USING MANIPULATIVES TO SUPPORT SECONDARY STUDENTS WITH HIGH-INCIDENCE DISABILITIES IN ALGEBRA

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ABSTRACT

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All students, including students with disabilities, are expected to learn mathematics at high levels. As algebra is considered to be the gatekeeper to higher level mathematics and beyond, it is essential for teachers to use effective practices, including the use of manipulatives, to support student learning. This alternative dissertation is comprised of three studies that explored teaching algebra to secondary students with high-incidence disabilities. The first was an evidence-based systematic review of literature investigating instructional practices to teach algebra to secondary students with high-incidence disabilities. Twenty studies published from 1999-2019 were reviewed and analyzed, of which 14 met the standards of high quality set by the Council for Exceptional Children (CEC), and five practices earned the label of potentially evidence-based (i.e., concrete-representational-abstract framework, manipulatives, enhanced anchor instruction, schema based instruction, and peer-assisted learning strategies). The second study used an alternating treatment design to compare the effectiveness of concrete algebra tiles to virtual algebra tiles to support middle school students with disabilities as they solved linear equations. Students were successful solving linear equations regardless of the type of manipulative they used, but preferred using the virtual tool. The final study used a multiple probe across behaviors, replicated across participant design to examine the effectiveness of the VA framework to support secondary students with high-incidence disabilities in their acquisition of algebra skills. A functional relationship existed between the VA framework and student performance on algebra probes, and students scored better on maintenance probes compared to baseline data. While there

remains a need for more high-quality research examining effective practices in supporting secondary students with high-incidence disabilities in the area of algebra, the studies in this alternative dissertation suggest manipulatives, both as a stand-alone tool and as part of a process is an effective, and efficient intervention.

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INTRODUCTION

Mathematics is an essential component of education and helps empower students so they can improve their lives and better understand the world around them (Gravemeijer, 2017). To align these expectations with public education, states have implemented the Common Core State Standards (CCSS), which help educators set targets for learning experiences and ensures teachers are using rigorous academic standards (Dennis et al., 2016; Every Student Succeeds Act [ESSA], 2015). In mathematics, the CCSS (2010) provide clarity and specificity to mathematics teaching and learning expectations, including a focus on depth, coherence, and rigor, so that students learn concepts in an organized way throughout the year and across grade levels (Kamin, 2016). The CCSS emphasize conceptual understanding and developing the ability to apply learning to real world situations including emphasis on the relevance of mathematics, like algebraic thinking, in everyday life (CCSS, 2010).

Algebra

Algebra is a strand of mathematics embedded throughout the K-12 curriculum (CCSS, 2010). Algebra teaches students to think critically by identifying and working with patterns, making generalizations, and interpreting change (Ketterlin-Geller et al., 2019; National Council of Teachers of Mathematics, 2000). Algebra may also include the manipulation of numbers and symbols to solve for an unknown and using deductive and inductive reasoning to problem solve complex situations (Jeannotte & Kieran, 2017). Through these algebraic experiences, students are challenged to apply concepts and procedures across various topics and levels of difficulty (Ralston et al., 2018).

As students enter middle school, their mathematics learning becomes less focused on numbers and operations and increasingly focuses on algebra (CCSS, 2010; Ketterlin-Geller et al., 2019). Some of the algebra concepts taught include real numbers, complex numbers, matrices and vectors (Witzel, 2016), and students are challenged to identify functions as linear or nonlinear and contrast their properties from tables, graphs, and equations (NCTM, 2014). Students are also required to represent and analyze mathematical situations and structures using algebraic symbols, recognize and generate equivalent expressions and equations, and to use mathematical models to represent and understand quantitative relationships (Ketterlin-Geller et al., 2019; NCTM, 2014). Middle school teachers guide students through these new experiences and help them build a strong algebraic foundation in order to prepare them for the mathematical demands of high school (CCSS, 2010).

Across the United States, proficiency in algebra is needed to meet high school graduation requirements, attend and succeed in postsecondary education, and effectively compete in an increasingly technologically and data driven world (Ketterlin-Geller, 2019; Watt et al., 2016). The skills learned in algebra classes, including the development of abstract reasoning skills and advanced cognitive demands, are the foundation for higher-level mathematics and science courses (NCTM, 2014; Ralston et al., 2018). In addition, researchers stressed the importance of developing algebra skills, as mathematics is an essential component for a diverse collection of occupations that do not necessarily require a two- or four-year college degree (Hwang & Riccomini, 2016). Developing problem-solving and critical-thinking skills as well as understanding the core concepts of algebra can help individuals better handle complex problems involving unknown variables in real life situations (Gravemeijer, 2017). Establishing a strong foundation in algebra creates additional opportunities for students and is an essential part of college and career readiness (Kamin, 2016).

