K-RATIONAL PREPERIODIC POINTS AND HYPERSURFACES ON PROJECTIVE SPACE

By

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ABSTRACT

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The present thesis has two main parts. In the first one, we study bounds for the number of rational preperiodic points of an endomorphism of $\mathbb{P}^1$. Let K be a number field and ϕ be an endomorphism of $\mathbb{P}^1$ over K of degree $d \geq 2$. Let S be the set of places of bad reduction for ϕ (including the archimedean places). Let $\mathrm{Per}(\phi, K)$, $\mathrm{PrePer}(\phi, K)$, and $\mathrm{Tail}(\phi, K)$ be the set of K-rational periodic, preperiodic, and purely preperiodic points of ϕ, respectively.

If we assume that $|\mathrm{Per}(\phi, K)| \geq 4$ (resp. $|\mathrm{Tail}(\phi, K)| \geq 3$), we prove bounds for $|\mathrm{Tail}(\phi, K)|$ (resp. $|\mathrm{Per}(\phi, K)|$) that depend only on the number of places of bad reduction $|S|$ (and not on the degree d). We show that the hypotheses of this result are sharp, giving counterexamples to any possible result of this form when $|\mathrm{Per}(\phi, K)| < 4$ (resp. $|\mathrm{Tail}(\phi, K)| < 3$). The key tool involved in these results is a bound for the number of solutions of S-unit equations.

Using bounds for the number of solutions of the celebrated Thue-Mahler equation, we obtain bounds for $|\mathrm{Per}(\phi, K)|$ and $|\mathrm{Tail}(\phi, K)|$ in terms of the number of places of bad reduction $|S|$ and the degree d of the rational function ϕ. Bounds obtained in this way are a significant improvement to previous result given by J. Canci and L. Paladino.

In the second part of the thesis, we study the set of K-rational purely preperiodic hypersurfaces of $\mathbb{P}^n$ of a given degree for an endomorphism of $\mathbb{P}^n$. Let ϕ be an endomorphism of $\mathbb{P}^n$ over K, S be the set of places of bad reduction for ϕ and $\mathrm{HTail}(\phi, K, e)$ be the set of K-rational purely preperiodic hypersurfaces of $\mathbb{P}^n$ of degree e.

We give a strong arithmetic relation between K-rational purely preperiodic hypersurfaces and K-rational periodic points. If we consider $N = \binom{e+n}{e} - 1$ and assume that ϕ has at least $2N+1$ K-rational periodic points such that no $N+1$ of them lie in a hypersurface of degree e then we give an effective bound on a large subset of $\mathrm{HTail}(\phi, K, e)$ depending on e and the number of places of bad reduction $|S|$. Finally, we prove that the set $\mathrm{HTail}(\phi, K, e)$ is finite if we assume that ϕ is an endomorphism of $\mathbb{P}^2$.

To my daughter Emma, my son Sebastian and my wife Yira.
I love you all.
I would not be here without you.

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KEY TO SYMBOLS

1. $\mathbb{N}$ the set of natural numbers.
2. $\mathbb{N}_0$ the set of non-negative integers.
3. $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ the set of integers, rational, real and complex numbers respectively.
4. $\subset$ means subset.
5. $\subsetneq$ means a proper subset.
6. $|A|$ the cardinality of a set A.
7. $\mathbf{i} = (i_0, \ldots, i_n) \in \mathbb{N}_0^{n+1}$ an $n+1$-dimensional multi-index.
8. $|\mathbf{i}| = i_0 + \cdots + i_n$
9. $\mathbf{X} = (X_0, \ldots, X_n)$ where $X_0, \ldots, X_n$ are $n+1$ variables.
10. $\mathbf{X}^{\mathbf{i}} = X_0^{i_0} \cdots X_n^{i_n}$
11. R^* the group of units of a ring R.
12. K a number field.
13. $\bar{K}$ an algebraic closure of K.
14. $\mathcal{O}$ the ring of integers of K.
15. $\mathfrak{p}$ a non-zero prime ideal of $\mathcal{O}$.
16. $v_{\mathfrak{p}}$ the $\mathfrak{p}$-adic valuation on K corresponding to the prime ideal $\mathfrak{p}$ (we always assume $v_{\mathfrak{p}}$ to be normalized so that $v_{\mathfrak{p}}(K^*) = \mathbb{Z}$).
17. If the context is clear, we will also use $v_{\mathfrak{p}}(I)$ for the $\mathfrak{p}$-adic valuation of a fractional ideal I of K.
18. S a fixed finite set of places of K including all archimedean places.
19. $|S| = s$ the cardinality of S.
20. $\mathcal{O}_S = \{x \in K : v_{\mathfrak{p}}(x) \geq 0$ for every prime ideal $\mathfrak{p} \notin S\}$ the ring of S-integers.
21. $\mathcal{O}_S^* = \{x \in K : v_{\mathfrak{p}}(x) = 0$ for every prime ideal $\mathfrak{p} \notin S\}$ the group of S-units.

- Let ϕ be an endomorphism of $\mathbb{P}^n$ defined over K.

22. $\text{Per}(\phi, K)$ the set of K-rational periodic points.
23. $\text{Tail}(\phi, K)$ the set of K-rational tail points.
24. $\text{PrePer}(\phi, K)$ the set of K-rational preperiodic points.
25. $\text{HPer}(\phi, K, e)$ the set of K-rational periodic hypersurfaces of degree e.
26. $\text{HTail}(\phi, K, e)$ the set of K-rational tail hypersurfaces of degree e.
27. $\text{HPrePer}(\phi, K, e)$ the set of K-rational preperiodic hypersurfaces of degree e.
28. $\text{HPer}(\phi, K)$ the set of K-rational periodic hypersurfaces.
29. $\text{HTail}(\phi, K)$ the set of K-rational tail hypersurfaces.
30. $\text{HPrePer}(\phi, K)$ the set of K-rational preperiodic hypersurfaces.

Chapter 1

Introduction

Let $\mathcal{S}$ be a set and $\phi\colon \mathcal{S} \to \mathcal{S}$ a function mapping the set $\mathcal{S}$ to itself. A (discrete) dynamical system is a pair consisting of the set $\mathcal{S}$ and the function ϕ. We denote by ϕ^n the nth iterate of ϕ under composition and by ϕ^0 the identity map. The *orbit* of $P \in \mathcal{S}$ under ϕ is the set $O_\phi(P) = \{\phi^n(P) : n \geq 0\}$.

The set $\mathcal{S}$ could be simply a set with no additional structure but most frequently we study dynamics when the set $\mathcal{S}$ has some additional structure. In arithmetic dynamics we are interested when the set $\mathcal{S}$ is an arithmetic set such as $\mathbb{Z}$, $\mathbb{Q}$, number fields K, quasi-projective variety, K-rational points, etc. and the function ϕ is a polynomial, a rational map, an endomorphism, etc. In this arithmetic context the **Principal Goal of Dynamics** is to classify the points P in $\mathcal{S}$ according to the behavior of their orbits $O_\phi(P)$ when $\mathcal{S}$ is an arithmetic set.

Let K be a number field. Our study in arithmetic dynamics will be when $\mathcal{S}$ is $\mathbb{P}^N(K)$ and ϕ an endomorphism of $\mathbb{P}^N$ of degree $d \geq 2$. A point $P \in \mathbb{P}^N(K)$ is called *periodic* under ϕ if there is an integer $n > 0$ such that $\phi^n(P) = P$. It is called *preperiodic* under ϕ if there is an integer $m \geq 0$ such that $\phi^m(P)$ is periodic. A point that is preperiodic but not periodic is called a *tail* point. Let $\mathrm{Tail}(\phi, K)$, $\mathrm{Per}(\phi, K)$ and $\mathrm{PrePer}(\phi, K)$ be the sets of K-rational tail, periodic and preperiodic points of ϕ, respectively.

A first objective of this thesis is to study the cardinality of the sets $\mathrm{Tail}(\phi, K)$, $\mathrm{Per}(\phi, K)$

and $\text{PrePer}(\phi, K)$. We start by asking

- Is the set of K-rational preperiodic points finite or infinite?.
- If finite, can we give an effective bound?.

Northcott [Nor50] proved in 1950 that the total number of K-rational preperiodic points of ϕ is finite. In fact, from Northcott's proof, an explicit bound can be found in terms of the coefficients of ϕ, the number field K and the dimension N.

Even when Northcott answered both questions, a bound for $\text{PrePer}(\phi, K)$ in terms of only a few basic parameters is desired. In 1994, Morton and Silverman [MS94] conjectured the celebrated Uniform Boundedness Conjecture (UBC) which predicts the existence of such a bound depending only on d, the dimension of the projective space and the degree of K.

Conjecture 1.0.1 (Uniform Boundedness Conjecture)**.**
Let K be a number field with $[K : \mathbb{Q}] = D$, and let ϕ be an endomorphism of $\mathbb{P}^N$, defined over K. Let $d \geq 2$ be the degree of ϕ. Then there is $C = C(D, N, d)$ such that ϕ has at most C preperiodic points in $\mathbb{P}^N(K)$.

This conjecture is an extremely strong uniformity conjecture. For example, the UBC on maps of degree 4 on $\mathbb{P}^1$ defined over $\mathbb{Q}$ implies Mazur's theorem that the torsion subgroup of an elliptic curve $E/\mathbb{Q}$ is bounded independently of E. More generally, the UBC for maps of degree 4 on $\mathbb{P}^1$ defined over K implies Merel's theorem that the size of the torsion subgroup of an elliptic curve over a number field K is bounded only in terms of the degree of $[K : \mathbb{Q}]$. The conjecture can also be applied to Lattès maps and abelian varieties, for more detail see [Fak01], [Maz77] and [Mer96].

Poonen [Poo98] later stated a sharper version of the conjecture for the special case of quadratic polynomials over $\mathbb{Q}$. Since every such quadratic polynomial map is conjugate to a polynomial of the form $\psi_c(z) = z^2 + c$ with $c \in \mathbb{Q}$ we can state Poonen's conjecture as follows:

Conjecture 1.0.2 (Poonen's conjecture)**.**

Let $\psi_c \in \mathbb{Q}[z]$ be a polynomial of degree 2 of the form $\psi_c(z) = z^2 + c$ with $c \in \mathbb{Q}$. Then

$$|\operatorname{PrePer}(\psi_c, \mathbb{Q})| \leq 9.$$

Even though Poonen's Conjecture is arguably the simplest case of the UBC, a proof of Poonen's Conjecture seems to be very far off at this time. If we consider polynomials of the form $\psi_c(z) = z^2 + c$ with $c \in \mathbb{Q}$, B. Hutz and P. Ingram [HI13] have shown that Poonen's conjecture holds when the numerator and denominator of c don't exceed 10^8. For more information on quadratic rational functions see [BCH$^+$14], [Can10], [FHI$^+$09], [Man07], [MN06], [Poo98].

Even though the UBC or Poonen's conjecture are impossible to prove at the moment, if we allow the bound from the UBC to depend on one more parameter, then effective results can be given in the case of $\mathbb{P}^1$.

In the first half of the thesis we work in the case $N = 1$, so from now we assume that ϕ is an endomorphism of $\mathbb{P}^1$. Let S be the set of places of K at which ϕ has bad reduction, including all archimedean places of K. The nonarchimedean places of bad reduction are those in which the degree of the reduction of ϕ in the residue field decreases. In other words, a place is said to be a place of good reduction if ϕ has a good behavior in the residue field associated with the place. Then the extra parameter needed to give effective results is the cardinality of S.

The first main result of this thesis [[Tro], Corollary 1.3.] gives a bound for $|\operatorname{PrePer}(\phi, K)|$

in terms of the number of places of bad reduction $|S|$ and the degree of the rational function ϕ. This bound significantly improves a previous bound given by J. Canci and L. Paladino [CP16].

In the second result, assuming that $|\operatorname{Tail}(\phi, K)| \geq 3$ (resp. $|\operatorname{Per}(\phi, K)| \geq 4$), we prove bounds for $|\operatorname{Per}(\phi, K)|$ (resp. $|\operatorname{Tail}(\phi, K)|$) that depend only on the number of places of bad reduction $|S|$ and $[K : \mathbb{Q}]$ (and not on the degree of ϕ). We show that the hypotheses of this result are sharp. Example 3.2.4 and Example 3.2.5 give counterexamples to any possible result of this form when $|\operatorname{Tail}(\phi, K)| < 3$ (resp. $|\operatorname{Per}(\phi, K)| < 4$).

Theorem 1.0.3. *Let K be a number field and S a finite set of places of K containing all the archimedean ones. Let ϕ be an endomorphism of $\mathbb{P}^1$, defined over K, and $d \geq 2$ the degree of ϕ. Assume ϕ has good reduction outside S.*

(a) If there are at least three K-rational tail points of ϕ then

$$|\operatorname{Per}(\phi, K)| \leq 2^{16|S|} + 3.$$

(b) If there are at least four K-rational periodic points of ϕ then

$$|\operatorname{Tail}(\phi, K)| \leq 4(2^{16|S|}).$$

Using the previous theorem, we can deduce a bound for $|\operatorname{PrePer}(\phi, K)|$ in terms of $|S|$ and the degree of ϕ for any endomorphism of $\mathbb{P}^1$.

Corollary 1.0.4. *Let K be a number field and S a finite set of places of K containing all the archimedean ones. Let ϕ be an endomorphism of $\mathbb{P}^1$, defined over K, and $d \geq 2$ the degree of ϕ. Assume ϕ has good reduction outside S. Then* *(a)* $|\operatorname{Per}(\phi, K)| \leq 2^{16|S|d^3} + 3.$

(b) $|\operatorname{Tail}(\phi, K)| \leq 4(2^{16|S|d^3}).$

(c) $|\operatorname{PrePer}(\phi, K)| \leq 5(2^{16|S|d^3}) + 3.$

These bounds depend, ultimately, on a reduction to S-unit equations. Using a reduction to Thue-Mahler equations instead, we obtain a better bound for $|\operatorname{Tail}(\phi, K)|$ in terms of $|S|$ and d.

Theorem 1.0.5. *Let K be a number field and S a finite set of places of K containing all the archimedean ones. Let ϕ be an endomorphism of $\mathbb{P}^1$, defined over K, and $d \geq 2$ the degree of ϕ. Assume ϕ has good reduction outside S. Then*

$$|\operatorname{Tail}(\phi, K)| \leq d \max\left\{(5 \cdot 10^6(d^3+1))^{|S|+4}, 4(2^{64(|S|+3)})\right\}.$$

To get a similar bound for $|\operatorname{Per}(\phi, K)|$ we need to assume that ϕ has at least one K-rational tail point. Under this assumption, using Theorem 1.0.3 and results about Thue-Mahler equations, we can get:

Theorem 1.0.6. *Let K be a number field and S a finite set of places of K containing all the archimedean ones. Let ϕ be an endomorphism of $\mathbb{P}^1$, defined over K, and $d \geq 2$ the degree of ϕ. Assume ϕ has good reduction outside S. If ϕ has at least one K-rational tail point then*

$$|\operatorname{Per}(\phi, K)| \leq \max\left\{(5 \cdot 10^6(d-1))^{|S|+3}, 4(2^{128(|S|+2)})\right\} + 1.$$

While the work described so far was being carried out, Canci and Vishkautsan [CV] proved a bound for $|\operatorname{Per}(\phi, K)|$, just assuming that ϕ has good reduction outside S. Their bound on $|\operatorname{Per}(\phi, K)|$ is roughly of the order of $d2^{16|S|} + 2^{2187|S|}$ where $d \geq 2$ is the degree of ϕ.

Now let's go through previous bounds for $|\operatorname{PrePer}(\phi, K)|$ which are relevant for our work. In 2007, Canci [Can07] proved for rational functions with good reduction outside S that the length of finite orbits is bounded by:

$$\left[e^{10^{12}}(|S|+1)^8(\log(5(|S|+1)))^8\right]^{|S|}. \tag{1.1}$$

Note that this bound depends only on the cardinality of S.

