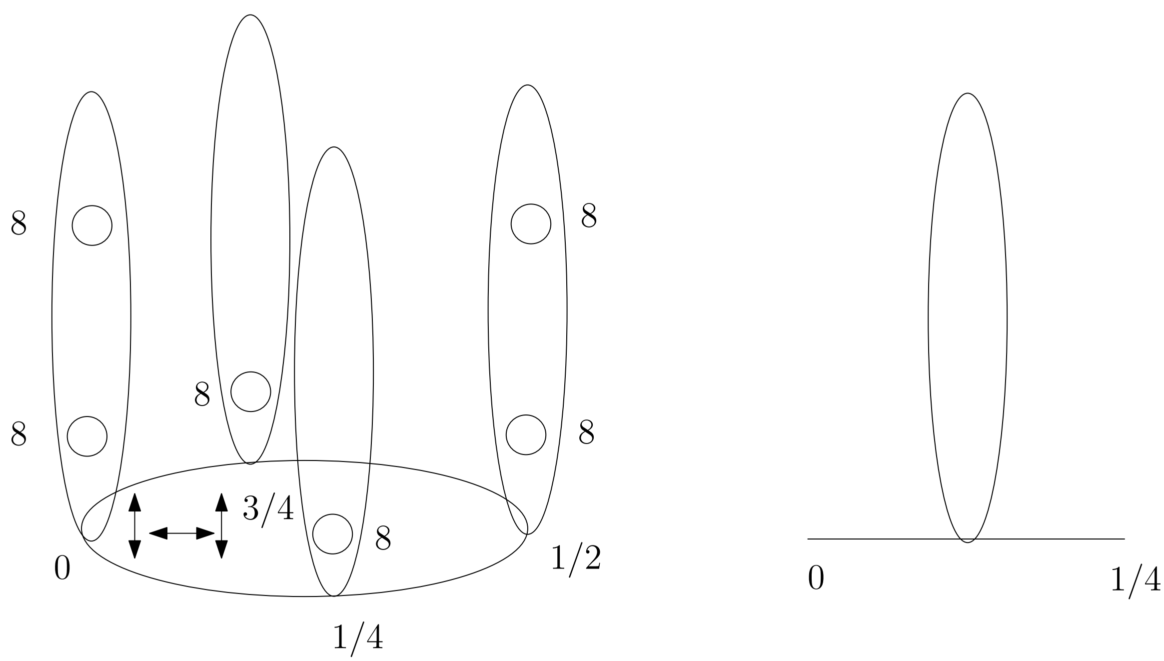

Figure 4.12: Base S^1 has x_7 coordinate, $b_2 = 1/2$

4.3 Summary of results

To sum up the results so far, as we mentioned in example 3.3.5 this work has covered the example of our joint work with Akbulut and Salur [AES], in which we obtained a pair of Borcea-Voisin manifolds with Hodge numbers $h^{1,1} = h^{2,1} = 19$, as special cases of sections 4.2.1 and 4.2.4. In addition, we have obtained another Borcea-Voisin manifold with Hodge numbers $h^{1,1} = h^{2,1} = 11$, as well as Calabi-Yau manifolds $\mathbb{T}^2 \times K3$ and $\mathbb{T}^6$ as submanifolds of Joyce manifolds of the first kind. For example, if we consider $J(1/2, 0, 0, 1/2, 1/2)$, x_1, x_2, and x_4 directions give Borcea-Voisin manifolds with Hodge numbers $(19, 19)$, x_3 direction gives a Borcea-Voisin manifold with Hodge numbers $(11, 11)$, x_5 and x_6 directions give $\mathbb{T}^2 \times K3$, and x_7 direction gives $\mathbb{T}^6$. So all of these Calabi-Yau 3-folds are mirror pairs as in the definition 3.3.3, not all pairs are mirror symmetric, but each 3-fold is self-mirror.

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