A TOPOLOGICAL STUDY OF TOROIDAL DYNAMICS

By

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ABSTRACT

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This dissertation focuses on developing theoretical tools in the field of Topological Data Analysis and more specifically, in the study of toroidal dynamical systems. We make contributions to the development of persistent homology by proving Künneth-type theorems, to topological time series analysis by further developing the theory of sliding window embeddings, and to multiscale data coordinatization in topological spaces by proving stability theorems.

First, in classical algebraic topology, the Künneth theorem relates the homology of two topological spaces with that of their product. We prove Künneth theorems for the persistent homology of the categorical and tensor product of filtered spaces. That is, we describe the persistent homology of these product filtrations in terms of that of the filtered components. Using these theorems, we also develop novel methods for algorithmic and abstract computations of persistent homology. One of the direct applications of these results is the abstract computation of Rips persistent homology of the N-dimensional torus.

Next, we develop the general theory of sliding window embeddings of quasiperiodic functions and their persistent homology. We show that the sliding window embeddings of quasiperiodic functions, under appropriate choices of the embedding dimension and time delay, are dense in higher dimensional tori. We also explicitly provide methods to choose these parameters. Furthermore, we prove lower bounds on Rips persistent homology of these embeddings. Using one of the persistent Künneth formulae, we provide an alternate algorithm to compute the Rips persistent homology of the sliding window embedding, which outperforms the traditional methods of landmark sampling in both accuracy and time. We also apply our theory to music, where using sliding windows and persistent homology, we

characterize dissonant sounds as quasiperiodic in nature.

Finally, we prove stability results for sparse multiscale circular coordinates. These coordinates on a data set were first created to aid non-linear dimensionality reduction analysis. The algorithm identifies a significant integer persistent cohomology class in the Rips filtration on a landmark set and solves a linear least squares optimization problem to construct a circled valued function on the data set. However, these coordinates depend on the choice of the landmarks. We show that these coordinates are stable under Wasserstein noise on the landmark set.

To my family & friends

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CHAPTER 1

INTRODUCTION

The story begins with the need to understand time series data. Such data prevails in this world and makes for an important object to study. As a result, there are many methods to study it. Some of these include the Discrete Fourier Transform, Wavelet analysis, correlation methods, etc. One method for detecting recurrence in time series data was recently developed [59] using ideas from geometry and topology. This novel framework used two key ingredients, sliding window embeddings and persistent homology.

The first of these ingredients, *sliding window embeddings*, were originally used in the study of dynamical systems to reconstruct the topology of underlying dynamics, under smoothness conditions given by Takens [67], from generic observation functions. It was shown in [59] that the sliding window embedding of a periodic function is homeomorphic to a circle. We study a closely related class of functions, namely *quasiperiodic functions*, which are defined as a superposition of periodic functions with non-commensurate harmonics. Such dynamics appear naturally in biphonation phenomena in mammals [74], as well as in the transition to chaos in rotating fluids [37].

An important point conveyed by Takens' theorem [67] is that the topological shape of the sliding window embedding can be measured and leveraged for applications. To measure this shape, we turn to our second ingredient, *persistent homology* [28, 76]. It is an algebraic and computational tool widely used in Topological Data Analysis (TDA) to quantify multiscale features of shapes. The input to a persistent homology computation is a sequence of topological spaces called a *filtration*, while the output is a multiset of intervals called a *barcode*, which constitutes a finite topological invariant of the coarse geometry governing the shape of a given point cloud.

Some of its applications to science and engineering include recurrence detection in time series data [53, 57, 59], coverage problems in sensor networks [23], the identification of spaces of fundamental features from natural scenes [15], inferring spatial properties of unknown environments via biobots [26], and many more. Some of the underlying ideas of persistence are also starting to permeate pure mathematics, most notably symplectic geometry [61, 70].

Although persistent homology is informative about the topology of data sets, one can further use it to perform nonlinear dimensionality reduction, for example on the point cloud obtained from sliding window embeddings, for the purposes of visualization and reconstruction of the underlying dynamics. This recoordinatization process, called *Eilenberg-MacLane coordinatization*, turns persistent cohomology classes into maps from point clouds into various Eilenberg-MacLane spaces, like the circle [25, 55], real or complex projective space [54], or lens spaces [60].

This dissertation focuses on developing theoretical tools in the field of TDA and more specifically, in the study of dynamical systems on tori. There are three major contributions:

1. **Theoretical development of persistent homology:** We develop two persistent Künneth formulae, that is, results relating persistent homology of two filtered spaces to persistent homology of their products. Leveraging these formulae, we present novel methods for algorithmic and abstract computations of persistent homology.

2. **Theoretical foundations of sliding window embeddings:** We prove foundational results in the theory of sliding window embeddings for quasiperiodic functions and develop a theoretical framework to detect quasiperiodicity in time series data using sliding window embeddings and a persistent Künneth formula for Rips complexes.

3. **Stability results for circular coordinates:** The circular coordinatization algorithm identifies a significant Rips persistent cohomology class on a subset of the point cloud called the *landmarks*, and solves a linear least squares optimization problem to construct a circle valued function on the data set. We prove that circular coordinates are stable under Wasserstein noise on the landmark set.

1.1 Outline & Overview of the dissertation

Due to the diverse nature of research problems discussed here, a wide range of mathematical concepts were used. While a comprehensive review for all of them is beyond the scope of this dissertation, we provide the necessary background in Chapter 2. We prove the two persistent Künneth formulae in Chapter 3 and revisit the persistent homology of an N-torus as an application. In Chapter 4, we develop the theory of sliding window embeddings for quasiperiodic functions. Using theorems from Chapter 3, we provide an alternate strategy to recover toroidal features from the sliding window point clouds. We also present an application where we characterize dissonance in music theory as a quasiperiodic feature. In Chapter 5, we prove stability results for the construction of sparse circular coordinates algorithm under Wasserstein noise on the landmarks. Next we provide an overview of the main results in this dissertation.

1.1.1 Persistent Künneth theorems

For a Principal Ideal Domain (PID) R and topological spaces X and Y, the classical Künneth formula given by the split natural short exact sequence

$$0 \to \bigoplus_i (H_i(X) \otimes_R H_{n-i}(Y)) \to H_n(X \times Y) \to \bigoplus_i (\mathsf{Tor}_R(H_i(X), H_{n-i-1}(Y))) \to 0 \quad (1.1)$$

relates the homology of the product space $X \times Y$ to the homology of spaces X and Y, with coefficients in R. As discussed above, the persistent homology algorithm takes a sequence of topological spaces or simplicial complexes, called a filtration

$$\mathcal{X} : X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \cdots \hookrightarrow X_d \hookrightarrow \cdots$$

where $d \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, as the input. It returns in homological dimension n, a barcode $\mathsf{bcd}_n(\mathcal{X})$ which is a collection of real intervals, or equivalently a persistence diagram $\mathsf{dgm}_n(\mathcal{X})$, which is a multiset of points in the plane $\mathbb{R}^2$. In this dissertation, we exploit the equivalence of these two invariants and use them interchangeably. A brief review is provided in section 2.3.

In Chapter 3, we develop Künneth formulae for persistent homology. For filtered topological spaces $\mathcal{X}$ and $\mathcal{Y}$, we consider two notions of product filtrations:

$$(\mathcal{X} \times \mathcal{Y})_d = X_d \times Y_d \qquad \text{and} \qquad (\mathcal{X} \otimes \mathcal{Y})_d = \bigcup_{i+j=d} X_i \times Y_j.$$

These choices are motivated by different observations. The first choice of filtration, which we refer to as the *categorical product*, is of interest for two reasons: (a) it is the product in the category of filtered topological spaces, and (b) it is relevant in certain computational settings. More precisely, if $(X, \mathbf{d}_X)$ is a metric space and

$$R_\epsilon(X, \mathbf{d}_X) = \{\sigma \subset X : |\sigma| < \infty, \mathsf{diam}(\sigma) \leq \epsilon\}$$

is its Rips complex at parameter ϵ, then $R_\epsilon(X \times Y, \mathbf{d}_{X\times Y}) = R_\epsilon(X, \mathbf{d}_X) \times R_\epsilon(Y, \mathbf{d}_Y)$ [1, Proposition 10.2]. Here $\mathbf{d}_{X\times Y}$ is the maximum metric

$$\mathbf{d}_{X\times Y}\big((x,y),(x',y')\big) = \max\{\mathbf{d}_X(x,x'), \mathbf{d}_Y(y,y')\},$$

and the product of Rips complexes is in the category of abstract simplicial complexes (see section 3.1). The other choice of filtration, which we call the *tensor product*, is relevant due to an algebraic resemblance between the short exact sequence it satisfies and the classical equation 1.1. In Figure 1.1, we see an example of a topological circle $\mathcal{X}$, while in Figure 1.2, we see two filtrations of a 2-torus defined by our products.

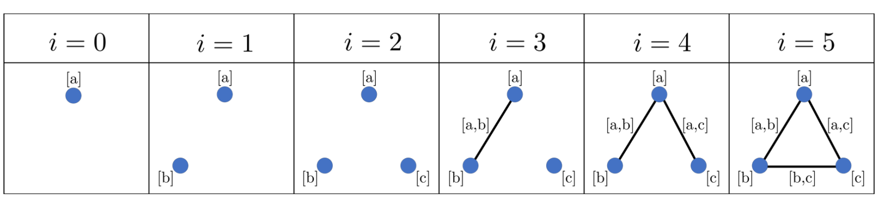


Figure 1.1: A filtered simplicial circle $\mathcal{X}$. The numbers next to simplex indicate when that corresponding simplex appeared in the filtration.

In the following theorem, we state the two Künneth formulae in terms of their barcode representation:

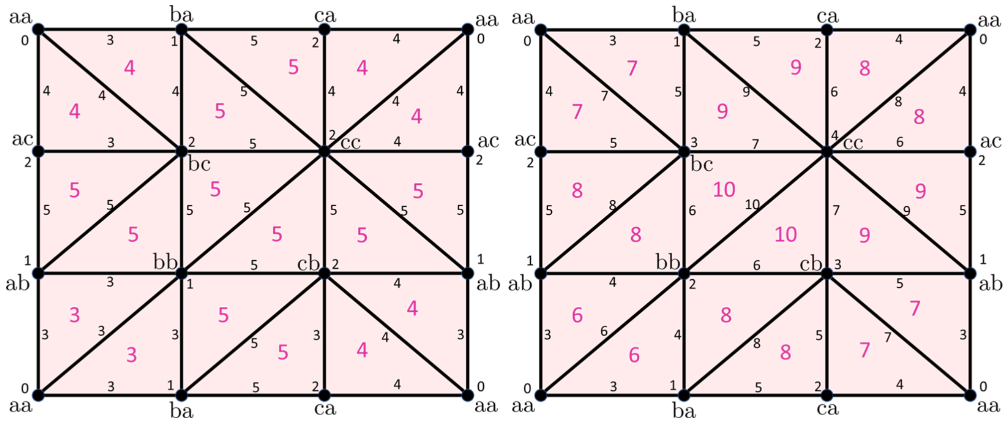


Figure 1.2: The product filtrations $\mathcal{X} \times \mathcal{X}$ (left) and $\mathcal{X} \otimes \mathcal{X}$ (right). The numbers next to simplex indicate when that corresponding simplex appeared in the filtration.

Theorem 1.1.1. *Let $\mathcal{X}$ and $\mathcal{Y}$ be pointwise finite filtrations. Then the barcodes of the categorical product $\mathcal{X} \times \mathcal{Y}$ are given by the (disjoint) union of multisets*

$$\mathsf{bcd}_n\left(\mathcal{X} \times \mathcal{Y}\right) = \bigcup_{i+j=n} \left\{ I \cap J \;\middle|\; I \in \mathsf{bcd}_i(\mathcal{X}),\ J \in \mathsf{bcd}_j(\mathcal{Y}) \right\}. \tag{1.2}$$

Similarly, the barcodes for the tensor product filtration $\mathcal{X} \otimes \mathcal{Y}$ satisfy

$$\mathsf{bcd}_n(\mathcal{X} \otimes \mathcal{Y}) = \bigcup_{i+j=n} \left\{ (\ell_J + I) \cap (\ell_I + J) \;\middle|\; I \in \mathsf{bcd}_i(\mathcal{X}),\ J \in \mathsf{bcd}_j(\mathcal{Y}) \right\}$$
$$\bigcup$$
$$\bigcup_{i+j=n-1} \left\{ (\rho_J + I) \cap (\rho_I + J) \;\middle|\; I \in \mathsf{bcd}_i(\mathcal{X}),\ J \in \mathsf{bcd}_j(\mathcal{Y}) \right\}$$

where ℓ_J and ρ_J denote, respectively, the left and right endpoints of the interval J.

The first Künneth formula implies the following corollary for the Rips persistent homology (i.e. persistent homology of Rips filtration) of product metric spaces:

Corollary 1.1.2. *Let $(X, \mathbf{d}_X), (Y, \mathbf{d}_Y)$ be finite metric spaces and let $\mathsf{bcd}_n^{\mathcal{R}}(X, \mathbf{d}_X)$ be the n-th dimensional barcode of the Rips filtration $\mathcal{R}(X, \mathbf{d}_X) := \{R_\epsilon(X, \mathbf{d}_X)\}_{\epsilon \geq 0}$. Then,*

$$\mathsf{bcd}_n^{\mathcal{R}}(X \times Y, \mathbf{d}_{X \times Y}) = \bigcup_{i+j=n} \left\{ I \cap J \;\middle|\; I \in \mathsf{bcd}_i^{\mathcal{R}}(X, \mathbf{d}_X),\ J \in \mathsf{bcd}_j^{\mathcal{R}}(Y, \mathbf{d}_Y) \right\}$$

for all $n \in \mathbb{N}_0$, where $\mathbf{d}_{X \times Y}$ is the maximum metric.

The above formulae also hold true when the inclusions $X_i \hookrightarrow X_{i+1}$ and $Y_i \hookrightarrow Y_{i+1}$ are replaced by continuous maps. The categorical product of $\mathbb{R}$-indexed diagrams is just the $\mathbb{R}$-indexed diagram defined using index-wise products. That is,

$$(\mathcal{X} \times \mathcal{Y})_r = X_r \times Y_r$$

for all $r \in \mathbb{R}$. The *generalized tensor product* $\otimes_{\mathbf{g}}$ of $\mathbb{N}_0$-indexed diagrams —equivalent to $\otimes$ for inclusions— is defined as follows: the space $(\mathcal{X} \otimes_{\mathbf{g}} \mathcal{Y})_k$ is the *homotopy colimit* (see section 2.2.2) of the functor from the poset $\triangle_k = \{(i,j) \in \mathbb{N}^2 : i+j \leq k\}$ to **Top** sending (i,j) to $X_i \times Y_j$, and $(i \leq i', j \leq j')$ to the map $X_i \times Y_j \to X_{i'} \times Y_{j'}$ induced by composite maps $X_i \to X_{i'}$ and $Y_j \to Y_{j'}$. The map $(\mathcal{X} \otimes_{\mathbf{g}} \mathcal{Y})_k \to (\mathcal{X} \otimes_{\mathbf{g}} \mathcal{Y})_{k+1}$ is the one induced at the level of homotopy colimits by the inclusion $\triangle_k \subset \triangle_{k+1}$.

1.1.1.1 Related work

Different versions of the algebraic Künneth theorem in persistent homology have appeared in the last three years. In [62, Proposition 2.9], the authors prove a Künneth formula for the tensor product of filtered chain complexes, while [12, Section 10] establishes Künneth theorems for the graded tensor product and the sheaf tensor product of persistence modules. In [14], the authors prove a Künneth formula relating the persistent homology of metric spaces $(X, \mathbf{d}_X), (Y, \mathbf{d}_Y)$ to the persistence of their cartesian product equipped with the L^1 (sum) metric $(X \times Y, \mathbf{d}_X + \mathbf{d}_Y)$, for homological dimensions $n = 0, 1$. The persistent Künneth theorems proven in this dissertation and [35] are the first to start at the level of filtered spaces, identifying the appropriate product filtrations and resulting barcode formulae. Compared to the sum metric [14], we remark that our results for the maximum metric hold in all homological dimensions.

1.1.2 Sliding window embeddings of quasiperiodic functions

The sliding window embedding of a function $f : \mathbb{R} \to \mathbb{C}$ for a fixed *embedding dimension* $d \in \mathbb{N}$ and *time delay* $\tau > 0$, at $t \in \mathbb{R}$ is defined as

$$SW_{d,\tau} f(t) = \begin{bmatrix} f(t) \\ f(t+\tau) \\ \vdots \\ f(t+d\tau) \end{bmatrix}.$$

This definition is motivated by Takens' embedding theorem which we cover in section 2.4.1. In its essence, the theorem provides conditions under which a smooth attractor can be reconstructed from a sequence of observations made with a generic observation function. While this reconstructed attractor might have a different geometric shape than the original, its topology is still preserved.

As discussed at the beginning of this chapter, if the set of harmonics controlling a function is incommensurate, that is if it is linearly independent over $\mathbb{Q}$, we call the function a *quasiperiodic function*. More rigorously, we define $f : \mathbb{R} \to \mathbb{C}$ to be *quasiperiodic with a given set of frequencies* $\{\omega_1, \dots, \omega_N\}$ if there is a function $F : \mathbb{R}^N \to \mathbb{C}$, called a *parent function*, such that F is periodic in the n-th variable with period $\frac{2\pi}{\omega_n}$, the restriction of F to the diagonal takes the values of f, and N is minimal. See Definition 4.1.1. Assuming that the parent function is continuously differentiable and using tools from multivariate harmonic analysis, we can write a Fourier-like series that converges everywhere to F and its restriction to the diagonal gives us the following series for the quasiperiodic function f:

$$f(t) = F(t, \dots, t) = \sum_{\mathbf{k} \in \mathbb{Z}^N} \hat{F}(\mathbf{k}) e^{it\langle \mathbf{k}, \omega \rangle}$$

$$\hat{F}(\mathbf{k}) = \frac{\prod_{n=1}^{N} \omega_n}{(2\pi)^N} \int_0^{\frac{2\pi}{\omega_1}} e^{-ik_1\omega_1 t_1} \int_0^{\frac{2\pi}{\omega_2}} \cdots \int_0^{\frac{2\pi}{\omega_N}} F(t_1, \dots, t_N) e^{-ik_N \omega_N t_N} dt_N \dots dt_1.$$

where ω is the frequency vector $(\omega_1, \dots, \omega_N)$ and $\langle \cdot, \cdot \rangle$ is the dot product in $\mathbb{R}^N$. As in single variable Fourier analysis, we can use a truncation parameter $Z \in \mathbb{N}_0$ to approximate f by the partial sum polynomials $S_Z f$:

$$S_Z f(t) = \sum_{\mathbf{k}\in I_Z^N} \hat{F}(\mathbf{k}) e^{it\langle \mathbf{k},\omega\rangle}$$

where $I_Z^N = \left\{\mathbf{k}\in\mathbb{Z}^N \;\middle|\; ||\mathbf{k}||_\infty \leq Z\right\}$. Since $S_Z f(t) \to f(t)$ everywhere as $Z\to\infty$ and since for fixed parameters d,τ, the map $f\to SW_{d,\tau}f$ is a bounded linear operator (see Proposition 2.4.8), we conclude that $SW_{d,\tau}S_Z f \to SW_{d,\tau}f$ everywhere as $Z\to\infty$. The next theorem gives us a series of bounds —for different levels of regularity of f— on the proximity of sliding window embeddings of f to that of its truncated polynomial $S_Z f$. Moreover, the more regular f is, the faster the convergence of sliding window embeddings will be. Then by the stability theorem for persistent homology [19], we have the following amalgamation of theorem 4.2.8 and corollary 4.2.9:

Theorem 1.1.3. *Let $r\in\mathbb{N}\setminus\{1\}$. If $F\in C^r\left(\mathbb{R}^N,\mathbb{C}\right)$, then for any bounded $T\subset\mathbb{R}$,*

$$\begin{aligned} d_B\left(\mathsf{bcd}_i^{\mathcal{R}}(SW_{d,\tau}f(T)),\mathsf{bcd}_i^{\mathcal{R}}(SW_{d,\tau}S_Z f(T))\right) &\leq 2d_H\left(SW_{d,\tau}f(T),SW_{d,\tau}S_Z f(T)\right) \\ &\leq 2\frac{\sqrt{N}^r\left(Area(S^{N-1})\right)^{1/2}}{\omega_{\mathsf{min}}^{r-1}\sqrt{2r-2}\left(\prod\limits_{n=1}^{N}|\omega_n|\right)^{1/2}}\left(\sum_{n=1}^{N}||R_Z(\partial_n^r F)||^2\right)^{1/2}\frac{\sqrt{d+1}}{Z^{r-1}} \end{aligned}$$

where $R_Z f := f - S_Z f$, and d_H and d_B are the Hausdorff and the Bottleneck distance respectively (see section 4.4).

The theorem tells us that as the sliding window point cloud $SW_{d,\tau}S_Z f(T)$ approximates $SW_{d,\tau}f(T)$ in the Hausdorff sense, the Rips persistent homology $\mathsf{bcd}_i^{\mathcal{R}}(SW_{d,\tau}S_Z f)$ converges to $\mathsf{bcd}_i^{\mathcal{R}}(SW_{d,\tau}f)$ in the Bottleneck sense. This suggests that one could replace the study of a function f with that of its approximation $S_Z f$. Accordingly, we study the geometric structure of the sliding window embedding of the truncated polynomial $S_Z f$. We see next that under appropriate conditions on d, τ, and Z, the closure of the sliding window embedding $SW_{d,\tau}S_Z f$ is homeomorphic to $\mathbb{T}^N$ wrapped around $\mathbb{T}^{d+1}$. Indeed, we have the following structure theorem:

Theorem 1.1.4. *The sliding window embedding* $SW_{d,\tau}S_Zf(\mathbb{Z}) = \left\{SW_{d,\tau}S_Zf(z) \mid z \in \mathbb{Z}\right\}$ *is dense in a space homeomorphic to* $\mathbb{T}^N$ *in* $\mathbb{T}^{d+1}$ *if*

- $d+1 \geq \alpha :=$ *the number of terms in* I_Z^N *with non zero coefficients, and*
- τ *satisfies that* $G(\tau) < (d+1)^2$ *for*

$$G(x) = \sum_{\mathbf{k}\neq\mathbf{l}\in I_Z^N} \left|1 + e^{ix\langle\mathbf{k}-\mathbf{l},\omega\rangle} + \cdots + e^{ix\langle\mathbf{k}-\mathbf{l},\omega\rangle d}\right|^2.$$

The condition on τ only guarantees the same topology as a torus. Unfortunately, the Rips persistent homology cares about the underlying geometry and that requires τ to satisfy an extra condition: for some practically reasonable large $\Upsilon \in \mathbb{R}^+$, τ satisfies

$$\tau = \underset{x\in[0,\Upsilon]}{\operatorname{argmin}}\ G(x).$$

The last main result we present here pertains to the Rips persistent homology of $SW_{d,\tau}S_Zf$. Let λ_{min} be the square of the minimum singular value of

$$\Omega_{Z,f} = \begin{bmatrix} e^{i\langle\mathbf{k}_1,\omega\rangle\tau 0} & \dots & e^{i\langle\mathbf{k}_\alpha,\omega\rangle\tau 0} \\ \vdots & \vdots & \vdots \\ e^{i\langle\mathbf{k}_1,\omega\rangle\tau d} & \dots & e^{i\langle\mathbf{k}_\alpha,\omega\rangle\tau d}, \end{bmatrix}$$

and $|\hat{F}(\mathbf{k})_{\mathsf{min}}|$ be the smallest non-zero Fourier coefficient in absolute value. Then we have the following theorem:

Theorem 1.1.5. *For* $i \leq N$, *there will be at least* $\binom{N}{i}$ *bars*

$$(0, d_1), (0, d_2), \ldots, (0, d_{\binom{N}{i}})$$

where $d_n \geq \sqrt{3}|\hat{F}(\mathbf{k})_{\mathsf{min}}|\sqrt{\lambda_{\mathsf{min}}}$ *in* $\mathsf{bcd}_i^{\mathcal{R}}\left(SW_{d,\tau}S_Zf, ||\cdot||_2\right)$.

We finish this section with an example:

Example 1.1.6. Assume that we are given a quasiperiodic time series $f(t) = \cos(t) + \cos(\sqrt{3}t)$. The sliding window embedding $SW_{d,\tau}f(\mathbb{R})$ of f with appropriate dimension and delay recovers the topology of $\mathbb{T}^2$. Rips persistent homology confirms that in the accompanying figure.

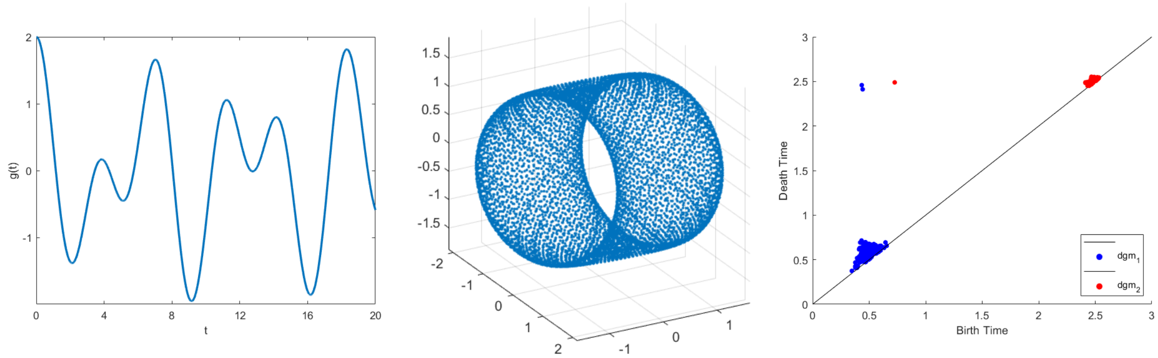


Figure 1.3: In the leftmost figure, we see a quasiperiodic function. In the middle figure, we see a PCA visualization of the reconstructed torus from sliding window embedding. In the rightmost figure, we see the 1 and 2-dimensional Rips persistence diagrams (blue and red points respectively) of the sliding window point cloud.

1.1.2.1 Related work

Since the inception of sliding windows and persistence technique for time series analysis in [59], there have been a few advancements to both theory and applications. The first results on sliding window embeddings for quasiperiodic functions [53] were restricted to the special case of sums of exponentials with incommensurate frequencies. Said results are proved in generality in this dissertation and [36]. While the sliding window embeddings of quasiperiodic functions trace tori densely, in [75], the authors construct families of time series whose sliding window embeddings trace Klein bottles, spheres, and projective planes. A recently published survey ties the worlds of dynamical systems and topological data analysis [57]. Among the applications of sliding window embeddings and persistence, a few are: periodicity quantification in gene expression data [58]; (quasi)periodicity detection in video data [69]; synthesis of slow-motion videos from recurrent movements [68]; detection of wheeze, which is an abnormal whistling sound produced while breathing [33]; and detection and classification of chatter (vibrations caused by machines) in machining [45, 46].

1.1.3 Stability of circular coordinates

Circular coordinates on a data set were developed to aid non-linear dimensionality reduction analysis [25]. Such reduction schemes address the problem of representing high dimensional data in terms of low dimensional coordinates. The authors used persistent cohomology to construct circular coordinates on a data set X. The algorithm identifies a significant integer persistent cohomology class on the Rips filtration and solves a linear least squares optimization problem to construct a circled valued function on the data set.

Using similar ideas and principal $\mathbb{Z}$-bundles, the *sparse* circular coordinates were constructed by using a landmark set $L = \{\ell_1, \ldots, \ell_N\}$ in lieu of the entire data set X [55]. The idea is as follows. For a fixed prime $q > 2$, compute the 1-dimensional Rips persistent cohomology $PH^1(\mathcal{R}(L); \mathbb{Z}_q)$. Choose a bar $[a, b)$ from the resultant barcode $\mathsf{bcd}^{\mathcal{R}}(L)$ satisfying $2a < b$. Let $2a < 2\alpha < b$ and let $\eta'' \in Z^1(R_{2\alpha}(L); \mathbb{Z}_q)$ be a cocycle representative for the persistent cohomology class corresponding to $[a, b)$. Lift η'' to an integer cocycle $\eta' \in Z^1(R_{2\alpha}(L); \mathbb{Z})$ such that $\eta'' - \eta' \mod q$ is a coboundary in $C^1(R_{2\alpha}(L); \mathbb{Z}_q)$. Lastly, define η to be the image of η' under $C^1(R_{2\alpha}(L); \mathbb{Z}) \to C^1(R_{2\alpha}(L); \mathbb{R})$ induced by the inclusion $\iota : \mathbb{Z} \hookrightarrow \mathbb{R}$. Choose positive weights for the vertices and edges of $R_{2\alpha}(L)$ and let $d_{2\alpha}^{+}$ be the weighted Moore-Penrose pseudoinverse of the coboundary map

$$d_{2\alpha} : C^0(R_{2\alpha}(L); \mathbb{R}) \to C^1(R_{2\alpha}(L); \mathbb{R}).$$

and define

$$\tau := -d_{2\alpha}^{+}(\eta) \quad \text{and} \quad \theta := \eta + d_{2\alpha}(\tau).$$

Denote $\tau_j := \tau([\ell_j])$ and $\theta_{jk} := \theta([\ell_j, \ell_k])$ when $[\ell_j, \ell_k] \in R_{2\alpha}(L)$, and 0 otherwise. Let $B_\alpha(\ell_j)$ denote the ball of radius α centered around $\ell_j \in L$, and $\{\varphi_j\}_j$ a partition of unity subordinate to the collection $\{B_\alpha(\ell_j)\}_j$. The sparse circular coordinates are defined as follows:

$$\begin{aligned} h_{\theta,\tau} : \bigcup_{r=1}^{N} B_\alpha(\ell_r) &\to S^1 \subset \mathbb{C} \\ B_\alpha(\ell_j) \ni b &\mapsto \exp\left\{2\pi i \left(\tau_j + \sum_{r=1}^{N} \varphi_r(b)\theta_{jr}\right)\right\}. \end{aligned}$$

We give a more detailed version of this algorithm in section 5.0.1.

Observe that the coordinates generated from this algorithm depend on the choice of landmarks. For any two finite landmark sets $L, \tilde{L}$ with probability weights ω and $\tilde{\omega}$, we obtain two circular coordinate functions h and $\tilde{h}$ respectively. Assume that

$$d_{\mathcal{W},\infty}((L,\omega),(\tilde{L},\tilde{\omega})) < \delta$$

where $d_{\mathcal{W},\infty}$ is the Wasserstein-Kantorovich-Rubinstein metric as in Definition 2.5.5. Our first result shows the stability of circular coordinates for a special case, that is when L and $\tilde{L}$ are in a weight preserving bijective correspondence. We call this the *geometric stability* of circular coordinates. The following theorem appears in more detail as theorem 5.2.7.

Theorem 1.1.7. *If (L,ω) and $(\tilde{L},\tilde{\omega})$ are landmark sets in a weight preserving bijective correspondence, and h and $\tilde{h}$ are the associated circle-valued functions, then*

$$||h - \tilde{h}||_\infty \le C\delta$$

where C is a term dependent on the choice of weights, the harmonic cocycle, and partitions of unity used in the construction of circular coordinates.

Let us look at the following example:

Example 1.1.8. We start with a dataset X with 1000 points sampled around a unit circle with Gaussian noise of 0.5. We first select the "blue" landmark set with 40 points by `maxmin` sampling. We define the "red" landmark set by choosing the respective nearest neighbours of the "blue" landmarks. See Figure 1.4.

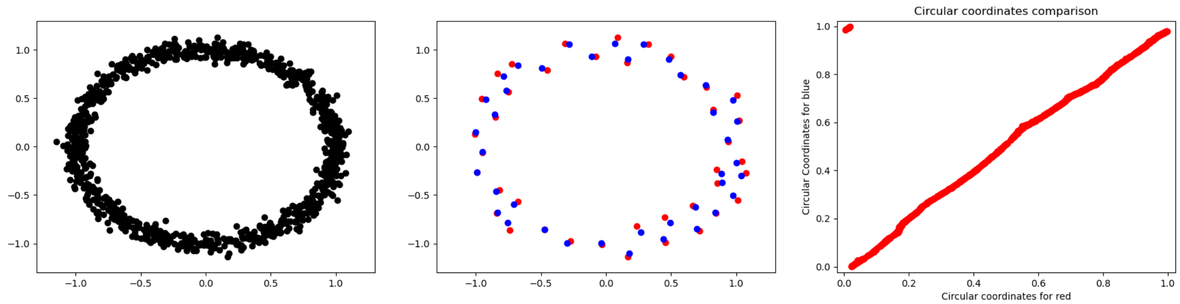


Figure 1.4: (Left) The data set X of 1000 points sampled around a circle, with two different sets of 40 landmarks (Middle). For each of these sets, we obtain a function that assigns to each data point in X a circular coordinate. (Right) We see a comparison of the two coordinate functions.

In a more general scenario, L and $\tilde{L}$ might have different cardinalities or vertex probability weights. In section 5.3.1, we introduce a way to create two new landmark sets (L_m, ω_m), $(\tilde{L}_m, \tilde{\omega}_m)$ such that they are in a weight preserving bijective correspondence, thereby reducing the case of *Wasserstein stability* to the case of *geometric stability*. Both landmark sets $L_m, \tilde{L}_m$ are just $L, \tilde{L}$ respectively, augmented with copies of points. The next result describes the relationship between the circular coordinates generated by L and the augmented L_m (and similarly for $\tilde{L}$ and $\tilde{L}_m$).

Theorem 1.1.9. *Let L_m be the landmark set augmented from L, and h_m and h be the circular functions associated to them. Then for all $b \in X$, $h_m(b) = h(b)$.*

See theorem 5.3.13 for more details. As a consequence, we can replace $h, \tilde{h}$ with $h_m, \tilde{h}_m$ to obtain

$$||h - \tilde{h}||_\infty = ||h_m - \tilde{h}_m||_\infty$$

where the right hand side can be bounded using bounds the geometric stability theorem. These arguments are presented in section 5.3.3, in particular summarized in Theorem 5.3.15.

1.1.3.1 Related work

In [25], a strategy for non-linear dimensionality reduction was developed using the first (non-sparse) circular coordinates. In [55], the theory of principal $\mathbb{Z}$-bundles was used to construct sparse circular coordinates by considering Rips filtration on a landmark set $L \subset X$. In [54], the multiscale projective coordinates were constructed, along with a non-linear dimensionality reduction scheme called Principal Projective Component Analysis. Using similar techniques, lens spaces coordinates were constructed in [60]. The authors also presented a non-linear dimensionality reduction scheme called LPCA or Lens Principal Component Analysis.

CHAPTER 2

BACKGROUND

This chapter deals with the necessary mathematical background for the later chapters of this dissertation. We provide a terse introduction to category theory in 2.1. In 2.2, we provide background material for an assortment of topics in topology. Using ideas from these two sections, we give a brief background for persistent homology in 2.3. In section 2.4, we discuss the basics of dynamical systems along with Takens' theorem and sliding window embeddings. In section 2.5, we review one variable Fourier series, Kronecker's approximation theorem, and Wasserstein metric on probability measures.

2.1 Category Theory

In this section, we give a succinct review of category theory. For a more comprehensive review, we refer the mathematician reader to [48] and the data scientist reader to [65]. Category theory was originally developed by Eilenberg and MacLane to bridge the gap between the fields of algebra and topology. More generally, category theory takes a bird eye's view of mathematics by considering abstractions of mathematical concepts. The advantage to this view is that a lot of intricate and subtle results can be proved by 'cleaner' methods. Over the last few decades, category theory has found itself useful in sciences, like quantum physics[2] and theoretical computer science [3].

2.1.1 Definitions and Examples

Definition 2.1.1. A *catgeory* $\mathbf{C}$ consists of a class of objects denoted by $\mathsf{Obj}(\mathbf{C})$ and a set of morphisms $\mathsf{Mor}(a, b)$ for any two objects $a, b \in \mathsf{Obj}(\mathbf{C})$ such that the following axioms hold:

1. If $f \in \mathsf{Mor}(a, b)$ and $g \in \mathsf{Mor}(b, c)$, then $g \circ f \in \mathsf{Mor}(a, c)$.

2. For each $a \in \mathsf{Obj}(\mathbf{C})$, there is a unique morphism $1_a \in \mathsf{Mor}(a, a)$ called the *identity*

morphism satisfying

$$f \circ 1_a = f = 1_b \circ f$$

for any $b \in \mathsf{Obj}(\mathbf{C})$ and any $f \in \mathsf{Mor}(a, b)$.

3. For any $f \in \mathsf{Mor}(a, b)$, $g \in \mathsf{Mor}(b, c)$, and $h \in \mathsf{Mor}(c, d)$,

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Some of the categories that we are interested in are:

Example 2.1.2. **Top** is an example of a category with topological spaces as objects and continuous maps as morphisms.

Example 2.1.3. Given a commutative ring R with unity, $\mathbf{Mod}_R$ is the category of all R-modules as objects and for any two R-modules M, N, the set of morphisms from M to N is the set of all R-module homomorphisms from M to N.

Example 2.1.4. Given a commutative ring R with unity, $\mathbf{Ch}_R$ is the category of all chain complexes of R-modules as objects and for any two such chain complexes C, C', the set of morphisms from C to C' is the set of all chain maps from C to C'.

Definition 2.1.5. A category $\mathbf{C}$ is *small* if the class of its objects, $\mathsf{Obj}(\mathbf{C})$, forms a set.

In some categories, like **Top**, the sets of morphisms may be very large in cardinality. For purposes of this document, we are interested in another special type of category that restricts the size of morphism classes:

Definition 2.1.6. A category is called *thin* if there is at most one morphism between any two objects.

Definition 2.1.7. A *subcategory* $\mathbf{D}$ of $\mathbf{C}$ consists of the following data:

1. A subclass $\mathsf{Obj}(\mathbf{D})$ of $\mathsf{Obj}(\mathbf{C})$,

2. For each $a, b \in \mathsf{Obj}(\mathbf{D})$, a subset of $\mathsf{Mor}_{\mathbf{C}}(a, b)$,

3. For each $a \in \mathsf{Obj}(\mathbf{D})$, the identity $1_a \in \mathsf{Mor}_{\mathbf{C}}(a, a)$ belongs to $\mathsf{Mor}_{\mathbf{D}}(a, a)$, and

4. For each $f : \mathsf{Mor}_{\mathbf{D}}(a, b)$ and $g \in \mathsf{Mor}_{\mathbf{D}}(b, c)$, the composition $g \circ f$ is in $\mathsf{Mor}_{\mathbf{D}}(a, c)$.

Definition 2.1.8. A *full subcategory* $\mathbf{D}$ of $\mathbf{C}$ satisfies

$$\mathsf{Mor}_{\mathbf{D}}(a, b) = \mathsf{Mor}_{\mathbf{C}}(a, b)$$

for any objects $a, b \in \mathsf{Obj}(\mathbf{D})$.

In fact, any subclass of $\mathsf{Obj}(\mathbf{C})$ yields a unique full subcategory of $\mathbf{C}$.

Definition 2.1.9. For a category $\mathbf{C}$, its opposite category $\mathbf{C}^{op}$ is defined as

$$\mathsf{Obj}(\mathbf{C}^{op}) := \mathsf{Obj}(\mathbf{C})$$

$$\mathsf{Mor}_{\mathbf{C}^{op}}(a, b) := \mathsf{Mor}_{\mathbf{C}}(b, a),$$

that is, all the morphisms are reversed.

2.1.2 Functors and natural transformations

Definition 2.1.10. Let $\mathbf{C}$ and $\mathbf{C}'$ be categories. A *covariant functor* D from $\mathbf{C}$ to $\mathbf{C}'$ associates to each object, $a \in \mathsf{Obj}(\mathbf{C})$ an object $D(a) \in \mathsf{Obj}(\mathbf{C}')$ and to each morphism $f \in \mathsf{Mor}_{\mathbf{C}}(a, b)$, a morphism $D(f) \in \mathsf{Mor}_{\mathbf{C}'}(D(a), D(b))$. Moreover, it is required that the functor respects morphism composition and preserves the identity morphisms, that is,

$$D(g \circ f) = D(g) \circ D(f) \quad \text{and} \quad D(1_a) = 1_{D(a)}$$

respectively.

Similarly, a *contravariant functor* D from $\mathbf{C}$ to $\mathbf{C}'$ associates to each object $a \in \mathsf{Obj}(\mathbf{C})$, an object $D(a) \in \mathsf{Obj}(\mathbf{C}')$ and to each morphism $f \in \mathsf{Mor}_{\mathbf{C}}(a, b)$, a morphism $D(f) \in \mathsf{Mor}_{\mathbf{C}'}(D(b), D(a))$. Moreover, it is required that the functor preserves the identity morphisms, however $D(g \circ f) = D(f) \circ D(g)$.

For example, if $\mathbf{C} = \mathbf{Top}$ and $\mathbf{C}' = \mathbf{Mod}_R$, then the functor D that takes a topological space X to the homology module $H_n(X; R)$ is covariant, while the functor D' that takes X to the cohomology module $H^n(X; R)$ is contravariant.

Definition 2.1.11. Let D, D' be covariant functors from C to C'. A *natural transformation* $\eta : D \to D'$ consists of a family of morphisms $\eta(a) : D(a) \to D'(a)$ for each object $a \in \mathsf{Obj}(\mathbf{C})$, such that for each morphism $f \in \mathsf{Mor}_{\mathbf{C}}(a, b)$, the diagram

$$\begin{array}{ccc} D(a) & \xrightarrow{\eta(a)} & D'(a) \\ {\scriptstyle D(f)}\big\downarrow & & \big\downarrow{\scriptstyle D'(f)} \\ D(b) & \xrightarrow[\eta(b)]{} & D'(b) \end{array}$$

commutes.

Definition 2.1.12. If $\mathbf{C}$ and $\mathbf{I}$ are categories, with $\mathbf{I}$ small, then we denote by $\mathbf{C}^{\mathbf{I}}$ the category whose objects are functors from $\mathbf{I}$ to $\mathbf{C}$, and whose morphisms are natural transformations between said functors. The objects of the functor category $\mathbf{C}^{\mathbf{I}}$ are often referred to in the literature as $\mathbf{I}$-*indexed diagrams in* $\mathbf{C}$. Two diagrams $D, D' \in \mathbf{C}^{\mathbf{I}}$ are said to be isomorphic, denoted $D \cong D'$, if they are naturally isomorphic as functors.

Here is an example describing the main type of indexing category we will consider throughout the dissertation.

Definition 2.1.13. A *partially ordered set*, or *poset*, is a set $\mathcal{P}$ with a relation $\preceq$ satisfying the following properties:

1. Reflexivity: for each $x \in \mathcal{P}$, we have $x \preceq x$.

2. Transitivity: for each $x, y, z \in \mathcal{P}$ satisfying $x \preceq y$ and $y \preceq z$, it is also true that $x \preceq z$.

3. Antisymmetry: for each $x, y \in \mathcal{P}$ satisfying $x \preceq y$ and $y \preceq x$, it is true that $x = y$.

Example 2.1.14. Let $(\mathcal{P}, \preceq)$ be a partially ordered set (i.e., a poset). Let $\mathbf{P}$ denote the category whose objects are the elements of $\mathcal{P}$, and a unique morphism $x \to y$ for each pair $x \preceq y$ in $\mathcal{P}$. We call $\mathbf{P}$ the *poset category* of $\mathcal{P}$.

This category is both small and thin and will be used for the purpose of indexing category **I**. As for the target category **C**, we will mostly be interested in spaces and their algebraic invariants.

2.1.3 Coequalizers

The contents of this section will be useful for the theory of homotopy colimits in section 2.2.2.

Definition 2.1.15. The *undercategory* of $c \in \mathbf{C}$, denoted $(c \downarrow \mathbf{C})$, is the category whose objects are pairs (c', σ) consisting of an object $c' \in \mathbf{C}$ and a morphism $\sigma : c \to c'$ in $\mathbf{C}$. A morphism from $(c', \sigma : c \to c')$ to $(c'', \beta : c \to c'')$ is a morphism $\tau : c' \to c''$ in $\mathbf{C}$ such that $\tau \circ \sigma = \beta$.

Remark 2.1.16. If $\sigma : c \to c'$ is a morphism in $\mathbf{C}$, then composing with σ induces covariant functors $\sigma^* : (c' \downarrow \mathbf{C}) \to (c \downarrow \mathbf{C})$ and $\sigma^*_{op} : (c' \downarrow \mathbf{C})^{op} \to (c \downarrow \mathbf{C})^{op}$.

Definition 2.1.17. Let $\{A_i\}_{i \in I}$ be a family of objects in a category $\mathbf{C}$. The *product* of this family, if it exists, is an object $P \in \mathbf{C}$ along with morphisms $p_i : P \to A_i$ such that for any $Y \in \mathbf{C}$ and any collection of morphisms $\{y_i : Y \to A_i\}_{i \in I}$, there exists a unique morphism $f : Y \to P$ so that $p_i \circ f = y_i$ for each i. The product, since when it exists it is unique up to isomorphism, will be denoted $\displaystyle\bigtimes_{i \in A} A_i$. The existence of f for any Y and any $\{y_i\}_{i \in I}$, is referred to as *the universal property* defining the categorical product.

The dual notion of *coproduct* of $\{A_i\}_{i \in I}$ is the object $C \in \mathbf{C}$, when it exists, along with morphisms $c_i : A_i \to C$ such that for any object Y and any collection of morphisms $\{y_i : A_i \to Y\}_{i \in I}$, there exists a unique morphism $g : C \to Y$ such that $g \circ c_i = y_i$ for each i. The coproduct of $\{A_i\}_{i \in I}$ is denoted $\displaystyle\coprod_{i \in I} A_i$, and the existence of g is the universal property defining it.

Example 2.1.18. If $A_i \in \mathbf{Top}$, then the product $\bigtimes_{i \in A} A_i$ is the space whose underlying set is the Cartesian product endowed with the product topology, while the coproduct $\coprod_{i \in I} A_i$ is their disjoint union.

Definition 2.1.19. Let $f, g : A \to B$ be morphisms in a category $\mathbf{C}$. The *coequalizer* of f and g, denoted $\mathsf{coeq}(A \rightrightarrows B)$ if it exists, is an object $C \in \mathbf{C}$ with a morphism $q : B \to C$ such that $q \circ f = q \circ g$. Moreover, if there is another pair (C', q') such that $q' \circ f = q' \circ g$, then there is a unique $u : C \to C'$ satisfying $u \circ q = q'$.

Example 2.1.20. If $f, g : A \to B$ are two morphisms in $\mathbf{Top}$, then $\mathsf{coeq}(A \rightrightarrows B)$ is the quotient space $B \Big/ \Big(f(a) \sim g(a),\ \forall a \in A \Big)$.

2.2 Algebraic Topology

In this section, we provide a brief review of the classical Künneth formula for the product of topological spaces with coefficients in a Principal Ideal Domain (PID) in 2.2.1, an explicit model for the homotopy colimit of a diagram of topological spaces in 2.2.2, and the theory of principal bundles and Čech cohomology in 2.2.3.

2.2.1 The Classical Künneth Theorems

The classical Künneth theorem in algebraic topology relates the homology of the product of two spaces to the homology of its factors. The relation is via a split natural short exact sequence, and the proof is a combination of the algebraic Künneth formula for chain complexes, and the Eilenberg-Zilber theorem. Both theorems are stated next, but first here are two relevant definitions:

Definition 2.2.1. Let R be a commutative ring, and let $C, C' \in \mathbf{Ch}_R$. The *tensor product chain complex* $C \otimes_R C'$ consists of R-modules $(C \otimes_R C')_n = \bigoplus_i (C_i \otimes_R C'_{n-i})$ and boundary morphisms

$$\begin{array}{rcl} C_i \otimes_R C'_{n-i} & \longrightarrow & (C \otimes_R C')_n \\ c \otimes_R c' & \mapsto & \partial_i c \otimes_R c' + (-1)^i c \otimes_R \partial'_{n-i} c'. \end{array}$$

Definition 2.2.2. An R-module M is called *flat*, if for every short exact sequence $0 \to A \to A' \to A'' \to 0$ of R-modules, the induced sequence

$$0 \to A \otimes_R M \to A' \otimes_R M \to A'' \otimes_R M \to 0$$

is also exact.

In particular, all free modules are flat.

Theorem 2.2.3 (The Algebraic Künneth Formula)**.** *If R is a PID and at least one of the chain complexes C, C' is flat (i.e., the constituent modules are flat), then for each $n \in \mathbb{N}_0$ there is a natural short exact sequence*

$$0 \longrightarrow \bigoplus_{i+j=n} (H_i(C) \otimes_R H_j(C')) \longrightarrow H_n(C \otimes_R C') \longrightarrow \bigoplus_{i+j} \mathsf{Tor}_R(H_i(C), H_{j-1}(C')) \longrightarrow 0$$

which splits but not naturally.

See for instance [40, Chapter 5, Theorem 2.1]. It is well known that when C and C' are the singular chain complexes of two topological spaces X and Y, denoted by $S_*(X;R)$ and $S_*(Y;R)$ respectively, then the Eilenberg-Zilber theorem [32] provides a link between the algebraic Künneth theorem and the homology of $X \times Y$:

Theorem 2.2.4 (Eilenberg-Zilber)**.** *For topological spaces X and Y, there is a natural chain equivalence $\zeta : S_*(X;R) \otimes_R S_*(Y;R) \longrightarrow S_*(X \times Y;R)$, and thus*

$$H_n(X \times Y; R) \cong H_n\Big(S_*(X;R) \otimes_R S_*(Y;R)\Big)$$

for all $n \geq 0$.

The existence of ζ is in fact an application of the acyclic models theorem of Eilenberg and MacLane [30]. This theorem provides conditions under which two functors to the category of chain complexes produce naturally isomorphic homology theories. Later, we will use similar ideas to prove a persistent Eilenberg-Zilber theorem (Theorem 3.3.10), which then yields our persistent Künneth formula for the tensor product (Theorem 3.3.12). For now, here is the topological Künneth theorem:

Corollary 2.2.5 (The topological Künneth formula)**.** *Let X, Y be topological spaces and let R be a PID. Then, there exists a natural short exact sequence*

$$0 \longrightarrow \bigoplus_{i+j=n} (H_i(X;R) \otimes_R H_j(Y;R)) \longrightarrow H_n(X \times Y;R) \longrightarrow \\ \bigoplus_{i+j=n} \mathsf{Tor}_R(H_i(X;R), H_{j-1}(Y;R)) \longrightarrow 0 \tag{2.1}$$

which splits, though not naturally.

For the sake of completeness, here is an example using the topological Künneth formula to compute the homology of a product of topological spaces:

Example 2.2.6. Let $X = S^1 = Y$ and $R = \mathbb{Z}$. Then the homology modules are $H_i(X;\mathbb{Z}) = \mathbb{Z} = H_i(Y;\mathbb{Z})$ for $i = 0, 1$ and $\{0\}$ otherwise. Since $\mathsf{Tor}_{\mathbb{Z}}$ is $\{0\}$ for any free input, we obtain that

$$H_0(X \times Y;\mathbb{Z}) \cong H_0(X;\mathbb{Z}) \otimes_{\mathbb{Z}} H_0(Y;\mathbb{Z}) = \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \cong \mathbb{Z}$$
$$H_1(X \times Y;\mathbb{Z}) \cong H_0(X;\mathbb{Z}) \otimes_{\mathbb{Z}} H_1(Y;\mathbb{Z}) \oplus H_1(X;\mathbb{Z}) \otimes_{\mathbb{Z}} H_0(Y;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$$
$$H_2(X \times Y;\mathbb{Z}) \cong H_1(X;\mathbb{Z}) \otimes_{\mathbb{Z}} H_1(Y;\mathbb{Z}) = \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \cong \mathbb{Z}$$

which is indeed the homology of a 2-torus.