In Canci's recent work (2014) with Paladino [CP16] a sharper bound for the length of finite orbits was found:

$$\max\left\{(2^{16|S|-8}+3)\left[12|S|\log(5|S|)\right]^{[K:\mathbb{Q}]}, \left[12(|S|+2)\log(5|S|+5)\right]^{4[K:\mathbb{Q}]}\right\}. \tag{1.2}$$

In our work we are interested in the number of K-rational tail points and K-rational periodic points, $|\operatorname{Tail}(\phi, K)|$ and $|\operatorname{Per}(\phi, K)|$ respectively.

The bounds mentioned in (1.1) and (1.2) can be used to deduce bounds on $|\operatorname{PrePer}(\phi, K)|$. For instance, if we assume that every finite orbit has cardinality given by (1.1) and using that every point could have at most d preimages under ϕ we obtain a bound for $|\operatorname{PrePer}(\phi, K)|$ that is roughly of the order of $d^{(|S|\log|S|)^{8|S|}}$ where $d \geq 2$ is the degree of ϕ. Similarly, the bound deduced from (1.2) is roughly of the order of $d^{2^{16|S|}(|S|\log(|S|)^{[K:\mathbb{Q}]}}$, where $d \geq 2$ is the degree of ϕ. These bounds are polynomial in the degree of ϕ, however they will be rather large in terms of $|S|$.

In 2007, Benedetto [Ben07] proved for the case of polynomial maps of degree $d \geq 2$ that

$|\operatorname{PrePer}(\phi, K)|$ is bounded by $O(|S| \log |S|)$, where S is the set of places of K at which ϕ has bad reduction, including all archimedean places of K. The big-O is essentially $\frac{d^2-2d+2}{\log d}$ for large $|S|$.

Results in positive characteristic have also been found. For instance, in 2007 Ghioca [Ghi07] proved a bound for the number of torsion points of a Drinfeld module. In this case, torsion points are preperiodic points under the action of an additive polynomial of degree larger than one.

Another result in characteristic different from 0 is the work of Canci and Paladino [CP16] which gives a bound for the length of finite orbits under an endomorphism of $\mathbb{P}^1$.

The second part of this thesis provides quantitative and finiteness results for the set of K-rational tail curves of degree e for a given endomorphism of $\mathbb{P}^2$. Compared to the 1-dimensional case, a primary difficulty in proving higher-dimensional results comes from the limited availability of arithmetic tools in higher dimensions. Indeed, arithmetic tools used frequently in the one-dimensional setting include Siegel's theorem, Faltings' theorem, and Roth's theorem. Higher-dimensional conjectural generalizations of these results remain largely open, even for surfaces (e.g., Bombieri-Lang conjecture, Vojta's conjecture). A secondary difficulty comes from the more complicated geometry possible in higher dimensions. For instance, general position conditions (which appear, for example, in Vojta's conjecture) are rather trivial and uninteresting on curves. For these reasons, any progress towards the UBC in higher dimensions is highly valuable.

Even though the UBC in $\mathbb{P}^N$ is very hard there are some results on the literature. For instance, Hutz [Hut15] provides an algorithm to find $\mathbb{Q}$-rational preperiodic points for endomorphisms of $\mathbb{P}^n$. His techniques may be used to find a bound for the cardinality of the set of $\mathbb{Q}$-rational periodic points, depending on the smallest prime of good reduction.

Another important study in dimension bigger than one is the papers by J. Bell, D. Ghioca, and T. Tucker [BGT15], [BGT16]. In these papers we can find an example of infinitely many fixed curves for an endomorphism of $\mathbb{P}^2$. Indeed if f is a homogeneous two-variable polynomial of degree n, then the morphism $\mathbb{P}^2 \to \mathbb{P}^2$ given by $[x : y : z] \to [f(x, z) : f(y, z) : z^n]$ has infinitely many f-invariant curves of the form $[xz^{n^k-1} : f^k(x, z) : z^{n^k}]$, where f^k is the homogenized kth iterate of the dehomogenized one-variable polynomial $x \to f(x, 1)$.

Motivated by the example of J. Bell, D. Ghioca, and T. Tucker and the Silverman-Morton Conjecture I study the set of K-rational preperiodic hypersurfaces of $\mathbb{P}^N$ under an endomorphism of $\mathbb{P}^N$. Let ϕ be an endomorphism of $\mathbb{P}^N$, defined over K, of degree d and H an irreducible K-rational hypersurface of $\mathbb{P}^N$ of degree e. We say that H is *periodic* under ϕ if there is an integer $n > 0$ such that $\phi^n(H) = H$. It is called *preperiodic* under ϕ if there is an integer $m \geq 0$ such that $\phi^m(H)$ is periodic. If H is preperiodic but not periodic it is called a *tail* hypersurface. Let $\mathrm{HTail}(\phi, K, e)$, $\mathrm{HPer}(\phi, K, e)$ and $\mathrm{HPrePer}(\phi, K, e)$ be the sets of K-rational tail, periodic and preperiodic hypersurfaces of degree e of ϕ, respectively.

It is important to notice that the degree of the preperiodic hypersurface will be involved in our study. This new parameter does not come up for points because the degree of a (geometric) point is always 1. However, this extra parameter is a natural condition because similar examples to the one given by J. Bell, D. Ghioca, and T. Tucker could be given if we consider subschemes in place of subvarieties. For instance, if instead of subvarieties we consider more generally integral closed K-subschemes, then a curve does have infinitely many periodic K-integral closed subschemes, because we can just take K-components of the subscheme of periodic points of period n. However, if we bound the degree of the K-subschemes, then once again we get finiteness by Northcott's theorem.

The main idea of my results on $\mathbb{P}^1$ [Tro] lies in an arithmetic relation between K-rational tail points and K-rational periodic points. Using a generalization of the $\mathfrak{p}$-adic logarithmic distance in $\mathbb{P}^1$, I was able to generalize the relation between K-rational tail points and K-rational periodic points to a relation between K-rational tail hypersurfaces and K-rational periodic points.

Theorem 1.0.7. *Let ϕ be an endomorphism of $\mathbb{P}^n$, defined over K. Suppose ϕ has good reduction outside S. Let H be a K-rational tail hypersurface, m the period of the periodic part of the orbit of H and H' the periodic hypersurface such that $H' = \phi^{m_0 m}(H)$ for some $m_0 > 0$. Let $P \in \mathbb{P}^n(K)$ be any periodic point such that $P \notin supp\{H'\}$. Then $\delta_v(P; H) = 0$ for every $v \notin S$.*

In [GTZ11] Bell, Ghioca and Tucker also propose the following question

Question: *Is there a constant $C = C(N, K, d)$ such that for any periodic K-rational subvariety V of $\mathbb{P}^N$, we have $\mathrm{Per}_\Phi(V) \leq C$?*

Using the previous arithmetic relation together with a result from Ru and Wong [RW91] we give a result that implies a partial answer to the previous question for curves on the projective plane. In fact, we provide a bound for the number of K-rational tail hypersurfaces of degree e in the backwards orbit of a given periodic K-rational hypersurface of $\mathbb{P}^n$.

Theorem 1.0.8. *Let ϕ be an endomorphism of $\mathbb{P}^n$, defined over K and suppose ϕ has good reduction outside S. Consider $N = \binom{e+n}{e} - 1$ and let $\{P_i\}_{i=1}^{2N+1}$ be a set of K-rational periodic points of $\mathbb{P}^n$ such that no $N+1$ of them lie in a curve of degree e. Consider $\mathcal{B} = \{H' \in \mathrm{HPer}(\phi, K)\colon \forall 1 \leq i \leq 2N+1, \quad P_i \notin supp\ H'\}$ and $\mathcal{A} = \{q \in \mathrm{HTail}(\phi, K, e)\colon$ there is $H' \in \mathcal{B}$ and $l \geq 0, \quad \phi^{l n_q}(q) = H'$ where n_q is the period of the periodic part of $q\}$. Then*

$$|\mathcal{A}| \leq \left(2^{33} \cdot (2N+1)^2\right)^{(N+1)^3(s+2N+1)}$$

In 2016 B. Hutz [Hut16] proved that the set of K-rational preperiodic subvarieties of $\mathbb{P}^n$ is finite. His proof is based on the theory of canonical height functions. In the special case of K-rational preperiodic curves of $\mathbb{P}^2$ we were able to give an alternative proof than the one given by Hutz. This alternative proof is based in a strong result of dynamical systems ([Fak03], Corollary 5.2) which states that if ϕ is an endomorphism of $\mathbb{P}^n$ then the set $\mathrm{Per}(\phi, \bar{K})$ for an endomorphism ϕ is Zariski dense in $\mathbb{P}^n$.

Theorem 1.0.9. *Let K be a number field and ϕ be an endomorphism of $\mathbb{P}^2$, defined over K. Then for every $e \in \mathbb{N}$ the set $\mathrm{HTail}(\phi, K, e)$ is finite.*

We end this introduction with a brief outline of the rest of the thesis. Chapter 2 introduces some classical notations and definitions from arithmetic dynamics, arithmetic geometry and number theory. We also prove some propositions needed for the main theorems of this manuscript.

Chapter 3 presents the proof of our results on $\mathbb{P}^1$. This chapter has three sections: the first section gives all the propositions and lemmas needed for the next two sections, the second section uses S-unit equations to get bounds for the set of K-rational preperiodic points and the third section uses Thue-Mahler equations to gives different bounds for the set of K-rational preperiodic points.

Finally, Chapter 4 presents definitions and results on $\mathbb{P}^N$. This chapter has four sections: the first one gives definitions and propositions on $\mathbb{P}^N$. The second section give effective results for a large subset of the set of K-rational tail hypersurfaces of $\mathbb{P}^N$ of a given degree. The third section prove finiteness of the set of K-rational tail curves of degree e of $\mathbb{P}^2$. The last section gives examples of K-rational tail and periodic hypersurfaces of $\mathbb{P}^N$.

Chapter 2

Background Material

This chapter contains the background information which will be used throughout the thesis. We'll discuss, define and prove many of the basics of arithmetic dynamics.

For the rest of the chapter/thesis we assume that K is a number field, $\bar{K}$ an algebraic closure of K, $\mathcal{O}$ its integer ring, $\mathfrak{p}$ a non-zero prime ideal of $\mathcal{O}$ and $v_{\mathfrak{p}}$ a $\mathfrak{p}$-adic valuation on K corresponding to the prime ideal $\mathfrak{p}$ (we always assume $v_{\mathfrak{p}}$ to be normalized so that $v_{\mathfrak{p}}(K^*) = \mathbb{Z}$), and S a fixed finite set of places of K including all archimedean places.

We also adopt the multi-index notation, which means that $\mathbf{X}$ is $(X_0, \ldots, X_n)$, $\mathbf{i}$ is $(i_0, \ldots, i_n)$, $\mathbf{X}^{\mathbf{i}} = X_0^{i_0} \cdots X_n^{i_n}$ and $|\mathbf{i}| = i_0 + \ldots + i_n$.

We say that $P \in \mathbb{P}^n(K)$, or P is defined over K, if we can find a representation of P such that every coordinate lies in K. We write $P = [x_0 : \cdots : x_n] \in \mathbb{P}^n(K)$ meaning that every $x_i \in K$.

We say that ϕ is an endomorphism of $\mathbb{P}^n$, defined over K if $\phi = [F_0 : \cdots : F_n]$ where $F_0, \ldots, F_n \in K[\mathbf{X}]$ are homogeneous polynomials of the same degree with no common factors and $F_0, \ldots, F_n$ does not have a common zero on $\mathbb{P}^n$.

2.1 Basics of arithmetic dynamics

We start this section with the idea of normalized forms with respect to $\mathfrak{p}$.

Definition 2.1.1. *1. We say that a point $P = [x_0 : \cdots : x_n] \in \mathbb{P}^n(K)$ is in normalized form with respect to $\mathfrak{p}$ if*

$$\min\{v_{\mathfrak{p}}(x_0), \ldots, v_{\mathfrak{p}}(x_n)\} = 0.$$

2. Let ϕ be an endomorphism of $\mathbb{P}^n$, defined over K. Assume ϕ is given by

$$\phi = [F_0 : \cdots : F_n]$$

where $F_0, \ldots, F_n \in K[\boldsymbol{X}]$ are homogeneous polynomials with no common factors of degree d. We say that $\phi = [F_0 : \cdots : F_n]$ is normalized with respect to $\mathfrak{p}$ or that ϕ is in normalized form with respect to $\mathfrak{p}$ if $F_i \in \mathcal{O}_{\mathfrak{p}}[\boldsymbol{X}\ ]$ for every $i \in \{0, \ldots, n\}$ and at least one coefficient of F_i is not in the maximal ideal of $\mathcal{O}_{\mathfrak{p}}$ for some $i \in \{0, \ldots, n\}$.

3. Let H be a hypersurface of $\mathbb{P}^n$ defined over K of degree d. Suppose that H is defined by an homogeneous polynomial $f = \sum_{|\boldsymbol{i}|=d} a_{\boldsymbol{i}} \boldsymbol{X}^{\boldsymbol{i}} \in K[\boldsymbol{X}]$. We say that H is in normalized form with respect to $\mathfrak{p}$ if

$$\min_{|\boldsymbol{i}|=d}\{v_{\mathfrak{p}}(a_{\boldsymbol{i}})\} = 0.$$

Let $P \in \mathbb{P}^n(K)$ and ϕ be an endomorphism of $\mathbb{P}^n$, defined over K. Since $\mathcal{O}_{\mathfrak{p}}$ is a discrete valuation ring, we can always find a representation of P and ϕ in normalized form with respect to $\mathfrak{p}$. However, it is not always true that the same representation is normalized for every $\mathfrak{p}$. For this reason we need a more global definition of normalized forms.

Definition 2.1.2. *1. We say that $P = [x_0 : \cdots : x_n] \in \mathbb{P}^n(K)$ is normalized with respect to S if $[x_0 : \cdots : x_n]$ is in normalized form with respect to $\mathfrak{p}$ for every $\mathfrak{p} \notin S$.*

2. *Let* $\phi = [F_0 : \cdots : F_n]$ *be an endomorphism of* $\mathbb{P}^n$*, defined over* K*. We say that* ϕ *is in normalized form with respect to* S *if* $[F_0 : \cdots : F_n]$ *is normalized with respect to* $\mathfrak{p}$ *for every* $\mathfrak{p} \notin S$.

3. *Let* H *be a hypersurface of* $\mathbb{P}^n$ *defined over* K *of degree* d*. Suppose that* H *is defined by an homogeneous polynomial* $f = \sum_{|\boldsymbol{i}|=d} a_{\boldsymbol{i}} \boldsymbol{X}^{\boldsymbol{i}} \in K[\boldsymbol{X}]$*. We say that* H *is in normalized form with respect to* S *if*

$$\min_{|\boldsymbol{i}|=d}\{v_{\mathfrak{p}}(a_{\boldsymbol{i}})\} = 0 \quad \textit{for every} \quad \mathfrak{p} \notin S$$

The following remark is a characterization of the previous definition and it will be a useful idea to keep in mind in the next chapters.

Remark 2.1.3. *A point* $P = [x_0 : \cdots : x_n] \in \mathbb{P}^n(K)$ *admits a normalized form with respect to* S *if and only if the* $\mathcal{O}_S$*-fractional ideal* $(x_0, \ldots, x_n)$ *is principal.*

Similarly, an endomorphism ϕ *or a hypersurface* H *admit a normalized form with respect to* S *if and only if the coefficients of the polynomials defined by* ϕ *or* H *generate an* $\mathcal{O}_S$*-principal ideal.*

Now we define the important concept of good reduction with respect to a prime $\mathfrak{p}$.

Definition 2.1.4. *Let* ϕ *be an endomorphism of* $\mathbb{P}^n$*, defined over* K *and write* $\phi = [F_0 : \cdots : F_n]$ *in normalized form with respect to* $\mathfrak{p}$*. We say that* ϕ *has good reduction at* $\mathfrak{p}$ *if* $\tilde{F}_0(\boldsymbol{X}) = \ldots = \tilde{F}_n(\boldsymbol{X}) = 0$ *has no solutions in* $\mathbb{P}^n(\bar{k})$*, where* $\tilde{F}_0, \ldots, \tilde{F}_n$ *are the reductions of* $F_0, \ldots, F_n$ *modulo* $\mathfrak{p}$ *respectively and* k *is the residue field of* $\mathcal{O}_{\mathfrak{p}}$.

