SAMPLE PATH PROPERTIES OF GAUSSIAN RANDOM FIELDS AND STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

By

Cheuk Yin Lee

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ABSTRACT

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Gaussian random fields are studied and applied in a wide range of scientific areas. In particular, the solutions of stochastic partial differential equations (SPDEs) form an important class of random fields and it is of interest to study the properties of their sample paths. The objective of this dissertation is to develop some methods for studying Gaussian random fields and to use these methods to investigate the sample path properties of SPDEs. We study the existence of multiple points for a general class of Gaussian random fields including fractional Brownian sheets, systems of stochastic heat equations and systems of stochastic wave equations. We also study the regularity of local times and the Hausdorff measure of level sets of Gaussian random fields and give an application to the stochastic heat equation. Moreover, for the stochastic wave equation, we examine further properties including local nondeterminism, the exact modulus of continuity, and the propagation of singularities.

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Chapter 1

Introduction

Gaussian random fields are studied extensively in probability, and have useful applications in a wide range of scientific areas such as statistics, physics, engineering, biology, economics and finance. Random fields are generalization of stochastic processes in the sense that they are indexed not only by a single time variable $t \in \mathbb{R}_+$, but by multi-dimensional variables such as spatial position $x \in \mathbb{R}^n$ ($n = 1, 2$ or 3) or even time and space variables $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$.

In particular, a large class of random fields arise naturally as solutions of stochastic partial differential equations (SPDEs). In this thesis, SPDEs are partial differential equations that are subject to random perturbations such as a white noise. Some of them are motivated from physics and can be used to model randomness in physical phenomena. In both mathematic and scientific point of view, it is interesting and meaningful to investigate properties of solutions of SPDEs.

For fundamental Gaussian random fields such as the Brownian motion, fractional Brownian motion and Brownian sheet, many sample path properties have been studied in the literature. Moreover, some unified methods for anisotropic Gaussian random fields with general assumptions were developed (see e.g. Xiao [70]). These methods allow us to study many properties including modulus of continuity, small ball probabilities, hitting probabilities, fractal properties of ranges, graphs, and level sets, existence and regularity of local times, etc.

There are different approaches for studying SPDEs. One of them is the random field approach based on Walsh's theory of stochastic integration. This approach emphasizes solutions as real-valued random fields, as opposed to other approaches that consider solutions that take values in certain infinite dimensional spaces. The links between different approaches are discussed by Dalang [12]. In this thesis, we focus on the random field approach. As we will see, the methods of random fields are useful for obtaining precise results on analytic and geometric properties of the sample paths of SPDEs.

The main purpose of this thesis is to develop some methods of Gaussian random fields and to use these methods to study sample path properties of the solutions of SPDEs.

This thesis is organized as follows. In Chapter 2, we begin with some preliminaries and overview of Gaussian random fields and SPDEs. We introduce some important examples of Gaussian random fields, and then review Walsh's theory of stochastic integration and SPDEs. We also introduce two important examples of SPDEs, namely the stochastic heat equation and wave equation.

In Chapter 3, we study the multiple points (or self-intersections) of the sample paths of Gaussian random fields. Based on a covering argument, we prove that for a large class of Gaussian random fields, multiple points do not exist in critical dimensions. We apply this result to the fractional Brownian sheet, systems of stochastic heat equations and systems of stochastic wave equations.

Chapter 4 is devoted to the study of the local times and level sets of a class of anisotropic Gaussian random fields that satisfies the property of strong local nondeterminism. We prove joint continuity and Hölder conditions for their local times, and discuss the Hausdorff dimension and Hausdorff measure of their level sets. Our results can be applied to the solution of stochastic heat equation, which satisfies strong local nondeterminism. We also determine the gauge function for the Hausdorff measure of its level sets.

In Chapters 5 and 6, we examine further properties of the stochastic wave equation driven by an additive Gaussian noise that is white in time and colored in space. In Chapter 5, we prove a property of local nondeterminism for the solution of the stochastic wave equation and apply this property to derive the exact uniform modulus of continuity for the solution. In Chapter 6, we discuss the notion of singularity for the stochastic wave equation and study the existence and propagation of singularities based on a simultaneous law of the iterated logarithm.

Throughout the thesis, we use C and K denote constants whose value may vary in each appearance, and we use $C_1, C_2, K_1, \dots$ for specific constants. We let $\mathbb{R}_+$ denote the set of all non-negative real numbers. Also, $|x|$ is the absolute value of x if $x \in \mathbb{R}$, and the Euclidean norm of x if $x \in \mathbb{R}^n$.

Chapter 2

Preliminaries

The purpose of this chapter is to give an overview of Gaussian random fields and stochastic partial differential equations (SPDEs). We first define Gaussian random fields and give some important examples. Then we give a self-contained introduction to Walsh's theory on stochastic integration and SPDEs. The stochastic heat equation and stochastic wave equation are the most important examples of SPDEs. We state some existence results and regularity properties of their solutions. The materials in this chapter are known in the literature, and they will provide sufficient preliminary knowledge for understanding the rest of the thesis.

2.1 Gaussian Random Fields

An *N-parameter d-dimensional random field*, or *(N, d)-random field* is a stochastic process $u = \{u(x) : x \in T\}$ that is indexed by a subset T of $\mathbb{R}^N$ and takes values in $\mathbb{R}^d$, i.e. a family of random variables $u(x) = (u_1(x), \dots, u_d(x)) : \Omega \to \mathbb{R}^d$ indexed by $x \in T$. We say that u is a *Gaussian random field* if the nd-dimensional random vector $(u(x_1), \dots, u(x_n))$ is Gaussian for all $n \ge 1$ and all $x_1, \dots, x_n \in T$. The probability distributions of the collection of all these random vectors are called the *finite dimensional distributions*.

The function $m : T \to \mathbb{R}^d$ defined by $m(x) = \mathbb{E}(u(x))$ is called the *mean function* and the function $C : T \times T \to \mathbb{R}^{d\times d}$, $C = (C_{ij})_{1\le i,j\le d}$, defined by $C_{ij}(x, y) = \mathrm{Cov}(u_i(x), u_j(y))$, is called the *covariance function*. We say that u is *centered* if $\mathbb{E}(u(x)) = 0$ for all $x \in T$.

We can define a Gaussian random field by specifying the mean function and covariance function because a Gaussian random field is determined by its finite dimensional distributions, which in turn are determined by the mean function and the covariance function. More precisely, if we are given a function $m : T \to \mathbb{R}^d$ and a function $C : T \times T \to \mathbb{R}^{d\times d}$ which is symmetric i.e. $C(x, y) = C(y, x)$, and nonnegative definite in the sense that

$$\sum_{i,j=1}^{d} \sum_{k,l=1}^{n} a_{i,k} a_{j,l} \, C_{i,j}(x_k, x_l) \geq 0$$

for all $n \geq 1$, for all $x_1, \ldots, x_n \in T$ and all $a_{i,k} \in \mathbb{R}$ $(i = 1, \ldots, d,\ k = 1, \ldots, n)$, then there exists an (N, d)-Gaussian random field $\{u(x) : x \in T\}$ whose mean function is m and covariance function is C.

The following are two fundamental examples of Gaussian random fields.

Example 2.1.1. *Multiparameter fractional Brownian motion.*

The (N, d)-fractional Brownian motion with Hurst index $H \in (0, 1)$ is defined as a centered (N, d)-Gaussian random field $\{B(t) : t \in \mathbb{R}^N\}$ with covariance function

$$\mathbb{E}(B_i(t)B_j(s)) = \delta_{ij} \, \frac{|t|^{2H} + |s|^{2H} - |t - s|^{2H}}{2}$$

for $t, s \in \mathbb{R}^N$, where $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ otherwise. It follows that the coordinate components $B_1, \ldots, B_d$ of B are independent and identically distributed (i.i.d.).

This Gaussian random field has stationary increments in the sense of Yaglom: for any $h \in \mathbb{R}^N$, $\{B(t + h) - B(h) : t \in \mathbb{R}^N\}$ and $\{B(t) - B(0) : t \in \mathbb{R}^N\}$ are equal in finite dimensional distributions.

When $N = 1$ and $H = 1/2$, it is the standard d-dimensional Brownian motion. When $N = 1$ and $0 < H < 1$, it is the d-dimensional fractional Brownian motion of Hurst index H. When $N > 1$ and $H = 1/2$, it is known as Levy's multiparameter Brownian motion.

Example 2.1.2. *Fractional Brownian sheet.*

The (N, d)-fractional Brownian sheet of Hurst indices $H_1, \ldots, H_N \in (0, 1)$ is defined as a centered (N, d)-Gaussian random field $\{B(t) : t \in \mathbb{R}^N\}$ with covariance function

$$\mathbb{E}(B_i(t)B_j(s)) = \delta_{ij} \prod_{\ell=1}^{N} \frac{|t_\ell|^{2H_\ell} + |s_\ell|^{2H_\ell} - |t_\ell - s_\ell|^{2H_\ell}}{2}$$

for $t, s \in \mathbb{R}^N$. When $H_1 = \cdots = H_N = 1/2$, it is called the (N, d)-Brownian sheet.

The fractional Brownian sheet has the property of being anisotropic in the sense that it can have different regularities and sample properties along different directions.

Also, the fractional Brownian sheet does not have stationary increments and it has subtle properties that are different from those of the fractional Brownian motion. For example, they have different form of small ball probabilities and Chung's law of the iterated logarithm.

For a Gaussian random field $u = \{u(t) : t \in T\}$, it will be convenient to use the notation

$$\sigma_u(t, s) := (\mathbb{E}|u(t) - u(s)|^2)^{1/2}$$

to denote the canonical metric (or pseudo-metric) on T. In many examples, we can find $\alpha_1, \ldots, \alpha_N \in (0, 1)$ and positive finite constants C_1, C_2 such that

$$C_1 \sum_{j=1}^{N} |t_j - s_j|^{\alpha_j} \le \sigma_u(t, s) \le C_2 \sum_{j=1}^{N} |t_j - s_j|^{\alpha_j} \tag{2.1}$$

for all $t, s \in T$. The parameters $\alpha_1, \ldots, \alpha_N$ play important roles in characterizing the sample path properties of $u(t)$ e.g. regularity, fractal properties and hitting probabilities (see [70]).

In Example 2.1.1, the (N, d)-fractional Brownian motion of Hurst index H satisfies (2.1) with $\alpha_1 = \cdots = \alpha_N = H$. In Example 2.1.2, the (N, d)-fractional Brownian sheet of Hurst indices $H_1, \ldots, H_N$ satisfies (2.1) with $\alpha_i = H_i$ for $i = 1, \ldots, N$.

In the next section, we will see more examples of Gaussian and non-Gaussian random fields that arise as solutions of stochastic partial differential equations.

2.2 Stochastic Partial Differential Equations

As explained in the Introduction, solutions of SPDEs form a large class of random fields and we are interested in studying their sample path properties. We follow [64] and give an introduction to Walsh's theory of stochastic integration, which allows us to construct random field solutions to SPDEs. We will then discuss two of the most important examples of SPDEs, namely the stochastic heat equation and stochastic wave equation.

Consider a differential operator $\mathscr{D}$ with constant coefficients and the SPDE

$$\mathscr{D}u(t,x) = \sigma(u(t,x))\dot{W}(t,x) + b(u(t,x)), \quad t \geq 0,\, x \in \mathbb{R}^k, \tag{2.2}$$

where $\sigma : \mathbb{R} \to \mathbb{R}$ and $b : \mathbb{R} \to \mathbb{R}$ are Lipschitz functions, and $\dot{W}$ is a Gaussian noise, whose definition will be given later.

From the theory of partial differential equations, the differential operator $\mathscr{D}$ always has a fundamental solution G, namely a distribution G that solves $\mathscr{D}G = \delta_0$, where δ_0 is the Dirac measure at $0 \in \mathbb{R}^{1+k}$, and it follows that $u = G * \varphi$ solves the equation $\mathscr{D}u = \varphi$ for any φ in $\mathcal{S}(\mathbb{R}_+ \times \mathbb{R}^k)$, the space of smooth rapidly decreasing functions (Schwartz space), where $G * \varphi$ is the convolution of G and φ in (t,x)-variables. See [55, Ch. 8].

In view of this, a *mild solution* to the SPDE (2.2) is a jointly measurable real-valued random field $\{u(t,x) : t \geq 0, x \in \mathbb{R}^k\}$ that satisfies

$$\begin{aligned} u(t,x) &= G * \Big(\sigma(u)\dot{W} + b(u)\Big) \\ &= \iint_{[0,t]\times\mathbb{R}^k} G(t-s, x-y)\Big(\sigma(u(s,y))\dot{W}(s,y) + b(u(s,y))\Big)\, ds\, dy \end{aligned}$$

and is adapted to a filtration generated by the noise $\dot{W}$ (defined in (2.5) below). To explain the meaning of the above stochastic integral, let us introduce Walsh's approach [64] of martingale measures and stochastic integration.

2.2.1 Walsh's Stochastic Integration

We will consider spatially homogeneous (centered) Gaussian noise W that is white in time and has spatial covariance f, which is a non-negative definite function. The Gaussian noise is defined as a centered Gaussian process $\{W(\varphi) : \varphi \in C_c^\infty(\mathbb{R}^{1+k})\}$ that is indexed by φ in $C_c^\infty(\mathbb{R}^{1+k})$, the space of real-valued smooth functions on $\mathbb{R}^{1+k}$ of compact support, and has covariance

$$E(W(\varphi)W(\psi)) = \int_{\mathbb{R}_+} ds \int_{\mathbb{R}^k} dy \int_{\mathbb{R}^k} dy' \varphi(s,y) f(y-y') \psi(s,y')$$

for all $\varphi, \psi \in C_c^\infty(\mathbb{R}^{1+k})$. Formally, we write

$$E(\dot{W}(s,y)\dot{W}(s',y')) = \delta_0(s-s') f(y-y'). \tag{2.3}$$

We say that $\dot{W}$ is a space-time white noise if $f = \delta_0$, the Dirac delta function. Another spatial covariance that is commonly used is $f(y) = |y|^{-\beta}$, where $0 < \beta < k$.

Let $\mathscr{B}_b(\mathbb{R}^k)$ denote the set of all bounded Borel sets in $\mathbb{R}^k$. The *martingale measure* induced by the noise $\dot{W}$ is the stochastic process $\{M_t(A) : t \in \mathbb{R}_+, A \in \mathscr{B}_b(\mathbb{R}^k)\}$ defined by

$$M_t(A) = \lim_{n\to\infty} W(\varphi_n), \tag{2.4}$$

where the right-hand side is the limit of a sequence $\{W(\varphi_n) : n \geq 1\}$ in $L^2(\Omega, \mathscr{F}, \mathbb{P})$, and φ_n is any sequence in $C_c^\infty(\mathbb{R}^{1+k})$ such that $\varphi_n \downarrow \mathbf{1}_{[0,t]\times A}$. It follows that for each $A \in \mathscr{B}_b(\mathbb{R}^k)$, the stochastic process $\{M_t(A) : t \geq 0\}$ is a martingale with respect to the filtration

$$\mathscr{F}_t = \sigma\{M_s(B) : 0 \leq s \leq t, B \in \mathscr{B}_b(\mathbb{R}^k)\}, \quad t \geq 0. \tag{2.5}$$

Let us define an elementary process as a stochastic process $g(t,x) : \Omega \to \mathbb{R}$, with $t \geq 0$ and $x \in \mathbb{R}^k$, of the form

$$g(t,x,\omega) = X(\omega)\mathbf{1}_{(a,b]}(t)\mathbf{1}_B(x),$$

where $0 \leq a < b$, $B \in \mathscr{B}_b(\mathbb{R}^k)$, and X is a bounded, $\mathscr{F}_a$-measurable random variable. For an elementary process, we can naturally define its stochastic integral as

$$\int_{[0,t]\times\mathbb{R}^k} g(s,y)W(ds\,dy) := X\big(M_{t\wedge b}(B) - M_{t\wedge a}(B)\big).$$

By linearity, we can then extend the definition of stochastic integration to the class $\mathscr{S}$ of all linear combinations of elementary processes, which we will call simple processes.

For the martingale measure M, we can define a function Q_M by $Q_M((s,t] \times B \times C) =$

$\langle M.(B), M.(C)\rangle_t - \langle M.(B), M.(C)\rangle_s$, for any $0 \le s < t$ and $B, C \in \mathscr{B}(\mathbb{R}^k)$.

We say that the martingale measure M is *worthy* if there exists a random σ-finite measure $K_M(A \times B \times C, \omega)$, where $A \in \mathscr{B}(\mathbb{R}_+)$, $B, C \in \mathscr{B}(\mathbb{R}^k)$ and $\omega \in \Omega$, such that

1. $B \times C \mapsto K_M(A \times B \times C, \omega)$ is nonnegative definite and symmetric;

2. $\{K_M((0,t] \times B \times C) : t \ge 0\}$ is a $\sigma(\mathscr{S})$-measurable process for all $B, C \in \mathscr{B}(\mathbb{R}^k)$;

3. For all $t > 0$ and compact sets $B, C \in \mathscr{B}(\mathbb{R}^k)$, $\mathbb{E}[K_M((0,t] \times B \times C)] < \infty$;

4. For all $t > 0$ and $B, C \in \mathscr{B}(\mathbb{R}^k)$, $|Q_M((0,t] \times B \times C)| \le K_M((0,t] \times B \times C)$ a.s.

Consider $t \in [0,T]$, where $T > 0$ is fixed. If M is a worthy martingale measure, then for any $t \in [0,T]$, the stochastic integral defines a linear map

$$g \mapsto \int_{[0,t]\times\mathbb{R}^k} g(s,y) W(ds\,dy), \tag{2.6}$$

from $\mathscr{S}$ to $L^2(\Omega, \mathscr{F}_T, \mathbb{P})$, which is continuous with respect to the norm $\|\cdot\|_M$ on $\mathscr{S}$ and the L^2-norm on $L^2(\Omega, \mathscr{F}_T, \mathbb{P})$, where $\|\cdot\|_M$ is defined by

$$\|g\|_M^2 := \mathbb{E} \iiint_{[0,T]\times\mathbb{R}^k\times\mathbb{R}^k} |g(s,y)g(s,y')| K_M(ds\,dy\,dy'). \tag{2.7}$$

Example 2.2.1. *Suppose that $\dot{W}$ is a space-time white noise i.e. $f = \delta_0$ in (2.3). Then $Q_M((0,t] \times B \times C) = t\lambda_k(B \cap C)$, where λ_k is the k-dimensional Lebesgue measure. Take $K_M(A \times B \times C) = \lambda_1(A)\lambda_k(B \cap C)$. It follows that M is a worthy martingale measure and*

$$\|g\|_M^2 = \mathbb{E} \int_0^T \int_{\mathbb{R}^k} |g(s,y)|^2\, ds\, dy.$$

Example 2.2.2. *Suppose that the Gaussian noise $\dot{W}$ satisfies* (2.3) *with $f(y) = |y|^{-\beta}$ and $0 < \beta < k$. Then*

$$Q_M((0,t] \times B \times C) = t \int_B \int_C |y - y'|^{-\beta}\, dy\, dy'.$$

We can take

$$K_M(A \times B \times C) = \lambda_1(A) \int_B \int_C |y - y'|^{-\beta}\, dy\, dy',$$

and the martingale measure M is worthy and

$$\|g\|_M^2 = \mathbb{E} \int_0^T \int_{\mathbb{R}^k} \int_{\mathbb{R}^k} |g(s,y)g(s,y')||y - y'|^{-\beta}\, ds\, dy\, dy'.$$

Let $\mathscr{P}_M$ be the set of all $\sigma(\mathscr{S})$-measurable processes g such that $\|g\|_M < \infty$. Then $(\mathscr{P}_M, \|\cdot\|_M)$ is a Banach space and $\mathscr{S}$ is dense in $\mathscr{P}_M$. It follows that (2.6) extends to a continuous linear map from $\mathscr{P}_M$ to $L^2(\Omega, \mathscr{F}_T, \mathbb{P})$. Therefore, we are now able to define the stochastic integral

$$\int_{[0,t]\times\mathbb{R}^k} g(s,y) W(ds\, dy)$$

as the image of g under this map, for a large class of processes g in $\mathscr{P}_M$.

2.2.2 Stochastic Heat Equation

Consider the stochastic heat equation

$$\begin{cases} \dfrac{\partial}{\partial t} u(t,x) - \Delta u(t,x) = \dot{W}(t,x), \quad t \geq 0,\, x \in \mathbb{R}^k, \\ u(0,x) = u_0(x), \end{cases} \tag{2.8}$$

where $\dot{W}$ is a Gaussian noise. The fundamental solution for the heat equation is

$$G(t,x) = \frac{1}{(4\pi t)^{1/2}}\, e^{-\frac{|x|^2}{4t}}.$$

For the moment, suppose that the spatial dimension is 1 (i.e. $k = 1$) and $\dot{W}$ is a space-time white noise. Then the mild solution to (2.8) is the real-valued Gaussian random field $\{u(t,x) : t \geq 0, x \in \mathbb{R}\}$ defined by

$$u(t,x) = (G * u_0)(t,x) + \int_{[0,t]\times\mathbb{R}} G(t-s, x-y) W(ds\, dy).$$

However, for $k \geq 2$, the stochastic integral above is not well-defined because the norm $\|\cdot\|_M$ of the integrand defined in (2.7) is infinite:

$$\|G(t-\cdot, x-\cdot)\|_M^2 = \int_0^t \int_{\mathbb{R}^k} |G(t-s, x-y)|^2\, ds\, dy = \infty.$$

As a consequence, there is no real-valued process solution for (2.8) when $k \geq 2$. The solution is a random Schwartz distribution, but we will not discuss this kind of solution in this thesis.

To obtain real-valued solutions when $k \geq 2$, Dalang's approach [11] is to replace the space-time white noise by a Gaussian noise that in white in time but correlated in space. Consider the nonlinear stochastic heat equation

$$\begin{cases} \dfrac{\partial}{\partial t} u(t,x) - \Delta u(t,x) = \sigma(u(t,x))\dot{W}(t,x) + b(u(t,x)), \quad t \geq 0,\, x \in \mathbb{R}^k, \\ u(0,x) = u_0(x). \end{cases} \tag{2.9}$$

Suppose that the Gaussian noise $\dot{W}$ satisfies (2.3):

$$\mathbb{E}[\dot{W}(s,y)\dot{W}(s',y')] = \delta_0(s-s')f(y-y').$$

Recall the natural filtration $\{\mathscr{F}_t\}$ of the noise defined in (2.5). By a mild solution to (2.9) we mean a jointly measurable, $\{\mathscr{F}_t\}$-adapted, real-valued random field $\{u(t,x) : t \geq 0, x \in \mathbb{R}^k\}$ that satisfies the integral equation

$$\begin{aligned} u(t,x) = (G * u_0)(t,x) &+ \int_{[0,t]\times\mathbb{R}^k} G(t-s,x-y)\sigma(u(s,y))W(ds\,dy) \\ &+ \int_{[0,t]\times\mathbb{R}^k} G(t-s,x-y)b(u(s,y))\,ds\,dy. \end{aligned} \tag{2.10}$$

Suppose that $f \geq 0$ and f is a non-negative definite function, i.e.

$$\int_{\mathbb{R}^k} (\varphi * \tilde{\varphi})(x)f(x)dx \geq 0$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^k)$, and $\tilde{\varphi}(x) := \varphi(-x)$. Let μ be a nonnegative measure on $\mathbb{R}^k$ whose Fourier transform is $f(x)$. For example, if $f(x) = |x|^{-\beta}$, where $0 < \beta < k$, then $\mu(d\xi) = c_{k,\beta}|\xi|^{\beta-k}d\xi$ for some constant $c_{k,\beta}$ depending on k and β.

The following is an existence and uniqueness result: if u_0 is measurable and bounded, σ and b are Lipschitz, and μ satisfies Dalang's condition

$$\int_{\mathbb{R}^k} \frac{\mu(d\xi)}{1+|\xi|^2} < \infty,$$

then there exists a unique solution to (2.9) which is L^2-continuous and satisfies

$$\sup_{0\le t\le T}\sup_{x\in\mathbb{R}^k}\mathbb{E}(|u(t,x)|^p)<\infty$$

for any $T<\infty$ and $p\ge 1$. See [11] (cf. [49, 50]).

The solution can be obtained by Picard iteration. Define $u_0(t,x)=u_0(x)$ and

$$\begin{aligned}u_{n+1}(t,x)=(G*u_0)(t,x)&+\int_{[0,t]\times\mathbb{R}^k}G(t-s,x-y)\sigma(u_n(s,y))W(ds\,dy)\\&+\int_{[0,t]\times\mathbb{R}^k}G(t-s,x-y)b(u_n(s,y))\,ds\,dy\end{aligned}$$

for $n\ge 1$. One can verify that $u_n(t,x)$ converges in L^2 using Gronwall's lemma, and show that the limit $u(t,x)$ satisfies the integral equation (2.10).

Here is a regularity result for the solution: u_0 is a bounded, ρ-Hölder continuous function for some $\rho\in(0,1)$, σ and b are Lipschitz, and

$$\int_{\mathbb{R}^k}\frac{\mu(d\xi)}{(1+|\xi|^2)^\eta}<\infty \tag{2.11}$$

for some $\eta\in(0,1)$, then the solution $u(t,x)$ of (2.9) is a.s. β_1-Hölder continuous in t and β_2-Hölder continuous in x, for any $0<\beta_1<\frac{1}{2}(\rho\wedge(1-\eta))$ and $0<\beta_2<\rho\wedge(1-\eta)$. Indeed, for any $T>0$, $p\ge 2$, $0<\beta_1<\frac{1}{2}(\rho\wedge(1-\eta))$ and $0<\beta_2<\rho\wedge(1-\eta)$, there exists C such that

$$\mathbb{E}\big(|u(t,x)-u(s,y)|^p\big)\le C\big(|t-s|^{\beta_1 p}+|x-y|^{\beta_2 p}\big)$$

for all $t,s\in[0,T]$ and $x,y\in\mathbb{R}^k$. In particular, if $f(y)=|y|^{-\beta}$, then (2.11) is satisfied if and only if $0<\beta<2\eta\wedge k$. See [58].

2.2.3 Stochastic Wave Equation

Consider the stochastic wave equation

$$\begin{cases} \dfrac{\partial^2}{\partial^2 t} u(t,x) - \Delta u(t,x) = \dot{W}(t,x), \quad t \geq 0,\, x \in \mathbb{R}^k, \\ u(0,x) = 0, \quad \dfrac{\partial}{\partial t} u(0,x) = 0, \end{cases} \tag{2.12}$$

with an additive Gaussian noise $\dot{W}$.

Suppose that if $k = 1$, $\dot{W}$ is either a space-time white noise or satisfies

$$\mathbb{E}[\dot{W}(s,y)\dot{W}(s',y')] = \delta_0(s-s')|y-y'|^{-\beta}, \quad 0 < \beta < 1;$$

and if $k \geq 2$, $\dot{W}$ satisfies

$$\mathbb{E}[\dot{W}(s,y)\dot{W}(s',y')] = \delta_0(s-s')|y-y'|^{-\beta}, \quad 0 < \beta < 2.$$

Let G be the fundamental solution of the wave equation. Recall that if $k = 1$, $G(t,x) = \frac{1}{2}\mathbf{1}_{\{|x|<t\}}$; if $k \geq 2$ and k is even,

$$G(t,x) = c_k \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(k-2)/2} (t^2 - |x|^2)_+^{-1/2};$$

if $k \geq 3$ and k is odd,

$$G(t,x) = c_k \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(k-3)/2} \frac{\sigma_t^k(dx)}{t},$$

where σ_t^k denotes the uniform surface measure on the sphere $\{x \in \mathbb{R}^k : |x| = t\}$, see [22, Chapter 5].

Since G is a function when $k = 1$ or 2, the solution of (2.12) can be defined by

$$u(t,x) = \int_0^t \int_{\mathbb{R}^k} G(t-s, x-y) W(ds\, dy) \tag{2.13}$$

in the sense of Walsh. For $k \geq 3$, G is not a function but a distribution. It is not straightforward to define the stochastic integral (2.13) in this case.

However, for all dimensions, the Fourier transform of G in variable x is still a function:

$$\mathscr{F}(G(t,\cdot))(\xi) = \frac{\sin(t|\xi|)}{|\xi|}, \quad t \geq 0, \xi \in \mathbb{R}^k. \tag{2.14}$$

Based on this observation, Dalang [11] extended the definition of Walsh's stochastic integration so that the integrand can be taken from certain class of distributions whose Fourier transform in x is a function. As a result, for all dimensions we can obtain real-valued process solutions of equation (5.1):

$$u(t,x) = \int_0^t \int_{\mathbb{R}^k} G(t-s, x-y)\, W(ds\, dy).$$

The range of β has been chosen so that the stochastic integral exists, and the solution is a Gaussian random field.

Dalang and Sanz-Solé [20, Proposition 4.1] proved that for all $0 < a < a' < \infty$ and $0 < b < \infty$, there exist positive finite constants C_1, C_2 such that

$$C_1(|t-s|^{2-\beta} + |x-y|^{2-\beta}) \leq \mathbb{E}(|u(t,x) - u(s,y)|^2) \leq C_2(|t-s|^{2-\beta} + |x-y|^{2-\beta})$$

for all $(t,x),(s,y) \in [a,a'] \times [-b,b]^k$. By the Kolmogorov continuity theorem, the solution $u(t,x)$ is a.s. Hölder continuous in (t,x) of any exponent $< (2-\beta)/2$.

For the nonlinear stochastic wave equation

$$\frac{\partial^2}{\partial t^2} u(t,x) - \Delta u(t,x) = \sigma(u(t,x))\dot{W}(t,x) + b(u(t,x)), \quad t \geq 0,\ x \in \mathbb{R}^k, \tag{2.15}$$

the case $k=1$ was studied by many authors, see e.g. [8, 9, 39, 45, 64]. For $k=2$, the existence and regularity of the solution were studied by Dalang and Frangos [13], and Millet and Morien [43]. For $k=3$, the Hölder-Sobolev regularity of the solution was studied by Dalang and Sanz-Solé [19]. Not much is known for the nonlinear stochastic wave equation in dimension $k \geq 4$.

Chapter 3

Multiple Points of Gaussian Random Fields

3.1 Introduction

In this chapter, we study the non-existence of multiple points of Gaussian random fields. Let $v = \{v(x) : x \in \mathbb{R}^k\}$ be an $\mathbb{R}^d$-valued Gaussian random field. For a set $T \subset \mathbb{R}^k$ and an integer $m \geq 2$, we say that $z \in \mathbb{R}^d$ is an *m-multiple point* of the sample path $v(\cdot, \omega)$ on T if there are m distinct points $x^1, \ldots, x^m \in T$ such that $z = v(x^1, \omega) = \cdots = v(x^m, \omega)$. This chapter is based on [16].

The existence of multiple points of Gaussian random fields have been studied by several authors. Sufficient or necessary conditions for the (N, d)-fractional Brownian motion $B^H = \{B^H(t) : t \in \mathbb{R}^k\}$ with Hurst index H to have multiple points were obtained by Kôno [32], Goldman [24] and Rosen [54]. Their results show that if $km > (m-1)Hd$, then B^H has m-multiple points on any interval $T \subseteq \mathbb{R}^k$; and if $km < (m-1)Hd$, then B^H has no m-multiple points on $\mathbb{R}^k \backslash \{0\}$. The multiple points of the Brownian sheet was also studied by Rosen [54] via self-intersection local times.

For B^H, the critical dimension is $km = (m-1)Hd$. In general, the problem for proving the non-existence of multiple points of a random field in the critical dimensions is more difficult than the non-critical case. The critical case for the fractional Brownian motion and the Brownian sheet has been solved by different methods. The former was solved by Talagrand [61] and the latter was solved by Dalang et al. [15] and Dalang and Mueller [17].

Our study is motivated by the interest in the intersection problems for the solutions of linear systems of stochastic heat and wave equations, where the method in [15, 17] fails in general. Based on the framework in [18], we extend Talagrand's approach in [61] to a large class of Gaussian random fields including fractional Brownian sheets and the solutions of systems of stochastic heat and wave equations with constant coefficients. Moreover, our theorem provides an alternative proof for the results in [15, 17] with the use of general Gaussian principles and the harmonizable representation of the Brownian sheet.

The chapter is organized as follows. In Section 3.2, we state our assumptions and main result (Theorem 3.2.4). In Section 3.3, we establish some necessary lemmas and the main estimate Proposition 3.3.6 for proving the main theorem and, in Section 3.4, we prove the theorem. In Section 3.5, we provide some examples of Gaussian random fields to which the theorem can be applied, including the Brownian sheet, fractional Brownian sheets, and the solutions of systems of stochastic heat and wave equations.

3.2 Assumptions and Main Result

Throughout this chapter, we assume that $v = \{v(x) : x \in \mathbb{R}^k\}$ is a centered, continuous $\mathbb{R}^d$-valued Gaussian random field defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with i.i.d. components. Write $v(x) = (v_1(x), \dots, v_d(x))$ for $x \in \mathbb{R}^k$. We will study the existence problem of multiple points of $v(x)$ on a set $T \subset \mathbb{R}^k$.

By a closed interval in $\mathbb{R}^k$ we mean a set I of the form $\prod_{j=1}^k [c_j, d_j]$, where $c_j < d_j$.

We assume that $T \subset \mathbb{R}^k$ is a fixed index set that can be written as a countable union of compact intervals. To avoid trivial multiple points, we will take, for example, $T = \mathbb{R}^k \backslash \{0\}$ or $T = (0, \infty)^k$.

We consider the following two assumptions, which are slight modification of Assumptions 2.1 and 2.4 in [18].

Assumption 3.2.1. *There exists a centered Gaussian random field* $\{v(A, x), A \in \mathscr{B}(\mathbb{R}_+), x \in T\}$, *where* $\mathscr{B}(\mathbb{R}_+)$ *is the Borel* σ*-algebra on* $\mathbb{R}_+ = [0, \infty)$, *such that the following hold:*

(a) For all $x \in T$, $A \mapsto v(A, x)$ *is an* $\mathbb{R}^d$*-valued white noise (or, more generally, an independently scattered Gaussian noise with a control measure* μ*) with i.i.d. components,* $v(\mathbb{R}_+, x) = v(x)$, *and* $v(A, \cdot)$ *and* $v(B, \cdot)$ *are independent whenever* A *and* B *are disjoint.*

(b) There exist constants $\gamma_j > 0$, $j = 1, \ldots, k$ *with the following properties: For every compact interval* $F \subset T$, *there exist constants* $c_0 > 0$ *and* $a_0 \geq 0$ *such that for all* $a_0 \leq a \leq b \leq \infty$ *and* $x, y \in F$,

$$\|v([a, b), x) - v(x) - v([a, b), y) + v(y)\|_{L^2} \leq c_0 \left(\sum_{j=1}^{k} a^{\gamma_j} |x_j - y_j| + b^{-1} \right), \tag{3.1}$$

and

$$\|v([0, a_0), x) - v([0, a_0), y)\|_{L^2} \leq c_0 \sum_{j=1}^{k} |x_j - y_j|. \tag{3.2}$$

In the above, $\|X\|_{L^2} = \left(\mathbb{E}|X|^2\right)^{1/2}$ *for a random vector* X.

Notice that in Assumption 3.2.1 the constants a_0 and c_0 may depend on F, but γ_j ($j = 1, \ldots, k$) do not. As shown by Dalang et al. [18], the parameters γ_j ($j = 1, \ldots, k$) play important roles in characterizing sample path properties (e.g., regularity, fractal properties, hitting probabilities) of the random field $\{v(x),\, x \in T\}$.

Let $\alpha_j = (\gamma_j + 1)^{-1}$ and $Q = \sum_{j=1}^{k} \alpha_j^{-1}$. Define the metric Δ on $\mathbb{R}^k$ by

$$\Delta(x, y) = \sum_{j=1}^{k} |x_j - y_j|^{\alpha_j}. \tag{3.3}$$

Assumption 3.2.2. *For every compact interval $F \subset T$, there are positive constants ε_0, C and $\delta_j \in (\alpha_j, 1]$, $j = 1, \ldots, k$, such that the following holds:*

For all closed intervals $I \subset F$, $x \in I$ and $0 < \rho \leq \varepsilon_0$, there is $x' \in I^{(\rho)}$ (here and below, $I^{(\rho)}$ denotes the ρ-neighbourhood of I in the Euclidean norm) such that for all $y, \bar{y} \in I^{(\rho)}$ with $\Delta(x, y) \leq 2\rho$ and $\Delta(x, \bar{y}) \leq 2\rho$,

$$\left|\mathbb{E}((v_i(y) - v_i(\bar{y}))v_i(x'))\right| \leq C \sum_{j=1}^{k} |y_j - \bar{y}_j|^{\delta_j}, \quad i = 1, \ldots, d. \tag{3.4}$$

The constants ε_0 and C may depend on F.

In addition, we impose a non-degeneracy assumption.

Assumption 3.2.3. *For any m distinct points $x^1, \ldots, x^m$ in T, the random variables $v_1(x^1), \ldots, v_1(x^m)$ are linearly independent, or equivalently, the Gaussian distribution of $(v_1(x^1), \ldots, v_1(x^m))$ is non-degenerate.*

The main result of this chapter is the following.

Theorem 3.2.4. *Let $m \geq 2$. Suppose that Assumptions 3.2.1, 3.2.2 and 3.2.3 hold. If $mQ \leq (m-1)d$, then $\{v(x), x \in T\}$ has no m-multiple points almost surely.*

3.3 Preliminaries

In this section, we provide some lemmas and a main estimate that will be used for proving Theorem 3.2.4. It suffices to prove that if $mQ \leq (m-1)d$ then, for every compact interval $F \subset T$, $\{v(x), x \in F\}$ has no m-multiple points. Therefore, from now on, we will assume that T is a compact interval.

For $x \in T$ and $r > 0$, denote by $S(x,r) = \{y \in \mathbb{R}^k : \Delta(x,y) \leq r\}$ the closed ball with center x and radius r in the metric Δ in (3.3) and let $B_r(x) = \prod_{j=1}^{k}[x_j - r^{1/\alpha_j},\ x_j + r^{1/\alpha_j}]$. Notice that $S(x,r) \subseteq B_r(x)$ and $B_{r/k}(x) \subseteq S(x,r)$.

Fix $m \geq 2$. Given any m distinct points $t^1, \ldots, t^m \in T$, we can find an integer $n \geq 1$ such that $\Delta(t^i, t^j) \geq 1/n$ for $i \neq j$. For $\rho > 0$, let $B^i_\rho = B_\rho(t^i)$ $(i = 1, \ldots, m)$.

Consider the random set

$$M_{t^1,\ldots,t^m;\rho} = \Big\{ z \in \mathbb{R}^d : \exists\, (x^1, \ldots, x^m) \in \prod_{i=1}^{m} B^i_\rho \qquad \text{such that } z = v(x^1) = \cdots = v(x^m) \Big\}, \tag{3.5}$$

which is the intersection of the images $v(B^i_\rho)$ for $i = 1, \ldots, m$. By the continuity of the process $v(x)$, the set of m-multiple points of $\{v(x) : x \in T\}$ can be written as a countable union

$$\bigcup_{n \geq 1} \ \bigcup_{(t^1,\ldots,t^m) \in A_n} \ \bigcup_{\rho_0 \in (0,1/n) \cap \mathbb{Q}} \ \bigcup_{\rho \in (0,\rho_0) \cap \mathbb{Q}} M_{t^1,\ldots,t^m;\rho} \tag{3.6}$$

where $A_n = \{(t^1, \ldots, t^m) : t^i \in T \cap \mathbb{Q}^k, \Delta(t^i, t^j) \geq 1/n \text{ for } i \neq j\}$.