Challenges in Learning Algebra

While developing algebra skills is important, certain characteristics specific to algebra make it difficult for students to master (Hwang, & Riccomini, 2016; Ketterlin-Geller et al., 2019). To begin, algebra is one of the most abstract strands in mathematics and thus moving from arithmetic to algebra is a difficult transition for many students (Witzel, 2016). Many effective instructional practices, like graphic organizers and manipulatives, help by creating concrete representations for abstract ideas (Jitendra et al., 2018; Watt et al., 2016). Second, mathematics can seem like a foreign language for students (Riccomini et al., 2015). Algebra, in particular, requires students to use and understand a whole new set of terminology whose correct application to novel ideas creates additional challenges (Witzel, 2016). Misunderstanding the semantics of the language may cause students to make errors that have little to do with their understanding of the concepts. To help, students need to be given frequent opportunities to reason with and talk about mathematical concepts, procedures, and strategies using algebraic language (Riccomini et al., 2015; Star et al., 2015). Finally, the structural features of algebra are difficult for many students to recognize and understand (Star et al, 2015). Failure to develop an understanding of algebraic structure negatively impacts a student's ability to apply appropriate strategies and thus work with both simple and advanced algebra concepts (Jitendra et al., 2018). While these characteristics create challenges for all students, they are especially demanding for students with disabilities (Star et al., 2015).

Students with Disabilities and Algebra

An estimated 7.6% of the school-aged population is identified as having a disability that may negatively impact their ability to develop mathematical skills (e.g., autism, emotional disturbance, intellectual disability, learning disability; National Center for Education Statistics, 2018). Students with disabilities often perform significantly below their same-aged peers without disabilities in mathematics (National Assessment of Educational Progress [NAEP], 2019). According to the NAEP (2019), students with disabilities scored an average of 31 points below their peers without disabilities on the mathematics assessment. In fact, only 17% of fourth-grade students with disabilities and 9% of eighth-grade students with disabilities met or exceeded the expected mathematics standards for their grade level (NAEP, 2019). These results are consistent with previous NAEP assessments, where eighth-grade students with disabilities scored an average of 46 points lower as compared to their same-aged peers without disabilities when only considering questions involving algebra (i.e., understanding of patterns, using variables, algebraic representation, and functions (NAEP, 2017).

The difficulties experienced by students with disabilities as they develop algebraic skills stem from a variety of areas. Some students possess a poor understanding of foundational mathematical concepts, including difficulty with number sense and operations (Geary, 2013; Hinton, 2014). Without number sense, students are unable to understand numbers and their relationships, and thus experience challenges in content that applies these skills such as algebra. This coupled with a failure to understand basic operations causes students to fall further behind in mathematics as they advance through the curriculum (Hinton, 2014). Other students may know and understand isolated facts and ideas but have a poor or inaccurate conceptual understanding, which leads to confusion and error patterns especially when concepts are to be communicated using symbols (Ketterlin-Geller, 2019). For example, students with disabilities may believe an equals sign is simply an indicator of operations to perform or that a negative symbol only means subtraction and does not modify a term (Hinton, 2014). When this occurs, students may be unsure of the necessary procedures, and this lack of knowledge prevents them from identifying any procedural errors they may make (Jitendra et al., 2018). Further, some students with disabilities experience deficits in working memory or lack the ability to use retrieval-based skills (Geary, 2013). Students may know and understand algebraic content, but be unable to retrieve the correct steps or ideas from memory (Geary, 2013). These challenges create additional barriers to accessing grade-level curriculum for students with disabilities.

Current legislation mandates students with disabilities have access to the general education curriculum and that Individual Education Program (IEP) goals align to state standards (ESSA, 2015; IDEA, 2004). Thus, it is important to teach grade-appropriate standards to all students, including those with disabilities. These standards promote rigorous accountability and gains toward college and career readiness for all students (Marita & Hord, 2017). As students with disabilities are increasingly held to higher mathematical standards, it is imperative teachers use evidence-based strategies and approaches for teaching, assessing skills, and monitoring progress in mathematics, including algebra (Strickland, 2016). As educators we must strive to give all students, including those with disabilities, a rich and meaningful educational experience (NCTM, 2014).

Manipulatives and Algebra

One intervention or educational practice supported in the literature for students with and without disabilities is the use of manipulatives (Bouck & Park, 2018; Peltier et al., 2019; Spooner et al., 2019). Manipulatives are used to introduce new concepts, practice skills, and to provide support during re-teaching and can be concrete or virtual (Lafay, et al., 2019). Concrete manipulatives are physical objects that students can handle and maneuver during mathematical thinking (Bouck & Park, 2018). Concrete manipulatives can be made from anything (e.g., beans, sticks), and common ones include tiles, geoboards, base ten blocks, fraction pieces, and algebra tiles (Carbonneau et al., 2013; Lafay et al., 2019). By utilizing these tools students are better able to understand and apply abstract ideas, which is a common challenge when working with algebraic concepts (Carbonneau at al., 2013).