We say that ϕ *has good reduction outside* S *if* ϕ *has good reduction at* $\mathfrak{p}$ *for every* $\mathfrak{p} \notin S$.

We define $\tilde{P}$ as the reduction of P modulo $\mathfrak{p}$ after P is written in normalized form with respect to $\mathfrak{p}$. Similarly, we define $\tilde{\phi}$ as the reduction of ϕ modulo $\mathfrak{p}$ after ϕ is written in normalized form with respect to $\mathfrak{p}$

Proposition 2.1.5. *Let ϕ, φ be two endomorphism of $\mathbb{P}^n$ defined over K. In addition, assume that ϕ and φ have good reduction at $\mathfrak{p}$. If $\tilde{*}$ is the reduction modulo $\mathfrak{p}$ then*

1. $\tilde{\phi}(\tilde{P}) = \widetilde{\phi(P)}$ *for all $P \in \mathbb{P}^n(K)$.*

2. *The composition $\phi \circ \varphi$ has good reduction at $\mathfrak{p}$ and*

$$\widetilde{\phi \circ \varphi} = \tilde{\phi} \circ \tilde{\varphi}$$

Proof. The proof for the one dimentional case can be found in [[Sil07], p.59.]. The higher dimensional case follows exactly as in $\mathbb{P}^1$. □

From the previous proposition we emphasize two properties that we will use in future chapters.

Remark 2.1.6. *Let ϕ be an endomorphism of $\mathbb{P}^n$, defined over K and assume ϕ has good reduction at $\mathfrak{p}$. We write $\phi = [F_0 : \cdots : F_n]$ in normalized form with respect to $\mathfrak{p}$ and let $P = [x_0 : \cdots : x_n] \in \mathbb{P}^1(K)$ be in normalized form with respect to $\mathfrak{p}$.*

1. *ϕ^k has good reduction at $\mathfrak{p}$ for every $k \geq 2$. Even more, $\phi^k = [G_{0,k} : \cdots : G_{n,k}]$ is in normalized form with respect to $\mathfrak{p}$, where*

   $G_{i,k}(X_0, \ldots, X_n) = F_i(G_{0,k-1}(X_0, \ldots, X_n), \ldots, G_{n,k-1}(X_0, \ldots, X_n))$,

   $G_{i,1}(X_0, \ldots, X_n) = F_i(X_0, \ldots, X_n)$ *and $i \in \{0, \ldots, n\}$.*

2. $\phi(P) = [F_0(x_0 : \ldots : x_n) : \cdots : F_n(x_0, \ldots, x_n)]$ *is in normalized form with respect to* $\mathfrak{p}$.

Let us define the most important concept of this work: K-rational tail, periodic and preperiodic subvarieties of an endomorphism of $\mathbb{P}^n$.

Definition 2.1.7. *Let V be subvariety of $\mathbb{P}^n(K)$ and ϕ be an endomorphism of $\mathbb{P}^n$ defined over K.*

1. *The orbit of V is the set $\{\phi^i(V)\}_{i \geq 0}$.*

2. *We say that a subvariety W $\mathbb{P}^n$ is in the backward orbit of V if $\phi^m(W) = V$ for some $m > 0$. The backward orbit of V is denoted by $\mathcal{O}^{-1}(V)$.*

3. *We say that V is a K-rational periodic subvariety if $\phi^m(V) = V$ for some $m \geq 1$.*

4. *We say that V is a K-rational preperiodic subvariety if there is $n_0 \geq 0$ such that $\phi^m(V) = \phi^{n_0}(V)$ for some $m > n_0$. Equivalent, V is a preperiodic subvariety if its orbit is finite.*

5. *Given V' a K-rational periodic subvariety we say that V is in the tail of V' if V is a preperiodic but not a periodic subvariety and V' is in the orbit of P.*

6. *We say that V is a K-rational tail subvariety if V is in the tail of some periodic subvariety.*

Notation 2.1.8. *Let ϕ be an endomorphism of $\mathbb{P}^n$ defined over K and $e \in \mathbb{N}$ then*

- *We denote by* $\mathrm{Per}(\phi, K)$ *the set of K-rational periodic points,* $\mathrm{Tail}(\phi, K)$ *the set of K-rational tail points and* $\mathrm{PrePer}(\phi, K)$ *the set of K-rational preperiodic points.*

- *We denote by* $\mathrm{Per}(\phi, K, e)$ *the set of* K*-rational periodic hypersurfaces of degree* e*,* $\mathrm{Tail}(\phi, K, e)$ *the set of* K*-rational tail hypersurfaces of degree* e *and* $\mathrm{PrePer}(\phi, K, e)$ *the set of* K*-rational preperiodic hypersurfaces of degree* e*.*

- *We denote by* $\mathrm{Per}(\phi, K) = \bigcup_{e>0} \mathrm{Per}(\phi, K, e)$ *the set of* K*-rational periodic hypersurfaces,* $\mathrm{Tail}(\phi, K) = \bigcup_{e>0} \mathrm{Tail}(\phi, K, e)$ *the set of* K*-rational tail hypersurfaces and* $\mathrm{PrePer}(\phi, K) = \bigcup_{e>0} \mathrm{PrePer}(\phi, K, e)$ *the set of* K*-rational preperiodic hypersurfaces.*

Definition 2.1.9. *Let* V *be subvariety of* $\mathbb{P}^n(K)$ *and* ϕ *an endomorphism of* $\mathbb{P}^n$ *defined over* K*.*

1. *We say that* V *has period* n *if* $\phi^n(V) = V$*.*

2. *We say that* V *has exact (or primitive) period* n *if* $\phi^n(V) = V$ *and* $\phi^i(V) \neq V$ *for all* $1 \leq i < n$*.*

A point of period one is called a fixed point, an irreducible hypersurface of degree 1 is called a hyperplane and an irreducible hypersurface of degree 2 is called a quadratic hypersurface.

Theorem 2.1.10 (Bezout's Theorem)**.** *Suppose that* X *and* Y *are two plane projective curves defined over* K *that do not have a common component. Then the total number of intersection points of* X *and* Y *with coordinates in* $\bar{K}$*, counted with their multiplicities, is equal to the product of the degrees of* X *and* Y*.*

Now we define the standard absolute values of $\mathbb{Q}$. There is an archimedean absolute value on $\mathbb{Q}$ defined by

$$|x|_\infty = \max\{x, -x\}.$$

This is just the restriction to $\mathbb{Q}$ is the usual absolute value on $\mathbb{R}$. Further, for each prime number p there is a nonarchimedean (or p-adic) absolute value defined as follows. For any nonzero rational number $x \in \mathbb{Q}$, let $ord_p(x)$ be the unique integer such that x can be written in the form

$$x = p^{ord_p(x)} \cdot \frac{a}{b} \qquad \text{with } a, b \in \mathbb{Z} \text{ and } p \nmid ab.$$

(If $x = 0$, we set $ord_p(x) = \infty$ by convention.) Then the p-adic absolute value of $x \in \mathbb{Q}$ is the quantity

$$|x|_p = p^{-ord_p(x)}.$$

The set of standard absolute values on $\mathbb{Q}$ is the set $M_{\mathbb{Q}}$ consisting of the archimedean absolute value $|\cdot|_\infty$ and the p-adic absolute values $|\cdot|_p$ for every prime p. We denote M_K the set of all absolute values on K whose restriction to $\mathbb{Q}$ is one of the standard absolute values on $\mathbb{Q}$. We define the local degree of $v \in M_K$ by $n_v = [K_v : \mathbb{Q}_v]$ where K_v and $\mathbb{Q}_v$ are the completions of K and $\mathbb{Q}$ at v, respectively. To simplify notation, we write the absolute value corresponding to $v \in M_K$ as $|\cdot|_v$.

Definition 2.1.11. *Let $P \in \mathbb{P}^N(K)$ be a point with homogeneous coordinates*

$$P = [x_0 : \cdots : x_N] \qquad x_0, \ldots, x_N \in K.$$

The height of P (relative to K) is the quantity

$$H_K(P) = \prod_{v \in M_K} \max\{|x_0|_v, \ldots, |x_n|_v\}^{n_v}.$$

The (absolute) height of P is the quantity

$$H(P) = H_K(P)^{1/[K:\mathbb{Q}]}.$$

Note that $H_K(P)$ and $H(P)$ are independent of the choice of homogeneous coordinates.

Theorem 2.1.12 (Northcott's Theorem). *Let K be a number field and let B be any constant. Then the set of points*

$$\{P \in \mathbb{P}^N(K) \colon H_K(P) \le B\} \quad \textit{is finite.}$$

More generally, for any constants B and D, the set of points

$$\{P \in \mathbb{P}^N(\bar{\mathbb{Q}}) \colon H(P) \le B \textit{ and } [\mathbb{Q}(P) : \mathbb{Q}] \le D\} \quad \textit{is finite.}$$

2.2 Definitions and results for $\mathbb{P}^1$

We start by defining the $\mathfrak{p}$-adic logarithmic distance between two points in $\mathbb{P}^1$.

Definition 2.2.1. *Let $P_1 = [x_1 : y_1]$ and $P_2 = [x_2 : y_2]$ be points in $\mathbb{P}^1(K)$. We will denote by*

$$\delta_{\mathfrak{p}}(P_1, P_2) = v_{\mathfrak{p}}(x_1 y_2 - x_2 y_1) - \min\{v_{\mathfrak{p}}(x_1), v_{\mathfrak{p}}(y_1)\} - \min\{v_{\mathfrak{p}}(x_2), v_{\mathfrak{p}}(y_2)\}$$

the $\mathfrak{p}$-adic logarithmic distance between the points P_1 and P_2.

Note that $\delta_{\mathfrak{p}}(P_1, P_2)$ is independent of the choice of homogeneous coordinates. We use the convention that $v_{\mathfrak{p}}(0) = \infty$.

Remark 2.2.2. *Note that if* $P = [x_1 : x_2]$ *and* $Q = [y_1 : y_2]$ *are in normalized form with respect to* $\mathfrak{p}$ *then* $\delta_{\mathfrak{p}}(P_1, P_2) = v_{\mathfrak{p}}(x_1 y_2 - x_2 y_1)$.

Next we will give the definition and some results on the n^{th} dynatomic polynomial associated to an endomorphism ϕ of $\mathbb{P}^1$ defined over K.

Definition 2.2.3. *Let* ϕ *be an endomorphism of* $\mathbb{P}^1$ *defined over* K *of degree* d. *For any* $n \geq 0$ *write*

$$\phi^n(X, Y) = [F_n(X, Y) : G_n(X, Y)]$$

with homogeneous polynomials $F_n, G_n \in K[X, Y]$ *of degree* d^n. *The n-period polynomial of* ϕ *is the polynomial*

$$\Phi_{\phi,n}(X, Y) = Y F_n(X, Y) - X G_n(X, Y).$$

$\Phi_{\phi,n}$ *is well defined up to a constant. Notice that* $\Phi_{\phi,n}(P) = 0$ *if and only if* $\phi^n(P) = P$.

The n^{th} *dynatomic polynomial of* ϕ *is the polynomial*

$$\Phi^*_{\phi,n}(X, Y) = \prod_{k|n} (Y F_k(X, Y) - X G_k(X, Y))^{\mu(n/k)} = \prod_{k|n} \Phi_{\phi,k}(X, Y)^{\mu(n/k)}$$

where μ *is the Möbius function. If* ϕ *is fixed, we write* Φ_n *and* Φ^*_n *for* $\Phi_{\phi,n}$ *and* $\Phi^*_{\phi,n}$ *respectively.*

The following remark will give us the degree of the dynatomic polynomial which will be useful in the end of the next chapter.

Remark 2.2.4. *The degree of the* n^{th} *dynatomic polynomial is given by*

$$\deg(\Phi^*_{\phi,n}) = \sum_{k|n} \mu\left(\frac{n}{k}\right)(d^k + 1).$$

In particular, if $n = 1$ the degree of $\Phi^*_{\phi,n}$ is $d + 1$ and if n is a prime number then the degree of $\Phi^*_{\phi,n}$ is $d^n - d$.

Definition 2.2.5. *Let ϕ be an endomorphism of $\mathbb{P}^1$ defined over K of degree $d \geq 2$ and let $P \in \mathbb{P}^1(\bar{K})$ be a periodic point for ϕ.*

1. *We say that P has formal period n if $\Phi^*_n(P) = 0$.*

2. *Suppose that P has primitive period, we say that P a (primitive) n-periodic point.*

The relation between the three previous definition is

$$\text{primitive period } n \Longrightarrow \text{formal period } n \Longrightarrow \text{period } n.$$

Definition 2.2.6. *Consider $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$. Let $\phi \in \bar{K}(z)$, and $\alpha \in \mathbb{P}^1$ a periodic point of exact period n. We define the multiplier of ϕ at α by*

$$\lambda(\alpha) = \begin{cases} (\phi^n)'(\alpha) & \text{if } \alpha \neq \infty \\ \lim\limits_{\alpha \to 0} \dfrac{\alpha^{-2}\phi'(\alpha^{-1})}{\phi(\alpha^{-1})^2} & \text{if } \alpha = \infty \end{cases} \tag{2.1}$$

where ϕ' is the derivative of ϕ with respect to the variable z.

The following theorem will guarantee that $\Phi^*_{\phi,n}$ is a polynomial. Also, the theorem will give us some useful tools for the next chapter.

Theorem 2.2.7 ([Sil07], p.151)**.** *Let ϕ be an endomorphism of $\mathbb{P}^1$ defined over K of degree $d \geq 2$. For each $P \in \mathbb{P}^1(\bar{K})$, let*

$$a_P(n) = \text{ord}_P(\Phi_{\phi,n}(X, Y)) \quad \textit{and} \quad a^*_P(n) = \text{ord}_P(\Phi^*_{\phi,n}(X, Y))$$

where $\operatorname{ord}_P(\Phi_{\phi,n}(X,Y))$ *and* $\operatorname{ord}_P(\Phi^*_{\phi,n}(X,Y))$ *are the order of zero or pole at* P *of* $\Phi_{\phi,n}(X,Y)$ *and* $\Phi^*_{\phi,n}(X,Y)$*, respectively. Then*

(a) $\Phi^*_{\phi,n} \in K[X,Y]$*, or equivalently,*

$$a^*_P(n) \geq 0 \text{ for all } n \geq 1 \text{ and all } P \in \mathbb{P}^1.$$

(b) Let P *be a point of primitive period* m *and let* $\lambda(P) = (\phi^m)'(P)$ *be the multiplier of* P*. Then* P *has formal period* n*, i.e.,* $a^*_P(n) > 0$*, if and only if one of the following is true:*

(i) $n = m$

(ii) $n = mr$ *and* $\lambda(P)$ *is a primitive* r^{th} *root of unity.*

In particular, $a^*_P(n)$ *is nonzero for at most two values of* n*.*

Now we state a weak version of the Riemann-Hurwitz formula.

Theorem 2.2.8 (Weak Riemann-Hurwitz Formula [Sil07], p.15)**.** *Let* ϕ *be an endomorphism of* $\mathbb{P}^1$ *defined over* K *of degree* d*. Then*

$$2d - 2 = \sum_{\alpha \in \mathbb{P}^1} \left(d - |\phi^{-1}(\alpha)| \right)$$

Using the Weak Riemann-Hurwitz Formula we deduce that an endomorphism of $\mathbb{P}^1$ of degree greater than one has at most two totally ramified points.

The following result proves the existence of n-periodic points for an endomorphism of $\mathbb{P}^1$.

Theorem 2.2.9 (Baker [Bak64]). *Let ϕ be an endomorphism of $\mathbb{P}^1$ defined over K of degree $d \geq 2$. Suppose that ϕ has no primitive n-periodic points. Then (n,d) is one of the pairs*

$$(2,2),(2,3),(3,2),(4,2).$$

Baker's theorem is extremely strong because it guarantees $\bar{K}$-rational n-periodic points for every (n,d) but the four exceptions described in the theorem. However, these four exceptions are fully known as it can be seen in the next remark.