For the rest of this section, we fix n and $(t^1, \ldots, t^m) \in A_n$. Let $\rho_0 \in (0, 1/n)$ be a small number which may depend on $t^1, \ldots, t^m$ and will be determined in Lemma 3.3.8 below. For simplicity of notation, we assume that $B_{\rho_0}(t^i) \subseteq T$ for $i = 1, \ldots, m$ (otherwise we take the intersection with T), and we omit the subscripts $t^1, \ldots, t^m$ in (3.5) and write M_ρ.

Recall from [18] that, under Assumption 3.2.1, Δ provides an upper bound for the L^2-norm of the increments of $\{v(x), x \in T\}$ and in particular $v(x)$ is continuous in $L^2(\Omega, \mathscr{F}, \mathbb{P})$.

Lemma 3.3.1. *[18, Proposition 2.2] Under Assumption 3.2.1, for all $x, y \in T$ with $\Delta(x, y) \leq \min\{a_0^{-1}, 1\}$, we have $\|v(x) - v(y)\|_{L^2} \leq 4c_0\Delta(x, y)$.*

Assumption 3.2.1 suggests that for any $s \in T$ and x that is close to s, the increment $v(x) - v(s)$ can be approximated well by $v([a, b), x) - v([a, b), s)$ if we choose a and b carefully. The following lemma from [18] quantifies the approximation error on $S(s, cr)$.

Lemma 3.3.2. *Let $c > 0$ be a constant. Consider $b > a > 1$, $\varepsilon_0 > r > 0$ and set*

$$A = \sum_{j=1}^{k} a^{\alpha_j^{-1} - 1} r^{\alpha_j^{-1}} + b^{-1}.$$

There are constants A_0, $\tilde{K}$ and $\tilde{c}$ (depending on c_0 in Assumption 3.2.1 and c) such that if $A \leq A_0 r$ and

$$u \geq \tilde{K} A \log^{1/2}\left(\frac{r}{A}\right), \tag{3.7}$$

then for any $s \in T$,

$$\mathbb{P}\left\{\sup_{x \in S(s,\, cr)} |v(x) - v(s) - (v([a, b), x) - v([a, b), s))| \geq u\right\} \leq \exp\left(-\frac{u^2}{\tilde{c}A^2}\right).$$

Remark 3.3.3. The constant c in Lemma 3.3.2 and Proposition 3.3.6 below is not important. It merely helps to simplify the presentation in Section 3.4, where sometimes we switch back and forth between a ball $S(s, r)$ and an interval $B_r(x)$.

For describing the contribution of the main part $v([a,b],x) - v([a,b],s)$, we will apply the small ball probability estimate given in Lemma 3.3.5 below. We refer to Lemma 2.2 of [60] for a general lower bound on the small ball probability of Gaussian processes. However, it was pointed out by Slobodan Krstic (personal communication) that the condition of that lemma is not correctly stated. Indeed, the lemma fails if we consider S consisting of two points and independent standard normal random variables indexed by the two points. We will make use of the following reformulation of the presentation of Talagrand's lower bound given by Ledoux [34, (7.11)–(7.13) on p. 257].

Lemma 3.3.4. *Let $\{X(t), t \in S\}$ be a separable, vector-valued, centered Gaussian process indexed by a bounded set S with the canonical metric $d_X(s,t) = (\mathbb{E}|X(s) - X(t)|^2)^{1/2}$. Let $N_\varepsilon(S)$ denote the smallest number of d_X-balls of radius ε needed to cover S. If there is a decreasing function $\psi : (0,\delta] \to (0,\infty)$ such that $N_\varepsilon(S) \le \psi(\varepsilon)$ for all $\varepsilon \in (0,\delta]$ and there are constants $c_2 \ge c_1 > 1$ such that*

$$c_1\psi(\varepsilon) \le \psi(\varepsilon/2) \le c_2\psi(\varepsilon) \tag{3.8}$$

for all $\varepsilon \in (0,\delta]$, then there is a constant K depending only on c_1 and c_2 such that for all $u \in (0,\delta)$,

$$\mathbb{P}\bigg(\sup_{s,t\in S} |X(s) - X(t)| \le u\bigg) \ge \exp\big(-K\psi(u)\big). \tag{3.9}$$

Let $\rho \in (0, \rho_0/3)$, recall that $B^1_{2\rho}, \dots, B^m_{2\rho}$ are the rectangles centered at $t^1, \dots, t^m$. By applying Assumption 3.2.1 and Lemma 3.3.4, we derive the following lemma.

Lemma 3.3.5. *Suppose that Assumption 3.2.1 holds and $\rho \in (0, \rho_0/3)$ is a constant. Then there exist constants K and $0 < \eta_0 < \rho_0/3$, depending on c_0 in Assumption 3.2.1, such that* *for all* $(s^1, \ldots, s^m) \in B^1_{2\rho} \times \cdots \times B^m_{2\rho}$, *for all* $0 < a < b$ *and* $0 < u < r < \eta_0$, *we have*

$$\mathbb{P}\left(\sup_{1 \le i \le m} \sup_{x^i \in S(s^i, r)} |v([a,b), x^i) - v([a,b), s^i)| \le u \right) \ge \exp\left(-K \frac{r^Q}{u^Q} \right). \tag{3.10}$$

Proof. As suggested by the proof of (3.3) in Talagrand [61], (3.10) can be derived from Lemma 3.3.4. However, there was a typo in the exponent in (3.3) in [61] (the ratio $\frac{r}{u^{1/\alpha}}$ there should be raised to the power N) and the suggested proof by introducing the auxiliary process Z does not give the correct power for $\frac{r}{u^{1/\alpha}}$ in (3.3) in [61], which is needed for proving Proposition 3.4 in [61]. Hence we give a proof of (3.10).

For $(s^1, \ldots, s^m) \in B^1_{2\rho} \times \cdots \times B^m_{2\rho}$ and $r < \rho_0/3$, define $S = \bigcup_{i=1}^m S(s^i, r)$. Under our assumption, we have $S(s^i, r) \subseteq T$ for $i = 1, \ldots, m$. Thus, $S \subseteq T$. It follows from Assumption 3.2.1 that for all x, $y \in S$,

$$\begin{aligned} \|v([a,b), x) - v([a,b), y)\|^2_{L^2} &= \|v(x) - v(y)\|^2_{L^2} - \|v(\mathbb{R}_+ \setminus [a,b), x) - v(\mathbb{R}_+ \setminus [a,b), y)\|^2_{L^2} \\ &\le \|v(x) - v(y)\|^2_{L^2}. \end{aligned}$$

By Lemma 3.3.1, we have that the canonical metric for $\{v([a,b), x), x \in S\}$ satisfies

$$d_v(s,t) := \|v([a,b), x) - v([a,b), y)\|_{L^2} \le 4c_0 \Delta(x, y)$$

for all x, $y \in S$ with $\Delta(x, y)$ small. Hence there is a constant $\eta_0 \in (0, \rho_0/3)$ such that for all $r \in (0, \eta_0)$ and $\varepsilon \le r$, the minimal number of d_v-balls of radius ε needed to cover S is

$$N_\varepsilon(S) \le \psi(\varepsilon) := C_{N,Q}(r/\varepsilon)^Q.$$

Note that this function $\psi(\varepsilon)$ satisfies (3.8) with the constants $c_1 = c_2 = 2^Q$ which are greater than 1. It follows from Lemma 3.3.4 that there is a constant K such that (3.10) holds. This proves Lemma 3.3.5. □

The following is the main estimate. It is an extension of Proposition 3.4 in [61].

Proposition 3.3.6. *Let $c > 0$ be a constant and suppose that Assumption 3.2.1 holds. Then there are constants K_1 and $0 < \eta_1 < 1$ such that for all $0 < r_0 < \eta_1$, $\rho \in (0, \rho_0/3)$, and $(s^1, \dots, s^m) \in B^1_{2\rho} \times \cdots \times B^m_{2\rho}$, we have*

$$\mathbb{P}\left(\exists\, r \in [r_0^2, r_0],\ \sup_{1 \le i \le m} \sup_{x^i \in S(s^i, cr)} |v(x^i) - v(s^i)| \le K_1 r \left(\log\log \frac{1}{r} \right)^{-1/Q} \right)$$
$$\ge 1 - \exp\left(- \left(\log \frac{1}{r_0} \right)^{1/2} \right).$$

Proof. The method of proof is similar to that of Proposition 3.4 in Talagrand [61]. But the latter contains several typos. For completeness we provide a proof of Proposition 3.3.6 here. The main ingredients are the small ball probability estimate in Lemma 3.3.5 and the estimate of the approximation error in Lemma 3.3.2,

As in [60, 61] and [18], let $U > 1$ be fixed for now and its value will be chosen later. Set $r_\ell = r_0 U^{-2\ell}$ and $a_\ell = U^{2\ell - 1}/r_0$. Consider the largest integer ℓ_0 such that

$$\ell_0 \le \frac{\log(1/r_0)}{2 \log U}. \tag{3.11}$$

Then for $\ell \le \ell_0$, we have $r_\ell \ge r_0^2$. It suffices to show that, for some large constant K_1,

$$\mathbb{P}\left(\exists 1 \le \ell \le \ell_0,\ \sup_{1\le i\le m}\ \sup_{x^i\in S(s^i, cr_\ell)} |v(x^i) - v(s^i)| \le K_1 \frac{r_\ell}{(\log\log\frac{1}{r_\ell})^{1/Q}}\right)$$
$$\ge 1 - \exp\left(-\left(\log\frac{1}{r_0}\right)^{1/2}\right).$$

It follows from Lemma 3.3.5 that, for K_1 large enough so that $K/K_1^Q \le 1/4$,

$$\begin{aligned}
&\mathbb{P}\left(\sup_{1\le i\le m}\ \sup_{x^i\in S(s^i, cr_\ell)} |v([a_\ell, a_{\ell+1}), x^i) - v([a_\ell, a_{\ell+1}), s^i)| \le K_1 \frac{r_\ell}{(\log\log\frac{1}{r_\ell})^{1/Q}}\right)\\
&\ge \exp\left(-\frac{K}{K_1^Q}\log\log\frac{1}{r_\ell}\right)\\
&\ge \left(\log\frac{1}{r_\ell}\right)^{-1/4}.
\end{aligned} \tag{3.12}$$

Thus, by the independence of the Gaussian processes $v([a_\ell, a_{\ell+1}), \cdot)$ ($\ell = 1, \ldots, \ell_0$), we have

$$\begin{aligned}
&\mathbb{P}\left(\exists \ell \le \ell_0,\ \sup_{1\le i\le m}\ \sup_{x^i\in S(s^i, cr_\ell)} |v([a_\ell, a_{\ell+1}), x^i) - v([a_\ell, a_{\ell+1}), s^i)| \le K_1 \frac{r_\ell}{(\log\log\frac{1}{r_\ell})^{1/Q}}\right)\\
&= 1 - \prod_{\ell=1}^{\ell_0}\left\{1 - \mathbb{P}\Bigg(\sup_{1\le i\le m}\ \sup_{x^i\in S(s^i, cr_\ell)} |v([a_\ell, a_{\ell+1}, x^i) - v([a_\ell, a_{\ell+1}), s^i)|\right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad\left.\le K_1 \frac{r_\ell}{(\log\log\frac{1}{r_\ell})^{1/Q}}\Bigg)\right\}.
\end{aligned}$$

By (3.12), we see that the last expression is greater than or equal to

$$1-\prod_{\ell=1}^{\ell_0}\left\{1-\left(\log\frac{1}{r_\ell}\right)^{-1/4}\right\}\geq 1-\left\{1-\left(\log\frac{1}{r_0^2}\right)^{-1/4}\right\}^{\ell_0}$$
$$\geq 1-\exp\left(-\ell_0\left(\log\frac{1}{r_0^2}\right)^{-1/4}\right). \tag{3.13}$$

Set

$$A_\ell=\sum_{j=1}^{k}a_\ell^{\alpha_j^{-1}-1}r_\ell^{\alpha_j^{-1}}+a_{\ell+1}^{-1}.$$

Notice that $r_\ell a_\ell = U^{-1}$ and $r_\ell a_{\ell+1}=U$. Then

$$A_\ell r_\ell^{-1}=\sum_{j=1}^{k}(a_\ell r_\ell)^{\alpha_j^{-1}-1}+(a_{\ell+1}r_\ell)^{-1}=\sum_{j=1}^{k}U^{-(\alpha_j^{-1}-1)}+U^{-1}\leq(k+1)U^{-\beta}, \tag{3.14}$$

with $\beta=\min\{1,\min_{j=1,\ldots,k}(\alpha_j^{-1}-1)\}>0$ since $\alpha_j<1$ for $j=1,\ldots,k$. Therefore, for U large enough, $A_\ell\leq A_0 r_\ell$, and for $u\geq\tilde{K}r_\ell U^{-\beta}\sqrt{\log U}$, (3.7) is satisfied. Hence, by Lemma 3.3.2 and (3.14),

$$\begin{aligned}
&\mathbb{P}\left(\sup_{1\leq i\leq m}\ \sup_{x^i\in S(s^i,cr_\ell)}\left|v(x^i)-v(s^i)-v([a_\ell,a_{\ell+1},x^i)+v([a_\ell,a_{\ell+1},s^i)\right|\geq u\right)\\
&\qquad\leq\exp\left(-\frac{u^2}{\tilde{c}A_\ell^2}\right)\\
&\qquad\leq\exp\left(-\frac{u^2}{\tilde{c}(k+1)^2r_\ell^2}U^{2\beta}\right).
\end{aligned}$$

Now we take $u=K_1r_\ell(\log\log\frac{1}{r_0})^{-1/Q}$, which is allowed provided

$$K_1r_\ell\left(\log\log\frac{1}{r_0}\right)^{-1/Q}\geq\tilde{K}r_\ell U^{-\beta}\sqrt{\log U}.$$

This is equivalent to

$$U^{\beta}(\log U)^{-1/2} \geq \frac{\tilde{K}}{K_1}\left(\log\log\frac{1}{r_0}\right)^{1/Q}, \tag{3.15}$$

which holds if U is large enough. It follows from the above that

$$\begin{aligned}
&\mathbb{P}\Bigg(\sup_{1\leq i\leq m}\sup_{x^i\in S(s^i,cr_\ell)}\left|v(x^i)-v(s^i)-v([a_\ell,a_{\ell+1}),x^i)+v([a_\ell,a_{\ell+1}),s^i)\right| \geq \frac{K_1 r_\ell}{(\log\log\frac{1}{r_0})^{1/Q}}\Bigg) \\
&\quad\leq \exp\left(-\frac{U^{2\beta}}{\tilde{c}(k+1)^2(\log\log\frac{1}{r_0})^{2/Q}}\right).
\end{aligned} \tag{3.16}$$

Let

$$\begin{aligned}
F_\ell &= \left\{\sup_{1\leq i\leq m}\sup_{x^i\in S(s^i,cr_\ell)}|v([a_\ell,a_{\ell+1}),x^i)-v([a_\ell,a_{\ell+1},s^i)| \leq \frac{K_1}{2}\frac{r_\ell}{(\log\log\frac{1}{r_\ell})^{1/Q}}\right\}, \\
G_\ell &= \Bigg\{\sup_{1\leq i\leq m}\sup_{x^i\in S(s^i,cr_\ell)}|v(x^i)-v(s^i)-v([a_\ell,a_{\ell+1},x^i)+v([a_\ell,a_{\ell+1},s^i)| \\
&\qquad\qquad \geq \frac{K_1}{2}\frac{r_\ell}{(\log\log\frac{1}{r_\ell})^{1/Q}}\Bigg\}.
\end{aligned}$$

Then

$$\begin{aligned}
&\mathbb{P}\left(\exists 1\leq \ell\leq \ell_0,\ \sup_{1\leq i\leq m}\sup_{x^i\in S(s^i,cr_\ell)}|v(x^i)-v(s^i)| \leq K_1\frac{r_\ell}{(\log\log\frac{1}{r_\ell})^{1/Q}}\right) \\
&\quad\geq \mathbb{P}\left(\bigcup_{\ell=1}^{\ell_0}(F_\ell\cap G_\ell^c)\right) \\
&\quad\geq \mathbb{P}\left(\left(\bigcup_{\ell=1}^{\ell_0}F_\ell\right)\cap\left(\bigcup_{\ell=1}^{\ell_0}G_\ell\right)^c\right) \\
&\quad\geq \mathbb{P}\left(\bigcup_{\ell=1}^{\ell_0}F_\ell\right)-\mathbb{P}\left(\bigcup_{\ell=1}^{\ell_0}G_\ell\right).
\end{aligned} \tag{3.17}$$

By (3.13), we have

$$\mathbb{P}\left(\bigcup_{\ell=1}^{\ell_0} F_\ell\right) \geq 1 - \exp\left(-\ell_0 \left(\log \frac{1}{r_0^2}\right)^{-1/4}\right),$$

and by (3.16),

$$\mathbb{P}\left(\bigcup_{\ell=1}^{\ell_0} G_\ell\right) \leq \ell_0 \exp\left(-\frac{U^{2\beta}}{\tilde{c}(k+1)^2(\log\log \frac{1}{r_0})^{2/Q}}\right).$$

Combining this with (3.17), we get

$$\begin{aligned}
&\mathbb{P}\left(\exists 1 \leq \ell \leq \ell_0,\ \sup_{1\leq i\leq m} \sup_{x^i\in S(s^i, cr_\ell)} |v(x^i) - v(s^i)| \leq K_1 \frac{r_\ell}{(\log\log \frac{1}{r_\ell})^{1/Q}}\right) \\
&\geq 1 - \exp\left(-\ell_0 \left(\log \frac{1}{r_0^2}\right)^{-1/4}\right) - \ell_0 \exp\left(-\frac{U^{2\beta}}{\tilde{c}(k+1)^2(\log\log \frac{1}{r_0})^{2/Q}}\right).
\end{aligned}$$

Therefore, the proof will be completed provided

$$\begin{aligned}
&\exp\left(-\ell_0 \left(\log \frac{1}{r_0^2}\right)^{-1/4}\right) + \ell_0 \exp\left(-\frac{U^{2\beta}}{\tilde{c}(k+1)^2(\log\log \frac{1}{r_0})^{2/Q}}\right) \\
&\quad \leq \exp\left(-\left(\log \frac{1}{r_0}\right)^{1/2}\right).
\end{aligned} \tag{3.18}$$

Recall the condition (3.15), and the definition of ℓ_0 in (3.11). If we set

$$U = \left(\log \frac{1}{r_0}\right)^{1/(2\beta)},$$

then for r_0 small enough, by (3.11),

$$\ell_0 > \frac{\beta}{2}\left(\log \frac{1}{r_0}\right)\left(\log\log \frac{1}{r_0}\right)^{-1} > 1.$$

Therefore, the left-hand side of (3.18) is bounded above by

$$\exp\left(-\frac{(\log\frac{1}{r_0})^{3/4}}{\tilde{c}(k+1)^2\log\log\frac{1}{r_0}}\right)+\left(1+\log\frac{1}{r_0}\right)\exp\left(-\frac{\log\frac{1}{r_0}}{\tilde{c}(k+1)^2(\log\log\frac{1}{r_0})^{2/Q}}\right)$$
$$\leq \exp\left(-\left(\log\frac{1}{r_0}\right)^{1/2}\right)$$

provided r_0 is small enough. This completes the proof of Proposition 3.3.6. $\square$

For each small $\rho > 0$, by Assumption 3.2.2, there are $(\hat{t}^1, \ldots, \hat{t}^m) \in B^1_{3\rho} \times \cdots \times B^m_{3\rho}$ such that for all $i = 1, \ldots, m$ and all $x, y \in B^i_{2\rho}$,

$$\left|\mathbb{E}\big((v(x) - v(y)) \cdot v(\hat{t}^i)\big)\right| \leq C \sum_{j=1}^{k} |x_j - y_j|^{\delta_j}. \tag{3.19}$$

The points $\hat{t}^1, \ldots, \hat{t}^m$ are fixed.

Let Σ_2 denote the σ-algebra generated by $v(\hat{t}^1), \ldots, v(\hat{t}^m)$. Define

$$v^2(x) = \mathbb{E}\big(v(x)|\Sigma_2\big), \quad v^1(x) = v(x) - v^2(x). \tag{3.20}$$

The Gaussian random fields $v^1 = \{v^1(x), x \in T\}$ and $v^2 = \{v^2(x), x \in T\}$ are independent.

Lemma 3.3.7. *There is a constant K_2 depending on $\hat{t}^1, \ldots, \hat{t}^m$ and the constant C in Assumption 3.2.2 such that for all $i = 1, \ldots, m$ and all $x, y \in B^i_{2\rho}$,*

$$\left|v^2(x) - v^2(y)\right| \leq K_2 \sum_{j=1}^{k} |x_j - y_j|^{\delta_j} \max_{1 \leq i \leq m} \left|v(\hat{t}^i)\right|.$$

Proof. By Assumption 3.2.3, the subspace in $L^2(\Omega; \mathbb{R}^d)$ of random vectors $\Omega \to \mathbb{R}^d$ spanned by $v(\hat{t}^1), \ldots, v(\hat{t}^m)$, has dimension $m \geq 2$. Let $\{\sum_{i=1}^m a_{i,j} v(\hat{t}^i) : j = 1, \ldots, m\}$ be an orthonormal basis of this subspace, where $a_{i,j}$ are constants that depend on $\hat{t}^1, \ldots, \hat{t}^m$. Then

$$v^2(x) = \sum_{j=1}^{m} \mathbb{E}\Big[\sum_{i=1}^{m} a_{i,j} v(\hat{t}^i) \cdot v(x)\Big]\Big(\sum_{\ell=1}^{m} a_{\ell,j} v(\hat{t}^\ell)\Big).$$

By (3.19), we have

$$\begin{aligned}\big|v^2(x) - v^2(y)\big| &= \left|\sum_{\ell=1}^{m}\left(\sum_{i=1}^{m}\sum_{j=1}^{m} a_{i,j} a_{\ell,j} \mathbb{E}\left[(v(x) - v(y)) \cdot v(\hat{t}^i)\right]\right) v(\hat{t}^\ell)\right| \\ &\le K \sum_{j=1}^{k} |x_j - y_j|^{\delta_j} \max_{1 \le \ell \le m} \big|v(\hat{t}^\ell)\big|.\end{aligned}$$

This completes the proof. □

Lemma 3.3.8. *Suppose Assumptions 3.2.1, 3.2.2 and 3.2.3 are satisfied. Then there exist constants K and $\rho_0 > 0$ depending on $t^1, \ldots, t^m$ such that for all $\rho \in (0, \rho_0)$, $a_2, \ldots, a_m \in \mathbb{R}^d$, $r > 0$, and all $(x^1, \ldots, x^m) \in B_\rho^1 \times \cdots \times B_\rho^m$,*

$$\mathbb{P}\left(\sup_{2 \le i \le m} |v^2(x^1) - v^2(x^i) - a_i| \le r\right) \le K r^{(m-1)d}.$$

Proof. We first assume $d = 1$. We claim that if ρ_0 is small then $v^2(x^1), \ldots, v^2(x^m)$ are linearly independent for all $\rho \in (0, \rho_0)$ and $(x^1, \ldots, x^m) \in B_\rho^1 \times \cdots \times B_\rho^m$. Indeed, by Assumption 3.2.3, we can find $K > 0$ such that $\mathrm{Var}(\sum_{i=1}^m b_i v(t^i)) \ge K|b|^2$ for all $b \in \mathbb{R}^m$.

By the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
\left[\mathbb{E}\Big(\sum_{i=1}^m b_i(v(t^i)-v^2(x^i))\Big)^2\right]^{1/2} &\le |b|\left[\mathbb{E}\left(\sum_{i=1}^m \Big(v(t^i)-v^2(x^i)\Big)^2\right)\right]^{1/2}\\
&\le |b|\sum_{i=1}^m\left(\left[\mathbb{E}\left(v(t^i)-v(\hat{t}^i)\right)^2\right]^{1/2}+\left[\mathbb{E}\left(\mathbb{E}(v(\hat{t}^i)-v(x^i)|\Sigma_2)\right)^2\right]^{1/2}\right)\\
&\le |b|\sum_{i=1}^m\Big(\|v(t^i)-v(\hat{t}^i)\|_{L^2}+\|v(\hat{t}^i)-v(x^i)\|_{L^2}\Big).
\end{aligned}$$

It follows that

$$\begin{aligned}
\left[\mathbb{E}\Big(\sum_{i=1}^m b_i v^2(x^i)\Big)\right]^{1/2} &\ge \left[\mathbb{E}\Big(\sum_{i=1}^m b_i v(t^i)\Big)^2\right]^{1/2}-\left[\mathbb{E}\Big(\sum_{i=1}^m b_i(v(t^i)-v^2(x^i))\Big)^2\right]^{1/2}\\
&\ge \left(K^{1/2}-\sum_{i=1}^m\Big(\|v(t^i)-v(\hat{t}^i)\|_{L^2}+\|v(\hat{t}^i)-v(x^i)\|_{L^2}\Big)\right)|b|.
\end{aligned}$$

Notice that, Assumption 3.2.1 implies the $L^2(\mathbb{P})$-continuity of $v(x)$ [cf. Lemma 3.3.1], we can find a small constant $\rho_0>0$ depending on $t^1,\dots,t^m$ so that the above is $\ge C|b|$ for all $\rho\in(0,\rho_0)$ and $(x^1,\dots,x^m)\in B^1_\rho\times\cdots\times B^m_\rho$, where $C>0$. It follows that $v^2(x^1),\dots,v^2(x^m)$ are linearly independent, and so are $v^2(x^1)-v^2(x^2), v^2(x^1)-v^2(x^3),\dots,v^2(x^1)-v^2(x^m)$.

Denote the determinant of the covariance matrix of the last random vector by

$$\det\operatorname{Cov}(v^2(y^1)-v^2(y^2), v^2(y^1)-v^2(y^3),\dots,v^2(y^1)-v^2(y^m)).$$

Then the map $(y^1,\dots,y^m)\mapsto\det\operatorname{Cov}(v^2(y^1)-v^2(y^2), v^2(y^1)-v^2(y^3),\dots,v^2(y^1)-v^2(y^m))$ is continuous and positive on the compact set $B^1_{\rho_0}\times\cdots\times B^m_{\rho_0}$, so it is bounded from below by a positive constant depending on $t^1, \ldots, t^m$. This and Anderson's theorem [3] imply that

$$\mathbb{P}\left(\sup_{2\le i\le m} |v^2(x^1) - v^2(x^i) - a_i| \le r\right) \le \mathbb{P}\left(\sup_{2\le i\le m} |v^2(x^1) - v^2(x^i)| \le r\right) \le K r^{m-1}.$$

Since $v(x)$ has i.i.d. components, the case $d > 1$ follows readily. □

We end this section with the following lemma which is obtained by applying Theorem 2.1 and Remark 2.2 of [28] to the metric space (T, Δ). It provides nested families of "cubes" sharing most of the good properties of dyadic cubes in the Euclidean spaces. For this reason, we call the sets in $\mathscr{Q}_q$ generalized dyadic cubes of order q. Their nested property will help us to construct an efficient covering for M_ρ.

Lemma 3.3.9. *There exist constants c_1, c_2, and a family $\mathscr{Q}$ of Borel subsets of T, where $\mathscr{Q} = \bigcup_{q=1}^{\infty} \mathscr{Q}_q$, $\mathscr{Q}_q = \{I_{q,\ell} : \ell = 1, \ldots, n_q\}$, such that the following hold.*

(i) $T = \bigcup_{\ell=1}^{n_q} I_{q,\ell}$ *for each* $q \ge 1$.

(ii) Either $I_{q,\ell} \cap I_{q',\ell'} = \varnothing$ *or* $I_{q,\ell} \subset I_{q',\ell'}$ *whenever* $q \ge q'$, $1 \le \ell \le n_q$, $1 \le \ell' \le n_{q'}$.

(iii) For each q, ℓ, *there exists* $x_{q,\ell} \in T$ *such that* $S(x_{q,\ell}, c_1 2^{-q}) \subset I_{q,\ell} \subset S(x_{q,\ell}, c_2 2^{-q})$ *and* $\{x_{q,\ell} : 1, \ldots, n_q\} \subset \{x_{q+1,\ell} : \ell = 1, \ldots, n_{q+1}\}$ *for all* $q \ge 1$.

3.4 Proof of Theorem 3.2.4

Recall that, by (3.6), it suffices to show that for all integers n and all points $t^1, \ldots, t^m \in T$ such that $\Delta(t^i, t^j) \ge 1/n$ for $i \ne j$, we can find a small $\rho_0 > 0$ depending on $t^1, \ldots, t^m$ so that for all $\rho \in (0, \rho_0)$, M_ρ is empty with probability 1. When $mQ < (m-1)d$ (we refer this as the sub-critical case), the last statement can be proved easily by using a standard covering argument based on the uniform modulus of continuity of $v = \{v(x), x \in T\}$ on compact intervals. In the following we provide a unified proof for both the critical and subcritical cases.

Now let $t^1, \dots, t^m \in T$ be m distinct points such that $\Delta(t^i, t^j) \geq 1/n$ for $i \neq j$ and some integer $n \geq 1$. They are fixed in the rest of the proof. We choose a constant $\rho_0 > 0$ such that both Lemma 3.3.8 and Assumption 3.2.2 hold for all $\rho \leq \rho_0$ (e.g., we take $\rho_0 \leq \varepsilon_0$). Hence we can find $(\hat{t}^1, \dots, \hat{t}^m) \in B^1_{3\rho} \times \cdots \times B^m_{3\rho}$ such that (3.19) holds. Furthermore, we assume that there is a compact interval $F \subset T$ such that the $B^j_{3\rho_0} \subset F$ for all $1 \leq j \leq m$.

Fix $\rho \in (0, \rho_0)$. For each integer $p \geq 1$, consider the random set

$$R_p = \Bigg\{(s^1, \dots, s^m) \in B^1_{2\rho} \times \cdots \times B^m_{2\rho} : \exists\, r \in [2^{-2p}, 2^{-p}] \text{ such that}$$
$$\sup_{1 \leq i \leq m} \sup_{x^i \in S(s^i, 4c_2 r)} |v(x^i) - v(s^i)| \leq K_1 r \left(\log\log\frac{1}{r}\right)^{-1/Q}\Bigg\},$$

where c_2 is the constant given by Lemma 3.3.9. Let $\beta = \min\{\beta^*, 1\}/2$, where $\beta^* = \min\{\delta_j/\alpha_j - 1 : j = 1, \dots, k\}$. Let λ denote the Lebesgue measure on $\mathbb{R}^{mk}$. Consider the events

$$\Omega_{p,1} = \Big\{\lambda(R_p) \geq \lambda(B^1_{2\rho} \times \cdots \times B^m_{2\rho})(1 - \exp(-\sqrt{p}/4))\Big\},$$
$$\Omega_{p,2} = \Big\{\max_{1 \leq i \leq m} |v(\hat{t}^i)| \leq 2^{\beta p}\Big\}.$$

By applying Proposition 3.3.6 with $c = 4c_2$ and Fubini's theorem, we derive that for p sufficiently large,

$$\mathbb{P}\Big((s^1, \dots, s^m) \in R_p\Big) \geq 1 - \exp(-\sqrt{p}/2)$$

for all $(s^1, \ldots, s^m) \in B^1_{2\rho} \times \cdots \times B^m_{2\rho}$. Then by Fubini's theorem, $\sum_{p=1}^{\infty} \mathbb{P}(\Omega^c_{p,1}) < \infty$. Moreover, it is clear that $\sum_{p=1}^{\infty} \mathbb{P}(\Omega^c_{p,2}) < \infty$.

Denote by $\mathscr{Q} = \bigcup_{p=1}^{\infty} \mathscr{Q}_p$ the family of generalized dyadic cubes given by Lemma 3.3.9 that intersect the compact interval F. Consider the event

$$\Omega_{p,3} = \left\{ \forall\, I \in \mathscr{Q}_{2p},\ \sup_{x,y \in I} |v(x) - v(y)| \le K_3 2^{-2p} p^{1/2} \right\}.$$

For every $I \in \mathscr{Q}_{2p}$, Lemma 3.3.1 implies that the diameter of I under the canonical metric $d_v(x,y) = \|v(x) - v(y)\|_{L^2}$ is at most $c_3\, 2^{-2p}$. By applying Lemma 2.1 in Talagrand [60] (see also Lemma 3.1 in [18]) we see that for any positive constant K_3 and p large,

$$\mathbb{P}\left(\sup_{x,y \in I} |v(x) - v(y)| \ge K_3 2^{-2p} p^{1/2} \right) \le \exp\left(- \left(\frac{K_3}{c_3} \right)^2 p \right).$$

Notice that the cardinality of the family $\mathscr{Q}_{2p}$ of generalized dyadic cubes of order $2p$ is at most $K 2^{2pQ}$. We can verify directly that $\sum_{p=1}^{\infty} \mathbb{P}(\Omega^c_{p,3}) < \infty$ provided K_3 is chosen to satisfy $K_3 > 2c_3 Q \ln 2$.

Let $\Omega_p = \Omega_{p,1} \cap \Omega_{p,2} \cap \Omega_{p,3}$ and

$$\Omega^* = \bigcup_{\ell \ge 1} \bigcap_{p \ge \ell} \Omega_p.$$

It follows that the event Ω^* occurs with probability 1. We will show that, for every $\omega \in \Omega^*$, we can construct families of balls in $\mathbb{R}^d$ that cover M_ρ.

For each $p \ge 1$, we first construct a family $\mathscr{G}_p$ of subsets in $\mathbb{R}^{mk}$ (depending on ω). Denote by $\mathscr{C}_p$ the family of subsets of T^m of the form $C = I_{q,\ell_1} \times \cdots \times I_{q,\ell_m}$ for some integer $q \in [p, 2p]$, where $I_{q,\ell_i} \in \mathscr{Q}_q$ are the generalized dyadic cubes of order q in Lemma

3.3.9.

We say that a dyadic cube $C = I^1 \times \cdots \times I^m$ of order q is *good* if it has the property that

$$\sup_{1\leq i\leq m} \sup_{x,y\in I^i} |v^1(x) - v^1(y)| \leq d_q, \tag{3.21}$$

where

$$d_q = 2(K_1 + K_2 \sum_{j=1}^{k} (2c_2)^{\delta_j/\alpha_j}) 2^{-q} (\log\log 2^q)^{-1/Q}. \tag{3.22}$$

For each $x \in B^1_{2\rho} \times \cdots \times B^m_{2\rho}$, consider the good dyadic cube C containing x (if any) of smallest order q, where $p \leq q \leq 2p$. By property (ii) of Lemma 3.3.9, we obtain in this way a family of disjoint good dyadic cubes of order $q \in [p, 2p]$ that meet the set $B^1_{2\rho} \times \cdots \times B^m_{2\rho}$. We denote this family by $\mathscr{G}^1_p$.

Let $\mathscr{G}^2_p$ be the family of dyadic cubes in T^m of order $2p$ that meet $B^1_\rho \times \cdots \times B^m_\rho$ but are not contained in any cube of $\mathscr{G}^1_p$. Let $\mathscr{G}_p = \mathscr{G}^1_p \cup \mathscr{G}^2_p$. Notice that for each $C \in \mathscr{C}_p$, the events $\{C \in \mathscr{G}^1_p\}$ and $\{C \in \mathscr{G}^2_p\}$ are in the σ-algebra $\Sigma_1 := \sigma(v^1(x) : x \in T)$.

Now we construct a family $\mathscr{F}_p$ of balls in $\mathbb{R}^d$ (depending on ω) as follows. For each $C \in \mathscr{C}_p$, we choose a distinguished (non-random) point $x_C = (x^1_C, \ldots, x^m_C)$ in $C \cap (B^1_{2\rho} \times \cdots \times B^m_{2\rho})$. If C is a cube of order q, then we define the ball $B_{p,C}$ as follows.

(i) If $C \in \mathscr{G}^1_p$, take $B_{p,C}$ as the Euclidean ball of center $v(x^1_C)$ of radius $r_{p,C} = 4d_q$. Recall that d_q is defined in (3.22).

(ii) If $C \in \mathscr{G}^2_p$, take $B_{p,C}$ as the Euclidean ball of center $v(x^1_C)$ of radius $r_{p,C} = 2K_3 2^{-2p} p^{1/2}$.

(iii) Otherwise, take $B_{p,C} = \varnothing$ and $r_{p,C} = 0$.

Note that for each $p \geq 1, C \in \mathscr{C}_p$, the random variable $r_{p,C}$ is Σ_1-measurable. Consider the event

$$\Omega_{p,C} = \left\{ \omega \in \Omega : \sup_{2 \le i \le m} \left| v(x_C^1, \omega) - v(x_C^i, \omega) \right| \le r_{p,C}(\omega) \right\}.$$

If $\omega \in \Omega^* \cap \Omega_{p,C}$, define $\mathscr{F}_p(\omega) = \{B_{p,C} : C \in \mathscr{G}_p(\omega)\}$. Otherwise, define $\mathscr{F}_p(\omega) = \varnothing$.

Choose an integer p_0 such that $2c_2 2^{-p} \le \rho$ and

$$p^{mQ/2} (\log p)^m \exp(-\sqrt{p}/4) \le \rho^{mQ} \tag{3.23}$$

for all $p \ge p_0$. We now show that $\mathscr{F}_p(\omega)$ covers $M_\rho(\omega)$ whenever $p \ge p_0$ and $\omega \in \Omega_p$.

Let $\omega \in \Omega_p$ and $z \in M_\rho(\omega)$. By definition, we can find a point $(y^1, \ldots, y^m) \in B_\rho^1 \times \cdots \times B_\rho^m$ such that $z = v(y^1, \omega) = \cdots = v(y^m, \omega)$. By the definitions of $\mathscr{G}_p^1$ and $\mathscr{G}_p^2$, the family $\mathscr{G}_p(\omega)$ of dyadic cubes covers $B_\rho^1 \times \cdots \times B_\rho^m$, thus the point $(y^1, \ldots, y^m)$ is contained in some $C = I^1 \times \cdots \times I^m \in \mathscr{G}_p(\omega)$. We will show that $z \in B_{p,C}$ and $\omega \in \Omega_{p,C}$. To this end, we distinguish two cases.