2.2.2 Homotopy Colimits

The machinery of homotopy colimits will be used in Section 3.1 to define the generalized tensor product $\otimes_{\mathbf{g}} : \mathbf{Top}^{\mathbf{N}} \times \mathbf{Top}^{\mathbf{N}} \longrightarrow \mathbf{Top}^{\mathbf{N}}$ appearing in Theorem 3.3.12. We provide here a basic review, but direct the interested reader to [27] or [73] for a more comprehensive presentation. If $\mathbf{C}$ is a category, then let $\mathcal{B}\mathbf{C}$ denote its classifying space—i.e., the geometric realization of the nerve of $\mathbf{C}$.

Definition 2.2.7. For a small indexing category $\mathbf{I}$, let $D : \mathbf{I} \longrightarrow \mathbf{Top}$ be a functor. The *colimit* of D is defined as:

$$\mathsf{colim}(D) := \mathsf{coeq}\left[\coprod_{i\to j} D(i) \rightrightarrows \coprod_{i} D(i)\right] \tag{2.2}$$

where the top map sends $D(i)$ to itself via the identity, the bottom map sends $D(i)$ to $D(j)$ via $D(i \to j)$, and the first coproduct is indexed over all morphisms in $\mathbf{I}$. Similarly, the *homotopy colimit* of D is defined as:

$$\mathsf{hocolim}(D) := \mathsf{coeq}\left[\coprod_{i\to j} D(i) \times \mathcal{B}(j \downarrow \mathbf{I})^{op} \rightrightarrows \coprod_{i} D(i) \times \mathcal{B}(i \downarrow \mathbf{I})^{op}\right] \tag{2.3}$$

The top map $D(i) \times \mathcal{B}(j \downarrow \mathbf{I})^{op} \to D(j) \times \mathcal{B}(j \downarrow \mathbf{I})^{op}$ is $D(i \to j)$ times the identity, while the bottom map $D(i) \times \mathcal{B}(j \downarrow \mathbf{I})^{op} \to D(i) \times \mathcal{B}(i \downarrow \mathbf{I})^{op}$ is the identity of $D(i)$ times the map on classifying spaces induced by $(i \to j)^*_{op} : (j \downarrow \mathbf{I})^{op} \to (i \downarrow \mathbf{I})^{op}$.

It is worth noting that if $D, D' \in \mathbf{Top}^{\mathbf{I}}$, then any morphism $D \to D'$ induces (functorially) a map $\mathsf{hocolim}(D) \to \mathsf{hocolim}(D')$, and that if $f \in \mathbf{J}^{\mathbf{I}}$ and $D \in \mathbf{Top}^{\mathbf{J}}$, then f induces a natural map $\phi_f : \mathsf{hocolim}(D \circ f) \to \mathsf{hocolim}(D)$ called the *change of indexing category*. Moreover, we have the following theorem from [27]:

Proposition 2.2.8. *Let $f : \mathbf{I} \to \mathbf{J}$, $g : \mathbf{J} \to \mathbf{K}$ and $D : \mathbf{K} \to \mathbf{Top}$ be functors. Then $\phi_{g\circ f} = \phi_g \circ \phi_f$.*

While both the colimit and the homotopy colimit of a diagram of spaces yield methods for functorially gluing spaces along maps, only the latter preservers homotopy equivalences. Indeed, [73, Proposition 3.7] says:

Theorem 2.2.9. *Let $D, D' \in \mathbf{Top}^{\mathbf{I}}$. If $D \to D'$ is a natural (weak) homotopy equivalence—i.e. each $D(i) \to D'(i)$ is a (weak) homotopy equivalence—then the induced map $\mathsf{hocolim}(D) \to \mathsf{hocolim}(D')$ is a (weak) homotopy equivalence.*

When the indexing category has a *terminal object*, that is, a unique object z such that for all $i \in \mathbf{I}$ there is a unique morphism $i \to z$, then the homotopy colimit is particularly simple [27, Lemma 6.8].

Theorem 2.2.10. *Suppose that* $\mathbf{I}$ *has a terminal object* z. *Then for all* $D \in \mathbf{Top}^{\mathbf{I}}$, *the collapse map* $\mathsf{hocolim}(D) \to D(z)$ *is a weak homotopy equivalence.*

Notice that by collapsing the classifying spaces $\mathcal{B}(j \downarrow \mathbf{I})^{op}$ and $\mathcal{B}(i \downarrow \mathbf{I})^{op}$ in the definition of homotopy colimit to a point, we recover the definition of the colimit. Next, we will see specific conditions under which this collapse defines a homotopy equivalence from $\mathsf{hocolim}(D)$ to $\mathsf{colim}(D)$. In order to state the result, we recall the notion of a cofibration.

Definition 2.2.11. A continuous map $f : A \to X$ is called a *cofibration* if it satisfies the *homotopy extension property* with respect to all spaces Y. That is, given a homotopy $h_t : A \to Y$ and a map $H_0 : X \to Y$ such that $H_0 \circ f = h_0$, then there is a homotopy $H_t : X \to Y$ such that $H_t \circ f = h_t$ for all t. If $f(A)$ is a closed subset of X, then f is called a *closed cofibration*.

Remark 2.2.12. Let $\mathcal{X} \in \mathbf{Top}^{\mathbf{N}}$, and let $\mathcal{T}(\mathcal{X}) \in \mathbf{Top}^{\mathbf{N}}$ be the functor defined as follows. For $j \in \mathbb{N}_0$, let $\mathcal{T}_j(\mathcal{X})$ be the *mapping telescope* of

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{j-2}} X_{j-1} \xrightarrow{f_{j-1}} X_j.$$

Explicitly, $\mathcal{T}_j(\mathcal{X})$ is the quotient space

$$\mathcal{T}_j(\mathcal{X}) = X_j \times \{j\} \sqcup \left(\bigsqcup_{i<j} X_i \times [i, i+1] \right) \Big/ (x, i+1) \sim (f_i(x), i+1).$$

The functor $\mathcal{T}(\mathcal{X}) \in \mathbf{Top}^{\mathbf{N}}$ satisfies that $\mathcal{T}_i(\mathcal{X}) \hookrightarrow \mathcal{T}_j(\mathcal{X})$ is a closed cofibration for all $i \leq j$ [10, Chapter VII, Theorem 1.5].

Here is one situation where colim and $\mathsf{hocolim}$ provide similar answers,

Lemma 2.2.13 (Projection Lemma)**.** *Let $\mathcal{P}$ be a partially ordered set and $\mathbf{P}$ its poset category. Let $D \in \mathbf{Top}^{\mathbf{P}}$ be such that $D(i \preceq j) : D(i) \to D(j)$ is a closed cofibration for each pair $i \preceq j$ in $\mathcal{P}$. Then the collapse map*

$$\mathsf{hocolim}(D) \to \mathsf{colim}(D)$$

is a homotopy equivalence.

Proof. See [73, Proposition 3.1]. □

2.2.3 Principal bundles and Čech cohomology

We present here a review of the theory of principal bundles and their connection with Čech cohomology. For details, see [43]. Throughout this section, we assume that B is a connected and paracompact topological space with basepoint $b_0 \in B$.

Definition 2.2.14. A pair (p, E) with E a topological space and $p : E \to B$ a continuous map, is called a *fiber bundle* over B with fiber $F = p^{-1}(b_0)$, if the following are satisfied:

1. p is surjective,
2. Every point $b \in B$ has an open neighborhood $U \subset B$ and a homeomorphism $\rho_U : U \times F \to p^{-1}(U)$, called a local trivialization around b, such that $p \circ \rho_U(b', e) = b'$ for every $(b', e) \in U \times F$.

The spaces E and B are called the *total space* and the *base space* of the bundle respectively, and p is called the *projection map*.

Let $(G, +)$ be an additive abelian topological group, that is, a topological space G with a continuous additive group structure.

Definition 2.2.15. A *(topological) right action* of a topological group G on a topological space X is a continuous map

$$\cdot : X \times G \to X$$
$$(x, g) \to x \cdot g$$

satisfying

1. $x \cdot 1_G = x$ for all $x \in X$, and
2. $(x \cdot g_1) \cdot g_2 = x \cdot (g_1 + g_2)$ for all $x \in X$ and $g_1, g_2 \in G$.

The action is called *transitive* if for every $x, y \in X$, $\exists g \in G$, such that $y = x \cdot g$. It is called *free* if for all $g \neq h \in G$, $\exists x \in X$, such that $x \cdot g \neq x \cdot h$.

Definition 2.2.16. A *principal G-bundle* is a fiber bundle (p, E) so that E is equipped with a right G-action satisfying

1. $\cdot : E \times G \to E$ is fiberwise preserving, that is,
$$p(e \cdot g) = p(e) \quad \forall (e, g) \in E \times G.$$
2. The induced fiberwise G-action $p^{-1}(b) \times G \to p^{-1}(b)$ is free and transitive for every b in the base space B.
3. The local trivializations $\rho_U : U \times F \to p^{-1}(U)$ can be chosen to be G-equivariant, that is, they satisfy
$$\rho_U(b, e \cdot g) = \rho_U(b, e) \cdot g$$
for every $(b, e, g) \in U \times F \times G$.

Definition 2.2.17. Two principal G-bundles $p_j : E_j \to B$ (for $j = 1, 2$) are called *isomorphic*, if there exists a G-equivariant homeomorphism $\Phi : E_1 \to E_2$ such that $p_2 \circ \Phi = p_1$.

This defines an equivalence relation on the set of principal G-bundles over B, and the set of its equivalence classes (isomorphism classes) is denoted by $\mathsf{Prin}_G(B)$.

Remark 2.2.18. Given a principal G-bundle $p : E \to B$ and a system of G-equivariant local trivializations

$$\{\rho_j : U_j \times F \to p^{-1}(U_j)\}_{j\in J},$$

we have that $\rho_k^{-1} \circ \rho_j : (U_j \cap U_k) \times F \to (U_j \cap U_k) \times F$ is a G-equivariant homeomorphism whenever $U_j \cap U_k \neq \emptyset$. Since the G-action on E is free and transitive on each fiber, the composition $\rho_k^{-1} \circ \rho_j$ induces a well defined continuous map $\rho_{jk} : U_j \cap U_k \to G$ for each $j, k \in J$, defined as:

$$\rho_k^{-1} \circ \rho_j(b, e) = (b, e \cdot \rho_{jk}(b)) \tag{2.4}$$

for all $(b, e) \in (U_j \cap U_k) \times F$. The maps ρ_{jk} are called the *transition functions* corresponding the system of local trivializations $\{\rho_j\}_{j\in J}$.

Now that we have the basics of principal bundles in place, we review some sheaf theory:

Definition 2.2.19. A *presheaf* $\mathcal{F}$ of groups over a topological space X consists of the following data:

1. A group $\mathcal{F}(U)$ for each open set $U \subset X$, such that $\mathcal{F}(\emptyset) = \{0\}$, and
2. A collection of group homomorphisms, called *restriction maps*, $\gamma_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ for each $V \subset U$ such that $\gamma_U^U = 1_{\mathcal{F}(U)}$ and if $W \subset V \subset U$, then $\gamma_W^U = \gamma_W^V \circ \gamma_V^U$.

If further, $\mathcal{F}$ satisfies the following gluing axiom, then it is called a *sheaf*:
If $U \subset X$ is open, $\{U_j\}_{j\in J}$ is an open cover of U, and there exists a collection $\{s_j, \mathcal{F}(U_j)\}_{j\in J}$ where $s_j \in \mathcal{F}(U_j)$, satisfying

$$\gamma_{U_j\cap U_k}^{U_j}(s_j) = \gamma_{U_j\cap U_k}^{U_k}(s_k)$$

for all $j, k \in J$, then $\exists\, s \in \mathcal{F}(U)$ such that

$$\gamma_{U_j}^{U}(s) = s_j$$

for each $j \in J$.

We now present a relevant example of a sheaf:

Example 2.2.20. If G is a topological group, and $\mathsf{Maps}(U, G)$ denotes the set of continuous maps from U to G, then the sheaf $\mathscr{C}_G$ of continuous G-valued functions over a topological space X is defined as:

$$\mathscr{C}_G(U) := \mathsf{Maps}(U, G).$$

The restriction maps $\mathscr{C}_G(U) \to \mathscr{C}_G(V)$ are induced by inclusions $V \hookrightarrow U$. This satisfies the gluing axiom because for an open cover $\{U_j\}_{j \in J}$ of an open set $U \subset X$, a family of continuous functions $\{U_j \to G\}_{j \in J}$ that agree on intersection of their domains can be glued together to define a global continuous function $U \to G$.

Definition 2.2.21 (Čech cohomology)**.** Let $\mathcal{F}$ be a presheaf of abelian groups on a topological space X and $\mathcal{U} = \{U_j\}_{j \in J}$ be an open cover of X. Define the n-th Čech cochain groups as

$$\check{C}^n(\mathcal{U}; \mathcal{F}) = \prod_{(j_0, j_1, \ldots, j_n) \in J^{n+1}} \mathcal{F}(U_{j_0 \cdots j_n})$$

and the n-th coboundary map as

$$d^n : \check{C}^n(\mathcal{U}; \mathcal{F}) \to \check{C}^{n+1}(\mathcal{U}; \mathcal{F})$$

where

$$d^n(f)_{j_0 \cdots j_{n+1}} = \sum_{k=0}^{n+1} (-1)^k f_{j_0 \cdots \hat{j}_k \cdots j_{n+1}} |_{U_{j_0 \cdots j_{n+1}}}.$$

It can be checked that $d^n \circ d^{n-1} = 0$ for each $n \geq 1$, that is each d^n is indeed a coboundary map. Define two subgroups of $\check{C}^n(\mathcal{U}; \mathcal{F})$, namely

$$\check{Z}^n(\mathcal{U}; \mathcal{F}) := \mathsf{Ker}(d^n)$$

$$\check{B}^n(\mathcal{U}; \mathcal{F}) := \mathsf{Img}(d^{n-1})$$

called the set of Čech n-cocycles and Čech n-coboundaries respectively. Their quotient is the *Čech n-th cohomology* of the covering $\mathcal{U}$:

$$\check{H}^n(\mathcal{U};\mathcal{F}) := \check{Z}^n(\mathcal{U};\mathcal{F})/\check{B}^n(\mathcal{U};\mathcal{F}).$$

The Čech cohomology of the topological space X is defined using direct limits of refinements of open covers, which we define next:

Definition 2.2.22. Let $\mathcal{U} = \{U_j\}_{j\in J}$ and $\mathcal{V} = \{V_r\}_{r\in R}$ be open covers of X. We say that $\mathcal{V}$ is a *refinement* of $\mathcal{U}$, written as $\mathcal{V} \leq \mathcal{U}$ if $\exists \tau : R \to J$ such that $V_r \subset U_{\tau(r)}$ for all $r \in R$.

Remark 2.2.23. The refinement function τ induces a chain map between Čech complexes

$$\tau^{\#} : \check{C}^n(\mathcal{U};\mathcal{F}) \to \check{C}^n(\mathcal{V};\mathcal{F})$$

and hence a map between the cohomology groups

$$\tau^* : \check{H}^n(\mathcal{U};\mathcal{F}) \to \check{H}^n(\mathcal{V};\mathcal{F}).$$

Furthermore, any other refinement function $\tilde{\tau} : R \to J$ induces the same homomorphism between homology groups as τ. The Čech cohomology of a topological space X is defined as the direct limit

$$\check{H}^n(X;\mathcal{F}) := \varinjlim_{\mathcal{U}} \check{H}^n(\mathcal{U};\mathcal{F})$$

over all covers with respect to refinement. In other words, this cohomology group is equal to the quotient $\coprod_{\mathcal{U}} \check{H}^n(\mathcal{U};\mathcal{F})/\sim$ where

$$\check{H}^n(\mathcal{U};\mathcal{F}) \ni [f] \sim [g] \in \check{H}^n(\mathcal{V};\mathcal{F})$$

if and only if $\exists$ a common refinement $\mathcal{W} \leq \mathcal{U},\mathcal{V}$ such that $\tau^*_{\mathcal{U}}[f] = \tau^*_{\mathcal{V}}[g]$.

We now transition (wordplay intended) back to principal bundles. The transition functions $\{\rho_{jk}\}$ defined in equation 2.4 also define an element $\rho = \{\rho_{jk}\} \in \check{C}^1(\mathcal{U};\mathscr{C}_G)$ in the Čech 1-cochains of the cover $\mathcal{U} = \{U_j\}_{j\in J}$ with coefficients in the sheaf $\mathscr{C}_G$ defined in example 2.2.20. Furthermore, the element ρ is a Čech cocycle:

Proposition 2.2.24. *The transition functions ρ_{jk} satisfy the following cocycle condition:*

$$\rho_{jl}(b) = (\rho_{jk} + \rho_{kl})(b)$$

for all $b \in U_j \cap U_k \cap U_l$. In other words, $\{\rho_{jk}\} \in \check{Z}^1(\mathcal{U}; \mathscr{C}_G)$ is a Čech cocycle.

For any open cover $\mathcal{U} = \{U_j\}_{j\in J}$ of a paracompact B and a Čech cocycle $\eta := \{\eta_{jk}\} \in \check{Z}^1(\mathcal{U}; \mathscr{C}_G)$, we can construct a principal G-bundle $p : E_\eta \to B$ as

$$E_\eta = \left(\bigcup_{j\in J} U_j \times \{j\} \times G \right) \Big/ (b, j, g) \sim (b, k, \eta_{jk}(b))$$

for $b \in U_j \cap U_k$. The projection p takes the class $[(b, j, g)]$ to b. Furthermore,

Theorem 2.2.25. *The function*

$$\check{H}^1(B, \mathscr{C}_G) \to \mathsf{Prin}_G(B)$$

$$[\eta] \to [E_\eta]$$

is a natural bijection.

Proof. See 2.4 and 2.5 in [55]. □

For each topological group G, we have a weakly contractible space EG along with a group action of G on it. The quotient space defined as

$$BG := EG/G$$

is called the *classifying space* of G, and the quotient map $p : EG \to BG$ defines a principal G bundle called the *universal bundle*. For example, if $G = \mathbb{Z}$, then $E\mathbb{Z} \simeq \mathbb{R}$ and $B\mathbb{Z} \simeq S^1$, and if $G = \mathbb{Z}_q$, then $E\mathbb{Z}_q \simeq \mathbb{C}^\infty$ and $B\mathbb{Z}_q \simeq L_q^\infty$. There are many constructions of the space EG, for example by Milnor [50] or Segal [63], but they are homotopy equivalent.

Furthermore, if we let $h : B \to BG$ be a continuous map, then

$$h^*EG := \{(b, e) \in B \times EG \mid h(b) = p(e)\} \to B$$

$$(b, e) \to b$$

is another principal G bundle, called the *pullback bundle*. Then we have another bijection:

Theorem 2.2.26. *The function*

$$[B, BG] \to \mathsf{Prin}_G(B)$$
$$[h] \to [h^* EG]$$

is a natural bijection.

Proof. See [43, Chapter 4: Theorems 12.2 and 12.4]. □

Putting things together, we obtain that there is a natural bijection between $\check{H}^1(B, \mathscr{C}_G)$ and $[B, BG]$. Furthermore, if G is abelian and B is a CW complex, then the 1-Čech cohomology group $\check{H}^1(B, \mathscr{C}_G)$ is isomorphic to the singular cohomology group $H^1(B; G)$. Before we finish this section, we quickly want to describe Eilenberg-MacLane spaces:

Definition 2.2.27. A connected CW complex $K(G, n)$ is called an *Eilenberg-MacLane space* if its homotopy type is uniquely determined by its j^{th} homotopy groups:

$$\pi_j(K(G, n)) \cong \begin{cases} 0 & \text{if} \quad j \neq n \\ G & \text{if} \quad j = n. \end{cases}$$

For example, $K(\mathbb{Z}, 1) \simeq S^1$, $K(\mathbb{Z}_2, 1) \simeq \mathbb{R}\mathbf{P}^\infty$, $K(\mathbb{Z}_q, 1) \simeq L_q^\infty$, and $K(\mathbb{Z}, 2) \simeq \mathbb{C}\mathbf{P}^\infty$. If $n = 1$, then for an abelian discrete G, we have $BG \simeq K(G, 1)$. This has the following consequence: Given a CW complex B, the Brown representability theorem for CW complexes and singular cohomology (in dimension 1) gives a natural bijection $H^1(B; G) \cong [B, K(G, 1)]$ between the 1-st cohomology of B with coefficients in G and homotopy classes of continuous maps from B to $K(G, 1)$. This bijection was exploited for $G = \mathbb{Z}$ to construct the sparse circular coordinates [55] and we study their stability in Chapter 5.

2.3 Persistent Homology

Persistent homology is an algebraic and computational tool from topological data analysis [56]. Broadly speaking, it is used to quantify multiscale features of shapes. The successes of persistent homology stem in part from a strong theoretical foundation [76, 24, 52] and a focus on efficient algorithmic implementations [5, 6, 9, 20, 72].

The framework we use here is that of diagrams in a category. Such a framework for persistence was first introduced by Bubenik and Scott in [13]. Perhaps the most important takeaways from this section are the classification via barcodes, and that the persistent homology of a diagram of spaces is isomorphic to the standard homology of the associated persistence chain complex (see [76] and Theorem 2.3.21).

The following is an example of a functor category defined in Definition 2.1.12:

Example 2.3.1. If **Top** is the category of topological spaces and continuous maps, and **I** is a thin category, then each object of $\mathbf{Top}^{\mathbf{I}}$ is a collection $\mathcal{X} = \{f_{j,i} : X_i \to X_j\}_{i\to j\in \mathbf{I}}$ of topological spaces X_i, and continuous maps $f_{j,i} : X_i \longrightarrow X_j$ for each morphism $i \to j$ in **I** satisfying: $f_{i,i}$ is the identity of X_i, and $f_{k,j} \circ f_{j,i} = f_{k,i}$ for any $i \to j \to k$. Similarly, a morphism ϕ from $\mathcal{X} = \{f_{j,i} : X_i \to X_j\}$ to $\mathcal{Y} = \{g_{j,i} : Y_i \to Y_j\}$ is a family of maps $\phi_i : X_i \to Y_i$ such that $g_{j,i} \circ \phi_i = \phi_j \circ f_{j,i}$ for every $i \to j$.

Example 2.3.2. Let $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ with its usual (total) order, and let **N** be its poset category. **N**-indexed diagrams in **Top** arise in Topological Data Analysis (TDA) as follows. Let $\mathbb{M}$ be a metric space, let $\mathbb{X} \subset \mathbb{M}$ be a subspace and let $X \subset \mathbb{M}$ be finite. In applications X is the data one observes (e.g., images, text documents, molecular compounds, etc), obtained by sampling from/around $\mathbb{X}$ (the ground truth, which is unknown in practice), both sitting in an ambient space $\mathbb{M}$. With the goal of estimating the topology of $\mathbb{X}$ from X, one starts by letting $X^{(\epsilon)}$ be the union of open balls in $\mathbb{M}$ of radius $\epsilon \geq 0$ centered at points of X. Hence $X^{(\epsilon)}$ provides a—perhaps rough—approximation to the topology of $\mathbb{X}$ for each $\epsilon \geq 0$, and any discretization $0 = \epsilon_0 < \epsilon_1 < \cdots$ of $[0, \infty)$ yields an object in $\mathbf{Top}^{\mathbf{N}}$ as follows: $\mathbb{N}_0 \ni i \mapsto X^{(\epsilon_i)}$, and the inclusion $X^{(\epsilon_i)} \hookrightarrow X^{(\epsilon_j)}$ is the map associated to $i \leq j$. Such objects allow one to avoid optimizing the choice of a single ϵ, leading to several recovery theorems and algorithms for topological inference [52].

Remark 2.3.3. If each $f_i : X_i \to X_{i+1}$ in $\mathcal{X} \in \mathbf{Top}^{\mathbf{N}}$ is an inclusion, e.g. as in Example 2.3.2, then $\mathcal{X}$ defines a *filtered topological space.* Recall that a filtered topological space consists of a space X together with a filtration $X_0 \subset X_1 \subset \cdots \subset X$.

This remark motivates the following definition.

Definition 2.3.4. We call $\mathcal{X} \in \mathbf{Top}^{\mathbf{N}}$ a *filtered space* if all its maps are inclusions. The collection of all filtered spaces forms a full subcategory (see definition 2.1.8) of $\mathbf{Top}^{\mathbf{N}}$ denoted **FTop**.

Example 2.3.5. Since the functor $\mathcal{T}(\mathcal{X}) \in \mathbf{Top}^{\mathbf{N}}$, defined in Remark 2.2.12, satisfies $\mathcal{T}_j(\mathcal{X}) \subset \mathcal{T}_k(\mathcal{X})$ whenever $j \leq k$, it defines an object in **FTop** called the *telescope filtration* of $\mathcal{X}$.

As is typical in algebraic topology, one is interested in the interplay between diagrams of spaces and diagrams of algebraic objects.

Example 2.3.6. Let R be a commutative ring with unity, and let $\mathbf{Mod}_R$ be the category of (left) R-modules and R-morphisms as in Example 2.1.3. The typical objects in $\mathbf{Mod}_R^{\mathbf{I}}$ one encounters in TDA arise from objects $\mathcal{X} \in \mathbf{Top}^{\mathbf{I}}$ by fixing $n \in \mathbb{N}_0$ and taking singular homology in dimension n with coefficients in R. Indeed, $H_n(X_i; R)$ is an R-module for each $i \in \mathbf{I}$, and for each morphism $i \to j$ in $\mathbf{I}$, the map $\mathcal{X}(i \to j) : X_i \longrightarrow X_j$ induces—in a functorial manner—a well-defined R-morphism from $H_n(X_i; R)$ to $H_n(X_j; R)$. The resulting object in $\mathbf{Mod}_R^{\mathbf{I}}$ will be denoted $H_n(\mathcal{X}; R)$.

Definition 2.3.7. Let $\mathcal{M}, \mathcal{N} \in \mathbf{Mod}_R^{\mathbf{I}}$, and let $\mathcal{M} \oplus \mathcal{N} : \mathbf{I} \longrightarrow \mathbf{Mod}_R$ be the functor sending $i \in \mathbf{I}$ to $(\mathcal{M} \oplus \mathcal{N})(i) := M_i \oplus N_i$, and each morphism $i \to j$ in $\mathbf{I}$ to

$$(\mathcal{M} \oplus \mathcal{N})(i \to j) := \mathcal{M}(i \to j) \oplus \mathcal{N}(i \to j).$$

We say that $\mathcal{M} \in \mathbf{Mod}_R^{\mathbf{I}}$ is *indecomposable* if $\mathcal{M} \cong \mathcal{N} \oplus \mathcal{O}$ only when either $\mathcal{N}$ or $\mathcal{O}$ is the zero functor. Similarly, let $\mathcal{M} \otimes \mathcal{N} : \mathbf{I} \longrightarrow \mathbf{Mod}_R$ be the functor sending $i \in \mathbf{I}$ to $M_i \otimes_R N_i$

and $i \to j$ to $\mathcal{M}(i \to j) \otimes_R \mathcal{N}(i \to j)$, where the tensor product $\otimes_R$ is the usual one for R-modules and R-morphisms.

Example 2.3.8. Recall that an *interval* in a poset $\mathcal{P}$ is a set $I \subset \mathcal{P}$ for which $i, k \in I$ and $i \preceq j \preceq k$ always imply $j \in I$. An interval $I \subset \mathcal{P}$ defines an object $\mathbb{1}_I$ in $\mathbf{Mod}_R^{\mathbf{P}}$ as follows: for $j \preceq j'$ in $\mathcal{P}$, let

$$\mathbb{1}_I(j) := \begin{cases} R & \text{if } j \in I \\ 0 & \text{else} \end{cases} \qquad \text{and} \qquad \mathbb{1}_I(j \preceq j') := \begin{cases} id_R & \text{if } j, j' \in I \\ 0 & \text{else} \end{cases}$$

The $\mathbb{1}_I$'s are called *interval diagrams*, and if $\mathcal{P}$ is totally ordered, then they are indecomposable in $\mathbf{Mod}_R^{\mathbf{P}}$ [13, Lemma 4.2]. Moreover, if $I, J \subset \mathcal{P}$ are intervals, then the tensor product of the corresponding interval diagrams satisfies

$$\mathbb{1}_I \otimes \mathbb{1}_J \cong \mathbb{1}_{I \cap J}.$$

Indeed, $\mathbb{1}_I \otimes \mathbb{1}_J(r) \cong 0$ if $r \notin I \cap J$, and $\mathbb{1}_I \otimes \mathbb{1}_J(r) \cong R \otimes_R R \cong R$ if $r \in I \cap J$, where the last isomorphism is given by multiplication in R. Since applying $id_R \otimes_R id_R$ and then multiplying equals multiplying and then applying id_R, the result follows.

Definition 2.3.9. An object $\mathcal{V} \in \mathbf{Mod}_{\mathbb{F}}^{\mathbf{I}}$ satisfying the condition $\dim_{\mathbb{F}} \mathcal{V}(j) < \infty$ *for all* $j \in \mathbf{I}$ is said to be *pointwise finite dimensional.*

Interval diagrams are fundamental building blocks in $\mathbf{Mod}_{\mathbb{F}}^{\mathbf{P}}$ [22, Theorem 1.1]:

Theorem 2.3.10. *Let $\mathbb{F}$ be a field and let $(\mathcal{P}, \preceq)$ be a totally ordered set. Suppose that $\mathcal{P}$ is separable with respect to the order topology, that is it contains a countable dense subset, and that $\mathcal{V} \in \mathbf{Mod}_{\mathbb{F}}^{\mathbf{P}}$ is pointwise finite dimensional. Then, there exists a multiset of intervals $I \subset \mathcal{P}$ called the* barcode *of $\mathcal{V}$, denoted* $\mathsf{bcd}(\mathcal{V})$, *and so that*

$$\mathcal{V} \cong \bigoplus_{I \in \mathsf{bcd}(\mathcal{V})} \mathbb{1}_I.$$

Moreover, $\mathsf{bcd}(\mathcal{V})$ *only depends on—and uniquely determines—the isomorphism type of $\mathcal{V}$.*

Definition 2.3.11. Let $\mathbb{F}$ and $\mathcal{P}$ be as in Theorem 2.3.10, and let $\mathcal{X} \in \mathbf{Top}^{\mathbf{P}}$ be so that $H_n(\mathcal{X};\mathbb{F})$ is pointwise finite. The barcode of $H_n(\mathcal{X};\mathbb{F})$, denoted $\mathsf{bcd}_n(\mathcal{X};\mathbb{F})$, is the unique multiset of intervals in $\mathcal{P}$ such that

$$H_n(\mathcal{X};\mathbb{F}) \cong \bigoplus_{I \in \mathsf{bcd}_n(\mathcal{X};\mathbb{F})} \mathbb{1}_I.$$

It follows that $\mathsf{bcd}_n(\mathcal{X};\mathbb{F})$ provides a succinct description of the isomorphism type of $H_n(\mathcal{X};\mathbb{F})$. The design of algorithms for the computation of barcodes, at least in the $\mathbb{N}_0$-indexed case, leverages a more concrete description of $H_n(\mathcal{X};\mathbb{F})$ which we describe next.

Definition 2.3.12. For $\mathcal{M} = \{h_{j,i} : M_i \longrightarrow M_j\}_{i \preceq j \in \mathcal{P}}$ an object in $\mathbf{Mod}_R^{\mathbf{P}}$, let

$$P\mathcal{M} := \bigoplus_{i \in \mathcal{P}} M_i. \tag{2.5}$$

$P\mathcal{M}$ is called the *persistence module* associated to $\mathcal{M}$.

The word module stems from the following observation: if $\mathcal{M} \in \mathbf{Mod}_R^{\mathbf{N}}$, then $P\mathcal{M}$ is a graded module over $R[t]$, the graded ring of polynomials in a variable t. Indeed, the $R[t]$-module structure is defined through multiplication by t as follows: for $\mathsf{m} = (m_0, m_1, \ldots) \in P\mathcal{M}$ let

$$t \cdot \mathsf{m} := \Big(0, h_0(m_0), h_1(m_1), \ldots\Big)$$

and extend the action to $R[t]$ in the usual way. The graded nature of the multiplication comes from noticing that $t^k \cdot M_i \subset M_{i+k}$ for every $i, k \in \mathbb{N}_0$.

If $\phi : \mathcal{M} \longrightarrow \mathcal{N}$ is a morphism in $\mathbf{Mod}_R^{\mathbf{N}}$, then it can be readily checked that $P\phi := \oplus_i \phi_i : P\mathcal{M} \longrightarrow P\mathcal{N}$ is a graded $R[t]$-morphism, and that P defines a functor satisfying:

Theorem 2.3.13 (Correspondence [76])**.** *Let $\mathbf{gMod}_{R[t]}$ denote the category of graded $R[t]$-modules and graded $R[t]$-morphisms. Then, $P : \mathbf{Mod}_R^{\mathbf{N}} \longrightarrow \mathbf{gMod}_{R[t]}$ is an equivalence of categories.*

A generalization of this result appeared in [21] which drops the condition on commutativity of R and asserts that the correspondence is an isomorphism instead of an equivalence.

Remark 2.3.14. Let $\ell < \rho$ in $\mathbb{N}_0 \cup \{\infty\}$, and let $\mathbb{1}_{[\ell,\rho)} \in \mathbf{Mod}_R^{\mathbf{N}}$ be the resulting interval diagram. Then,

$$P\mathbb{1}_{[\ell,\rho)} \cong \left(t^\ell \cdot R[t]\right) / (t^\rho)$$

as graded $R[t]$-modules, with the convention $t^\infty = 0$. These are called *interval modules*.

Let $\mathcal{X} \in \mathbf{Top}^{\mathbf{N}}$—for instance, encoding multiscale approximations to the topology of an underlying unknown space (see Example 2.3.2). Applying $H_n(\,\cdot\,; R)$ as in Example 2.3.6 yields an object in $\mathbf{Mod}_R^{\mathbf{N}}$, and applying the functor P from the Correspondence Theorem (2.3.13) defines a graded $R[t]$-module. Explicitly:

Definition 2.3.15. Given an object $\mathcal{X} \in \mathbf{Top}^{\mathbf{N}}$, its n-dimensional *persistent homology* with coefficients in R is the persistence module $PH_n(\mathcal{X}; R) \in \mathbf{gMod}_{R[t]}$,

$$PH_n(\mathcal{X}; R) := \bigoplus_{i \in \mathbb{N}_0} H_n(X_i; R).$$

Remark 2.3.16. If $\mathbb{F}$ is a field and $H_n(\mathcal{X}; \mathbb{F})$ is pointwise finite, then

$$PH_n(\mathcal{X}; \mathbb{F}) \cong \bigoplus_{[\ell,\rho) \in \mathsf{bcd}_n(\mathcal{X};\mathbb{F})} \left(t^\ell \cdot \mathbb{F}[t]\right) / (t^\rho)$$

as graded $\mathbb{F}[t]$-modules. A graded version of the structure theorem for modules over a PID implies that if $PH_n(\mathcal{X}; \mathbb{F})$ is finitely generated over $\mathbb{F}[t]$, then $\mathsf{bcd}_n(\mathcal{X}; \mathbb{F})$ can be recovered from the (graded) invariant factor decomposition of $PH_n(\mathcal{X}; \mathbb{F})$. This is how the first general persistent homology algorithms were implemented [76].

Example 2.3.17. Recall that a *simplicial complex* is a collection K of finite nonempty sets—called simplices—so that if $\sigma \in K$ and $\emptyset \neq \tau \subset \sigma$, then $\tau \in K$. Let $K^{(n)} \subset K$ be the collection of simplices of K with $n+1$ elements—these are called n-simplices and we write $v \in K^{(0)}$ instead of $\{v\} \in K^{(0)}$. A function $f : K \longrightarrow K'$ between simplicial complexes is called a *simplicial map* if $f(\{v_0, \ldots, v_n\}) = \{f(v_0), \ldots, f(v_n)\}$ for every $\{v_0, \ldots, v_n\} \in K$.

Because simplicial complexes carry more structure than arbitrary topological spaces, they are easier to code in computers. As a result in TDA, instead of diagrams in **Top**, diagrams in **Simp** are used. Using isomorphisms of singular and simplicial homology, one can show that the entire persistent homology algorithm works for the case of simplicial complexes (as we will see in remark 3.2.1). The following example illustrates the theory of persistent homology:

Example 2.3.18. Consider the filtration of a simplicial circle defined in figure 2.1. The

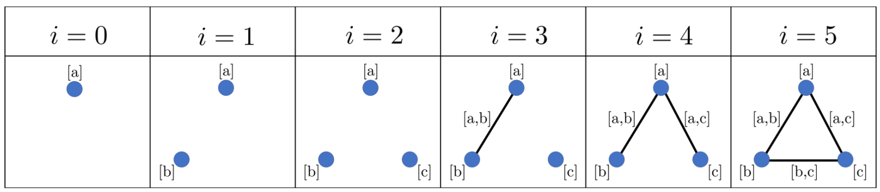


Figure 2.1: A filtered simplicial circle $\mathcal{K} = \{K_i\}_i$, with $K_i = K_5$ for $i \geq 5$.

barcodes are $\mathsf{bcd}(\mathcal{K}) = \{[0, \infty)_0, [1, 3)_0, [2, 4)_0, [5, \infty)_1\}$, where the subscripts denote homological dimension.

Thus far we have described persistent homology in terms of barcodes and $\mathbb{F}[t]$-graded modules. The next, and final description, is in terms of the standard homology of the *persistence chain complex*. Indeed, let $\mathbf{Ch}_R$ denote the category of ($\mathbb{N}_0$-graded) chain complexes of R-modules, and chain maps. Recall that given two chain complexes C_* and C'_*, their direct sum is given by

$$C_* \oplus C'_* = \left\{\partial_i \oplus \partial'_i : C_i \oplus C'_i \longrightarrow C_{i-1} \oplus C'_{i-1}\right\}_{i \in \mathbb{N}_0}.$$

Definition 2.3.19. Let $\mathcal{C}_* = \{f_j : C_{*j} \longrightarrow C_{*j+1}\}$ be an object in $\mathbf{Ch}_R^{\mathbf{N}}$. That is, each C_{*j} is a chain complex of R-modules, and the f_j's are chain maps. Then, the *persistence chain complex* of $\mathcal{C}_*$ is

$$P\mathcal{C}_* := \bigoplus_{j \in \mathbb{N}_0} C_{*j}$$

where each $P\mathcal{C}_i = \bigoplus_{j \in \mathbb{N}_0} C_{i,j} \in \mathbf{gMod}_R$, and therefore $P\mathcal{C}_*$ is an object in the category $\mathbf{gCh}_{R[t]}$ of chain complexes of graded $R[t]$-modules.

Remark 2.3.20. Since homology commutes with direct sums of chain complexes, then

$$H_n(P\mathcal{C}_*) \cong \bigoplus_{j\in\mathbb{N}_0} H_n(C_{*j}) = PH_n(\mathcal{C}_*)$$

as $R[t]$ modules.

For $X \in \mathbf{Top}$, let $S_*(X;R) \in \mathbf{Ch}_R$ denote the chain complex of singular chains in X with coefficients in R. Then, given $\mathcal{X} \in \mathbf{Top}^{\mathbf{N}}$, composition of functors yields an object $S_*(\mathcal{X};R)$ in $\mathbf{Ch}_R^{\mathbf{N}}$. The associated persistence chain complex is thus an object $PS_*(\mathcal{X};R)$ in $\mathbf{gMod}_{R[t]}$ and its homology—by Remark 2.3.20—recovers the persistent homology of $\mathcal{X}$. In other words,

Theorem 2.3.21 ([76])**.** *Let $\mathcal{X} \in \mathbf{Top}^{\mathbf{N}}$. Then, its persistent homology is isomorphic over $R[t]$ to the homology of $PS_*(\mathcal{X},R)$:*

$$PH_n(\mathcal{X};R) \cong H_n(PS_*(\mathcal{X};R)). \tag{2.6}$$

For a general (small) indexing category $\mathbf{I}$ the barcode is no longer available as a discrete invariant for pointwise finite objects in $\mathbf{Mod}_{\mathbb{F}}^{\mathbf{I}}$ [17]. However, it is always possible to consider the rank invariant:

Definition 2.3.22. Let $\mathcal{M} \in \mathbf{Mod}_{\mathbb{F}}^{\mathbf{I}}$. The *rank invariant* of $\mathcal{M}$ is the function on morphisms of $\mathbf{I}$ given by $\rho^{\mathcal{M}}(i \to j) = \mathsf{rank}(\mathcal{M}(i \to j)) \in \mathbb{N}_0 \cup \{\infty\}$. For an object $\mathcal{X} \in \mathbf{Top}^{\mathbf{I}}$ we let $\rho_n^{\mathcal{X}} := \rho^{H_n(\mathcal{X};\mathbb{F})}$.

Remark 2.3.23. Note that $\rho^{\mathcal{M}}$ is an invariant of the isomorphism type of $\mathcal{M}$. It is in fact computable in polynomial time when $\mathbf{I} = \mathbf{N}^k$ and $P\mathcal{M}$ is finitely generated as an $\mathbb{F}[t_1, \dots, t_k]$ module [16]. If $\mathcal{M} \in \mathbf{Mod}_{\mathbb{F}}^{\mathbf{R}}$ is pointwise finite where $\mathbf{R}$ is the poset category of $\mathbb{R}$, then $\rho^{\mathcal{M}}$ and $\mathsf{bcd}(\mathcal{M})$ can be recovered from each other [17, Theorem 12]. Furthermore, $\rho^{\mathcal{M}}$ is a complete invariant for the isomorphism type of $\mathcal{M}$ if $\mathbf{I}$ is a subcategory of $\mathbf{R}$.

2.4 Dynamical Systems

In this section, we cover the necessary basics of dynamical system needed for this dissertation. Simply put, a dynamical system consists of two components: a state space M and an evolution rule ϕ that describes how these states change as a function of time. Perhaps, one of the most popular examples in the study of dynamical systems is the Lorenz attractor. It is a system of the following ordinary differential equations,

$$\frac{dx}{dt} = \sigma(y - x)$$
$$\frac{dy}{dt} = x(\rho - z) - y$$
$$\frac{dz}{dt} = xy - \beta z$$

where x, y, and z are functions of time t and σ, β, and ρ are positive constants. Here, the state space M is the space of all 3-tuples, that is $\mathbb{R}^3$, and the evolution ϕ is dictated by the solution to the ODE system. It was originally developed by Edward Lorenz as a simplified

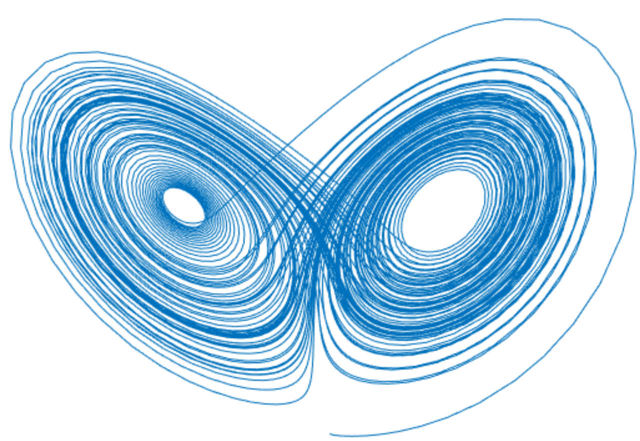


Figure 2.2: The Lorenz attractor for parameters $\sigma = 10$, $\beta = \frac{8}{3}$, and $\rho = 28$.

mathematical model for atmospheric convention [47], and became widely known for having chaotic solutions for certain parameters and initial conditions.

Next, we give a definition of dynamical system on topological spaces:

Definition 2.4.1. A *continuous time dynamical system* is a topological space M along with an evolution rule consisting of a continuous map $\phi : M \times \mathbb{R} \to M$ satisfying the condition

$$\phi(\phi(x,t),s) = \phi(x,t+s)$$

for all $x \in M$ and $t, s \in \mathbb{R}$.

Note that in the above definition, the topological space M serves as the space of states mentioned at the beginning of this section. Next, we give another example of a dynamical system:

Example 2.4.2. Recall that the 2-torus $\mathbb{T}^2$ can be defined as the quotient $\mathbb{R}^2/\mathbb{Z}^2$. An important example of a dynamical system on $\mathbb{T}^2$ is that of the irrational flow:

$$\phi : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{T}^2$$
$$((x,y),t) \to (x+t, y+\omega t)$$

where ω is irrational.

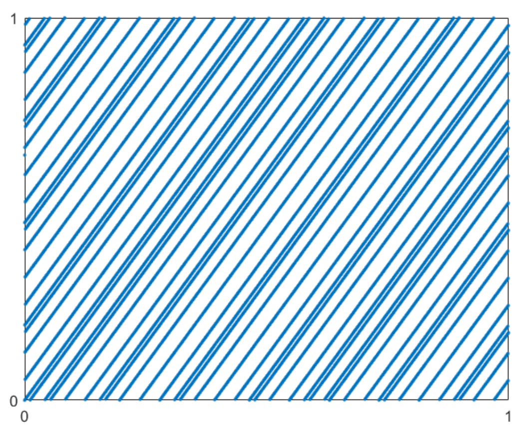


Figure 2.3: The evolution of $(0,0)$ under the irrational flow ϕ on $\mathbb{T}^2$ as defined in Example 2.4.2 for $\omega = \sqrt{3}$.

If M is a differentiable manifold with a vector field X on it, then the flow associated to X gives us an evolution rule. Restriction to a discrete subset of $\mathbb{R}$, say $\mathbb{N}$, yields a *discrete time dynamical system*. Another example is of a manifold M and a diffeomorphism φ. In this case,

$$\begin{aligned} \phi : M \times \mathbb{N} &\to M \\ (x, n) &\to \phi(x, n) := \varphi^n(x). \end{aligned}$$

In practice, dynamical systems are present everywhere and often, the evolution of states is captured via temporal data, i.e. a time series. To be more precise, given the evolution function ϕ of a dynamical system, a real (or complex) valued function $f : M \to \mathbb{R}$ called an *observation function*, and a state x, a time series can be defined

$$\begin{aligned} F : \mathbb{R} &\to \mathbb{R} \\ t &\to f\left(\phi(x, t)\right). \end{aligned}$$

For example, for the irrational flow in Example 2.4.2, define an observation function

$$\begin{aligned} f : \mathbb{T}^2 &\to \mathbb{R} \\ (x, y) &\to \cos(x) + \cos(y). \end{aligned}$$

Then for the observation of the evolution of $(0, 0)$ under ϕ gives us a time series $F(t) = \cos(t) + \cos(\omega t)$. See Figure 2.4.

So far, we have discussed that given a dynamical system, one can obtain relevant measurements in the form of a time series. The next section discusses the conditions under which given a time series, some information about the underlying dynamical system can be recovered.

2.4.1 Takens's embedding theorem

In 1981, Floris Takens proved a delay embedding theorem [67, Theorem 1] which provides conditions under which a smooth attractor can be reconstructed from a sequence of obser-

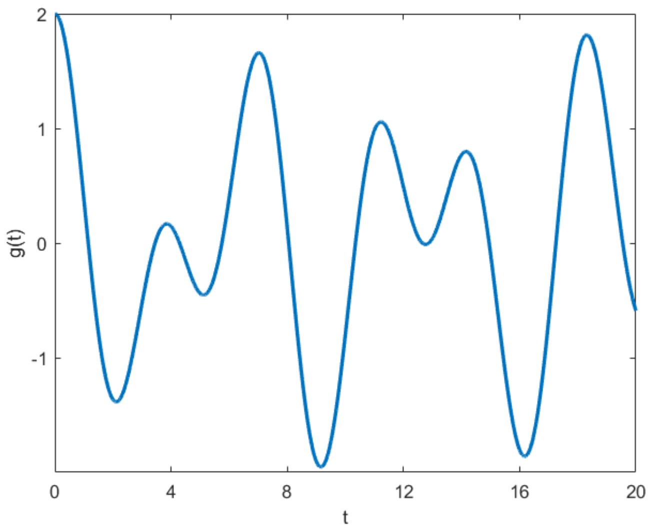


Figure 2.4: Time series associated to the observation f of the evolution of $(0,0)$ for the irrational flow ϕ.

vations made with a generic observation function. While this reconstructed attractor might have a different geometric shape than the original, its topology is still preserved. We present next the version of Takens' theorem which is the most relevant to us, in particular, when the manifold M is compact and the dynamical system is discrete in nature.

Theorem 2.4.3 ([67], Theorem 1)**.** *Let M be a m-dimensional compact Riemannian manifold. For pairs (φ, f), where $\varphi \in C^2(M,M)$ and $f \in C^2(M,\mathbb{R})$, it is a generic property that the map $\Phi_{\varphi,f} : M \to \mathbb{R}^{2m+1}$ defined as*

$$\Phi_{\varphi,f}(x) = \big(f(x), f(\varphi(x)), f(\varphi^2(x)), \ldots, f(\varphi^{2m}(x))\big)$$

is an embedding.

There are two things to note here. The first is that the genericity of the pair (φ, f) is in C^2-Whitney topology on the function space $C^2\left(M, M \times \mathbb{R}\right)$, in which roughly speaking, two functions f and g are close to each other if and only if their derivatives are close to each other. For a more rigorous version, we refer the reader to [41]. The other is that Takens also provides similar theorems for continuous dynamical systems and in the case when the manifold M is non-compact. Takens' theorem motivates the definition of the sliding window embedding which is what we describe next.

Definition 2.4.4. For a function $f : \mathbb{R} \to \mathbb{C}$, an integer $d > 0$ called the *embedding dimension* and a real number $\tau > 0$ called the *time delay*, the *sliding window embedding* of f at t is given by:

$$SW_{d,\tau} f(t) = \begin{bmatrix} f(t) \\ f(t+\tau) \\ \vdots \\ f(t+d\tau) \end{bmatrix}$$

If this function is indeed a time series generated by observing some underlying dynamical system, Takens' theorem implies that by studying the shape of the sliding window point cloud, we can recover some information about the dynamical system. Let us look at some examples, with a simple function $g(t) = \cos(t)$ first.

Example 2.4.5. Assume a particle at $(1, 0)$ on a unit circle S^1 is moving counter-clockwise with the function $\phi(t) = (\cos(t), \sin(t))$. Let $p : S^1 \to \mathbb{R}$ defined as $p((x, y)) = x$ be the observation. Then we get a time series

$$g(t) = p(\phi(t)) = \cos(t).$$

The sliding window embedding $SW_{d,\tau} g(\mathbb{R})$ of g with appropriate dimension and delay recovers the topology of S^1.

While the figure on the right is topologically a circle, it looks different from a standard circle geometrically. The reader might ponder why it would be a problem—because in real applications, we only have a discrete time series and the hope is that the sliding window embedding of a continuous approximation to this time series recovers the topology of the underlying dynamical system. In [58], Perea and Harer used Rips persistent homology to study finite sliding window point clouds generated from general periodic functions and provided

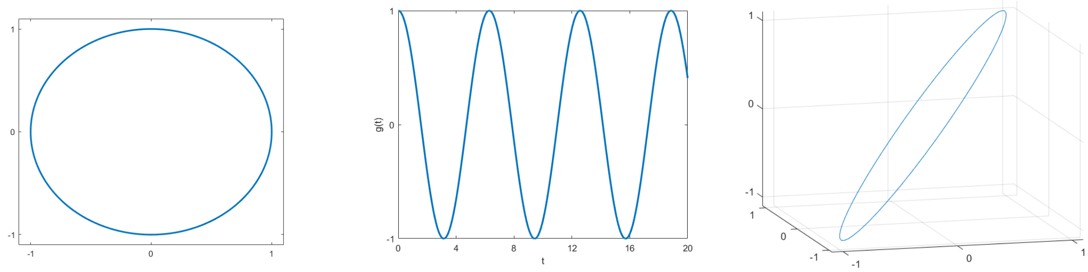


Figure 2.5: Dynamical System to Sliding Window Reconstruction for a counterclockwise flow on S^1: In the leftmost figure, we see a standard circle. In the middle, we see the time series generated $g(t)$. In the rightmost figure, we see the reconstructed circle from sliding window embedding.

conditions on parameters d and τ under which the geometry of the recovered topological circle is a good geometric approximation. In particular, we will use persistent homology of Rips complexes on $SW_{d,\tau}g(T) \subset SW_{d,\tau}g(\mathbb{R}) \subset \mathbb{C}^{d+1}$ for some $T \subset \mathbb{R}$.