Although researchers found positive results for students with disabilities using concrete manipulatives to support learning of algebra (e.g., Satsangi et al., 2016; Strickland & Macini, 2013), there are drawbacks to using concrete manipulatives especially at the secondary level where students may find them stigmatizing (Satsangi & Miller, 2017). With an increase in the use and availability of technology in schools, researchers directed attention towards virtual manipulatives (Bouck et al., 2019; Satsangi et al., 2018). Virtual manipulatives are interactive, digital representations of concrete manipulates that can be accessed through computers or mobile devices as either internet-based on app-based (Bouck, Mathews & Peltier, 2019). Researchers found virtual manipulatives possess certain features that make them more appealing compared to concrete manipulatives, such as their portable nature, social desirability, and ability to represent more complex mathematics (Bouck, Mathews & Peltier, 2019). Some students simply enjoy using technology, while older students often prefer virtual manipulatives because they seem less childish and stigmatizing compared to concrete manipulatives (Satsangi & Miller, 2017). Virtual manipulatives also have built-in features that allow for differentiation so students may have various levels of support, but they do not appear different than their peers (Bouck, Working & Bone, 2018). Likewise, these features promote independence as students receive immediate feedback on their work without requiring teacher guidance (Satsangi et al., 2018b). In addition, virtual manipulatives do not take up storage space and can be easily accessed using school-issued Chromebooks and iPads through online programs or purchased apps (Bouck, Working & Bone, 2018).

Manipulatives – concrete or virtual – can be used on their own to teach or support students with disabilities in learning mathematics (Bouck, Chamberlain & Park, 2017; Satsangi et al., 2016). When comparing concrete and virtual manipulatives, students with disabilities preferred using manipulatives to working through problems without assistance (Bouck, Chamberlain & Park, 2017), and were successful using both virtual and concrete balance scales to solve single-variable linear equations (Satsangi et al., 2016). Using the algebra balance scales students were able to bridge the gap between the abstract ideas and concrete processes of algebra as demonstrated by their ability to solve problems with 90% or greater accuracy. Secondary students with disabilities were also successful solving multi-step algebraic equations using virtual manipulatives paired with explicit instruction (Satsangi, Hammer, & Evmenova, 2018a; Satsangi, Hammer, & Hogan, 2018b). Using virtual manipulatives helped students improve accuracy, maintain skills over time, and several students were able to generalize their skills with some success.

Manipulatives can also be used as a vital component of graduated instructional sequences like the concrete-representational-abstract framework (CRA; Watt et al., 2016). The CRA framework is an evidence-based practice for teaching mathematics to students with disabilities, and specifically students with learning disabilities (Agrawal & Morin, 2016; Bouck, Satsangi, & Park, 2018). Researchers demonstrated the effectiveness of the CRA framework to support students with disabilities acquisition of algebra skills (e.g., Maccini & Hughes, 2000; Strickland

& Maccini, 2012; Witzel, 2005). These researchers found middle school students learned algebraic skills related to linear expressions when they were taught via the CRA framework.

With research demonstrating the effectiveness of virtual manipulatives, there is also emerging research supporting adaptations of the CRA, in which virtual manipulatives are used in place of concrete manipulatives (Bouck & Sprick, 2019). The virtual-representational-abstract (VRA) framework is effective in teaching various mathematical concepts to students with disabilities (e.g., Bouck et al., 2017) and can be modified to exclude the representational phase when appropriate (e.g., Bouck et al., 2019). Though limited in quantity, recently researchers demonstrated the efficacy of using the virtual-abstract (VA) framework to support students with disabilities in mathematics (Bouck et al., 2020), including using the VA framework to teach algebra concepts to students with disabilities (Bouck, Park et al., 2019).

Purpose of the Study

Algebra is often viewed as a gatekeeper to higher-level mathematics, with proficiency considered essential for all students (Watt at al., 2016). Yet, students with disabilities regularly struggle to develop grade-appropriate algebra skills (NAEP, 2017). Many students with high-incidence disabilities receive a majority of their supports and services within the general education setting. In order to support students with high-incidence disabilities, teachers need to access and use evidence-based practices (EBP). EBP are instructional supports and teaching methods shown by high-quality, methodological-sound research studies to have a positive impact on student learning (Cook & Cook, 2013). Researchers found the use of manipulatives, both as a stand-alone mathematical tool and as a part of an instructional framework, are an effective practice to teach algebra-related content to secondary students with disabilities (Agrawal & Morin, 2016; Bouck & Park, 2018). However, the use of concrete manipulatives may be stigmatizing to older students (Satsangi, et al., 2016). More research is needed to explore the use of virtual manipulatives, including using the tool as part of the Virtual-Abstract framework to support secondary students with disabilities.

Study 1 – Evidence-based systematic review of literature

Study 1 was an evidence-based synthesis focused on research regarding algebraic instruction and students with high-incidence disabilities published between 1999 and 2019. The researchers coded studies based on the research design, mathematical practice utilized, and whether the study met the quality indicators and practice standards set by the Council for Exceptional Children (CEC, 2014; Cook et al., 2014). From there, researchers identified studies with the required quantity of high-quality research to be categorized as evidence-based or potentially evidence-based. The results were analyzed and discussed to inform the current state of literature on teaching algebra to secondary student with high-incidence disabilities.

