DIGITAL RESOURCES AND MATHEMATICS:
TEACHERS' CONCEPTIONS AND NOTICING

By

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## ABSTRACT

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Mathematics teachers at all levels are expected ever more frequently to integrate use of emergent technologies (e.g., calculators, software, online tools, device applets) in mathematics teaching and learning. Too often technology is used for technology's sake, rather than in intentional ways to support mathematical reasoning, sense-making, and understanding. Teacher decision-making is necessarily complex as they consider the complex interactions of mathematics, technology, and learning, especially attending to affordances and limitations of technologies in relation to mathematical goals, rigor, representation, and interaction. It is critical to consider ways in which teachers can be supported in developing decision-making in locating, designing, adapting, and integrating technologies for effective mathematics teaching and learning. In this study, I explore teachers' conceptions of the nature of mathematics, and mathematics teaching and learning, teachers' noticing of characteristics of online tools and resources, and potential relationships between their conceptions and noticings. I also describe implications for mathematics education and directions for future research.

This study is a qualitative analysis of cases, with six focus teachers: Breanna, Melinda, Jana, Nicole, Heidi, and Alison. Breanna, Melinda, and Jana were grouped in a lower elementary grades team in the course, Nicole and Heidi in an upper elementary team, and Melinda in a middle grades team. Teachers varied in length of teaching experience: Breanna (2 years), Melinda (3), Jana (16), Nicole (5), Heidi (6), and Alison (4). Nicole, Heidi, and Alison had a mathematics credential, while Breanna, Melinda, and Jana had not.

Regarding the three conceptions of the nature of mathematics described by Thompson (1992) and Ernest (1989b), findings of the study included the necessity of expanding meanings of aspects of conceptions and focusing expanded aspects to include school mathematics. Meanings of aspects

of mathematical conceptions should reflect teachers' meanings of those aspects as well as reflecting researchers' meanings. All teachers showed some evidence of an Instrumentalist conception, but Breanna and Melinda most strongly showed alignment with an Instrumentalist conception. Jana, Nicole, Alison, and Heidi showed close alignment with a Platonist conception. Alison additionally indicated alignment with a Problem-Driven view. Teachers showed rich understanding of aspects of the nature of mathematics that are excluded from Ernest's framework.

In considering teachers' noticing of mathematical aspects of online tools, patterns across levels of sophistication and alignment with conceptions emerged. Noticing patterns emerged that supported past research on levels of sophistication in noticing structures of mathematical tasks and aspects of mathematics curricula. For example, teachers with fewer years of experience attended to general goals more easily than specific goals as described by M. G. Sherin and Drake (2009). Teacher with fewer years of experience also attended to non-mathematical aspects (e.g., visuals, reasons for engagement) of online tools more frequently than mathematical aspects (B. Sherin & Star, 2011; M. G. Sherin & van Es, 2009). All teachers expressed expectations for more online tools to provide feedback on correctness than to provide opportunities for interaction with mathematical consequences.

Relationships between teachers' conceptions of the nature of mathematics and their noticing of aspects of online tools emerged through teachers' focus on three aspects of online tools: learning goals, strategies for learning, and knowledge structures. For each, language used by teachers was considered in tandem with descriptions of relationships between Instrumentalist and Platonist conceptions of the nature of mathematics and approaches to teaching (Thompson, 1992; Ernest, 1989b). Breanna's noticing showed close alignment with an Instrumentalist conceptions, while Nicole's and Alison's showed close alignment with a Platonist conception. Jana, Melinda, and Heidi showed some variation in their alignments and showed some balance between alignment with Instrumentalist and with Platonist conceptions.

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# CHAPTER 1

# INTRODUCTION

Gaps between the educational promise and practical reality of classroom use of emergent technologies have been noted and analyzed (e.g., Angeli & Valanides, 2005, 2009; Niess, 2005; Reich, Murnane, & Willett, 2012). Barriers obstructing teachers in elementary, middle, and secondary grades when attempting to begin integrating technologies into teaching and learning have been identified and explored (e.g., Hew & Brush, 2007). To support teachers in overcoming these gaps and barriers, efforts have been made to both (a) supply classrooms with necessary technologies and (b) supply teachers with appropriate professional development opportunities. New strategies continue to evolve as teachers of mathematics, mathematics educators, and mathematics education researchers work to develop new understandings of the complex interaction of teachers' knowledge and beliefs about mathematics, technology, and pedagogy.

With each new wave of electronic technologies comes a new cycle in which a few pioneers first assert educational possibilities and then "poor progression of the use in schools has to be recognised and unexpected difficulties become evident... But, as the viability of technology is questioned, a new generation of technology appears and pioneers can again work on new assumptions" (Lagrange, Artigue, Laborde, & Trouche, 2003). These waves of promise and disappointment have occurred in an ongoing way over almost a full century (Lagrange et al., 2003). Most recently, the promise of classroom Internet use has arisen. One clear barrier to its use was lack of access to the Internet and to Internet-ready devices. Cuban (2001) found that access to Internet technology has increased overall for teachers and classrooms in the United States, even if, as Bracey Sutton (2014) reported, in many schools this access is not available in all classrooms or is not seamlessly available. Despite the increase in access to Internet technology, researchers have noted that teachers continue to struggle to integrate technology successfully into their teaching (Angeli & Valanides, 2005; Niess, 2005; Angeli & Valanides, 2009; Reich et al., 2012).

Hew and Brush (2007) identified six types of barriers that can keep teachers in elementary, middle, and secondary grades from integrating technology successfully in their teaching: resources, institutional, subject culture, attitudes and beliefs, knowledge and skills, and assessment. Two strategies Hew and Brush proposed for overcoming these barriers are to conduct professional development to support teachers' development of knowledge and skills and to support changes in teachers' attitudes and beliefs. Despite such professional development, many K-12 teachers still struggle to identify and plan for changes brought by the complex interaction of technology, mathematical content, and mathematics pedagogy that result from integrating digital resources (e.g., instructional resources such as lesson plans or instructional activities available online even if their implementation does not require a device) or digital tools (e.g., applets, apps, software, or other lesson plans or instructional activities whose implementation depends on a device such as a computer or iPad) (Bos, 2011; De Olviera, 2010; Harris, Mishra, & Koehler, 2009; Kramarski & Michalsky, 2010; Richardson, 2009).

One concern has been that professional development supporting teachers' use of technology has, at times, focused on only one type of knowledge (e.g., developing teachers' technical skills) but the multiple knowledge types required for successful integration do not develop as individual elements; rather, they develop as integrated components of a complex system (Angeli & Valanides, 2005; Koehler, Mishra, Hershey, & Peruski, 2004; Koehler & Mishra, 2005a, 2005b; Niess, 2005). To address teachers' needs for better targeted PD opportunities, many research projects began focusing on developing "deeper understandings of the nuances and complexities of technology education" (Koehler & Mishra, 2005a, p.95).

To focus attention on the different types of knowledge and the complexity involved, several research groups proposed extensions of Shulman's (1986) teacher knowledge framework to include additional types of knowledge that teachers needed support in developing: technology knowledge, technology content knowledge, technological pedagogical knowledge, and technological pedagogical content knowledge (Angeli & Valanides, 2005; Koehler et al., 2004; Koehler & Mishra, 2005a, 2005b; Niess, 2005). This framework is now referred to as the *technological pedagogical content knowledge (TPACK) framework*, and researchers have used it for over a decade to draw attention to the complex interaction of technology, pedagogy, and content in teaching, especially planning for and taking advantage of changes that occur in pedagogy and content when technology is used as a classroom resource.

Many researchers have argued, however, that the TPACK categories are too "fuzzy" and the framework must be clarified or yoked with other frameworks to support its practical use for teachers' professional development (Angeli & Valanides, 2009; Archambault & Barnett, 2010; Cox & Graham, 2009; Graham, 2011). One source of such frameworks is research on teachers' decision-making in choice, analysis, and adaptation of curricular resources (e.g., M. G. Sherin & Drake, 2009; Brown, 2009; Dietiker, Males, Amador, Earnest, & Stohlmann, 2014). To better understand the nature and development of teachers' abilities to critically choose and strategically use instructional resources, other frameworks have been introduced in educational research, and in mathematics education research, including frameworks focused on teachers' and students' noticing (Erickson, 2011; Lobato, Hohensee, & Rhodehamel, 2013; B. Sherin & Star, 2011; M. G. Sherin & Drake, 2009) and teachers' adaptation or construction of curricula (Brown, 2009; Lloyd, Remillard, & Herbel-Eisenmann, 2010; Stein & Kim, 2009). Researchers have proposed that teachers' decision-making is impacted by characteristics of instructional resources that they notice which in turn is impacted by teachers' beliefs and conceptions (M. G. Sherin & Drake, 2009; Brown, 2009; Dietiker et al., 2014).

## Purpose

The purpose of this study is to explore individual teachers' decision-making with respect to integration of technology in mathematics teaching and learning, by focusing on (a) expressed conceptions of mathematics, mathematics teaching and learning, and educational use of technology, (b) noticing mathematical and pedagogical affordances and constraints of online tools, and (c) relationships between expressed conceptions and noticing.

In this project, I consider practicing teachers' evaluations of online tools to understand factors that may impact teachers' professional decision-making with respect to integration of technology in mathematics teaching and learning, namely: their conceptions of the nature of mathematics, learning mathematics, and teaching mathematics, and what they notice about the structures in online tools with respect to mathematics, pedagogy, or mathematics pedagogy. The teachers that I study were enrolled in an online masters-level course that I will refer to as the *Mathematics, Technology, and Education* course. The teachers' conceptions and the structures or features they noticed likely changed over the course of the semester. These changing conceptions and changes in what they noticed may help delineate the relationships between these conceptions, but the change itself is not the focus of this study. .

I chose to explore relationships between teachers' noticing of mathematical or pedagogical structures in online tools and teachers' conceptions of mathematics, mathematics teaching and learning, and use of technology to support mathematics teaching and learning. I chose this focus in response to a call for more research on relationships between teachers' conceptions of the nature of mathematics and their decision-making about the use of technologies in mathematics teaching and learning (e.g., Kim, Kim, Lee, Spector, & DeMeester, 2013). I argue that teachers' conceptions impact their decision-making by influencing the characteristics of technologies that they notice. I extend the argument given by Brown (2009) and Drake, Cirillo, and Herbel-Eisenmann (2009) that beliefs impact teachers' perceptions of affordances of instructional materials: I argue that teachers' conceptions of the nature of mathematics impact their noticing of mathematical or pedagogical structures in online tools.

## Research Questions

I address these research questions:

1. What conceptions of the nature of mathematics, the nature of teaching and learning mathematics, and the role of instructional use of technology are evidenced through teachers' participation in course activities?

2. What characteristics, aspects, or features of online tools emerge through teacher noticings (as evidenced through evaluation activities), either general, mathematical, or pedagogical?

3. What potential relationships emerge between teachers' conceptions and teachers' noticing as described in the previous questions?

## Overview of the Study

The study focuses on data gathered from an online course, *Mathematics, Technology, and Education,* and post-course interviews. The course was taught by myself and co-instructor: together we developed and implemented course activities, and shared grading responsibilities. Students in the course included practicing teachers who teach mainly mathematics (e.g., secondary teachers or elementary math specialists) and teachers who teach mathematics much less (i.e., as one of several broad content areas) and perhaps with less preparation (e.g., elementary teachers or special education teachers). The six central participants in this study are lower elementary, upper elementary, and middle grades teachers enrolled in one of three online Master of Teaching programs offered through a large research university. Three of the participants taught lower elementary and had not earned a mathematics credential in their teacher preparation program; the other three participants had earned mathematics credentials. Each participant participated in a 90-minute interview within two weeks of the end of the course. During the interviews, participants were asked questions about mathematics and teaching and learning mathematics. Interviews were audio and video recorded, and transcribed. Data were coded and analyzed according to frameworks discussed in the Literature Reviews sections above. Data from the interviews, and from course discussions and reflections, were analyzed according to explore emergent teachers' conceptions of the nature of mathematics and of mathematics teaching and learning based statements they made in the course and interview. Data from evaluations submitted throughout the course were analyzed for mathematical noticing with respect to teachers' responses to unstructured evaluation prompts and structured evaluation templates.

## Conceptual Issues

Conceptions and noticing are complex and not transparent. Participants, as any humans, are likely unaware of every conception or every feature that they notice and find important. Even when they are aware of a conception or a feature, the description or words they use to describe it may not communicate perfectly their thought or they may not describe it at all. Then, the act of communicating their conceptions and noticings, in response to discussion, reflection, or evaluation questions, is mediated by their use of written language, by the course context—ideas that they are "trying out" in response to readings or other course experiences, and by their beliefs about the expectations of the course, their instructors, or their classmates.

I rely on participants' language in this study. Participants vary in their access to language, however. Some participants may use more precise language at times. Some participants may have had more supportive past experiences that allow them more flexibility and accuracy in their use of language to describe external and internal phenomena. Participants also vary in their meanings and use of vocabulary; one participant may write "create mathematics" and mean something different than a classmate who wrote the same words.

Context of the course and individual units matters; we chose experiences in the course intentionally to support ways of thinking and responding. Teasing apart those expected responses from a teacher's actual conceptions or noticings is not possible. It is likely that participants responded to discussion, reflection, or evaluation prompts in ways that allow them to play around with their new ideas, rather than clearly reflective of deeply held conceptions or spontaneous noticings. Even when participants describe accurately their conceptions or noticings, previous research has shown that conceptions are complex structures that can be connected or disconnected and that different conceptions arise in different contexts. The different contexts of a course (in which a participant is a student), an interview (in which a participant is a respondent to questions), and a classroom (in which a participant is a teacher) should be expected to bring out different conceptions. It is similarly likely that the different participant-as-student and participant-as-teacher contexts will bring out different noticings.

Students enrolled in a course naturally react to their perception of expectations within a course, without necessarily reacting intentionally to those perceptions. That is, a participant may respond to prompts in ways she thinks are "correct"—that is, responses that will be met with approval—rather than ways that reflect her internal conceptions or noticings.

## Organization of the Dissertation

This dissertation is organized into eight chapters. Chapter 1 provides an overview of the research problem and study purpose. Chapter 2 is a review of related literature. Chapter 3 describes the study design and methodology. Findings are presented in Chapters 5, 6, and 7, each describing findings from the corresponding research question. Chapter 5 describes teachers' conceptions of the nature of mathematics, a modification of the original research question. Chapter6 presents teachers' noticing of mathematical aspects of online tools. Chapter 7 examines potential relationships between teachers' conceptions and teachers' noticing. Finally, in Chapter 8, I present brief summaries of findings along with discussion about implications of the findings for research and teacher education.

Chapter 1 introduces the study. The first section explores technology, conceptions, and noticing in mathematics education. In second section, the three research questions are presented along with an overview of the study, significance and limitations of the study, and this description of the organization of the dissertation.

Chapter 2 provides an overview of relevant literature. A discussion of technology is followed by literature describing the construct *conceptions*. I describe conceptions generally and then in the context of the nature of mathematics. In the next section, literature describing the evolution of noticing within mathematics education is presented. Throughout both sections, attention is given to potential relationships between teachers' conceptions and teachers' noticing, as discussed by mathematics education researchers.

Chapter 3 sets out the design and methodology of the study. I describe the *Mathematics, Technology, and Education* course, with a detailed look at the course structure and activities. I describe the study participants briefly, and each of the six central participants in detail. I describe the process of data collection from the online course and post-course interviews. I describe data analysis methods.

In Chapter 5, I present findings in response to the first research question: *What conceptions of the nature of mathematics, the nature of teaching and learning mathematics, and the role of instructional use of technology are evidenced through teachers' participation in course activities?* Due to the nature of the data and analysis, I revised this question to: *What conceptions of the nature of mathematics in the context of mathematics teaching and learning are evidenced through teachers participation in course activities and post-course interview?* I explore participants' conceptions of the nature of mathematics and mathematics teaching and learning, by analyzing statements made by participants in the course and in the post-course interview, with respect to mathematical connections, mathematical problem-solving and facts, skills, and rules, and mathematical correctness. I share emergent frameworks for each section based on teacher statements.

I present findings in response to the second research question in Chapter 6: *What characteristics, aspects, or features of online tools emerge through teacher noticings (as evidenced through evaluation activities), either general, mathematical, or pedagogical?* Due to the scope of the study, I revised the question to: *What mathematical characteristics, aspects, or features of digital tools and resources do teachers notice, as evidenced through responses to explicit evaluation questions and open-ended evaluations?* I focus on mathematical, pedagogical mathematical, and technological pedagogical mathematical noticings. I organize these findings in the chapter by considering teachers' descriptions of mathematical focus, mathematical representations, intended mathematical understanding, learning activity types, and aspects that support learning. I report findings from the participants' responses to these questions in each section and describe emergent themes and patterns.

In Chapter 7, I present findings in response to the third research question; that is, I present potential relationships between teachers' conceptions of the nature of mathematics and mathematics teaching and learning and teachers' noticing of mathematical aspects of online tools. Building on the descriptions of relationships between Instrumentalist, Platonist, and Problem-Driven conceptions and views of teaching and learning, described in Thompson (1992) and Ernest (1989b), I propose a revised framework for relationships between teachers' conceptions and noticing. I report findings of teachers' attention to these aspects, and across aspects, in relation to conclusions about their conceptions.

In Chapter 8, I offer brief summaries of key findings from Chapters 5, 6, and 7. I discuss contributions of the study, implications for teacher professional development, limitations of the study, and future directions for research.

# CHAPTER 2

# REVIEW OF LITERATURE

In this Review of Literature, I briefly describe research, frameworks, and definitions of: (a) technology and instructional use of technology; (b) teachers' conceptions of mathematics and mathematics learning and teaching; (c) teachers' noticing of mathematical and pedagogical characteristics of online tools; and (d) potential relationships between teachers' conceptions and teachers' noticing. Due to length considerations, I am selective of the examples of literature that I provide.

## Meaning of Technology

Technology as a concept encompasses a wide range of meanings, from systematic ways of knowing or thinking to artifacts created from practical application of that knowledge or thinking. For example, Merriam-Webster (2016) defined technology as: "The practical application of knowledge especially in a particular area," "a capability given by the practical application of knowledge," and "the specialized aspects of a particular field of endeavor."

Merriam-Webster (2016) added that technology is "a machine, piece of equipment, method, etc., that is created by technology." The popular meaning of technology has developed to focus for the most part on this last part of the definition: technological artifacts. Cox and Graham (2009) argued that, over time, technologies cease being thought of as technology. They offered books as an example: "books were once considered technology—a tool that was easier to use and had more capacity than a scroll" but that "after several hundred years, they are now so ubiquitous that no one thinks of them as a technology" (p. 63). Cox and Graham proposed replacing the term *technologies* with *emerging technologies* to emphasize the dynamic nature of the categorization of technologies.

In mathematics education, technology can refer to electronic devices, computer software, and internet resources. More narrowly, graphing calculators, calculator- or computer-based sensors, *Mathematica*, *MATLAB*, GeoGebra, and NCTM Illuminations Interactives applets could all be considered mathematics-specific technologies.

In this study, my meaning of technology includes both general and math-specific technologies, but I tighten my focus to include only *online tools and resources*. By online tools and resources, I mean to broadly include software, tools, and resources available for use on the World Wide Web through use of Internet-capable devices. By online tools, I mean those that can be used directly for mathematical activity such as virtual manipulatives (e.g., National Library of Virtual Manipulative) or applets (e.g., Desmos Marble Slides). By online resources, I mean those that can be used by teachers to develop their instruction; such as, online lesson plans (e.g., lessons at illuminations.nctm.org/Lessons-Activities.aspx), blogs (e.g., blog.mrmeyer.com written by Dr. Dan Meyer), or videos (e.g., GeoGebra tube).

Availability of computer technology, specifically Internet availability, for mathematics teachers in the United States has grown rapidly over the past few decades (Cuban, 2001; Gray, Thomas, & Lewis, 2010). As access has increased, so too has the recognition of a need to support teachers in learning to make decisions about their use of technology to support mathematics teaching and learning (Voogt, Fisser, Pareja Roblin, Tondeur, & van Braak, 2013; Koehler & Mishra, 2008).

New technologies have regularly been touted as having the potential to "provide unprecedented opportunities" to mathematics classrooms (Roberts, Leung, & Lins, 2013, p. 540). As research on technology use in mathematics teaching has progressed, researchers have identified gaps between how technology is used in everyday classrooms and how technology is used in researchers' classrooms (Lagrange et al., 2003; Taylor, 2003). One explanation for the gap is that researchers and advocates of new technologies focus on the affordances, offering "a wealth of ideas and propositions that are stimulating" but at the same time giving "little consideration to possible difficulties" with classroom implementation (Lagrange et al., 2003, p. 258). With each new wave of electronic technologies comes a new cycle in which a few pioneers first assert educational possibilities and then "poor progression of the use in schools has to be recognised and unexpected difficulties become evident" (Lagrange et al., 2003, p. 255). Before the difficulties can be addressed, "a new generation of technology appears and pioneers can again work on new assumptions" (Lagrange et al., 2003, p. 255). This continuing cycle of promise and defeat widens gaps between research and classroom use as researchers turn their focus to the promise of new technologies, leaving practitioners to choose between the cost of the new and the difficulties of the old.

I argue that mathematics educators and mathematics education researchers must support teachers in developing their professional decision-making for evaluating and planning use of online tools to support mathematics teaching and learning. Support for teachers must attend to the beliefs they hold about the nature of mathematics and mathematics teaching and learning. Teacher support must also attend to characteristics of online tools to which teachers attend. I describe literature on teacher noticing and how it may be extended to online instructional tools and resources in the second section below.

## Conceptions

Understanding teachers' beliefs, measuring their impact on practice, and investigating how both teachers themselves and teacher educators might intentionally affect practice by altering beliefs are all generally acknowledged to be worthwhile, though complex, pursuits. There has, however, been a great deal of disagreement about the nature and definition of beliefs. That is, even though many researchers agree that beliefs should be studied, few researchers agree on a common meaning for beliefs (Ball, Lubienski, & Mewborn, 2001; Putnam & Borko, 1997; Borko & Putnam, 1996; Ernest, 1989a; Fives & Buehl, 2012; Schoenfeld, 2011; Thompson, 1992).

In their attempt to consolidate definitions of teachers' beliefs, Fives and Buehl (2012) synthesized decades of beliefs-focused research. Based on their synthesis, they proposed five basic characteristics of the nature of beliefs. Each characteristic provides a range, describing beliefs as: (a) implicit or explicit in nature, (b) stable or dynamic over time, (c) situated in contexts or generalizable across situations, (d) different or the same as knowledge, or (e) existing individually or as systems (or clusters).

For example, Fives and Buehl (2012) analyzed Thompson's (1992) definition of beliefs: "Belief systems are dynamic, permeable mental structures, susceptible to change in light of experience...

The relationship between beliefs and practice is a dialectic, not a simple cause-and-effect relationship" (p. 140). Fives and Buehl argued that this definition described beliefs as dynamic and as existing as systems of structures.

These characteristics impact how a researcher might expect to gather data on and measure teacher beliefs, and they influence how (or if) results can support intentionally changing beliefs to affect practice (Fives & Buehl, 2012). For example, if beliefs are defined as implicit in nature, then researchers should not ask teachers directly about their beliefs but should observe their practice to infer teachers' beliefs through analysis of "actual teacher actions, planned actions, or talk" (Fives & Buehl, 2012, p. 474). If beliefs are defined as being situated in context, then a researcher might observe seemingly contradictory beliefs across data-gathering sessions. In that case, a researcher should carefully consider and describe when and where data is gathered.

Based on their synthesis of beliefs-focused research, Fives and Buehl (2012) concluded that teachers' beliefs can be both implicit and explicit, "exist along a continuum of stability" (p. 474), "are activated by context demands" (p. 475), "are individually held conceptions that are in constant relation to the context and teachers' experiences" (p. 476), "Teachers' knowledge and beliefs are interwoven" (p. 476). "Beliefs are best understood as integrated systems" (p. 477).

## Beliefs and Conceptions in Mathematics Education

To consider beliefs, or conceptions, of teachers about the nature of mathematics, I first describe attention to these concepts in mathematics education research. Researchers in mathematics education have variously referred to beliefs, views, conceptions, or orientations as mental structures that are related and include each other as smaller mental structures. For example, Ernest (1989a) defined *beliefs* as including "beliefs, conceptions, values and ideology" (p. 20). Thompson (1992) defined *conceptions* as including "beliefs, meanings, concepts, perceptions, rules, mental images, preferences" (p. 130). Schoenfeld (2011) defined *orientations* as including "beliefs, values, preferences, and tastes" (p. 460). Philipp (2007) described beliefs as "lenses through which one looks when interpreting the world" (p. 258). He later defined beliefs more formally as, "Psychologically held understandings, premises, or propositions about the world that are thought to be true. Beliefs are more cognitive, are felt less intensely, and are harder to change than attitudes" (p. 259). Philipp defined conceptions as, "a general notion or mental structure encompassing beliefs, meanings, concepts, propositions, rules, mental images, and preferences" (p. 259). Philipp based these definition on the definition of conceptions given by Thompson (1992).

Philipp (2007) distinguished between beliefs, values, and knowledge. In distinguishing between values and beliefs, Philipp offered an example of a person who *believes* that mathematics is fun but not useful while chemistry is useful but not fun. If this person *values* fun as occupying a more important role in their life and decision-making, then they may choose to improve their abilities in doing mathematics rather than chemistry. If the person *values* utility over fun, then they may choose to focus on chemistry instead. Thus, Philipp defined *belief* as occupying more of a "true/false" dichotomy while *value* occupies more of a "desirable/undesirable" dichotomy. In distinguishing knowledge and belief, Philipp acknowledged that the distinction has been (and will be!) much debated. Philipp focused instead on how to distinguish the two in a way that is useful for educational research, arguing "two people who hold contradictory *beliefs* about something may have more in common with each other than with a person who holds one of these notions as *knowledge*" (p.267, emphasis in original). Philipp used the word *conception* as a generic mental construct that might be a belief or a conception. Philipp explained that

> a conception is a *belief* for an individual if he or she could respect a position that is in disagreement with the conception as reasonable and intelligent, and it is *knowledge* for that individual if he or she could not respect a disagreeing position with the conception as reasonable or intelligent (p. 267, emphasis in original).

That is, Philipp described a person holding a conception may be able or unable to question it. If able, then the conception is a belief. If unable, then the conception is knowledge.

In this study, I use the term *conceptions* as defined by Philipp (2007) as "a general notion or mental structure encompassing beliefs, meanings, concepts, propositions, rules, mental images, and preference" (p. 259). I acknowledge the fuzzy boundaries between knowledge and belief in Philipp's description. I use the term *conceptions* as a construct that includes both beliefs and knowledge.

## Nature of Mathematics and Mathematics Teaching and Learning

Thompson (1992) described a teacher's conceptions of the nature of mathematics as, "that teacher's conscious or subconscious beliefs, concepts, meanings, rules, mental images, and preferences concerning the discipline of mathematics" (p. 132). Using evidence from other mathematics education researchers, Thompson (1992) argued that teachers' beliefs about mathematics as a field influence their teaching. Her argument was that what it means to know and learn mathematics depends on what mathematics is believed to be. Thompson argued that the meaning of effective teaching in mathematics changes depending on choice of academic goals; the goals themselves depend on what teachers believe mathematics is and what knowing mathematics means. In a similar vein, other researchers have emphasized the need to examine, and support the development of, teachers' conceptions of the nature of mathematics because of its impact on their teaching practice (e.g., Philipp, 2007; Shilling-Traina & Stylianides, 2013). In this section, I focus on three issues central to understanding conversations about the nature of mathematics: First, I focus on the gap, or the difference in meanings, between school mathematics and disciplinary mathematics. I describe this gap to argue that both mathematicians and mathematics educators acknowledge such a gap and recommend similar strategies to support students of all levels in bridging that gap. I then move to aspects of the nature of mathematics as it has been described by mathematics educators and mathematicians. I describe these aspects to emphasize similarities and distinctions in views of the nature of mathematics. The different aspects and views motivate the final section in which I describe categorizations of conceptions of the nature of mathematics.

### *School and Disciplinary Mathematics: A Gap*

For the remainder of the paper, I separate school and disciplinary mathematics. By *school mathematics*, I refer to mathematics as it is experienced or perceived by K-12 teachers and students; *disciplinary mathematics* refers to mathematics as experienced or perceived by mathematicians, scientists, and other professionals that interact with mathematics in their disciplines. Mathematicians and mathematics education researchers, including those who research K-12 mathematics education, have described a gap between the nature of *school mathematics* and the nature of *disciplinary mathematics*. In the paragraphs below, I share points of view about such a gap from both mathematicians and mathematics educators. I include two mathematicians who wrote almost a century apart. I chose these individuals to emphasize the persistence of this gap and the importance of addressing it. I also include three mathematics educators who are known for their focus on K-12 mathematics education with a special focus on mathematics in the younger grades. I choose these mathematicians and mathematics educators to emphasize that those with a focus on what is important mathematically and what is important developmentally have explicitly or implicitly acknowledged a gap and proposed similar solution strategies for narrowing the gap.

Alfred North Whitehead was a mathematician interested in the educational experiences of mathematics students. Guershon Harel is a mathematician whose research interests focus on pedagogical implications of cognition and epistemology of mathematics. Both Whitehead (1911) and Harel (2008) argued that a gap exists and needs to be narrowed. They each proposed that students need experiences to be mathematicians, and to do mathematics in ways similar to mathematicians and scientists. Deborah Ball, Mary Kay Stein, and Margaret Schwan Smith, researchers of K-12 mathematics education, have also argued that K-12 students need experiences doing mathematics and being mathematicians to develop the deep understandings of mathematical ideas necessary later in life.

Over a century ago, Whitehead (1911) described a gap between mathematics as experienced by students (and teachers) and mathematics as experienced by mathematicians and scientists. Whitehead explained that, despite the promise of the "important applications of the science, the theoretical interest of its ideas, and the logical rigour of its methods," many times students are buried in a "mass of details which are not illuminated by any general conception" (p. 8). In this description, Whitehead suggested that the mathematics seen by students and teachers is fundamentally different than the mathematics seen by mathematicians and scientists. He wrote that students need to have opportunities to see the elegance and power of mathematics without being buried in skills and tools, and that they need to see fundamental ideas "disentangled from the technical procedure" and to use these fundamental ideas to illuminate the "mass of details" that they must learn (p. 8).

More recently, Harel (2008) pointed to a similar gap. Harel argued that teachers are not always able to provide complete cognitive and epistemological reasons for teaching the mathematical topics that are taught in the way they are taught because the institutional reasons for many mathematical topics and the way they are taught are social rather than cognitive or epistemological. In a way similar to Whitehead (1911), Harel turned to mathematicians' ways of knowing and interacting with mathematics to inform his theory of how mathematics students should understand and mentally interact with mathematics. Harel differentiated between mathematics-as-subject-matter and mathematics-as-mathematical-practices. Harel specifically argued that "Mathematicians, the practitioners of the discipline of mathematics, practice mathematics by carrying out mental acts with particular characteristics (ways of thinking) to produce particular constructs (ways of understanding). Accordingly, mathematics consists of these two categories of knowledge" (p. 272). Harel argued that there are public ways of understanding, as well as private and idiosyncratic ways of understanding that can be valid without being "assumed-to-be-shared" by all members of the community. Harel explained that mathematics students should have opportunities to develop ways of understanding and ways of thinking by engaging with mathematics in ways similar to mathematicians, but that teachers are responsible to "help students gradually refine and modify them" to become valid (compatible and consistent with institutionalized ways of thinking and ways of understanding; p. 274).

Both Whitehead (1911) and Harel (2008) argued that, to close the gap between school mathematics and disciplinary mathematics, students need to experience mathematics in the same (or at least similar) ways as mathematicians and scientists. Students need to *do* mathematics and *be* mathematicians, confronting mathematical complexity and struggle. Both Whitehead and Harel wrote from their positions as mathematicians, and perhaps did not speak from a place of experience with K-12 mathematics. Mathematics educators have echoed Whitehead's and Harel's arguments (Stein, Grover, & Henningsen, 1996; Henningsen & Stein, 1997; Stein & Smith, 1998; Stein, Engle, Smith, & Hughes, 2008). Ball et al. (2001) drew on Whitehead (1911) to describe problematic views of mathematics. Stein et al. (1996) built on arguments from mathematics educators and philosophers (e.g., Lakatos, 1976; Kitcher, 1984; Schoenfeld, 1992) to describe a difference between students' understanding mathematics content and students' "capacity to 'do mathematics'" (p. 456). Stein et al. described *doing mathematics* as, "doing what makers and users of mathematics do: framing and solving problems, looking for patterns, making conjectures, examining constraints, making inferences from data, abstracting, inventing, explaining, justifying, challenging, and so on" (p. 456). Here, Stein et al. argued that students of mathematics should *be mathematicians* or at least engage in activities similar to those activities in which mathematicians engage.

### *Aspects of the Nature of Mathematics*

As mathematics educators and mathematicians wrote about school mathematics and disciplinary mathematics, they included descriptions of important aspects of mathematics. Whitehead (1911) listed characteristics of mathematics that students ought to experience, including: elegance and power of mathematics, importance of mathematical applications, theoretical interest of mathematical ideas, and logical rigor of mathematical methods. Most of all, Whitehead emphasized the abstract nature of mathematics. He explained his meaning of *abstract* as dealing "with properties and ideas which are applicable to things just because they are things, and apart from any particular feelings, or emotions, or sensations, in any way connected with them" (p. 9).

Harel (2008) explained that mathematics is "like a living organism," "it grows continually," and that mathematics is "a human endeavor, not a predetermined reality" (p. 272). As mentioned above, Harel differentiated between the nature of mathematical understanding and the nature of mathematical practice. Harel listed a number of mental acts, or mathematical practices: "interpreting, conjecturing, inferring, proving, explaining, structuring, generalizing, applying, predicting, classifying, searching, and problem-solving" (p. 267). Harel argued that students of mathematics must interact with mathematics through these mental acts to develop appropriate mathematical practice.

Ball et al. (2001) confirmed characteristics listed by Whitehead and argued further that, although skills and procedures are important to mathematics, also important is seeing mathematics as a system of thought and seeing the "power, elegance, and beauty" of mathematics (p. 435). Ball et al. explained that school mathematics shape perceptions of mathematical authority as external to the students: "a body of precepts and facts to be remembered 'because the teacher said so'" (p. 434). Ball et al. further described a perception of mathematics as "no more than a set of arbitrary rules and procedures to be memorized" (p. 434). Ball et al. argued that these perceptions of mathematics as set of rules and procedures, as arbitrarily assigned by external authority, and as something to be simply memorized without question.

Stein et al. (1996) argued that "students should not view mathematics as a static, bounded system of facts, concepts, and procedures to be absorbed but, rather, as a dynamic process of 'gathering, discovering and creating knowledge in the course of some activity having a purpose' (Romberg, 1992, p. 61)" (p. 456).

From these descriptions, some common and repeated themes can be found. Ball et al. (2001) explicitly cited Whitehead (1911), quoting his description of the power, beauty, and elegance of mathematics, as well as seeing mathematics as a system of thought. All mentioned that students of mathematics must see the importance and relevance of mathematics in the world and to see mathematics as a human endeavor or as a way that humans interpret the world around them. Ball also pointed out the importance for mathematical authority to shift from seeming external to students or the classroom, to being internalized. Ball et al. and Stein et al. (1996) emphasized the importance of students seeing mathematics as more than simply facts, skills, or procedures that are given by an external authority and expected to be absorbed. Harel (2008) and Stein and Smith (1998) described the importance for students to create and discover mathematics, and to see mathematics as continually growing. Considering the aspects of mathematics that these researchers found important especially for students of mathematics, I next describe different ways of categorizing conceptions or views of the nature of mathematics.

### *Categorizing Conceptions of the Nature of Mathematics*

To explore potential significant and different conceptions of the nature of mathematics, as listed above, I turn to Ernest (1988) and Thompson (1992). Thompson (1992) synthesized research available at the time to discuss four different ways of categorizing conceptions of mathematics. I limit my focus to two of those categorizations. One set of views was proposed by Skemp (1978) as Instrumental and Relational Mathematics. A second set of views that Thompson (1992) described relied on Ernest's (1988) description of three views or conceptions of the nature of mathematics: the Instrumentalist view, the Platonist view, and the Problem-driven view. This set remains relevant in current mathematics education research (e.g., Shilling-Traina & Stylianides, 2013; Swan, Pead, Doorman, & Mooldijk, 2013; Liljedahl & Hannula, 2016). I describe this set of views below, but first describe Instrumental and Relational Mathematics.

Thompson (1992) described a set of views proposed by Skemp (1978): instrumental and relational mathematics. These two views of mathematics are based on Skemp's description of relational and instrumental understanding: Skemp described instrumental understanding as "rules without reasons" or "the possession of such a rule and the ability to use it" (Skemp, p. 9). Skemp described relational understanding, on the other hand, as "knowing both what to do and why" (p. 9). Based on these definitions, Skemp uses an analogy of learning to play music to differentiate instrumental and relational mathematics. In Skemp's descriptions, one group of children encountered music as only memorizing procedures for writing down notes to copy a piece of music, identifying the time of a piece, identifying the key, and transposing a piece of music from one key to another. His analogy implies that instrumental mathematics is entirely copying procedures, proofs, or explanations, identifying superficial features of the mathematical content or reasoning, and then applying the procedure to a slightly different situation. That is, there are rules to learn, to follow, and to apply to new situations with no (or little) adaptation. A second group of children, according to Skemp, encountered music as connecting sounds with written notes, making the sounds themselves on instruments, learning to associate sequences of written notes with melodies and harmonies, exploring relationships between musical keys, and playing around with their own compositions. In this description, Skemp described relational mathematics as connecting written mathematics to something more than what is on the page. Because Skemp did not explain this relational mathematics in great detail, I turn to Lockhart (2009) who proposed a similar analogy differentiating two types of mathematics; although he calls them *pseudo-mathematics* and *real mathematics,* there is a clear mapping between pseudo and instrumental, real and relational. Lockhart described pseudo-mathematics as being about facts and "about formulas and definitions and memorizing algorithms" (p. 6). He explained pseudo-mathematics further as an "emphasis on the accurate yet mindless manipulation of symbols" (p. 6). On the other hand, Lockhart described real mathematics in multiple ways, emphasizing: ideas instead of facts; creative process with results instead of results alone; the why instead of only the what; the explanation and argument instead of simply "truth." Lockhart explained that the art of mathematics is making patterns of ideas about imaginary and simple things and "asking simple and elegant questions about our imaginary creations, and crafting satisfying and beautiful explanations" (p. 4). Lockhart stated, "Mathematics is *the art of explanation*" (emphasis in original, p. 5). He wrote, "Mathematics is *the music of reason*" emphasis in original, p. 8). He described mathematics further as an activity in which students (or mathematicians) have opportunities "to pose their own problems, make their own conjectures and discoveries, to be wrong, to be creatively frustrated, to have an inspiration, and to cobble together their own explanations and proofs" (p. 5).

Thompson (1992) described the three conceptions of mathematics, Instrumentalist, Platonist, and Problem-Driven, as proposed by Ernest (1988) and Ernest (1989b).

### ***Instrumentalist Conception***

Ernest (1989b) described having an Instrumentalist view as viewing mathematics as "an accumulation of facts, rules and skills to be used in pursuance of some external end. Thus mathematics is a set of unrelated but utilitarian rules and facts" (p. 250). Ernest (1988) described the Instrumentalist view as mathematics "like a bag of tools, is made up of an accumulation of facts, rules and skills to be used by the trained artisan skillfully in the pursuance of some external end" (p. 10). Ernest chose the term *artisan*, which describes a person who works in a skilled trade to produce objects. An artisan is not an artist: an artisan can produce fine objects but not works of art. The emphasis is on the production of objects rather than on the creative process. Ernest described the artisan as *trained* and as *skilled.* Trained implies an externally imposed model of behavior; skilled implies that the artisan is valued.

Ernest chose the words *unrelated*, *accumulation*, and *external end* to describe the Instrumentalist view. In his definitions, Ernest emphasized the disconnected nature in his descriptions: "unrelated but utilitarian rules and facts" (p. 250 Ernest, 1989b), and "bag of tools" (p. 10 Ernest, 1988). Ernest used the word *accumulation*. The word accumulation can mean "something that has accumulated or has been accumulated," where *accumulating* means "to gather or acquire (something) gradually as time passes" ("Accumulation", 2016). This term is neutral regarding the motivating force behind the gathering and regarding the origin of mathematical facts, rules, and skills: the choice of term avoids a stance on whether mathematics is intentionally gathered by humans, or a natural process like leaves blown against the side of a house. It avoids a stance on whether the facts and rules are created by humans or discovered in whole form. Because *accumulation* can mean something that "has accumulated" or "the action of accumulating," the choice of this term also remains neutral on whether the accumulation of facts, rules, and skills is static or dynamic. That is to say, whether the facts, rules, and skills are changing or are continuously newly created or discovered; or whether the facts, skills, and rules are static and exist in completed form.

In Ernest's descriptions of the Instrumentalist view, mathematics is a collection of facts, rules, and skills that are: unrelated, utilitarian, accumulated. The Instrumentalist view is neutral with respect to whether mathematics was created or discovered, continually changing or static.

### *Platonist Conception*

A Platonist view is described by Ernest (1989b) as viewing mathematics as "as a static but unified body of certain knowledge. Mathematics is discovered, not created" (p. 250). Ernest (1989a) described it further as "consisting of interconnecting structures and truths" (p. 21). Ernest (1988) described the Platonist view as mathematics "as a static but unified body of knowledge, a crystalline realm of interconnecting structures and truths, bound together by filaments of logic and meaning. Thus mathematics is a monolith, a static immutable product" (p. 10). In these descriptions, Ernest emphasized characteristics of mathematics including: mathematics as static, unified, organized, deeply connected, and discovered. His use of the word *truths* and the phrase *certain knowledge* both imply aspects of mathematical knowledge as unquestionable and independently existent. His choice of the name *Platonist* draws on Platonism as it is described outside of mathematics: that our named everyday objects are imperfect copies of the pure and transcendent concept for which they are named. That is, by using the term *Platonist,* Ernest drew on this philosophy implying that mathematics exists in a pure and unchanging form, so we discover small parts of the larger and purer structure of the whole of mathematics as we develop our understandings. That is, as humans, we cannot create mathematics but instead we discover different forms of the perfect and preformed mathematics. An important point is that mathematics exists whether we use it or not, but the goal of learning mathematics can be simply to discover new mathematics. Learning mathematics does not focus on its utility.

### *Problem-driven Conception*

In describing a Problem-Driven view, Ernest (1989b) said mathematics is "a dynamic, continually expanding field of human creation and invention, a cultural product. Mathematics is a process of enquiry and coming to know, not a finished product, for its results remain open to revision" (p. 250). Ernest (1989a) described "a dynamic, problem-driven view of mathematics as a continually expanding field of human inquiry. Mathematics is not a finished product, and its results remain open to revision" (p. 21). Ernest (1988) wrote "there is a dynamic, problem-driven view of mathematics as a continually expanding field of human creation and invention, in which patterns are generated and then distilled into knowledge. Thus mathematics is a process of enquiry and coming to know, adding to the sum of knowledge. Mathematics is not a finished product, for its results remain open to revision" (p. 10). Here, Ernest focused on characteristics of mathematics such as: dynamic, incomplete, open to revision, problem-driven, inquiry-based, requiring sense-making, and created. In the Problem-Driven view of mathematics, the name *problem-driven* implies there is some motivating force, whether practical or abstract, that pushes mathematical doing forward. The descriptions given by Ernest, however, focus more on a *process of enquiry* and *coming to know* rather than on utility or finding clear answers.

### *Impact of Three Conceptions on Teaching Beliefs*

To summarize, I provide a table to simplify the descriptions of the three conceptions along consistent categories.

Table 2.1

*Summary of Ernest's Three Nature of Mathematics Conceptions*

| Mathematics is... | Instrumentalist | Platonist | Problem-Driven |
|---|---|---|---|
| In brief: | Bag of tools | Crystalline monolith | Cultural product |
| Composed of? | Facts, rules, skills | Certain knowledge | Human inquiry process |
| Connected? | No | Yes | Yes |
| Utilitarian? | Yes | No | No |
| Origin? | NA | Discovered | Created |
| Growth? | NA | Static | Continually expanding |
| Solutions? | Right or wrong | Open (strat., repres.) | Open (interp., soln.) |

*Note.* Summary across consistent categories of Ernest's three conceptions of the nature of mathematics. Based on Ernest (1988, 1989a, 1989b).

Thompson (1992) emphasized that an individual teacher's view can include "aspects of more than one [of the three views]—even seemingly conflicting aspects" (p. 132). That is, a teacher might believe mathematics is a bag of unrelated tools and also a coherent structure of connected ideas. Thompson (1992) explained that teachers with fewer years of experience, especially, may show contradictory beliefs.

Ernest (1989b) connected teachers' conceptions of mathematics with likely conceptions of teaching and learning. Thompson (1992) reinterpreted findings from Thompson (1984) in light of these three categories, establishing a mapping between three teachers focal to her study and Ernest's three views of mathematics and mathematics teaching and learning. I describe Ernest's and Thompson's mappings and show a visual of these mappings in Figure 2.1.

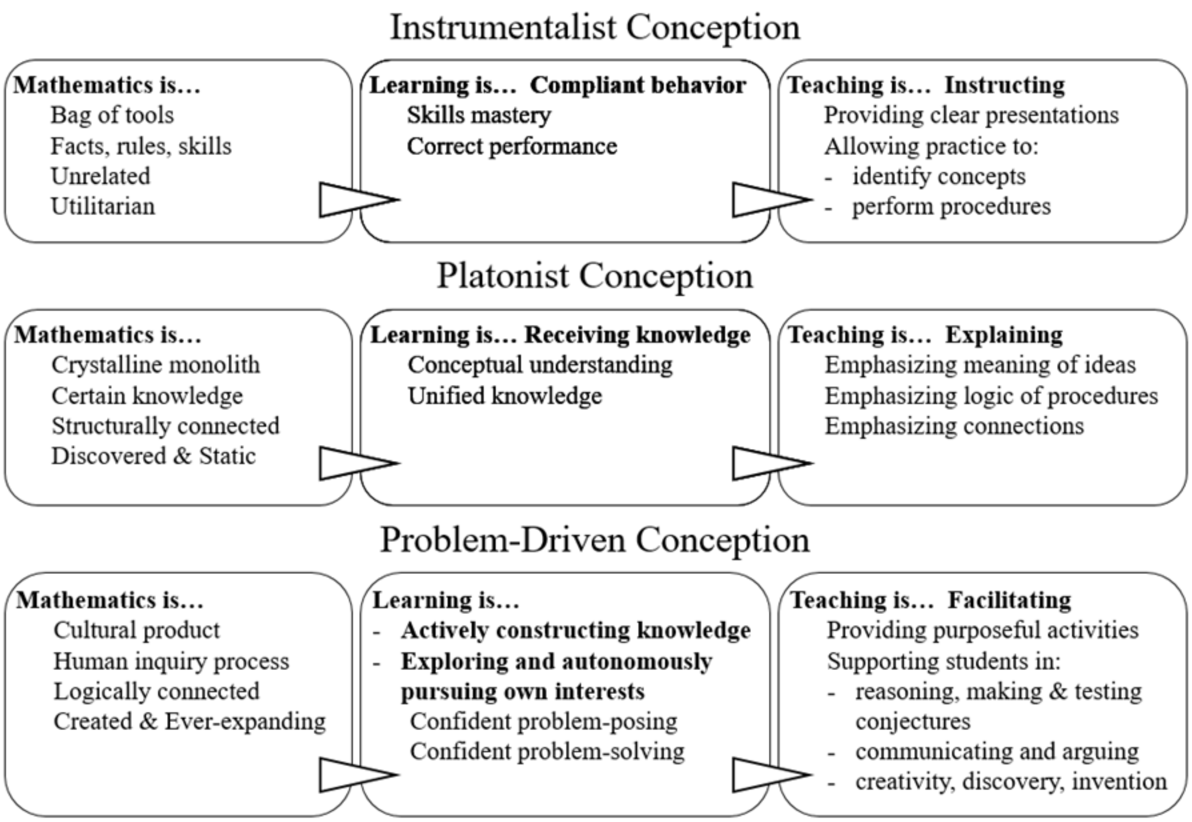


Figure 2.1: Conceptions of Mathematics, Learning, and Teaching

Ernest (1989b) described a potential mapping from an Instrumentalist view of mathematics to corresponding views of teaching and learning mathematics. He described a teacher with an Instrumentalist view as an Instructor, supporting students in mastering skills and correct performance. Ernest described such a model of learning as "compliant behavior and mastering skills" (p.

252). Thompson (1992) described one mapping between the nature of mathematics, the meaning of knowing mathematics, and methods of teaching mathematics: a teacher who saw mathematics as “a discipline characterized by accurate results and infallible procedures” might view knowing mathematics as “being skillful in performing procedures and being able to identify the basic concepts of the discipline” (p. 127). Such a teacher might reasonably approach teaching mathematics such that “concepts and procedures are presented in a clear way and opportunities are afforded the students to practice identifying concepts and performing procedures” (p. 127). Hence, an Instrumentalist conception of the nature of mathematics in which mathematics is a collection of facts, skills, and rules with goal of consistently calculating correct answers leads to an instructional approach where students memorize facts and practice procedures to calculate accurate results.

A teacher with a Platonist view of mathematics, according to Ernest (1989b), would likely be an Explainer. Such a teacher would expect students to develop “conceptual understanding and unified knowledge” (p. 252). Ernest described such a model of learning as “reception of knowledge” (p. 252). Thompson (1992) argued that a teacher with a conception of mathematics “as a coherent subject consisting of logically interrelated topics” (p. 134) might view knowing mathematics as conceptual understanding of the meaning of and connections between facts, rules, and representations. Such a teacher might focus on emphasizing “the mathematical meaning of concepts and the logic of mathematical procedures” (Thompson, 1992, p. 134) in her teaching. Based on this reasoning, a Platonist conception of the nature of mathematics in which mathematics exists in a pure form as structure of meaningfully and logically connected ideas, objects, and their relationships leads to an instructional approach where students explore the deeper meaning of facts, rules, and relationships to understand larger mathematical structures.

Ernest (1989b) a teacher with a Problem-Driven view of mathematics as a Facilitator. Such a teacher would support students in becoming “confident problem-posers and problem-solvers” (p. 252). She would hold a model of learning as “active construction of knowledge” and possibly as “exploration and autonomous pursuity of own interests” (p. 252). Thompson (1992) described a conception of the nature of mathematics as mathematics consisting of abstract ideas and their representations created in an ongoing way by humans as they explore situations and confront problems. Thompson (1992) described knowing mathematics in this context means creating mathematics, or mathematics “in the making” (p. 128). A teacher with this perspective on mathematics and knowing mathematics would teach in ways in which “students engage in purposeful activities that grow out of problem situations, requiring reasoning and creative thinking, gathering and applying information, discovering, inventing, and communicating ideas, and testing those ideas through critical reflection and argumentation” (p. 128). Hence, a Problem-Driven view of mathematics in which mathematics is continuously created in an organic and logically connected way, would support a view of teaching in which students develop ways of approaching situations or problems to create, discover, and invent new mathematics.

### *Conclusion*

The idea of school versus disciplinary mathematics is an important undercurrent when considering Ernest’s (1989) three views of mathematics. As Thompson (1992) wrote, “For the majority of the teachers whose mathematical beliefs have been reported in the United States, mathematics is the mathematics of the school curriculum” (p. 133). She also reported that, in studies based on Ernest’s three categories, most preservice and inservice teachers showed evidence of an Instrumentalist view. Thompson argued that teachers’ views of mathematics as school mathematics may explain “the preponderance of ... the instrumentalist and Platonist views” (p. 134).

In considering the way mathematics was described by mathematicians and mathematics educators, as reported above, it is clear that each considers different aspects of mathematics as important. It is also clear that the mathematics educators and mathematicians seemed to describe disciplinary mathematics—the mathematics that students should experience—as closer to Ernest’s Problem-Driven view than to his Instrumentalist view. For example, Ball et al. (2001) and Stein et al. (1996) described mathematics as more than only facts, skills, and procedures. Harel (2008) and Stein et al. (1996) described mathematics as including creation and discovery of new mathematics. Several aspects, however, that these researchers included as important are not included in Ernest’s three categories. For example, viewing mathematics as a system of thought (Ball et al., 2001; Whitehead, 1911) or focusing on internalizing rather than externalizing mathematical authority (Ball et al., 2001) are also important.

## Noticing

The idea of *noticing* is common in everyday language, and is used in a similar way in educational research. to clarify its usage in this study, I describe the common meaning here. Merriam-Webster (2017) included several potential meanings for the verb *to notice*, including: "to become aware of" and "to pay attention to." In mathematics education, teacher noticing is used similarly to refer aspects of mathematics tasks, student thinking, or instructional materials to which a teacher pays attention (or becomes aware of) as she makes in-the-moment decisions about supporting students' mathematical thinking.

Many mathematics education researchers have investigated teacher noticing, and the literature on noticing reflects several distinct (but similar) perspectives. I share three such perspectives here: expert-novice noticing, teachers' noticing in support of in-the-moment decision-making (e.g., in teaching episodes), and teachers' noticing in support of making decisions about instructional resources.

### Mathematical Tasks

Several decades ago, researchers described differences between the way experts and novices noticed mathematical tasks: Experts are able to notice deep structures of mathematical tasks while novices attend to surface structure features (Chi, Feltovich, & Glaser, 1981; Schoenfeld & Herrmann, 1982). Schoenfeld and Herrmann (1982) described deep and surface structures of mathematical problems as the "mathematical principles necessary for solution" (p. 486) and "most prominent mathematical objects or the general subject area it comes from" (p. 486), respectively. For example, a novice might notice similarities and differences in contexts of mathematical problems, while an expert might notice similarities or differences in potential solution strategies or mathematical structures.

## In-the-Moment Decision-Making

More recently, the importance of teacher noticing has been explored, especially its effect on teachers' planning and in-the-moment decision-making (B. Sherin & Star, 2011; M. G. Sherin & van Es, 2009). Teacher noticing refers to noticing student behavior during a lesson, within the lesson focus as well as external to the lesson (e.g., classroom management, student health) (Erickson, 2011). M. G. Sherin and van Es (2009) described an aspect of noticing as selective attention, or "how the teacher decides where to pay attention at a given moment" (p. 22). Similar to expert and novice noticing of deep or superficial mathematical structures of tasks, M. G. Sherin and van Es described differences in expert and novice teachers' noticing of substantive or surface-level features of classroom interactions. They found that teacher participants began (as novices) by paying attention almost entirely to pedagogical issues at first, and over time paid more attention to mathematical issues, especially mathematical student thinking.

## Adaptation and Construction of Curricula

Drake et al. (2009) described three dimensions of a teacher's strategy for curricular use: reading, evaluating, and adapting. M. G. Sherin and Drake (2009) found that their teachers noticed different types of information in curricula, attending to: overall ideas, details, or both (but only one at a time). When considering teachers' evaluation of a curriculum, they found that teachers evaluated for a particular audience (i.e., for herself, her students, or others), and that the audience might change depending on whether the teacher was evaluating before, during, or after instruction. Sherin and Drake found that teachers adapted curricula by omitting or replacing components of a lesson, or creating new components of a lesson. They also found that teachers tended to adapt instructional components of a curriculum before teaching it, but adapted mathematical content during instruction. Sherin and Drake explained that "teachers were less able to envision *before* *instruction* what kind of changes in the mathematics of the lesson they wanted to make" (emphasis in original, p. 487).

## Instructional Resources

The teacher noticing construct can be extended from supporting in-the-moment and curricular decision-making to supporting teachers' evaluation of and decision-making about instructional resources. Brown (2009) discussed the differences in teachers' skills in "perceiving the affordance of the materials and making decisions about how to use them to craft instructional episodes that achieve her goals" (p. 29). Brown (2009) argued that teachers' ability to skillfully engage in this process depended on "(a) subject matter knowledge; (b) pedagogical content knowledge (Shulman, 1986); and (c) goals and beliefs" (p. 29).

Dietiker et al. (2014) defined curricular noticing as "how teachers make sense of the complexity of content and pedagogical opportunities of written curriculum materials" (p. 4). Each of the conference presenters engaged in a curricular noticing activity with preservice teachers (PSTs) in a mathematics methods course, and reported their findings. Two conference presenters (Males and Stohlmann) described the use of a framework to support PSTs' noticing. Males asked her students to use three curriculum analysis tools to analyze quadratics content in three curricula. These tools included rubrics for assessing *CCSSM* content alignment, *CCSSM* practices alignment, and inclusion of equity, assessment, and technology. Stohlmann asked his students to develop and use a framework, with three themes suggested by Stohlmann and all others suggested by PSTs, to identify strengths and weaknesses of a similar lessons found in two distinct sets of curricula.

## Online Instructional Resources

Focusing noticing characteristics of instructional resources more narrowly on those resources that are digital or online, researchers have described teachers' development from identifying superficial characteristics of a digital tool (e.g., bright and engaging) to deeper characteristics (e.g., support of particular mathematics content or particular visualization of a mathematics concept).

I argue that the construct of *teacher noticing*, with respect to characteristics of online tools that will be used to support mathematics lessons, draws on ideas from the teacher noticing constructs described above. First, online tools may be used to support a mathematical task or mathematical tasks may be embedded in the tools. Teachers may notice different levels of aspects of mathematical tasks: superficial features (e.g., surface level mathematical content, type of virtual manipulative) or deeper structures (e.g., mathematical structures or representations, structures of potential solution strategies).

Second, teachers may attend to general, pedagogical, or mathematical aspects of online tools. For example, teacher may attend only to general technology characteristics, such as, whether the tool is bright and engaging or usable on students' devices. A teacher might attend only to pedagogocial aspects, such as whether the tool provides immediate feedback, whether the teacher must model the tool before students use it, or whether students can use the tool independently to learn about a concept. Finally, a teacher might attend to mathematical aspects of online tools (see previous section).

Next, teachers may attend to different types of curricular information: overall ideas, details, or both (but one at a time). To attend to deeper mathematical features and ways in which mathematics embedded in the tool can support mathematical ideas, teachers may need to use the online tool in their classroom through an iterative process of: first, identifying mathematical ideas they hope to convey, then planning the integration of the tool into a lesson, implementing the lesson, and reflecting on the tool and the mathematical ideas that emerged through its use.

Finally, teachers may need the support of a template to think through the complex characteristics of online tools. They may need to explicitly compare tools as they begin to develop their noticing of the general, pedagogical, and mathematical aspects of each tool. Teachers may need to discuss with others about which aspects they find important as they develop their noticing; that is, as they learn which aspects are most helpful to them in determining which tool will most successfully support students' conceptualizations of particular concepts.

# CHAPTER 3

# METHOD

This study is a qualitative analysis consisting of a case study developed based on data from participants in the context of a graduate-level *Mathematics, Technology, and Education* course. In the following sections I describe the study design, data collection, the course and its structure and activities, the participants, the post-course interview, and the procedures for data analysis. I describe sources of data and an overview of data analysis for each research question. A more detailed description of each analysis is given in the corresponding findings chapter to explain adaptations necessitated by the data and individual elements of each analytic frame.

## Study Design

This study explores elements of teacher decision-making for use of instructional technology in the learning and teaching of mathematics. In this study, I focus on teacher conceptions, teacher noticing, and potential relationships between teacher conceptions and noticing. More specifically, I explore how teachers enrolled in the graduate-level *Mathematics, Technology, and Education* course expressed (a) conceptions of the nature of mathematics, the nature of mathematics teaching and learning, and the role of technology in instruction and (b) noticing of general, mathematical, and pedagogical features of digital tools and resources. I apply the findings from the first two research questions to explore (c) potential relationships between teacher conceptions and teacher noticing. In this study, I explore the case of an online course focused on use of technologies in mathematics teaching and learning. Case study development in this project relied on data collected through course activities from the course and interview data from course participant interviews. I describe the process of data collection and participant selection in sections below. This project was approved as Exempt by Michigan State University's Institutional Review Board (IRB) with IRB Number: x14-1309e.

## Data Collection

The primary data for exploring teachers' conceptions—their beliefs about the nature of mathematics, the nature of mathematics teaching and learning, and the role of technology in mathematics teaching and learning—consists of their interviews, their description of themselves in their personal profile, excerpts from grade-level team and whole-class discussions, and reflections throughout the course. The primary data for examining teachers' noticing—what they attended to when evaluating and considering use of digital tools and resources—consists of their initial and final evaluations, the evaluation framework each team developed in Unit 2, the course evaluation framework developed at least in part by the whole class that was introduced in Unit 3 and finalized in Unit 7, and teachers' final reflections in which they compared their initial and final evaluations. I then use findings about teachers' conceptions and noticing to inform an analysis of potential relationships between conceptions and noticing.

This study does not focus on how teachers changed or were influenced by the course, course activities, or interaction with other teachers. I do, however, describe the course and teacher groups to give context to teachers' responses. I acknowledge external factors that also influence their thinking; for example, mathematics methods courses from teacher preparation, interactions with colleagues, and professional development experiences. To triangulate, to explore teachers' thinking in multiple contexts and hence from multiple perspectives, I consider data from the course that includes discussions between teachers as well as teachers' individual reflections and I gathered data from individual interviews after the course had ended (Creswell, 2008).

In the following sections, I first describe the course and course activities. I then describe the participants: the teachers who gave consent for use of their course data and the teachers that further consented to individual interviews. I describe the data and its collection. Finally, I describe the individual interviews and the structure of data gathered.

## *Mathematics, Technology, and Education* Course

The course within which this research took place is a semester-long online master's course I refer to as the *Mathematics, Technology, and Education* course. Individuals enrolled in the course typically have a background in teaching and were enrolled in one of three online masters-level programs (described in the next section). I was one of two co-instructors of the course.

Before the course began, and throughout the course, the co-instructor and I discussed online learning environments that would most appropriately support the learning experiences we intended to provide to enrolled students. We relied on the learning management system, Desire2Learn from D2L Corporation, for course organization, communication, and discussion forums. We used Google Classroom for creating, assigning, and collecting individual reflections. We used Google Drive folders and Google Docs as a collaborative workspace for enrolled students. We used the Michigan State University College of Education Wiki provided through Wikispaces by TES as a way for enrolled students to easily share their evaluations by embedding their Google Docs, and also as a way to tag and organize resources so other teachers could draw on and build on work that had previously been completed. Finally, at the end of the course, enrolled students made their own choice of a platform to share their evaluations from the course. Students chose platforms such as Weebly Website Builder or WordPress from Automattic, Inc.