Remark 2.2.10 (Kisaka [Kis95]). *Kisaka completely classifies all the endomorphisms associated to the exceptional pairs (n,d) mentioned in Baker's Theorem. Each of these exceptional endomorphisms have at least two distinct fixed points in $\bar{K}$.*

2.3 General results

In this section we will cite strong theorems that can be found in the literature. Each of these results will be useful in one way or another in future chapters.

The next two theorems will give bounds for the number of solutions of two important equations, the S-unit equation and the Thue-Mahler equation.

First we start with the S-unit equation. We can consider the S-unit equation $ax+by=1$ where $a,b \in K^*$ and x,y are S-units. Bounds on the number of solutions of this equation give powerful consequences in different areas of mathematics. Among many studies on the S-unit equation, one of the best bounds is the following:

Theorem 2.3.1 (Beukers and Schlickewei [BS96]). *Let Γ be a subgroup of $(K^*)^2 = K^* \times K^*$*

of rank r. Then the equation

$$x + y = 1 \qquad in \quad (x, y) \in \Gamma$$

has at most $2^{8(r+1)}$ solutions.

Corollary 2.3.2. *Let Γ_0 be a subgroup of K^* of rank r. Consider $\Gamma = \Gamma_0 \times \Gamma_0$ and assume $a, b \in K^*$. Then the equation*

$$ax + by = 1 \qquad in \quad (x, y) \in \Gamma$$

has at most $2^{8(2r+2)}$ solutions.

Now we gives bound for the Thue-Mahler equation.

Let $F(X, Y)$ be a binary form of degree $r \geq 3$ with coefficients in $\mathcal{O}_S$. An $\mathcal{O}_S^*$-coset of solutions of

$$F(x, y) \in \mathcal{O}_S^* \qquad \text{in} \quad (x, y) \in \mathcal{O}_S^2 \tag{2.2}$$

is a set $\{\epsilon(x, y) : \epsilon \in \mathcal{O}_S^*\}$, where (x, y) is a fixed solution of (2.2).

Theorem 2.3.3 (Evertse [Eve97])**.** *Let $F(X, Y)$ be a binary form of degree $r \geq 3$ with coefficients in $\mathcal{O}_S$ which is irreducible over K. Then the set of solutions of*

$$F(x, y) \in \mathcal{O}_S^* \qquad in \quad (x, y) \in \mathcal{O}_S^2$$

is the union of at most

$$(5 \cdot 10^6 r)^s$$

$\mathcal{O}_S^*$*-cosets of solutions.*

Now we will give some results to bound equations in more than two variables.

First we say that a point $x = (x_1, \ldots, x_n) \in K^n$ is said to be an S-integral point if $x_i \in \mathcal{O}_S$ for all $1 \leq i \leq n$.

Let $(D_i)_{i=1}^l$ be distinct K-rational hyperplanes of $\mathbb{P}^n$. Let $D = \sum_{i=1}^{l} D_i$. We call a set $\mathcal{R} \subset \mathbb{P}^n(K) \backslash D$ a set of (D, S)-integral points if there exists an affine embedding $\mathbb{P}^n \backslash D \subset \mathcal{A}_K^n$ such that every $P \in \mathcal{R}$ has S-integral coordinates.

Before stating the theorem we will recall the definition of general position for hyperplanes.

Definition 2.3.4. *A set* $\mathcal{H}$ *of hyperplanes of* $\mathbb{P}^n$ *is in* general position *if, for any* $1 \leq k \leq n$, *the intersection of* j *hyperplanes in* $\mathcal{H}$ *is of dimension* $n - j$, *and the intersection of any* $n + 1$ *hyperplanes in* $\mathcal{H}$ *is empty.*

Now we state an important result from Ru and Wong.

Theorem 2.3.5 (Ru and Wong [RW91])**.** *Let* $(D_i)_{i=1}^l$ *be distinct* K*-rational hyperplanes of* $\mathbb{P}^n$ *that are in general position. Let* $D = \sum_{i=1}^{l} D_i$. *If* $l > 2n$ *then any set of* (D, S)*-integral points of* $\mathbb{P}^n(K)$ *is finite.*

A generalization of the previous theorem can be found in [NW02].

Next we will state a result coming from decomposable form equations.

Let $F(X) = l_1(X) \ldots l_r(X) \in \mathcal{O}_S[X]$ be a decomposable form of degree r, where $X = (X_1, \ldots, X_n)$ and $l_1, \ldots, l_r$ are linear form with coefficients in some extension of K such that

$$\{x \in K^n \colon l_1(x) = 0, \ldots, l_r(x) = 0\} = \{0\}.$$

Theorem 2.3.6 ([Eve95]). *Assume that the number of $\mathcal{O}_S^*$-cosets of solutions of*

$$F(x) \in \mathcal{O}_S^* \text{ in } x \in \mathcal{O}_S^n \tag{2.3}$$

is finite. Then this number is at most $(2^{33}r^2)^{n^3 s}$.

Finally we end this subsection with a strong consequence of Dirichlet's Theorem on primes in arithmetic progression.

Theorem 2.3.7 ([Rib01], p.527). *If I is a fractional ideal of $\mathcal{O}_S$, then there is a prime ideal P_0 of $\mathcal{O}_S$ such that $[I] = [P_0]$ as $\mathcal{O}_S$-ideal classes i.e. there is a $\lambda \in K$ such that $I = (\lambda)P_0$.*

The next proposition shows that after slightly enlarging any given set S, we can always write a map (or a point) in normalized form with respect to S.

Proposition 2.3.8. *Let $\phi = [F_0 : \ldots : F_n]$ be an endomorphism of $\mathbb{P}^n$, defined over K of degree d with*

$$F_j(X) = \sum_{|\boldsymbol{i}|=d} a_{\boldsymbol{i},j} \boldsymbol{X}^{\boldsymbol{i}} \qquad \text{where} \qquad 0 \le j \le n.$$

Then there exists a prime ideal $\mathfrak{p}_0$ of K and an element $\alpha \in K$ such that $\phi = [\alpha^{-1}F_0 : \ldots : \alpha^{-1}F_n]$ is in normalized form with respect to $S' = S \cup \{\mathfrak{p}_0\}$.

Proof. Consider the fractional ideal $I = (a_{\mathbf{i},j})_{\mathbf{i},j}\mathcal{O}_S$. Then by Theorem 2.3.7 there is a prime $\mathfrak{p}_I$ of K and $\alpha_I \in K$ such that $I = (\alpha_I)\mathfrak{p}_I\mathcal{O}_S$.

Consider the representation of ϕ given by $\phi = [\alpha_I^{-1}F_0 : \ldots : \alpha_I^{-1}F_n]$ and let $S' = S \cup \{\mathfrak{p}_I\}$. Then $v_{\mathfrak{p}}((\alpha_I^{-1}a_{\mathbf{i},j})_{\mathbf{i},j}) = 0$ for every $\mathfrak{p} \notin S'$. In other words, $[\alpha_I^{-1}F_0 : \ldots \alpha_I^{-1}F_n]$ is normalized with respect to S'. □

Proposition 2.3.9. *For every* $P = [x_0 : \ldots : x_n] \in \mathbb{P}^n(K)$ *there exists a prime ideal* $\mathfrak{p}_0$ *of* K *and an element* $\alpha \in K$ *such that* $P = [\alpha^{-1}x_0 : \ldots \alpha^{-1}x_0]$ *is in normalized form with respect to* $S' = S \cup \{\mathfrak{p}_0\}$.

Proof. The proof follows the proof of the previous proposition. □

Chapter 3

Arithmetic Dynamics on $\mathbb{P}^1$

This chapter will be divided in three sections. The first section will state and prove a fundamental arithmetic relation between K-rational tail points and K-rational periodic points. We will also prove some results needed for the rest of the chapter.

The last two sections will give two different approaches to bound the cardinality of $\mathrm{PrePer}(\phi, K)$ for an endomorphism ϕ of $\mathbb{P}^1$ defined over K. The first approach is using S-unit equations and the second one is using Thue-Mahler equations.

3.1 Main propositions on $\mathbb{P}^1$

First we will prove the key arithmetic relation between K-rational tail and periodic points, which will used throughout the chapter.

Proposition 3.1.1. *Let ϕ be an endomorphism of $\mathbb{P}^1$, defined over K. Suppose ϕ has good reduction outside S. Let $P \in \mathbb{P}^1(K)$ be a periodic point, $Q \in \mathbb{P}^1(K)$ a fixed point with $P \neq Q$ and $R \in \mathbb{P}^1(K)$ a tail point of Q. Then $\delta_{\mathfrak{p}}(P, R) = 0$ for every $\mathfrak{p} \notin S$.*

Proof. Let $\mathfrak{p} \notin S$ be a prime of good reduction. Consider $P = [p_1 : p_2], Q = [q_1 : q_2], R = [r_1 : r_2]$ and $\phi = [F(x, y) : G(x, y)]$ all in normalized form with respect to $\mathfrak{p}$. Let n be the period of P and $L_Q(x, y) = q_2 x - q_1 y$ a linear form defining Q .

Given $N > 1$ consider $\phi^N = [F_N(x, y) : G_N(x, y)]$ where $F_N(x, y) = F_{N-1}(F(x, y), G(x, y))$,

$G_N(x,y) = G_{N-1}(F(x,y), G(x,y))$, $F_1(x,y) = F(x,y)$ and $G_1(x,y) = G(x,y)$. By Remark 2.1.6, $\phi^N = (F_N, G_N)$ is in normalized form with respect to $\mathfrak{p}$ and $[F_N(p_1,p_2) : G_N(p_1,p_2)]$ is in normalized form with respect to $\mathfrak{p}$.

Therefore for every $m > 0$ we can find $\lambda \in \mathcal{O}_{\mathfrak{p}}^*$ such that $F_{nm}(p_1,p_2) = \lambda p_1$ and $G_{nm}(P) = \lambda p_2$. We conclude

$$v_{\mathfrak{p}}(L_Q(F_{nm}(p_1,p_2), G_{nm}(p_1,p_2))) = v_{\mathfrak{p}}(L_Q(p_1,p_2)) + v_{\mathfrak{p}}(\lambda) = v_{\mathfrak{p}}(L_Q(p_1,p_2)). \tag{3.1}$$

Pick m big enough so that $\phi^{mn}(R) = Q$. Then $L_Q(F_{nm}(r_1,r_2), G_{nm}(r_1,r_2)) = 0$.

Let $L_R(x,y) = r_2x - r_1y$ be a linear form defining R, and notice that $L_Q(x,y)$, $L_R(x,y)$ are factors of $L_Q(F_{nm}(x,y), G_{nm}(x,y))$. By Gauss's lemma, we can find a polynomial $H(x,y) \in (\mathcal{O}_S)_{\mathfrak{p}}[x,y]$ such that

$$L_Q(F_{nm}(x,y), G_{nm}(x,y)) = L_R(x,y)L_Q(x,y)H(x,y).$$

Hence

$$v_{\mathfrak{p}}(L_Q(F_{nm}(p_1,p_2), G_{nm}(p_1,p_2))) = v_{\mathfrak{p}}(L_R(p_1,p_2)) + v_{\mathfrak{p}}(L_Q(p_1,p_2)) + v_{\mathfrak{p}}(H(p_1,p_2)).$$

So by (3.1)

$$0 = v_{\mathfrak{p}}(L_R(p_1,p_2)) + v_{\mathfrak{p}}(H(p_1,p_2)).$$

Since $v_{\mathfrak{p}}(L_R(p_1,p_2)) \geq 0$ and $v_{\mathfrak{p}}(H(p_1,p_2)) \geq 0$ we get $v_{\mathfrak{p}}(L_R(p_1,p_2)) = 0$. Finally, since R and P are in normalized form with respect to $\mathfrak{p}$, we have $v_{\mathfrak{p}}(L_R(p_1,p_2)) = \delta_{\mathfrak{p}}(P,R) = 0$ by Remark 2.2.2.

□

Theorem 3.1.2. *Let ϕ be an endomorphism of $\mathbb{P}^1$, defined over K. Suppose ϕ has good reduction outside S. Let $R \in \mathbb{P}^1(K)$ be a tail point and let n be the period of the periodic part of the orbit of R. Let $P \in \mathbb{P}^1(K)$ be any periodic point that is not $\phi^{mn}(R)$ for some m. Then $\delta_{\mathfrak{p}}(P, R) = 0$ for every $\mathfrak{p} \notin S$.*

Proof. Take the minimum $m > 0$ such that $\phi^{mn}(R)$ is a periodic point. By Remark 2.1.6, ϕ^n also has good reduction outside S.

Now apply the previous proposition using ϕ^n for ϕ, $\phi^{mn}(R)$ for the fixed point and P as the periodic point different from $\phi^{mn}(R)$. □

The last theorem tells us that R is an S-integral point with respect to P (and vice versa). For instance, if $P = [x_1 : y_1]$ and $R = [x_2 : y_2]$ are written with coprime S-integral coordinates, then $x_1y_2 - x_2y_1$ is an S-unit.

3.2 S-unit equation approach

The goal of this section is to prove two statements. The first gives a bound for $|\operatorname{Per}(\phi, K)|$ and $|\operatorname{Tail}(\phi, K)|$ depending only on the amount of places of bad reduction, provided that $|\operatorname{Tail}(\phi, K)| \geq 3$ and $|\operatorname{Per}(\phi, K)| \geq 4$, respectively.

Theorem 3.2.1. *Let K be a number field and S a finite set of places of K containing all the archimedean ones. Let ϕ be an endomorphism of $\mathbb{P}^1$, defined over K, and $d \geq 2$ the degree of ϕ. Assume ϕ has good reduction outside S.*

(a) If there are at least three K-rational tail points of ϕ then

$$|\operatorname{Per}(\phi, K)| \leq 2^{16s} + 3.$$

(b) If there are at least four K-rational periodic points of ϕ then

$$|\operatorname{Tail}(\phi, K)| \leq 4(2^{16s}).$$

The second theorem gives bounds for $|\operatorname{Tail}(\phi, K)|$, $|\operatorname{Per}(\phi, K)|$ and $|\operatorname{PrePer}(\phi, K)|$ in terms of $|S|$ and the degree of ϕ for any endomorphism of $\mathbb{P}^1$.

Theorem 3.2.2. *Let K be a number field and S a finite set of places of K containing all the archimedean ones. Let ϕ be an endomorphism of $\mathbb{P}^1$, defined over K, and $d \geq 2$ the degree of ϕ. Assume ϕ has good reduction outside S. Then*

(a) $|\operatorname{Per}(\phi, K)| \leq 2^{16sd^3} + 3$.

(b) $|\operatorname{Tail}(\phi, K)| \leq 4(2^{16sd^3})$.

(c) $|\operatorname{PrePer}(\phi, K)| \leq 5(2^{16sd^3}) + 3$.

In the last part of this section we will provide examples to show the necessity of the hypotheses of Theorem 3.2.1.

3.2.1 Proof of Theorem 3.2.1

For this subsection, we will state a notation presented in [Can07].

Let $\mathbf{a_1},...,\mathbf{a_h}$ be a full system of integral representatives for the ideal classes of $\mathcal{O}_S$. Hence, for each $i \in \{1,..,h\}$ there is an S-integer $\alpha_i \in \mathcal{O}_S$ such that

$$\mathbf{a_i}^h = \alpha_i \mathcal{O}_S.$$

Let L be the extension of K given by

$$L = K(\zeta, \sqrt[h]{\alpha}_1, ..., \sqrt[h]{\alpha}_h)$$

where ζ is a primitive h-th root of unity. Consider the following subgroups of L^*:

$$\sqrt{K^*} := \{a \in L^* : \exists m \in \mathbb{Z}_{>0} \text{ with } a^m \in K^*\}$$

and

$$\sqrt{\mathcal{O}_S^*} := \{a \in L^* : \exists m \in \mathbb{Z}_{>0} \text{ with } a^m \in \mathcal{O}_S^*\}.$$

Denote by $\mathbb{S}$ the set of places of L lying above the places in S and by $\mathcal{O}_\mathbb{S}$ and $\mathcal{O}_\mathbb{S}^*$ the ring of $\mathbb{S}$-integers and the group of $\mathbb{S}$-units, respectively in L. By definition $\mathcal{O}_\mathbb{S}^* \cap \sqrt{K^*} = \sqrt{\mathcal{O}_S^*}$ and $\sqrt{\mathcal{O}_S^*}$ is a subgroup of L^* of free rank $s-1$ by Dirichlet's unit theorem.