Case 1. If $C \in \mathscr{G}_p^1(\omega)$, then it is a good dyadic cube of order $q \in [p, 2p]$ such that

$$\sup_{1 \le i \le m} |v^1(x_C^i, \omega) - v^1(y^i, \omega)| \le d_q.$$

By Lemma 3.3.9, $x_C^i, y^i \in I^i \subset S(x^*, c_2 2^{-q})$ for some $x^* \in T$, so we have

$$\sum_{j=1}^k |x_{C,j}^i - y_j^i|^{\delta_j} \le \sum_{j=1}^k (2c_2)^{\delta_j/\alpha_j} 2^{-q(1+\beta^*)}, \tag{3.24}$$

recall that $\beta^* = \min_{1\le j\le k} \left\{\frac{\delta_j}{\alpha_j} - 1\right\}$. Since $\omega \in \Omega_{p,2}$, Lemma 3.3.7 and (3.24) imply that

$$\sup_{1\le i\le m} \left|v^2(x_C^i) - v^2(y^i)\right| \le K_2 \sum_{j=1}^{k} (2c_2)^{\delta_j/\alpha_j} 2^{-q(1+\beta^*-\beta)} \le d_q. \tag{3.25}$$

It follows that

$$\sup_{1\le i\le m} \left|v(x_C^i, \omega) - z\right| = \sup_{1\le i\le m} \left|v(x_C^i, \omega) - v(y^i, \omega)\right| \le 2d_q,$$

which implies that $z \in B_{p,C}$ and $\omega \in \Omega_{p,C}$.

Case 2. Now we assume $C \in \mathscr{G}_p^2(\omega)$. Since $\omega \in \Omega_{p,3}$, we have

$$\sup_i |v(x_C^i, \omega) - z| = \sup_i |v(x_C^i, \omega) - v(y^i, \omega)| \le K_3 2^{-2p} p^{1/2},$$

hence $z \in B_{p,C}$ and $\omega \in \Omega_{p,C}$.

Therefore, for every $\omega \in \Omega^*$, $\mathscr{F}_p(\omega)$ covers $M_\rho(\omega)$ when p is large enough. We claim that, with probability 1, the family $\mathscr{F}_p$ is empty for infinitely many p. This will imply that M_ρ is empty with probability 1 and the proof will then be complete.

We prove the aforementioned claim by contradiction. Suppose the claim is not true. Then the event Ω' that $\mathscr{F}_p$ is nonempty for all large p has positive probability and the event $\Omega' \cap \Omega^* = \bigcup_{\ell\ge 1} \bigcap_{p\ge \ell} (\Omega' \cap \Omega_p)$ also has positive probability. Denote

$$\phi(r) = r^{mQ-(m-1)d} (\log\log(1/r))^m, \qquad f(r) = r^{mQ} (\log\log(1/r))^m,$$

and consider the random variables X_p defined by

$$X_p := \mathbf{1}_{\Omega' \cap \Omega_p} \sum_{B_{p,C} \in \mathscr{F}_p} \phi(r_{p,C}) = \mathbf{1}_{\Omega' \cap \Omega_p} \sum_{C \in \mathscr{C}_p} f(r_{p,C}) r_{p,C}^{-(m-1)d} \mathbf{1}_{\{C \in \mathscr{G}_p\}} \mathbf{1}_{\Omega_{p,C}}. \tag{3.26}$$

Let $X := \liminf_p X_p$. Since $mQ \le (m-1)d$, we have $\phi(r) \to \infty$ as $r \to 0+$. Moreover, for every $\omega \in \Omega' \cap \Omega^*$, $\mathscr{F}_p(\omega)$ is not empty for all large p. This and the definition of X_p in (3.26) imply that $X(\omega) = \infty$ on $\Omega' \cap \Omega^*$. In particular, $\mathbb{E}(X) = \infty$.

On the other hand, notice that $\mathscr{G}_p^1$ covers R_p on the event Ω_p for all $p \ge p_0$. Indeed, if $\omega \in \Omega_p$, $s = (s^1, \ldots, s^m) \in R_p(\omega)$, and $C = I^1 \times \cdots \times I^m$ is the dyadic cube of order q in $\mathscr{G}_p^1$ containing s, then there exists $r \in [2^{-2p}, 2^{-p}]$ that satisfies the condition in the definition of R_p and we can find q such that $2^{-q-1} < r \le 2^{-q}$, $p \le q \le 2p$, and

$$\sup_{1 \le i \le m} \sup_{x^i \in S(s^i, 2c_2 2^{-q})} |v(x^i) - v(s^i)| \le K_1 2^{-q} (\log\log 2^q)^{-1/Q}. \tag{3.27}$$

By the property that $I^i \subset S(x', c_2 2^{-q})$ for some x' and by Lemma 3.3.7, it follows from (3.25) and (3.27) that (3.21) holds. Thus C is a good dyadic cube. This proves that $\mathscr{G}_p^1(\omega)$ covers $R_p(\omega)$.

By the choice of p_0, the cubes in $\mathscr{G}_p^2$ are contained in $B_{2\rho}^1 \times \cdots \times B_{2\rho}^m$, thus in $B_{2\rho}^1 \times \cdots \times B_{2\rho}^m \setminus R_p$, whose Lebesgue measure is at most $\exp(-\sqrt{p}/4)$ on Ω_p. For any $C = I^1 \times \cdots \times I^m \in \mathscr{G}_p^2$ of order $2p$, each I^i contains a set $S(x^i, c_1 2^{-2p})$ for some x^i and the set has Lebesgue measure $K 2^{-2pQ}$, so Ω_p is contained in the event $\widetilde{\Omega}_p$ that the cardinality of $\mathscr{G}_p^2$ is at most $K 2^{2pmQ} \exp(-\sqrt{p}/4)$.

Recall that both $\mathscr{G}_p^1$ and $\mathscr{G}_p^2$ depend on Σ_1. We see that $\widetilde{\Omega}_p$ belongs to the σ-algebra Σ_1.

Hence for $p \geq p_0$,

$$\begin{aligned}\mathbb{E}(X_p) &\leq \mathbb{E}\bigg(\mathbf{1}_{\widetilde{\Omega}_p} \sum_{C \in \mathscr{C}_p} f(r_{p,C}) r_{p,C}^{-(m-1)d} \mathbf{1}_{\{C \in \mathscr{G}_p\}} \mathbf{1}_{\Omega_{p,C}}\bigg) \\ &= \mathbb{E}\bigg(\mathbf{1}_{\widetilde{\Omega}_p} \sum_{C \in \mathscr{C}_p} f(r_{p,C}) r_{p,C}^{-(m-1)d} \mathbf{1}_{\{C \in \mathscr{G}_p\}} \mathbb{P}(\Omega_{p,C} | \Sigma_1)\bigg) \\ &\leq K\mathbb{E}\bigg(\mathbf{1}_{\widetilde{\Omega}_p} \sum_{C \in \mathscr{C}_p} f(r_{p,C}) \mathbf{1}_{\{C \in \mathscr{G}_p\}}\bigg),\end{aligned} \tag{3.28}$$

where the last inequality follows from Lemma 3.3.8 and independence of v^1 and v^2.

Now consider any dyadic cube $C \in \mathscr{C}_p$ of order q. If $C \in \mathscr{G}_p^1$, then $f(r_{p,C}) \leq K2^{-qmQ} \leq K\lambda(C)$ (where $\lambda(\cdot)$ denotes Lebesgue measure); if $C \in \mathscr{G}_p^2$, then $f(r_{p,C}) \leq K2^{-2pmQ} p^{mQ/2} (\log p)^m$. Moreover, for $p \geq p_0$ the dyadic cubes in $\mathscr{G}_p^1$ are disjoint and contained in $B_{3\rho}^1 \times \cdots \times B_{3\rho}^m$. These observations, together with (3.28) and (3.23), imply that for all $p \geq p_0$,

$$\mathbb{E}(X_p) \leq K\mathbb{E}\bigg(\sum_{C \in \mathscr{C}_p} \lambda(C) \mathbf{1}_{\{C \in \mathscr{G}_p^1\}} + p^{mQ/2} (\log p)^m \exp(-\sqrt{p}/4)\bigg) \leq K\rho^{mQ}.$$

By Fatou's lemma, we derive $\mathbb{E}(X) \leq K\rho^{mQ} < \infty$. This is a contradiction. The proof of Theorem 3.2.4 is complete. □

3.5 Examples

In this section we provide some examples where Theorem 3.2.4 is applicable. These include fractional Brownian sheets, and the solutions to systems of stochastic heat and wave equations.

3.5.1 Fractional Brownian Sheets

The (N, d)-fractional Brownian sheet with Hurst parameter $H = (H_1, \ldots, H_N) \in (0,1)^N$ is an $\mathbb{R}^d$-valued continuous Gaussian random field $\{v(x), x \in \mathbb{R}_+^N\}$ with mean zero and covariance

$$\mathbb{E}(v_j(x)v_\ell(y)) = \delta_{j,\ell} \prod_{i=1}^N \frac{1}{2}\left(|x_i|^{2H_i} + |y_i|^{2H_i} - |x_i - y_i|^{2H_i}\right).$$

When $N = 1$, it is the fractional Brownian motion and the non-existence of multiple points in the critical dimension was proved by Talagrand [61]. So we focus on the case $N \geq 2$.

Let $\alpha \in (0,1)$ be a constant. We start with the identity that any $x \in \mathbb{R}$,

$$|x|^{2\alpha} = c_\alpha^2 \int_{\mathbb{R}} \frac{1-\cos x\xi}{|\xi|^{2\alpha+1}}\, d\xi, \quad \text{where } c_\alpha = \left(\int_{\mathbb{R}} \frac{1-\cos\xi}{|\xi|^{2\alpha+1}} d\xi\right)^{-1/2},$$

which can be obtained by a change of variable in the integral. It implies that for any $x, y \in \mathbb{R}$,

$$\frac{1}{2}\left(|x|^{2\alpha} + |y|^{2\alpha} - |x-y|^{2\alpha}\right) = c_\alpha^2 \int_{\mathbb{R}} \left[\frac{(1-\cos x\xi)(1-\cos y\xi)}{|\xi|^{2\alpha+1}} + \frac{\sin x\xi \sin y\xi}{|\xi|^{2\alpha+1}}\right] d\xi.$$

It follows that for $H \in (0,1)^N$ and $x, y \in \mathbb{R}^N$, we can write

$$\prod_{i=1}^N \frac{1}{2}\left(|x_i|^{2H_i} + |y_i|^{2H_i} - |x_i - y_i|^{2H_i}\right) = c_H^2 \sum_{p \in \{0,1\}^N} \int_{\mathbb{R}^N} \prod_{i=1}^N \frac{f_{p_i}(x_i\xi_i) f_{p_i}(y_i\xi_i)}{|\xi_i|^{2H_i+1}}\, d\xi, \tag{3.29}$$

where $f_0(t) = 1 - \cos t$ and $f_1(t) = \sin t$. It gives a representation for the fractional Brownian sheet: If W_p, $p \in \{0,1\}^N$, are independent $\mathbb{R}^d$-valued Gaussian white noises on $\mathbb{R}^N$ and

$$v(x) := c_H \sum_{p \in \{0,1\}^N} \int_{\mathbb{R}^N} \prod_{i=1}^N \frac{f_{p_i}(x_i\xi_i)}{|\xi_i|^{H_i+1/2}}\, W_p(d\xi), \tag{3.30}$$

then (a continuous modification of) $\{v(x), x \in \mathbb{R}^N_+\}$ is an (N,d)-fractional Brownian sheet with Hurst index H. In particular, when $H_i = \frac{1}{2}$ for $i = 1, \ldots, k$, the Gaussian random field $\{v(x), x \in \mathbb{R}^N\}$ is the Brownian sheet and (3.30) provides a harminozable representation for it.

We take $T = (0,\infty)^N$ [since $v(x) = 0$ for all $x \in \partial\mathbb{R}^N_+$ a.s., the existence of multiple points is trivial on $\partial\mathbb{R}^N_+$]. We use the representation (3.30) to show that the fractional Brownian sheet satisfies the assumptions of Theorem 3.2.4 on T.

Define the random field $\{v(A,x), A \in \mathscr{B}(\mathbb{R}_+), x \in T\}$ by

$$v(A,x) = c_H \sum_{p\in\{0,1\}^N} \int_{\{\max_i |\xi_i|^{H_i} \in A\}} \prod_{i=1}^N \frac{f_{p_i}(x_i\xi_i)}{|\xi_i|^{H_i+1/2}} W_p(d\xi).$$

Lemma 3.5.1. *For any $n \geq 1$, let $F_n = [1/n, n]^N$, $\varepsilon_0 = (2n)^{-1}$, $a_0 = 0$ and $\gamma_i = H_i^{-1} - 1$. There is a constant $c_0 > 0$ depending on n such that for all $0 \leq a < b \leq \infty$ and $x, y \in F_n$,*

$$\big\|(v(x) - v([a,b),x)) - (v(y) - v([a,b),y))\big\|_{L^2} \leq c_0\bigg(\sum_{i=1}^N a^{\gamma_i}|x_i - y_i| + b^{-1}\bigg). \tag{3.31}$$

Proof. Without loss of generality, we may assume $d = 1$. For any $0 \leq a < b \leq \infty$, let $B = \{\xi \in \mathbb{R}^N : \max_i |\xi_i|^{H_i} \in [a,b)\}$. Then we can express its complement as

$$\mathbb{R}^N \setminus B = \big\{|\xi_k| < a_k, \forall 1 \leq k \leq N\big\} \cup \bigcup_{k=1}^N \big\{|\xi_k| \geq b_k\big\},$$

where $a_i = a^{1/H_i}$ and $b_i = b^{1/H_i}$.

Note that

$$
\begin{aligned}
&\prod_{i=1}^{N} \frac{f_{p_i}(x_i\xi_i)}{|\xi_i|^{H_i+1/2}} - \prod_{i=1}^{N} \frac{f_{p_i}(y_i\xi_i)}{|\xi_i|^{H_i+1/2}} \\
&= \sum_{i=1}^{N} \left(\frac{f_{p_i}(x_i\xi_i) - f_{p_i}(y_i\xi_i)}{|\xi_i|^{H_i+1/2}} \prod_{1\le j<i} \frac{f_{p_j}(y_j\xi_j)}{|\xi_j|^{H_j+1/2}} \prod_{i<j\le N} \frac{f_{p_j}(x_j\xi_j)}{|\xi_j|^{H_j+1/2}} \right).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\|(v(x) - v([a,b),x)) - (v(y) - v([a,b),y))\|_{L^2} \\
&\le c \sum_{p\in\{0,1\}^N} \left[\int_{\{|\xi_k|<a_k,\forall k\}} \left(\prod_{i=1}^{N} \frac{f_{p_i}(x_i\xi_i)}{|\xi_i|^{H_i+1/2}} - \prod_{i=1}^{N} \frac{f_{p_i}(y_i\xi_i)}{|\xi_i|^{H_i+1/2}} \right)^2 d\xi \right]^{1/2} \\
&\quad + c \sum_{p\in\{0,1\}^N} \sum_{k=1}^{N} \left[\int_{\{|\xi_k|\ge b_k\}} \left(\prod_{i=1}^{N} \frac{f_{p_i}(x_i\xi_i)}{|\xi_i|^{H_i+1/2}} - \prod_{i=1}^{N} \frac{f_{p_i}(y_i\xi_i)}{|\xi_i|^{H_i+1/2}} \right)^2 d\xi \right]^{1/2} \\
&\le c \sum_{p} \sum_{i=1}^{N} \left[\int_{\{|\xi_k|<a_k,\forall k\}} \left(\frac{f_{p_i}(x_i\xi_i) - f_{p_i}(y_i\xi_i)}{|\xi_i|^{H_i+1/2}} \prod_{1\le j<i} \frac{f_{p_j}(y_j\xi_j)}{|\xi_j|^{H_j+1/2}} \prod_{i<j\le N} \frac{f_{p_j}(x_j\xi_j)}{|\xi_j|^{H_j+1/2}} \right)^2 d\xi \right]^{\frac{1}{2}} \\
&\quad + c \sum_{p} \sum_{k=1}^{N} \sum_{i=1}^{N} \left[\int_{\{|\xi_k|\ge b_k\}} \left(\frac{f_{p_i}(x_i\xi_i) - f_{p_i}(y_i\xi_i)}{|\xi_i|^{H_i+1/2}} \prod_{1\le j<i} \frac{f_{p_j}(y_j\xi_j)}{|\xi_j|^{H_j+1/2}} \prod_{i<j\le N} \frac{f_{p_j}(x_j\xi_j)}{|\xi_j|^{H_j+1/2}} \right)^2 d\xi \right]^{\frac{1}{2}}
\end{aligned}
$$

Using the bounds $|f_{p_i}(x\xi) - f_{p_i}(y\xi)| \le |x-y||\xi|$ and $|f_{p_i}(x\xi) - f_{p_i}(y\xi)| \le 2$ for $p_i = 0$ and 1, we see that the above is at most

$$
\begin{aligned}
&c \sum_{p} \sum_{i=1}^{N} \left[\int_{\{|\xi_i|<a_i\}} \frac{|x_i-y_i|^2}{|\xi_i|^{2H_i-1}} \left(\prod_{1\le j<i} \frac{f_{p_j}(y_j\xi_j)}{|\xi_j|^{H_j+1/2}} \prod_{i<j\le N} \frac{f_{p_j}(x_j\xi_j)}{|\xi_j|^{H_j+1/2}} \right)^2 d\xi \right]^{1/2} \\
&\quad + c \sum_{p} \sum_{k=1}^{N} \sum_{i=1}^{N} \left[\int_{\{|\xi_k|\ge b_k\}} \frac{4}{|\xi_i|^{2H_i+1}} \left(\prod_{1\le j<i} \frac{f_{p_j}(y_j\xi_j)}{|\xi_i|^{H_i+1/2}} \prod_{i<j\le N} \frac{f_{p_j}(x_j\xi_j)}{|\xi_j|^{H_j+1/2}} \right)^2 d\xi \right]^{1/2}.
\end{aligned}
$$

Then by (3.29) the above is bounded from above by

$$c\sum_p\sum_{i=1}^N\left[a_i^{2-2H_i}|x_i-y_i|^2\prod_{1\le j<i}|y_j|^{2H_j}\prod_{i<j\le N}|x_j|^{2H_j}\right]^{1/2}$$
$$+c\sum_p\sum_{k=1}^N\sum_{i=1}^N\left[b_k^{-2H_k}\prod_{1\le j<i}|y_j|^{2H_j}\prod_{i<j\le N}|x_j|^{2H_j}\right]^{1/2}.$$

Since $|x_j|,|y_j|\le n+(2n)^{-1}$, we obtain (3.31) for some c_0 depending on n. □

Lemma 3.5.2. *For any $n\ge 1$, there is $\tilde c>0$ such that for all $x\in[1/n,n]^N$, $\|v_j(x)\|_{L^2}\ge\tilde c$ for all j. There is $C>0$ such that for all $x\in[1/n,n]^N$ and $y,\bar y$ with $|x_i-y_i|\le 1/2n$ and $|x_i-\bar y_i|\le 1/2n$,*

$$\left|\mathbb{E}((v_j(y)-v_j(\bar y))v_j(x))\right|\le C\sum_{i=1}^N\left|y_i-\bar y_i\right|^{\delta_i}$$

for all j, where $\delta_i=\min\{2H_i,1\}$.

Proof. The first statement is obvious because $\|v_j(x)\|_{L^2}\ge(\prod_{i=1}^N|x_i|^{2H_i})^{1/2}$. For the second statement, it suffices to show that

$$\left|\prod_{i=1}^N(|x_i|^{2H_i}+|y_i|^{2H_i}-|x_i-y_i|^{2H_i})-\prod_{i=1}^N(|x_i|^{2H_i}+|\bar y_i|^{2H_i}-|x_i-\bar y_i|^{2H_i})\right|\le K\sum_{i=1}^N\left|y_i-\bar y_i\right|^{\delta_i}.$$

For $1\le\ell\le N$, let $A_\ell=U_\ell-V_\ell$, where

$$U_\ell=\prod_{i=1}^\ell\left(|x_i|^{2H_i}+|y_i|^{2H_i}-|x_i-y_i|^{2H_i}\right),\quad V_\ell=\prod_{i=1}^\ell\left(|x_i|^{2H_i}+|\bar y_i|^{2H_i}-|x_i-\bar y_i|^{2H_i}\right).$$

When $\ell=1$, we have $|A_1|\le\left||y_1|^{2H_1}-|\bar y_1|^{2H_1}\right|+\left||x_1-y_1|^{2H_1}-|x_1-\bar y_1|^{2H_1}\right|$. If $2H_1\le 1$, then by the triangle inequality, $|A_1|\le 2|y_1-\bar y_1|^{2H_1}$; if $2H_1>1$, then we can use the mean value theorem to get $|A_1| \le K|y_1 - \bar{y}_1|$. Thus $|A_1| \le K|y_1 - \bar{y}_1|^{\delta_1}$. For $2 \le \ell \le N$,

$$
\begin{aligned}
A_\ell &= U_{\ell-1}(|x_\ell|^{2H_\ell} + |y_\ell|^{2H_\ell} - |x_\ell - y_\ell|^{2H_\ell}) - V_{\ell-1}(|x_\ell|^{2H_\ell} + |\bar{y}_\ell|^{2H_\ell} - |x_\ell - \bar{y}_\ell|^{2H_\ell}) \\
&= A_{\ell-1}(|x_\ell|^{2H_\ell} + |y_\ell|^{2H_\ell} - |x_\ell - y_\ell|^{2H_\ell}) \\
&\quad + V_{\ell-1}(|y_\ell|^{2H_\ell} - |\bar{y}_\ell|^{2H_\ell} + |x_\ell - \bar{y}_\ell|^{2H_\ell} - |x_\ell - y_\ell|^{2H_\ell}).
\end{aligned}
$$

Then $|A_\ell| \le K(|A_{\ell-1}+|y_\ell-\bar{y}_\ell|^{\delta_\ell})$ and by induction we obtain $|A_N| \le K\sum_{\ell=1}^N |y_\ell-\bar{y}_\ell|^{\delta_\ell}$. □

The following lemma verifies Assumption 3.2.3 for fractional Brownian sheets. The sectorial local nondeterminism in Theorem 1 of Wu and Xiao [65] provides more information on the conditional variances among $v(x_1), \dots, v(x_m)$.

Lemma 3.5.3. *If $x^1, \dots, x^m \in (0,\infty)^N$ are distinct points, then the random variables $v(x_1), \dots, v(x_m)$ are linearly independent.*

Proof. Suppose that $a_1, \dots, a_m$ are real numbers such that $\sum_{\ell=1}^m a_\ell v(x^\ell) = 0$ a.s. Recalling the representation (3.30) for $v(x)$, we have

$$
0 = \mathbb{E}\bigg(\sum_{\ell=1}^m a_\ell v(x^\ell)\bigg)^2 = c_H^2 \sum_{p\in\{0,1\}^N} \int_{\mathbb{R}^N} \left(\sum_{\ell=1}^m a_\ell \prod_{j=1}^N \frac{f_{p_j}(x_j^\ell \xi_j)}{|\xi_j|^{H_j+1/2}}\right)^2 d\xi.
$$

Then for each $p \in \{0,1\}^N$, $\sum_{\ell=1}^m a_\ell \prod_{j=1}^N f_{p_j}(x_j^\ell \xi_j) = 0$ and, equivalently,

$$
\sum_{\ell=1}^m a_\ell \prod_{j=1}^N \tilde{f}_{p_j}(x_j^\ell \xi_j) = 0
$$

for all $\xi \in \mathbb{R}^N$, where $\tilde{f}_0(t) = 1 - \cos t$ and $\tilde{f}_1(t) = -i \sin t$. It follows that

$$\sum_{\ell=1}^{m} a_\ell \prod_{j=1}^{N} \left(1 - \exp(ix_j^\ell \xi_j)\right) = \sum_{p\in\{0,1\}^N} \sum_{\ell=1}^{m} a_\ell \prod_{j=1}^{N} \tilde{f}_{p_j}(x_j^\ell \xi_j) = 0 \tag{3.32}$$

for all $\xi \in \mathbb{R}^N$. We claim that $a_1 = 0$. Let $L_{1,1}, \ldots, L_{1,k_1}$ be partitions of $\{1, \ldots, m\}$ obtained from the equivalence classes of the equivalence relation $\sim_1$ defined by $\ell \sim_1 k$ if and only if $x_1^\ell = x_1^k$. We may assume $1 \in L_{1,1}$. Let $\hat{x}_1^1, \ldots, \hat{x}_1^{m_1}$ be such that $x_1^\ell = \hat{x}_1^k$ for all $\ell \in L_{1,k}$, $k = 1, \ldots, m_1$. Let $\xi_2, \ldots, \xi_N \in \mathbb{R}$ be arbitrary and define $c_{1,1}, c_{1,2}, \ldots, c_{1,m_1}$ by

$$c_{1,k} = \sum_{\ell\in L_{1,k}} a_\ell \prod_{j=2}^{N} \left(1 - \exp(ix_j^\ell \xi_j)\right).$$

Then by (3.32), we have, for all $\xi_1 \in \mathbb{R}$,

$$c_{1,1} \exp(i\hat{x}_1^1 \xi_1) + \cdots + c_{1,m_1} \exp(i\hat{x}_1^{m_1} \xi_1) + (c_{1,1} + \cdots + c_{1,m_1}) = 0.$$

Since $\hat{x}_1^1, \ldots, \hat{x}_1^{m_1}$ are non-zero and distinct, the functions $\exp(i\hat{x}_1^1\xi), \ldots, \exp(i\hat{x}_1^{m_1}\xi), 1$ are linearly independent over $\mathbb{C}$, we have $c_{1,1} = \cdots = c_{1,m_1} = 0$. In particular, we have

$$\sum_{\ell\in L_{1,1}} a_\ell \prod_{j=2}^{N} \left(1 - \exp(ix_j^\ell \xi_j)\right) = 0$$

for all $\xi_2, \ldots, \xi_N \in \mathbb{R}$. Next we consider the partitions $L_{2,1}, \ldots, L_{2,m_2}$ of $\{1, \ldots, m\}$ obtained from equivalence classes of $\sim_2$ defined by $\ell \sim_2 k$ iff $x_2^\ell = x_2^k$ (with $1 \in L_{2,1}$). Then the argument above yields

$$\sum_{\ell \in L_{1,1} \cap L_{2,1}} a_\ell \prod_{j=3}^{N} \Big(1 - \exp(i x_j^\ell \xi_j)\Big) = 0.$$

By induction, we obtain

$$\sum_{\ell \in L_{1,1} \cap \cdots \cap L_{N,1}} a_\ell = 0.$$

Note that $L_{1,1} \cap \cdots \cap L_{N,1} = \{1\}$ because $x^1, \ldots, x^m$ are distinct. Hence $a_1 = 0$. Similarly, we can show that $a_\ell = 0$ for $\ell = 2, \ldots, m$. □

Proposition 3.5.4. *Let $v = \{v(x), x \in \mathbb{R}_+^N\}$ be an (N, d)-fractional Brownian sheet with Hurst parameter $H \in (0,1)^N$. If $mQ \le (m-1)d$ where $Q = \sum_{i=1}^N H_i^{-1}$, then v has no m-multiple points on $(0,\infty)^N$ almost surely.*

Proof. By the three lemmas above, $\{v(x),\, x \in [1/n,\, n]^N\}$ satisfies the assumptions of Theorem 3.2.4 with $Q = \sum_{i=1}^N H_i^{-1}$ for every $n \ge 1$. Hence the result follows immediately from the theorem. □

We remark that for the case of Brownian sheet i.e. $H_i = 1/2$ for all i, the above result provides an alternative proof for the main results in [15, 17] .

3.5.2 System of Stochastic Heat Equations

Let $k \ge 1$ and $\beta \in (0, k \wedge 2)$, or $k = 1 = \beta$. Consider the $\mathbb{R}^d$-valued random field $\{v(t,x), (t,x) \in \mathbb{R}_+ \times \mathbb{R}^k\}$ defined by

$$v(t,x) = \int_{\mathbb{R}} \int_{\mathbb{R}^k} e^{-i\xi\cdot x} \frac{e^{-i\tau t} - e^{-t|\xi|^2}}{|\xi|^2 - i\tau} |\xi|^{-(k-\beta)/2}\, W(d\tau, d\xi),$$

where W is a $\mathbb{C}^d$-valued space-time Gaussian white noise on $\mathbb{R}^{1+k}$ i.e. $W = W_1 + iW_2$ and W_1, W_2 are independent $\mathbb{R}^d$-valued space-time Gaussian white noises on $\mathbb{R}^{1+k}$. According to Proposition 7.2 of [18], the process $\hat{v}(t,x) := \operatorname{Re} v(t,x)$, $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^k$, has the same law as the mild solution to the system of stochastic heat equations

$$\begin{cases} \dfrac{\partial}{\partial t}\hat{v}_j(t,x) = \Delta \hat{v}_j(t,x) + \dot{\hat{W}}_j(t,x), \quad j = 1, \dots, d, \\ \hat{v}(0,x) = 0, \end{cases} \tag{3.33}$$

where $\hat{W}$ is an $\mathbb{R}^d$-valued spatially homogeneous Gaussian noise that is white in time with spatial covariance $|x-y|^{-\beta}$ if $k \geq 1$ and $\beta \in (0, k \wedge 2)$; it is an $\mathbb{R}^d$-valued space-time Gaussian white noise when $k = 1 = \beta$. Note that, in this case, we take $T = (0, \infty) \times \mathbb{R}^k$.

The Hölder exponents of $v(t,x)$ are $\alpha_1 = (2-\beta)/4$ in time and $\alpha_2 = \cdots = \alpha_{1+k} = (2-\beta)/2$ in space. See [18, §7] or [14]. In this case, we have $Q = (4+2k)/(2-\beta)$.

The following lemma can also be found in [52, Lemma A.5.3].

Lemma 3.5.5. *Let $(t^1, x^1), \dots, (t^m, x^m)$ be distinct points in $(0,\infty) \times \mathbb{R}^k$. Then the random variables $\hat{v}_1(t^1, x^1), \dots, \hat{v}_1(t^m, x^m)$ are linearly independent.*

Proof. Suppose that $a_1, \dots, a_m$ are real numbers such that $\sum_{j=1}^m a_j \hat{v}_1(t^j, x^j) = 0$ a.s. Then

$$0 = \mathbb{E}\left(\sum_{j=1}^m a_j \hat{v}_1(t^j, x^j)\right)^2 = \int_{\mathbb{R}} \int_{\mathbb{R}^k} \left| \sum_{j=1}^m a_j e^{-i\xi \cdot x^j} (e^{-i\tau t^j} - e^{-t^j |\xi|^2}) \right|^2 \frac{d\tau \, d\xi}{(|\xi|^4 + \tau^2)|\xi|^{k-\beta}}$$

and thus $\sum_{j=1}^m a_j e^{-i\xi \cdot x^j}(e^{-i\tau t^j} - e^{-t^j|\xi|^2}) = 0$ for all $\tau \in \mathbb{R}$ and $\xi \in \mathbb{R}^k$. We claim that $a_j = 0$ for all $j = 1, \dots, m$. Let $\hat{t}^1, \dots, \hat{t}^p$ be all distinct values of the t^j's. Fix an arbitrary

$\xi \in \mathbb{R}^k$. Then for all $\tau \in \mathbb{R}$, we have

$$\sum_{\ell=1}^{p} \Bigg(\sum_{j:t^j=\hat{t}^\ell} a_j e^{-i\xi\cdot x^j} \Bigg) e^{-i\tau\hat{t}^\ell} - \sum_{j=1}^{m} a_j e^{-i\xi\cdot x^j - t^j|\xi|^2} = 0.$$

Since the functions $e^{-i\tau\hat{t}^1}, \ldots, e^{-i\tau\hat{t}^p}, 1$ are linearly independent over $\mathbb{C}$, it follows that for all $\xi \in \mathbb{R}^k$, for all $\ell = 1, \ldots, p$,

$$\sum_{j:t^j=\hat{t}^\ell} a_j e^{-i\xi\cdot x^j} = 0. \tag{3.34}$$

Since $(t^1, x^1), \ldots, (t^n, x^n)$ are distinct, the x^j's in the sum in (3.34) are distinct for any fixed ℓ. By linear independence of the functions $e^{-i\xi\cdot x^j}$, we conclude that $a_j = 0$ for all j. □

The following result solves the existence problem of m-multiple points for (3.33).

Proposition 3.5.6. *If $m(4+2k)/(2-\beta) \le (m-1)d$, then $\{\hat{v}(t,x), t \in (0,\infty),\ x \in \mathbb{R}^k\}$ has no m-multiple points a.s.*

Proof. Assumptions 3.2.1 and 3.2.2 are satisfied with $Q = (4+2k)/(2-\beta)$ by Lemma 7.3 and 7.5 of [18]. Assumption 3.2.3 is also satisfied by Lemma 3.5.5 above. The result follows from Theorem 3.2.4. □

3.5.3 System of Stochastic Wave Equations

Let $k \ge 1$ and $\beta \in [1, k \wedge 2)$, or $k = 1 = \beta$. Consider the $\mathbb{R}^d$-valued random field $\{v(t,x), (t,x) \in \mathbb{R}_+ \times \mathbb{R}^k\}$ defined by

$$v(t,x) = \int_{\mathbb{R}} \int_{\mathbb{R}^k} F(t,x,\tau,\xi)|\xi|^{-(k-\beta)/2}\, W(d\tau, d\xi),$$

where W is a $\mathbb{C}^d$-valued space-time Gaussian white noise on $\mathbb{R}^{1+k}$ and

$$F(t,x,\tau,\xi) = \frac{e^{-i\xi\cdot x - i\tau t}}{2|\xi|}\left[\frac{1-e^{it(\tau+|\xi|)}}{\tau+|\xi|} - \frac{1-e^{it(\tau-|\xi|)}}{\tau-|\xi|}\right].$$

By Proposition 9.2 of [18], the process $\hat{v}(t,x) = \operatorname{Re} v(t,x)$, $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^k$, has the same law as the mild solution to the system of stochastic wave equations

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}\hat{v}_j(t,x) = \Delta \hat{v}_j(t,x) + \dot{\hat{W}}_j(t,x), \quad j = 1,\ldots,d, \\ \hat{v}(0,x) = 0, \quad \dfrac{\partial}{\partial t}\hat{v}(0,x) = 0, \end{cases}$$

where $\hat{W}$ is the spatially homogeneous $\mathbb{R}^d$-valued Gaussian noise as in (3.33).

The Hölder exponents of $v(t,x)$ are $\alpha_1 = \alpha_2 = \cdots = \alpha_{1+k} = (2-\beta)/2$ in both time and space. See [18, §9] or [20]. In this case, we have $Q = (2+2k)/(2-\beta)$.

Lemma 3.5.7. *Let $(t^1,x^1),\ldots,(t^m,x^m)$ be distinct points in $T = (0,\infty)\times\mathbb{R}^k$. Then the random variables $\hat{v}_1(t^1,x^1),\ldots,\ \hat{v}_1(t^m,x^m)$ are linearly independent.*

Proof. Suppose that $a_1,\ldots,a_m$ are real numbers such that $\sum_{j=1}^m a_j \hat{v}_1(t^j,x^j) = 0$ a.s. Then

$$0 = \mathbb{E}\left(\sum_{j=1}^m a_j \hat{v}_1(t^j,x^j)\right)^2 = \int_{\mathbb{R}}\int_{\mathbb{R}^k}\left|\sum_{j=1}^m a_j F(t^j,x^j,\tau,\xi)\right|^2 \frac{d\tau\, d\xi}{|\xi|^{k-\beta}}.$$

It follows that $\tau \in \mathbb{R}$ and $\xi \in \mathbb{R}^k$, $\sum_{j=1}^m a_j F(t^j,x^j,\tau,\xi) = 0$ and thus

$$\sum_{j=1}^m b_j e^{-i\tau t^j} + c_1\tau + c_2 = 0,$$

where $b_j = -2a_j|\xi|e^{-i\xi\cdot x^j}$,

$$c_1 = -\sum_{j=1}^{m} a_j e^{-i\xi\cdot x^j}(e^{it^j|\xi|} - e^{-it^j|\xi|})$$

and

$$c_2 = \sum_{j=1}^{m} a_j|\xi|e^{-i\xi\cdot x^j}(e^{it^j|\xi|} + e^{-it^j|\xi|}).$$

We claim that $a_j = 0$ for all $j = 1, \ldots, m$. Let $\hat{t}^1, \ldots, \hat{t}^p$ be all distinct values of the t^j's. If we take arbitrary $\xi \in \mathbb{R}^k$ and take derivative with respect to τ, we see that

$$\sum_{\ell=1}^{p}\Big(-i\hat{t}^\ell \sum_{j:t^j=\hat{t}^\ell} b_j\Big)e^{-i\tau\hat{t}^\ell} + c_1 = 0$$

for all $\tau \in \mathbb{R}$. Since the functions $e^{-i\tau\hat{t}^1}, \ldots, e^{-i\tau\hat{t}^p}, 1$ are linearly independent over $\mathbb{C}$, we have

$$-i\hat{t}^1 \sum_{j:t^j=\hat{t}^\ell} b_j = 0$$

for all $\ell = 1, \ldots, p$. It implies that for all $\xi \in \mathbb{R}^k$, for all $\ell = 1, \ldots, p$,

$$\sum_{j:t^j=\hat{t}^\ell} a_j e^{-i\xi\cdot x^j} = 0. \tag{3.35}$$

Since $(t^1, x^1), \ldots, (t^m, x^m)$ are distinct, the x^j's that appear in the sum in (3.35) are distinct for any fixed ℓ. By linear independence of the functions $e^{-i\xi\cdot x^j}$, we conclude that $a_j = 0$ for all j. $\square$

Proposition 3.5.8. *If* $m(2+2k)/(2-\beta) \leq (m-1)d$*, then* $\{\hat{v}(t,x), t \in (0,\infty),\ x \in \mathbb{R}^k\}$ *has no* m*-multiple points a.s.*

Proof. Assumptions 3.2.1 and 3.2.2 are satisfied with $Q = (2+2k)/(2-\beta)$ by Lemmas 9.3 and 9.6 of [18]. Assumption 3.2.3 is also satisfied by Lemma 3.5.7. Hence the result follows from Theorem 3.2.4. □

Chapter 4

Local Times and Level Sets of Gaussian Random Fields

4.1 Introduction

The purpose of this chapter is to study the local times and level sets of anisotropic Gaussian random fields satisfying strong local nondeterminism with respect to an anisotropic metric. We will prove joint continuity for the local times in Section 4.2 and Hölder condition in Section 4.3. Then we discuss the Hausdorff dimension and Hausdorff measure of the level sets in Section 4.4. As an example, we apply these results to the stochastic heat equation in Section 4.5.

Let $Y = \{Y(t) : t \in \mathbb{R}^N\}$ be a real-valued centered Gaussian random field. Let us consider the (N, d)-Gaussian random field $X = \{X(t) : t \in \mathbb{R}^N\}$ defined by

$$X(t) = (X_1(t), \ldots, X_d(t)),$$

where $X_1, \ldots, X_d$ are i.i.d. copies of Y. We will study the regularities of the local times of X and the Hausdorff measure of the level sets $\{t \in \mathbb{R}^N : X(t) = x\}$.

Consider a fixed closed bounded cube $T \subset \mathbb{R}^N$. Suppose there is a constant vector

$H = (H_1, \ldots, H_N) \in (0, 1)^N$ (not depending on T) and two positive finite constants C_1 and C_2 such that

$$C_1 \rho(t, s)^2 \leq \mathbb{E}[(Y(t) - Y(s))^2] \leq C_2 \rho(t, s)^2 \tag{4.1}$$

for all $t, s \in T$, where ρ is the metric defined by

$$\rho(t, s) = \sum_{j=1}^{N} |t_j - s_j|^{H_j}.$$

Suppose that Y satisfies *strong local nondeterminism* in the following sense: there is a positive finite constant C_3 such that for all integers $n \geq 1$, for all $t, t^1, \ldots, t^n \in T$,

$$\mathrm{Var}(Y(t)|Y(t^1), \ldots, Y(t^n)) \geq C_3 \min_{0 \leq k \leq n} \rho(t, t^k)^2, \tag{4.2}$$

where $t^0 = 0$.