The course comprised eight two-week units exploring issues of mathematics teaching and learning through instructional use of technology. Each unit focused on a strand of mathematics content, as organized by NCTM's *Principles and Standards for School Mathematics* (*PSSM*) (National Council of Teachers of Mathematics [NCTM], 2000). Within each unit were several phases: first, teachers spent some time individually exploring a topic or issue; second, teachers discussed or collaborated in small dedicated teams; third, teachers discussed as a whole class; and, fourth, teachers wrote a brief reflection at the end of each unit. During the individual exploration phase, teachers read articles, excerpts from policy documents (e.g., *PSSM* and *Common Core State Standards Mathematics* (*CCSSM*) (Council of Chief State School Officers [CCSSO], 2010)), and searched for, interacted with, and evaluated digital tools and resources. During the small group discussions and whole class discussions, teachers used online discussion forums to share and talk about what they learned in response to a small set of questions. In their small groups, teachers also worked together in online discussion forums or on collaborative Google Docs to create concept maps, to evaluate digital tools or resources, and to give feedback to each other on evaluations completed during the exploration phase. At the end of each unit, teachers individually reflected across the unit, by writing a written response to a set of questions given by the instructors. The questions varied, but generally asked teachers to take a position on the readings, to reflect on how their thinking about mathematics, teaching and learning mathematics, and use of instructional technologies may have changed with respect to the readings and activities of the unit. A brief synopsis of unit activities across the course is provided in Appendix A.

After the course ended and grades were submitted, I gathered course data. The data collection included course discussions, reflections, and evaluations. For each type of data, I describe the collection process, deidentification, its addition to the NVivo master file, and its preparation for analysis (QSR International Pty, Ltd., 2012).

For discussions, I returned to the course Desire2Learn site and downloaded each discussion into a common file. I gathered all whole class discussions into one file. For each course team, I also gathered their discussions into one file. My choice to create one file per team was motivated by my desire to minimize the number of source files I would later use in NVivo, and also to simplify the deidentification process (QSR International Pty, Ltd., 2012). When I had compiled the five files, I used the copy and replace function of Microsoft Word to replace participant names with identifying codes. Because of misspellings and other inconsistencies, the copy and replace by itself was not sufficient. In a second pass, I skimmed through each file, replacing names with identifying codes by hand, replacing posts from non-participants with a note that the post had been deleted, and replacing other identifying information (e.g., city names, school names) with generic names. I made a third and final pass to double-check the deidentification. I also used the search function of Microsoft Word to search for first names, last names, and other identifying information that I believed was deleted. I then saved each file as a PDF and imported the files into the master

NVivo project.

Similar to whole class and smaller team files, I chose to create one file for each participant. Each file included the participant's reflections and evaluations. Reflections were saved in labeled folders on Google Drive. I downloaded reflections for each participant individually. The collection of evaluation data involved separate strategies for evaluations created as a part of smaller unit assignments and for evaluations created solely for the final course project. The smaller unit evaluations were in labeled folders on Google Drive and I downloaded participants' evaluations individually. Guidelines for their final project, an online resource library, required each student to create a resource library that included 20 annotated links to online tools and resources with 10 evaluations written by the student. As they created their final projects, participants used an assortment of evaluations they had created throughout the semester, evaluations that classmates had created, and new evaluations that they completed for the final project. Some students also revised or rewrote evaluations they had created for a previous unit. Many evaluations were stored in folders on Google Drive, but evaluations created for the final project were stored in different ways, depending on the individual. Most of the evaluations were labeled with the creator's name, but not always. The occasional mislabeling meant that I needed to research certain evaluations to determine whose it was. I created a spreadsheet for each participant that included the name of the online tool or resource, a link to its website, the version of the evaluation format that was used, and for which unit it was created. I created a second spreadsheet for each participant to capture information about the links he or she included in the online resource library. This second spreadsheet included: the participant's name, a link to the participant's online resource library, the name of each online tool or resource with a link to its website, and, if an evaluation had been created, a link to the evaluation and the name of the individual who had created the evaluation. If no name appeared on the evaluation, I highlighted that cell and explored other collections to find the creator of the evaluation. At the end of this process, I am confident of each participant's collection of evaluations. A list of online tools evaluated by participants is shown in Appendix B.

After gathering reflections and evaluations, I organized one file for each participant. I created sections for each unit, and organized each participant's reflections and evaluations according to the course units. If an evaluation created for the online resource library was significantly different than an evaluation for the same tool or resource created as a unit assignment, then I included both evaluations. Otherwise, I included only the evaluation created as the unit assignment. Once each file was created, I followed a similar process described above to deidentify the file. I then converted it to a PDF and imported it to the NVivo project (QSR International Pty, Ltd., 2012).

To prepare the data documents for analysis, I used NVivo to code the data according to: type of activity (i.e., discussion, evaluation, or reflection), unit (e.g., Unit 1), unit activity (e.g., Unit 1 - Personal Profile), and participant code (e.g., P03-Nicole) (QSR International Pty, Ltd., 2012). For evaluations, I further coded each with the name of the tool or resource (e.g., IXL - Fractions). I then coded within the evaluation according to the structured evaluation sections (e.g., 1-Learning Activity Types and LA-Assess).

## Participants

Initially, 16 students were enrolled in the *Mathematics, Technology, and Education* course for the semester in which this study took place; 14 students completed the course on time. All students were pursuing one of three online masters-level programs. For purposes of identification in this study, I use the identifier given in parends. The three programs, listed in no particular order, include the following focuses: one program focuses on uses of technology in education (EdTech), a second program focuses on education more broadly (Educ), and a third program focuses on supporting the learning of students and teachers (T&L). Noting the program is useful because students in the EdTech program had identified themselves as interested in educational technology, while students in the other two programs may have held similar interests in educational technology but had not committed to a similar focus. Practicing teachers enrolled in the course taught with a primary focus on mathematics (e.g., secondary teachers or elementary math specialists) or taught mathematics along with other content areas (e.g., elementary teachers or special education teachers).

Near the end of the course, I sent an email to the enrolled teachers. In the email, I described my proposed study and asked if any would consent to my use of their data or, in addition to use of data, to a post-course interview. Consent was given, or declined, through an online consent form given as a brief online survey. Teachers that consented answered additional questions. As a co-instructor of the course, I did not have access to the survey's password until after grades were submitted. The description, consent form, and brief survey are shown in Appendix C. Of the 14 teachers who completed the course, 13 consented to share their evaluations and other coursework as data for this study, and six consented to participate in a post-course interview. Descriptions of participants who consented to the study appear in Table 3.1; participants who additionally agreed to the post-course interview are indicated with boldface. I refer to these participants as *central* or *focus* participants for the remainder of this dissertation. Note that no participants in the high school group consented to the post-course interview; all central participants were in Elementary or Middle Grades groups. Details provided in Table 3.1 come from participants' responses to the consent survey and their responses to the introductory course Personal Profile. For remainder of the dissertation, pseudonyms shown in Table 3.1 will be used for participants.

In Chapter 4, I provide a few details about each participant to provide context, though the core analyses focus on only the six participants that consented to an interview. The additional context is important because teachers in the course discussed readings and worked together in their small grade-level groups in each unit of the course. This focused and ongoing interaction likely supported participants' development shared understandings of ideas that would not necessarily reach the class as a whole. Participants' backgrounds and the constraints and affordances within their current positions would also likely influence these interactions and the ideas they shared.

Participant descriptions in Table 3.1 are organized by grade-level teams; each participant was assigned to a grade-level team in the course based on the teacher's description of their teaching experience. Grade-level teams were structured around grade-level bands: Lower Elementary (preK-2), Upper Elementary (3-5), Middle Grades (6-8), High School (9-12). Table 3.1 also includes the pseudonym for each participant, their position, years of experience, grade levels taught, and the Master of Arts program in which they were enrolled. All information is based on responses

Table 3.1

*Participant Overview*

| Participant | Position | Yrs. Exp. | Grade Lvls. | Master's Program |
|---|---|---|---|---|
| Lower Elementary Group | | | | |
| **Breanna** | Kindergarten | 2 | PreK-2; 3-5 | Educ |
| Kristine | No position | 1 | PreK-2; 3-5 | T&L |
| **Melinda** | Second Grade | 3 | PreK-2; 6-8 | EdTech |
| **Jana** | No position | 16 | All Lvls. | EdTech |
| Upper Elementary Group | | | | |
| Jason | Fifth Grade | 4 | 3-5 | EdTech |
| **Nicole** | No position | 5 | 3-5 | T&L |
| **Heidi** | Fifth Grade | 6 | 3-5; 6-8 | Educ |
| Middle Grades Group | | | | |
| Ashley | Seventh Grade | 3 | 6-8 | Educ |
| **Alison** | Seventh Grade | 4 | 6-8 | EdTech |
| High School Group | | | | |
| Brandon | Learning Ctr. Dir. | 2 | 9-12 | EdTech |
| Ben | Instructional Asst. | 3 | 3-5; 9-12 | Educ |
| Denise | Resource Teacher | 3 | 9-12 | EdTech |
| Heather | Algebra, Geometry | 6 | 9-12 | Educ |

*Note.* Descriptions of participants include: pseudonym, position (or most recent position, if no position), years of experience teaching (not including student teaching), grade levels they had taught (including student teaching), and the Master of Arts program in which they were enrolled. All descriptions valid at time of study. Focus participants indicated with boldface.

given by the participants at the time the study was conducted.

Most of the participants (12 out of 13) had fewer than seven years of teaching experience. This number makes sense for this course, because the course was part of a Master of Arts program, and, although promotion schedules vary widely across different states and districts, many teachers choose to complete a Master of Arts degree relatively early in their career to develop more focused knowledge of teaching and to qualify for increases in status and pay. At the time of the study, Kristine, Jana, and Nicole were not employed as teachers but were working full-time on Master of Arts programs. Over half of the participants (7 out of 13) were working in Michigan. Again, this number makes sense because the Master of Arts programs were offered online, through a Michigan university. Regarding the Masters of Arts programs, six participants were enrolled in the

EdTech Master of Arts, five in Educ, and two in T&L. Each small group had at least one teacher enrolled in the EdTech Master of Arts and at least one enrolled in the Educ Master of Arts; each elementary group had one member enrolled in the T&L Master of Arts program. The variation in programs could allow teachers to bring different ideas into the discussions because they would have been taking different courses and encountering different perspectives on educational theory and instructional technologies.

## Interview Structure

I designed the interview to provide additional opportunities for teachers to reveal their conceptions about the nature of mathematics, the nature of mathematics teaching and learning, and the role of technology in mathematics teaching and learning. The interviews were semi-structured with some questions that I asked all participants and others that were chosen for individuals based on their course data. My guiding rationale for most questions was in response to Speer (2005), as she called for additional attention to differences in researchers' and teachers' assumed meanings of terms: "reported inconsistencies [between teacher beliefs and practice] may be the result of a lack of *shared understanding* among researchers and teachers about what descriptive terms mean" [emphasis in original] (Speer, 2005, p.362). After considering teacher statements from previous semesters, and assumptions that I and the instructor of the course made about teachers' use of terms from course activities and discussions, it seemed likely that teachers in the course used particular terms as if the meanings were shared, but that individual meanings of a particular term varied (Watson & Mason, 2007).

After the course ended and grades were submitted, I was given access to the Qualtrics survey and, hence, the list of participants who had agreed to be interviewed (Qualtrics Labs, Inc., 2013). Over the next few days, I read the interview participants evaluations across the semester, their reflections, and skimmed their contributions to grade-level team discussions. I explored the initial and final evaluations for all participants, whether they had agreed to interviews or not. Based on the data, I chose individualized questions to probe into ideas that differed from participant to participant. I also chose questions to ask all interview participants to get at their meanings for language that was used in the course.

I wrote a draft of the interview protocol several months before the course began. I piloted the interview protocol with a graduate student who was teaching an elementary mathematics content course. Based on her responses, and based on data from the participants described above, I revised the interview protocol after grades were submitted and before interviewing the first participant. A sample interview protocol is provided in Appendix D. One week after submitting grades for the semester, I conducted the first interview, and two weeks after the first interview, I completed the final interview. My goal in waiting one week to begin interviewing was to allow a small buffer for the teachers to close the book on the course; I hoped this buffer would allow their responses in the interview to be more reflective of their own conceptions than an attempt to answer correctly based on the themes of the course. My goal in completing all interviews within two weeks was to limit the length of the buffer to provide some consistency in participants' recall and distance from the course.

Each interview was scheduled for 90 minutes, and, on average, interviews were completed at about 90 minutes (but ranging from 76 minutes to 104 minutes). In the request for participants, I asked whether each would prefer to meet via video-conferencing software (i.e., Zoom Video Communications) or in person. One of the six participants (Breanna) chose to meet face-to-face, while the others elected to use the Zoom Video Communications software. For each of the five remote participants, I recorded the video and audio using Zoom Video Communications. As a backup recording, I recorded only the audio using a digital recorder. For the face-to-face participant, I recorded the video and audio using built-in software on my laptop computer, along with the digital recorder as back-up. I recorded the screen during the card sort using Camtasia screen-capturing video software. In addition to video and audio recordings, each participant (with the exception of Alison) also manipulated an online card sort and I saved the final versions for each. One participant (Alison) was able to only view (not manipulate) the Google Slides, and so I manipulated the slides for her in response to her verbal instructions.

I transcribed each interview in multiple passes. In a first pass, I used the Nuance Dragon Naturally-Speaking 13 voice recognition software to transcribe the audio recording as well as it was able (Nuance Communications, Inc., 2012). In a second pass, I transcribed in real time (without pausing or slowing down the audio) as well as possible. In a third pass, I took more time, pausing the audio to correct errors and add content as needed. I made additional passes as needed. In a final pass, I read the transcript while listening to the recording, checking that no errors or missing content remained. Once I completed the final pass of the audio transcription, I watched the video recording, especially focusing on the participant's actions during the card sort. I captured the card sort actions (e.g., "selects 'always open to discussion and diverse views' and drags to top, to the left of 'ideas that are all connected'" and inserted images of the card sort into the transcript. I inserted one image of the completed sort, taken immediately before the participant switched slides to the next card sort. I also included images taken at the end of a dialogue turn (e.g., when a participant asked a question). Including images of the changing card sort along with descriptions of actions added context to statements made by participants that clarified their meanings and intentions. Finally, I saved each transcript with images as a PDF and imported it into NVivo Qualitative Analysis software (QSR International Pty, Ltd., 2012).

## Data Analysis

This study analysis is a qualitative analysis of course and interview data gathered from individuals enrolled in the *Mathematics, Technology, and Education* course described above. I use the method of developing case studies to describe individual participant's conceptions and noticing, and conduct cross-case analysis of the case studies to identify patterns of similar and dissimilar results to deepen understanding of the theories and individuals. In the sections below, I first describe my role in the course as a teaching assistant and researcher, and follow with a description of case study research methods and coding approaches that I use in this analysis.

## My Role

As a teaching assistant for the course, I interacted with individuals enrolled in the course by responding to emails, answering questions, and providing feedback on some assignments in tandem with the instructor of the course. As described in the Participants section above, I did not know which individuals had agreed to be participants until after the course ended and grades were submitted. In the first week after the grades were submitted, I re-read all participants' reflections and evaluations from a research perspective rather than a teaching perspective to make my shift from viewing the participants as learners and educators to viewing them as educators and professionals. My rationale was that providing a brief period of time between the end of the course and the first interview would allow students to make a similar shift in perspective.

## Case Study Method

Yin (2014) defined a case study as a form of empirical inquiry that "investigates a contemporary phenomenon (the 'case') in depth and within its real-world context, especially when the boundaries between phenomenon and context may not be clearly evident" (p. 16). Yin argued that a case study allows a researcher to use the case and rich description to further understanding of theories. Although case studies are not generalizable to populations, they can be "generalizable to theoretical propositions" (p. 21), that is to extend and explain theories rather than populations. This generalization is referred to as analytic generalization or theoretical elaboration, and the cases are used to "support, contest, refine, or elaborate a theory, model, or concept" (Schwandt, 2007, p.5). The choice of case, according to Schwandt, is based on its relevance to the theoretical account. Creswell (2013) named this type as an *instrumental case*, where the case is selected for its potential to support deeper understanding of an issue or theory.

This study is a qualitative case study analysis with the goal of examining conceptions and noticing to understand the individuals more deeply and to understand and extend theoretical frameworks of teacher conceptions and noticing with respect to mathematics teaching and learning through use of instructional technologies.

The unit of analysis (or case) will be defined as the *Mathematics, Technology, and Education* course, bounded by the context (teaching and learning mathematics through use of instructional technologies) and time interval (semester) of the course, with a focus on statements made by participants through their interaction with the course. In this study, the boundaries between the phenomena being studied (teacher conceptions and noticing) and the context of the course (*Mathematics, Technology, and Education*) are not clear because research on both conceptions and noticing has acknowledged that both are context-dependent.

I use *within-case analysis* which, according to Creswell (2013), means to analyze multiple units within the case. In this study, participating teachers are considered units within this case. In this study, I examine conceptions and noticing of individual teachers by analyzing the statements they made within the context of the course and, for comparison, statements made in a post-course interview. I analyze using themes for "understanding the complexity of the case" (Creswell, p. 101). I coded data using NVivo. I describe the coding in more detail below.

## Approach to Coding

A within-case analysis was used to identify patterns or themes within the data, based on examination of teachers' statements (Creswell, 2013). I used existing literature and theoretical frameworks to develop analytical frames. Due to recognized "dynamic interplay between theory (concept) and observations (evidence)" (Schwandt, 2007, p.295), my method allowed for emergent themes from observations that I used to extend and explain pre-existing theories. I used thematic analysis, which is an exploratory analysis in which "the analyst codes (marks or indexes) sections of a text (e.g., a transcript, field notes, and documents) according to whether they appear to contribute to emerging themes" (p. 291).

Using NVivo, I coded data according to broad themes for each research question. In examining data within each theme, I used research memos, different forms of visualizations, and matrices of categories and evidence, following recommendations for case study analysis from Yin (2014). I identified emergent patterns based on Yin's description of case analysis, and considered patterns across all focus teachers as well as across individual teachers' data. The enactment of this analysis was consistent for research question one a large-scale, but details varied. I describe the approach to coding and analysis for each research question below.

## Teacher Conceptions

To answer the first research question, *What conceptions of the nature of mathematics, the nature of teaching and learning mathematics, and the role of instructional use of technology are evidenced through teachers' participation in course activities?*, I examined participants' statements across course and interview data. Participant statements were elicited in response to course activities and questions asked in discussions, reflections, and the interview. I describe below the data sources, the relevant contexts in which statements were elicited, and analysis methods.

### *Data Sources*

To explore statements teachers made about the nature of mathematics, the nature of mathematics teaching and learning, and the role of instructional technologies in mathematics teaching and learning, I considered course data and post-course interview data. The course data included team discussions and individual reflections. The post-course interview data included responses to two types of questions: unstructured questions and structured questions. In answering the first research question, the goal is to explore teachers' statements to understand more deeply an individual teacher's conceptions; hence, these sources were intentionally chosen to provide validity through triangulating data sources (Creswell, 2008).

From the course data, I focused on discussions and reflections from the first two units of the course. I considered statements from Units 1 and 2 for two main reasons: first, we (the co-instructors) asked teachers to focus on mathematics and mathematics teaching and learning more directly in these units than in other units; and, second, we asked teachers to consider mathematics and mathematics teaching and learning as a whole in these units. In later units, teachers considered similar questions within narrowed contexts (e.g., Algebra). Because the research question considers the whole of mathematics, the first two units hold more relevance than later units. Readings and discussions were intentionally structured to problematize the nature of mathematics and the nature of mathematics teaching and learning, especially with respect to the role of technology in mathematics teaching and learning.

### *Brief Overview of Unit 1*

In Unit 1, teachers familiarized themselves with the Desire2Learn site and the expectations of the course. They completed a "pre-test" set of three evaluations. They introduced themselves to each other in a first discussion where they posted a Personal Profile describing themselves, their backgrounds, and their hopes for the course. They also read Erlwanger's (1973) *Benny's conception of rules and answers in Individually Prescribed Instruction (IPI) mathematics*. In this reading, Erlwanger presented the case study of a student who was seen as successful in IPI mathematics, but showed troubling misconceptions about the facts and rules he had been using. We as instructors chose this reading as a way to support teachers' thinking about the nature of mathematics and the nature of mathematics teaching and learning, but also to support their thinking about this particular form of independent learning because it is often reflected in expectations and development of computer software that supports students' independent mathematical learning. We asked teachers to discuss the following questions:

> What was Benny learning? Was Benny an active learner? In what sense? What's your evidence? For Benny, what is mathematics? For Benny, how do you decide whether something is true or correct mathematically? Why was remediation difficult with Benny? Was Benny's teacher teaching?

In their individual reflections, we asked teachers to consider the questions: "What insights or new ways of thinking about teaching and learning did you have from reading and discussing the Benny article? Does it raise any issues about technology's role in mathematics teaching and learning?"

***Brief Overview of Unit 2***

In Unit 2, teachers watched a presentation overview of the big ideas of the course (mathematics, teaching, learning, and technology), read (and later discussed) Skemp's (1978) *Relational understanding and instrumental understanding*, and watched (and later discussed) a snippet from *Ma and Pa Kettle Back on the Farm*, a 1951 Universal Pictures film. As described above, Skemp made visible two types of understanding and two types of mathematics: relational and instrumental. Briefly, Skemp defined relational as knowing facts and procedures as well as how and why to use them while instrumental meant "rules without reason" (p. 9). We asked teachers to discuss, in their small teams,

> Which view of knowing and understanding are you drawn to and why? Do you think that both senses of understanding mathematics are valuable goals for teachers to pursue and for students to gain or is one much more important than the other? If you see merit in both, how do you see their relationship in the classroom? Is it possible to pursue both at the same time or are they inherently opposed visions and goals?

The snippet of video from *Ma and Pa Kettle Back on the Farm* included Pa (Percy Kilbride) and Ma (Marjorie Main) proving to Billy (Emory Parnell) that 25 divided by 5 is 14; Billy argued that 25 divided by 5 is simply 5. Ma and Pa used arguments that revealed misconceptions about place value when multiplying, dividing, and adding. Billy attempted to answer their arguments, but was ultimately unsuccessful in convincing Ma and Pa. We asked teachers to discuss, "What's going on with the Kettles? How does their computation make a certain kind of sense when they explain it, but be incorrect?"

### *Brief Overview of Interview*

From the interviews of the six focus participants, I considered responses to both unstructured and structured questions about the nature of mathematics and mathematics teaching and learning. The unstructured questions included:

> First, I want you take a moment and imagine all of mathematics. Mathematics can mean math in the everyday world. It can mean math at school, or math as it's used by professionals: like marketing research, politicians, or scientists. Just all of mathematics everywhere. Describe mathematics to me.

The structured questions were online card sorts to complete the sentences: "Mathematics is ...," "Doing mathematics means ...," and "Learning mathematics means learning..." The sorting task was given online in the form of a Google Slides presentation where the participant had access to manipulate the cards on the slide: to move them around, delete them, add new cards, or edit existing cards. I asked participants to delete any cards they disagreed with, to edit any they felt needed editing, to add any cards if something was missing, and to choose (and move to the top) four or five of the cards they felt completed the sentence best.

### *Analysis of Conceptions*

In the Literature Review section above, I describe ways in which mathematicians and mathematics education researchers have described critical aspects of the nature of mathematics, nature of mathematics teaching and learning, and the role of technology in mathematics teaching and learning. My goal for analysis is to describe each participant's conceptions, based on statements they made in the course and in the interview. I use coding to analyze the data so that I am able to systematically construct these descriptions. Coding, as defined in Creswell (2008), is a process that "involves aggregating the text or visual data into small categories of information, seeking evidence for the code from different databases being used in a study, and then assigning a label to the code" (p. 184). Coding can be done using a grounded theory approach or using preexisting codes. In this study, I use preexisting codes while remaining "open to additional codes emerging during the analysis" (Creswell, p. 185).

I used NVivo 10 (QSR International Pty, Ltd., 2012) to create nodes and code data based on analytic frameworks that emerged from literature discussed in the Literature Review section above. I describe them in more detail at the beginning of each Findings section because some choices that I have made regarding the codes that I focus on here are based on preliminary coding passes. In the following sections, I briefly describe my strategies in a first pass of coding, to tag teacher statements as describing the nature of mathematics, the nature of mathematics teaching and learning, and the role of technology in mathematics teaching and learning. Note that, as described in the Literature Review, these categories are not intended to be mutually exclusive; statements tagged as describing aspects of the nature of mathematics, for example, may also be tagged as describing the nature of mathematics teaching and learning.

For each of the six focus participants, I considered each of the statements made by each participant in Units 1 and 2 discussions and reflections. I also considered statements made by participants in the unstructured and structured sections of their interview. I began by tagging any statements made by teachers that included explicit reference to the nature of mathematics; that is, statements that began "math is ...," "learning math is...," "understanding math is...," etc. were tagged as describing the nature of mathematics. For example, Melinda wrote, "I agree with you Breanna when you say that math is about presenting a set of rules, demonstrating how to use and apply them in different scenarios, and being able to check your answers" (Melinda, Unit 1, Discussion of Benny). I tagged this statement because Melinda wrote "math is about..." In the next several passes, I tagged additional statements that described the nature of mathematics less explicitly. For these statements, I relied on preexisting aspects of the nature of mathematics, especially statements that seemed to indicate whether mathematics is created or discovered, connected or disconnected, complete and finished or always changing, problem-driven or a collection of facts, skills, and rules, always right or wrong, or always open to discussion or interpretation. I tagged these statements over several passes, and thought about potential emergent codes; for example, whether mathematics is about the process and strategies or about the results, what are goals for learning mathematics, and what are mathematical actions.

As I read the statements and attempted to categorize them, I drew conclusions about meaningful codes that I use in the final analysis. For example, I found from course data teachers did not talk about mathematics as created or discovered, complete or changing. I describe my final decisions regarding an analytic framework in Chapter 5, because those choices are part of the report of findings.

## Teacher Noticing

To answer the second research question, *What characteristics, aspects, or features of digital tools and resources emerge through teacher noticings (as evidenced through evaluation activities), either general, mathematical, or pedagogical?*, I explored participants' evaluation templates and evaluations of online tools across the course. I describe below the data sources, the contexts of the evaluations, and analysis methods.

### *Data Sources*

To describe the characteristics of online tools that participants noticed, I analyze evaluation templates and individual evaluations of online tools. Throughout the course, participants explored, discussed, and revised frameworks for evaluating online tools and resources. They used different frameworks to evaluate between 10 and 20 online tools. I describe these activities and their contexts briefly below.

### *Brief Overview of Evaluations*

In the first units of the course, participants created evaluation template frameworks to evaluate online tools that might be used to support mathematical thinking and learning. These templates were designed as lists of questions that participants answered or qualities that participants rated, based on their exploration of a particular online tool.

In Units 1 and 8, participants were given three online tools to evaluate without use of templates as a pre- and post-test. Participants were asked to

> assume you are considering using these tech tools in your teaching. Evaluate the three tools to decide whether you would recommend using them. Write an evaluation of the three tech tools for an audience of other teachers who might be considering using them. Make your recommendation clear: which (if any) would you recommend? Argue your position. You will need to decide which elements, features, or characteristics of each tech tool to use in supporting your argument.

In Unit 2, participants in each group read group members' initial three evaluations. They discussed and developed a common evaluation template and used it individually to evaluate an online tool.

In Unit 3, each group shared the group template with the whole class. All members of the course discussed important questions and characteristics and developed a class template. Each participant then used the whole class template to evaluate an online tool focused on "Numbers & Operations" and "Computational Power" of technologies.

In Units 4 and 5, the instructors presented participants with a revised whole class template. Participants evaluated an online tool in each of the two units. In Unit 5, participants reflected on their use of the whole class evaluation template and proposed how they would revise it.

In Unit 6, participants evaluated two online tools with their revised template. For some participants, the revised template was very similar to the whole class template, and for others it was quite different.

In Unit 7, instructors presented participants with an updated evaluation template based on comments and whole class discussion. Participants completed an evaluation and began developing their final course project, an Online Resource Library. For the online resource library, participants were asked to include 20 annotated links to online tools and resources. Ten of the 20 annotated links were required to be the participants' own evaluations. Teachers were allowed to use evaluations they had previously written from any unit of the course. The remaining annotated links were allowed to be other completed evaluations, newly written evaluations, or a brief description of an online tool or resource.

Participants interpreted course requirements in slightly different ways, and so ended with a variety of different types of evaluations. All participants wrote two evaluations, at least, for online tools from Unit 1 and 8. Some participants wrote three or more evaluations for a single tool using different templates and for different units. I show a total count of evaluations written by each participant and a total count of tools evaluated by participants. A classification of each focus participants' evaluations appear in Table 3.2. Participants completed unstructured initial and final evaluations at the beginning and end of the course (Units 1 and 8). They completed an evaluation using a team-developed template in Unit 2. Participants used whole class templates in Units 3, 4, 5, and 7. These whole class templates were revised three times during the course (Class-1, Class-2, and Class-3). In Unit 6, they used their own revision of the whole class evaluation template. For her Online Resource Library, Jana re-evaluated four technology tools using the Class-3 template: three were the tools she evaluated for her initial and final evaluations, and one tool was the same that she evaluated in Unit 2. With the exception of Jana, and excepting initial and final evaluations, participants evaluated each tool only once. A full list of the technology tools evaluated by focus participants is shown in Appendix B. Depending on the analysis, at times I consider the number of teachers' evaluations and at other times I consider the number of teachers' tools that they evaluated.

Participants evaluated online tools and resources that might be appropriate in support of mathematics teaching and learning at their grade level. In the context of the course, we defined *technology tools* as software, apps, applets, or website content that would directly support doing mathematics or mathematics teaching and learning; for example, videos, quizzes, virtual manipulatives, computational or graphical tools. We defined *technology resources* as website content that would indirectly support doing mathematics or mathematics teaching and learning; for example, data repositories, online lesson plans, blogs, textbooks, printables, videos of implementation.

Table 3.2

*Participant Evaluation Types and Viable Tool Evaluations*

| Participant Name | Initial Unit 1 | Team 2 | Class-1 3 | Class-2 4, 5 | Revised 6 | Class-3 7, ORL | Final 8 | Eval Total | Tool Total |
|---|---|---|---|---|---|---|---|---|---|
| Breanna | 3 | 1 | 1 | 3 (2) | 2 | 10 (5) | 3 | 23 (17) | 20 |
| Melinda | 3 | 1 | 1 | 3 (2) | 2 | 9 (5) | 3 | 22 (17) | 19 |
| Jana | 3 | 1 | 1 | 3 | 2 (1) | 10 (3) | 3 | 23 (15) | 15 |
| Nicole | 3 | 1 | 1 | 3 (2) | 2 | 10 (9) | 3 | 23 (21) | 20 |
| Heidi | 2 (1) | 1 | 1 | 3 | 2 | 9 (6) | 3 | 21 (17) | 20 |
| Alison | 8 | 1 | 1 | 3 (2) | 2 | 0 | 3 | 18 (17) | 18 |

*Note.* Values in each column, except the rightmost, represent the number of evaluations completed by teachers. In the rightmost column, the number of individual tools evaluated is shown. Values in parends indicate the number of evaluations for online tools with viable links at the time of analysis.

In Unit 1, for the initial evaluations, we provided four sets of three online tools each and asked participants to choose one group of three tools to evaluate. (Note: Alison encountered some difficulty accessing some tools and so she attempted to evaluate all 12 tools in her initial evaluation.) The instructors intended each set of three tools to be appropriate to one grade band and to focus on one mathematical topic. For example, all participants but Heidi evaluated the first set of online tools: Maths is Fun: *Math Trainer,* NCTM Illuminations: *Ten Frame Addition,* and NLVM: *Base Blocks Addition.* In Unit 8, for the final evaluations, participants evaluated the same set of three tools they evaluated in their initial evaluations. (Alison was assigned one group of tools.)

For other units, we provided a list of online tools appropriate to the focus of the unit. Unit focus and structure is shown in Appendix A. We also encouraged participants to find and evaluate their own online tools throughout the course; however, as the course continued, we asked more frequently and explicitly for participants to search for and choose their own technology tools and resources. For purposes of this study, I focus on participants' evaluations of technology tools. In Table 3.2 above, all evaluations are shown. To show the number of tool evaluations, specifically, any changes are shown in parentheses. Not all sites remain available, and so tool evaluations with viable links at the time of writing are counted.

### *Analysis of Noticing*

Analysis of evaluation templates provides one perspective on the features of an online tool that participants notice as important. Each participant may interpret the meaning of a question or quality differently than other participants in their response to an evaluation template, however, so analyzing completed evaluations both in response to formal templates, to personalized templates, or to an open question, reveals perspectives on characteristics participants' saw and felt were important in a tool. I used NVivo 10 (QSR International Pty, Ltd., 2012) to create nodes and code data.

In analyzing completed evaluations, I focused on a participant's description of mathematical purpose, mathematical representation, mathematical learning activity (e.g., exploration or practice), potential type of mathematical understanding (i.e., instrumental or relational), and student engagement. In the whole class evaluation templates, participants answered explicit questions to describe such features. For example, in the final version of the whole class evaluation template (see Appendix E), questions 2, 3a, 3c, 4a, 4c, and 5b ask for a participant to address these features. In questions 2 (learning activity types), 3a (relational or instrumental), and 3b (proficiency strands, process standards, or mathematical practice standards), I coded the technology tool precisely as the participant did. For example, Melinda described the Smarty Games: *Learn Money* tool as:

2: Practice, Explore, Apply

3a: Both instrumental and relational

*Smarty Games Learn Money technology tool teaches the concept of money. It includes everything from identifying a coin amount, counting coins in various fashions, coin identification, and matching coins to an amount. Both instrumental and relational understanding are used because students are not just counting money laid out for them. They are having to think about the amount that they need, determine and count the coins that they need to equal that amount, and decipher between which coins combination they wish to use to match up to the price tag on each item. The teacher could even expand this tool and explain to the students that they need to use a few coins as possible to create the amount on the price tag. Students* *not only have to have basic knowledge of money, the coins, and their amounts, but they also have to count and develop a combination that adds up to the amount needed to buy each item, which can vary.*

3c: (did not respond)

4a: *Students must have basic understanding of how to manipulate a mouse, clicking, dragging, dropping, etc. to use this tech tool. Otherwise it is fairly simple to use for second graders.*

4c: *Mathematics is represented with the use of coins including: quarters, nickels, dimes, and pennies. It is also represented on the price tag of each item where an amount is expressed using the cents sign. Students must count out the amount of coins that equals that amount on the price tag, therefore their coin amount also represents mathematics.*

5b: *I believe the tool is attractive to students and seems rather engaging. There are brighter colors and different clipart used, but its not too distracting from the actual coin counting practice, which is the purpose of the tool. I wouldnt say that it is overly attractive or one of the sharpest tools in the form of aesthetics, however, I do think its appealing enough for students to be engaged and to remain engaged through the duration of using the tool.*

Hence, Melinda coded the tool as Learning Activity (Practice, Explore, Apply) and Mathematical Understanding (Instrumental, Relational).

## Potential Relationships between Conceptions and Noticing

In order to examine data in response to the third research question, *What potential relationships emerge between teachers' conceptions and teachers' noticing as described in the previous questions?*, I first completed analyses and findings for the first two research questions. Following Yin's (2014) analytic strategies, I wrote research memos as I completed analyses for the first two research questions. In the research memos, I wrote hypotheses and questions about potential relationships.

In the memos, I considered how conceptions of the nature of mathematics and mathematics teaching and learning might inform the aspects of online tools that teachers might attend to in their evaluations.

As a summary of the noticing findings, I attempted a preliminary examination of potential relationships. Based on that preliminary examination of data, I returned to Ernest (1989b) and Thompson (1992) their descriptions of potential relationships between a teacher's conceptions and her approach to teaching and student learning. I created a framework based on their descriptions. Based on the framework and my reading of this data, I attempted to extend the framework to teachers' noticing of online tools. I describe the initial framework below after describing sources of data.

### *Data Sources*

To analyze data for potential relationships between teachers' conceptions and noticing, I used Ernest (1989b) and Thompson (1992). Building from their proposed relationships between teachers' conceptions and teachers' approaches to mathematics teaching and learning, I examined participants' statements from their evaluations of online tools for alignment with conceptions. I describe the relationship framework next.

### *Analysis of Relationships*

Based on Ernest (1989b) and Thompson (1992), I created an analytic framework. I used it to create a coding scheme using NVivo 10 (QSR International Pty, Ltd., 2012). I describe Ernest's and Thompson's framework, adding in my extensions to teachers' noticing of online tools. By examining the data through Ernest's and Thompson's lens, I test and extend their theory.

Ernest (1989b) and Thompson (1992) proposed relationships between teachers' conceptions of the nature of mathematics and their views (and practice) of mathematics teaching and learning. Thompson described Ernest's framework in her own terms and with her own examples. Ernest justified his framework by referencing earlier research by Thompson. Because of the clear interaction between their descriptions of the conceptions framework, I attribute ideas described by both to both researchers. In this section, I use the preliminary examination of data along with Ernest's and Thompson's descriptions, to extend those relationships to teachers' noticing of features or affordances of online tools.

Thompson (1992) and Ernest (1989b) proposed characteristics of ways in which a teacher holding an Instrumentalist conception would view learning and approach teaching. Each argued that such a teacher would view learning as engaging in compliant behavior and outcomes learning as skills mastery and correct performance. They argued that such a teacher would view the action of teaching as instructing, and would value teaching actions such as providing clear presentations and allowing practice of identifying concepts and performing simple procedures. My extension of this framework to teachers' noticing of aspects of online tools changed little. It focused on clearly presenting facts, concepts, or examples of procedures; allowing independent practice of identifying concepts and performing simple procedures; and giving immediate feedback on correctness.

Thompson (1992) and Ernest (1989b) similarly described extensions to the framework, indicating characteristics of how a teacher holding a Platonist conception would view learning and approach teaching. They argued that such a teacher would viewing the action of learning as receiving knowledge and learning outcomes as conceptual understanding and unified knowledge. They argued that such a teacher would view the action of teaching as explaining, and emphasizing the meanings of ideas, logic of procedures, and connections between concepts. My extension to noticed aspects of online tools included: providing conceptual explanations of meanings and relationships, providing opportunities for exploration to understand deeper meanings of concepts and procedures, using representations and mathematical consequences in response to student actions.

My expectations for what teachers with a Problem-Driven conception would notice in a tool were not reflected in the data. First, the word *problem* was used inconsistently by teachers. They used the term *problem* to refer to any level of mathematical question, whether it was a simple procedural or identification task or a more complex procedural or investigative task. Second, because no teacher held a purely Problem-Driven conception, the impact of other conceptions on their noticing would have obscured any potential patterns. Third, the evaluated online tools rarely included rich tasks, complex mathematical situations, or ill-defined and open questions. Thompson's (1992) and Ernest's (1989b) descriptions of student activity in a Problem-Driven teacher's classroom focused on students themselves driving the learning, communicating, conjecturing, etc., within a fairly open and complex mathematical situation, with a teacher providing guidance and loose structure. In such a classroom, I would imagine a teacher making multiple online tools available to students to use in support of larger goals. Indeed, several online tools evaluated by teachers provided tasks that could be fairly open and allowed student investigation of ideas, but not to the extent described by Thompson and Ernest. In addition, teachers who evaluated such tools included statements that indicated their concern that students would be confused, would not achieve the intended discovery, or would play around in an unfocused way. These teachers (i.e., Nicole, Heidi, and Alison) described their perceived need for sets of questions or worksheets to guide students through the task. For example, Alison wrote about one such tool, "I would definitely need to have students follow through an exploration or something." Heidi wrote about a similar tool, "teachers may also need to provide a worksheet or lesson to help guide students with using the tool. There is a lot to explore, which can be overwhelming." Their proposed solutions to such open exploratory tasks would lower the complexity and demand of the tasks. For these reasons, I focus on considering relationships between teachers' noticing and the Instrumentalist and Platonist conceptions, and exclude the Problem-Driven conception.

Figure 3.1 shows a extension of Figure 2.1, of potential ways that holding an Instrumentalist, Platonist, or Problem-Driven view may impact the affordances or constraints that a teacher notices in an online tool.

Considering the Instrumentalist view of mathematics, Figure 3.1 extends the views on teaching and learning to views on valuable affordances or features of online tools. For example, Thompson (1992) proposed a teacher with an Instrumentalist view would, in her teaching, value clear presentations of concepts and procedures and allow students opportunities to practice identifying concepts and performing procedures. Ernest (1989b) proposed that a teacher with an Instrumentalist

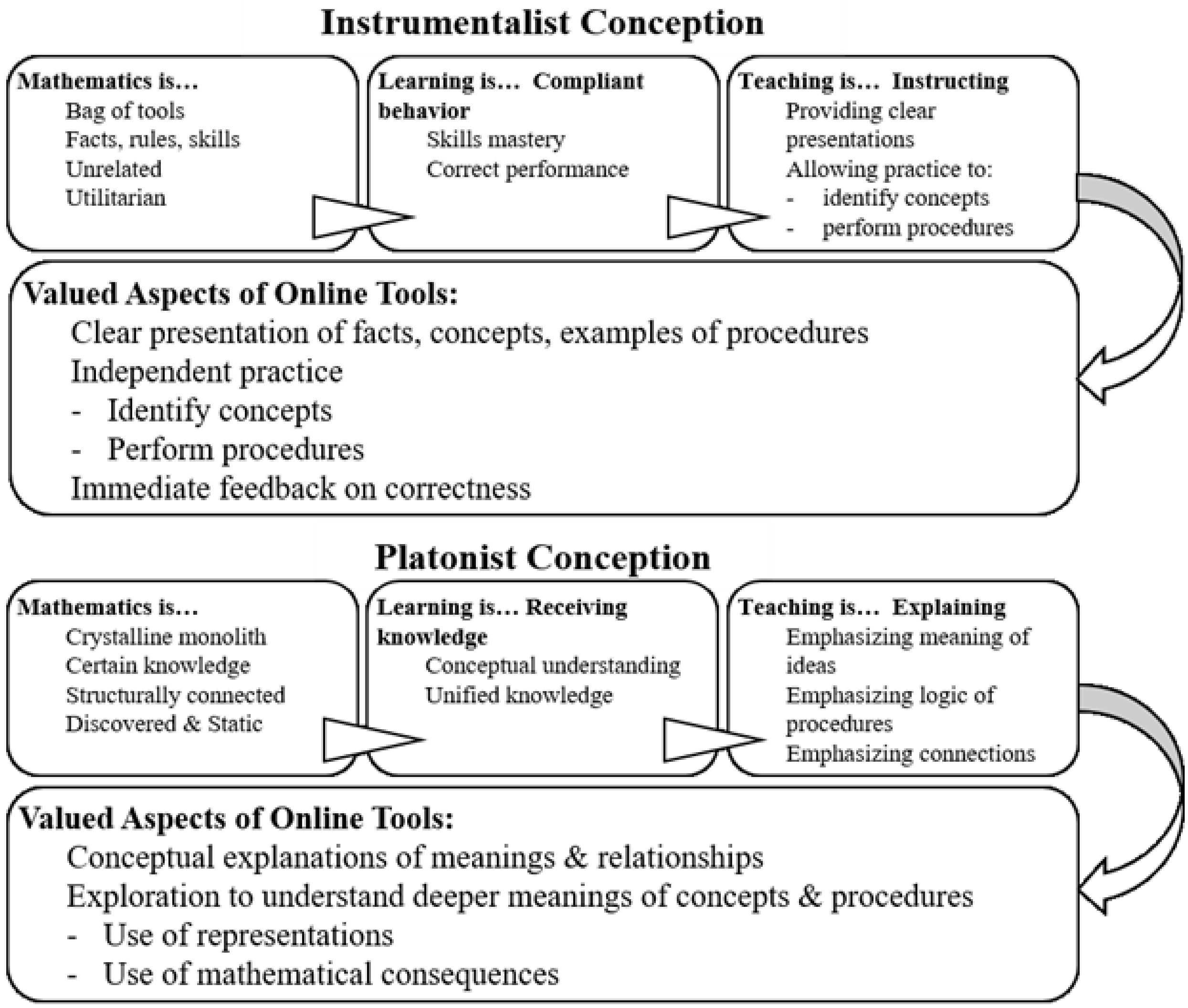


Figure 3.1: Relationships Framework: Adapted from Thompson (1992) and Ernest (1989b)

conception would act as an Instructor, and students' learning would be exemplified by compliant behavior, correct performance, and mastery of skills. I propose that such a view would direct a teacher's attention, in evaluating online tools and resources for use in such a classroom, to opportunities provided for a student to practice procedures and skills independently, with immediate feedback indicating correct or incorrect results. A teacher would attend to features that support this type of practice, or that would present clear examples, or would present facts and procedures in a clear way that could support a student's practice.

Figure 3.1 similarly summarizes views of teachers with Platonist and Problem-Driven conceptions of mathematics. I propose that teachers with a Platonist view would focus their attention on features of online tools that allow students to explore deeper meaning of procedures or concepts and provide conceptual explanations of meanings and relationships. Finally, I propose that a teacher with a Problem-Driven conception would attend to online tools and resources that support the type of teaching shown in Figure 3.1; that is, the teacher would attend to affordances of a tool that allow a student to explore, create, discover, investigate information mathematically, and develop arguments and reflection to support communication and critiquing other arguments. The teacher might also attend to constraints of a tool that limit a student's creativity, exploration, or critical investigation of a situation or information.

# CHAPTER 4

# THE TEACHERS

In this chapter, I describe the participant teachers, their backgrounds, and (when applicable) the schools in which they taught at the time of the study. Not all teachers were teaching at the time of the study, but, when available, a brief description of the school can provide important context that helps situate them. I provide an overview of all participants, first, and then I tell the stories of the focus participants. Because teachers worked closely with each other in small grade-level teams, an overview of all team members who consented can help provide additional context as part of each of the focus participants' stories.

Breanna, Melinda, Jana, Nicole, Heidi, and Alison each consented to a post-course, semi-structured interview. This study focuses on their conceptions and noticing, so I describe these teachers briefly based on details they shared in their pre-course Personal Profile introduction, in discussions and reflections, and in the post-course interview. These backgrounds provide some context because participants' past experiences with mathematics, technology, and teaching inform their perspectives and reactions to experiences in the course. Before describing each participant in more depth, however, I briefly summarize in Table 4.1 details about the schools for which teachers worked. I include details about all participants, not only the focal participants, because the backgrounds of team members can help situate focal participants.

The details about each school can give additional context to the differences in the teachers' experiences with teaching and learning mathematics, especially regarding their administrators, community, students, and resources. I use details provided by National Center for Education Statistics from the Common Core of Data, based on enrollment data from 2014-2015 school year (for Education Statistics (NCES), 2014-2015). I include the grade levels served by the school, the student to teacher ratio, the percentage of students who qualified for free or reduced school lunch, and race/ethnicity enrollment percentages.

Table 4.1 shows the locale of each school, its grade span, student-to-teacher ratio, percentage

Table 4.1

*Overview of School Context for Each Participant*

| Participant | Locale | Grade Span | S:T Ratio | % Free/Red. Lunch | Title 1 |
|---|---|---|---|---|---|
| Lower Elementary Group | | | | | |
| **Breanna** | Lg. City[a] | preK-5 | 21:1 | 85.74% | Yes |
| Kristine | – | – | – | – | – |
| **Melinda** | Dist. Town[d] | preK-2 | 14:1 | 62.73% | Yes |
| **Jana** | – | – | – | – | – |
| Upper Elementary Group | | | | | |
| Jason | Lg. Suburb[c] | K-5 | 20:1 | 12.63% | No |
| **Nicole** | – | – | – | – | – |
| **Heidi** | Sm. City[b] | preK-5 | 17:1 | 36.93% | Yes |
| Middle Grades Group | | | | | |
| Ashley | Fringe Rural[e] | 6-8 | 23:1 | 46.03% | No |
| **Alison** | Sm. City[b] | 6-8 | 17:1 | 18.56% | Yes |
| High School Group | | | | | |
| Brandon | – | – | – | – | – |
| Ben | Lg. Suburb[c] | 7-12 | 11:1 | 29.27% | No |
| Denise | Sm. City[b] | 9-12 | 22:1 | 71.56% | Yes (Yes) |
| Heather | Lg. Suburb[c] | 9-12 | 22:1 | 10.40% | No (–) |

*Note.* Overview of schools in which participants' held a position. Enrollment statistics collected from National Center for Education Statistics' (NCES) Common Core of Data (CCD) for each school (2014-2015). Focus participants are in boldface. Participants without placements at the time of the study are listed without data.
[a] inside Principal City w/ pop. $\geq$ 250,000 [b] inside Principal City w/ $250,000 \leq$ pop. $\leq 100,000$
[c] outside Principal City & inside Urbanized Area w/ pop. $\leq$ 250,000 [d] inside Urban Cluster > 10 miles & $\leq$ 35 miles from Urbanized Area [e] $\leq$ 5 miles from Urbanized Area or $\leq$ 2.5 miles from Urban Cluster

of students who qualified for free or reduced lunch, and whether the school was a Title 1 school or had a school-wide Title 1 program. Breanna, Melinda, and Denise worked for schools with over half of the students qualifying for reduced or free lunch. Their schools also had school-wide Title 1 programs. Heidi and Alison worked for Title 1 schools that did not have school-wide Title 1 programs. NCES designates a locale as City, Suburb, Town, and Rural. NCES further subdivides locales as follows: "For city and suburb, these are gradations of size  large, midsize, and small. Towns and rural areas are further distinguished by their distance from an urbanized area. They can be characterized as fringe, distant, or remote" (for Education Statistics (NCES), 2014-2015).

The teachers worked for schools in a variety of locales, from Ashley in a “Rural: Fringe” locale to Breanna in a “City: Large.” A Large City has 250,000 residents or more; a Small City has fewer than 100,000 residents. Teachers also worked at schools with a range of student-to-teacher ratios, from 11 students per teacher at Ben’s high school, to 22 at Ashley’s middle school and Heather’s high school.

Table 4.2

*School Demographics*

| Participant | Stu. Total | Asian[a] | Black | Hispanic | White | Multiracial |
|---|---|---|---|---|---|---|
| Lower Elementary Group | | | | | | |
| **Breanna** | 680 | 0.00% | 99.41% | 0.29% | 0.29% | 0.00% |
| Kristine | – | – | – | – | – | – |
| **Melinda** | 601 | 0.00% | 2.50% | 1.16% | 96.01% | 0.33% |
| **Jana** | – | – | – | – | – | – |
| Upper Elementary Group | | | | | | |
| Jason | 483 | 6.42% | 8.07% | 4.14% | 79.30% | 2.07% |
| **Nicole** | – | – | – | – | – | |
| **Heidi** | 834 | 2.52% | 11.51% | 25.54% | 55.52% | 4.80% |
| Middle Grades Group | | | | | | |
| Ashley | 554 | 0.54% | 4.33% | 5.78% | 81.77% | 6.14% |
| **Alison** | 738 | 32.79% | 7.18% | 3.79% | 52.71% | 3.39% |
| High School Group | | | | | | |
| Brandon | – | – | – | – | – | – |
| Ben | 714 | 1.12% | 0.98% | 6.58% | 89.50% | 1.54% |
| Denise | 320 | 1.56% | 66.25% | 26.25% | 5.00% | 0.29% |
| Heather | 1645 | 7.54% | 2.98% | 2.80% | 84.56% | 0.63% |

*Note.* School demographics for each participant at time of study include number of students enrolled and percentage of students identified as Asian, Black, Hispanic, White, or Multiracial. Only race/ethnicity categories with at least 2% in at least one of the schools were included in the table. Nonresident aliens and race/ethnicity unknown categories were not included. Enrollment statistics were collected from National Center for Education Statistics’ Common Core of Data for each school (2014-15). Focus participants are in boldface. Participants without placements at the time of the study are listed without data.

[a] Combined Asian and Native Hawaiian / Pacific Islander categories

Table 4.2 shows additional differences in context for the teachers, including the size of each school and demographic information for students enrolled in each school. Five of the teachers worked at schools with over 80% of the students identified as white. The four schools that were not Title 1 (as shown in Table 4.1) are included in these five schools with over 80% white students. Melinda's school was unique in that almost all students identify as white, and the school was located in a town distant from an urban cluster, but it had a school-wide Title 1 program with a high percentage of students qualifying for free or reduced lunch. Denise and Breanna worked at schools where over two-thirds of the students identified as black. Their schools both had school-wide Title 1 programs and had high percentages of students qualifying for free or reduced lunch (85.74% and 71.56%, respectively). One quarter of the students enrolled at Denise's school identified as Hispanic. At Heidi's school, over a quarter of the students identified as Hispanic. Alison worked at a school where almost one third of students identified as Asian (which includes Native Hawaiian and Pacific Islander categories).

## Breanna

Breanna wrote in her Personal Profile that she "loved math growing up and I still love math now." Of her own experiences learning mathematics with technology, she mentioned only that she had enjoyed using *MyMathLab* in her university mathematics course. In her preparation to become a teacher, Breanna had hoped to earn a mathematics credential but felt the mathematics courses became too difficult. She took Trigonometry and two courses for Mathematics for Elementary Teachers as her highest mathematics courses. She was certified to teach K-5 and was in her second year of teaching; for both years she had taught Kindergarten, and had taught in the 3-5 grade-level band for her student teaching. Breanna said she was using *Go Math* (Houghton Mifflin Harcourt, 2011) and that she did not have a choice of which curriculum she should use, but she could choose materials (e.g., tools, resources, activities) used in her mathematics teaching.

At the time of the study, Breanna taught Kindergarten and, in addition to teaching her class, she was enrolled in two courses as part of her online Master's of Education program. She also was lead teacher in her grade, despite being the newest teacher to teaching. Her position meant additional responsibility with no additional pay; for example, she was responsible for creating kindergarten events, sending notes home with students, and facilitating communication both ways between the other kindergarten teachers and administrators. She explained,

> I say I do 90% of the work. They [the other teachers] do 10% of the work. And I even have to leave sometimes. On Thursday at 9 o'clock, they told me 'You have to leave by 9:45. You have to leave and go somewhere.' and I was like 'What?' So they even make me go off campus and do stuff.

Breanna also worked a second job as a waitress to make ends meet. She said, "I actually work seven days a week. I don't know how I do it. ... I just get up and go."

## Melinda

In her Personal Profile, Melinda wrote that she "had a decent overall experience in mathematics throughout school. I enjoyed my teachers and felt they were very knowledgeable." She said what she felt her experience lacked was

> the amount of hands on experiences. I feel like there are so many people that struggle or are even intimidated by math because, at times, it can be abstract. But if you create real-world, hands on experiences to create concrete experiences that relate, concepts are more easily understood.

In preparing to become a teacher, Melinda's highest mathematics course was College Algebra and she had not earned a mathematics credential. She enrolled in this course as part of her online Master's of Educational Technology program.

At the time of the study, Melinda taught second grade and was in her third year of teaching; she had taught two years of seventh grade Communication Arts and was currently teaching her first year of Second Grade. In addition to teaching her class and taking master's level courses, she also served as the technology representative for her building. She described this position as focal for other teachers in her building: "colleagues are constantly looking to me for a great web tool to incorporate into their teaching." Her district did not use a mathematics curriculum, but teachers at each school and grade level created their own. She served on her second grade curriculum committee to develop a mathematics curriculum, with a curriculum coach as support. She explained that she enjoyed this experience

> because we have created our own curriculum, we are able to take a deeper approach to what students in second grade should understand. With that being said, we accomplish this by using web tools, manipulatives, games, projects, etc. to reach that conceptual understanding that many of the standards discuss.

She said she did not have a choice in mathematics curriculum, but that she could choose materials (e.g., tools, resources, activities) used in her mathematics teaching.

## Jana

Jana had earned a Master's in Biochemistry, and had taken some mathematics as part of that but was mostly focused on science. She was enrolled in an online Master's of Educational Technology program. Jana had learned English as an additional language. Jana's perspective in her grade-level group (Lower Elementary) was unique, likely because of her many years of experience at different grade levels and in different countries. To prevent the participant from being easily identified, I do not include the details of which countries or languages Jana spoke.

At the time of the course, Jana was not teaching. She explained that she was on a one-year leave from her school board. She normally taught in an English-speaking country, but was living in the United States at the time of the course. She was certified to teach K-12 and had 16 years of teaching experience. She had taught at all grade band levels, and had taught in three different countries. First, she had taught in her home country "as a Grade 7, 8 Math and Science teacher" and then as a "Grade 9, 10 Physics, Chemistry, Maths and Science" teacher. She continued there by working as a "5-10 Math, Science, Biology and Chemistry teacher as well as K-12 vice Principal." Next, she moved to an English-speaking country and taught grades K-4. She then briefly taught in a Michigan private school. She had worked briefly with younger children in Kindergarten, First, and Second grades.

Throughout her teaching, she had also taught Special Education classes, especially Special Education in Mathematics. In the English-speaking country where she normally teaches, *CCSSM* is not used; Jana used curriculum documents from that country instead of *CCSSM*. Jana had a daughter in Second Grade, and, because she was not currently teaching, often described her daughter's homework and experiences as she responded to course activities. She reported that she did not have a choice in the mathematics curriculum, but could choose materials (e.g., tools, resources, activities) used in her mathematics teaching.

## Nicole

Nicole said that, as a student, "I always enjoyed my math classes in school." She said problem-solving was her favorite part of mathematics, explaining "every problem was like a puzzle that I had to work through to find the answer. When working on math problems, I always enjoyed the 'aha!' feeling of solving the problem and finding the answer." She said that even though she felt mathematics was always one of her stronger subjects, she also "had to put forth effort and work hard throughout all of my classes." In her teacher preparation program, she had specialized in middle school mathematics. She had taken three elementary mathematics content courses, and two middle school mathematics content courses. Because of her mathematics specialization, she took additional mathematics courses, which included courses such as "Finite Math, Discrete Math, Calculus," culminating in Calculus II.

Although Nicole was working full-time on her Master's program in Teaching and Curriculum full-time at the time of the study, she had taught in Wisconsin for the five years previous. She taught as a substitute for one year, as a long-term substitute teacher the next year, and then a full-time classroom teacher for three years. She had taught in the 3-5 and 6-8 grade-level bands. She taught fifth grade mathematics and science for one year, using *Everyday Mathematics* (University of Chicago School Mathematics Project (UCSMP), 2005) for her mathematics curriculum. She then taught two years of middle school sixth grade mathematics using the *Connected Mathematics Project 2* (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2009). She said that she did not have a choice in mathematics curriculum, nor a choice in materials (e.g., tools, resources, activities) used in her mathematics teaching.

## Heidi

Heidi wrote in her Personal Profile that she had "had wonderful math teachers from primary through my undergrad study that kept me both engaged and enthused to teach this content myself." Even though she had positive experience overall, she mentioned in her interview a powerful experience from her second college mathematics course. She said,

> I remember getting an answer that was completely different than what my classmates were and the professor getting really, really upset. And then finally someone else chimed in that they had the same response and actually ours was the correct one but we had not solved the problem the way she had described it. And it was weird to me to be in a liberal institution where I was almost being penalized for my right answer because I hadn't done it her way.

At the time of the study, Heidi was a Fifth Grade Math / Science Teacher for Gifted and Talented Students, and it was her "sixth year teaching, and my sixth year using the *Everyday Mathematics* program." Heidi's teaching centered on project-based learning. She said that she did not have a choice in mathematics curriculum, but was able choose materials (e.g., tools, resources, activities) used in her mathematics teaching. In addition to teaching, she was enrolled in an online Master's in Education program, with an emphasis in technology. In her teacher preparation program, Heidi had earned a mathematics major for elementary education. This preparation meant she had taken mathematics courses through Calculus II. Heidi mentioned that she and Nicole had really bonded and had found they shared many similar perspectives as they worked together and discussed ideas in their grade-level group (Upper Elementary) in this course.

## Alison

Alison wrote in her Personal Profile that her "dominant experience of mathematics in school was this: desks in rows, textbooks open, notebooks out, teacher up front, chalkboard filled to the ends of every corner with problems, then homework at home." She said mathematics always came easily for her, "but I was never engaged. Ever. All of the math teachers were usually weird geeky type people with not much of a personality and I never really felt that math really mattered in the world." She explained that through her teaching program and her mathematics courses, she began to see that mathematics "didn't have to be boring and awful." She saw "that math literally was EVERYWHERE!" Alison had earned a mathematics credential with her teaching degree, majoring in mathematics and language arts. Her highest mathematics course was Calculus.

At the time of the study, Alison taught Seventh Grade Algebra, and she had been teaching for four years; two years at the current school and two years previously at a private school. At the time of the interview, she was teaching "Algebra and two support level courses for struggling students." She taught using the *Big Ideas for Common Core edition* (Larson & Boswell, 2014). She said that she did not have a choice in mathematics curriculum, but could choose materials (e.g., tools, resources, activities) used in her mathematics teaching. Her hope for earning a Master's in Educational Technology was to become the director of technology for a district.

# CHAPTER 5

# TEACHER CONCEPTIONS

The nature of mathematics and mathematics teaching and learning is complex and can be viewed from many perspectives, whether by mathematicians, mathematics educators or educational researchers, teachers or their students. Educational researchers have attempted to describe potential perspectives that may impact the way K-12 teachers choose to teach mathematics (e.g., Ernest, 1989a; Thompson, 1992; Williams, 2012). I described these perspectives in Chapter 2; here I describe their use in this study.

Ernest (1989a) described three views of the nature of mathematics that encompassed potential understandings of characteristics of the nature of mathematics; such as, whether mathematical ideas are connected or disconnected, ever-changing or complete and final, or created or discovered.

In the card sorting activity of the interview, I asked teachers to consider their view of the nature of mathematics. I created cards based on characteristics of Ernest's (1989) three views, as discussed in Chapter 2. Table 5.1 shows the number of teachers that agreed or disagreed with a particular characteristic, along with the number that edited each card by altering the wording or deleting it entirely and the number that chose the characteristic as one of the most important characteristics of mathematics.

Based on Table 5.1, most teachers (three or more each) agreed that mathematics is: both created and discovered, both connected and not isolated, continually changing or always the same, all three: abstract and concrete and practical, both about problem-solving and a collection of facts, skills, and rules, and both open to discussion and not just right or wrong.

Teachers seemed to view the nature of mathematics through their experiences in teaching and learning school mathematics; that is, they viewed mathematics as a school subject more often than as a larger set of ideas or ways of thinking. That they viewed mathematics narrowly through their teacher lens is reasonable and unsurprising, but led to surprising responses that illustrate the danger of taken-as-shared meanings for these terms. For example, in discussing whether mathematics is

Table 5.1

*Overview of Card Sort Results: Mathematics is...*

| Mathematics is ... | Agree | Disagree | Edited | Deleted | Top Choice |
|---|---|---|---|---|---|
| Created | 4 | 2 | 3 | 1 | 1 |
| Discovered | 5 | 1 | 3 | 2 | 2 |
| Ideas that are all connected | 6 | 0 | 1 | 0 | 4 |
| Made of many isolated topics | 1 | 4 | 4 | 4 | 1 |
| Always going to be the same | 3 | 3 | 2 | 2 | 1 |
| Continually changing | 5 | 1 | 3 | 1 | 3 |
| Abstract | 4 | 1 | 5 | 2 | 0 |
| Concrete | 5 | 0 | 3 | 1 | 0 |
| Practical | 4 | 0 | 3 | 0 | 1 |
| All about problem-solving | 4 | 0 | 1 | 0 | 4 |
| A collection of facts, skills, and rules | 5 | 0 | 0 | 0 | 4 |
| Either right or wrong | 1 | 4 | 4 | 3 | 0 |
| Always open (discussion, diverse views) | 5 | 0 | 0 | 0 | 5 |

*Note.* Number of teachers who agreed or disagreed with characteristics in response to the "Mathematics is..." prompt in the interview card sort. Shows the number of teachers responding to each characteristic, and how many edited, deleted, or chose each as a top choice.

created or discovered, or is always the same or continually changing, some teachers agreed with seemingly opposing views. When they explained their responses, it was clear that the teachers interpreted these words and phrases in unexpected ways. In the next paragraph, I illustrate teachers' focus on school mathematics along with the potential for diversity in meanings of these ideas.