Study 2 – Comparing virtual and concrete manipulatives

Manipulatives are an effective mathematical tool to support students with disabilities in the area of mathematics (Bouck & Park, 2018; Peltier et al., 2019; Spooner et al., 2019). Currently, there is limited research comparing the effectiveness of concrete and virtual manipulatives to support students in the area of algebra, and no research exists making this comparison with algebra tiles (Satsangi et al., 2016). Of the existing literature, students perform better when they use manipulatives regardless of the type, but there are inconsistencies in terms of which type is more effective and which type is preferred by students (Bouck, Shurr et al., 2018; Satsangi et al., 2016). This study used a single-subject adaptive alternating treatment design to compare the efficacy of concrete and virtual algebraic tiles as instructional tools to support middle school students with disabilities as they solved two-step algebra problems. The dependent variables for the study included (a) accuracy, defined as the percentage of two-step algebra problems answered correctly out of five, (b) independence, defined as the percentage of steps in the task analysis of solving the two-step algebra problems participants completed without any prompting, and (c) task completion time, defined as the time needed to complete the five-problem assessment

Study 3 – Virtual-Abstract instructional framework

This study aimed to build on the limited understanding of how virtual manipulatives support students with disabilities by investigating their use within the VA framework. The VA framework is based on the evidence-based practice CRA (Agrawal & Morin, 2016), but concrete manipulatives are replaced with virtual manipulatives and the representational phase is removed from the process (Bouck et al., 2019). This study employed a multiple probe across behaviors, replicated across participants, design. Mathematical behaviors based on grade-level content standards included: one-step equations with positive and negative numbers, two-step equations with positive numbers, and two-step equations with positive and negative numbers. The dependent variable was accuracy, defined as the number of algebra problems answered correctly out of five.

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CHAPTER 2

An Evidence-Based Systematic Review of the Literature on Algebra Instruction and Interventions for Students with High-Incidence Disabilities

Algebra is considered by many to be the mathematical gatekeeper, and mastering algebra skills gives students a passport to educational opportunities and an expansive job market (Ralston et al., 2018). Although often conceptualized as a stand-alone course, algebra is a strand of mathematics requiring a set of skills that can be used across topics (Ralston et al., 2018; Stephens et al., 2015). Basic algebra problems may be as simple as 3+2 = ___, with more complex algebra involving multiple steps and imaginary numbers [e.g., 4(3m – 7) = 2(6 + 9m)]. Algebra may include the manipulation of numbers and symbols to solve for an unknown, identifying and analyzing patterns, examining relationships, making generalizations, and interpreting change (National Council of Teachers of Mathematics, 2000; Stephens et al., 2015).

According to the National Council of Teachers of Mathematics (NCTM) Standards (2000) and the Common Core State Standards for Mathematics (CCSSM, 2010), algebraic thinking should be incorporated into all grade levels. Foundations of algebra are introduced upon entering school and students begin by developing fluency with numbers, exploring structure in operations, and describing relationships (Kieran, 2014; Stephens et al., 2015). Students build on these skills every year and should learn to express algebraic relationships symbolically using appropriate mathematical language by middle school. This includes solving basic expressions and equations, analyzing patterns between independent and dependent variables, and solving both real-life and mathematical problems using numerical and algebraic knowledge (CCSSM. 2010). Once students enter high school, they are challenged to create and reason with equations, inequalities, and systems of equations at an even more advanced level (Kieran, 2014).

There are several unique challenges associated with learning algebra. To begin, algebra requires a considerable amount of abstract thinking and in order to further advance their mathematical understanding, students must learn to navigate the gap from concrete to abstract reasoning (Stephens et al., 2015; Witzel, 2016). Without a solid mathematical foundation, students struggle to manage multiple representations of algebraic objects (Kieran, 2014). Second, language plays an integral part in gaining proficiency in mathematics, and the novelty associated with algebra creates additional challenges (Witzel, 2016). Specifically, assigning appropriate meaning to symbols is difficult for some students. For this reason, it is important to generate opportunities for students to practice using algebraic language as they converse about strategies, concepts, and mathematical procedures (Star et al., 2015). Finally, many students struggle to recognize and understand the structural characteristics of algebra (Star et al., 2015). For example, often students believe a variable can only stand for one number, instead of recognizing it could represent an infinitely large set of values. Each of these challenges create barriers as students work to develop their algebraic thinking and strategies. While these challenges can be difficult for all students, they are especially demanding for students with disabilities (Star et al., 2015).

Algebra and Students with Disabilities

According to the National Assessment of Educational Progress (NAEP, 2017), only nine percent of students with disabilities meet or exceed mathematics standards by eighth grade. When analyzing questions involving algebra (i.e., understanding of patterns, using variables, algebraic representation, and functions), students with disabilities scored an average of 46 points lower as compared to their same-aged peers without disabilities (NAEP, 2017). In order to close the gap, it is imperative teachers use effective teaching strategies to give students with disabilities access to and support with the general education curriculum, such as algebra (CCSSM, 2010; Stevens et al., 2018). Ways to determine effective – or evidence-based practices – for teaching algebra are through systematic reviews of the literature or evidence-based practices syntheses (Witzel, 2016; Cook et al., 2014).