The property of local nondeterminism (LND) is useful for investigating sample paths of Gaussian random fields. This terminology was first introduced by Berman [6] for Gaussian processes and extended by Pitt [51] for Gaussian random fields to study their local times. Later, the property of strong local nondeterminism was developed to study exact regularity of local times, small ball probability and other sample paths properties for Gaussian random fields (see, e.g., [68, 69]).

For example, the multiparameter fractional Brownian motion satisfies strong local nondeterminism with $\rho(t, s) = |t - s|^H$ (see Pitt [51]). Sufficient conditions in terms of spectral measures for Gaussian random fields with stationary increments to satisfy strong LND can be found in [70, 37]. In Section 4.5, we will show that the stochastic heat equation satisfies strong local nondeterminism.

The Hölder conditions for the local times and the Hausdorff measure of the level sets of strongly locally nondeterministic Gaussian random fields with stationary increments were studied by Xiao [67]. The case of anisotropic Gaussian random fields satisfying a weaker form of LND called sectorial local nondeterminism was considered by Wu and Xiao [66]. An example of Gaussian random field that satisfies sectorial local nondeterminism is the fractional Brownian sheet. The Gaussian random field X that we consider here satisfies strong local nondeterminism (4.2) with respect to an anisotropic metric, but it does not necessarily have stationary increments.

Let $S \subset \mathbb{R}^N$ be a Borel set. We say that an $\mathbb{R}^d$-valued random field $X = \{X(t) : t \in \mathbb{R}^N\}$ has a *local time* on S if the occupation measure $\mu_S(A) = \lambda_N\{t \in S : X(t) \in A\}$, $A \in \mathscr{B}(\mathbb{R}^d)$, is absolutely continuous with respect to the Lebesgue measure λ_d on $\mathbb{R}^d$. In this case, the local time is defined as (a version of) the Radon–Nikodym derivative

$$L(x, S) = \frac{d\mu_S}{d\lambda_d}(x), \quad x \in \mathbb{R}^d.$$

Note that if X has local time on S, then it also has local time on any Borel set $B \subseteq S$.

By Theorem 6.4 of [23], the local time satisfies the following *occupation density formula*: for any Borel set $B \subset S$ and any nonnegative measurable function $f : \mathbb{R}^d \to \mathbb{R}$,

$$\int_B f(X(t))\, dt = \int_{\mathbb{R}^d} f(x) L(x, B)\, dx. \tag{4.3}$$

By Theorem 8.1 of [70], if condition (4.1) holds on T and if $d < \sum_{j=1}^N H_j^{-1}$, then X has a local time $L(\cdot, S) \in L^2(\mathbb{R}^d)$ on any Borel set $S \subseteq T$, with the representation

$$L(x, S) = (2\pi)^{-d} \int_{\mathbb{R}^d} du\, e^{-i\langle u,x\rangle} \int_S dt\, e^{i\langle u,X(t)\rangle} \tag{4.4}$$

for almost every $x \in \mathbb{R}^d$.

4.2 Joint Continuity of Local Times

Let $T = \prod_{j=1}^N [\tau_j, \tau_j + h_j] \subset \mathbb{R}^N$ be a closed bounded cube, where $h_j > 0$ for $j = 1, \ldots, N$. Suppose X has a local time $L(x, \cdot)$ on T. We say that the local time is *jointly continuous* on T if we can find a version of the process

$$\left\{ L\Big(x, \prod_{j=1}^N [\tau_j, \tau_j + s_j]\Big) : x \in \mathbb{R}^d, s \in \prod_{j=1}^N [0, h_j] \right\}$$

such that with probability 1, the sample function

$$(x, s) \mapsto L\Big(x, \prod_{j=1}^N [\tau_j, \tau_j + s_j]\Big)$$

is continuous in all variables on the domain $\mathbb{R}^d \times \prod_{j=1}^N [0, h_j]$. If the local time is jointly continuous on T, then for each x, $L(x, \cdot)$ is a well-defined measure on the Borel sets in T, supported on the level set $X^{-1}(x) \cap T = \{t \in T : X(t) = x\}$ (see [1], Theorem 8.6.1).

The goal of this section is to prove the following:

Theorem 4.2.1. *Suppose* (4.1) *and* (4.2) *hold on the closed bounded cube* $T \subset \mathbb{R}^N$. *Suppose* $d < Q$, *where* $Q = \sum_{j=1}^N H_j^{-1}$. *Then the Gaussian random field* X *has a jointly continuous local time on* T *almost surely.*

The key of the proof is to derive moment bounds for the increments of the local time. We follow Lemma 2.5 of Xiao [67].

For $a \in \mathbb{R}^N$ and $r > 0$, let $B_\rho(a, r) = \{t \in \mathbb{R}^N : \rho(t, a) \le r\}$ denote the anisotropic ball at a of radius r under the metric ρ.

Lemma 4.2.2. *Let T be a closed bounded cube in $\mathbb{R}^N$. Suppose $0 < d \le \beta \le \beta_0 < Q$, where $Q = \sum_{j=1}^N H_j^{-1}$. Then there exists a positive finite constant C depending on N, d, H and β_0 only such that for all subset S of T, for all integers $j \ge 1$, for all $t^1, \ldots, t^j \in T$, we have*

$$\int_S \left[\min_{0 \le k \le j-1} \rho(t, t^k)\right]^{-\beta} dt \le C j^{\beta/Q} \lambda_N(S)^{1-\beta/Q}. \tag{4.5}$$

In particular, for all $a \in \mathbb{R}^N$, $0 < r < 1$, with $D := B_\rho(a, r) \subseteq T$, for all integers $j \ge 1$, for all $t^1, \ldots, t^j \in T$, we have

$$\int_D \left[\min_{0 \le k \le j-1} \rho(t, t^k)\right]^{-\beta} dt \le C j^{\beta/Q} r^{Q-\beta}. \tag{4.6}$$

Proof. Let I denote the integral in (4.6). For $l = 0, \ldots, j-1$, define

$$\Gamma_l = \left\{ t \in S : \rho(t, t^l) = \min_{0 \le k \le j-1} \rho(t, t^k) \right\}.$$

Then $S = \bigcup_{l=0}^{j-1} \Gamma_l$ and

$$\begin{aligned} I &= \sum_{l=0}^{j-1} \int_{\Gamma_l} \rho(t, t^l)^{-\beta} dt \\ &= \sum_{l=0}^{j-1} \int_{\Gamma_l} \left(\sum_{m=1}^N |t_m - t_m^l|^{H_m} \right)^{-\beta} dt. \end{aligned}$$

Fix $l \in \{0, 1, \ldots, j-1\}$. Let us consider a change of variables on Γ_l:

$$\begin{aligned}
t_1 &= t_1^l + h^{1/H_1}[\cos(\theta_1)]^{2/H_1}, \\
t_2 &= t_2^l + h^{1/H_2}[\sin(\theta_1)\cos(\theta_2)]^{2/H_2}, \\
&\vdots \\
t_{N-1} &= t_{N-1}^l + h^{1/H_{N-1}}[\sin(\theta_1)\ldots\sin(\theta_{N-2})\cos(\theta_{N-1})]^{2/H_{N-1}}, \\
t_N &= t_N^l + h^{1/H_N}[\sin(\theta_1)\ldots\sin(\theta_{N_2})\sin(\theta_{N-1})]^{2/H_N},
\end{aligned}$$

for $\theta = (\theta_1, \ldots, \theta_{N-1}) \in A := [0, 2\pi] \times [0, \pi]^{N-2}$ and $h \in [0, h_l(\theta)]$, where $[x]^p := \mathrm{sgn}(x)|x|^p$. We may write the integral

$$\int_{\Gamma_l} \left(\sum_{m=1}^N |t_m - t_m^l|^{H_m} \right)^{-\beta} dt$$

into a sum of 2^N terms, each of which is an integral over the intersection of Γ_l with one of the 2^N open quadrants centered at t^l. Then the Jacobian exists on each open quadrant and the absolute value of its determinant is $h^{Q-1}\varphi(\theta)$ for some bounded function φ. We can use the change of variables formula for each term and then recombine the terms to get

$$\begin{aligned}
I &= N^{-\beta} \sum_{l=0}^{j-1} \int_A d\theta\, \varphi(\theta) \int_0^{h_l(\theta)} h^{Q-1-\beta} dh \\
&= \frac{N^{-\beta}}{Q-\beta} \sum_{l=0}^{j-1} \int_A h_l(\theta)^{Q-\beta}\, \varphi(\theta)\, d\theta.
\end{aligned}$$

Note that the Lebesgue measure of Γ_l is

$$\begin{aligned}\lambda(\Gamma_l) &= \int_A d\theta\, \varphi(\theta) \int_0^{h_l(\theta)} h^{Q-1} dh \\ &= \frac{C_N}{Q} \int_A h_l(\theta)^Q\, C_N^{-1} \varphi(\theta)\, d\theta,\end{aligned} \tag{4.7}$$

where $C_N := \int_A \varphi(\theta) d\theta$. Since $0 < \beta < Q$, the function $x \mapsto x^{1-\beta/Q}$ is concave on $[0, \infty)$. Then by Jensen's inequality and (4.7),

$$\begin{aligned} I &\le \frac{C_N N^{-d}}{Q-\beta_0} \sum_{l=0}^{j-1} \int_A \Big(h_l(\theta)^Q \Big)^{1-\beta/Q}\, C_N^{-1} \varphi(\theta)\, d\theta \\ &\le \frac{C_N N^{-d}}{Q-\beta_0} \sum_{l=0}^{j-1} \left(\int_A h_l(\theta)^Q C_N^{-1} \varphi(\theta)\, d\theta \right)^{1-\beta/Q} \\ &= \frac{C_N N^{-d}}{Q-\beta_0} \sum_{l=0}^{j-1} \left(\frac{Q}{C_N} \lambda(\Gamma_l) \right)^{1-\beta/Q} \\ &\le \frac{C_N N^{-d}}{Q-\beta_0} \left(\frac{Q}{C_N} \vee 1 \right)^{1-d/Q} \sum_{l=0}^{j-1} \lambda(\Gamma_l)^{1-\beta/Q}. \end{aligned}$$

Then by Jensen's inequality again,

$$\begin{aligned} I &\le C\, j \left(\frac{1}{j} \sum_{l=0}^{j-1} \lambda(\Gamma_l)^{1-\beta/Q} \right) \\ &\le C\, j \left(\frac{1}{j} \sum_{l=0}^{j-1} \lambda(\Gamma_l) \right)^{1-\beta/Q} \\ &= C\, j^{\beta/Q} \lambda(S)^{1-\beta/Q}, \end{aligned}$$

where C depends on N, d, Q and β_0. Hence we obtain (4.5). This implies (4.6) immediately, since $\lambda_N(B_\rho(a, r)) \le C r^Q$. $\square$

The following proposition gives a moment estimate for the local time on anisotropic balls.

Proposition 4.2.3. *Suppose (4.1) and (4.2) hold on the closed bounded cube $T \subset \mathbb{R}^N$. Suppose $d < Q$, where $Q = \sum_{j=1}^{N} H_j^{-1}$. Then there exists a positive finite constant C such that for all subset S of T, for all $x \in \mathbb{R}^d$ and all integers $n \geq 1$, we have*

$$\mathbb{E}[L(x,S)^n] \leq C^n (n!)^{d/Q} \lambda_N(S)^{n(1-d/Q)}.$$

In particular, for all $a \in \mathbb{R}^N$, $r \in (0,1)$ with $D := B_\rho(a,r) \subseteq T$, for all $x \in \mathbb{R}^d$ and all integers $n \geq 1$, we have

$$\mathbb{E}[L(x,D)^n] \leq C^n (n!)^{d/Q} r^{n(Q-d)}.$$

Proof. By (4.4), we have

$$\mathbb{E}[L(x,S)^n] = (2\pi)^{-nd} \int_{\mathbb{R}^{nd}} d\bar{u} \int_{S^n} d\bar{t}\, e^{-i\sum_{j=1}^{n}\langle u^j, x\rangle}\, \mathbb{E}\left[e^{i\sum_{j=1}^{n}\langle u^j, X(t^j)\rangle}\right],$$

where $\bar{u} = (u^1, \dots, u^n)$ and $\bar{t} = (t^1, \dots, t^n)$. Since $X_1, \dots, X_d$ are i.i.d. copies of Y, we have

$$\begin{aligned}
\mathbb{E}[L(x,S)^n] &\leq (2\pi)^{-nd} \int_{S^n} d\bar{t} \prod_{k=1}^{d} \int_{\mathbb{R}^n} d\bar{u}_k\, e^{-\frac{1}{2}\mathrm{Var}(\sum_{j=1}^{n} u_k^j Y(t^j))} \\
&= (2\pi)^{-nd/2} \int_{S^n} \left[\det \mathrm{Cov}(Y(t^1), \dots, Y(t^n))\right]^{-d/2} d\bar{t}
\end{aligned}$$

where $\bar{u}_k = (u_k^1, \dots, u_k^n)$. Since

$$\det \mathrm{Cov}(Y(t^1), \dots, Y(t^n)) = \mathrm{Var}(Y(t^1)) \prod_{j=2}^{n} \mathrm{Var}(Y(t^j) | Y(t^1), \dots, Y(t^{j-1})),$$

it follows from assumption (4.2) that

$$\mathbb{E}[L(x,S)^n] \le C(2\pi)^{-nd/2} \int_{S^n} \prod_{j=1}^{n} \left[\min_{0\le k\le j-1} \rho(t^j, t^k) \right]^{-d} d\bar{t}. \tag{4.8}$$

If we integrate (4.8) in the order of $dt^n, dt^{n-1}, \ldots, dt^1$, and apply Lemma 4.2.2 (with $\beta = d$) repeatedly, we deduce that

$$\mathbb{E}[L(x,S)^n] \le C^n (n!)^{d/Q} \lambda_N(S)^{n(1-d/Q)}.$$

This yields the first statement of the proposition. The last statement follows immediately since $\lambda_N(D) \le Cr^Q$. $\square$

Next, we would like to extend the moment estimate in the above proposition to moment estimates for the increments of the local time. To this end, we need some lemmas. The following lemma is taken from [10, Lemma 2].

Lemma 4.2.4. *Let $Y_1, \ldots, Y_n$ be mean zero Gaussian random variables that are linearly independent and assume that $\int_{\mathbb{R}} g(v) e^{-\varepsilon v^2} dv < \infty$ for all $\varepsilon > 0$. Then*

$$\int_{\mathbb{R}^n} g(v_1) \exp\left[-\frac{1}{2} \mathrm{Var}\left(\sum_{l=1}^{n} v_l Y_l \right) \right] dv_1 \ldots dv_n = \frac{(2\pi)^{(n-1)/2}}{\det \mathrm{Cov}(Y_1, \ldots, Y_n)^{1/2}} \int_{\mathbb{R}} g(v/\sigma_1) e^{-v^2/2} dv$$

where $\sigma_1 = \mathrm{Var}(Y_1 | Y_2, \ldots, Y_n)$.

Let us recall the following version of Besicovitch's covering theorem for cubes in $\mathbb{R}^N$. See [26], Theorem 1.1.

Lemma 4.2.5. *There exists a positive integer $M = M(N)$ depending only on N with the following property. For any bounded subset A of $\mathbb{R}^N$ and any family $\mathscr{B} = \{Q(x) : x \in A\}$*

of closed cubes such that $Q(x)$ is centered at x for every $x \in A$, there exists a sequence $\{Q_i\}$ in $\mathscr{B}$ such that:

(i) $A \subset \bigcup_i Q_i$*;*

(ii) the cubes of $\{Q_i\}$ can be distributed in M families of disjoint cubes.

We will use the covering theorem to prove Lemma 4.2.6 below. Before that, let us introduce the notation:

$$\tilde{\rho}(t,s) = \max_{1 \le j \le N} |t_j - s_j|^{H_j}.$$

Note that

$$\tilde{\rho}(t,s) \le \rho(t,s) \le N\tilde{\rho}(t,s) \tag{4.9}$$

for all $t, s \in \mathbb{R}^N$. Then under the assumption (4.2), for any closed bounded cube T in $\mathbb{R}^N$, there exists a positive finite constant C_3 (depending on T) such that for all integers $n \ge 1$ and all $t, t^1, \ldots, t^n \in T$,

$$\mathrm{Var}(Y(t)|Y(t^1), \ldots, Y(t^n)) \ge C_3 \min_{0 \le l \le n} \tilde{\rho}(t, t^l)^2. \tag{4.10}$$

Lemma 4.2.6. *There exists a positive integer $K = K(N)$ depending only on N such that for any integer $n \ge 1$, for any distinct points $s^0, s^1, \ldots, s^n \in \mathbb{R}^N$, the cardinality of the set of all $j \in \{1, \ldots, n\}$ such that*

$$\tilde{\rho}(s^j, s^0) = \min\{\tilde{\rho}(s^j, s^i) : 0 \le i \le n, i \ne j\} \tag{4.11}$$

is at most K.

Proof. Without loss of generality, we may assume that (4.11) is satisfied for $j = 1, \ldots, k$.

Note that for $s \in \mathbb{R}^N$ and $r > 0$, the ball $B_{\tilde{\rho}}(s, r) := \{t \in \mathbb{R}^N : \tilde{\rho}(t, s) \le r\}$ under the metric $\tilde{\rho}$ is the closed cube centered at s with side lengths $2r^{1/H_1}, \dots, 2r^{1/H_N}$. We will use Lemma 4.2.5 to show that $k \le K$ for some positive integer $K = K(N)$ that depends on N only.

To this end, let

$$\delta_0 = \min \left\{ \frac{\tilde{\rho}(s^i, s^0)}{\tilde{\rho}(s^j, s^0)} : i, j \in \{1, \dots, k\} \right\}.$$

Note that $0 < \delta_0 \le 1$. Take a small $0 < \varepsilon_0 < 1$ such that $(1 - \varepsilon_0)^{1/H_p}(1 + \delta_0^{1/H_p}) \ge 1$ for all $p \in \{1, \dots, N\}$. Let $\varepsilon = \varepsilon_0 \min\{\tilde{\rho}(s^i, s^0) : 1 \le i \le k\}$. Let $A = \{s^1, \dots, s^k\}$ and consider the family $\mathscr{B} = \{B_{\tilde{\rho}}(s^1, r_1), \dots, B_{\tilde{\rho}}(s^k, r_k)\}$ of closed cubes, where $r_i = \tilde{\rho}(s^i, s^0) - \varepsilon$. By Lemma 4.2.5, we can find $\mathscr{B}_1, \dots, \mathscr{B}_M \subset \mathscr{B}$, with $\mathscr{B}_i = \{B_{\tilde{\rho}}(s^{i,1}, r_{i,1}), \dots, B_{\tilde{\rho}}(s^{i,J(i)}, r_{i,J(i)})\}$ such that

$$A = \{s^1, \dots, s^k\} \subset \bigcup_{i=1}^{M} \bigcup_{j=1}^{J(i)} B_{\tilde{\rho}}(s^{i,j}, r_{i,j})$$

and for each i, the cubes of $\mathscr{B}_i$ are pairwise disjoint, where $M = M(N)$ is a positive integer which depends on N only. For each $1 \le j \le k$, by the assumption (4.11), if $\ell \ne j$, then $\tilde{\rho}(s^j, s^\ell) \ge \tilde{\rho}(s^j, s^0) > r_j$. In other words, for each $1 \le j \le k$, the cube $B_{\tilde{\rho}}(s^j, r_j)$ does not contain any other s^ℓ, where $\ell \ne j$. It means that we need at least k cubes to cover the set A, and hence $k \le J(1) + \cdots + J(M)$.

Let us fix i and estimate the cardinality $J(i)$ of the family $\mathscr{B}_i$. Let us consider the family $\mathscr{B}_i^* = \{B_{\tilde{\rho}}(s^{i,1}, r^*_{i,1}), \dots, B_{\tilde{\rho}}(s^{i,J(i)}, r^*_{i,J(i)})\}$, where $r^*_{i,j} = \tilde{\rho}(s^{i,j}, s^0)$. Since the cubes of $\mathscr{B}_i$ are pairwise disjoint, for any pair $s^{i,\ell} \ne s^{i,j}$ we can find $p \in \{1, \dots, N\}$ such that

$$|s^{i,\ell}_p - s^{i,j}_p| > r_{i,\ell}^{1/H_p} + r_{i,j}^{1/H_p}.$$

Then by the definition of ε, ε_0 and δ_0,

$$
\begin{aligned}
|s_p^{i,\ell} - s_p^{i,j}| &> (\tilde{\rho}(s^{i,\ell}, s^0) - \varepsilon)^{1/H_p} + (\tilde{\rho}(s^{i,j}, s^0) - \varepsilon)^{1/H_p} \\
&\geq (1-\varepsilon_0)^{1/H_p} \left(\tilde{\rho}(s^{i,\ell}, s^0)^{1/H_p} + \tilde{\rho}(s^{i,j}, s^0)^{1/H_p} \right) \\
&= (1-\varepsilon_0)^{1/H_p} (1 + \delta_0^{1/H_p}) \tilde{\rho}(s^{i,j}, s^0)^{1/H_p} \\
&\geq \tilde{\rho}(s^{i,j}, s^0)^{1/H_p} \\
&= (r_{i,j}^*)^{1/H_p}.
\end{aligned}
$$

It follows that $\tilde{\rho}(s^{i,\ell}, s^{i,j}) > r_{i,j}^*$, which means that the cube $B(s^{i,j}, r_{i,j}^*)$ does not contain any other $s^{i,\ell}$ where $\ell \neq j$. On the other hand, every cube of $\mathscr{B}_i^*$ contains s^0, so these cubes are not pairwise disjoint. Then another application of Lemma 4.2.5 to the set $\{s^{i,1}, \ldots, s^{i,J(i)}\}$ and the family $\mathscr{B}_i^*$ implies that $J(i) \leq M$. Therefore, we have $k \leq M^2$ and we may take $K = M^2$. □

We use the previous lemma to prove the lemma below, which provide a correction to the estimate (2.20) in Xiao [67].

Lemma 4.2.7. *Let $0 < \gamma < 1$. Let $t^0 = 0$ and $t^1, \ldots, t^n$ be distinct points in $\{t \in \mathbb{R}^N : |t| \leq R\} \setminus \{0\}$, where $R > 0$. Then there exist a positive integer $K = K(N)$ depending on N only, a positive finite constant C depending only on R, H, N, and a permutation π of $\{0, 1, \ldots, n\}$ with $\pi(0) = 0$ such that*

$$
\prod_{j=1}^{n} \frac{1}{\min\{\tilde{\rho}(t^j, t^i)^\gamma : 0 \leq i \leq n, i \neq j\}} \leq C^n \prod_{j=1}^{n} \frac{1}{\tilde{\rho}(t^{\pi(j)}, t^{\pi(j-1)})^{2K\gamma}}.
$$

Proof. Since $(\min_j c^{-H_j})\tilde{\rho}(t,s) \leq \tilde{\rho}(c^{-1}t, c^{-1}s) \leq (\max_j c^{-H_j})\tilde{\rho}(t,s)$, it suffices to prove the result for $R = 1/2$. In this case, $\tilde{\rho}(t^i, t^j) \le 1$ for all i, j. Let $\pi(0) = 0$. Define $\pi(1), \ldots, \pi(n)$ inductively such that

$$\tilde{\rho}(t^{\pi(1)}, 0) = \min\{\tilde{\rho}(t^i, 0) : i = 1, \ldots, n\}$$

and for $j \ge 2$,

$$\tilde{\rho}(t^{\pi(j)}, t^{\pi(j-1)}) = \min\left\{\tilde{\rho}(t^i, t^{\pi(j-1)}) : i \in \{1, \ldots, n\} \setminus \{\pi(1), \ldots, \pi(j-1)\}\right\}.$$

Then π is a permutation of $\{0, 1, \ldots, n\}$ with $\pi(0) = 0$ and we have

$$\prod_{j=1}^{n} \min\{\tilde{\rho}(t^j, t^i)^\gamma : 0 \le i \le n, i \ne j\} = \prod_{j=1}^{n} \min\{\tilde{\rho}(t^{\pi(j)}, t^i)^\gamma : 0 \le i \le n, i \ne \pi(j)\}.$$

For $1 \le j \le n$ and $0 \le m \le n$ with $m \ne \pi(j)$, let us define

$$I_{j,m} = \begin{cases} 1 & \text{if } \tilde{\rho}(t^{\pi(j)}, t^m) = \min\{\tilde{\rho}(t^{\pi(j)}, t^i) : 0 \le i \le n, i \ne \pi(j)\}, \\ 0 & \text{otherwise.} \end{cases}$$

For each j, by the definition of π, $\tilde{\rho}(t^{\pi(j)}, t^m) \ge \tilde{\rho}(t^{\pi(j)}, t^{\pi(j+1)})$ for all m such that $\pi^{-1}(m) > j + 1$. This implies

$$\min\{\tilde{\rho}(t^{\pi(j)}, t^i)^\gamma : 0 \le i \le n, i \ne \pi(j)\} \ge \prod_{\substack{0 \le m \le n, \\ m \ne \pi(j), \\ \pi^{-1}(m) \le j+1}} \tilde{\rho}(t^{\pi(j)}, t^m)^{I_{j,m}\gamma}.$$

Indeed, if there is a unique m such that $\tilde{\rho}(t^{\pi(j)}, t^m) = \min\{\tilde{\rho}(t^{\pi(j)}, t^i) : 0 \le i \le n, i \ne \pi(j)\}$, then the equality holds; if there are more than one m such that the minimum is attained (i.e. $I_{j,m} = 1$ for more than one m), then we get the above inequality by the condition that

$\tilde{\rho}(t^i, t^j) \leq 1$ for all i, j. It follows that

$$\prod_{j=1}^{n} \min\{\tilde{\rho}(t^j, t^i)^\gamma : 0 \leq i \leq n, i \neq j\} \geq \prod_{1 \leq j \leq n} \prod_{\substack{0 \leq m \leq n, \\ m \neq \pi(j), \\ \pi^{-1}(m) \leq j+1}} \tilde{\rho}(t^{\pi(j)}, t^m)^{I_{j,m}\gamma}$$

$$= \prod_{0 \leq m \leq n} \prod_{\substack{1 \leq j \leq n, \\ \pi(j) \neq m, \\ \pi^{-1}(m) \leq j+1}} \tilde{\rho}(t^{\pi(j)}, t^m)^{I_{j,m}\gamma}.$$

Putting $m = \pi(\ell)$ with $0 \leq \ell \leq n$, we have

$$\prod_{j=1}^{n} \min\{\tilde{\rho}(t^j, t^i)^\gamma : 0 \leq i \leq n, i \neq j\} \geq \prod_{0 \leq \ell \leq n} \prod_{\substack{1 \leq j \leq n, \\ j \neq \ell, \, j+1 \geq \ell}} \tilde{\rho}(t^{\pi(j)}, t^{\pi(\ell)})^{I_{j,\pi(\ell)}\gamma}.$$

By the definition of π, for $0 \leq \ell \leq n$ and $1 \leq j \leq n$ with $j \neq \ell$, $j \geq \ell - 1$, we have

$$\tilde{\rho}(t^{\pi(j)}, t^{\pi(\ell)}) \geq \min\left\{\tilde{\rho}(t^{\pi(\ell-1)}, t^{\pi(\ell)}), \tilde{\rho}(t^{\pi(\ell+1)}, t^{\pi(\ell)})\right\},$$

with the notation $\pi(-1) := \pi(1)$ and $\pi(n+1) := \pi(n-1)$. Then

$$\begin{aligned}
&\prod_{j=1}^{n} \min\{\tilde{\rho}(t^j, t^i)^\gamma : 0 \leq i \leq n, i \neq j\} \\
&\geq \prod_{0 \leq \ell \leq n} \prod_{\substack{1 \leq j \leq n, \\ j \neq \ell, \, j+1 \geq \ell}} \left(\min\left\{\tilde{\rho}(t^{\pi(\ell-1)}, t^{\pi(\ell)}), \tilde{\rho}(t^{\pi(\ell+1)}, t^{\pi(\ell)})\right\}\right)^{I_{j,\pi(\ell)}\gamma} \\
&= \prod_{0 \leq \ell \leq n} \left(\min\left\{\tilde{\rho}(t^{\pi(\ell-1)}, t^{\pi(\ell)}), \tilde{\rho}(t^{\pi(\ell+1)}, t^{\pi(\ell)})\right\}\right)^{K_\ell \gamma},
\end{aligned}$$

where

$$K_\ell = \sum_{\substack{1 \le j \le n, \\ j \neq \ell, j+1 \ge \ell}} I_{j,\pi(\ell)}.$$

For each fixed ℓ, we have $\sum_{j:\, j\neq \ell} I_{j,\pi(\ell)} \le K(N) = K$ by Lemma 4.2.6. Hence $K_\ell \le K$ and

$$\Big(\min\big\{\tilde\rho(t^{\pi(\ell-1)}, t^{\pi(\ell)}), \tilde\rho(t^{\pi(\ell+1)}, t^{\pi(\ell)})\big\}\Big)^{K_\ell \gamma} \ge \tilde\rho(t^{\pi(\ell-1)}, t^{\pi(\ell)})^{K\gamma} \tilde\rho(t^{\pi(\ell+1)}, t^{\pi(\ell)})^{K\gamma}.$$

Therefore, we get that

$$\prod_{j=1}^n \min\big\{\tilde\rho(t^j, t^i)^\gamma : 0 \le i \le n, i \neq j\big\} \ge \prod_{1 \le \ell \le n} \tilde\rho(t^{\pi(\ell)}, t^{\pi(\ell-1)})^{2K\gamma}.$$

This proves the lemma. □

Now, we prove a moment estimate for the increments of the local time.

Proposition 4.2.8. *Suppose* (4.1) *and* (4.2) *hold on the closed bounded cube $T \subset \mathbb{R}^N$. Suppose $d < Q$, where $Q = \sum_{j=1}^N H_j^{-1}$. Then there is a positive integer K depending only on N, and there is a positive finite constant C depending only on T, N, d and H such that for all small $0 < \gamma < 1$, for all Borel set $S \subseteq T$, for all $x, y \in \mathbb{R}^d$, for all even integers $n \ge 2$, we have*

$$\mathbb{E}[(L(x,S) - L(y,S))^n] \le C^n |x-y|^{n\gamma} (n!)^{d/Q + (1+2K/Q)\gamma} \lambda_N(S)^{n(1-(d+2K\gamma)/Q)}.$$

In particular, for all small $0 < \gamma < 1$, for all $a \in \mathbb{R}^N$ and $0 < r < 1$ with $D := B_\rho(a, r) \subseteq T$, for all $x, y \in \mathbb{R}^d$, for all even integers $n \ge 2$, we have

$$\mathbb{E}[(L(x,D) - L(y,D))^n] \le C^n |x-y|^{n\gamma} (n!)^{d/Q + (1+2K/Q)\gamma} r^{n(Q-d-2K\gamma)}.$$

Proof. Recall that for any even integer $n \geq 2$ and $x, y \in \mathbb{R}^d$,

$$\begin{aligned}
&\mathbb{E}[(L(x,S) - L(y,S))^n] \\
&= (2\pi)^{-nd} \int_{S^n} d\bar{t} \int_{\mathbb{R}^{nd}} d\bar{u} \prod_{j=1}^{n} \left[e^{-i\langle u^j, x\rangle} - e^{-i\langle u^j, y\rangle} \right] \mathbb{E}\left[e^{i\sum_{l=1}^{n}\langle u^l, X(t^l)\rangle} \right],
\end{aligned} \tag{4.12}$$

where $\bar{u} = (u^1, \ldots, u^n) \in \mathbb{R}^{nd}$ and $\bar{t} = (t^1, \ldots, t^n) \in S^n$. Note that

$$\mathbb{E}\left[e^{i\sum_{l=1}^{n}\langle u^l, X(t^l)\rangle} \right] = \exp\left[-\frac{1}{2} \sum_{k=1}^{d} \mathrm{Var}\Big(\sum_{l=1}^{n} u_k^l Y(t^l) \Big) \right] \tag{4.13}$$

for all $u^1, \ldots, u^n \in \mathbb{R}^d$ and $t^1, \ldots, t^n \in S$. For any $0 < \gamma < 1$, we have $|e^{iu} - 1| \leq 2^{1-\gamma}|u|^\gamma$ and $|u+v|^\gamma \leq |u|^\gamma + |v|^\gamma$ for all $u, v \in \mathbb{R}$. It follows that

$$\prod_{j=1}^{n} \left| e^{-i\langle u^j, x\rangle} - e^{-i\langle u^j, y\rangle} \right| \leq 2^{(1-\gamma)n} |x-y|^{n\gamma} \sum_{\bar{k}} \prod_{j=1}^{n} |u_{k_j}^j|^\gamma \tag{4.14}$$

for all $u^1, \ldots, u^n, x, y \in \mathbb{R}^d$, where the summation is taken over all $\bar{k} = (k_1, \ldots, k_n) \in \{1, \ldots, d\}^n$. Then (4.12), (4.13) and (4.14) imply that

$$\mathbb{E}[(L(x,S) - L(y,S))^n] \leq (2\pi)^{-nd}\, 2^n |x-y|^{n\gamma} \sum_{\bar{k}} \int_{S^n} J(\bar{t}, \bar{k})\, d\bar{t}, \tag{4.15}$$

where

$$J(\bar{t}, \bar{k}) = \int_{\mathbb{R}^{nd}} \Big(\prod_{j=1}^{n} |u_{k_j}^j|^\gamma \Big) \exp\left[-\frac{1}{2} \sum_{k=1}^{d} \mathrm{Var}\Big(\sum_{l=1}^{n} u_k^l Y(t^l) \Big) \right] d\bar{u}$$

for $\bar{t} = (t^1, \ldots, t^n) \in S^n$ and $\bar{k} = (k_1, \ldots, k_n) \in \{1, \ldots, d\}^n$. By the generalized Hölder inequality,

$$J(\bar{t},\bar{k}) \le \prod_{j=1}^{n} \left\{ \int_{\mathbb{R}^{nd}} |u_{k_j}^j|^{n\gamma} \exp\left[-\frac{1}{2}\sum_{k=1}^{d} \mathrm{Var}\Big(\sum_{l=1}^{n} u_k^l Y(t^l)\Big)\right] d\bar{u} \right\}^{1/n}.$$

If we fix $\bar{t}$, $\bar{k}$ and j, then by Lemma 4.2.4, we have

$$\begin{aligned}
&\int_{\mathbb{R}^{nd}} |u_{k_j}^j|^{n\gamma} \exp\left[-\frac{1}{2}\sum_{k=1}^{d} \mathrm{Var}\Big(\sum_{l=1}^{n} u_k^l Y(t^l)\Big)\right] d\bar{u} \\
&= \frac{(2\pi)^{(nd-1)/2}}{\det \mathrm{Cov}(Y(t^1),\dots,Y(t^n))^{d/2}} \int_{\mathbb{R}} \left|\frac{v}{\sigma_j}\right|^{n\gamma} e^{-v^2/2} dv,
\end{aligned}$$

where $\sigma_j = \mathrm{Var}(Y(t^j)|Y(t^l) : 1 \le l \le n, l \ne j)$. By Jensen's inequality and the moments of the standard Gaussian,

$$\int_{\mathbb{R}} |v|^{n\gamma} e^{-v^2/2} dv \le \sqrt{2\pi}\left(\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} |v|^n e^{-v^2/2} dv\right)^{\gamma} \le \sqrt{2\pi}(n!)^{\gamma}.$$

It follows that

$$J(\bar{t},\bar{k}) \le \frac{C^n (n!)^{\gamma}}{\det \mathrm{Cov}(Y(t^1),\dots,Y(t^n))^{d/2}} \prod_{j=1}^{n} \frac{1}{\sigma_j^{\gamma}} \tag{4.16}$$

for some C depending on d. By (4.10),

$$\begin{aligned}
\prod_{j=1}^{n} \frac{1}{\sigma_j^{\gamma}} &\le \prod_{j=1}^{n} \frac{1}{C_3^{\gamma/2} \min\{\tilde{\rho}(t^j,t^l)^{\gamma} : 0 \le l \le n, l \ne j\}} \\
&\le (C_3 \wedge 1)^{-n/2} \prod_{j=1}^{n} \frac{1}{\min\{\tilde{\rho}(t^j,t^l)^{\gamma} : 0 \le l \le n, l \ne j\}}.
\end{aligned}$$

Then by Lemma 4.2.7 and (4.9),

$$\prod_{j=1}^{n} \frac{1}{\sigma_j^{\gamma}} \leq C^n \prod_{j=1}^{n} \frac{1}{\rho(t^{\pi(j)}, t^{\pi(j-1)})^{2K\gamma}}$$

where K is a positive integer that depends only on N, and C is a positive finite constant that depends on T, N and H. By assumption (4.1),

$$\mathrm{Var}(Y(t^{\pi(j)})|Y(t^{\pi(l)}) : l = 1, \ldots, j-1) \leq C_2 \rho(t^{\pi(j)}, t^{\pi(j-1)})^2.$$

Hence

$$\begin{aligned} \prod_{j=1}^{n} \frac{1}{\sigma_j^{\gamma}} &\leq C^n \prod_{j=1}^{n} \frac{C_2^{K\gamma}}{\mathrm{Var}(Y(t^{\pi(j)})|Y(t^{\pi(l)}) : l = 1, \ldots, j-1)^{K\gamma}} \\ &= C^n \frac{(C_2 \vee 1)^{Kn}}{\det \mathrm{Cov}(Y(t^1), \ldots, Y(t^n))^{K\gamma}}. \end{aligned} \tag{4.17}$$

Then (4.16), (4.17) and assumption (4.2) imply that

$$\begin{aligned} J(\bar{t}, \bar{k}) &\leq \frac{C^n (n!)^{\gamma}}{\det \mathrm{Cov}(Y(t^1), \ldots, Y(t^n))^{d/2 + K\gamma}} \\ &\leq C^n (n!)^{\gamma} \prod_{j=1}^{n} \left[\min_{0 \leq l \leq j-1} \rho(t^j, t^l) \right]^{-d-2K\gamma} \end{aligned} \tag{4.18}$$

where C depends on T, N, H and d. Since $d < Q$, we can choose and fix some $d < \beta_0 < Q$. If $0 < \gamma < 1$ is chosen small enough such that

$$d < d + 2K\gamma \leq \beta_0 < Q,$$

then by Lemma 4.2.2 with $\beta = d + 2K\gamma$, we have

$$\int_{S^n} \prod_{j=1}^{n} \left[\min_{0 \le l \le j-1} \rho(t^j, t^l) \right]^{-d-2K\gamma} d\bar{t} \le C^n (n!)^{(d+2K\gamma)/Q} \lambda_N(S)^{n(1-(d-2K\gamma)/Q)},$$

which gives

$$\int_{S^n} J(\bar{t}, \bar{k})\, d\bar{t} \le C^n (n!)^{d/Q+(1+2K/Q)\gamma} \lambda_N(S)^{n(1-(d+2K\gamma)/Q)}. \tag{4.19}$$

Note that this bound does not depend on $\bar{k}$. Combining (4.15), (4.18) and (4.19), we have

$$\mathbb{E}[(L(x,S) - L(y,S))^n] \le C^n |x-y|^{n\gamma} (n!)^{d/Q+(1+2K/Q)\gamma} \lambda_N(S)^{n(1-(d+2K\gamma)/Q)}.$$

This completes the proof of Proposition 4.2.8. $\square$

Now, we turn to the proof of Theorem 4.2.1.