For example, teachers responded in the card sort activity as if mathematics is both created and discovered, but spoke about student creation or discovery of mathematical ideas as part of learning (school mathematics focus) in overlapping ways with creation or discovery of mathematics as it exists in the world (external mathematics focus). One teacher, Alison, explained that "we can create math and we can kind of make it what we want," indicating an external mathematics focus: mathematics used in jobs and life. She went on to say that we "are always still discovering topics... even our students are continually discovering things." The first phrase is unclear, but the second phrase referred to student discovery of mathematical ideas as a part of learning (school mathematics focus). Jana explained that we discover strategies, tools, and formulas, but said, "I don't think so, it's created. That's natural around us in our life. ...It is already there." Similarly, as teachers explained their views on whether mathematics is continually changing or always going to be the same, their meanings seemed embedded in school mathematics. Heidi, Melinda, Breanna, and Jana mentioned the strategies, solutions, or ways of doing mathematics are continually changing. Nicole talked about strategies for teaching mathematics as changing. Heidi and Jana talked about policies, standards, and strands as changing. Heidi, Nicole, Breanna, and Jana referred to mathematics, or at least some mathematics concepts, as unchanging, even if strategies or policies change. Alison and Nicole explicitly discussed mathematical concepts as changing. In fact, Nicole began by stating that mathematical concepts do not change, but ended by saying mathematics must change "or why are there mathematicians?"

## Modifications to Research Question 1

Based on participants' responses, and the differences in their descriptions, I chose to refine my original research question to focus on a subset of conception characteristics. I discussed this choice with co-chairs of my dissertation and they approved my plan. As described above, findings suggested that teachers did not consistently view mathematics as distinct from mathematics teaching and learning. Hence, I focus on answering a revised question: *What conceptions of the nature of mathematics in the context of mathematics teaching and learning are evidenced through teachers' participation in course activities and post-course interview?* Interview questions were based on what I saw from course activities and what I hoped to learn based on the Ernest framework, so my findings reflect both course activities and responses to interview questions. I asked teachers about their views of mathematics through an open question as well as a more structured question.

I describe findings around the following themes: (a) connectivity of mathematical ideas; (b) focus of mathematics on problem-solving versus facts, skills, and rules; and (c) mathematical problems have right or wrong answers, or are open to discussion and diverse views.

## Overview of Data, Analysis, and Findings

To answer this part of the first research question, I analyzed discussion and reflection data from Units 1 and 2, along with data from the interview card sort task. I coded data using NVivo by first coding statements made by participants in Units 1 and 2 according to the large themes from Ernest (1989a) that were relevant: (a) mathematical connectivity; (b) focus of mathematics as problem-solving; (c) focus of mathematics as facts, skills, or rules; (d) correctness of mathematical answers; and (e) openness of mathematics and solutions. I collected teacher statements for each and used Yin's (2014) analytic strategies to look for patterns: creating an array of emergent themes with relevant statements, creating visualizations, and coding subthemes.

I focus my analysis on data collected in Units 1 and 2 because participants had read Erlwanger (1973) in Unit 1 and Skemp (1978) in Unit 2. We as co-instructors prompted participants to discuss ideas that emerged in these readings about the nature of mathematics and the nature of mathematics teaching and learning. In later units, teachers focused more narrowly on particular content areas of mathematics and mathematics teaching and learning. In Units 1 and 2, however, they focused broadly on mathematics as a whole. The larger focus on mathematics as a whole provides relevant data to support answering the first research question.

In referencing statements that participants made, I abbreviate source material as follows: Interview Card Sort (Int.), Unit 1 Personal Profile (1PP), Unit 1 Discussion of Erlwanger (1973) (1D), Unit 1 Reflection (1R), Unit 2 Discussion of Skemp (1978) (2D), and Unit 2 Reflection (2R).

## Connections

Overall, across Unit 1 and 2 discussions and reflections, as well as in the interview card sort, teachers made many statements about the connectivity or isolation of mathematical ideas. In the interview card sort, all six teachers spoke positively of mathematics being connected. All but Melinda chose "ideas that are all connected" as one of their top five in the "Math is ..." card sort. All but Breanna emphasized that mathematical ideas are not disconnected. In the "Doing math means ..." section of the interview card sorts, all six participants agreed that connecting is an important part of doing mathematics. Their agreement about mathematical connections, however, may only be evidence that teachers have been told about mathematical connections in through their teacher preparation, professional development, or curriculum use. Despite uniform agreement that connections are important, teachers' statements referencing mathematical connections provide context and meaning for their differing conceptualizations of the meaning of mathematical connectivity.

To understand the teachers' individual meanings for mathematical connectivity, I considered each statement that included attention to connectivity. I found that teacher statements revealed different conceptions about the *extent* of mathematical connections, the *nature* of connections, and the *utility* of connections. I use examples below to describe what I mean by these terms and to differentiate the teachers' conceptions based on their statements.

## Extent of Connections

In teachers' statements that addressed the connectivity of mathematical ideas, teachers indicated different conceptions of the *extent* to which mathematical ideas are connected or isolated. That is, whether connections among mathematical ideas are sparse or dense, or whether connectivity is simply a characteristic of mathematics.

When discussing the isolation of mathematical ideas in the interview card sort, four teachers (Heidi, Nicole, Jana, and Melinda) simply deleted the "made of many isolated topics" card from the card sort, indicating that they disagreed that many mathematical ideas are isolated. Jana explained, "when I was doing the stats they were not all—they were not isolated."

Breanna chose "made of many isolated topics" as one of her top five cards, along with "ideas that are all connected." She gave the example of connecting addition with multiplication to support her stance that math is connected, but also explained that math is "made up of many different topics Because, like I say, you have addition, subtraction. You've got geometry. You've got statistics."

Of the six teachers, four teachers (Heidi, Alison, Nicole, and Breanna) chose "ideas that are all connected" as one of their top five cards. Jana edited it to "ideas that are all almost connected," but did not choose it as a "top five." Jana gave examples from biostatistics courses that she took as part of her Master's in Biochemistry. She explained that she thought "ideas are all almost connected" and explained that symbols are the same in different math areas, that operations, formulas, and strategies can be similar even if goals are different. She explained,

> Then we come to the elementary or the middle grade maths, we are using the same type of addition, subtraction. Some of the formulas were little bit easier and so I think so maybe the complexity is different but still we are using the same type of maths. Like the conformational analysis, then comes to biostats, the statistics are there, the findings are there. So everything is connected there.

Heidi said that the course activities of digging into standards and strands helped her see connections: "[Dissecting] the different strands [of standards]... then you look at the other ones and you dig in. You see how it's all kind of connected—there's that connective tissue." In her Unit 2 Reflection, Heidi similarly referenced "mathematics being complex with many concepts and relationships." Nicole also mentioned the connectivity of mathematics in her Unit 2 Reflection, referencing "the interconnectedness of math concepts." In the interview card sort, Melinda emphasized that students can make connections between any two math topics, saying "number sense or geometry or anything can be connected in some way, I think. In math."

In Table 5.2, I show the source of statements that teachers made. I considered the course of statements made to consider the robustness of an idea: if a teacher mentions a particular idea frequently without being prompted, it may be more likely part of that teacher's belief system. Another way the table might be used is to consider if a particular reading or discussion in the course prompted teachers to voice a particular idea.

In this case, the only time a teacher described mathematical ideas as isolated was when explicitly asked in the interview card sort, but teachers did mention the connectivity of mathematical ideas at different times in the course. Three teachers mentioned connections in their Unit 2 Reflection, which could indicate the belief is not as robust, but simply prompted from the Unit 2 course

Table 5.2

*Attention to Mathematical Connectivity: Extent*

| Participant | Some Isolated | Some Connected | All Connected |
|---|---|---|---|
| Lower Elementary | | | |
| Breanna | Int. | Int. | – |
| Melinda | – | 2R | Int. |
| Jana | – | 1D, 2D, Int. | – |
| Upper Elementary | | | |
| Nicole | – | Int. | 2R |
| Heidi | – | 2R | Int. |
| Middle Grades | | | |
| Alison | – | 1PP, Int. | – |

*Note.* Extent of mathematical connectivity, whether some concepts are isolated, some concepts are connected, or all mathematics is interconnected. Source of statements about extent of connectivity: Interview Card Sort (Int.), Unit 1 Personal Profile (1PP), Unit 1 Discussion (1D), Unit 1 Reflection (1R), Unit 2 Discussion (2D), Unit 2 Reflection (2R).

activities or discussions. This conclusion may be especially likely for Nicole and Heidi who were in the same small group.

## Nature of Connections

In statements about the connectivity of mathematical ideas, especially in the interview card sort, differences in views of the nature of mathematical connections emerged. Teachers talked about connections between mathematical ideas at different levels of focus: between concepts and operations, between representations, between standards or strands, between school mathematics and mathematical ideas in disciplines that rely on mathematics, and between mathematics and the real world. In Table 5.3, I indicate which teachers made statements about the nature of mathematical connections along with the placement of their statement in the course or interview.

All teachers talked about connections between mathematical concepts, although their references differed somewhat in focus: some were general statements and others were focused on particular examples. For example, Nicole made statements about general connectedness of concepts, writing about the "interconnectedness of math concepts" (Unit 2 Reflection) and about helping students connect their learning to "to their prior knowledge or other math concepts" (Unit 2 Discussion). Although Melinda, Breanna, and Jana also mentioned connections between mathematical concepts in general terms, they also gave more focused examples. Melinda gave the example of connecting "number sense or geometry or anything" in her interview card sort, which is still fairly general. In their interview card sorts, Breanna gave the example of connecting addition and multiplication, while Jana described connecting operations at different levels of complexity.

As will be described later, all teachers discussed mathematical representations more generally; however, Jana specifically wrote about connections between representations in her Unit 1 Discussion, saying Erlwanger had recommended that Benny needed "remedial work emphasizing relationships between numerals and physical quantities especially with manipulatives."

Jana and Heidi both described connections between larger mathematical areas through strands or standards. Heidi mentioned in her interview card sort that she had become more aware of connections between standards and strands as she dissected different strands. Jana mentioned in the Unit 2 discussion that teachers should consider how mathematical concepts might be connected to later mathematical areas to decide which concepts to teach relationally or instrumentally. Jana further described her own experiences in making connections between elementary school mathematics and the mathematics she used in a masters-level pure mathematics course and in biostatistics courses:

> When I was doing research about that, everything was mathematical formulas there. But again, in the formulas, the basic signs that we were using—the sigma, addition, subtraction—but even these are diverse. Then we come to the elementary or the middle grade maths, we are using the same type of addition, subtraction. Some of the formulas were little bit easier and so I think so maybe the complexity is different but still we are using the same type of maths.

Jana and Melinda mentioned connecting mathematical ideas to real-world mathematics in the course and in their interview card sorts. In her Personal Profile, Melinda described the importance of creating "real-world, hands-on experiences to create concrete experiences" that connected to mathematical ideas. Jana (in Unit 2 Discussion) and Melinda (in Unit 2 Reflection) both described the importance of students connecting school mathematics to support them in solving real world problems.

Table 5.3

*Attention to Mathematical Connectivity: Nature*

| Participant | Concepts | Reprs | Standards | Disciplines | Real World |
|---|---|---|---|---|---|
| Lower Elementary | | | | | |
| Breanna | Int. | – | – | – | – |
| Melinda | 1PP, 2R, Int. | – | – | – | 1PP, 2R, Int. |
| Jana | 1D, 2D, Int. | 1D | 2D | Int. | 2D, Int. |
| Upper Elementary | | | | | |
| Nicole | 2D, 2R | – | – | – | – |
| Heidi | 1D, 2R | – | Int. | – | – |
| Middle Grades | | | | | |
| Alison | 1PP | – | – | – | – |

*Note.* Nature of mathematical connectivity: between mathematical concepts, representations, standards or strands, disciplines, or between mathematics and the real world. Source of statements about nature of connectivity: Interview Card Sort (Int.), Unit 1 Personal Profile (1PP), Unit 1 Discussion (1D), Unit 1 Reflection (1R), Unit 2 Discussion (2D), Unit 2 Reflection (2R).

Table 5.3 helps illustrate some patterns. First, the table shows the contexts in which each of the six teachers made statements attending to connections between mathematical concepts: Melinda, Jana, Nicole, and Heidi all mentioned connections at multiple times across the course and interview card sort. Each mentioned connections between mathematical concepts in Unit 2 Discussions or Reflections or both. Second, the table shows that most teachers focused on connections between concepts. Jana, specifically, mentioned several different types of mathematical connections, while Heidi and Melinda mentioned one additional connection type.

## Utility of Connections

Across the interview and course data, in reading teachers' statements that spoke to the connectedness or isolation of mathematical ideas, two themes of the utility of connections emerged: first, that a focus on connections is important because new mathematical knowledge builds on prior mathematical knowledge and, second, that a focus on connections is important to illustrate the nature of mathematical concepts as interconnected or having many relationships.

In Table 5.4, I show the source of statements teachers made referencing the utility of mathematical connections in supporting building new concepts on previously learned concepts and in illustrating mathematical relationships. In Table 5.5, I provide examples of statements each participant made about the two big ideas.

Table 5.4

*Attention to Mathematical Connectivity: Utility*

| Participant | Building on | Relationships |
|---|---|---|
| Lower Elementary | | |
| Breanna | – | – |
| Melinda | 2R | – |
| Jana | 2D | 1D |
| Upper Elementary | | |
| Nicole | 2D, 2R, Int. | 2R |
| Heidi | 1D, 1R | 1R, 2R |
| Middle Grades | | |
| Alison | – | – |

*Note.* Utility of mathematical connectivity, whether building on previous knowledge or gaining knowledge about the concept of mathematical connectivity. Source of statements about utility of connectivity: Interview Card Sort (Int.), Unit 1 Personal Profile (1PP), Unit 1 Discussion (1D), Unit 1 Reflection (1R), Unit 2 Discussion (2D), Unit 2 Reflection (2R).

Table 5.4 shows that Jana, Nicole, and Heidi each made statements indicating that students learn mathematical connections to build new understanding on the foundations of prior understanding and to understand relationships between mathematical ideas. Melinda made statements referencing the former idea but not the latter. Table 5.4 shows that statements from Nicole and

Heidi come from multiple sources, which is an indication of some consistency of this conception across contexts.

In Table 5.5, I provide examples of statements each participant made about the two big ideas.

Table 5.5

*Mathematical Interconnectedness or Building on: Examples*

| Participant | Description |
|---|---|
| Build on | |
| Melinda[a] | *apply the methods I teach them to different problems* |
| Jana[a] | *connections with new materials, new setting[s]* |
| Nicole[b] | *math builds on and connects prior knowledge* |
| Heidi[b] | *building math background knowledge* |
| Relationships | |
| Jana | *math's concepts and relationships* |
| Nicole | *interconnectedness of math concepts* |
| Heidi | *relationships and patterns ... essential in [math]* |

[a] Lower Elementary [b] Upper Elementary [c] Middle Grades

Regarding the "building" idea, Heidi and Nicole explicitly used the term "build on." Heidi referred to "fundamental concepts that we build on in math" (Unit 1 Discussion). Nicole explained "math builds on and connects prior knowledge" (Unit 2 Discussion). Jana and Melinda did not explicitly use the words "build on," but they each described their students using previously learned mathematics with "new materials, new setting[s]" (Jana, Unit 2 Discussion) or in application to different problems (Melinda, Unit 2 Reflection).

Heidi, Nicole, and Jana used different language to describe relationships between mathematical ideas. Heidi and Jana explicitly referred to relationships between mathematical ideas: Heidi referred to mathematics as "complex with many concepts and relationships" (Unit 2 Reflection) and Jana referenced "math's concepts and relationships" (Unit 1 Discussion). Nicole referred simply to the "interconnectedness of math concepts" (Unit 2 Discussion).

## Summary of Participant Conceptions About Connectivity

Jana made many statements across both units and in the interview card sort about connectivity. She described a particular way mathematical ideas are connected ("symbols, operations, formulas, strategies") across grade levels, disciplines, and views of the nature of connections. Melinda, Nicole, and Heidi each stated that all mathematical ideas are connected, but only Melinda gave an explicit example (number sense and geometry) of two big ideas. Lack of examples does not mean they were unable to provide any, however. For example, Heidi did explain that she "connective tissue" as she analyzed standards and strands and so perhaps she could have given explicit examples of connections given two mathematical ideas.

Breanna and Alison made so few statements about connectivity that it is difficult to draw conclusions about their conceptions. Breanna did give one example of a connection (between addition and multiplication), and also gave a list of different topics in mathematics. This list suggests that she conceived of mathematics as a loose collection of topics rather than a complex, interconnected structure.

# Problem-Solving and Facts, Rules, and Skills

Teachers' statements from the course and from the interview card sorts seemed to characterize mathematics simply as "a collection of facts, skills, and rules." Considering Ernest's three views (Instrumentalist, Platonist, Problem-driven), this characterization implies an Instrumentalist View of mathematics. In considering each teacher statement, teachers mentioned facts, skills, procedures, or rules as necessities in supporting mathematical problem-solving. Each of the six teachers stated, however, that more than rules alone are needed for at least some problems. Such a perspective could align with a Problem-Driven View of mathematics, which, in part, focuses on the development of mathematics in response to problems (whether abstract and theoretical or concrete and immediate). After reading their statements, I saw that separating "collection" statements from "problem-solving" statements would not only be difficult but would likely result in losing insight into the teachers' views. Hence, I combined these two sets of statements regarding teachers' views of mathematics as problem-solving and their views of mathematics as a collection of facts, procedures, skills, and rules. As in previous sections, teachers' statements rarely described mathematics as a discipline, but instead described mathematics through the lens of teaching and learning mathematics. Although teachers' statements are not purely about the nature of mathematics, which is the focus of this section, I argue that their descriptions of mathematical learning reveals a great deal about how they view the nature of mathematics.

Teachers made many statements about problem-solving and the use of rules, facts, and skills in mathematics across the Unit 1 discussions and reflections about Erlwanger (1973), the Unit 2 discussions and reflections about Skemp (1978), and the interview card sorts. As is shown in Table 5.1, in the "Mathematics is..." card-sorting activity of the interview, Melinda, Breanna, and Jana chose both "a collection of facts, skills, and rules" and "all about problem-solving" as one of their top five cards. Heidi chose "a collection of facts, skills, and rules" but not "all about problem-solving" in her top five. Nicole edited "all about problem-solving" to read "problem-solving" and included the edited version in her top five. In the interview card sort, participants did not define what they meant by problems or problem-solving and most did not explain their choices of "a collection" in their top five.

Teachers made collection and problem-solving statements in both Unit 1, as they discussed or reflected on Benny's learning and use of mathematics (Erlwanger, 1973), and Unit 2, as they discussed or reflected on instrumental and relational understandings (Skemp, 1978). As always, teachers' statements may strongly reflect (or be reflected by) the readings. For example, Erlwanger wrote, "Benny's case indicates that a mastery of content and skill does not imply understanding" (p. 7). Heidi, Melinda, Alison, and Jana each used this quote, or paraphrased it, in their Unit 1 discussions or reflections. Their use of the quote may indicate its influence on their conception of mathematics as more than content and skill, or it may indicate that they held this belief prior to the reading and used it because the quote matched their conception. As I have mentioned earlier, because of the nature of conceptions, it is not likely that I can separate the teachers' conceptions from the context of the course; however, I have tried to compare how teachers voiced similar and different conceptions in reaction to this common context.

To examine their statements to understand the conceptions revealed, I considered all statements made by teachers in their Unit 1 and Unit 2 discussions and reflections. I found that teachers made statements about the importance of facts, skills, and rules in mathematics, especially indicating that following rules leads to successful mathematical problem-solving. These statements indicate an Instrumentalist view of mathematics. I found that teachers also made statements about the dangers inherent in knowing only facts, skills, or rules without deeper understanding; teachers especially emphasized knowing why or when facts, skills, or rules can be applied. These statements indicate a view of mathematical rules as making sense and being in place for understandable reasons, rather than being created by mathematical authorities for reasons that are either arbitrary or not understandable for common users of mathematics. Such statements imply that a teacher may have a more nuanced view of mathematics, rather than a solely Instrumentalist view. Teachers described one danger of focusing only on rules as increased inflexibility or rigidity. Teachers argued that solving mathematical problems often requires flexibility in use of strategies and representations. These statements indicate views that are closer to a Problem-Driven view than Instrumental. While an Instrumental view often includes assumptions that rules are given by external authority, these statements indicate assumptions that anyone can change mathematical rules to fit a problem situation; understanding rules more deeply helps problem-solvers adapt rules in valid ways without relying on external authority. Teachers described another danger of over-reliance on facts, skills, and rules: that superficial knowledge of facts, skills, or rules might block important habits of mind or higher-order thinking that is needed for problem-solving. These statements more strongly indicate a Problem-Driven view as teachers describe the need for students to create their own understandings and their own rules in response to problem situations.

In the paragraphs below, I separate teachers' statements into four sections expressing teachers' views on: the value of facts, skills, and rules; the need for a deeper understanding of facts, skills, and rules; the need for flexibility in problem-solving; and the need for higher-order thinking in problem-solving.

## Value of Facts, Skills, and Rules

All six teachers made statements about the importance of facts, skills, and rules in mathematics. Five of the six teachers described rules as composing at least part of mathematics, mentioning that students need to know how to use facts, skills, and rules in mathematics. Each of the six teachers made statements indicating that following mathematical rules correctly yields correct answers. Each teacher also made statements indicating that mathematics is more than only a collection of facts, skills, and rules.

In the interview card sort, only Breanna, Nicole, and Heidi attempted to explain why they felt mathematics is "a collection of facts, skills, and rules." Breanna said,

> So when somebody tells me a step-by-step how to do a problem, and it actually works, that's what math is. It has rules. And your skills in being able to complete those rules in order—or steps I should say. Then that's what it is.

Nicole and Heidi both acknowledged that mathematics includes such a collection, but both argued that mathematics is more than a collection of facts, skills, and rules. For example, Nicole responded, "I do think there are facts, skills, rules that make up math... I don't think necessarily that you should only stick to that." Similarly, Heidi said, "And then the 'collection of facts, skills, and rules.' I think that that has a time and a place."

As one example of viewing mathematics as rule-based, I describe Breanna and Melinda's discussions. Breanna described Benny and his view of mathematics, writing "He knows that every math concept has rules and if you apply the same rules to the problems, you should be able to get the right answer every single time." Breanna recommended that Benny's teacher "could prepare a study guide sheet that lists the steps for the correct way to complete each math concept" if she was not able to give him "in the moment" feedback. In their Unit 1 and Unit 2 discussions, Breanna and Melinda described mathematics in a way similar to Breanna's statement above. In their Unit 1 discussion, Breanna began by writing, "A part of teaching math is presenting a set of rules, demonstrating how to apply them to multiple mathematical scenarios, and being able to check your answers on your own." Melinda agreed with this statement. The statement indicates a view of mathematics as composed of sets of rules, and thus teaching mathematics is "presenting a set of rules."

In a later discussion, both Breanna and Melinda discussed a rule-based view that extended beyond mathematical problem-solving to their lives more generally. In their Unit 2 discussion, Breanna wrote,

> I've always been the type of person to 'follow the rules', not just in mathematics but in life in general. I like to have a clear cut list of things that need to be done, and do them. I like to have everything planned out and then I follow that plan. I can see how that kind of thinking gets me overwhelmed sometimes because if things don't 'work out', I fill as if it's because I didn't plan correctly or that I didn't follow the plan exactly as I should have.

Melinda agreed with Breanna, writing,

> Like you, I was always the one that followed the rules and believed in a path to the goals set forth prior, however, I was also taught to be flexible and adapt to changes in the path. Was I always quick to react? No. Did straying from the path or plan fluster me? Definitely!

This exchange indicates how a teacher's general approach to life may inform her view of mathematics, or her view of mathematics may inform her approach to life. It also indicates a slight difference between Breanna and Melinda; despite both desiring to follow rules, Melinda argued that she could be flexible to adapt to a path.

On the other hand, Breanna and Melinda both made statements that indicated mathematics must be more than rules alone. In discussing instrumental and relational thinking in the Unit 2 discussion on Skemp, Breanna argued that "teachers use the instrumental approach the most is because for one, students are able to see a set of rules, follow them, and produce correct answers." This statement seemed to support her earlier statements described above about presenting rules and following them. She continued her argument, however, to recommend that a relational approach would be better. She described the importance of students forming "their own ways of thinking to solve problems the best way that fits them.... [allowing] a mind to take that concept and run with it, so to speak." She argued that, "Relational thinking allows people to use more than one approach to solve a problem, not just the cookie cutter version that was presented to them." In this argument, she revealed a view of mathematics as more than rigidly following the same rules; instead, Breanna recommended that each individual's mind could create their own ways of problem-solving and used the term "cookie cutter version" to describe a set of rules given to students in class. Her argument indicates her view that individuals should create their own rules rather than being given rules. Later in the same discussion entry, however, she seems to contradict her own argument by imagining a potential classroom activity in which some students explore (and share) steps to a problem solution while other students explore (and share) a deeper understanding of the problem situation. She described splitting students into four groups to "jig-saw a math concept. Maybe two groups could explain the 'why' and 'how' and the other two groups could show how to complete the steps to a problem." Her recommendation seems to emphasize the need for steps or rules in problem-solving, and the need for someone to demonstrate them, while also emphasizing the importance of individuals creating and making sense of their own steps and rules.

Melinda's statements also indicated some conflict in her views. Although in the discussion described above she agreed with Breanna about following rules to reach answers, she qualified her agreement immediately by emphasizing the need for flexibility and adaptability. The fourth member of the Lower Elementary group, Kristine, suggested in her Unit 2 discussion that teachers should balance instrumental and relational thinking: "You could have an exploratory practice before teaching the rules. You could teach the rules and then have a discovery project that demanded relational understanding to use the rules in the correct context." Melinda responded to Kristine, writing, "I like your idea of exploratory practice prior to teaching the rules in the classroom as well as the discovery project." In these statements, Kristine and Melinda both seem to argue that students need to explore and understand more deeply, but that giving students the rules is always the goal.

Other teachers' statements also indicated conflict in their views. Alison made many statements challenging a rules-based view of mathematics, especially drawing on Bloom's taxonomy to explain why memorization alone is dangerous for students. On the other hand, as part of her arguments against rules-based mathematics, she explained, "It is very simple to teach students rules to follow, procedures to go through to get the answer" (Unit 2, Discussion on Skemp) which implied that mathematical problem-solving can be rules-based and following procedures does support students in finding answers. On the other hand, in her Unit 1 Reflection she stated, "Rules are great each time they work, until you come to a situation where they don't." Her statement implied that there are situations where rule-based problem-solving fails, and so problem-solvers need more. She acknowledged in her Unit 2 Reflection that, although her philosophy is more in line with supporting relational understanding, she felt she too often teaches instrumentally. Her statements indicate some conflict between seeing mathematics as rules-based, on the one hand, and, on the other, seeing mathematics as requiring "the use of understanding, applying, analyzing, evaluating, and creating (all of the upper levels of Bloom's Taxonomy)" (Unit 2, Discussion on Skemp).

Jana consistently described mathematics as more than following rules. In only one statement, in her Unit 2 reflection, she described a potential value in following rules or procedures, arguing they are useful for students to "do a particular question swiftly." As I will describe below, in many statements, Jana argued that students need to use critical thinking, and need to see the reason or sense in mathematics. She used the phrase "he developed an inflexible, rule-oriented attitude toward Maths" in both her Unit 1 discussion and reflection as a condemnation of the mathematics teaching program used in Benny's classroom.

### *Summary of Values*

Table 5.6 shows which teachers (and in which units) made statements about (a) the importance in mathematics of knowing how to use facts, skills, and rules; (b) appropriate use of rules in mathematical problem-solving always results in correct answers; and (c) mathematics being more than just a collection of facts, skills, and rules.

Table 5.6

*Attention to Importance of Facts, Skills, Rules*

| Participant | Knowing How | Yields Correct Answers | Math is More |
|---|---|---|---|
| Lower Elementary | | | |
| Breanna | 1D, 2D, Int. | 1D, 2D | 1D, 1R, 2D, 2R |
| Melinda | 1D, 2D, 2R | 1D, 2D, 2R | 1D, 2D, 2R |
| Jana | 2R | 2R | 1D, 1R, 2D, 2R |
| Upper Elementary | | | |
| Nicole | 2D, Int. | 2D | 2D, 2R, Int. |
| Heidi | Int. | 1D | 1D, 2R, Int. |
| Middle Grades | | | |
| Alison | – | 1D, 1R, 2D | 1D, 1R, 2D |

*Note.* Teachers' reference to importance of *knowing how* to use facts, skills, and rules in mathematics, and especially that following rules correctly *yields correct answers* for mathematical problems. Then, mathematics *is more* than a collection of facts, skills, and rules. Source of statements: Interview Card Sort (Int.), Unit 1 Personal Profile (1PP), Unit 1 Discussion (1D), Unit 1 Reflection (1R), Unit 2 Discussion (2D), Unit 2 Reflection (2R).

Every teacher but Alison stated at least once that knowing facts, skills, and rules, and how to use them, has a place in mathematics. Of all teachers, Breanna spoke most strongly about mathematics being a set of rules or steps to follow. Every teacher stated at least once that following rules results in correct answers. And every teacher stated at least once that mathematics is more than just a collection of facts, skills, and rules. It is interesting to note that teachers more frequently and across more contexts argued that mathematics is more than a collection than argued that students should know how to use facts, skills, and rules. In the next section, I explore more deeply what teachers described when they argued that "mathematics is more" than the collection.

In the next three sections, I share findings that explore teachers' statements about what mathematics is in addition to a collection of facts, skills, and rules.

## Understanding Facts, Skills, and Rules

In discussing mathematical problem-solving and the need for rules, facts, and skills, teachers argued that knowing rules, facts, and skills alone is not enough. Teachers emphasized that mathematics is more than just rules, facts, and skills; it also includes understanding reasoning and processes that justify and validate rules, facts, and skills. Teachers argued that knowing how and why they work is important, as well as knowing why or when (in which situations) facts, skills, or rules can be applied. Teachers made statements about the dangers (blocking flexibility, blocking habits of mind or higher-order thinking) inherent in knowing only facts, skills, or rules without deeper understanding. As teachers described these potential dangers, they revealed a more nuanced view of mathematics. Exploring these statements reveals more about teachers' views of the nature of mathematics, because their arguments for understanding imply that, not only can mathematical rules and facts be understood, but that any user of mathematics can make sense of the reasoning behind the rules and facts.

### *Need for More Understanding*

Each of the six teachers stated there is more to mathematics than simply knowing facts, skills, and rules. Breanna's statements are a good example of how teachers argued generally, or without explicitly giving reasons, for a deeper understanding. Across the two units, Breanna consistently emphasized the need for students to "fully understand the mathematical processes that underlie the rules to each concept" (Unit 1, Discussion of Benny). Similar to other teachers, Breanna did not explain what she meant by these underlying processes. As mentioned above, Heidi, Melinda, and Jana each quoted, or paraphrased, Erlwanger's statement that "a mastery of content and skill does not imply understanding" (p. 7).

Jana argued at times that students would enjoy mathematics if they were given opportunities to understand the reasoning behind the rules, facts, and skills: "Now how can you expect a child to like a subject/topic if they don't know the process, or don't know why they are doing this then how can we get the deeper understanding" (Jana, Unit 2, Discussion on Skemp). Melinda agreed with

Jana, stating that “they don’t necessarily understand the reasoning behind what they are doing” (Melinda, Unit 2, Discussion on Skemp). Nicole explained a similar view, arguing “With the [relational] view, students are not committing facts and formulas to memory, but rather gaining a deeper understanding of how math ‘works’” (Nicole, Unit 2, Discussion on Skemp).

Breanna and Melinda both drew on Skemp’s definitions of instrumental and relational understanding to explain the importance of understanding how and why, but their descriptions of what should be explained was slightly different. Breanna wrote about “how those rules really worked” (Breanna, Unit 1, Discussion of Benny), while Melinda wrote about “why rules or algorithms are in place” (Melinda, Unit 1, Discussion of Benny), “why a method or rule works and how you can adapt it” (Melinda, Unit 2, Reflection), and “why or how rules apply [in a mathematical situation]” (Melinda, Unit 2, Reflection).

Breanna explained that if students understand “the depth of why they need to compute math problems the way they do, then maybe it could help students understand the breakdown of the math processes that goes along with problem-solving” (Breanna, Unit 1 Reflection). She also recommended that teachers could break students into groups, with some groups exploring how and why problems are solved and then explaining their findings to the rest of the class.

Between Breanna and Melinda, then, their arguments revealed a more nuanced understanding of mathematical rules. Breanna’s argument that students should understand how rules work seems in line with her statements about the underlying processes of rules. Melinda’s arguments reveal that there is some choice involved in rules and algorithms (why they are in place), that a method or rule is not set in stone but can be adjusted based on the situation (why it works and how to adapt), and that there are reasons to choose to use rules in particular situations.

Several of the teachers argued that not only is there more to mathematics than facts, skills, and rules, but that an over-reliance on facts, skills, and rules can block users of mathematics from using mathematics effectively. For example, Nicole argued, “If students rely on these rules and do not know the meaning behind them, they can end up incorrectly applying it without even realizing it” (Nicole, Unit 2, Reflection). Teachers argued that the dangers of not understanding why, when, or how rules work, or of over-reliance on rules, include being inflexible and being blocked from engaging in important mathematical activity such as communicating, reasoning, or critical thinking.

### *'Rules Alone' Blocks Flexibility*

Five of the six teachers referred to the importance of flexibility or adaptability in mathematical problem-solving, and each of the five emphasized that a simplistic view of mathematics as only a collection of facts, skills, and rules can block students from being flexible. As mentioned above, in her Unit 1 discussion and reflection, Jana emphasized that because "IPI [individually prescribed instruction] was forcing Benny into the passive role of writing answers to get them marked right. Therefore, he developed an inflexible, rule-oriented attitude towards Maths" (Jana, Unit 1, Reflection). Similarly, Nicole described Benny's "stubbornness" and compared him to other students she had known:

> How do you think a teacher could 'break' him of this stubbornness and his thinking of mathematics and the rules he had previously learned? ... It reminds me of students who don't have the right answer, but are so set in their ways that they don't believe they are wrong (Nicole, Unit 1, Discussion on Benny).

Alison argued in her Unit 2 discussion, "Instrumental understanding does not allow our students to be flexible with what they know" (Alison, Unit 2, Discussion on Skemp). As mentioned earlier, the Lower Elementary group spoke of different types of problem or strategies: some that were "cookie cutter" or "fit the mold" and others that did not. Breanna, Melinda, and Jana each described the potential for a focus on rules without understanding to block flexibility. For example, Melinda wrote "Many students in mathematics are able to use the rule to determine the answer to cookie cutter math problems, but cannot adapt the rules when need be or explain why or how the rule applies" (Melinda, Unit 2, Reflection).

### *'Rules Alone' Blocks Habits of Mind*

I will describe in more depth what other skills and reasoning that teachers argued can be blocked by a focus on rules without understanding. Five of the six teachers described such an emphasis blocking students' deeper understanding or reasoning. As an example, in imagining students who had learned mathematics relationally, Melinda asked, "would they be more apt to develop their thinking deeper, where these abstract concepts are not as difficult to handle? Would we see more students excel? Would the stigma associated with mathematics begin to lessen? What about the students' confidence?" (Melinda, Unit 2, Discussion on Skemp). Nicole argued that for her students, "They didn't want to take time exploring and learning the meaning behind something they already knew how to do. I think that is another example of what happens if there is too much focus on the instrumental understanding" (Nicole, Unit 2, Discussion on Skemp). Heidi stated similarly about her students,

> I'm finding another challenge when asking students to explain their reasoning or justifying their work. ... Communication in math I feel comes from the [relational] understanding and making that shift or turning that corner isn't easy for student or teacher when they are set in their ways like you and Nicole have talked about (Heidi, Unit 2, Discussion on Skemp).

### *Summary of Understanding*

In Table 5.7 below, I show which teachers spoke to one of these themes, and in which sources of data. Teachers argued that students need more understanding of facts, skills, and rules: generally and, more specifically, about why, when, or how rules work in mathematical situations. Teachers also argued that knowing 'rules without understanding' can block flexibility and other reasoning in problem-solving.

All teachers described the need for more understanding of facts, skills, and rules. Teachers did not give explicit examples of a particular rule and its meaning, but did explain reasons that too

Table 5.7

*Attention to Deeper Understanding of Facts, Skills, and Rules*

| Participant | Need Understanding | Blocks Flexibility | Blocks Habits of Mind |
|---|---|---|---|
| Lower Elementary | | | |
| Breanna | 1D, 1R, 2D, 2R | 2D | – |
| Melinda | 1D, 1R, 2D, 2R | 2D, 2R | 2D |
| Jana | 1D, 2D, 2R | 1D, 1R | 2D, 2R |
| Upper Elementary | | | |
| Nicole | 2D, 2R | 1D | 1D, 2D |
| Heidi | 1D, 2R | – | 2D |
| Middle Grades | | | |
| Alison | 1D, 1R, 2D, 2R | 1D, 2D | 2D |

*Note.* Teachers' attention (with data source) to students' need for deeper understanding of facts, skills, and rules, and to the idea that knowing 'rules without understanding' can block flexibility and other reasoning in problem-solving. Source of statements: Interview Card Sort (Int.), Unit 1 Personal Profile (1PP), Unit 1 Discussion (1D), Unit 1 Reflection (1R), Unit 2 Discussion (2D), Unit 2 Reflection (2R).

superficial understanding was dangerous for mathematical problem-solvers. All teachers except Heidi mentioned the danger of adaptability or flexibility being blocked. All teachers except Breanna mentioned some example of a habit of mind or higher-order thinking that would be blocked.

## Flexibility for Problem-Solving

Although five of the six teachers argued that focus on rules without understand might block flexibility or adaptability, teachers described different strategies for achieving flexibility and adaptability. The strategies they describe are certainly focused on teaching and learning mathematics, but exploring these strategies helps elucidate their meanings of flexibility and adaptability in mathematical problem-solving.

Breanna and Melinda described students understanding why or when rules work to support students in adapting rules to new mathematical situations or problems. For example, Melinda explained that understanding why a method or rule works helps students use "the tools or steps to complete the problems" but also equips them to "apply these procedures to problems that don't quite fit the mold" (Melinda, Unit 2, Discussion on Skemp). In her reflection, she explained more, writing that

> Relational understanding occurs when you've learned a particular method or rule, are able to use it in solving the problem, but you also are aware and understand why it works and how you can adapt it to situations that may not necessarily follow the same pattern or fit into the same mold (Melinda, Unit 2, Reflection).

Breanna wrote that teaching relationally can help students "to form their own ways of thinking to solve problems the best way that fits them." (Breanna, Unit 2, Discussion on Skemp)

Breanna, Jana, and Alison recommended asking students to use multiple strategies to complete mathematical problems. Breanna described a need for multiple strategies because "not all students learn the same and so teaching only one way to solve a problem could be detrimental for some students" (Breanna, Unit 2, Discussion on Skemp). Jana explained that she asked her students to find answers in multiple ways because she believed "it will help to elevate the students critical thinking and problem-solving skills" (Jana, Unit 2, Discussion on Skemp).

Heidi and Alison both recommended supporting students' use of multiple ways of representing solutions. Heidi grouped multiple representations and strategies together as she explained,

> Good teaching and learning values multiple strategies and solution methods and presenting students with these options provides opportunities for a deeper understanding of the mathematical concepts.... Multiple solution strategies and models to show patterns and make generalizations would support Benny to become a mathematical thinker (Heidi, Unit 1, Reflection).

Similarly, Alison grouped multiple solution strategies and representations to help students be able "to explain their thinking to others" (Alison, Unit 1, Reflection).

***Summary of Flexibility***

In Table 5.8, I show which teachers recommended each strategy, and in which source of data. Although five of the six teachers argued that rules without understanding can block flexibility in problem-solving, only Breanna and Melinda explicitly recommended a deeper understanding of why or when to use rules and procedures to develop flexibility.

Table 5.8

*Flexibility in Problem-Solving*

| Participant | Why or When | Multiple Representations | Multiple Strategies |
|---|---|---|---|
| Lower Elementary | | | |
| Breanna | 1R, 2D | – | 1D, 2D |
| Melinda | 2R | – | – |
| Jana | – | – | 2D |
| Upper Elementary | | | |
| Nicole | – | – | 2D |
| Heidi | – | 1D, 1R | 1R |
| Middle Grades | | | |
| Alison | – | 1R | 1R |

*Note.* Participants' reference to aspects flexibility for problem-solving: why or when (to adapt), multiple representations, multiple strategies. Source of statements: Interview Card Sort (Int.), Unit 1 Personal Profile (1PP), Unit 1 Discussion (1D), Unit 1 Reflection (1R), Unit 2 Discussion (2D), Unit 2 Reflection (2R).

Four of the six teachers described using multiple strategies to support students in developing flexibility in problem-solving, and two recommended using multiple representations. The teachers described multiple strategies differently, however: Breanna seemed to advocate teaching multiple strategies, or at least leaving choice of strategies to students, as opposed to Jana, Nicole, and Heidi who talked about expectations for students to solve in multiple ways leading to deeper understanding or critical thinking.

## Habits of Mind for Problem-Solving

Teachers described a variety of mathematical habits of mind or higher-order thinking skills that may be blocked by focusing too much on rules without understanding; communicating and critical thinking are two large themes that I chose to focus on, based on teachers' statements. Communicating is an important theme as it highlights a view of mathematics as a language. Critical thinking is another important theme because it speaks to mathematics as a way of thinking. This second theme also indicates a view of mathematics that puts the authority in the individual rather than with an external authority. Each theme includes multiple mathematical habits of mind.

### *Communicating*

All six teachers described some form of communicating as an important component of doing mathematics, whether they made statements in which they used a more general term (e.g., communicating, writing), or a more specific term (e.g., explaining, justifying, discussing), or some combination of general and specific terms. Based on the teachers' statements, mathematical communication seems to serve one (or more) of three purposes: (a) provide evidence of a student's understanding (assessment), (b) act as a pathway to deeper understanding of the mathematics being used (instructional strategy), or (c) act as a critical component of problem-solving (mathematical habit of mind).

Breanna judged Benny to be an active learner because he was "able to fully explain his answers with the rules he has come up with on his own" and "able to justify his answer, although it is incorrect, with carefully followed step-by-step procedures" (Breanna, Unit 1, Discussion of Benny). In her description, Breanna describes the ability to explain or justify a process to an answer as a positive, even if the explanation reveals that the process is not mathematically valid.

In her Unit 2 discussion, Nicole described her curriculum as "requiring students to justify and explain their thinking" and described her students as able to "learn a rule, remember it, apply it, and move on. They've found success in that, which is why they are probably resistant with writing and digging deeper" (Nicole, Unit 2, Discussion of Skemp). Nicole's description may indicate that she sees that her view of understanding mathematics conflicts with her students' views.

Heidi similarly described a challenge: "when asking students to explain their reasoning or justifying their work. I get responses like 'just multiply top and bottom' or the 'butterfly method' for comparing fractions" (Heidi, Unit 2, Discussion on Skemp). Heidi's description may indicate that she, similar to Nicole, is noticing that her meaning of reasoning and justifying is in conflict with her students' meaning of reasoning and justifying.

Both Nicole and Heidi seem to describe communication as a way to see students' thinking, but describe their (and their students') frustration when the students are unable to explain the reasoning behind the method. One reason for this frustration may be that, while students can use the method to successfully find correct answers, they do not really understand how or why the method works. Heidi explained this struggle, saying "Communication in math I feel comes from the [relational] understanding and making that shift or turning that corner isn't easy for student or teacher when they are set in their ways like you and Nicole have talked about" (Heidi, Unit 2, Discussion on Skemp). A second reason may be that the students have a different view of what reasoning means: The teachers, Heidi and Nicole, expressed the view that reasoning and communicating that reasoning should be based in higher-level thinking and a deep understanding why and how procedures work. If their students have a view of reasoning based in lower-level thinking, then communicating that reasoning may mean simply writing each step rather than writing a rationale for each step. Conflicting views and values can support teachers' and students' frustration as they bring different meanings of mathematical reasoning, rationale, and communication. All six teachers mentioned in their Unit 2 discussion or reflection a concern about students being unable to explain mathematical processes.

Melinda, Jana, Alison, and Nicole explicitly mentioned communication as a critical part of a mathematical problem-solving process, and so I refer to communication in this context as a mathematical habit of mind. For example, Alison described mathematics as "analyzing, reflecting, and discussing real-world situations" and described the need to discuss and talk with others to get through and work through math problems (Alison, Unit 1, Discussion of Benny).

To explore their meanings in a more focused way, I turn to their interview card sort responses. In the "Doing mathematics means..." card sort portion of the interview, one card choice was communicating. Four teachers (Heidi, Nicole, Melinda, and Jana) chose communicating as one of their top five cards. Nicole connected mathematical communication to problem-solving and sense-making, saying, “I think that’s very important for doing math: Talking about it with others and solving problems.” She explained her view of the close connections between sense-making, reasoning, communicating, and problem-solving:

> communicating to others your own reasoning and your own sense-making. And then having that argument to even disprove maybe what you thought made sense actually doesn’t anymore once you listen to other people’s reasonings or critique of your work. But it could also lead to a deeper sense-making. A deeper understanding of the concept.

Jana also connected mathematical communication to problem-solving and sense-making, saying, “Communication of our ideas. Communication of our knowledge, skills, understanding. Making sense of something will communicate. Sometimes we communicate to prove. Sometimes we communicate to reason. Sometimes we communicate to solve a problem.” Heidi explained that communicating, which includes constructing and critiquing arguments, is an important instructional strategy and habit of mind: “You know, with proving and disproving other peoples solution strategies. I think it’s so powerful for them and it helps with the understanding” (Heidi, Interview Card Sort).

On the other hand, Melinda described communicating as more focused on assessment and giving feedback:

> I think if you just give a grade as a teacher and you see that as a student, you’re not communicating whats expected or maybe what they should try next time or any changes in their behavior or processes. If you’re constantly communicating like I talked about with those small groups, it’s much, much easier to correct behavior.

She went on to describe how student-to-teacher communication and peer-to-peer communication can support student learning:

> Peer-to-peer communication. Making sure that those arguments are building on each other and that they're even taking place to begin with. If kids aren't allowed to talk about their learning or talk about the processes of math or whatever the case may be, I don't think that's a good thing because they can learn through those arguments and conversations.

Melinda also connected communicating with constructing and constructively critiquing arguments of others, reasoning, and defining.

### *Critical Thinking*

Each of the six teachers mentioned that thinking, especially critical or higher-order thinking, is an important aspect of doing mathematical work, and used words such as critically thinking, exploring, analyzing, applying, creating, evaluating, reasoning, and reflecting to explain the types of thinking that should happen. The way that teachers used these words indicated that they have varying definitions across teachers, and even within an individual teacher's statements. For example, in the Unit 1 discussion of Benny, Breanna and Melinda wrote about the need for students to apply a set of rules to multiple mathematical situations. In her Unit 2 discussion of Skemp, Melinda wrote "they present the rules and expect the students to follow them.... When students were actually asked to apply the information they've learned, if they ever were, they were unsure on how to tackle the problem" and "There's not a real-world application step within the process that forces the learner to use those rules, but in a way that requires deeper understanding of the concepts" (Melinda, Unit 2, Discussion on Skemp). In these statements, it seems that Melinda was using the word *apply* in slightly different ways, depending on whether students were directly applying rules to familiar situations or applying rules by adapting them for "to problems that don't quite fit the mold" (Melinda, Unit 2, Discussion on Skemp).

Exploring, analyzing, applying, creating, and reasoning were all cards in the "doing mathematics means..." card sort portion of the interview. Teachers described the connections between these ways of doing mathematics in complex ways, often using different words to define or distinguish each from another.

Heidi explained that Exploring can start out as just moving things around, but that exploration can become experimentation,

> But I feel like if you connect it to science, it really ends up with an experiment because they've asked a question and then they're testing it. They're making decisions. And they're kind of collecting data in a way too. And even revamping their question that theyre asking.

Nicole said "When you're exploring part of it could be experimenting or testing—there you go, testing theories or ideas." Nicole created a new card, interacting, that she said connected to exploring. She said interacting could be with tools, people, and math concepts, and that by interacting a student is exploring. Melinda explained that exploring connects to "being able to learn by doing and also making mistakes, going back to drawing board, understanding why the mistake was made, and moving forward."

Alison described reasoning and analyzing as different

> because I feel like analyzing starts very open-ended. It's kind of like 'Analyze this' based on just a little snip of information that you might have. But I feel like reasoning is used for when an argument has already been presented. And then you are backing it up or giving arguments to make your case.

She described analyzing as,

> Looking at the big picture and really looking at every single facet around what you're looking at. Whether it be a graph. Looking at every single part of the graph and, you know, making sense of it. Maybe connecting it to something else. Maybe drawing conclusions from it. ...I feel like analyzing is never really done. There's a lot of different ways that you can analyze something. You can continue to analyze something for years, if need be.

Heidi described Applying as connecting multiple concepts to real-world situations, saying

> they have to take several concepts and its almost like a real-world problem and they have to use—They have to pool their knowledge... it's either a problem or a project where they're kind of coming together with a real world problem.

She also described applying as a problem that students must develop their own strategy to solve, and where multiple valid strategies could be applied, saying, "they're having to connect several pieces of their mathematical knowledge to solve one problem." Nicole argued that Applying is almost a justification of learning math "because if you can't apply it then why are you learning it?" She also said that Applying is Connecting, because it includes "connecting with either your prior knowledge to analyze or apply the new math you've learned or... you have to connect to other areas of math to solve the problem." Although I did not probe deeply into what Melinda, Breanna, or Jana meant by Applying, all three mentioned that students gain particular knowledge or learn a particular rule or skill and then apply it.

Teachers connected reasoning to sense-making, analyzing, proving, communicating, and testing theories and ideas. Alison explained that reasoning comes at the end of the process to "make your case." Nicole described reasoning as similar to Alison: that an argument is what happens when you communicate your reasoning. Breanna's description of reasoning seemed to have three parts: reasoning as following steps to get to an answer, proving is showing your reasoning by showing that the steps work, reasoning is also figuring out why the steps work. So, reasoning is following steps to find an answer, constructing a proof that the steps worked, and determining why the steps worked. On the other hand, Nicole also explained that reasoning is the process used to make sense of mathematics. Reasoning is a way to first make sense of a situation and then to deepen understanding by sharing reasoning, hearing others' reasonings, and critiquing reasoning. Jana's description of reasoning is providing understanding to other people and to yourself, especially when she was explaining differences in reasoning as communication or as sense-making, seems to show she cannot separate the three: "making sense to provide the understanding to other people and providing understanding to yourself while communicating."

### *Summary of Habits*

Table 5.9 shows which teachers (and in which units) made statements about mathematical actions, categorized as supporting mathematical communication or higher-level mathematical thinking. Note from the table that every teacher advocated for some form of mathematical communicating and some form of higher-level mathematical thinking. As discussed above, however, teachers did not always have the same goals for these actions and often did not share the same meanings.

Table 5.9

*Attention to Problem-Driven Mathematical Actions*

| Action | Breanna | Melinda | Jana | Nicole | Heidi | Alison |
|---|---|---|---|---|---|---|
| Communicating | – | – | – | 2D | 2D | – |
| Discussing | – | 2R | – | – | – | 1D, 1R |
| Explaining | 1D, 2D | 2R | 2D | 2D | 2D | 1R |
| Justifying | 1D | – | – | 2D | 2D | – |
| Higher-level Thinking | – | 2D | 2D, 1PP | – | – | - |
| Exploring | – | 2D | 2D, 2R | 2D | 1D | – |
| Analyzing | – | – | – | – | – | 1D, 2D |
| Applying | 1D | 1D, 2D, 2R | – | 2D, 2R | – | 1D, 2D |
| Creating | – | – | – | 1D | – | 1D, 2D |
| Reasoning | 2D, 2R | 2D | 2D, 2R | 1D | 2D | 1D |

*Note.* Participants' references to mathematical actions: communicating (discussing, explaining, justifying) and higher-level thinking (exploring, analyzing, applying, creating, reasoning). Source of statements: Interview Card Sort (Int.), Unit 1 Personal Profile (1PP), Unit 1 Discussion (1D), Unit 1 Reflection (1R), Unit 2 Discussion (2D), Unit 2 Reflection (2R). To allow all actions in one table, actions are listed in the first column with participants' source of statements organized in columns.

As an example, the most stated mathematical actions were: explaining, exploring, applying, and reasoning. Explaining and reasoning were actions stated by all participants as important in mathematics. All teachers pointed to students' struggles in communicating their mathematical reasoning. In describing student explanations and communication, Breanna seemed to focus on students' abilities to explain their steps through a procedure to find the correct answer. Nicole and Heidi, on the other hand, seemed to focus on students' abilities to explain their reasoning in a deeper way: why certain choices or steps made sense in the context of the problem. As teacher wrote and spoke about mathematical thinking and problem-solving, Breanna and Melinda seemed to express views that were Instrumental, while Heidi, Nicole, and Jana seemed to express Problem-Driven views. Alison wrote about higher-level thinking in mathematics, but it was unclear whether she felt that was an important skill in doing mathematics or only for learning mathematics.

## Right, Wrong, or Open

In their interview card sorts, Heidi, Nicole, Melinda, Breanna, and Jana chose “mathematics is always open to discussion and diverse views” as one of their top five choices in the card sort. Heidi, Melinda, and Jana deleted the card “mathematics is either right or wrong,” while Nicole replaced it with a new card saying “mathematics has many solution pathways.” Only Breanna explicitly stated that she agreed with the statement “mathematics is either right or wrong,” explaining “Because if you say 2+2 = 6. Well, no. So I actually do believe it’s either right or wrong.” In the interview card sort, Melinda explained her disagreement, “I think a lot of kids have in their mind that my answer is either right or it’s wrong but I think if you look through the process and—Mistakes are just as valuable as correct answers in teaching form and so I definitely disagree with that one.” In her Unit 1 discussion on Benny, Jana argued that “reaching the right answer is not the goal of learning” (Jana, Unit 1, Discussion on Benny) and in her interview card sort, she said, “There are many things in addition to the right and wrong” and later that “Either right and wrong. Math is not like that.” In the paragraphs below, I describe the different ways teachers considered solutions and solution strategies for mathematical problems, including their views of answer keys (in response to Erlwanger (1973)), mathematical validity and the ability to self-check (or prove) answers, problem-solving process versus solution, and sharing (and learning from) multiple strategies and solutions.

Several teachers described reliance on an answer key alone as problematic, especially when with respect to solution representations, i.e., *correct answers* versus *desired answers*. Similar to her response in the interview card sort, Breanna wrote, “I too use the answer key to check my students’ math test. The answer is either correct or it’s not” (Breanna, Unit 1, Discussion on Benny). She later wrote a potentially contradicting question, however, wondering, ‘The teachers know the answers to the answer key. Does knowing the answers to the answer key impact their way of teaching?" (Breanna, Unit 1, Discussion on Benny). Heidi expressed that teachers and students should rely on more than simply an answer key to determine truth. She also expressed that students should not be forced to represent their answers in a particular form, just to match an answer key: "Finding if an answer was true or not was as simple as checking the key" (Heidi, Unit 1, Discussion on Benny). Here Heidi seems to indicate that a teacher or student should rely on more than the answer key.

> A prime example is having multiple representations for a solution. Benny had strong opinions in that he knew there can be multiple ways to show a solution and yet he was forced to give an answer that the key would accept (Heidi, Unit 1, Reflection).

Again, here Heidi seems to indicate that forcing students to make their answers in the form provided by an answer key is problematic. Similarly, Alison expressed that relying only on the key to determine truth or correctness is problematic because a student may value the answer key over testing for mathematical validity: "Benny decides when something is true by looking at the 'key' to see if he got the correct answer or not" (Alison, Unit 1, Discussion on Benny). Nicole expressed that relying only on the key can teach students a kind of helplessness, trying to match the "correct" answer when many forms of an answer exist. And that rely only on the key to determine truth or mathematical correctness is problematic (see validity below). "He also knew that there would be only one correct answer on the key and seemed to search to find that answer." (Nicole, Unit 1, Reflection)

Due to the agreement among teachers that over-reliance on answer keys can be problematic, teachers expressed their views of mathematical validity that ranged from a simple ability to check mathematical work to being able to prove a solution is correct. Note that teachers most frequently used the words *correct* or *right*, rather than *valid* when discussing solutions. Breanna argued, and Melinda stated that she agreed, that "A part of teaching math is presenting a set of rules, demonstrating how to apply them to multiple mathematical scenarios, and *being able to check your answers on your own*" (emphasis added, Breanna, Unit 1, Discussion on Benny). Alison and Nicole seemed to most closely describe checking mathematical validity. For example, Alison expressed that students could analyze "information with another student" and evaluate whether or not "answers are true in every context" (Alison, Unit 1, Reflection), while Nicole expressed that, rather than relying on an answer key to determine when "something is true or correct mathematically" (Nicole, Unit 1, Discussion on Benny), they should be required to "prove to others that their answer is correct" (Nicole, Unit 2, Discussion on Skemp).

Breanna, Alison, Melinda, and Jana all argued that, rather than relying on checking a solution as correct or incorrect, they valued seeing the solution process and whether a student was thinking correctly. Breanna expressed that she must have her students explain their process to understand their thinking.

> I need to ask the students who are struggling to explain how they got their answer so I can see where it is that they are confused. Like Benny's teacher and her assistant, I too use the answer key to check my students' math test. The answer is either correct or it's not. But I need to see why the student may have gotten the answer wrong. I need to see where did the student get confused on the concept. The only way to do that is to have the student explain their answers to me (Breanna, Unit 1, Reflection).

Jana expressed that she expects her students to share their reasoning. "In the classroom, I always ask my students can you please tell me the reason you got this answer" (Jana, Unit 2, Discussion on Skemp). Alison said students should learn "ways to explain their thinking to others" (Alison, Unit 2, Reflection). Melinda focused more on the value of mistakes and paying attention to thinking more than the final answer, saying "their thought process and their thinking" could be correct.

Each of the six teachers except Melinda argued for the importance of students learning from each other through discussion and sharing solution methods and strategies. In her interview card sort, Breanna explained that discussion was important to mathematics because "you find new ways to do things." Nicole referenced the Common Core practice of "critiquing the arguments of others" and said she felt that students needed to "really interact and talk" to understand mathematical problem-solving processes. She said,

> I know with the Common Core Mathematical Practices—you know, critiquing the arguments of others and just having that discussion and thinking about 'Okay, they solved it this way but I solved it this way. What do I think about doing it your way?' ... And, you know, they could both be right. They could both have solved it. Just in different ways. And that's awesome.

Heidi explained that she stays open to multiple solutions, not just solution strategies:

> I really don't like to discount any student's responses unless—I have a few jokers in my class that will often just kind of put something on a homework page. But, I mean, typically within our math discussions and our whole group discussions and small-group discussions, there is always something to learn and gain from each other's methods and solutions.

In discussing answers and solutions, and what *correct* means, teachers may value comparing answers to answer keys, checking one's own answers, and proving one's own answers. Teachers may also value thinking and strategy over an answer alone; they may ask students to explain their process to the teacher or to other students. In Table 5.10 below, I show which teachers spoke to one of these themes, and in which sources of data.

Table 5.10 indicates that Nicole, Heidi, and Alison each stated (after reading Erlwanger (1973)) that over-reliance on an answer key can be problematic; each also stated that she reasoned thus because solutions come in multiple forms (see Table 5.11 below). Breanna, Melinda, Nicole, and Alison stated the importance of checking your work; although, Nicole specifically mentioned "proving" and Alison mentioned "analyzing." Every teacher felt that students should share their thinking and reasoning through the problem-solving process with either (or both) the teacher herself or others in the class, because mathematics is more than just an answer but also a process.

In discussing mathematical answers, teachers may recognize that solutions can represented in different ways and that more than one strategy may be utilized to reach a solution. Teachers may also recognize that problem situations (whether abstract or concrete) may be interpreted in more

Table 5.10

*Meanings for Mathematical Correctness*

| Participant | Answer Correct When: | | |
|---|---|---|---|
| | Matches Key | Self-Checked | Verified |
| Lower Elementary | | | |
| Breanna | Yes (1D) | 1D | – |
| Melinda | – | 1D | – |
| Jana | – | – | – |
| Upper Elementary | | | |
| Nicole | Need more (1D, 1R) | – | Proving (2D) |
| Heidi | Need more (1D) | – | – |
| Middle Grades | | | |
| Alison | Need more (1D) | – | Analyzing (1R) |

*Note.* Participants' statements (and data sources) indicating they value: checking answers using answer keys, checking one's own answers, verifying one's own answers, and sharing a solution process with teacher or others. Source of statements: Interview Card Sort (Int.), Unit 1 Personal Profile (1PP), Unit 1 Discussion (1D), Unit 1 Reflection (1R), Unit 2 Discussion (2D), Unit 2 Reflection (2R).

than one way and, hence, yield more than one valid solution. In Table 5.11 below, I show which teachers spoke to each of these themes, and in which sources of data.

Table 5.11 shows that teachers indicated problems may have more than one valid strategy or solution representation; only Heidi explicitly stated that her students may have more than valid solution for a problem. No teachers explicitly stated that a problem situation may be validly interpreted in multiple ways, thus yielding multiple solutions.

## Summary of Participant Conceptions

Ernest (1989b) presented three broad conceptions of the nature of mathematics: Instrumentalist, Platonist, and Problem-Driven. Each conception consisted of a number of aspects. Table 5.12 shows the finalized framework for conceptions of the nature of mathematics.

In their discussions, reflections, and interviews, teachers revealed varying conceptions of each aspect. Hence, considering an individual element of Ernest's (1989b) framework (e.g., mathematics is connected) as a whole, taken-as-shared concept proved problematic. Rather, in this chapter, I

Table 5.11

*Multiple Representations, Strategies, Solutions, Interpretations*

| Participant | Each Problem Has More Than One: Form of Solution | Strategy | Solution | Interpretation |
|---|---|---|---|---|
| Lower Elementary | | | | |
| Breanna | – | 1D, 2D | – | – |
| Melinda | – | – | – | – |
| Jana | – | 2D | – | – |
| Upper Elementary | | | | |
| Nicole | 1R | Int | – | – |
| Heidi | 1D, 1R | Int | Int | – |
| Middle Grades | | | | |
| Alison | 1R | 1R | – | – |

*Note.* Participants' statements (and data sources) indicating that they recognize that mathematical problems may have more than one: solution form (representation of solution), strategy, solution, or interpretation. Source of statements: Interview Card Sort (Int.), Unit 1 Personal Profile (1PP), Unit 1 Discussion (1D), Unit 1 Reflection (1R), Unit 2 Discussion (2D), Unit 2 Reflection (2R).

addressed potential complexities of each aspect by proposing sub-aspects (e.g., extent, nature, and utility of connections).