Within the last decade, researchers conducted two systematic reviews focused on effective mathematical practices to teach algebra-related content to students with disabilities (Hughes et al., 2014; Watt et al., 2016). Hughes and colleagues (2014) limited their analysis to quasi-experimental and experimental designs, and included more than peer-reviewed articles (e.g., dissertations). They analyzed 12 manuscripts including 13 different studies spanning from 1983-2013. Participants included students with disabilities, with three studies involving elementary students and 10 studies focused on secondary students. Researchers identified six intervention categories: (a) cognitive/model based instruction, (b) co-teaching, (c) concrete-representational-abstract (CRA) framework, (d) graphic organizer, (e) single-sex interventions; and, (d) technology. All intervention categories had a positive effect on algebra achievement except single-sex interventions. However, only two categories – cognitive/model-based instruction and the CRA framework – had enough information to calculate a weighted effect size. Both practices indicated moderate effects on students' algebra achievement and included systematic and explicit instruction as part of the intervention.

Watt et al. (2016) included only students with identified learning disabilities and expanded on the previous review by Hughes et al. (2014) by including single-case design along with quasi-experimental and experimental design. Although their search for articles meeting criteria spanned between 1980 and 2014, the publication dates of studies that met the inclusion criteria were limited to between 2000 and 2014. In all, they reviewed 15 studies, including five single-case and 10 group design. Watt et al. identified five interventions that constituted the majority of research regarding algebra and students with learning disabilities. Some interventions were consistent with the findings by Hughes et al. (2014), such as the CRA framework, cognitive strategy or modeling-based instruction, and graphic organizers. In addition, Watt et al. also suggested researchers have examined enhanced anchor instruction and tutoring to support the learning of algebra by students with learning disabilities. All of the studies included in the review by Watt et al. used explicit instruction as part of the intervention, and all but three used some type of visual representation.

Purpose of the Study

Due to the importance of using evidence-based practices (EBP) to teach mathematics to students with disabilities, there has been an increase in systematic reviews exploring interventions used to support students with disabilities in the area of mathematics (e.g., Bouck et al., 2018; Hughes et al., 2014; Martia & Hord, 2017; Watt et al., 2016). However, there exist only two reviews in the last ten years that focus explicitly on effective interventions in the area of algebra for students with disabilities (Hughes et al, 2014; Watt et al., 2016). Both of these reviews included studies involving elementary students, while Hughes et al. (2014) excluded single-case research designs and Watt et al. (2016) focused exclusively on students with learning disabilities. Single-case research is commonly used in special education research and should be included when reviewing practices being used with students with disabilities. The aim of this evidence-based synthesis was to identify and critically analyze the practices for teaching algebra-related concepts to secondary students with high-incidence disabilities. By applying the quality indicators and practice standards set by the Council for Exceptional Children (CEC, 2014; Cook et al., 2014) to the current research base, the authors sought to identify EBD in teaching algebra to secondary students with high-incidence disabilities. The research questions include: (a) What educational practices have been used to teach algebra to secondary students with high-incidence disabilities? (b) According to CEC standards, which of these educational practices used to teach algebra to secondary students with high-incidence disabilities are evidence-based?

Method

Literature Search

This evidence-based synthesis focused on research regarding algebraic instruction and students with high-incidence disabilities. To begin, the author conducted a keyword search of existing databases. Specifically, the author searched three databases: ProQuest, EBSCOhost, and Google Scholar. Search terms were chosen to identify studies focused on practices used to teach algebraic content to students with high-incidence disabilities. The author used a combination of different search terms in an effort to obtain all available research in this area. Each search included one word or phrase referring to algebra content (e.g., algebra, linear equations, equations, systems of equations, expressions, and multi-step equations), language describing a high-incidence disability (e.g., disabilities, high-incidence disabilities, emotional disabilities, ADHD, learning disabilities, and cognitive disabilities), and a term related to instruction (e.g., teach, learn, support, intervention, and instruction). During the initial search more than 150 combinations of phrases, using a word from each category (e.g., algebra interventions for students with high-incidence disabilities), were used in each of the three identified databases (authors will provide a full list upon request).

The search was restricted to articles published in English in a peer-reviewed journal between 1999 and 2019. The authors limited articles to the last twenty years to ensure practices being evaluated were still relevant to current educational standards (Sahlberg, 2016). Sixty-seven articles were initially identified. Each was then screened for adherence to four inclusion criteria: (a) had at least one dependent variable relative to algebra learning or skill acquisition described under the Expressions and Equations (Grades 6–8) or Algebra (Grades 9–12) domains of the Common Core State Standards; (b) had the target population as students with high-incidence disabilities, including those with emotional and/or behavioral disorders (EBD), learning disabilities (LD), mild intellectual disability (MID), high-functioning autism, and attention-deficit hyperactivity disorder (ADHD); (c) were conducted with students enrolled in sixth through 12th grade; and (d) involved a single case design or a group comparison design. The author included group comparison and single case designs because both are used in educational research and when planned and conducted appropriately, researchers can infer causality using these designs (Ledford & Gast, 2018).

After applying the screening procedures, 20 studies met all inclusion criteria. Once the studies were identified, the researchers recorded the study characteristics (see Table 2.1). Study characteristics included: (a) study descriptions (e.g., title, author, and date of publication), (b) sample (e.g., number of participants, age or grade), (c) participants' identified disability or inclusion criteria, (d) the mathematical content, (e) the intervention used to teach the algebraic concept, (f) the measure used to evaluate the effectiveness of the intervention (design), and (g) the results of the study.