Lemma 3.2.3. *Assume the notation above. There exist fixed representations* $[x_P : y_P] \in \mathbb{P}^1(L)$ *for every rational point* $P \in \mathbb{P}^1(K)$ *satisfying the following two conditions.*

(a) For every $P \in \mathbb{P}^1(K)$*, we have* $x_P, y_P \in \sqrt{K^*}$ *and*

$$x_P \mathcal{O}_\mathbb{S} + y_P \mathcal{O}_\mathbb{S} = \mathcal{O}_\mathbb{S}.$$

(b) If $P, Q \in \mathbb{P}^1(K)$ then

$$x_P y_Q - y_P x_Q \in \sqrt{K^*}.$$

Proof. Let $P = [x : y]$ be a representation of P in $\mathbb{P}^1(K)$ and consider $\mathbf{b} \in \{\mathbf{a_1}, ..., \mathbf{a_h}\}$ a representative of $x\mathcal{O}_S + y\mathcal{O}_S$. We can find $\beta \in K^*$ such that $\mathbf{b}^h = \beta\mathcal{O}_S$. Then there is $\lambda \in K^*$ such that

$$(x\mathcal{O}_S + y\mathcal{O}_S)^h = \lambda^h \beta \mathcal{O}_S. \tag{3.2}$$

We define in L

$$x' = \frac{x}{\lambda \sqrt[h]{\beta}} \qquad y' = \frac{y}{\lambda \sqrt[h]{\beta}}$$

and with this definition, it is clear that $x', y' \in \sqrt{K^*}$ such that $x'\mathcal{O}_{\mathbb{S}} + y'\mathcal{O}_{\mathbb{S}} = \mathcal{O}_{\mathbb{S}}$. Furthermore, let $P = [x'_1, y'_1]$ and $Q = [x'_2 : y'_2]$ where

$$x'_i = \frac{x_i}{\lambda_i \sqrt[h]{\beta_i}} \qquad y'_i = \frac{y_i}{\lambda_i \sqrt[h]{\beta_i}}$$

and λ_i, β_i are as the ones described in equation (3.2) for $i \in \{1, 2\}$. Then

$$(x'_1 y'_2 - y'_1 x'_2)^h = \frac{(x_1 y_2 - y_1 x_2)^h}{\lambda_1^h \lambda_2^h \beta_1 \beta_2} \in K^*.$$

□

The previous representation is given the name *S-radical coprime coordinates* in recent paper of J. Canci and L. Paladino [CP16] and J. Canci and S. Vishkautsan [CV].

Proof of Theorem 3.2.1 part (a). Let P_1, P_2, P_3 be three different K-rational tail points and let n_i be the period of the periodic part of the orbit of P_i with $i \in \{1, 2, 3\}$. Let P be a

K-rational periodic point such that $\phi^{mn_i}(P_i) \neq P$ for every $m \in \mathbb{Z}_{\geq 0}$ and $i \in \{1,2,3\}$ (if such a P does not exist then $|\operatorname{Per}(\phi, K)| \leq 3$ and the proof will be complete).

By Lemma 3.2.3, for every $i \in \{1,2,3\}$ there exist $P = [x : y]$, $P_i = [x_i : y_i]$ with $x, y, x_i, y_i \in L$ such that

(a) $x_i \mathcal{O}_{\mathbb{S}} + y_i \mathcal{O}_{\mathbb{S}} = \mathcal{O}_{\mathbb{S}}$,

(b) $x \mathcal{O}_{\mathbb{S}} + y \mathcal{O}_{\mathbb{S}} = \mathcal{O}_{\mathbb{S}}$,

(c) $x_i y - y_i x \in \sqrt{K^*}$.

By (a) and (b) we have $\delta_{\mathfrak{p}'}(P, P_i) = v_{\mathfrak{p}'}(x_i y - y_i x)$ for every $\mathfrak{p}' \notin \mathbb{S}$ and every $i \in \{1,2,3\}$. Using Corollary 3.1.2 we can find $\mathbb{S}$-units $u_1, u_2, u_3 \in \mathcal{O}_{\mathbb{S}}^*$ such that

$$x_1 y - y_1 x = u_1, \tag{3.3}$$

$$x_2 y - y_2 x = u_2, \tag{3.4}$$

$$x_3 y - y_3 x = u_3. \tag{3.5}$$

Notice that by (c), $u_i \in \sqrt{K^*} \cap \mathcal{O}_{\mathbb{S}}^* = \sqrt{\mathcal{O}_S^*}$ for each $i \in \{1,2,3\}$.

Using equations (3.3) and (3.4) we get

$$x = \frac{u_1 x_2}{y_2 x_1 - y_1 x_2} - \frac{u_2 x_1}{y_2 x_1 - y_1 x_2} \quad \text{and} \quad y = \frac{u_1 y_2}{y_2 x_1 - y_1 x_2} - \frac{u_2 y_1}{y_2 x_1 - y_1 x_2}.$$

Then by (3.5) we get

$$(x_3 y_2 - y_3 x_2) u_1 + (y_3 x_1 - x_3 y_1) u_2 = (y_2 x_1 - y_1 x_2) u_3.$$

Thus

$$Au + Bv = 1$$

where $A = \frac{(x_3y_2 - y_3x_2)}{(y_2x_1 - y_1x_2)}$, $B = \frac{(y_3x_1 - x_3y_1)}{(y_2x_1 - y_1x_2)}$, $u = u_1u_3^{-1}$ and $v = u_2u_3^{-1}$.

Notice that $A, B \neq 0$ since $P_2 \neq P_3, P_1 \neq P_3$ and the denominator is not 0 since $P_1 \neq P_2$.

Hence by Corollary 2.3.2 with $\Gamma_0 = \sqrt{\mathcal{O}_S^*}$, the total number of solutions $(u, v) \in \sqrt{\mathcal{O}_S^*} \times \sqrt{\mathcal{O}_S^*}$ of $Au + Bv = 1$ is bounded by $2^{8(2s)}$.

From equations (3.3) and (3.5), we can solve for x/y in terms of x_1, y_1, x_3, y_3, u. Therefore there are $2^{8(2s)}$ possible $[x : y]$. Finally notice that there are at most three periodic points P such that $\phi^{mn_i}(P_i) = P$ for some $m \in \mathbb{Z}_{\geq 0}$ and some $i \in \{1, 2, 3\}$. Therefore

$$|\operatorname{Per}(\phi, K)| \leq 2^{16s} + 3.$$

□

The proof of Theorem 3.2.1 part (b) is similar and requires only minor changes at the start and conclusion of the proof.

Proof of Theorem 3.2.1 part (b). Let P_1, P_2, P_3, P_4 be 4 different K-rational periodic points and let n_i be the period of P_i with $i \in \{1, 2, 3, 4\}$. Let P be a K-rational tail point such that $\phi^{mn_i}(P) \neq P_i$ for every $m \in \mathbb{Z}_{\geq 0}$ and $i \in \{1, 2, 3\}$.

By Lemma 3.2.3, for every $i \in \{1, 2, 3\}$ we can take $P = [x : y]$, $P_i = [x_i : y_i]$ with $x, y, x_i, y_i \in L$ such that

(a) $x_i\mathcal{O}_{\mathbb{S}} + y_i\mathcal{O}_{\mathbb{S}} = \mathcal{O}_{\mathbb{S}}$,

(b) $x\mathcal{O}_{\mathbb{S}} + y\mathcal{O}_{\mathbb{S}} = \mathcal{O}_{\mathbb{S}}$,

(c) $x_i y - y_i x \in \sqrt{K^*}$.

Using the same argument of proof of Theorem 3.2.1 part (a), we get that there are at most $2^{8(2s)}$ possible $[x : y]$.

Now for the K-rational tail points given by $\phi^{mn_1}([x : y]) = P_1$, $\phi^{mn_2}([x : y]) = P_2$, $\phi^{mn_3}([x : y]) = P_3$ we use the same argument with the triples (P_2, P_3, P_4), (P_1, P_3, P_4) and (P_1, P_2, P_4), respectively. In each case we get the same bound 2^{16s}.

Therefore,

$$|\operatorname{Tail}(\phi, K)| \leq 4(2^{16s}).$$

□

3.2.2 Proof of Theorem 3.2.2

Proof of Theorem 3.2.2. We claim that we can find a field extension of K to a field E such that ϕ has at least three E-rational tail points (resp. four E-rational periodic points) and $[E : K] \leq d^3$. Suppose the claim is true and let S' be the set of places of E lying above the places of S. Then Theorem 3.2.2 follows by applying Theorem 3.2.1 to get

$$|\operatorname{Per}(\phi, K)| \leq |\operatorname{Per}(\phi, E)| \leq 2^{16|S'|} + 3 = 2^{16|S|d^3} + 3$$

and

$$|\operatorname{Tail}(\phi, K)| \leq |\operatorname{Tail}(\phi, E)| \leq 4(2^{16|S'|}) = 4(2^{16|S|d^3})$$

respectively.

Now we just have to prove our claim.

Part (a). Assume ϕ has at least three periodic points; otherwise the bound trivially holds.

By the weak Riemann-Hurwitz formula a rational function has at most two totally ramified points. Therefore at least one of our periodic points admits a non-periodic preimage. Let P_1 be one possible preimage of such a point and consider E_1 the field of definition of P_1 over K. Notice that $[E_1 : K] \leq d$.

Consider $P_2, P_3 \in \mathbb{P}^1(\bar{K})$ a preimage of P_1 and P_2, respectively. Let E_2 be the field of definition of P_2 over E_1 and E the field of definition of P_3 over E_2. Notice that $[E_2 : E_1] \leq d$, $[E : E_2] \leq d$, $P_2 \in \mathbb{P}^1(E_2)$ and $P_3 \in \mathbb{P}^1(E)$.

So $[E : K] \leq d^3$ and ϕ has at least three E-rational tail points.

Part (b). If $|\operatorname{Per}(\phi, K)| > 4$ then we can apply Theorem 3.2.1 to get the desired bound. Now assume $1 \leq |\operatorname{Per}(\phi, K)| \leq 3$.

Case 1: Suppose there exists a point $P \in \mathbb{P}^1(K)$ of period 3 under ϕ. Considering the field extension $E = K(Q)$ of K where Q is a fixed point of ϕ. Notice that $[E : K] \leq d + 1 \leq d^3$ by Remark 2.2.4 and ϕ has at least four E-rational periodic points.

Case 2: Suppose there exists no periodic point of period 3 in $\mathbb{P}^1(K)$ but there is a point $P \in \mathbb{P}^1(\bar{K}) - \mathbb{P}^1(K)$ of period 3 under ϕ. Considering the field extension $E = K(P)$ of K we have that ϕ has a 3-periodic point in $\mathbb{P}^1(E)$. Notice that $[E : K] \leq d^3 - d \leq d^3$ by Remark 2.2.4 and ϕ has at least four E-rational periodic points since $1 \leq |\operatorname{Per}(\phi, K)|$.

Case 3: Suppose there exists no point $P \in \mathbb{P}^1(\bar{K})$ of period 3 under ϕ. Then by Theorem 2.2.9 and Remark 2.2.10, ϕ admits a point $P_1 \in \mathbb{P}^1(\bar{K})$ of period 2 and two distinct fixed points $P_2, P_3 \in \mathbb{P}^1(\bar{K})$. Since $1 \leq |\operatorname{Per}(\phi, K)| \leq 3$ we can assume that at least one of P_1, P_2, P_3 is K-rational. Let $E = K(P_1, P_2, P_3)$. Notice that $[E : K] \leq d^3$ by Remark 2.2.4 and $|\operatorname{Per}(\phi, E)| \geq 4$.

□

3.2.3 Examples

In this subsection we will present two examples that show the sharpness of the hypotheses of Theorem 3.2.1 part (a) and (b).

The first example gives a family of rational functions with exactly two $\mathbb{Q}$-rational tail points and a fixed set of places of bad reduction. However the size of the set of $\mathbb{Q}$-rational periodic points grows with the degree of the rational functions in the family. This proves that the hypothesis of Theorem 3.2.1 part (a) is necessary.

Example 3.2.4. *Consider*

$$f_d(x) = \frac{1}{x} + \frac{(x-2^{-d})(x-2^{-d+1})...(x-1)...(x-2^{d-1})(x-2^d)}{x^{2d+1}} \quad \in \mathbb{Q}(x).$$

If we take $S = \{\infty, 2\}$ then $f_d(x)$ has good reduction outside S.

Now notice that 0 and ∞ are tail points and 1 is a fixed point with orbit $0 \to \infty \to 1 \to 1$. Also for every $i \in \{-d, .., -1, 1, .., d\}$ the points 2^i are $\mathbb{Q}$-rational periodic points of period 2.

Finally by Theorem 3.2.1 if $d > 2^{31}+1$, then $\mathrm{Tail}(f_d, \mathbb{Q}) = \{0, \infty\}$. Thus, this gives an example of a family of rational functions f_d such that each rational function f_d has exactly two $\mathbb{Q}$-rational tail points, good reduction outside of a fixed finite set of places S, and the number of $\mathbb{Q}$-rational periodic points grows with the degree of $f_d(x)$.

The second example gives a family of rational functions with exactly three $\mathbb{Q}$-rational periodic points and a fixed set of places of bad reduction. However the size of the set of $\mathbb{Q}$-rational tail points grows with the degree of the rational functions in the family. This proves that the hypothesis of Theorem 3.2.1 part (b) is necessary.

Example 3.2.5. *Consider*

$$f_d(x) = \frac{(x-1)(x-2)(x-2^2)...(x-2^{d-1})}{x^d} \quad \in \mathbb{Q}(x).$$

If we take $S = \{\infty, 2\}$ *then* $f_d(x)$ *has good reduction outside* S.

Now we notice that 0 is a periodic point with orbit $0 \to \infty \to 1 \to 0$ *and that* $2, .., 2^{d-1}$ *are in the tail of 0.*

Finally by Theorem 3.2.1 if $d > 2^{34}+1$, *then* $\mathrm{Per}(f_d, \mathbb{Q}) = \{0, 1, \infty\}$. *Thus, this gives an example of a family of rational functions* f_d *such that each rational function* f_d *has exactly three* $\mathbb{Q}$*-rational periodic points, good reduction outside of a fixed finite set of places* S, *and the number of* $\mathbb{Q}$*-rational tail points grows with the degree of* $f_d(x)$.

3.3 Thue-Mahler approach

The goal of this section is to improve Theorem 3.2.1 and Theorem 3.2.2. In order to do so we will use Thue-Mahler equations instead of S-unit equations.

The first result will improve the bound for $|\mathrm{Tail}(\phi, K)|$ given in the previous section.

Theorem 3.3.1. *Let* K *be a number field and* S *a finite set of places of* K *containing all the archimedean ones. Let* ϕ *be an endomorphism of* $\mathbb{P}^1$, *defined over* K, *and* $d \geq 2$ *the degree of* ϕ. *Assume* ϕ *has good reduction outside* S. *Then*

$$|\mathrm{Tail}(\phi, K)| \leq d \max\left\{(5 \cdot 10^6(d^3+1))^{s+4}, 4(2^{64(s+3)})\right\}.$$

Similarly, the second result improves the bound for $|\mathrm{Per}(\phi, K)|$ given in the previous section. This improvement needs the extra assumption that ϕ has at least one K-rational tail point.

Theorem 3.3.2. *Let K be a number field and S a finite set of places of K containing all the archimedean ones. Let ϕ be an endomorphism of $\mathbb{P}^1$, defined over K, and $d \geq 2$ the degree of ϕ. Assume ϕ has good reduction outside S. If ϕ has at least one K-rational tail point then*

$$|\operatorname{Per}(\phi, K)| \leq \max\left\{(5 \cdot 10^6(d-1))^{s+3}, 4(2^{128(s+2)})\right\} + 1.$$

3.3.1 Proof of Theorem 3.3.1

In this subsection, we assume all the hypotheses of Theorem 3.3.1.