I describe each teacher in terms of Ernest's (1989a) three views of knowledge: Instrumentalist, Platonist, and Problem-Driven. I use as evidence the findings of teachers' views of mathematics as connected, rules-driven, problem-driven, and as right, wrong, or open. All teachers showed some evidence of an Instrumentalist conception, but Breanna and Melinda most strongly showed alignment with an Instrumentalist conception. Jana, Nicole, Alison, and Heidi showed close alignment with a Platonist conception. Alison additionally indicated alignment with a Problem-Driven view.

## Breanna

Although she made statements that did not align with any one category, Breanna's conceptions seemed to fit most closely with an Instrumentalist conception. She stated that mathematics *was* created and is unchanging. Her statements indicated that sees some mathematical ideas as connected, but mathematics in general as a loose collection of ideas. Her responses indicated a fairly

Table 5.12

*Conceptions of Mathematics Framework*

| Aspect of Mathematics | Brief Listing of Framework Elements |
|---|---|
| Connections | |
| Extent | *Sparse (some ideas connected or isolated), Dense (structurally connected)* |
| Nature (Connecting) | *Concepts, Representations, Standards, Disciplines, Real world* |
| Utility | *Building on prior knowledge, Illustrating connective nature* |
| Problem-Solving | |
| Lower-Level Thinking | *Knowing how to use, Using to find correct answers, Understanding facts & rules* |
| Flexibility | *Knowing why or when to use, Using multiple representations, Using multiple strategies* |
| Communicating | *Discussing, Explaining, Justifying* |
| Higher-Level Thinking | *Exploring, Analyzing, Applying, Creating, Reasoning* |
| Correctness Means: | *Matched to key, Self-checked, Verified (Proof, Analysis)* |
| Openness to Multiple: | *Interpretations, Strategies, Solutions, Solution Forms* |

*Note.* Brief description of finalized framework describing aspects of conceptions of the nature of mathematics, based on Ernest (1989b) and expanded through exploration of teacher statements.

rigid rule-based view of mathematics that focused on strict views of mathematical answers as right or wrong. In addition, her view of mathematical authority seemed far from a typical mathematics user or learner: that mathematics was created at some point indicates that users no longer create or discover mathematics. Also, her description of an answer key indicated she saw the key as more of an authority than the teacher or student. Breanna explained that she valued students' thinking when they made mistakes, so she could identify their confusion and help them. She indicated that students should understand rules to be able to solve mathematical problems that were not "cookie cutter" problems, and that students should be able to create their own solution strategies and methods. For example, she described a jigsaw approach to a mathematics topic where some small groups would create a solution method and share it with the class. Other small groups would explore the problem situation to understand the "how" and "why" and share their understanding with the class. Despite indicating such a view, her description of the jigsaw activity separates students' understanding of a situation from students' creation of a solution strategy. Such a description could indicate that Breanna does see the connection between these two processes. Or, the description might indicate that Breanna sees value in students' understanding and creating solution processes, but it might be a relatively new value that did not have time to mature in the brief moments of discussion. Breanna made arguments about the importance of students' understanding any meaning behind mathematical processes, but she did not explain what she meant by "understand" or what she meant by "process that underlie the rules."

## Melinda

Throughout the course, Melinda and Breanna shared similar views, although Melinda indicated her value of flexibility and higher-order thinking more frequently than did Breanna. Melinda's views seemed to fit into the Instrumentalist conception, as did Breanna's. Melinda stated that mathematics is both created and discovered, in the sense that students create mathematical questions and that discovery is essential in their problem-solving. In discussing mathematics as continually changing or always the same, Melinda focused on mathematical strategies rather than mathematics itself.

She emphasized her view that any two mathematical ideas are connected. She indicated a flexible rule-based view of mathematics, agreeing with Breanna that mathematics is following rules. She consistently acknowledged the need for rules. She argued consistently understanding how and why rules work is essential for students, so they can be flexible and adapt rules to non-"cookie cutter" problems and situations. Melinda persistently valued mistakes and their usefulness in developing understanding. She valued student thought processes over correct answers, saying that students' correct thinking was more important than displaying a correct answer alone. She indicated that any answer is right or wrong, but exploring mistakes and student thinking is more valuable than simply indicating correctness. Her statements indicated that she valued students' flexibility and their creation of their own solution strategies. In her definition of mathematics, she included both aspects: that mathematics is about exploring thinking (higher-level thinking) and also about procedurally finding answers to questions (lower-level thinking).

I argue that both Breanna and Melinda expressed an Instrumentalist conception of mathematics.

In comparing Breanna and Melinda, similarities and differences are apparent. Breanna expressed a fairly rigid rules-based view of mathematics where mathematical authority (e.g., who determines the topics, strategies, and correctness of answers) is external to the teacher and students. Melinda expressed a view of mathematics where deeper understanding is valued to support more flexible use of rules and procedures, and where mathematical authority rests more with students as they create strategies and communicate their thinking.

## Jana

Despite being in the same grade-level group as Melinda and Breanna, Jana differed in her perspectives. Jana's conceptions seemed to fall in the Instrumentalist and Platonist categories, weighted more toward Platonist than Instrumentalist. I claim Instrumentalist because at times she described the utility of mathematics as finding answers. On the one hand, she described the mathematics of young children and of scientists as connected because both groups make calculations based in the same operations. On the other hand, I claim her conceptions are Platonist because her statements indicated a sense of mathematics as complete and whole, and as existing in an abstract way outside of human activity. She stated that mathematics exists without being created or discovered and that mathematics is unchanging. These views indicate a Platonist conception.

Jana described the need for students to explore the world of numbers, which sets her apart from Breanna and Melinda in that Jana valued students' conceptual exploration. That is, she valued an activity where students explored the big ideas of numbers without necessarily finding correct answers or understanding procedures. Jana's focus on conceptual understanding over procedural fluency, and on exploration of mathematical ideas, indicates a Platonist conception of mathematics.

Jana described mathematics as connected because she viewed mathematics at every level from young children to professional. She stated that, even though goals and strategies become more complex, basic operations and symbols stay the same. She expressed value for students' higher-level thinking, and only described rules or procedures as valuable if students needed to quickly calculate something. Otherwise, she argued that students should be supported in their reasoning and thinking more than following rules. She also argued that mathematics is not all about right or wrong answers, but that sharing thinking and making sense were more important.

Jana's views of mathematics set her apart from Melinda and Breanna because of her expressed value for meaning, sense-making, and creativity, over simply following procedures and finding correct answers. Because of these differences, I argue that Jana expressed a Platonist conception of mathematics more strongly than Instrumentalist.

## Nicole

As is true of every teacher, Nicole's statements did not easily align with one category. However, I claim her views most strongly expressed a Platonist conception. She described mathematics as "brought out" of situations. Her description implies she viewed mathematics existing in an abstract form outside human thought and activity and, thus, is discovered rather than created. This description indicates a Platonist conception. On the other hand, she spent some time choosing the phrase "brought out," and rejecting "created" and "discovered." She specifically described teaching strategies as changing, but aspects of mathematics itself as stable. Her expressed view that mathematics is stable, even if strategies change, supports that she held a Platonist view of mathematics. A minute later, however, she described mathematics as continually changing because otherwise "why are there mathematicians?" Hence, her conception based on these descriptions alone is ambiguous.

Nicole described openness to discussion and diverse views in terms of allowing students to create and use their own solution strategies. She described mathematics as characterized by its structural interconnectedness. She disagreed that mathematics was entirely problem-driven or entirely rules-driven. She argued that lower-level students may need rules-based mathematics, while higher-level students ought to engage in reasoning, conceptual understanding, and more creative problem-solving. Nicole seemed conflicted at times between valuing reasoning, but arguing that bringing it in to her teaching increased difficulty for teachers and students. She explained that, although higher-level thinking was important to her, her students resisted.

In her two most recent years of teaching, she had used the *Connected Mathematics Project 2* curriculum (Lappan et al., 2009). Using such a curriculum, that emphasizes conceptual understanding and inquiry-based learning, may have influenced Nicole to change her views of mathematics and mathematics teaching and learning. Her arguments of "why it won't work" as a counterpoint to her desire to teach mathematics this way could indicate her valid perception of barriers. Her arguments might also indicate that, despite her values, she felt hesitant or inadequate in implementing such a teaching style. Her additional argument that lower-level students would not be able to succeed in such a classroom, while higher-level students would be able to succeed (if they stopped resisting it), might reveal her own internal struggle to reconcile her changing views of mathematics with changing views of teaching and learning mathematics.

## Heidi

Similar to Breanna and Melinda, Heidi and Nicole shared views to some extent. Heidi even mentioned in the interview that she and Nicole had bonded over shared views and hoped to continue to correspond about their teaching. Based on her statements, Heidi's views fell in the Instrumentalist and Platonist conceptions of mathematics. In responding to questions about creation or discovery, changing nature or stability of mathematics, Heidi focused on the content or strategies of school mathematics: how students create or discover as they learn and how school mathematics changes continually. For example, she explained that mathematics continually changes because student strategies change and because what is valued in mathematics changes. On the other hand, she explained that mathematics is unchanging because some of what is valued does not change, such as problem-solving, and because mathematics will always be in students' lives. Heidi's focus in these statements indicates her view of mathematics as school mathematics.

Heidi argued that the collection of facts, skills, and rules has a important place in mathematics, although she acknowledged the collection was not the whole focus of mathematics. Her statements indicate contradiction. On the one hand, she argued that mathematics was more than only facts, skills, and rules. On the other hand, she argued that students who did not master required facts and rules would struggle to complete future tasks where such knowledge was assumed. Similar to the contradiction in Nicole's views about facts and skills for some and deeper thinking for others, Heidi's descriptions of mathematics may be an acknowledgment of the type of thinking valued in school mathematics, but opposed to the type of thinking Heidi valued. Heidi may have believed that students need facts and rules, but also need critical thinking and other mathematical practices built on those facts and skills.

She described a structural "connective tissue" of mathematics. Heidi described mathematical openness as valuing multiple solutions, solution strategies, and solution representations. She argued against simple memorization of rules, and argued for students' creation of multiple solution methods and representations to support their flexibility in problem-solving. She explained that she valued students proving or disproving each other's solution strategies to support deeper understanding. Heidi described her value of complex problems where students had to draw on multiple concepts, to "pool their knowledge," to create solutions.

I argue that Heidi's views indicate a Platonist conception, although her focus on facts and skills in mathematics also indicates a leaning toward an Instrumentalist conception. Nicole similarly showed some contradiction in her views of mathematics. It is possible for both Nicole and Heidi to have been exhibiting contradiction due to the contradiction in expectations of school mathematics and assessments. It is also possible that Nicole and Heidi exhibited contradictions due to a change from one set of conceptions to another.

## Alison

Alison's statements indicated some views consistent with an Instrumentalist or Platonist conception, but her views seemed more consistently aligned with a Problem-Driven conception. I say Problem-Driven, because she described mathematics as a school of thought, a way of thinking, a way of making sense of the world and abstract topics, and a way of communicating. She described mathematics as both created and discovered. She described users of mathematics creating mathematics and making it "what we want." She also described users discovering mathematical topics, connections, and simply “discovering things and finding things out.” She explained that mathematics is continually changing because of this creation and discovery.

She indicated that noticing mathematical connections is new to her, but that she sees more connections as she teaches. Although she consistently argued for students’ high-level thinking, referencing the upper levels of Bloom’s taxonomy, she also indicated a rules-driven view of mathematics. For example, she said rules are easy to teach and support students in finding answers. On the other hand, Alison argued that rules alone can block students from essential flexibility for situations where rules do not work. She explained that she asked her students to solve problems in multiple ways and also to explain their thinking in multiple ways. Alison argued that communication was a critical part of problem-solving, that talking through problems supports sense-making.

Alison described mathematical authority as within students, explaining that students could analyze their solution strategies to see if strategies were valid. She valued students’ thinking and solution process over a right or wrong answer alone. She valued students explaining their thinking to each other and learning from each other through discussion.

I argue that Alison’s views are most consistent with a Problem-Driven conception, because of her focus on mathematical creation and discovery, mathematics as always changing, mathematics as communication, and mathematics as a system of thought. Her views show weaker alignment with Instrumentalist or Platonist conceptions. She acknowledged some of her views, such as mathematical connections, as recently developed. Her persistent references to Bloom’s taxonomy might indicate that the taxonomy was providing an external scaffold of her views. That is, the taxonomy may have influenced her so that she wrote statements more indicative of its structure than of her own views. On the other hand, in her description of mathematics in response to the unstructured interview question, her expressed definition (key elements listed above) strongly aligned with a Problem-Drive conception.

# CHAPTER 6

# TEACHER NOTICING

Teacher decision-making about online tools and resources and their use in mathematics classrooms is complex and can be overwhelming. Teachers must attend to multiple aspects of online tools, including general technological, pedagogical, and mathematical aspects, whether tools were designed to support mathematics education or designed for broader educational or mathematical utility. The design and use of the tool, a teacher's instructional strategies, and mathematical concepts and tasks all interact and impact each other in complex ways. Teachers must be able to quick and informed decisions to fit already overloaded schedules.

One goal of the *Mathematics, Technology, and Education* course was to support teachers in developing the evaluation aspect of their decision-making by using a detailed template to think through aspects of online tools. The instructors intended the template to support teachers in developing an internal framework for quick but thoughtful evaluation of future tools. The template structure (shown in Appendix E) included attention to: general technological characteristics (technology access and logistics), pedagogical characteristics (learners, learning instruction support), and mathematical characteristics (mathematical focus and representation). Although the template had separate categories, such aspects of online tools are neither well-defined nor cleanly separated; they interact, overlap, and impact each other in messy ways. Acknowledging this complexity, I share findings based on selected aspects of teachers' evaluations to describe their attention to mathematical characteristics.

## Modifications to Research Question 2

In this section, I share findings to answer the second research question. Initially the second research question had a broad focus to explore teachers' noticing of all types of tool characteristics. In consideration of space and time limitations of this study, of the large number of potential characteristics, and of the amount of data, I chose to narrow the focus of the study by using a mathematical lens. Hence, I modified the second research question to read: *What mathematical characteristics, aspects, or features of digital tools and resources do teachers notice, as evidenced through responses to explicit evaluation questions and open-ended evaluations?* To answer this broad question, I attend to four subquestions: *How do teachers describe mathematical learning goals and representations? How do teachers describe intended mathematical understandings? How do teachers describe intended learning activity types? In considering how tools support mathematical learning, which aspects of tools do teachers describe as affordances and which do they describe as supporting student engagement?* I align the four questions with four sections of the evaluation template: mathematical focus and representation, intended mathematical understandings, types of learning activities, and aspects of tools that supported learning.

## Overview of Data, Analysis, and Findings

I share findings about participants' patterns of noticing from my analysis of evaluations written throughout the course. Some evaluations were well-structured through use of a detailed evaluation template (see Appendix E). Others were more open. Teachers chose online tools to evaluate. In early units, instructors provided lists from which to choose. In later units, teachers were encouraged to find online tools themselves. A list of evaluated tools is provided in Appendix B. First, I share findings in response to the first subquestion by considering participants' statements about mathematical foci, learning goals, and representations of the online tools. To answer the second subquestion, I considered participants' attention to instrumental and relational understanding, strands of mathematical proficiency, mathematical process standards, and standards of mathematical practice. By sharing teachers' rationales for use of different labels, I share their varied meanings for different types of mathematical understanding. To share findings for the third subquestion, I considered participants' characterizations of online tools' potentials for supporting presentation, exploration, practice, or application activities. I share teachers' portrayals of learning activity types. Finally, I share findings for fourth subquestion by considering participants' attention to affordances of online tools, suggested by instructors and teachers, and their attention to aspects that support student engagement.

## Mathematical Focus

Teachers attended to mathematical focus and learning goals, in structured and unstructured evaluations, by describing their views of intended mathematical learning goals. To make sense of their attention to mathematics in online tools, I considered their descriptions of learning goal types: general or specific, learning or performance, and facts-and-skills or habits-of-mind.

### Intended Mathematical Learning

To explore teachers' statements about what mathematics could be learned through interaction with specific online tools, I gathered statements from their evaluations. For evaluations based on whole-class templates (as shown in Appendix B), I tagged all statements in response to three questions: (3a) *What mathematics is being learned?* and (3b) *Content Standards.* I also tagged mathematics-focused statements in response to question (5a) *What do learners need to know?*. Even though most relevant statements were written in response to (3a), I included (3b) and (5a) because teachers wrote statements, in each of the three sections, describing the mathematics students would learn and the mathematics students should have learned before using the online tool. For evaluations based on other templates or no template (i.e., those written in Units 1, 2, 3, 6, and 8), I considered the entire evaluation. If a teacher included her own explicit sections addressing mathematical focus, I tagged all statements in that section. I tagged any statements in the entire evaluation that attended to mathematical focus, learning goals, or standards.

I coded goal-like statements for each online tool, following Schunk's (1996) categories of *performance goals* and *learning goals*. I use Schunk's definitions of these terms, where *performance goals* refer to teachers' intentions for actions that students will take or tasks that students will complete. Schunk described *learning goals* as teachers' intentions about content knowledge or skills that students will develop. To illustrate the difference between the two types of goals, I provided examples in the form of statements made by Nicole. Nicole made several goal-like statements about NCTM Illuminations: *Turtle Pond.* Two statements that I coded as performance goals were "getting the turtle to the pond" and "measuring the distance between two objects... on a coordinate grid." I labeled these goals as *performance goals* because they addressed the actions students would perform through interaction with the online tool (i.e., reaching the pond, measuring distances on a coordinate grid). Two goals that I coded as *learning* were: "learning how to use visualization and spatial reasoning" and "learning how to mentally change the position and orientation of the turtle as it moves across the grid." For these, Nicole explicitly used the term *learning* and hence I coded them as learning goals. More importantly, she described the knowledge and skills she intended that students would gain from interacting with the tool. Note that the second statement supports the first; that is, if students learned to *mentally change position and orientation*, it would support the big idea of *being able to use visualization and spatial reasoning.* Note that mentally changing the position of the turtle can be described as an action, and so if Nicole had written, "Students will mentally change the position of the turtle," then I would code this statement as a performance goal. Because she wrote instead that students were "learning how to mentally change the position," I coded it as a learning goal. That is, in this statement, Nicole's stated intent was not just for students to complete an action (performance) but for students to learn how to complete an action (learning).

Learning and performance goals might be stated in the same sentence. For example, the statement "deepen their understanding of counting by allowing them to exchange coins to get to the same amount" shared the learning goal, *deepen understanding of counting*, and the performance goal, *exchange coins for the same amount*. I considered each goal-like statement, focusing on the mathematics referenced and verbs used, to label it *performance*, *learning*. Note that teachers did not write goal-like statements in every evaluation. Figure 6.1 shows the overview of frequency of teachers' use of goal types, performance or learning, for online tools across their evaluations.

Figure 6.1 shows that all teachers wrote at least one goal-like statement for 70% or more of the online tools they evaluated. Breanna and Nicole alone wrote goal-like statements for every tool they evaluated; Melinda, Jana, Heidi, and Alison did not write goal-like statements in some

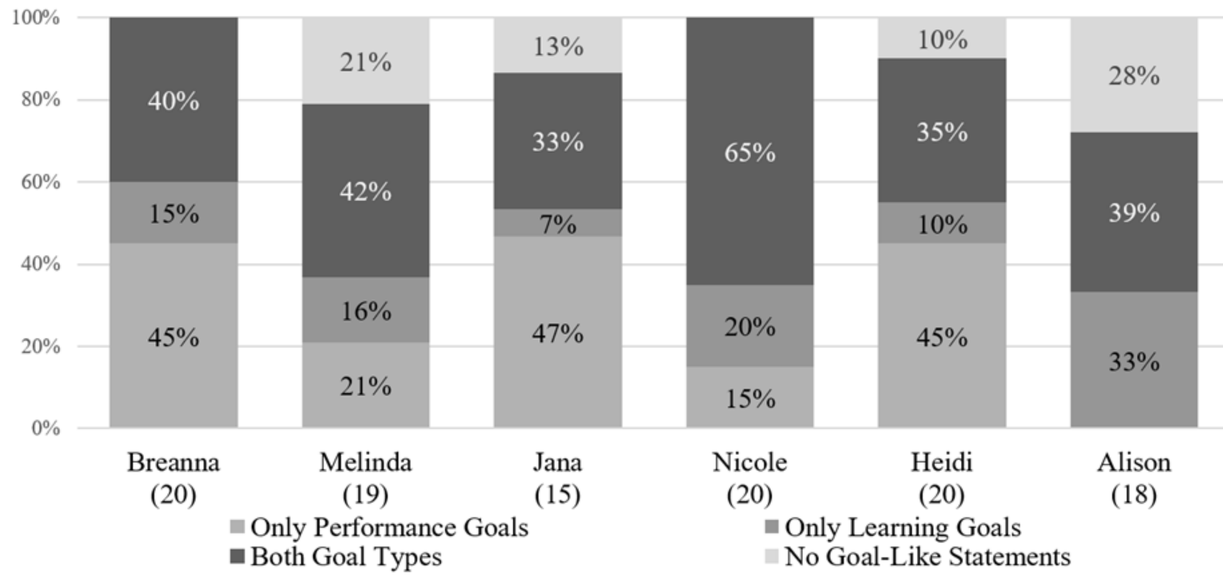


Figure 6.1: Performance and Learning Goals

of their evaluations. Melinda and Nicole attended to learning and performance goals in almost perfect balance; Melinda wrote performance goals for slightly more tools than learning, while Nicole wrote learning goals for slight more tools than performance. Breanna, Jana, and Heidi wrote performance goals for more online tools than they wrote learning goals, while Alison wrote learning goals for more tools than performance. Teachers wrote both performance and learning goals in evaluations. For example, Alison described one goal of Math Playground: *Spider Match* saying that students "start with an answer and find two numbers that work," a performance goal. She also wrote a learning goal about the same tool, writing that it supported an "understanding of how integers work when added together." Nicole wrote both learning and performance goals for 65% of her tools, while the other teachers ranged from 33% to 42%.

Within learning and performance goal categories, teachers described goals with varying levels of focus and varying types of expected performance or thinking. I first describe teachers' use of focus; that is, whether they wrote specific or general goals in their descriptions of tools. I then describe teachers' attention to the nature of knowledge; that is, whether they wrote goals that described students' development of facts and skill knowledge or habits of mind knowledge. In considering teachers' goals for a particular online tool, I considered the overall picture of types of goal, levels of focus, and nature of thinking.

### *Goals: Levels of Focus*

First, levels of focus varied. In addition to considering whether they attended to student performance or learning, goals tended to vary in focus. A less focused learning goal attended to a big idea, for example, to "gain a deeper understanding of numbers in general." A more focused learning goal attended a narrowly defined goal, potentially able to be addressed in a lesson, such as "learning about place value." Similarly, performance goals varied in focus from a less focused, "explore concepts of area and perimeter," to a more focused, "order fractions from least to greatest."

In considering levels of focus of goal-like statements, I use only two levels: specific and general. For both performance and learning goals, I consider a specific goal to be written so that someone outside of the classroom could read it and have reasonable expectations for what a student would learn or how they would interact with mathematics through use of an online tool. A general goal provides information about broad expectations more appropriate for describing expectations for student interaction within a larger unit or even across the school year. For performance goals, an example of a specific performance goal is: "order fractions from least to greatest." An example of a general performance goal is: "explore concepts of area and perimeter." That is, I designated a performance goal as *specific,* when a teacher was explicit about specific expectations for actions or tasks. Similarly, examples of general learning goals would be "further learning or understanding" or "learning about algebra." In these examples, the teacher expresses an expectation for learning in general or for learning about a broad topic. Examples of specific learning goals would be "learning that a tessellation is a repeating pattern of polygons that cover a plane with no gaps or overlaps" or "shape recognition," where an explicit lesson focus is given. General learning goals, on the other hand, are broader, as in these examples: "train their minds to remember certain math problems" or "data and data representation."

Figure 6.2 shows the overview of teachers' level of focus in goal-like statements across online tools evaluated.

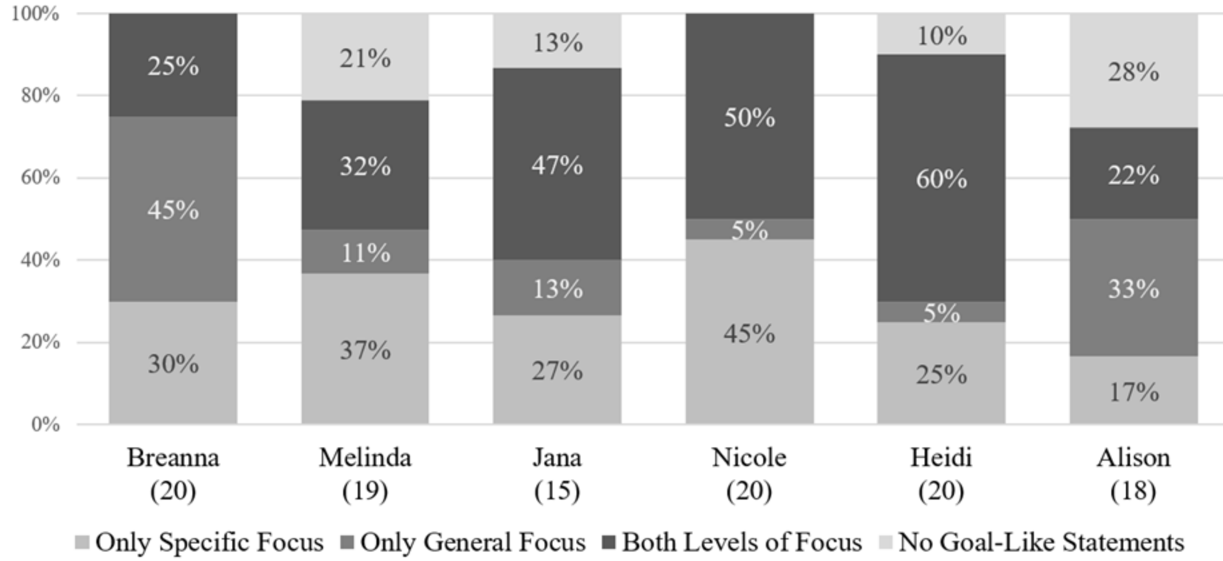


Figure 6.2: Overview of Focus Levels

Figure 6.2 shows that, except Alison and Breanna, all teachers described goals with a specific focus for more tools than they described a general focus. Breanna and Alison both described general goals for more tools than specific; for almost half (45%) of Breanna's tools and about one third (33%) of Alison's tools, each described only general goals and no specific goals. Nicole and Heidi wrote specific goals for 95% and 85% of their tools, respectively. I next consider teachers' level of focus for performance goals and then learning goals.

As examples of specific performance goals, Nicole wrote that students would use one tool to "practice identifying how many symbols are in the 10-frame" and Melinda wrote that students would use a tool to "locate the shapes that are hidden in animated, real-world pictures." In these examples, Nicole and Melinda clearly described their expectations for students' mathematical interaction with the tool: identifying symbols and locating shapes. Hence, I coded the performance goals as indicating a specific level of focus. As examples of general performance goals, Heidi wrote that students would use a tool to "view a visual and audio presentation that will demonstrate this method" and Jana wrote that students would use a tool to "apply mathematics to problems and situations." In these goals, Heidi and Jana described students' mathematical interaction, but only generally alluded to the mathematics without specifying particular topics or ideas; thus, I coded such goals as indicating a general level of focus. Teachers often wrote several goal-like statements for a single online tool. Figure 6.3 shows the frequency of online tools for which teachers described both specific and general performance goals, only specific performance goals, or only general performance goals.

Teachers wrote specific and general learning goals. Alison wrote that a tool would "solidify the fact that perimeter is AROUND an object" (emphasis in original), which indicates a specific idea that she expected students would learn through their interaction. Breanna wrote her expectations about a general idea that students would learn, writing that the tool would "train their minds to remember certain math problems." Figure 6.4 shows the frequency of online tools for which teachers described learning goals with both specific and general focus, only specific focus, or only general focus.

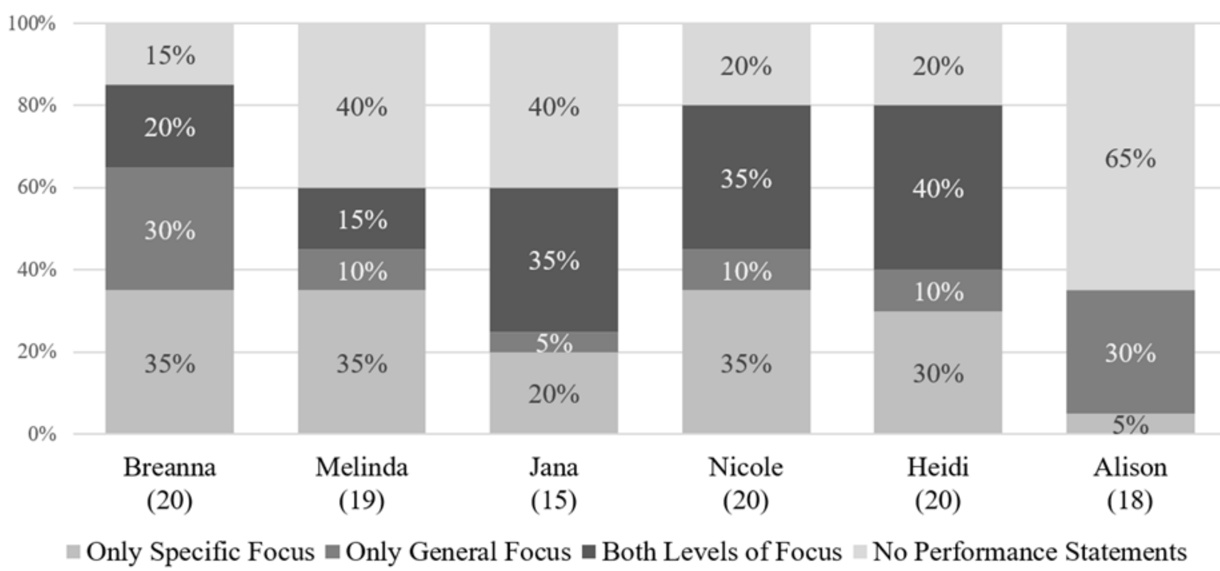


Figure 6.3: Performance Goals: Focus Levels

Figure 6.3 and Figure 6.4 confirm, as a reminder to the reader, that Breanna, Jana, and Heidi wrote performance goals for more online tools than learning goals, Melinda and Nicole were relatively balanced, and Alison wrote learning goals for more tools than performance.

Figure 6.3 shows that, for at least half of their online tool evaluations, teachers, except Alison,

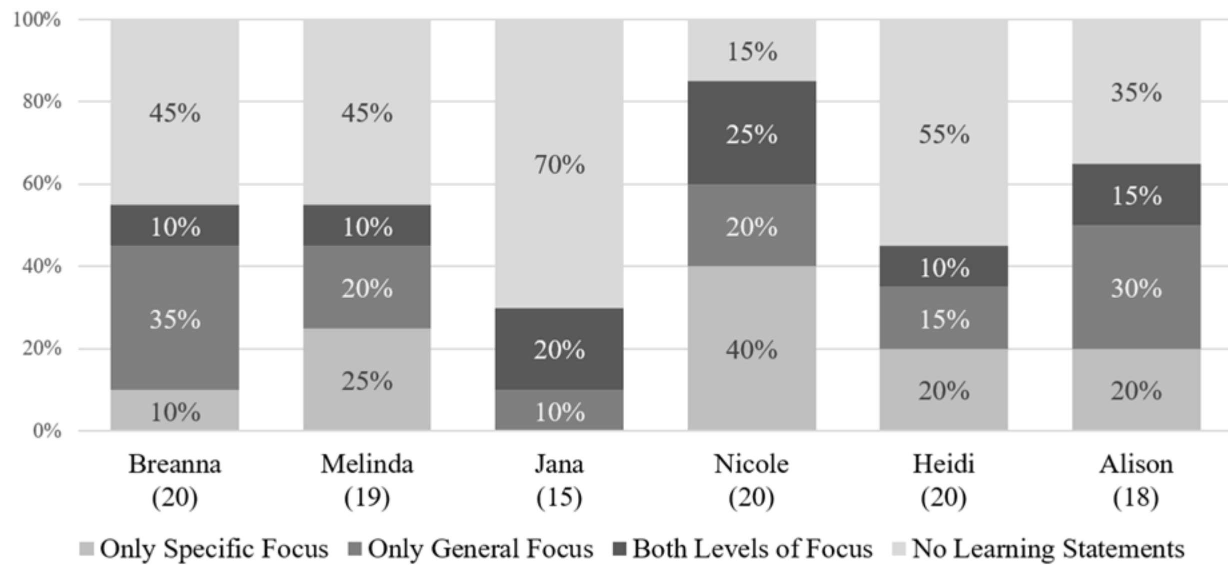


Figure 6.4: Learning Goals: Focus Levels

wrote performance goals with a specific focus. Nicole and Heidi wrote specific performance goals for 70% of their online tools. Breanna, Nicole, and Heidi wrote general specific goals for about half of their online tools (50%, 45%, and 50%, respectively). Breanna and Alison wrote general-level-only performance goals for more of their online tools than any other teacher.

Figure 6.4 shows more variation in teachers' attention to specific or general learning goals than performance goals. Overall, Breanna wrote performance goals for more online tools than learning goals. She showed little difference in her attention to general learning goals compared to general performance (45% and 50%, respectively), but a great difference in her attention to specific learning versus performance goals (20% and 55%, respectively). Melinda and Nicole showed a larger difference in their attention to general learning goals than general performance goals.

In comparing teachers' attention to general or specific goals across learning or performance goals, only Alison attended to more general performance goals than specific performance goals. All other teachers attended to specific performance goals in more tools than they did general. For learning goals, however, only Nicole and Heidi attended to specific learning goals in more online tools than general learning goals. Breanna, Melinda, Jana, and Alison all described general for more online tools than specific learning goals.

### ***Goals: Nature of Knowledge***

The nature of knowledge, or expectations for student thinking varied. In addition to performance and learning goals, or general and specific goals, expectations tended to express a facts-and-skills nature or a habits-of-mind nature. To explore the nature of thinking described in teachers' statements, I inspected each goal-like statement for a focus on *facts and skills* or *habits of mind.* As I explored goal-like statements, I looked for facts and skills key words, such as: *practice, algorithm, fluency, memorization, train, rules, exercises, facts,* and *steps*. For example, Jana wrote that a tool would support students' learning of "basic number facts by using ten frames" and Melinda wrote that a tool would help students "become fluent in using base ten blocks." I similarly looked for habits of mind key words, such as: *think, create, analyze, explore, how and why, interpret, investigate, apply, reason, proof, connect, discover, understand, relationships,* and *conceptual*. For example, Heidi wrote that a tool would help students "get past the notion that an '=' is there for just answering questions instead of showing equality as a relationship between two expressions." Melinda wrote that students would use a tool to "create and identify sets and patterns."

Not every goal-like statement indicated a focus on habits of mind or on facts and skills. For example, Heidi described about one online tool: "Students must manipulate the shapes to meet the specified criteria." This performance goal indicates attention to neither facts and skills nor habits of mind. Other statements that Heidi wrote about the same tool did attend to the nature of student thinking. Figure 6.5 shows the overview of teachers' attention to nature of knowledge as described in all goals across online tools evaluated.

Figure 6.5 shows that no teachers wrote goals for every tool that indicated the targeted nature of knowledge, but Jana, Nicole, and Heidi gave descriptions that indicated the nature of knowledge for at least 65% of their tools. All teachers, except Melinda and Breanna, described more tools using goal-like statements focused on habits of mind than facts and skills. Melinda's attention to nature of knowledge was in balance, describing habits of mind goals for 32% of her tools and facts and skills goals for the same percentage. Breanna described facts and skills goals for 40% of her online tools, and habits of mind goals for 30%. Jana and Nicole wrote both facts and skills goals

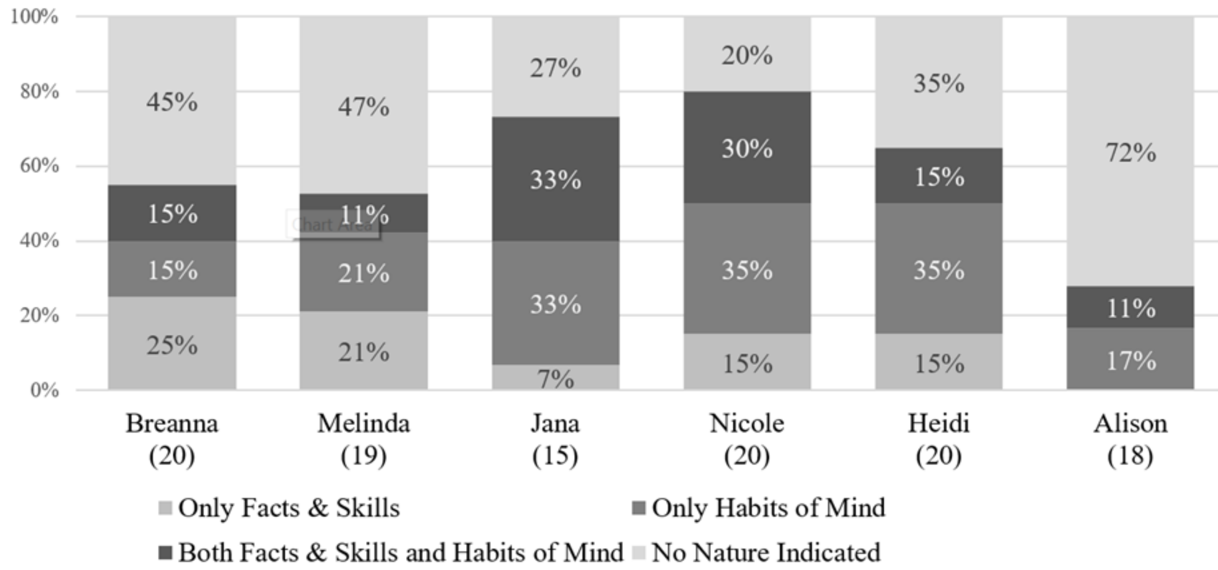


Figure 6.5: Overview of Nature of Knowledge

and habits of mind goals for about a third of their tools (33% and 30%, respectively).

I next consider the relationship between types of goals (performance or learning) that attend to nature of knowledge (facts and skills or habits of mind). First, I consider performance goals. Teachers wrote performance goals that expressed a facts and skills nature or a habits of mind nature. Nicole wrote her expectation for students to use a tool to "explore the connections between fractions/decimals/percents/diagrams and what happens when the numerator or denominator changes." This performance goal also indicates a focus on a habits of mind nature of knowledge, because of the focus on explore connections and seeing what happens. Melinda wrote a performance goal expressing that a tool provided "drill-and-practice methods for memorizing number recognition and fact families up to ten." This performance goal indicates Melinda's expectation that students would gain knowledge of facts. Figure 6.6 shows the overview of teachers' attention to nature of knowledge as described in performance goals across online tools evaluated.

I then consider learning goals, and how teachers described learning goals with a facts and skills nature or habits of mind nature. Breanna wrote a learning goal, expressing that students would use a tool to "deepen their understanding of counting," which indicates a focus on developing students' deeper understanding. I describe this expectation as part of developing students' habits of mind.

A learning goal written by Jana expressed her expectation that students would use the tool to "reinforce the already known topic or to practice skills." This learning goal focused on a facts and skills nature of knowledge. Figure 6.7 shows the overview of teachers' attention to nature of knowledge as stated in learning goals across online tools evaluated.

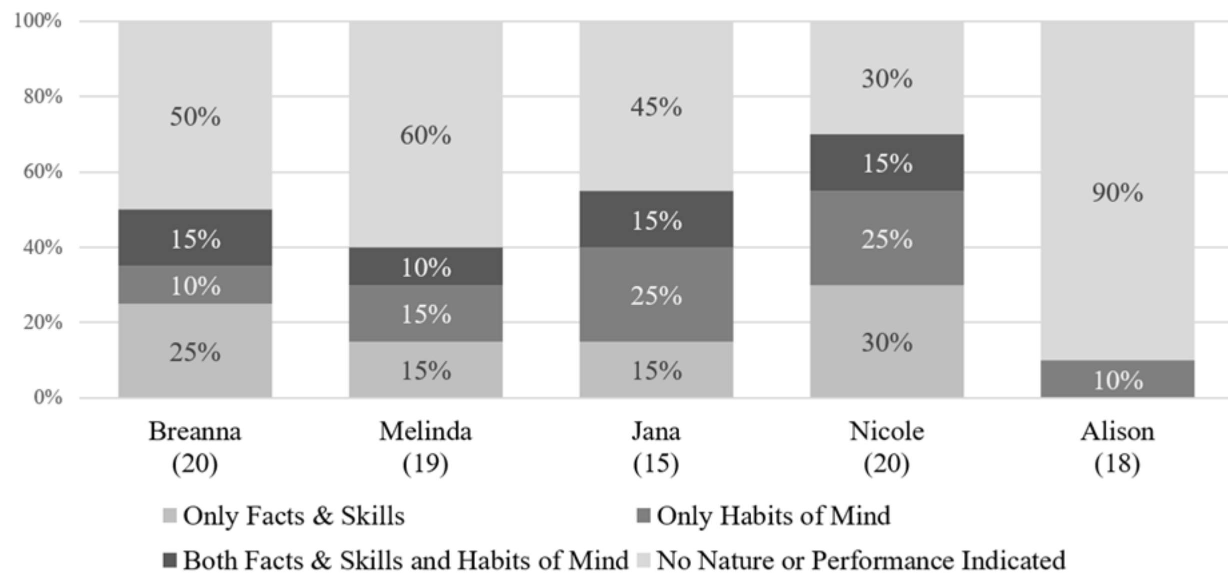


Figure 6.6: Performance Goals: Nature of Learning

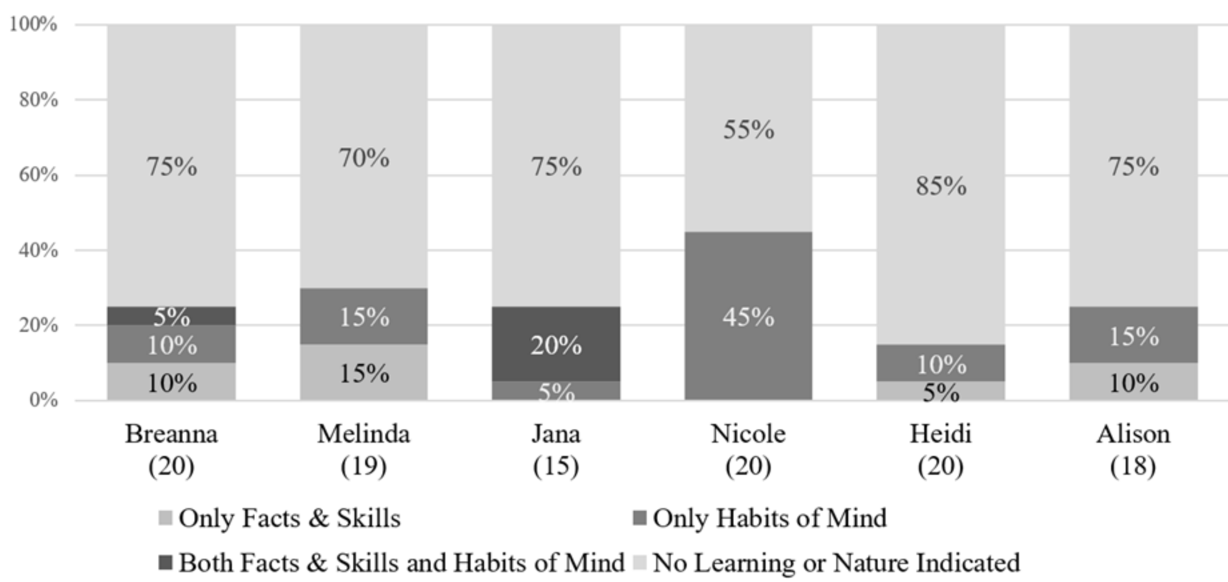


Figure 6.7: Learning Goals: Nature of Learning

Figure 6.7 and Figure 6.6 shows that Jana, Heidi, and Alison attended to habits of mind in both performance and learning goals for more online tools than facts and skills. Breanna, Melinda, and Nicole described more facts and skills performance goals. For learning goals, Nicole only described habits of mind goals, while Breanna and Melinda described facts and skills goals for the same number of online tools as habits of mind goals.

## Summary

In considering teachers' attention to mathematical goals and representation embedded in online tools, I attend to emergent characteristics that may help distinguish teachers as "experts" and "novices" across these aspects. Regarding mathematical goals, teachers generally wrote a number of different types of goals. That is, goals were a collection of types: learning or performance, specific or general, facts & skills or habits of mind.

Looking across the goals,

- Breanna favored, first, general performance goals with a facts & skills nature, and, second, specific performance goals with a habits of mind nature.
- Melinda favored, first, specific performance goals with a habits of mind nature, and, second, specific performance goals with a facts & skills nature.
- Jana favored, first, specific performance goals with a habits of mind nature, and, second, general performance goals with a habits of mind nature.
- Nicole favored specific performance goals with a habits of mind nature. (She had no clear second preference.)
- Heidi favored, first, specific performance goals with a habits of mind nature, and, second, general performance goals with a habits of mind nature.
- Alison favored, first, general learning goals with a habits of mind nature, and, second, specific learning goals with a habits of mind nature.

That is, except Alison, teachers favored performance goals over learning goals. They favored performance goals with a habits of mind nature, whether specific or general, slightly more than they favored performance goals with a facts & skills nature.

In considering, how teachers balanced the types of goals, I considered the number of tools for which teachers wrote both learning and performance goals. Of tools for which they wrote goals, Melinda, Nicole, and Alison all wrote both learning and performance goals for over half of their tools. I then considered both general and specific: Jana, Nicole, and Heidi all wrote both general and specific goals for over half of the tools for which they wrote any goals. Finally, I considered both facts & skills and habits of mind goals. No teacher wrote both within the majority of their tools. Nicole and Jana had the highest relative frequency, both over 30% of the tools for which they wrote goals showing both facts & skills goals and habits of mind goals. Hence, some teachers did balance their goals within one tool between learning and performance or specific and general. But no teacher balanced their goals within tools between habits of mind and facts & skills.

## Mathematical Representation

In this section, I describe teachers' noticing of mathematical representation in online tools by exploring statements they wrote in evaluations about those tools. Teachers attended to mathematical representation of online tools in response to explicit question in evaluation templates and as a common theme that arose as they described aspects of online tools in other sections or in evaluations for which the question remained unasked. They described multiple types of mathematical representation. Among the types of mathematical representation described by teachers, I focus on three important types: symbolic representation, visual representation, and dynamic representation.

Hence, I begin the section by describing teachers' attention to mathematical representation in response to the explicit question about mathematical representation, in response to other questions or prompts, and as a theme when they chose their own evaluation structure or questions. I describe the types of mathematical representation to which they attended.

I then describe, in some detail, their attention to symbolic representation, visual representation, and dynamic representation. Symbolic, visual, and dynamic representations are not distinct types of representation. Symbolic representation is a way of visualizing mathematical ideas. Dynamic representation is a particular form of visual representation that involves movement rather than simply a static image. In presenting teachers' attention to these representations as separate ideas in separate sections, I do not intend to argue that the types of representation are mutually exclusive, or even distinct. I instead argue for the utility of these three names as allowing the researcher to shine a light on different aspects of representation. I focus on aspects of each type of representation as emergent themes from exploring teachers' statements about representation. I describe these aspects below.

In describing teachers' attention to symbolic representation, I focus on their use of words to indicate a representation that, to them, is "the standard representation" or "the real representation." I also focus on how they distinguish (or not) between the symbolic representation of a mathematical object and the mathematical object itself.

In the section on visual representation, I focus on teachers' attention to visual representation, their attention to mathematical or non-mathematical aspects of visuals of the online tool, and, finally, their use of explicit names for particular mathematical representations, their description of mathematical learning goals supported by that representation, and their description of the mathematical ideas or objects being represented.

In describing teachers' attention to dynamic aspects of representation, I focus on their attention to movement of or interaction with the online tool, followed by an analysis of teachers' expectations of what students would do as they interacted with the tool and what the tool would provide to the student. I then describe teachers' attention to types of dynamic representation: dynamically or statically linked representations, manipulatives, and movement of static images. Finally, I describe teachers' attention to mathematical consequences or feedback supported by online tools.

## Attention to Mathematical Representation

In the whole class evaluation templates, a question was, "How is the mathematics represented?" as a part of the "What role does technology play?" section. In addition to responding to this question in many of their evaluations, teachers described mathematical representations throughout each evaluation as they described the online tool, its mathematical focus, and potential student interaction. Table 6.1 shows the number of statements about mathematical representation found across a participant's evaluations compared to their frequency of response to an explicit question about mathematical representation.

Table 6.1

*Frequency of Attention to Representations*

| Participant | Explicit Questions | | Other Questions | | Total Count | |
|---|---|---|---|---|---|---|
| | *n* | % | *n* | % | *n* | % |
| Lower Elementary Group | | | | | | |
| Breanna | 18 | 90 | 14 | 70 | 20 | 100 |
| Melinda | 11 | 58 | 14 | 74 | 18 | 95 |
| Jana | 14 | 93 | 13 | 87 | 14 | 93 |
| Upper Elementary Group | | | | | | |
| Nicole | 14 | 70 | 18 | 90 | 20 | 100 |
| Heidi | 13 | 65 | 16 | 80 | 19 | 95 |
| Middle Grades Group | | | | | | |
| Alison | 6 | 33 | 10 | 56 | 11 | 61 |

*Note.* Number of tools for which participants described mathematical representation. Counts given according to: response to an explicit question or response to other question. Percentages of total tools evaluated given.

Table 6.1 shows that, in looking across all of the online tools that were evaluated, only Breanna and Nicole made at least one mathematical representation statement about every one of their online tools. All participants except Alison wrote at least one statement attending to mathematical representation about every tool, or all but one tool. Note in Table 6.1 that Alison wrote mathematical representation statements for the least number of tools. Breanna and Nicole made at least one statement about every tool. Breanna and Jana made statements about mathematical representation in response to explicit questions for the highest percentage of tools, while Melinda and Alison for the lowest percentage of tools. Jana and Nicole made representations statements in response to other questions for the highest percentage of tools, while Alison, Breanna, and Melinda the lowest percentage. Melinda, Nicole, Heidi, and Alison each described mathematical representation in response to other questions or their own choice for more online tools than for which they responded to the explicit question.

There seems to be no strong pattern across grade bands, as far as making statements in response to explicit or other questions. This finding may be counter-intuitive, because it might make sense for upper grades and middle grades teachers to focus more on mathematical representation than the lower grades teachers. It may be, however, that the *types* of representation described may be different across grade bands.

In four units, teachers had the opportunity to choose the questions they asked themselves, or the characteristics they described. That is, in Units 1 and 8, teachers were asked to make an argument in support or against each tool, supporting their argument with description of characteristics. In Unit 2, small grade-level groups looked at their teammates' evaluations from Unit 1 and collaboratively wrote a set of questions or characteristics they would focus on. Each individual then used their small group evaluation template to evaluate an online tool. In Unit 6, after teachers had used the whole class evaluation template in Unit 5 and then individually identified the questions they felt were most important, each teacher used their own subset of questions to evaluate two online tools. One way to explore how highly teachers value mathematical representation is to consider whether they chose to address representation when they had that choice. Table 6.2 shows a frequency count of online tools for which teachers chose to describe aspects of mathematical representation in each unit, regardless of questions, compared to the total number of tools they evaluated in that unit ("O" and "T", respectively). When teachers responded to explicit questions about mathematical representation (that is, they chose to ask themselves the question and then responded to it), the count is shown as "E".

Table 6.2 shows that teachers rarely included explicit questions in these units where they had

Table 6.2

*Choosing Mathematical Representation*

| Participant | Unit 1 | | | Unit 2 | | | Unit 6 | | | Unit 8 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | E | O | T | E | O | T | E | O | T | E | O | T |
| Lower Elementary Group | | | | | | | | | | | | |
| Breanna | 0 | 3 | 3 | 1 | 1 | 1 | 2 | 2 | 2 | 0 | 3 | 3 |
| Melinda | 0 | 3 | 3 | 0 | 1 | 1 | 0 | 2 | 2 | 0 | 2 | 2 |
| Jana | 0 | 3 | 3 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 3 | 3 |
| Upper Elementary Group | | | | | | | | | | | | |
| Nicole | 0 | 3 | 3 | 0 | 1 | 1 | 0 | 2 | 2 | 0 | 3 | 3 |
| Heidi | 0 | 0 | 2 | 0 | 1 | 1 | 0 | 2 | 2 | 3 | 3 | 3 |
| Middle Grades Group | | | | | | | | | | | | |
| Alison | 0 | 4 | 8 | 0 | 0 | 1 | 0 | 2 | 2 | 0 | 3 | 3 |

*Note.* Number of tools for which participants described mathematical representation when they chose their evaluation questions or focus, separated according to whether the description was in response to an explicit question ("E") or other question ('O") compared to total tools evaluated in each unit ("T").

choice. Some did; for example, Jana included an explicit question about mathematical representation in Units 2, 6, and 8. In Units 2 and 8, she asked herself the same question about Universal Design for Learning (UDL) principles. In Unit 2, she had suggested the inclusion of UDL in her group's evaluation template as one of eight big questions. In the description of UDL in their evaluation template, the description includes that "The tool should provide a multiple means of representation and expression." Note that Jana and Breanna included a description of their online tool's representation in response to this question, but Melinda focused on levels of learners and did not include representation in her response to this question. Breanna included explicit question in her Units 2 and 6 evaluations, and Heidi included explicit questions about representation in her Unit 8 evaluations.

Even though not all teachers chose to include explicit questions in their evaluations, all described aspects of mathematical representation in their evaluations. Only Heidi and Alison did not include some description of representation in every tool. Heidi did not describe the aspects of mathematical representation in her Unit 1 online tools, while Alison only four of the eight tools that she evaluated. Alison did not include mathematical representation description in her Unit 2 evaluation; in her Unit 6 evaluations, she asked herself about mathematical representation but only responded, “NA - repeat question.”

## Types of Representation

So far, I have described participants’ use of mathematical representation to describe online tools as a simple *yes* or *no* question. Within description of representation, teachers attended to different aspects. In part, their attention was directed by the mathematical representation apparent in the online tool. For example, if a tool only included graphical and symbolic representations, the teacher might describe the two types of representation. At times, teachers also described what additional representation they thought should be included, or were critical about the efficacy of the tool’s use of representation. Within participant’s descriptions of particular types of mathematical representation, differences emerged. First, I give an overview of participants’ attention to the various types of mathematical representation. Following that overview, I explore more deeply a few selected types of representation: symbolic, visual, and interactive. Figure 6.8 shows the types of representation to which teachers attended as a frequency of descriptions of: symbolic, verbal, visual, graphical, diagrammatic, tabular, dynamic, or other types of representations.

Figure 6.8 shows that participants paid more attention to symbolic, visual, and dynamic representations than other forms of representation. Participants’ attention to symbolic representation was not always explicit and rarely included the word *symbolic,* but did reference alternate descriptors that indicated a view of symbolic representation as the actual mathematical object, rather than one of many possible representations of a mathematical object. The greater attention to visual and dynamic representations may be due, in part, to the ambiguous nature of these descriptors. At times, participants explicitly mentioned visual representation as including concrete, graphical, tabular, diagrammatic, and other types of representation. Participants’ use of dynamic representation was similarly broad; it included attention to movement within an online tool, whether a user of the tool could interact with the representation, whether representations were dynamically linked,

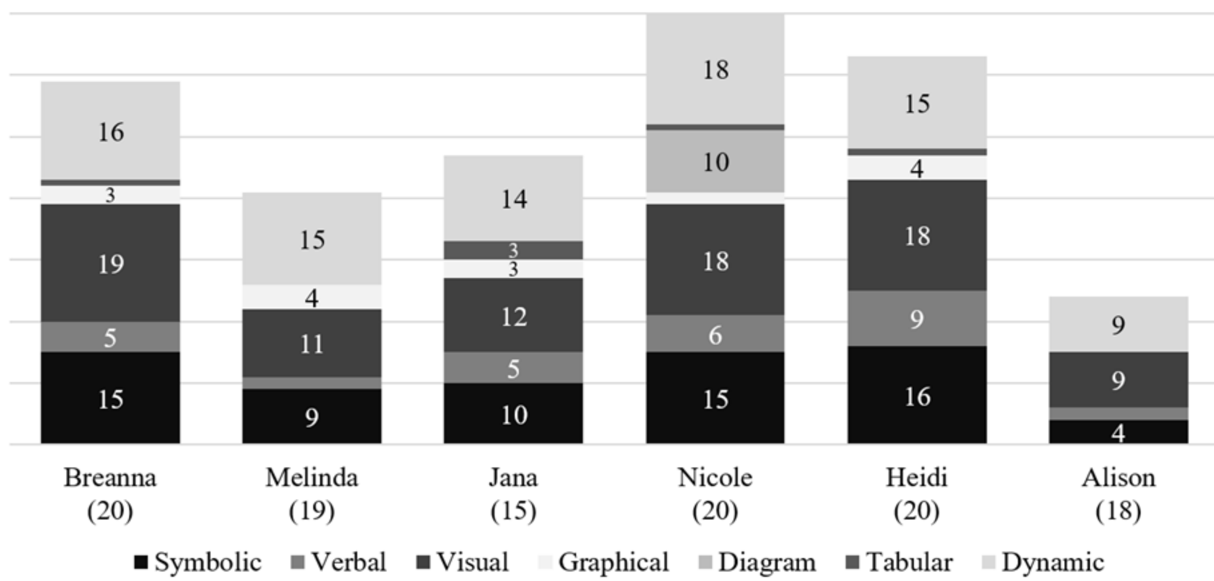


Figure 6.8: Types of Mathematical Representation

and whether users could explore mathematical consequences resulting from their interaction with dynamically linked and interactive representations.

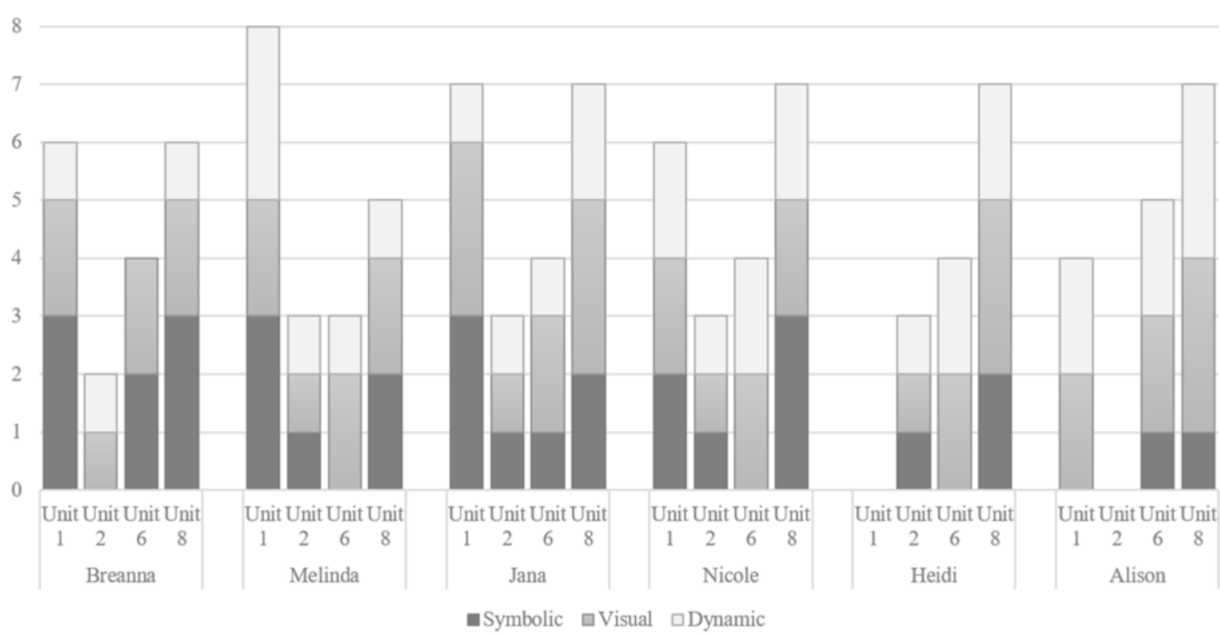


Figure 6.9: Choice of Symbolic, Visual, or Dynamic

In Figure 6.9, I show an overview of teachers' attention to symbolic, visual, or dynamic types of representation when they had opportunity to make choices about aspects to focus on. In Units 1, 2, 6, and 8, teachers made choices about the characteristics they focused on in their evaluations. In Unit 1 and Unit 8, teachers were asked an open question about the online tool and could respond using any characteristics or arguments they chose. In Unit 2, teachers in grade-level groups agreed on important characteristics to create an evaluation template. Each teacher within the group used that template to evaluate an online tool. In Unit 6, teachers individually chose aspects or elements of the class evaluation template that they felt were important. They evaluated two online tools according to that subset of the evaluation template.

The figure shows the number of tools for which teachers made statements attending to these aspects. Most teachers evaluated 3 online tools in Unit 1, 1 in Unit 2, 2 in Unit 6, and 3 in Unit 8. Heidi and Alison were exceptions, evaluating 2 and 8 in Unit 1. Note that when teachers made choices about how they would describe online tools, they did consistently attend to symbolic, visual, and dynamics aspects of the mathematical representation. In Heidi's initial evaluations, and in Alison's Unit 2 evaluations, they did not attend to any of the three, but both attended to at least visual and dynamic aspects of mathematical representation for at least two of the final evaluations (which were the same tools). I describe below the words and descriptions participants used as they attended to symbolic, visual, and dynamic types of representation.

## Symbolic Representation

Overall, participants attended to symbolic representation using both explicit and implicit descriptions. I will focus on participants' explicit attention to symbolic representation; that is, when they used explicit terms that mean representation such as "representation" or "form" in conjunction with explicit terms that are easily identified as indicating symbolic representation. For example, I include expressions such as "numerical form," "number sentences," "number representation," "represented as an expression." I also consider teacher responses to explicit questions about mathematical representation or sentences that explicitly state the teacher's focus on representation. For example, Heidi wrote a description of mathematical representation that is unambiguously referring to symbolic representation: "Mathematics is represented through models and symbolic representations of fractions. Students are manipulating math symbols to make true mathematical number sentences." Heidi used the phrases "symbolic representation," "math symbols," and "mathematical number sentences." Each phrase is clearly indicating her attention to symbolic representation. In other statements, the language used by Heidi and her classmates was less clear. As one example, Breanna responded to an explicit question about mathematical representation by writing, "Math is represented in the form of addition facts that rapidly keep coming as soon as the student answers each question." In this example, Breanna explicitly stated "math is represented" followed by "in the form of addition facts." Hence, I collected the expression "addition facts" as a type of symbolic representation. As a second example, I included the following statement written by Melinda: "only the ability to view numbers in the form of fractions." Melinda used the phrase "to view numbers in the form" which I interpret as indicating her attention to representation. Further, because she wrote "numbers in the form of fractions," I collected the term "fractions" as a type of symbolic representation. In both of these examples, I interpret the teachers' words as describing a type of representation that I call *symbolic representation*. My interpretation is based on the context of the statement in the evaluation as well as my knowledge of the online tool.