Coding for Quality

In quantitative research, methodological rigor refers to the precision of a study in terms of design, data collection, analysis, and distribution of results (Cook & Cook, 2013; Cook et al., 2014). To establish methodological rigor in the 20 studies that met inclusion criteria, the author used specific quality indicators identified and categorized by the CEC (Cook et al., 2014). The categories included: (1) context and setting; (2) participants; (3) intervention agent; (4) description of practice; (5) implementation fidelity; (6) internal validity; (7) outcome data/dependent variable; (8) data analysis. Each category included anywhere from one to nine specific quality indicators based on the type of study (i.e., single-case or group). A majority of the quality indicators were the same for both single-case and group design, however there were some variation. For instance, internal validity was assessed using different quality indicators for each design, and group design also had an additional quality indicator related to outcomes and data analysis. Thus, 22 quality indicators applied to single case research studies and 24 quality indicators applied to group studies (Cook et al., 2014). Using these standards, a study is considered methodologically sound only if it meets all of the quality indicators for the specified research design (Cook et al., 2014).

When assessing the study for quality using the indicators proposed by the CEC, the reviewers first determined whether the authors provided sufficient information when describing the setting. This includes describing the general location like the geographic location and community characteristics such as socioeconomic status, as well as more specifics about the space being used for the intervention. For a school this may include whether it is public or private, the type of program and/or physical layout of the classroom (Cook et al., 2014). Next, the reviewers assessed whether authors included adequate information to describe the participants. This includes relevant demographic information such as specific disability diagnosis and/or whether the student is at-risk in a particular area (Cook et al., 2014). In a quality study, authors describe the method used to determine status (e.g., national or state assessments, teacher nomination, curriculum-based measurement probes, etc.).

The reviewers then assessed the presence of the intervention agent, description of practice, and fidelity of implementation. A quality study reports on critical characteristics of the person or people implementing the intervention including pertinent demographics, their background as relevant to the study, and proof they are appropriately qualified to implement the intervention (Cook et al., 2014). In addition, each study should provide sufficient information regarding the critical features of the intervention in a way that it could be replicated by those reading the description. This includes detailed intervention procedures, intervention agents' actions, and detailed explanation of the materials (Cook et al., 2014). If this information is not included, the study should indicate how the information can be accessed (e.g., cite original source). The study should use direct reliable measures to evaluate and document procedures for implementation, including frequency and intensity, regularly throughout each component of the intervention and for each participant.

Reviewers then checked that authors established internal validity. This is demonstrated when the researcher manipulates the variable in a consistent manner, and participants have very limited or no access to the intervention. For single-case research, researchers should also describe the baseline, and for group studies details of the control and comparison conditions need to be explained (Cook et al., 2014). In group studies, assignment to group must be clearly established and described using one of the following methods: (a) randomly; (b) nonrandomly, but comparison and interventions groups are matched (c) nonrandomly, but using techniques to measure and control statistical differences; or (d) nonrandomly, but using a practical cutoff (Cook et al., 2014). Quality group studies should also demonstrate low attrition both across groups (e.g., <30% in a 1-year study) and between groups (e.g., ≤10%) or controlled for by adjusting for those who do not complete the study. In single-case research the design selected should control for common threats to internal validity (e.g., history, maturation, testing), document three demonstrations of experimental effect at three different points in time, and include a minimum of three data points during baseline indicating unfavorable results in the absence of an intervention (Cook et al., 2014).

Quality studies demonstrate adequate psychometrics as they appropriately apply measures to determine the effect of the intervention on study outcomes. In quality research, the outcomes are socially important and the study establishes reliability (e.g., internal reliability, interobserver reliability, test-retest reliability, parallel-form reliability; Cook et al., 2014). In addition, authors clearly define the study and adequately describe the measurements of the dependent variable, the time and frequency of data collection are considered appropriate, and all effects of the intervention are reported and not just those with positive results (Cook et al., 2014). For group methods, the study should also provide sufficient evidence of validity (e.g., content, construct, criterion, social validity).

Finally, in a quality study, authors use appropriate data analysis and report effect size. For group studies this includes using techniques capable of analyzing change in performance of two or more groups and either reporting effect size or providing the necessary information to calculate effect size (Cook et al., 2014). For single-case research, a graph or graphs that clearly represent all data collected is necessary in order to determine the effect of the intervention using standard visual analysis procedures (Cook et al., 2014).

Determination of Evidence-Based Practice

To determine whether identified interventions met criteria to be considered evidence-based, the author assessed the status of each category of practice. Based on the CEC standards, interventions are classified as (a) evidence-based; (b) potentially evidence-based; (c) mixed effects; (d) insufficient evidence; or (e) negative effects (Cook et al., 2014). For each category, there is set criteria that must be met and in order to classify practices in special education research, methodologically sound studies need to report positive, neutral/mixed, or negative effects. Dependent on the research design, number of participants, and ratio of positive to neutral outcomes, an educational practice may be considered evidence-based with as few as two methodologically sound studies (Cook et al., 2014).