Proof of Theorem 4.2.1. By Proposition 4.2.8 and the multiparameter version of Kolmogorov's continuity theorem (see e.g. [29], Theorem 4.3), for any anisotropic ball $B_\rho(a,r) \subset T$, X has a local time $L(x, B_\rho(a,r))$ that is continuous in x.

Next, we prove the joint continuity. Let $T = \prod_{j=1}^N [\tau_j, \tau_j + h_j]$. For simplicity, denote $[\tau, \tau + s] = \prod_{j=1}^N [\tau_j, \tau_j + s_j]$ for any $s_1, \ldots, s_N \ge 0$. For all $x, y \in \mathbb{R}^d$, $s, t \in \prod_{j=1}^N [0, h_j]$ and all even integers $n \ge 2$, we have

$$\begin{aligned}
&\mathbb{E}[(L(x,[\tau,\tau+s]) - L(y,[\tau,\tau+t]))^n] \\
&\le 2^{n-1} \big\{ \mathbb{E}[(L(x,[\tau,\tau+s]) - L(y,[\tau,\tau+s]))^n] + \mathbb{E}[(L(y,[\tau,\tau+s]) - L(y,[\tau,\tau+t]))^n] \big\}.
\end{aligned}$$

By Proposition 4.2.8, we can find some small $\gamma > 0$ such that the first term is

$$\mathbb{E}[(L(x,[\tau,\tau+s]) - L(y,[\tau,\tau+s]))^n] \le C|x-y|^{n\gamma}.$$

For the second term, by considering the symmetric difference of the cubes $[\tau,\tau+s]$ and $[\tau,\tau+t]$, we see that $L(y,[\tau,\tau+s]) - L(y,[\tau,\tau+t])$ can be written as a sum of M terms of the form $\pm L(y,T_i)$, where M is a finite number depending only on N and T_i is a closed bounded cube in $T = [\tau,\tau+h]$ with at least one edge length $\le |s-t|$. Then by Proposition 4.2.3, the second term is

$$\begin{aligned}
\mathbb{E}[(L(y,[\tau,\tau+s]) - L(y,[\tau,\tau+t]))^n] &\le M^{n-1}\sum_{i=1}^{M}\mathbb{E}[L(y,T_i)^n] \\
&\le M^n C^n (n!)^{d/Q}\lambda_N(T_i)^{n(1-d/Q)} \\
&\le C|s-t|^{n(1-d/Q)}.
\end{aligned}$$

Combining the two terms, we have

$$\mathbb{E}[(L(x,[\tau,\tau+s]) - L(y,[\tau,\tau+t]))^n] \le C(|x-y|^{n\beta} + |s-t|^{n\beta})$$

for some small $\beta > 0$. Therefore, by Kolmogorov's continuity theorem, X has a jointly continuous local time on T. □

4.3 Hölder conditions of Local Times

In the previous section, we derive joint continuity for the local time. In fact, we can also derive a Hölder condition for the local time.

Theorem 4.3.1. *Suppose X satisfies (4.1) and (4.2) on a closed bounded cube $T \subset \mathbb{R}^N$. Suppose $d < Q$ where $Q = \sum_{j=1}^N H_j^{-1}$. For any $x \in \mathbb{R}^d$, let $L(x, \cdot)$ be a (joint continuous) local time of X, a random measure on T that is supported on the level set $X^{-1}(x) \cap T = \{t \in T : X(t) = x\}$. Then there exists a positive finite constant C such that for any $x \in \mathbb{R}^d$, with probability 1, for $L(x, \cdot)$-almost every $t \in T$,*

$$\limsup_{r \to 0+} \frac{L(x, B_\rho(t, r) \cap T)}{\varphi(r)} \le C, \tag{4.20}$$

where $\varphi(r) = r^{Q-d}(\log\log(1/r))^{d/Q}$.

Proof. For any $x \in \mathbb{R}^d$ and any integer $k \ge 1$, consider the random measure $L_k(x, \cdot)$ on Borel subsets C of T defined by

$$\begin{aligned} L_k(x, C) &= \int_C \left(\frac{k}{2\pi}\right)^{d/2} \exp\left(-\frac{k|X(t) - x|^2}{2}\right) dt \\ &= \int_C \int_{\mathbb{R}^d} \frac{1}{(2\pi)^d} \exp\left(-\frac{|u|^2}{2k} + i\langle u, X(t) - x\rangle\right) du\, dt. \end{aligned} \tag{4.21}$$

By the occupation density formula (4.3) and the continuity of $y \mapsto L(y, C)$ for all rectangles C in T, one can verify that a.s. for all C, $L_k(x, C) \to L(x, C)$ as $k \to \infty$. It follows that a.s., $L_k(x, \cdot)$ converges weakly to $L(x, \cdot)$.

For each $m \ge 1$, define $f_m(t) = L(x, B_\rho(t, 2^{-m}))$. By Proposition 4.2.3, 4.2.8 and the multiparameter version of Kolmogorov's continuity theorem [29], $f_m(t)$ is a.s. bounded and continuous on T. Then by the a.s. weak convergence of $L_k(x, \cdot)$, for all $m, n \ge 1$,

$$\int_T [f_m(t)]^n L(x, dt) = \lim_{k \to \infty} \int_T [f_m(t)]^n L_k(x, dt) \quad \text{a.s.}$$

Hence, by the dominated convergence theorem, (4.21) and (4.4), we have

$$
\begin{aligned}
&\mathbb{E}\int_T [L(x, B_\rho(t, 2^{-m}))]^n L(x, dt) \\
&= \frac{1}{(2\pi)^d} \lim_{k\to\infty} \mathbb{E}\int_T dt \int_{\mathbb{R}^d} du \, \exp\Big(- \frac{|u|^2}{2k} + i\langle u, X(t) - x\rangle\Big) [L(x, B(t, 2^{-m}))]^n \\
&= \frac{1}{(2\pi)^d} \int_T dt \int_{\mathbb{R}^d} du \, e^{i\langle u, X(t)-x\rangle} \mathbb{E}[L(x, B(t, 2^{-m}))^n] \\
&= \frac{1}{(2\pi)^{(n+1)d}} \int_T \int_{B_\rho(t,2^{-m})^n} d\bar{s} \int_{\mathbb{R}^{(n+1)d}} d\bar{u} \, e^{-i\sum_{\ell=1}^{n+1}\langle x, u^\ell\rangle} \mathbb{E}\Big(e^{i\sum_{\ell=1}^{n+1}\langle u^\ell, X(s^\ell)\rangle}\Big),
\end{aligned}
$$

where $\bar{u} = (u^1, \ldots, u^{n+1}) \in \mathbb{R}^{(n+1)d}$, $\bar{s} = (t, s^2, \ldots, s^{n+1})$ and $s^1 = t$. Similar to the proof of Proposition 4.2.3, we can deduce that

$$
\begin{aligned}
&\mathbb{E}\int_T [L(x, B_\rho(t, 2^{-m}))]^n L(x, dt) \\
&\le C^n \int_{T\times B_\rho(t,2^{-m})^n} \frac{d\bar{s}}{[\det \mathrm{Cov}(Y(t), Y(s^2), \ldots, Y(s^{n+1}))]^{d/2}} \\
&\le C^n (n!)^{d/Q} 2^{-nm(Q-d)}.
\end{aligned}
\tag{4.22}
$$

Let $A > 0$ be a constant to be determined. Consider the random set

$$
B_m = \{t \in T : L(x, B_\rho(t, 2^{-m})) > A\varphi(2^{-m})\}.
$$

Consider the random measure μ on T defined by $\mu(B) = L(x, B)$ for any $B \in \mathscr{B}(T)$. Take $n = \lfloor \log m \rfloor$, the integer part of $\log m$. Then by (4.22) and Stirling's formula,

$$
\begin{aligned}
\mathbb{E}\,\mu(B_m) &\le \frac{\mathbb{E}\int_T [L(x, B_\rho(t, 2^{-m}))]^n L(x, dt)}{[A\varphi(2^{-m})]^n} \\
&\le \frac{C^n (n!)^{d/Q} 2^{-nm(Q-d)}}{A^n 2^{-nm(Q-d)} (\log m)^{nd/Q}} \le m^{-2}
\end{aligned}
$$

provided $A > 0$ is chosen large enough. This implies that

$$\mathbb{E} \sum_{m=1}^{\infty} \mu(B_m) < \infty.$$

By the Borel–Cantelli lemma, with probability 1, for μ-a.e. $t \in T$, we have

$$\limsup_{m\to\infty} \frac{L(x, B_\rho(t, 2^{-m}))}{\varphi(2^{-m})} \leq A. \tag{4.23}$$

For any $r > 0$ small enough, there exists an integer m such that $2^{-m} \leq r < 2^{-m+1}$ and (4.23) can be applied. Since $\varphi(r)$ is increasing near $r = 0$, we can use a monotonicity argument to obtain (4.33). □

4.4 Hausdorff Measure of Level Sets

Let us consider the class $\mathscr{C}$ of functions $\varphi : [0, \delta_0] \to \mathbb{R}_+$ such that φ is nondecreasing, continuous, $\varphi(0) = 0$, and satisfies the doubling condition, i.e. there exists a positive finite constant c_0 such that

$$\frac{\varphi(2s)}{\varphi(s)} \leq c_0 \tag{4.24}$$

for all $s \in (0, \delta_0/2)$.

For any Borel set A in $\mathbb{R}^N$, the *Hausdorff measure* of A with respect to the function $\varphi \in \mathscr{C}$ in the metric ρ is defined by

$$\mathcal{H}_\rho^\varphi(A) = \lim_{\delta\to 0+} \inf \left\{ \sum_{n=1}^{\infty} \varphi(2r_n) : A \subseteq \bigcup_{n=1}^{\infty} B_\rho(t^n, r_n) \text{ where } t^n \in \mathbb{R}^N,\ r_n \leq \delta \text{ for all } n \right\}.$$

When $\varphi(s) = s^\beta$, where β is a positive real number, $\mathcal{H}_\rho^\varphi(A)$ is called the β-dimensional

Hausdorff measure of A in the metric ρ, and the Hausdorff dimension of A in the metric ρ is defined as

$$\dim_H^\rho(A) = \inf\{\beta > 0 : \mathcal{H}_\rho^\varphi(A) = 0\}.$$

Suppose X satisfies (4.1) and (4.2) on a closed bounded cube $T \subset \mathbb{R}^N$. Let $Q = \sum_{j=1}^N H_j^{-1}$ and consider the level set $X^{-1}(x) \cap T = \{t \in T : X(t) = x\}$. By Theorem 7.1 of [70], if $d < Q$, then $X^{-1}(x) \cap T = \varnothing$ a.s.; if $Q > d$, then the Hausdorff dimension of $X^{-1}(x) \cap T$ in the Euclidean metric is (assuming that $0 < H_1 \le H_2 \le \cdots \le H_N < 1$)

$$\dim_H(X^{-1}(x) \cap T) = \min\left\{ \sum_{j=1}^k \frac{H_k}{H_j} + N - k - H_k d : 1 \le k \le N \right\} \tag{4.25}$$

which is also equal to

$$\sum_{j=1}^\tau \frac{H_\tau}{H_j} + N - \tau - H_\tau d$$

where τ is the unique integer between 1 and N such that $\sum_{j=1}^{\tau-1} H_j^{-1} \le d < \sum_{j=1}^\tau H_j^{-1}$. On the other hand, Theorem 4.2 of [66] implies that if $d < Q$, then the Hausdorff dimension of $X^{-1}(x) \cap T$ in the metric ρ is

$$\dim_H^\rho(X^{-1}(x) \cap T) = Q - d.$$

It would be interesting to determine the exact *gauge function* for the Hausdorff measure of the level set, that is, to find a function φ such that

$$0 < \mathcal{H}_\rho^\varphi(X^{-1}(x) \cap T) < \infty \quad \text{a.s.}$$

The following theorem is a partial result that gives a lower bound for the Hausdorff measure.

Theorem 4.4.1. *Suppose X satisfies* (4.1) *and* (4.2) *on a closed bounded cube $T \subset \mathbb{R}^N$. Suppose $d < Q$ where $Q = \sum_{j=1}^N H_j^{-1}$. Then there is a positive finite constant C such that*

$$CL(x,T) \le \mathcal{H}_\rho^\varphi(X^{-1}(x) \cap T) \quad a.s.,$$

where $\varphi(r) = r^{Q-d}(\log\log(1/r))^{d/Q}$.

Proof. Recall that there exists a positive constant $c \ge 1$ depending only on c_0 in (4.24) such that for any finite Borel measure μ on $\mathbb{R}^N$ and any Borel set $E \subset \mathbb{R}^N$,

$$c^{-1}\mathcal{H}_\rho^\varphi(E) \inf_{t\in E} \overline{D}_\mu^{\varphi,\rho}(t) \le \mu(E) \le c\,\mathcal{H}_\rho^\varphi(E) \sup_{t\in E} \overline{D}_\mu^{\varphi,\rho}(t) \tag{4.26}$$

where

$$\overline{D}_\mu^{\varphi,\rho}(t) := \limsup_{r\to 0+} \frac{\mu(B_\rho(t,r))}{\varphi(r)}$$

is called the ρ-upper φ-density of μ at the point t (see Theorem 4.1 of [66]). We can take $\mu = L(x, \cdot \cap T)$, which is a.s. a finite Borel measure on $\mathbb{R}^N$ supported on $X^{-1}(x) \cap T$. Then by Theorem 4.3.1, there exists a positive finite constant C such that

$$\sup_{t\in E} \overline{D}_\mu^{\varphi,\rho}(t) \le C \quad \text{a.s.}$$

This and the upper bound of (4.26) with $E = X^{-1}(x) \cap T$ yields the desired result. $\square$

4.5 Stochastic Heat Equation and Strong Local Non-determinism

Consider the system of stochastic heat equations

$$\begin{cases} \frac{\partial}{\partial t}u_j(t,x) - \Delta u_j(t,x) = \dot{W}_j(t,x), & t \geq 0, x \in \mathbb{R}^N, \\ u_j(0,x) = 0, & j = 1, \dots, d, \end{cases}$$

where $\dot{W}_j$, $j = 1, \dots, d$, are i.i.d. Gaussian noises that are white in time and colored in space with covariance

$$\mathbb{E}[\dot{W}_j(t,x)\dot{W}_j(s,y)] = \delta_0(t-s)|x-y|^{-\beta}$$

where $0 < \beta < 2 \wedge N$. Let $u(t,x) = (u_1(t,x), \dots, u_d(t,x))$. Then $\{u(t,x) : t \geq 0, x \in \mathbb{R}^N\}$ is a $(1+N,d)$-Gaussian random field and $u_1, \dots, u_d$ are i.i.d.

Recall that for any $0 < a < b < \infty$, there exist positive finite constants C_1, C_2 such that

$$C_1\rho((t,x),(s,y)) \leq \mathbb{E}[(u_1(t,x) - u_1(s,y))^2]^{1/2} \leq C_2\rho((t,x),(s,y)) \tag{4.27}$$

for all $(t,x),(s,y) \in [a,b] \times [-b,b]^N$, where $\rho((t,x),(s,y)) = |t-s|^{(2-\beta)/4} + |x-y|^{(2-\beta)/2}$. See e.g. Lemma 4.2 of [14]. It shows that u satisfies condition (4.1) on any closed bounded cube in $(0,\infty) \times \mathbb{R}^N$.

Recall Section 3.4.2. We may assume that $u_1(t,x)$ has the following representation:

$$u_1(t,x) = \int_{\mathbb{R}} \int_{\mathbb{R}^N} e^{-i\langle \xi, x\rangle} \frac{e^{-i\tau t} - e^{-t|\xi|^2}}{|\xi|^2 - i\tau} |\xi|^{-(N-\beta)/2}\, \tilde{W}(d\tau\, d\xi), \tag{4.28}$$

where $\tilde{W}$ is a $\mathbb{C}$-valued space-time Gaussian white noise on $\mathbb{R}^{1+N}$. The following proposition shows that u also satisfies the condition (4.2) of strong local nondeterminism.

Proposition 4.5.1. *For any $0 < a < b < \infty$, there exists a positive finite constant C such that for all integers $n \geq 1$, for all $(t, x), (t^1, x^1), \ldots, (t^n, x^n) \in [a, b] \times [-b, b]^N$,*

$$\mathrm{Var}(u_1(t, x)|u_1(t^1, x^1), \ldots, u_1(t^n, x^n)) \geq C \min_{1 \leq i \leq n} \rho((t, x), (t^i, x^i))^2. \tag{4.29}$$

Proof. Since u is Gaussian, the conditional variance in (4.29) is the squared L^2-distance of $u_1(t, x)$ from the linear subspace of $L^2(\mathbb{P})$ spanned by $u_1(t^1, x^1), \ldots, u_1(t^n, x^n)$, that is,

$$\mathrm{Var}(u_1(t, x)|u_1(t^1, x^1), \ldots, u_1(t^n, x^n)) = \inf_{a_1, \ldots, a_n \in \mathbb{R}} \mathbb{E}\left[\left(u_1(t, x) - \sum_{j=1}^n a_j u_1(t^j, x^j)\right)^2\right].$$

Therefore, it suffices to show that there exists a positive constant C such that

$$\mathbb{E}\left[\left(u_1(t, x) - \sum_{j=1}^n a_j u_1(t^j, x^j)\right)^2\right] \geq C r^{2-\beta},$$

for any $n \geq 1$, any $(t, x), (t^1, x^1), \ldots, (t^n, x^n) \in [a, a'] \times [-b, b]^N$, and any $a_1, \ldots, a_n \in \mathbb{R}$, where

$$r = \min_{1 \leq j \leq n} (|t - t^j|^{1/2} \vee |x - x^j|).$$

From (4.28), we have

$$\begin{aligned} &\mathbb{E}\left[\left(u_1(t, x) - \sum_{j=1}^n a_j u_1(t^j, x^j)\right)^2\right] \qquad (4.30) \\ &\geq C \int_{\mathbb{R}} d\tau \int_{\mathbb{R}^N} d\xi \left| e^{-i\langle \xi, x \rangle}(e^{-i\tau t} - e^{-t|\xi|^2}) - \sum_{j=1}^n a_j e^{-i\langle \xi, x^j \rangle}(e^{-i\tau t^j} - e^{-t^j|\xi|^2}) \right|^2 \frac{|\xi|^{\beta - N}}{|\xi|^4 + |\tau|^2}. \end{aligned}$$

Let M be a positive finite constant depending on a' and b such that $|t-t'|^{1/2} \vee |x-x'| \leq M$ for all $(t,x),(t',x') \in [a,a'] \times [-b,b]^N$. Let $\rho = \min\{a/M^2, 1\}$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R}^N \to \mathbb{R}$ be nonnegative smooth bump functions that vanish outside $[-\rho,\rho]$ and the unit ball respectively and satisfy $\varphi(0) = \psi(0) = 1$. Let $\varphi_r(\tau) = r^{-2}\varphi(r^{-2}\tau)$ and $\phi_r(\xi) = r^{-N}\psi(r^{-1}\xi)$. Let us consider the integral

$$I := \int_{\mathbb{R}} d\tau \int_{\mathbb{R}^N} d\xi \Big[e^{-i\langle \xi, x\rangle}(e^{-i\tau t} - e^{-t|\xi|^2}) - \sum_{j=1}^{n} a_j e^{-i\langle \xi, x^j\rangle}(e^{-i\tau t^j} - e^{-t^j|\xi|^2}) \Big] \\ \times e^{i\langle \xi, x\rangle} e^{i\tau t} \widehat{\varphi}_r(\tau)\widehat{\psi}_r(\xi).$$

By inverse Fourier transform, we have

$$I = (2\pi)^{1+N} \Big[\varphi_r(0)\psi_r(0) - \varphi_r(t)(p_t * \psi_r)(0) \\ - \sum_{j=1}^{n} a_j \Big(\varphi_r(t-t^j)\psi_r(x-x^j) - \varphi_r(t)(p_{t^j} * \psi_r)(x-x^j) \Big) \Big],$$

where $p_t(x)$ is the heat kernel

$$p_t(x) = \frac{1}{(4\pi t)^{N/2}} e^{-|x|^2/(4t)}.$$

By the definition of r, for every j, we have either $|t-t^j| \geq r^2$ or $|x-x^j| \geq r$, thus $\varphi_r(t-t^j)\psi_r(x-x^j) = 0$. Moreover, since $t/r^2 \geq a/M^2 \geq \rho$, we have $\varphi_r(t) = 0$ and hence

$$I = (2\pi)^{1+N} r^{-2-N}. \tag{4.31}$$

On the other hand, by the Cauchy–Schwarz inequality and (4.30),

$$I^2 \le C\,\mathbb{E}\left[\left(u_1(t,x) - \sum_{j=1}^n a_j u_1(t^j,x^j)\right)^2\right] \int_{\mathbb{R}} \int_{\mathbb{R}^N} \left|\widehat{\varphi}_r(\tau)\widehat{\psi}_r(\xi)\right|^2 \left(|\xi|^4 + |\tau|^2\right) |\xi|^{N-\beta} d\tau\, d\xi.$$

Note that $\widehat{\varphi}_r(\tau) = \widehat{\varphi}(r^2\tau)$ and $\widehat{\psi}_r(\xi) = \widehat{\psi}(r\xi)$. Then by a scaling of variables, we have

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{\mathbb{R}^N} \left|\widehat{\varphi}_r(\tau)\widehat{\psi}_r(\xi)\right|^2 \left(|\xi|^4 + |\tau|^2\right) |\xi|^{N-\beta} d\tau\, d\xi \\
&= r^{-6+\beta-2N} \int_{\mathbb{R}} \int_{\mathbb{R}^N} \left|\widehat{\varphi}(\tau)\widehat{\psi}(\xi)\right|^2 \left(|\xi|^4 + |\tau|^2\right) |\xi|^{N-\beta} d\tau\, d\xi,
\end{aligned}$$

where the last integral is finite since $\widehat{\varphi}$ and $\widehat{\psi}$ are Schwartz functions. It follows that

$$I^2 \le C_0 r^{-6+\beta-2N}\, \mathbb{E}\left[\left(u(t,x) - \sum_{j=1}^n a_j u(t^j,x^j)\right)^2\right] \tag{4.32}$$

for some finite constant C_0 (depending on a, a' and b). Combining (4.31) and (4.32), we obtain

$$\mathbb{E}\left[\left(u(t,x) - \sum_{j=1}^n a_j u(t^j,x^j)\right)^2\right] \ge (2\pi)^{2+2N} C_0^{-1} r^{2-\beta}.$$

The proof is complete. □

With (4.27) and Proposition 4.5.1, the following result is a direct consequence of Theorem 4.2.1 and 4.3.1.

Theorem 4.5.2. *Suppose $d < Q$, where $Q = (4+2N)/(2-\beta)$. Let T be any closed bounded cube in $(0,\infty)\times\mathbb{R}^N$. Then $\{u(t,x) : t \ge 0, x \in \mathbb{R}^N\}$ has a jointly continuous local time $L(\cdot, T)$ on T that satisfies the following Hölder condition: there exists a positive finite* *constant C such that for any $x \in \mathbb{R}^d$, with probability 1,*

$$\limsup_{r\to 0+} \frac{L(x, B_\rho(t,r))}{r^{Q-d}(\log\log(1/r))^{d/Q}} \le C \tag{4.33}$$

for $L(x,\cdot)$-almost every $t \in T$.

The theorem below identifies the correct gauge function for the Hausdorff measure (in the metric ρ) of the level sets $u^{-1}(z) = \{(t,x) \in (0,\infty)\times\mathbb{R}^N : u(t,x) = z\}$ of the stochastic heat equation.

Theorem 4.5.3. *Suppose $d < Q := (4+2N)/(2-\beta)$. Then for any $z \in \mathbb{R}^d$ and any closed bounded cube $T \subset (0,\infty)\times\mathbb{R}^N$, there exists a positive finite constant C such that*

$$CL(z,T) \le \mathcal{H}_\rho^\varphi(u^{-1}(z)\cap T) < \infty \quad a.s., \tag{4.34}$$

where $\varphi(r) = r^{Q-d}(\log\log(1/r))^{d/Q}$.

Remark 4.5.4. *We conjecture that there exist positive finite constants C_1 and C_2 such that*

$$C_1 L(z,T) \le \mathcal{H}_\rho^\varphi(u^{-1}(z)\cap T) \le C_2 L(x,T) \quad a.s.$$

We also conjecture that the gauge function for the Hausdorff measure in the Euclidean metric of the level set is $\varphi(r) = r^\beta(\log\log(1/r))^{d/Q}$, where β is the Hausdorff dimension in (4.25).

Proof of Theorem 4.5.3. The lower bound of (4.34) follows immediately from Theorem 4.4.1.

To prove that the Hausdorff measure is finite, we use the method in [67], which is similar to Talagrand's covering argument in [61] and Chapter 3 of this thesis. To this end, note that we may assume $T = B((t_0, x_0), \eta_0)$, where $\eta_0 > 0$ small and $(t_0, x_0) \in T$ are fixed. Let

$$\begin{aligned} u^1(t,x) &= u(t,x) - u^2(t,x), \\ u^2(t,x) &= \mathbb{E}(u(t,x)|u(t_0,x_0)) \end{aligned}$$

Note that u^1 and u^2 are independent.

By Proposition 3.3.6, there exists $\eta_1 > 0$ small such that for all $0 < r_0 < \eta_1$, and all $(t,x) \in T$, we have

$$\begin{aligned} \mathbb{P}\Bigg(\exists\, r \in [r_0^2, r_0], \sup_{(s,y)\in B_\rho((t,x),2c_2 r)} |u(t,x) - u(s,y)| \le K_1 r\Big(\log\log\frac{1}{r}\Big)^{-1/Q}\Bigg) \\ \ge 1 - \exp\Bigg(-\Big(\log\frac{1}{r_0}\Big)^{1/2}\Bigg). \end{aligned} \tag{4.35}$$

Moreover, recall that Assumption 3.2.2 is satisfied with $\delta_j = 1$ (see [18], Lemma 7.5). Then by Lemma 3.3.7, for all $(t,x), (s,y) \in T$,

$$|u^2(t,x) - u^2(s,y)| \le K_2\big(|t-s| + \sum_{j=1}^{N} |x_j - y_j|\big)|u(t_0,x_0)|. \tag{4.36}$$

Let

$$\begin{aligned} R_p = \Bigg\{(t,x) \in T : \exists\, r \in [2^{-2p}, 2^{-p}] \text{ such that} \\ \sup_{(s,y)\in B_\rho((t,x),2c_2 r)} |u(t,x) - u(s,y)| \le K_1 r\Big(\log\log\frac{1}{r}\Big)^{-1/Q}\Bigg\}. \end{aligned}$$

Consider the events

$$\Omega_{p,1} = \Big\{\omega : \lambda_N(R_p) \geq \lambda_N(T)(1 - \exp(-\sqrt{p}/4))\Big\},$$
$$\Omega_{p,2} = \Big\{\omega : |u(t_0, x_0)| \leq 2^{pb}\Big\},$$

where $b > 0$ is chosen and fixed such that $\frac{2}{2-\beta} - b > 1$. By (4.35), $\mathbb{P}((t,x) \in R_p) \geq 1 - \exp(-\sqrt{p/2})$. Then by Fubini's theorem, $\sum_{p=1}^{\infty} \mathbb{P}(\Omega_{p,1}^c) < \infty$. Moreover, it is easy to see that $\sum_{p=1}^{\infty} \mathbb{P}(\Omega_{p,2}^c) < \infty$. Let $\mathscr{Q} = \bigcup_{p=1}^{\infty} \mathscr{Q}_p$ be the family of (generalized) dyadic cubes in T given by Lemma 3.3.9. Consider the event

$$\Omega_{p,3} = \Big\{\omega : \forall\, I \in \mathscr{Q}_{2p}, \sup_{(t,x),(s,y) \in I} |u(t,x) - u(s,y)| \leq K_3 2^{-2p} p^{1/2}\Big\}.$$

It is shown in the proof of Theorem 3.2.4 (Section 3.4) that $\sum_{p=1}^{\infty} \mathbb{P}(\Omega_{p,3}^c) < \infty$ provided K_3 is a large enough constant. Let $\Omega_p = \Omega_{p,1} \cap \Omega_{p,2} \cap \Omega_{p,3}$ and

$$\Omega^* = \bigcup_{\ell \geq 1} \bigcap_{p \geq \ell} \Omega_p.$$

Then Ω^* is an event of probability 1.

We are going to construct a random covering of the level set $u^{-1}(z) \cap T$. For any $p \geq 1$ and $(t,x) \in T$, let $I_p(t,x) \in \mathscr{Q}_p$ be the unique dyadic cube of order p containing (t,x). We say that $I_q(t,x)$ is a good dyadic cube of order q if it satisfies the following property:

$$\sup_{(s,y),(s',y') \in I_q(t,x)} |u^1(s,y) - u^1(s',y')| \leq K_1 2^{-q} (\log\log 2^q)^{-1/Q}. \tag{4.37}$$

For each $(t,x) \in R_p$, since I_q is contained in some ball $B_\rho(c_2 2^{-q})$ by Lemma 3.3.9 (iii), there is a good dyadic cube $I \in \mathscr{Q}$ containing (t, x) of smallest order q, where $p \leq q \leq 2p$. By property (ii) of Lemma 3.3.9, we obtain in this way a family $\mathscr{G}_p^1$ of disjoint dyadic cubes that cover R_p. On the other hand, we cover $T \setminus R_p$ by a family $\mathscr{G}_p^2$ of dyadic cubes in $\mathscr{Q}_{2p}$ of order $2p$ that are not contained in any cube of $\mathscr{G}_p^1$. Let $\mathscr{G}_p = \mathscr{G}_p^1 \cup \mathscr{G}_p^2$. Note that $\mathscr{G}_p$ depends only on the random field $\{u^1(t, x) : (t, x) \in T\}$.

For any dyadic cube $I \in \mathscr{Q}$, choose a distinguished point $(t_I, x_I) \in I \cap T$. Fix $p \geq 1$. For any $I \in \mathscr{Q}_q$ of order q, where $p \leq q \leq 2p$, consider the event

$$\Omega_{p,I} = \{\omega : |u(t_I, x_I) - z| \leq 2r_{p,I}\}$$

where

$$r_{p,I} = \begin{cases} K_1 2^{-q} (\log\log 2^q)^{-1/Q} & \text{if } I \in \mathscr{G}_p^1, \\ K_1 2^{-2p} p^{1/2} & \text{if } I \in \mathscr{G}_p^2. \end{cases}$$

Let $\mathscr{F}_p$ be the subcover of $\mathscr{G}_p$ (depending on ω) defined by

$$\mathscr{F}_p(\omega) = \{I \in \mathscr{G}_p(\omega) : \omega \in \Omega_{p,I}\}.$$

We claim that for p large, on the event Ω_p, $\mathscr{F}_p$ covers the set $u^{-1}(z) \cap T$. Suppose Ω_p occurs and $(t, x) \in u^{-1}(z) \cap T$. Since $\mathscr{G}_p$ covers T, the point (t, x) is contained in some dyadic cube I and either $I \in \mathscr{G}_p^1$ or $I \in \mathscr{G}_p^2$.

Case 1: if $I \in \mathscr{G}_p^1$, then $I = I_q(t, x)$ is a good dyadic cube of order q, where $p \leq q \leq 2p$, and (4.37) holds. Note that I is contained in some ball $B_\rho(c_2 2^{-q})$ by Lemma 3.3.9 (iii).

Since $\Omega_{p,2}$ occurs, it follows that from (4.36) and (4.37) that

$$
\begin{aligned}
|u(t_I, x_I) - z| &\le |u^1(t_I, x_I) - u^1(t, x)| + |u^2(t_I, x_I) - u^2(t, x)| \\
&\le K_1 2^{-q} (\log\log 2^q)^{-1/Q} + K_2 \Big((2c_2^{\frac{4}{2-\beta}} + 2Nc_2^{\frac{2}{2-\beta}}) 2^{-q\frac{2}{2-\beta}} \Big) 2^{pb}.
\end{aligned}
$$

This is $\le 2r_{p,I}$ for p large because b is chosen such that $\frac{2}{2-\beta} - b > 1$. Hence $I \in \mathscr{F}_p$.

Case 2: if $I \in \mathscr{G}_p^2$, since $\Omega_{p,3}$ occurs, we have

$$
|u(t_I, x_I) - z| = |u(t_I, x_I) - u(t, x)| \le K_3 2^{-2p} p^{1/2}.
$$

In this case, $I \in \mathscr{F}_p$. Hence the claim is proved.

Let Σ_1 be the σ-field generated by $\{u^1(t, x) : (t, x) \in T\}$. To estimate the conditional probability $\mathbb{P}(\Omega_{p,I} | \Sigma_1)$, note that by (4.27), for all $(t, x) \in T = B_\rho((t_0, x_0), \eta_0)$,

$$
\begin{aligned}
\operatorname{Var}(\mathbb{E}(u(t,x) | u(t_0, x_0))) &= \operatorname{Var}(u(t,x)) - \mathbb{E}[\operatorname{Var}(u(t,x) | u(t_0, x_0))] \\
&\ge \inf_{(t,x)\in T} \operatorname{Var}(u(t,x)) - C_2 \sup_{(t,x)\in T} \rho((t,x), (t_0, x_0))^2 \\
&\ge K > 0
\end{aligned}
$$

provided $\eta_0 > 0$ is chosen small enough. Then $\mathbb{P}(|u^2(t,x) - v| \le r) \le K r^d$ for all $(t, x) \in T$, $v \in \mathbb{R}^d$ and $r > 0$. It follows from the independence of u^1 and u^2 that $\mathbb{P}(\Omega_{p,I} | \Sigma_1) \le K r_{p,I}^d$.

Let $\tilde{\Omega}_p$ denote the event that the cardinality of $\mathscr{G}_p^2$ is at most $K 2^{2pQ} \exp(-\sqrt{p}/4)$. Note that $\tilde{\Omega}_p \in \Sigma_1$. Since $T \setminus R_p$ has Lebesgue measure $\le \exp(-\sqrt{p}/4)$ on $\Omega_{p,1}$ and each I of order $2p$ has Lebesgue measure $\sim K 2^{-2pQ}$ by Lemma 3.3.9 (iii), it follows that the event

$\Omega_{p,1}$ is contained in $\tilde{\Omega}_p$. Let $q[I]$ denote the order of $I \in \mathscr{F}_p$. Then

$$\mathbb{E}\left[\mathbf{1}_{\Omega_p} \sum_{I \in \mathscr{F}_p} \varphi(2c_2 2^{-q[I]})\right] \le \mathbb{E}\left[\mathbf{1}_{\tilde{\Omega}_p} \sum_{q=p}^{2p} \sum_{I \in \mathscr{Q}_q} \varphi(2c_2 2^{-q}) \mathbf{1}_{\{I \in \mathscr{G}_p\}} \mathbf{1}_{\Omega_{p,I}}\right]$$
$$= \mathbb{E}\left[\mathbb{E}\left(\mathbf{1}_{\tilde{\Omega}_p} \sum_{q=p}^{2p} \sum_{I \in \mathscr{Q}_q} \varphi(2c_2 2^{-q}) \mathbf{1}_{\{I \in \mathscr{G}_p\}} \mathbf{1}_{\Omega_{p,I}} \middle| \Sigma_1\right)\right]$$

$$= \mathbb{E}\left[\mathbf{1}_{\tilde{\Omega}_p} \sum_{q=p}^{2p} \sum_{I \in \mathscr{Q}_q} \varphi(2c_2 2^{-q}) \mathbf{1}_{\{I \in \mathscr{G}_p\}} \mathbb{E}\Big(\mathbf{1}_{\Omega_{p,I}} | \Sigma_1\Big)\right]$$
$$\le K\mathbb{E}\left[\mathbf{1}_{\tilde{\Omega}_p} \sum_{q=p}^{2p} \sum_{I \in \mathscr{Q}_q} \varphi(2c_2 2^{-q}) r_{p,I}^d \mathbf{1}_{\{I \in \mathscr{G}_p\}}\right].$$

If $I \in \mathscr{G}_p^1$ is of order q, then

$$\varphi(2c_2 2^{-q}) r_{p,I}^d \le K 2^{-q(Q-d)} (\log\log 2^q)^{d/Q} 2^{-qd} (\log\log 2^q)^{-d/Q} \le K\lambda_N(I),$$

and these I's are disjoint sets contained in T. If $I \in \mathscr{G}_p^2$, then

$$\varphi(2c_2 2^{-2p}) r_{p,I}^d \le K 2^{-2pQ} (\log\log 2^{2p})^{d/Q} p^{d/2}$$

and there are at most $K2^{2pQ}\exp(-\sqrt{p}/4)$ many such I's on $\tilde{\Omega}_p$. It follows that

$$\begin{aligned}
\mathbb{E}\left[\mathbf{1}_{\Omega_p}\sum_{I\in\mathscr{F}_p}\varphi(2^{-q[I]})\right] &\le K\mathbb{E}\left[\sum_{I\in\mathscr{G}_p^1}\lambda_N(I)+\mathbf{1}_{\tilde{\Omega}_p}\sum_{I\in\mathscr{G}_p^2}2^{-pQ}(\log\log 2^{2p})^{d/Q}p^{d/2}\right]\\
&\le K\Big(\lambda_N(T)+(\log\log 2^{2p})^{d/Q}p^{d/2}\exp(-\sqrt{p}/4)\Big)\\
&\le K\big(\lambda_N(T)+1\big)
\end{aligned}$$

provided p is large. Recall that $\mathscr{F}_p$ is a cover for $u^{-1}(z)\cap T$ on Ω_p for large p and each I is contained in a ρ-ball of radius $c_2 2^{-q[I]}$. Therefore, by Fatou's lemma,

$$\begin{aligned}
\mathbb{E}\Big[\mathcal{H}_\rho^\varphi(u^{-1}(z)\cap T)\Big] &= \mathbb{E}\Big[\mathbf{1}_{\Omega^*}\,\mathcal{H}_\rho^\varphi(u^{-1}(z)\cap T)\Big]\\
&\le \mathbb{E}\left[\liminf_{p\to\infty}\mathbf{1}_{\Omega_p}\sum_{I\in\mathscr{F}_p}\varphi(2c_2 2^{-q[I]})\right]\\
&\le \liminf_{p\to\infty}\mathbb{E}\left[\mathbf{1}_{\Omega_p}\sum_{I\in\mathscr{F}_p}\varphi(2c_2 2^{-q[I]})\right]\\
&\le K\big(\lambda_N(T)+1\big)<\infty.
\end{aligned}$$

This completes the proof of Theorem 4.5.3. $\square$

Chapter 5

Local Nondeterminism and the Exact Modulus of Continuity for the Stochastic Wave Equation

5.1 Introduction

Let $k \geq 1$ and $0 < \beta < k \wedge 2$, or $k = 1 = \beta$. Let us consider the linear stochastic wave equation in arbitrary spatial dimension $k \geq 1$:

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} u(t,x) = \Delta u(t,x) + \dot{W}(t,x), \quad t \geq 0, x \in \mathbb{R}^k, \\ u(0,x) = \dfrac{\partial}{\partial t} u(0,x) = 0. \end{cases} \tag{5.1}$$

Here, $\dot{W}$ is the space-time Gaussian white noise if $k = 1 = \beta$; and is a Gaussian noise that is white in time and colored in space with covariance

$$\mathbb{E}(\dot{W}(t,x)\dot{W}(s,y)) = \delta_0(t-s)|x-y|^{-\beta}$$

if $k \geq 1$ and $0 < \beta < k \wedge 2$. The purpose of this chapter is to study the exact modulus of continuity for the solution. This part is based on [35].