Table 6.3 shows the number of tools for which participants explicitly described symbolic representation or made any reference to symbolic representation. Participants varied in their attention to symbolic representation, with Alison referencing symbolic representation least. Considering use of explicit terms, only Nicole used such terms to describe more than half of her tools, while Breanna, Melinda, Heidi, and Alison used such terms to describe a third or fewer of their tools.

Exploring the language teachers used when attending to symbolic representation may illuminate issues in how symbolic representation is used in online tools, as well as how teachers attended to and communicated such use. Attention to language highlights whether teachers attend to tools' representations of quantities through linked symbolic forms. Attention to teachers' language also illuminates teachers' precision in use of mathematical language. That is, mathematical concepts and their representations may be referenced with the same words, or teachers' communication of

Table 6.3

*Symbolic Representation*

| Participant | Explicit Reference | | Other Reference | | Total Count | |
|---|---|---|---|---|---|---|
| | *n* | % | *n* | % | *n* | % |
| Lower Elementary Group | | | | | | |
| Breanna | 5 | 25 | 15 | 75 | 15 | 75 |
| Melinda | 3 | 16 | 9 | 47 | 9 | 47 |
| Jana | 6 | 40 | 10 | 67 | 10 | 67 |
| Upper Elementary Group | | | | | | |
| Nicole | 10 | 50 | 15 | 75 | 15 | 75 |
| Heidi | 7 | 35 | 15 | 75 | 16 | 80 |
| Middle Grades Group | | | | | | |
| Alison | 1 | 6 | 3 | 17 | 4 | 22 |

*Note.* Number of tools for which participants referenced symbolic representations (whether implicitly or explicitly).

their noticing may be supported or constrained by their access to precise language. to explain these ideas, first I show the words each participant used across tools to describe symbolic representation; second, I will explore more specific instances of their attention to symbolic representation.

Table 6.3 shows the variety of words used by teachers to describe symbolic representation in an online tool. I present some different ways that teachers used language as they attended to symbolic representation. I use the distinction between an abstract mathematical idea and its symbolic representation, and teachers' use of language to attend to this distinction, to illuminate varying levels of sophistication in their noticing.

In considering the distinction between an abstract mathematical idea and its symbolic representation, the teachers' levels of sophistication range from no distinction to explicit distinction. Consideration of multiple issues are important here: teachers' access to language, use of language, and mathematical understanding. That is, to write statements that reveal a sophisticated way of thinking about symbolic representation, teachers must have access to words with the potential to precisely communicate their thinking. In addition to having access to the words, teachers must be able to use them in precise, mathematical ways that reflect their thinking. Finally, teachers must

Table 6.4

*Use of Language in Symbolic Representation*

| Participant | Words Used as Synonyms of Symbolic Representation |
|---|---|
| Lower Elementary Group | |
| Breanna | answers, how a student sees a math problem on paper, coin's worth, subtraction problems, addition facts, math sentence, math problem |
| Melinda | fractions, numbers, amount expressed on price tag |
| Jana | fractions, numbers, numeral, digit, equation, mathematical expression |
| Upper Elementary Group | |
| Nicole | number sentences, symbols, multiplication facts, answers, fractions, percents, whole numbers, decimals, numbers, number form |
| Heidi | form of numbers, numerical form, symbolic representations, number sentences, math symbols, standard representation, number model |
| Middle Grades Group | |
| Alison | how a number is written |

*Note.* Participant word choice in their descriptions of symbolic representation of online tools. Complete list of words used for each teacher.

be able to notice different ways symbolic representation can be used. Hence, I argue that it is the interaction of vocabulary, language use, and mathematical understanding that contribute to the revealed level of sophistication in noticing I describe here.

Heidi wrote, "using a visual model and the standard representation of the fraction" in her description of an online tool. Her use of the phrase "standard representation" implies that a symbolic representation is the standard, or default, representation of an abstract mathematical idea. Her use of language sets the stage for examples of a lower level of noticing of symbolic representation: If teachers view symbolic representation as the standard representation of mathematical objects or ideas, then this form of representation may be less visible for them. That is, teachers may ignore symbolic representation because they expect symbolic representation to be used. Some teachers wrote statements that indicated the invisible nature of the distinction between a mathematical idea or object and its symbolic representation.

In Breanna's evaluation of one online tool, she described a student's interaction with the tool in which they would choose a symbolic representation (addition sentence) that matched a graphical representation (movement on number line): "quickly pick the math problem that displays the same problem on the number line." In this statement, Breanna uses "math problem" as a referent of an abstract mathematical idea that is being represented in two ways. At the same time, she uses "math problem" as a referent of the abstract idea's symbolic representation. Other teachers also used "problem" or "facts" to indicate both the representation and the concept. Melinda, for example, wrote: "This tool [provides] only the ability to view numbers in the form of fractions and match them together." When she uses the phrase "in the form of fractions" she may be referring to the abstract mathematical idea of fractions as a representation of numbers or she may be using the word fractions to indicate the symbolic representation.

Other teachers, however, clearly distinguished between an abstract idea and its representation. Nicole, specifically, attended to the way her online tools presented multiple symbolic representations of a quantity; specifically, linking fraction, percent, and decimal forms of the quantity. She also attended to which representation an online tool allowed when a quantity might potentially be represented symbolically in multiple ways; specifically, considering proper fractions, improper fractions, and mixed numbers. I chose Nicole's attention to these issues because in her discussion of multiple symbolic forms, it was clear that she separated the abstract mathematical idea (quantity) from its potential symbolic forms. Nicole described mathematical ideas as being separate from their representations in other evaluations as well. For example, she stated this distinction clearly in one evaluation: "The mathematics (fraction) is represent[ed] in number form and visual form." In another evaluation, Nicole wrote, "a number or amount can mean more than one thing and can be represented in more than one way." These statements show her attention to the distinction between number as an abstract idea and the representations of number. On the other hand, Nicole did not make this distinction clear in all evaluations. In one evaluation, she seemed to use the term fraction to indicate a representation: "Students practice matching whole numbers, shapes, fractions, or multiplication facts to their equivalent representations."

Heidi also seemed to clearly distinguish between a mathematical object and its symbolic representation: She was the sole participant to explicitly refer to symbolic representation (shown in the quote above). She used the phrase “number model” as well. She wrote, “Both a number model and a picture model are provided to give support.” Her use of this phrase may have been prompted by the name of the online tool IXL: *Subtracting Fractions with a Model;* she used the same phrase, however, in a later evaluation of a different tool. Her use of this phrase is interesting because the meaning is somewhat unclear. In Heidi’s attention to symbolic representation in her Math Playground: *Dividing Fractions* evaluation, she wrote, “When students have given an incorrect answer a number model showing the correct solution is given.” In this case, it seems likely she is using the term model as a synonym for representation, but it also seems to indicate “an example for imitation or emulation” (“Model”, 2017). In the quote above, in conjunction with *picture model*, the use of *number model* seems more likely to indicate representation. Confusion with multiple meanings of the term model in mathematics has been noted by other researchers (J. P. Smith, 2015).

## Visual Representation

As mentioned above, visual representation was used by teachers as a generic term that might include almost any other type of representation including symbolic representation. In the next section, I focus on teachers’ attention to dynamic aspects of visual representation. In this section, I focus on the static aspects, as possible, while I explore teachers’ meaning when they use the phrase *visual* and teachers’ attention when they describe visual aspects of an online tool. I acknowledge fuzziness in the boundary between visual and dynamic representations, but attend to different aspects of representation in the two sections.

To capture teacher statements, I examine their use of words with *visual* as a root (e.g., visual, visuals, visualization), along with words that indicate attention to visual, including: *display, image, picture.* Table 6.5 shows teachers’ attention to visual representation across their evaluated tools, in explicit use of the word *visual* or implicit attention to visual representation through other words.

Table 6.5 shows that Jana and Nicole explicitly referenced visual representations most often and Breanna, Melinda, and Alison the least often with 30% or fewer explicit references. All teachers

Table 6.5

*Visual Representation*

| Participant | Explicit Reference | | Other Reference | | Total Count | |
|---|---|---|---|---|---|---|
| | *n* | % | *n* | % | *n* | % |
| Lower Elementary Group | | | | | | |
| Breanna | 5 | 25 | 19 | 95 | 19 | 95 |
| Melinda | 5 | 26 | 11 | 58 | 11 | 58 |
| Jana | 10 | 67 | 12 | 80 | 12 | 80 |
| Upper Elementary Group | | | | | | |
| Nicole | 13 | 65 | 18 | 90 | 18 | 90 |
| Heidi | 7 | 35 | 17 | 85 | 18 | 90 |
| Middle Grades Group | | | | | | |
| Alison | 5 | 28 | 9 | 50 | 9 | 50) |

*Note.* Number of tools for which participants referenced visual representations (whether implicitly or explicitly).

referenced some aspect of visual representation in at least half of the tools they evaluated; Alison and Melinda had the lowest frequencies of over reference to visual representation aspects of online tools.

I attend to teachers' use of language as well as the types of visual representation that they described. When teachers used the term *visual,* the term did not always seem to have the same meaning. In their attention to visuals, teachers described (a) a visual representation of a mathematical idea or object with the focus on the mathematics, (b) a visual representation with the focus on nonmathematical aspects, and (c) the purpose of the visual representation (what users will do with it or how they will benefit from it). I use three statements from Heidi as examples.

First, she wrote "The tool ... visually shows a balance that moves to display equal and unequal expressions." A second statement read: "Students can use this tool as [a] visual on how we can make equivalent fractions." Finally, a third statement was written: "Students are provided a visual and audio tutorial on how to model the concept." In each of the three tools, visual representations are used. For the first, Heidi explicitly attended to and describes that representation: a pan balance that moves to show equal and unequal symbolic expressions. In the second, NCTM Illuminations:

*Playing Fraction Track,* several fraction lines are shown, each partitioned into a different number of equal parts. Heidi does indicate that it can be used as a visual for making equivalent fractions, but she did not describe the visual itself; that is, she described the goal without describing the actual visual representation and how it supports equivalent fractions.

I considered each statement, across all participants, that had been previously coded as capturing an aspect of visual representation for these categories. The statements addressed different aspects of visual representation, including attention to nonmathematical aspects and mathematical aspects. When teachers attended to mathematical aspects of visual representation, they varied naming one or more of: (a) a larger mathematical context in which the representation was situated, (b) a mathematical representation, (c) expectations of mathematical learning resulting from use of the representation, and (d) mathematics made visible through use of the representation. When attending to visual representation, teachers also varied in the number of tools for which visual representation was described and in the number of sentences they used when describing visual representation. I share these findings using the figures below. First, I show teachers' attention to mathematical aspects compared to nonmathematical aspects of visual representation across the total number of sentences written. Then, I share teachers' attention to mathematical or nonmathematical aspects as represented across the tools they evaluated; that is, whether they attend to only mathematical, only nonmathematical, or both mathematical and nonmathematical aspects of online tools in their evaluations. Finally, I share teachers' attention to particular mathematical aspects of visual representations in the online tools evaluated. I share teachers' attention across all tools, in the relative frequencies of written sentences, and within tools, in the number of tools for which teachers attended to the four mathematical aspects of visual representations.

Figure 6.10 shows the total number of sentences addressing visual representation that each participant wrote across all of their evaluations. Percentages of their total number of sentences focusing on mathematical or nonmathematical aspects are shown. To code each sentence as *focus on mathematical aspects* or *focus on nonmathematical aspects,* I read each sentence that had previously been coded as attending to visual representation. I considered whether the sentence focused

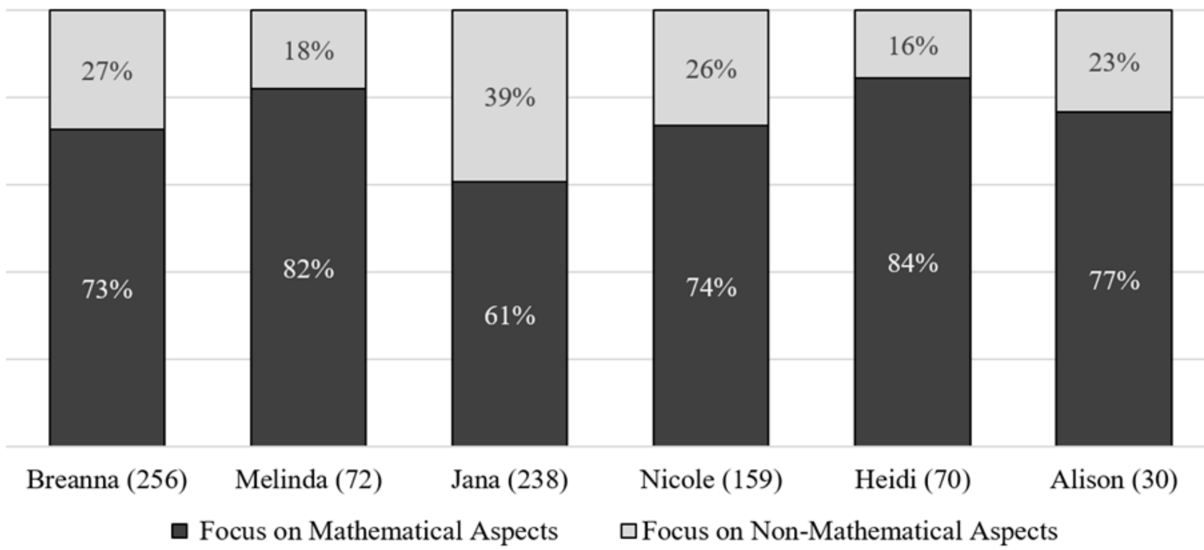


Figure 6.10: Mathematical and Nonmathematical Visual Aspects—Sentences

on mathematical aspects or not. For example, in Alison's evaluation of Charlotte Mecklenburg Library: *Story Place – I Spy Shapes,* she wrote a sentence that I coded as focused on mathematical aspects: "Mathematics is represented in the animated pictures of the beach, outdoors, living space, etc. so students to identify and recognize shapes within the pictures." She also wrote a sentence that I coded as focused on nonmathematical aspects: "It has very well illustrated pictures and characters that would definitely keep students engaged in their learning." Note that all teachers wrote some sentences focused on mathematical aspects and some sentences focused on nonmathematical aspects. Nicole and Melinda wrote a majority of sentences focused on nonmathematical aspects of visual representation, while the majority of sentences written by other teachers focused on mathematical aspects of visual representation. With over 200 sentences each, Jana and Breanna wrote the majority of sentences focused on visual representation. Although they wrote fewer sentences overall, Heidi and Melinda focused on mathematical aspects in their sentences relatively more frequently than other teachers. Jana focused least frequently with only 61% of sentences focused on mathematical aspects; other teachers had at least 72% of their sentences focusing on mathematical aspects.

After considering the number of sentences that teachers wrote across all evaluations, for each tool a teacher evaluated, I noted whether visual representation statements within the evaluation focused only on mathematical aspects, only on nonmathematical aspects, or on both mathematical and nonmathematical aspects of the visual representations. Figure 6.11 shows the number of tools for which teachers wrote sentences focused on only mathematical aspects, only nonmathematical aspects, or both mathematical and nonmathematical aspects. After considering the number of sentences that teachers wrote across all evaluations, for each tool a teacher evaluated, I noted whether visual representation statements within the evaluation focused only on mathematical aspects, only on nonmathematical aspects, or on both mathematical and nonmathematical aspects of the visual representations. Figure 6.11 shows the number of tools for which teachers wrote sentences focused on only mathematical aspects, only nonmathematical aspects, or both mathematical and nonmathematical aspects.

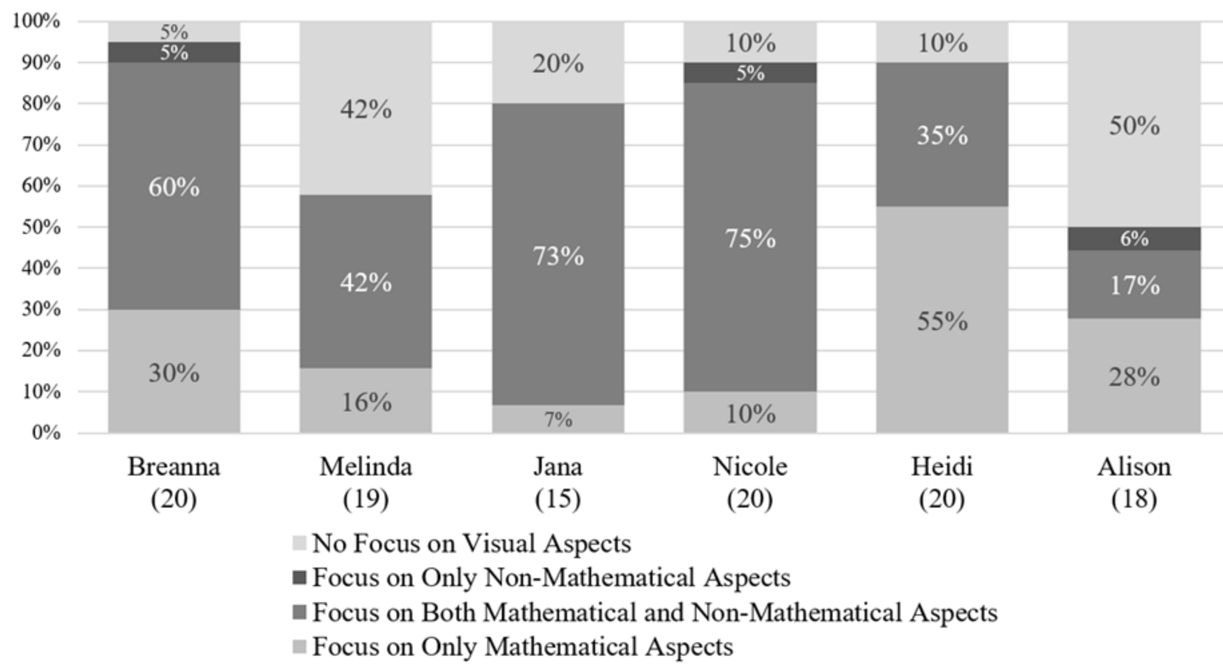


Figure 6.11: Mathematical and Nonmathematical Visual Aspects—Tools

Figure 6.11 shows that each teacher had at least one evaluation for which they focused on only mathematical aspects of visual representation. Jana, Melinda, and Heidi had no tools for which they focused on only nonmathematical aspects; that is, they wrote at least some sentences focused on mathematical aspects of visual representation for every tool. All teachers except Heidi and

Alison had a balance between attention to mathematical and nonmathematical aspects, focusing on both for the majority of their tools. Alison and Melinda attended to visual representation least frequently with such sentences written for slightly more than half of their evaluations.

Figures 6.10 and 6.11 showed how frequently teachers were attending to mathematical aspects of visual representation, but do not inform the question: When teachers are focused on mathematical aspects of a visual representation, on which mathematical aspects do they focus? I give some examples to explain four emergent aspects on which I focus. Teachers wrote a variety of sentences, some containing a great deal of information and other containing very little. For example, Heidi wrote, "Mathematics is represented in a visual way." I considered this sentence as being focused on mathematical aspects of visual representation and hence included it in the figures above. Heidi's sentence, however, conveys almost no information about the mathematical aspects of the visuals she was considering. In the next sentence of the same evaluation, Heidi added information: "The learner is presented with a model for the operations of addition and subtraction of mixed numbers with unlike denominators." I noted that in this sentence, Heidi named the mathematical context of the visual model, without naming or describing the representation itself. Breanna wrote a more detailed statement about a visual aspect of her tool: "In the event that there is not enough space to fit all the same color bugs into the appropriate column, students have to change the scale of the graph to fit all data (bugs) onto it." For this sentence, I noted that Breanna included mathematical context of the tool [scales of graph to fit data], a name and description of the visual representation [graph with colored bugs as data], mathematical learning goals for students through use of the visual [confront situation where no room means scale must change], and a description of mathematics made visible through use of the representation [too many bugs in one column motivates need for re-scaling the graph]. Figure 6.12 shows the frequency of sentences in which teachers explicitly named a particular mathematical representation, stated mathematical learning goals for that representation, or described the mathematics being represented.

Figure 6.12 shows variation among the teachers. First, note that Alison, Heidi, and Melinda wrote the fewest sentences attending to mathematical aspects of visual representation with 23, 59,

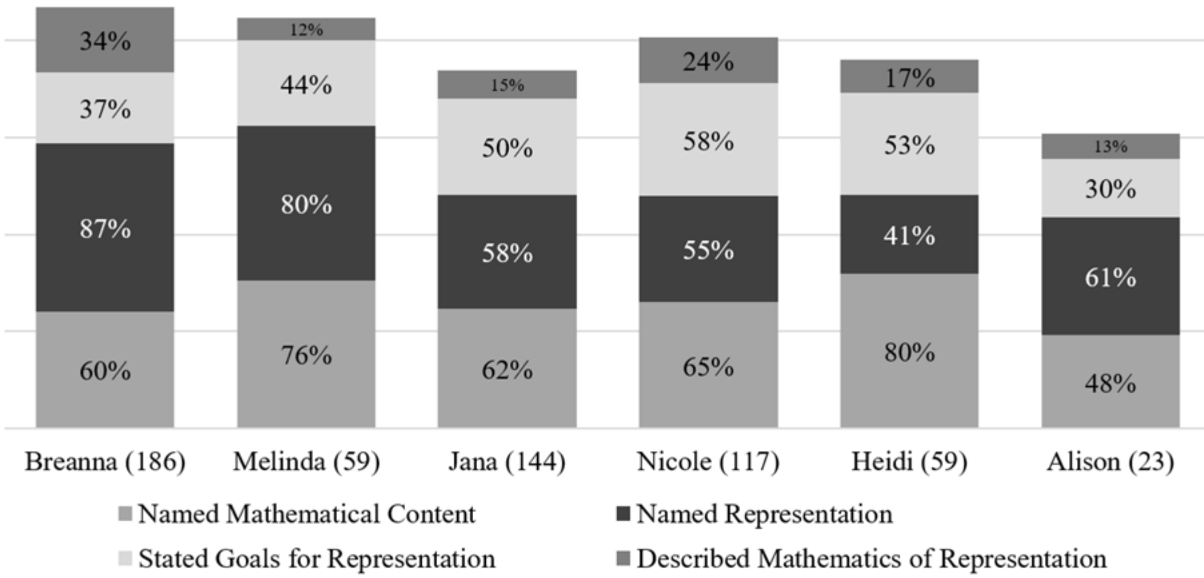


Figure 6.12: Mathematical Aspects of Visual Representation—Sentences

and 59, respectively. Breanna, Jana, and Nicole almost doubled those numbers, writing 186, 144, and 117 sentences, respectively. Teachers with a high number of statements may have used many sentences to describe the visual representation of a single tool. to consider the work that teachers' statements did collectively in description of each tool, Figure 6.13 shows the number of tools for which teachers attended to these aspects, including the number of tools for which teachers attended to all four aspects.

Figure 6.13 shows that, except for Alison, teachers attended to mathematical content, name or description of representation, and learning goals for representation for more than half of their tools. Alison, Heidi, and Melinda described the mathematics made visible by visual representations for fewer than half of their tools. Nicole and Breanna provided descriptions of each of the four aspects for half or more of their tools. Breanna, especially, focused heavily on the mathematical aspects of visual representation, giving detailed examples of the mathematics students might see and interact with, and how the visualizations would support their mathematical thinking.

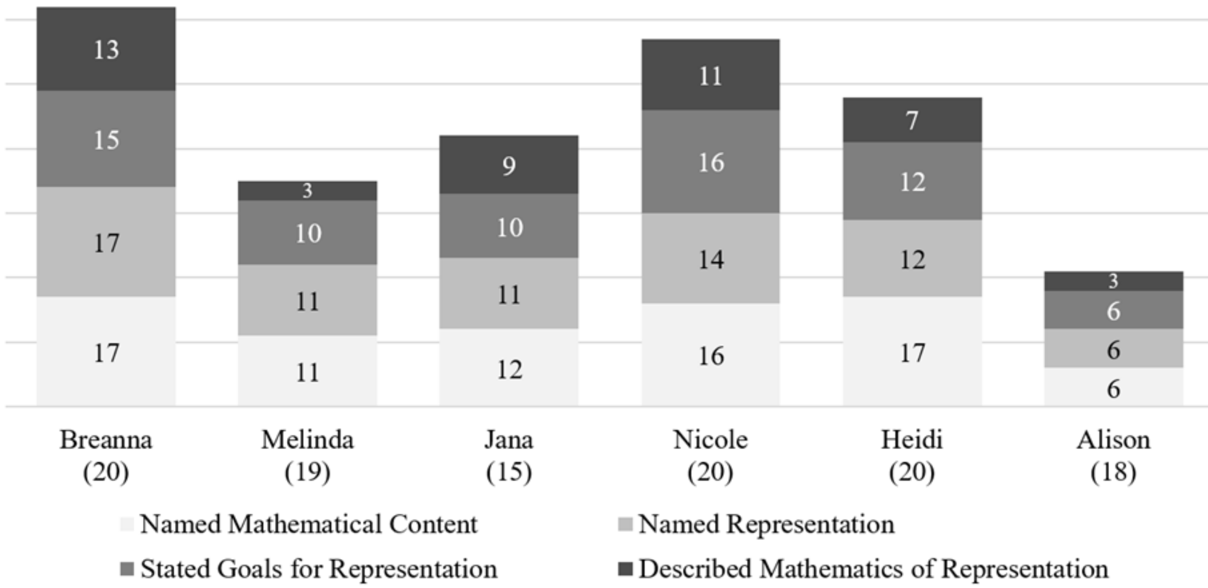


Figure 6.13: Mathematical Aspects of Visual Representation—Tools

## Dynamic Representation

In considering teachers' attention to the dynamic aspect of representation in their evaluations of online tools, I considered (a) teachers' attention to dynamic representation generally, especially the potential for interactivity or passivity; (b) their attention to expectations they had for students' and tools; (c) their attention to the nature of dynamic representation; and (d) their attention attention to mathematical feedback or consequences of student interaction with the tools. As a reminder, a constraint of the data I am working with is that not all teachers evaluated the same tools which means one teacher may attend to interactivity for every tool simply because the tools she chose happened to include more opportunities for interaction. That is, a teacher may attend to interaction when it is present but find that interaction is rarely possible in a tool. At the same time, a teacher who is interested in opportunities for students to interact, may report on the lack of opportunity for non-interactive tools. Because my interest is in teachers' noticing and not necessarily in the tools themselves, I include any statements about the dynamic or interactive aspect of a tool, whether they indicate is dynamic or is not dynamic.

### *Dynamic Nature of Tools: Interactive or Passive*

I attend to the dynamic aspects of online tools first by sharing the number of tools for which each teacher referenced the dynamic, interactive, or passive nature. To code a statement as referencing a dynamic aspect of an online tool, I simply considered if movement was in any way indicated. Movement might be indicated through word choice or through description. For example, Melinda explicitly used the word movement about a tool: "The movement and hands-on capabilities". Teachers might also reference video, animation, interaction, manipulation, or any of many other words that indicate movement. For example, Heidi wrote "Number Rights is a video showing" and Alison wrote "the animation aspect of this short skit." Through description, Breanna indicated movement, saying, "the kangaroo falls into the water." I also considered changes in otherwise static screens as dynamic; for example, describing manipulation of objects on the screen or describing pop-ups or other changes in the screen. For example, Nicole wrote about certain tools, "They can click on the numbers" and "They can drag symbols". Because my focus is on how the teachers are attending to these aspects, I also coded statements that indicated lack of movement in a tool. For example, Alison wrote, "it has no animations, no color, and no graphics".

After coding teacher statements as attending to the dynamic nature (or lack thereof) of online tools, I considered whether teachers indicated a tool was interactive or passive in nature. First, I found and coded any explicit use of the words interact or manipulate (or words for which these are roots). Second, I found any statements in which teachers referenced a user's actions with respect to the online tool and coded these as interactive. Other statements, in which a teacher indicated a user would simply watch, I coded as passive. Statements that did not provide enough information to indicate its interactivity, I coded as "not enough information to determine attention to interactivity." Figure 6.14 shows the number of tools evaluated for which teachers attended to dynamic, interactive, or passive aspects.

Figure 6.14 shows that every teacher included dynamic aspects in descriptions of over half of their tools. As a reminder, when coding their statements, I considered multiple statements written by participants for each tool. Breanna, Jana, and Heidi made at least one statement indicating

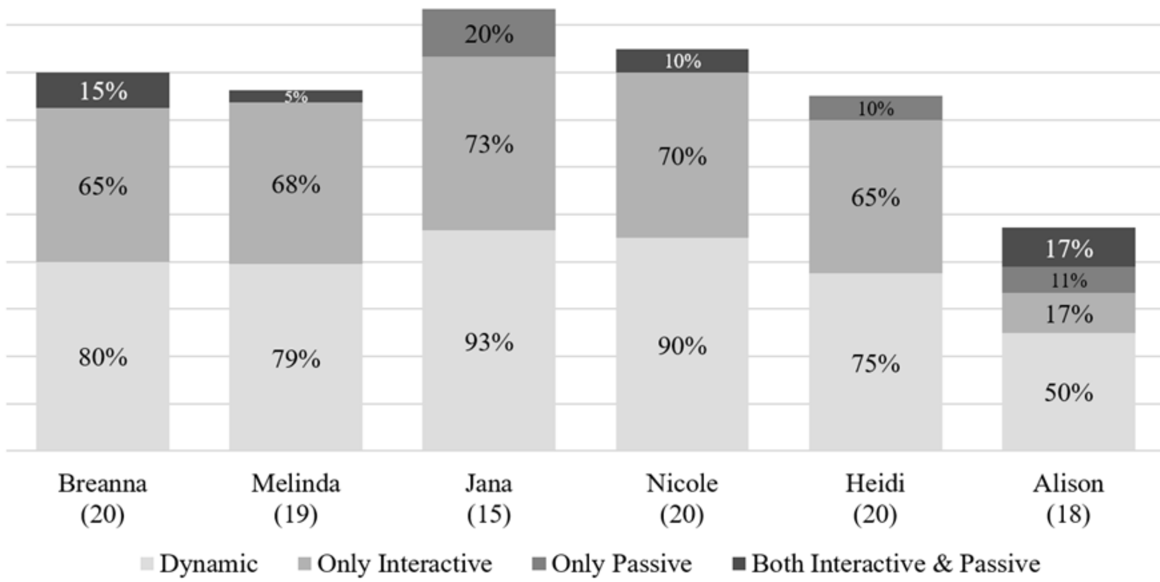


Figure 6.14: Dynamic Aspects of Online Tools

interactivity or passivity for every tool for which they attended to dynamic aspects of representation. That is, their frequencies in Figure 6.14 are balanced between Dynamic and Only Interactive / Only Passive / Both Interactive and Passive. With the exception of Alison, all participants described most of their tools as only interactive. Only Alison (17%) and Jana (20%) described 15% or more of their online tools as only passive. Breanna (15%) and Alison (33%) were the only participants who described 15% or more of their online tools as both passive and interactive.

In the next sections, I share the aspects of dynamic representation that participants attended to, whether passive, interactive, or both. First, I share teachers' attention to their expectations of students' interaction with online tools. Second, I share the nature of dynamic representation as described by teachers: dynamic singular representation, statically linked representations, or dynamically linked representations. Finally, I share the nature of representations of mathematical consequences or feedback.

### *Expectations for Dynamic Representation*

I started my coding with the four categories of video or tutorial, game, manipulative, or interactive model (graph or diagram). I expected some emergent categories as I read statements. It soon was clear that few teachers wrote statements about dynamic representation that explicitly described the tool in those terms. It was also clear that those categories did not describe what I had hoped to capture. I deleted the coding and began exploring the statements once more. I captured verbs teachers used to describe what a student would do to interact with the dynamic representation and what a tool would as it provided the dynamic representation. I then considered the verbs and sorted them into larger themes of "Student will" and "Tool will," with the following subthemes. I list the subthemes with the verbs used by teachers.

1. Student will ...

    - Do Mathematics

      *identify an object, find an answer, perform operations, meet specified criteria, solve a problem, create a mathematical object, customize or create a problem, support reasoning*

    - Interact with Mathematics in Tool

      *pop balloons, shoot at target, race, dive, match representations, manipulate objects, view video or animation*

    - Learn Mathematics

      *practice a skill, use mental math, use math knowledge, develop better understanding, see how something changes, see equality or equivalence*

2. Tool will ...

    - Provide Feedback

      *indicate correctness, respond even when incorrect, provide explanations*

- Provide Learning Opportunity

  *generate problems, show how or demonstrate, ensure accuracy, draw mathematical objects, provide learning opportunities*

Figure 6.15 shows the results of coding according to these aspects: Student Will and Tool Will.

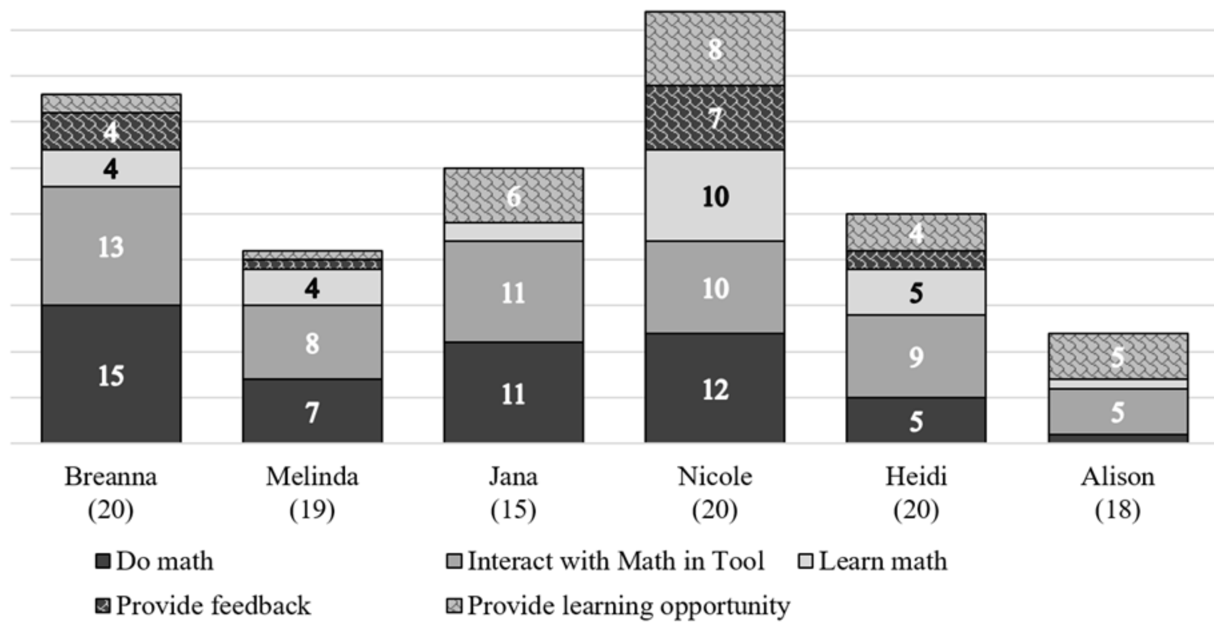


Figure 6.15: Interaction with Dynamic Representation

Figure 6.15 shows that all teachers described how the dynamic representation in a tool would support students in doing mathematics, interacting with mathematics, and learning mathematics. Teachers' statements categorized in the "learning mathematics" category showed evidence that teachers' expectations of the purpose of interaction can be valid, yet different from other teachers' expectations. I give an example of Nicole and Breanna and how they viewed the NLVM: *Base-Ten Blocks* tool. Nicole explained,

> I could move the blocks around and use them to assist with solving the problem, but I did not know where to type my answer. There, I discovered that I could use my mouse to group/ungroup the blocks to help me solve the problem and get the answer.... After the tool automatically grouped the blocks for me, it gave the answer and I could move

> on to the next problem. Basically, I could just sit there and use my mouse to have the tool do all the work for me, find the answer, and I did not have to actively participate with solving the problem. Because of this, I do not think this is a beneficial math tech tool for working with base-ten blocks.

She indicated that she wanted students to "actively participate with solving the problem" and that the tool took that potential away from the students because it regrouped and ungrouped blocks automatically. Breanna, on the other hand, described the automatic grouping and ungrouping in favorable terms, writing

> I loved it. ... Once a student reaches 10 ones in the first column, they can gather the 10 cubes in the ones column together to make a block of ten cubes strung together. Once you've made a ten block, students can drag it over to the left. When cubes are dragged to the left to a higher place value, you are regrouping. When cubes are dragged to the right, the student is exchanging place values to break a number down into pieces more favorable to the student. Once the student has their cubes in the correct columns, they will see the answer they have come up with to the right of their screen if done correctly.

Breanna focused on students' understanding of grouping and ungrouping while Nicole focused on students' participation in finding the answer.

From Figure 6.15, descriptions of student interaction with mathematics in a tool was most consistently attended to by teachers; often the interaction involved students making connections between mathematical representations. I describe that interaction in more depth below. I also describe teachers' attention to feedback below. All teachers described how students would do mathematics using dynamic representation, and Breanna, Jana, and Nicole described doing mathematics for over half of their tools. Figure 6.16 shows the results of coding according to the verbs included in the "Students will do mathematics" theme.

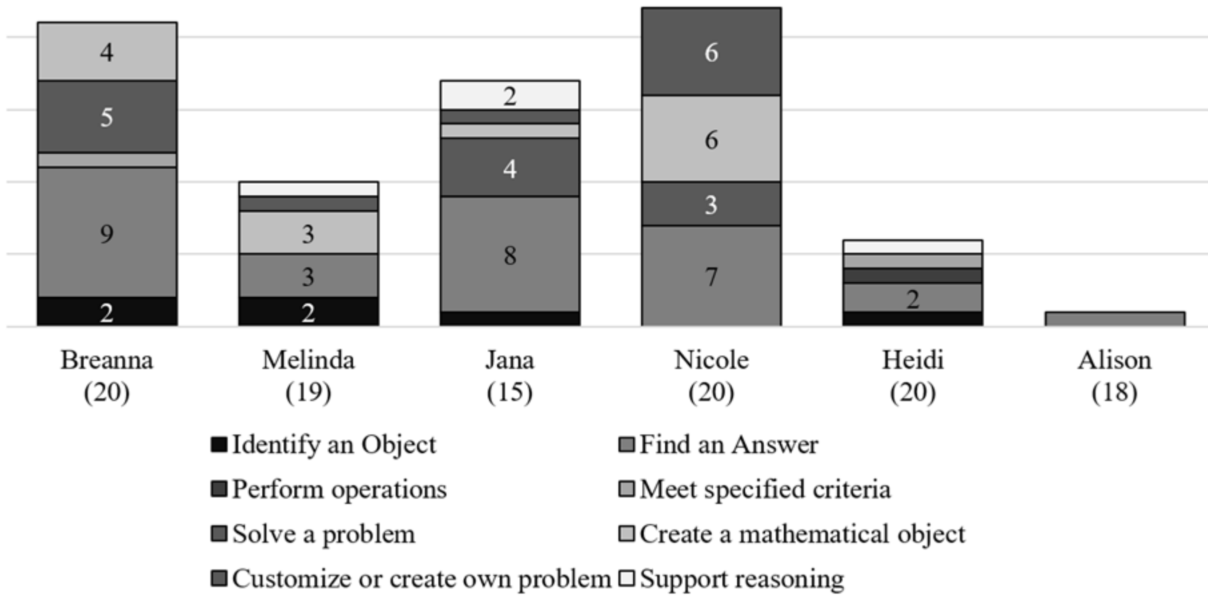


Figure 6.16: Students Will Do Mathematics

Figure 6.16 shows that all teachers made statements about students finding answers or solving problems through use of the dynamic representation. I attended to teachers' use of words to distinguish between the two categories; it is unclear whether solving problems means attending to truly problematic situations or whether it is a second way of saying the students found answers. For example, Alison wrote, "Helps students manipulate questions to get an answer." Nicole wrote, "They can drag symbols into the 10-frame to solve the problems." I included Alison's statement in "find an answer" and Nicole's in "solve a problem" based on their words. It seems likely that the two interactions in this case are not different in a practical sense, and both focus on simply finding answers. On the other hand, another interaction that Nicole described with the term problem involved students writing steps of a plan for a turtle to move from one spot to another, avoiding obstacles along the way. It included students measuring steps and specifying rotations of the turtle. In this case, Nicole wrote "Every student can work through their own problem." In this example, the word problem seems justified as interacting with a problematic situation. Because my analysis is based on teachers' attention through their descriptions, I accept the words they chose rather than my interpretation of those words. Hence, when a teacher used the word "problem," I accepted that word choice.

Breanna, Melinda, Jana, and Nicole each described students using the dynamic representation to create a mathematical object. The objects varied: tesselations (Nicole), graphs (Nicole, Melinda, Breanna), representations of numbers and operations on numbers (Melinda, Nicole, Breanna, Jana), patterns (Breanna, Jana), angles (Breanna).

***Nature of Dynamic Representation***

In considering the nature of dynamic representation, I noted two common descriptions of dynamic representation were multiple dynamically linked representations and single representations, often in the form of a manipulative. Some linked representations were static; such as static images of a value on a number line, the same value represented in an area model, and the symbolic representation of the same value. Static images can be part of a dynamic representation in different ways. For example, Nicole described a type of memory matching game where a student would match different representations of a mathematical object by clicking on blank cards which would then reveal the representation on "the other side." In this case, even though the representations themselves are static, they are embedded in a dynamic environment. I explored teacher statements to compare their attention to multiple linked representations (dynamically linked, linked manipulatives, statically linked) and single representations (dynamic or changeable, manipulatives, static images). Figure 6.17 shows the results of coding according to these aspects. Note that I attended to the type of representation described in the teachers' individual statements. Hence, statements about one tool might focus on one representation and on multiple representations and the tool would then be counted in both sets.

Figure 6.17 shows that all participants described linked and single manipulatives. Many teachers described statically linked manipulatives. As in the example above, many statements describing statically linked representations gave the context as students matching two representations in a game. In considering linked representations, especially in the context of online tools where some connections are provided by the tool and other connections are created by students in quiz-like

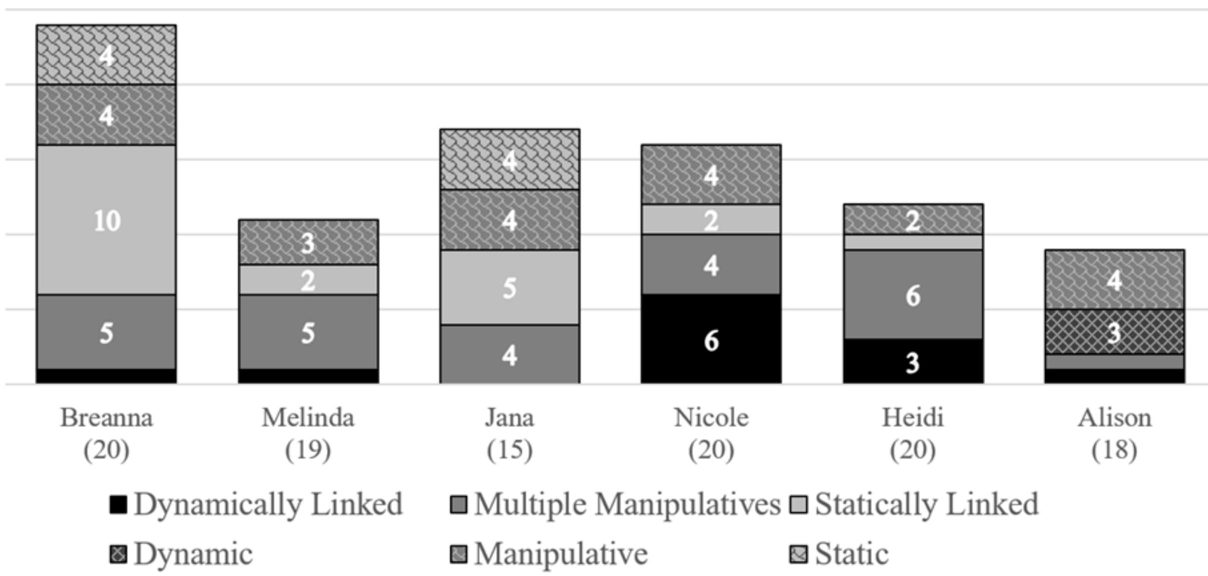


Figure 6.17: Nature of Dynamic Representation

environments, a question arises: Does the tool reveal connections or does the student choose connections? Figure 6.18 considers multiple linked representations and whether the tool provided the link between representations or whether the student connected representations themselves.

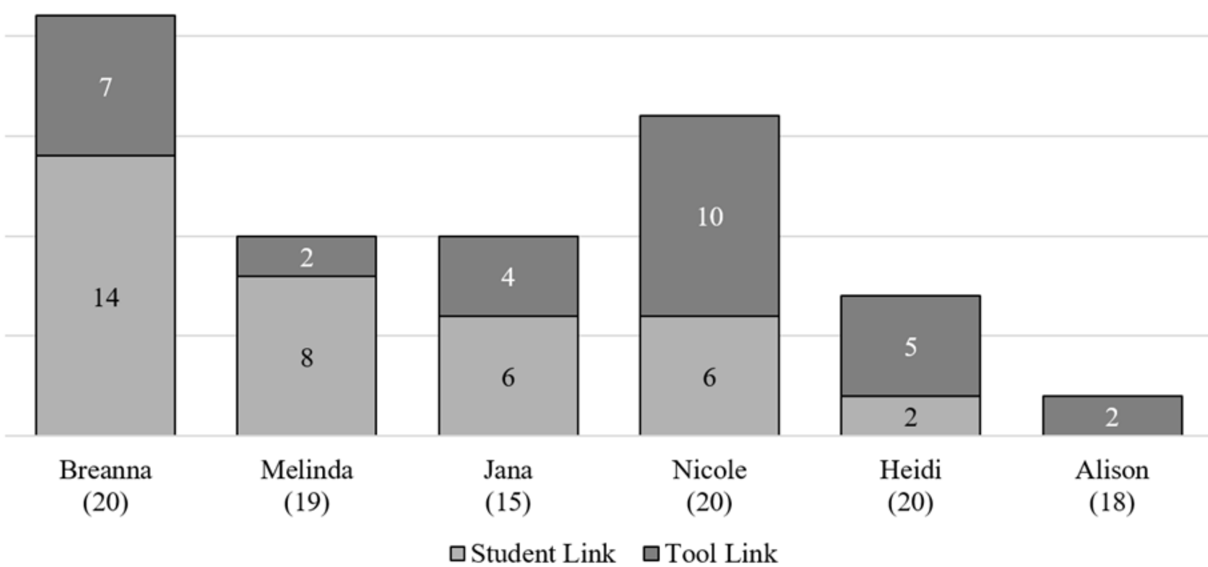


Figure 6.18: Connections between Representations

Figure 6.18 shows that the early elementary teachers described students making connections more often than the tool providing connections. The upper elementary and middle grades teachers, on the other hand, described connections in tools more frequently than students making those connections. Many tools that required students to make connections were games or quizzes, while tools that provided connections were often exploratory tools where representations were dynamically linked. It seems reasonable that online tools intended for older children might provide more opportunities for students to experiment with dynamically linked representations.

### *Mathematical Consequences or Feedback*

Teachers described the nature of linked representations; for example, whether students made connections or tools provided connections. As they described the linking of representations, teachers also described the way an online tool responded to student interaction, showing mathematical consequences or feedback of mathematical correctness.

Breanna wrote, "If the student incorrectly tries to put a bug in the wrong column, the tool will release the bug back into the cybrary." In this example, the tool gives feedback to a student in the sense that the bug is not allowed in the wrong column. This feedback is different than mathematical consequences, because the tool does not show the student what happens mathematically when the bug is placed in the wrong column. Within dynamic representation statements, especially, I saw some teachers described the potential for students to encounter mathematical consequences of their actions. By *mathematical consequences*, I follow the meaning Dick and Burrill (2016) described in their *Action-Consequence Principle*: "The technology should afford an opportunity for the student to take a *mathematically meaningful action.* The action taken by the student should result in an *immediate, visual* and *mathematically meaningful consequence.*" I noted that other teachers described their expectation that tools would provide feedback. Teachers' meanings of feedback differed from mathematical consequences.

To clarify my meaning and coding, I offer the following examples of statements that distinguish mathematical consequences from feedback of mathematical correctness. Nicole wrote, "Because students are able to change the numerator/denominator, this can provide a learning opportunity in that students can see what happens to the decimal, percent, and diagram when something changes."

I view this statement as describing mathematical consequences, because Nicole describes a mathematical action ("students ... change the numerator/denominator") and the mathematical consequence ("see what happens to the decimal, percent, and diagram"). In this statement, Nicole described an activity that seemed exploratory without any necessary answer or correctness. Jana wrote,

> For example students will think that if I am going to put cocoon at the corner it will make more weight/or bird feeder will bend to one side and if I am going to put a bird here it will balance the other side.

This statement describes mathematical consequences, because Jana describes a hypothetical student considering the mathematical consequence ("bend to one side... balance the other side") of placing objects along a balance. In this description, Jana explained a situation where a student was looking for an answer in the form of which objects in which locations would create balance. I share the two examples to illustrate that mathematical consequences can support a student's exploration of a situation or a student's problem-solving activity. I describe mathematical consequences as exhibiting *internal authority* because the consequences themselves do not necessarily indicate correctness or incorrectness. A student explores or experiments within the situation and develops her own sense of correctness or incorrectness (as applicable).

In evaluations of online tools, teachers also indicated feedback from tools that indicated to the student, in different ways, that the student's action or answer was correct or incorrect. For example, Heidi described, "They shoot at their targets and are awarded points for correct answers and deducted for incorrect answers." Nicole wrote,

> The tool also lets students know if they are right or wrong. If a student correctly selects the balloon it makes a sound and pops. If a student incorrectly selects a balloon it makes a 'beep' sound and does not pop.

Breanna wrote, "If the student gets the answer incorrect, the tool will give an explanation in a box below the problem." In each of these examples, the tool indicates to a user when answer is correct or incorrect but not with a *mathematical meaningful consequence.* Even in Breanna's statement, when the tool provides a mathematical explanation of the incorrect answer, the response is not mathematically meaningful because it is not an effect that follows the internal logic of a mathematical object. Instead, the response comes from outside of the mathematical object.

Figure 6.19 shows the number of tools for which teachers made statements attending to mathematical consequences of student actions or feedback provided by the tool indicating mathematical correctness, and the number of tools for which teachers made some statements about consequences and other statements about feedback.

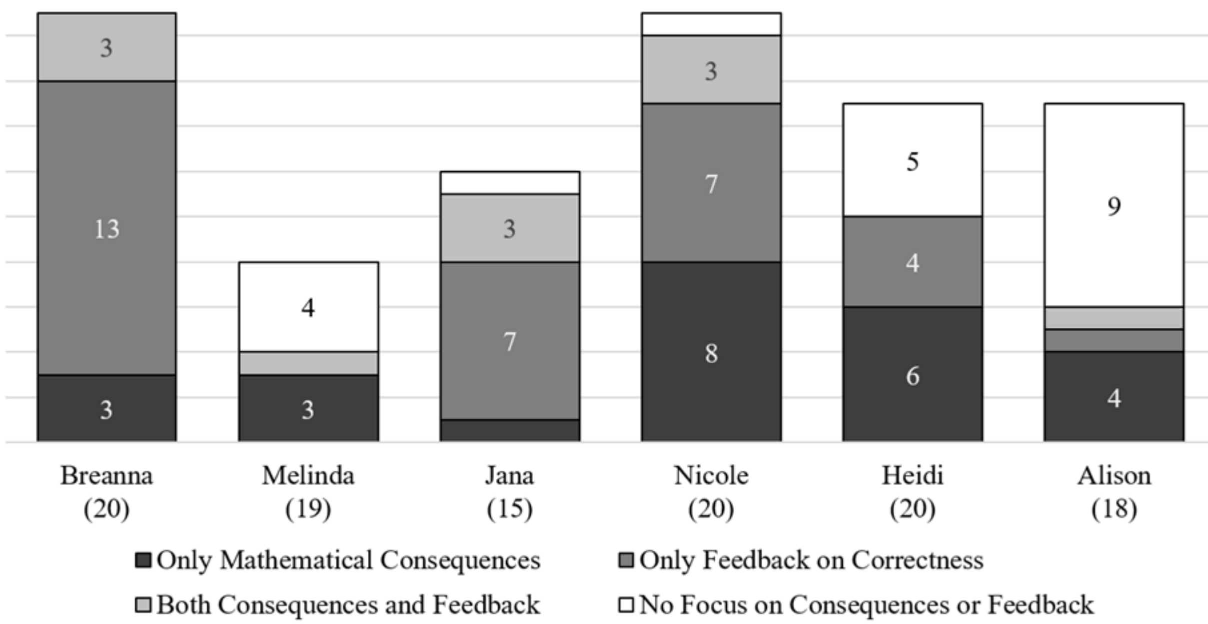


Figure 6.19: Mathematical Consequences through Dynamic Representation

Figure 6.19 shows that all teachers made some statements about mathematical consequences and other statements about mathematical feedback, provided through dynamic representation of an online tool. Breanna, Melinda, Nicole, and Alison each had at least one tool for which they wrote sentences attending to both mathematical consequences and feedback on correctness; Breanna had the most with four. Every teacher attended to mathematical consequences alone for at least a quarter of the tools they evaluated. Every participant attended to feedback on correctness for at least one tool; Breanna and Jana attended to correctness for more tools than consequences.

Nicole and Melinda each described a tool's feedback in a negative way; otherwise, all teachers described feedback from tools as a positive of the tool. Nicole described answer entry as problematic because "if the answer were two-digits, but you thought it was only one-digit or even three-digits, the tool would not let you type in your answer." Melinda pointed to a similar issue, writing "The tool does not respond to the student's manipulations other than when the student selected the correct answer." In both cases, the teachers pointed out that when students have an unexpected incorrect answer, the tool will not respond.

In describing dynamic representations of tools, Nicole addressed the need for additional constraints on student actions. For example, Nicole described a ten frame tool, saying,

> First, it did not matter where students placed the symbols in the 10-frame, as long as there was the correct number of symbols in it. If I remember correctly, students should start with the top left box, fill the first row, then move to the bottom left box and fill the second row. With this tool, students could randomly place the symbols in the 10-frame, and it would still be considered correct.

From these statements, Nicole indicates that the tool should constrain the student to follow a systematic placement procedure. These statements attend to mathematical consequences, if indirectly. This example illustrates the conflict between rules that are conventional, rules that are mathematically necessary, and rules that are intended to support mathematical thinking and learning. Nicole described a rule: "students should start with the top left box, fill the first row, then move to the bottom left box and fill the second row." In this case, the rule might support students in adding two ten frames if the counters are placed systematically. On the other hand, allowing students to place the counters anywhere, may support students in discovering the rule themselves as they use the ten frames.

In Figure 6.20, I show the frequency of mathematical feedback types described by participants: immediate feedback indicating whether an answer was correct or incorrect, an explanation of a correct solution, or a summary of correct or incorrect responses.

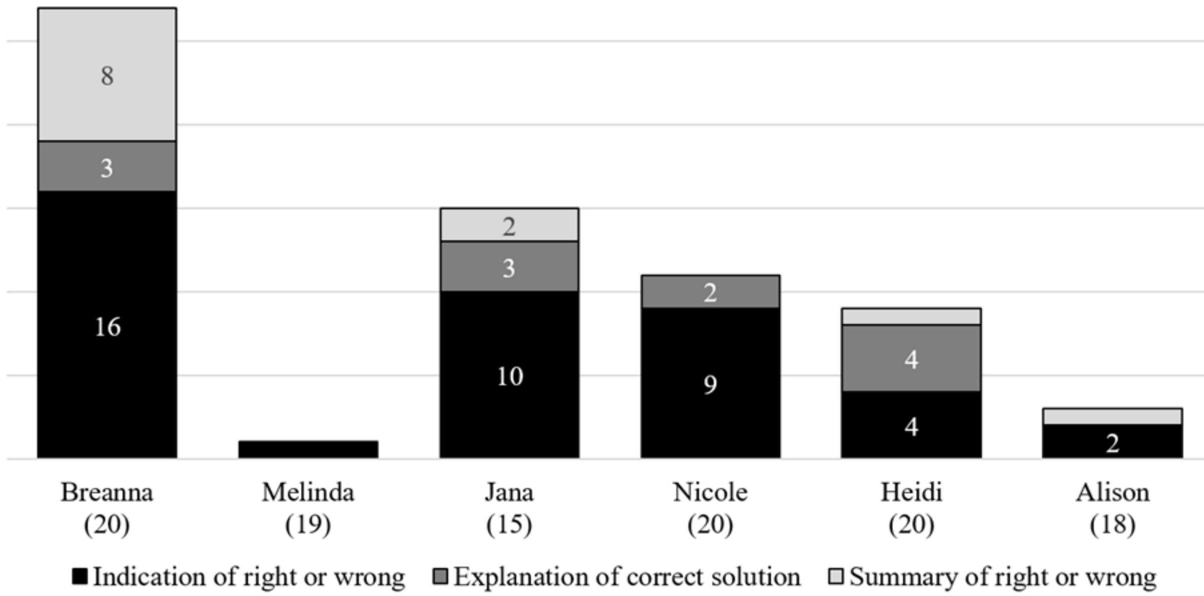


Figure 6.20: Mathematical Feedback

Figure 6.21 shows that teachers attended to online tools' feedback that indicated an answer was correct or incorrect. Although their descriptions depend on the tool, Breanna, Jana, and Nicole additionally indicated when a tool gave immediate feedback without an explanation. For example, Nicole wrote "However, there is no explanation showing students why they were wrong if they entered the incorrect answer." Similar statement were made by all three in describing several of their tools. Their attention indicates they felt online tools should provide students with explanations rather than simply indicating correctness.

As Nicole wrote about a tool that provided internal consequences, "Students are not really using this tool to check right or wrong answers. Instead, they are using the tool to show an answer and gain a deeper understanding of division problems." That is, the tools that provide students with opportunities to take a mathematical action and then see the mathematical consequences of that action are not necessarily telling the student whether the action was right or wrong but instead allowing them to develop their own sense of consequences.

In exploring statements written by the teachers, I noted some emergent themes. These are not intended to describe all potential mathematical consequences, but only to describe those that the teachers noticed. I list them here:

1. Change a value or representation and view immediate changes in linked values or representations

   *Example (Heidi):* "manipulate each shape and see how the perimeter and area changes as the shapes are changed"

   *Example (Nicole):* "see what happens to the pulley system. The larger the fraction, the heavier the fraction, so the more it drops."

2. Compare and manipulate non-linked representations to support reasoning

   *Example (Breanna):* "fill the grid with however many different coin amounts, but as long as they equal $1."

   *Example (Melinda):* "match the fraction they were given in the beginning, to the fraction on the number line"

3. Construct a mathematical object by transformation or translation of other objects

   *Example (Nicole):* "creating their own tessellation and adding/changing/deleting shapes that would lead to gaps or overlaps"

4. Construct a plan, enact a plan, make revisions based on enactment

   *Example (Nicole):* "create a plan to get the turtle to a pond"

   *Example (Alison):* "manipulate the values to get two cars to go the same speed"

5. Manipulate a constrained representation to discover hidden attributes or properties

   *Example (Heidi):* "manipulate shapes to help in determining which shape maker is each type of quadrilateral"

   *Example (Breanna):* "cannot fit all the bugs into the specified column... [forces student] to change the scale of the graph"

In Figure 6.21, I show the types of mathematical consequences described by participants based on the list above.

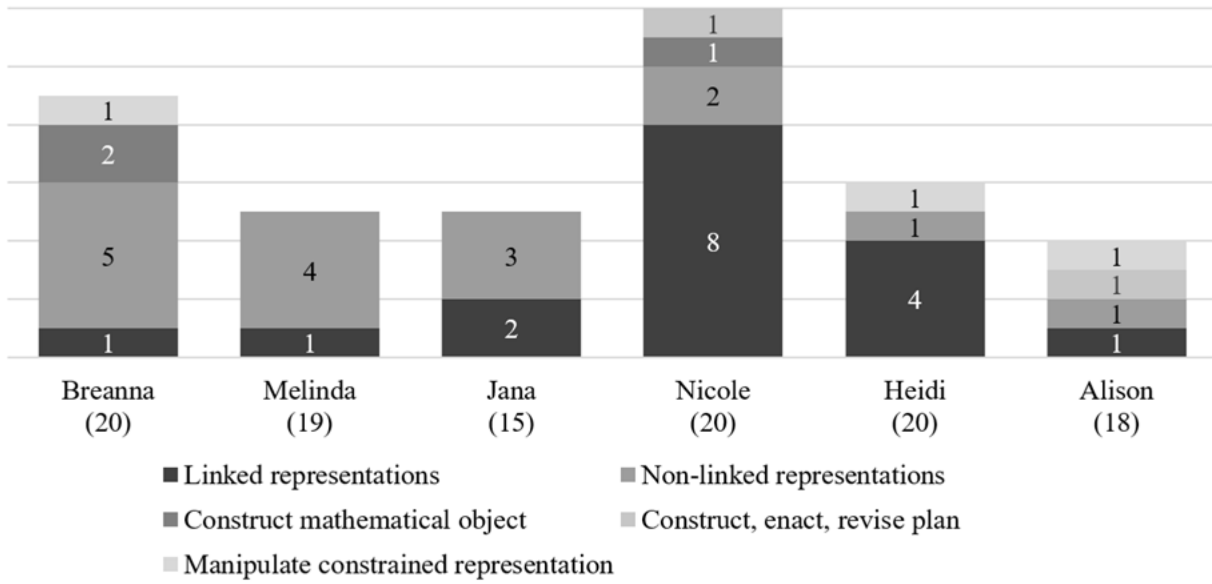


Figure 6.21: Mathematical Consequences

Figure 6.21 shows each participant described at least one tool as allowing the student to either change a value or representation and view immediate changes in linked values or representations or to compare and manipulate non-linked representations to support reasoning. Of course, the teacher's descriptions of these aspects are dependent on the tool they evaluated. Figure 6.21 shows that teacher descriptions of tools (and potentially the tools themselves) focused on manipulating a representation and comparing it to corresponding linked or unlinked representations to support reasoning. Rarely did teachers describe the construction of mathematical objects, creation and enactment of plans, or manipulating a constrained representation.