Notice that if ϕ has at least four K-rational periodic points, then by Theorem 3.2.1

$$|\operatorname{Tail}(\phi, K)| \leq 4(2^{16s}).$$

Therefore until the end of the proof of Theorem 3.3.1 we assume $|\operatorname{Per}(\phi, K)| \leq 3$.

If $|\operatorname{Per}(\phi, K)| = 0$ then $|\operatorname{Tail}(\phi, K)| = 0$. So there is nothing to prove in this case. The remaining possibilities can be divided into two cases: when $|\operatorname{Per}(\phi, K)| = 2$ or 3 and when $|\operatorname{Per}(\phi, K)| = 1$.

Before we start analyzing these two cases, we will prove a proposition that will be useful in both.

Proposition 3.3.3. *Let K be a number field and S a finite set of places of K containing all the archimedean ones. Let ϕ be an endomorphism of $\mathbb{P}^1$, defined over K and $d \geq 2$ the degree of ϕ. Assume ϕ has good reduction outside S and ϕ admits a normalized form with respect to S. Let $\mathscr{A} \subset \operatorname{Tail}(\phi, K)$ be such that every point in $\mathscr{A}$ admits a normalized form* *with respect to S. Then*

$$|\mathscr{A}| \leq \max\left\{(5\cdot 10^6(d^3+1))^{s+1}, 4(2^{64s})\right\}.$$

Proof. Suppose that there exists a $\bar{K}$-rational periodic point P_* of period $1, 2$ or 3 such that $[E:K]\geq 3$ where $E = K(P_*)$. Notice that $[E:K]\leq d^3$.

Let S_E be the set of places of E lying above places in S. Applying Proposition 2.3.9 to P_* and S_E, we can find a prime $\mathfrak{p}_E$ in E such that P_* can be written in normalized form with respect to $S_E\cup\{\mathfrak{p}_E\}$. Consider $S' = S\cup\{\mathfrak{p}_K\}$ where $\mathfrak{p}_K$ is the prime of K lying below $\mathfrak{p}_E$ and let S'_E be the set of places in E lying above places in S'.

Let $P = [x:y]\in\mathscr{A}$ be in normalized form with respect to S' and $P_* = [a:b]\in\mathbb{P}^1(E)$ in normalized form with respect to S'_E. Notice that P_* is not in the orbit of P since it is not K-rational.

For every prime $\mathfrak{p}'_E\notin S'_E$, $\delta_{\mathfrak{p}'_E}(P,P_*)=0$. Then for every $\mathfrak{p}'_E\notin S'_E$

$$v_{\mathfrak{p}'_E}(ay-bx)=0. \tag{3.6}$$

Denote by $N_{E/K}$ the norm from E to K and consider $F(X,Y) = N_{E/K}(aY-bX)\in K[X,Y]$ where the embedding of E over K act trivially on X and Y. Since P_* is in normalized form with respect to S'_E, we have that $a,b\in\mathcal{O}_{E,S'_E}$. Hence $F(X,Y)\in\mathcal{O}_{K,S'}[X,Y]$. Notice that the degree of F is $[E:K]$. Since P_* is a root of $F(X,Y)$ and E is the field of definition of P_* we have that $F(X,Y)$ is irreducible over K. Finally using that every $P=[x:y]\in\mathscr{A}$ is in normalized form with respect to S' and equation (3.6) we have $F(x,y)\in\mathcal{O}^*_{K,S'}$.

Now we have all the hypotheses to apply Theorem 2.3.3. Therefore in this case we get

$$|\mathscr{A}| \leq (5 \cdot 10^6 [E:K])^{s+1} \leq (5 \cdot 10^6 d^3)^{s+1}.$$

Now suppose that for every $\bar{K}$-periodic point P of period $1, 2$ or 3, we have $[K(P) : K] \leq 2$. We claim that in this case we can find a field E of degree $[E : K] \leq 4$ such that ϕ has at least 4 distinct E-rational periodic points. To prove the claim we just need to use Theorem 2.2.9 and Remark 2.2.10 as follows.

Case 1: There exists a point $P \in \mathbb{P}^1(\bar{K})$ of period 3 under ϕ. Let $Q \in \mathbb{P}^1(\bar{K})$ be a fixed point of ϕ and $E = K(P, Q)$. Then by assumption $[E : K] \leq 4$ and we have $|\operatorname{Per}(\phi, E)| \geq 4$.

Case 2: There does not exist a point $P \in \mathbb{P}^1(\bar{K})$ of period 3 under ϕ. By Theorem 2.2.9 and Remark 2.2.10, ϕ admits a point $P_1 \in \mathbb{P}^1(\bar{K})$ of period 2 and two distinct fixed points $P_2, P_3 \in \mathbb{P}^1(\bar{K})$. Since $1 \leq |\operatorname{Per}(\phi, K)| \leq 3$, we can assume that at least one of P_1, P_2, P_3 is K-rational. Let $E = K(P_1, P_2, P_3)$. Then again we have $[E : K] \leq 4$ and $|\operatorname{Per}(\phi, E)| \geq 4$.

Then by Theorem 3.2.1

$$|\mathscr{A}| \leq |\operatorname{Tail}(\phi, K)| \leq |\operatorname{Tail}(\phi, E)| \leq 4(2^{16(4(s))}) = 4(2^{64s}).$$

In any case

$$|\mathscr{A}| \leq \max\left\{(5 \cdot 10^6 (d^3 + 1))^{s+1}, 4(2^{64s})\right\}.$$

□

Notice that if $\mathcal{O}_S$ is a PID then Theorem 3.3.1 follows immediately from Proposition 3.3.3.

Proof of Theorem 3.3.1. **Case 1:** $|\operatorname{Per}(\phi, K)| \in \{2, 3\}$

By Proposition 2.3.8 we can assume ϕ is in normalized form with respect to S_1, for some S_1 with $|S_1| = |S| + 1$ and $S \subset S_1$.

Let $P_1 = [x_1 : y_1], P_2 = [x_2 : y_2]$ be two different K-rational periodic points. For every $P = [x_P : y_P] \in \operatorname{Tail}(\phi, K)$ there is $i_P \in \{1, 2\}$ such that

$$\delta_{\mathfrak{p}}(P, P_{i_P}) = 0 \quad \text{for every} \quad \mathfrak{p} \notin S_1.$$

Then

$$(x_P y_{i_P} - y_P x_{i_P})\mathcal{O}_{K,S_1} = (x_P, y_P)(x_{i_P}, y_{i_P})\mathcal{O}_{K,S_1} \quad \text{for every} \quad P \in \operatorname{Tail}(\phi, K).$$

Applying Proposition 2.3.9 on P_1, P_2 and S_1, we can find a representation of P_1 and P_2 such that $P_1 = [x'_1 : y'_1]$ and $P_2 = [x'_2 : y'_2]$ are in normalized form with respect to S_2, for some S_2 with $S_1 \subset S_2$ and $|S_2| = |S_1| + 2$. Hence, for every $P \in \operatorname{Tail}(\phi, K)$

$$(x_P y'_{i_P} - y_P x'_{i_P})\mathcal{O}_{K,S_2} = (x_P, y_P)\mathcal{O}_{K,S_2}$$

and x_P and y_P generate a principal $\mathcal{O}_{K,S_2}$-ideal. Therefore, for every $P \in \operatorname{Tail}(\phi, K)$ we can find a representation of P that is normalized with respect to S_2, namely $P = [\alpha_P^{-1} x_P : \alpha_P^{-1} y_P]$, where $\alpha_P = x_P y'_{i_P} - y_P x'_{i_P}$ (Remark 2.1.3).

Every point $P \in \operatorname{Tail}(\phi, K)$ admits a normalized form with respect to S_2 and ϕ is in normalized form with respect to S_2 with good reduction outside S_2. Applying Proposition 3.3.3 gives

$$|\operatorname{Tail}(\phi, K)| \leq \max\left\{(5 \cdot 10^6(d^3 + 1))^{s+4}, 4(2^{64(s+3)})\right\}.$$

Case 2: $|\operatorname{Per}(\phi, K)| = 1$

By Proposition 2.3.8 we can assume ϕ is in normalized form with respect to S_1, for some S_1 with $|S_1| = |S| + 1$ and $S \subset S_1$. Let $Q \in \mathbb{P}^1(K)$ be the only K-rational periodic point. Applying Proposition 2.3.9 on Q and S_1, we can find a representation of Q such that $Q = [q_1 : q_2]$ is in normalized form with respect to S_2, for some S_2 with $S_1 \subset S_2$ and $|S_2| = |S_1| + 1$.

Let $P = [x_P : y_P] \in \operatorname{Tail}(\phi, K)$. Since $\phi = [F, G]$ is in normalized form with respect to S_2 and ϕ has good reduction outside S_2. Thus

$$v_{\mathfrak{p}}((F(x_P, y_P), G(x_P, y_P))) = v_{\mathfrak{p}}((x_P, y_P)^d) \quad \text{for every} \quad \mathfrak{p} \notin S_2.$$

Therefore,

$$(F(x_P, y_P), G(x_P, y_P)) = (x_P, y_P)^d \quad \text{for every} \quad \mathcal{O}_{K,S_2}\text{-ideals.} \tag{3.7}$$

Applying the last equality repeatedly we get that the $\mathcal{O}_{K,S_2}$-ideal class $[(x_P, y_P)^{d^n}] = [(q_1, q_2)] = [1]$ is trivial for some $n > 0$ depending on P.

Assume the notation of Theorem 2.2.7. By Theorem 2.2.7 there are at most two values of n such that

$$a^*_Q(n) = \operatorname{ord}_Q(\Phi^*_{\phi,n}(X, Y)) \neq 0.$$

Since $a^*_Q(1) \neq 0$ we get that either $a^*_Q(2) = 0$ or $a^*_Q(3) = 0$. Set $l = \min\{i : a^*_Q(i) = 0\}$.

Consider $\Phi^*_{\phi,l}(X, Y)$ and notice that every root of $\Phi^*_{\phi,l}$ is a periodic point of period 1 or l, different from Q. Let

$$\Phi^*_{\phi,l}(X,Y) = cf_1(X,Y)^{\alpha_1} \cdots f_i(X,Y)^{\alpha_i} \cdots f_r(X,Y)^{\alpha_r}$$

be the irreducible factorization of $\Phi^*_{\phi,l}(X,Y)$ over K and $c \in K^*$. Let $e_i = \deg f_i$ for $i = 1, ..., r$. Note that the degree of $\Phi^*_{\phi,l}$ is $d^l - d$.

Fix $i \in \{1, ..., r\}$. Let $Q_i = [a_i : b_i] \in \mathbb{P}^1(\bar{K})$ be a root of $f_i(X,Y)$. Consider $E_i = K(Q_i)$ the field of definition of Q_i and $e_i = [E_i : K]$. Let S_{E_i} be the set of places of E_i lying above places of S_2.

Denote by $N_{E_i/K}$ the norm from E_i to K and notice that $f_i(X,Y) = N_{E_i/K}(a_iY - b_iX) \in K[X,Y]$ up to a constant. For every $P \in \mathrm{Tail}(\phi, K)$ and for every $\mathfrak{p}_{E_i} \notin S_{E_i}$ we have $\delta_{\mathfrak{p}_{E_i}}(P, Q_i) = 0$. Then

$$(x_P b_i - y_P a_i) = (a_i, b_i)(x_P, y_P) \text{ as } \mathcal{O}_{E_i, S_{E_i}}\text{-ideals.} \tag{3.8}$$

Applying $N_{E_i/K}$ to (3.8) we get

$$(f_i(x_P, y_P))\mathcal{O}_{K,S_2} = I_i(x_P, y_P)^{e_i}\mathcal{O}_{K,S_2} \tag{3.9}$$

where $I_i = N_{E_i/K}((a_i, b_i))$ is an $\mathcal{O}_{K,S_2}$-ideal. Taking appropriate powers and multiplying over all i gives

$$(\Phi^*_{\phi,l}(x_P, y_P))\mathcal{O}_{K,S_2} = I(x_P, y_P)^{\sum_i \alpha_i e_i}\mathcal{O}_{K,S_2} \tag{3.10}$$

where $I = \Pi_i I_i^{\alpha_i}$ is an $\mathcal{O}_{K,S_2}$-ideal.

By Theorem 2.3.7 applied to the $\mathcal{O}_{K,S_2}$-ideal I, there is a prime ideal $\mathfrak{p}_0$ in K and $\beta_I \in K$ such that $(\beta_I)I = \mathfrak{p}_0\mathcal{O}_{K,S_2}$. Consider $S_2' = S_2 \cup \{\mathfrak{p}_0\}$. Then multiplying (3.10) by β_I we get

$$\beta_I(\Phi^*_{\phi,l}(x_P,y_P))\mathcal{O}_{K,S_2} = \beta_I I(x_P,y_P)^{d^l-d}\mathcal{O}_{K,S_2} = \mathfrak{p}_0\mathcal{O}_{K,S_2}(x_P,y_P)^{d^l-d}\mathcal{O}_{K,S_2}.$$

Notice that $\mathfrak{p}_0\mathcal{O}_{K,S_2}$ is the trivial ideal in $\mathcal{O}_{K,S'_2}$. Therefore

$$\beta_I(\Phi^*_{\phi,l}(x_P,y_P))\mathcal{O}_{K,S'_2} = (x_P,y_P)^{d^l-d}\mathcal{O}_{K,S'_2}. \tag{3.11}$$

Thus, the ideal class of $(x_P,y_P)^{d^l-d}$ in $\mathcal{O}_{K,S'_2}$ is trivial. Then the ideal class of $(x_P,y_P)^{d^n}$ in $\mathcal{O}_{K,S'_2}$ is trivial since the ideal class of $(x_P,y_P)^{d^n}$ in $\mathcal{O}_{K,S_2}$ is trivial . Taking the g.c.d. of d^l-d and d^n we get that the ideal class of $(x_P,y_P)^d$ in $\mathcal{O}_{K,S'_2}$ is trivial.

Let $\mathscr{A}$ be the set of all K-rational tail points excluding the initial point in each maximal orbit. Using equation (3.7) and Remark 2.1.3 every point $P\in\mathscr{A}$ admits a normalized form with respect to S'_2.

Now applying Proposition 3.3.3 to $\mathscr{A}$ and S'_2, we get

$$|\mathscr{A}| \le \max\left\{(5\cdot 10^6(d^3+1))^{s+4}, 4(2^{64(s+3)})\right\}.$$

This gives us

$$|\operatorname{Tail}(\phi,K)| \le d|\mathscr{A}| \le d\max\left\{(5\cdot 10^6(d^3+1))^{s+4}, 4(2^{64(s+3)})\right\}.$$

□

3.3.2 Proof of Theorem 3.3.2

In this subsection, we assume the hypotheses in Theorem 3.3.2. Hence $|\operatorname{Tail}(\phi, K)| \geq 1$.

Notice that if ϕ has at least three K-rational tail points, then by Theorem 3.2.1 we have that

$$|\operatorname{Per}(\phi, K)| < 2^{16s} + 3.$$

Therefore in the rest of the section we assume $|\operatorname{Tail}(\phi, K)| \in \{1, 2\}$.

As before, we will need to prove a proposition to use in the proof of Theorem 3.3.2.

Proposition 3.3.4. *Let ϕ be an endomorphism of $\mathbb{P}^1$, defined over K. Let $d \geq 2$ be the degree of ϕ. Assume ϕ has good reduction outside S and ϕ is in normalized form with respect to S. Let $\mathscr{A} \subset \operatorname{Per}(\phi, K)$ such that every point in $\mathscr{A}$ admits a normalized form with respect to S. Then*

$$|\mathscr{A}| \leq \max\left\{(5 \cdot 10^6(d-1))^{s+1}, 4(2^{128s})\right\}.$$

Proof. Suppose that for every tail point $P_* \in \mathbb{P}^1(\bar{K}) - \mathbb{P}^1(K)$ such that $\phi(P_*)$ is a K-rational periodic point, $[K(P_*) : K] \geq 3$ where $E = K(P_*)$ is the field of definition of P_*. Then the same proof as the first part of the proof of Proposition 3.3.3 yields the desired result (notice that $[E : K] \leq d - 1$).