In Chapter 4, we considered Gaussian random fields with the property of strong local nondeterminism and we showed that the stochastic heat equation satisfies this property. In this chapter, we will show that the stochastic wave equation satisfies a different form of local nondeterminism. As an application, we use this property to determine the exact uniform modulus of continuity of the solution in time and space variables jointly.

It is well known that the Brownian sheet $\{B(t) : t \in \mathbb{R}_+^N\}$ does not satisfy strong local nondeterminism, but it satisfies sectorial local nondeterminism (see [31], Proposition 4.2). Namely, for all $\varepsilon > 0$, there exists $C > 0$ such that for all $n \geq 1$, for all $t, t^1, \ldots, t^n \in [\varepsilon, \infty)^N$,

$$\mathrm{Var}(B(t)|B(t^1), \ldots, B(t^n)) \geq C \sum_{j=1}^{N} \min_{1\leq i\leq n} |t_j - t_j^i|.$$

Recall from Theorem 3.1 of [64] that if $\dot{W}$ is the space-time white noise, the solution $u(t,x)$ of (5.1) has the representation

$$u(t,x) = \frac{1}{2}\,\hat{W}\left(\frac{t-x}{\sqrt{2}}, \frac{t+x}{\sqrt{2}}\right), \tag{5.2}$$

where $\hat{W}$ is a modified Brownian sheet (cf. [64, p.281]). In this special case, $u(t,x)$ shares many properties with the Brownian sheet. It is therefore natural to study whether the stochastic wave equation satisfies local nondeterminism.

In this chapter, we investigate the property of local nondeterminism for the solution of (5.1) and use this property to study the uniform modulus of continuity of its sample functions. The main results are Proposition 5.2.1 and Theorem 5.3.1. Proposition 5.2.1 shows that for any spatial dimension k, the solution $u(t,x)$ satisfies an integral form of local nondeterminism. When $k = 1$ and $\beta = 1$, this property (see (5.6) below) can also be derived from the sectorial local nondeterminism for the Brownian sheet in [31, Proposition 4.2] after a change of coordinates. While for $k = 1$ and $\beta \in (0, 1)$, property (5.6) is similar to the sectorial local nondeterminism in [65, Theorem 1] for a fractional Brownian sheet, which suggests that the sample function $u(t, x)$ may have subtle properties that are different from those of Gaussian random fields with stationary increments (an important example of the latter is fractional Brownian motion). We believe that Proposition 5.2.1 is useful for studying precise regularity and other sample path properties of $u(t, x)$. In Theorem 5.3.1, we apply Proposition 5.2.1 to derive the exact uniform modulus of continuity of $u(t, x)$.

The exact modulus of continuity provides precise information about the regularity and oscillation of sample paths. General conditions for uniform and local exact moduli of continuity of Gaussian processes were studied by Marcus and Rosen [40]. The exact moduli of continuity for anisotropic Gaussian random fields were studied by Meerschaert, Wang and Xiao [42], with applications to fractional Brownian sheets and one-dimensional stochastic heat equation driven by the space-time white noise. Similar results for the stochastic heat equation driven by fractional-colored noise can be found in [62, 27].

5.2 Local Nondeterminism

Let G be the fundamental solution of the wave equation. Recall from Section 2.2.3 that for $k \geq 3$, G is not a function but a distribution. Also recall that for any dimension $k \geq 1$, the Fourier transform of G in variable x is given by

$$\mathscr{F}(G(t, \cdot))(\xi) = \frac{\sin(t|\xi|)}{|\xi|}, \quad t \geq 0, \xi \in \mathbb{R}^k. \tag{5.3}$$

In [11], Dalang extended Walsh's stochastic integration and proved that the real-valued process solution of equation (5.1) is given by

$$u(t,x) = \int_0^t \int_{\mathbb{R}^k} G(t-s, x-y)\, W(ds\, dy),$$

where W is the martingale measure induced by the noise $\dot{W}$. The range of β has been chosen so that the stochastic integral is well-defined. Recall from Theorem 2 of [11] that

$$\mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^k} H(s,y) W(ds\, dy)\right)^2\right] = c_{k,\beta} \int_0^t ds \int_{\mathbb{R}^k} d\xi\, |\xi|^{\beta-k} |\mathscr{F}(H(s,\cdot))(\xi)|^2 \tag{5.4}$$

provided that $s \mapsto H(s,\cdot)$ is a deterministic function with values in the space of nonnegative distributions with rapid decrease and

$$\int_0^t ds \int_{\mathbb{R}^k} d\xi\, |\xi|^{\beta-k} |\mathscr{F}(H(s,\cdot)(\xi)|^2 < \infty.$$

The following result shows that the solution $u(t,x)$ satisfies an integral form of local nondeterminism.

Proposition 5.2.1. *Let $0 < a < a' < \infty$ and $0 < b < \infty$. There exist constants $C > 0$ and $\delta > 0$ depending on a, a' and b such that for all integers $n \geq 1$ and all $(t,x), (t^1,x^1), \ldots, (t^n,x^n)$ in $[a,a'] \times [-b,b]^k$ with $|t-t^j| + |x-x^j| \leq \delta$, we have*

$$\mathrm{Var}\,(u(t,x)|u(t^1,x^1), \ldots, u(t^n,x^n)) \geq C \int_{\mathbb{S}^{k-1}} \min_{1 \leq j \leq n} |(t-t^j) + (x-x^j)\cdot w|^{2-\beta}\, dw, \tag{5.5}$$

where dw is the surface measure on the unit sphere $\mathbb{S}^{k-1}$.

Remark 5.2.2. *When $k = 1$, the surface measure dw in (5.5) is supported on $\{-1,1\}$. It* *follows that $u(t,x)$ satisfies sectorial local nondeterminism:*

$$\begin{aligned} &\mathrm{Var}(u(t,x)|u(t^1,x^1),\ldots,u(t^n,x^n)) \\ &\geq C\left(\min_{1\leq j\leq n}|(t-t^j)+(x-x^j)|^{2-\beta}+\min_{1\leq j\leq n}|(t-t^j)-(x-x^j)|^{2-\beta}\right). \end{aligned} \tag{5.6}$$

As we pointed out in the introduction in Section 5.1, property (5.6) *is similar to the sectorial local nondeterminism but different from the strong local nondeterminism of Gaussian random fields with stationary increments. It indicates that $u(t,x)$ may have properties that are different from those of Gaussian random fields with stationary increments such as fractional Brownian motion.*

Proof of Proposition 5.2.1. Take $\delta = a/2$. For each $w \in \mathbb{S}^{k-1}$, let

$$r(w) = \min_{1\leq j\leq n}|(t^j-t)-(x^j-x)\cdot w|.$$

Since u is a centered Gaussian random field, the conditional variance $\mathrm{Var}(u(t,x)|u(t^1,x^1),\ldots,u(t^n,x^n))$ is the squared distance of $u(t,x)$ from the linear subspace spanned by $u(t^1,x^1),\ldots,u(t^n,x^n)$ in $L^2(\mathbb{P})$. Thus, it suffices to show that there exist constants $C>0$ and $\delta>0$ such that for all $(t,x),(t^1,x^1),\ldots,(t^n,x^n)$ in $[a,a']\times[-b,b]^k$ with $|t-t^j|+|x-x^j|\leq\delta$, we have

$$\mathbb{E}\left[\left(u(t,x)-\sum_{j=1}^n \alpha_j u(t^j,x^j)\right)^2\right] \geq C\int_{\mathbb{S}^{k-1}} r(w)^{2-\beta}\,dw \tag{5.7}$$

for any choice of real numbers $\alpha_1,\ldots,\alpha_n$. Using (5.3), (5.4) and spherical coordinate $\xi = \rho\, w$, we have

$$
\begin{aligned}
&\mathbb{E}\Bigg[\bigg(u(t,x) - \sum_{j=1}^n \alpha_j u(t^j, x^j)\bigg)^2\Bigg] \\
&= c_{k,\beta} \int_0^\infty ds \int_{\mathbb{R}^k} \frac{d\xi}{|\xi|^{2+k-\beta}} \times \\
&\qquad\qquad \bigg| \sin((t-s)|\xi|)\mathbf{1}_{[0,t]}(s) - \sum_{j=1}^n \alpha_j e^{-i(x^j - x)\cdot\xi} \sin((t^j - s)|\xi|)\mathbf{1}_{[0,t^j]}(s)\bigg|^2 \\
&\geq c_{k,\beta} \int_0^{a/2} ds \int_0^\infty \frac{d\rho}{\rho^{3-\beta}} \int_{\mathbb{S}^{k-1}} dw \bigg| \sin((t-s)\rho) - \sum_{j=1}^n \alpha_j e^{-i\rho(x^j - x)\cdot w} \sin((t^j - s)\rho)\bigg|^2 \\
&= \frac{c_{k,\beta}}{8} \int_0^{a/2} ds \int_{-\infty}^\infty \frac{d\rho}{|\rho|^{3-\beta}} \int_{\mathbb{S}^{k-1}} dw \bigg| \left(e^{i(t-s)\rho} - e^{-i(t-s)\rho}\right) \\
&\qquad\qquad - \sum_{j=1}^n \alpha_j e^{-i\rho(x^j - x)\cdot w} \left(e^{i(t^j - s)\rho} - e^{-i(t^j - s)\rho}\right) \bigg|^2 \\
&=: \frac{c_{k,\beta}}{8} \int_{\mathbb{S}^{k-1}} A(w)\, dw.
\end{aligned}
$$

Let $\lambda = \min\{1, a/[2(a' + 2\sqrt{k}b)]\}$ and consider the bump function $\varphi : \mathbb{R} \to \mathbb{R}$ defined by

$$
\varphi(y) = \begin{cases} \exp\left(1 - \frac{1}{1 - |\lambda^{-1} y|^2}\right), & |y| < \lambda, \\ 0, & |y| \geq \lambda. \end{cases}
$$

Let $\varphi_r(y) = r^{-1}\varphi(y/r)$. For each $w \in \mathbb{S}^{k-1}$ such that $r(w) > 0$, consider the integral

$$
\begin{aligned}
I(w) := \int_0^{a/2} ds \int_{-\infty}^\infty d\rho \Bigg[&\left(e^{i(t-s)\rho} - e^{-i(t-s)\rho}\right) \\
&- \sum_{j=1}^n \alpha_j e^{-i\rho(x^j - x)\cdot w} \left(e^{i(t^j - s)\rho} - e^{-i(t^j - s)\rho}\right) \Bigg] e^{-i(t-s)\rho} \widehat{\varphi}_{r(w)}(\rho).
\end{aligned}
$$

By the inverse Fourier transform (or one can apply the Plancherel theorem), we have

$$\begin{aligned}I(w) = 2\pi \int_0^{a/2} ds \Big[&\varphi_{r(w)}(0) - \varphi_{r(w)}\big(2(t-s)\big) \\ &- \sum_{j=1}^{n} \alpha_j \Big(\varphi_{r(w)}\big((x^j - x)\cdot w - (t^j - t)\big) - \varphi_{r(w)}\big((x^j - x)\cdot w - (t^j - t) + 2(t^j - s)\big)\Big)\Big].\end{aligned}$$

Note that $r(w) \le |t^j - t| + |x^j - x| \le a' + 2\sqrt{k}b$. For any $s \in [0, a/2]$, we have $2(t-s)/r(w) \ge a/[(a' + 2\sqrt{k}b)]$ and $|(x^j - x)\cdot w - (t^j - t)|/r(w) \ge 1$, thus

$$\varphi_{r(w)}\big(2(t-s)\big) = 0 \quad \text{and} \quad \varphi_{r(w)}\big((x^j - x)\cdot w - (t^j - t)\big) = 0 \quad \text{for } j = 1, \dots, n.$$

Also, $[(x^j - x)\cdot w - (t^j - t) + 2(t^j - s)]/r(w) \ge (-\delta + a)/[(a' + 2\sqrt{k}b)] \ge \lambda$, thus

$$\varphi_{r(w)}\big((x^j - x)\cdot w - (t^j - t) + 2(t^j - s)\big) = 0.$$

It follows that

$$I(w) = a\pi \, r(w)^{-1}.$$

On the other hand, by the Cauchy–Schwarz inequality and scaling, we obtain

$$\begin{aligned}(a\pi)^2 r(w)^{-2} = |I(w)|^2 &\le A(w) \times \int_0^{a/2} ds \int_{-\infty}^{\infty} d\rho \, |\widehat{\varphi}(r(w)\rho)|^2 |\rho|^{3-\beta} \\ &= (a/2) A(w) r(w)^{\beta-4} \int_{-\infty}^{\infty} d\rho \, |\widehat{\varphi}(\rho)|^2 |\rho|^{3-\beta} \\ &= C A(w) r(w)^{\beta-4}\end{aligned}$$

for some finite constant C. Hence we have

$$A(w) \geq C' r(w)^{2-\beta} \tag{5.8}$$

and this remains true if $r(w) = 0$. Integrating both sides of (5.8) over $\mathbb{S}^{k-1}$ yields (5.7). $\square$

5.3 The Exact Uniform Modulus of Continuity

Let $f : I \to \mathbb{R}$ be a function with $I \subset \mathbb{R}^N$. Let $\phi : [0, \infty) \to [0, \infty)$ be a function such that $\lim_{\varepsilon \to 0+} \phi(\varepsilon) = \phi(0) = 0$. Recall that ϕ is called a modulus of continuity for f on I if there exists a finite constant C such that

$$|f(x) - f(y)| \leq C\phi(|x - y|)$$

for all $x, y \in I$.

In order to identify the optimal modulus function, Marcus and Rosen [40] introduced the following definition. Let σ be a metric on I. We say that ϕ is an *exact modulus of continuity* for f on (I, σ) if there exists a positive finite constant C such that

$$\lim_{\varepsilon \to 0+} \sup_{\substack{x,y \in I: \\ 0 < \sigma(x,y) \leq \varepsilon}} \frac{|f(x) - f(y)|}{\phi(\sigma(x, y))} = C.$$

For example, Lévy's theorem of modulus of continuity shows that the exact modulus of continuity for the Brownian motion is $\phi(\varepsilon) = \sqrt{\varepsilon \log(1/\varepsilon)}$ with $C = \sqrt{2}$ (and σ being the Euclidean metric).

It is known that sectorial local nondeterminism is useful for proving the exact uniform modulus of continuity for Gaussian random fields [42]. In this section we show that the integral form of local nondeterminism in Proposition 5.2.1 can serve the same purpose for deriving the exact uniform modulus of continuity of the solution $u(t,x)$ to (5.1).

Let us denote

$$\sigma\big[(t,x),(t',x')\big] = \mathbb{E}[(u(t,x)-u(t',x'))^2]^{1/2}.$$

Recall from [20, Proposition 4.1] that for any $0 < a < a' < \infty$ and $0 < b < \infty$, there are positive constants C_1 and C_2 such that

$$C_1\left(|t-t'|+\sum_{j=1}^{k}|x_j-x_j'|\right)^{2-\beta} \le \sigma[(t,x),(t',x')]^2 \le C_2\left(|t-t'|+\sum_{j=1}^{k}|x_j-x_j'|\right)^{2-\beta} \tag{5.9}$$

for all $(t,x),(t',x') \in [a,a'] \times [-b,b]^k$.

The following result establishes the exact uniform modulus of continuity of $u(t,x)$ in the time and space variables (t,x).

Theorem 5.3.1. *Let $I = [a,a'] \times [-b,b]^k$, where $0 < a < a' < \infty$ and $0 < b < \infty$. Let*

$$\gamma\big[(t,x),(t',x')\big] = \sigma\big[(t,x),(t',x')\big]\sqrt{\log\left(1+\sigma\big[(t,x),(t',x')\big]^{-1}\right)}.$$

Then there is a positive finite constant K such that

$$\lim_{\varepsilon\to 0+} \sup_{\substack{(t,x),(t',x')\in I,\\ 0<\sigma[(t,x),(t',x')]\le\varepsilon}} \frac{|u(t,x)-u(t',x')|}{\gamma\big[(t,x),(t',x')\big]} = K, \quad \text{a.s.} \tag{5.10}$$

Proof. For any $\varepsilon > 0$, let

$$J(\varepsilon) = \sup_{\substack{(t,x),(t',x')\in I,\\ 0<\sigma[(t,x),(t',x')]\le\varepsilon}} \frac{|u(t,x) - u(t',x')|}{\gamma\big[(t,x),(t',x')\big]}.$$

Since $\varepsilon \mapsto J(\varepsilon)$ is non-decreasing, we see that the limit $\lim_{\varepsilon\to 0+} J(\varepsilon)$ exists a.s. In order to prove (5.10), we prove the following statements: there exist positive and finite constants K^* and K_* such that

$$\lim_{\varepsilon\to 0+} J(\varepsilon) \le K^*, \qquad \text{a.s.} \tag{5.11}$$

and

$$\lim_{\varepsilon\to 0+} J(\varepsilon) \ge K_*, \qquad \text{a.s.} \tag{5.12}$$

Then the conclusion of Theorem 5.3.1 follows from Lemma 7.1.1 of [40] where τ is chosen to be the Euclidean metric and d is the canonical metric $\sigma[(t,x),(t',x')]$. [It is a 0-1 law for the modulus of continuity which is obtained by applying Kolmogorov's 0-1 law to the Karhunen–Loève expansion of $u(t,x)$.]

The proof of the upper bound (5.11) is standard. For any $\varepsilon > 0$, denote by $N(I,\varepsilon,\sigma)$ the smallest number of balls of radius ε in the canonical metric $\sigma\big[(t,x),(t',x')\big]$ that are needed to cover the compact interval I. By the upper bound in (5.9), we have $N(I,\varepsilon,\sigma) \le C\varepsilon^{-(1+k)/(2-\beta)}$ and thus

$$\int_0^\varepsilon \sqrt{\log N(I,\tilde\varepsilon,\sigma)}\, d\tilde\varepsilon \le C\varepsilon\sqrt{\log(1+\varepsilon^{-1})}.$$

By Theorem 1.3.5 of [2], there is a positive finite constant K^* such that

$$\limsup_{\varepsilon\to 0+} \sup_{\substack{(t,x),(t',x')\in I,\\ 0<\sigma[(t,x),(t',x')]\le\varepsilon}} \frac{|u(t,x)-u(t',x')|}{\varepsilon\sqrt{\log(1+\varepsilon^{-1})}} \le K^* \qquad \text{a.s.}$$

From this we can deduce (5.11) by considering $\varepsilon_{n+1} \le \sigma[(t,x),(t',x')] \le \varepsilon_n$ where $\varepsilon_n = 1/n$, and using the fact that the function $\varepsilon \mapsto \varepsilon\sqrt{\log(1+\varepsilon^{-1})}$ is increasing for ε small, and

$$\lim_{n\to\infty} \frac{\varepsilon_n\sqrt{\log(1+\varepsilon_n^{-1})}}{\varepsilon_{n+1}\sqrt{\log(1+\varepsilon_{n+1}^{-1})}} = 1.$$

Next we prove the lower bound (5.12). This is accomplished by applying Proposition 5.2.1, a conditioning argument and the Borel–Cantelli lemma. We first choose δ according to Proposition 5.2.1 and let $\delta' = \min\{\delta/(1+\sqrt{k}), a'-a, 2b\}$. Note that δ' depends only on a, a' and b. For each $n \ge 1$, let

$$\varepsilon_n = [C_2((1+k)\delta')^{2-\beta}2^{-(2-\beta)n}]^{1/2}.$$

For $i = 0, 1, \dots, 2^n$, let $t^{n,i} = a + i\delta' 2^{-n}$ and $x_j^{n,i} = -b + i\delta' 2^{-n}$. Then

$$\begin{aligned}
\lim_{\varepsilon\to 0+} J(\varepsilon) &= \lim_{n\to\infty} \sup_{\substack{(t,x),(t',x')\in I,\\ 0<\sigma[(t,x),(t',x')]\le\varepsilon_n}} \frac{|u(t,x)-u(t',x')|}{\gamma[(t,x),(t',x')]} \\
&\ge \liminf_{n\to\infty} \max_{1\le i\le 2^n} \frac{|u(t^{n,i},x^{n,i})-u(t^{n,i-1},x^{n,i-1})|}{\varepsilon_n\sqrt{\log(1+\varepsilon_n^{-1})}} \\
&=: \liminf_{n\to\infty} J_n.
\end{aligned}$$

To obtain the inequality, we have used the fact that $\sigma[(t^{n,i},x^{n,i}),(t^{n,i-1},x^{n,i-1})] \le \varepsilon_n$ and that the function $\varepsilon \mapsto \varepsilon\sqrt{\log(1+\varepsilon^{-1})}$ is increasing for ε small.

Let $K_* > 0$ be a constant whose value will be determined later. Fix n and write $t^{n,i} = t^i$, $x^{n,i} = x^i$ to simplify notations. By conditioning, we can write

$$\begin{aligned}
&\mathbb{P}\left(J_n \le K_*\right) \\
&= \mathbb{P}\left(\max_{1\le i\le 2^n} \frac{|u(t^i, x^i) - u(t^{i-1}, x^{i-1})|}{\varepsilon_n\sqrt{\log(1+\varepsilon_n^{-1})}} \le K_* \right) \\
&= \mathbb{E}\left[\mathbf{1}_A \mathbb{P}\left(\frac{|u(t^{2^n}, x^{2^n}) - u(t^{2^n-1}, x^{2^n-1})|}{\varepsilon_n\sqrt{\log(1+\varepsilon_n^{-1})}} \le K_* \middle| u(t^i, x^i) : 0 \le i \le 2^n - 1 \right) \right],
\end{aligned} \tag{5.13}$$

where A is the event defined by

$$A = \left\{ \max_{1\le i\le 2^n - 1} \frac{|u(t^i, x^i) - u(t^{i-1}, x^{i-1})|}{\varepsilon_n\sqrt{\log(1+\varepsilon_n^{-1})}} \le K_* \right\}.$$

Since $|t^{2^n} - t^i| + |x^{2^n} - x^i| \le \delta$, by Proposition 5.2.1 we have

$$\begin{aligned}
&\mathrm{Var}\left(u(t^{2^n}, x^{2^n}) | u(t^i, x^i) : 0 \le i \le 2^n - 1\right) \\
&\ge C \int_{\mathbb{S}^{k-1}} \min_{0\le i\le 2^n-1} |(t^{2^n} - t^i) + (x^{2^n} - x^i)\cdot w|^{2-\beta}\, dw \\
&\ge C \int_{\{w\in\mathbb{S}^{k-1} :\, (1,\dots,1)\cdot w \ge 0\}} \min_{0\le i\le 2^n-1} |\delta'(2^n - i)2^{-n} + \delta'(2^n - i)2^{-n}(1,\dots,1)\cdot w|^{2-\beta}\, dw \\
&\ge C(\delta')^{2-\beta}\, 2^{-(2-\beta)n} \int_{\{w\in\mathbb{S}^{k-1} :\, (1,\dots,1)\cdot w \ge 0\}} dw \\
&= C_0\, \varepsilon_n^2
\end{aligned} \tag{5.14}$$

for some constant $C_0 > 0$ depending on a, a' and b.

Since the conditional distribution of $u(t^{2^n}, x^{2^n})$, given $u(t^i, x^i)$, $(0 \le i \le 2^n - 1)$, is Gaussian with conditional variance $\mathrm{Var}\left(u(t^{2^n}, x^{2^n}) | u(t^i, x^i) : 0 \le i \le 2^n - 1\right)$, it follows from Anderson's inequality [3] and (5.14) that

$$\mathbb{P}\left(\frac{|u(t^{2^n}, x^{2^n}) - u(t^{2^n-1}, x^{2^n-1})|}{\varepsilon_n \sqrt{\log(1+\varepsilon_n^{-1})}} \le K_* \middle| u(t^i, x^i) : 0 \le i \le 2^n - 1 \right)$$
$$\le \mathbb{P}\left(|Z| \le K_* \sqrt{C_0^{-1} \log\left(1 + \varepsilon_n^{-1}\right)} \right)$$

where Z is a standard normal random variable. Using $\mathbb{P}(|Z| > x) \ge (\sqrt{2\pi})^{-1} x^{-1} \exp(-x^2/2)$ for $x \ge 1$ and $1+\varepsilon^{-1} < 2/\varepsilon$ for ε small, we deduce that when n is large the above probability is bounded from above by

$$1 - \frac{C(\varepsilon_n/2)^{K_*^2/(2C_0)}}{K_* \sqrt{\log\left(2/\varepsilon_n\right)}} \le \exp\left(-\frac{C(\varepsilon_n/2)^{K_*^2/(2C_0)}}{K_* \sqrt{\log\left(2/\varepsilon_n\right)}} \right) \le \exp\left(- \frac{C_{K_*} 2^{-\frac{(2-\beta)K_*^2}{4C_0} n}}{\sqrt{n}} \right)$$

where $C_{K_*} > 0$ is a constant depending on K_*. Then by (5.13) and induction, we have

$$\mathbb{P}\big(J_n \le K_*\big) \le \exp\left(- 2^n \frac{C_{K_*} 2^{-\frac{(2-\beta)K_*^2}{4C_0} n}}{\sqrt{n}} \right).$$

We can now choose $K_* > 0$ to be a sufficiently small constant such that

$$1 - \frac{(2-\beta)K_*^2}{4C_0} > 0.$$

Then $\sum_{n=1}^{\infty} \mathbb{P}\big(J_n \le K_*\big) < \infty$. Hence, by the Borel–Cantelli lemma, $\liminf_n J_n \ge K_*$ a.s. and the proof is complete. $\square$

Chapter 6

Propagation of Singularities for the Stochastic Wave Equation

6.1 Introduction

In this chapter, we consider the stochastic wave equation in one spatial dimension:

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}u(t,x) - \dfrac{\partial^2}{\partial x^2}u(t,x) = \dot{W}(t,x), \quad t \geq 0, x \in \mathbb{R}, \\ u(0,x) = 0, \quad \dfrac{\partial}{\partial t}u(0,x) = 0, \end{cases} \tag{6.1}$$

where $\dot{W}$ is a Gaussian noise that is white in time and colored in space with spatial covariance

$$\mathbb{E}[\dot{W}(t,x)\dot{W}(s,y)] = \delta_0(t-s)|x-y|^{-\beta} \tag{6.2}$$

with $0 < \beta < 1$. The purpose of this chapter is to study the singularities of the solution $\{u(t,x) : t \geq 0, x \in \mathbb{R}\}$. This chapter is based on [36].

In this context, singularity is related to exceptionally large increments of a stochastic process. By singularity we mean a random point at which the process has local oscillations that are much larger than those specified by the law of the iterated logarithm (LIL). For the Brownian motion, this phenomenon was first studied by Orey and Taylor [48]. It is well known that at a fixed time, the increments of a Brownian path satisfies the LIL almost surely. However, it is not true that the LIL holds simultaneously for all time points with probability one. Indeed, according to Lévy's modulus of continuity, we can find random points at which the LIL fails and the increments are exceptionally large, and therefore we can define these exceptional points as singularities. Similarly, we can define singularities for other general random fields.

The singularities of the Brownian sheet and the one-dimensional stochastic wave equation driven by the space-time white noise were studied by Walsh [63, 64], and those of semi-fractional Brownian sheet was studied by Blath and Martin [7]. Based on a simultaneous law of the iterated logarithm, Walsh [63] showed that the singularities of the Brownian sheet propagate parallel to the coordinate axis. Moreover, Walsh [64] found an interesting relation between the Brownian sheet and the solution $u(t,x)$ to (6.1) driven by the space-time white noise. Specifically, Theorem 3.1 in [64] shows that the solution can be written as the sum of three components:

$$u(t,x) = \frac{1}{2}\left[B\Big(\frac{t-x}{\sqrt{2}}, \frac{t+x}{\sqrt{2}}\Big) + \hat{W}\Big(\frac{t-x}{\sqrt{2}}, 0\Big) + \hat{W}\Big(0, \frac{t+x}{\sqrt{2}}\Big)\right], \tag{6.3}$$

where the main component B is a Brownian sheet and $\hat{W}$ is the modified Brownian sheet defined in Chapter 1 of Walsh [64], and the processes $\{B(s,t) : s,t \geq 0\}$, $\{\hat{W}(s,0) : s \geq 0\}$ and $\{\hat{W}(0,t) : t \geq 0\}$ are independent. This relation implies that the singularities of $u(t,x)$ propagate along the characteristic curves $t - x = c$ and $t + x = c$.

Later, Carmona and Nualart [9] extended the study of singularities of the solution to the linear stochastic wave equation (6.1) driven by space-time white noise in [63, 64] to the case of one-dimensional nonlinear stochastic wave equations driven by a space-time white noise.

Their approach is based on the general theory of semimartingales and two-parameter strong martingales. In particular, they proved the law of the iterated logarithm for a semimartingale by the LIL of Brownian motion and a time change. They also proved that, for a class of two-parameter strong martingales, the law of the iterated logarithm in one variable holds simultaneously for all values of the other variable.

The main objective of this chapter is to study the existence and propagation of singularities of the solution to (6.1) driven by a Gaussian noise that is white in time and colored in space with spatial covariance given by (6.2) with $0 < \beta < 1$. In this case, the solution shares some similarity with the fractional Brownian sheet, but it seems to us that there is not a natural relation like (6.3) between the solution and the fractional Brownian sheet. Also, the method in Carmona and Nualart [9] based on semimartingales and two-parameter strong martingales is not applicable in the case of colored noise. Our approach is based on a simultaneous LIL for the solution and general methods for Gaussian processes.

This chapter is organized as follows. First, we establish a simultaneous LIL for the solution of the stochastic wave equation. We prove that after a rotation, the LIL in one variable holds simultaneously for all values of the other variable. The proof consists of two parts: proving the upper bound and lower bound. The upper bound is proved in Section 6.2 and the lower bound is proved in Section 6.3. In Section 6.4, we introduce the definition of singularity for the stochastic wave equation and apply the simultaneous LIL to study the propagation of singularities. The main result Theorem 6.4.3 shows that singularities propagate along the characteristic curves.

6.2 Simultaneous Law of Iterated Logarithm: Upper Bound

The noise in (6.1) is defined as the mean zero Gaussian process $W(\varphi)$ indexed by compactly supported smooth functions $\varphi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R})$ with covariance function

$$\begin{aligned} \mathbb{E}[W(\varphi)W(\psi)] &= \int_{\mathbb{R}_+} ds \int_{\mathbb{R}} dy \int_{\mathbb{R}} dy'\, \varphi(s,y)|y-y'|^{-\beta}\psi(s,y') \\ &= \frac{1}{2\pi}\int_{\mathbb{R}_+} ds \int_{\mathbb{R}} \mu(d\xi)\, \mathscr{F}(\varphi(s,\cdot))(\xi)\overline{\mathscr{F}(\psi(s,\cdot))(\xi)} \end{aligned} \tag{6.4}$$

for all $\varphi, \psi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R})$, where μ is the measure whose Fourier transform is $|\cdot|^{-\beta}$ and $\mathscr{F}(\varphi(s,\cdot))(\xi)$ is the Fourier transform of the function $y \mapsto \varphi(s,y)$ in the following convention:

$$\mathscr{F}(\varphi(s,\cdot))(\xi) = \int_{\mathbb{R}} e^{-i\xi y}\varphi(s,y)dy.$$

Note that $\mu(d\xi) = C_\beta |\xi|^{-1+\beta} d\xi$, where

$$C_\beta = \frac{\pi^{1/2} 2^{1-\beta}\Gamma(\frac{1}{2}-\frac{\beta}{2})}{\Gamma(\frac{\beta}{2})}$$

see [59, p.117]. We assume that W is defined on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Following [11, 13], for any bounded Borel set A in $\mathbb{R}_+ \times \mathbb{R}$, we can define

$$W(A) = \lim_{n\to\infty} W(\varphi_n)$$

in the sense of $L^2(\mathbb{P})$-limit, where (φ_n) is a sequence in $C_c^\infty(\mathbb{R}_+ \times \mathbb{R})$ with a compact set K such that $\operatorname{supp}\varphi_n \subset K$ for all n and $\varphi_n \to \mathbf{1}_A$. From (6.4), it follows that for any bounded

Borel sets A, B in $\mathbb{R}_+ \times \mathbb{R}$, we have

$$\begin{aligned}\mathbb{E}[W(A)W(B)] &= \int_{\mathbb{R}_+} ds \int_{\mathbb{R}} dy \int_{\mathbb{R}} dy' \, \mathbf{1}_A(s,y)|y-y'|^{-\beta}\mathbf{1}_B(s,y') \\ &= \frac{1}{2\pi}\int_{\mathbb{R}_+} ds \int_{\mathbb{R}} \frac{C_\beta d\xi}{|\xi|^{1-\beta}} \, \mathscr{F}(\mathbf{1}_A(s,\cdot))(\xi)\overline{\mathscr{F}(\mathbf{1}_B(s,\cdot))(\xi)}.\end{aligned} \tag{6.5}$$

In dimension one, the fundamental solution of the wave equation is $\frac{1}{2}\mathbf{1}_{\{|x|\le t\}}$, so the mild solution of (6.1) is

$$u(t,x) = \frac{1}{2}\int_0^t \int_{\mathbb{R}} \mathbf{1}_{\{|x-y|\le t-s\}}(s,y)\, W(ds\, dy) = \frac{1}{2}\, W(\Delta(t,x)), \tag{6.6}$$

where $\Delta(t,x) = \{(s,y) \in \mathbb{R}_+ \times \mathbb{R} : 0 \le s \le t,\ |x-y| \le t-s\}$.

Consider a new coordinate system (τ, λ) obtained by rotating the (t,x)-coordinates by -45°. In other words,

$$(\tau,\lambda) = \Big(\frac{t-x}{\sqrt{2}}, \frac{t+x}{\sqrt{2}}\Big) \quad \text{and} \quad (t,x) = \Big(\frac{\tau+\lambda}{\sqrt{2}}, \frac{-\tau+\lambda}{\sqrt{2}}\Big).$$

For $\tau \ge 0$, $\lambda \ge 0$, let us denote

$$\tilde{u}(\tau,\lambda) = u\Big(\frac{\tau+\lambda}{\sqrt{2}}, \frac{-\tau+\lambda}{\sqrt{2}}\Big).$$

We are going to prove a simultaneous LIL for the Gaussian random field $\{\tilde{u}(\tau,\lambda) : \tau \ge 0, \lambda \ge 0\}$. The following result shows an upper bound for the LIL in λ, which holds simultaneously for all values of τ. By a symmetric argument, we can also prove that the LIL in τ holds simultaneously for all λ.

Proposition 6.2.1. *For any $\lambda > 0$, we have*

$$\mathbb{P}\left(\limsup_{h\to 0+} \frac{|\tilde{u}(\tau,\lambda+h)-\tilde{u}(\tau,\lambda)|}{\sqrt{(\tau+\lambda)h^{2-\beta}\log\log(1/h)}} \le K_\beta \text{ for all } \tau\in[0,\infty)\right) = 1, \tag{6.7}$$

where

$$K_\beta = \left(\frac{2^{(1-\beta)/2}}{(2-\beta)(1-\beta)}\right)^{1/2}.$$

Lemma 6.2.2. *For any $0<\beta<1$, $a<b$ and $c<d$, we have*

$$C_\beta \int_{-\infty}^{\infty} |\mathscr{F}\mathbf{1}_{[a,b]}(\xi)|^2 \frac{d\xi}{|\xi|^{1-\beta}} = \frac{4\pi}{(2-\beta)(1-\beta)}(b-a)^{2-\beta} \tag{6.8}$$

and

$$\begin{aligned} &C_\beta \int_{-\infty}^{\infty} \mathscr{F}\mathbf{1}_{[a,b]}(\xi)\overline{\mathscr{F}\mathbf{1}_{[c,d]}(\xi)} \frac{d\xi}{|\xi|^{1-\beta}} \\ &= \frac{2\pi}{(2-\beta)(1-\beta)}\Big(|c-b|^{2-\beta}+|d-a|^{2-\beta}-|c-a|^{2-\beta}-|d-b|^{2-\beta}\Big). \end{aligned}$$

Proof. The Fourier transform of the function $\mathbf{1}_{[a,b]}$ is

$$\mathscr{F}\mathbf{1}_{[a,b]}(\xi) = \frac{e^{-i\xi a}-e^{-i\xi b}}{i\xi}.$$

It follows that

$$\begin{aligned} C_\beta \int_{-\infty}^{\infty} |\mathscr{F}\mathbf{1}_{[a,b]}(\xi)|^2 \frac{d\xi}{|\xi|^{1-\beta}} &= C_\beta \int_{-\infty}^{\infty} |e^{i\xi(b-a)}-1|^2 \frac{d\xi}{|\xi|^{3-\beta}} \\ &= C_\beta (b-a)^{2-\beta} \int_{-\infty}^{\infty} |e^{i\xi}-1|^2 \frac{d\xi}{|\xi|^{3-\beta}}. \end{aligned}$$

The last equality follows by scaling. The proof of Proposition 7.2.8 of [56] shows that

$$\int_{-\infty}^{\infty} |e^{i\xi} - 1|^2 \frac{d\xi}{|\xi|^{3-\beta}} = \frac{2\pi}{(2-\beta)\Gamma(2-\beta)\sin(\frac{\pi\beta}{2})}.$$

Also, using the relations $\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma(z+1)$, $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ and $z\Gamma(z) = \Gamma(z+1)$ (cf. [25, p.895–896]), we can show that

$$C_\beta = \frac{2\Gamma(2-\beta)\sin(\frac{\pi\beta}{2})}{1-\beta}.$$

Hence (6.8) follows.