## Summary

I shared findings across symbolic, visual, and dynamic aspects of representation. For symbolic, teachers' attention to the distinction between an abstract idea and its symbolic representation would indicate a higher level of expert noticing. For visual, expert noticing would attend to mathematical aspects of visuals at least as often as non-mathematical aspects, and would describe clear mathematical learning and mathematics made visible through use of visuals. For dynamic representation, expert noticing would attend to mathematical aspects of motion or interaction, ways in which these aspects support higher-level thinking, and ways in which students can encounter mathematical consequences.

For symbolic representation, Nicole and Heidi clearly indicated a distinction between an abstract idea and its symbolic representation. Jana and Alison may have indicated a distinction, but their language was unclear. Breanna and Melinda did not indicate such a distinction.

In considering visual representation, all teachers focused on mathematical aspects in the majority of their sentences addressing the visual aspects of tools. All but Alison included statements focused on mathematical aspects for over half of their tools. In fact, except for Alison and Melinda, all teachers wrote such statements for at least 80% of their tools. Teachers focused much more on their learning expectations for students' interaction with the visual representation than they did on the mathematics being made visible by the representation. Except for Alison, every teacher wrote learning expectations for the visual representation for over half of their tools; Nicole and Breanna had the highest frequency, writing such for 75% or more of their tools. Only Nicole, Jana, and Breanna described the mathematics in the visuals for over half of their tools.

For dynamic representation, all teachers wrote statements about interaction or movement for half or more of their tools. Only Breanna, Jana, and Nicole wrote statements describing doing mathematics, interacting with mathematics of the tool, and learning mathematics statements for more than half of their tools. Most attention was on students finding an answer or solving a problem (the two expectations may be mostly synonymous) which seems to be a lower level expectation of students' interaction with dynamic representation. However, teachers' attention was also on students creating mathematical objects (Breanna, Melinda, Jana, Nicole) and customizing or creating their own problem (Melinda, Jana, Nicole), both of which seem to involve higher level thinking. Only Nicole focused on the potential for students to encounter mathematical consequences through interaction with dynamic representations for over half of their tools.

## Intended Mathematical Understanding

In considering potentials of online tools for supporting student understanding, teachers responded to evaluation questions about the type of understanding a tool could support. Teachers rated online tools on whether a tool emphasized instrumental or relational understanding (Skemp, 1978). They also chose one of three frameworks, and rated tools according to the elements of that framework; either *Adding it Up's* proficiency strands (National Research Council, 2001), *Principles and Standards of School Mathematics' (PSSMs')* mathematical process standards (National Council of Teachers of Mathematics [NCTM], 2000), or *Common Core State Standards in Mathematics' (CCSSMs')* standards of mathematical practice (Council of Chief State School Officers [CCSSO], 2010).

### Instrumental or Relational

As described in Chapter 3, participants read Skemp's (1978) *Relational understanding and instrumental understanding* in Unit 1. We used these two types of understanding in whole class evaluation templates. As described in more depth in Chapter 2, Skemp described instrumental understanding as "rules without reasons" or "the possession of such a rule and the ability to use it" (Skemp, p. 9) and relational understanding as "knowing both what to do and why" (p. 9). Figure 6.22 shows the frequency for which participants rated online tools as supporting instrumental or relational understanding, and the frequency they rated tools as supporting both types of understanding.

In Figure 6.22, note that participants did not always use this categorization. After reading Skemp (1978) in Unit 1, some participants described online tools as supporting instrumental or relational mathematical understanding while others did not. A question explicitly asking participants to describe tools in such a way was introduced in the Unit 4 whole class evaluation template. Even after its introduction, participants chose to use this categorization less frequently than Learning Activity types. I included a category to show when participants marked tools as both Instrumental

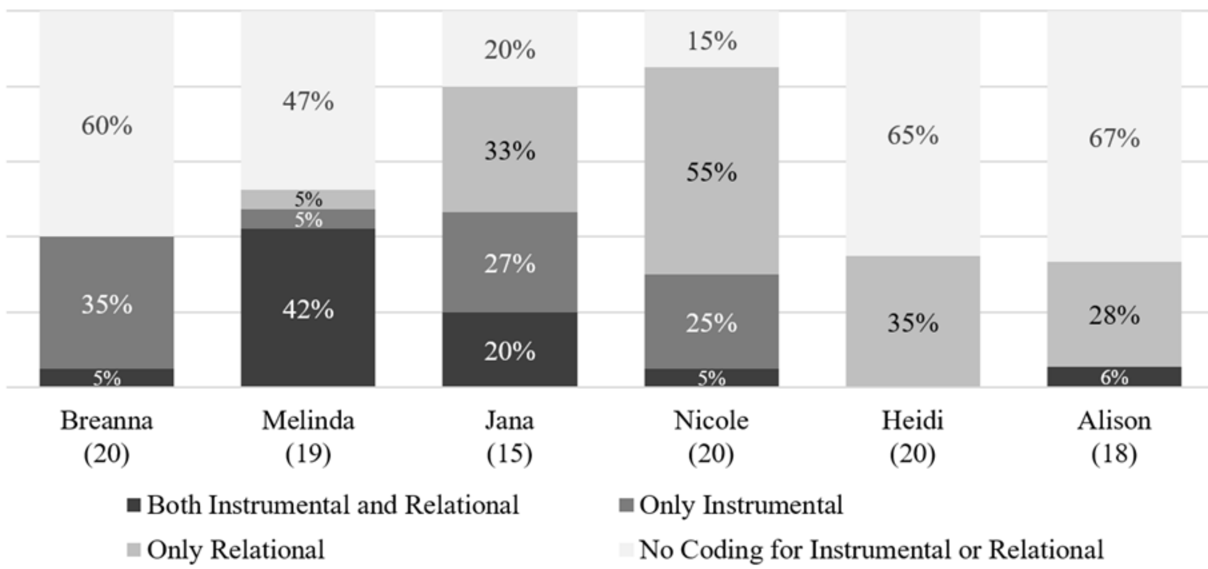


Figure 6.22: Instrumental or Relational Understanding

and Relational.

Figure 6.22 shows that Breanna only marked tools as Instrumental, while Heidi marked all as Relational. One explanation might be that Breanna was a Kindergarten teacher and so the tools for that age group might only be intended to support Instrumental understanding, while Heidi taught Fifth grade for gifted and talented and so she might have looked at tools that would mostly support Relational understanding. Melinda, also an early grades teacher, marked the majority of her tools as Instrumental. From the Both category, it is seen that Melinda marked tools as Instrumental and Relational almost two-thirds of the time. All other participants—Jana, Nicole, Heidi, and Alison—marked tools more often Relational than Instrumental and seldom marked tools as both Instrumental and Relational.

In considering teachers' descriptions of instrumental and relational, some patterns emerge. First, Breanna, who only rated tools as supporting instrumental understanding, described instrumental as "knowing the rules of the concept and knowing how to demonstrate it using mathematical situations." For this example, she went on to explain her definition's relevance to her tool, "There is no explanation about what patterns are because it assumes that the student has learned that concept already. Students are to use what they know about patterns to answer the problems correctly to obtain a high score." In response to other tools she rated as supporting instrumental understanding, she wrote similar justifications: that the student would demonstrate what they knew to answer questions correctly.

Melinda described support of instrumental understanding in a similar way, writing that it "applies when students have to identify and understand." Across the tools that she rated as supporting instrumental understanding, her descriptions indicated she viewed it as supporting students in demonstrating a basic understanding or knowledge of facts and procedures. When she described how tools supported relational understanding, she seemed to indicate those tools require students to engage in higher-level thinking, such as having students "apply what they know about the shapes and be able to classify, sort, define similarities and differences between shapes, and so on."

Jana's descriptions indicated a similar view, although she wrote about the instrumental or relational support of a tool, and followed that section with a separate section in which she described opportunities for "higher-order thinking." She described support of higher-order thinking as allowing students to "think, analyze, and question" as they interact with the tool.

Nicole wrote relatively detailed descriptions of her views of instrumental and relational understanding. She wrote that tools that support instrumental understanding ask students to follow a "specific rule" or "step-by-step procedure," or to practice simple problems. She used words such as discovering, exploring, and applying and described students being supported in developing deeper understanding. For example, in comparing a tool she felt supported instrumental and not relational understanding, Nicole wrote that a tool's "main purpose is for students to practice problems, not really to gain a deeper understanding of a concept, how it works, and how its connected to other ideas/disciplines." Her justification for using the instrumental understanding rating sheds additional light on her view of relational understanding as including deeper understanding, understanding how something works, and understanding connections between the mathematical idea and "other ideas/disciplines."

Heidi did not provide detailed descriptions of her reasoning, but the description of relational understanding that she provided indicated that she may view support of relational understanding as including visual support of abstract ideas. In two evaluations, she wrote "The use of models and pictures helps to reinforce a relational understanding of the mathematics" and "This tool supports a relational understanding as they are modeling how to represent fractions in a picture." Without explicit attention to her reasoning, her views of relational understanding are unclear, but seem to focus on the use of visual or concrete representations of abstract mathematical ideas.

Alison justified her choice for relational understanding as "due to the application of concepts," because students would be "manipulating the shapes and working with looking at all sides," and because students would be "visually seeing how the formulas have came about." Her statements indicate a view of relational understanding that includes applying past knowledge to new situations, being able to play around with shapes or other mathematical objects to be able to move them around and look at them from different perspectives, and to use visual representation to understand mathematical formulas.

Overall, the teachers seemed to have multiple perspectives on the meaning of relational understanding, and also multiple perspectives on how online tools might support students in developing relational understanding. Their descriptions indicate that if students engage in higher-level thinking such as analysis, exploration, and discovery, and if students are able to interact visually and dynamically with mathematical objects and ideas, then they will develop relational understanding. Their descriptions indicate a view of relational understanding (a) as more than steps and procedures, (b) as an application or building on of past knowledge, and (c) as a deeper understanding how mathematics works and how it connects to other ideas.

## Proficiency / Processes / Practices

As part of describing the "mathematical content and learning goals" of a technology tool, the instructors asked participants to evaluate tools using their choice of one, or some combination, of three frameworks: *Adding it Up's* proficiency strands (National Research Council, 2001), *Principles and Standards of School Mathematics' (PSSMs')* mathematical process standards (National Council of Teachers of Mathematics [NCTM], 2000), and *Common Core State Standards in Math-*

*ematics' (CCSSMs')* standards of mathematical practice (Council of Chief State School Officers [CCSSO], 2010). I describe each framework briefly below due to time and space limitations, but I acknowledge each framework element is much more complex than my descriptions indicate.

In Unit 3, participants read about the proficiency strands, process standards, and mathematical practices. Teams collaboratively explored the meanings of the frameworks by building a representation to show connections between them. Teams shared their representations in a whole class discussion and commented on each other's. Starting in Unit 4, the evaluation template contained a question asking participants to choose one of the three frameworks (or to use more than one) as they evaluated their technology tools. Figure 6.23 shows teachers' frequency of choices of strands, process, or practices as well as their subsequent ratings.

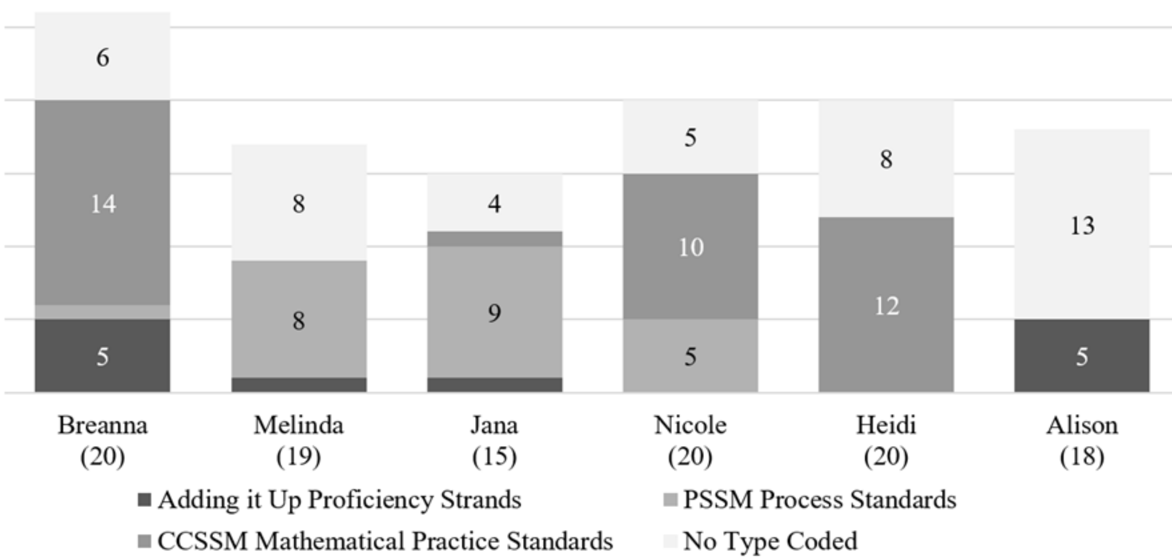


Figure 6.23: Strands / Processes / Practices

Figure 6.23 shows that participants varied in their choices, sometimes choosing one framework and sometimes another. Breanna, Melinda, and Jana used each of the three frameworks for at least one online tool. Heidi and Alison used only one of the three frameworks. Nicole used two of the frameworks varied across their evaluations. Breanna, Nicole, and Heidi each used the *CCSSM* standards of mathematical practice for most of their evaluations, while Jana and Melinda used *PSSM* mathematical process standards for most of theirs. I show the participants' choices of categories below. For each, I only include participants who chose a particular framework for at least one evaluation; for example, Alison will only be included in Figure 6.24 because she only used the *Adding it Up* proficiency strands framework in evaluations.

Figure 6.24 shows the frequency of participants' choice of each category within the *Adding it Up* proficiency strands framework in their evaluations (Ball, 2003). Ball advocated for these five strands of mathematical proficiency to be used as a guiding framework across K-12 mathematics education. The intention is that the strands are interwoven in complex ways to support and describe the ways students should develop mathematically. As a brief overview, the five proficiency strands are: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition.

- *conceptual understanding*: "comprehension of mathematical concepts, operations, and relations"
- *procedural fluency*: "skill in carrying out procedures flexibly, accurately, efficiently, and appropriately"
- *strategic competence*: "ability to formulate, represent, and solve mathematical problems"
- *adaptive reasoning*: "capacity for logical thought, reflection, explanation, and justification"
- *productive disposition*: "habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in the value of diligence and in ones own efficacy" (Ball, 2003, p. 9).

Of the four, only Alison used the productive disposition category. Melinda used this framework to evaluate only one technology tool, and categorized it with all categories except productive disposition. Breanna used procedural fluency most often (in three of her four evaluations), while Alison used conceptual fluency most often.

National Council of Teachers of Mathematics [NCTM] (2000) introduced Process Standards to focus attention and classroom time on developing mathematical skills beyond simply knowledge of mathematics content that are part of a solid mathematical foundation. As a brief overview, the

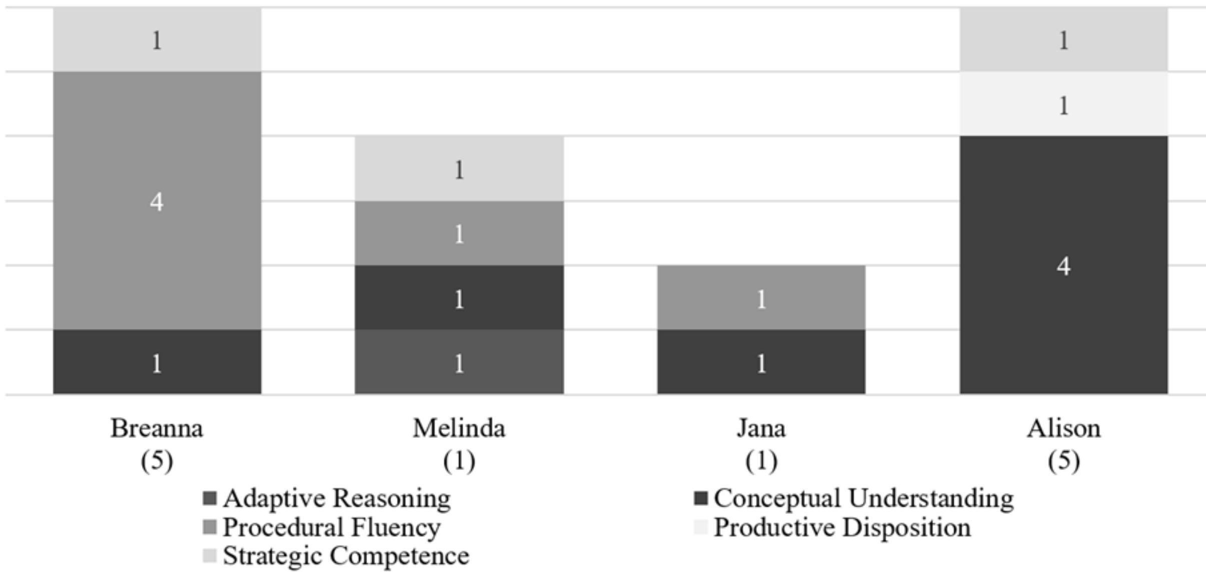


Figure 6.24: Proficiency Strands

five process standards are: problem-solving, reasoning and proof, communication, connections, and representation (National Council of Teachers of Mathematics [NCTM], 2000).

- *problem-solving*: "engaging in a [complex task] for which the solution method is not known in advance" (p. 52)
- *reasoning and proof*: part of reasoning is to "note patterns, structure, or regularities in both real-world situations and symbolic objects"; proving is a way to formalize reasoning and justification (p. 56)
- *communication*: involves using the language of mathematics in sharing one's own thinking and reasoning as well as in clear and convincing ways, as well as "listening to other's explanations" through "speaking, writing, reading, and listening" (p. 60)
- *connections*: become visible in "rich interplay among mathematical topics, in contexts that relate mathematics to other subjects, and in their own interests and experience" (p. 64)
- *representation*: "refers to both process and product – in other words, to the act of capturing a mathematical concept or relationship in some form and to the form itself" both externally and internally (National Council of Teachers of Mathematics [NCTM], 2000, p. 67)

Figure 6.25 shows the frequency of participants' choice of each category within the *PSSM* mathematical process standards framework in their evaluations.

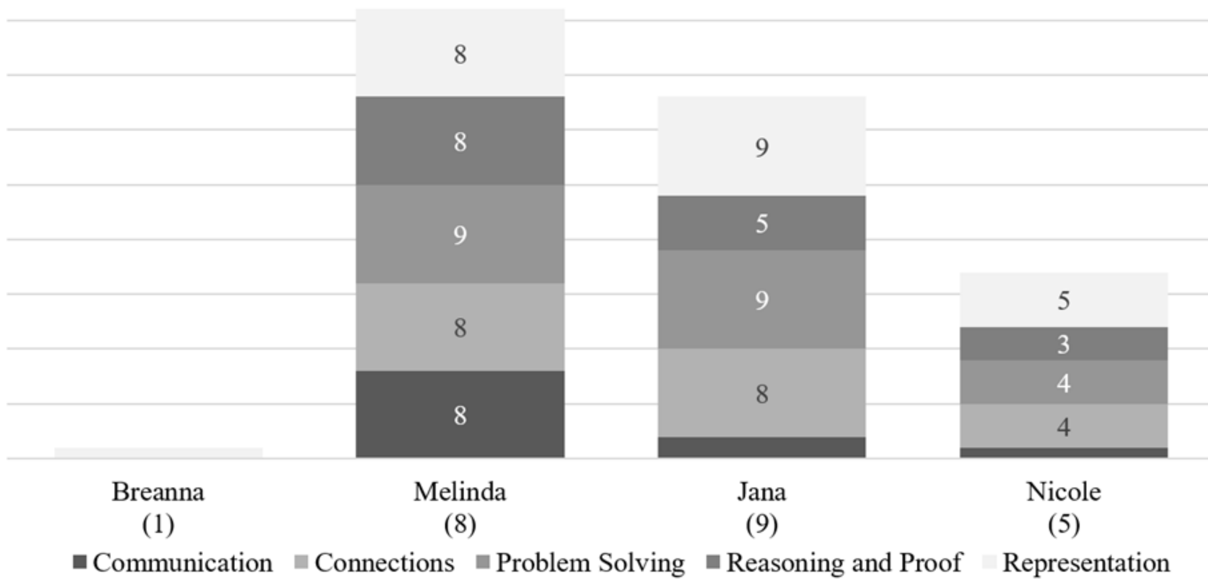


Figure 6.25: Mathematical Processes

Breanna used the framework for only one tool, so only used the representations category. Melinda, Jana, and Nicole used every category at least once. Melinda and Nicole used the communications category least often, while both used problem-solving, connections, and representations most often.

Breanna, Jana, Nicole, and Heidi used the *CCSSM* standards of mathematical practice framework in their evaluations. Writers of the *CCSSM* standards of mathematical practice developed them by reflecting on the strands of mathematical proficiency and mathematical process standards frameworks (Council of Chief State School Officers [CCSSO], 2010). There are eight practices:

- MP1: Make sense of problems and persevere in solving them
- MP2: Reason abstractly and quantitatively
- MP3: Construct viable arguments and critique the reasoning of others
- MP4: Model with mathematics

- MP5: Use appropriate tools strategically
- MP6: Attend to precision
- MP7: Look for and make use of structure
- MP8: Look for and express regularity in repeated reasoning

Figure 6.26 shows the frequency of their choice of each category within the *CCSSM* standards of mathematical practice framework.

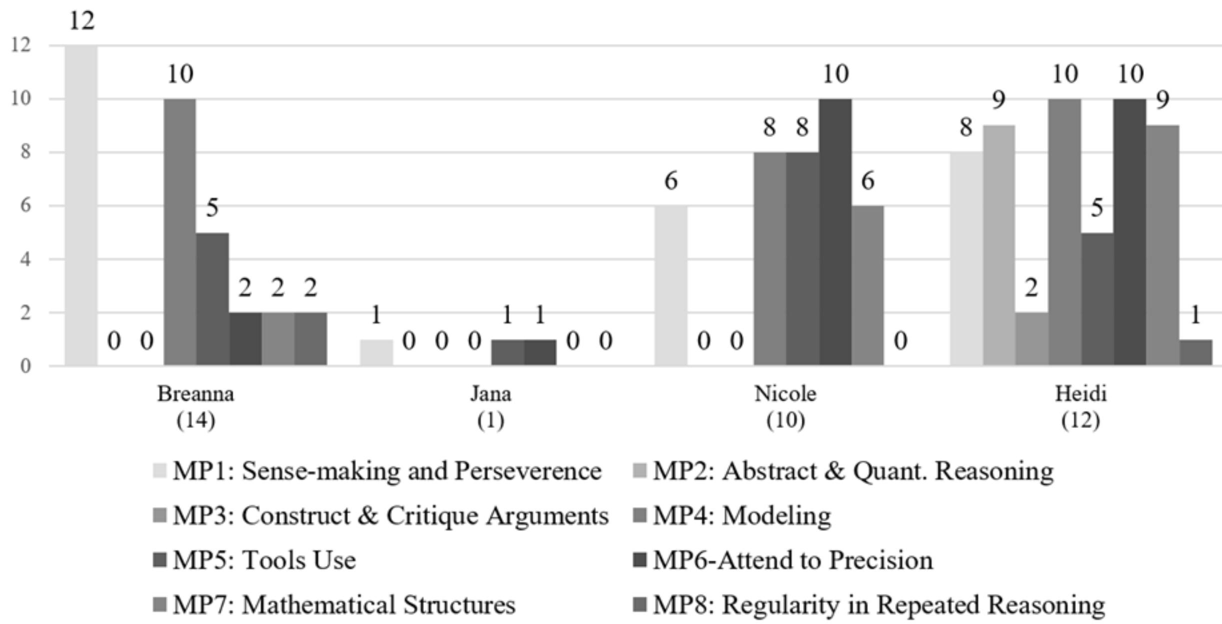


Figure 6.26: Mathematical Practices

Figure 6.26 shows that only Heidi chose MP2 (Reason abstractly and quantitatively) and MP3 (Construct viable arguments and critique the reasoning of others) as describing tools she evaluated. Breanna used MP1 (sense-making and problem-solving) and MP4 (modeling) most frequently. Nicole focused on MP4 (modeling), MP5 (strategic use of tools), and MP6 (attend to precision). Across the participants, MP3 and MP8 were least used.

## Summary

In considering teachers' attention to types of mathematical understanding supported by online tools, I attend to emergent characteristics that may help distinguish teachers as "experts" and "novices" across these aspects. In considering teachers' attention to aspects of online tools that support instrumental or relational understanding, expert noticing would reveal a clear understanding of relational understanding and provide reasonable rationale for a choice with respect to aspects of tools.

Teachers varied in how they rated tools as Instrumental or Relational. From Breanna who rated all as Instrumental, to Melinda who rated mostly as both Relational and Instrumental, to Jana and Nicole who were more balanced between Instrumental and Relational, or to Heidi and Alison who rated most as Relational. Some teachers described Instrumental understanding as knowing (able to demonstrate use of) facts, rules, or procedures (Breanna, Melinda, Nicole). Some teachers described Relational understanding as a deeper understanding (Nicole, Heidi, Alison) and using representations to gain that deeper understanding (Nicole, Heidi, Alison), and as involving higher-level thinking (Melinda, Nicole) or using multiple strategies (Jana, Nicole).

In considering teachers' attention to mathematical proficiency strands, mathematical process standards, or standards of mathematical practice, expert noticing would reveal clear understanding of the meaning of a strand, process, or practice, and provide reasonable rationale for a choice with respect to aspects of tools. Teachers used these frameworks in a fairly scattered way and often did not report their rationale for choosing particular ratings. For example, Breanna, Jana, Nicole, and Heidi each used one framework to rate nine tools. Each of the teachers either gave rote responses for a rationale or did not give a rationale for their ratings. Hence, too few tools were rated by one teacher using any particular framework to be useful. And few rationales of choices keeps their ratings from being useful.

## Learning Activity Types

Grandgenett, Harris, and Hofer (2011) described a taxonomy of *mathematics learning activity types* to support teachers to thoughtfully and intentionally integrate mathematics, pedagogy, and technology in their teaching. The instructors adapted this taxonomy to the current course, focusing on mathematics learning activity types that might be supported by online tools: presentation (demonstration or explanation), practice for fluency, exploration / investigation, application, review, and assessment. We gave the following descriptions of each.

- Present: (read or attend to) presentation of new content/ideas
  - Present-Demo: demonstration
  - Present-Explain: explanation
- Explore: exploring/investigating mathematical ideas
- Apply: applying mathematics to problems and situations
- Practice: practicing for fluency
- Review: reviewing
- Assess: assessment

For this analysis, within Learning Activity types, I focus on four distinct types: Present, Explore, Practice, and Apply. I consolidate the Present categories (rather than distinguishing Demonstration and Explanation). I focus on Present, Explore, Practice, and Apply because the four types seemed most distinct, and we provided more detailed descriptions for these four than for the others. Review seemed to be a special case of Practice. Although the Assess type may have been interesting, Assess was used by only one participant (Melinda).

Figure 6.27 shows teachers' frequency of referencing particular Learning Activity (LA) types across their online tool evaluations.

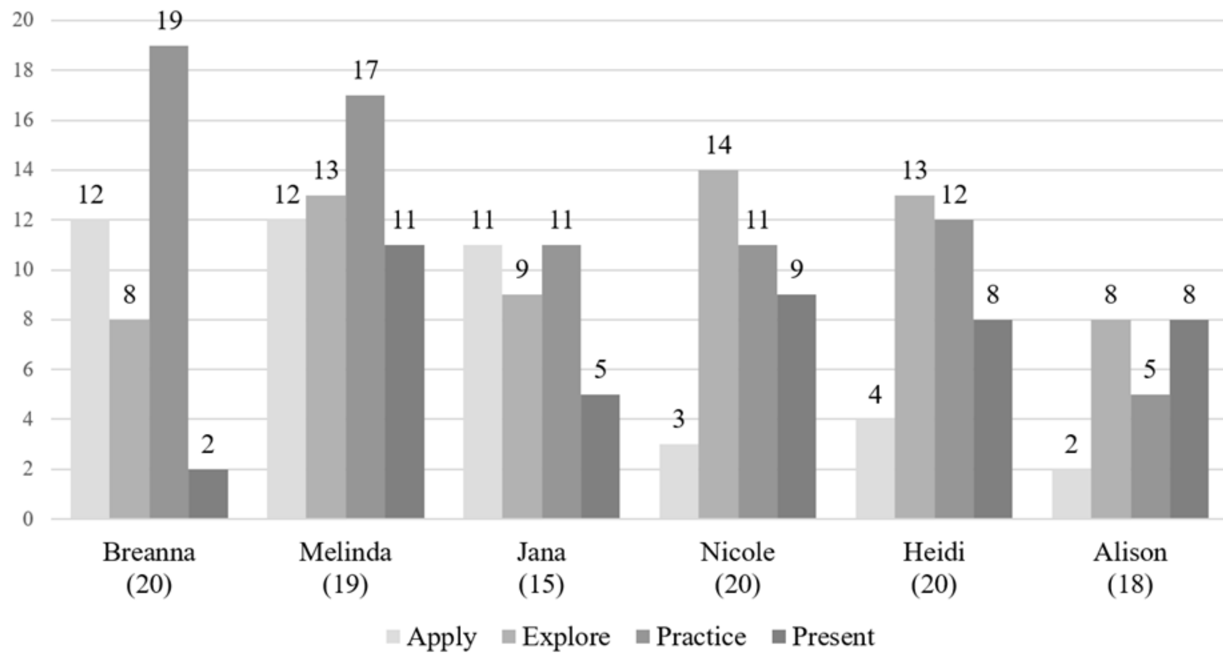


Figure 6.27: Learning Activity Types

Figure 6.27 shows that participants did not use this categorization for every evaluation. Some participants (e.g., Nicole) used these terms to describe online tools even before being explicitly introduced to this framework in Unit 4. Figure 6.27 makes some patterns visible. Some of these patterns seem to divide the Lower Elementary teachers from Upper Elementary and Middle Grades teachers. Considering evaluations in which they used LA types, Breanna and Melinda both labeled every technology tool as a *practice* tool. Breanna, Melinda, and Jana all labeled tools as *practice* more frequently than any other, considering the number of tools evaluated. Nicole, Heidi, and Alison labeled tools most frequently as *explore*. Alison also labeled tools most frequently as *present*. Nicole, Heidi, and Alison also labeled tools as *apply* least frequently. Breanna, Melinda, and Jana used the label *present* least frequently.

Teachers often did not provide explanatory text for their ratings in this section. Melinda and Nicole did not provide explanations for any LA ratings; Breanna, Jana, Heidi, and Alison explained ratings for a few of their online tools. I describe patterns in the way they explained their use of LA types to develop this framework and understand how these teachers saw the elements in different ways.

In each of her statements for present, practice, and explore, Breanna wrote (with differing levels of detail) “how to collect, sort, and graph data.” In the apply statement, she used slightly different wording but it seems relatively consistent. Her consistent wording seems to suggest that she sees one correct way of “how to” that is supported by the tool and that all students would use. The teacher demonstrates the one method. Then students practice following the method the teacher demonstrated. Students can explore the method, perhaps to understand it better. Despite the instructors’ prompt for explore as “exploring / investigating mathematical ideas,” Breanna’s wording does not suggest the students are exploring mathematical ideas, or exploring the situation to develop their own strategies or to refine the method. Breanna’s wording does not suggest the students are exploring the mathematical ideas and concepts of data and graphing.

Jana included some explanation of her reasoning for present, explore, and apply for two tools: PBS Kids: *Sid the Science Kid – Balancing Act* and Turtle Diary: *Subtraction Matrix.* She also gave some explanation for her apply rating of the online tool, Math Playground: *Island Chase Subtraction.* In her explanation of present for Balancing Act, Jana described the teacher’s explanation as including questions for the students directing them to wonder what happens when different items are placed in different ways along the balance. For her explanation of explore, she described students considering the meaning of “equality and an equal sign” as they watch the presentation. For her explanation of apply, she gave a brief view of students’ background understanding and how a teacher might ask questions to give the students opportunities to apply their developing understanding. Her descriptions of present, explore, and apply indicate that her focus is on students’ understanding and thinking. Her description of explore suggests a view of explore as mediated by a teacher during presentation of the online tool.

Heidi gave brief descriptions for her LA ratings of LearnZillion: *Adding mixed numbers with unlike denominators by creating area models* and NCTM Illuminations: *Fraction Track.* For *Fraction Track,* Heidi’s descriptions are interesting because they seemed to focus on different mathematical ideas that support a central theme. That is, for practice, Heidi wrote “practice adding fractions with like and unlike denominators.” For explore, she wrote “explore how to add fractions to get to one whole." And for apply, Heidi wrote "apply what they know about equivalent fractions to add fractions to get to one whole." Based on her statements, she sees the tool as supporting students in applying knowledge of equivalent fractions (e.g., fractions in different forms, such as $\frac{1}{2} = \frac{2}{4} = \frac{3}{6}$) to practice addition of fractions with like and unlike denominators and to explore how fractions can add to a whole. The descriptions indicate a coherent view of students' past knowledge and support of new knowledge.

Unlike other teachers, Alison did include some explanatory text for several of her tools, when rating them with LA types. She described her use of the present rating as including use of a tool to present a new topic, to introduce a mathematical idea, to initiate a whole class discussion, to play around with mathematical objects, to demonstrate or explain particular aspects of an object or concept. For example, for GeoGebra: *Area of a Circle,* she wrote that present meant "Play around with the different types of options; demonstration of the concept of circumference and area; explain the concepts of how the formula came about." For the explore rating of NCTM Illuminations: *Shape Balance,* Alison described allowing students to discover uses of variables and then to share those discoveries in a whole class discussion. For her use of the apply rating of the same tool, Alison described her reasoning as follows: "After enough fluency is gained in the understanding of this topic, students could write equations in the form of Xs to represent the shapes." Her description of these ratings indicates her openness to allowing students to explore and discover their own ideas, even in her presentations. It also indicates her use of whole class discussion to share findings and explore ideas. Her description of practice indicates a willingness to develop and use worksheets to support students' interaction with online tools, rather than relying on the tool to provide the practice.

The four teachers seemed to use the apply rating in similar ways, but showed different ways of interpreting explore and present. Heidi and Alison showed, for at least one online tool, that their reasons for choosing ratings could create a narrative of some sort, describing how different ways of using the tool could support a coherent view of the mathematical ideas.

## Summary

In considering teachers' attention to mathematical learning activity types enacted through online tools, I attend to emergent characteristics that may help distinguish teachers as "experts" and "novices" across these aspects. Expert noticing would reveal clear understanding of a chosen type and a clear rationale with respect to aspects of the tool.

When they used this framework, Breanna and Melinda rated every tool as supportive of practice activities. In fact, when they used this framework, every teacher, except Alison, rated the majority of tools as supporting practice; and, except Breanna, all rated the majority of tools as supporting exploration.

Teachers often did not provide explanatory text for their ratings in this section. Melinda and Nicole did not explain any LA ratings; Breanna, Jana, Heidi, and Alison explained ratings for a very few of their online tools. Teachers did not seem to explain what they meant by practice, simply indicated students would practice a particular skill or concept. When teachers explained exploration, they did seem to indicate that students would explore a mathematical idea; Breanna wrote instead that students would explore a given method.

# Aspects that Support Learning

In this section, I share findings of teachers's noticing of affordances and constraints of online tools and online tools' aesthetics and engagement potentials. The focus is on affordances and constraints that support (or potentially inhibit) mathematics teaching and learning. In considering aesthetics and engagement, I consider how teachers view student engagement of online tools used for mathematics teaching and learning, and especially whether they see engagement as only a matter of aesthetics or as a complex interaction of aesthetics, motivation, and mathematics content and representation.

## Affordances and Constraints

In considering instructional technologies for use in mathematics teaching and learning, it is valuable to consider what a particular technology allows a user to do that would be difficult or impossible to do without that particular technology. An affordance of a particular technology refers to an action that is made possible by the technology, or a new type of resource or support that it provides. A constraint of a particular technology, on the other hand, refers to new limitations or restrictions imposed by the use of that technology. Any software or online tool has countless *affordances* and *constraints* which are context-dependent. In Unit 2 of the course, participants watched a video presentation about one framework that can be used to categorize affordances of instructional technologies for use in mathematics teaching and learning. In this framework are five broad categories of affordances: computing & automating, representing ideas & thinking, accessing information, communicating & collaborating, and capturing & creating. I briefly describe these categories below.

- Computing & Automating: used as a practical expedient to reduce mental load (e.g., complex calculations, organization of data) and increase efficiency and precision (e.g., using software to quickly create many examples)
- Representing Ideas & Thinking: used to represent, record, manipulate, and interact with mathematical ideas and to support mathematical thinking in new or different ways
- Accessing Information: used to gain access to information, data, media, classroom resources, reference materials, etc.
- Communicating & Collaborating: used to communicate or collaborate synchronously (e.g., video chat, instant messaging, GoogleDocs) or asynchronously (e.g., email, blogs, online video) with distant classmates, peers, colleagues, professionals; also used to communicate or collaborate with others in the same room (e.g., SmartBoards, presentation software, wikis)
- Capturing & Creating: used to capture or create images, video or audio recording, data, representations, etc.

In a first pass of the data, I noted participants' explicit use of categories from the framework. Figure 6.28 shows the frequency of participants' explicit use of categories of this framework to evaluate tools.

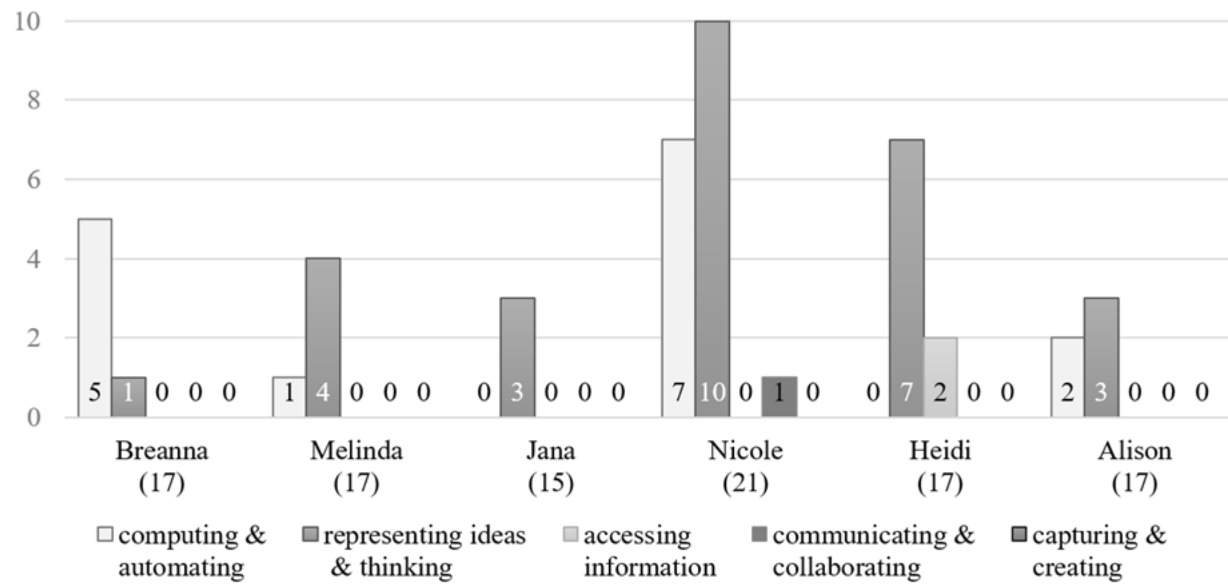


Figure 6.28: Affordances

Figure 6.28 shows that all participants explicitly referred to representation as an affordance of at least one of the tools they evaluated. Breanna and Nicole explicitly referenced computing or automating as an affordance of all (Breanna) or most (Nicole) of their tools. Nicole was the only participant to mention communicating and collaborating as an affordance; she wrote, "the one player versus two player option allows for communication and collaboration between students." Heidi was the only participant to notice accessing information as an affordance, noting that students could access a tutorial or play an animation on their own computer and as often as needed.

In a second pass of the data, I broke up their response into a list of affordances and looked for themes. On the one hand, this pass allowed me to look more deeply at how participants used categories from the framework; for example, Breanna stated several tools used computing and automating because, "The games are already prepared for the student. All students have to do is apply what they know about subtraction and play the game." Considering these explanations helped me better understand the meanings that participants brought to the framework categories. On the other hand, this pass allowed me to consider other affordances that participants discussed and consider how those might fit into the framework, or how the framework might need to be expanded. For example, Melinda, Heidi, and Nicole all mentioned that a user's ability to choose options (e.g., level of difficulty) was an affordance of different tools. Nicole wrote, "Students can customize and select whatever fractions they want. This allows for more of a variety and exploration catered to whatever the student wants to explore." Although different categories from the framework might work together to make this freedom possible, this affordance does not seem to reside within a particular category. That is, because of the computing & automating affordance, the applet will represent any choice of fraction. An argument might be made that the accessing information or capturing & creating affordances are also at play in supporting a user's customization of fractions.

### *Use of Framework Categories*

By exploring the way participants used these categories, and how they described the categories, the categories can be developed to be more useful for teachers. I focus on the *computing & automating* and *representing ideas & thinking* categories as they were used most often by participants.

Within the *computing & automating* category, participants described affordances as: (a) automatically setting up an environment for students to allow students to focus on applying their knowledge (Breanna), (b) providing or generating multiple opportunities with intentional variation or scaffolding (Melinda, Nicole, Alison), (c) providing or generating precise diagrams and other representations (Nicole and Alison), and (d) computing measurements or enacting routine calculations (Alison). One additional affordance that participants used in this category was (e) that these other actions would happen quickly and easily, allowing the teacher or student to focus on other aspects of the mathematics.

Within the *representing ideas & thinking* category, participants described affordances as: supporting student visualization and development of mental images (Jana, Nicole, Heidi, and Alison), allowing students to view dynamic mathematical objects and processes in motion (Nicole and

Alison), allowing the students to compare multiple representations (Breanna, Nicole, Heidi, and Alison), allowing the student to interact (effect changes) in dynamic ways with the mathematical objects and ideas (Breanna, Melinda, Nicole, Heidi, and Alison), and allowing the student to confront mathematical consequences of their actions (Nicole, Heidi, and Alison).

Across the two categories, participants discussed insights of how computing & automating aspects of technologies can support the efficient creation of precise and accurate representations. Representations of mathematical ideas are valuable, but can be created statically in textbook illustrations or worksheets. The participants pushed further into how online tools can allow different types of interaction: from dynamic representations that a student views to dynamic representations that a student interacts with. That is, representations with which a student effects change and might confront the mathematical consequences of that change to deepen their understanding of the mathematical concepts and relationships.

### *New Affordance Categories*

In addition to considering what teachers wrote that fit within the original framework of affordances, this second pass allowed me to consider additional affordances presented by participants. Across the six participants, I saw evidence of 12 additional categories. I list them in Table 6.6, each with an example explanation, in order of popularity: first, ordered by number of participants using each and, second, by number of instances.

As discussed above, the two categories *computing & automating* and *representing & thinking* appeared most often as noted affordances, both in number of participants (four and six, respectively) and in number of instances (15 and 28, respectively). Table 6.6 shows that the three additional categories that were most popular were practice, feedback & hints, and independence. The popularity of these categories suggests that teachers were considering the use of tools as supporting students in independent practice with immediate feedback. Table 6.6 shows that only Nicole gave a mathematical reason as an affordance. That mathematical reasons were few may be explained by the mathematical focus of the evaluation in other areas than this one. Some might argue that the

Table 6.6

*Affordance Categories from Participants, with Examples.*

| Affordance | Number of Participants | Number of Instances | Examples |
|---|---|---|---|
| practice | 4 | 10 | *allows students to practice a mathematical skill* (Heidi) |
| feedback & hints | 4 | 8 | *allows for immediate feedback or hints during a struggle* (Melinda) |
| independence | 4 | 8 | *kids can get around the program and play games on their own* (Jana) |
| choice & differentiation | 3 | 5 | *can customize and select whatever fractions they want* (Nicole) |
| entertaining & fun | 2 | 3 | *adds an entertaining component to the activity* (Nicole) |
| attractive | 2 | 2 | *very attractive and increases engagement* (Melinda) |
| ease of use | 1 | 8 | *very simple with easy to navigate system* (Jana) |
| familiar & comforting | 1 | 6 | *Computers are a medium that most students have familiarity and comfort with as learners* (Jana) |
| mathematics reason given | 2 | 6 | *encourages simplifying, which many of the other tools did not* (Nicole) |
| teacher resources | 1 | 1 | *provides a video for the teachers to watch for ideas and reference when using* (Jana) |
| records progress | 1 | 1 | *can record [progress] so that the teacher can keep track of it* (Jana) |
| friendly | 1 | 1 | *friendly sounds* (Jana) |

three most popular categories here (practice, feedback & hints, and independence) are mathematical reasons because they allow students to focus on practicing and improving their mathematics; that argument holds true for some views of mathematics.

***Constraints***

Although the instructors asked participants to address any constraints of tools as well, fewer constraints were raised overall. Constraints that were raised are shown in Table 6.7.

Table 6.7

*Constraint Categories from Participants, with Examples*

| Constraints | Number of Participants | Number of Instances | Examples |
|---|---|---|---|
| math reasons | 2 | 6 | *does not allow for students to change improper fractions into mixed numbers* (Nicole) |
| need computer skills | 1 | 2 | *able to complete basic computer techniques such as clicking, drag and drop, etc.* (Melinda) |
| need to model use | 1 | 2 | *must be modeled by the teacher before a student is able to work individually* (Melinda) |
| lack of choice | 1 | 1 | *A constraint to the tool is that you cannot customize* (Nicole) |
| limited opportunities | 1 | 1 | *same groups of fractions appear every time you visit the site... restricts the number of times this tool can be used* (Nicole) |

Table 6.7 shows that some constraints may fall into the framework the instructors gave participants for affordances. For example, the limited opportunities might fall under *computing & automating*, as multiple opportunities fell in that category above. So, a tool may seem to have a constraint when a user expects a particular affordance and finds it lacking.

Nicole and Heidi pointed to the lack of support for particular mathematical conceptualizations. For example, in Table 6.7, the example shows that Nicole wrote that she wanted her students to change improper fractions into mixed numbers; Nicole noticed that as a constraint in two evaluations. In other tools, she wrote mathematical constraints were the lack of simplification in final answers (2 evaluations) and focus on proper fractions so mixed numbers or improper fractions were not supported (1 evaluation). Heidi wrote a similar constraint in her evaluation, saying that the tool did not support students' understanding of the relationship between division and multiplication of fractions. These constraints are important to note as an important goal for teachers to recognize how different tools may support or obscure particular mathematical conceptualizations.

## Aesthetics and Engagement

The whole class evaluation template included explicit questions about whether or not a tool was aesthetically pleasing or engaging. Figure 6.29 shows, in parentheses, the number of online tools for which each teacher described the aesthetics or engagement of a tool out of the total number of tools evaluated. I include aesthetics and engagement in one figure to allow comparison of teachers' attention to each, especially whether *aesthetics* is a sufficient reason for a tool to be considered *engaging*. Teachers responded at times to the evaluation template question without explicitly addressing the aesthetics or engagement of a tool and so Figure 6.29 shows the frequency that participants explicitly addressed aesthetics or engagement, along with the number of tools a participant described as "yes, engaging" or "yes, attractive." Participants often wrote their reasoning in response to this question, so I provide additional analysis of their responses below to further explore whether they may have considered a tool's *aesthetics* as a sufficient reason for it to be considered *engaging*.

Other than Breanna and Alison, who explicitly addressed a tool's aesthetics more frequently (or with the same frequency) than its engagement, Figure 6.29 shows all participants explicitly addressed the engagement of more tools than they did the aesthetics. Alison, Melinda, and Heidi rated the same number of tools as attractive and engaging, while Breanna, Jana, and Nicole rated

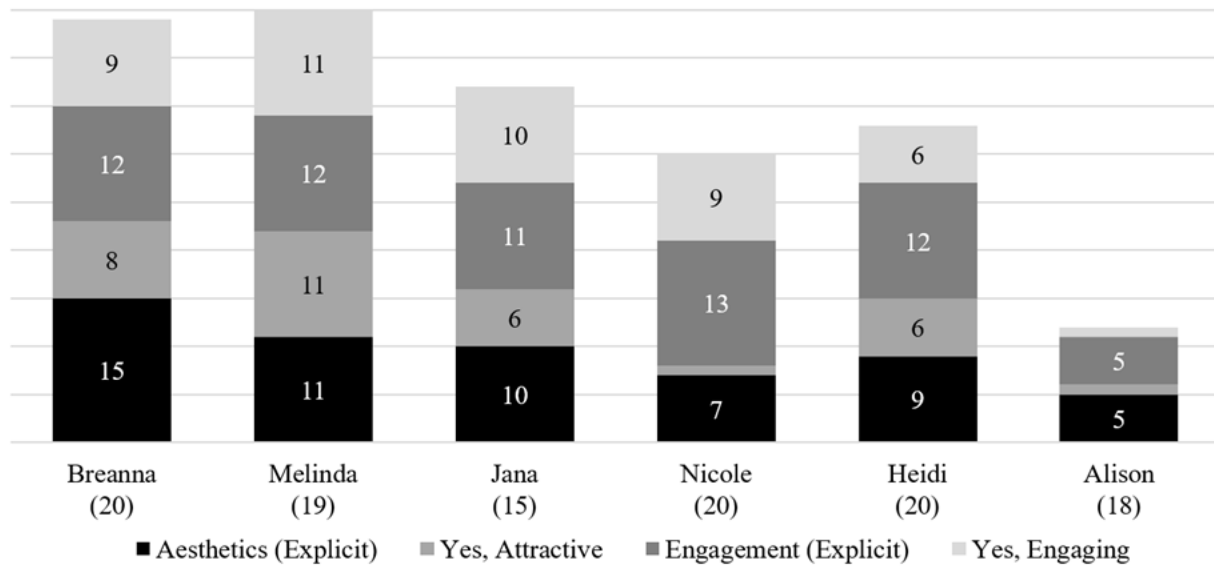


Figure 6.29: Aesthetics or Engagement

more tools as engaging than aesthetically pleasing.

In teachers' reasoning about engagement, they referenced mathematical content or actions along with other reasons. Teachers used words and phrases referencing mathematical content or actions, such as: challenging but achievable, allowing creating or building mathematical objects, customizable or personalizable content levels or manipulatives, ability to explore or discover, and possible movement or interaction. Other reasons included: aesthetic reasons: appearance and sounds, competitive nature of the tool or ability to play with others, ease of use, encouragement from the tool, fun or game aspects, ability to work at one's own pace, an lack of distractions.

Figure 6.30 shows the overall frequencies of teachers' use of reasons, when they explicitly rated a tool as engaging: mathematical, other, or no reasons given.

Of all the teachers, Figure 6.30 shows that Breanna described mathematical reasons for many more tools than she gave a non-mathematical reason. Heidi and Melinda are the opposite, with more other reasons than mathematical. Melinda also gave several one word ("Yes") answers to the question, so she has a relatively high number of no reason responses. Nicole and Jana are about balanced equally between reasons that referenced mathematical content or actions or other reasons.

Only Breanna referenced the reason, challenging but achievable. She wrote twice, explicitly, "I

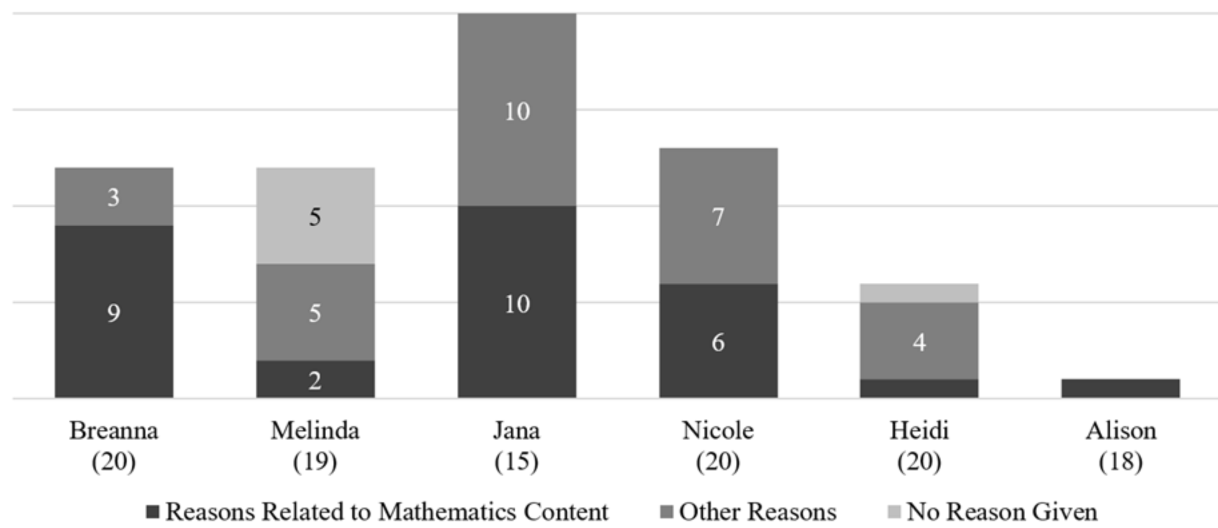


Figure 6.30: Reasons for Engagement

believe students would be engaged while playing the games within the app because the games are challenging but also achievable." In the other two instances, she instead mentioned a tool would be engaging for students who "like a challenge" or "don't mind a challenge."

Breanna, Jana, and Alison used the reason, creating or building. Breanna stated that a tool would be more engaging, if it allowed "them to build their own patterns or at least be able to manipulate patterns/objects." In her use of creating or building, Jana also referenced personalized learning as she wrote, "It also provides personalized learning by providing them opportunity to create and solve the problem." Alison wrote, "It is colorful and engaging as it allows students to build their own 3D objects." Across these three examples, teachers described creating or building mathematical objects (patterns and 3D objects) and mathematical problems.

Nicole and Jana used the reason, customizable or personalizable content levels or manipulatives. Jana wrote almost word for word, in all but one case, "It also provides personalized learning so that students will be engaged in their own learning process." For customizable or personalizable, Nicole wrote "It can also be customized to match a level that students want to play."

Nicole, Melinda, and Jana used the reason, ability to explore or discover, to explain a tool's engagement. Nicole wrote that a tool was "more of an exploration/discovery lesson, which will interest students" and that students "can explore and work at their own pace, which is likely to motivate students." In reference to exploration or discovery, Melinda mentioned "The tool is very engaging due to the options that it allows for exploration." Jana mentioned the ability for "little learners to explore the numbers in a welcoming way."

A reason of movement or interaction, was used by five teachers to explain engagement: Heidi, Nicole, Melinda, Breanna, and Jana. Breanna wrote that four of her tools were engaging because the student is constantly moving manipulatives to find answers or responding to the tool's movement to find answers. For example, she wrote "It is engaging by constantly have the student create number bonds to clear the board." Melinda wrote that the "movement and hands-on capabilities allow students to physically maneuver the shape" and "There is movement within the tool that will definitely keep the students engaged." When she used the reason, movement or interaction, Jana referenced "the competitive nature of the human beings" as contributing to their engagement in successfully answering mathematical questions: to move a canoe faster, to shoot a geometrical shape faster, and to dive at a target number faster. Nicole wrote about student actions, animations, hands-on capabilities, and descriptions of interaction. For example, Nicole wrote "students can edit/delete the data and witness what happens to the graph." Heidi wrote simply "The tutorial itself is engaging with the use of colors and movement through the process of this skill."

## Summary

In considering teachers' attention to types of affordances and constraints, or the relationship between aesthetics and engagement, supported by online tools, I attend to emergent characteristics that may help distinguish teachers as "experts" and "novices" across these aspects. In exploring teachers' responses to the affordances and constraints framework, and in reading their additional contributions to what they perceive as affordances or constraints of online tools, expert noticing would attend to mathematical aspects with clear rationale of how those aspects support mathematical thinking.

Nicole and Breanna rated most of their tools as displaying the "computing & automating" affordance; Breanna, Melinda, Nicole, and Alison rated tools with this affordance. All six teachers rated at least one tool as displaying the "representing ideas & thinking" affordance; Nicole rated almost every tool with this affordance. Heidi was the only teacher to rate tools using "accessing information." Nicole was the only teacher to use "communicating & collaborating."

With respect to aesthetics and engagement of online tools, expert noticing would acknowledge the complexity of aspects of a tool that support student engagement, but would also focus on aspects of the mathematics and mathematical representation that support engagement.

In attending to aesthetics and engagement, Breanna provided mathematical reasons for engagement more frequently than non-mathematical. Alison also provided mathematical reasons for the sole tool she described as engaging. Jana provided a balance between mathematical and non-mathematical, and Melinda, Nicole, and Heidi all focused on non-mathematical reasons for engagement.

Teachers described online tools as engaging when they allowed student interaction with movement (all six teachers), were challenging but achievable (Breanna), allowed student exploration or discovery (Melinda, Jana, Nicole), allowed students to customize mathematics (Jana, Nicole), or allowed students to create or build (Jana).

## Summary of Teacher Noticing

The purpose of investigating teachers' descriptions of these aspects of online tools was to answer the question *What mathematical characteristics, aspects, or features of digital tools and resources emerge through participants' noticings as evidenced through responses to explicit evaluation questions and open-ended evaluations?*

Figure 6.31 shows a listing of categories for comparing teacher noticing. On the left, a listing of categories for comparing teacher noticing from this chapter. Each pair of comparison terms represents a potential continuum between two preferences. Each pair describes an aspect of a teacher's level of sophistication or alignment with a conception. The pairs are organized into the two categories on the right.

| Levels of Sophistication | Conception Alignment |
|---|---|
| • Mathematical Learning Goals: *General vs. Specific*<br>• Symbolic Representation: *Implicit vs. Explicit*<br>• Visual Representation: *Non-Mathematical vs. Mathematical*<br>• Visual Representation: *Mathematical Context vs. Mathematics in Representation*<br>• Engagement: *Non-Mathematical vs. Mathematical* | • Mathematical Learning Goals: *Facts & Skills vs. Habits of Mind*<br>• Dynamic Representation: *Finding Answers / Identifying Object vs. Creating Objects / Basis for Reasoning*<br>• Dynamic Representation: *Feedback on Correctness vs. Mathematical Consequences*<br>• Mathematical Understanding: *Instrumental vs. Relational*<br>• Learning Activity Types: *Practice vs. Explore* |

Figure 6.31: Summary: Noticing Framework

Each emergent pair of terms indicates a potential spectrum for comparing teachers' noticing of that aspect of a tool. These comparisons are intended to suggest potential future directions for supporting teachers in developing their noticing. Some comparisons support levels of sophistication as described in the Review of Literature of this study; for example, teachers may attend to general goals more easily than specific goals as described by M. G. Sherin and Drake (2009). Teachers may attend to non-mathematical aspects of online tools before attending to mathematical aspects (B. Sherin & Star, 2011; M. G. Sherin & van Es, 2009). They may more easily attend to certain levels of mathematical structure (Chi et al., 1981; Schoenfeld & Herrmann, 1982). Following these hypotheses, I compared teachers on spectra with aspects that would potentially indicate less sophisticated noticing on the left and more sophisticated on the right in the list shown in Figure 6.31 and within the radar graphs shown in Figure 6.32.

Other comparisons may provide insight to a teacher's noticing as it aligns with a teacher's conception of the nature of mathematics (e.g., practicing as aligned with an Instrumentalist conception vs. exploring as aligned with a Platonist conception). Based on emergent categories, I loosely grouped aspects that potentially align with an Instrumentalist view of the nature of mathematics on the left and with Platonist on the right (in the list shown in Figure 6.31 and within the radar graphs shown in Figure 6.32).

Using these pairs of comparisons, I show in Figures 6.32 and 6.33 below two ways of summarizing findings described in this chapter.

## Levels of Sophistication

As described in the Literature Review chapter, researchers have used *expert* and *novice* labels in noticing research. In this study, I view this construct as more of a spectrum than a set of two bins; hence, I refer to *levels of sophistication.* Figure 6.32 shows a comparison of noticed aspects of online tools, attending to potential levels of sophistication between two sets of teachers, grouped according to their years of experience. The radar graph is based on the levels of sophistication aspects as listed in Figure 6.31 above. Moving in clockwise direction, on the upper and right-hand side of the radar graph, aspects of online tools that potentially indicate higher levels of sophistication are listed. Their counter-parts or opposites are listed on the bottom and left-hand side (starting with General Math'l Learning Goals). These aspects potentially indicate lower levels of sophistication. On the left radar graph, I compare Breanna, Melinda, and Alison who had 2, 3, and 4 years of experience, respectively. On the right radar graph, I compare Jana, Nicole, and Heidi who had 16, 5, and 6 years of experience, respectively.

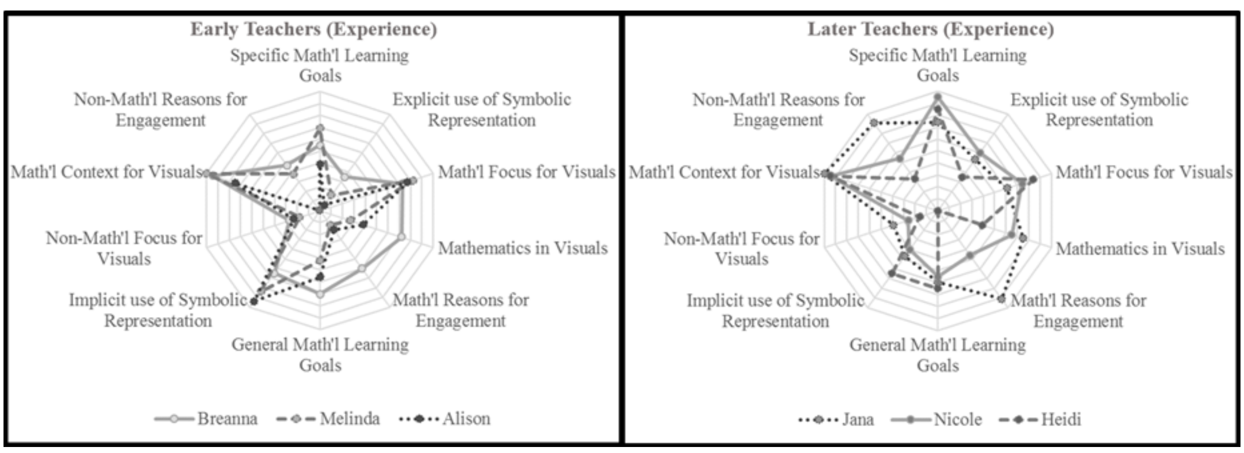


Figure 6.32: Noticing Framework: Levels of Sophistication

Figure 6.32 reveals some patterns. Both groups of teachers described general learning goals, but the more experienced teachers described more specific mathematical learning goals overall. The more experienced teachers used more explicit language in their attention to symbolic representation than the less experienced teachers. Breanna used explicit language for more tools than other members of her group, while Heidi used explicit language for fewer tools than other members of her group. Both groups focused on mathematical aspects of visual in their evaluations. Both groups described the larger mathematical context of visuals in tools, but the more experienced teachers described the mathematics in visual representations for more tools than the less experienced teachers, with one exception in each group. Heidi described mathematics in visuals for fewer of her tools, and Breanna for more, than other members of their groups.