Now suppose that for every tail point $P_* \in \mathbb{P}^1(\bar{K}) - \mathbb{P}^1(K)$ such that $\phi(P_*)$ is a K-rational periodic point, $[K(P_*) : K] < 3$. In this case, assume we can find three different K-rational periodic points Q_1, Q_2, Q_3 which are not totally ramified (otherwise $\operatorname{Per}(\phi, K)$ is trivially bounded). We can find three different tail points $P_i \in \mathbb{P}^1(\bar{K}) - \mathbb{P}^1(K)$ such that $\phi(P_i) = Q_i$ and $1 \leq [K(P_i) : K] \leq 2$ where $1 \leq i \leq 3$. Applying Theorem 3.2.1 gives

$$|\mathscr{A}| \leq 4(2^{128s}).$$

Therefore we get

$$|\mathscr{A}| \leq \max\left\{(5 \cdot 10^6(d-1))^{s+1}, 4(2^{128s})\right\}.$$

□

Notice that if $\mathcal{O}_S$ is a PID then every point in $\mathrm{Per}(\phi, K)$ admits a normalized form with respect to S. Thus, in this case Proposition 3.3.4 gives a bound for $\mathrm{Per}(\phi, K)$ and the hypothesis on the existence of a K-rational tail point will not be required in Theorem 3.3.2.

Proof of Theorem 3.3.2. This proof follows the proof of Case I of Theorem 3.3.1 except that we use Proposition 3.3.4 instead of Proposition 3.3.3. After this change we obtain

$$|\mathrm{Per}(\phi, K)| \leq \max\left\{(5 \cdot 10^6(d-1))^{s+3}, 4(2^{128(s+2)})\right\} + 1.$$

□

Chapter 4

Arithmetic Dynamics on $\mathbb{P}^n$

This chapter consists of four sections. The first one gives a generalization of the logarithmic $\mathfrak{p}$-adic chordal distance and we prove the main tools needed for our study of dynamics in $\mathbb{P}^n$. In the second section we prove some effective results for K-rational tail hypersurfaces. In the third section we prove finiteness of K-rational tail curves on $\mathbb{P}^2$. Finally, in the last section we provide examples of preperiodic hypersurfaces.

4.1 Main definitions and propositions

We start this section by giving a generalization of the logarithmic $\mathfrak{p}$-adic chordal distance between two points in $\mathbb{P}^1$ to a logarithmic v-adic distance between a K-rational hypersurface and a point in $\mathbb{P}^n$ with respect to a nonarchimedean place v.

Definition 4.1.1. *Let H be a hypersurface of $\mathbb{P}^n$ defined over K of degree e. Further, suppose H is defined by an homogeneous polynomial $f = \sum_{|\boldsymbol{i}|=e} a_{\boldsymbol{i}} \boldsymbol{X}^{\boldsymbol{i}} \in K[\boldsymbol{X}]$ of degree e. Let v be a nonarchimedean place of K and $P = [x_0 : \cdots : x_n]$ a point in $\mathbb{P}^n(K)$ such that $P \notin supp(H)$.*

We define the logarithmic v-adic distance between P and H with respect to v by

$$\delta_v(P; H) = v(f(x_0, \ldots, x_n)) - d \min_{0 \le i \le n}\{v(x_i)\} - \min_{|\boldsymbol{i}|=e}\{v(a_{\boldsymbol{i}})\}. \tag{4.1}$$

Note that $\delta_v(P; H)$ is independent of the choice of homogeneous coordinates for P and the choice of defining polynomial for H.

Now we will prove some important properties of our logarithmic distance. We would like to emphasize that in the following two propositions there is no assumption of irreducibility or preperiodicity on the hypersurface.

Proposition 4.1.2. *Let H, H' be two hypersurfaces of $\mathbb{P}^n$ defined over K and v be a nonarchimedean place of K. Consider $x, y \in \mathbb{P}^n(K)$ such that $x \notin supp\{H\}$ and $y \notin supp\{H+H'\}$ then*

$$\delta_v(x; H) \geq 0 \tag{4.2}$$

and

$$\delta_v(y; H + H') = \delta_v(y; H) + \delta_v(y; H'). \tag{4.3}$$

Proof. Suppose H is defined by $f(\mathbf{X}) = \sum_{|\mathbf{i}|=e} a_{\mathbf{i}}\mathbf{X}^{\mathbf{i}}$ a homogeneous polynomial of degree e and H' is defined by $g(\mathbf{X}) = \sum_{|\mathbf{i}|=e'} b_{\mathbf{i}}\mathbf{X}^{\mathbf{i}}$ a homogeneous polynomial of degree e'. Without loss of generality assume that $\min_{|\mathbf{i}|=e}\{v(a_{\mathbf{i}})\} = 0$ and $\min_{|\mathbf{i}|=e'}\{v(b_{\mathbf{i}})\} = 0$. Assume that $x = [x_0 : \cdots : x_n]$ and $y = [y_: \cdots : y_n]$ are in normalized form with respect to v.

Proof of (4.2):After our assumptions (4.2) is just $v(f(x)) \geq 0$ which is trivially true.

Proof of (4.3): Notice that $H + H'$ is defined by $f(\mathbf{X})g(\mathbf{X}) = \sum_{|\mathbf{i}|=e+e'} c_{\mathbf{i}}\mathbf{X}^{\mathbf{i}}$, where $\min_{|\mathbf{i}|=e+e'}\{v(c_{\mathbf{i}})\} = 0$.

Now (4.3) is just $v(f(y)g(y)) = v(f(y)) + v(g(y))$ which is trivially true. □

The definition of distance above can also be defined for archimedean places; we will just have to change the notation of valuation to absolute values. For our study we will be working with nonarchimedean places so the valuation notation is more convenient in our case.

Proposition 4.1.3. *Let ϕ be an endomorphism of $\mathbb{P}^n$, defined over K and S a set finite set of places, including the archimedean ones. Suppose ϕ has good reduction outside S. Let H be a hypersurface of $\mathbb{P}^n$ defined over K and $v \notin S$. If $x \in \mathbb{P}^n(K)$ such that $\phi(x) \notin supp\{H\}$ then*

$$\delta_v(z; \phi^* H) = \delta_v(\phi(z); H). \tag{4.4}$$

Proof. Suppose H is defined by $f(\mathbf{X}) = \sum_{|\mathbf{i}|=e} a_{\mathbf{i}} \mathbf{X}^{\mathbf{i}}$ a homogeneous polynomial of degree e. Without loss of generality assume that $\min_{|\mathbf{i}|=e}\{v(a_{\mathbf{i}})\} = 0$ and $x = [x_0 : \cdots : x_n]$ is in normalized form with respect to v.

Since ϕ is in normalized form with respect to v and it has good reduction at v we have that $\phi(z)$ is in normalized form with respect to v.

Notice that $\phi^* H$ is defined by $h(\mathbf{X}) = f(\phi(\mathbf{X})) = \sum c_{\mathbf{j}} \mathbf{X}^{\mathbf{j}}$. Further, since f, ϕ are in normalized form with respect to v and ϕ has good reduction outside S, we have that $\min\{v(c_{\mathbf{j}})\} = 0$.

Now (4.4) is just $v(h(x)) = v(f(\phi(x)))$ which is trivially true.

□

The following result is a key arithmetic relation between a tail hypersurface and a periodic point. We will be using this property for the rest of the chapter.

Proposition 4.1.4. *Let ϕ be an endomorphism of $\mathbb{P}^n$, defined over K. Suppose ϕ has good reduction outside S. Let $P \in \mathbb{P}^n(K)$ be a periodic point of ϕ, and consider H' a K-rational fixed hypersurface and H a K-rational tail hypersurface in the tail of H'. Suppose that $P \notin supp\{H'\}$. Then $\delta_v(P; H) = 0$ for every $v \notin S$.*

Proof. Let m be the exact period of P, $v \notin S$ and $M = ml$ be the smallest natural number such that $\phi^M(H) = H'$. Since P is periodic of period m we have $\phi^M(P) = P$. Suppose H' is defined by f an irreducible homogeneous polynomial with coefficients in K.

Let $\phi^{*M}(H')$ be the pullback of H' by ϕ^M and notice that $\phi^{*M}(H')$ is a hypersurface defined by $F = f(\phi^M)$. Take the irreducible decomposition of F (over K) given by $\prod f_i^{k_i}$ and let C_i be the prime divisor defined by f_i. Then $\phi^{*M}(H') = \sum k_i C_i$. Even more, since H' is a fixed hypersurface and $\phi^M(H) = H'$ we have that $\phi^{*M}(H') = H' + H + D$ for some K-rational hypersurface D (not necessarily irreducible).

Using the properties of the logarithmic v-adic distance we get

$$\delta_v(P; H') = \delta_v(\phi^M(P); H') = \delta_v(P; \phi^{*M} H') = \delta_v(P; H' + H + D) = \delta_v(P; H') + \delta_v(P; H) + \delta_v(P; D).$$

Then

$$0 = \delta_v(P; H) + \delta_v(P; D)$$

Since $\delta_v(P; D) \geq 0$ we conclude that $\delta_v(P; H) = 0$. □

Theorem 4.1.5. *Let ϕ be an endomorphism of $\mathbb{P}^n$, defined over K. Suppose ϕ has good reduction outside S. Let H be a K-rational tail hypersurface, m the period of the periodic part of the orbit of H and H' the periodic hypersurface such that $H' = \phi^{m_0 m}(H)$ for some $m_0 > 0$. Let $P \in \mathbb{P}^n(K)$ be any periodic point such that $P \notin supp\{H'\}$. Then $\delta_v(P; H) = 0$ for every $v \notin S$.*

Proof. By Remark 2.1.6, ϕ^m also has good reduction outside S.

Now apply the previous proposition using ϕ^m for ϕ, H' for the fixed hypersurface and P as the periodic point. □

The following two lemmas will normally be used together with Proposition 2.3.9. We also point out that H is not necessarily an irreducible hypersurface and neither P nor H are assumed to be preperiodic.

Lemma 4.1.6. *Let H be a K-rational hypersurface of $\mathbb{P}^n$ and P a point of $\mathbb{P}^n(K)$ that does not lie on H. Assume that P admits a normalized form with respect to S and $\delta_v(P;H) = 0$ for every $v \notin S$. Then H admits a normalized form with respect to S.*

Proof. Suppose H is defined by an homogeneous polynomial $f = \sum_{|\mathbf{i}|=e} a_{\mathbf{i}} \mathbf{X}^{\mathbf{i}}$. Take $P = [x_0 : \cdots : x_n]$ in normalized form with respect to S.

By definition, for every $v \notin S$, $v(f(x_0, \ldots, x_n)) = \min_{|\mathbf{i}|=e} \{v(a_{\mathbf{i}})\}$ since $\delta_v(P;H) = 0$. Then

$$(f(x_0, \ldots, x_n))\mathcal{O}_S = (a_{\mathbf{i}})_{\mathbf{i}}\mathcal{O}_S.$$

Therefore, the ideal generated by $(a_{\mathbf{i}})_{\mathbf{i}}$ is a principal $\mathcal{O}_S$-ideal which implies that H admits a normalized form with respect to S by Remark 2.1.3. More explicitly, H given by $g = \sum_{|\mathbf{i}|=d} (f(x_0, \ldots, x_n))^{-1} a_{\mathbf{i}} \mathbf{X}^{\mathbf{i}}$ is in normalized form with respect to S. □

To have a similar result in the other direction we have to use that H is a hyperplane.

Lemma 4.1.7. *Let H be a K-rational hyperplane of $\mathbb{P}^n$ and P a point of $\mathbb{P}^n(K)$ that does not lie on H. Assume that H admits a normalized form with respect to S and $\delta_v(P;H) = 0$ for every $v \notin S$. Then P admits a normalized form with respect to S.*

Proof. The proof is analogous to the previous one.

□

4.2 Effective results

For the rest of the section we fix $e \in \mathbb{N}$ and $N(e) = N = \binom{e+n}{e} - 1$.

Consider the e-th veronese embedding

$$\begin{aligned}\Psi_e \colon \mathbb{P}^n &\to \mathbb{P}^N \\ \mathbf{x} &\mapsto \left(\mathbf{x}^{\mathbf{i}}\right)_{|\mathbf{i}|=e} = (w_{\mathbf{i}})_{|\mathbf{i}|=e}\,.\end{aligned}$$

With this embedding in mind we can associate a point of $\mathbb{P}^n$ to a hyperplane of $\mathbb{P}^N$ as follows:

$$P = \mathbf{x} \mapsto H_P : \sum_{|\mathbf{i}|=e} \mathbf{x}^{\mathbf{i}} w_{\mathbf{i}}.$$

Similarly, we associate a degree e hypersurface of $\mathbb{P}^n$ to a point of $\mathbb{P}^N$ as follows:

$$C : \sum_{|\mathbf{i}|=e} c_{\mathbf{i}} \mathbf{X}^{\mathbf{i}} \mapsto P_C = (c_{\mathbf{i}})_{|\mathbf{i}|=e}\,.$$

We notice that

$$H_P(P_C) = C(P). \tag{4.5}$$

Let $\mathcal{V}$ be the subvariety of $(\mathbb{P}^n)^{N+1}$ defined by the determinant of the square matrix $\mathcal{M}$, where the i-th row of $\mathcal{M}$ consists of all monomials of degree e in the coordinate variables of the i-th copy of $\mathbb{P}^n$ for $0 \leq i \leq N$.

For the rest of the section we assume the above notation for $\mathcal{M}$, $\mathcal{V}$ and for the association of points and hypersurfaces of degree e to hyperplanes and points, respectively. Our objective now is to use this association together with (4.5) to study $|\operatorname{HTail}(\phi, K, e)|$ for an endomorphism ϕ of $\mathbb{P}^n$, defined over K. More precisely, we will give an effective bound for the cardinality of a large subset of HTail(ϕ, K, e).

First, we need to establish some condition on a set of points of $\mathbb{P}^n$, in order to guarantee general position for the hyperplanes of $\mathbb{P}^N$ associated to those points.

Proposition 4.2.1. *Let $\mathcal{A} = \{P_i\}_{i=1}^{l}$ be a set of points of $\mathbb{P}^n$ with $l \geq N+1$. Then the associated set of hyperplanes $\mathcal{B} = \left\{H_{P_i}\right\}_{i=1}^{l}$ is not in general position if and only if there is a point $(Q_0, \ldots, Q_N) \in \mathcal{A}^{N+1} \subset (\mathbb{P}^n)^{N+1}$ with different coordinates that lies in $\mathcal{V}$ if and only if there are $N+1$ points of $\mathcal{A}$ that lie in hypersurface of degree e.*

Proof. To simplify notation we write H_i for H_{P_i} for $1 \leq i \leq l$. Notice that:

$\mathcal{B}$ is not in general position

if and only if there are $N+1$ hyperplanes $H_{i_0}, \ldots, H_{i_N}$ of $\mathcal{B}$ such that

$$\det \begin{pmatrix} \Psi_e(P_{i_0}) \\ \Psi_e(P_{i_1}) \\ \vdots \\ \Psi_e(P_{i_N}) \end{pmatrix} = 0$$

if and only if the point $(P_{i_0}, \ldots, P_{i_N})$ lies in $\mathcal{V}$

if and only if there is a non-zero vector $v = (v_{\mathbf{i}})_{|\mathbf{i}|=e}$ such that

$$\begin{pmatrix} \Psi_e(P_{i_0}) \\ \Psi_e(P_{i_1}) \\ \vdots \\ \Psi_e(P_{i_N}) \end{pmatrix} v = 0$$

if and only if the points $P_{i_0}, \ldots, P_{i_N}$ are zeros of $\sum_{|\mathbf{i}|=e} v_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}$, where $\mathbf{x}$ consists of the coordinate variables of $\mathbb{P}^n$. $\square$

Just for the next theorem we assume the following notation: Let ϕ be an endomorphism of $\mathbb{P}^n$, defined over K. For $T \in \mathrm{HTail}(\phi, K, e)$ we denote by n_T the the exact period of the periodic part of the orbit of T.