Next, we have another example of sliding window embedding of a periodic function $g(t) = \cos(t) + \cos(3t)$.

Example 2.4.6. Assume a particle at $(0, 0)$ on $\mathbb{T}^2$ is moving with the function $\phi(t) = (t, 3t)$. Let $p : \mathbb{T}^2 \to \mathbb{R}$ defined as $p((x, y)) = \cos(x) + \cos(y)$ be the observation. Then we get a time series

$$g(t) = p(\phi(t)) = \cos(t) + \cos(3t).$$

The sliding window embedding $SW_{d,\tau}g(\mathbb{R})$ of g with appropriate dimension and delay recovers the topology of S^1. Rips persistent homology confirms that in Figure 2.6.

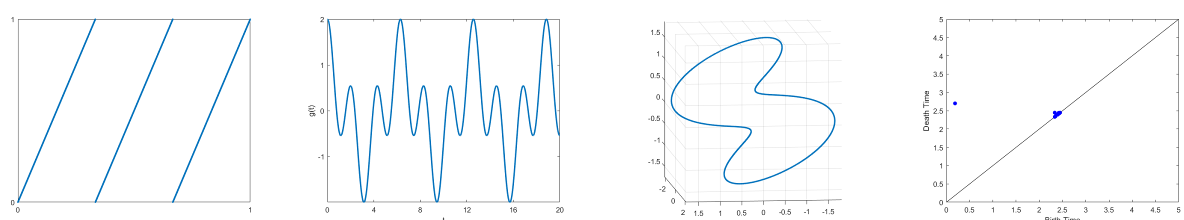


Figure 2.6: Dynamical System to Persistence for a rational flow on $\mathbb{T}^2$: In the leftmost figure, we see a rational flow on $\mathbb{T}^2$. In the second, we see the time series generated $g(t)$. In the third figure, we see a PCA visualization of the reconstructed circle from sliding window embedding. In the rightmost figure, we see the 1- dimensional Rips persistence diagram of a sliding window point cloud sampled from the third figure.

A closely related function to the periodic function in the example above is $g(t) = \cos(t) + \cos(\sqrt{3}t)$, which is an example of a *quasiperiodic* function, which we will study in its generality in Chapter 4. In simpler terms, a quasiperiodic function is controlled by two or more harmonics and these harmonics are linearly independent over $\mathbb{Q}$. A special first case was studied by Perea in [53].

Example 2.4.7. Assume a particle at $(0,0)$ on $\mathbb{T}^2$ is moving with the irrational flow function $\phi(t) = (t, \sqrt{3}t)$. Let $p : \mathbb{T}^2 \to \mathbb{R}$ defined as $p((x,y)) = \cos(x) + \cos(y)$ be the observation. Then we get a time series

$$g(t) = p(\phi(t)) = \cos(t) + \cos(\sqrt{3}t).$$

The sliding window embedding $SW_{d,\tau}g(\mathbb{R})$ of g with appropriate dimension and delay recovers the topology of $\mathbb{T}^2$. Rips persistent homology confirms that in Figure 2.7.

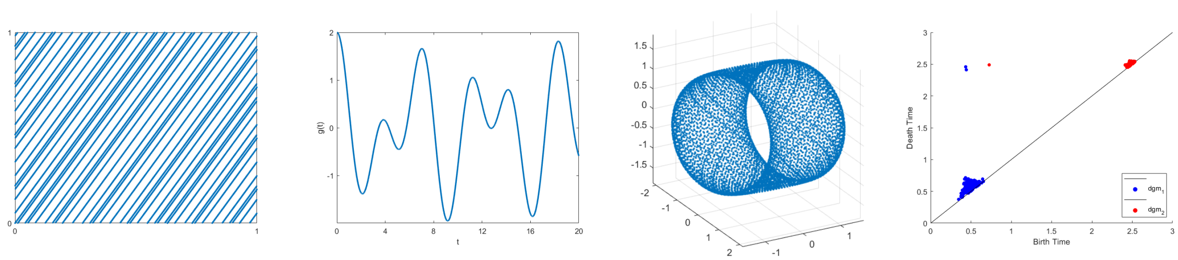


Figure 2.7: Dynamical System to Persistence for an irrational flow on $\mathbb{T}^2$: In the leftmost figure, we see an irrational flow on $\mathbb{T}^2$. In the second, we see the time series generated $g(t)$. In the third figure, we see a PCA visualization of the reconstructed torus from sliding window embedding. In the rightmost figure, we see the Rips persistence diagrams in dimensions $1, 2$ of a sliding window point cloud sampled from the third figure.

Before finishing this section, we want to mention a property that we will use later. The following was shown in [59].

Proposition 2.4.8. *For all $d \in \mathbb{N}$ and $\tau > 0$, the mapping $SW_{d,\tau} : C(\mathbb{R}, \mathbb{C}) \to C(\mathbb{R}, \mathbb{C}^{d+1})$ is a bounded linear operator with norm*

$$\| SW_{d,\tau} \| \leq \sqrt{d+1}.$$

2.5 Analytical tools

In this section, we provide a brief review of Fourier series in 2.5.1, Kronecker's approximation theorem in 2.5.2, and Wasserstein metric on probability measures in 2.5.3.

2.5.1 Fourier Series

Fourier series were introduced by Joseph Fourier in the 1800s for the purpose of solving the heat equation in a metal plate. Using the ideas that Fourier introduced, it was shown that every arbitrary real valued continuous periodic function can be approximated by a sum of sines and cosines. Since then Fourier series have had many applications in electrical engineering, acoustics, optics, signal & image processing, etc. Let us review:

Let $\mathbb{T}$ be the quotient space $\mathbb{R}/(2\pi\mathbb{Z})$. For a single variable periodic function $g \in L^2(\mathbb{T})$ and $Z \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, its Fourier series and its Z-truncated Fourier polynomial are written as

$$g(t) = \sum_{k=-\infty}^{\infty} \hat{g}(k)e^{ikt} \quad \text{and} \quad S_Z g(t) = \sum_{k=-Z}^{Z} \hat{g}(k)e^{ikt}$$

respectively, where

$$\hat{g}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)e^{-ikt}dt$$

are the Fourier coefficients of g and the series given by partial sums $S_Z g$ converges to g almost everywhere as $Z \to \infty$. If additionally, g is continuously differentiable, then pointwise convergence is guaranteed by the following pipeline of results in analysis:

Proposition 2.5.1. 1. *If $f \in C^1([a,b])$, then f, f' are both continuous on $[a,b]$ by definition. Then f is Lipschitz.*

2. *If f is Lipschitz on $[a,b]$, then f is uniformly continuous and has bounded variation, with the variation bounded by $L_f(b-a)$ where L_f is the Lipschitz constant.*

3. *(Jordan's Criterion) If f is a function of bounded variation in a neighbourhood of x, then*

$$\lim_{Z\to\infty} S_Z f(x) = \frac{1}{2}[f(x+) + f(x-)].$$

From this proposition, we can conclude that if $g \in C^1([a,b])$, then $S_Z g(x)$ converges to $g(x)$ for all $x \in [a,b]$. A proof of these propositions can be found in [29, Chapter 10].

2.5.2 Kronecker's theorems

Recall that $\mathbb{R}$ can be viewed as an infinite dimensional vector space over $\mathbb{Q}$ with an uncountable basis. A finite subset $\{\beta_1, \dots, \beta_N\} \subset \mathbb{R}$ of reals (vectors) is called *incommensurate* if $\beta_1, \dots, \beta_N$ are linearly independent over $\mathbb{Q}$, that is,

$$r_1\beta_1 + \cdots + r_N\beta_N = 0 \Rightarrow r_1 = \cdots = r_N = 0$$

for $r_1, \dots, r_N \in \mathbb{Q}$. The following theorem by Kronecker is a result about diophantine approximations applying to several real numbers. A proof can be found in [42, Chapter 2].

Theorem 2.5.2 (Kronecker's theorem)**.** *Let $\{\beta_1, \dots, \beta_N\}$ be a set of real numbers, $r_1, \dots, r_N$ be arbitrary real numbers, and ϵ, L be arbitrary positive numbers, then one can find integers l and $p_1, \dots, p_N$ with*

$$|\beta_n l - r_n - p_n| < \epsilon \quad \text{for} \quad 1 \le n \le N$$

where $|l| \ge L$ if and only if $\{\beta_1, \dots, \beta_N\}$ is incommensurate. As a consequence, $\{\beta_1, \dots, \beta_N\}$ is incommensurate over $\mathbb{Q}$ if and only if the collection $\{(z\beta_1, \dots, z\beta_N) \mid z \in \mathbb{Z}\}$ of N-tuples is dense in $\mathbb{T}^N = (\mathbb{R}/(2\pi\mathbb{Z}))^N$, which is equivalent to the collection $\{(e^{i\beta_1 z}, \dots, e^{i\beta_N z}) | z \in \mathbb{Z}\}$ being dense in $\mathbb{T}^N = S^1 \times \cdots \times S^1$.

In section 4.3, we will use Kronecker's theorem to prove:

Corollary 4.3.1: *An α-tuple $\beta = (\beta_1, \dots, \beta_\alpha)$ of real numbers spans a N-dimensional vector space over $\mathbb{Q}$ if and only if the collection $\{z\beta \mid z \in \mathbb{Z}\}$ is dense in an embedding of $\mathbb{T}^N$ in*

$\mathbb{T}^\alpha$.

which in turn will be essential to proving the structure theorem:

Theorem 4.3.5: *The sliding window embedding $SW_{d,\tau}S_Z f(\mathbb{Z})$ is dense in a space homeomorphic to $\mathbb{T}^N$ in $\mathbb{T}^{d+1}$ for appropriate conditions on τ, d, and Z.*

2.5.3 Wasserstein metric on probability measures

In this section, we review the definition and relevant properties of the Wasserstein metric. We will use this metric to measure the noise on the landmark sets in Chapter 5, for which we prove the stability theorems.

To begin, consider a compact metric space $(\mathbb{M}, \mathbf{d})$.

Definition 2.5.3. A *correspondence* (or *relation*) between non-empty subsets $A, B \subset \mathbb{M}$ is a subset $R \subset A \times B$ such that

- for all $a \in A$, there exists $b \in B$ such that $(a, b) \in R$, and
- for all $b \in B$, there exists $a \in A$ such that $(a, b) \in R$.

Denote by $\mathcal{R}(A, B)$ the set of all possible correspondences between A and B. Furthermore, define the *Hausdorff distance* as

$$d_H(A, B) = \inf_{R \in \mathcal{R}(A,B)} \sup_{(a,b) \in R} \mathbf{d}(a, b).$$

Definition 2.5.4. Given $A, B \subset \mathbb{M}$ and measures μ_A, μ_B on $\mathbb{M}$ with $\mathsf{Supp}(\mu_A) = A$ and $\mathsf{Supp}(\mu_B) = B$, a measure μ on $A \times B$ is a *coupling* of μ_A and μ_B if and only if

$$\mu(A_0 \times B) = \mu_A(A_0) \quad \forall A_0 \subset A \quad \text{measureable and}$$
$$\mu(A \times B_0) = \mu_B(B_0) \quad \forall B_0 \subset B \quad \text{measureable.}$$

Denote by $\mathcal{M}(\mu_A, \mu_B)$, the collection of all couplings of μ_A and μ_B. Denote by $R(\mu) = \mathsf{Supp}(\mu)$, the correspondence defined by the support of a coupling. For $A = \{a_i\}$ and $B = \{b_j\}$ finite, we represent a coupling as a $|A| \times |B|$ matrix $[\mu_{ij}]$ with $\mu_{ij} = \mu(a_i, b_j)$.

Definition 2.5.5. For $p \geq 1$ or $p = \infty$, the following

$$d_{\mathcal{W},p}(\mu_A, \mu_B)^p = \inf_{\mu \in \mathcal{M}(\mu_A, \mu_B)} \int_{A \times B} \mathbf{d}(a,b)^p d\mu(a,b) \quad \text{and,}$$

$$d_{\mathcal{W},\infty}(\mu_A, \mu_B) = \inf_{\mu \in \mathcal{M}(\mu_A, \mu_B)} \sup_{(a,b) \in R(\mu)} \mathbf{d}(a,b)$$

define the *Wasserstein–Kantorovich–Rubinstein metrics* between measures on $\mathbb{M}$.

Since the support of a coupling yields a relation, we have

Lemma 2.5.6. *Let $A, B \subset (\mathbb{M}, d)$ compact metric space and let μ_A and μ_B be measures on $\mathbb{M}$. Then,*

$$d_H(\mathsf{Supp}(\mu_A), \mathsf{Supp}(\mu_B)) \leq d_{\mathcal{W},\infty}(\mu_A, \mu_B).$$

Proof. See Corollary 2.1 [49]. □

CHAPTER 3

PERSISTENT KÜNNETH THEOREMS

In this chapter, we prove our two Künneth formulae in persistent homology. In section 3.1, we define two products of (diagrams of) spaces, namely the *categorical product* in Definition 3.1.1 and the *generalized tensor product* in Definition 3.1.13, study their properties, and give examples. In sections 3.2 and 3.3, we provide the proofs of our persistent Künneth formulae for the categorical and generalized tensor products, respectively. In section 3.4, as an application of the categorical Künneth formula for Rips complexes (Corollary 3.2.6), we revisit the theoretical results about the Rips persistent homology of the N-torus. Parts of this chapter appeared in [35].

3.1 Products of Diagrams of Spaces

Given a category $\mathbf{S}$ of spaces, such as **Top**, **Simp**, or **Met**, and a small indexing category $\mathbf{I}$, the first objective is to identify relevant products between $\mathbf{I}$-indexed diagrams $\mathcal{X}, \mathcal{Y} \in \mathbf{S}^{\mathbf{I}}$. For a particular product, the goal is to describe its persistent homology in terms of the persistent homology of $\mathcal{X}$ and $\mathcal{Y}$. This is what we call a *persistent Künneth formula*. We begin with the first product construction: the *categorical product* in $\mathbf{S}^{\mathbf{I}}$.

Definition 3.1.1. Let $\mathbf{S}$ be a category having all pairwise products (e.g., finitely complete), let $\mathcal{X}, \mathcal{Y} \in \mathbf{S}^{\mathbf{I}}$, and let $\mathcal{X} \times \mathcal{Y} : \mathbf{I} \longrightarrow \mathbf{S}$ be the functor taking each object $i \in \mathbf{I}$ to the (categorical) product $X_i \times Y_i \in \mathbf{S}$, and each morphism $i \to j$ in $\mathbf{I}$ to the unique morphism $\mathcal{X} \times \mathcal{Y}(i \to j)$ making the following diagram commute:

$$\begin{array}{ccccc} & & X_i \times Y_i & & \\ & \swarrow^{\mathcal{X}(i\to j)\circ p_i^X} & \downarrow & \searrow^{\mathcal{Y}(i\to j)\circ p_i^Y} & \\ X_j & \xleftarrow[p_j^X]{} & X_j \times Y_j & \xrightarrow[p_j^Y]{} & Y_j \end{array} \tag{3.1}$$

where p_i^X and p_j^Y are canonical projections of $X_i \times Y_j$ onto X_i and Y_j respectively. The existence of $\mathcal{X} \times \mathcal{Y}(i \to j)$ follows from the universal property defining $X_j \times Y_j$.

We have the following observation:

Proposition 3.1.2. *Let* $\mathbf{S}$ *be a category with all pairwise products. Then,* $\mathcal{X} \times \mathcal{Y} \in \mathbf{S}^{\mathbf{I}}$ *is the categorical product of* $\mathcal{X}, \mathcal{Y} \in \mathbf{S}^{\mathbf{I}}$.

Proof. First, note that the existence of $p^{\mathcal{X}} : \mathcal{X} \times \mathcal{Y} \longrightarrow \mathcal{X}$ and $p^{\mathcal{Y}} : \mathcal{X} \times \mathcal{Y} \longrightarrow \mathcal{Y}$ follows from that of $p_i^X : X_i \times Y_i \longrightarrow X_i$ and $p_i^Y : X_i \times Y_i \longrightarrow Y_i$ for each $i \in \mathbf{I}$, and the commutativity of (3.1) for each $i \to j$.

Let $\mathcal{Z} \in \mathbf{S}^{\mathbf{I}}$, and let $\mu : \mathcal{Z} \to \mathcal{X}$, $\nu : \mathcal{Z} \to \mathcal{Y}$ be morphisms. For each $i \in \mathbf{I}$, let $\mu_i \times \nu_i : Z_i \to X_i \times Y_i$ be the unique morphism so that $p_i^X \circ (\mu_i \times \nu_i) = \mu_i$ and $p_i^Y \circ (\mu_i \times \nu_i) = \nu_i$. It readily follows that $\mu \times \nu : \mathcal{Z} \longrightarrow \mathcal{X} \times \mathcal{Y}$, given by $(\mu \times \nu)(i) = \mu_i \times \nu_i$, is the unique morphism of $\mathbf{I}$-indexed diagrams such that $p^{\mathcal{X}} \circ (\mu \times \nu) = \mu$, and $p^{\mathcal{Y}} \circ (\mu \times \nu) = \nu$. □

Let us describe in more detail the categories we have in mind for $\mathbf{S}$, as well as what the categorical products are in each case.

Example 3.1.3 (Topological spaces)**.** Recall that **Top** denotes the category of topological spaces and continuous maps. For $X, Y \in \mathbf{Top}$, their Cartesian product equipped with the product topology is the categorical product of X and Y in **Top**.

Example 3.1.4 (Metric Spaces)**.** Let **Met** denote the category of metric spaces and non-expansive maps. That is, its objects are pairs $(X, \mathbf{d}_X)$—a set and a metric— and its morphisms are functions $f : X \longrightarrow Y$ satisfying $\mathbf{d}_Y(f(x), f(x')) \leq \mathbf{d}_X(x, x')$ for all $x, x' \in X$.

Definition 3.1.5. Given $(X, \mathbf{d}_X), (Y, \mathbf{d}_Y) \in \mathbf{Met}$, let $X \times Y$ denote the Cartesian product of the underlying sets, and let $\mathbf{d}_{X \times Y}$ be the *maximum metric* on $X \times Y$:

$$\mathbf{d}_{X \times Y}\Big((x, y), (x', y')\Big) := \max\{\mathbf{d}_X(x, x'), \mathbf{d}_Y(y, y')\}.$$

Since the coordinate projections $p^X : X \times Y \longrightarrow X$ and $p^Y : X \times Y \longrightarrow Y$ are non-expansive, it readily follows that $(X \times Y, \mathbf{d}_{X \times Y})$ is the categorical product of $(X, \mathbf{d}_X)$ and $(Y, \mathbf{d}_Y)$ in **Met**.

Example 3.1.6 (Simplicial Complexes)**.** Let **Simp** be the category of simplicial complexes and simplicial maps.

Definition 3.1.7. For $K, K' \in$ **Simp**, let $K \times K'$ be the smallest simplicial complex containing all Cartesian products $\sigma \times \sigma'$, for $\sigma \in K$ and $\sigma' \in K'$.

In other words, $\tau \in K \times K'$ if and only if there exist $\sigma \in K$ and $\sigma' \in K'$ with $\tau \subset \sigma \times \sigma'$. Let $p : \tau \to \sigma$ and $p' : \tau \to \sigma'$ be the projection maps onto the first and second coordinate, respectively. Since $p(\tau) \subset \sigma$, then $p(\tau) \in K$, and similarly $p'(\tau) \in K'$. Hence p and p' define simplicial maps $K \xleftarrow{p} K \times K' \xrightarrow{p'} K'$, and it readily follows that $K \times K'$ is the categorical product of K and K' in **Simp**.

The *Rips* complex at scale $\epsilon \in \mathbb{R}^+$ is a simplicial complex which can be associated to any metric space $(X, \mathbf{d}_X)$. It is widely used in TDA—when $(X, \mathbf{d}_X)$ is the observed data set—and it is defined as

$$R_\epsilon(X) := \left\{ \{x_0, \ldots, x_n\} \subset X : \max_{0 \le i,j \le n} \mathbf{d}_X(x_i, x_j) < \epsilon \right\}. \tag{3.2}$$

Notice that any non-expansive function $f : X \to Y$ between metric spaces extends to a simplicial map $R_\epsilon(f) : R_\epsilon(X) \to R_\epsilon(Y)$, and thus R_ϵ defines a functor from **Met** to **Simp**. This functor is in fact compatible with the categorical products:

Lemma 3.1.8. *Let* $(X, \mathbf{d}_X)$*,* $(Y, \mathbf{d}_Y)$ *be metric spaces, and let* $\epsilon \in \mathbb{R}^+$*. Then,*

$$R_\epsilon(X \times Y) = R_\epsilon(X) \times R_\epsilon(Y).$$

Proof. See the proof of Proposition 10.2 in [1]. □

One drawback of the categorical product in **Simp** is that it is not well-behaved with respect to geometric realizations. Indeed, recall that the geometric realization of a simplicial complex K—denoted $|K|$—is the collection of all functions $\varphi : K^{(0)} \to [0, 1]$ so that $\{v \in K^{(0)} : \varphi(v) \neq 0\} \in K$, and for which $\sum_{v \in K^{(0)}} \varphi(v) = 1$. Moreover, if $\varphi, \psi \in |K|$, then

$$\mathbf{d}_{|K|}(\varphi, \psi) := \sum_{v \in K^{(0)}} |\varphi(v) - \psi(v)|$$

defines a metric on $|K|$. Every simplicial map $f : K \to L$ induces a function $|f| : |K| \to |L|$ given by

$$|f|(\varphi)(w) = \sum_{\substack{v \in K^{(0)} \\ f(v)=w}} \varphi(v) \qquad , \qquad |f|(\varphi)(w) = 0 \;\text{ if }\; w \notin f(K^{(0)})$$

which, as one can check, is non-expansive. In other words, geometric realization defines a functor $| \cdot | : \mathbf{Simp} \longrightarrow \mathbf{Met}$. To see the incompatibility of $| \cdot |$ and the product in **Simp**, let $K = L = \{0, 1, \{0, 1\}\}$. It follows that $|K \times L|$ is (homeomorphic to) the geometric 3-simplex $\Big\{(t_0, \ldots, t_3) \in \mathbb{R}^4 : t_j \geq 0,\ t_0 + \cdots + t_3 = 1\Big\}$, while $|K| \times |L|$ is the unit square $[0, 1] \times [0, 1] \subset \mathbb{R}^2$. Hence, $|K \times L|$ is in general not equal, nor homeomorphic to $|K| \times |L|$. This leads one to consider other alternatives; specifically, the category of *ordered simplicial complexes.*

Example 3.1.9 (Ordered Simplicial Complexes)**.** If $K^{(0)}$ comes equipped with a partial order so that each simplex is totally ordered, then we say that K is an ordered simplicial complex. A simplicial map between ordered simplicial complexes is said to be order-preserving, if it is so between 0-simplices. Let **oSimp** denote the resulting category. For data analysis applications, and particularly for the Rips complex construction, **oSimp** is perhaps more relevant than **Simp**. Indeed, a data set $(X, \mathbf{d}_X)$ stored in a computer comes with an explicit total order on X. If $K, K' \in \mathbf{oSimp}$, with partial orders $\preceq, \preceq'$, respectively, and $\sigma \in K$, $\sigma' \in K'$, then the Cartesian product $\sigma \times \sigma'$ has a partial order $\preceq^\times$ given by $(v, v') \preceq^\times (w, w')$ if and only if $v \preceq w$ and $v' \preceq' w'$.

Definition 3.1.10. For $K, K' \in \mathbf{oSimp}$, let $K \oslash K'$ be defined as follows: $\tau \in K \oslash K'$ if and only if there exist $\sigma \in K$ and $\sigma' \in K'$ so that τ is a totally ordered subset of $\sigma \times \sigma'$, with respect to $\preceq^\times$.

Just like for simplicial complexes, the coordinate projections p and p' induce order-preserving simplicial maps $p : K \oslash K' \to K$ and $p' : K \oslash K' \to K'$. Moreover,

Lemma 3.1.11. *Let K and K' be ordered simplicial complexes. Then,*

1. *$K \oslash K'$ is their categorical product in* **oSimp**.

2. *$K \oslash K' \subset K \times K'$, where the right hand side is the categorical product in* **Simp**. *Moreover, $|K \oslash K'|$ is a deformation retract of $|K \times K'|$.*

3. *The product map $|p| \times |p'| : |K \oslash K'| \longrightarrow |K| \times |K'|$ is a homeomorphism.*

Proof. See [31], specifically Definitions 8.1, 8.8 and Lemmas 8.9, 8.11. □

The above are the categories of spaces and the products we will consider moving forward. Going back to diagrams in **Top**, recall that a CW-complex is a topological space X together with a CW-structure. That is, a filtration $\mathcal{X} = \{X^{(k)}\}_{k\in\mathbb{N}_0}$ of X, so that $X^{(0)}$ has the discrete topology, $X^{(k)}$ is obtained from $X^{(k-1)}$ by attaching k-dimensional cells $D^k \cong e^k_\alpha \subset X$ via continuous maps $\varphi_\alpha : \partial D^k \to X^{(k-1)}$, and so that the topology on X coincides with the weak topology induced by $\mathcal{X}$. It follows that $\mathcal{X} \in \mathbf{FTop} \subset \mathbf{Top}^{\mathbf{N}}$. In this context, a cellular map between CW-complexes—i.e. a continuous function $f : X \to Y$ so that $f\left(X^{(k)}\right) \subset Y^{(k)}$ for all k—is exactly a morphism in $\mathbf{Top}^{\mathbf{N}}$. Thus, if **CW** denotes the category of locally compact CW-complexes (we will see in a moment why this restriction) and cellular maps, then **CW** is a full subcategory of **FTop**.

If X and Y are CW-complexes, then their topological product $X \times Y$ can also be written as a union of cells. Specifically, the k-cells of $X \times Y$ are the products $e^i_\alpha \times e^j_\beta$ with $k = i + j$, for e^i_α an i-cell of X, and e^j_β a j-cell of Y. Thus, if

$$(X \times Y)^{(k)} := \bigcup_{i+j=k} X^{(i)} \times Y^{(j)} \quad , \quad k \in \mathbb{N}_0 \tag{3.3}$$

then $\left\{(X \times Y)^{(k)}\right\}_{k\in\mathbb{N}_0}$ will be a CW-structure for $X \times Y$ provided the product topology coincides with the weak topology induced by (3.3). If either X or Y are locally compact, then this will be the case (see Theorem A.6 in [39]).

In summary: the categorical product in $\mathbf{Top}^{\mathbf{N}}$ does not recover the usual product of CW-complexes, which suggests the existence of other useful (non-categorial) products in **FTop** and $\mathbf{Top}^{\mathbf{N}}$. The product of CW-complexes suggests the following definition,

Definition 3.1.12. If $\mathcal{X}, \mathcal{Y} \in \mathbf{FTop}$, then their tensor product is the filtered space

$$\mathcal{X} \otimes \mathcal{Y} := \left\{ \bigcup_{i+j=k} X_i \times Y_j \right\}_{k \in \mathbb{N}_0}.$$

The tensor product of filtered spaces is sometimes regarded in the literature as the de facto product filtration; see for instance [8, 11, 64] or [66, Chapter 27]. For general objects in $\mathbf{Top}^{\mathbf{N}}$—e.g., when the internal maps are not inclusions—taking the union of, say, $X_{i+1} \times Y_j$ and $X_i \times Y_{j+1}$ does not make sense. Instead, the corresponding operation would be to glue these spaces along the images of the product maps

$$X_{i+1} \times Y_j \longleftarrow X_i \times Y_j \longrightarrow X_i \times Y_{j+1}.$$

Here is where the machinery of homotopy colimits comes in as a means to defining a generalized tensor product in $\mathbf{Top}^{\mathbf{N}}$:

Definition 3.1.13. For $k \in \mathbb{N}_0$, let $\triangle_k$ be the poset category of $\{(i,j) \in \mathbb{N}_0^2 : i + j \leq k\}$ with its usual product order, and for $\mathcal{X}, \mathcal{Y} \in \mathbf{Top}^{\mathbf{N}}$, let

$$\mathcal{X} \boxtimes \mathcal{Y} : \mathbf{N}^2 \longrightarrow \mathbf{Top}$$

be the functor sending (i,j) to $X_i \times Y_j$, and $(i,j) \leq (s,t)$ to $\mathcal{X}(i \leq s) \times \mathcal{Y}(j \leq t)$. The *generalized tensor product* of $\mathcal{X}$ and $\mathcal{Y}$ is the object $\mathcal{X} \otimes_{\mathbf{g}} \mathcal{Y} \in \mathbf{Top}^{\mathbf{N}}$ given by

$$\begin{aligned} \mathcal{X} \otimes_{\mathbf{g}} \mathcal{Y}(k) &:= \mathsf{hocolim}(\mathcal{X} \boxtimes \mathcal{Y}|_{\triangle_k}) \\ \mathcal{X} \otimes_{\mathbf{g}} \mathcal{Y}(k \leq k') &:= \mathsf{hocolim}\left(\mathcal{X} \boxtimes \mathcal{Y}|_{\triangle_k} \to \mathcal{X} \boxtimes \mathcal{Y}|_{\triangle_{k'}}\right) \end{aligned} \tag{3.4}$$

where the latter is the continuous map associated to the change of indexing categories induced by the inclusion $\triangle_k \subset \triangle_{k'}$.

Remark 3.1.14. The definition of the generalized tensor product can be extended to diagrams indexed by other subcategories of $\mathbf{R}$. For example, to the poset categories of $\mathbb{R}$, $\mathbb{R}_+$ (the set of non negative reals) and $\mathbb{R}_*$ (the set of positive reals). In general, for $\mathbf{I} = \mathbf{N}$, $\mathbf{R}$, $\mathbf{R}_+$, or $\mathbf{R}_*$, we let $\triangle_r = \{(i,j) \in \mathbf{I}^2 : i + j \leq r\}$ and define $\mathcal{X} \otimes_{\mathbf{g}} \mathcal{Y} : \mathbf{I} \longrightarrow \mathbf{Top}$ as in (3.4).

Although $\triangle_r$ and $\mathcal{X} \boxtimes \mathcal{Y}$ are different for each choice of $\mathbf{I}$, we will use the same notation for aesthetic purposes. The choice will be clear from the context as we will specify the $\mathbf{I}$ in $\mathbf{Top}^{\mathbf{I}}$.

We have the following relation between $\otimes$ and $\otimes_{\mathbf{g}}$,

Lemma 3.1.15. *Let $\mathcal{X}, \mathcal{Y} \in \mathbf{Top}^{\mathbf{N}}$, and let $\mathcal{T}(\mathcal{X}), \mathcal{T}(\mathcal{Y}) \in \mathbf{FTop}$ be their telescope filtrations. Then, $\mathcal{X} \otimes_{\mathbf{g}} \mathcal{Y}$ is naturally homotopy equivalent to $\mathcal{T}(\mathcal{X}) \otimes \mathcal{T}(\mathcal{Y})$.*

Proof. We will show that there is a morphism $\mathcal{X} \otimes_{\mathbf{g}} \mathcal{Y} \longrightarrow \mathcal{T}(\mathcal{X}) \otimes \mathcal{T}(\mathcal{Y})$, so that each map $\big(\mathcal{X} \otimes_{\mathbf{g}} \mathcal{Y}\big)_k \longrightarrow \big(\mathcal{T}(\mathcal{X}) \otimes \mathcal{T}(\mathcal{Y})\big)_k$, $k \in \mathbb{N}_0$, is a homotopy equivalence. Indeed, since $\mathcal{T}_j(\mathcal{X})$ deformation retracts onto X_j, then each inclusion $X_j \hookrightarrow \mathcal{T}_j(\mathcal{X})$ is a homotopy equivalence, and thus (by Theorem 2.2.9) so is the induced map

$$\big(\mathcal{X} \otimes_{\mathbf{g}} \mathcal{Y}\big)_k = \mathsf{hocolim}\left(\mathcal{X} \boxtimes \mathcal{Y}\Big|_{\triangle_k}\right) \longrightarrow \mathsf{hocolim}\left(\mathcal{T}(\mathcal{X}) \boxtimes \mathcal{T}(\mathcal{Y})\Big|_{\triangle_k}\right).$$

Since each inclusion $\mathcal{T}_i(\mathcal{X}) \hookrightarrow \mathcal{T}_j(\mathcal{X})$ is a closed cofibration (Remark 2.2.12), then the projection Lemma 2.2.13 implies that the collapse map

$$\mathsf{hocolim}\left(\mathcal{T}(\mathcal{X}) \boxtimes \mathcal{T}(\mathcal{Y})\Big|_{\triangle_k}\right) \longrightarrow \mathsf{colim}\left(\mathcal{T}(\mathcal{X}) \boxtimes \mathcal{T}(\mathcal{Y})\Big|_{\triangle_k}\right)$$

is a homotopy equivalence. Moreover, since all the maps in $\mathcal{T}(\mathcal{X}) \boxtimes \mathcal{T}(\mathcal{Y})$ are inclusions, and the colimit of such a diagram is essentially the union of the spaces, then there is a natural homeomorphism

$$\mathsf{colim}\left(\mathcal{T}(\mathcal{X}) \boxtimes \mathcal{T}(\mathcal{Y})\Big|_{\triangle_k}\right) \cong \bigcup_{i+j \leq k} \mathcal{T}_i(\mathcal{X}) \times \mathcal{T}_j(\mathcal{Y}) = \big(\mathcal{T}(\mathcal{X}) \otimes \mathcal{T}(\mathcal{Y})\big)_k$$

which completes the proof. $\square$

It follows that,

Corollary 3.1.16. *If $\mathcal{X}, \mathcal{Y} \in \mathbf{Top}^{\mathbf{N}}$, then $PH_n(\mathcal{X} \otimes_{\mathbf{g}} \mathcal{Y}; \mathbb{F}) \cong PH_n\big(\mathcal{T}(\mathcal{X}) \otimes \mathcal{T}(\mathcal{Y}); \mathbb{F}\big)$ as graded $\mathbb{F}[t]$-modules.*

Later on—when studying the persistent singular chains of $\mathcal{T}(\mathcal{X})\otimes\mathcal{T}(\mathcal{Y})$—it will be useful to have Proposition 3.1.17 below. The setup is as follows: observe that if $\mathcal{X} \in \mathbf{Top}^{\mathbf{N}}$ and $0 < \epsilon < 1$, then

$$\mathcal{T}_j^{\epsilon}(\mathcal{X}) := X_j \times [j, j+\epsilon) \sqcup \left(\bigsqcup_{i<j} X_i \times [i, i+1] \right) \Big/ (x, i+1) \sim (f_i(x), i+1)$$

is an open neighborhood of $\mathcal{T}_j(\mathcal{X})$ in $\mathcal{T}_{j+1}(\mathcal{X})$, which deformation retracts onto $\mathcal{T}_j(\mathcal{X})$. Indeed, any element $\mathbf{x} \in \mathcal{T}_j^{\epsilon}(\mathcal{X})$ can be written uniquely as $\mathbf{x} = (x, t)$ where either $x \in X_j$ and $t \in [j, j+\epsilon)$, or $x \in X_i$ for $i < j$ and $t \in [i, i+1)$. In these coordinates $\mathcal{T}_j(\mathcal{X}) = \{(x,t) \in \mathcal{T}_j^{\epsilon}(\mathcal{X}) : t \leq j\}$, and a deformation retraction $h_j^X : \mathcal{T}_j^{\epsilon}(\mathcal{X}) \times [0,1] \to \mathcal{T}_j^{\epsilon}(\mathcal{X})$ can be defined as

$$h_j^X\big((x,t),\lambda\big) = \begin{cases} (x,t) & \text{if} \quad t \leq j \\ \big(x, \lambda j + (1-\lambda)t\big) & \text{if} \quad t \geq j \end{cases} \tag{3.5}$$

Let $\mathcal{T}^{\epsilon}(\mathcal{X}) \in \mathbf{FTop}$ be $i \mapsto \mathcal{T}_i^{\epsilon}(\mathcal{X})$ and $(i \leq j) \mapsto \left(\mathcal{T}_i^{\epsilon}(\mathcal{X}) \hookrightarrow \mathcal{T}_j^{\epsilon}(\mathcal{X})\right)$. Then,

Proposition 3.1.17. *If $\mathcal{X}, \mathcal{Y} \in \mathbf{Top}^{\mathbf{N}}$ and $k \in \mathbb{N}_0$, then $\mathcal{T}_k^{\epsilon}(\mathcal{X})$ deformation retracts onto $\mathcal{T}_k(\mathcal{X})$, and $\left(\mathcal{T}^{\epsilon}(\mathcal{X}) \otimes \mathcal{T}^{\epsilon}(\mathcal{Y})\right)_k$ deformation retracts onto $\left(\mathcal{T}(\mathcal{X}) \otimes \mathcal{T}(\mathcal{Y})\right)_k$.*

Proof. Given $k \in \mathbb{N}_0$, our goal is to define a deformation retraction

$$H_k : \bigcup_{i+j=k} \mathcal{T}_i^{\epsilon}(\mathcal{X}) \times \mathcal{T}_j^{\epsilon}(\mathcal{Y}) \times [0,1] \longrightarrow \bigcup_{i+j=k} \mathcal{T}_i^{\epsilon}(\mathcal{X}) \times \mathcal{T}_j^{\epsilon}(\mathcal{Y}).$$

To this end, we subdivide $(\mathcal{T}^{\epsilon}(\mathcal{X}) \times \mathcal{T}^{\epsilon}(\mathcal{Y}))_k$ as follows: For $i + j = k$, let

$$\begin{aligned} A_k &= (\mathcal{T}(\mathcal{X}) \times \mathcal{T}(\mathcal{Y}))_k \\ B_{i,j} &= X_i \times [i, i+\epsilon) \times Y_j \times [j, j+\epsilon) \\ C_{i,j-1} &= X_i \times (i, i+\epsilon) \times Y_{j-1} \times [j-1+\epsilon, j) \\ D_{i,j-1} &= X_i \times [i+\epsilon, i+1) \times Y_{j-1} \times (j-1, j-1+\epsilon) \\ E_{i,j-1} &= X_i \times (i, i+\epsilon) \times Y_{j-1} \times (j-1, j-1+\epsilon) \end{aligned}$$

We further write $E_{i,j-1} = E^+_{i,j-1} \cup E^-_{i,j-1}$ where $\big((x,t),(y,s)\big) \in E^+_{i,j-1}$ (sim. $E^-_{i,j-1}$) if and only if $t - i \geq s + 1 - j$ (sim. $t - i \leq s + 1 - j$). Figure 3.1 below visually represents the subsets defined above for $k = 3$.

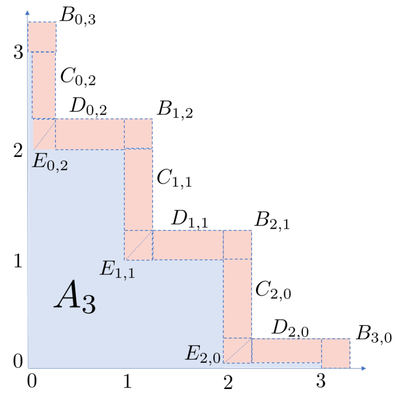


Figure 3.1: Visual representation for the deformation retract H_3.

Let us now define the map H_k. Let $H_k : A_k \times [0,1] \longrightarrow A_k$ be the projection onto the first coordinate. On each $B_{i,j} \times [0,1]$ we define

$$H_k\big((\mathbf{x},\mathbf{y}),\lambda\big) = \big(h^X_i(\mathbf{x},\lambda), h^Y_j(\mathbf{y},\lambda)\big)$$

where h^X_i, h^Y_j are given by (3.5). For elements in $\big(C_{i,j-1} \cup E^+_{i,j-1}\big) \times [0,1]$, H_k takes $\big(\mathbf{x},\mathbf{y},\lambda\big) = \big((x,t),(y,s),\lambda\big)$ to

$$\left(h^X_i(\mathbf{x},\lambda), \left(y, (1-\lambda)s + \lambda\left(j - \frac{j-s}{1+i-t}\right)\right)\right)$$

And finally, for elements in $\big(D_{i,j-1} \cup E^-_{i,j-1}\big) \times [0,1]$, H_k takes $(\mathbf{x},\mathbf{y},\lambda)$ to

$$\left(\left(x, (1-\lambda)t + \lambda\left(i - \frac{1+i-t}{j-s}\right)\right), h^Y_{j-1}(\mathbf{y},\lambda)\right)$$

Continuity readily follows by construction. Furthermore, the marginal function $H_k(-,0)$ is the identity everywhere and $H_k(-,1) \in A_k$. This finishes the proof. □

3.1.1 A comparison theorem

Next we show that the persistent homology of the categorical and generalized tensor products are interleaved in the log scale. While a full characterization of stability for each product filtration is beyond the scope of this dissertation, this section showcases some of the techniques one can employ when addressing this question.

Definition 3.1.18. For $\delta \in \mathbb{R}$, the δ-*shift functor* $T_\delta : \mathbf{C}^{\mathbf{R}} \to \mathbf{C}^{\mathbf{R}}$ is defined as follows: For $\mathcal{M} \in \mathbf{C}^{\mathbf{R}}$, let $T_\delta(\mathcal{M}) \in \mathbf{C}^{\mathbf{R}}$ be the functor sending $r \in \mathbb{R}$ to $\mathcal{M}(r+\delta)$, and $r \le r' \in \mathbb{R}$ to $\mathcal{M}(r+\delta \le r'+\delta)$. For a morphism $\varphi : \mathcal{M} \to \mathcal{N}$, we get a morphism $T_\delta(\varphi) : T_\delta(\mathcal{M}) \to T_\delta(\mathcal{N})$, and if $\delta \ge 0$, then there is a morphism $T_\delta^{\mathcal{M}} : \mathcal{M} \to T_\delta(\mathcal{M})$ satisfying

$$\left(T_\delta^{\mathcal{M}}\right)_r = \mathcal{M}\left(r \le r+\delta\right)$$

for all $r \in \mathbb{R}$. We call $T_\delta^{\mathcal{M}}$ the δ-*transition morphism.*

The notion of interleavings, first introduced in [18], allows one to compare objects in $\mathbf{Mod}_{\mathbb{F}}^{\mathbf{R}}$. Specifically:

Definition 3.1.19. Let $\mathcal{M}, \mathcal{N} \in \mathbf{C}^{\mathbf{R}}$ and $\delta \ge 0$. A δ-*interleaving* between $\mathcal{M}$ and $\mathcal{N}$ consists of morphisms $\varphi : \mathcal{M} \to T_\delta(\mathcal{N})$ and $\psi : \mathcal{N} \to T_\delta(\mathcal{M})$ so that $T_\delta(\varphi) \circ \psi = T_{2\delta}^{\mathcal{N}}$ and $T_\delta(\psi) \circ \varphi = T_{2\delta}^{\mathcal{M}}$. We define the *interleaving distance* d_I between $\mathcal{M}$ and $\mathcal{N}$ as:

$$d_I\left(\mathcal{M}, \mathcal{N}\right) := \inf\Big\{\delta \mid \mathcal{M} \text{ and } \mathcal{N} \text{ are } \delta\text{-interleaved } \Big\}.$$

If there are no interleavings between $\mathcal{M}$ and $\mathcal{N}$, we say that $d_I(\mathcal{M}, \mathcal{N}) = \infty$.

Let $\mathbf{R}_*$ denote the poset category associated to the set of positive reals, $\mathbb{R}_*$, and define the logarithm functor $\mathbf{ln} : \mathbf{Top}^{\mathbf{R}_*} \to \mathbf{Top}^{\mathbf{R}}$ as

$$\mathbf{ln}(\mathcal{X})(r) = \mathcal{X}(e^r)$$
$$\mathbf{ln}(\mathcal{X})(r \le r') = \mathcal{X}(e^r \le e^{r'}).$$

Then,

Theorem 3.1.20. *If* $\mathcal{X}, \mathcal{Y} \in \mathbf{Top}^{\mathbf{R}_*}$, *then*

$$d_I\Big(H_n\Big(\mathbf{ln}(\mathcal{X} \times \mathcal{Y}); \mathbb{F}\Big)\,,\, H_n\Big(\mathbf{ln}(\mathcal{X} \otimes_{\mathbf{g}} \mathcal{Y}); \mathbb{F}\Big)\Big) \;\le\; \ln(2).$$

Proof. For each $r \in \mathbb{R}_*$, let $\square_r = \{(i,j) \in \mathbf{R}^2_* : \max\{i,j\} \le r\}$. We have inclusions $\triangle_r \hookrightarrow \square_r \hookrightarrow \square_{2r}$ and $\square_r \hookrightarrow \triangle_{2r}$, and thus by Proposition 2.2.8, the triangles in the diagram

$$\begin{array}{ccc} \mathsf{hocolim}(\mathcal{X} \boxtimes \mathcal{Y}|_{\triangle_{2r}}) & \longrightarrow & \mathsf{hocolim}(\mathcal{X} \boxtimes \mathcal{Y}|_{\triangle_{8r}}) \\ \uparrow & \searrow & \uparrow \\ \mathsf{hocolim}(\mathcal{X} \boxtimes \mathcal{Y}|_{\square_r}) & \longrightarrow & \mathsf{hocolim}(\mathcal{X} \boxtimes \mathcal{Y}|_{\square_{4r}}) \end{array} \tag{3.6}$$

commute. Since $\square_r$ has a terminal object, namely (r,r), then the natural map

$$\mathsf{hocolim}(\mathcal{X} \boxtimes \mathcal{Y}\big|_{\square_r}) \longrightarrow X_r \times Y_r$$

is a weak homotopy equivalence (Theorem 2.2.10). Such maps induce isomorphisms at the level of homology [39, Theorem 4.21], and thus applying $\mathbf{ln}$ followed by $H_n(\,\cdot\,;\mathbb{F})$ turns (3.6) into a ln(2)-interleaving of the desired objects in $\mathbf{Mod}_{\mathbb{F}}^{\mathbf{R}}$. $\square$

If $\mathcal{M}, \mathcal{N} \in \mathbf{Mod}_{\mathbb{F}}^{\mathbf{R}}$ are pointwise finite, then—in addition to their interleaving distance $d_I(\mathcal{M}, \mathcal{N})$—one can compute the *bottleneck distance* d_B between $\mathsf{bcd}(\mathcal{M})$ and $\mathsf{bcd}(\mathcal{N})$ as follows. A matching between two multisets M and N is a multiset bijection $\varphi : S_M \longrightarrow S_N$ between some $S_M \subset M$ and $S_N \subset N$. Given φ, we say that $I \in S_M$ and $\varphi(I) \in S_N$ are matched, and all other elements of $(M \smallsetminus S_M) \cup (N \smallsetminus S_N)$ are called unmatched. Let $\delta > 0$. A matching between $\mathsf{bcd}(\mathcal{M})$ and $\mathsf{bcd}(\mathcal{N})$ is called a δ-matching if the following conditions hold:

1. If I and J are matched, then

$$\max\{|\ell_I - \ell_J|, |\rho_I - \rho_J|\} \le \delta.$$

   Recall that ℓ_I and ρ_I are, respectively, the left and right endpoints of the interval I.

2. If I is unmatched, then $|\ell_I - \rho_I| \le 2\delta$.

The bottleneck distance d_B between $\mathsf{bcd}(\mathcal{M})$ and $\mathsf{bcd}(\mathcal{N})$ is defined as:

$$d_B\Big(\mathsf{bcd}(\mathcal{M}), \mathsf{bcd}(\mathcal{N})\Big) := \inf\Big\{\delta \mid \mathsf{bcd}(\mathcal{M}) \text{ and } \mathsf{bcd}(\mathcal{N}) \text{ are } \delta\text{-matched }\Big\}$$

If no δ-matchings exist, then the bottleneck distance is ∞.

The *Isometry Theorem* [7] implies that if $\mathcal{M}, \mathcal{N} \in \mathbf{Mod}_{\mathbb{F}}^{\mathbf{R}}$ are pointwise finite, then

$$d_I(\mathcal{M}, \mathcal{N}) = d_B\left(\mathsf{bcd}(\mathcal{M}), \mathsf{bcd}(\mathcal{N})\right).$$

The following corollary is a direct consequence of the above equality and Theorem 3.1.20.

Corollary 3.1.21. *Let $\mathcal{X}, \mathcal{Y} \in \mathbf{Top}^{\mathbf{R}_*}$ be so that $\mathcal{X} \times \mathcal{Y}$ and $\mathcal{X} \otimes_{\mathbf{g}} \mathcal{Y}$ are pointwise finite. Then*

1. *If under a matching realizing $d_B\Big(\mathsf{bcd}_n(\mathcal{X} \times \mathcal{Y}), \mathsf{bcd}_n(\mathcal{X} \otimes_{\mathbf{g}} \mathcal{Y})\Big)$—which exists by [19, Theorem 5.12]—we have that I is matched to J, then*

$$\frac{1}{2} \leq \frac{\ell_I}{\ell_J}, \frac{\rho_I}{\rho_J} \leq 2.$$

2. *If I is unmatched, then $\frac{\rho_I}{\ell_I} \leq 4$.*

3.2 A Persistent Künneth formula for the categorical product

Let $\mathbf{I}$ be a small indexing category and let $\mathbf{S}$ be any of the categories of spaces from Section 3.1 (that is $\mathbf{S} = \mathbf{Top}$, $\mathbf{Met}$, $\mathbf{Simp}$, or $\mathbf{oSimp}$). In this section, we relate the rank invariants of two objects in $\mathbf{S}^{\mathbf{I}}$ to that of their categorical product, and prove the persistent Künneth formula for the categorical product in $\mathbf{S}^{\mathbf{P}}$, where $\mathbf{P}$ is the poset category of a separable totally ordered poset $\mathcal{P}$.

Remark 3.2.1. Recall that the topological Künneth formula (Corollary 2.2.5) holds for objects $X, Y \in \mathbf{Top}$ and singular homology. Every $(X, \mathbf{d}_X) \in \mathbf{Met}$ can be viewed as a topological space via the metric topology, and since the maximum metric induces the product topology, then the Künneth formula holds for $\mathbf{Met}$. Let $K, L \in \mathbf{oSimp}$, and let $|\cdot| : \mathbf{oSimp} \to \mathbf{Top}$ be the geometric realization functor. From the homeomorphism $|K \oslash L| \to |K| \times |L|$ and the natural isomorphism $H_n(|K|; R) \cong H_n(K; R)$ between singular and simplicial homology, the formula holds in $\mathbf{oSimp}$. Using (2) of lemma 3.1.11, the formula also holds for $\mathbf{Simp}$.

Let $\mathbb{F}$ be a field and let $\mathcal{X}$ and $\mathcal{Y}$ be objects in $\mathbf{S}^{\mathbf{I}}$. Using the topological Künneth formula and the observation that $H_n(X_r; \mathbb{F})$ and $H_n(Y_r; \mathbb{F})$ are vector spaces for each $n \in \mathbb{N}_0$, $r \in \mathbf{I}$, we obtain isomorphisms

$$\bigoplus_{i+j=n} H_i(X_r; \mathbb{F}) \otimes_{\mathbb{F}} H_j(Y_r; \mathbb{F}) \xrightarrow{\cong} H_n(X_r \times Y_r; \mathbb{F}).$$

Using naturality, this yields an isomorphism in $\mathbf{Mod}_{\mathbb{F}}^{\mathbf{I}}$ between $H_n(\mathcal{X} \times \mathcal{Y}; \mathbb{F})$ and $\mathcal{M}_n$, where

$$\mathcal{M}_n(r) := \bigoplus_{i+j=n} H_i(X_r; \mathbb{F}) \otimes_{\mathbb{F}} H_j(Y_r; \mathbb{F}) \tag{3.7}$$

and the homomorphism $\mathcal{M}_n(r \to r')$ is the sum of the ones induced between tensor products by the maps $X_r \to X_{r'}$ and $Y_r \to Y_{r'}$. In other words, and using the notation of tensor product and direct sum of objects in $\mathbf{Mod}_{\mathbb{F}}^{\mathbf{I}}$ (see Definition 2.3.7), we obtain the following:

Lemma 3.2.2. *If $\mathcal{X}, \mathcal{Y} \in \mathbf{S}^{\mathbf{I}}$, then for every $n \in \mathbb{N}_0$ and every field $\mathbb{F}$*

$$H_n(\mathcal{X} \times \mathcal{Y}; \mathbb{F}) \cong \bigoplus_{i+j=n} H_i(\mathcal{X}; \mathbb{F}) \otimes H_j(\mathcal{Y}; \mathbb{F})$$

in $\mathbf{Mod}_{\mathbb{F}}^{\mathbf{I}}$.