Providing mathematical or non-mathematical reasons for student engagement with online tools varied: Heidi described mathematical or non-mathematical reasons for student engagement for few tools, while Nicole gave mathematical and non-mathematical reasons for about half of her tools and Jana for almost every tool. Melinda and Alison gave mathematical reasons for few of their tools. Breanna gave mathematical and non-mathematical reasons for about half of her tools, while Melinda gave non-mathematical for about half and Alison for none. This finding may indicate that some teachers (namely, Jana, Nicole, and Breanna) were more likely to notice any aspects (mathematical or non-mathematical) of online tools that would support student engagement, while some teachers do not (namely, Heidi and Alison).

## Alignment with Conceptions

Figure 6.33 shows a comparison of noticing, attending to potential alignment with conceptions of teachers, grouped according to their placements as lower elementary teachers (Breanna, Melinda, and Jana) or as upper elementary and middle grades (Nicole, Heidi, and Alison). I group the teachers by grade-level, because potentially they would be looking at online tools that are similar in nature because the online tools focused on younger or older students. The radar graph is based on the alignment with conceptions aspects as listed in Figure 6.31 above. Moving in clockwise direction, on the upper and right-hand side of the radar graph, aspects of online tools that potentially indicate an Instrumentalist conception of mathematics are listed. Their counter-parts, sometimes opposites, are listed on the bottom and left-hand side (starting with Goals: Habits of Mind) and potentially indicate a Platonist conception of mathematics.

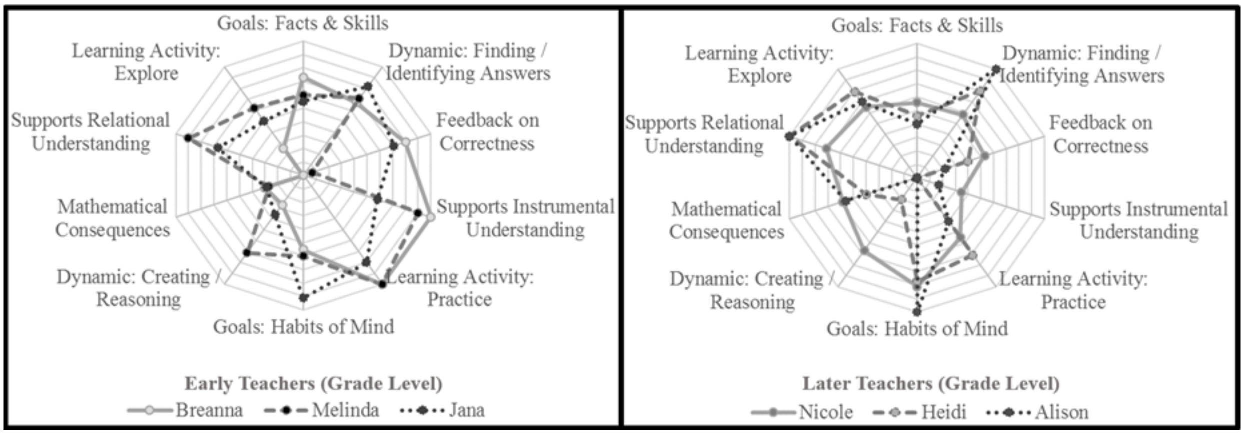


Figure 6.33: Noticing Framework: Alignment with Conceptions

Figure 6.33 shows that only Breanna and Jana were medium or high in all Instrumentalist aspects; only Nicole was medium or high in all Platonist aspects. Goals showed little variation. All teachers wrote facts and skills goals for around half of their online tools, while all teachers also wrote habits of mind goals for around half or more of their online tools. Jana and Alison wrote habits of mind goals for most of their tools. Descriptions of what mathematical actions students would perform varied. Alison, Heidi, and Jana wrote students would find answers or identify objects for almost all of their online tools (Nicole and Breanna about half), while Jana and Breanna described students creating objects or using tools to support reasoning for almost all of their tools (Nicole and Melinda for about half). Breanna, Jana, Nicole, and Heidi described the value of feedback in over half of their tools, while Nicole and Alison described mathematical consequences for about half of their tools. Considering almost all of their online tools, Breanna and Melinda described as supporting instrumental understanding (Jana about half); Alison, Heidi, and Melinda described as supporting relational (Nicole and Jana about half). Note that Melinda described most of her tools as supporting both relational and instrumental understanding. Finally, in considering teachers' use of learning activity types, Breanna and Melinda described most of their tools as supporting practice, while Heidi, Nicole, and Jana described about half as supporting practice. All teachers but Breanna described about half of their tools as supporting exploration.

## Conclusion

Findings from these descriptors are inconclusive, of course, without all teachers evaluating the same set of tools or without considering a sample with many more teachers. The goal of this exercise was not to validate the instruments, but to explore teachers' noticing of online tools and develop a theoretical framework for noticing. The framework for levels of sophistication seems promising, but the framework for conceptions needs further development. In the next chapter, I explore relationships between teachers' conceptions and teachers' noticing at a greater depth and propose a theoretical framework to explore those relationships.

# CHAPTER 7

# RELATIONSHIPS BETWEEN CONCEPTIONS AND NOTICING

I have shared findings of teachers' conceptions and teachers' noticing. In this chapter, I present potential relationships between teachers' conceptions of the nature of mathematics and mathematics teaching and learning and teachers' noticing of affordances and aspects of online tools. First, I share an extension of Figure 2.1, a framework of relationships between teachers' views of mathematics and their views of teaching and learning, that was based on relationships proposed by Thompson (1992) and Ernest (1989b). I then describe my revision of this framework based on teachers' statements, and share findings based on an analysis using the revised framework. Finally, I summarize the emergent relationships and refine the framework.

## Applying the Relationships Framework

In a first pass of data, I applied the Relationships Framework, shown in Figure 3.1, to teachers' statements from their online tool evaluations. As I considered the evaluations through the Relationships Framework, I noted that teachers often did not include enough detail to differentiate between categories. For example, in considering teachers' descriptions of online tools that presented mathematical ideas or algorithms, I noted that teachers' statements did not explicitly indicate whether they saw the tool as supplying a "clear presentation of facts or procedures" or a "conceptual explanation of meanings and relationships."

To prepare for a second pass, I returned to my interpretation of Figure 3.1, the Relationships Framework. Considering the meaning behind the categories showed two main categories of noticing compared across Instrumentalist and Platonist conceptions: first, attending to aspects of an online tool supporting teacher presentation or explanation, and, second, attending to aspects supporting student actions. Using NVivo 10 qualitative analysis software, I compiled participant statements that included teacher presentation words (explaining, presenting, demonstrating) and student action words (memorizing or practicing, and applying, discovering, or exploring) (QSR

International Pty, Ltd., 2012).

The presentation words (explaining, presenting, demonstrating) required context to classify their use as aligned with an Instrumentalist or Platonist conception; in a pass through the data, I found that few teachers provided adequate context so I focused on the student action words. Based on my understanding of the Relationships Framework, I categorized the student action words aligned with a conception as: Instrumentalist (memorizing, practicing) and Platonist (applying, discovering, exploring). To consider relationships between each teacher's noticing and her conceptions, and to avoid projecting my own interpretation onto teachers' meanings, I considered her explicit use of these student action words across her online tool evaluations.

Considering student action words aligned with an Instrumentalist conception (memorizing, practicing), Breanna used them for every tool she evaluated, while Melinda, Jana, Nicole, and Heidi used one of the two words for over half of their tools. Alison used them only for two tools. For use of Platonist student action words, Nicole used them to describe almost all of her tools. Heidi used them for about half of her tools and Jana and Melinda for about a third each. The word search, and reading teachers' statements that included such words, revealed inconsistencies in the ways that teachers used these words.

To explore participants' noticing in relation to their conceptions of mathematics, I focused on each teacher's statements in their use of the words listed above. I read each of their statements and summarized each using generalized key words. For example, Breanna wrote a statement about Math Playground: Kangaroo Hop: "I wouldnt mind letting my Kindergarteners explore this tool to practice fluency on identifying geometrical shapes." In the statement, I noted that Breanna used the word "explore" to describe how her students would "practice fluency" to "identify objects." As a second example, Nicole wrote statements describing NCTM Illuminations: Concentration: "Students play the game concentration and explore the concept of equivalency. Students practice matching whole numbers, shapes, fractions, or multiplication facts to their equivalent representations." In these statements, Nicole described "exploring a concept," "practicing identifying objects," and "practicing facts and representations." I kept teachers' original words for the most part, but generalized some terms: so "identifying geometrical shapes" (in Breanna's example) and "matching whole numbers, shapes, fractions, or multiplication facts to their equivalent representations" (in Nicole's example) both became "identifying objects."

Figure 7.1 provides a visual of frequencies of teachers' use of standardized themes based on their statements. Figure 7.1 shows themes and theme frequencies used by participants in descriptions of online tools that referenced Demonstrating, Explaining, Presenting, Memorizing, Practicing, Applying, Discovering, Exploring. I share these findings first, and then explain the revised framework that emerged from this activity.


Figure 7.1: Frequency of Words

The word cloud in Figure 7.1 illustrates some patterns in teachers' use of themes. For example, although Breanna, Melinda, and Jana each used "practicing" most often, they used other themes with different frequencies. Breanna used fluency, operations, skills, knowledge most frequently. Melinda used skills, objects, and understanding. Jana used skills, demonstrating, and explaining.

Similarly, Nicole and Alison used the theme *exploring* most often, while Heidi used *practicing* most often, with *exploring* almost as frequently. Nicole's top themes, after exploring, included *practicing*, *objects*, *representations*, and *operations*. Heidi's most used themes included: *operations, representations*, and *skills*. Alison's included *presenting*, *objects*, *explaining*, *practicing*, and *operations*. Not all statements from the word search were useful, because not all statements provided context or rationale for use of the words; for example, "good tool for exploration."

After summarizing each statement for Figure 7.1, above, themes emerged based on participants' statements, especially their explanations describing why a tool was instrumental but not relational, or was higher-level thinking but not lower-level thinking. As I read and coded statements, revised and then read and coded again, larger themes emerged. I describe first my focus on relationships between noticing and Instrumentalist or Platonist conceptions (excluding the Problem-Driven conception). I then share my revised framework of learning goals, learning strategies, and knowledge structures.

As I read teachers' statements, and referenced Thompson's (1992) and Ernest's (1989b) descriptions, consistent themes emerged that aligned with Thompson's (1992) and Ernest's (1989b) descriptions. The themes fell into categories of mathematical knowing: goals of learning (type of knowing that teachers expectd students to develop), strategies for learning (student actions that teachers expected students to engage in), and knowledge structures (fundamental form of the mathematics teachers described). To illustrate, I share statements from Heidi's evaluation of Math Playground: Fraction Bars. In one statement, Heidi wrote "Through this tool student[s] can explore the relationships of fractions." Her choice of words in this statement indicate her attention to a knowledge structure (relationships) and to *exploring* as a strategy for learning. In a second statement, Heidi wrote " It would be best used for practicing the skill of comparing fractions." which indicates her attention to *skill* as a knowledge structure. In a third statement, Heidi wrote "This tool supports a relational understand[ing]," which indicates her attention to *relational understanding* as a goal of learning.

In the sections below, I describe the framework and rationale for each of the three categories of mathematical knowing.

## Goals of Learning

In reading teachers' statements, I noted that they described intended learning outcomes in their online tool evaluations. I refer to these descriptions as *goals of learning*. Thompson (1992) and Ernest (1989b) described distinctions between teachers' expectations for student learning when teachers hold an Instrumentalist conceptions and when they hold a Platonist conception. For Instrumentalist and Platonist conceptions, I first remind the reader of the descriptions of learning and teaching from Thompson (1992) and Ernest (1989b). I then describe my selection of key terms based on emergent themes from the data. I use these key terms as indicators of teachers' attention to aspects of online tools that support mathematics teaching and learning aligned with an Instrumentalist or Platonist conception of mathematics.

### Learning Goals and Instrumentalist Conceptions

Ernest (1989b) described the expected outcomes of mathematical learning, through the lens of an Instrumentalist conception, to be "skills mastery with correct performance" (p. 251). From the context of the course, and definitions of teachers based on readings, I selected key words that indicate teachers' expectations of similar outcomes from their students' interaction with online tools. The key words are: instrumental understanding, fluency, and mastery. Please note that, despite the similarity of the terms, instrumental understanding is a construct developed by Skemp and the instrumentalist conception is a construct developed by Ernest: they are separate constructs.

I describe my rationale for choosing the key words instrumental understanding, fluency, and mastery as indicators of Ernest's (1989b) goals of learning that align with an Instrumentalist conception. I chose word *mastery* from Ernest's description, and describe my rationale for the other terms below.

Teachers read Skemp (1978) in Unit 1. As described in Chapter 6, teachers described their meanings of instrumental understanding in evaluations. For example, Breanna wrote "knowing the rules of the concept and knowing how to demonstrate it using mathematical situations." Melinda described instrumental understanding as basic knowledge of facts and procedures. In discussions, reflections, and evaluations, teachers similarly described instrumental understanding as a goal of learning in which students know procedures and are able to use them, and know basic facts and rules. I argue that the teachers' use of the term instrumental understanding as a mathematical learning goal indicates alignment with an Instrumentalist conception. I argue for this alignment because of the implication that students become skillful users of unrelated mathematical facts and rules that students accept as useful without understanding their logic or meaning.

In Unit 2, teachers read about *Adding it Up's* strands of mathematical proficiency (National Research Council, 2001). One strand is *procedural fluency*. Procedural fluency, as described in *Adding it Up* is one valid and valuable part of mathematical learning. When explaining the meaning of procedural fluency, teachers consistently referenced the quote "carry out procedures flexibly, accurately, efficiently, and appropriately" (p. 116 National Research Council, 2001). I use the term *fluency* rather than *procedural fluency* because teachers truncated the term and also used it more broadly to indicate identifying objects, matching representations, and other expressions of efficient and accurate skill in addition to carrying out procedures.

Teachers indicated that fluency comes after students have learned a fact or procedure. For example, Breanna wrote "Its basically a math drill/practice kind of applet that is really for students to gain fluency in answer addition math problems mentally." Within the context of the course and online tool evaluations, teachers consistently used the term *fluency* in a way that aligns this learning goal with *instrumental understanding.* In one evaluation, Nicole explicitly described her perception of relationships between these terms, writing in a reflection that she saw most of her group's tools focused on instrumental understanding rather than relational, explaining "my groups tech tools lent themselves to instrumental understanding rather than [relational]... the tools focused on fluency of operations or practicing of mathematical algorithms rather than exploring math concepts and relating them to other concepts learned." I argue that teachers' use of the term fluency aligns with Ernest's (1989b) description of learning goals through the Instrumentalist lens because the focus is on students' ability to use procedures or identify objects without a focus on student understanding of the procedures or facts.

## Learning Goals and Platonist Conceptions

From the perspective of a Platonist conception, argued Ernest (1989b), learning goals include "conceptual understanding with unified knowledge" (p. 251). In considering teachers' statements, I chose key terms: relational understanding, deeper understanding, and conceptual understanding.

In a similar vein to teachers' use of instrumental understanding as defined by Skemp (1978) to indicate learning goals that align with an Instrumentalist conception, teachers' use of *relational understanding* indicates learning goals aligned with a Platonist conception. Teachers' use of the term in the context of coursework reflected a meaning of relational understanding as a type of mathematical knowledge of the meanings or logic of concepts, procedures, and ideas. In addition to understanding the logic or meaning behind procedures, teachers focused on relational understanding as an understanding mathematical objects in a deeper or more complex way. For example, Nicole described relational understanding in the context of a tool as "understanding that a number or amount can mean more than one thing and can be represented in more than one way." In her description, Nicole captured a sentiment other teachers indicated as well: that relational understanding is more complex knowledge of a mathematical object that includes multiple perspectives. Similarly, Breanna explained a difference between the depth of definitions that would support instrumental and relational understanding. Breanna wrote that a tool supported instrumental understanding rather than relational, because it showed an example of a square, "but it does not fully break down the definition of a square. For example, a square and rectangle has four vertices. Therefore, real world objects that are squares and rectangles must also have four vertices." In her explanation, Breanna emphasized that relational understanding would involve more formal understanding of a definition than instrumental.

I include *conceptual understanding* as a term as it is used by Ernest. I acknowledge, however, that teachers' use of the term is was established by their reading of *Adding it Up's* strands of mathematical proficiency (National Research Council, 2001). In their use of the term, the teachers' meanings of conceptual understanding is, I believe, similar to Ernest's intentions for the term. For example, Breanna wrote that "Conceptual understanding is the comprehension of mathematical concepts, operations and relations." Teachers additionally used the term *deeper understanding* synonymously with conceptual or relational understanding; thus, I include it as a key term.

## Findings: Learning Goals

Figure 7.2 compares teachers' frequencies of referencing learning goals aligned with Instrumentalist or Platonist conception across their online tool evaluations. That is, learning goals aligned with Instrumentalist conceptions include Instrumental understanding, fluency, and mastery. Goals aligned with a Platonist conception include Relational understanding and deeper understanding. I give explicit examples of each below.

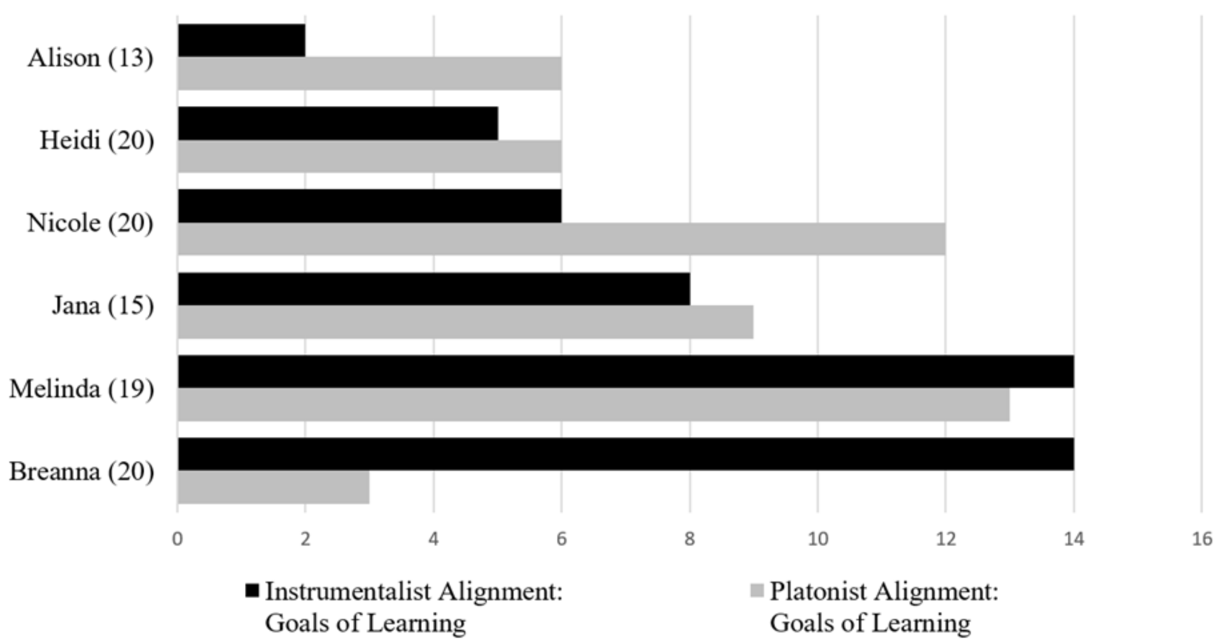


Figure 7.2: Learning Goals Comparison: Instrumentalist or Platonist

Figure 7.2 shows that Breanna attended to learning goals that aligned with an Instrumentalist conception in far more evaluations than Platonist. Nicole and Alison both attended to Platonist goals for far more tools than Instrumentalist. Melinda attended to Instrumentalist goals for one more tool than Platonist, while Jana and Heidi reversed that pattern.

## Strategies for Learning

In their online tool evaluations, teachers used phrases to reference particular strategies used in the tool to support students in achieving intended outcomes. I refer to these phrases as referencing *strategies for learning*. Thompson (1992) and Ernest (1989b) described distinctions between teachers' strategies for supporting student learning when teachers held an Instrumentalist conception and when they held a Platonist conception. For each of Instrumentalist and Platonist conceptions, I first remind the reader of the descriptions of teaching expectations described by Thompson (1992) and Ernest (1989b). I then provide a rationale for my use of particular key terms as indicators of teachers' attention to aspects of online tools that support mathematics teaching and learning in ways that align with an Instrumentalist or Platonist conception of mathematics.

### Learning Strategies and Instrumentalist Conceptions

Ernest (1989b) designated a teacher with an Instrumentalist conception of mathematics as *instructor*. In considering his description and the word's definition, Ernest's use of the term *instructor* implies giving knowledge to another through training or drilling. Thompson (1992) proposed that strategies for learning mathematics through the lens of an Instrumentalist conception included: clear presentation of topics and allowing students to practice identifying concepts and performing procedures.

In exploring teachers' statements, and referencing Ernest's (1989b) and Thompson's (1992) descriptions, I chose key terms that indicated strategies for learning that aligned with an Instrumentalist perspective: identifying or recognizing, memorizing or drilling, and presentation (modeling rules or procedures). I feel that these choices are in close agreement with Thompson's descriptions of strategies. I describe below my rationale and the meaning of the terms as they were used by teachers in the course.

In considering teachers' descriptions of how online tools would support student learning, especially when accompanied by expectations of learning outcomes that aligned with an Instrumentalist conception, I noticed that teachers described identifying or recognizing mathematical objects, expressions, or representations. For example, Melinda explained that this type of activity would support instrumental understanding rather than relational, "Instrumental understanding applies when students have to identify and understand the shapes, but relational comes into play when students have to apply what they know about the shapes." Hence, identifying shapes supports instrumental understanding (which I have argued is a learning outcome aligned with an Instrumentalist conception) while using that knowledge for more complex activity supports relational understanding (which I have argued is an outcome aligned with a Platonist conception.)

Teachers described online tools that supported drilling or memorizing as *drill and practice* or *drill and kill.* For example, Breanna wrote that a tool "possesses drill-and-practice methods for memorizing number recognition and other math related facts" and Melinda wrote that a tool was "meant to 'drill and kill' their skills." In these descriptions, teachers emphasized that students would gain fluency by the opportunities to drill on facts, skills, and solving problems. For example, Melinda wrote that a tool was "a great tool for drill-and-practice math problems and fluency of those math problems."

At times teachers described an online tool that showed an animation or other activity that could be used to demonstrate or model rules or procedures. Heidi described a presentation, "Students will view a visual and audio presentation that will demonstrate this method." Nicole described a teacher using a tool to present a method, writing that the teacher "models/explains using an overhead."

### Learning Strategies and Platonist Conceptions.

A teacher with a Platonist conception of mathematics was described by Ernest (1989b) as an *explainer.* Ernest's use of this term implies telling or showing another person the meaning of something; in particular, explaining implies clarifying an unclear idea or procedure. Strategies for learning mathematics through the lens of an Instrumentalist conception proposed by Thompson

(1992) included: emphasizing meanings of ideas and procedures, and emphasizing connections between mathematical concepts.

I chose the following terms to indicate strategies for learning that aligned with a Platonist conception, based on Ernest's (1989b) and Thompson's (1992) descriptions and my reading of teachers' statements: exploring or discovering, making sense or meaning, and critical or high-level thinking. I describe my rationale and meanings for these terms based on teachers' usage in their coursework.

First, I chose to focus on teachers' descriptions of students exploring or discovering because often those terms implied, or explicitly stated, support of learning goals that aligned with a Platonist conception. For example, Heidi wrote of one tool that it was suited for an exploration activity and later explained further that "students are exploring the relationships of equality through expressions and equations." Because she described students exploring relationships of a concept, her use of the term *exploring* implies that the meaning of concepts and their connections is the goal of the activity. This goal aligns with Thompson's description of activities that emphasize meanings of ideas and procedures. Because of this alignment between exploring and making meaning, I felt both were valid additions to key terms. Teachers described students exploring (or making sense of) relationships, ideas, concepts, and how procedures worked or had developed.

Another common theme for teachers in descriptions was the idea of students engaging in high-level thinking or critical thinking. Alison explicitly distinguished between mathematical *skills* and *thinking* when she wrote "It is more than just math skills though because students have to think backwards to find an original problem." This distinction implies some qualitative difference between an approach to mathematical learning as training skills (Instrumentalist) versus learning as developing mathematical thinking and understanding of meaning (Platonist). It may be debatable whether high-level or critical thinking is better aligned with a Problem-Driven conception (where we create mathematics in response to situations) as opposed to Platonist (where mathematics exists and we discover meaning). Based on teachers' use of the word *thinking*, I argue the latter is more appropriate. For example, Melinda tended to use the term *critical thinking,* and one example of her meaning is a description of students' critical thinking about "the amount that they need... and decipher between which coins combination they wish to use to match up to the price tag on each item." In this example, Melinda described as critical thinking that students make decisions based on their knowledge of coins, values of coins, and relationships between quantities and amounts. On the one hand, students may create their own strategies and approaches to making sense of the situation and, hence, the activity may align with a Problem-Driven view of mathematics. On the other hand, the activity is strictly bounded to a student responding to the tool's proposal of an amount to find. Students create their own strategies, but the activity seems more narrowly focused on students making sense of the coins and strategies than students investigating a complex or ill-defined situation, or a real mathematical problem. Hence, I argue that her use of critical thinking is more in alignment with a Platonist conception than Problem-Driven view.

## Findings: Learning Strategies

Figure 7.3 compares teachers' frequency of referencing learning strategies aligned with Instrumentalist or Platonist conceptions across their online tool evaluations. That is, learning strategies aligned with Instrumentalist conceptions include identifying or recognizing mathematical objects or facts, memorizing or drilling, and teachers' modeling of rules or procedures. Strategies aligned with a Platonist conception include exploring or discovering mathematical ideas, sense-making or making meaning, and critical or higher-level thinking. I give explicit examples of each below.

Figure 7.3 shows that Melinda attended to Instrumentalist strategies for more tools than Platonist; Breanna attended to Instrumentalist strategies for one more tool than Platonist. Jana and Heidi stated equal numbers of Platonist and Instrumentalist strategies. Nicole and Alison attended to Platonist strategies for more tools than Instrumentalist.

## Knowledge Structures

I have described above the ways that I have categorized teachers' descriptions of learning goals or outcomes and strategies for learning. In reading teachers' statements, I noted that the third

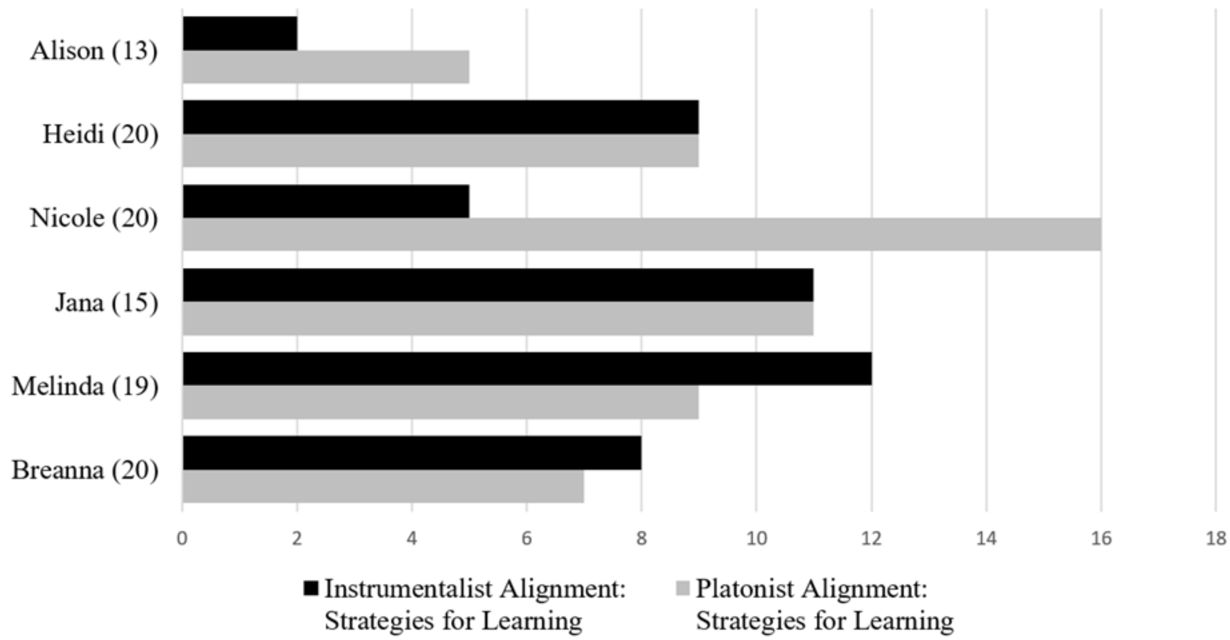


Figure 7.3: Learning Strategies Comparison: Instrumentalist or Platonist

form of knowing that teachers described was the type of knowledge structure that they wanted students to develop. Ernest (1989b) and Thompson (1992) did not explicitly address knowledge structures in their descriptions of alignment between teachers' conceptions of mathematics and of teaching and learning, however I argue that such structures are tied closely in their descriptions of the three conceptions of mathematics. For example, Ernest defined an Instrumentalist conception as including a view of mathematics as unrelated facts, rules, and skills. He described mathematics from a Platonist conception as consisting of structurally interconnected truths and concepts.

Based on that description, I chose as key words that would indicate knowledge structures aligned with an Instrumentalist conception of mathematics: facts, rules or procedures, and skills. And key words aligned with Platonist conceptions: ideas, and connections or relationships. As with the key words for learning goals or strategies for learning, teachers may intend different meanings for the terms, and so I considered the contexts of statements as well as the term itself. Based on context, teachers did use these terms fairly consistently: the former often combined with practice or drill and the latter with exploration and meaning. For example, Nicole wrote that a tool allowed "students to practice basic math facts" and another tool allowed students to match "multiplication facts with answers and diagrams." Initially, I included *concepts* with *concepts or ideas* because it seemed parallel to *facts*, but somehow more *conceptual* in nature. After reading teacher statements that included references to concepts, I removed the term. Teachers used it inconsistently to refer to mathematical constructs that could be: practiced for fluency, mastered, explored, investigated, connected, and more. Because of the inconsistency, I determined it was not a useful indicator of teachers' conception of mathematics.

## Findings: Knowledge Structures

Figure 7.4 compares teachers' frequency of referencing knowledge structures aligned with Instrumentalist or Platonist conceptions, as listed in Table 7.1, across their online tool evaluations. That is, knowledge structures aligned with Instrumentalist conceptions include facts, rules or procedures, and skills. Goals aligned with a Platonist conception include ideas, connections, and relationships.

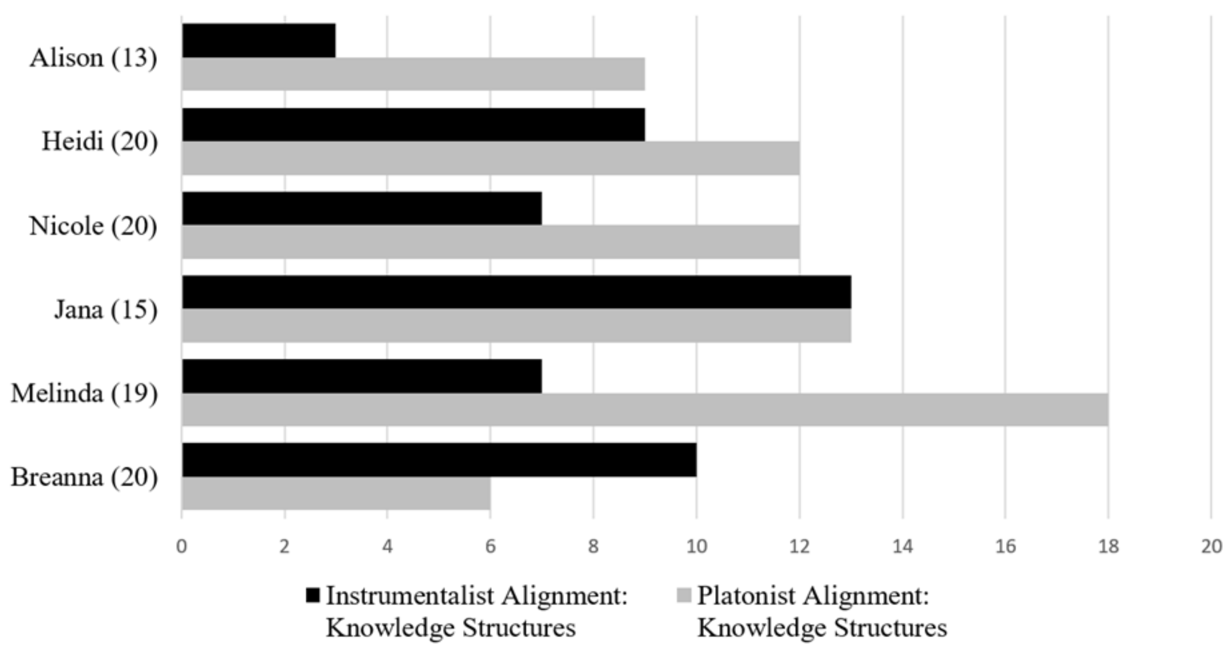


Figure 7.4: Knowledge Structures Comparison: Instrumentalist or Platonist

Figure 7.4 shows that every teacher except Breanna and Jana referenced knowledge structures that aligned with a Platonist conception for more tools than Instrumentalist. Jana was perfectly balanced, while Breanna had more Instrumentalist leanings.

## Summary of Relationships Between Conceptions and Noticing

Table 7.1 shows the finalized framework for the aspects of noticing: goals of learning, strategies for learning, and knowledge structures. The three aspects are described according to an Instrumentalist or Platonist conception, based on the sections above.

Table 7.1

*Finalized Alignment Framework: Conceptions and Noticing*

| Theme | Instrumentalist | Platonist |
|---|---|---|
| Goals of Learning | Instrumental Understanding | Relational Understanding |
| | Fluency | Deeper Understanding |
| | Mastery | |
| Strategies for Learning | Identifying, Recognizing | Exploring, Discovering |
| | Memorizing, Drilling | Sense-making, making meaning |
| | Modeling Rules or Procedures | Critical or Higher-Level Thinking |
| Knowledge Structures | Facts | Concepts or Ideas |
| | Rules or Procedures | Connections or Relationships |
| | Skills | |

*Note.* Brief description of aspects of teachers' noticing as they relate to Instrumentalist or Platonist conceptions of the nature of mathematics.

In the Chapter 5 summary, I concluded that Breanna and Nicole seemed to most clearly hold an Instrumentalist and Platonist conception, respectively. In the sections above, based on the aspects listed in Table 7.1, Breanna strongly leaned Instrumentalist and Nicole strongly leaned Platonist for goals of learning, strategies for learning, and knowledge structures. Melinda leaned heavily to an Instrumentalist conception, but used language that indicated aspects of a Platonist conception. Melinda showed some alignment with an Instrumentalist conception for the goals and strategies, but leaned strongly Platonist for knowledge structures.

Heidi leaned most strongly toward a Platonist conception, but used language that indicated leanings toward an Instrumentalist conception. Jana seemed closely aligned with a Platonist conception, and so did Alison, but Alison used language indicating aspects of a Problem-Driven conception. Jana and Heidi were perfectly balanced, or slightly leaning toward Platonist, for each of the three aspects. Alison strongly leaned toward Platonist conception for the three aspects.

Figure 7.5 compares teachers' frequency of referencing aspects listed in Table 7.1 aligned with Instrumentalist or Platonist conceptions across their online tool evaluations.

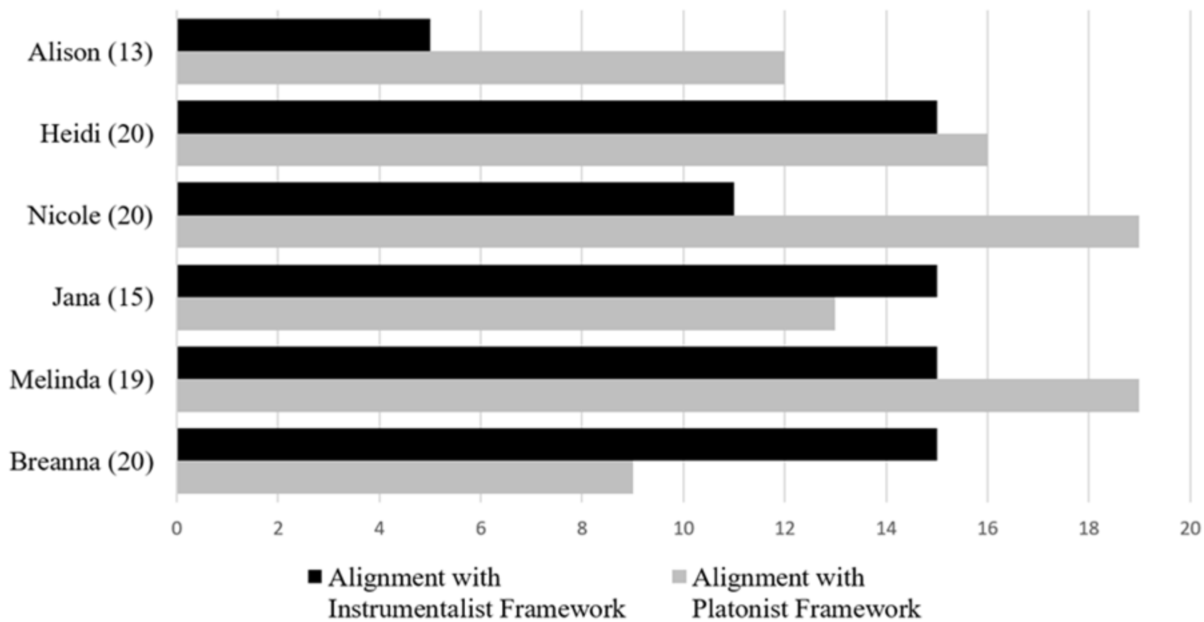


Figure 7.5: Comparison of Noticing: Instrumentalist or Platonist

Figure 7.5 shows that teachers' attention to aspects aligned with Instrumentalist or Platonist conceptions varied. As an overview, I give a brief description of these findings. Breanna attended to Instrumentalist aspects of more online tools than Platonist, and her Instrumentalist focus was consistent across learning goals, learning strategies, and knowledge structures. Melinda attended to Platonist aspects of online tools more than Instrumentalist. In referencing knowledge structures, she used terms that align with a Platonist conception for more than twice the tools than Instrumentalist. For learning goals and strategies, however, Melinda more frequently attended to Instrumentalist aspects.

Nicole attended to Platonist aspects for almost twice as many tools as Instrumentalist. She referenced Platonist learning goals and knowledge structures for about twice as many tools as Instrumentalist, but attended to Platonist learning strategies for many more tools than Instrumentalist. Alison consistently attended to Platonist aspects for more online tools than Instrumentalist; more than twice as many for overall aspects, learning goals, and knowledge structures.

Jana referenced Instrumentalist aspects for one more tool than Platonist, so she was nearly balanced in her attention overall. She was perfectly balanced in her attention to Instrumentalist and Platonist learning goals and knowledge structures, but attended to Platonist learning structures for a few more tools than Instrumentalist. Heidi attended to Platonist aspects for one more tool than Instrumentalist, so she was nearly balanced in her attention overall. She was perfectly balanced in her attention to learning strategies, but attended to Platonist aspects of learning goals and knowledge structures for a few more tools than Instrumentalist.

In 7.2, I show a summary of participants' alignments from each of the sections above, considering their attention to learning goals, strategies for learning, knowledge structures, and overall.

Table 7.2

*Summary of Noticing and Conceptions*

| Participant | Learning Goals | Strat. for Learning | Knowledge Structures | Overall |
|---|---|---|---|---|
| Breanna | Instrumentalist | Instrumentalist | Instrumentalist | Instrumentalist |
| Melinda | Instrumentalist | Instrumentalist | Platonist | Platonist |
| Jana | Platonist | Both equally | Both equally | Instrumentalist |
| Nicole | Platonist | Platonist | Platonist | Platonist |
| Heidi | Platonist | Balanced | Platonist | Platonist |
| Alison | Platonist | Platonist | Platonist | Platonist |

*Note.* Patterns in teachers' alignments between their noticing and conceptions, considering the aspects: learning goals, strategies for learning, knowledge structures, and overall.

Using the summary in 7.2 shows reasonable coherence in Breanna, Nicole, Heidi, and Alison. Melinda showed some alignment with Instrumentalist conceptions (in learning goals and strategies for learning) and with Platonist conceptions (in knowledge structures and overall). Jana leaned slightly Platonist for learning goals and showed equal attention to both conceptions for strategies and structures. Her overall leaning showed as Instrumentalist. These alignments are similar to my findings of teacher conceptions in Chapter 5, despite the complexity of conceptions and noticing.

# CHAPTER 8

# DISCUSSION

In this chapter, I briefly summarize my findings, presented in detail in Chapters 5, 6, and 7, relative to the research questions of the study. I first describe major findings of teachers' conceptions about the nature of mathematics and mathematics teaching and learning, as revealed through course discussions, individual reflections, and a post-course interview. I discuss issues that arose with respect to meanings of researcher constructs, and how revealing teacher meanings can help support future researchers and teacher educators. I then summarize findings about teachers' noticing of the mathematical aspects of online tools as evidenced by their descriptions in structured and unstructured evaluations. Next, I summarize what the study revealed about the relationships between teachers' conceptions and their noticing. In each of these three sections, I consider my results in the context of what other scholars have found and reported. I conclude with a description of contributions of the study, limitations, and future directions.

## Teachers' Conceptions

My first research question aimed to describe *teachers' conceptions of the nature of mathematics in the context of mathematics teaching and learning.* To address this question, I considered course and interview statements with respect to five aspects of the three conceptions of the nature of mathematics proposed by Ernest (1989b) and further described in Thompson (1992)—the Instrumentalist, Platonist, and Problem-Driven conceptions. Based on a close reading of their descriptions of these conceptions, the aspects of the nature of mathematics that I used for analysis were mathematical connectivity; mathematics as a collection of facts, skills, and rules; mathematics as problem-solving; mathematics being always right or wrong; and mathematics being always open to discussion and diverse views. The six focus teachers referenced these ideas in discussions and reflections during the first two units of the course. In individual post-course interviews, I asked teachers to work through a think-aloud card-sorting activity that included these aspects.

Before beginning the data analysis, I was prepared to see complexity in teachers' conceptions. On the one hand, Thompson (1992) described the likelihood that teachers, especially early in their careers, show inconsistencies in conceptions about mathematics and sometimes hold aspects in alignment with more than one of the three conceptions. On the other hand, Speer (2005) argued that teachers' beliefs are complex and mediated by language. Speer suggested that researchers sometimes assume incorrectly that meanings for words are shared, leading to additional confusion and complexity in analyses of teachers' beliefs.

In response to Thompson's (1992) and Speer's (2005) arguments, I examined teachers' statements for inconsistencies and for alternate meanings of words and constructs. I used their statements to understand their conceptions and make claims about alignment within Ernest's conceptions framework.

Overall, in the card-sorting activity, all six teachers agreed that most of the listed aspects could describe at least some part of mathematics. All six teachers agreed that mathematics consists of ideas that are all connected. At least three of the six teachers agreed that mathematics is created, discovered, continually changing, always the same, abstract, concrete, a collection of facts, skills, and rules, all about problem-solving, and always open to discussion and diverse views. As teachers indicated agreement with aspects, I noticed they accepted contradictory aspects at times. For example, Heidi and Breanna agreed that mathematics is always changing and always going to be the same. Alison and Melinda agreed that we create mathematics and we discover mathematics.

Teachers' acceptance of seemingly contradictory aspects could imply, on the one hand, that teachers had inconsistent conceptions of mathematics (e.g., Thompson, 1992). On the other hand, acceptance of contradictory aspects could imply that teachers used unanticipated meanings for words (e.g., Speer, 2005). If teachers' meanings of words differed from the meanings I had anticipated, then it could be reasonable for them to see no contradiction. That is, their meanings may have resolved the contradiction created by my meanings.

In the next few sections, I give a synthesis of these findings about teachers' agreement with constructs from Ernest's (1989) framework. I highlight emergent aspects of such constructs (e.g., connectivity) that I felt were essential in understanding teachers' conceptions of mathematics. I propose the implications for teacher educators and future research. On implication is that Ernest's framework is one lens on teachers' conceptions of mathematics; that it reduces the complexity of their conceptions in a way that can be helpful to researchers and to teacher educators—perhaps to the teachers themselves. The reductive nature of such a framework, however, can also be misleading. It necessarily simplifies a complex situation, which can lead to a deficit model of teachers' conceptions rather than a deeper understanding.

## Mathematics

I describe briefly some unanticipated lenses used by teachers in describing their views of the nature of mathematics in order to provide context to their descriptions of aspects of Ernest's (1989) conceptions of mathematics.

Before I introduced teachers to aspects of conceptions, they described definitions of mathematics in response to an open question: *Imagine all of mathematics... Describe mathematics to me.* Teachers' definitions clearly referenced different conceptions of the nature of mathematics that did not clearly fit into Ernest's (1989) framework of conceptions. Four teachers spoke of mathematics as everywhere, everyday, and an inescapable part of life. Three teachers spoke of mathematics as a way of thinking critically. Two teachers described a language or a way of communicating. Two teachers spoke of a way to support decision-making or problem-solving. Three teachers spoke of a way of making sense of the world. Although these conceptions are valid conceptions of mathematics, they are not explicitly aspects of Ernest's proposed conceptions.

Later, in the card-sorting activity, all six teachers spoke about mathematics as they sorted the cards and reasoned through their choices. In their reasoning, addition aspects of what they meant by *mathematics* emerged. For example, Heidi described mathematics as continually changing because student strategies change and because the content that is valued (e.g., appears in standards) changes. To explain why she felt mathematics is always the same, Heidi described that mathematics will always include problem-solving and will always be present in students' lives.

Her description of mathematics indicated her view of problem-solving strategies as a continually changing aspect of mathematics, even while problem-solving in general is an unchanging aspect of mathematics. Other teachers described similarly some elements of mathematics as changeable while other elements are always the same.

In the definitions they gave, all six teachers spoke of *disciplinary* mathematics; that is, mathematics as structures or patterns that emerge from real-world or abstract situations, useful in mathematics or in other disciplines. But all six also spoke of mathematics as what is valued or what is taught in *school* mathematics. Different teachers described different aspects; taken together they included as mathematics: the mathematical knowledge that is assessed, lists of content areas and topics in curricula and in state standards, mathematical habits of mind or practices, approaches to problems or solution strategies, and strategies for supporting mathematical thinking and development. Each teacher that I interviewed mentioned two or three of these conceptions of mathematics.

Teachers' descriptions of mathematics showed that they can see mathematics through different lenses. It is reasonable that aspects of mathematics change depending on the lens used. For example, aspects of mathematics when using a lens of *mathematics as standardized assessment* or *mathematics needed for success on an assessment* will not the same as when using a lens of *mathematics as a way of making sense of the world* or *mathematics as a way of communication.* Acknowledging the different conceptions of mathematics that teachers must internalize and navigate may help both mathematics education researchers and teacher educators. As researchers or teacher educators externalize and make visible multiple potential conceptions of mathematics, the process may help build more precise research instruments or learning experiences to make sense of how teachers switch between conceptions as they speak with researchers, teacher educators, colleagues, or students. The process can help researchers and teacher educators develop more precise language to talk about what mathematics means across different contexts and realities.

## Mathematical Connectivity

All six teachers agreed that mathematics consists of ideas that are all connected. In exploring statements made by teachers about mathematical connectivity, however, different conceptions about mathematical connectivity were revealed. Three large themes emerged from teachers' statements about connectivity: extent, nature, and utility. *Extent* of mathematical connectivity referred to whether some mathematical ideas are connected or mathematics is structurally connected such that all ideas, objects, or structures could be connected in many ways. *Nature* of mathematical connections referred to what specifically was being connected. Teachers referred to connections between concepts, operations, representations, standards (or large topic areas), disciplines, and between mathematical ideas and real world situations. Finally, teachers described the *utility* of connections, or the importance for students to see connections. One way that teachers described the utility was to take advantage of connectivity to make learning easier by building new mathematical knowledge on previous knowledge. Hence, students should be able to add layers on previously learned concepts or adapt old understandings to new situations. A second way that teachers described utility of connectivity was simply to support students' development of a conception of mathematics as interconnected.

Separating understanding of mathematical connectivity into extent, nature, and utility of connections can provide researchers and teacher educators ways to determine the robustness of a teacher's conception of mathematical connectivity. Teacher educators may tell teachers that all mathematical ideas are connected. Teachers may be interested when they see for themselves a particular connection between two mathematical ideas. An interesting result from this analysis was that the only time a teacher described mathematical ideas as isolated was when explicitly asked in the interview (and only one—Breanna—did so then). This lack of attention to isolated ideas may indicate that the teachers did not believe mathematical ideas are isolated, or (more likely) that isolated ideas are not interesting to point out. That is, counter-intuitively teachers may have made statements about mathematical connectivity because the idea is emphasized in standards but still new and visible enough that it feels interesting to point out when a connection is seen. Teacher educators may need to support teachers in internalizing the idea through exploration of different ways two mathematical ideas or objects can be connected and how different connections may reveal different aspects of mathematical ideas.

Future research could use extent, nature, and utility to examine further how teachers at different points in their careers may view connectivity in more or less sophisticated ways. Teacher educators could use these aspects to structure tasks that help teachers develop more sophisticated conceptions of connectivity by exploring individual connections as well as regularly reflecting on or discussing connections between concepts and areas of focus.

## Mathematics as Facts, Rules, and Problem-Solving

Although all six teachers agreed that there is more to mathematics than only facts and rules, they all also agreed that the collection of facts and rules is important for students. They all mentioned that knowing only facts and rules can block students' flexibility and higher-level problem-solving skills. I saw from teachers' statements that some focused on the need for students to know how to use facts and rules in their problem-solving, and some focused on the idea that correct use of facts and rules yields correct answers.

In their statements, all teachers indicated that students need to understand more deeply why and how facts and rules work to flexibly approach more complex or surprising problem situations. For students to develop this deeper understanding and flexibility, some teachers described asking students to solve problems in more than one way and to use more than one representation. Teachers also explained that communicating and higher-level thinking was important: Asking students to discuss their solutions, to explain or justify their approaches, and to engage in exploring, analyzing, applying, creating, and reasoning in their problem-solving activities were important activities to show that students could use what they knew.

The three lower elementary teachers agreed that students innately desire to understand mathematics more deeply and would enjoy mathematics more if they had opportunities to develop such an understanding. The three upper elementary and middle grades teachers agreed that, although they wanted to teach in such a way, their students resist this type of deeper learning. Alison (middle grades) wrote that, although she wished all her students could develop relational understanding, she did not believe that it was realistic. Heidi (upper elementary) wrote that her students regularly searched for, and found, rules to follow that allowed them to avoid relational understanding. She said her students showed boredom if she asked them to dig deeper. Similarly, Nicole (upper elementary) wrote that relational understanding was met with hesitation and resistance because her students did not see why they should understand the meaning behind something they already could do. Nicole stated her belief that only gifted students had the potential to learn relationally. She reflected that her gifted students, however, struggled when she gave them experiences that would support such learning. She said their frustration made her worry about turning these students away from mathematics. In contrast, Melinda (lower elementary) described her belief that relational understanding would help her struggling students make sense of mathematics by gaining the deeper meaning before understanding how to successfully use a rule. Four teachers wrote that they felt teaching deeper understanding is too difficult for both teachers and students. These teacher statements indicate apparent conflict between, on the one hand, expressing a conception of mathematics as solving problems using relational understanding and, on the other hand, expressing misgivings about students' abilities to understand mathematics relationally. This conflict could indicate a reasonable evaluation of classroom realities. Or, the conflict could indicate a teachers' concern about her own abilities to support students in successfully developing relational understanding. It could also indicate a teachers' movement from holding one conception (e.g., Instrumentalist) to another (e.g., Platonist). Thompson (1992) explained the potential for teachers to hold multiple conceptions at one time, especially early in their careers.

For teacher educators and researchers, understanding such different views of mathematical problem-solving may support discussion about the role, utility, and limitations of different views. Teacher educators already address these views held by preservice and inservice teachers by promoting group-worthy tasks (e.g., Lotan, 2003) and high cognitive demand tasks (e.g., Stein et al., 1996). Tasks such as these include mathematical connections, multiple entry points, and meaningful contexts (Lotan, 2003; Stein et al., 1996). Use of these tasks has been shown to support all students: traditionally high-performing students (e.g., Stein et al., 1996), students from high poverty areas (e.g., Isenbarger & Baroody, 2001), students with identified learning disabilities (e.g., Miller & Hudson, 2006), and English language learners (e.g., Wiest, 2008). This study confirms that it is essential to continue discussions about the realities of teaching and how to embed rich tasks despite or because of those realities. Teacher educators can support teachers in confronting contradictions between what research suggests students need, what teachers believe are useful aspects of the nature of mathematics, what teachers think is possible in their classrooms, and what an assessment or curriculum implies are necessary aspects. Such honest discussions may help teachers enact lessons that are more consistent with their internal beliefs, but they may continue to struggle with pressures of the reality in which they live. Some teacher educators have proposed ways to support teachers in *creative insubordination* (Gutiérrez, 2013), an idea that has been included in the recent *Association of Mathematics Teacher Educators* standards for teacher education (Association of Mathematics Teacher Educators [AMTE], 2017).

## Mathematics is Right, Wrong, or Open

From the interview, five teachers agreed that mathematics is always open to discussion and diverse views, and four teachers disagreed that mathematics is always right or wrong. In examining their statements from the course, teachers revealed much more focus on mathematical correctness than mathematical validity, and seemed to indicate a view of mathematics as always right or wrong. That is, in mathematics, a *correct* solution matches the answer key or matches the solution that a teacher expected. A *valid* solution, however, might not be what a teacher expects but could still be mathematically correct. For example, a different form of the same solution (e.g., 6/8 instead of 3/4) can be a valid solution. As a second example, base assumptions made about mathematical problem can also result in different, but valid, solutions (e.g., “fair sharing” in a mathematical problem can mean “equal sharing” or it can mean “equitable sharing,” resulting in two different but valid solutions). In their evaluations, for example, some teachers expressed concern when an online tool did not tell a student whether an answer was correct or incorrect. The question of mathematical authority with respect to correctness, or *who decides* whether an answer is right or wrong, yielded three basic views: The answer key determines correctness, students self-check their own answers, or students must prove answers or analyze approaches. The focus on an answer key excluded students, and even teachers at times, from the authority of determining mathematical correctness. Some teachers described students checking their own answers, or even proving answers and analyzing approaches. These latter views seemed closest to acknowledging mathematical validity. Some teachers described requiring students to discuss solution strategies with the teacher or with each other for different reasons: to enable the teacher to understand and correct the student's thinking, or to enable students to understand each other's thinking. In describing how mathematics can be open to discussion or diverse views, teachers expressed support for multiple strategies and solution forms (or solution representations). No teacher explicitly stated that they would be open to multiple interpretations of a problem situation that would lead to multiple valid solutions.

For researchers and teacher educators, understanding the struggle teachers may be engaged in with the authority of curricula and assessments may support profitable discussions about mathematical validity. Teacher educators could support teachers with experiences that allow them to confront correctness of multiple solutions for a problem, when correctness depends on initial assumptions and must be mathematically and contextually valid. Such experiences could support teachers in acknowledging students' valid solutions by allowing discussion of assumptions and validity. In this study, teachers expressed concern that students would not know when an answer was correct or incorrect. Indeed, supporting teachers in challenging mathematical correctness can be dangerous because they still must accept the correctness imposed by standardized assessments. Researchers and teacher educators already advocate student discussion of strategies and solutions (e.g., Stein & Smith, 2011). I argue that teacher educators could explicitly address this concern in rigorous mathematical discussion about mathematical *wrongness* compared to mathematical *validity.* Future research could explore how to best support teachers in developing their sense of correctness and validity through use of complex tasks and discussion, and through exploration of mathematical consequences. In addition, future research could explore how best to support teachers to support students in similar ways, as they navigate mathematical validity and assumptions made by curriculum or assessment creators.

## Conclusion

Supporting explicit teacher discussion about different meanings of mathematics and aspects of mathematics could help them refine their own conceptualizations and develop precision in their use of language with other teachers and with their own students. Indeed, Heidi requested access to the card sort activity at the end of her interview. She explained that she was leading a professional development session with her colleagues and thought it would be a helpful activity in directing their thinking and discussion about their own meanings of mathematics.

Research has shown that, although the influence is complex and difficult to precisely identify, teachers' beliefs and conceptions are important to research because of their influence on teachers' planning and practice (Philipp, 2007; Brown, 2009; Schoenfeld, 2011). Speer (2005) argued that there may be times when researchers' assumed meanings contribute to revealed inconsistencies between teachers' expressed beliefs and their practice. This analysis has confirmed that teachers' meanings for taken-as-shared terms such as mathematical connectivity can differ from researcher expectations. Rather than viewing the alternate meanings as invalid or less sophisticated than my own, I found that teachers' meanings were rich and complex in nature. As I explored mathematicians' meanings (e.g., Harel, 2008; Whitehead, 1911), teacher educators' meanings (e.g., Ball et al., 2001; Stein et al., 1996), and teachers' meanings (in this study), I saw that the nature of mathematics is simply complex. I noticed that teacher statements had many commonalities with mathematicians' and teacher educators'. Teachers' statements indicated they engage with the construct of *mathematics* in multiple contexts, expressing multiple meanings individually and across the group of six teachers. I argue that this complexity is inherent in the construct itself, and that teachers' expression of multiple meanings shows a rich, diverse, and contextually-dependent understanding of the term. This study also supported that teachers' conceptions exist in environments they do not control, within which teachers may feel unable to make choices that align with their conceptions. For example, teachers described their frustration with external pressures such as standardized assessment designed to focus on facts and rules rather than habits of mind.

For this construct, and others, I proposed frameworks to express these ideas in ways that can support both research and teacher education. The creation of any framework must simplify the complexity of a messy, complex situation in order to make sense of useful aspect(s) of that situation. The simplification reveals some patterns and structures otherwise lost in the complexity, but it also obscures other patterns and structures. Assumptions about the utility of a framework are based on an initial understanding of the situation and intended use of the framework. Understanding conceptions of the nature of mathematics or different lenses through which to view its nature can help both researchers and teacher educators understand teachers' decision-making in practice. Using the emergent frameworks from this study can likewise allow researchers and teacher educators to refine their language to be more precise in discussing teachers' understanding of aspects of mathematics. Researchers can use and build on frameworks that emerged from this study focused on connectivity, facts, rules, and problem-solving, authority for correctness and validity, and openness of mathematics. The detail allowed by use of these additional frameworks move the study of teachers' conceptions beyond what Ernest (1989b) and Thompson (1992) presented. Future research could focus on testing these frameworks and their utility in predicting teachers' decision-making. Teacher educators can use these frameworks to identify strengths and needs of teachers to build experiences and tasks that help teachers develop their understanding of these aspects of mathematics.

The danger in using any framework is the potential to forget that, while it reveals some aspects, it obscures others. I argue that it may be essential to step back from conceptions framework proposed by (Ernest, 1989b) in order to examine the many meanings of the nature of mathematics that teachers must use in order to navigate between their personal stance and the stances of researchers and teacher educators that engage them, as well as the stances of creators of standards, curricula, and assessments. Examining these meanings and making them visible to teachers may help them be more intentional as they develop experiences to support their students' development of similar meanings. I argue that reducing the complexity of the different faces of the nature of mathematics, without balancing the reduction with consideration of the overall complexity, does teachers a disservice.

## Teachers' Noticing

The second research question focused on *mathematical characteristics, aspects, or features of digital tools and resources that teachers noticed.* To answer this question, I analyzed four main aspects of the tools and resources that teachers examined: mathematical focus and representation, intended mathematical understandings, types of learning activities, and aspects of tools that supported learning. I used the structure of the whole class evaluation template used in the course to choose and analyze these areas. I first summarize findings from the four areas, and then discuss implications for teacher educators and future research.

### Mathematical Focus and Representation

In the evaluation template, teachers were asked: *What mathematics is being learned? How is the mathematics represented?* Teachers responded to these questions in different ways, in both structured and unstructured evaluations. In considering teachers' attention to the potential mathematical focuses of an online tool, I examined their statement of goals of the tool. I categorized their goals using level of focus (general or specific), type of goal (performance or learning), and nature of knowledge (facts and skills or habits of mind). All teachers wrote multiple goals for student learning based in the context of individual tools, and often those goals represented coherent statements of intended learning opportunities. Teachers with fewer years of experience (Breanna, Melinda, and Alison) tended to write general learning goals for tools, while more experienced teachers (Jana, Nicole, and Heidi) wrote specific learning goals. The attention to specific goals by the more experienced teachers may result from their greater experience in teaching and planning lessons or analyzing mathematical content.

In considering teachers' noticing of mathematical representation, teachers attended much more to symbolic, visual, and dynamic aspects of representation than to tabular, diagrammatical, or verbal representations, so I chose the former three aspects as a focus for exploring their noticing. In considering symbolic representation, I found that more experienced teachers explicitly referenced symbolic representation, while newer teachers rarely did. Teachers indicated different levels of awareness of the distinction between the symbolic representation of quantity and the quantity itself as an abstraction. For example, Heidi indicated her awareness of the abstraction *fractions* as distinct from their symbolic representations through use of precise language that explicitly referenced *symbolic representation*, "Mathematics is represented through models and symbolic representations of fractions. Students are manipulating math symbols to make true mathematical number sentences." Breanna used less explicit and less precise language in a description of representation, stating that students would "quickly pick the math problem that displays the same problem on the number line." This analysis of symbolic representation provides some insight of a teacher's use of precise mathematical language and her level of sophistication in understanding representation. The finding that Jana, Nicole, and Heidi, the more experienced teachers, more frequently used explicit language and distinguished between a quantity and its representation may simply indicate that this noticing develops through experience.

Regarding any visual features (including but not limited to mathematical visual representation) of online tools, all teachers referenced visual aspects explicitly and explained mathematical aspects of these visual features for most, if not all, of their online tools. *Visual representation*, as used by teachers in their evaluations, is a broad term that may include images of concrete objects (e.g., beach balls), images of manipulatives (e.g., base-ten blocks), graphical representation, or diagrammatic representation. In describing visual features of online tools, all teachers gave a mathematical context for the visual features of most of their tools. Breanna, Nicole, and Jana consistently described the mathematics revealed through use of the visual representation of most tools. For example, Breanna described in detail how images of base-ten blocks in NLVM: *Base Blocks Addition* were connected to the symbolic representation of operations embedded in the addition algorithm. This finding is interesting because these three teachers have different levels of experience and teach in different grade band levels. The analysis highlights multiple aspects of visual representation that teachers must attend to multiple aspects of visual representation. In addition to content focus and mathematical validity of representations, teachers in this study indicated the pressure they felt to attend to pedagogical appropriateness and aesthetic aspects.