For the second part,

$$\int_{-\infty}^{\infty} \mathscr{F}\mathbf{1}_{[a,b]}(\xi)\overline{\mathscr{F}\mathbf{1}_{[c,d]}(\xi)}\frac{d\xi}{|\xi|^{1-\beta}} = \int_{-\infty}^{\infty} \big(e^{i\xi(c-a)} + e^{i\xi(d-b)} - e^{i\xi(c-b)} - e^{i\xi(d-a)}\big)\frac{d\xi}{|\xi|^{3-\beta}}.$$

Note that this integral is real, so we have

$$\begin{aligned}\int_{-\infty}^{\infty} \mathscr{F}\mathbf{1}_{[a,b]}(\xi)\overline{\mathscr{F}\mathbf{1}_{[c,d]}(\xi)}\frac{d\xi}{|\xi|^{1-\beta}} = \frac{1}{2}\int_{-\infty}^{\infty} \big(&e^{i\xi(c-a)} + e^{-i\xi(c-a)} + e^{i\xi(d-b)} + e^{-i\xi(d-b)} \\ &- e^{i\xi(c-b)} - e^{-i\xi(c-b)} - e^{i\xi(d-a)} - e^{-i\xi(d-a)}\big)\frac{d\xi}{|\xi|^{3-\beta}}.\end{aligned}$$

Since $|e^{i\xi(x-y)} - 1|^2 = 2 - e^{i\xi(x-y)} - e^{-i\xi(x-y)}$ for all $x, y \in \mathbb{R}$, we have

$$\begin{aligned}&\int_{-\infty}^{\infty} \mathscr{F}\mathbf{1}_{[a,b]}(\xi)\overline{\mathscr{F}\mathbf{1}_{[c,d]}(\xi)}\frac{d\xi}{|\xi|^{1-\beta}} \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \big(-|e^{i\xi(c-a)} - 1|^2 - |e^{i\xi(d-b)} - 1|^2 + |e^{i\xi(c-b)} - 1|^2 + |e^{i\xi(d-a)} - 1|^2\big)\frac{d\xi}{|\xi|^{3-\beta}}.\end{aligned}$$

Now the result follows from the first part of the proof. $\square$

Lemma 6.2.3. *For any $\tau, \lambda, h > 0$,*

$$\mathbb{E}[(\tilde{u}(\tau,\lambda+h) - \tilde{u}(\tau,\lambda))^2] = \frac{1}{2}\, K_\beta^2 \left[(\tau+\lambda)h^{2-\beta} + (3-\beta)^{-1}h^{3-\beta}\right],$$

where

$$K_\beta = \left(\frac{2^{(1-\beta)/2}}{(2-\beta)(1-\beta)}\right)^{1/2}.$$

Proof. Note that

$$\begin{aligned}
&\mathbb{E}[(\tilde{u}(\tau,\lambda+h) - \tilde{u}(\tau,\lambda))^2] \\
&= \mathbb{E}\left[\left(u\Big(\frac{\tau+\lambda+h}{\sqrt{2}}, \frac{-\tau+\lambda+h}{\sqrt{2}}\Big) - u\Big(\frac{\tau+\lambda}{\sqrt{2}}, \frac{-\tau+\lambda}{\sqrt{2}}\Big)\right)^2\right] \\
&= \frac{1}{4}\,\mathbb{E}\left[\left(W\Big(\Delta\Big(\frac{\tau+\lambda+h}{\sqrt{2}}, \frac{-\tau+\lambda+h}{\sqrt{2}}\Big)\backslash\Delta\Big(\frac{\tau+\lambda}{\sqrt{2}}, \frac{-\tau+\lambda}{\sqrt{2}}\Big)\Big)\right)^2\right].
\end{aligned}$$

Then by (6.5) and Lemma 6.2.2,

$$\begin{aligned}
&\mathbb{E}[(\tilde{u}(\tau,\lambda+h) - \tilde{u}(\tau,\lambda))^2] \\
&= \frac{1}{8\pi}\left\{\int_0^{\frac{\tau+\lambda}{\sqrt{2}}} ds \int_{-\infty}^{\infty} \frac{C_\beta d\xi}{|\xi|^{1-\beta}} \left|\mathscr{F}\mathbf{1}_{[\sqrt{2}\lambda-s,\sqrt{2}(\lambda+h)-s]}(\xi)\right|^2\right. \\
&\qquad \left. + \int_{\frac{\tau+\lambda}{\sqrt{2}}}^{\frac{\tau+\lambda+h}{\sqrt{2}}} ds \int_{-\infty}^{\infty} \frac{C_\beta d\xi}{|\xi|^{1-\beta}} \left|\mathscr{F}\mathbf{1}_{[-\sqrt{2}\tau+s,\sqrt{2}(\lambda+h)-s]}(\xi)\right|^2\right\} \\
&= \frac{1}{2(2-\beta)(1-\beta)}\left\{\int_0^{\frac{\tau+\lambda}{\sqrt{2}}} \left(\sqrt{2}h\right)^{2-\beta} ds + \int_{\frac{\tau+\lambda}{\sqrt{2}}}^{\frac{\tau+\lambda+h}{\sqrt{2}}} \left(\sqrt{2}(\tau+\lambda+h) - 2s\right)^{2-\beta} ds\right\} \\
&= \frac{1}{2}\, K_\beta^2 \left[(\tau+\lambda)h^{2-\beta} + (3-\beta)^{-1}h^{3-\beta}\right].
\end{aligned}$$

$\square$

Recall a standard result for large deviation (cf. [33, 41]): If $\{Z(t) : t \in T\}$ is a continuous centered Gaussian random field which is a.s. bounded, then

$$\lim_{\gamma\to\infty} \frac{1}{\gamma^2} \log \mathbb{P}\left(\sup_{t\in T} Z(t) > \gamma\right) = -\frac{1}{2\sup_{t\in T}\mathbb{E}(Z(t)^2)}. \tag{6.9}$$

By symmetry of the distribution of $\{Z(t) : t \in T\}$, we have

$$\lim_{\gamma\to\infty} \frac{1}{\gamma^2} \log \mathbb{P}\left(\sup_{t\in T} |Z(t)| > \gamma\right) = -\frac{1}{2\sup_{t\in T}\mathbb{E}(Z(t)^2)}. \tag{6.10}$$

Now, we prove Proposition 6.2.1.

Proof of Proposition 6.2.1. It suffices to show that for any $0 \le a < b < \infty$ and any $0 < \varepsilon < 1$,

$$\mathbb{P}\left(\limsup_{h\to 0+} \frac{|\tilde{u}(\tau,\lambda+h) - \tilde{u}(\tau,\lambda)|}{\sqrt{(\tau+\lambda)h^{2-\beta}\log\log(1/h)}} \le (1+\varepsilon)K_\beta \text{ for all } \tau \in [a,b]\right) = 1. \tag{6.11}$$

Let $c \in [a,b]$, $\delta = (a+\lambda)\varepsilon/2$ and $d = c+\delta$. Take $0 < \theta < 1$ such that $\theta(1+\varepsilon) > 1$. Choose a real number q such that $1 < q < [\theta(1+\varepsilon)]^{1/(2-\beta)}$. Consider the event

$$A_n = \left\{ \sup_{\tau\in[0,d]} \sup_{h\in[0,q^{-n}]} \left|\tilde{u}(\tau,\lambda+h) - \tilde{u}(\tau,\lambda)\right| > \gamma_n \right\},$$

where

$$\gamma_n = (1+\varepsilon)K_\beta\sqrt{(c+\lambda)(q^{-n-1})^{2-\beta}\log\log q^n}.$$

By Lemma 6.2.3,

$$\mathbb{E}\left[(\tilde{u}(\tau,\lambda+h) - \tilde{u}(\tau,\lambda))^2\right] = \frac{1}{2}K_\beta^2\left[(\tau+\lambda)h^{2-\beta} + (3-\beta)^{-1}h^{3-\beta}\right].$$

By (6.10), for all large n,

$$\frac{1}{\gamma_n^2} \log \mathbb{P}(A_n) \leq -\frac{\theta}{K_\beta^2[(d+\lambda)(q^{-n})^{2-\beta} + (3-\beta)^{-1}(q^{-n})^{3-\beta}]}.$$

It follows that

$$\mathbb{P}(A_n) \leq \exp\left(-\frac{\theta(1+\varepsilon)^2(c+\lambda)}{q^{2-\beta}[(d+\lambda)+(3-\beta)^{-1}q^{-n}]} \log(n \log q)\right) = (n \log q)^{-p_n},$$

where

$$p_n = \frac{\theta(1+\varepsilon)^2}{q^{2-\beta}[(1+\frac{\delta}{c+\lambda}) + (3-\beta)^{-1}(c+\lambda)^{-1}q^{-n}]}.$$

Recall that $\delta = (c+\lambda)\varepsilon/2$. If n is sufficiently large, then $(3-\beta)^{-1}(c+\lambda)^{-1}q^{-n} \leq \varepsilon/2$, which implies that

$$p_n \geq \frac{\theta(1+\varepsilon)}{q^{2-\beta}} > 1.$$

Hence $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$ and by the Borel–Cantelli lemma, we have $\mathbb{P}(A_n \text{ i.o.}) = 0$. It follows that with probability 1,

$$\sup_{h \in [q^{-n-1}, q^{-n}]} \sup_{\tau \in [c,d]} \frac{|\tilde{u}(\tau, \lambda+h) - \tilde{u}(\tau, \lambda)|}{\sqrt{(c+\lambda)(q^{-n-1})^{2-\beta} \log\log q^n}} \leq (1+\varepsilon)K_\beta$$

eventually for all large n. Hence

$$\mathbb{P}\left(\limsup_{h \to 0+} \frac{|\tilde{u}(\tau, \lambda+h) - \tilde{u}(\tau, \lambda)|}{\sqrt{(\tau+\lambda)h^{2-\beta} \log\log(1/h)}} \leq (1+\varepsilon)K_\beta \text{ for all } \tau \in [c,d]\right) = 1.$$

From this, we can deduce (6.11) by covering the interval $[a,b]$ by finitely many intervals $[c,d]$ of length δ. $\square$

6.3 Simultaneous Law of Iterated Logarithm: Lower Bound

In this section, we prove the lower bound for the simultaneous LIL:

Proposition 6.3.1. *For any $\lambda > 0$,*

$$\mathbb{P}\left(\limsup_{h\to 0+} \frac{|\tilde{u}(\tau, \lambda+h) - \tilde{u}(\tau,\lambda)|}{\sqrt{(\tau+\lambda)h^{2-\beta}\log\log(1/h)}} \geq K_\beta \text{ for all } \tau \in [0,\infty)\right) = 1, \tag{6.12}$$

where K_β is the same constant as in Proposition 6.2.1, i.e.

$$K_\beta = \left(\frac{2^{(1-\beta)/2}}{(2-\beta)(1-\beta)}\right)^{1/2}.$$

Recall the following version of Borel–Cantelli lemma [55, p.391].

Lemma 6.3.2. *Let $\{A_n : n \geq 1\}$ be a sequence of events. If*

(i) $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$ *and*

(ii) $\liminf_{n\to\infty} \dfrac{\sum_{j=1}^{n}\sum_{k=1}^{n} \mathbb{P}(A_j \cap A_k)}{[\sum_{j=1}^{n} \mathbb{P}(A_j)]^2} = 1,$

then $\mathbb{P}(A_n \text{ i.o.}) = 1$.

We will also use the following lemma, which is essentially proved in [57]. For the sake of completeness, we provide a proof for this result.

Lemma 6.3.3. *Let Z_1 and Z_2 be jointly Gaussian random variables with $\mathbb{E}(Z_i) = 0$, $\mathbb{E}(Z_i^2) = 1$ and $\mathbb{E}(Z_1Z_2) = r$. Then for any $\gamma_1, \gamma_2 > 0$, there exists a number r^* between 0 and r such that*

$$\mathbb{P}(Z_1 > \gamma_1, Z_2 > \gamma_2) - \mathbb{P}(Z_1 > \gamma_1)\mathbb{P}(Z_2 > \gamma_2) = rg(\gamma_1, \gamma_2; r^*),$$

where $g(x,y;r)$ is the standard bivariate Gaussian density with correlation r, i.e.

$$g(x,y;r) = \frac{1}{2\pi(1-r^2)^{1/2}} \exp\left(-\frac{x^2+y^2-2rxy}{2(1-r^2)}\right).$$

Proof. Let $\gamma_1, \gamma_2 > 0$ and $p(r) = \int_{\gamma_1}^{\infty} \int_{\gamma_2}^{\infty} g(x,y;r)\,dx\,dy$. Define the Fourier transform of a function $f(x,y)$ as $\mathscr{F} f(\xi,\zeta) = \iint_{\mathbb{R}^2} e^{-i(x\xi+y\zeta)} f(x,y)\,dx\,dy$. Note that

$$g(x,y;r) = \frac{1}{(2\pi)^2} \iint_{\mathbb{R}^2} e^{i(x\xi+y\zeta)} [\mathscr{F} g(*\,;r)](\xi,\zeta)\,d\xi\,d\zeta$$

and

$$[\mathscr{F} g(*\,;r)](\xi,\zeta) = e^{-\frac{1}{2}(\xi^2+2r\xi\zeta+\zeta^2)}.$$

By the dominated convergence theorem,

$$\partial_r g(x,y;r) = \frac{-1}{(2\pi)^2} \iint_{\mathbb{R}^2} e^{i(x\xi+y\zeta)} \xi\zeta [\mathscr{F} g(*\,;r)](\xi,\zeta)\,d\xi\,d\zeta.$$

Since $(i\xi)(i\zeta) \cdot \mathscr{F} f(\xi,\zeta) = [\mathscr{F} \partial_x \partial_y f](\xi,\zeta)$, we have

$$\partial_r g(x,y;r) = \frac{1}{(2\pi)^2} \iint_{\mathbb{R}^2} e^{i(x\xi+y\zeta)} [\mathscr{F} \partial_x \partial_y g(*;r)](\xi,\zeta)\,d\xi\,d\zeta = \partial_x \partial_y g(x,y;r).$$

Therefore,

$$\partial_r p = \int_{\gamma_1}^{\infty} \int_{\gamma_2}^{\infty} \partial_x \partial_y g(x,y;r)\,dx\,dy = g(\gamma_1,\gamma_2;r).$$

The mean value theorem implies that $p(r) - p(0) = r g(\gamma_1,\gamma_2;r^*)$ for some r^* between 0 and r, and hence the result. $\square$

Let σ and $\tilde{\sigma}$ be the canonical metric on $\mathbb{R}_+ \times \mathbb{R}$ for u and $\tilde{u}$, respectively, i.e.

$$\sigma[(t, x), (t', x')] = \mathbb{E}[(u(t, x) - u(t', x'))^2]^{1/2},$$

$$\tilde{\sigma}[(\tau, \lambda), (\tau', \lambda')] = \mathbb{E}[(\tilde{u}(\tau, \lambda) - \tilde{u}(\tau', \lambda'))^2]^{1/2}.$$

For a rectangle $I = [a, a'] \times [-b, b]$, where $0 < a < a' < \infty$ and $0 < b < \infty$, recall from [20, Proposition 4.1] that there exist positive finite constants C_1 and C_2 such that

$$C_1\big(|t - t'| + |x - x'|\big)^{(2-\beta)/2} \leq \sigma[(t, x), (t', x')] \leq C_2\big(|t - t'| + |x - x'|\big)^{(2-\beta)/2} \tag{6.13}$$

for all $(t, x), (t', x') \in I$.

The proof of the following lemma is based on the method in [46, 47].

Lemma 6.3.4. *Let $\tau > 0$, $\lambda > 0$ and $q > 1$. Then for all $0 < \varepsilon < 1$,*

$$\mathbb{P}\left(\frac{\tilde{u}(\tau, \lambda + q^{-n}) - \tilde{u}(\tau, \lambda + q^{-n-1})}{\tilde{\sigma}[(\tau, \lambda + q^{-n}), (\tau, \lambda + q^{-n-1})]} \geq (1 - \varepsilon)\sqrt{2 \log \log q^n} \text{ infinitely often in } n\right) = 1. \tag{6.14}$$

Proof. For $n \geq 1$, let $A_n = \{Z_n > \gamma_n\}$, where

$$Z_n = \frac{\tilde{u}(\tau, \lambda + q^{-n}) - \tilde{u}(\tau, \lambda + q^{-n-1})}{\tilde{\sigma}[(\tau, \lambda + q^{-n}), (\tau, \lambda + q^{-n-1})]}$$

and

$$\gamma_n = (1 - \varepsilon)\sqrt{2 \log \log q^n}.$$

We will complete the proof by showing that (i) and (ii) of Lemma 6.3.2 are satisfied. For (i), by using the standard estimate

$$\mathbb{P}(Z > x) \geq (2\sqrt{2\pi})^{-1} x^{-1} \exp(-x^2/2), \quad x > 1, \tag{6.15}$$

for a standard Gaussian random variable Z, we derive that for large n,

$$\mathbb{P}(Z_n > \gamma_n) \geq \frac{C}{n^{(1-\varepsilon)^2} \sqrt{\log n}}$$

and hence $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$.

Next, we show that (ii) is satisfied. Since

$$\sum_{j=1}^{n} \sum_{k=1}^{n} \left[\mathbb{P}(A_j \cap A_k) - \mathbb{P}(A_j)\mathbb{P}(A_k)\right] = \mathbb{E}\left[\left(\sum_{j=1}^{n} \left(\mathbf{1}_{A_j} - \mathbb{P}(A_j)\right)\right)^2\right] \geq 0$$

and $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$, it is enough to prove that

$$\liminf_{n\to\infty} \frac{\sum_{1 \leq j < k \leq n} [\mathbb{P}(A_j \cap A_k) - \mathbb{P}(A_j)\mathbb{P}(A_k)]}{[\sum_{j=1}^{n} \mathbb{P}(A_j)]^2} \leq 0. \tag{6.16}$$

We are going to use Lemma 6.3.3 to estimate the difference in the numerator. First, we estimate the correlation r_{jk} between Z_j and Z_k for $j < k$. Let $(t_n, x_n) = (\frac{\tau+\lambda+q^{-n}}{\sqrt{2}}, \frac{-\tau+\lambda+q^{-n}}{\sqrt{2}})$.

Let $\Delta_j = \Delta(t_j, x_j)$. By (6.5), we have

$$
\begin{aligned}
&\mathbb{E}\big[(\tilde{u}(\tau,\lambda+q^{-j}) - \tilde{u}(\tau,\lambda+q^{-j-1}))(\tilde{u}(\tau,\lambda+q^{-k}) - \tilde{u}(\tau,\lambda+q^{-k-1}))\big] \\
&= \frac{1}{4}\,\mathbb{E}\big[W\big(\Delta_j\backslash\Delta_{j+1}\big)W\big(\Delta_k\backslash\Delta_{k+1}\big)\big] \\
&= \frac{1}{8\pi}\int_0^\infty ds\int_{\mathbb{R}}\frac{C_\beta d\xi}{|\xi|^{1-\beta}}\mathscr{F}(\mathbf{1}_{\Delta_j\backslash\Delta_{j+1}}(s,\cdot))(\xi)\overline{\mathscr{F}(\mathbf{1}_{\Delta_k\backslash\Delta_{k+1}}(s,\cdot))(\xi)} \\
&= \frac{1}{8\pi}\int_0^{t_{k+1}} ds\int_{\mathbb{R}}\frac{C_\beta d\xi}{|\xi|^{1-\beta}}\mathscr{F}(\mathbf{1}_{[x_{k+1}+t_{k+1}-s,\,x_k+t_k-s]})(\xi)\overline{\mathscr{F}(\mathbf{1}_{[x_{j+1}+t_{j+1}-s,\,x_j+t_j-s]})(\xi)} \\
&\quad+\frac{1}{8\pi}\int_{t_{k+1}}^{t_k} ds\int_{\mathbb{R}}\frac{C_\beta d\xi}{|\xi|^{1-\beta}}\mathscr{F}(\mathbf{1}_{[x_k-t_k+s,\,x_k+t_k-s]})(\xi)\overline{\mathscr{F}(\mathbf{1}_{[x_{j+1}+t_{j+1}-s,\,x_j+t_j-s]})(\xi)}.
\end{aligned}
$$

Note that (6.5) also implies that this covariance is nonnegative. Then by Lemma 6.2.2,

$$
\begin{aligned}
&\mathbb{E}\big[(\tilde{u}(\tau,\lambda+q^{-j}) - \tilde{u}(\tau,\lambda+q^{-j-1}))(\tilde{u}(\tau,\lambda+q^{-k}) - \tilde{u}(\tau,\lambda+q^{-k-1}))\big] \\
&= Ct_{k+1}\Big[(q^{-j-1}-q^{-k})^{2-\beta} - (q^{-j-1}-q^{-k-1})^{2-\beta} \\
&\qquad\qquad +(q^{-j}-q^{-k-1})^{2-\beta} - (q^{-j}-q^{-k})^{2-\beta}\Big] \\
&\quad+C\int_{q^{-k-1}}^{q^{-k}}\Big[(q^{-j-1}-q^{-k})^{2-\beta} - (q^{-j-1}-s)^{2-\beta} + (q^{-j}-s)^{2-\beta} - (q^{-j}-q^{-k})^{2-\beta}\Big]\,ds \\
&=: J_1 + J_2. \qquad (6.17)
\end{aligned}
$$

Let us consider the first term J_1. By the mean value theorem, we can find some a and b such that

$$
q^{-j-1}-q^{-k} \le a \le q^{-j-1}-q^{-k-1} < q^{-j}-q^{-k} \le b \le q^{-j}-q^{-k-1}.
$$

and

$$
\begin{aligned}
&(q^{-j-1} - q^{-k})^{2-\beta} - (q^{-j-1} - q^{-k-1})^{2-\beta} + (q^{-j} - q^{-k-1})^{2-\beta} - (q^{-j} - q^{-k})^{2-\beta} \\
&= (2-\beta)(b^{1-\beta} - a^{1-\beta})(q^{-k} - q^{-k-1}) \\
&\leq (2-\beta)[(q^{-j} - q^{-k-1})^{1-\beta} - (q^{-j-1} - q^{-k})^{1-\beta}]q^{-k}.
\end{aligned}
$$

Suppose $j \leq k-2$. By the mean value theorem again, we can find some ξ between $q^{-j}-q^{-k-1}$ and $q^{-j-1} - q^{-k}$ such that

$$
\begin{aligned}
(q^{-j} - q^{-k-1})^{1-\beta} - (q^{-j-1} - q^{-k})^{1-\beta} &= (1-\beta)\xi^{-\beta}[(q^{-j} - q^{-k-1}) - (q^{-j-1} - q^{-k})] \\
&\leq (1-\beta)(q^{-j-1} - q^{-k})^{-\beta} q^{-j} \\
&\leq (1-\beta)(q^{-j-1} - q^{-j-2})^{-\beta} q^{-j} \\
&\leq (1-\beta)(q^{-1} - q^{-2})(q^{-j})^{1-\beta}.
\end{aligned}
$$

It follows that

$$
J_1 \leq C(q^{-j})^{1-\beta} q^{-k}.
$$

Next, we consider the term J_2 in (6.17). For every $s \in [q^{-k-1}, q^{-k}]$, we can find some $\tilde{a}$ and $\tilde{b}$ (depending on s) such that

$$
q^{-j-1} - q^{-k} \leq \tilde{a} \leq q^{-j-1} - s < q^{-j} - q^{-k} \leq \tilde{b} \leq q^{-j} - s
$$

and

$$
\begin{aligned}
&(q^{-j-1} - q^{-k})^{2-\beta} - (q^{-j-1} - s)^{2-\beta} + (q^{-j} - s)^{2-\beta} - (q^{-j} - q^{-k})^{2-\beta} \\
&= (2-\beta)(\tilde{b}^{1-\beta} - \tilde{a}^{1-\beta})(q^{-k} - s) \\
&\le (2-\beta)[(q^{-j} - s)^{1-\beta} - (q^{-j-1} - q^{-k})^{1-\beta}]q^{-k}.
\end{aligned}
$$

For $j \le k-2$, by the mean value theorem again, there exists η between $q^{-j} - s$ and $q^{-j-1} - q^{-k}$ such that

$$
\begin{aligned}
(q^{-j} - s)^{1-\beta} - (q^{-j-1} - q^{-k})^{1-\beta} &= (1-\beta)\eta^{-\beta}[(q^{-j} - s) - (q^{-j-1} - q^{-k})] \\
&\le (1-\beta)(q^{-j-1} - q^{-j-2})^{-\beta} q^{j} \\
&\le (1-\beta)(q^{-1} - q^{-2})(q^{-j})^{1-\beta}.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
J_2 = C \int_{q^{-k-1}}^{q^{-k}} &\left[(q^{-j-1} - q^{-k})^{2-\beta} - (q^{-j-1} - s)^{2-\beta} + (q^{-j} - s)^{2-\beta} - (q^{-j} - q^{-k})^{2-\beta}\right] ds \\
&\le C(q^{-j})^{1-\beta}(q^{-k})^2 \le C(q^{-j})^{1-\beta} q^{-k}.
\end{aligned}
$$

Therefore, by combining (6.17), the upper bounds for J_1, J_2, and recalling (6.13), we see that for $j \le k-2$, the correlation r_{jk} between Z_j and Z_k satisfies

$$
0 \le r_{jk} = \mathbb{E}(Z_j Z_k) \le \frac{C(q^{-j})^{1-\beta} q^{-k}}{(q^{-j})^{1-\beta/2}(q^{-k})^{1-\beta/2}} = C_0 (q^{-(k-j)})^{\beta/2} =: \xi_{jk}. \tag{6.18}
$$

By (6.18), we can choose a fixed $l \ge 2$ such that $r := \sup\{r_{jk} : j \le k - l\} < 1$. Since

$\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$, in order to prove (6.16), it suffices to prove that for any $\delta > 0$, there exists m such that

$$\liminf_{n\to\infty} \frac{\sum_{k=m}^{n} \sum_{j=1}^{k-l} [\mathbb{P}(A_j \cap A_k) - \mathbb{P}(A_j)\mathbb{P}(A_k)]}{[\sum_{j=1}^{n} \mathbb{P}(A_j)]^2} \leq \delta. \tag{6.19}$$

Let $\delta > 0$ be given and let m be a large integer that will be chosen appropriately depending on δ. Let $\rho_k = \frac{4}{(\beta/2)\log q} \log \gamma_k$, so that for $1 \leq j \leq k - \rho_k$,

$$\xi_{jk} \leq C_0 \gamma_k^{-4}. \tag{6.20}$$

Provided m is large, $1 < k - \rho_k < k - l$ for all $k \geq m$. By Lemma 6.3.3, we have

$$\sum_{k=m}^{n} \sum_{j=1}^{k-l} [\mathbb{P}(A_j \cap A_k) - \mathbb{P}(A_j)\mathbb{P}(A_k)] \leq \left(\sum_{k=m}^{n} \sum_{j=1}^{\lfloor k-\rho_k \rfloor} + \sum_{k=m}^{n} \sum_{j=\lfloor k-\rho_k \rfloor}^{k-l} \right) r_{jk} g(\gamma_j, \gamma_k; r_{jk}^*), \tag{6.21}$$

where r_{jk}^* is a number such that $0 \leq r_{jk}^* \leq r_{jk}$ for each j, k. Let us consider the two sums on the right-hand side of (6.21) separately. By (6.18), the first sum is

$$\begin{aligned}
&\sum_{k=m}^{n} \sum_{j=1}^{\lfloor k-\rho_k \rfloor} \frac{r_{jk}}{2\pi(1 - r_{jk}^{*2})^{1/2}} \exp\left(- \frac{\gamma_j^2 + \gamma_k^2 - 2r_{jk}^* \gamma_j \gamma_k}{2(1 - r_{jk}^{*2})} \right) \\
&\leq \sum_{k=m}^{n} \sum_{j=1}^{\lfloor k-\rho_k \rfloor} \frac{\xi_{jk} \gamma_j \gamma_k}{2\pi(1 - \xi_{jk}^2)^{1/2}} \exp\left(\frac{-r_{jk}^{*2}(\gamma_j^2 + \gamma_k^2) + 2r_{jk}^* \gamma_j \gamma_k}{2(1 - r_{jk}^{*2})} \right) \gamma_j^{-1} e^{-\gamma_j^2/2} \gamma_k^{-1} e^{-\gamma_k^2/2}.
\end{aligned}$$

Note that $\gamma_j < \gamma_k$ for $j < k$. Then by (6.15), (6.18) and (6.20), the sum is

$$\leq 4 \sum_{k=m}^{n} \sum_{j=1}^{\lfloor k-\rho_k \rfloor} \frac{C_0 \gamma_k^{-2}}{(1 - C_0^2 \gamma_k^{-8})^{1/2}} \exp\left(\frac{C_0 \gamma_k^{-2}}{1 - C_0^2 \gamma_k^{-8}} \right) \mathbb{P}(A_j)\mathbb{P}(A_k).$$

Since $\gamma_k \to \infty$, we may choose m to be large enough such that this sum is $\le \delta[\sum_{j=1}^n \mathbb{P}(A_j)]^2$.

By (6.15), the second sum on the right-hand side of (6.21) is

$$\begin{aligned}
&\sum_{k=m}^{n} \sum_{j=\lfloor k-\rho_k \rfloor}^{k-l} \frac{r_{jk}}{2\pi(1-r_{jk}^{*2})^{1/2}} \exp\left(-\frac{\gamma_j^2+\gamma_k^2-2r_{jk}^*\gamma_j\gamma_k}{2(1-r_{jk}^{*2})}\right) \\
&\le \sum_{k=m}^{n} \sum_{j=\lfloor k-\rho_k \rfloor}^{k-l} \frac{r_{jk}\gamma_j}{2\pi(1-r_{jk}^{*2})^{1/2}} \exp\left(-\frac{(\gamma_k-r_{jk}^*\gamma_j)^2}{2(1-r_{jk}^{*2})}\right) \gamma_j^{-1} e^{-\gamma_j^2/2} \\
&\le \frac{2}{\sqrt{2\pi}} \sum_{k=m}^{n} \sum_{j=\lfloor k-\rho_k \rfloor}^{k-l} \frac{\gamma_k}{(1-r_{jk}^{*2})^{1/2}} \exp\left(-\frac{(1-r_{jk}^*)^2\gamma_k^2}{2(1-r_{jk}^{*2})}\right) \mathbb{P}(A_j).
\end{aligned}$$

Recall that $r = \sup\{r_{jk} : j \le k-l\} < 1$. Moreover, if m is large enough, then

$$\frac{\gamma_k \log \gamma_k}{(1-r^2)^{1/2}} \exp\left(-\frac{(1-r)\gamma_k^2}{2(1+r)}\right) \le \delta$$

and $k - \rho_k > k/2$ for all $k \ge m$, so that the last sum above is

$$\begin{aligned}
&\le \frac{2}{\sqrt{2\pi}} \sum_{k=m}^{n} \frac{\rho_k\gamma_k}{(1-r^2)^{1/2}} \exp\left(-\frac{(1-r)\gamma_k^2}{2(1+r)}\right) \mathbb{P}(A_{\lfloor k-\rho_k \rfloor}) \\
&\le C \sum_{k=m}^{n} \frac{\gamma_k \log\gamma_k}{(1-r^2)^{1/2}} \exp\left(-\frac{(1-r)\gamma_k^2}{2(1+r)}\right) \mathbb{P}(A_{\lfloor k/2 \rfloor}) \\
&\le 2C\delta \sum_{k=1}^{n} \mathbb{P}(A_k).
\end{aligned}$$

We get that

$$\sum_{k=m}^{n} \sum_{j=1}^{k-l} [\mathbb{P}(A_j \cap A_k) - \mathbb{P}(A_j)\mathbb{P}(A_k)] \le \delta\left(\sum_{j=1}^{n} \mathbb{P}(A_j)\right)^2 + 2C\delta \sum_{j=1}^{n} \mathbb{P}(A_j).$$

Hence (6.19) follows and the proof of Lemma 6.3.4 is complete. □

We now come to the proof of Proposition 6.3.1.

Proof of Proposition 6.3.1. Fix $\lambda > 0$. It suffices to show that for any $0 \le a < b < \infty$ and $0 < \varepsilon < 1$,

$$\mathbb{P}\left(\limsup_{h\to 0+} \frac{|\tilde{u}(\tau,\lambda+h) - \tilde{u}(\tau,\lambda)|}{\sqrt{(\tau+\lambda)h^{2-\beta}\log\log(1/h)}} \ge (1-\varepsilon)K_\beta \text{ for all } \tau \in [a,b]\right) = 1. \tag{6.22}$$

To this end, let us fix a, b and ε for the rest of the proof.

Note that when q is large, $q^{-(2-\beta)/2}(1 + \frac{q^{-n-1}}{\tau+\lambda})^{1/2} < \varepsilon/4$ uniformly for all $\tau \in [a,b]$. So we can choose and fix a large $q > 1$ such that

$$(1-\varepsilon/4)\left(\frac{q-1}{q}\right)^{(2-\beta)/2} - q^{-(2-\beta)/2}\left(1 + \frac{q^{-n-1}}{\tau+\lambda}\right)^{1/2} - (1-\varepsilon) > \varepsilon/4 \tag{6.23}$$

for all $\tau \in [a,b]$. We also choose $\delta > 0$ small such that

$$\frac{\lambda(\varepsilon/4)^2}{\delta} > 1. \tag{6.24}$$

Since we can cover $[a,b]$ by finitely many intervals $[c,d]$ of length δ, we only need to show (6.22) for $\tau \in [c,d]$, where $[c,d] \subset [a,b]$ and $d = c + \delta$.

Let us define the increment of $\tilde{u}$ over a rectangle $(\tau,\tau'] \times (\lambda,\lambda']$ by

$$\Delta\tilde{u}((\tau,\tau'] \times (\lambda,\lambda']) = \tilde{u}(\tau',\lambda') - \tilde{u}(\tau,\lambda') - \tilde{u}(\tau',\lambda) + \tilde{u}(\tau,\lambda).$$

Then for all $\tau \in [c, d]$ we can write

$$\begin{aligned} \tilde{u}(\tau, \lambda + q^{-n}) - \tilde{u}(\tau, \lambda) = {} & \tilde{u}(d, \lambda + q^{-n}) - \tilde{u}(d, \lambda + q^{-n-1}) \\ & + \tilde{u}(\tau, \lambda + q^{-n-1}) - \tilde{u}(\tau, \lambda) \\ & - \Delta\tilde{u}((\tau, d] \times (\lambda + q^{-n-1}, \lambda + q^{-n}]). \end{aligned} \tag{6.25}$$

By Lemma 6.3.4, we have

$$\frac{|\tilde{u}(d, \lambda + q^{-n}) - \tilde{u}(d, \lambda + q^{-n-1})|}{\tilde{\sigma}[(d, \lambda + q^{-n}), (d, \lambda + q^{-n-1})]} \geq (1 - \varepsilon/4)\sqrt{2 \log \log q^n}$$

infinitely often in n with probability 1. By Lemma 6.2.3,

$$\begin{aligned} & \tilde{\sigma}[(d, \lambda + q^{-n}), (d, \lambda + q^{-n-1})] \\ & = \frac{K_\beta}{\sqrt{2}} \sqrt{(d + \lambda + q^{-n-1})(q^{-n} - q^{-n-1})^{2-\beta} + (3 - \beta)^{-1}(q^{-n} - q^{-n-1})^{3-\beta}}, \end{aligned}$$

so we have

$$|\tilde{u}(d, \lambda + q^{-n}) - \tilde{u}(d, \lambda + q^{-n-1})| \geq (1 - \varepsilon/4) K_\beta \sqrt{(d + \lambda)(q^{-n} - q^{-n-1})^{2-\beta} \log \log q^n} \tag{6.26}$$

infinitely often in n with probability 1. Also, by Proposition 6.2.1, with probability 1, for all $\tau \in [c, d]$ simultaneously,

$$|\tilde{u}(\tau, \lambda + q^{-n-1}) - \tilde{u}(\tau, \lambda)| \leq K_\beta \sqrt{(\tau + \lambda + q^{-n-1})(q^{-n-1})^{2-\beta} \log \log q^n} \tag{6.27}$$

eventually for all large n.