A practice is considered evidence-based if it meets specific criteria established by the CEC. When reviewing group designs, a practice must be supported by at least two methodologically sound studies including random assignment, positive results, and at least 60 total participants across students. If non-random assignment is used, there needs to be at least four studies and at least 120 total participants across studies. When reviewing single case research designs, a practice must be supported by at least five methodologically sound studies with positive effects and at least 20 total participants across studies (Cook et al., 2014). If various research designs are used to evaluate a practice, it is considered evidence-based if it meets at least 50% of the criteria for two or more of the study designs. In addition, there must be at least a 3:1 ratio of studies conducted with positive results to studies yielding neutral or mixed results. If any of the studies result in negative effects on students, the practice will not be considered evidence-based (Cook et al., 2014).

A practice may be considered potentially evidence-based if there are positive results, but too few high-quality studies. When reviewing group designs, a practice must be supported by at least one methodologically sound group design with random assignment and positive effects. If nonrandom assignment is used, the practice must be supported by a minimum of two or three methodologically sound group studies. When reviewing single case research, the practice must be supported by two to four methodologically sound single case research with positive effects (Cook et al., 2014). If various research designs are used to evaluate a practice, it is considered potentially evidence-based if it meets at least 50% of criteria for each of the study designs. In addition, there must be at least a 2:1 ratio of studies conducted yielding positive results to studies yielding neutral or mixed results. Further, if even one study yields negative results, the practice will not be considered potentially evidence-based (Cook et al., 2014).

Interobserver Agreement

Agreement for inclusion of articles in the study and coding of study characteristics were through the consensus of the first author and a doctoral student. Twenty-five of the original 67 studies (i.e., 37.3%) were independently coded based on the set criteria. Interobserver agreement (IOA) was determined by dividing the number of agreements by the number of agreements plus disagreement. The IOA was calculated as 97%. When a disagreement occurred, the researchers reviewed criteria and discussed until 100% agreement was reached for both inclusion and study characteristics. Due to the acceptable IOA, the first author independently coded the remaining studies to determine inclusion for the review.

All twenty studies that met inclusion criteria were coded independently by the first and third author for the application of quality indicators (i.e., 100%). The researcher chose to have all of the studies coded by two people because failure to meet all standards eliminated the study from being categorized as methodologically sound research. When the coders believed there was not a meaningful threat to validity and that the design issue was addressed adequately, a study was considered to have satisfied a quality indicator (Cook et al., 2014). Each coder recorded a Y when the quality indicator was met, and a N when the quality indicator was not met. A disagreement was highlighted red to indicate a need for further discussion. The researcher divided the number of agreements by the total number of indicators and then multiplied the quotient by 100 to determine interrater reliability. The IOA for quality indicators was 98.1% with the coders disagreeing on nine indicators. Once the coders discussed the indicators of disagreement they came to agreement on 100% of the indicators and sixteen studies were classified as methodologically sound.

Results

Twenty studies met the criteria to be a part of this systematic review spanning from 1999-2019. One study was published in the 1990s, 13 in the 2000s, and six in the 2010s. All studies focused on algebra-related concepts such as word problems including algebraic processes, operations with integers, and solving linear equations. All studies had neutral or positive results, and statistical results indicated moderate to large effects.

Participant Characteristics

Participants were in sixth through twelfth grade. The total number of participants across all studies reported was 930. Of that total, 506 participants were typically-performing peers used in group designs for comparison, 419 participants were identified as having a disability, and 5 participants were low performing or at risk for mathematics failure, but did not have a disability diagnosis. Specifically, participants were identified as having various high-incidence disabilities including: learning disabilities, disabilities in mathematics, emotional or behavioral disabilities, language disabilities, ADHD, and mild intellectual disability.

Study Designs

Of the 20 studies, 12 involved single-case research methodology and 8 used group design methodology (see Table 2.1). Within the single case studies, one used alternating treatment design, eight used a multiple probe design, and three used a multiple baseline design. The group design methodology included two experimental designs and six quasi-experimental design. For the quasi-experimental designs, participants were assigned to either a treatment or control group based on their class at school.

Categories of Mathematical Practice

After analyzing the mathematical practices described in each study, eight intervention categories emerged: (a) concrete-representational-abstract (CRA) framework (b) schema-based instruction, (c) enhanced anchor instruction, (d) manipulatives, (e) peer-assisted learning strategies, (f) virtual-abstract (VA) framework (g) graphic organizers and diagrams; and, (h) explicit inquiry routine. Seven studies investigated the concrete-representational-abstract (CRA) framework, with two of these studies using a graduated instructional sequence and two implementing graphic organizers. Three studies investigated schema-based instruction (SBI) and two studies investigated enhanced anchored instruction (EAI). Three studies investigated manipulatives separate from an instructional framework, with one study comparing virtual and concrete manipulatives, and two pairing virtual manipulatives with explicit instruction. One study explored peer-assisted learning strategies (PALS), and one study explored the virtual abstract (VA) framework. Three studies explored the impact of visual displays like graphic organizers and diagrams, and one study investigated explicit inquiry routine.