An immediate consequence is the following calculation at the level of rank invariants (see Definition 2.3.22):

Proposition 3.2.3. *Let $\mathcal{X}, \mathcal{Y} \in \mathbf{S}^{\mathbf{I}}$ and let $\rho^{\mathcal{X}}$, $\rho^{\mathcal{Y}}$ be their rank invariants. Then*

$$\rho_n^{\mathcal{X} \times \mathcal{Y}}(r \to r') = \sum_{i+j=n} \rho_i^{\mathcal{X}}(r \to r') \cdot \rho_j^{\mathcal{Y}}(r \to r')$$

for every $r \to r'$ in $\mathbf{I}$, with the convention that $\infty \cdot 0 = 0 = 0 \cdot \infty$.

Proof. This follows by direct inspection of the homomorphism

$$\begin{aligned} f_{r',r} \otimes_{\mathbb{F}} g_{r',r} : H_i(X_r; \mathbb{F}) \otimes_{\mathbb{F}} H_j(Y_r; \mathbb{F}) &\longrightarrow H_i(X_{r'}; \mathbb{F}) \otimes_{\mathbb{F}} H_j(Y_{r'}; \mathbb{F}) \\ a \otimes_{\mathbb{F}} b &\mapsto f_{r',r}(a) \otimes_{\mathbb{F}} g_{r',r}(b) \end{aligned}$$

for each $r \to r' \in \mathbf{I}$. Indeed, since $\mathsf{Img}(f_{r',r} \otimes_{\mathbb{F}} g_{r',r}) = \mathsf{Img}(f_{r',r}) \otimes_{\mathbb{F}} \mathsf{Img}(g_{r',r})$, then the result follows from the fact that the dimension of the tensor product of two vector spaces equals the product of the dimensions. □

We are now ready to prove the main result of this section: the Künneth formula for the categorical product $\mathcal{X} \times \mathcal{Y}$.

Theorem 3.2.4. *Let* $\mathbf{P}$ *be the poset category of a separable (with respect to the order topology) totally ordered set. Let* $\mathcal{X}, \mathcal{Y} \in \mathbf{S}^{\mathbf{P}}$ *be* $\mathbf{P}$*-indexed diagrams of spaces, and assume that* $H_i(\mathcal{X};\mathbb{F})$ *and* $H_j(\mathcal{Y};\mathbb{F})$ *are pointwise finite for each* $0 \le i, j \le n$. *Then* $H_n(\mathcal{X} \times \mathcal{Y};\mathbb{F})$ *is pointwise finite, and its barcode satisfies:*

$$\mathsf{bcd}_n\left(\mathcal{X} \times \mathcal{Y};\mathbb{F}\right) = \bigcup_{i+j=n} \left\{ I \cap J \;\middle|\; I \in \mathsf{bcd}_i(\mathcal{X};\mathbb{F}),\; J \in \mathsf{bcd}_j(\mathcal{Y};\mathbb{F}) \right\}$$

where the union on the right is of multisets (i.e., repetitions may occur).

Proof. The fact that $H_n(\mathcal{X} \times \mathcal{Y};\mathbb{F})$ is pointwise finite follows directly from Lemma 3.2.2. As for the barcode formula, and removing the field $\mathbb{F}$ from the notation, we have the following sequence of isomorphisms

$$\begin{aligned}
H_n(\mathcal{X} \times \mathcal{Y}) &\cong \bigoplus_{i+j=n} H_i(\mathcal{X}) \otimes H_j(\mathcal{Y}) \\
&\cong \bigoplus_{i+j=n} \left(\bigoplus_{I \in \mathsf{bcd}_i(\mathcal{X})} \mathbb{1}_I \right) \otimes \left(\bigoplus_{J \in \mathsf{bcd}_j(\mathcal{Y})} \mathbb{1}_J \right) \\
&\cong \bigoplus_{i+j=n} \bigoplus_{\substack{I \in \mathsf{bcd}_i(\mathcal{X}) \\ J \in \mathsf{bcd}_j(\mathcal{Y})}} \mathbb{1}_I \otimes \mathbb{1}_J \\
&\cong \bigoplus_{i+j=n} \bigoplus_{\substack{I \in \mathsf{bcd}_i(\mathcal{X}) \\ J \in \mathsf{bcd}_j(\mathcal{Y})}} \mathbb{1}_{I \cap J}.
\end{aligned}$$

The direct sum distributes with respect to the tensor product in $\mathbf{Mod}_{\mathbb{F}}^{\mathbf{P}}$ since it does so in $\mathbf{Mod}_{\mathbb{F}}$. The last isomorphism is the one from Example 2.3.8, and the theorem follows from the uniqueness of the barcode. □

This result generalizes to the product of k objects in $\mathbf{S}^{\mathbf{P}}$ by inductively using $\mathsf{bcd}_j(\mathcal{X}_k)$ and $\mathsf{bcd}_i(\mathcal{X}_1 \times \cdots \times \mathcal{X}_{k-1})$ to compute $\mathsf{bcd}_{i+j}(\mathcal{X}_1 \times \cdots \times \mathcal{X}_k)$:

Corollary 3.2.5. *Let $\mathcal{X}_1, \ldots, \mathcal{X}_k \in \mathbf{S}^{\mathbf{P}}$. Assume that for each $1 \leq j \leq k$, $0 \leq n_j \leq n$, $H_{n_j}(\mathcal{X}_j)$ is pointwise finite. Then*

$$\mathsf{bcd}_n(\mathcal{X}_1 \times \cdots \times \mathcal{X}_k) = \left\{ I_1 \cap \cdots \cap I_k \;\middle|\; I_j \in \mathsf{bcd}_{n_j}(\mathcal{X}_j),\; \sum_{j=1}^{k} n_j = n \right\}.$$

The result which is perhaps most relevant to persistent homology computations with Rips complexes is as follows:

Corollary 3.2.6. *Let $(X, \mathbf{d}_X), (Y, \mathbf{d}_Y)$ be finite metric spaces and let $\mathsf{bcd}^{\mathcal{R}}_n(X, \mathbf{d}_X)$ be the barcode of the Rips filtration $\mathcal{R}(X, \mathbf{d}_X) := \{R_\epsilon(X, \mathbf{d}_X)\}_{\epsilon \geq 0}$. Then,*

$$\mathsf{bcd}^{\mathcal{R}}_n(X \times Y, \mathbf{d}_{X \times Y}) = \bigcup_{i+j=n} \left\{ I \cap J \;\middle|\; I \in \mathsf{bcd}^{\mathcal{R}}_i(X, \mathbf{d}_X),\; J \in \mathsf{bcd}^{\mathcal{R}}_j(Y, \mathbf{d}_Y) \right\}$$

for all $n \in \mathbb{N}_0$, if $\mathbf{d}_{X \times Y}$ is the maximum metric.

Proof. Fix total orders on X and Y. Then, their ϵ-Rips complexes are ordered simplicial complexes, and by lemmas 3.1.8 and 3.1.11 we have that:

$$|R_\epsilon(X \times Y)| = |R_\epsilon(X) \times R_\epsilon(Y)| \simeq |R_\epsilon(X) \oslash R_\epsilon(Y)| \cong |R_\epsilon(X)| \times |R_\epsilon(Y)|.$$

Since the equivalences commute with inclusions of Rips complexes, the result follows. □

3.3 A Persistent Künneth Formula for the generalized tensor product

In this section, we will establish the Künneth formula for the generalized tensor product $\mathcal{X} \otimes_{\mathbf{g}} \mathcal{Y}$ of two objects $\mathcal{X}, \mathcal{Y} \in \mathbf{Top}^{\mathbf{N}}$. We start with a brief discussion on the grading of the tensor product and Tor of two graded modules over a graded ring R. The discussion then specializes to $R = \mathbb{F}[t]$ in order to establish the $\mathbb{F}[t]$-isomorphisms

$$\begin{aligned} \left(P\mathbb{1}_{I=[\ell_I,\rho_I)}\right) \otimes_{\mathbb{F}[t]} \left(P\mathbb{1}_{J=[\ell_J,\rho_J)}\right) &\cong P\mathbb{1}_{(\ell_J+I)\cap(\ell_I+J)} \\ \mathsf{Tor}_{\mathbb{F}[t]}\left(P\mathbb{1}_{I=[\ell_I,\rho_I)}\,,\; P\mathbb{1}_{J=[\ell_J,\rho_J)}\right) &\cong P\mathbb{1}_{(\rho_J+I)\cap(\rho_I+J)} \end{aligned} \tag{3.8}$$

for intervals $I, J \subset \mathbb{N}_0$. We end with a persistent Eilenberg-Zilber theorem, from which the aforementioned persistent Künneth theorem will follow.

3.3.1 The tensor product and Tor of graded modules

Recall that a commutative ring R with unity is called *graded* if it can be written as $R = R_0 \oplus R_1 \oplus \cdots$ where each R_i is an additive subgroup of R and $R_i R_j \subset R_{i+j}$ for every $i, j \in \mathbb{N}_0$. Similarly, an R-module A is graded if it can be written as $A = A_0 \oplus A_1 \oplus \cdots$, where the A_i's are subgroups of A and $R_i A_j \subset A_{i+j}$ for all $i, j \in \mathbb{N}_0$. An element $a \in A$ (resp. in R) is called *homogeneous* of degree $i \in \mathbb{N}_0$ if $a \in A_i$ (resp. in R_i), and we will use the notation $\mathsf{deg}(a) = i$.

Let A, B be graded modules over a graded ring R. Our first goal is to describe a direct sum decomposition of the R-module $A \otimes_R B$ inducing the structure of a graded R-module. Indeed, since A_i and B_j are R_0-modules for all $i, j \in \mathbb{N}_0$, then so are A, B and we have the R_0-isomorphism

$$A \otimes_{R_0} B \cong \bigoplus_{k \in \mathbb{N}_0} \left(\bigoplus_{i+j=k} A_i \otimes_{R_0} B_j \right).$$

Fix $k \in \mathbb{N}_0$ and let J_k be the R_0-submodule of $A \otimes_{R_0} B$ generated by elements of the form $(ra) \otimes b - a \otimes (rb)$, where $a \in A$, $b \in B$ and $r \in R$ are homogeneous elements with $\mathsf{deg}(a) + \mathsf{deg}(r) + \mathsf{deg}(b) = k$. It follows that J_k is a submodule of

$$\left(A \otimes_{R_0} B\right)_k := \bigoplus_{i+j=k} A_i \otimes_{R_0} B_j$$

and thus $J_k \cap J_\ell = \{0\}$ if $k \neq \ell$. Let $J = J_0 \oplus J_1 \oplus \cdots$. Any homogeneous element $r \in R$ of degree ℓ induces a well-defined R_0-homomorphism

$$\begin{array}{ccc} (A \otimes_{R_0} B)_k / J_k & \longrightarrow & (A \otimes_{R_0} B)_{k+\ell} / J_{k+\ell} \\ a \otimes b \, + \, J_k & \mapsto & (ra) \otimes b \, + \, J_{k+\ell} \, = \, a \otimes (rb) \, + \, J_{k+\ell} \end{array}$$

and therefore

$$A \otimes_{R_0} B / J \cong \bigoplus_{k \in \mathbb{N}_0} (A \otimes_{R_0} B)_k / J_k \tag{3.9}$$

inherits the structure of a graded R-module.

The inclusion $R_0 \hookrightarrow R$ induces an R_0-epimorphism $\iota : A \otimes_{R_0} B \longrightarrow A \otimes_R B$ with $\mathsf{ker}(\iota) = J$. Indeed, the elements of J are exactly the relations missing between $A \otimes_{R_0} B$ and $A \otimes_R B$ when defining the latter as a quotient module. It follows that ι induces an isomorphism

$$\iota_* : A \otimes_{R_0} B/J \longrightarrow A \otimes_R B$$

of R-modules, and the grading on the codomain induced by (3.9) turns ι_* into a graded isomorphism of graded R-modules. We summarize this analysis as follows:

Proposition 3.3.1. *Let A, B be graded modules over a graded ring R. Then $A \otimes_R B$ decomposes as the direct sum of the subgroups*

$$(A \otimes_R B)_k = \left\{ \sum_\alpha a_\alpha \otimes b_\alpha \;\middle|\; a_\alpha \in A, b_\alpha \in B \textit{ homogeneous, } \mathsf{deg}(a_\alpha) + \mathsf{deg}(b_\alpha) = k \right\}$$

and $R_\ell \cdot (A \otimes_R B)_k \subset (A \otimes_R B)_{k+\ell}$ for all $k, \ell \in \mathbb{N}_0$. In other words, $A \otimes_R B$ is a graded R-module.

And now we prove the corresponding proposition for $\mathsf{Tor}_R(A, B)$:

Proposition 3.3.2. *If A, B are graded R-modules, then so is $\mathsf{Tor}_R(A, B)$.*

Proof. Fix a graded free resolution

$$\cdots \to \mathcal{F}_2(A) \to \mathcal{F}_1(A) \to \mathcal{F}_0(A) \to \mathcal{F}_{-1}(A) = A \to 0$$

defined inductively as follows. Let $\mathcal{F}_0(A)$ be the free R-module generated by the homogeneous elements of A, and define for each $i \in \mathbb{N}_0$ the additive subgroup

$$\mathcal{F}_{0,i}(A) = \left\{ \sum_\alpha r_\alpha \cdot a_\alpha \;\middle|\; r_\alpha \in R, a_\alpha \in A \text{ homogeneous with } \mathsf{deg}(r_\alpha) + \mathsf{deg}(a_\alpha) = i \right\}.$$

Then $\mathcal{F}_0(A) = \mathcal{F}_{0,0}(A) \oplus \mathcal{F}_{0,1}(A) \oplus \cdots$, and with this decomposition, the natural map $\mathcal{F}_0(A) \to \mathcal{F}_{-1}(A)$ is a surjective graded homomorphism of graded R-modules. In particular, the kernel of this homomorphism is a graded R-submodule of $\mathcal{F}_0(A)$. We proceed inductively by letting $\mathcal{F}_{j+1}(A)$ be the graded free R-module generated by the homogeneous elements in the kernel of $\mathcal{F}_j(A) \to \mathcal{F}_{j-1}(A)$.

Taking the tensor product with B over R yields

$$\cdots \to \mathcal{F}_2(A) \otimes_R B \to \mathcal{F}_1(A) \otimes_R B \to \mathcal{F}_0(A) \otimes_R B \to A \otimes_R B \to 0$$

and we note that:

1. Each $\mathcal{F}_j(A) \otimes_R B$ is a graded R-module, by Proposition 3.3.1, and each $\mathcal{F}_j(A) \otimes_R B \to \mathcal{F}_{j-1}(A) \otimes_R B$ is a graded R-homomorphism.

2. $\mathsf{ker}\Big(\mathcal{F}_j(A) \otimes_R B \to \mathcal{F}_{j-1}(A) \otimes_R B\Big)$ decomposes as the direct sum

$$\bigoplus_{k \in \mathbb{N}_0} \mathsf{ker}\Big(\big(\mathcal{F}_j(A) \otimes_R B\big)_k \to \big(\mathcal{F}_{j-1}(A) \otimes_R B\big)_k\Big)$$

and similarly, $\mathsf{Img}\Big(\mathcal{F}_{j+1}(A) \otimes_R B \to \mathcal{F}_j(A) \otimes_R B\Big)$ decomposes as

$$\bigoplus_{k \in \mathbb{N}_0} \mathsf{Img}\Big(\big(\mathcal{F}_{j+1}(A) \otimes_R B\big)_k \to \big(\mathcal{F}_j(A) \otimes_R B\big)_k\Big)$$

with the image being a graded R-submodule of the kernel.

3. $\mathsf{Tor}_R(A, B)$ inherits the structure of a graded R-module from the decomposition

$$\mathsf{Tor}_R(A, B) \cong \bigoplus_{k \in \mathbb{N}_0} \frac{\mathsf{ker}\Big(\big(\mathcal{F}_1(A) \otimes_R B\big)_k \to \big(\mathcal{F}_0(A) \otimes_R B\big)_k\Big)}{\mathsf{Img}\Big(\big(\mathcal{F}_2(A) \otimes_R B\big)_k \to \big(\mathcal{F}_1(A) \otimes_R B\big)_k\Big)}$$

finishing the proof. □

We now specialize to the case $R = \mathbb{F}[t]$ and present explicit computations for interval modules. For $m \in \mathbb{N}_0, k \in \mathbb{N}_0 \cup \{\infty\}$, let $\mathcal{I}_{m,k}$ denote the graded $\mathbb{F}[t]$-module $P\mathbb{1}_{[m,m+k)}$. For brevity, we will drop the subscript k if it is ∞. That is, we will use $\mathcal{I}_m$ to denote $P\mathbb{1}_{[m,\infty)}$. Then, $\mathcal{I}_{m,k} \cong t^m\mathbb{F}[t]\big/(t^{m+k})$ and $\mathcal{I}_m \cong t^m\mathbb{F}[t]$.

Proposition 3.3.3. *Let $m, n \in \mathbb{N}_0$, $k, l \in \mathbb{N}_0 \cup \{\infty\}$. Then $\mathcal{I}_{m,k} \otimes_{\mathbb{F}[t]} \mathcal{I}_{n,l} \cong \mathcal{I}_{m+n,\min\{k,l\}}$ and $\mathsf{Tor}_{\mathbb{F}[t]}(\mathcal{I}_{m,k}, \mathcal{I}_{n,l}) \cong \mathcal{I}_{m+n+\max\{k,l\},\min\{k,l\}}$.*

Proof. There is a graded $\mathbb{F}[t]$-isomorphism $\mathcal{I}_m \otimes_{\mathbb{F}[t]} \mathcal{I}_n \to \mathcal{I}_{m+n}$, which takes the generator $t^m \otimes_{\mathbb{F}[t]} t^n$ to t^{m+n}. Since $\mathcal{I}_{m,k}$ and $\mathcal{I}_{n,l}$ are quotient modules of $\mathcal{I}_m$ and $\mathcal{I}_n$ respectively, their tensor product is isomorphic to a quotient module of $\mathcal{I}_{m+n}$, say $\mathcal{I}_{m+n,q}$ for some appropriate $1 \leq q \leq \infty$. The isomorphism takes $t^m \otimes_{\mathbb{F}[t]} t^n$ to t^{m+n} and the action of t^p gives $t^p \cdot \left(t^m \otimes_{\mathbb{F}[t]} t^n\right) \to t^{m+n+p}$. The left hand side is zero if either t^{p+m} is zero in $\mathcal{I}_{m,k}$ or if t^{p+n} is zero in $\mathcal{I}_{n,l}$. This implies that if $p \geq \min\{k,l\}$, then t^{m+n+p} is zero in $\mathcal{I}_{m+n,q}$. The minimum of all such p is $\min\{k,l\}$ and therefore, $q = \min\{k,l\}$.

To compute $\mathsf{Tor}_{\mathbb{F}[t]}\left(\mathcal{I}_{m,k}, \mathcal{I}_{n,l}\right)$, we fix a graded free resolution for $\mathcal{I}_{m,k}$:

$$0 \to \mathcal{I}_{m+k} \to \mathcal{I}_m \to \mathcal{I}_{m,k} \to 0.$$

Taking the tensor product of the sequence with $\mathcal{I}_{n,l}$ gives us:

$$\mathcal{I}_{m+k} \otimes_{\mathbb{F}[t]} \mathcal{I}_{n,l} \to \mathcal{I}_m \otimes_{\mathbb{F}[t]} \mathcal{I}_{n,l} \to \mathcal{I}_{m,k} \otimes_{\mathbb{F}[t]} \mathcal{I}_{n,l} \to 0.$$

Using the definition of Tor and the tensor product of intervals, we get that

$$\begin{aligned}\mathsf{Tor}_{\mathbb{F}[t]}(\mathcal{I}_{m,k}, \mathcal{I}_{n,l}) &\cong \mathsf{Ker}\Big(\mathcal{I}_{m+k} \otimes_{\mathbb{F}[t]} \mathcal{I}_{n,l} \to \mathcal{I}_m \otimes_{\mathbb{F}[t]} \mathcal{I}_{n,l}\Big) \\ &\cong \mathsf{Ker}\Big(\mathcal{I}_{m+k+n,l} \to \mathcal{I}_{m+n,l}\Big) \\ &\cong \mathcal{I}_{m+n+\max\{k,l\},\min\{k,l\}}\end{aligned}$$

where the last isomorphism is justified by the following argument: the element $t^{m+p+n} \in \mathcal{I}_{m+k+n,l}$ goes to $t^p t^{m+n} \in \mathcal{I}_{m+n,l}$ where $p \geq k$. The latter is zero if and only if $p \geq l$, or said equivalently $p \geq \max\{k,l\}$. This finishes the proof. □

Remark 3.3.4. This last proposition recovers the formulas shown in (3.8). Moreover, using bilinearity, these isomorphisms can be used to compute the tensor product and Tor of any pair of finitely generated graded $\mathbb{F}[t]$-modules.

3.3.2 The Graded Algebraic Künneth Formula

The Algebraic Künneth Formula for graded modules over a graded PID R is proved in exactly the same way as Theorem 2.2.3, by noticing that all the steps can be carried out in a degree-preserving fashion. A few versions of this have appeared recently, see for instance [62, Proposition 2.9] or [12, Theorem 10.1], and we include it next for completeness:

Theorem 3.3.5. *Let C and C' be two chain complexes of graded modules over a graded PID R, and assume that one of C, C' is flat. Then, for each $n \in \mathbb{N}_0$, there is a natural short exact sequence*

$$0 \longrightarrow \bigoplus_{i+j=n} H_i(C) \otimes_R H_j(C') \xrightarrow{\ \mu\ } H_n(C \otimes_R C') \xrightarrow{\ \nu\ }$$
$$\bigoplus_{i+j=n} \mathsf{Tor}_R(H_i(C), H_{j-1}(C')) \longrightarrow 0$$

which splits, though not naturally.

It is at this point that we depart from the existing literature on persistent Künneth formulas for the tensor product of filtered chain complexes. Indeed, next we will turn these algebraic results into theorems at the level of filtered topological spaces. We begin with the following result:

Theorem 3.3.6. *Let $\mathcal{X}, \mathcal{Y} \in \mathbf{Top}^{\mathbf{N}}$. Then, for all $n \in \mathbb{N}_0$, we have a natural short exact sequence*

$$0 \to \bigoplus_{i+j=n} PH_i(\mathcal{X}) \otimes_{\mathbb{F}[t]} PH_j(\mathcal{Y}) \to H_n\left(PS_*(\mathcal{T}(\mathcal{X})) \otimes_{\mathbb{F}[t]} PS_*(\mathcal{T}(\mathcal{Y}))\right) \to$$
$$\bigoplus_{i+j=n} \mathsf{Tor}_{\mathbb{F}[t]}\left(PH_i(\mathcal{X}), PH_{j-1}(\mathcal{Y})\right) \to 0$$

which splits, but not naturally.

Proof. Since $\mathcal{T}(\mathcal{X})$ and $\mathcal{T}(\mathcal{Y})$ are filtered spaces, then $PS_*(\mathcal{T}(\mathcal{X}))$ and $PS_*(\mathcal{T}(\mathcal{Y}))$ are chain complexes of free graded $\mathbb{F}[t]$-modules, and hence flat. Now, each inclusion $\iota_i : X_i \hookrightarrow \mathcal{T}_i(\mathcal{X})$ is a homotopy equivalence, and thus induces an isomorphism at the level of homology. One thing to note is that if $i < j$, then

$$\begin{array}{ccc} \mathcal{T}_i(\mathcal{X}) & \hookrightarrow & \mathcal{T}_j(\mathcal{X}) \\ \iota_i \uparrow & & \uparrow \iota_j \\ X_i & \xrightarrow[\mathcal{X}(i<j)]{} & X_j \end{array}$$

commutes only up to homotopy, but this is enough to conclude that $\iota = \{\iota_i\}_{i\in\mathbb{N}_0}$ induces natural isomorphisms $\iota_* : H_n(\mathcal{X};\mathbb{F}) \longrightarrow H_n(\mathcal{T}(\mathcal{X});\mathbb{F})$, $n \in \mathbb{N}_0$. The result follows from plugging $C = PS_*(\mathcal{T}(\mathcal{X}))$ and $C' = PS_*(\mathcal{T}(\mathcal{Y}))$ into Theorem 3.3.5, and using the natural $\mathbb{F}[t]$-isomorphisms $P\iota_* : PH_i(\mathcal{X};\mathbb{F}) \longrightarrow PH_i(\mathcal{T}(\mathcal{X});\mathbb{F})$. □

3.3.3 A Persistent Eilenberg-Zilber theorem

Our next objective is to show that the chain complexes $PS_*(\mathcal{T}(\mathcal{X})) \otimes_{\mathbb{F}[t]} PS_*(\mathcal{T}(\mathcal{Y}))$ and $PS_*(\mathcal{X} \otimes_{\mathbf{g}} \mathcal{Y})$ have naturally isomorphic homology. The result follows, as we will show in Theorem 3.3.11, from applying the machinery of acyclic models [30] to the functors

$$\begin{aligned} E : \mathbf{FTop}_0 \times \mathbf{FTop}_0 &\longrightarrow \mathbf{gCh}_{\mathbb{F}[t]} \\ (\mathcal{X},\mathcal{Y}) &\mapsto PS_*(\mathcal{X}) \otimes_{\mathbb{F}[t]} PS_*(\mathcal{Y}) \\ F : \mathbf{FTop}_0 \times \mathbf{FTop}_0 &\longrightarrow \mathbf{gCh}_{\mathbb{F}[t]} \\ (\mathcal{X},\mathcal{Y}) &\mapsto \overline{PS}_*(\mathcal{X} \otimes \mathcal{Y}) \end{aligned}$$

Here $\mathbf{FTop}_0$ is the full subcategory of $\mathbf{FTop}$ comprised of those filtered spaces $\mathcal{X} = \{X_j\}_{j\in\mathbb{N}_0}$ where X_j is an open subset of X_{j+1} for all $j \in \mathbb{N}_0$, and $\overline{PS}_n(\mathcal{X}\otimes\mathcal{Y})$ is the graded $\mathbb{F}[t]$-module whose degree-k component is the sum of vector spaces

$$S_n(X_k \times Y_0;\mathbb{F}) \; + \; S_n(X_{k-1} \times Y_1;\mathbb{F}) \; + \cdots + \; S_n(X_1 \times Y_{k-1};\mathbb{F}) \; + \; S_n(X_0 \times Y_k;\mathbb{F}),$$

where each $S_n(X_{k-\ell}\times Y_\ell;\mathbb{F})$ is regarded as a linear subspace of $S_n\big((\mathcal{X}\otimes\mathcal{Y})_k;\mathbb{F}\big)$. The models in question are a family $\mathcal{M}$ of objects from $\mathbf{FTop}_0 \times \mathbf{FTop}_0$ satisfying certain properties (Lemmas 3.3.7 and 3.3.8) implying our Persistent Eilenberg-Zilber Theorem 3.3.10. We start by defining $\mathcal{M}$.

For $X \in \mathbf{Top}$, let $\Sigma_e X \in \mathbf{FTop}_0$ be the filtered space with the empty set $\emptyset$ at indices $0 \le i < e$, and X for $i \ge e$:

$$\Sigma_e X : \; \emptyset \subset \cdots \subset \emptyset \subset X \subset X \subset \cdots$$

Let $\Delta^p \subset \mathbb{R}^{p+1}$ be the standard geometric p-simplex, and let $\mathcal{M}$ be the set of objects $(\Sigma_e \Delta^p, \Sigma_f \Delta^q)$ in $\mathbf{FTop}_0 \times \mathbf{FTop}_0$ for $p, q, e, f \in \mathbb{N}_0$. We have the following:

Lemma 3.3.7. *Each $M \in \mathcal{M}$ is both E-acyclic and F-acyclic. That is, the homology groups $H_n(E(M))$ and $H_n(F(M))$ are trivial for all $n > 0$.*

Proof. If $M = \left(\Sigma_e \Delta^p, \Sigma_f \Delta^q\right)$, then

$$F(M) = \overline{PS}_* \left(\Sigma_e \Delta^p \otimes \Sigma_f \Delta^q\right) \cong PS_*(\Sigma_{e+f} \Delta^p \times \Delta^q)$$

as chain complexes of graded $\mathbb{F}[t]$-modules. Since $\Delta^p \times \Delta^q$ is convex, then

$$H_n(F(M)) \cong PH_n(\Sigma_{e+f} \Delta^p \times \Delta^q) = 0$$

for all $n > 0$, showing that M is F-acyclic.

To show that M is E-acyclic, notice that $PH_n(\Sigma_e \Delta^p) = 0$ for $n > 0$, and that $PH_0(\Sigma_e \Delta^p) \cong t^e \mathbb{F}[t]$ is a free graded $\mathbb{F}[t]$-module. By the graded algebraic Künneth Theorem 3.3.5, with $C = PS_*(\Sigma_e \Delta^p)$ and $C' = PS_*(\Sigma_f \Delta^q)$, we get that

$$H_n(E(M)) = H_n(PS_*(\Sigma_e \Delta^p) \otimes_{\mathbb{F}[t]} PS_*(\Sigma_f \Delta^q)) = 0$$

for all $n > 0$, since the tensor products and Tor modules in the short exact sequence are all zero. □

Lemma 3.3.8. *For each $n \in \mathbb{N}_0$, the functors F_n and E_n are free with base in $\mathcal{M}$.*

Proof. Let $n \in \mathbb{N}_0$. The functor F_n (resp. E_n) being free with base in $\mathcal{M}$ means that for all $(\mathcal{X}, \mathcal{Y}) \in \mathbf{FTop}_0 \times \mathbf{FTop}_0$, the graded $\mathbb{F}[t]$-module $F_n(\mathcal{X}, \mathcal{Y})$ (resp. $E_n(\mathcal{X}, \mathcal{Y})$) is freely generated over $\mathbb{F}[t]$ by some subset of

$$\bigcup_{\substack{M \in \mathcal{M} \\ u : M \to (\mathcal{X}, \mathcal{Y})}} \mathsf{Img}\Big(F_n(u) : F_n(M) \longrightarrow F_n(\mathcal{X}, \mathcal{Y})\Big)$$

where u ranges over all morphisms in $\mathbf{FTop}_0 \times \mathbf{FTop}_0$ from $M \in \mathcal{M}$ to $(\mathcal{X}, \mathcal{Y})$.

Starting with F, we will first construct a free $\mathbb{F}[t]$-basis for

$$F_n(\mathcal{X},\mathcal{Y}) = \overline{PS}_n(\mathcal{X}\otimes\mathcal{Y}) = \bigoplus_{k\in\mathbb{N}_0}\sum_{i+j=k} S_n(X_i\times Y_j)$$

consisting of singular n-simplices on the spaces $X_i \times Y_j$. We proceed inductively as follows: Let $\mathcal{S}_0$ be the collection of all singular n-simplices $\Delta^n \to X_0 \times Y_0$, and assume that for a fixed $k \geq 1$ and every $0 \leq \ell < k$, we have constructed a set $\mathcal{S}_\ell$ of simplices $\Delta^n \to X_p \times Y_q$, $p + q = \ell$, so that

$$\mathcal{A}_{k-1} \;=\; t^{k-1}\mathcal{S}_0 \,\cup\, t^{k-2}\mathcal{S}_1 \,\cup\, \cdots \,\cup\, t\mathcal{S}_{k-2} \,\cup\, \mathcal{S}_{k-1}$$

is a basis, over $\mathbb{F}$, for

$$\overline{PS}_n(\mathcal{X}\otimes\mathcal{Y})_{k-1} = \sum_{p+q=k-1} S_n(X_p\times X_q).$$

Since $\overline{PS}_n(\mathcal{X}\otimes\mathcal{Y})$ is an $\mathbb{F}[t]$-submodule of $PS_n(\mathcal{X}\otimes\mathcal{Y})$ and the latter is torsion free, then $t\mathcal{A}_{k-1}$ is an $\mathbb{F}$-linearly independent subset of $\overline{PS}_n(\mathcal{X}\otimes\mathcal{Y})_k$. We claim that $t\mathcal{A}_{k-1}$ can be completed to an $\mathbb{F}$-basis $\mathcal{A}_k$ of $\overline{PS}_n(\mathcal{X}\otimes\mathcal{Y})_k$, consisting of singular n-simplices $\Delta^n \to X_i \times Y_j$, $i + j = k$. Indeed, if $\mathscr{C}$ is the collection of all sets C of simplices $\Delta^n \to X_i \times Y_j$, $i+j = k$, so that C is $\mathbb{F}$-linearly independent in $\overline{PS}_n(\mathcal{X}\otimes\mathcal{Y})_k$ and $t\mathcal{A}_{k-1} \subset C$, then $\mathscr{C}$ can be ordered by inclusion, and any chain in $\mathscr{C}$ is bounded above by its union. By Zorn's lemma $\mathscr{C}$ has a maximal element $\mathcal{A}_k$, which yields the desired basis. If we let $\mathcal{S}_k = \mathcal{A}_k \smallsetminus t\mathcal{A}_{k-1}$, then

$$\mathcal{S} := \bigcup_{k\in\mathbb{N}_0} \mathcal{S}_k$$

is a free $\mathbb{F}[t]$-basis for $\overline{PS}_n(\mathcal{X}\otimes\mathcal{Y})$.

We claim that each $\sigma \in \mathcal{S}$ is in the image of $F_n(u)$ for some model $M \in \mathcal{M}$ and some morphism $u : M \to (\mathcal{X},\mathcal{Y})$ in $\mathbf{FTop}_0^2$. Indeed, since σ is of the form $\sigma : \Delta^n \to X_i \times Y_j$ for some $i, j \in \mathbb{N}_0$, then there are unique singular simplices $\sigma_i^X : \Delta^n \to X_i$ and $\sigma_j^Y : \Delta^n \to Y_j$, so that if $d : \Delta^n \to \Delta^n \times \Delta^n$ is the diagonal map $d(a) = (a, a)$, then $\sigma = \left(\sigma_i^X, \sigma_j^Y\right) \circ d$. Let $\Sigma\sigma_i^X : \Sigma_i\Delta^n \to \mathcal{X}$ be the morphism in $\mathbf{FTop}_0$ defined as the inclusion $\emptyset \hookrightarrow X_\ell$ for $\ell < i$, and as the composition $\Delta^n \xrightarrow{\sigma_i^X} X_i \hookrightarrow X_\ell$ for $\ell \geq i$. Define $\Sigma\sigma_j^Y : \Sigma_j\Delta^n \to \mathcal{Y}$ in a similar fashion. Then, $\left(\Sigma\sigma_i^X, \Sigma\sigma_j^Y\right) : \left(\Sigma_i\Delta^n, \Sigma_j\Delta^n\right) \longrightarrow (\mathcal{X}, \mathcal{Y})$ is a morphism in $\mathbf{FTop}_0^2$, and if

$$d_{i,j} \in S_n\Big((\Sigma_i\Delta^n)_i \times (\Sigma_j\Delta^n)_j\Big) = S_n(\Delta^n \times \Delta^n)$$

is the singular n-simplex corresponding to the diagonal map d, then

$$F_n\left(\Sigma\sigma_i^X, \Sigma\sigma_j^Y\right)(d_{i,j}) = \sigma.$$

As for the freeness of

$$E_n(\mathcal{X}, \mathcal{Y}) = \bigoplus_{p+q=n} PS_p(\mathcal{X}) \otimes_{\mathbb{F}[t]} PS_q(\mathcal{Y}),$$

we note that each direct summand $PS_p(\mathcal{X}) \otimes_{\mathbb{F}[t]} PS_q(\mathcal{Y})$ is freely generated over $\mathbb{F}[t]$ by elements which can be written as the tensor product of a homogeneous element from $PS_p(\mathcal{X})$ and a homogeneous element from $PS_q(\mathcal{Y})$. That is, by elements of the form $\sigma_p^X \otimes_{\mathbb{F}[t]} \sigma_q^Y$, where $\sigma_p^X : \Delta^p \to X_e$ and $\sigma_q^Y : \Delta^q \to Y_f$ are singular simplices. We let $\Sigma_e\sigma_p^X : \Sigma_e\Delta^p \to \mathcal{X}$ and $\Sigma_f\sigma_q^Y : \Sigma_f\Delta^q \longrightarrow \mathcal{Y}$ be defined as in the analysis of F_n above. Let $i_p \in PS_p(\Sigma_e\Delta^p)$ be the homogeneous element of degree e corresponding to the identity of Δ^p, and define $i_q \in PS_q(\Sigma_f\Delta^q)$ in a similar fashion. Then, $i_p \otimes_{\mathbb{F}[t]} i_q \in E_n(\Sigma_e\Delta^p, \Sigma_f\Delta^q)$ and

$$\sigma_p^X \otimes_{\mathbb{F}[t]} \sigma_q^Y \;=\; E_n\left(\Sigma_e\sigma_p^X, \Sigma_f\sigma_q^Y\right)(i_p \otimes_{\mathbb{F}[t]} i_q),$$

thus completing the proof. ☐

Lemma 3.3.9. *The functors H_0F and H_0E are naturally equivalent.*

Proof. Given $(\mathcal{X}, \mathcal{Y}) \in \mathbf{FTop}_0 \times \mathbf{FTop}_0$, our goal is to define a natural isomorphism

$$\Psi : H_0\left(PS_*(\mathcal{X}) \otimes_{\mathbb{F}[t]} PS_*(\mathcal{Y})\right) \longrightarrow H_0\left(\overline{PS}_*(\mathcal{X} \otimes \mathcal{Y})\right).$$

The first thing to note is that the graded algebraic Künneth formula (Theorem 3.3.5) yields a natural isomorphism

$$H_0\left(PS_*(\mathcal{X}) \otimes_{\mathbb{F}[t]} PS_*(\mathcal{Y})\right) \cong PH_0(\mathcal{X}) \otimes_{\mathbb{F}[t]} PH_0(\mathcal{Y}).$$

Moreover, since each $X_{k-\ell} \times Y_\ell$ is open in $(\mathcal{X} \otimes \mathcal{Y})_k$, then the inclusion

$$\sum_{i+j=k} S_n(X_i \times Y_j) \hookrightarrow S_n((\mathcal{X} \otimes \mathcal{Y})_k)$$

is a chain homotopy equivalence (see [39, Proposition 2.21]), and thus

$$H_n\left(\overline{PS}_*(\mathcal{X} \otimes \mathcal{Y})\right) \cong PH_n(\mathcal{X} \otimes \mathcal{Y}).$$

Therefore, it is enough to construct a natural isomorphism

$$\Psi : PH_0(\mathcal{X}) \otimes_{\mathbb{F}[t]} PH_0(\mathcal{Y}) \longrightarrow PH_0(\mathcal{X} \otimes \mathcal{Y})$$

and we will do so on each degree k as follows. Recall that (see Proposition 3.3.1)

$$\left(PH_0(\mathcal{X}) \otimes_{\mathbb{F}[t]} PH_0(\mathcal{Y})\right)_k \cong \left(\bigoplus_{i+j=k} H_0(X_i) \otimes_{\mathbb{F}} H_0(Y_j)\right) \Big/ J_k$$

where J_k is the linear subspace of $\bigoplus\limits_{i+j=k} H_0(X_i) \otimes_{\mathbb{F}} H_0(Y_j)$ generated by elements of the form $(t^\ell a) \otimes_{\mathbb{F}} b - a \otimes_{\mathbb{F}} (t^\ell b)$ with $\mathsf{deg}(a) + \mathsf{deg}(b) + \ell = k$. Let

$$\begin{array}{rccc} \psi_k : & \bigoplus\limits_{i+j=k} H_0(X_i) \otimes_{\mathbb{F}} H_0(Y_j) & \longrightarrow & H_0((\mathcal{X} \otimes \mathcal{Y})_k) \\ & \Gamma_i^X \otimes_{\mathbb{F}} \Gamma_j^Y & \mapsto & \Gamma \end{array}$$

where Γ_i^X (resp. Γ_j^Y) is a path-connected component of X_i (resp. Y_j), $i+j=k$, and Γ is the unique path-connected component of $(\mathcal{X} \otimes \mathcal{Y})_k$ containing $\Gamma_i^X \times \Gamma_j^Y$. This definition on basis elements uniquely determines ψ_k as an $\mathbb{F}$-linear map, and we claim that it is surjective with $\mathsf{ker}(\psi_k) = J_k$. Surjectivity follows from observing that if $(x, y) \in \Gamma$, then $(x, y) \in X_i \times Y_j$ for some $i + j = k$, and that if Γ_i^X and Γ_j^Y are the path-connected components in X_i and Y_j, respectively, so that $(x, y) \in \Gamma_i^X \times \Gamma_j^Y$, then path-connectedness of $\Gamma_i^X \times \Gamma_j^Y$ and maximality of Γ imply $\Gamma_i^X \times \Gamma_j^Y \subset \Gamma$.

The inclusion $J_k \subset \mathsf{ker}(\psi_k)$ is immediate, since it holds true for elements of the form $\left(t^\ell \Gamma_{k-j}^X \otimes_{\mathbb{F}} \Gamma_{j-\ell}^Y - \Gamma_{k-j}^X \otimes_{\mathbb{F}} t^\ell \Gamma_{j-\ell}^Y\right)$, and these generate J_k over $\mathbb{F}$. In order to show that $\mathsf{ker}(\psi_k) \subset J_k$, we will first establish the following:

<u>Claim 1</u>: If $\Gamma_i^X \times \Gamma_j^Y \subset X_i \times Y_j$ and $\Gamma_p^X \times \Gamma_q^Y \subset X_p \times Y_q$ are path-connected components, $p+q = i+j = k$, such that $\psi_k\left(\Gamma_i^X \otimes_{\mathbb{F}} \Gamma_j^Y\right) = \psi_k\left(\Gamma_p^X \otimes_{\mathbb{F}} \Gamma_q^Y\right)$, then

$$\left(\Gamma_i^X \otimes_{\mathbb{F}} \Gamma_j^Y - \Gamma_p^X \otimes_{\mathbb{F}} \Gamma_q^Y\right) \in J_k.$$

Indeed, let $\gamma : [0,1] \longrightarrow (\mathcal{X} \otimes \mathcal{Y})_k$ be a path with $\gamma(0) \in \Gamma_i^X \times \Gamma_j^Y$ and $\gamma(1) \in \Gamma_p^X \times \Gamma_q^Y$. Since each $X_{k-\ell} \times Y_\ell$ is open in $(\mathcal{X} \otimes \mathcal{Y})_k$, then the Lebesgue's number lemma yields a partition $0 = r_0 < r_1 < \cdots < r_N = 1$ such that $\gamma\left([r_n, r_{n+1}]\right) \subset X_{i_n} \times Y_{j_n}$ and $i_n + j_n = k$ for all $0 \leq n < N$. Let $(x_n, y_n) = \gamma(r_n)$, and let $\Gamma_{i_n}^X \times \Gamma_{j_n}^Y$ be its path-connected component in $X_{i_n} \times Y_{j_n}$. Since $(i,j) = (i_0, j_0)$ and $(p,q) = (i_N, j_N)$, then it is enough to show that

$$\left(\Gamma_{i_n}^X \otimes_{\mathbb{F}} \Gamma_{j_n}^Y - \Gamma_{i_{n+1}}^X \otimes_{\mathbb{F}} \Gamma_{j_{n+1}}^Y\right) \in J_k \quad \text{for all} \quad n = 0, \ldots, N-1.$$

Fix $0 \leq n < N$, assume that $i_n \leq i_{n+1}$, and let $\ell = i_{n+1} - i_n = j_n - j_{n+1}$ (the argument is similar if $i_n \geq i_{n+1}$). Since the projection $\pi^{\mathcal{X}}\left(\gamma\left([r_n, r_{n+1}]\right)\right) \subset X_{i_n}$ is a path in X_{i_n} from x_n to x_{n+1}, then $t^\ell \Gamma_{i_n}^X = \Gamma_{i_{n+1}}^X$ in $H_0(X_{i_{n+1}})$. Similarly, the projection $\pi^{\mathcal{Y}}\left(\gamma\left([r_n, r_{n+1}]\right)\right) \subset Y_{j_n}$ is a path in Y_{j_n} from y_n to y_{n+1}, and since $j_{n+1} \leq j_n$, then $t^\ell \Gamma_{j_{n+1}}^Y = \Gamma_{j_n}^Y$ in $H_0(Y_{j_n})$. This calculation shows that

$$\left(\Gamma_{i_n}^X \otimes_{\mathbb{F}} \Gamma_{j_n}^Y - \Gamma_{i_{n+1}}^X \otimes_{\mathbb{F}} \Gamma_{j_{n+1}}^Y\right) = \left(\Gamma_{i_n}^X \otimes_{\mathbb{F}} t^\ell \Gamma_{j_{n+1}}^Y - t^\ell \Gamma_{i_n}^X \otimes_{\mathbb{F}} \Gamma_{j_{n+1}}^Y\right) \in J_k$$

and therefore $\left(\Gamma_i^X \otimes_{\mathbb{F}} \Gamma_j^Y - \Gamma_p^X \otimes_{\mathbb{F}} \Gamma_q^Y\right) \in J_k$. (*end of Claim 1*)

Let us now show that $\mathsf{ker}(\psi_k) \subset J_k$. To this end, let $c \in \mathsf{ker}(\psi_k)$ and write

$$c = \sum_{n=1}^{N} c_n \cdot \left(\Gamma_{i_n}^X \otimes_{\mathbb{F}} \Gamma_{j_n}^Y\right)$$

where $c_n \in \mathbb{F}$, and where the indices (i_n, j_n) have been ordered such that there are integers $0 = n_0 < n_1 < \cdots < n_D = N$, and distinct path-connected components $\Gamma_1, \ldots, \Gamma_D$ of $(\mathcal{X} \otimes \mathcal{Y})_k$ so that if $n_{d-1} < n \leq n_d$, then $\psi_k\left(\Gamma_{i_n}^X \otimes_{\mathbb{F}} \Gamma_{j_n}^Y\right) = \Gamma_d$ for all $d = 1, \ldots, D$. Since the Γ_d's are linearly independent in $H_0((\mathcal{X} \otimes \mathcal{Y})_k)$, then $\psi_k(c) = 0$ implies

$$c_{n_d} = \sum_{n=1+n_{d-1}}^{n_d - 1} -c_n \quad , \quad d = 1, \ldots, D$$

and therefore

$$c = \sum_{d=1}^{D} \sum_{n=1+n_{d-1}}^{n_d - 1} c_n \cdot \left(\Gamma^X_{in} \otimes_{\mathbb{F}} \Gamma^Y_{jn} - \Gamma^X_{in_d} \otimes_{\mathbb{F}} \Gamma^Y_{jn_d} \right)$$

where each summand is an element of J_k, by Claim 1 above. Thus, $\mathsf{ker}(\psi_k) = J_k$, ψ_k induces a natural $\mathbb{F}$-isomorphism

$$\Psi_k : \left(PH_0(\mathcal{X}) \otimes_{\mathbb{F}[t]} PH_0(\mathcal{Y}) \right)_k \longrightarrow H_0((\mathcal{X} \otimes \mathcal{Y})_k)$$

and letting $\Psi = \Psi_0 \oplus \Psi_1 \oplus \cdots$ completes the proof. □

We now move onto the main results of this section.

Lemma 3.3.10 (Persistent Eilenberg-Zilber)**.** *For objects $\mathcal{X}, \mathcal{Y} \in \mathbf{FTop_0}$ and coefficients in a field $\mathbb{F}$, there is a natural graded chain equivalence*

$$\zeta : PS_*(\mathcal{X}) \otimes_{\mathbb{F}[t]} PS_*(\mathcal{Y}) \longrightarrow \overline{PS}_*(\mathcal{X} \otimes \mathcal{Y})$$

with $H_0(\zeta) = \Psi$ (Lemma 3.3.9), and unique up to a graded chain homotopy.

Proof. The proof follows exactly that of Theorem 2.2.4 [32], using acyclic models [30] (see also the proof of Theorem 5.3 in [71] for a more direct comparison). Indeed, the functors in question are acyclic (Lemma 3.3.7), free (Lemma 3.3.8), and naturally equivalent at the level of zero-th homology (Lemma 3.3.9). □

Theorem 3.3.11. *If $\mathcal{X}, \mathcal{Y} \in \mathbf{Top^N}$, then there is a natural $\mathbb{F}[t]$-isomorphism*

$$H_n \left(PS_*(\mathcal{T}(\mathcal{X})) \otimes_{\mathbb{F}[t]} PS_*(\mathcal{T}(\mathcal{Y})) \right) \;\cong\; PH_n(\mathcal{X} \otimes_{\mathbf{g}} \mathcal{Y}).$$

Proof. Lemma 3.1.15 implies that $PH_n(\mathcal{X} \otimes_{\mathbf{g}} \mathcal{Y}) \cong PH_n(\mathcal{T}(\mathcal{X}) \otimes \mathcal{T}(\mathcal{Y}))$, and Proposition 3.1.17 shows that if $0 < \epsilon < 1$, then

$$PH_n(\mathcal{T}(\mathcal{X}) \otimes \mathcal{T}(\mathcal{Y})) \cong PH_n(\mathcal{T}^\epsilon(\mathcal{X}) \otimes \mathcal{T}^\epsilon(\mathcal{Y})).$$

Moreover, since $\mathcal{T}^\epsilon(\mathcal{X}) \in \mathbf{FTop}_0$, then the inclusion

$$\overline{PS}_*(\mathcal{T}^\epsilon(\mathcal{X}) \otimes \mathcal{T}^\epsilon(\mathcal{Y})) \hookrightarrow PS_*(\mathcal{T}^\epsilon(\mathcal{X}) \otimes \mathcal{T}^\epsilon(\mathcal{Y}))$$

is a graded chain homotopy equivalence (see [39, Proposition 2.21]), and thus Lemma 3.3.10 shows that

$$PH_n(\mathcal{X} \otimes_{\mathbf{g}} \mathcal{Y}) \cong H_n\left(PS_*(\mathcal{T}^\epsilon(\mathcal{X})) \otimes_{\mathbb{F}[t]} PS_*(\mathcal{T}^\epsilon(\mathcal{Y}))\right).$$

The ϵ can be removed since $\mathcal{T}_k^\epsilon(\mathcal{X})$ deformation retracts onto $\mathcal{T}_k(\mathcal{X})$ (Proposition 3.1.17), which completes the proof. □

Finally, we obtain

Theorem 3.3.12. *Let $\mathcal{X}, \mathcal{Y} \in \mathbf{Top}^{\mathbf{N}}$. Then there is a natural short exact sequence of graded $\mathbb{F}[t]$-modules*

$$0 \to \bigoplus_{i+j=n} \left(PH_i(\mathcal{X};\mathbb{F}) \otimes_{\mathbb{F}[t]} PH_j(\mathcal{Y};\mathbb{F})\right) \to PH_n\left(\mathcal{X} \otimes_{\mathbf{g}} \mathcal{Y}\right) \to \bigoplus_{i+j=n} \mathsf{Tor}_{\mathbb{F}[t]}\left(PH_i(\mathcal{X};\mathbb{F}), PH_{j-1}(\mathcal{Y};\mathbb{F})\right) \to 0$$

which splits, though not naturally.

Remark 3.3.13. If $\mathbf{S}$ is either **Top**, **Met**, **oSimp**, or **Simp**, then following Remark 3.2.1 yields a similar Persistent Künneth theorem in each case.