Next, we derive a bound for the term $\Delta\tilde{u}((\tau, d] \times (\lambda + q^{-n-1}, \lambda + q^{-n}])$. For $\tau \in [c, d]$, let

$$\phi(\tau) = (1-\varepsilon/4)\left(\frac{q-1}{q}\right)^{(2-\beta)/2}(d+\lambda)^{1/2} - q^{-(2-\beta)/2}(\tau+\lambda+q^{-n-1})^{1/2} - (1-\varepsilon)(\tau+\lambda)^{1/2}.$$

Consider the events

$$A_n = \left\{ \sup_{\tau\in[c,d]} |\Delta\tilde{u}((\tau,d]\times(\lambda+q^{-n-1},\lambda+q^{-n}])| > \gamma_n \right\},$$

where

$$\gamma_n = K_\beta\,\phi(d)\sqrt{(q^{-n})^{2-\beta}\log\log q^n}.$$

Note that $\Delta\tilde{u}((\tau,d]\times(\lambda+q^{-n-1},\lambda+q^{-n}]) = \frac{1}{2}W(Q)$, where Q is the image of the rectangle $(\tau,d]\times(\lambda+q^{-n-1},\lambda+q^{-n}]$ under the rotation $(\tau,\lambda)\mapsto(\frac{\tau+\lambda}{\sqrt{2}},\frac{-\tau+\lambda}{\sqrt{2}})$. Provided n is large, we have $Q = Q_1\cup Q_2\cup Q_3$, where

$$Q_1 = \left\{(t,x) : \frac{\tau+\lambda+q^{-n-1}}{\sqrt{2}} < t \le \frac{\tau+\lambda+q^{-n}}{\sqrt{2}}, \sqrt{2}(\lambda+q^{-n-1}) - s < x < -\sqrt{2}\tau + s\right\},$$

$$Q_2 = \left\{(t,x) : \frac{\tau+\lambda+q^{-n}}{\sqrt{2}} < t \le \frac{d+\lambda+q^{-n-1}}{\sqrt{2}}, \right.$$
$$\left. \sqrt{2}(\lambda+q^{-n-1}) - s < x \le \sqrt{2}(\lambda+q^{-n}) - s\right\},$$

$$Q_3 = \left\{(t,x) : \frac{d+\lambda+q^{-n-1}}{\sqrt{2}} < t \le \frac{d+\lambda+q^{-n}}{\sqrt{2}}, -\sqrt{2}d + s \le x \le \sqrt{2}(\lambda+q^{-n}) - s\right\}.$$

By (6.5), it follows that

$$
\begin{aligned}
&\mathbb{E}\big[(\Delta\tilde{u}((\tau,d]\times(\lambda+q^{-n-1},\lambda+q^{-n}]))^2\big] = \frac{1}{4}\,\mathbb{E}\big[W(Q)^2\big]\\
&= \frac{1}{8\pi}\Bigg\{\int_{\frac{\tau+\lambda+q^{-n-1}}{\sqrt{2}}}^{\frac{\tau+\lambda+q^{-n}}{\sqrt{2}}} ds\int_{-\infty}^{\infty}\frac{C_\beta d\xi}{|\xi|^{1-\beta}}\big|\widehat{\mathbf{1}}_{[\sqrt{2}(\lambda+q^{-n-1})-s,\,-\sqrt{2}\tau+s]}(\xi)\big|^2\\
&\quad+\int_{\frac{\tau+\lambda+q^{-n}}{\sqrt{2}}}^{\frac{d+\lambda+q^{-n-1}}{\sqrt{2}}} ds\int_{-\infty}^{\infty}\frac{C_\beta d\xi}{|\xi|^{1-\beta}}\big|\widehat{\mathbf{1}}_{[\sqrt{2}(\lambda+q^{-n-1})-s,\,\sqrt{2}(\lambda+q^{-n})-s]}(\xi)\big|^2\\
&\quad+\int_{\frac{d+\lambda+q^{-n-1}}{\sqrt{2}}}^{\frac{d+\lambda+q^{-n}}{\sqrt{2}}} ds\int_{-\infty}^{\infty}\frac{C_\beta d\xi}{|\xi|^{1-\beta}}\big|\widehat{\mathbf{1}}_{[-\sqrt{2}d+s,\,\sqrt{2}(\lambda+q^{-n})-s]}(\xi)\big|^2\Bigg\}.
\end{aligned}
$$

Then by Lemma 6.2.2,

$$
\begin{aligned}
&\mathbb{E}\Big[\big(\Delta\tilde{u}((\tau,d]\times(\lambda+q^{-n-1},\lambda+q^{-n}])\big)^2\Big]\\
&= \frac{1}{2(2-\beta)(1-\beta)}\Bigg\{\int_{\frac{\tau+\lambda+q^{-n-1}}{\sqrt{2}}}^{\frac{\tau+\lambda+q^{-n}}{\sqrt{2}}}\big(2s-\sqrt{2}(\tau+\lambda+q^{-n-1})\big)^{2-\beta}ds\\
&\qquad\qquad+\int_{\frac{\tau+\lambda+q^{-n}}{\sqrt{2}}}^{\frac{d+\lambda+q^{-n-1}}{\sqrt{2}}}\big(\sqrt{2}(q^{-n}-q^{-n-1})\big)^{2-\beta}ds\\
&\qquad\qquad+\int_{\frac{d+\lambda+q^{-n-1}}{\sqrt{2}}}^{\frac{d+\lambda+q^{-n}}{\sqrt{2}}}\big(\sqrt{2}(d+\lambda+q^{-n})-2s\big)^{2-\beta}ds\Bigg\}\\
&= \frac{1}{2(2-\beta)(1-\beta)}\Bigg\{2\cdot\frac{2^{\frac{1-\beta}{2}}}{3-\beta}(q^{-n}-q^{-n-1})^{3-\beta}\\
&\qquad\qquad+2^{\frac{1-\beta}{2}}(q^{-n}-q^{-n-1})^{2-\beta}\big(d-\tau-(q^{-n}-q^{-n-1})\big)\Bigg\}\\
&= \frac{1}{2}\,K_\beta^2(q^{-n}-q^{-n-1})^{2-\beta}\Bigg\{(d-\tau)-\frac{1-\beta}{3-\beta}(q^{-n}-q^{-n-1})\Bigg\}.
\end{aligned}
$$

Since $d-\tau \le d-c=\delta$, we have

$$\sup_{\tau\in[c,d]} \mathbb{E}\big[(\Delta\tilde{u}((\tau,d]\times(\lambda+q^{-n-1},\lambda+q^{-n}]))^2\big] \le \frac{1}{2}\,K_\beta^2(q^{-n}-q^{-n-1})^{2-\beta}\delta.$$

By (6.10), for all large n,

$$\frac{1}{\gamma_n^2}\log\mathbb{P}(A_n) \le -\frac{1}{K_\beta^2(q^{-n}-q^{-n-1})^{2-\beta}\delta}.$$

It follows that

$$\mathbb{P}(A_n) \le \exp\left(-\frac{\phi(d)^2(q^{-n})^{2-\beta}\log\log q^n}{(q^{-n}-q^{-n-1})^{2-\beta}\delta}\right) = (n\log q)^p,$$

where

$$p=\frac{1}{\delta}\left(\frac{q}{q-1}\right)^{2-\beta}\phi(d)^2.$$

By (6.23) and (6.24),

$$\begin{aligned} p &\ge \frac{d+\lambda}{\delta}\left[(1-\varepsilon/4)\left(\frac{q-1}{q}\right)^{(2-\beta)/2} - q^{-(2-\beta)/2}\left(1+\frac{q^{-n-1}}{d+\lambda}\right)^{1/2} - (1-\varepsilon)\right]^2 \\ &> \frac{\lambda(\varepsilon/4)^2}{\delta} > 1. \end{aligned}$$

Hence $\mathbb{P}(A_n \text{ i.o.})=0$ by the Borel–Cantelli lemma. Then the symmetry of u and the monotonic decreasing property of ϕ imply that with probability 1, simultaneously for all $\tau\in[c,d]$,

$$\left|\Delta\tilde{u}((\tau,d]\times(\lambda+q^{-n-1},\lambda+q^{-n}])\right| \le K_\beta\phi(\tau)\sqrt{(q^{-n})^{2-\beta}\log\log q^n} \tag{6.28}$$

eventually for all large n. By (6.25) and the triangle inequality,

$$\begin{aligned}\left|\tilde{u}(\tau,\lambda+q^{-n})-\tilde{u}(\tau,\lambda)\right| \geq & \left|\tilde{u}(d,\lambda+q^{-n})-\tilde{u}(d,\lambda+q^{-n-1})\right| \\ & -\left|\tilde{u}(\tau,\lambda+q^{-n-1})-\tilde{u}(\tau,\lambda)\right| \\ & -\left|\Delta\tilde{u}((\tau,d]\times(\lambda+q^{-n-1},\lambda+q^{-n}])\right|.\end{aligned}$$

Then (6.26), (6.27) and (6.28) together imply that with probability 1, for all $\tau \in [c,d]$ simultaneously,

$$\begin{aligned}&\left|\tilde{u}(\tau,\lambda+q^{-n})-\tilde{u}(\tau,\lambda)\right| \\ &\geq \left[(1-\varepsilon/4)\left(\frac{q-1}{q}\right)^{(2-\beta)/2}(d+\lambda)^{1/2}-q^{-(2-\beta)/2}(\tau+\lambda+q^{-n-1})^{1/2}-\phi(\tau)\right] \\ &\qquad \times K_\beta\sqrt{(q^{-n})^{2-\beta}\log\log q^n} \\ &\geq (1-\varepsilon)K_\beta\sqrt{(\tau+\lambda)(q^{-n})^{2-\beta}\log\log q^n}\end{aligned}$$

infinitely often in n. This yields (6.22) for $\tau \in [c,d]$ and concludes the proof of Proposition 6.3.1. □

6.4 Singularities and Their Propagation

In this section, we study the existence and propagation of singularities of the stochastic wave equation (6.1). The main result is Theorem 6.4.3.

Let us first discuss the interpretation of singularities and how they may arise. Proposition

6.2.1 and 6.3.1 imply that LIL holds at any fixed point (t, x):

$$\limsup_{h\to 0+} \frac{|u(t+\frac{h}{\sqrt{2}}, x+\frac{h}{\sqrt{2}}) - u(t,x)|}{\sqrt{h^{2-\beta}\log\log(1/h)}} = K_\beta(\sqrt{2}t)^{1/2} \quad \text{a.s.}$$

It indicates the size of oscillation of u when (t, x) is fixed. However, the behavior will be different when (t, x) is not fixed. Indeed, from the modulus of continuity in Theorem 5.3.1, we know that for $I = [a, a'] \times [-b, b]$, where $0 < a < a'$ and $b > 0$, there exists a positive finite constant K such that

$$\lim_{h\to 0+} \sup_{\substack{(t,x),(t',x')\in I:\\ 0<\sigma[(t,x),(t',x')]\le h}} \frac{|u(t',x') - u(t,x)|}{\sigma[(t,x),(t',x')]\sqrt{\log(1+\sigma[(t,x),(t',x')]^{-1})}} = K \quad \text{a.s.}$$

Recalling (6.13), this result shows that the largest oscillation in I is of order $\sqrt{h^{2-\beta}\log(1/h)}$, which is larger than $\sqrt{h^{2-\beta}\log\log(1/h)}$ specified by the LIL. It suggests that the LIL does not hold simultaneously for all $(t, x) \in I$ and there may exist random exceptional points with much larger oscillation. Therefore, we can define singularities as such points where the LIL fails. More precisely, we say that (τ, λ) is a *singular point* of $\tilde{u}$ in the λ-direction if

$$\limsup_{h\to 0+} \frac{|\tilde{u}(\tau, \lambda+h) - \tilde{u}(\tau,\lambda)|}{\sqrt{h^{2-\beta}\log\log(1/h)}} = \infty$$

and a *singular point* in the τ-direction if

$$\limsup_{h\to 0+} \frac{|\tilde{u}(\tau+h, \lambda) - \tilde{u}(\tau,\lambda)|}{\sqrt{h^{2-\beta}\log\log(1/h)}} = \infty.$$

Our goal is to justify the existence of random singular points and study their propagation.

Fix $\tau_0 > 0$. Let us decompose $\tilde{u}$ into $\tilde{u}_1 + \tilde{u}_2$, where

$$\tilde{u}_i(\tau, \lambda) = u_i\Big(\frac{\tau + \lambda}{\sqrt{2}}, \frac{-\tau + \lambda}{\sqrt{2}}\Big), \quad i = 1, 2,$$

and

$$\begin{aligned} u_1(t, x) &= \frac{1}{2} W\Big(\Delta(t, x) \cap \big\{0 \le t < \tau_0/\sqrt{2}\big\}\Big), \\ u_2(t, x) &= \frac{1}{2} W\Big(\Delta(t, x) \cap \big\{t \ge \tau_0/\sqrt{2}\big\}\Big). \end{aligned}$$

Let $\mathscr{F}_{\tau_0}$ be the σ-field generated by $\{W\big(B \cap \{0 \le t < \tau_0/\sqrt{2}\}\big) : B \in \mathscr{B}_b(\mathbb{R}^2)\}$ and the $\mathbb{P}$-null sets. Note that $\mathscr{F}_{\tau_0}$ is independent of the process $\tilde{u}_2$.

Following the approach of Walsh [63] and Blath and Martin [7], we will use Meyer's section theorem to prove the existence of a random singularity. Let us recall Meyer's section theorem ([21], Theorem 37, p.18):

Let $(\Omega, \mathscr{G}, \mathbb{P})$ be a complete probability space and S be a $\mathscr{B}(\mathbb{R}_+) \times \mathscr{G}$-measurable subset of $\mathbb{R}_+ \times \Omega$. Then there exists a $\mathscr{G}$-measurable random variable T with values in $(0, \infty]$ such that

(a) the graph of T, denoted by $[T] := \{(t, \omega) \in \mathbb{R}_+ \times \Omega : T(\omega) = t\}$, is contained in S;

(b) $\{T < \infty\}$ is equal to the projection $\pi(S)$ of S onto Ω.

Lemma 6.4.1. *Let $\tau_0 > 0$. Then there exists a positive, finite, $\mathscr{F}_{\tau_0}$-measurable random variable Λ such that*

$$\limsup_{h \to 0+} \frac{|\tilde{u}_1(\tau_0, \Lambda + h) - \tilde{u}_1(\tau_0, \Lambda)|}{\sqrt{h^{2-\beta} \log\log(1/h)}} = \infty \quad a.s.$$

Proof. Note that

$$\limsup_{h\to 0+} \frac{|\tilde{u}_1(\tau_0, \Lambda+h) - \tilde{u}_1(\tau_0, \Lambda)|}{\sqrt{h^{2-\beta}\log\log(1/h)}} = \limsup_{h\to 0+} \frac{|\tilde{v}_1(\tau_0, \Lambda+h) - \tilde{v}_1(\tau_0, \Lambda)|}{\sqrt{h^{2-\beta}\log\log(1/h)}},$$

where $\tilde{v}_1(\tau_0, \lambda) = \tilde{u}_1(\tau_0, \lambda) - \tilde{u}_1(\tau_0, 0)$. The covariance for the process $\{\tilde{v}_1(\tau_0, \lambda) : \lambda \geq 0\}$ is

$$\mathbb{E}[\tilde{v}_1(\tau_0, \lambda)\tilde{v}_1(\tau_0, \lambda')] = \frac{1}{4}\,\mathbb{E}[W(A_\lambda)W(A_{\lambda'})]$$

for $\lambda, \lambda' \geq 0$, where $A_\lambda = \{(t, x) : 0 \leq t < \tau_0/\sqrt{2}, -t < x \leq \sqrt{2}\lambda - t\}$. By (6.5) and Lemma 6.2.2,

$$\begin{aligned}
&\mathbb{E}[\tilde{v}_1(\tau_0, \lambda)\tilde{v}_1(\tau_0, \lambda')] \\
&= \frac{1}{8\pi}\int_0^{\tau_0/\sqrt{2}} ds \int_{-\infty}^{\infty} \frac{C_\beta d\xi}{|\xi|^{1-\beta}} \mathscr{F}\mathbf{1}_{[-s,\sqrt{2}\lambda - s]}(\xi)\overline{\mathscr{F}\mathbf{1}_{[-s,\sqrt{2}\lambda' - s]}(\xi)} \\
&= \frac{1}{4(2-\beta)(1-\beta)}\int_0^{\tau_0/\sqrt{2}} \Big(|\sqrt{2}\lambda|^{2-\beta} + |\sqrt{2}\lambda'|^{2-\beta} - |\sqrt{2}\lambda - \sqrt{2}\lambda'|^{2-\beta}\Big) ds \\
&= \frac{2^{-(3+\beta)/2}\,\tau_0}{(2-\beta)(1-\beta)}\Big(|\lambda|^{2-\beta} + |\lambda'|^{2-\beta} - |\lambda - \lambda'|^{2-\beta}\Big).
\end{aligned}$$

It follows that $\{C_0\tilde{v}_1(\tau_0, \lambda) : \lambda \geq 0\}$ is a fractional Brownian motion of Hurst parameter $(2-\beta)/2$ for some constant C_0 depending on τ_0 and β.

Let

$$S = \left\{(\lambda, \omega) \in \mathbb{R}_+ \times \Omega : \limsup_{h\to 0+} \frac{|\tilde{v}_1(\tau_0, \lambda+h)(\omega) - \tilde{v}_1(\tau_0, \lambda)(\omega)|}{\sqrt{h^{2-\beta}\log\log(1/h)}} = \infty\right\}.$$

Then S is $\mathscr{B}(\mathbb{R}_+) \times \mathscr{F}_{\tau_0}$-measurable. Using Meyer's section theorem, we can find a positive $\mathscr{F}_{\tau_0}$-measurable random variable Λ such that (a) $[\Lambda] \subset S$, and (b) $\pi(S) = \{\Lambda < \infty\}$.

We claim that $\Lambda < \infty$ a.s. Indeed, by the modulus of continuity for fractional Brownian motion (cf. [30], Theorem 1.1), for any $0 \leq a < b$,

$$\limsup_{h \to 0+} \sup_{\lambda \in [a,b]} \frac{|\tilde{v}_1(\tau_0, \lambda + h) - \tilde{v}_1(\tau_0, \lambda)|}{\sqrt{h^{2-\beta} \log(1/h)}} = C_0^{-1} \sqrt{2} \quad \text{a.s.} \tag{6.29}$$

We now use an argument with nested intervals (cf. [48], Theorem 1) to show the existence of a random λ^* such that

$$\limsup_{h \to 0+} \frac{|\tilde{v}_1(\tau_0, \lambda^* + h) - \tilde{v}_1(\tau_0, \lambda^*)|}{\sqrt{h^{2-\beta} \log\log(1/h)}} = \infty \tag{6.30}$$

with probability 1. First take an event Ω^* of probability 1 such that (6.29) holds for all intervals $[a, b]$, where a and b are rational numbers. Let

$$\varphi(h) = \frac{1}{2} C_0^{-1} \sqrt{2h^{2-\beta} \log(1/h)}.$$

Let $h_0 > 0$ be small such that φ is increasing on $[0, h_0]$. For an $\omega \in \Omega^*$, we define two sequences $(\lambda_n), (\lambda_n')$ as follows. By (6.29), we can choose λ_1, λ_1', say in $[1, 2]$, with $\lambda_1 < \lambda_1'$ such that $\lambda_1' - \lambda_1 < h_0$ and

$$|\tilde{v}_1(\tau_0, \lambda_1') - \tilde{v}_1(\tau_0, \lambda_1)| > \varphi(\lambda_1' - \lambda_1).$$

Suppose $n \geq 1$ and λ_n and λ_n' are chosen with $\lambda_n < \lambda_n'$ and

$$|\tilde{v}_1(\tau_0, \lambda_n') - \tilde{v}_1(\tau_0, \lambda_n)| > \varphi(\lambda_n' - \lambda_n).$$

Since $\tilde{v}_1$ is continuous and $\varphi(h)$ is increasing for h small, we can find some $\tilde{\lambda}_n$ such that $\lambda_n < \tilde{\lambda}_n < \min\{\lambda'_n, \lambda_n + 2^{-n}\}$ and

$$|\tilde{v}_1(\tau_0, \lambda'_n) - \tilde{v}_1(\tau_0, \lambda)| > \varphi(\lambda'_n - \lambda) \quad \text{for all } \lambda \in [\lambda_n, \tilde{\lambda}_n]. \tag{6.31}$$

Then we can apply (6.29) for a rational interval $[a, b] \subseteq [\lambda_n, \tilde{\lambda}_n]$ to find λ_{n+1} and λ'_{n+1} such that $\lambda_n \le \lambda_{n+1} < \lambda'_{n+1} \le \tilde{\lambda}_n$ and

$$|\tilde{v}_1(\tau_0, \lambda'_{n+1}) - \tilde{v}_1(\tau_0, \lambda_{n+1})| > \varphi(\lambda'_{n+1} - \lambda_{n+1}).$$

We obtain a sequence of nested intervals $[\lambda_1, \lambda'_1] \supset [\lambda_2, \lambda'_2] \supset \cdots$ with lengths $\lambda'_n - \lambda_n \le 2^{-n+1}$. Therefore, the intervals contain a common point $\lambda^* \in [1, 2]$ such that $\lambda'_{n+1} \downarrow \lambda^*$. Since $\lambda^* \in [\lambda_n, \tilde{\lambda}_n]$ for all n, by (6.31) we have

$$|\tilde{v}_1(\tau_0, \lambda'_n) - \tilde{v}_1(\tau_0, \lambda^*)| > \varphi(\lambda'_n - \lambda^*).$$

Hence, for each $\omega \in \Omega^*$, there is at least one $\lambda^* > 0$ (depending on ω) such that (6.30) holds. It implies that $\Omega^* \subset \pi(S)$. Then from (b) we deduce that $\Lambda < \infty$ a.s., and from (a) we conclude that

$$\limsup_{h \to 0+} \frac{|\tilde{v}_1(\tau_0, \Lambda + h) - \tilde{v}_1(\tau_0, \Lambda)|}{\sqrt{h^{2-\beta} \log\log(1/h)}} = \infty \quad \text{a.s.}$$

The proof of Lemma 6.4.1 is complete. $\square$

Lemma 6.4.2. *For any $\tau_0 > 0$ and $\lambda > 0$,*

$$\mathbb{P}\left(\limsup_{h\to 0+} \frac{|\tilde{u}_2(\tau,\lambda+h) - \tilde{u}_2(\tau,\lambda)|}{\sqrt{h^{2-\beta}\log\log(1/h)}} = K_\beta(\tau - \tau_0 + \lambda)^{1/2} \text{ for all } \tau \geq \tau_0\right) = 1.$$

Proof. By Proposition 6.2.1 and 6.3.1,

$$\mathbb{P}\left(\limsup_{h\to 0+} \frac{|\tilde{u}(\tau,\lambda+h) - \tilde{u}(\tau,\lambda)|}{\sqrt{h^{2-\beta}\log\log(1/h)}} = K_\beta(\tau + \lambda)^{1/2} \text{ for all } \tau \geq 0\right) = 1.$$

Then the result can be obtained by the observation that $\{\tilde{u}_2(\tau_0+\tau,\lambda) : \tau,\lambda \geq 0\}$ has the same distribution as $\{\tilde{u}(\tau,\lambda) : \tau,\lambda \geq 0\}$. Indeed, for any bounded Borel sets A, B in $\mathbb{R}_+ \times \mathbb{R}$ and $c = (c_1, c_2) \in \mathbb{R}_+ \times \mathbb{R}$, by (6.5) and change of variables we have

$$\begin{aligned}
&\mathbb{E}\big[W(A+c)W(B+c)\big] \\
&= \int_{c_1}^{\infty} ds \int_{\mathbb{R}} dy \int_{\mathbb{R}} dy'\, \mathbf{1}_A(s-c_1, y-c_2)|y-y'|^{-\beta}\mathbf{1}_B(s-c_1, y-c_2) \\
&= \int_0^{\infty} ds \int_{\mathbb{R}} dy \int_{\mathbb{R}} dy'\, \mathbf{1}_A(s,y)|y-y'|^{-\beta}\mathbf{1}_B(s,y) \\
&= \mathbb{E}\big[W(A)W(B)\big].
\end{aligned}$$

Since

$$\Delta\Big(\frac{\tau_0+\tau+\lambda}{\sqrt{2}}, \frac{-\tau_0-\tau+\lambda}{\sqrt{2}}\Big) \cap \big\{t \geq \tau_0/\sqrt{2}\big\} = \Delta\Big(\frac{\tau+\lambda}{\sqrt{2}}, \frac{-\tau+\lambda}{\sqrt{2}}\Big) + c,$$

where $c = (\frac{\tau_0}{\sqrt{2}}, -\frac{\tau_0}{\sqrt{2}})$, it follows that for any $\tau, \lambda, \tau', \lambda' \geq 0$,

$$\begin{aligned}
&\mathbb{E}\big[\tilde{u}_2(\tau_0+\tau,\lambda)\tilde{u}_2(\tau_0+\tau',\lambda')\big] \\
&= \frac{1}{4}\,\mathbb{E}\Big[W\Big(\Delta\Big(\frac{\tau_0+\tau+\lambda}{\sqrt{2}}, \frac{-\tau_0-\tau+\lambda}{\sqrt{2}}\Big) \cap \big\{t \geq \tau_0/\sqrt{2}\big\}\Big) \\
&\qquad \times W\Big(\Delta\Big(\frac{\tau_0+\tau'+\lambda'}{\sqrt{2}}, \frac{-\tau_0-\tau'+\lambda'}{\sqrt{2}}\Big) \cap \big\{t \geq \tau_0/\sqrt{2}\big\}\Big)\Big] \\
&= \frac{1}{4}\,\mathbb{E}\Big[W\Big(\Delta\Big(\frac{\tau+\lambda}{\sqrt{2}}, \frac{-\tau+\lambda}{\sqrt{2}}\Big)\Big)W\Big(\Delta\Big(\frac{\tau'+\lambda'}{\sqrt{2}}, \frac{-\tau'+\lambda}{\sqrt{2}}\Big)\Big)\Big] \\
&= \mathbb{E}[\tilde{u}(\tau,\lambda)\tilde{u}(\tau',\lambda')].
\end{aligned}$$

The result follows immediately. □

We are now in a position to state and prove our main theorem. The first part of the theorem justifies the existence of a random singularity. It shows that if we fix $\tau_0 > 0$, then based on the information from the σ-field $\mathscr{F}_{\tau_0}$, we can actually find a random variable Λ such that (τ_0, Λ) is a singularity in the λ-direction. The second part says that if (τ_0, Λ) is a singularity in the λ-direction, then (τ, Λ) is also a singularity for all $\tau \geq \tau_0$. In other words, singularities in the λ-direction propagate orthogonally, towards the τ-direction. By a symmetric argument, one can show that singularities in the τ-direction propagate towards the λ-direction. These are the directions of the characteristic curves $t + x = c$ and $t - x = c$.

Theorem 6.4.3. *Let $\tau_0 > 0$. The following statements hold.*

(i) There exists a positive, finite, $\mathscr{F}_{\tau_0}$-measurable random variable Λ such that

$$\limsup_{h\to 0+} \frac{|\tilde{u}(\tau_0, \Lambda+h) - \tilde{u}(\tau_0,\Lambda)|}{\sqrt{h^{2-\beta}\log\log(1/h)}} = \infty \quad a.s.$$

(ii) If Λ is any positive, finite, $\mathscr{F}_{\tau_0}$-measurable random variable, then on an event of probability 1, we have

$$\limsup_{h\to 0+} \frac{|\tilde{u}(\tau_0, \Lambda + h) - \tilde{u}(\tau_0, \Lambda)|}{\sqrt{h^{2-\beta} \log\log(1/h)}} = \infty \Leftrightarrow \limsup_{h\to 0+} \frac{|\tilde{u}(\tau, \Lambda + h) - \tilde{u}(\tau, \Lambda)|}{\sqrt{h^{2-\beta} \log\log(1/h)}} = \infty$$

for all $\tau > \tau_0$ simultaneously.

Proof. To simplify notations, let

$$L(\tau, \lambda) = \limsup_{h\to 0+} \frac{|\tilde{u}(\tau, \lambda + h) - \tilde{u}(\tau, \lambda)|}{\sqrt{h^{2-\beta} \log\log(1/h)}}$$

and

$$L_i(\tau, \lambda) = \limsup_{h\to 0+} \frac{|\tilde{u}_i(\tau, \lambda + h) - \tilde{u}_i(\tau, \lambda)|}{\sqrt{h^{2-\beta} \log\log(1/h)}}$$

for $i = 1, 2$. As in [63, 7], we will use the property that for any two functions f and g,

$$\limsup_{h\to 0} |f(h)| - \limsup_{h\to 0} |g(h)| \le \limsup_{h\to 0} |f(h) + g(h)| \le \limsup_{h\to 0} |f(h)| + \limsup_{h\to 0} |g(h)| \quad (6.32)$$

provided that $\limsup_{h\to 0} |g(h)| < \infty$.

(i). By Lemma 6.4.1, we can find a positive, finite, $\mathscr{F}_{\tau_0}$-measurable random variable Λ such that

$$L_1(\tau_0, \Lambda) = \infty \quad \text{a.s.}$$

Since Λ is independent of the process $\tilde{u}_2$, Lemma 6.4.2 implies that

$$L_2(\tau_0, \Lambda) = K_\beta \Lambda^{1/2}$$

which is finite a.s. Since $\tilde{u} = \tilde{u}_1 + \tilde{u}_2$, it follows from the lower bound of (6.32) that

$$L(\tau_0, \Lambda) \geq L_1(\tau_0, \Lambda) - L_2(\tau_0, \Lambda) = \infty \quad \text{a.s.}$$

This proves (i).

(ii). Suppose Λ is a positive, finite, $\mathscr{F}_{\tau_0}$-measurable random variable. By (6.32), we have

$$L_1(\tau, \Lambda) - L_2(\tau, \Lambda) \leq L(\tau, \Lambda) \leq L_1(\tau, \Lambda) + L_2(\tau, \Lambda) \tag{6.33}$$

for all $\tau \geq \tau_0$, provided that $L_2(\tau, \Lambda) < \infty$. Note that for $\tau \geq \tau_0$,

$$\tilde{u}_1(\tau, \Lambda + h) - \tilde{u}_1(\tau, \Lambda) = \tilde{u}_1(\tau_0, \Lambda + h) - \tilde{u}_1(\tau_0, \Lambda),$$

hence $L_1(\tau, \Lambda) = L_1(\tau_0, \Lambda)$. Also, by Lemma 6.4.2 and independence between Λ and $\tilde{u}_2$, we have

$$\mathbb{P}\Big(L_2(\tau, \Lambda) = K_\beta(\tau - \tau_0 + \Lambda)^{1/2} \text{ for all } \tau \geq \tau_0\Big) = 1.$$

Since Λ is finite a.s., it follows from (6.33) that

$$\mathbb{P}\Big(L_1(\tau_0, \Lambda) - K_\beta(\tau - \tau_0 + \Lambda)^{1/2} \leq L(\tau, \Lambda) \leq L_1(\tau_0, \Lambda) + K_\beta(\tau - \tau_0 + \Lambda)^{1/2} \text{ for all } \tau \geq \tau_0\Big) = 1,$$

and it implies

$$\mathbb{P}\Big(L(\tau_0, \Lambda) = \infty \Leftrightarrow L(\tau, \Lambda) = \infty \text{ for all } \tau \geq \tau_0\Big) = 1. \qquad \square$$

BIBLIOGRAPHY

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[1] R. J. Adler, *The geometry of random fields.* John Wiley & Sons, Ltd., Chichester, 1981

[2] R. J. Adler and J. E. Taylor, *Random fields and geometry.* Springer, New York, 2007.

[3] T. W. Anderson, The integral of a symmetric unimodal function over a symmetric convex set and some probability inequalities. *Proc. Amer. Math. Soc.* **6** (1955), 170–176.

[4] A. Ayache and Y. Xiao, Asymptotic properties and Hausdorff dimensions of fractional Brownian sheets. *J. Fourier Anal. Appl.* **11** (2005) 407–439.

[5] R. M. Balan, L. Quer-Sardanyons and J. Song, Existence of density for the stochastic wave equation with space-time homogeneous Gaussian noise. *Electron. J. Probab.* **24** (2019), no.106, 1–43.

[6] S. M. Berman, Local nondeterminism and local times of Gaussian processes. *Indiana Univ. Math. J.* **23** (1973), 69–94.

[7] J. Blath and A. Martin, Propagation of singularities in the semi-fractional Brownian sheet. *Stochastic Process. Appl.* **118** (2008) 1264–1277.

[8] E. Cabaña, On barrier problems for the vibrating string. *Z. Wahrsch. Verw. Gebiete* **22** (1972) 13–24.

[9] R. Carmona and D. Nualart, Random nonlinear wave equations: propagation of singularities. *Ann. Probab.* **16** (1988) no. 2, 730–751.

[10] J. Cuzick and J. P. DePreez, Joint continuity of Gaussian local times. *Ann. Probab.* **10** (1982) no. 3, 810–817.

[11] R. C. Dalang, Extending the martingale measures stochastic integral with applications to spatially homogeneous SPDE's. *Electron. J. Probab.* **4** (1999), no. 6, 1–29.

[12] R. C. Dalang, Stochastic integrals for spde's: a comparison. *Expo. Math.* **29** (2011), no. 1, 67–109.

[13] R. C. Dalang and N. E. Frangos, The stochastic wave equation in two spatial dimensions. *Ann. Probab.* **26** (1998) no. 1, 187–212.

[14] R. C. Dalang, D. Khoshnevisan and E. Nualart, Hitting probabilities for systems of non-linear stochastic heat equations with additive noise. *Latin Amer. J. Probab. Statist. (ALEA)* **3** (2007), 231–271.

[15] R. C. Dalang, D. Khoshnevisan, E. Nualart, D. Wu and Y. Xiao, Critical Brownian sheet does not have double points. *Ann. Probab.* **40** (2012), no. 4, 1829–1859.

[16] R. C. Dalang, C. Y. Lee, C. Mueller and Y. Xiao, Multiple points of Gaussian random fields. Preprint. arXiv:1911.09793.

[17] R. C. Dalang and C. Mueller, Multiple points of the Brownian sheet in critical dimensions. *Ann. Probab.* **43** (2015), no. 4, 1577–1593.

[18] R. C. Dalang, C. Mueller and Y. Xiao, Polarity of points for Gaussian random fields. *Ann. Probab.* **45** (2017), no. 6B, 4700–4751.

[19] R. C. Dalang and M. Sanz-Solé, Hölder-Sobolev regularity of the solution to the stochastic wave equation in dimension three. *Mem. Amer. Math. Soc.* **199** (2009) no. 931, vi+70 pp.

[20] R. C. Dalang and M. Sanz-Solé, Criteria for hitting probabilities with applications to systems of stochastic wave equations. *Bernoulli* **16** (2010), no. 4, 1343–1368.

[21] C. Dellacherie, *Capacités et processus stochastiques*, Springer-Verlag, 1972.

[22] G. B. Folland, *Introduction to Partial Differential Equations*. Second Edition. Princeton University Press, Princeton, NJ, 1995.

[23] D. Geman and J. Horowitz, Occupation densities. *Ann. Probab.* **8** (1980), 1–67.

[24] A. Goldman, Points multiples des trajectoires de processus Gaussiens. *Z. Wahrsch. Verw. Gebiete.* **75** (1981), 481–494.

[25] I. S. Gradshteyn and I. M. Ryzhik, *Tables of integrals, series, and products*. Seventh Edition, Academic Press, Amsterdam, 2007.

[26] M. de Guzmán, *Differentiation of integrals in R^n.* Lecture Notes in Math. **481** Springer-Verlag, 1975.

[27] R. Herrell, R. Song, D. Wu and Y. Xiao, Sharp space-time regularity of the solution to stochastic heat equation driven by fractional-colored noise. Preprint. arXiv:1810.00066.

[28] A. Käenmäki, T. Rajala and V. Suomala, Existence of doubling measures via generalised nested cubes. *Proc. Amer. Math. Soc.* **140** (2012), no. 9, 3275–3281.

[29] D. Khoshnevisan, A primer on stochastic partial differential equations. *A minicourse on stochastic partial differential equations*, 1–38, Lecture Notes in Math. **1962**, Springer, Berlin, 2009.

[30] D. Khoshnevisan and Z. Shi, Fast sets and points for fractional Brownian motion. *Séminaire de Probabilités, XXXIV*, 393–416, Lecture Notes in Math. **1729**, Springer, Berlin, 2000.

[31] D. Khoshnevisan and Y. Xiao, Images of the Brownian sheet. *Trans. Amer. Math. Soc.* **359** (2007), no. 7, 3125–3151.

[32] N. Kôno, Double points of a Gaussian sample path. *Z. Wahrsch. Verw. Gebiete.* **45** (1978), 175–180.

[33] H. Landau and L. A. Shepp, On the supremum of a Gaussian process, *Sankhya Ser. A* **32** (1970) 369–378.

[34] M. Ledoux, Isoperimetry and Gaussian analysis. *Lectures on Probability Theory and Statistics (Saint-Flour, 1994). Lecture Notes in Math.* **1648** 165–294. Springer, Berlin, 1996.

[35] C. Y. Lee and Y. Xiao, Local nondeterminism and the exact modulus of continuity for stochastic wave equation. *Electron. Commun. Probab.* **24** (2019) no. 52, 1–8.

[36] C. Y. Lee and Y. Xiao, Propagation of singularities for the stochastic wave equation. (2020) Submitted.

[37] N. Luan and Y. Xiao, Spectral conditions for strong local nondeterminism and exact Hausdorff measure of ranges of Gaussian random fields. *J. Fourier Anal. Appl.* **18** (2012), 118–145.

[38] M. B. Marcus, Hölder conditions for Gaussian processes with stationary increments. *Trans. Amer. Math. Soc.* **134** (1968) 29–52.

[39] M. Marcus and V. J. Mizel, Stochastic hyperbolic systems and the wave equation. *Stochastics Stochastics Rep.* **36** (1991) no. 3–4, 225–244.

[40] M. B. Marcus and J. Rosen, *Markov Processes, Gaussian Processes, and Local Times.* Cambridge University Press, Cambridge, 2006.

[41] M. B. Marcus and L.A. Shepp, Sample behaviour of Gaussian processes, *Proc. Sixth Berkeley Symp. Math. Statist. Probab.* **2** 423–442. Univ. California Press, Berkeley, 1972.

[42] M. M. Meerschaert, W. Wang and Y. Xiao, Fernique-type inequalities and moduli of continuity for anisotropic Gaussian random fields. *Trans. Amer. Math. Soc.* **365** (2013), no. 2, 1081–1107.

[43] A. Millet and P.-L. Morien, On a stochastic wave equation in two space dimensions: regularity of the solution and its density. *Stoch. Processes Appl.* **86** (2000) 141–162.

[44] C. Mueller and R. Tribe, Hitting properties of a random string. *Electron. J. Probab.* **7** (2002), no. 10, 29 pp.

[45] E. Orsingher, Randomly forced vibrations of a string. *Ann. Inst. H. Poincaré Sect. B (N.S.)* **18** (1982) no. 4, 367–394.

[46] J. Ortega, On the size of the increments of nonstationary Gaussian processes. *Stochastic Process. Appl.* **18** (1984) 47–56.

[47] J. Ortega, Upper classes for the increments of fractional Wiener processes. *Probab. Th. Rel. Fields.* **80** (1989) 365–379.

[48] S. Orey and S. J. Taylor, How often on a Brownian path does the law of iterated logarithm fail? *Proc. London Math. Soc* **3** (1974) no. 28, 174–192.

[49] S. Peszat and J. Zabczyk, Stochastic evolution equations with a spatially homogeneous Wiener process. *Stoch. Processes AppI.* **72** (1997) 187–204.

[50] S. Peszat and J. Zabczyk, Nonlinear stochastic wave and heat equations. *Probab. Theory ReI. Fields.* **116** (2000) 421–443.

[51] L. D. Pitt, Local times for Gaussian vector fields. *Indiana Univ. Math. J.* **27** (1978), no. 2, 309–330.

[52] F. Pu, The stochastic heat equation: hitting probabilities and the probability density function of the supremum via Malliavin calculus. Ph.D. thesis no. 8695 (2018), Ecole Polytechnique Fédérale de Lausanne, Switzerland.

[53] A. Renyi, *Probability Theory*, North-Holland Publishing Co., Amsterdam-London, 1970.

[54] J. Rosen, Self-intersections of random fields. *Ann. Probab.* **12** (1984), 108–119.

[55] W. Rudin, *Functional analysis*, second edition, McGraw-Hill, 1991.

[56] G. Samorodnitsky, M. Taqqu, *Stable non-Gaussian random processes*, Chapman & Hall, 1994.

[57] D. Slepian, The one-sided barrier problem for Gaussian noise, *Bell System Tech. J.* **41** (1962), 463–501.

[58] M. Sanz-Solé and M. Sarrà, Hölder continuity for the stochastic heat equation with spatially correlated noise. *Seminar on Stochastic Analysis, Random Fields and Applications, III (Ascona, 1999)*, 259–268, Progr. Probab. **52**, Birkhuser, Basel, 2002.

[59] E. M. Stein, *Singular integrals and differentiability properties of functions.* Princeton University Press, 1970.

[60] M. Talagrand, Hausdorff measure of trajectories of multiparameter fractional Brownian motion. *Ann. Probab.* **23** (1995) no. 2, 767–775.

[61] M. Talagrand, Multiple points of trajectories of multiparameter fractional Brownian motion. *Probab. Theory Relat. Fields* **112** (1998), 545–563.

[62] C. A. Tudor and Y. Xiao, Sample paths of the solution to the fractional-colored stochastic heat equation. *Stoch. Dyn.* **17** (2017), no. 1, 1750004, 20 pp.

[63] J. B. Walsh, Propagation of singularities in the Brownian sheet. *Ann. Probab.* **10** (1982) no. 2, 279–288.

[64] J. B. Walsh, An introduction to stochastic partial differential equations. *École d'été de probabilités de Saint-Flour*, XIV-1984, pp.265–439, Lecture Notes in Math. **1180**, Springer, Berlin, 1986.

[65] D. Wu and Y. Xiao, Geometric properties of fractional Brownian sheets. *J. Fourier Anal. Appl.* **13** (2007), 1–37.

[66] D. Wu and Y. Xiao, On local times of anisotropic Gaussian random fields. *Commun. Stoch. Anal.* **5** (2011) no. 1, 15–39.

[67] Y. Xiao, Hölder conditions for the local times and the Hausdorff measure of the level sets of Gaussian random fields. *Probab. Theory Relat. Fields.* **109** (1997) 129–157.

[68] Y. Xiao, Properties of local nondeterminism of Gaussian and stable random fields and their applications. *Ann. Fac. Sci. Toulouse Math.* **XV** (2006), 157–193.

[69] Y. Xiao, Strong local nondeterminism and sample path properties of Gaussian random fields. In: *Asymptotic Theory in Probability and Statistics with Applications*, pp.136–176, Adv. Lect. Math. **2**. Int. Press, Somerville, MA, 2008.

[70] Y. Xiao, Sample path properties of anisotropic Gaussian random fields. *A minicourse on stochastic partial differential equations,* (D. Khoshnevisan and F. Rassoul-Agha, editors), *Lecture Notes in Math*, **1962**, Springer, New York (2009), 145–212