Theorem 4.2.2. *Let ϕ be an endomorphism of $\mathbb{P}^n$, defined over K and suppose ϕ has good reduction outside S. Let $\{P_i\}_{i=1}^{2N+1}$ be a set of K-rational periodic points of $\mathbb{P}^n$. Assume that no $N+1$ of them lie in a hypersurface of degree e. Consider $\mathcal{B} = \{H' \in \mathrm{HPer}(\phi, K) \colon \forall 1 \leq i \leq 2N+1, \quad P_i \notin \mathrm{supp}\ H'\}$ and $\mathcal{A} = \{T \in \mathrm{HTail}(\phi, K, e) \colon$ there is $l \geq 0, \quad \phi^{l n_T}(T) \in \mathcal{B}\}$. Then*

$$|\mathcal{A}| \leq \left(2^{33} \cdot (2N+1)^2\right)^{(N+1)^3 (s+2N+1)}$$

Proof. We can assume by Proposition 2.3.9 that every point of $\{P_i\}_{i=1}^{2N+1}$ is in normalized form with respect to S_1, for some S_1 with $|S_1| = |S| + 2N + 1$ and $S \subset S_1$. Since $\{P_i\}_{i=1}^{2N+1}$ are in normalized form with respect to S_1 then the associated hyperplanes $\{H_{P_i}\}_{i=1}^{2N+1}$ are in normalized form with respect to S_1. To simplify notation, for the rest of the proof we write H_i for H_{P_i}.

By Lemma 4.1.6 every $T \in \mathcal{A}$ admits a normalized form with respect to S_1 because P_1 is in normalized form with respect to S_1 and $\delta_v(P_1; T) = 0$ for every $v \notin S_1$ (by Theorem 4.1.5).

Take $T \in \mathcal{A}$ in normalized form with respect to S_1 and P_T be the associated point in $\mathbb{P}^N$. Then by (4.5) and Theorem 4.1.5 we have that $H_i(P_T) \in \mathcal{O}_{S_1}^*$ for every $1 \leq i \leq 2N+1$.

Hence P_T is a solution of the system

$$\begin{cases} H_1(x_0,\ldots,x_N) \in \mathcal{O}_{S_1}^* \\ \vdots \\ H_{2N+1}(x_0,\ldots,x_N) \in \mathcal{O}_{S_1}^* \end{cases} \quad \text{with} \quad (x_0,\ldots,x_N) \in \mathcal{O}_{S_1}^{N+1} \tag{4.6}$$

Notice that the solutions of the set of $\mathcal{O}_{S_1}^*$-cosets of solution of the system (4.6) are $(\sum_{1}^{2N+1} H_i, S_1)$-integral point. Then by Theorem 2.3.5 the set of $\mathcal{O}_{S_1}^*$-cosets of solution of the system (4.6) is finite.

Finally, by Theorem 2.3.6 the number of $\mathcal{O}_{S_1}^*$-cosets of solutions of

$$F(x_0,\ldots,x_N) = H_1(x_0,\ldots,x_N)\cdots H_{2N+1}(x_0,\ldots,x_N) \in \mathcal{O}_{S_1}^* \quad \text{with} \quad (x_0,\ldots,x_N) \in \mathcal{O}_{S_1}^{N+1}$$

is bounded by $\left(2^{33}\cdot(2N+1)^2\right)^{(N+1)^3(s+2N+1)}$. This implies that

$$|\mathcal{A}| \leq \left(2^{33}\cdot(2N+1)^2\right)^{(N+1)^3(s+2N+1)}.$$

□

4.3 Finiteness

Finiteness for the set of K-rational preperiodic subvarieties of $\mathbb{P}^n$ has been proven by B. Hutz [Hut16] using the theory of canonical height functions. Our goal in this section is to give an alternative proof to the one given by Hutz. We divide this section in two subsections. In the first one we give a result on finiteness for the set of K-rational tail hypersurfaces on the backward orbit of a K-rational periodic hypersurface. In the second subsection we prove finiteness of the set of K-rational preperiodic curves for an endomorphism of $\mathbb{P}^2$.

4.3.1 Finiteness on $\mathbb{P}^n$

For this subsection we assume that e is a fixed natural number and $N(e) = N = \binom{e+n}{e} - 1$.

We start recalling a strong result from dynamical systems.

Theorem 4.3.1 ([Fak03], Corollary 5.2)**.**

Let ϕ be an endomorphism of $\mathbb{P}^n$ over $\bar{K}$ of degree $d \geq 2$. Then the set $\mathrm{Per}(\phi, \bar{K})$ is Zariski dense in $\mathbb{P}^n$.

Now we can prove a lemma which will have many application in this section.

Lemma 4.3.2. *Let ϕ be an endomorphism of $\mathbb{P}^n$, defined over K and $Y \subsetneq \mathbb{P}^n$ a closed subset of $\mathbb{P}^n$. For any $l \in \mathbb{N}$, there are l periodic points in $\mathbb{P}^n(\bar{K}) \setminus Y$ such that no $N+1$ lie on a curve of degree e.*

Proof. First notice that it is enough to prove the result for $l > N$, so Without loss of generality assume $l > N$.

Let I be a subset of $\{1, \ldots, l\}$ of size $N+1$. Let $\phi_I \colon (\mathbb{P}^n)^l \to (\mathbb{P}^n)^{N+1}$ be the projection onto $(\mathbb{P}^n)^{N+1}$ corresponding to the elements of I.

Consider the proper algebraic subset of $(\mathbb{P}^n)^l$ given by $X = \bigcup_I \phi_I^{-1}(\mathcal{V})$ where $\mathcal{V}$ is the subvariety of $(\mathbb{P}^n)^{N+1}$ defined at the beginning of the section.

Take a point $P = (P_1, \ldots, P_l) \in (\mathbb{P}^n)^l$ and notice that by Proposition 4.2.1 P lies on X if and only if no subset of $N+1$ distinct points in $\{P_1, \ldots, P_l\}$ lie on a curve of degree e.

For every $1 \leq i \leq l$ take $\pi_i \colon (\mathbb{P}^n)^l \to \mathbb{P}^n$ the projection of the i-coordinate onto $\mathbb{P}^n$. Consider the proper algebraic subset of $(\mathbb{P}^n)^l$ given by $X' = X \cup \bigcup_{1 \leq i \leq l} \pi_i^{-1}(Y)$. Since

$\mathrm{Per}(\phi, \bar{K})$ is a Zariski dense subset of $\mathbb{P}^n$ then $\left(\mathrm{Per}(\phi, \bar{K})\right)^l$ is Zariski dense in $(\mathbb{P}^n)^l$. Hence, we can find a point $(Q_1, \ldots, Q_l) \in \left(\mathrm{Per}(\phi, \bar{K})\right)^l \setminus X'$. So $\{Q_1, \ldots, Q_l\}$ is the desired set of points.

□

Theorem 4.3.3. *Let ϕ be an endomorphism of $\mathbb{P}^n$, defined over K and H be a K-rational periodic hypersurface. If $\mathcal{A} = \{T \in \mathrm{HTail}(\phi, K, e) \colon \exists m > 0 \quad \phi^m(T) = H\}$ then $|\mathcal{A}|$ is finite. In other words, the set $\mathrm{HTail}(\phi, K, e) \cap \mathcal{O}^{-1}(H)$ is finite.*

Proof. First notice that it is enough to prove the theorem for H a fixed hypersurface. Without loss of generality assume that H is a fixed hypersurface.

Let S be the set of places of bad reduction of ϕ (including the archimedean places). By Lemma 4.3.2 we can find $2N + 1$ periodic points such that none lie on H and no $N + 1$ of them lie on a curve of degree e. Let E be the field of definition of our $2N + 1$ points and S_E the set of places of E lying above the places of S. Now applying Theorem 4.2.2 we get that the cardinality of

$$\mathcal{B} = \{T \in \mathrm{HTail}(\phi, E, e) \colon \exists m > 0 \quad \phi^m(T) = H\}$$

is bounded. Since $\mathcal{A} \subset \mathcal{B}$ we get that $\mathcal{A}$ is finite. □

4.3.2 Finiteness on $\mathbb{P}^2$

Now we have all of the tools to prove finiteness of K-rational tail curves of a given degree.

Theorem 4.3.4. *Let K be a number field and ϕ be an endomorphism of $\mathbb{P}^2$, defined over K. Then for every $e \in \mathbb{N}$ the set $\mathrm{HTail}(\phi, K, e)$ is finite.*

Proof. Let e be a fixed natural number, S be the set of places of bad reduction of ϕ (including the archimedean places) and $N = \binom{e+2}{2} - 1$. We split this proof in two main cases.

CASE 1: Suppose that all K-rational periodic curves have degree at most e.

For this case we assume the following notation: If $T \in \mathrm{HTail}(\phi, K, e)$ we denote by n_T the exact period of the periodic part of the orbit of T.

By Lemma 4.3.2 we can find $2N+1+e^2$ periodic points such that no $N+1$ of them lie on a curve of degree e. We denote these points by $\mathcal{P} = \{P_i\}_1^l$ where $l = 2N+1+e^2$. Let E be the field of definition of the points $\mathcal{P}$ and S_E the set of places of E lying above the places of S.

Let I be a subset of $\{1, \ldots, l\}$ of size $2N+1$. Consider $\mathcal{B}_I = \{H' \in \mathrm{HPer}(\phi, E) \colon \forall i \in I \quad P_i \notin \mathit{supp}\ H'\}$ and $\mathcal{A}_I = \{T \in \mathrm{HTail}(\phi, E, e) \colon \text{there is } l \geq 0 \quad \phi^{l n_T}(T) \in \mathcal{B}_I\}$. Notice that $\mathcal{A}_I$ is finite by Theorem 4.2.2.

Consider $\mathcal{D} = \{H' \in \mathrm{HPer}(\phi, E) \colon$ at least e^2+1 points of $\mathcal{P}$ are in $\mathit{supp}\ H'\}$ and $\mathcal{C} = \{T \in \mathrm{HTail}(\phi, E, e) \colon \text{there is } l \geq 0 \quad \phi^{l n_T}(T) \in \mathcal{D}\}$. By Bezout's Theorem $|\mathcal{D}| \leq \binom{l}{e^2+1}$ and applying Theorem 4.3.3 we have that the set $\mathcal{C}$ is finite.

Finally we notice that $\mathrm{HTail}(\phi, K, e) \subset \mathcal{C} \cup \bigcup_I \mathcal{A}_I$. So $|\mathrm{HTail}(\phi, K, e)| < \infty$.

CASE 2: Suppose there is a K-rational periodic curve H that has degree $f > e$.

Let n be the exact period of H, $\psi = \phi^n$ and $\mathcal{B} = \{T \in \mathrm{HTail}(\phi, K, e) \colon \exists m > 0 \quad \phi^{mn}(T) = H\}$.

Let $T \in \mathrm{HTail}(\phi, K, e) \setminus \mathcal{B}$, n_T the period of the periodic part of T and $H_T = \phi^{m n_T}(T) \in \mathrm{HPer}(\phi, K)$ for some $m > 0$. Then we claim that every point in $T \cap H$ is preperiodic with respect to ψ. Indeed, notice that if $P \in T \cap H$ then all but finitely many points of $\mathcal{O}_{\psi^{n_T}}(P)$ will lie on $H \cap H_T$ and by Bezout's theorem H and H_T intersect in finitely many points. Therefore $\mathcal{O}_{\psi^{n_T}}(P)$ is finite which implies that $\mathcal{O}_{\psi}(P)$ is finite, i.e. every point of $T \cap H$ is preperiodic. We also notice that every point in $T \cap H$ has degree bounded by ef over K.

Let $\mathcal{R}$ be the set of preperiodic points of ψ restricted to H of degree at most ef over K. This set is finite by Northcott's Theorem.

We claim that the set of curves T of degree e such that $T \cap H \subset \mathcal{R}$ is finite.

First notice that up to multiplication by constants, there are finitely many functions on H of degree at most ef that are regular on $H \setminus \mathcal{R}$ and their multiplicative inverses are also regular on $H \setminus \mathcal{R}$ (i.e. no zeros or poles).

Fix a curve T_0 of degree e such that $T_0 \cap H \subset \mathcal{R}$ and let t_0 be an homogeneous polynomial of degree e defining T_0. If T is a curve of degree e such that $T \cap H \subset \mathcal{R}$, then let t be a homogeneous polynomial of degree e defining T. Then the function $\frac{t}{t_0}$ on H will be a function of degree at most ef with no zeros or poles on $H \setminus \mathcal{R}$.

Now consider T_1 and T_2 two distinct curves of degree e such that $T_1 \cap H \subset \mathcal{R}$ and $T_2 \cap H \subset \mathcal{R}$. Let t_1 and t_2 be two homogeneous polynomials of degree e defining T_1 and T_2 respectively. Then the functions $\frac{t_1}{t_0}$ and $\frac{t_2}{t_0}$ on H do not differ by multiplication by a constant because if there were a constant c such that $\frac{t_1}{t_0} = c\frac{t_2}{t_0}$ then $t_1 - ct_2$ will be divisible by the polynomial defining H and this is impossible since $f > e$. This finishes our claim.

Therefore, the set $\text{HTail}(\phi, K, e) \setminus \mathcal{B}$ is finite and by Theorem 4.3.3 on H and ψ we have that $\mathcal{B}$ is finite. In other words,

$$|\,\mathrm{HTail}(\phi, K, e)| < \infty.$$

□

4.4 Examples

We will give three examples of preperiodic hypersurfaces of an endomorphism of $\mathbb{P}^n$.

Example 4.4.1. *Let S_{n+1} be the symmetric group on $n+1$ elements, $\sigma \in S_{n+1}$ and $\phi_{\sigma,d} = [x^d_{\sigma(0)}, ..., x^d_{\sigma(n)}]$ a degree d endomorphism of $\mathbb{P}^n$. Then all the canonical hyperplanes H_i defined by x_i are periodic. Further, their orbits are given by the decomposition into disjoint cycles of σ.*

Example 4.4.2. *Consider the morphism*

$$\psi_d \colon \mathbb{P}^2 \to \mathbb{P}^2$$
$$(x : y : z) \mapsto (x^d : y^d : z^d).$$

For any $i \in \mathbb{N}$ take the curve $C_i \subset \mathbb{P}^2$ defined by $z^{d^i-1}y = x^{d^i}$. Notice that C_i is an irreducible curve of degree d^i. Further C_i is fixed under ψ_d.

If ρ is a root of unity then the curve C_ρ defined by $z^{d^i-1}y = \rho x^{d^i}$ is preperiodic under ψ_d. Further, if ρ is a d^k-root of unity for some $k \in \mathbb{N}$ then C_ρ is in the tail of C_i. Otherwise, C_ρ is periodic.

Therefore, ψ_d has infinitely many $\mathbb{Q}$-rational periodic curves. Notice that the degree of the fixed curves go to infinity.

It is important to notice that a number field contains only finitely many roots of unity. Therefore, in this way we can construct only finitely many tail curves that are rational over a given number field.

This example illustrates the importance of including the degree of a hypersurface as a parameter in order to obtain finiteness results for K-rational preperiodic hypersurfaces.

Example 4.4.3 (Bell, Ghioca and Tucker [BGT15])**.** *Let f be a homogeneous two-variable polynomial of degree d and consider the morphism*

$$\psi\colon \mathbb{P}^2 \to \mathbb{P}^2$$
$$(x : y : z) \mapsto [f(x,z) : f(y,z) : z^d].$$

Then ψ has infinetely many fixed curves. Indeed, all curves of the form $[xz^{d^k-1} : f^k(x,z) : z^{d^k}]$ are fixed under ψ, where f^k is the homogenized kth iterate of the dehomogenized one-variable polynomial $x \to f(x,1)$.

Like the previous example, we have infinitely many fixed curves. However just like before, the degree of the the fixed curves go to infinity.

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