3.3.4 Barcode Formulae

We now give the barcode formula for the generalized tensor product of objects in $\mathbf{S}^{\mathbf{N}}$. Recall from Theorem 2.3.10 that pointwise finite objects in $\mathbf{Mod}_{\mathbb{F}}^{\mathbf{N}}$ can be uniquely decomposed into interval diagrams. In Proposition 3.3.3, the tensor product and Tor of the associated interval modules was computed, and the following is a consequence of Theorem 3.3.12.

Theorem 3.3.14. *Let $\mathcal{X}, \mathcal{Y} \in \mathbf{S}^{\mathbf{N}}$. Assume that for $0 \le i, j \le n$, the diagrams $H_i(\mathcal{X})$ and*

$H_j(\mathcal{Y})$ are pointwise finite. Then $H_n(\mathcal{X} \otimes_{\mathbf{g}} \mathcal{Y})$ is pointwise finite with barcode

$$\mathsf{bcd}_n(\mathcal{X} \otimes_{\mathbf{g}} \mathcal{Y}) = \bigcup_{i+j=n} \left\{ (\ell_J + I) \cap (\ell_I + J) \;\middle|\; I \in \mathsf{bcd}_i(\mathcal{X}), J \in \mathsf{bcd}_j(\mathcal{Y}) \right\}$$

$$\bigcup$$

$$\bigcup_{i+j=n} \left\{ (\rho_J + I) \cap (\rho_I + J) \middle| \; I \in \mathsf{bcd}_i(\mathcal{X}), J \in \mathsf{bcd}_{j-1}(\mathcal{Y}) \right\}.$$

In the above formula, the first union of intervals is associated with the tensor part of the sequence in theorem 3.3.12, while the second union is associated with the Tor part. The results are easy to generalize to 0-th persistent homology in the case of finite tensor product of objects in **Top**.

Corollary 3.3.15 (0-th persistent homology)**.** *Let $\mathcal{X}_1, \ldots, \mathcal{X}_p \in \mathbf{S}^{\mathbf{N}}$, and assume that $H_0(\mathcal{X}_i)$ is pointwise finite for all $i \in \mathbb{N}_0$. Let $\mathsf{bcd}_0(\mathcal{X}_i)$ be the 0-dimensional barcode of $\mathcal{X}_i$, then*

$$\mathsf{bcd}_0\left(\mathcal{X}_1 \otimes_{\mathbf{g}} \cdots \otimes_{\mathbf{g}} \mathcal{X}_p\right) = \left\{ \bigcap_{i=1}^{p} \left(\left(\ell - \ell_{I_i}\right) + I_i \right) \middle| \; \ell = \sum_{i=1}^{p} \ell_{I_i}, \; I_i \in \mathsf{bcd}_0(\mathcal{X}_i) \right\}.$$

In other words, a bar in $\mathcal{X}_1 \otimes_{\mathbf{g}} \cdots \otimes_{\mathbf{g}} \mathcal{X}_p$ corresponding to p bars, one from each $\mathsf{bcd}_0(\mathcal{X}_i)$, starts at the sum of the starting points and lives as long as the shortest one.

Topological inference deals with estimating the homology of a metric space $\mathbb{X}$ from a finite point cloud X, as discussed in Example 2.3.2. Usually, longer bars in a barcode represent significant topological features. The following corollary allows us to count the number of bars longer than a threshold $\epsilon > 0$ in the barcode of the tensor product.

Corollary 3.3.16 (On high persistence)**.** *For $n \geq 0$ and $\epsilon > 0$, let $\mathbf{c}_n^{\mathcal{X}}(\beta)$ and $\mathbf{c}_n^{\mathcal{X}}(\beta \geq \epsilon)$ denote the number of bars with length exactly β and at least ϵ respectively in $\mathsf{bcd}_n(\mathcal{X})$. Assume that the lengths of longest bars in $\mathsf{bcd}_n(\mathcal{X})$ and $\mathsf{bcd}_n(\mathcal{Y})$ are $\beta_n^{\mathcal{X}}$ and $\beta_n^{\mathcal{Y}}$ respectively, then*

$$\beta_n^{\mathcal{X} \otimes_{\mathbf{g}} \mathcal{Y}} = \max \left\{ \max_{k+l=n} \left\{ \min\{\beta_k^{\mathcal{X}}, \beta_l^{\mathcal{Y}}\} \right\}, \max_{k+l=n} \left\{ \min\{\beta_k^{\mathcal{X}}, \beta_{l-1}^{\mathcal{Y}}\} \right\} \right\}$$

and

$$\mathbf{c}_n^{\mathcal{X} \otimes_{\mathbf{g}} \mathcal{Y}}(\beta \geq \epsilon) = \sum_{k+l=n} \mathbf{c}_l^{\mathcal{Y}}(\beta \geq \epsilon) \mathbf{c}_k^{\mathcal{X}}(\beta \geq \epsilon) + \sum_{k+l=n-1} \mathbf{c}_l^{\mathcal{Y}}(\beta \geq \epsilon) \mathbf{c}_k^{\mathcal{X}}(\beta \geq \epsilon). \tag{3.10}$$

By replacing ϵ with $\beta_n^{\mathcal{X}\otimes_{\mathbf{g}}\mathcal{Y}}$, the formula in equation 3.10 can be used to compute the number of bars with the longest length in $\mathsf{bcd}_n\left(\mathcal{X}\otimes_{\mathbf{g}}\mathcal{Y}\right)$.

3.4 Application: Persistent Homology of N-torus

The Rips persistence diagrams of a standard N-torus equipped with the maximum metric can be computed from the following two propositions 3.4.1 and 3.4.2. The first is a result by Adams and Adamaszek [1] that theoretically computes the homotopy type of the Rips complex of a circle S^1, at each scale ϵ. In the original result, the circle was a equipped with the geodesic metric but we present a remetrized version with the Euclidean metric:

Proposition 3.4.1 ([1])**.** *The Vietoris-Rips complex $R_\epsilon(S^1_r)$ of a circle of radius r (equipped with Euclidean metric) is homotopy equivalent to S^{2l+1} if*

$$2r\sin\left(\pi\frac{l}{2l+1}\right) < \epsilon \leq 2r\sin\left(\pi\frac{l+1}{2l+3}\right).$$

Recall the homology of an m-sphere, with coefficients in some field $\mathbb{F}$, is $\mathbb{F}$ in dimensions $i = 0, m$ and $\{0\}$ otherwise. Now, Proposition 3.4.1 implies that for a circle of radius r, the Rips persistence diagrams in odd dimensions are singleton as multisets and empty in positive even dimensions:

$$\mathsf{bcd}_i^{\mathcal{R}}\left(S^1_r\right) = \begin{cases} \left\{\left[2r\sin\left(\pi\frac{l}{2l+1}\right), 2r\sin\left(\pi\frac{l+1}{2l+3}\right)\right)\right\} & \text{if } i = 2l+1 \\ \{[0,\infty)\} & \text{if } i = 0 \\ \emptyset & \text{if } i = 2l+2 \end{cases} \tag{3.11}$$

for $l \in \mathbb{N}_0$. The second proposition is an extension of Corollary 3.2.6 to an N-fold product of metric spaces:

Proposition 3.4.2. *Let $(X_1, \mathbf{d}_1), \ldots, (X_N, \mathbf{d}_N)$ be finite metric spaces and let $\mathsf{bcd}_n^{\mathcal{R}}(X, \mathbf{d}_X)$ be the n-th dimensional persistence diagram of the Rips filtration $\mathcal{R}(X, \mathbf{d}_X) := \{R_\epsilon(X, \mathbf{d}_X)\}_{\epsilon\geq 0}$.*

Then,

$$\mathsf{bcd}_n^{\mathcal{R}}(X_1 \times \cdots \times X_N, \mathbf{d}_{\mathsf{max}}) = \bigcup_{\sum_{m=1}^{N} i_m = n} \left\{ \bigcap_{m=1}^{N} I_m \;\middle|\; I_m \in \mathsf{bcd}_{i_m}^{\mathcal{R}}(X_m, \mathbf{d}_m) \right\}$$

for all $n \in \mathbb{N}_0$, where $\mathbf{d}_{\mathsf{max}}$ is the maximum metric.

Consider the Vietoris-Rips complexes on the N-torus $\mathbb{T}^N := S^1_{r_1} \times \cdots \times S^1_{r_N}$ equipped with the maximum metric. Then by Propositions 3.4.1 and 3.4.2, we obtain the following theorem about the Rips persistence of a N-torus:

Theorem 3.4.3. *Let $\mathbb{T}^N := S^1_{r_1} \times \cdots \times S^1_{r_N}$ be the N-torus equipped with the maximum metric $\mathbf{d}_{\mathsf{max}}$. Then for each homological dimension p, the Rips persistence barcode of $\mathbb{T}^N$ can be computed as:*

$$\mathsf{bcd}_p^{\mathcal{R}}\left(\mathbb{T}^N, \mathbf{d}_{\mathsf{max}}\right) = \bigcup_{\sum_{m=1}^{N} i_m = p} \left\{ \bigcap_{m=1}^{N} I_m \;\middle|\; I_m \in \mathsf{bcd}_{i_m}^{\mathcal{R}}\left(S^1_{r_m}\right) \right\}.$$

More explicitly,

$$\mathsf{bcd}_1^{\mathcal{R}}\left(\mathbb{T}^N, \mathbf{d}_{\mathsf{max}}\right) = \left\{[0, \sqrt{3} r_n] \;\middle|\; n = 1, \ldots, N\right\}$$

in dimension 1, and

$$\mathsf{bcd}_2^{\mathcal{R}}\left(\mathbb{T}^N, \mathbf{d}_{\mathsf{max}}\right) = \left\{ \left[0, \sqrt{3}\min\{r_n, r_m\}\right] \;\middle|\; n, m = 1, \ldots, N\right\}$$

in dimension 2. Let $r = \min\{r_1, \ldots, r_N\}$. Then,

Corollary 3.4.4. *There are exactly $\binom{N}{p}$ bars in $\mathsf{bcd}_p^{\mathcal{R}}\left(\mathbb{T}^N, \mathbf{d}_{\mathsf{max}}\right)$ that are born at 0 and live longer than $\sqrt{3}r$.*

This is true because all the bars used in the persistent Künneth formula are either infinite, or of length atleast $\sqrt{3}r$, or shorter, depending on the dimension.

CHAPTER 4

QUASIPERIODICITY

In this chapter, we develop the general theory of sliding window embeddings of quasiperiodic functions and their persistent homology. We define quasiperiodic functions in a rigorous manner and study their properties in section 4.1 and write a Fourier-like decomposition in section 4.2. In section 4.3, we show that the sliding window embeddings of quasiperiodic functions, under appropriate choices of the embedding dimension d and time delay τ, are dense in higher dimensional tori. We also explicitly provide methods to choose these parameters. In section 4.4, we prove lower bounds on Rips persistent homology of these embeddings. The last section of this chapter is dedicated to computational results and an application to music. Parts of this chapter will appear in [36].

4.0.1 Notation

Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z} \simeq S^1$, that is, $[0, 2\pi]$ with the endpoints identified. Similarly for $N \in \mathbb{N}$, let $\mathbb{T}^N = (\mathbb{R}/2\pi\mathbb{Z})^N \simeq S^1 \times \cdots \times S^1$. Let $\mathbf{v} = (v_1, \cdots, v_M) \in \mathbb{R}^M$ such that none of the coordinates v_m are 0, then let $\mathbb{T}^{\mathbf{v}}$ define

$$\mathbb{T}^{\mathbf{v}} = \mathbb{R}\Big/\left(\frac{2\pi}{v_1}\mathbb{Z}\right) \times \cdots \times \mathbb{R}\Big/\left(\frac{2\pi}{v_M}\mathbb{Z}\right).$$

If S^1_c denotes the circle of radius c, then for $\mathbf{c} = (c_1, \ldots, c_N) \in \mathbb{C}^N$ where none of the c_n are 0, let $\mathbb{T}^N_{\mathbf{c}}$ denote

$$\mathbb{T}^N_{\mathbf{c}} := S^1_{|c_1|} \times \cdots \times S^1_{|c_N|}.$$

4.1 Quasiperiodic functions

In this section, we provide a general definition of a quasiperiodic function for a given incommensurate set of harmonics and give some examples.

Definition 4.1.1. Let $f : \mathbb{R} \to \mathbb{C}$ be a function and let $\omega = \{\omega_1, \dots, \omega_N\} \subset \mathbb{R} \setminus \{0\}$ be an incommensurate set, that is, ω is linearly independent over $\mathbb{Q}$. Then we call f *a quasiperiodic function with the set of harmonics* ω if there exists a function $F : \mathbb{R}^N \to \mathbb{C}$

$$(t_1, , \dots, t_N) \to F(t_1, , \dots, t_N)$$

such that

1. $F(t, \dots, t) = f(t)$ for all $t \in \mathbb{R}$,

2. for all $1 \leq n \leq N$, and $\forall \left(t_1^*, \dots, t_{n-1}^*, t_{n+1}^*, \dots, t_N^*\right) \in \mathbb{R}^{N-1}$, the one variable function

$$t_n \to F((t_1^*, \dots, t_{n-1}^*, t_n, t_{n+1}^*, \dots, t_N^*))$$

   is periodic with time period $\frac{2\pi}{\omega_n}$, and

3. N is the lowest such integer.

We will call F *a parent function* of f.

The condition that N must be minimal is necessary. In its absence, $f(t) = e^{it}e^{i\pi t}$ will have two parent functions: $F(t_1, t_2) = e^{it_1}e^{i\pi t_2}$ and $G(t_1) = e^{i(\pi+1)t_1}$. The function f is clearly periodic with a single frequency $\pi + 1$.

Remark 4.1.2. Note that every periodic function can be viewed as a quasiperiodic function. Indeed, if f is periodic with time period t_f, then it is quasiperiodic with the set of harmonics $\omega = \left\{\frac{2\pi}{t_f}\right\}$ and the associated parent function $F(t_1) := f(t_1)$. However, the incommensurateness of the harmonics implies that not all quasiperiodic functions are periodic. For example, $f(t) = e^{it} + e^{i\pi t}$ is quasiperiodic with $\{1, \pi\}$ and parent function $F(t_1, t_2) = e^{it_1} + e^{i\pi t_2}$, but it is not periodic because $\mathsf{lcm}\left(\frac{2\pi}{1}, \frac{2\pi}{\pi}\right)$ of the time periods of individual terms of f does not exist.

The following example demonstrates that for a quasiperiodic function f with a set of harmonics ω, its parent function may not be unique.

Example 4.1.3. The function $f(t) = e^{it} + e^{i\pi t}$ is quasiperiodic with the set of harmonics $\{1, \pi\}$. Let $F, G : \mathbb{R}^2 \to \mathbb{C}$ be defined as

$$F(t_1, t_2) = e^{it_1} + e^{i\pi t_2}$$

$$G(t_1, t_2) = e^{it_2} + e^{i\pi t_1}.$$

Clearly, f is the restriction to the diagonal of the functions F and G. Moreover, $F(t_1, t_2^*)$ and $G(t_1, t_2^*)$ are periodic in t_1 with periods 2π and 2 respectively for all choices of $t_2^\star$, and $F(t_1^*, t_2)$ and $G(t_1^*, t_2)$ are periodic in t_2 with periods 2 and 2π respectively for all choices of t_1^*. Therefore, F and G are distinct parent functions of f.

In the next proposition, we show that for a continuous quasiperiodic f associated to a given set ω, its continuous parent function is unique up to a permutation of variables:

Proposition 4.1.4. *If f is a continuous quasiperiodic function with a set $\omega = \{\omega_1, \dots, \omega_N\}$, then its continuous parent function is unique up to a permutation of variables.*

Proof. Let $f : \mathbb{R} \to \mathbb{C}$ be a quasiperiodic function with the incommensurate set $\{\omega_1, \dots, \omega_N\}$ and let there be two continuous parent functions $F, G : \mathbb{R}^N \to \mathbb{C}$. After a permutation of variables, assume that the functions F and G are periodic with the same time period $\frac{2\pi}{\omega_n}$ in t_n. By definition, F and G agree on the diagonal $\triangle_{\mathbf{0}} = \{(t, \dots, t)\,|t \in \mathbb{R}\}$. Therefore, they agree on the lines $\triangle_{\mathbf{k}} = \left\{\left(k_1\frac{2\pi}{\omega_1} + t, \dots, k_N\frac{2\pi}{\omega_N} + t\right)\Big|t \in \mathbb{R}\right\}$ for each $\mathbf{k} = (k_1, \dots, k_N) \in \mathbb{Z}^N$. We claim that the set of all $\triangle_{\mathbf{k}}$ is dense in the set of all lines parallel to the diagonal. For an arbitrary $(\overline{t_1}, \dots, \overline{t_N}) \in \mathbb{R}^N$, consider the hyperplane $V : t_1 = \overline{t_1}$. We will show that there is a line $\triangle_{\mathbf{k}}$ intersecting V at a point arbitrarily close to $(\overline{t_1}, \dots, \overline{t_N})$. Any line $\triangle_{\mathbf{k}}$ intersects V at a point $\left(\overline{t_1}, k_2\frac{2\pi}{\omega_2} + t, \dots, k_N\frac{2\pi}{\omega_N} + t\right)$ which yields that $t_1 = \overline{t_1}$ and for $2 \leq n \leq N$, $t_n = k_n\frac{2\pi}{\omega_n} + \overline{t_1} - k_1\frac{2\pi}{\omega_1}$. Without loss of generality, we can assume that $\overline{t_1} = 0$. We want to show that the set of the intersection points I is dense in V. We can rewrite $t_n = \frac{2\pi}{\omega_n}\left(k_n - k_1\frac{\omega_n}{\omega_1}\right)$. Define $s_n = \frac{\omega_n}{2\pi}t_n$ for $2 \leq n \leq N$. Call this rescaled hyperplane $\overline{V}$. Then the intersection points satisfy $s_1 = t_1 = \overline{t_1}$ and $s_n = k_n - k_1\frac{\omega_n}{\omega_1}$ for $2 \leq n \leq N$.

Clearly, $\left\{\frac{\omega_2}{\omega_1}, \ldots, \frac{\omega_N}{\omega_1}\right\}$ is incommensurate. Then by Kronecker's theorem (theorem 2.5.2), for any arbitrary $(\overline{t_1}, \overline{s_2}, \ldots, \overline{s_N}) \in \overline{V}$, we can find an intersection point arbitrarily close to it. Therefore, we can find an intersection point arbitrarily close to $(\overline{t_1}, \ldots, \overline{t_N})$. This proves our claim. Therefore, F and G agree on a dense subset of the set of lines parallel to the diagonal. By their continuity, they agree everywhere. □

Before we move on to writing a Fourier like series for a quasiperiodic function f, we want to mention that some functions can be quasiperiodic with different sets of harmonics:

Example 4.1.5. Observe that the function $f(t) = \left(e^{it} + 1\right) e^{i\pi t}$ is quasiperiodic with sets $\{1, \pi\}$ and $\{1, \pi + 1\}$: it is the restriction to the diagonal of two parent functions $F, G : \mathbb{R}^2 \to \mathbb{C}$ defined as

$$F(t_1, t_2) = \left(e^{it_1} + 1\right) e^{i\pi t_2}$$
$$G(t_1, t_2) = e^{i(\pi+1)t_1} + e^{i\pi t_2}.$$

$F(t_1, t_2^*)$ and $G(t_1, t_2^*)$ are periodic in t_1 with periods 2π and $\frac{2\pi}{\pi+1}$ respectively for all choices of $t_2^\star$, and $F(t_1^*, t_2)$ and $G(t_1^*, t_2)$ are both periodic in t_2 with periods 2 for all choices of t_1^*.

4.2 "Fourier series" of a quasiperiodic function

In this section, we adapt the theory of multivariate Fourier analysis to write a Fourier-like series for the function f. For a more comprehensive review, see [38, Chapter 3].

4.2.1 The iterated Fourier series

Let $f : \mathbb{R} \to \mathbb{C}$ be a quasiperiodic function with the incommensurate set $\{\omega_1, \ldots, \omega_N\}$ and parent function $F : \mathbb{R}^N \to \mathbb{C}$. Let ω denote the frequency vector $(\omega_1, \ldots, \omega_N)$ and recall the notation (section 4.0.1) for $\mathbb{T}^\omega$:

$$\mathbb{T}^\omega = \mathbb{R} \Big/ \left(\frac{2\pi}{\omega_1}\mathbb{Z}\right) \times \cdots \times \mathbb{R} \Big/ \left(\frac{2\pi}{\omega_N}\mathbb{Z}\right).$$

Furthermore assume that the parent function F belongs to $C^1(\mathbb{T}^\omega)$. Recall that by definition of a quasiperiodic function, each marginal

$$F_n(t_n) := F\left(t_1^*, \ldots, t_{n-1}^*, t_n, t_{n+1}^*, \ldots, t_N^*\right)$$

is a one variable periodic function in t_n with the period $\frac{2\pi}{\omega_n}$.

Lemma 4.2.1. *If $g : X \to \mathbb{C}$ is a continuously differentiable function on a finite measure space $X \subset \mathbb{R}^m$, then it is square integrable on X.*

The next proposition shows that for each such F, all its marginal functions F_n are continuously differentiable.

Proposition 4.2.2. *Let $\mathbb{T}^{\omega N} = \mathbb{R}\Big/\left(\frac{2\pi}{\omega_N}\mathbb{Z}\right)$. If $F \in C^1\left(\mathbb{T}^\omega\right)$, then $F_N \in C^1\left(\mathbb{T}^{\omega N}\right)$.*

Proof. Since F is continuously differentiable on $\mathbb{T}^\omega$, all its partial derivatives exist and are continuous, which yields that F_N is continuously differentiable with respect to t_N. Therefore, $F_N \in C^1\left(\mathbb{T}^{\omega N}\right)$. □

For each choice of $\left(t_1^*, \ldots, t_{N-1}^*\right) \in \mathbb{T}^{\omega 1} \times \cdots \times \mathbb{T}^{\omega N-1}$, we obtain a Fourier series

$$F\left(t_1^*, \ldots, t_{N-1}^*, t_N\right) = \sum_{k_N \in \mathbb{Z}} \hat{F}\left(t_1^*, \ldots, t_{N-1}^*, k_N\right) e^{ik_N \omega_N t_N}$$

where

$$\hat{F}\left(t_1^*, \ldots, t_{N-1}^*, k_N\right) = \frac{\omega_N}{2\pi} \int_0^{\frac{2\pi}{\omega_N}} F\left(t_1^*, \ldots, t_{N-1}^*, t_N\right) e^{-ik_N \omega_N t_N} dt_N$$

are its Fourier coefficients. By proposition 2.5.1, the series converges pointwise to F_N. The next proposition shows that the coefficient functions $\hat{F}\left(t_1, \ldots, t_{N-1}, k_N\right)$ for each choice of $k_N \in \mathbb{Z}$ are square integrable on $\mathbb{T}^{\omega 1} \times \cdots \times \mathbb{T}^{\omega N-1}$.

Proposition 4.2.3. *If $F \in C^1\left(\mathbb{T}^\omega\right)$, then for all $k_N \in \mathbb{Z}$, the $N-1$ variable coefficient functions $\hat{F}(t_1, \ldots, t_{N-1}, k_N) \in L^2\left(\mathbb{T}^{\omega 1} \times \cdots \times \mathbb{T}^{\omega N-1}\right)$.*

Proof. The proof follows directly from the definition:

$$\int_{\mathbb{T}^{\omega_1}\times\cdots\times\mathbb{T}^{\omega_{N-1}}} \left|\hat{F}(t_1,\ldots,t_{N-1},k_N)\right|^2 dt_{N-1}\cdots dt_1$$

$$= \int_{\mathbb{T}^{\omega_1}\times\cdots\times\mathbb{T}^{\omega_{N-1}}} \left|\frac{\omega_N}{2\pi}\int_0^{\frac{2\pi}{\omega_N}} F(t_1,\ldots,t_N)e^{-ik_N\omega_N t_N}dt_N\right|^2 dt_{N-1}\cdots dt_1$$

$$\leq \frac{\omega_N}{2\pi}\int_{\mathbb{T}^{\omega_1}\times\cdots\times\mathbb{T}^{\omega_{N-1}}}\int_0^{\frac{2\pi}{\omega_N}} |F(t_1,\ldots,t_N)|^2 dt_N dt_{N-1}\cdots dt_1 < \infty$$

where the last term is finite because $F \in C^1(\mathbb{T}^\omega)$ and Lemma 4.2.1. □

Now observe that for each $(t_1^*,\ldots,t_{N-2}^*) \in \mathbb{T}^{\omega_1}\times\cdots\times\mathbb{T}^{\omega_{N-2}}$ and each $k_N \in \mathbb{Z}$, the one variable function defined by

$$\hat{F}(t_1^*,\ldots,t_{N-2}^*.t_{N-1},k_N) = \frac{\omega_N}{2\pi}\int_0^{\frac{2\pi}{\omega_N}} F(t_1^*,\ldots,t_{N-2}^*,t_{N-1},t_N)e^{-ik_N\omega_N t_N}dt_N$$

is periodic in t_{N-1} with period $\frac{2\pi}{\omega_{N-1}}$.

Proposition 4.2.4. *For each $(t_1^*,\ldots,t_{N-2}^*) \in \mathbb{T}^{\omega_1}\times\cdots\times\mathbb{T}^{\omega_{N-2}}$ and each $k_N \in \mathbb{Z}$, the function $\hat{F}(t_1^*,\ldots,t_{N-2}^*,t_{N-1},k_N) \in L^2\left(\mathbb{T}^{\omega_{N-1}}\right)\cap C^1\left(\mathbb{T}^{\omega_{N-1}}\right)$.*

Proof. Similar to the proof of proposition 4.2.2. □

Therefore, we can write $\hat{F}(t_1^*,\ldots,t_{N-2}^*,t_{N-1},k_N)$ as a Fourier series which converges pointwise to it:

$$\hat{F}(t_1^\star,\ldots,t_{N-2}^\star,t_{N-1},k_N) = \sum_{k_{N-1}\in\mathbb{Z}} \hat{F}(t_1^\star,\ldots,t_{N-2}^\star,k_{N-1},k_N)e^{ik_{N-1}\omega_{N-1}t_{N-1}}$$

which is now equal the following double sum:

$$\sum_{k_N\in\mathbb{Z}}\sum_{k_{N-1}\in\mathbb{Z}} \hat{F}(t_1^\star,\ldots,t_{N-2}^\star,k_{N-1},k_N)e^{ik_{N-1}\omega_{N-1}t_{N-1}}e^{ik_N\omega_N t_N},$$

where $\hat{F}(t_1^*, \ldots, t_{N-2}^* . k_{N-1}, k_N)$ is given by

$$\frac{\omega_{N-1}\omega_N}{(2\pi)^2} \int_0^{\frac{2\pi}{\omega_{N-1}}} e^{-ik_{N-1}\omega_{N-1}t_{N-1}} \int_0^{\frac{2\pi}{\omega_N}} F(t_1^*, \ldots, t_{N-2}^*, t_{N-1}, t_N) e^{-ik_N\omega_N t_N} dt_N dt_{N-1}.$$

Similar to Proposition 4.2.3, we can show that if

$$\hat{F}(t_1, \ldots, t_{N-1}, k_N) \in L^2(\mathbb{T}^{\omega_1} \times \cdots \times \mathbb{T}^{\omega_{N-1}}),$$

then $\hat{F}(t_1, \ldots, t_{N-2}, k_{N-1}, k_N) \in L^2(\mathbb{T}^{\omega_1} \times \cdots \times \mathbb{T}^{\omega_{N-2}})$ for all pairs $(k_{N-1}, k_N) \in \mathbb{Z}^2$:

$$\begin{aligned}
&\int\limits_{\mathbb{T}^{\omega_1}\times\cdots\times\mathbb{T}^{\omega_{N-2}}} \left|\hat{F}(t_1, \ldots, t_{N-2}, k_{N-1}, k_N)\right|^2 dt_{N-2} \cdots dt_1 \\
&= \int\limits_{\mathbb{T}^{\omega_1}\times\cdots\times\mathbb{T}^{\omega_{N-2}}} \left|\frac{\omega_{N-1}}{2\pi} \int_0^{\frac{2\pi}{\omega_{N-1}}} \hat{F}(t_1, \ldots, t_{N-1}, k_N) e^{-ik_{N-1}\omega_{N-1}t_{N-1}} dt_{N-1}\right|^2 dt_{N-2} \cdots dt_1 \\
&\le \frac{\omega_{N-1}}{2\pi} \int\limits_{\mathbb{T}^{\omega_1}\times\cdots\times\mathbb{T}^{\omega_{N-2}}} \int_0^{\frac{2\pi}{\omega_{N-1}}} \left|\hat{F}(t_1, \ldots, t_{N-1}, k_N)\right|^2 dt_{N-1} \cdots dt_1 \\
&\le \frac{\omega_{N-1}}{2\pi} \int\limits_{\mathbb{T}^{\omega_1}\times\cdots\times\mathbb{T}^{\omega_{N-1}}} \left|\hat{F}(t_1, \ldots, t_{N-1}, k_N)\right|^2 dt_{N-1} \cdots dt_1 < \infty.
\end{aligned}$$

Repeating the above arguments, we obtain that for all $(k_{n+1}, \ldots, k_N) \in \mathbb{Z}^{N-n}$, the intermediate $\hat{F}(t_1, \ldots, t_n, k_{n+1}, \ldots, k_N)$ will be in $L^2(\mathbb{T}^{\omega_1} \times \cdots \times \mathbb{T}^{\omega_n})$ if $F \in L^2(\mathbb{T}^{\omega})$. Using induction, we obtain

$$F(t_1, \ldots, t_N) = \sum_{k_N \in \mathbb{Z}} \cdots \sum_{k_1 \in \mathbb{Z}} \hat{F}(k_1, \ldots, k_N) e^{i(k_1\omega_1 t_1 + \cdots + k_N\omega_N t_N)} \tag{4.1}$$

where the convergence of series to F is pointwise because all intermediate $\hat{F}$'s are continuously differentiable. Note that for $\mathbf{k} = (k_1, \ldots, k_N) \in \mathbb{Z}^N$, the coefficients $\hat{F}(\mathbf{k})$ are given by:

$$\hat{F}(\mathbf{k}) = \frac{\prod_{n=1}^{N} \omega_n}{(2\pi)^N} \int_0^{\frac{2\pi}{\omega_1}} e^{-ik_1\omega_1 t_1} \int_0^{\frac{2\pi}{\omega_2}} \cdots \int_0^{\frac{2\pi}{\omega_N}} F(t_1, \ldots, t_N) e^{-ik_N\omega_N t_N} dt_N \ldots dt_1.$$

To be able to rearrange the terms of the series in equation 4.1, we need it to be absolutely convergent. To prove this, we investigate the coefficients $\hat{F}(\mathbf{k})$ in the next section. We finish this section with a remark about the periodicity of the partial derivatives of the parent function which will be useful in proof of Theorem 4.2.6.

Remark 4.2.5. Let f be a quasiperiodic function and let $F : \mathbb{R}^N \to \mathbb{C}$ be its parent function. Suppose for some $1 \leq n \leq N$ and $r \in \mathbb{N}$, the r-th partial derivative $\partial_n^r F$ exists, then $\partial_n^r F$ will be periodic in t_n with the same period $\frac{2\pi}{\omega_n}$. If for $m \neq n$, $\partial_n^r F$ is dependent on t_m, then it will be periodic in t_m with $\frac{2\pi}{\omega_m}$.

4.2.2 Analyzing the coefficients $\hat{F}(\mathbf{k})$

For $Z \in \mathbb{N}_0$, let $I_Z^N = \left\{\mathbf{k} = (k_1, \dots, k_N) \in \mathbb{Z}^N \middle| \ \forall n \ , \ |k_n| \leq Z\right\}$ denote the box of lengths $2Z$ centered around $(0, \dots, 0) \in \mathbb{Z}^N$. Moreover, let $\mathbf{k} \odot \omega = (k_1\omega_1, \dots, k_N\omega_N)$ denote the Hadamard product of two vectors $\mathbf{k}$ and ω. The next theorem provides bounds on each coefficient $\hat{F}(\mathbf{k})$ in terms of its corresponding coefficient in the series decomposition of a certain partial derivative. Furthermore, it concludes that the sequence of coefficients is absolutely convergent.

Theorem 4.2.6. *Let $r \in \mathbb{N}$ and $\mathbf{k}, \mathbf{l} \in \mathbb{Z}^N$. Suppose that $\partial^{\mathbf{l}} F$ exist and are integrable for all $||\mathbf{l}||_1 \leq r$. Then*

$$\left|\hat{F}(\mathbf{k})\right| \leq \frac{\sqrt{N}^r}{||\mathbf{k} \odot \omega||_2^r} \left|\widehat{\partial_n^r F}(\mathbf{k})\right|$$

where n satisfies $|k_n\omega_n| = ||\mathbf{k} \odot \omega||_\infty$ and $\widehat{\partial_n^r F}$ is the r-th (partial) derivative of F with respect to t_n. As a consequence, the sequence of coefficients is absolutely convergent:

$$\sum_{\mathbf{k} \in \mathbb{Z}^N} \left|\hat{F}(\mathbf{k})\right| < \infty.$$

Proof. Fix $\mathbf{k} \in \mathbb{Z}^N \setminus \{0\}$ and let n be the coordinate such that $|k_n \omega_n| = ||\mathbf{k} \odot \omega||_\infty$. Then $k_n \neq 0$. Recall that

$$\hat{F}(\mathbf{k}) = \frac{\prod_{n=1}^{N} \omega_n}{(2\pi)^N} \int_0^{\frac{2\pi}{\omega_1}} e^{-ik_1\omega_1 t_1} \int_0^{\frac{2\pi}{\omega_2}} \cdots \int_0^{\frac{2\pi}{\omega_N}} F(t_1, \ldots, t_N) e^{-ik_N \omega_N t_N} dt_N \ldots dt_1.$$

Using Fubini's theorem, we can change the order so that the integral with respect to t_n is the inner most and looks like

$$\int_0^{\frac{2\pi}{\omega_n}} F(t_1, \ldots, t_N) e^{-ik_n \omega_n t_n} dt_n$$

Integrating by parts with respect to t_n yields

$$\int_0^{\frac{2\pi}{\omega_n}} F e^{-ik_n \omega_n t_n} dt_n = \left[\frac{e^{-ik_n \omega_n t_n}}{-ik_n \omega_n} F \right]_0^{\frac{2\pi}{\omega_n}} - \int_0^{\frac{2\pi}{\omega_n}} \frac{e^{-ik_n \omega_n t_n}}{-ik_n \omega_n} \partial_n F dt_n$$

where the first term on the right hand side vanishes. The second term can be further integrated using by parts to obtain

$$-\left[\frac{e^{-ik_n \omega_n t_n}}{(-ik_n \omega_n)^2} \partial_n F \right]_0^{\frac{2\pi}{\omega_n}} + \int_0^{\frac{2\pi}{\omega_n}} \frac{e^{-ik_n \omega_n t_n}}{(-ik_n \omega_n)^2} \partial_n^2 F dt_n$$

and the first term vanishes again. After a total of r such iterations, we obtain

$$\int_0^{\frac{2\pi}{\omega_n}} F e^{-ik_n \omega_n t_n} dt_n = (-1)^r \int_0^{\frac{2\pi}{\omega_n}} \frac{e^{-ik_n \omega_n t_n}}{(-ik_n \omega_n)^r} \partial_n^r F dt_n$$

which gives us upon rearrangement

$$\begin{aligned} \hat{F}(\mathbf{k}) =& \frac{\prod_{n=1}^{N} \omega_n}{(2\pi)^N} \frac{1}{(ik_n \omega_n)^r} \int_0^{\frac{2\pi}{\omega_1}} e^{-ik_1\omega_1 t_1} \int_0^{\frac{2\pi}{\omega_2}} \cdots \int_0^{\frac{2\pi}{\omega_N}} \partial_n^r F(t_1, \ldots, t_N) e^{-ik_N \omega_N t_N} dt_N \ldots dt_1 \\ =& \frac{1}{(ik_n \omega_n)^r} \widehat{\partial_n^r F}(\mathbf{k}) \end{aligned}$$

where the last step results from remark 4.2.5. Using the fact that $||\mathbf{k} \odot \omega||_2 \leq \sqrt{N} ||\mathbf{k} \odot \omega||_\infty = \sqrt{N} \mid k_n \omega_n \mid$, we obtain

$$\left| \hat{F}(\mathbf{k}) \right| \leq \frac{\sqrt{N}^r}{||\mathbf{k} \odot \omega||_2^r} \left| \widehat{\partial_n^r F}(\mathbf{k}) \right|, \tag{4.2}$$

which proves the first part of the theorem. Now, we will show that

$$\sum_{\mathbf{k}\in\mathbb{Z}^N} \left|\hat{F}(\mathbf{k})\right| < \infty,$$

By equation 4.2, we obtain

$$\sum_{\mathbf{k}\notin I_Z^N} \left|\hat{F}(\mathbf{k})\right| \leq \sum_{\mathbf{k}\notin I_Z^N} \frac{\sqrt{N}^r}{||\mathbf{k}\odot\omega||_2^r} |\widehat{\partial_n^r F}(\mathbf{k})|$$

Note that n depends on $\mathbf{k}$ and can be written as $n(\mathbf{k})$. Using Cauchy-Schwarz theorem and Parseval's identity, the right hand term can be bounded:

$$\sum_{\mathbf{k}\notin I_Z^N} \frac{\sqrt{N}^r}{|\mathbf{k}\odot\omega|^r} |\widehat{\partial_{n(\mathbf{k})}^r F}(\mathbf{k})| \leq \sqrt{N}^r \left(\sum_{\mathbf{k}\notin I_Z^N} \frac{1}{||\mathbf{k}\odot\omega||_2^{2r}}\right)^{1/2} \left(\sum_{\mathbf{k}\notin I_Z^N} |\widehat{\partial_{n(\mathbf{k})}^r F}(\mathbf{k})|^2\right)^{1/2}.$$

However, for each n, $\partial_n^r F$ is continuous, so it is square integrable and the coefficients are square summable:

$$\sum_{\mathbf{k}\in\mathbb{Z}^N} |\widehat{\partial_n^r F}(\mathbf{k})|^2 < \infty. \tag{4.3}$$

Hence, we can rearrange terms and get

$$\left(\sum_{\mathbf{k}\notin I_Z^N} |\widehat{\partial_{n(\mathbf{k})}^r F}(\mathbf{k})|^2\right)^{1/2} \leq \left(\sum_{\mathbf{k}\notin I_Z^N} \sum_{n=1}^{N} |\widehat{\partial_n^r F}(\mathbf{k})|^2\right)^{1/2} = \left(\sum_{n=1}^{N} \sum_{\mathbf{k}\notin I_Z^N} |\widehat{\partial_n^r F}(\mathbf{k})|^2\right)^{1/2}$$

where the last term is finite because of equation 4.3. Before proceeding further, we denote by J_Z^N, the box

$$\left\{\mathbf{x} = (x_1, \ldots, x_N) \in \mathbb{R}^N \;\middle|\; ||\mathbf{x}||_\infty \leq Z\right\},$$

by B_Z^N, the ball

$$\left\{\mathbf{x} = (x_1, \ldots, x_N) \in \mathbb{R}^N \;\middle|\; ||\mathbf{x}||_2 \leq Z\right\},$$

around $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^N$, and by E_Z^N, the ellipsoid-ball

$$\left\{\mathbf{y} = (y_1, \ldots, y_N) \in \mathbb{R}^N \;\middle|\; \left(\frac{y_1^2}{\omega_1^2} + \cdots + \frac{y_N^2}{\omega_N^2}\right)^{1/2} \leq Z\right\}$$

that we obtain upon a change of variables $\omega_n x_n = y_n$ for each n. Observe that

$$\left(\sum_{\mathbf{k}\notin I_Z^N} \frac{1}{||\mathbf{k}\odot\omega||_2^{2r}}\right)^{1/2} \leq \left(\int\limits_{\mathbf{x}\notin J_Z^N} \frac{1}{\left(x_1^2\omega_1^2+\cdots+x_N^2\omega_N^2\right)^r} dx_1 dx_2 \ldots dx_N\right)^{1/2}$$

$$\leq \left(\int\limits_{\mathbf{x}\notin B_Z^N} \frac{1}{\left(x_1^2\omega_1^2+\cdots+x_N^2\omega_N^2\right)^r} dx_1 dx_2 \ldots dx_N\right)^{1/2}$$

$$= \frac{1}{\left(\prod\limits_{n=1}^{N}|\omega_n|\right)^{1/2}} \left(\int\limits_{\mathbf{y}\notin E_Z^N} \frac{1}{\left(y_1^2+\cdots+y_N^2\right)^r} dy_1 dy_2 \ldots dy_N\right)^{1/2}.$$

Now, let $Z_m := Z\omega_{\mathsf{min}} = Z\min\{\omega_1,\ldots,\omega_N\}$. Then the above is bounded by

$$\frac{1}{\left(\prod\limits_{n=1}^{N}|\omega_n|\right)^{1/2}} \left(\int\limits_{\mathbf{y}\notin B_{Z_m}^N} \frac{1}{\left(y_1^2+\cdots+y_N^2\right)^r} dy_1 dy_2 \ldots dy_N\right)^{1/2},$$

which in higher dimensional spherical coordinates can be written as

$$\frac{1}{\left(\prod\limits_{n=1}^{N}|\omega_n|\right)^{1/2}} \left(\int\limits_{\sigma}\int\limits_{Z_m}^{\infty} \frac{1}{(\rho)^{2r}}\rho d\rho d\sigma\right)^{1/2}$$

where $d\sigma$ is the differential surface area of unit sphere S^{N-1} and ρ is the distance of a point from the center. Evaluating the integrals yields

$$\frac{1}{\left(\prod\limits_{n=1}^{N}|\omega_i|\right)^{1/2}} \left(Area(S^{N-1})\right)^{1/2} \left(\int\limits_{Z_m}^{\infty} \frac{1}{(\rho)^{2r-1}} d\rho\right)^{1/2}$$

$$= \frac{1}{\left(\prod\limits_{n=1}^{N}|\omega_i|\right)^{1/2}} \left(Area(S^{N-1})\right)^{1/2} \left(\left.\frac{\rho^{2-2r}}{2-2r}\right|_{Z_m}^{\infty}\right)^{1/2}$$

$$= \frac{1}{\left(\prod\limits_{n=1}^{N}|\omega_i|\right)^{1/2}} \left(Area(S^{N-1})\right)^{1/2} \frac{Z_m^{1-r}}{\sqrt{2r-2}}.$$

Summarizing, we obtain

$$\sum_{\mathbf{k}\notin I_Z^N} \left|\hat{F}(\mathbf{k})\right| \leq \frac{\sqrt{N}^r \left(Area(S^{N-1})\right)^{1/2}}{\omega_{\min}^{r-1}\sqrt{2r-2}\left(\prod_{n=1}^{N}|\omega_n|\right)^{1/2}} \left(\sum_{n=1}^{N}\sum_{\mathbf{k}\notin I_Z^N} |\widehat{\partial_n^r F}(\mathbf{k})|^2\right)^{1/2} \frac{1}{Z^{r-1}} \tag{4.4}$$

where the right hand side is finite if $Z > 0$ and $r \in \mathbb{N}$. Since I_Z^N has finitely many terms, we obtain that

$$\sum_{\mathbf{k}\in\mathbb{Z}^N} \left|\hat{F}(\mathbf{k})\right| < \infty. \tag{4.5}$$

□

Therefore, due to the absolute convergence of the coefficients, equation 4.1 can be rewritten as:

$$F(t_1, \dots, t_N) = \sum_{\mathbf{k}\in\mathbb{Z}^N} \hat{F}(\mathbf{k}) e^{i(k_1\omega_1 t_1 + \cdots + k_N\omega_N t_N)},$$

where $\mathbf{k} = (k_1, \dots, k_N) \in \mathbb{Z}^N$. Along the diagonal, we get an iterated Fourier series for the quasiperiodic function f:

$$f(t) = F(t, \dots, t) = \sum_{\mathbf{k}\in\mathbb{Z}^N} \hat{F}(\mathbf{k}) e^{it\sum_{n=1}^{N} k_n\omega_n} = \sum_{\mathbf{k}\in\mathbb{Z}^N} \hat{F}(\mathbf{k}) e^{it\langle \mathbf{k},\omega\rangle} \tag{4.6}$$

where $\omega = (\omega_1, \dots, \omega_N) \in \mathbb{R}^N$ and $\langle\cdot,\cdot\rangle$ is the inner product (dot product) on $\mathbb{R}^N$. Moreover, the series in equation 4.6 converges everywhere. Next, we move to sliding window embedding of the function f.

4.2.3 Approximation theorems for sliding window embeddings

From Definition 2.4.4 and Equation 4.6, the sliding window embedding of a quasiperiodic function f at t with parameters d and τ is:

$$SW_{d,\tau}f(t) = \begin{bmatrix} f(t) \\ f(t+\tau) \\ \vdots \\ f(t+d\tau) \end{bmatrix} = \begin{bmatrix} \sum\limits_{\mathbf{k}\in\mathbb{Z}^N} \hat{F}(\mathbf{k})e^{i\langle \mathbf{k},\omega\rangle t} \\ \sum\limits_{\mathbf{k}\in\mathbb{Z}^N} \hat{F}(\mathbf{k})e^{i\langle \mathbf{k},\omega\rangle (t+\tau)} \\ \vdots \\ \sum\limits_{\mathbf{k}\in\mathbb{Z}^N} \hat{F}(\mathbf{k})e^{i\langle \mathbf{k},\omega\rangle (t+d\tau)} \end{bmatrix} = \sum_{\mathbf{k}\in\mathbb{Z}^N} \hat{F}(\mathbf{k})e^{i\langle \mathbf{k},\omega\rangle t}\mathbf{u_k}$$

where $\mathbf{u_k}$ is the vector

$$\mathbf{u_k} = \begin{bmatrix} 1 \\ e^{i\langle \mathbf{k},\omega\rangle (\tau)} \\ \vdots \\ e^{i\langle \mathbf{k},\omega\rangle (d\tau)} \end{bmatrix}.$$

Remark 4.2.7. We make a few observations.

1. $\mathbf{u_k} = SW_{d,\tau}e^{it\langle \mathbf{k},\omega\rangle} \mid_{t=0}$.
2. If $\mathbf{k} = \mathbf{l}$, then $\langle \mathbf{u_k}, \mathbf{u_l}\rangle = d+1$.
3. $\langle \mathbf{k},\omega\rangle = 0$ if and only if $k_1 = \cdots = k_N = 0$
4. $\langle \mathbf{k},\omega\rangle = \langle \mathbf{l},\omega\rangle$ if and only if $\mathbf{k} = \mathbf{l}$.

For $Z \in \mathbb{N}_0$, we define the polynomial $S_Z f$ by truncating the series:

$$S_Z f(t) := \sum_{\mathbf{k}\in I_Z^N} \hat{F}(\mathbf{k})e^{i\langle \mathbf{k},\omega\rangle t},$$

where I_Z^N is the box

$$I_Z^N = \left\{ \mathbf{k} \in \mathbb{Z}^N \;\middle|\; ||\mathbf{k}||_\infty \le Z \right\}.$$

Then $S_Z f(t)$ converges to $f(t)$ for each $t \in \mathbb{R}$ as $Z \to \infty$. Since $SW_{d,\tau}$ is a bounded linear operator (and hence, continuous) by Proposition 2.4.8, $SW_{d,\tau} S_Z f$ converges to $SW_{d,\tau} f$ pointwise as well. Let

$$R_Z f(t) := f(t) - S_Z f(t) = \sum_{\mathbf{k} \notin I_Z^N} \hat{F}(\mathbf{k}) e^{i\langle \mathbf{k}, \omega \rangle t}$$

be the remainder term of polynomial $S_Z f(t)$. Then the following result shows that the bound on the sliding window embedding of the remainder term $R_Z f$ is controlled by the regularity of the function f. In particular, the more differentiable f is, the faster is the rate of convergence.

Theorem 4.2.8. *Let $r \in \mathbb{N} \setminus \{1\}$. If $F \in C^r\left(\mathbb{R}^N, \mathbb{C}\right)$, then for all $t \in \mathbb{R}$*

$$\begin{aligned}
&\left|\left|SW_{d,\tau} f(t) - SW_{d,\tau} S_Z f(t)\right|\right|_2 \\
&\leq \frac{\sqrt{N}^r \left(Area(S^{N-1})\right)^{1/2}}{\omega_{\min}^{r-1}\sqrt{2r-2}\left(\prod_{n=1}^{N}|\omega_n|\right)^{1/2}} \left(\sum_{n=1}^{N} ||R_Z(\partial_n^r F)||^2\right)^{1/2} \frac{\sqrt{d+1}}{Z^{r-1}}.
\end{aligned}$$

Proof. We begin with the remainder term $R_Z f$:

$$\begin{aligned}
|R_Z f(t)| &= \left|\sum_{\mathbf{k} \notin I_Z^N} \hat{F}(\mathbf{k}) e^{i\langle \mathbf{k}, \omega \rangle t}\right| \leq \sum_{\mathbf{k} \notin I_Z^N} \left|\hat{F}(\mathbf{k})\right| \\
&\leq \frac{\sqrt{N}^r \left(Area(S^{N-1})\right)^{1/2}}{\omega_{\min}^{r-1}\sqrt{2r-2}\left(\prod_{n=1}^{N}|\omega_n|\right)^{1/2}} \left(\sum_{n=1}^{N} \sum_{\mathbf{k} \notin I_Z^N} |\widehat{\partial_n^r F}(\mathbf{k})|^2\right)^{1/2} \frac{1}{Z^{r-1}} \\
&= \frac{\sqrt{N}^r \left(Area(S^{N-1})\right)^{1/2}}{\omega_{\min}^{r-1}\sqrt{2r-2}\left(\prod_{n=1}^{N}|\omega_n|\right)^{1/2}} \left(\sum_{n=1}^{N} ||R_Z(\partial_n^r F)||_2^2\right)^{1/2} \frac{1}{Z^{r-1}}
\end{aligned}$$

where the second inequality comes from Equation 4.4. Therefore,

$$\begin{aligned}
&\left|\left|SW_{d,\tau} f(t) - SW_{d,\tau} S_Z f(t)\right|\right|_2 \\
\leq &\sqrt{d+1}||R_Z f||_\infty \leq \frac{\sqrt{N}^r \left(Area(S^{N-1})\right)^{1/2}}{\omega_{\min}^{r-1}\sqrt{2r-2}\left(\prod_{n=1}^{N}|\omega_n|\right)^{1/2}} \left(\sum_{n=1}^{N} ||R_Z(\partial_n^r F)||_2^2\right)^{1/2} \frac{\sqrt{d+1}}{Z^{r-1}}.
\end{aligned}$$

□

Corollary 4.2.9. *If* $X = SW_{d,\tau}f(T)$ *and* $Y = SW_{d,\tau}S_Zf(T)$ *for* $T \subset \mathbb{R}$*, then*

$$d_B\left(\mathsf{bcd}_i^{\mathcal{R}}(X), \mathsf{bcd}_i^{\mathcal{R}}(Y)\right) \leq 2\xi$$

for

$$\xi = \frac{\sqrt{N}^r\left(Area(S^{N-1})\right)^{1/2}}{\omega_{\mathsf{min}}^{r-1}\sqrt{2r-2}\left(\prod_{n=1}^{N}|\omega_n|\right)^{1/2}}\left(\sum_{n=1}^{N}||R_Z(\partial_n^r F)||_2^2\right)^{1/2}\frac{\sqrt{d+1}}{Z^{r-1}}$$

Proof. Let $\epsilon > \xi$, then by theorem 4.2.8, $Y \subset X^\epsilon$ and $X \subset Y^\epsilon$ and therefore $d_H(X,Y) \leq \epsilon$. Since this is true for every $\epsilon > \xi$, this gives us $d_H(X,Y) \leq \xi$, which from the stability theorem [19] gives us

$$d_B\left(\mathsf{bcd}_i^{\mathcal{R}}(X), \mathsf{bcd}_i^{\mathcal{R}}(Y)\right) \leq 2\xi.$$

□

Corollary 4.2.9 implies that the Rips persistent homology barcodes $\mathsf{bcd}_i^{\mathcal{R}}(SW_{d,\tau}S_Zf)$ converge to $\mathsf{bcd}_i^{\mathcal{R}}(SW_{d,\tau}f)$ as $Z \to \infty$. This suggests that one could replace the study of a quasiperiodic function f with that of its approximation S_Zf. Accordingly, we study the geometric structure of the sliding window embedding of the truncated polynomial S_Zf next.