In considering dynamic aspects and representation in online tools, I explored teachers' descriptions from multiple perspectives. I found that most of the six teachers attended to interactions that students might have with aspects of dynamic representation in the tool. Tools with dynamic representation may allow either active interaction with the user (e.g., virtual manipulatives) or more passive experience (e.g., instructional videos). All teachers focused on interactive aspects or described how students might interact actively with the representations for most of their tools. As teachers described the dynamic interaction, they expressed expectations of students' interaction and expectations of what the tool ought to provide. I summarized these findings according to how students would *do* mathematics, *interact* with mathematics in the tool, or *learn* mathematics, while the tool would provide feedback and clear learning opportunities. I found that five teachers all described opportunities for students to find answers or identify objects for most of their tools for which they described students doing mathematics. Only two teachers focused on opportunities for students to create objects or questions, or to use the online tool to support mathematical reasoning. In considering teachers' attention to online tools providing mathematical feedback, I saw that teachers described feedback as a measure or indication of correctness in most cases. Some teachers described feedback as a response by the tool to a student action; in this case, such feedback can be called *mathematical consequences.* Five teachers described mathematical consequences for at least one tool; only one teacher described mathematical consequences consistently for the tools she evaluated with such features.

This analysis provides confirmation of the complexity and messiness that teachers must attend to as they evaluate online tools for alignment with learning and performance goals and for the appropriateness and validity of mathematical representation. In this study, I saw that teachers attended to multiple aspects of representation that included the general aesthetics of representations, appropriateness to grade level and topic, mathematical validity, appropriateness of interaction or explanation, as well as feedback and and mathematical consequences. Because teachers in this study attended to these multiple features of each online tool, this analysis is a confirmation of need for teachers to develop complex knowledge of the interaction between technology, mathematics, and pedagogy (e.g., Koehler, Mishra, Kereluik, Shin, & Graham, 2014). For teacher educators, the analysis and frameworks that emerged from the data may support teachers in deeper analysis of coherence between pedagogical goals for an online tool and its use of mathematical representation and interaction types, especially in consideration of the tension between immediate feedback for correctness and fidelity of mathematical consequences. Teacher educators already create experiences for teachers to engage in productive discussion of mathematical consequences, with correctness and its meaning a part of the discussion (e.g., Stein & Smith, 2011). Online tools that allow users to make any choices and confront mathematical consequences of those choices can support such discussion. Such discussions could allow teachers to become more intentional about when to use online tools that provide immediate feedback and when to use tools that allow engagement with consequences.

For researchers, this analysis of goals and representations supports insight into the complexity of teachers' knowledge types interaction (e.g., Koehler et al., 2014). Separating the knowledge types that teachers draw on, however, may inappropriately reduce the complexity. For example, a teacher considering a dynamic representation would need to be confident and flexible in drawing on the different types of knowledge quickly to support her internal sense of whether it is mathematically accurate and valid, will work for her students, and whether it will support a particular goal. Future research could consider how the evaluation structure can better support teachers in thinking through their evaluative process, especially in supporting deeper mathematical analysis.

## Intended Mathematical Understandings

Teachers had multiple opportunities to consider each online tool's support for developing mathematical understanding. Class evaluation templates included a question about instrumental or relational understanding, and then gave teachers the option of choosing a framework as a second question about understanding. For the first choice, teachers made a claim about an online tool's support of instrumental or relational understanding. The teachers often provided justification for their choice. Melinda chose both relational and instrumental understanding as supported by nearly all of her tools, while Breanna chose instrumental understanding and Heidi chose relational for almost every tool. Through their descriptions of instrumental or relational understanding, and how online tools might support such understanding, teachers showed they perceived tools focused on memorization or step-by-step procedures as supporting instrumental understanding while tools that supported creativity, reasoning, or higher-level thinking as supporting relational understanding.

In response to the second question, teachers chose from *Adding it Up's* proficiency strands (National Research Council, 2001), *Principles and Standards of School Mathematics' (PSSMs')* mathematical process standards (National Council of Teachers of Mathematics [NCTM], 2000), or *Common Core State Standards in Mathematics' (CCSSMs')* standards of mathematical practice (Council of Chief State School Officers [CCSSO], 2010). This second question was less informative than the first, because teachers did not consistently choose one framework to use and rarely provided a rationale for aspect choice.

## Learning Activity Types

In use of class evaluation templates, teachers were asked to choose a particular learning type that they felt the tool would support: presentation, practice, exploration, or application. Teachers rarely justified their choices. Breanna was unique in choosing the learning activity type *practice* for almost every tool she evaluated, and *explore* for very few. Other teachers seemed more balanced, often choosing both *explore* and *practice* for the same tools.

## Affordances of Online Tools and Student Engagement

Teachers described affordances of online tools and rated tools as supportive of student engagement, or not. For both affordances and engagement, teachers wrote descriptions with both mathematical and non-mathematical focuses. The instructors provided an initial set of affordances for teachers to choose. Although they did choose affordances provided by the instructors for some tools, these choices were rare. More common were affordances suggested by teachers. Some affordances were non-mathematical and included descriptions of the tool as: entertaining or fun, attractive, easy to use, familiar or comforting, friendly, providing additional teacher resources, and recording student progress. Teachers wrote implicitly about certain mathematical affordances, such as that the tool: allowed student practice, gave feedback or hints, allowed students to work independently, and provided students with choice and differentation. Teachers proposed explicitly mathematical affordances, such as Nicole describing the way a tool supported a particular fraction form.

Similarly, teachers suggested a number of possibilities for what aspects of online tools make them engaging to students. These reasons ranged from non-mathematical to mathematical in nature. As examples of non-mathematical aspects, teachers listed that the tool: was aesthetically pleasing, allowed competition or able to play with other person, was easy to use, provided encouragement, had a fun or a game aspect, was not distracting (e.g., no advertisements flashing), used good sounds, and allowed a student to work at their own pace. Teachers listed reasons that tended to be more mathematical as well, including that the tool: was challenging but achievable, allowed a student to create or build, allowed students to customize or personalize aspects, allowed exploration or discovery, and included movement or allowed interaction.

## Conclusion

As I considered teachers' evaluation statements addressing these subquestions, I also considered teachers' potential levels of sophistication and alignment with conceptions spectra. My goal was not to locate them on an objective spectrum for precise measurement. Instead, I considered emergent characteristics of noticing that may be used to distinguish teachers as they attend to different aspects of online tools. I created radar graphs, shown as Figures 6.33 and 6.32. These graphs allowed comparison of teachers' noticing across several aspects of tools (Chi et al., 1981; Schoenfeld & Herrmann, 1982; M. G. Sherin & Drake, 2009; B. Sherin & Star, 2011; M. G. Sherin & van Es, 2009), and showed promising variation that corresponded with expectations from noticing research or that corresponded to descriptions of alignment between conceptions and teaching practice (Thompson, 1992; Ernest, 1988).

For researchers, these initial frameworks can support future testing and development for more robust frameworks of levels of sophistication and alignment with conceptions. The analysis suggests additional research questions about potential gaps between teachers' expectations of online tools and how students interact with the tools or what students gain from the tools in practice. Following findings of M. G. Sherin and Drake (2009) that, in adapting curricular materials, teachers noticed different aspects and anticipated different adaptations before enactment and after enactment. Sherin and Drake claimed that teachers became more cognizant of the mathematical content and how it affected students after enactment. Future research could focus on teachers' evaluations of online tools before using them in the classroom and evaluations after enactment, especially in examining their attention to mathematics content and representation. Based on this research, I expect that teachers will be better equipped to notice aspects that support their students' learning after enactment than before.

In this study, teachers chose their own online tools to evaluate. Rarely did teachers choose the same tools, which makes it difficult to compare the teachers' noticing in meaningful ways. Future research could ask teachers to evaluate the same tools, which would allow for easier comparison. Teachers in this study came from multiple grade levels and online tools often were targeted for particular concepts useful in one grade level. Asking the teachers to evaluate the same tools would make the tools less relevant for some teachers, and those teachers would reasonably struggle to evaluate a tool that would not be relevant to their own students. Indeed, this struggle was seen in the pre- and post-test evaluations where teachers chose to evaluate a set of tools that was intended for a lower grade level than their own. To address these difficulties in choice of tools, researchers could limit research to groups of teachers within one grade band (e.g., lower elementary). Researchers could develop sets of comparable tools within each grade band to consider (a) what tools are available to teachers at different grade levels and (b) how teachers across grade bands attend to comparable aspects.

For teacher educators, this study is relevant in how teachers used the evaluation frameworks to internalize evaluation of online tools and how the teachers' experiences with those frameworks might be adapted for different learning goals. We, the co-instructors, had hoped teachers would analyze the mathematical content of online tools in a deeper way than we saw. In examining teachers' evaluations and the structure of the evaluation template, especially for their mathematical noticing, I saw that questions in the template directed teachers to particular aspects. At times these questions asked teachers to be evaluative, but most questions asked teachers to be descriptive. For example, teachers indicated types of learning activities supported by an online tool, but the question did not ask teachers to determine whether the type of learning activity was useful, appropriate, or effective. In reflecting on the course, I imagine teachers may have been overwhelmed with the frameworks that we asked them to engage with and respond to in their evaluations. They encountered the frameworks within the templates, used them multiple times, and were asked to reflect on them, but I wonder if the teachers could be better supported as they gain familiarity with the frameworks to keep thinking about them in more and more sophisticated ways. If I were to revise the course, I might try to scaffold teachers' engagement with the different frameworks by amending the evaluation template so that, first, they use them in descriptive ways, but then move toward more critical or evaluative stances.

We could also show them examples from previous courses of sections of teacher evaluations and allow the current teachers to critique and build on those responses. For example, continuing with learning activity types, teachers might read that a certain online tool was tagged as an *exploration* tool. The teachers might use the tool and then explain why they felt it does support exploration or what they might change. This experience could be done individually and then through discussion. A second way to support teachers in moving toward more evaluative stances could be to change the questions from unit to unit. For example, starting with *what learning activity type and why?* as a descriptive question. Then in the next unit, teachers might answer *explain how the learning activity type supports (or not) the mathematical goals you listed.* In another unit, teachers might respond to *assuming the creator of this online tool asked you how you would revise it, what learning activity type would you change it to and why?* Teachers might even use the new questions to revisit previously written evaluations, to revise those evaluations in each unit. The revision could give them opportunities to notice new aspects of online tools and to notice how their critical eye is developing.

## Relationships between Teachers' Conceptions and Noticing

To answer the third research question, to describe potential relationships between teachers' conceptions and their noticing of aspects of online tools, I considered findings from Chapters 5 and 6. I explored two frameworks for potential relationships between noticing and conceptions in this study. In the summary of Chapter 6, I sketched a potential framework considering alignments between noticing and conceptions. I briefly describe this framework, and its finding and implications, here.

I then summarize findings from use of a second emergent framework. In Chapter 7, I revisited teachers' evaluation data and considered their use of language in describing tools. From that analysis, I determined three potential categories to build a framework for relationships between conceptions and noticing. The three categories, or aspects of online tools, were: learning goals, strategies for learning, and knowledge structures.

### First Alignment Framework

In the summary of Chapter 6, I compared noticing findings across a subset of tool characteristics. Based on impacts of Instrumentalist and Platonist conceptions on teachers' practice described by Ernest (1989b) and further described in Thompson (1992), I claimed that certain tool characteristics aligned with one of the two conceptions. For the Instrumentalist conception, I claimed that a teacher would focus on *facts and skills* goals for online tools, support of *instrumental understanding*, *finding or identifying answers* interactions with dynamic representation, *feedback on correctness* affordances, and *practice* learning activity types. In contrast, aspects including a focus on *habits of mind* goals, support of *relational understanding*, *creating or reasoning* interactions with dynamic representation, *mathematical consequences* affordances, and *explore* learning activity types would align with a Platonist conception. I found that Breanna and Heidi most clearly showed alignment; that is, they attended to aspects aligned with Instrumentalist and Platonist conceptions, respectively.

It is possible that this finding confirms that holding an Instrumentalist or Platonist conception does impact what a teacher notices in online tools. If so, then this finding indicates that teacher decision-making about the use of online tools can be impacted by their conceptions of the nature of mathematics. I suggest future research test these potential alignments when teachers evaluate the same set of tools. Future research should also explore whether teacher education that supports teachers in developing Platonist or Problem-Driven conceptions impacts what they notice in online tools.

Breanna and Heidi showed the strongest alignment with a single conception in the Chapter 5 analysis. Other teachers' conceptions were less clear. Because of the lack of clarity, it may not be surprising that their noticing showed little alignment with one conception over another. That is, because I found indications that teachers held to aspects of more than one conception, it may not be surprising that their noticing would align with more than one conception. The seeming lack of alignment, thus, may be reasonable because teachers themselves were not clearly aligned with one conception. Hence, the lack of alignment could also indicate that the emergent framework would only be useful for teachers that clearly and consistently held one conception. Based on this study, where only two of the six teachers showed a strong alignment with a single conception, it may be that the alignment framework is not useful in understanding teachers' noticing through the lens of their conceptions of the nature of mathematics. It could also indicate problems with the framing and choice of conceptions.

## Second Alignment Framework

Because teachers' conceptions may not generally align with either an Instrumentalist or Platonist conception, the previous framework may not be useful for future researchers or teacher educators. Because of teachers' lack of alignment with one conception, the results from the first framework did not provide deeper understanding of the way teachers' conceptions may impact teachers' noticing and, hence, their decision-making about use of online tools in mathematics instruction. In order to find a framework that might provide more utility in understanding teachers' decision-making, I returned to teachers' data. I examined statements teachers made in their evaluations. I chose key words related to each of three core issues (learning goals, strategies for learning, and knowledge structures) based on descriptions of the impact of different conceptions on teachers' practice found in Ernest (1989b) and Thompson (1992).

### *Learning Goals*

In considering learning goals written by teachers in their evaluations, I distinguished between learning goals aligned with Instrumentalist conception and those aligned with Platonist conceptions. For learning goals aligned with an Instrumentalist conception, I focused on key words: instrumental understanding, fluency, and mastery. For learning goals aligned with a Platonist conception, I focused on key words: relational understanding, deeper understanding, and conceptual understanding. In examining teachers' evaluations, I found that the learning goals of Breanna aligned with an Instrumentalist conception, while those of Nicole and Alison aligned with a Platonist conception. Other teachers' use of terms was nearly equal between the two conceptions, with Melinda more aligned with Instrumentalist, and Jana and Heidi more with Platonist conceptions.

### *Strategies for Learning*

Teachers described how students would learn through use of each online tool in different ways. For strategies in alignment with Instrumentalist conceptions, I chose key terms (or descriptions): identifying or recognizing, memorizing or drilling, and presentation (modeling rules or procedures). For the opposing Platonist conception, I chose terms (or descriptions): exploring or discovering, making sense or meaning, and critical or high-level thinking. For strategies for learning, only Nicole and Alison, and possibly Melinda, used descriptions that aligned with one conception much more frequently than another. Nicole and Alison showed strong preference for descriptions of learning strategies that aligned with a Platonist conception, while Melinda for Instrumentalist. Jana and Heidi wrote, for equal numbers of tools, statements indicating Instrumentalist and Platonist alignments. Breanna showed slight preference for learning strategies that aligned with an Instrumentalist conception.

### *Knowledge Structures*

In considering teachers' attention to learning goals and strategies for learning, I noted that they used particular terms for the type of knowledge that students were developing. I considered descriptions of knowledge structures that align with an Instrumentalist conception as: facts, rules or procedures, and skills. For structures aligned with a Platonist conception, I chose key terms: ideas, and connections or relationships. Breanna used knowledge structures that I claim are aligned with Instrumentalist conceptions for many more tools than Platonist. Similarly, Melinda, Nicole, Heidi, and Alison used terms for many more tools that aligned with a Platonist conception. Jana's use of terms was equal; the number that aligned with an Instrumentalist conception was the equal to the number of those aligned with a Platonist conceptions.

### *Across Three Aspects*

Some consistency exists in how teachers' conceptions were reflected in these three aspects of learning. Across the three aspects, Breanna's noticing was aligned with an Instrumentalist conception for several more tools than Platonist; Melinda, Alison, and Nicole showed alignment with a Platonist conception for several more tools than Instrumentalist. Heidi's noticing was aligned with a Platonist conception for only one more tool than Instrumentalist; Jana's noticing was aligned with an Instrumentalist conception for two more tools than Platonist. Melinda's noticing strongly aligned with an Instrumentalist conception for strategies but a Platonist conception for knowledge structures. Similarly, Heidi aligned with a Platonist conception for learning goals and knowledge structures, but was aligned with Instrumentalist and with Platonist conceptions for learning strategies. If Heidi and Melinda's noticing, or even just the way they used language, showed some preference for strategies aligned with an Instrumentalist conception while their language for knowledge structures and learning goals tended to align with a Platonist conception, it may be that they would be frustrated in expecting their students to develop particular types of knowledge despite reliance on strategies that support development of different types of knowledge. That is, if a teacher expected students to develop deeper understanding of mathematical meaning but used strategies that supported superficial understanding, then it would be reasonable for both the teacher and students to be frustrated by inconsistencies between expectations and experiences.

In this study, the course provided ways for these six teachers to discuss different levels of mathematical understanding that they wanted their students to develop. This second framework provided insight into inconsistencies between teachers' noticing of goals, knowledge structures, and learning strategies. Researchers and teacher educators may both find this second framework useful, because such results could support discussion with teachers about strategic alignment between experiences and expectations. Future research could test the framework by more directly asking teachers to describe learning goals and learning strategies of online tools, and considering their use of knowledge structure terms. Research using this framework could support a deeper understanding of teachers' perspectives on the ways online tools support student mathematical learning and how those perspectives relate to their conceptions of the nature of mathematics. Understanding their perspectives and the relationships could help designers of online tools be more explicit about the learning goals of the tool, and more intentional about the teacher resources that are included with the tool. Researching these relationships could help teacher educators make intended learning goals and strategies more visible to teachers. Teacher educators could provide teachers with experiences to analyze online tools and reflect on how they support (or not) particular aspects of the nature of mathematics, particularly in connection with valued teaching practices.

## Conclusion

I have considered two potential frameworks of relationships between characteristics of online tools and teachers' conceptions of the nature of mathematics. One framework focused on categorizing teacher responses to particular questions within the online tool evaluations each completed. The other framework focused on categorizing teachers' attention to learning goals, learning strategies, and knowledge structures. Both frameworks may be impacted by uncontrolled aspects of the study, such as teachers' own choices of online tools, their interpretations of evaluation questions, and their motivation for responding to the questions. Both frameworks support a way of understanding how teachers' conceptions of the nature of mathematics may impact their noticing of mathematical aspects of online tools.

For the teachers in this study, their noticing may have been impacted by certain factors. For example, use of the first framework indicated that Breanna noticed most of her tools provided opportunities to *practice* while Heidi noticed most of her tools provided opportunities to *explore*. One factor that may have impacted teachers is their alignment with a particular conception. I claimed that Breanna was aligned with an Instrumentalist conception while Heidi was aligned with a Platonist conception. Use of the second framework indicated that Breanna and Nicole used language to describe learning goals, learning strategies, and knowledge structures that aligned, for most of their tools, with an Instrumentalist and Platonist conceptions, respectively. This alignment was consistent with my finding that Breanna's and Nicole's conceptions of mathematics seemed aligned with Instrumentalist and Platonist conceptions, respectively. They may have noticed aspects in tools that aligned with the corresponding conception. Other factors besides conceptions may have impacted their noticing, however. A second factor may have been their choice of tools. Teachers may have chosen online tools that aligned with their corresponding conception, but they may have simply chosen tools that were easily accessible. A third factor, thus, could be the type of online tool made available for the content area and grade level by creators of these tools. The type of online tool available, targeting each grade band, may have been different. If the tools available to Breanna, for Kindergarten, focused mostly on practice than it would make sense that she chose practice tools and saw that they supported practice. The tools available to Breanna may have had learning goals and strategies that naturally aligned with an Instrumentalist conception, and so she noticed those aspects. Heidi, teaching fifth grade gifted and talented, found tools targeted to her grade level were focused mostly on exploration. Similarly, Nicole may have found that online tools targeted for the middle school mathematics courses she had taught were naturally aligned with a Platonist conception. An interesting inconsistency is that Heidi described learning goals and strategies that aligned with an Instrumentalist conception for almost as many tools as goals and strategies that aligned with Platonist. If online tools targeted to her fifth grade gifted and talented mathematics course were more likely to emphasize exploration, then it seems contradictory that Heidi described almost equal numbers of online tools in ways that aligned with Instrumentalist and with Platonist, when considering her descriptions using the second framework. The contradiction may imply that there may be times when Heidi's meaning for *exploration* aligned with an Instrumentalist conception and times that it aligned with a Platonist conception.

Future research could investigate teachers' meanings of exploration and practice to distinguish different types of meaning. Teacher educators could use additional meanings from such research to support teachers' discussions about *exploration* and *practice* with respect to learning goals to help teachers use more precise language that might help them be more strategic in their choice and use of online tools.

With respect to the online tools that are available to teachers, future research could attend to the diversity (or lack of diversity) of online tools that targeted to certain grade levels or content areas. Results from such research could support teachers as they search for online tools that provide richer exploration and learning opportunities for students.

## Conclusion, Limitations, Future Directions

In closing, I summarize the complexities and challenges I found in studying teacher conceptions and noticing. I propose implications and contributions of the study to the research and teacher education communities. I follow with a presentation of limitations of the study. I end with future directions of research based on this study.

### Complexity and Challenges in Studying Teacher Conceptions and Noticing

I began this study with what seemed at the time to be a reasonable expectation of the complexity and challenges I would face in this study of teacher conceptions and noticing. I had reflected on and discussed my conception of the construct *conceptions*. I expected conceptions to be context-dependent and to be potentially contradictory as I had identified contradictory conceptions in my own thinking. I likewise expected an analysis of noticing to be difficult because I chose to rely on teachers' responses to homework assignments. Indeed, I found indications that teachers' conceptions were complex and context-dependent. I found that teachers' responses to their homework assignments (e.g., evaluations) did vary in depth and sometimes included copy-and-pasted responses from previous assignments; this inconsistency increased the difficulty of consistent noticing analyses. But, I also found unexpected challenges in analyzing their conceptions and noticing.

In the study itself, I enjoyed examining and exploring teachers' statements and attending to different descriptions of constructs that I read. But in the process, two themes emerged as particular generators of complexity: language and validity. Both themes made analysis of conceptions and noticing difficult in various ways.

By *language*, I refer to struggles in understanding what teachers meant when they described their thinking. This acknowledgment of the imprecision of language is by no means new, but

I continually reminded myself that words are symbols to which individuals assign meaning. At times I had to rely on teachers' word choice for my analysis, even though I know that teachers assigned meanings in different ways to the same words; even individual teachers would use the same word in multiple ways in a single evaluation. The imprecise use of language is a normal human activity, but it felt important that I acknowledge its contribution to the complexity of the study: Different teachers assigned different meanings and were differently skilled at using language as a tool for communication. In a future study, I might give teachers opportunities to draw pictures or create other artifacts to support different ways humans are able to communicate meaning. The imprecision of language was not limited to teachers, however, but also to the course instructors. In communicating our expectations and constructs to teachers, we were not always as precise with language as we could have been. Based on my reading of their responses, I saw indications that teachers interpreted our evaluation, discussion, and reflection questions in very different ways even from unit to unit. This varying interpretation resulted in clear differences in the data that resulted within and across individual teachers.

In considering teachers' statements for hints at conceptions and noticing, I often considered the tension between *conceptions* and *misconceptions.* In my experience as an instructor of pre-service elementary teachers, I have read their plans to "stamp out" and "destroy" mathematical misconceptions of students. I have seen that the meaning of *misconceptions* can be arbitrary: a valid conception today might be a misconception in the future, or a misconception today might be a valid conception in the future. Misconceptions usually have validity in restricted contexts, even as they are considered mistakes in the discipline of mathematics. I have tried to turn instead to supporting my future teachers in talking about *levels of sophistication* of students' conceptions and how to support students in reaching more sophisticated understandings. My personal view of sophisticated understanding is that it is a more flexible, complex understanding with potentially contradictory conceptions. What makes such an understanding more sophisticated, in my mind, is that the person is able to choose a conception that is useful for a particular situation. Sometimes one conception works as an efficient model in a particular context, but would be invalid or misleading in a different context. As I examined teachers' statements in this study, I caught myself considering how to "stamp out" my teachers' misconceptions about the utility of certain online tools or about mathematical correctness. I recognized that at times I viewed some teachers' conceptions or noticing as wrong or invalid. I struggled to reframe my reactions to ask instead, "What is useful about this conception or noticing to this teacher in this situation?" In future research or in future classes, I plan to have discussions with colleagues and teachers about the meanings of levels of sophistication with respect to conceptions and noticing. For example, if I follow my definition of sophistication shared above, then it can make sense for a teacher to have multiple conceptions of the nature of mathematics. That is, a teachers' conceptions may align with all three of Ernest's (1989b) views: Instrumentalist, Platonist, and Problem-driven. The goal of teacher educators should not be to "stamp out" the Instrumentalist conception, but to reinforce aspects of the Platonist and Problem-driven conceptions. A teacher's sophistication would be apparent when she is able to reflect on the needs of her students and her own needs, and strategically choose her use of aspects of conceptions. For example, for standardized assessments, a teacher may need to support her students in acquiring a collection of facts and rules. For certain content areas, or at regular intervals throughout a school year, she may support her students in engaging in creative application of their knowledge to solve more complex tasks. Teacher educators may be better equipped to support teachers in reflecting on their teaching and students in order to reach more consistent and sophisticated levels of conceptions, if they do not also feel obligated to "stamp out" teachers' misconceptions.

I claim that a large lesson that I learned from this study is the necessity of awareness of my own flexibility with language and validity as a researcher and teacher educator. In order to learn from teachers' meanings, I must be able to see my own assumptions and recognize how those assumptions can support communication and also how they can constrain communication. By making assumptions visible to myself, I believe I can be more open to assumptions that others bring. That is, I can more clearly distinguish between the meanings that I share with teachers and the meanings that teachers have developed based on their experiences in their lives, in teacher preparation, and in their classrooms. Just as teachers described the need for their students to develop deeper understanding of mathematical ideas in order to become more flexible in their use of those ideas, I feel that understanding assumptions can allow me to be more flexible with their use. I can use this understanding in my role as teacher educator and in my role as researcher. In addition, recognizing that teachers have unexamined assumptions about their own meanings for aspects of mathematics, for learning goals and strategies, and for mathematical practices, may help future research understand the challenge in analyzing teachers' responses in this type of study.

## Contributions of the Study

Despite agreement that teacher beliefs impact their teaching practice and their noticing of instructional materials, researchers have found contradictions among observed or asserted beliefs (Fives & Buehl, 2012). Speer (2005) argued that, in addition to the complexity of belief structures, such contradiction may result because researchers wrongly assume shared understandings of terms. Speer recommended that researchers explore teacher meanings of aspects of beliefs to develop more nuanced understanding of potential teacher meanings. This study explored teachers' expressed conceptions deeply to understand potential meanings that teachers may bring to this conversation. I found that reducing teachers' conceptions of the nature of mathematics to the three categories of Instrumentalist, Platonist, or Problem-driven inhibits understanding the richness with which they view mathematics across different contexts. I found indications that teachers did hold to aspects of multiple conceptions from Ernest's (1989b) framework. Future research could provide teachers more opportunities to talk about aspects of these conceptions to understand how a teacher might use aspects of multiple conceptions strategically to navigate the realities of her classroom. I found that teachers' meanings for aspects of mathematics, such as connectivity, problem-solving, and correctness, were more complex than I had anticipated. I found emergent frameworks that allowed a more nuanced understanding of their meanings. In future research, I hope to similarly examine teachers' meanings of *practicing* and *exploration* to understand how teachers may use these terms to describe different types of student activity. Understanding potential meanings in more depth will support future research of teacher beliefs and the creation of professional development activities to support teachers in understanding these constructs from multiple perspectives. Additional aspects of connectivity, problem-solving, and correctness that I found in this study can support both researchers and teacher educators in productive discussions with teachers about their different understandings.

As teachers are expected more and more to integrate technology in their teaching, to supplement or adapt current curricula, they need support in developing an internal framework to support meaningful choices about online tools, based on quick assessments of aspects of those tools. In this study, I explored teachers' use of a detailed evaluation template for noticing aspects of technology. In its way, the intentions of the detailed template were similar to the *Thinking Through a Lesson Protocol* described in M. S. Smith, Bill, and Hughes (2008). I found indications that, in response to the same questions in the template, teachers described aspects of online tools in using different levels of sophistication. A teacher's experience may contribute to her ability to notice and describe particular aspects on online tools. I found, for example, that the three teachers with less experience tended to use less precise language in describing an online tool's use of symbolic representation. I found that teachers did attend to mathematical features of online tools; almost all of the six teachers attended to mathematical aspects even when they were not explicitly asked. Despite recommendations that students confront mathematical consequences of "wrong" choices (e.g., Dick & Burrill, 2016), I found only one teacher that consistently attended to the availability of mathematical consequences (even for some tools that did not provide mathematical consequences). I also found that several teachers expressed concern when they noticed a tool provided mathematical consequences but not feedback on correctness, arguing that students needed feedback to recognize correctness. Before beginning this study, I hypothesized that teachers would conflate the potential for student *engagement* with the *aesthetics* of a tool. I found instead that most teachers attended to aspects of engagement as separate from the aesthetics of a tool. Understanding what teachers notice, as they are supported by such a framework and as they write open-ended evaluations, will support future research on supporting teachers in internalizing such frameworks, supporting teachers by directing their attention to critical characteristics, and supporting them in developing sophistication in their noticing.

Kim et al. (2013) argued that teacher conceptions of the nature of mathematics and mathematics teaching and learning is a gap in current research on technology integration; despite recognition that teachers beliefs impact teacher integration, current instructional technology research focuses on teacher beliefs about the value and use of classroom technology. This study explored relationships between teacher conceptions of the nature of mathematics and mathematics teaching and learning and teacher noticings of aspects of online tools. I found difficulty in aligning teachers' noticing with their conceptions using Ernest's (1989b) framework of conceptions of the nature of mathematics. In future research, I plan to use a different framework focusing on teachers' expressed conceptions of mathematics (e.g., mathematics as a way of communication) to test for better alignment. Testing alignments between teachers' conceptions and noticing is challenging, perhaps because of the complexity of teachers' conceptions and the complexity of features of online tools. I found that teachers' use of language, the dissimilar sets of online tools, and the different grade levels brought additional challenges to my analysis. Despite these challenges, I did find promising indications of relationships between teachers' conceptions and noticing using two different frameworks. In future research, I hope to refine these frameworks by providing grade-level groups of teachers with consistent sets of online tools. Understanding such relationships can help support development of learning opportunities for practicing teachers to confront their own assumptions about use of online tools in mathematics teaching and learning.

Recent teacher education recommendations, for inservice and preservice teachers, have emphasized teachers' use of *high-level tasks* to support all students in understanding mathematics more deeply and teachers' *transformation* of their teaching using technology (Association of Mathematics Teacher Educators [AMTE], 2017; Conference Board of the Mathematical Sciences, 2012). With or without technology, teachers can use high-level math tasks that are accessible, connect to students out-of-school experiences, and enable multiple interpretations, entry points, and solution strategies (González, Moll, & Amanti, 2006; M. S. Smith et al., 2008). First, students engage in the task and, then, teachers guide discussions where students share solution strategies and use them to develop their mathematical understanding. This method supports students of all levels by allowing students to engage in a task at their own level of sophistication. In order for students to make sense of the mathematics during the discussion, different strategies at different levels must be attempted. Although the goal is to support students in learning how to find correct answers, there is not *one correct answer* but many solutions and many strategies with different ways of being useful: The students use discussion to explore what makes solutions and strategies meaningful or not, valid or not, more efficient or not, more understandable or not (Stein et al., 2008). I found that teachers' concerns about students' success if students do not receive immediate feedback on correctness, or are allowed to reach wrong solutions and solutions based on unexpected assumptions, are not dissimilar from Stein et al.'s description of their teachers' concerns. In future research and in my future courses, I hope to find ways to support teachers in addressing these concerns by using mathematical consequences and discussion to support students in developing their mathematical creativity and reasoning. Using technology to support students' engagement in rich tasks relies on teachers' acceptance of the need to use high-level tasks as well as their acceptance of the potential for technology to support such tasks in multiple ways: facilitating communication and collaboration through multiple media (e.g., text, audio, visual), facilitating student interaction with "messy" but real data, facilitating student interaction with and use of mathematical representation, etc.

In this study, I found that most of the online tools that teachers evaluated involved small tasks; the depth and richness of tasks that teachers find in online tools may also inhibit their use of technologies that transform their teaching. There are online tools and resources that support teachers' use of online tools for more complex, rich tasks. At the end of each interview, I asked teachers what support they needed in finding online tools. All answered that they needed more support in finding good tools and resources online; most expressed hope that more tools would include teacher resources. For example, videos showing enactment of tasks in classrooms or descriptions of how tools might be used in a lesson. Nicole said that she wanted a sample lesson to follow the first time she used an online tool, even if later she would move past the first example to design her own lessons. Other teachers expressed similar needs. By researching teachers' conceptions of the nature of mathematics, how development of more sophisticated conceptions can be supported, and how conceptions inform teachers' noticing, researchers and teacher educators can support teachers in developing their own internal frameworks for choice and use of online tools in their classrooms.

## Limitations

As described in the introduction of this study, certain conceptual and methodological limitations must be acknowledged. First, conceptual limitations include the challenges that arise in translating theoretical frameworks into analytical frameworks. In analyzing teachers' written statements, I found that the initial frameworks had to evolve to include emergent meanings. In addition, my observation of teachers' conceptions and noticing was mediated by teachers' imprecise use of language. In considering teacher statements indicating conceptions, I used statements from three sources: group discussions and individual reflections within the course, and responses to a structured card-sorting activity in a post-course interview. T o understand teacher meanings of larger concepts, such as the meaning of "mathematics is ideas that are all connected," I considered different aspects of the larger concept. Considering words adjacent or supportive of the larger concept addressed this difficulty to some extent. At some level, however, I must take the meanings of some words used by teachers as shared and understood. The reliance on language is an unavoidable, but weighty, limitation of the study. Especially with respect to the consideration of relationships between conceptions and noticing, teachers' varying abilities to use language to describe aspects of online tools must impact my assumptions about alignment.

Second, methodological limitations include the sample for cases, the variation of online tool evaluation types, and variation of the online tools themselves. In the proposal of this study, I described different characteristics of teachers that would inform my decision of cases for the study. I hoped to choose at least six focus participants, and only six volunteered. Hence, my choices were restricted to the teachers who volunteered to participate in the post-course interview. Despite the inability to choose my focus participants, I found these six focus teachers brought interesting similarities and differences to the study. For example, it is helpful that I had multiple teachers from the lower and upper elementary groups because they brought two shared contexts to my analysis. In addition, the differences in experience and views assisted in illustrating and contradicting my initial theories in useful ways. I and my co-instructor intentionally gave opportunities to teachers to develop their own evaluation templates, because a goal of the course was for teachers to develop their own internal frameworks for evaluating online tools. In analyzing teachers' noticing, I faced challenges because teachers were not responding to the same questions for each tool they evaluated. The different in evaluation frameworks, however, opens the potential for future research to analyze their choices as a different perspective on their noticing. Finally, we asked teachers to choose their own online tools because a goal of the course was to support teachers in finding tools that would be appropriate for each teachers' classrooms. The variation of types of tools and their features, however, brought challenges in analyzing teacher noticing. That is, I faced challenges in comparing teachers' noticing of particular features because those features were not necessarily a part of each tool. As I have discussed elsewhere, Breanna may have described every tool as supporting Instrumental Understanding and practice because those were aspects of each tool she chose. Indeed, more complex tools or richer tasks may not have been available as online tools for Kindergarten topics.

## Future Directions

This study answered a call for additional research on teachers' conceptions of mathematics and mathematics teaching and learning as they correspond with teachers' integration of classroom technology in their lessons (e.g., Kim et al., 2013). Findings were suggestive that such relationships between conceptions and noticing do exist—although they are not in any way simple or easy to identify. Future research should focus on these potential relationships to support teacher education: As access to Internet-capable devices becomes universal across classrooms, teachers need support in finding, choosing, and integrating online tools and resources into rich tasks that allow students to act as mathematicians. I believe that findings from this study can support both future researchers and teacher educators in developing and testing ways in which preservice and early inservice teachers can be supported in developing their strategies for intentionally selecting, integrating, and implementing use of digital tools and technology to support mathematics teaching and learning. One aspect of supporting teachers' decision-making include exploring the type of support teachers need for, first, identifying particular mathematical gaps in their curriculum and, second, planning and implementing strategies for teaching and learning the identified mathematical concepts or conceptualizations through use of technology, when appropriate. A second aspect includes supporting teachers in (efficiently and productively) searching for and choosing digital tools and tasks to support their students concept development to bridge the gap identified previously. Searching, as resources continue to evolve and multiply, means that teachers must find reliable sources that provide relevant tools. There are collections available online. Teachers may also gain familiarity with the mathtwitterblogosphere (MTBoS), as recommended in new teacher education recommendations (AMTE, 2017). As they gain familiarity with the MTBoS, they may find new collections along with teacher resources provided informally by teachers or teacher educators. So that teachers are not simply searching online for answers from others, which may or may not fit their particular context, teachers must be supported in developing their own internal frameworks for evaluating characteristics and mathematical content of digital tools and tasks. Such internal frameworks can support bridging identified gaps in curriculum and their own conceptions of mathematics teaching and learning. That is, teachers need support in identifying particular digital tools and tasks that support their instructional strategies. Teachers can also be supported in adapting their students' interaction with tools to match their instructional style. For example, students might interact with tools and then discuss the strengths and limitations of the tools' representation or strategies. Finally, teachers need support in considering differences in mathematical values, representations, and practices enabled by various digital tools and tasks, and recognizing how those differences make some ideas more visible while hiding other ideas.

Based on this study, to support teachers' development in these areas, professional development must support their development in other areas as well. For example, teachers can be supported in developing their levels of sophistication in noticing pedagogical and mathematical aspects of online tools. Teachers can be supported in reflecting on their conceptions of mathematics and mathematics teaching and learning, learning about other perspectives, and, through discussion and reflection, noticing potential conflicts in their conceptions and what they are noticing in online tools.

# APPENDICES

# APPENDIX A

## COURSE STRUCTURE

- Unit 1: Getting Started.
  - Teachers introduce themselves by writing and commenting on personal profiles.
  - Teachers complete their first digital tool evaluations (pre-test).
  - Teachers read and discuss Erlwanger (1973).
- Unit 2: Learning, Mathematics, and Technology.
  - Exploring the questions: What do we learn when we learn mathematics? How do we learn it?
  - Teachers read and discuss Skemp (1978).
  - Teacher teams collaboratively develop an evaluation framework for digital tools and resources. Teachers then test out the framework by evaluating a digital tool, as well as sharing and discussing their frameworks with the other teacher teams.
- Unit 3: Computational Power (Number and Operations).
  - Technology focus on the power of technology to carry out various kinds of computations and to support the learning of computational skills and concepts.
  - Mathematical focus on the Number and Operations domain.
  - Teachers read and discuss excerpts:
    * related to Number and Operations from *PSSM* and *CCSSM*,
    * NCTM Learning and Technology Principles (NCTM, 2000, pp. 20-21, 24-27),
    * NCTM Process Standards,
    * CCSSM Standards of Mathematical Practice, and

        * Strands of Mathematical Proficiency from *Adding it Up* (National Research Council, 2001, pp. 115-145).

- Unit 4: Powerful Representations (Algebra).
  - Technology focus on how technology represents mathematical ideas and thinking in powerful ways.
  - Mathematical focus on algebraic reasoning and the development of conceptual understanding of algebraic ideas through early and later grades.
  - Teachers read and discuss excerpts related to Algebra and Algebraic Reasoning from *PSSM* and *CCSSM*, as well as article and book chapter readings about technology and mathematical representation; we provided one set for upper grades (Heid & Blume, 2008) and one for lower grades (Bremner, 2013; Kurz, 2013; National Research Council, 2001, pp. 71-102).
- Unit 5: The Wide Webbed World.
  - Technological focus on ways in which the Internet and the web can support teaching and learning, especially with respect to accessing information, communicating, and collaborating.
  - Teachers explore online repositories of mathematical tools and resources with a focus on the two questions: How can we as teachers access (and share) information and resources to support our teaching and professional lives? How we can help our students take advantage of available resources and support their learning through collaboration and communication with others?
- Unit 6: Exploratory Environments (Geometry).
  - Technological focus on interactive mathematical environments (e.g., Hohenwarter et al., 2013).

  - Mathematical focus on Geometry and aspects of Geometry that appear across all grade levels.
  - Teachers read and discuss excerpts related to Geometry from *PSSM* and *CCSSM*, as well as Schoenfeld (1988).
  - Teachers explore interactive Geometry tools and activities designed to use those tools, asking: How is the technology used to support learning? What assumptions about learning may the designer have held?

- Unit 7: Bridging the Gap (Data Analysis and Probability).
  - Technological focus on using digital tools and resources to make connections between formal school mathematics and mathematical situations that occur in the real world.
  - Mathematical focus on Data Analysis and Probability and how activities in this domain are similar or different across grade levels.
  - Teachers read and discuss excerpts related to Data Analysis and Probability from *PSSM* and *CCSSM*. Teachers explore online survey tools and data sources or repositories.

- Unit 8: Wrapping Up.
  - Teachers complete the final project of the course: an online resource library in which they share digital tools and resources organized along a planned theme, for a chosen audience (e.g., their students, teachers at their school or grade level, parents) along with course unit evaluations (or newly written evaluations).
  - Teachers complete final evaluations of the same digital tools from the first week ("post-test") and final reflections in which they compare their initial and final evaluations.

## APPENDIX B

## ONLINE TOOLS

1. Fuzz Bugs

   http://www.abcya.com/counting_sorting_comparing.htm

2. Annenberg Learner - Interactives - Geometry 3D Shapes

   http://www.learner.org/interactives/geometry/3d_pyramids.html

3. BBC - Fraction Frames

   http://www.bbc.co.uk/schools/teachers/ks2_activities/maths/activities/fractions.swf

4. BBC - Gumball Probability

   http://www.bbc.co.uk/schools/teachers/ks2_activities/maths/activities/probability.swf

5. Braining Camp - Fraction Division Manipulative

   https://www.brainingcamp.com/content/dividing-fractions/manipulative.php

6. Charlotte Mecklenburg Library Story Place - I Spy Shapes

   https://www.storyplace.org/activity/i-spy-shapes

7. College-Cram Algebra - Area of a Circle

   http://college-cram.com/study/algebra/geometric-shapes/area-of-a-circle/

8. Crickweb - String of Beads

   http://www.crickweb.co.uk/Early-Years.html#beads

9. EDC in Maine - Comparing Number Lines

http://maine.edc.org/file.php/1/tools/CompareNumberLines.html

10. Fun Brain - Math Baseball

    http://www.funbrain.com/math/

11. GeoGebra - Area of a Circle

    https://www.geogebra.org/m/SFE3bRrq

12. GeoGebra - Triangle Sum Theorem

    http://archive.geogebra.org/en/upload/files/english/Barbara_Perez/Triangle_Angle_Sum.html

13. GeoGebra - Which Quadrilateral Am I

    https://www.geogebra.org/material/show/id/51987

14. ICT Games - James Barrett - Shark Pool Place Value

    http://www.ictgames.com/sharknumbers.html

15. IXL - Recognize Patterns

    http://www.ixl.com/math/grade-1/recognize-patterns

16. IXL - Subtracting Fractions with a Model

    http://www.ixl.com/math/grade-5/subtract-fractions-with-unlike-denominators-using-models

17. KidsOLR - Kids, Fractions, Tutorials - Multiplying and Dividing Fractions

    http://www.kidsolr.com/math/fractions.html

18. Kristen Carvell - MSTE Illinois - Rectangle Applet

    http://mste.illinois.edu/carvell/rectperim/RectPerim.php

19. LearnZillion - Adding mixed numbers with unlike denominators by creating area models

    https://learnzillion.com/lesson_plans/7559-add-mixed-numbers-with-unlike-denominators-by-creating-area-models

20. Math Playground - Alien Angles

    http://www.mathplayground.com/alienangles.html

21. Math Playground - Builders Inc

    http://www.mathplayground.com/MathApprentice/Builders.html

22. Math Playground - Dividing Fractions

    http://www.mathplayground.com/fractions_div.html

23. Math Playground - Fraction Bars

    https://www.mathplayground.com/Fraction_bars.html

24. Math Playground - Island Chase - Subtraction

    http://www.mathplayground.com/ASB_IslandChaseSubtraction.html

25. Math Playground - Kangaroo Hop

    http://www.mathplayground.com/ASB_Kangaroo_Hop.html

26. Math Playground - Minus Missions

    http://www.mathplayground.com/ASB_MinusMission.html

27. Math Playground - Number Bonds

    http://www.mathplayground.com/number_bonds_10.html

28. Math Playground - Ratio Blaster

    http://www.mathplayground.com/ASB_RatioBlaster.html

29. Math Playground - Spider Match

    http://www.mathplayground.com/ASB_SpiderMatchIntegers.html

30. Math Playground - Tug Team Addition

    http://www.mathplayground.com/ASB_TugTeamAddition.html

31. Math Snacks - Bad Date

    http://mathsnacks.com/baddate-en.html

32. Math Snacks - Number Rights

    http://mathsnacks.com/numberrights.html

33. Math Snacks - Pearl Diver

    http://mathsnacks.com/pearl-diver-en.html

34. Maths is Fun - Match the Fraction - Fraction to Number Line

    http://www.mathsisfun.com/numbers/fractions-match-frac-line.html

35. Maths is Fun - Math Trainer - Addition

    http://www.mathsisfun.com/numbers/math-trainer-addition.html

36. NCTM Illuminations - Coin Box

    http://illuminations.nctm.org/coinbox/

37. NCTM Illuminations - Concentration

    http://illuminations.nctm.org/Activity.aspx?id=3563

38. NCTM Illuminations - Five Frame

    http://illuminations.nctm.org/Activity.aspx?id=3564

39. NCTM Illuminations - Fraction Model

    http://illuminations.nctm.org/Activity.aspx?id=3519

40. NCTM Illuminations - Fraction Track

    http://illuminations.nctm.org/Activity.aspx?id=4148

41. NCTM Illuminations - Geometric Solids

    http://illuminations.nctm.org/Activity.aspx?id=3521

42. NCTM Illuminations - How Many Under the Shell

    http://illuminations.nctm.org/Activity.aspx?id=3566

43. NCTM Illuminations - Pan Balance - Numbers

    http://illuminations.nctm.org/Activity.aspx?id=3530

44. NCTM Illuminations - Playing Fraction Track

    http://illuminations.nctm.org/Activity.aspx?id=6382

45. NCTM Illuminations - Shape Balance

    http://illuminations.nctm.org/Activity.aspx?id=3531

46. NCTM Illuminations - Ten Frame Addition

    http://illuminations.nctm.org/Activity.aspx?id=3565

47. NCTM Illuminations - Tessellation Creator

    http://illuminations.nctm.org/Activity.aspx?id=3533

48. NCTM Illuminations - Turtle Pond

    http://illuminations.nctm.org/Activity.aspx?id=3534

49. NCTM Illuminations - Two Terrains

    http://illuminations.nctm.org/Activity.aspx?id=3535

50. NLVM - Adding Fractions

    http://nlvm.usu.edu/en/nav/
    frames_asid_106_g_2_t_1.html?from=category_g_2_t_1.html

51. NLVM - Base Blocks Addition

    http://nlvm.usu.edu/en/nav/
    frames_asid_154_g_1_t_1.html?from=category_g_1_t_1.html

52. NLVM - Box Plot - Histogram

http://nlvm.usu.edu/en/nav/
frames_asid_200_g_3_t_5.html?open=instructions&from=category_g_3_t_5.html

53. NLVM - Multiplication of Fractions
http://nlvm.usu.edu/en/nav/
frames_asid_194_g_2_t_1.html?from=category_g_2_t_1.html

54. Oswego City School District - Mark Cogan - Give the Dog a Bone
http://resources.oswego.org/games/DogBone/gamebone.html

55. PBS Kids - CyberChase - Bugs in the System
http://pbskids.org/cyberchase/math-games/bugs-in-the-system/

56. PBS Kids - CyberChase - Feed That Dog
http://thinktv.pbslearningmedia.org/resource/
vtl07.math.number.ope.feedthatdo/feed-that-dog/

57. PBS - CyberChase - Poddle Weigh-in
http://pbskids.org/cyberchase/math-games/poddle-weigh-in/

58. PBS Kids - Sid the Science Kid - Balancing Act
http://pbskids.org/sid/balancingact.html

59. Sheppard Software - Balloon Pop Math
https://www.sheppardsoftware.com/mathgames/fractions/
Balloons_fractions3.htm

60. Sheppard Software - Math Man
https://www.sheppardsoftware.com/mathgames/fractions/
mathman_equivalent_fractions.htm

61. Sheppard Software - Purpys Shapes
http://www.sheppardsoftware.com/preschool/ngames/shapes.htm

62. Shodor - Interactivate - Area Explorer

http://www.shodor.org/interactivate/activities/AreaExplorer/

63. Smarty Games - One Dollar Store

http://www.smartygames.com/igre/math/learnMoney.html

64. Teach Mathematics - Transformations

http://www.teachmathematics.net/files/teachmaths/files/ALGEBRA/Transforming_functions_stretch/What_transformation.html

65. TeacherLED - Fraction Comparison Tool

http://www.teacherled.com/resources/equivfrac/equivfracload.html=

66. Turtle Diary - Number Line Addition

http://www.turtlediary.com/kindergarten-games/math-games/number-line-addition.html

67. Turtle Diary - Subtraction Matrix

http://www.turtlediary.com/grade-2-games/math-games/subtraction-matrix.html

68. XP Math - Call of Geometry - Quadrilateral Warfare

http://www.xpmath.com/forums/arcade.php?s=97b6a3ed64a494e27bb4d8f0be12b4e6&do=play&gameid=84#.UU-VlBycESt

## APPENDIX C

## CONSENT SURVEY

Before continuing, please read this formal statement detailing the terms and conditions of the study. The question about participation is at the bottom of this text:

I am asking you to participate in a research study of practicing teachers' use, beliefs, and noticing features of instructional technology in mathematics teaching. Your classwork and responses will help me understand how teachers' beliefs may impact what they notice about features of digital tools and resources.

I am asking you to allow me to analyze your coursework (e.g., certain online postings, reflections, and evaluations) that you have completed as part of the coursework of [this *Mathematics, Technology, and Education* course]. I will remove any identifying information in the coursework for the data analysis and for purposes of reporting research findings. Beyond that, I will also ask if I can interview you in the next few weeks (scheduled according to your needs) about course-related issues. If you consent to participate in the study by taking the brief survey and allowing your coursework to be used in the research analysis, you will be entered into a drawing for one of 3 $20 Amazon gift cards that will be emailed by May 30. If you additionally consent to participate in an interview after the course has ended, you will receive an email with the $40 Amazon gift card at the end of the interview. Participation in this research project is completely voluntary. The risks and benefits to participating in this study are minimal.You have the right to say no. You may change your mind at any time and withdraw. You may choose not to answer specific questions or to stop participating at any time. Your choice of whether or not to participate will in no way impact your standing or achievement in [this *Mathematics, Technology, and Education* course] or any other class, and the instructors will not know who has chosen or has not chosen to participate until after all final grades have been posted for the semester. No data will be analyzed until all final grades have been posted for the semester.

If you have concerns or questions about this study, please contact the researcher: Eryn Stehr (stehrery@msu.edu). Thank you so much for considering participating! I appreciate your time!

1. Do you consent to participate in this research?

   a) No, I do not consent to participate. [survey jumps to "Thank you" below]

   b) Yes, I consent to participate by allowing Eryn to analyze my coursework (e.g., online postings, reflections, and evaluations). I will be entered to win one of three $20 Amazon gift cards.

   c) Yes, I consent to participate by allowing Eryn to analyze my coursework (e.g., online postings, reflections, and evaluations). Additionally, I consent to a 90-minute interview with Eryn if needed. I will receive a $40 Amazon gift card after the interview.

2. Thank you for consenting! I will contact you by May 30 if you win one of the three $20 Amazon gift cards. Please enter your name here: ______________

3. Thank you for consenting! I will send you the $40 Amazon gift card within a week after the interview. So that I can schedule an interview, please enter your name here: __________ Please enter an email address that you check regularly: ______________

4. Please briefly describe your current position (e.g., teacher, math specialist):

5. How many years of experience do you have teaching in a classroom? (If you have not taught, just type '0'? if you include your student teaching then briefly say that? if you include other types of a teaching outside of a classroom, then briefly describe that)

6. What grade levels have you taught?

   - PreK - 2
   - 3 - 5
   - 6 - 8
   - 9 - 12

7. If you have taught a class with a specific mathematics focus (e.g., algebra), describe the class (or classes) here: ______________

8. What mathematics curriculum or textbook series do you currently use regularly in your classroom? (include its title and publisher if possible or state that you do not use any particular textbook or curriculum regularly? if you teach more than one class, then include the textbook for each class) ______________

9. Do you choose the mathematics curriculum or textbook that you use for mathematics teaching?

   - No, I do not have a choice in the mathematics curriculum or textbook.
   - Yes, I do choose the mathematics curriculum or textbook.
   - I do not use a mathematics curriculum or textbook regularly.

10. Do you choose the materials (e.g., tools, resources, activities) that you use for your mathematics teaching?

    - Yes
    - No

11. Thank you so much for taking the time to respond. I truly appreciate it!

## APPENDIX D

## INTERVIEW STRUCTURE

# Part I. General Information

[Start recording] Thank you very much for agreeing to be interviewed. Do you give permission for me to record this interview?

Today is May 19. My dissertation is focused on teachers' professional decision-making about instructional use of technology in mathematics teaching and learning. This interview isn't a test and I'm not planning to evaluate your knowledge or your teaching. I'd like to talk about various concepts that we focused on in [the graduate-level *Mathematics, Technology, and Education* course], to learn more about what was important to you in the course, and I'd like to hear more about your views on some of big ideas of the course. I'm hoping for the interview to be a little informal - I have some questions I'd like to ask but I also want you to bring up ideas that you want to talk about from the course. I'll try to keep asking you but please feel free to sidetrack us if you need to!

# Part II. Interview and Participant Information

1. Date of interview:

2. Location of interview:

3. Time begin:

4. Time end:

5. What is your current position?

6. How many years have you been teaching?

7. What grade levels have you taught? (preK-2, 3-5, 6-8, 9-12)

8. What mathematics have you taught? (if any)

9. What is your current MSU program of study?

10. Do you have a preference for a pseudonym?

11. What math curriculum do you currently use?

12. What is your mathematical background? That is, do you have a mathematics major or what was your highest mathematics course?

## Part III. Ice-breaker Question

1. Now that you've had a little down time after the course and maybe you've had time to decompress, how will you recognize a high quality mathematics tech tool? (Probes: What is important to you about how it represents the mathematics? What is important to you about how it supports your teaching needs?)

2. Thinking about the mathematics, what would you look for to choose a tech tool? (PP: What did the tech tool do mathematically that other tools could not? What do other tools do mathematically that you wish this tool would do? How does the representation support the mathematics? How would you change the way the tech tool works mathematically or represents the math, if you could?)

3. Thinking about your everyday teaching strategies, what would you look for to choose a tech tool? (Probes: What did it do instructionally that other tools could not? What do other tools do instructionally that you wish this tech tool would do? How would you change it to support your teaching better?)

4. What are some of the evaluation criteria that you thought were complex or difficult to define? (Probe: In case it has not come up, bring up particular issues from final evaluation & reflection here.)

5. How would you explain student engagement of a tech tool to a colleague at your school?

6. Do you view practicing and exploring with tech tools as different activities? How would you explain them?

7. How would you explain what mathematical representation with a tech tool means to a colleague at your school?

## Part IV. Core Questions Addressing Research Questions.

Something I noticed during the course was that we don't all mean the same things when we talk about mathematics and mathematics teaching and learning. I looked into research about some of the ideas that came up, and found there are a lot of ways to think about these ideas. So, I'm going to ask you some questions about some terms that came up in evaluations in just a minute, but I'm going to start by asking about your personal views on the nature of mathematics, teaching and learning math, and using technology to support math teaching and learning.

1. First, I want you take a moment and imagine all of mathematics - that can mean math in the everyday world. It can mean math at school, or math as it's used by professionals: like marketing research, politicians, or scientists. Just all of mathematics everywhere. Describe mathematics to me. (Probes: Overall, how do you hope your students will view mathematics? What is different or similar about how they view mathematics?)

2. Next, what is your favorite strategy for teaching math (with or without tools) - describe it to me? (Probes: Why? How does it help students? Do you often get to use that strategy? Why or why not?)

3. Now, in the course, we talked about a lot of different ways of thinking about mathematics, what it means to do mathematics, and what it means to learn mathematics - with and without technology. I took a list of words from evaluations and from some of our readings from this

semester. I'm sending a link through the Zoom chat window to a Google slides. Will you follow the link? I will also share screens but I'd like you to be able to interact with the slides.

a) I'm showing you a number of descriptions of what mathematics is. Please delete any that you really don't agree with. Feel free to edit any or add any that are missing. Please choose four or five of your favorite - and then tell me what they mean to you.

b) I'm showing you a number of descriptions of what doing mathematics looks like. Same process - Please delete any that you really don't agree with. Feel free to edit any or add any that are missing. Please choose four or five of your favorite - and then tell me what they mean to you.

c) Finally, I'm showing you a number of descriptions of what successful mathematics learning looks like. Same process - Please delete any that you really don't agree with. Feel free to edit any or add any that are missing. Please choose four or five of your favorite - and then tell me what they mean to you.

4. Munter (2014) Let's pretend: One of your friends who is a teacher asked you to come and observe them teaching a math lesson that uses technology. They asked you to give them feedback to help them make the lesson a high quality technology-integrated math lesson. Based on your thinking over the past semester, what kinds of things would you look for during the observation to decide whether the lesson is high quality? Think about: mathematics, teaching practices, and how the students interact with the technology and/or the mathematics. Depending on the participant's response, I asked the following questions:

    a) Why do you think it is important to use/do ______________ in a math classroom?

    b) Is there anything else you would look for? If so, what? Why?

    c) What math tasks that use technology do you think the teacher (or students) should be using for instruction to be of high quality?

    d) What are some of the things that the teacher (or students) should actually be doing with the technology for instruction to be of high quality?

e) How would the teacher or students interact? Can you please describe what classroom discussion would look and sound like if instruction was of high quality?

5. Additional questions based on evaluations & reflections: [different for different participants]

## Part V. Closing Comments

Thank you so much for taking the time to meet with me; I know this is a busy time of year for you! Is there anything you want to add? I will keep all of your information that I use for my research project, including your classwork and this interview, confidential. Do you have any questions for me? Would you like me to send you my findings? Thank you again!

## APPENDIX E

## FINAL EVALUATION TEMPLATE

1. **Intended Use**

   Describe briefly how will use the tool or activity. This use can be hypothetical, but we need a description to be able to address some of the evaluation criteria. The same applet, for example, could be used in different ways that might be Present, Explore, or Practice learning activities.

2. **Learning Activity Types**

   List here one or more activity types from list below that characterize the learning activity you described. (You can simply delete the activity types you are NOT using.) Add a few words of explanation if the choice is not obvious. As you respond to the subsequent evaluation questions, keep these activity types in mind, referring to them as needed.

   - Present - (read or attend to) presentation of new content/ideas
     - Present-Demo - demonstration
     - Present-Explain - explanation
   - Explore - exploring/investigating mathematical ideas
   - Apply - applying mathematics to problems and situations
   - Practice - practicing for fluency
   - Review - reviewing
   - Assess - assessment
   - other? If the existing activity types dont capture your activity, propose a new one. Well be refining and expanding the list as we go.

3. **Mathematics Content and Learning goals**

a) What mathematics is being learned?

Description of the mathematics being learned (or intended to be learned) through activity with the tech tool. Include whether relational or instrumental understanding is emphasized.

b) Content Standards

What NCTM content standards or CCSM standards are addressed? List key NCTM or CCSM standards addressed; you dont need to list both.

c) Proficiency Strands/NCTM Process Standards/CCSM Mathematical Practices

You may choose one of the three frameworks or use a mixture to discuss which proficiency strands addressed.

d) (opt.) How does activity/focus fit with your current curriculum/lesson goals?

This item is here to remind you to consider how the activity/tech tool fits with what you are already doing. Is it a first presentation of a topic that you will continue to focus on with other activities? Is it an application activity to complement more straightforward presentation of math content? There is clearly overlap here with the activity types and content items above; the key here is thinking about these in relation to the particular use you will make of the activity or tool. Omit this item if it isnt relevant because you arent planning an actual use.

4. **What role does technology play?**

a) Which affordances of technology are important? Are there any constraints that are important to note?

What does the technology do that makes the tool special? For example, does it represent the mathematics in ways not possible without the technology? Does it allow learners to interact with the content in particular ways? Does it provide access to content?

b) Are there other tools or activities (non-tech-heavy) that might support learning goals better?

c) How is the mathematics represented?

5. **Learners**

   a) What do learners need to know?

      What technology and other background knowledge/skills will learners need? How much non-math learning will be needed to be able to use the tool/activity?

   b) Aesthetics and Engagement

      Is the tool/site attactive to students? Is it likely to be engaging?

   c) Is the tool accessible/usable by all students?

      Consider issues of accessibility, UDL, learning styles, etc. here.

   d) Differentiation

      Does the tool allow for modifying/adjusting for different student needs?

6. **Learning/Instruction Support**

   Issues to address here will depend on what sort of learning activity is involved. Following are examples of questions that may be relevant for this tech tool/activity.

   a) Can students work at their own pace?

   b) Does the website/app give feedback?

   c) Does the student get multiple attempts?

   d) Does the website/app give hints if a student is struggling?

   e) Are there instructions on the app/website for the student?

7. **Teacher**

   a) Are there particular knowledge or start-up skills or time you (as the teacher) will need?

      Are there technology and other background knowledge/skills you will need? How much learning/set-up will be required? It is important to consider these issues when considering use in your classroom. For example, if you have to spend a lot of time learning how to use a tool or setting it up for your students, you may decide its not a good choice, or something to be used next year.

b) Managing and tracking student activity

Consider here features of the tool for managing and tracking student activity. If the tool doesnt have such features, will the teacher need to figure out how to manage/track student activity?

8. **Technology Access and Logistics**

This section is both to consider your own situation (will the tool work in your classroom?) as well as provide information to other teachers considering use of the tool/activity. This is the place to think about technical/logistic hitches before you use the tool with your students. Consider here issues such as:

a) Is it compatible with different operating systems?

b) What devices (smart phone, ipad, etc) does it work on?

c) Can the student gain access from multiple locations?

d) Do students need to create separate accounts? (and is this permitted by school and/or parents?)

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