4.3 The geometric structure of $SW_{d,\tau}S_Zf$

We dedicate this section to showing that under appropriate conditions on d, τ, and Z, the closure of the sliding window embedding of the truncated polynomial S_Zf, is homeomorphic to the N-dimensional torus. To begin, let α be the cardinality of support of the coefficient function

$$\hat{F} : I_Z^N \to \mathbb{C}$$
$$\mathbf{k} \to \hat{F}(\mathbf{k})$$

where I_Z^N is the box $\left\{\mathbf{k} \in \mathbb{Z}^N \;\middle|\; ||\mathbf{k}||_\infty \leq Z\right\}$. Observe that $0 \leq \alpha \leq (2Z+1)^N$. Choose an enumeration of points $\mathbf{k} \in \mathsf{Supp}(\hat{F}) \subset I_Z^N$, say $\mathbf{k}_1, \cdots, \mathbf{k}_\alpha$. Let $x_{Z,f}(t)$ be a $\alpha \times 1$ vector given by

$$x_{Z,f}(t) = \begin{bmatrix} \hat{F}(\mathbf{k}_1)e^{i\langle \mathbf{k}_1, \omega\rangle t} \\ \vdots \\ \hat{F}(\mathbf{k}_\alpha)e^{i\langle \mathbf{k}_\alpha, \omega\rangle t}. \end{bmatrix}.$$

Next, we present a corollary to Theorem 2.5.2 that we will utilize to prove the main result (Theorem 4.3.5) of this section.

Corollary 4.3.1 (a corollary to Kronecker's theorem)**.** *An α-tuple $\beta = (\beta_1, \ldots, \beta_\alpha)$ of real numbers spans a N-dimensional vector space over $\mathbb{Q}$ if and only if the collection $\{z\beta \mid z \in \mathbb{Z}\}$ is dense in an embedding of $\mathbb{T}^N$ in $\mathbb{T}^\alpha$, which is equivalent to $\left\{(e^{i\beta_1 z}, \cdots, e^{\beta_\alpha z}) \;\middle|\; z \in \mathbb{Z}\right\}$ being dense in the image of a homeomorphism from $\mathbb{T}^N$ to $\mathbb{T}^\alpha$.*

Proof. Let $\{\beta_1, \ldots, \beta_\alpha\}$ span a N dimensional vector space over $\mathbb{Q}$. Without loss of generality, let us assume that the $\beta_1, \ldots, \beta_N$ form a basis of said vector space. Then for $N+1 \leq j \leq \alpha$, β_j depends linearly on $\beta_1, \ldots, \beta_N$, i.e. for appropriate coefficients $c_{j,n} \in \mathbb{Q}$, we have

$$\beta_j = \sum_{n=1}^{N} c_{j,n}\beta_n.$$

We want to show that $\{z\beta \mid z \in \mathbb{Z}\}$ is dense in a space homeomorphic to $\mathbb{T}^N = (\mathbb{R}/2\pi\mathbb{Z})^N$ in $\mathbb{T}^\alpha = (\mathbb{R}/2\pi\mathbb{Z})^\alpha$. It is equivalent to showing that

$$\left\{(e^{i\beta_1 z}, \cdots, e^{i\beta_\alpha z}) \;\middle|\; z \in \mathbb{Z}\right\}$$

is dense in a space homeomorphic to $\mathbb{T}^N$ in $\mathbb{T}^\alpha$. Observe that

$$\beta_j = \sum_{n=1}^{N} c_{j,n}\beta_n$$

if and only if

$$e^{i\beta_j t} = (e^{i\beta_1 t})^{c_{j,1}} \cdots (e^{i\beta_N t})^{c_{j,N}}.$$

Define a map ϕ from $\mathbb{T}^N \to \mathbb{T}^\alpha$ by

$$\phi(z_1, \ldots, z_N) = (z_1, \ldots, z_N, \tilde{z}_{N+1}, \ldots, \tilde{z}_\alpha), \tag{4.7}$$

where $\tilde{z}_j = z_1^{c_{j,1}} z_2^{c_{j,2}} \cdots z_N^{c_{j,N}}$. Observe that the set

$$\left\{(e^{i\beta_1 z}, \cdots, e^{i\beta_\alpha z}) \;\middle|\; z \in \mathbb{Z}\right\} \subset \phi(\mathbb{T}^N) \subseteq \mathbb{T}^\alpha.$$

By definition, ϕ is injective, continuous, and it is vacuously surjective onto its own image. The inverse from the image $\phi(\mathbb{T}^N)$ to $\mathbb{T}^N$ is just the projection map and that is continuous. Therefore, ϕ is a homeomorphism. In particular, it is continuous and surjective (onto $\phi(\mathbb{T}^N)$) and since the set

$$\left\{(e^{i\beta_1 z}, \cdots, e^{i\beta_N z}) \;\middle|\; z \in \mathbb{Z}\right\}$$

is dense in $\mathbb{T}^N$ by Theorem 2.5.2, its image under ϕ,

$$\left\{(e^{i\beta_1 z}, \cdots, e^{\beta_\alpha z}) \;\middle|\; z \in \mathbb{Z}\right\}$$

is dense in $\phi(\mathbb{T}^N)$. This proves the 'if' part of our claim.

For the other direction, let $\{z\beta \mid z \in \mathbb{Z}\}$ be dense in $\phi(\mathbb{T}^N) \subset \mathbb{T}^\alpha$ for some homeomorphism ϕ and let $(\beta_1, \ldots, \beta_\alpha)$ span an L-dimensional vector space for $L \neq N$. From the first part of the proof, we obtain that $\{z\beta \mid z \in \mathbb{Z}\}$ is dense in $\varphi(\mathbb{T}^L) \subset \mathbb{T}^\alpha$ for some homeomorphism φ. Therefore,

$$\varphi\left(\mathbb{T}^L\right) = \overline{\{z\beta \mid z \in \mathbb{Z}\}} = \phi(\mathbb{T}^N)$$

and since homeomorphism is an equivalence relation, we get that $\mathbb{T}^L \simeq \mathbb{T}^N$, which is a contradiction. This finishes the proof. □

Remark 4.3.2. From the definition of ϕ in the proof of Corollary 4.3.1, we observe that for any two α-tuples β and $\tilde{\beta}$ spanning a N dimensional vector space over $\mathbb{Q}$, the closure of $\mathbb{Z}\beta$ and $\mathbb{Z}\tilde{\beta}$ depends only on how the dependent β_j's (and $\tilde{\beta}_j$'s) depend on the basis of N linear independent reals, instead of depending the actual values β_j's (respectively $\tilde{\beta}_j$'s). Assume that the first N reals in both tuples form linearly independent sets and for $N+1 \le j \le \alpha$,

$$\beta_j = \sum_{n=1}^{N} c_{j,n}\beta_n \quad \text{and} \quad \tilde{\beta}_j = \sum_{n=1}^{N} c_{j,n}\tilde{\beta}_n$$

for the same dependency coefficients $c_{j,n} \in \mathbb{Q}$. Then both $\mathbb{Z}\beta$ and $\mathbb{Z}\tilde{\beta}$ are dense in $\phi(\mathbb{T}^N)$ for

$$\phi(z_1, \dots, z_N) = (z_1, \dots, z_N, \tilde{z}_{N+1}, \dots, \tilde{z}_\alpha),$$

where $\tilde{z}_j = z_1^{c_{j,1}} z_2^{c_{j,2}} \cdots z_N^{c_{j,N}}$.

For a better intuition of Corollary 4.3.1, see figure 4.1. We begin with four triples of real numbers: the first is equivalent to a pair $\{\sqrt{2}, \sqrt{3}\}$, the second is a triple $\{\sqrt{2}, \sqrt{3}, \sqrt{2}+\sqrt{3}\}$, the third is a triple $\{\sqrt{2}, \sqrt{3}, 3\sqrt{2}+2\sqrt{3}\}$, and the last is the triple $\{\sqrt{2}, \sqrt{3}, \pi\}$. For each of these tuples β, we compute $\mathbb{Z}\beta$:

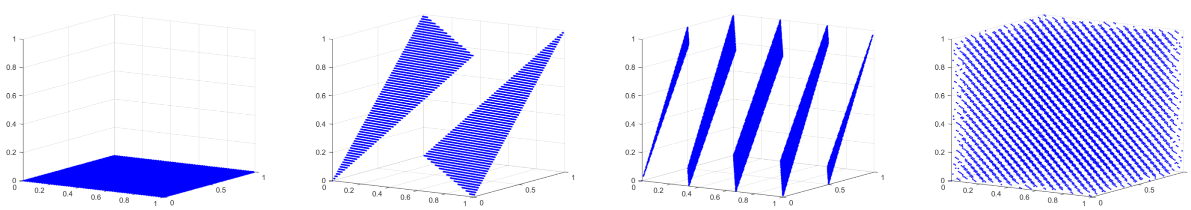


Figure 4.1: The sets $\mathbb{Z}\beta$ for tuples $\beta = (\sqrt{2}, \sqrt{3}) \equiv (\sqrt{2}, \sqrt{3}, 0)$ (left), $\beta = (\sqrt{2}, \sqrt{3}, \sqrt{2}+\sqrt{3})$ (second), $\{\sqrt{2}, \sqrt{3}, 3\sqrt{2}+2\sqrt{3}\}$ (third), and $\beta = (\sqrt{2}, \sqrt{3}, \pi)$ (right).

The next lemma shows that if $Z \in \mathbb{N}_0$ is large enough such that $\mathsf{Supp}(\hat{F})$ contains N linearly independent vectors, then the set $\mathbb{X}_{Z,f} = \{x_{Z,f}(s) \mid s \in \mathbb{Z}\}$ is dense in an embedding of $\mathbb{T}^N$ in $\mathbb{T}^\alpha$. For general Z, if $M \le N$ is the largest cardinality of a linearly independent set of vectors in $\mathsf{Supp}(\hat{F})$, then $\mathbb{X}_{Z,f}$ is dense an in embedding of $\mathbb{T}^M$ in $\mathbb{T}^\alpha$.

Lemma 4.3.3. *If $Z \in \mathbb{N}_0$ is large enough such that $\mathsf{Supp}(\hat{F})$ contains N linearly independent vectors, then the set $\mathbb{X}_{Z,f}$ is dense in a space homeomorphic to a torus $\mathbb{T}^N$ embedded in*

$$\mathbb{T}^\alpha_{\hat{F}} = \bigtimes_{\mathbf{k} \in \mathsf{Supp}(\hat{F})} S^1_{|\hat{F}(\mathbf{k})|}.$$

Proof. Recall that the exponential functions e^{iat} and e^{ibt} are equal if and only if $a = b$. Moreover, recall from Remark 4.2.7 that the inner product values $\langle \mathbf{k}, \omega \rangle$ are different for distinct vectors $\mathbf{k}$. By the condition on Z, there exist N linearly independent vectors in $\mathsf{Supp}(\hat{F})$, say $\mathbf{k}_1, \ldots, \mathbf{k}_N$. Then for all $j \in \mathsf{Supp}(\hat{F})$,

$$\mathbf{k}_j := \sum_{n=1}^{N} c_{j,n} \mathbf{k}_n$$

for $c_{j,n} \in \mathbb{Q}$. Observe that the linearity of inner products implies that for all $\mathbf{k}_j \in \mathsf{Supp}(\hat{F})$,

$$\langle \mathbf{k}_j, \omega \rangle = \sum_{n=1}^{N} c_{j,n} \langle \mathbf{k}_n, \omega \rangle.$$

Therefore the tuple $\beta = (\langle \mathbf{k}_1, \omega \rangle, \ldots, \langle \mathbf{k}_\alpha, \omega \rangle)$ spans an N-dimensional vector space over $\mathbb{Q}$. From Corollary 4.3.1, we get that the collection $\left\{ (e^{i\beta_1 z}, \cdots, e^{\beta_\alpha z}) \;\middle|\; z \in \mathbb{Z} \right\}$ is dense in an embedding of $\mathbb{T}^N$ in $\mathbb{T}^\alpha$. Since the multilinear scaling given by

$$\mathbb{T}^\alpha \to \bigtimes_{\mathbf{k} \in \mathsf{Supp}(\hat{F})} S^1_{|\hat{F}(\mathbf{k})|}$$

$$(z_1, \cdots, z_\alpha) \to (\hat{F}(\mathbf{k}_1) z_1, \cdots, \hat{F}(\mathbf{k}_\alpha) z_\alpha)$$

is a homeomorphism, it follows that $\mathbb{X}_{Z,f}$ is dense in an embedding of $\mathbb{T}^N$ in $\bigtimes_{\mathbf{k} \in \mathsf{Supp}(\hat{F})} S^1_{|\hat{F}(\mathbf{k})|}$. ☐

Now, let $\Omega_{Z,f}$ be the $(d+1) \times \alpha$ matrix

$$\Omega_{Z,f} = \begin{bmatrix} e^{i\langle \mathbf{k}_1, \omega \rangle \tau 0} & \ldots & e^{i\langle \mathbf{k}_\alpha, \omega \rangle \tau 0} \\ \vdots & \vdots & \vdots \\ e^{i\langle \mathbf{k}_1, \omega \rangle \tau d} & \ldots & e^{i\langle \mathbf{k}_\alpha, \omega \rangle \tau d} \end{bmatrix} \tag{4.8}$$

comprised of vectors $\mathbf{u}_{\mathbf{k}_j}$ as its columns. Observe that for each $t \in \mathbb{R}$,

$$SW_{d,\tau} S_Z f(t) = \Omega_{Z,f} \cdot x_{Z,f}(t).$$

When we go from $\mathbb{X}_{Z,f}$ to $SW_{d,\tau} S_Z f$, the toroidal geometry is minimally perturbed if the columns $\mathbf{u}_{\mathbf{k}_j}$ of $\Omega_{Z,f}$ are mutually orthogonal. Since $\alpha > N$ typically, it is not possible for all pairs to be orthogonal to each other. However, we can make these as orthogonal as possible by controlling their inner products. Suppose the embedding dimension d has been chosen. For any two $\mathbf{k}, \mathbf{l} \in \mathsf{Supp}(\hat{F})$, the inner product $\langle \mathbf{u_k}, \mathbf{u_l} \rangle$ is computed as

$$\langle \mathbf{u_k}, \mathbf{u_l} \rangle = 1 + e^{i\tau\langle \mathbf{k}-\mathbf{l}, \omega \rangle} + \cdots + e^{i\tau\langle \mathbf{k}-\mathbf{l}, \omega \rangle d}.$$

The mutual orthogonality is maximized when the following objective function $G : \mathbb{R} \to \mathbb{R}$ defined as

$$G(x) = \sum_{\mathbf{k} \neq \mathbf{l}} \left| 1 + e^{ix\langle \mathbf{k}-\mathbf{l}, \omega \rangle} + \cdots + e^{ix\langle \mathbf{k}-\mathbf{l}, \omega \rangle d} \right|^2$$

is minimized and ideally, we would like to choose $\tau = \arg\min_x G(x)$. There are two obstacles in solving this optimization problem. The first, which could be considered a blessing in disguise is that there might not be a global minimum, that is when there is a sequence of local minimas converging to $G(x) = 0$. The second is when the global minimum exists, but is too large to be ever useful in applications. The problem then reduces to finding a reasonable local minima, where the definition of "reasonable" depends on the size and nature of the data. In general, one can use peak finding algorithms to detect these local minima. For the purpose of this dissertation, we use the findpeaks($\cdot$) function in MATLAB. We add one more condition for τ to satisfy: $G(\tau) < (d+1)^2$. This condition is sufficient to argue that no two vectors $\mathbf{u_k}, \mathbf{u_l}$ will be parallel two each other, and will allow us to prove the structure theorem. Next, we provide an example for determining a good choice of τ:

Example 4.3.4. Let the quasiperiodic function of interest be $f(t) = e^{it} + e^{i\sqrt{3}t} + e^{i(\sqrt{3}+1)t}$. Clearly, the set $\{1, \sqrt{3}, 1+\sqrt{3}\}$ is not incommensurate and in the language developed above, one can write it as

$$f(t) = e^{i\langle (1,0),(1,\sqrt{3})\rangle t} + e^{i\langle (0,1),(1,\sqrt{3})\rangle t} + e^{i\langle (1,1),(1,\sqrt{3})\rangle t}.$$

Then the objective function corresponding to this f is

$$\begin{aligned} G(x) = &\left| 1 + e^{ix\sqrt{3}} + \cdots + e^{ix\sqrt{3}3} \right|^2 + \left| 1 + e^{ix} + \cdots + e^{ix3} \right|^2 \\ &+ \left| 1 + e^{ix(\sqrt{3}-1)} + \cdots + e^{ix(\sqrt{3}-1)3} \right|^2 \end{aligned}$$

where we have chosen the dimension $d = 3$, the reasons for which will be clear in Theorem 4.3.5. The graph of the objective function G in Figure 4.2 suggests several local minimas at $x \in [0, 40]$. To estimate these, we use the findpeaks$(\cdot)$ function in MATLAB on $-G(x)$. While in practice we will only concern ourselves in the global minima in the chosen range $[0, 40]$, we will analyze the top few choices for the purpose of this example. To make the process faster, one can insert a threshold on the function $-G(x)$. In table 4.1, we find all peaks of $-G$ that are higher than -0.5, i.e. all local minimas lower than 0.5, along with angles $\theta_{\mathbf{k},\mathbf{l}}$ (in degrees) between $\mathbf{u_k}$ and $\mathbf{u_l}$ for each of the values of x.

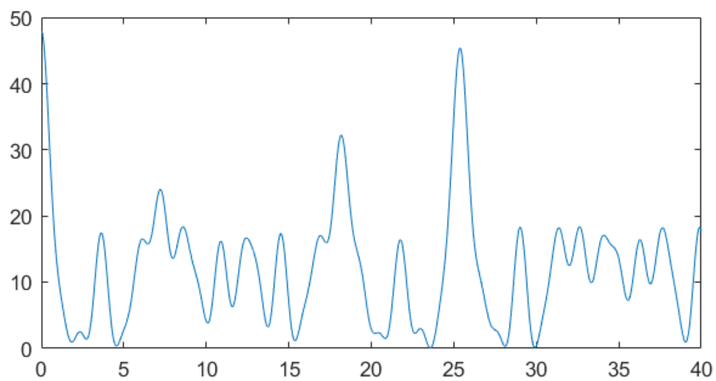


Figure 4.2: Example: The objective function $G(x)$ for finding an optimal τ, for the function $f(t) = e^{it} + e^{i\sqrt{3}t} + e^{i(\sqrt{3}+1)t}$.

x	$G(x)$	$\theta_{(1,0),(1,1)}$	$\theta_{(0,1),(1,1)}$	$\theta_{(1,0),(0,1)}$
4.5623	0.3204	88.0983	84.4038	84.4327
23.5718	0.0056	89.6226	89.5993	89.0784
28.1141	0.2717	89.9883	85.4674	84.0524
29.9150	0.0791	89.0968	87.0751	87.3789

Table 4.1: Example of τ estimation: Candidates for τ, along with the corresponding local minimums, and the angles $\theta_{\mathbf{k},\mathbf{l}}$ (in degrees) between $\mathbf{u_k}$ and $\mathbf{u_l}$

Clearly, the best choice for τ is 23.5718 out of the 4 values in the table. However, if in a scenario, it is too big to be useful, then one would have to use $\tau = 4.5623$. We can also see that for all x in the table, $G(x) \ll (d+1)^2 = 16$.

Having established how to choose the delay parameter τ, we now prove the main result of this section, that is, the structure theorem:

Theorem 4.3.5. *The sliding window embedding* $\left\{SW_{d,\tau}S_Z f(z) \mid z \in \mathbb{Z}\right\}$ *is dense in a space homeomorphic to* $\mathbb{T}^N$ *in* $\mathbb{T}^{d+1}$ *if* $d+1 \geq \alpha$ *and if* τ *satisfies that* $G(\tau) < (d+1)^2$.

Proof. Assume on the contrary that the matrix $\Omega_{Z,f}$ is not full rank. Let $L = \min(d+1, \alpha)$ and let A be the $L \times L$ upper left block of $\Omega_{Z,f}$. Since $\mathsf{rank}\left(\Omega_{Z,f}\right) < L$, we obtain that $\det(A) = 0$. This implies that the rows of A are linearly dependent, so for some $c_0, \ldots, c_{L-1} \in \mathbb{C}$, not all zero, we have

$$\sum_{l=0}^{L-1} c_l e^{i\langle \mathbf{k}_j, \omega\rangle \tau l} = 0$$

for all $1 \leq j \leq L$. In other words, for each $1 \leq j \leq L$, $e^{i\langle \mathbf{k}_j, \omega\rangle \tau}$ is a root of the polynomial $P(z) = c_0 + c_1 z + \cdots + c_{L-1} z^{L-1}$. By remark 4.2.7, all $\langle \mathbf{k}_j, \omega\rangle$ are distinct. We claim that the assumption on τ implies that all $\langle \mathbf{k}_j, \omega\rangle \tau$ are distinct modulo 2π, because if for some $\mathbf{k}_i \neq \mathbf{k}_j$, the following is true

$$e^{i\langle \mathbf{k}_i, \omega\rangle \tau} = e^{i\langle \mathbf{k}_j, \omega\rangle \tau}$$

then we would obtain $G(\tau) \geq \left|\left\langle \mathbf{u}_{\mathbf{k}_i}, \mathbf{u}_{\mathbf{k}_j}\right\rangle\right|^2 = (d+1)^2$ which would contradict our assumption. This implies that $P(z)$ has L distinct roots, and this further contradicts that P is a polynomial of degree $L-1$. Therefore, A is invertible and hence, $\Omega_{Z,f}$ has full rank. Concluding, if the embedding dimension $d+1 \geq \alpha$, then the sliding window embedding $SW_{d,\tau}S_Z f(\mathbb{Z})$ is dense in a space homeomorphic to $\mathbb{T}^N$ embedded inside $\mathbb{T}^{d+1}$. □

We would like to remark that the condition on τ in the above theorem only guarantees the topology of a N-torus. Preserving the geometric structure as much as possible requires choosing τ by optimizing G.

4.4 Persistent Homology

In this section, we present results about the Rips persistent homology of the sliding window embedding. The strategy we use is to construct a composite Lipschitz map

$$SW_{d,\tau}S_Z f \xrightarrow{\Omega_{Z,f}^{-1}} \mathbb{X}_{Z,f} \xrightarrow{\pi} \mathbb{T}^N$$

where the map $\pi : \mathbb{X}_{Z,f} \to \mathbb{T}^N$ is a projection onto a standard N-dimensional torus given by

$$\pi\left(\begin{bmatrix} \hat{F}(\mathbf{k}_1)e^{i\langle \mathbf{k}_1,\omega\rangle t} \\ \vdots \\ \hat{F}(\mathbf{k}_N)e^{i\langle \mathbf{k}_N,\omega\rangle t} \\ \hat{F}(\mathbf{k}_{N+1})e^{i\langle \mathbf{k}_{N+1},\omega\rangle t} \\ \vdots \\ \hat{F}(\mathbf{k}_\alpha)e^{i\langle \mathbf{k}_\alpha,\omega\rangle t} \end{bmatrix}\right) = \begin{bmatrix} e^{i\langle \mathbf{k}_1,\omega\rangle t} \\ \vdots \\ e^{i\langle \mathbf{k}_n,\omega\rangle t} \\ \vdots \\ e^{i\langle \mathbf{k}_N,\omega\rangle t} \end{bmatrix}$$

where after reordering $\{\mathbf{k}_1, \ldots, \mathbf{k}_N\}$ is a linearly independent set over $\mathbb{Q}$. We will first show that π is a Lipschitz function:

Proposition 4.4.1. *The projection map* π

$$\pi : \left(\mathbb{X}_{Z,f}, ||\cdot||_\infty\right) \to \left(\mathbb{T}^N, ||\cdot||_\infty\right)$$

is Lipschitz, with Lipschitz constant $\frac{1}{|\hat{F}(\mathbf{k})_{\mathsf{min}}|}$ *where* $|\hat{F}(\mathbf{k})_{\mathsf{min}}| := \min\limits_{\mathbf{k}\in \mathsf{Supp}(\hat{F})} \left|\hat{F}(\mathbf{k})\right|$.

Proof. For any two $s, t \in \mathbb{Z}$, we have

$$\begin{aligned}
\left\|\pi\left(x_{Z,f}(t)\right)-\pi\left(x_{Z,f}(s)\right)\right\|_\infty &= \max_{\mathbf{k}_1,\cdots,\mathbf{k}_N}\left|e^{i\langle\mathbf{k}_n,\omega\rangle t}-e^{i\langle\mathbf{k}_n,\omega\rangle s}\right| \\
&= \max_{\mathbf{k}_1,\cdots,\mathbf{k}_N}\frac{1}{|\hat{F}(\mathbf{k})_{\mathsf{min}}|}|\hat{F}(\mathbf{k})_{\mathsf{min}}|\left|e^{i\langle\mathbf{k}_n,\omega\rangle t}-e^{i\langle\mathbf{k}_n,\omega\rangle s}\right| \\
&\le \frac{1}{|\hat{F}(\mathbf{k})_{\mathsf{min}}|}\max_{\mathbf{k}_1,\cdots,\mathbf{k}_N}|\hat{F}(\mathbf{k})|\left|e^{i\langle\mathbf{k}_n,\omega\rangle t}-e^{i\langle\mathbf{k}_n,\omega\rangle s}\right| \\
&\le \frac{1}{|\hat{F}(\mathbf{k})_{\mathsf{min}}|}\max_{\mathbf{k}\in I_Z^N}|\hat{F}(\mathbf{k})|\left|e^{i\langle\mathbf{k}_n,\omega\rangle t}-e^{i\langle\mathbf{k}_n,\omega\rangle s}\right| \\
&= \frac{1}{|\hat{F}(\mathbf{k})_{\mathsf{min}}|}\left\|x_{Z,f}(t)-x_{Z,f}(s)\right\|_\infty .
\end{aligned}$$

This finishes the proof. □

It is important to note that the Lipschitz constant $\frac{1}{|\hat{F}(\mathbf{k})_{\mathsf{min}}|}$ is independent of choice of linearly independent vectors $\{\mathbf{k_1}, \cdots, \mathbf{k_N}\}$.

Next, we will use theorem 3.4.3 and proposition 4.4.1 to prove a lower bound on the 1-dimensional Rips persistence of $\mathbb{X}_{Z,f}$. Assume without a loss of generality that $\{\mathbf{k}_1, \cdots, \mathbf{k}_N\}$ is a linearly independent set over $\mathbb{Q}$. Hence, for all $1 \le j \le \alpha$, there exist rational coefficients $c_{j,n} \in \mathbb{Q}$ such that

$$\mathbf{k}_j = \sum_{n=1}^{N} c_{j,n}\mathbf{k}_n$$

and consequently,

$$\beta_j := \langle\mathbf{k}_j, \omega\rangle = \left\langle\sum_{n=1}^{N} c_{j,n}\mathbf{k}_n, \omega\right\rangle = \sum_{n=1}^{N} c_{j,n}\langle\mathbf{k}_n, \omega\rangle =: \sum_{n=1}^{N} c_{j,n}\beta_n.$$

Define a map from $\mathbb{T}^N$ to $\mathbb{T}^\alpha_{\hat{F}}$:

$$\phi : \mathbb{T}^N \to \mathbb{T}^\alpha_{\hat{F}}$$

$$(e^{i\beta_1 t_1}, \cdots, e^{i\beta_N t_N}) \to (z_1, \ldots, z_\alpha)$$

where $z_j = \hat{F}(\mathbf{k}_j)e^{i\sum_{n=1}^{N} c_{j,n}\beta_n t_n}$.

Lemma 4.4.2. *If the prime p in $\mathbb{Z}_p$ satisfies*

$$\max_n \mathsf{lcm}\left(\frac{1}{c_{1,n}}, \frac{1}{c_{2,n}}, \cdots, \frac{1}{c_{\alpha,n}}\right) < p,$$

then there will be at least N bars of the type

$$(0, \rho_1), (0, \rho_2), \ldots, (0, \rho_N)$$

where $\rho_n \geq \sqrt{3}|\hat{F}(\mathbf{k})_{\mathsf{min}}|$ in $\mathsf{bcd}_1^{\mathcal{R}}\left(\mathbb{X}_{Z,f}, ||\cdot||_\infty\right)$.

Proof. First, setting $t_2 = \cdots = t_N = 0$ yields a closed curve

$$\gamma_1(t_1) = \left(\hat{F}(\mathbf{k}_1)e^{i\beta_1 t_1}, \ldots, \hat{F}(\mathbf{k}_\alpha)e^{ic_{\alpha,1}\beta_1 t_1}\right)$$

in $\mathbb{T}^\alpha_{\hat{F}}$. The time period of this curve is $\frac{2\pi Q_1}{\beta_1}$ where $Q_1 = \mathsf{lcm}\left(1, \frac{1}{c_{2,1}}, \cdots, \frac{1}{c_{\alpha,1}}\right) \in \mathbb{N}$, and the least common multiple is taken over reciprocals of all non zero $c_{j,1}$. Let $\delta, \epsilon_1, \epsilon_2 > 0$ be such that

$$0 \leq \max_{j,n} \left|\hat{F}(\mathbf{k}_j)c_{j,n}\beta_n\right| \delta < \epsilon_1 < \epsilon_2 < \sqrt{3}. \tag{4.9}$$

Let there be a finite $T \subset \left[0, \frac{2\pi Q_1}{\beta_1}\right]$ of the form

$$T = \{0 = s_0 < s_1 < \cdots < s_M\}$$

satisfying $d_H\left(T, \left[0, \frac{2\pi Q_1}{\beta_1}\right]\right) < \delta$. For all $0 \leq m \leq M$, $|s_m - s_{m+1}| < \delta$. This implies that

$$\begin{aligned}||\gamma_1(s_{m+1}) - \gamma_1(s_m)||_\infty &= \max_j |\hat{F}(\mathbf{k}_j)e^{i\beta_1 c_{j,1} s_{m+1}} - \hat{F}(\mathbf{k}_j)e^{i\beta_1 c_{j,1} s_m}| \\ &\leq \max_j |\hat{F}(\mathbf{k}_j)\beta_1 c_{j,1}(s_{m+1} - s_m)| \\ &\leq \max_j |\hat{F}(\mathbf{k}_j)c_{j,1}\beta_1|\delta\end{aligned}$$

where the last term is less than ϵ_1 by equation 4.9. We obtain a 1-homology cycle in $R_{\epsilon_1}\left(\mathbb{T}^\alpha_{\hat{F}}\right)$ given by

$$\tilde{\nu}_1 = [\gamma_1(s_0), \gamma_1(s_1)] + [\gamma_1(s_1), \gamma_1(s_2)] + \cdots + [\gamma_1(s_M), \gamma_1(s_0)]$$

and therefore, we obtain a class $\pi_*([\tilde{\nu}_1]) \in H_1\left(R_{\frac{\epsilon_1}{|\hat{F}(\mathbf{k})_{\mathsf{min}}|}}\left(\mathbb{T}^N\right)\right)$.

Now, let $\{0 = u_0 < u_1 < \cdots < u_{M'}\} := T \mod \frac{2\pi}{\beta_1}$ and χ_1 be a closed curve in $\mathbb{T}^N$ defined as

$$\chi_1(u) = (e^{i\beta_1 u}, 1, \cdots, 1)$$

for $u \in \left[0, \frac{2\pi}{\beta_1}\right]$. Let $\tilde{m}$ be such that $u_{\tilde{m}}$ corresponds to s_m modulo $\frac{2\pi}{\beta_1}$. Then by Proposition 4.4.1,

$$||\chi_1(u_{\tilde{m}+1}) - \chi_1(u_{\tilde{m}})||_\infty \leq ||\tilde{\pi}\gamma_1(s_{m+1}) - \tilde{\pi}\gamma_1(s_m)||_\infty \leq \frac{1}{|\hat{F}(\mathbf{k})_{\mathsf{min}}|}\epsilon_1.$$

This defines a 1-dimensional cycle μ_1 in $R_{\frac{\epsilon_1}{|\hat{F}(\mathbf{k})_{\mathsf{min}}|}}(\mathbb{T}^N)$ given by

$$\mu_1 = [\chi_1(u_0), \chi_1(u_1)] + [\chi_1(u_1), \chi_1(u_2)] + \cdots + [\chi_1(u_{M'}), \chi_1(u_0)]$$

satisfying $\pi_*([\tilde{\nu}_1]) = Q_1[\mu_1]$. By the assumptions on the prime p, Q_1 is invertible in $\mathbb{F}_p$ and hence, $\iota_*\pi_*([\tilde{\nu}_1]) \neq 0$. From the commutativity of the diagram

$$\begin{array}{ccc} H_1\left(R_{\epsilon_1}\left(\phi\left(\mathbb{T}^N\right)\right)\right) & \xrightarrow{\iota_*} & H_1\left(R_{\epsilon_2}\left(\phi\left(\mathbb{T}^N\right)\right)\right) \\ \downarrow \pi_* & & \downarrow \pi_* \\ H_1\left(R_{\frac{\epsilon_1}{|\hat{F}(\mathbf{k})_{\mathsf{min}}|}}\left(\mathbb{T}^N\right)\right) & \xrightarrow{\iota_*} & H_1\left(R_{\frac{\epsilon_2}{|\hat{F}(\mathbf{k})_{\mathsf{min}}|}}\left(\mathbb{T}^N\right)\right) \end{array} \tag{4.10}$$

we obtain that $\iota_*([\tilde{\nu}_1]) \neq 0$. By denseness of $\mathbb{X}_{Z,f}$ in $\phi(\mathbb{T}^N)$, we can find $\alpha > 0$ and points $x_{Z,f}(t_0), \cdots, x_{Z,f}(t_M) \in \mathbb{X}_{Z,f}$ such that

$$||\gamma_1(s_m) - x_{Z,f}(t_m)||_\infty < \alpha$$

and

$$||\gamma_1(s_{m+1}) - \gamma_1(s_m)||_\infty + 2\alpha < \epsilon_1.$$

By an application of triangle inequality, we obtain

$$||x_{Z,f}(t_{m+1}) - x_{Z,f}(t_m)||_\infty < \epsilon_1$$

and this gives us another 1-homology cycle

$$\nu_1 = [x_{Z,f}(t_0), x_{Z,f}(t_1)] + [x_{Z,f}(t_1), x_{Z,f}(t_2)] + \cdots + [x_{Z,f}(t_M), x_{Z,f}(t_0)]$$

in both $R_{\epsilon_1}\left(\phi\left(\mathbb{T}^N\right)\right)$ and $R_{\epsilon_1}\left(\mathbb{X}_{Z,f}\right)$. We claim that ν_1 is just homologous to $\tilde{\nu}_1$. If we join each edge $[\gamma_1(s_m), x_{Z,f}(t_m)]$ as well, we get $\tilde{\nu}_1$ and ν_1 as top and bottom faces of a 2-dimensional prism of height α. Through subdivision of simplices, as in [39, Theorem 2.10], we finish the argument. It is also worth noting that a different ν satisfying the same properties but built with different values of $t_0, \cdots, t_M$ will be homologous to ν_1 as well.

This implies that $\iota_*([\nu_1]) \neq 0$. Therefore, it yields a bar $[\ell_1, \rho_1)$ such that $\ell_1 \leq \epsilon_1$ and $\rho_1 \geq \epsilon_2$. Since this is true for all $\epsilon_1 > \max_{j,n}\left|\hat{F}(\mathbf{k}_j)c_{j,n}\beta_n\right|\delta$ and all $\epsilon_2 < \sqrt{3}|\hat{F}(\mathbf{k})_{\mathsf{min}}|$, we get that $\ell_1 \leq \max_{j,n}\left|\hat{F}(\mathbf{k}_j)c_{j,n}\beta_n\right|\delta$ and $\rho_1 \geq \sqrt{3}|\hat{F}(\mathbf{k})_{\mathsf{min}}|$. This entire process works for all δ satisfying the equation and therefore, as we let $\delta \to 0$, we obtain a class alive for all $\epsilon_1 > 0$.

Similarly, we can obtain N 1-homology classes $[\nu_1], \ldots, [\nu_N]$ that map to $Q_1[\mu_1]$, $\ldots$, $Q_N[\mu_N]$ respectively under π_*, and correspond to bars $(0, \rho_n)$ such that $\rho_n \geq \sqrt{3}|\hat{F}(\mathbf{k})_{\mathsf{min}}|$ for all n. □

We will now use Lemma 4.4.2 along with cup products [39, Chapter 3] and the isomorphism of persistent homology and cohomology [24] to prove the following result.

Theorem 4.4.3. *For $i \leq N$, there will be at least $\binom{N}{i}$ bars of the type*

$$(0, \rho_1), (0, \rho_2), \ldots, (0, \rho_{\binom{N}{i}})$$

where $\rho_n \geq \sqrt{3}|\hat{F}(\mathbf{k})_{\mathsf{min}}|$ in $\mathsf{bcd}_i^{\mathcal{R}}\left(\mathbb{X}_{Z,f}, ||\cdot||_\infty\right)$.

Proof. By Lemma 4.4.2, $\exists$ N distinct 1-dimensional Rips persistent homology classes which take birth exiting 0 and die entering $\rho_n \geq \sqrt{3}|\hat{F}(\mathbf{k})_{\mathsf{min}}|$. By the isomorphism of persistent homology and cohomology [24], there are N distinct 1-dimensional persistent cohomology classes corresponding to $(0, \rho_n)$ respectively. Pick any $\epsilon \in \left(0, \sqrt{3}|\hat{F}(\mathbf{k})_{\mathsf{min}}|\right)$. Then we get N 1-dimensional cohomology classes

$$[\vartheta_1], \cdots, [\vartheta_N]$$

in $H^1\left(R_\epsilon\left(\mathbb{X}_{Z,f}\right)\right)$. Using cup products and any choice of i of these N classes, we can get i-th dimensional classes

$$[\vartheta_{l_1}] \smile \cdots \smile [\vartheta_{l_i}].$$

Varying ϵ and using that the cup product is functorial, we get

$$\begin{array}{ccc} H^1\left(R_{\epsilon_1}\left(\mathbb{X}_{Z,f}\right)\right) \times \cdots \times H^1\left(R_{\epsilon_1}\left(\mathbb{X}_{Z,f}\right)\right) & \longrightarrow & H^i\left(R_{\epsilon_1}\left(\mathbb{X}_{Z,f}\right)\right) \\ \uparrow & & \uparrow \\ H^1\left(R_{\epsilon_2}\left(\mathbb{X}_{Z,f}\right)\right) \times \cdots \times H^1\left(R_{\epsilon_2}\left(\mathbb{X}_{Z,f}\right)\right) & \longrightarrow & H^i\left(R_{\epsilon_2}\left(\mathbb{X}_{Z,f}\right)\right) \end{array} \tag{4.11}$$

is commutative. This implies that the i-th dimensional cohomology classes are alive at $\sqrt{3}|\hat{F}(\mathbf{k})_{\mathsf{min}}| \leq \min_n \rho_n$ and die entering 0. Therefore, there are at least $\binom{N}{i}$ persistent cohomology classes in dimension i with cohomological death at 0 and persistence at least $\sqrt{3}|\hat{F}(\mathbf{k})_{\mathsf{min}}|$. The isomorphism between persistent homology and cohomology finishes the proof. $\square$

We now use linear algebra to argue that $\Omega_{Z,f}^{-1}$ is Lipschitz. In the proposition below, $\Omega_{Z,f}^{H}$ is the conjugate transpose of $\Omega_{Z,f}$

Proposition 4.4.4. *Let λ_{min} and λ_{max} be the minimum and the maximum eigenvalues of $\Omega_{Z,f}^{H} \cdot \Omega_{Z,f}$ respectively. Then*

$$\frac{1}{\sqrt{(d+1)\lambda_{\mathsf{max}}}} ||\mathbf{v}||_2 \leq \left|\left|\Omega_{Z,f}^{-1} \cdot \mathbf{v}\right|\right|_\infty \leq \frac{1}{\sqrt{\lambda_{\mathsf{min}}}} ||\mathbf{v}||_2$$

for each $\mathbf{v} \in \mathbb{C}^\alpha$.

Therefore, the composition map $\pi \circ \Omega_{Z,f}^{-1}$ is Lipschitz with the constant $\frac{1}{\sqrt{\lambda_{\mathrm{min}}}|\hat{F}(\mathbf{k})_{\mathsf{min}}|}$.

Theorem 4.4.5. *For $i \leq N$, there will be at least $\binom{N}{i}$ bars*

$$(0, \rho_1), (0, \rho_2), \ldots, (0, \rho_{\binom{N}{i}})$$

where $\rho_n \geq \sqrt{3}|\hat{F}(\mathbf{k})_{\mathsf{min}}|\sqrt{\lambda_{\mathsf{min}}}$ in $\mathsf{bcd}_i^{\mathcal{R}}\left(SW_{d,\tau}S_Z f, ||\cdot||_2\right)$.

Proof. By Proposition 4.4.4, we have the following commutative diagram for each i

$$\begin{array}{ccc}
H_i\left(R_{\epsilon_1}\left(SW_{d,\tau}S_Z f, ||\cdot||_2\right)\right) & \longrightarrow & H_i\left(R_{\frac{\sqrt{(d+1)\lambda_{\mathsf{max}}}}{\sqrt{\lambda_{\mathsf{min}}}}\epsilon_1}\left(SW_{d,\tau}S_Z f, ||\cdot||_2\right)\right) \\
\downarrow & \nearrow & \downarrow \\
H_i\left(R_{\frac{\epsilon_1}{\sqrt{\lambda_{\mathsf{min}}}}}\left(\mathbb{X}_{Z,f}, ||\cdot||_\infty\right)\right) & \longrightarrow & H_i\left(R_{\frac{\sqrt{(d+1)\lambda_{\mathsf{max}}}}{\lambda_{\mathsf{min}}}\epsilon_1}\left(\mathbb{X}_{Z,f}, ||\cdot||_\infty\right)\right)
\end{array} \tag{4.12}$$

where the downward maps are induced by $\Omega_{Z,f}^{-1}$ and the upper diagonal map is induced by $\Omega_{Z,f}$. By [53, Lemma 2.3] and Theorem 4.4.3, there are at least $\binom{N}{i}$ bars with length at least $\sqrt{3}|\hat{F}(\mathbf{k})_{\mathsf{min}}|\sqrt{\lambda_{\mathsf{min}}}$ in $\mathsf{bcd}_i^{\mathcal{R}}\left(SW_{d,\tau}S_Z f, ||\cdot||_2\right)$. □

4.5 Computational Example

In this section, we apply the above established theory to quasiperiodic signals. We will first work through the entire pipeline for the special case where $\alpha = N$, i.e. when the quasiperiodic function is just the sum of exponential functions with non-commensurate frequencies:

$$f(t) = e^{i\omega_1 t} + \cdots + e^{i\omega_N t}.$$

In general, we begin with a signal sampled from f, use Fast Fourier Transform to estimate the frequencies, which we further use to estimate a good choice of τ as discussed in section 4.3, compute the sliding window point cloud and finally, compute Rips persistent homology of $SW_{d,\tau}f$.

One of the main challenges in computing $\mathsf{bcd}_i^{\mathcal{R}}(SW_{d,\tau}f(T);\mathbb{F})$ for a quasiperiodic time series f is that if $SW_{d,\tau}f$ fills out the torus at a slow rate — requiring a potentially large $T \subset \mathbb{Z}$ — then the persistent homology computation becomes prohibitively large. To combat this, we will use an alternate strategy leveraging a persistent Künneth formula introduced in [35]. The strategy is based on the categorical persistent Künneth formula (Corollary 3.2.6).

For our second example, we apply our theory to music. We start with a real music sample with *dissonance*, which is a characterization of simultaneous tones that causes unpleasantness. We show using the developed theory that dissonance can be characterized as quasiperiodicity in the signal.

4.5.1 An approximation strategy using a persistent Künneth formula

Let us revisit the Künneth approximation strategy from [35] and its comparison with the more direct technique using landmark approximation which we describe first:

Landmarks: We select a set $L \subset SW_{d,\tau}f$ of landmarks using `maxmin` sampling. That is, $\ell_1 \in SW_{d,\tau}f$ is chosen at random and the rest of the landmarks are selected inductively as

$$\ell_{j+1} = \underset{x \in SW_{d,\tau}f}{\operatorname{argmax}} \min \left\{ \|x - \ell_1\|_2, \dots, \|x - \ell_j\|_2 \right\}$$

We endow L with the Euclidean distance in $\mathbb{C}^d$, and compute the Rips barcodes $\mathsf{bcd}_i^{\mathcal{R}}(L, \|\cdot\|_2)$, for $i = 0, 1, \cdots, N$, directly using `Ripser` [4].

We describe the Künneth strategy next:

Künneth: For $1 \leq n \leq N$, Let $p_n : \mathbb{C}^N \longrightarrow \mathbb{C}$ be the projections onto the n-th coordinate. Let $SW_{d,\tau}f(t) = \Omega_{Z,f} \cdot \phi(t)$. We select N sets of landmarks, $X_n \subset p_n \circ \phi(T)$ with l points each using `maxmin` sampling, and endow the Cartesian product $X_1 \times \cdots \times X_N$ with the maximum metric in $\mathbb{C}^N$. The barcodes of X_n, for each n are computed using `Ripser`[4], and we deduce $\mathsf{bcd}_i^{\mathcal{R}}(X_1 \times \cdots \times X_N, \|\cdot\|_\infty)$ for $i = 0, 1, \dots, N$ using Corollary 3.4.2.

4.5.1.1 An example with 2 frequencies

The general strategy will be illustrated with a computational example: Let $c_1, c_2 \in \mathbb{C} \smallsetminus \{0\}$ with $|c_1|^2 + |c_2|^2 = 1$, and let $\omega \in \mathbb{R} \smallsetminus \mathbb{Q}$. If

$$f(t) = c_1 e^{it} + c_2 e^{i\omega t}$$

then $SW_{d,\tau} f(t) = \Omega \cdot \phi(t)$, where

$$\Omega = \frac{1}{\sqrt{d+1}} \begin{bmatrix} 1 & 1 \\ e^{i\tau} & e^{i\omega\tau} \\ \vdots & \vdots \\ e^{id\tau} & e^{id\omega\tau} \end{bmatrix} \quad , \qquad \phi(t) = \sqrt{d+1} \begin{bmatrix} c_1 e^{it} \\ c_2 e^{i\omega t} \end{bmatrix}.$$

An elementary calculation shows that if $d \geq 1$ and $\tau = \frac{2\pi}{(d+1)|\omega - 1|}$, then the columns of Ω are orthonormal in $\mathbb{C}^{d+1}$. Fix $d = 1$, $\omega = \sqrt{3}$, τ as above, and $c_1 = c_2 = 1/\sqrt{2}$. Figure 4.3 below shows the real part of f (left), and a visualization (right) of the *sliding window point cloud* $\mathbb{SW}_{d,\tau} f = SW_{d,\tau} f(T)$, $T = \{t \in \mathbb{Z} : 0 \leq t \leq 4{,}000\}$, using Principal Component Analysis (PCA) [44].

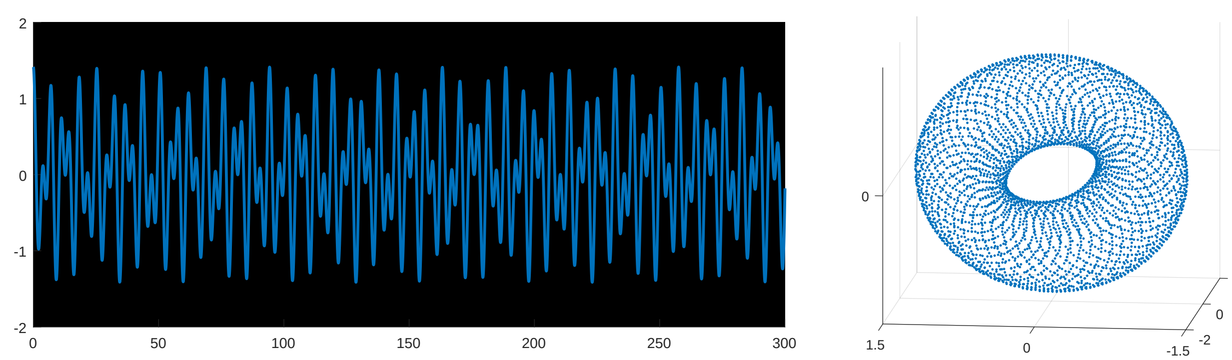


Figure 4.3: (Left) $\mathsf{real}\big(f(t)\big) = \frac{1}{\sqrt{2}}\cos(t) + \frac{1}{\sqrt{2}}\cos(\sqrt{3}t)$ for the quasiperiodic time series f, and (Right) a PCA projection into $\mathbb{R}^3$ of the point cloud $\mathbb{SW}_{d,\tau} f$.

Computing the exact barcodes $\mathsf{bcd}_i^{\mathcal{R}}(\mathbb{SW}_{d,\tau} f)$, $i = 0, 1, 2$, for point clouds of this size is already prohibitively large (e.g., with `Ripser` [4]). Thus, one strategy is to choose a smaller set of landmarks $L \subset \mathbb{SW}_{d,\tau} f$ and use the stability theorem [19]

$$d_B\left(\mathsf{bcd}_i^{\mathcal{R}}(L), \mathsf{bcd}_i^{\mathcal{R}}(\mathbb{SW}_{d,\tau} f)\right) \leq 2d_{GH}(L, \mathbb{SW}_{d,\tau} f)$$

to estimate the barcodes of $\mathbb{SW}_{d,\tau} f$ using those of L, up to an error (in the bottleneck sense) of twice the Gromov-Hausdorff distance d_{GH} between L and $\mathbb{SW}_{d,\tau} f$. Specifically, if $r = d_{GH}(L, \mathbb{SW}_{d,\tau} f)$ and $I = [\ell_I, \rho_I) \in \mathsf{bcd}_i^{\mathcal{R}}(L)$ satisfies $\rho_I - \ell_I > 4r$, then there is a unique $J \in \mathsf{bcd}_i^{\mathcal{R}}(\mathbb{SW}_{d,\tau} f)$ with $|\rho_I - \rho_J|, |\ell_I - \ell_J| \leq 2r$, and thus

$$\begin{aligned} \max\{0\ ,\ \rho_I - 2r\} \leq \rho_J \leq \rho_I + 2r \\ \max\{0\ ,\ \ell_I - 2r\} \leq \ell_J \leq \ell_I + 2r. \end{aligned} \tag{4.13}$$

We next apply the two strategies discussed above:

Landmarks: We select a set $L \subset \mathbb{SW}_{d,\tau} f$ of 400 landmarks (%10 of the data) using `maxmin` sampling. We endow L with the Euclidean distance in $\mathbb{C}^2$, and compute the barcodes $\mathsf{bcd}_i^{\mathcal{R}}(L, \|\cdot\|_2)$, $i = 0, 1, 2$, directly using `Ripser` [4].

Künneth: We select two sets of landmarks, $X_L \subset p_1 \circ \phi(T)$ and $Y_L \subset p_2 \circ \phi(T)$ with 70 points each using `maxmin` sampling, and endow the cartesian product $X_L \times Y_L$ with the maximum metric in $\mathbb{C}^2$. The barcodes of X_L and Y_L are computed using `Ripser`, and we deduce those of $\mathsf{bcd}_i^{\mathcal{R}}(X_L \times Y_L, \|\cdot\|_\infty)$ for $i = 0, 1, 2$ using Corollary 3.2.6.

4.5.1.2 Computational Results

Table 4.2 shows the computational times in each approximation strategy — i.e., via the landmark set L and the persistent Künneth theorem applied to $X_L \times Y_L$ — as well as the number of data points in each case.

As these results show, the Künneth approximation strategy is almost two orders of magnitude faster than just taking landmarks, and as we will see below, it is also more accurate.

	Landmarks	**Künneth**
Time (sec)	15.22	0.20
# of points	400	4,900

Table 4.2: Computational times, and number of points, for the landmark and Künneth approximations to $\mathsf{bcd}_i^{\mathcal{R}}(\mathbb{SW}_{d,\tau}f)$, $i \leq 2$.

Indeed, the inequalities in (4.13) can be used to generate a *confidence region* for the existence of an interval $J \in \mathsf{bcd}_i^{\mathcal{R}}(\mathbb{SW}_{d,\tau}f)$ given a large enough interval $I \in \mathsf{bcd}_i^{\mathcal{R}}(L)$. A similar region can be estimated from $\mathsf{bcd}_i^{\mathcal{R}}(X_L \times Y_L)$ as follows. Since the columns of Ω are orthonormal, then for every $z \in \mathbb{C}^2$ we have $\|z\|_\infty \leq \|z\|_2 = \|\Omega z\|_2 \leq \sqrt{2}\|z\|_\infty$ and thus

$$R_\epsilon(\mathbb{SW}_{d,\tau}f, \|\cdot\|_2) \subset R_\epsilon(\phi(T), \|\cdot\|_\infty) \subset R_{\sqrt{2}\epsilon}(\mathbb{SW}_{d,\tau}f, \|\cdot\|_2).$$

Moreover, (see 3.1.21) if $I \in \mathsf{bcd}_i^{\mathcal{R}}(X_L \times Y_L, \|\cdot\|_\infty)$ satisfies

$$\frac{\rho_I}{\sqrt{2}} - \sqrt{2}\ell_I \quad > \quad 4d_{GH}\big(X_L \times Y_L, \phi(T)\big)$$

where $\lambda = d_{GH}(X_L \times Y_L, \phi(T))$ is computed for subspaces of $(\mathbb{C}^2, \|\cdot\|_\infty)$, then there exists a unique $J \in \mathsf{bcd}_i^{\mathcal{R}}(\mathbb{SW}_{d,\tau}f, \|\cdot\|_2)$ so that

$$\begin{aligned} \max\left\{0\ ,\ \frac{\rho_I - 2\lambda}{\sqrt{2}}\right\} \ &\leq \rho_J \ \leq \ \sqrt{2}\cdot(\rho_I + 2\lambda) \\ \max\left\{0\ ,\ \frac{\ell_I - 2\lambda}{\sqrt{2}}\right\} \ &\leq \ell_J \ \leq \ \sqrt{2}\cdot(\ell_I + 2\lambda). \end{aligned} \tag{4.14}$$

Figure 4.4 shows the resulting confidence regions for both approximation strategies. Each interval $[\ell, \rho)$ is replaced by a point $(\ell, \rho) \in \mathbb{R}^2$, and the regions for $\mathsf{bcd}_i^{\mathcal{R}}(L)$ (4.13) and $\mathsf{bcd}_i^{\mathcal{R}}(X_L \times Y_L)$ (4.14) are shown in blue and red, respectively. We compare these regions by computing their area, and report the results in Table 4.3.

i	**Landmarks**	**Künneth**
1	0.7677	0.4680
1	0.7480	0.4709
2	0.9072	0.4704

Table 4.3: Areas for confidence regions from $\mathsf{bcd}_i^{\mathcal{R}}(L)$ (blue) and $\mathsf{bcd}_i^{\mathcal{R}}(X_L \times Y_L)$ (red), $i = 1, 2$ (see Figure 4.4). A smaller area suggests a better approximation.

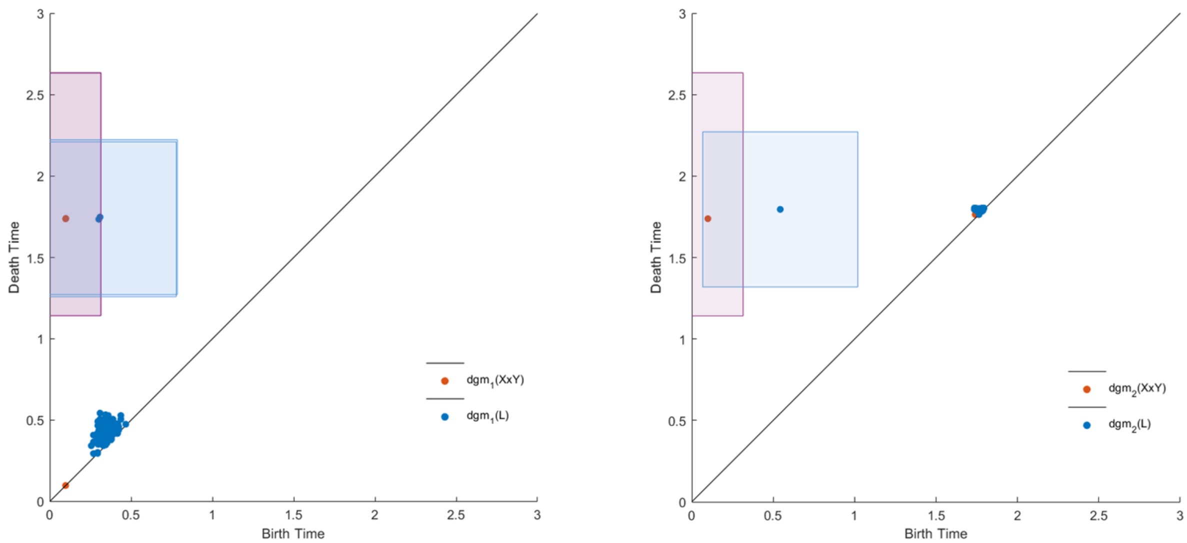


Figure 4.4: Confidence regions for $\mathsf{bcd}_i^{\mathcal{R}}(L)$ and $\mathsf{bcd}_i^{\mathcal{R}}(X_L \times Y_L)$, $i = 1, 2$. For a given color, each box (confidence region) contains a point (ℓ_J, ρ_J) corresponding to a unique $J \in \mathsf{bcd}_i^{\mathcal{R}}(\mathbb{SW}_{d,\tau} f)$.

These results suggest that the Künneth strategy provides a tighter approximation than using landmarks. Moreover, both strategies can be used in tandem — improving the quality of approximation — via intersection of confidence regions.

4.5.2 An application to music theory

In music theory, consonance and dissonance are classification of multiple simultaneous tones. While the former is associated with pleasantness, the latter creates tension as experienced by the listener. Perfect dissonance occurs when the audio frequencies are irrational with respect to each other. One of such instances is that of a *tritone*, which a musical interval that is halfway between two octaves. Mathematically, for a base frequency ω_1, the tritone is $\sqrt{2}\omega_1$. We will use the theory of sliding window embedding to quantify quasiperiodicity from a dissonant sample. For the purpose of this application, we use a 5-second audio recording of a brass horn provided by Adam Huston. It was read by audioread function in MATLAB and the signal plot is shown in Figure 4.5.

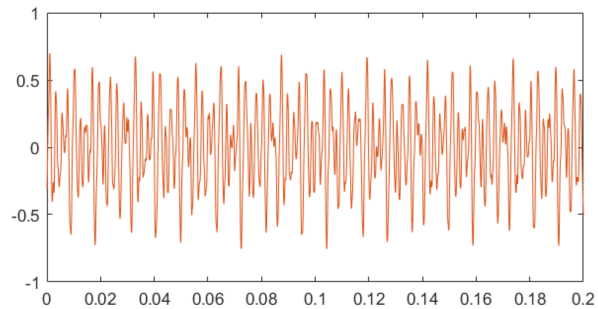


Figure 4.5: The audioread plot of a dissonant music sample created on a brass horn.

Like before, in order to perform sliding window analysis, we need to choose appropriate parameters d and τ. The first step is to use Fast Fourier Transform in MATLAB to obtain the amplitude-frequency spectrum. We use findpeaks with thresholds MinPeakHeight $= 0.06$

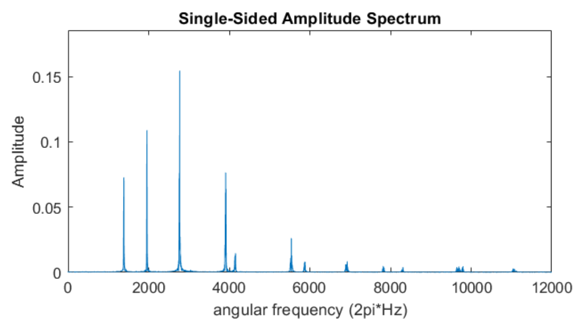


Figure 4.6: The amplitude-frequency spectrum of the music sample plot using Fast Fourier Transform.

and MinPeakDistance $= 100$ to detect frequencies which we will use for estimation of the embedding parameters. See Table 4.4.

Angular Frequencies	1384.93	1957.83	2769.86	3911.93
Frequencies (Hz)	220.41	311.59	440.83	622.60
Proportion	1	$1.4137 \approx \sqrt{2}$	2	$2.8246 \approx 2\sqrt{2}$

Table 4.4: List of frequencies in the amplitude-frequency spectrum: The first row is the list of frequencies that were detected. The second row is their conversion to Hertz. The third row is their ratio with respect to the first one.

Remark 4.5.1. While our entire theory has been developed for complex functions, it can be adjusted to work with real functions via the following identities:

$$\cos(at) = \frac{1}{2}e^{iat} + \frac{1}{2}e^{i(-a)t} \quad \text{and} \quad \sin(at) = \frac{1}{2i}e^{iat} - \frac{1}{2i}e^{-i(-a)t}$$

for any $a \in \mathbb{R}$, $t \in \mathbb{R}$.

For the purpose of our analysis, this would mean that instead of 4 frequencies, we count 8 of them (with their negatives). By the theory, the complex embedding dimension d should be at least 8, that would imply that the real embedding dimension should be at least 8. Therefore, we choose the embedding dimension $d = 9$, and the delay $\tau = 0.02858$ in accordance with the optimization discussion in section 4.3. We use cubic splines to compute the sliding window vectors and present the PCA representation of the point cloud, along with the persistence diagrams in Figure 4.7. We conclude the dissonance is indeed a quasiperiodic feature.

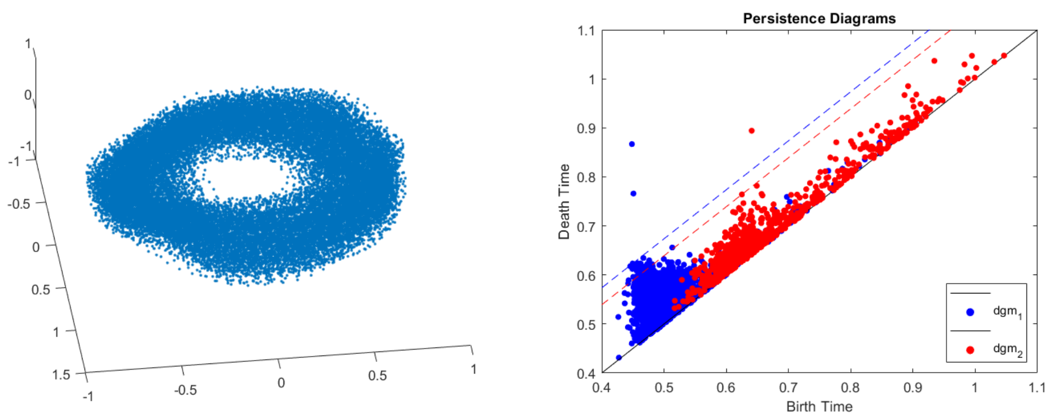


Figure 4.7: (Left) PCA representation of the sliding window point cloud generated from the dissonant sample with parameters $d = 9$ and $\tau = 0.02858$. (Right) Persistence Diagrams in homological dimension 1 and 2. The dotted lines separate the top 2 persistence points (sim. maximum persistence point) from the rest in dimension 1 (sim. dimension 2).

CHAPTER 5

STABILITY OF MULTISCALE CIRCULAR COORDINATES

In this chapter, we provide a stability characterization of circular coordinates. We set up the problem of stability with respect to the choice of landmarks in section 5.1. In section 5.2, we present stability results for the geometric noise model. In section 5.3, we approach the general case and prove stability for the Wasserstein noise model. The last section 5.4 is dedicated to examples. Parts of this chapter are to appear in [34]. We begin with a description of the sparse circular coordinates algorithm [55, Section 1.2] first.

5.0.1 The sparse circular coordinates algorithm

1. Let $(\mathbb{M}, \mathbf{d})$ be a finite metric space (i.e., the dataset) and select a set of landmarks $L = \{\ell_1, \ldots, \ell_N\} \subset \mathbb{M}$, e.g., randomly or via `maxmin` sampling.

2. Choose a prime $q > 2$ and compute the 1-dimensional Rips persistent cohomology $PH^1(\mathcal{R}(L); \mathbb{Z}_q)$ on $(L, \mathbf{d}|_L)$ with coefficients in $\mathbb{Z}_q$. Let $\mathtt{bcd}(L)$ denote the resulting persistence diagram.

3. Let $r_L = \max_{x \in \mathbb{M}} \min_{\ell \in L} \mathbf{d}(x, \ell)$ be the minimal cover radius. Observe that $r_L = d_H(L, \mathbb{M})$. If there exists $(a, b) \in \mathtt{bcd}(L)$ such that $\max\{a, r_L\} < \frac{b}{2}$, then let

$$\alpha = t \cdot \max\{a, r_L\} + \frac{1}{2}(1-t)b$$

for some $t \in (0,1)$ and $\eta'' \in Z^1(R_{2\alpha}(L); \mathbb{Z}_q)$ be a cocycle representative for the persistent cohomology class corresponding to (a, b).

4. Lift η'' to an integer cocycle $\eta' \in Z^1(R_{2\alpha}(L); \mathbb{Z})$ such that $\eta'' - \eta' \mod q$ is a coboundary in $C^1(R_{2\alpha}(L); \mathbb{Z}_q)$. In practice, we use the cochain

$$\eta'(\sigma) = \begin{cases} \eta''(\sigma) & \text{if } \eta''(\sigma) \leq \frac{q-1}{2} \\ \eta''(\sigma) - q & \text{if } \eta''(\sigma) > \frac{q-1}{2}. \end{cases}$$

Lastly, define $\eta = \iota_1 \eta'$ where $\iota_1 : C^1(R_{2\alpha}(L);\mathbb{Z}) \to C^1(R_{2\alpha}(L);\mathbb{R})$ is the homomorphism induced by the inclusion $\iota : \mathbb{Z} \hookrightarrow \mathbb{R}$.

5. Choose positive weights for the vertices and edges of $R_{2\alpha}(L)$ and let

$$d_{2\alpha}^{+} : C^1(R_{2\alpha}(L);\mathbb{R}) \to C^0(R_{2\alpha}(L);\mathbb{R})$$

be the weighted Moore-Penrose pseudoinverse of the coboundary map

$$d_{2\alpha} : C^0(R_{2\alpha}(L);\mathbb{R}) \to C^1(R_{2\alpha}(L);\mathbb{R}).$$

and define

$$\tau = -d_{2\alpha}^{+}(\eta) \quad \text{and} \quad \theta = \eta + d_{2\alpha}(\tau).$$

6. Denote $\tau_j := \tau([\ell_j])$ and $\theta_{jk} := \theta([\ell_j, \ell_k])$ when $[\ell_j, \ell_k] \in R_{2\alpha}(L)$ (it is zero otherwise). Let $B_\alpha(\ell_j)$ denote a ball of radius α centered around $\ell_j \in L$ and $\{\varphi_j\}_j$ denote the partition of unity subordinate to the collection $\{B_\alpha(\ell_j)\}_j$. The sparse circular coordinates are defined as follows:

$$\begin{aligned} h_{\theta,\tau} : \bigcup_{j=1}^{N} B_\alpha(\ell_j) &\to S^1 \subset \mathbb{C} \\ B_\alpha(\ell_j) \ni b &\mapsto \exp\left\{2\pi i \left(\tau_j + \sum_{r=1}^{N} \varphi_r(b)\theta_{jr}\right)\right\}. \end{aligned}$$

The circular coordinates algorithm can be used to reconstruct dynamical systems as in the following example:

Example 5.0.1. The quasiperiodic function $f(t) = \frac{1}{\sqrt{2}}e^{it} + \frac{1}{\sqrt{2}}e^{i\sqrt{3}t}$ in section 4.5.1.1 can be realized as an observation of the irrational winding on a 2-torus starting at $(0,0)$, with slope $\sqrt{3}$. The theory developed in the last chapter tells that the sliding window embedding of the function will be a 2-torus and persistence verifies that. Using the two 1-persistent cocycles, we can generate two sets of circular coordinates using the algorithm above, and together we reconstruct an approximation to the original winding. See figure 5.1 for the entire pipeline.

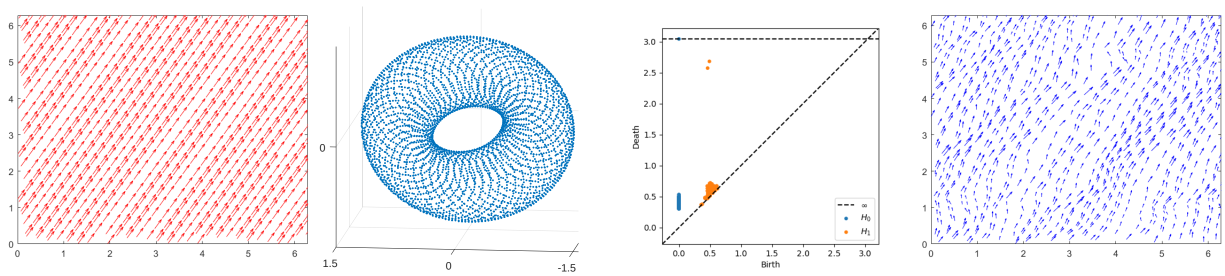


Figure 5.1: An example of reconstructed toroidal dynamics: (Left) The irrational winding on a 2-torus starting at $(0,0)$ with slope $\sqrt{3}$. (Second) The sliding window embedding of the observation $f(t) = \frac{1}{\sqrt{2}}e^{it} + \frac{1}{\sqrt{2}}e^{i\sqrt{3}t}$ of the winding. (Third) The Rips persistence diagram in dimension $0, 1$. The two 1-persistent cocycles can be used to create two sets of circular coordinates. (Right) The pair of coordinates can be used to reconstruct the toroidal winding. The average slope is estimated to be 1.9698.

Note that the algorithm depends on a choice of landmarks $L \subset \mathbb{M}$ and as one would expect, different choices of landmarks may yield different circular coordinates. One would also expect that two landmark sets that are "close" to each other in some sense yield circular coordinates that are also "close" in some sense. In the next section, we set up the problem of stability.

5.1 The stability set-up

We start with two finite landmark sets $L, \tilde{L}$ of the metric space $(\mathbb{M}, \mathbf{d})$ with probability weights ω and $\tilde{\omega}$ satisfying that

$$d_{\mathcal{W},\infty}((L,\omega),(\tilde{L},\tilde{\omega})) < \delta$$

where $d_{\mathcal{W},\infty}$ is the Wasserstein-Kantorovich-Rubinstein metric as in Definition 2.5.5. The goal is to obtain results on the proximity of the associated circular valued functions in the L^∞-metric. We first solve the problem for a special case, that is when L and $\tilde{L}$ are in a weight preserving bijective correspondence: We call this the *Geometric Noise Model*. We approach the general case by duplicating some landmarks in a well-defined manner to create a bijection and therefore, reducing it to the Geometric Noise Model. Next, we give explicit choices for partitions of unity and edge weights that we use in our stability analysis.

5.1.1 Partition of Unity

Let $\phi : \mathbb{R} \to [0, \infty)$ be increasing and Lipschitz, i.e. for all $a, b \in \mathbb{R}$, $\phi(a) \leq \phi(b)$ if $a \leq b$, and $|\phi(a) - \phi(b)| \leq L_\phi |a - b|$ where L_ϕ is the Lipschitz constant. Moreover, let $\phi(a) = 0$ for all $a \leq 0$.

Proposition 5.1.1. *Let $L = \{\ell_1, \ldots, \ell_N\} \subset (\mathbb{M}, \mathbf{d})$ be a finite set with weights $\{\omega_1, \ldots, \omega_N\}$. For an $\alpha > 0$, and the function ϕ defined above, the collection of functions $\left\{\varphi_j : \mathbb{M} \to \mathbb{R}\right\}$ defined as*

$$\varphi_j(b) = \frac{\omega_j \phi(\alpha - \mathbf{d}(b, \ell_j))}{\sum\limits_{k=1}^{N} \omega_k \phi(\alpha - \mathbf{d}(b, \ell_k))}$$

is a partition of unity subordinated to $\left\{B_\alpha(\ell_j)\right\}_{j=1}^{N}$.

Proof. By construction, we have $\sum\limits_{j=1}^{N} \varphi_j(b) = 1$. Thus, we only need to verify that $\mathsf{Supp}(\varphi_j) \subset B_\alpha(\ell_j)$. Clearly, if $b \notin B_\alpha(\ell_j)$, then $\mathbf{d}(b, \ell_j) \geq \alpha$, thus $\alpha - \mathbf{d}(b, \ell_j) \leq 0$ and by construction of ϕ, then $\phi(\alpha - \mathbf{d}(b, \ell_j)) = 0$. □

If there is another finite set $\tilde{L} = \left\{\tilde{\ell}_1, \ldots, \tilde{\ell}_N\right\} \subset (\mathbb{M}, \mathbf{d})$ with weights $\{\tilde{\omega}_1, \ldots, \tilde{\omega}_N\}$ satisfying $\mathbf{d}(\tilde{\ell}_i, \ell_i) < \delta$, then for $\alpha - \delta$ we can define another partition of unity $\left\{\psi_j : \mathbb{M} \to \mathbb{R}\right\}$ as

$$\psi_j(b) = \frac{\tilde{\omega}_j \phi(\alpha - \delta - \mathbf{d}(b, \tilde{\ell}_j))}{\sum\limits_{k=1}^{N} \tilde{\omega}_k \phi(\alpha - \delta - \mathbf{d}(b, \tilde{\ell}_k))}$$

We introduce notation:

$$\overline{\varphi}(b) = \sum_{k=1}^{N} \omega_k \phi(\alpha - \mathbf{d}(b, \ell_k))$$
$$\overline{\psi}(b) = \sum_{k=1}^{N} \tilde{\omega}_k \phi(\alpha - \delta - \mathbf{d}(b, \tilde{\ell}_k)).$$

If the weights satisfy $\omega_k = \tilde{\omega}_k$ for all k, then the following is true:

Proposition 5.1.2. *If $\omega_k = \tilde{\omega}_k$ for all k, and $\mathbf{d}(\tilde{\ell}_k, \ell_k) < \delta$ for all k, then $\overline{\psi}(b) \leq \overline{\varphi}(b)$ for all $b \in \mathbb{M}$.*

Proof. By triangle inequality, since $\left|\mathbf{d}(b,\ell_k)-\mathbf{d}(b,\tilde{\ell}_k)\right| \leq \mathbf{d}(\ell_k,\tilde{\ell}_k) < \delta$, then

$$\mathbf{d}(b,\ell_k) < \delta + \mathbf{d}(b,\tilde{\ell}_k).$$

By properties of ϕ, we obtain $\phi(\alpha-\delta-\mathbf{d}(b,\tilde{\ell}_k)) \leq \phi(\alpha-\mathbf{d}(b,\ell_k))$ for each k. Therefore, $\overline{\psi}(b) \leq \overline{\varphi}(b)$. ☐

Proposition 5.1.3. *For $b \in L^{(\alpha-\delta)}$, we have*

$$\sum_{j=1}^{N}\left|\varphi_j(b)-\psi_j(b)\right| \leq \frac{4\delta L_\phi}{\overline{\varphi}(b)}.$$

Proof. We will proceed as follows:

$$\begin{aligned}
\left|\varphi_j(b)-\psi_j(b)\right| &= \left|\frac{\omega_j\phi(\alpha-\mathbf{d}(b,\ell_j))}{\sum_{k=1}^{N}\omega_k\phi(\alpha-\mathbf{d}(b,\ell_k))} - \frac{\tilde{\omega}_j\phi(\alpha-\delta-\mathbf{d}(b,\tilde{\ell}_j))}{\sum_{k=1}^{N}\tilde{\omega}_k\phi(\alpha-\delta-\mathbf{d}(b,\tilde{\ell}_k))}\right| \\
&\leq \frac{\omega_j L_\phi}{\overline{\varphi}(b)}\left|\delta+\mathbf{d}(b,\tilde{\ell}_j)-\mathbf{d}(b,\ell_j)\right| + \frac{\tilde{\omega}_j\phi(\alpha-\delta-\mathbf{d}(b,\tilde{\ell}_j))}{\overline{\psi}(b)}\left|1-\frac{\overline{\psi}(b)}{\overline{\varphi}(b)}\right| \\
&\leq \frac{\omega_j L_\phi}{\overline{\varphi}(b)}2\delta + \frac{\tilde{\omega}_j\phi(\alpha-\delta-\mathbf{d}(b,\tilde{\ell}_j))}{\overline{\psi}(b)}\left|1-\frac{\overline{\psi}(b)}{\overline{\varphi}(b)}\right|.
\end{aligned}$$

where in the second step, we added and subtracted $\frac{\omega_j\phi(\alpha-\delta-\mathbf{d}(b,\tilde{\ell}_j))}{\sum_{k=1}^{N}\omega_k\phi(\alpha-\mathbf{d}(b,\ell_k))}$, and used triangle inequality, while keeping in mind that ω and $\tilde{\omega}$ are the same. Summing over all j yields

$$\sum_{j=1}^{N}\left|\varphi_j(b)-\psi_j(b)\right| \leq \frac{2\delta L_\phi}{\overline{\varphi}(b)}\sum_{j=1}^{N}\omega_j + \left|1-\frac{\overline{\psi}(b)}{\overline{\varphi}(b)}\right| \leq \frac{2\delta L_\phi}{\overline{\varphi}(b)} + \frac{2\delta L_\phi}{\overline{\varphi}(b)} = \frac{4\delta L_\phi}{\overline{\varphi}(b)}.$$

☐

5.1.2 Weights on edges

By a weight family on the edges of L, we mean a sequence of functions $\nu_\alpha : R_\alpha(L)^{(1)} \to [0,\infty)$ (parametrized by α) satisfying the following properties:

- ν_α is increasing in α, and

- $\nu_\alpha\left([\ell_i,\ell_j]\right)$, which we call the weight of the edge $[\ell_i,\ell_j]$, is 0 if and only if $0<\alpha\le\mathbf{d}(\ell_i,\ell_j)$.

For the purpose of this dissertation, we define these ν_α in the same spirit as the partitions of unity, i.e. controlled by an increasing and Lipschitz function $\rho:\mathbb{R}\to[0,\infty)$ satisfying $\rho(a)=0$ for all $a\le 0$, and by vertex weights ω in the following manner:

$$\nu_\alpha([\ell_i,\ell_j])=\omega_i\omega_j\rho(\alpha-\mathbf{d}(\ell_i,\ell_j)). \tag{5.1}$$

The vertex weights ω and the weight families ν_α define inner products $\langle\cdot,\cdot\rangle_\alpha$ on $C^0(R_\alpha(L);\mathbb{R})$ and $C^1(R_\alpha(L);\mathbb{R})$ respectively, by letting the indicator functions 1_σ on k-simplices $\sigma\in R_\alpha(L)$ (for $k=0,1$) be orthogonal, and setting

$$\langle 1_{[\ell_i]},1_{[\ell_i]}\rangle=\omega_i \quad\text{and}\quad \langle 1_\sigma,1_\sigma\rangle_\alpha=\nu_\alpha(\sigma).$$

Similarly, we define vertex weights $\tilde{\omega}$ and a weight family $\tilde{\nu}_\alpha$ on the edges of $\tilde{L}$

$$\tilde{\nu}_\alpha([\tilde{\ell}_i,\tilde{\ell}_j])=\tilde{\omega}_i\tilde{\omega}_j\rho(\alpha-\mathbf{d}(\tilde{\ell}_i,\tilde{\ell}_j)).$$

and the induced inner products on $C^0(R_\alpha(\tilde{L});\mathbb{R})$ and $C^1(R_\alpha(\tilde{L});\mathbb{R})$ follow.

5.2 Geometric noise model

Let $L=\{\ell_1,\dots,\ell_N\}$ and $\tilde{L}=\{\tilde{\ell}_1,\dots,\tilde{L}_N\}$ be subsets of $\mathbb{M}$ such that there is a bijection $\mu:\tilde{L}\to L$ satisfying the following conditions:

1. $\mathbf{d}(\tilde{\ell}_i,\mu(\tilde{\ell}_i))<\delta$
2. $\mu(\tilde{\ell}_i)=\ell_i$
3. $\tilde{\omega}_i:=\tilde{\omega}(\tilde{\ell}_i)=\omega(\ell_i)=:\omega_i$

for all $i\in\{1,\dots,N\}$. By an application of triangle inequality,

$$\mathbf{d}(\tilde{\ell}_i,\tilde{\ell}_j)<2\alpha-2\delta\Rightarrow\mathbf{d}\left(\ell_i,\ell_j\right)<2\alpha,$$

and therefore, μ extends to a simplicial map

$$\mu : R_{2\alpha-2\delta}(\tilde{L}) \to R_{2\alpha}(L),$$

where α is chosen so that $R_{2\alpha-2\delta}(\tilde{L})$, and hence $R_{2\alpha}(L)$, is connected. In case of disconnectedness, one can consider each connected component separately. Consider the induced commutative diagrams

$$\begin{array}{ccc} C^0(R_{2\alpha-2\delta}(\tilde{L});\mathbb{R}) & \xrightarrow{\tilde{d}_{2\alpha-2\delta}} & C^1(R_{2\alpha-2\delta}(\tilde{L});\mathbb{R}) \\ \mu_0\uparrow & & \mu_1\uparrow \\ C^0(R_{2\alpha}(L);\mathbb{R}) & \xrightarrow{d_{2\alpha}} & C^1\left(R_{2\alpha}(L);\mathbb{R}\right) \end{array}$$

where $\tilde{d}_{2\alpha-2\delta}, d_{2\alpha}$ are the coboundary maps, and μ_0, μ_1 are the maps induced by the simplicial map μ. In fact, μ_0 is an isometry since it preserves norms.

Let $\nu_\alpha, \tilde{\nu}_\alpha$ be the weight-families on the edges of $L, \tilde{L}$ respectively.

Proposition 5.2.1. *If the edge weights satisfy the condition in equation 5.1, then*

$$\left|\nu_{2\alpha}([\ell_i,\ell_j]) - \tilde{\nu}_{2\alpha-2\delta}([\tilde{\ell}_i,\tilde{\ell}_j])\right| \le 4\omega_i\omega_j L_\rho \delta.$$

Proof. By our choice in equation 5.1 and the assumption on vertex weights, we obtain

$$\left|\nu_{2\alpha}([\ell_i,\ell_j]) - \tilde{\nu}_{2\alpha-2\delta}([\tilde{\ell}_i,\tilde{\ell}_j])\right| = \omega_i\omega_j \left|\rho(2\alpha - \mathbf{d}(\ell_i,\ell_j)) - \rho(2\alpha - 2\delta - \mathbf{d}(\tilde{\ell}_i,\tilde{\ell}_j))\right|.$$

By Lipschitzness of ρ, we obtain

$$\left|\rho(2\alpha - \mathbf{d}(\ell_i,\ell_j)) - \rho(2\alpha - 2\delta - \mathbf{d}(\tilde{\ell}_i,\tilde{\ell}_j))\right| \le L_\rho \left|2\delta + \mathbf{d}(\tilde{\ell}_i,\tilde{\ell}_j) - \mathbf{d}(\ell_i,\ell_j)\right|$$

which is less than $L_\rho 4\delta$ by triangle inequality. This finishes the proof. ∎

Now, let $(\tilde{d}_{2\alpha-2\delta})^+$ and $(d_{2\alpha})^+$ be the weighted Moore-Penrose pseudoinverses

$$(\tilde{d}_{2\alpha-2\delta})^+ : C^1(R_{2\alpha-2\delta}(\tilde{L});\mathbb{R}) \to C^0(R_{2\alpha-2\delta}(\tilde{L});\mathbb{R})$$
$$(d_{2\alpha})^+ : C^1\left(R_{2\alpha}(L);\mathbb{R}\right) \to C^0\left(R_{2\alpha}(L);\mathbb{R}\right)$$

of the coboundary maps $\tilde{d}_{2\alpha-2\delta}$ and $d_{2\alpha}$ respectively. For the the image

$$0 \neq \eta \in \mathsf{Img}\{C^1\left(R_{2\alpha}(L);\mathbb{Z}\right) \to C^1\left(R_{2\alpha}(L);\mathbb{R}\right)\},$$

let

$$\begin{aligned} C^0(R_{2\alpha}(L);\mathbb{R}) \ni \tau = -d_{2\alpha}^{+}(\eta) \qquad & \tilde{\tau} = -(\tilde{d}_{2\alpha-2\delta})^{+}\mu_1(\eta) \in C^0(R_{2\alpha-2\delta}(\tilde{L});\mathbb{R}) \\ C^1(R_{2\alpha}(L);\mathbb{R}) \ni \theta = \eta + d_{2\alpha}(\tau) \qquad & \tilde{\theta} = \mu_1(\eta) + \tilde{d}_{2\alpha-2\delta}(\tilde{\tau}) \in C^1(R_{2\alpha-2\delta}(\tilde{L});\mathbb{R}). \end{aligned}$$

We want to bound the difference between the circle valued functions

$$\begin{aligned} \tilde{h} : \tilde{L}^{(\alpha-\delta)} &\to S^1 \\ B_{\alpha-\delta}(\tilde{\ell}_j) \ni b &\to \exp\left\{2\pi i\left(\tilde{\tau}_j + \sum_r \psi_r(b)\tilde{\theta}_{jr}\right)\right\} \end{aligned}$$

and

$$\begin{aligned} h : L^{(\alpha)} &\to S^1 \\ B_{\alpha}(\ell_j) \ni b &\to \exp\left\{2\pi i\left(\tau_j + \sum_r \varphi_r(b)\theta_{jr}\right)\right\} \end{aligned}$$

on their common domain $\tilde{L}^{(\alpha-\delta)} \subset L^{(\alpha)}$, where the partitions of unity $\{\varphi_r\}_r$ and $\{\psi_r\}_r$ are as defined above.

Lemma 5.2.2. *The functions $\tilde{h}$ and h defined as above differ on their common domain by*

$$|h(b) - \tilde{h}(b)| \leq 2\pi\left(\frac{\phi(\alpha-\delta)}{\overline{\psi}(b)}||\mu_0(\tau) - \tilde{\tau}||_\omega + ||\varphi(b) - \psi(b)||_1||\theta||_\infty\right),$$

where $||\varphi(b) - \psi(b)||_1 = \sum_r |\varphi_r(b) - \psi_r(b)|$.

Proof. From the definitions of θ and $\tilde{\theta}$, we have that

$$\tilde{\theta}_{jr} - \left(\tilde{\tau}_r - \tilde{\tau}_j\right) = \mu_1(\eta)_{jr} = \eta_{jr} = \theta_{jr} - \left(\tau_r - \tau_j\right)$$

if and only if the index pair (j,r) represents an edge in both $R_{2\alpha-2\delta}(\tilde{L})$ and $R_{2\alpha}(L)$. If $b \in B_{\alpha-\delta}(\tilde{\ell}_j) \subset B_\alpha\left(\ell_j\right)$, it is indeed the case and therefore,

$$\begin{aligned} |h(b) - \tilde{h}(b)| &= \left|\exp\left\{2\pi i\left(\tau_j + \sum_r \varphi_r(b)\theta_{jr}\right)\right\} - \exp\left\{2\pi i\left(\tilde{\tau}_j + \sum_r \psi_r(b)\tilde{\theta}_{jr}\right)\right\}\right| \\ &\leq 2\pi\left|\left(\tau_j - \tilde{\tau}_j\right) + \sum_r\left(\varphi_r(b)\theta_{jr} - \psi_r(b)\tilde{\theta}_{jr}\right)\right| \end{aligned}$$

We rewrite τ_j as $\sum_r \psi_r(b)\tau_j$. Observe that if $\psi_r(b) \neq 0$, then $B_{\alpha-\delta}(\tilde{\ell}_r) \cap B_{\alpha-\delta}(\tilde{\ell}_j)$. This is equivalent to saying that the index (j, r) represents an edge in both $R_{2\alpha-2\delta}(\tilde{L})$ and $R_{2\alpha}(L)$. Therefore,

$$\begin{aligned}
|h(b) - \tilde{h}(b)| &\leq 2\pi \left| \sum_r \psi_r(b) \left(\tau_r - \tilde{\tau}_r + \tilde{\theta}_{jr} - \theta_{jr} \right) + \sum_r \left(\varphi_r(b)\theta_{jr} - \psi_r(b)\tilde{\theta}_{jr} \right) \right| \\
&= 2\pi \left| \sum_r \psi_r(b) \left(\tau_r - \tilde{\tau}_r \right) + \sum_r \left(\varphi_r(b) - \psi_r(b) \right) \theta_{jr} \right| \\
&\leq 2\pi \left(\sum_r \psi_r(b) \cdot |\tau_r - \tilde{\tau}_r| + \sum_r |\theta_{jr}| \cdot |\varphi_r(b) - \psi_r(b)| \right) \\
&\leq 2\pi \left(\sum_r \frac{\omega_r}{\overline{\psi}(b)} \phi(\alpha-\delta) \cdot |\tau_r - \tilde{\tau}_r| + \sum_r |\theta_{jr}| \cdot |\varphi_r(b) - \psi_r(b)| \right) \\
&\leq 2\pi \left(\frac{\phi(\alpha-\delta)}{\overline{\psi}(b)} ||\mu_0(\tau) - \tilde{\tau}||_\omega + ||\varphi(b) - \psi(b)||_1 ||\theta||_\infty \right).
\end{aligned}$$

where the last step comes from Hölder's inequality. □

The ℓ^1 bound on the difference of partitions of unity was proved in lemma 5.1.3. The rest of this section is dedicated to finding an upper bound for $||\mu_0(\tau) - \tilde{\tau}||_\omega$. By definition, we have

$$(\tilde{\tau} - \mu_0(\tau)) = (\tilde{d}_{2\alpha-2\delta})^+ \mu_1(\eta) - \mu_0 d_{2\alpha}^+(\eta).$$

Using that $\eta + d_{2\alpha}(\tau) = \theta$, the commutativity relations, and the properties of the Moore-Penrose pseudoinverse, we get

$$\begin{aligned}
(\tilde{d}_{2\alpha-2\delta})^+ \mu_1(\eta) &= (\tilde{d}_{2\alpha-2\delta})^+ \left(-\mu_1 d_{2\alpha}(\tau) + \mu_1(\theta) \right) \\
&= -(\tilde{d}_{2\alpha-2\delta})^+ \tilde{d}_{2\alpha-2\delta} \mu_0(\tau) + (\tilde{d}_{2\alpha-2\delta})^+ \mu_1(\theta) \\
&= -\mathsf{Pr}_{\mathsf{Ker}(\tilde{d}_{2\alpha-2\delta})^\perp} (\mu_0(\tau)) + (\tilde{d}_{2\alpha-2\delta})^+ \mu_1(\theta)
\end{aligned}$$

where we have used that $(\tilde{d}_{2\alpha-2\delta})^+ \tilde{d}_{2\alpha-2\delta}$ projects surjectively onto the orthogonal complement of $\mathsf{Ker}(\tilde{d}_{2\alpha-2\delta})$. Using the connectedness of $R_{2\alpha-2\delta}(\tilde{L})$, we get

$$\begin{aligned}
(\tilde{\tau} - \mu_0(\tau)) &= -\mathsf{Pr}_{\mathsf{Ker}(\tilde{d}_{2\alpha-2\delta})^\perp} (\mu_0(\tau)) + (\tilde{d}_{2\alpha-2\delta})^+ \mu_1(\theta) + \mu_0(\tau) \\
&= \mathsf{Pr}_{\mathsf{Ker}(\tilde{d}_{2\alpha-2\delta})} (\mu_0(\tau)) + (\tilde{d}_{2\alpha-2\delta})^+ \mu_1(\theta) \\
&= (\tilde{d}_{2\alpha-2\delta})^+ \mu_1(\theta)
\end{aligned}$$

where the last step is true because $\mu_0(\tau) \perp \mathsf{Ker}(\tilde{d}_{2\alpha-2\delta})$.

Hence, it is enough to bound the 0-cochain $(\tilde{d}_{2\alpha-2\delta})^{+}\mu_1(\theta) \in C^0(R_{2\alpha-2\delta}(\tilde{L});\mathbb{R})$. First, since the Moore-Penrose pseudoinverse satisfies $A^+ = A^+AA^+$ for any matrix A, we obtain

$$\left\|(\tilde{d}_{2\alpha-2\delta})^{+}\mu_1(\theta)\right\|_{\omega}^{2} \leq \left\|(\tilde{d}_{2\alpha-2\delta})^{+}\right\|^{2} \left\|(\tilde{d}_{2\alpha-2\delta})(\tilde{d}_{2\alpha-2\delta})^{+}\mu_1(\theta)\right\|_{2\alpha-2\delta}^{2} \tag{5.2}$$

where the first norm is the weighted operator norm. We will bound both the norms separately. Let

$$(\tilde{d}_{2\alpha-2\delta})^{*} : C^1\left(R_{2\alpha-2\delta}(\tilde{L});\mathbb{R}\right) \to C^0\left(R_{2\alpha-2\delta}(\tilde{L});\mathbb{R}\right)$$

be the adjoint of $\tilde{d}_{2\alpha-2\delta}$ with respect to the inner products defined by the weights $\tilde{\nu}_{2\alpha-2\delta}$ and the probability measure $\tilde{\omega} = \omega$. Then it satisfies the following proposition:

Proposition 5.2.3. *Let K be a simplicial complex with edge weights*

$$\nu_K : K^{(1)} \to [0, \infty)$$

If $d : C^0(K,\mathbb{R}) \to C^1(K,\mathbb{R})$ is the coboundary operator, let $d^* : C^1(K,\mathbb{R}) \to C^0(K,\mathbb{R})$ be its adjoint corresponding to the inner product on $1-$chains induced by ν_K, then d^*d is the graph Hodge Laplacian on $K^{(1)}$.

An application of the spectral theorem to the graph Hodge Laplacian operator $\Delta_H = (\tilde{d}_{2\alpha-2\delta})^{*}\tilde{d}_{2\alpha-2\delta}$ yields an orthonormal eigenbasis $\tilde{\mathbf{x}}_1, \ldots, \tilde{\mathbf{x}}_N$ for $C^0(R_{2\alpha-2\delta}(\tilde{L});\mathbb{R})$ with eigenvalues $0 = (\lambda_1)^2 < (\lambda_2)^2 \leq \cdots (\lambda_N)^2$. The eigenvalue 0 has multiplicity 1 because $R_{2\alpha-2\delta}(\tilde{L})$ is connected.

Let $\mathcal{B}$ be another orthonormal basis of $C^0(R_{2\alpha-2\delta}(\tilde{L});\mathbb{R})$ defined as

$$\mathcal{B} = \left\{ \frac{1}{\sqrt{\omega_j}}[\tilde{\ell}_j] \;\middle|\; j = 1, \ldots, N \right\}.$$

Then with the respect to the basis $\mathcal{B}$, the matrix $\mathcal{L}_H$ of Δ_H can be computed as

$$\begin{aligned}
\mathcal{L}_H(j,k) &= \left\langle \Delta_H \left(\frac{1}{\sqrt{\omega_j}} [\tilde{\ell}_j] \right), \frac{1}{\sqrt{\omega_k}} [\tilde{\ell}_k] \right\rangle_{\omega} \\
&= \frac{1}{\sqrt{\omega_j \omega_k}} \left\langle \tilde{d}_{2\alpha-2\delta}([\tilde{\ell}_j]), \tilde{d}_{2\alpha-2\delta}([\tilde{\ell}_k]) \right\rangle_{2\alpha-2\delta} \\
&= \begin{cases} \frac{1}{\omega_j} \sum_{1 \le i \le N} \tilde{\nu}_{2\alpha-2\delta}([\tilde{\ell}_i, \tilde{\ell}_j]) & \text{if } j = k \\ -\frac{\tilde{\nu}_{2\alpha-2\delta}([\tilde{\ell}_j, \tilde{\ell}_k])}{\sqrt{\omega_j \omega_k}} & \text{if } j \neq k. \end{cases}
\end{aligned}$$

Note that since change of basis doesn't change the eigenvalues of the matrix, the smallest non-zero eigenvalue of $\mathcal{L}_H$ is also $(\lambda_2)^2$. This will be used while proving the Wasserstein stability later in this Chapter. We are now ready to bound the operator norm.

Lemma 5.2.4. *The weighted pseudoinverse of the coboundary map $\tilde{d}_{2\alpha-2\delta}$ satisfies*

$$\left|\left| (\tilde{d}_{2\alpha-2\delta})^+ \right|\right|^2 = \left(\frac{1}{\lambda_2} \right)^2$$

Proof. Since the eigenvalues of $(\tilde{d}_{2\alpha-2\delta})^* \tilde{d}_{2\alpha-2\delta}$ are $0 = (\lambda_1)^2 < (\lambda_2)^2 \le \cdots (\lambda_N)^2$, we get that λ_2 is the maximum eigenvalue of the pseudoinverse $(\tilde{d}_{2\alpha-2\delta})^+$. Therefore,

$$\left|\left| (\tilde{d}_{2\alpha-2\delta})^+ \right|\right| = \left(\frac{1}{\lambda_2} \right).$$

□

Lemma 5.2.5. *The weighted norm of $(\tilde{d}_{2\alpha-2\delta})(\tilde{d}_{2\alpha-2\delta})^+ \mu_1(\theta)$ satisfies*

$$\left|\left| (\tilde{d}_{2\alpha-2\delta})(\tilde{d}_{2\alpha-2\delta})^+ \mu_1(\theta) \right|\right|^2_{2\alpha-2\delta} \le 4\delta L_\rho \left|\left|\theta\right|\right|_\infty \left|\left| (\tilde{d}_{2\alpha-2\delta})^+ \mu_1(\theta) \right|\right|_\omega .$$

Proof. First, let $\mathbf{v} = \mu_0^{-1} (\tilde{d}_{2\alpha-2\delta})^+ \mu_1(\theta) \in C^0(R_{2\alpha}(L); \mathbb{R})$. Then

$$\begin{aligned}
\left|\left| (\tilde{d}_{2\alpha-2\delta})(\tilde{d}_{2\alpha-2\delta})^+ \mu_1(\theta) \right|\right|^2_{2\alpha-2\delta} &= \left\langle \mu_1(\theta), (\tilde{d}_{2\alpha-2\delta})(\tilde{d}_{2\alpha-2\delta})^+ \mu_1(\theta) \right\rangle_{2\alpha-2\delta} \\
&= \left\langle \mu_1(\theta), \mu_1 d_{2\alpha}(\mathbf{v}) \right\rangle_{2\alpha-2\delta}.
\end{aligned}$$

The last inner product can be computed as

$$
\begin{aligned}
&\sum_e \tilde{\nu}_{2\alpha-2\delta}(e)\theta(e)d_{2\alpha}(\mathbf{v})(e)\\
&=\sum_e \left(\tilde{\nu}_{2\alpha-2\delta}(e)-\nu_{2\alpha}(e)\right)\theta(e)d_{2\alpha}(\mathbf{v})(e)\\
&=\sum_{1\le j<k\le N}\left(\tilde{\nu}_{2\alpha-2\delta}\left([\tilde{\ell}_j,\tilde{\ell}_k]\right)-\nu_{2\alpha}\left([\ell_j,\ell_k]\right)\right)\theta_{jk}(\mathbf{v}_k-\mathbf{v}_j)
\end{aligned}
$$

Taking absolute values, we obtain

$$
\begin{aligned}
&\left|\left|(\tilde{d}_{2\alpha-2\delta})(\tilde{d}_{2\alpha-2\delta})^{+}\mu_1\left(\theta\right)\right|\right|^2_{2\alpha-2\delta}\\
\le&\sum_{1\le j<k\le N}\left|\tilde{\nu}_{2\alpha-2\delta}\left([\tilde{\ell}_j,\tilde{\ell}_k]\right)-\nu_{2\alpha}\left([\ell_j,\ell_k]\right)\right|\left|\theta_{jk}\right|\left(|\mathbf{v}_k|+\left|\mathbf{v}_j\right|\right)\\
\le&\, 4\delta L_\rho\left|\left|\theta\right|\right|_\infty\sum_{1\le j<k\le N}\omega_j\omega_k\left(|\mathbf{v}_k|+\left|\mathbf{v}_j\right|\right)\\
\le&\, 4\delta L_\rho\left|\left|\theta\right|\right|_\infty\sum_{1\le j,k\le N}\omega_j\omega_k\left(\left|\mathbf{v}_j\right|\right)\\
=&\, 4\delta L_\rho\left|\left|\theta\right|\right|_\infty\sum_{1\le j\le N}\omega_j\left|\mathbf{v}_j\right|\\
\le&\, 4\delta L_\rho\left|\left|\theta\right|\right|_\infty\left(\sum_{1\le j\le N}\omega_j\left|\mathbf{v}_j\right|^2\right)^{\frac{1}{2}}\\
=&4\delta L_\rho\left|\left|\theta\right|\right|_\infty\left|\left|\mathbf{v}\right|\right|_\omega.
\end{aligned}
$$

Since μ_0 is an isometry, we obtain

$$
\left|\left|(\tilde{d}_{2\alpha-2\delta})(\tilde{d}_{2\alpha-2\delta})^{+}\mu_1\left(\theta\right)\right|\right|^2_{2\alpha-2\delta}\le 4\delta L_\rho\left|\left|\theta\right|\right|_\infty\left|\left|(\tilde{d}_{2\alpha-2\delta})^{+}\mu_1(\theta)\right|\right|_\omega. \tag{5.3}
$$

□

Now, we can put together the two lemmas to bound the difference between $\tilde{\tau}$ and $\mu_0(\tau)$:

Proposition 5.2.6. *The difference between cochains $\tilde{\tau}$ and $\mu_0(\tau)$ is bounded by:*

$$
\left|\left|(\tilde{\tau}-\mu_0(\tau))\right|\right|_\omega\le\frac{\|\theta\|_\infty 4L_\rho}{(\lambda_2)^2}\delta.
$$

Proof. Recall equation 5.2:

$$\left\|(\tilde{d}_{2\alpha-2\delta})^{+}\mu_1\left(\theta\right)\right\|_{\omega}^{2} \leq \left\|(\tilde{d}_{2\alpha-2\delta})^{+}\right\|^{2}\left\|(\tilde{d}_{2\alpha-2\delta})(\tilde{d}_{2\alpha-2\delta})^{+}\mu_1\left(\theta\right)\right\|_{2\alpha-2\delta}^{2}.$$

By Lemmas 5.2.4 and 5.2.5, we obtain

$$\left\|(\tilde{d}_{2\alpha-2\delta})^{+}\mu_1\left(\theta\right)\right\|_{\omega}^{2} \leq \left(\frac{1}{\lambda_2}\right)^{2} 4\delta L_\rho \left|\left|\theta\right|\right|_\infty \left\|(\tilde{d}_{2\alpha-2\delta})^{+}\mu_1(\theta)\right\|_{\omega}.$$

This finishes the proof since the above implies

$$||(\tilde{\tau}-\mu_0(\tau))||_\omega \leq \frac{\|\theta\|_\infty 4L_\rho}{(\lambda_2)^2}\delta.$$

□

We now use Lemma 5.2.2 and Proposition 5.2.6 to achieve the following result:

Theorem 5.2.7. *If (L,ω) and $(\tilde{L},\tilde{\omega})$ are landmark sets in a weight preserving bijective correspondence, and h and $\tilde{h}$ are the associated circle-valued functions, then*

$$||h-\tilde{h}||_\infty \leq \frac{8\pi\|\theta\|_\infty}{\min\limits_{j}\omega_j}\delta\left(\frac{\phi(\alpha-\delta)}{\phi\left(\alpha-\delta-r_{\tilde{L}}\right)}\frac{L_\rho}{(\lambda_2)^2}+\frac{L_\phi}{\phi\left(\alpha-r_L\right)}\right).$$

Proof. Recall that Lemma 5.1.3 gives us the bounds on partitions and therefore,

$$||\varphi(b)-\psi(b)||_1 \leq \frac{4\delta L_\phi}{\bar{\varphi}(b)}.$$

By Lemma 5.2.2 and Proposition 5.2.6, we get that

$$|h(b)-\tilde{h}(b)| \leq 8\pi\|\theta\|_\infty\delta\left(\frac{\phi(\alpha-\delta)}{\overline{\psi}(b)}\frac{L_\rho}{(\lambda_2)^2}+\frac{L_\phi}{\bar{\varphi}(b)}\right). \tag{5.4}$$

The average terms $\overline{\varphi}(b)$ and $\overline{\psi}(b)$ depend on the choice of b. Since $L^{(\alpha)}$ covers $\mathbb{M}$, then there exists j such that $\mathbf{d}(b,\ell_j) \leq r_L$ (the cover radius of L). Therefore,

$$\phi(\alpha-r_L) \leq \phi(\alpha-\mathbf{d}(b,\ell_j)) \leq \sum_{k=1}^{N}\phi(\alpha-\mathbf{d}(b,\ell_k)).$$

Similarly, since $\tilde{L}^{(\alpha-\delta)}$ also covers $\mathbb{M}$,

$$\phi(\alpha-\delta-r_{\tilde{L}}) \leq \sum_{k=1}^{N} \phi(\alpha-\delta-\mathbf{d}(b,\tilde{\ell}_k)).$$

Consequently,

$$\overline{\varphi}(b) = \sum_{k=1}^{N} \omega_k \phi(\alpha-\mathbf{d}(b,\ell_k)) \geq \min_j \omega_j \left(\sum_{k=1}^{N} \phi(\alpha-\mathbf{d}(b,\ell_k))\right) \geq \min_j \omega_j \phi(\alpha-r_L)$$
$$\overline{\psi}(b) = \sum_{k=1}^{N} \omega_k \phi(\alpha-\delta-\mathbf{d}(b,\tilde{\ell}_k)) \geq \min_j \omega_j \left(\sum_{k=1}^{N} \phi(\alpha-\delta-\mathbf{d}(b,\tilde{\ell}_k))\right) \geq \min_j \omega_j \phi(\alpha-\delta-r_{\tilde{L}}).$$

This implies that

$$||h-\tilde{h}||_\infty \leq \frac{8\pi\|\theta\|_\infty}{\min_j \omega_j}\delta\left(\frac{\phi(\alpha-\delta)}{\phi\left(\alpha-\delta-r_{\tilde{L}}\right)}\frac{L_\rho}{(\lambda_2)^2}+\frac{L_\phi}{\phi\left(\alpha-r_L\right)}\right).$$

□

Remark 5.2.8. Before we proceed to the general case, it is worth noting that since h and $\tilde{h}$ are functions taking values in a circle of unit radius, the ℓ^∞ distance on the left can not exceed 2. Since the expression on the right is a real number, the bound in Theorem 5.2.7 only tells us something about stability, if

$$\delta \leq \left[\frac{8\|\theta\|_\infty}{\min_j \omega_j}\left(\frac{\phi(\alpha-\delta)}{\phi\left(\alpha-\delta-r_{\tilde{L}}\right)}\frac{L_\rho}{(\lambda_2)^2}+\frac{L_\phi}{\phi\left(\alpha-r_L\right)}\right)\right]^{-1}.$$

5.3 The general Wasserstein noise model

Let $L, \tilde{L}$ be finite sets with different cardinalities, say N and $\tilde{N}$ respectively. Also, assume that ω and $\tilde{\omega}$ are the probability weights on the two sets respectively. Assume that these sets satisfy

$$d_{\mathcal{W},\infty}((L,\omega),(\tilde{L},\tilde{\omega})) < \delta.$$

In Section 5.3.1, we will construct sets (L_m, ω_m) and $(\tilde{L}_m, \tilde{\omega}_m)$ such that they are in a weight preserving bijection. As a result, the coordinates generated by (L_m, ω_m) and $(\tilde{L}_m, \tilde{\omega}_m)$ will be compared using bounds from the geometric stability case.

5.3.1 Landmark Augmentation

Let $(\mathbb{M}, \mathbf{d})$ be a metric space. Let $L, \tilde{L} \subset \mathbb{M}$ be finite sets with $N = |L|$ and $\tilde{N} = \left|\tilde{L}\right|$ together with measures ω and $\tilde{\omega}$ on $\mathbb{M}$ supported in L and $\tilde{L}$ respectively. The goal of this section is to build extended landmark sets L_m and $\tilde{L}_m$, and measures ω_m and $\tilde{\omega}_m$ supported on L_m and $\tilde{L}_m$ respectively, such that L_m is in a weight preserving bijection with $\tilde{L}_m$.

The first step in our construction is to provide an algorithm to extend one landmark set in a controlled manner. The following lemma showcases the basic tools for such construction.

Lemma 5.3.1. *For any matrix $A \in \mathbb{R}^{n\times m}$, there exists a matrix B such that*

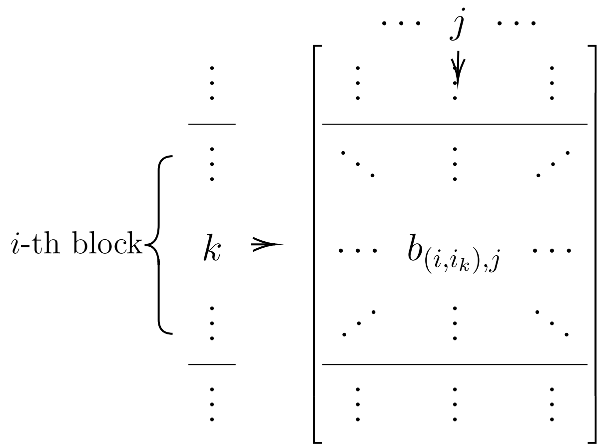
1. *B has size $n' \times m$ where n' is the number of non-zero entries in A,*
2. *B has at most one non-zero entry per row, and*
3. *for each $1 \leq j \leq m$, the non-zero entries of $A_{\cdot,j}$ and $B_{\cdot,j}$ are the same.*

Proof. See [34].

□

Remark 5.3.2. Conditions (1) and (2) imply that there is exactly one non-zero entry per row in B. Therefore, the number of non-zero entries in B is the same as in A.

Remark 5.3.3. The matrix B can be indexed using a block-wise construction. From here on, we will denote by $b_{(i,i_k),j}$ the entry of B in the k-th row of the i-th block corresponding to the j-th column.



Let d_δ be the metric on $\mathbb{N}$ such that for any $(s,t) \in \mathbb{N}^2$, $d_\delta(s,t) = \delta_{s,t}$, where $\delta_{s,t}$ is 1 if $s = t$ and 0 otherwise.

Theorem 5.3.4. *Let $\beta > 0$, let $d_\beta = \mathbf{d} + \beta d_\delta$ be a metric on $\mathbb{M} \times \mathbb{N}$ and let ω and $\tilde{\omega}$ be measures on $(\mathbb{M}, d)$ supported on $L, \tilde{L}$ respectively. If $d_{\mathcal{W},\infty}((L,\omega),(\tilde{L},\tilde{\omega})) < \delta$, then there exist functions $m : L \to \mathbb{N}$ and $\tilde{m} : \tilde{L} \to \mathbb{N}$ and measures ω_m and $\tilde{\omega}_m$ on $(\mathbb{M}, d_\beta)$ with supports*

$$L_m = \{(\ell, k) \in L \times \mathbb{N} : 1 \leq k \leq m(\ell)\}$$
$$\tilde{L}_m = \left\{(\tilde{\ell}, s) \in \tilde{L} \times \mathbb{N} : 1 \leq s \leq \tilde{m}(\tilde{\ell})\right\}$$

such that

1. *$\omega(\ell) = \sum_{k=1}^{m(\ell)} \omega_m(\ell, k)$ for all $\ell \in L$ and $\tilde{\omega}(\tilde{\ell}) = \sum_{s=1}^{\tilde{m}(\tilde{\ell})} \omega_m(\tilde{\ell}, s)$ for all $\tilde{\ell} \in \tilde{L}$.*
2. *There is a bijection $\vartheta : L_m \to \tilde{L}_m$ such that $\omega_m(\ell, k) = \tilde{\omega}_m(\vartheta(\ell, k))$ and*

$$d_\beta((\ell, k), \vartheta(\ell, k)) < \delta + \beta$$

for all $(\ell, k) \in L_m$.

Proof. Since $d_{\mathcal{W},\infty}((L,\omega),(\tilde{L},\tilde{\omega})) < \delta$, there exists $\mu \in \mathcal{M}(\omega, \tilde{\omega})$ such that

$$\sup_{(a,b) \in R(\mu)} \mathbf{d}(a, b) < \delta$$

where R is the relation associated to the coupling measure μ as in section 2.5.3. Let M^0 be the $N \times \tilde{N}$ matrix where $m_{j,k} = \mu(\ell_j, \tilde{\ell}_k)$ for $\ell_j \in L$ and $\tilde{\ell}_k \in \tilde{L}$. An application of Lemma 5.3.1 to M^0 yields a matrix M^1 such that

- M^1 has size $N' \times \tilde{N}$ where N' is the number of non-zero entries in M^0,
- M^1 has at most one non-zero entry per row and
- the non-zero entries in each column of M^0 and M^1 are the same.

Remark 5.3.3 showcases the fact that we can index the entries in M^1 as $m^1_{(i,i_k),j}$, where $1 \le i \le N$, $1 \le j \le m$ and $1 \le k \le n_i$ where n_i is the number of non-zero elements in the i-th row of M^0. We now apply Lemma 5.3.1 to $(M^1)^T$ to obtain a matrix M^2 such that

- M^2 has size $\tilde{N}' \times N'$ where $\tilde{N}'$ is the number of non-zero entries in $(M^1)^T$,
- M^2 has at most one non-zero entry per row, and
- the non-zero entries in each column of $(M^1)^T$ and M^2 are the same.

Observe that $\tilde{N}'$ is the number of non-zero entries in M^1, and by Remark 5.3.2, it is the same as the number of non-zero entries in M^0. Thus $\tilde{N}' = N'$, making M^2 a square matrix.

Since the sets of non-zero entries for each column of M^2 and $(M^1)^T$ coincide and M^1 has at most one non-zero entry per row, we conclude that M^2 has at most one non-zero entry in each column and row.

By using the convention in 5.3.3, we can index the elements of M^2 as $m^2_{(j,j_s),(i,i_k)}$ with $1 \le j \le \tilde{N}$, $1 \le s \le \tilde{n}_j$, where $\tilde{n}_j$ is the number of non-zero elements in the j-th column of M and $1 \le i \le N$, $1 \le k \le n_i$ where n_i is the number of non-zero elements in the i-th row of M. If we define

$$L_m = \bigcup_{i=1}^{N} \{(\ell_i, k) \in \mathbb{M} \times \mathbb{N} : 1 \le k \le n_i\}$$
$$\tilde{L}_m = \bigcup_{j=1}^{\tilde{N}} \left\{(\tilde{\ell}_j, s) \in \mathbb{M} \times \mathbb{N} : 1 \le s \le \tilde{n}_j\right\}$$

then $M := (M^2)^T$ defines a measure on $(\mathbb{M} \times \mathbb{N}) \times (\mathbb{M} \times \mathbb{N})$ supported on $L_m \times \tilde{L}_m$. Let

$$\omega_m(\ell_i, k) = \sum_{1 \leq j \leq \tilde{N}} \sum_{1 \leq s \leq \tilde{n}_j} m_{(i,i_k),(j,j_s)}$$

$$\tilde{\omega}_m(\tilde{\ell}_j, s) = \sum_{1 \leq i \leq N} \sum_{1 \leq k \leq n_j} m_{(i,i_k),(j,j_s)}.$$

Notice that the projection on the first component $\pi_1 : \mathbb{M} \times \mathbb{N} \to \mathbb{M}$ is such that $\pi_1(L_m) = \{\ell_i \in \mathbb{M} : 1 \leq i \leq N\} = L$ and $\pi_1(\tilde{L}_m) = \left\{\tilde{\ell}_j \in \mathbb{M} : 1 \leq j \leq \tilde{N}\right\} = \tilde{L}$. And since $\pi_1^{-1}(\ell_i) = \{l_i\} \times \mathbb{N}$, then $\pi_1^{-1}(\ell_i) \cap L_m = \{(\ell_i, k) \in \mathbb{M} \times \mathbb{N} : 1 \leq k \leq n_i\}$ which implies that

$$\begin{aligned}
\sum_{k=1}^{n_i} \omega_m(\ell_i, k) &= \sum_{1 \leq k \leq n_i} \sum_{1 \leq j \leq \tilde{N}} \sum_{1 \leq s \leq \tilde{n}_j} m_{(i,i_k),(j,j_s)} \\
&= \sum_{1 \leq j \leq \tilde{N}} \sum_{1 \leq s \leq \tilde{n}_j} \underbrace{\left(\sum_{1 \leq k \leq n_i} m_{(i,i_k),(j,j_s)} \right)}_{\text{entries in the } i\text{-th row of } M} \\
&= \sum_{1 \leq j \leq \tilde{N}} m_{i,j} = \omega(\ell_i).
\end{aligned}$$

An analogous argument shows that

$$\sum_{s=1}^{\tilde{n}_i} \tilde{\omega}_m(\tilde{\ell}_i, s) = \tilde{\omega}(\tilde{\ell}_j).$$

Furthermore since M has only one non-zero entry per row and column, it defines a bijection ϑ between L_m and $\tilde{L}_m$ such that $\omega_m(\ell_i, k) = \tilde{\omega}_m(\vartheta(\ell_i, k))$. Finally, consider the following metric on $\mathbb{M} \times \mathbb{N}$:

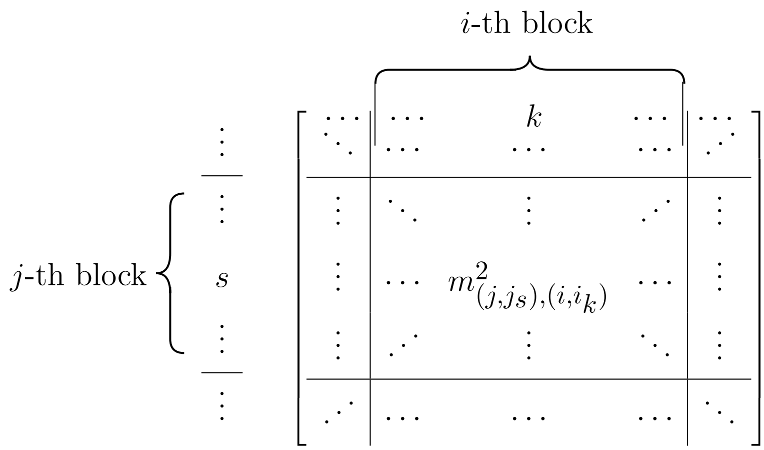
$$d_\beta((x, s), (y, t)) = \mathbf{d}(x, y) + \beta d_\delta(s, t).$$

Using this metric we obtain,

$$d_\beta((\ell_i, k), (\tilde{\ell}_j, s)) = \mathbf{d}(\ell_i, \tilde{\ell}_j) + \beta d_\delta(k, s) < \delta + \beta.$$

This finishes the proof. □

Remark 5.3.5. Index convention for M^2 as defined in theorem 5.3.4.



Notation 5.3.6. From this point on we will denote the element $(\ell_j, k) \in L_m$ as $\ell_{j,k} \in \mathbb{M}$. This notation allows us to make explicit the fact that we will be treating L_m as a multiset of $(\mathbb{M}, \mathbf{d})$. Furthermore $\mathbf{d}\left(\ell_{j,k}, \ell_{j,k'}\right) = \mathbf{d}(\ell_j, \ell_j) = 0$ for any $\ell_j \in L$ and $k, k' \in \mathbb{N}$.

Using the projection π_1 in Theorem 5.3.4 we have well defined maps

$$\begin{aligned} \iota : L &\to L_m & \quad p : L_m &\to L \\ \ell_j &\mapsto \ell_{j,1} & \quad \ell_{j,s} &\mapsto \pi_1(l_j, s) = \ell_j. \end{aligned}$$

Moreover, they induce simplicial maps for the Rips complex at scale $\alpha > 0$

$$\iota : R_\alpha(L) \to R_\alpha(L_m) \quad \text{and} \quad p : R_\alpha(L_m) \to R_\alpha(L).$$

Definition 5.3.7. Given simplicial maps $s, t : K \to K'$, they are said to be *contiguous* if for any simplex $\sigma \in K$ then $s(\sigma) \cup t(\sigma) \in K'$.

Proposition 5.3.8. *Let $s, t : K \to K'$ be contiguous simplicial maps between simplicial complexes K and K'. Then $s_* = t_* : H_q(K) \to H_q(K')$ for all $q \geq 0$.*

Proof. See [51, Theorem 12.5].

□

Proposition 5.3.9. *$\iota p : R_\alpha(L_m) \to R_\alpha(L_m)$ is contiguous to $\mathbf{I}_{R_\alpha(L_m)} : R_\alpha(L_m) \to R_\alpha(L_m)$. And $p\iota : R_\alpha(L) \to R_\alpha(L)$ is contiguous to $\mathbf{I}_{R_\alpha(L)} : R_\alpha(L) \to R_\alpha(L)$.*

Proof. Observe that $p\iota\left([\ell_{j_1},\dots,\ell_{j_q}]\right) = p\left([\ell_{j_1,1},\dots,\ell_{j_q,1}]\right) = [\ell_{j_1},\dots,\ell_{j_q}]$. Thus, $p\iota = \mathbf{I}_{R_\alpha(L)}$. On the other hand

$$\iota p\left([\ell_{j_1,s_1},\cdots,\ell_{j_q,s_q}]\right) = \iota\left([\ell_{j_1},\cdots,\ell_{j_q}]\right) = [\ell_{j_1,1},\cdots,\ell_{j_q,1}].$$

We want to show that $[\ell_{j_1,1},\cdots,\ell_{j_q,1},\ell_{j_1,s_1},\cdots,\ell_{j_q,s_q}] \in R_\alpha(L_m)$. Since both ℓ_{j_1,s_1} and $\ell_{j_1,1}$ are copies of ℓ_{j_1}, we have $\mathbf{d}(\ell_{j_1,s_1},\ell_{j_1,1}) = 0$. Then

$$\mathsf{diam}\left(\left\{\ell_{j_1,1},\cdots,\ell_{j_q,1},\ell_{j_1,s_1},\cdots,\ell_{j_q,s_q}\right\}\right) = \mathsf{diam}\left(\left\{\ell_{j_1,s_1},\cdots,\ell_{j_q,s_q}\right\}\right) < \alpha$$

since $[\ell_{j_1,s_1},\cdots,\ell_{j_q,s_q}] \in R_\alpha(L_m)$. This finishes the proof. ◻

Proposition 5.3.9 together with Proposition 5.3.8 implies that $\iota_* p_* = \mathbf{I}_{H_*(R_\alpha(L_m))}$ and $p_*\iota_* = \mathbf{I}_{H_*(R_\alpha(L))}$, thus $H_*(R_\alpha(L_m)) \cong H_*(R_\alpha(L))$ for all $\alpha > 0$. We end this part with a Corollary:

Corollary 5.3.10. *If L_m is the augmented version of a landmark set $L \subset \mathbb{M}$, then their Rips persistent homology satisfy*

$$\mathsf{bcd}_n^{\mathcal{R}}(L_m) = \mathsf{bcd}_n^{\mathcal{R}}(L)$$

in all dimensions n.

5.3.2 Coordinates old and new

We show next that (L,ω) and (L_m,ω_m) create the same coordinates. For brevity, we will denote by ω_{js}, the weight $\omega_m(\ell_{j,s})$. Let $p_i^\#$ be the cochain map from $C^i(R_\alpha(L);\mathbb{R})$ to $C^i(R_\alpha(L_m);\mathbb{R})$ induced by $p: L_m \to L$. Then for a $\eta \in C^1(R_\alpha(L);\mathbb{R})$, its image $p_1^\#(\eta) \in C^1(R_\alpha(L_m);\mathbb{R})$ is

$$p_1^\#(\eta)_{(j,s),(j',t)} := p_1^\#(\eta)([\ell_{j,s},\ell_{j',t}]) = \begin{cases} \eta\left([\ell_j,\ell_{j'}]\right) & \text{if } j \neq j' \\ 0 & \text{if } j = j', \end{cases}$$

and for a $\zeta \in C^0(R_\alpha(L);\mathbb{R})$, its image $p_0^\#(\zeta) \in C^0(R_\alpha(L_m);\mathbb{R})$ is

$$p_0^\#(\zeta)_{(j,s)} := p_0^\#(\zeta)([\ell_{j,s}]) = \zeta\left([\ell_j]\right)$$

for all $1 \leq s \leq M_j$ and j. If ω is the probability weight on the set L, then the induced weights ω_m on L_m are defined as in Section 5.3.1. Let ν_α be the weight family on the edges of L. This induces the weight family ν_α^m on the edges of L_m defined as

$$\nu_\alpha^m(\ell_{j,s}, \ell_{j',t}) = \begin{cases} \frac{\omega_{js}\omega_{j't}}{\omega_j\omega_{j'}}\nu_\alpha([\ell_j, \ell_{j'}]) & \text{if } j \neq j' \\ \omega_{js}\omega_{jt}\rho(\alpha) & \text{if } j = j' \text{ and } s \neq t \end{cases}.$$

These weight families induce inner products on $C^0(R_\alpha(L_m);\mathbb{R})$ and $C^1(R_\alpha(L_m);\mathbb{R})$ respectively. Recall that both $\tau = -d_\alpha^+(\eta)$ and $\tau_m = -(d_\alpha^m)^+(p_1^\#(\eta))$ are minimum norm solutions to minimization problems. In particular, they minimize the pairs of objective functions (F, N) and (F_m, N_m) respectively:

$$F(\chi) = \sum_{[\ell_j,\ell_{j'}]} \nu_\alpha([\ell_j, \ell_{j'}])\left(\eta_{j,j'} - d_\alpha\chi_{j,j'}\right)^2$$

$$N(\chi) = \sum_{\ell_j} \omega_j(\chi_j)^2$$

and

$$F_m(\chi) = \sum_{[\ell_{j,s},\ell_{j',t}]} \nu_\alpha^m([\ell_{j,s}, \ell_{j',t}])\left(p_1^\#(\eta)_{(j,s),(j',t)} - d_\alpha^m\chi_{(j,s),(j',t)}\right)^2$$

$$N_m(\chi) = \sum_{\ell_{j,s}} \omega_{js}(\chi_{(j,s)})^2$$

Lemma 5.3.11. *For every $\zeta \in C^0(R_\alpha(L);\mathbb{R})$, the objective functions $F(\zeta)$ and $N(\zeta)$ are equal to $F_m(p_0^\#(\zeta))$ and $N_m(p_0^\#(\zeta))$ respectively.*

Proof. For any $\zeta \in C^0(R_\alpha(L);\mathbb{R})$, the objective function F_m is evaluated at its image

$p_0^{\#}(\zeta) \in C^0\left(R_\alpha(L_m);\mathbb{R}\right)$ as

$$F_m(p_0^{\#}(\zeta)) = \sum_{\left[\ell_{j,s},\ell_{j',t}\right]} \nu_\alpha^m([\ell_{j,s},\ell_{j',t}])\left(p_1^{\#}(\eta)_{(j,s),(j',t)} - (p_0^{\#}(\zeta)_{(j',t)} - p_0^{\#}(\zeta)_{(j,s)})\right)^2$$
$$= \sum_{[\ell_j,\ell_{j'}]}\left[\sum_{s=1}^{M_j}\sum_{t=1}^{M_{j'}}\frac{\omega_{js}\omega_{j't}}{\omega_j\omega_{j'}}\nu_\alpha([\ell_j,\ell_{j'}])\left(\eta_{j,j'} - (\zeta_{j'} - \zeta_j)\right)^2\right]$$

where the outermost sum is over all distinct edges $[\ell_j, \ell_{j'}]$ and the contribution of all terms of the form $p_1^{\#}(\eta)_{(j,s),(j,t)}$ and $p_0^{\#}(\zeta)_{(j,t)} - p_0^{\#}(\zeta)_{(j,s)}$ is zero. The above sum becomes

$$\sum_{[\ell_j,\ell_{j'}]}\nu_\alpha([\ell_j,\ell_{j'}])\left(\eta_{j,j'} - (\zeta_{j'} - \zeta_j)\right)^2$$

which is just $F(\zeta)$. Similarly,

$$N_m(p_0^{\#}(\zeta)) = \sum_{\ell_{j,s}}\omega_{js}(p_0^{\#}(\zeta)_{(j,s)})^2 = \sum_{\ell_{j,s}}\omega_{js}(\zeta_j)^2 = \sum_{\ell_j}\omega_j(\zeta_j)^2$$

which is just $N(\zeta)$. □

Proposition 5.3.12. *The solution τ_m to the minimization of F_m is just $p_0^{\#}(\tau)$. That is,*

$$(\tau_m)_{(j,s)} := (\tau_m)([\ell_{j,s}]) = \tau\left([\ell_j]\right)$$

for all j and $1 \le s \le M_j$. Consequently, $\theta_m = p_1^{\#}(\eta) + (d_\alpha^m)(\tau_m)$ is

$$(\theta_m)_{(j,s),(j',t)} := \theta_m([\ell_{j,s},\ell_{j',t}]) = \begin{cases}\theta\left([\ell_j,\ell_{j'}]\right) & \text{if } j \neq j' \\ 0 & \text{if } j = j'.\end{cases}$$

Proof. To show that they are the same, we need to show that the two minimization problems are equivalent. Recall that

$$\tau_m = \operatorname*{argmin}_{\chi\in C^0(R_\alpha(L_m);\mathbb{R})}\sum_{[\ell_{j,s},\ell_{j',t}]}\nu_\alpha^m([\ell_{j,s},\ell_{j',t}])\left(p_1^{\#}(\eta)_{(j,s),(j',t)} - (\chi_{(j',t)} - \chi_{(j,s)})\right)^2.$$

For each $\chi \in C^0(R_\alpha(L_m);\mathbb{R})$, observe that the objective function evaluation

$$\begin{aligned} F_m(\chi) &= \sum_{[\ell_{j,s},\ell_{j',t}]} \nu_\alpha^m([\ell_{j,s},\ell_{j',t}]) \left(p_1^{\#}(\eta)_{(j,s),(j',t)} - d_\alpha^m \chi_{(j,s),(j',t)}\right)^2 \\ &= \sum_{[\ell_{j,s},\ell_{j',t}]} \nu_\alpha^m([\ell_{j,s},\ell_{j',t}]) \left(p_1^{\#}(\eta)_{(j,s),(j',t)} - (\chi_{(j',t)} - \chi_{(j,s)})\right)^2 \\ &\geq \sum_{[\ell_j,\ell_{j'}]} \left(\sum_{s=1}^{M_j} \sum_{t=1}^{M_{j'}} \frac{\omega_{js}\omega_{j't}}{\omega_j \omega_{j'}} \nu_\alpha([\ell_j,\ell_{j'}]) \left(\eta_{j,j'} - (\chi_{(j',t)} - \chi_{(j,s)})\right)^2 \right) \end{aligned}$$

Expanding the square term give us

$$\begin{aligned} &\sum_{[\ell_j,\ell_{j'}]} \nu_\alpha([\ell_j,\ell_{j'}]) \left(\eta_{j,j'}\right)^2 \\ &+ \sum_{[\ell_j,\ell_{j'}]} \nu_\alpha([\ell_j,\ell_{j'}]) \left(\sum_{s=1}^{M_j} \sum_{t=1}^{M_{j'}} \frac{\omega_{js}\omega_{j't}}{\omega_j \omega_{j'}} (\chi_{(j',t)} - \chi_{(j,s)})^2 \right) \\ &- \sum_{[\ell_j,\ell_{j'}]} \nu_\alpha([\ell_j,\ell_{j'}]) 2\eta_{j,j'} \left(\sum_{s=1}^{M_j} \sum_{t=1}^{M_{j'}} \frac{\omega_{js}\omega_{j't}}{\omega_j \omega_{j'}} (\chi_{(j',t)} - \chi_{(j,s)}) \right). \end{aligned}$$

Now for the second term, we use Lyapunov's inequality; in particular, that the expected value of a squared random variable is at least the square of the expected value of the random variable.

$$\begin{aligned} \geq &\sum_{[\ell_j,\ell_{j'}]} \nu_\alpha([\ell_j,\ell_{j'}]) \left(\eta_{j,j'}\right)^2 \\ &+ \sum_{[\ell_j,\ell_{j'}]} \nu_\alpha([\ell_j,\ell_{j'}]) \left(\sum_{s=1}^{M_j} \frac{\omega_{js}}{\omega_j} \chi_{(j,s)} - \sum_{t=1}^{M_{j'}} \frac{\omega_{j't}}{\omega_{j'}} \chi_{(j',t)} \right)^2 \\ &- \sum_{[\ell_j,\ell_{j'}]} \nu_\alpha([\ell_j,\ell_{j'}]) 2\eta_{j,j'} \left(\sum_{s=1}^{M_j} \frac{\omega_{js}}{\omega_j} \chi_{(j,s)} - \sum_{t=1}^{M_{j'}} \frac{\omega_{j't}}{\omega_{j'}} \chi_{(j',t)} \right) \end{aligned}$$

where the third term is a mere rearrangement. Define a 0-cochain $\zeta \in C^0\left(R_\alpha(L);\mathbb{R}\right)$ as

$$\zeta_j := \zeta([\ell_j]) = \sum_{s=1}^{M_j} \frac{\omega_{js}}{\omega_j} \chi_{(j,s)}.$$

Then its image $p_0^{\#}(\zeta) \in C^0\left(R_\alpha(L_m); \mathbb{R}\right)$ is

$$p_0^{\#}(\zeta)_{(j,s)} := p_0^{\#}(\zeta)\left([\ell_{j,s}]\right) = \sum_{s=1}^{M_j} \frac{\omega_{js}}{\omega_j} \chi_{(j,s)}$$

for all j and $1 \leq s \leq M_j$. Therefore, for each χ, we have a found a ζ satisfying

$$F_m(\chi) \geq F_m(p_0^{\#}(\zeta)) = F(\zeta) \geq F(\tau) = F_m(p_0^{\#}(\tau))$$

where the two equalities follow from lemma 5.3.11. Since $p_0^{\#}(\tau)$ is independent of choice of χ, it is indeed the minimizer of F_m. Furthermore,

$$N_m(\chi) = \sum_{\ell_{j,s}} \omega_{js}(\chi_{(j,s)})^2 = \sum_{\ell_j} \omega_j \left(\sum_{s=1}^{M_j} \frac{\omega_{js}}{\omega_j} (\chi_{(j,s)})^2 \right).$$

By Lyapunov's inequality,

$$N_m(\chi) \geq \sum_{\ell_j} \omega_j \left(\sum_{s=1}^{M_j} \frac{\omega_{js}}{\omega_j} (\chi_{(j,s)}) \right)^2 = \sum_{\ell_j} \omega_j (\zeta_j)^2 = N(\zeta) \geq N(\tau) = N_m(p_0^{\#}(\tau))$$

where ζ is as defined above. Thus, $p_0^{\#}(\tau)$ indeed minimizes N_m as well. Therefore,

$$\tau_m := (d_\alpha^m)^+ p_1^{\#}(\eta) = p_0^{\#}(\tau).$$

Consequently,

$$\begin{aligned}
(\theta_m)_{(j,s),(j',t)} &= p_1^{\#}(\eta)_{(j,s),(j',t)} + d_\alpha^m(\tau_m)_{(j,s),(j',t)} \\
&= \eta_{j,j'} + p_1^{\#}(d_\alpha(\tau))_{(j,s),(j',t)} \\
&= \eta_{j,j'} + d_\alpha(\tau)_{j,j'} = \theta_{j,j'}
\end{aligned}$$

as desired.

□

We define a partition of unity $\{\varphi^m_{(j,s)}\}_{(j,s)}$ subordinate to the cover $\left\{B_{\frac{\alpha}{2}}(\ell_{(j,s)})\right\}_{(j,s)}$ in the same fashion the original partition $\{\varphi_j\}_j$ was defined:

$$\varphi^m_{(j,s)}(b) = \frac{\omega_{js}\phi(\alpha - \mathbf{d}(\ell_{j,s}, b))}{\sum_{j=1}^{N} \sum_{s=1}^{M_j} \omega_{js}\phi(\alpha - \mathbf{d}(\ell_{j,s}, b))}.$$

for all j and $1 \leq s \leq M_j$. Note that if the increasing Lipschitz function used here is the same as the one in φ_j, the right hand side is just $\frac{\omega_{js}}{\omega_j}\varphi_j(b)$. With cocycle $p_1^{\#}(\eta)$, τ_m, θ_m, and partition of unity $\{\varphi^m_{(j,s)}\}_{(j,s)}$, we get circular coordinates:

$$B_{\frac{\alpha}{2}}(\ell_{(j,s)}) \ni b \to \exp\left\{2\pi i\left((\tau_m)_{(j,s)} + \sum_{(r,t)} \varphi^m_{(r,t)}(b)(\theta_m)_{(j,s),(r,t)}\right)\right\}.$$

We conclude in the next theorem that these coordinates agree with the coordinates generated from the landmark set L.

Theorem 5.3.13. *For all $b \in B_{\frac{\alpha}{2}}(\ell_j)$, every j, and every $1 \leq s \leq M_j$ we have that*

$$\exp\left\{2\pi i\left((\tau_m)_{(j,s)} + \sum_{(r,t)} \varphi^m_{(r,t)}(b)(\theta_m)_{(j,s),(r,t)}\right)\right\} = \exp\left\{2\pi i\left(\tau_j + \sum_r \varphi_r(b)\theta_{j,r}\right)\right\}$$

That is, the circular coordinates induced by the landmark sets L and L_m are equal.

Proof. By Proposition 5.3.12, $(\tau_m)_{(j,s)} = \tau_j$ and $(\theta_m)_{(j,s),(j',t)} = \theta_{j,j'}$. By our choices of partition of unity, we have that

$$\sum_{(r,t)} \varphi^m_{(r,t)}(b)(\theta_m)_{(j,s),(r,t)} = \sum_r \sum_{t=1}^{M_r} \frac{\omega_{rt}}{\omega_r}\varphi_r(b)\theta_{j,r} = \sum_r \varphi_r(b)\theta_{j,r}$$

for all r and $1 \leq t \leq M_r$. This finishes the proof. □

5.3.3 Wasserstein stability

In this section, we combine the previous results to prove Wasserstein stability theorem 5.3.15 for circular coordinates. We begin this section with a lemma relating the Hodge Laplacian matrix $\mathcal{L}_H$ to another such matrix defined as:

$$\mathcal{L}^m_H((j,s),(k,t)) = \begin{cases} \frac{1}{\omega_{js}} \sum\limits_{1\leq i\leq N} \sum\limits_{t=1}^{M_i} \tilde{\nu}^m_{2\alpha-2\delta}([\tilde{\ell}_{i,t}, \tilde{\ell}_{j,s}]) & \text{if } j = k \\ -\frac{\tilde{\nu}_{2\alpha-2\delta}([\tilde{\ell}_{j,s}, \tilde{\ell}_{k,t}])}{\sqrt{\omega_{js}\omega_{kt}}} & \text{if } j \neq k. \end{cases}$$

For any matrix A, let $e(A)$ denote the set of distinct eigenvalues.

Lemma 5.3.14. *Let L_m be the augmented landmark set associated to any L. If $\mathcal{L}_H$ and $\mathcal{L}_H^m$ are Hodge Laplacians for $C^0(R_\alpha(L);\mathbb{R})$ and $C^0(R_\alpha(L_m);\mathbb{R})$ respectively for any α, then*

$$e\left(\mathcal{L}_H^m\right) \subset e\left(\mathcal{L}_H\right).$$

Proof. Let Λ be an eigenvalue of $\mathcal{L}_H^m$. Then

$$\mathcal{L}_H^m \mathbf{X} = \Lambda \mathbf{X} \qquad \text{for some} \qquad \mathbf{X} = \begin{bmatrix} x_{11} \\ \vdots \\ x_{1M_1} \\ \vdots \\ x_{N1} \\ \vdots \\ x_{NM_N} \end{bmatrix}, \tag{5.5}$$

where $\mathbf{X}$ is an eigenvector for Λ. Therefore, for all $1 \leq j \leq N$, and $1 \leq s \leq M_j$, and $i = M_1 + \cdots + M_{j-1} + s$,

$$\left(\mathcal{L}_H^m \mathbf{X}\right)_i = (\Lambda \mathbf{X})_i = \Lambda x_{js}. \tag{5.6}$$

By definition of $\mathcal{L}_H^m$ and $\mathbf{X}$, we obtain that Λx_{js} is equal to the terms on the left hand side:

$$\begin{aligned} &-\sum_{k\neq j}\sum_{t=1}^{M_k}\left(\omega_{js}\omega_{kt}\right)^{\frac{1}{2}}\rho\left(\alpha-\mathbf{d}(\ell_{j,s},\ell_{k,t})\right)x_{kt}-\sum_{t\neq s}\left(\omega_{js}\omega_{jt}\right)^{\frac{1}{2}}\rho\left(\alpha\right)x_{jt} \\ &+\sum_{k\neq j}\sum_{t=1}^{M_k}\omega_{kt}\rho\left(\alpha-\mathbf{d}(\ell_{j,s},\ell_{k,t})\right)x_{js}+\sum_{t\neq s}\omega_{jt}\rho\left(\alpha\right)x_{js}. \end{aligned}$$

Adding and subtracting the term $x_{js}\omega_{js}\rho(\alpha)$, the above can be compactly written as

$$-\sum_{k=1}^{N}\sum_{t=1}^{M_k}\left(\omega_{js}\omega_{kt}\right)^{\frac{1}{2}}\rho\left(\alpha-\mathbf{d}(\ell_{j,s},\ell_{k,t})\right)x_{kt}+\sum_{k=1}^{N}\sum_{t=1}^{M_k}\omega_{kt}\rho\left(\alpha-\mathbf{d}(\ell_{j,s},\ell_{k,t})\right)x_{js}$$

. Multiplying this term by $\left(\frac{\omega_{js}}{\omega_j}\right)^{\frac{1}{2}}$ and adding for all $1 \leq s \leq M_j$, we obtain

$$\sum_{s=1}^{M_j} \left(\frac{\omega_{js}}{\omega_j}\right)^{\frac{1}{2}} \Lambda x_{js} = \sum_{s=1}^{M_j} \left(\frac{\omega_{js}}{\omega_j}\right)^{\frac{1}{2}} \left[-\sum_{k=1}^{N} \sum_{t=1}^{M_k} \left(\omega_{js}\omega_{kt}\right)^{\frac{1}{2}} \rho\left(\alpha - \mathbf{d}(\ell_{j,s}, \ell_{k,t})\right) x_{kt} \right] + \sum_{s=1}^{M_j} \left(\frac{\omega_{js}}{\omega_j}\right)^{\frac{1}{2}} \left[\sum_{k=1}^{N} \sum_{t=1}^{M_k} \omega_{kt} \rho\left(\alpha - \mathbf{d}(\ell_{j,s}, \ell_{k,t})\right) x_{js} \right]$$

Algebraic rearrangement allows the right hand side to be written as

$$-\sum_{s=1}^{M_j} \left[\sum_{k=1}^{N} \left(\omega_{js}\right)^{\frac{1}{2}} \left(\frac{\omega_{js}}{\omega_j}\right)^{\frac{1}{2}} \rho\left(\alpha - \mathbf{d}(\ell_{j,s}, \ell_{k,t})\right) (\omega_k)^{\frac{1}{2}} \left(\sum_{t=1}^{M_k} x_{kt} \left(\frac{\omega_{kt}}{\omega_k}\right)^{\frac{1}{2}} \right) \right] + \sum_{s=1}^{M_j} \left[\sum_{k=1}^{N} x_{js} \left(\frac{\omega_{js}}{\omega_j}\right)^{\frac{1}{2}} \rho\left(\alpha - \mathbf{d}(\ell_{j,s}, \ell_{k,t})\right) \left(\sum_{t=1}^{M_k} \omega_{kt} \right) \right]$$

This is equal to

$$-\left(\sum_{s=1}^{M_j} \frac{\omega_{js}}{(\omega_j)^{\frac{1}{2}}} \right) \left[\sum_{k=1}^{N} \rho\left(\alpha - \mathbf{d}(\ell_{j,s}, \ell_{k,t})\right) (\omega_k)^{\frac{1}{2}} \left(\sum_{t=1}^{M_k} x_{kt} \left(\frac{\omega_{kt}}{\omega_k}\right)^{\frac{1}{2}} \right) \right] + \left(\sum_{s=1}^{M_j} x_{js} \left(\frac{\omega_{js}}{\omega_j}\right)^{\frac{1}{2}} \right) \left[\sum_{k=1}^{N} \rho\left(\alpha - \mathbf{d}(\ell_{j,s}, \ell_{k,t})\right) \left(\sum_{t=1}^{M_k} \omega_{kt} \right) \right]$$

Since $\sum_{s=1}^{M_j} \omega_{js} = \omega_j$ and $\sum_{t=1}^{M_k} \omega_{kt} = \omega_k$, letting $y_j := \sum_{s=1}^{M_j} x_{js} \left(\frac{\omega_{js}}{\omega_j}\right)^{\frac{1}{2}}$ for each j, makes this

$$-(\omega_j)^{\frac{1}{2}} \sum_{k=1}^{N} \rho\left(\alpha - \mathbf{d}(\ell_{j,s}, \ell_{k,t})\right) (\omega_k)^{\frac{1}{2}} y_k + \sum_{k=1}^{N} y_j \rho\left(\alpha - \mathbf{d}(\ell_{j,s}, \ell_{k,t})\right) \omega_k$$

$$= -\sum_{k \neq j} \left(\omega_j \omega_k\right)^{\frac{1}{2}} \rho\left(\alpha - \mathbf{d}(\ell_{j,s}, \ell_{k,t})\right) y_k + \sum_{k=1}^{N} \omega_k \rho\left(\alpha - \mathbf{d}(\ell_{j,s}, \ell_{k,t})\right) y_j = (\mathcal{L}_H \mathbf{Y})_j$$

where $\mathbf{Y}$ is the $N \times 1$ vector

$$\mathbf{Y} = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}. \tag{5.7}$$

This yields

$$(\mathcal{L}_H \mathbf{Y})_j = \Lambda \left(\sum_{s=1}^{M_j} x_{js} \left(\frac{\omega_{js}}{\omega_j}\right)^{\frac{1}{2}} \right) = \Lambda y_j = \Lambda \mathbf{Y}_j$$

which implies that Λ is an eigenvalue for $\mathcal{L}_H$. Therefore,

$$e\left(\mathcal{L}_H^m\right) \subset e\left(\mathcal{L}_H\right).$$

□

Now, we prove the Wasserstein stability theorem for circular coordinates.

Theorem 5.3.15. *If (L, ω) and $(\tilde{L}, \tilde{\omega})$ are two weighted landmarks sets satisfying*

$$d_{\mathcal{W},\infty}((L,\omega),(\tilde{L},\tilde{\omega})) < \delta,$$

and if h and $\tilde{h}$ are the associated circular valued functions, then

$$||h-\tilde{h}||_\infty \leq \frac{8\pi\|\theta\|_\infty}{\min\left\{\min\limits_j \omega_j, \min\limits_j \tilde{\omega}_j\right\}}\delta\left(\frac{\phi(\alpha-\delta)}{\phi\left(\alpha-\delta-r_{\tilde{L}}\right)}\frac{L_\rho}{(\lambda_2)^2}+\frac{L_\phi}{\phi\left(\alpha-r_L\right)}\right).$$

Proof. By theorem 5.3.13, we have

$$|h(b)-\tilde{h}(b)| = |h_m(b)-\tilde{h}_m(b)|$$

for each $b \in \mathbb{M}$. The new landmark sets (L_m, ω_m) and $(\tilde{L}_m, \tilde{\omega}_m)$ are in a weight preserving bijective correspondence. Hence, by equation 5.4 in the proof of the geometric stability theorem 5.2.7,

$$|h_m(b)-\tilde{h}_m(b)| \leq 8\pi\|\theta_m\|_\infty\delta\left(\frac{\phi(\alpha-\delta)}{\overline{\psi}^m(b)}\frac{L_\rho}{(\Lambda_2)^2}+\frac{L_\phi}{\overline{\varphi}^m(b)}\right)$$

where Λ_2 is the smallest non-zero eigenvalues of $\mathcal{L}_H^m$. Recall that $||\theta_m||_\infty = ||\theta||_\infty$, and observe that $\overline{\varphi}^m(b) = \overline{\varphi}(b)$ and $\overline{\psi}^m(b) = \overline{\psi}(b)$. Moreover, by Lemma 5.3.14, if Λ_2 are λ_2 are the smallest non-zero eigenvalues of $\mathcal{L}_H^m$ and $\mathcal{L}_H$ respectively, then $\Lambda_2 \geq \lambda_2$. Combining the above, we obtain

$$|h(b)-\tilde{h}(b)| \leq 8\pi\|\theta\|_\infty\delta\left(\frac{\phi(\alpha-\delta)}{\overline{\psi}(b)}\frac{L_\rho}{(\lambda_2)^2}+\frac{L_\phi}{\overline{\varphi}(b)}\right).$$

An argument analogous to the proof of Theorem 5.2.7 yields uniform bounds for all b

$$||h-\tilde{h}||_\infty \leq \frac{8\pi\|\theta\|_\infty}{\min\left\{\min\limits_j \omega_j, \min\limits_j \tilde{\omega}_j\right\}}\delta\left(\frac{\phi(\alpha-\delta)}{\phi\left(\alpha-\delta-r_{\tilde{L}}\right)}\frac{L_\rho}{(\lambda_2)^2}+\frac{L_\phi}{\phi\left(\alpha-r_L\right)}\right)$$

This finishes the proof. □

5.4 Example & Discussion

In this section, we demonstrate circular coordinates through examples. The first example compares circular coordinates generated from two landmark sets that are close in the Wasserstein sense, while the second example is designed to clearly motivate the use of Wasserstein distance to compare landmarks sets.

Example 5.4.1. We start with a dataset X with 1000 points sample around a unit circle with Gaussian noise of 0.5. We first select the "blue" landmark set with 40 points by `maxmin` sampling. We define the "red" landmark set by choosing the respective nearest neighbours of the "blue" landmarks. These two landmark sets can be used to create the two associated circle-valued functions h_{red} and h_{blue}. See Figure 1.4 for the setup and the results. In the rightmost figure, we plot the pair $(h_{\mathsf{red}}(b), h_{\mathsf{blue}}(b))$ for each $b \in X$.The closer the curve is to the perfect diagonal, the closer the two functions are.

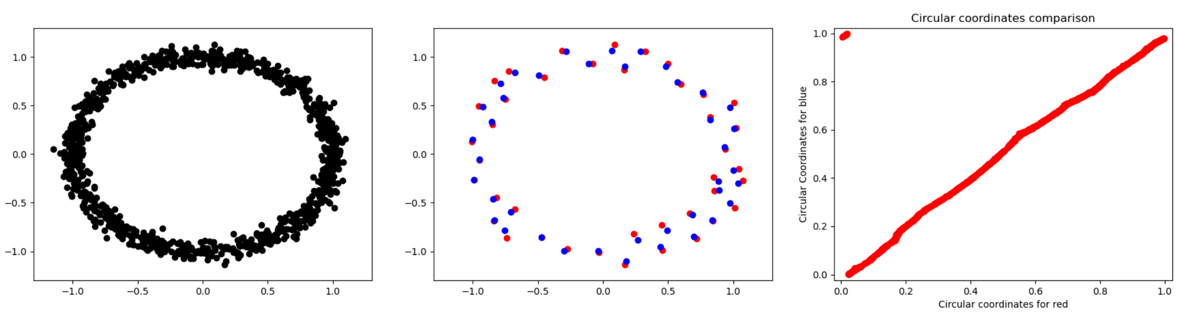


Figure 5.2: An example demonstrating stability of Circular Coordinates: (Left) The data set X of 1000 points sampled around a circle, with two different sets of 40 landmarks (Middle). For each of these sets, we obtain a function that assigns to each data point in X a circular coordinate. (Right) We see a comparison of the two coordinate functions.

Example 5.4.2. To demonstrate the drawbacks of comparing landmarks with Hausdorff distance, we present a simple example in which landmarks are replicated at a single spot, so as to have zero Hausdorff distance. Our dataset is generated to lie uniformly around the unit circle, perturbed by some small Gaussian noise. See Figure 5.3. The first set of landmarks L is selected via `maxmin`, say $L = \left\{\ell_j\right\}_{j=1}^{16}$. The second set of landmarks $\tilde{L}$ is created the following way:

$$\tilde{\ell}_j = \ell_j \text{ for } j = 1, ..., 15$$
$$\tilde{\ell}_j = \ell_{16} \text{ for } j = 16, ..., 15 + n$$

that is, the final landmark ℓ_{16} is replicated $n-1$ times. For each of the landmark sets, we construct circular coordinates in two fashions to compare them.

In the first case, we simply ignore any notion of weights and calculate the circular coordinates as originally described in [55].

In the second case, landmarks are weighted so that Wasserstein distance $d_{\mathcal{W},\infty}((L,\omega),(\tilde{L},\tilde{\omega}))$ is 0. In particular, $\omega_j = 1/16$ for every landmark ℓ_j, while $\tilde{\omega}_j = 1/16$ for $j = 1, ...15$ and $\tilde{\omega}_j = 1/(16n)$ for $i = 16, ..., 15 + n$.

In both cases, the circular coordinates built on L are the same, but the coordinates on $\tilde{L}$ change between the two scenarios. In Fig. 5.3, we see that the first case yields growing error between the pair of circular coordinates, while in the second case the circular coordinates remain static regardless of how many replicated landmarks are made.

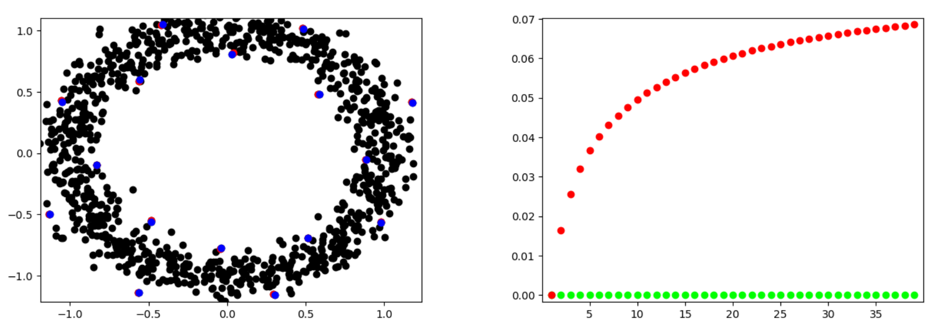


Figure 5.3: Comparison between Hausdorff and Wasserstein noise: (Left) Example of the dataset in black and 16 landmarks shown in blue. (Right) Comparison of the number of replicates against the incurred error between the circular coordinates. Red shows the initial 'Zero-Hausdorff-distance' pair, while green shows the second 'Zero-Wasserstein-distance' pair.

Example 5.4.2 demonstrates that Wasserstein distance is the appropriate way to compare landmarks.

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