Applying Quality Indicators

After applying the criteria established by the CEC, six studies did not meet the requirements to be considered methodologically sound research. Four of the eight group design studies had all twenty-four quality indicators (See Table 2.2) and ten of the twelve single-case studies had all twenty-two quality indicators (see Table 2.3). The study conducted by Witzel et al. (2003) failed to demonstrate adequate evidence of reliability while Witzel (2005) and

Scheuermann et al. (2009) failed to include adequate implementation fidelity information. Ives (2007) reported two studies in one publication and both were missing several key indicators including adequate procedural information, proof of fidelity of implementation, and adequate internal validity. Finally, Stickland and Maccini (2012) only included two baseline data points for one of their participants. Although they provided a rationale for this decision, a minimum of three baseline points was needed in order for a single case design to be methodologically sound (Cook et al., 2014).

Based on the CEC standards and the 14 studies that met all quality indicators for algebraic instruction and students with high-incidence disabilities, no interventions met the criteria to be considered evidence-based. All studies yielded neutral or positive results, but the research base lacked adequate quantity of high-quality studies for each practice. Five mathematical practices can be considered potentially evidence-based including: CRA, manipulatives, EAI, SBI, and PALS (see Table 2.4).

In total, five quality studies explored the CRA framework and all yielded positive results. Two studies paired the CRA framework with the problem-solving strategy STAR (search, translate, answer, review; Maccini & Hughes, 2000; Maccini & Ruhl, 2000), and one modified the CRA strategy to include a graphic organizer during the abstract phase (Strickland & Maccini, 2013). When only including studies that explored manipulatives as a stand-alone mathematical tool, and not part of a framework, three single case studies with nine participants existed. One study compared concrete and virtual manipulatives and found both to be effective for secondary students with disabilities (Satsangi et al., 2016), and two paired virtual manipulatives with explicit instruction and reported a functional relation (Satsangi et al., 2018a, 2018b).

This review included two group designs with 142 participants that investigated EAI. One of these studies yielded positive results (Bottge et al., 2007), while the other had neutral results where participants in the EAI outperformed the control group and students with disabilities scored better on word problems using EAI, but their computation skills were lower (Bottge et al, 2002). One group design study with 22 participants (Xin et al., 2005) and two single case design studies with eight participants (Jitendra et al., 1999; Jitendra et al., 2002) investigated SBI. One group design with 92 participants explored PALS (Calhoon & Fuchs, 2003). All studies yielded positive results categorizing these practices as potentially evidence-based.

Discussion

This evidence-based synthesis analyzed the literature on teaching algebra to secondary students with high-incidence disabilities. Twenty studies were reviewed and analyzed, of which 14 met the CEC standards of high quality (Cook et al., 2014). Across the 20 studies, researchers investigated eight different mathematical interventions to teach algebra related content to secondary students with high-incidence disabilities. The main result of the review was that none of these eight interventions met the necessary criteria to be considered evidence-based for this particular demographic and mathematical content. However, five mathematical practices (i.e., CRA, manipulatives, EAI, SBI, and PALS) were found to be potentially evidence-based. From this systematic review, educators can make informed decisions about the instructional practices they use to teach algebra to students with high-incidence disabilities, and researchers can plan future studies to fill the gaps in literature.

When focusing specifically on interventions to support secondary students with high-incidence disabilities in the area of algebra, no interventions met the standard of evidence-based, per CEC (2014). This is due to a lack of literature on the topic, as well as too few of the published studies being designed to meet the quality standards. There have only been 20 studies in the past 20 years focused on algebra interventions and practices to support secondary students with high-incidence disabilities. This number is relatively small compared to research pertaining to interventions and instructional practices to support students in reading (Wood et al., 2018) and mathematics interventions focused on more foundational content (e.g., early numeracy, basic operations; Dennis et al., 2016; Stevens et al., 2018). As such, the results of this review should be taken as a contribution to an ever-expanding empirical base regarding more advanced mathematics content and students with disabilities. This review demonstrates the need for more high-quality research to be conducted to provide practitioners with evidence-based practices for teaching algebra to secondary students with high-incidence disabilities.

Nearly one-third of the studies analyzed in this review failed to meet the standards established by the CEC to be considered methodologically sound (Cook et al., 2014). CEC's standards were published in 2014, resulting in four-fifths of included research studies being published before the standards were established. While quality indicators and standards existed previous to this date (e.g., Horner et al., 2005 for single-case and Gersten et al., 2005 for group), nearly half of the included publications even proceeded these earlier quality indicators and standards. In this evidence-based synthesis, we applied indicators and standards ex post facto to studies that were published prior to such guidelines. Furthermore, the CEC quality indicators and standards applied here may be more rigorous than other options, because they require studies to meet all quality indicators in order to be considered methodologically sound (Cook et al., 2014; Cook & Cook, 2013). Applying quality indicators with such high standards means only the most credible studies are included when determining whether a practice is evidence-based (Cook et al., 2014; Cook & Cook, 2013). Thus, when a practice meets evidence standards, practitioners can use it with confidence.

Five mathematical practices met the criteria for potentially evidence-based: CRA, manipulatives, EAI, SBI, and PALS. Consistent with previous reviews, the implementation of these interventions yielded positive results for students with disabilities acquiring algebra skills (e.g., Martia & Hord, 2017; Watt et al., 2016), and some are even considered an EBP for students with disabilities more generally or across more general mathematical areas (e.g., CRA; Bouck et al., 2018). A classification of potentially evidence-based means there were too few studies and/or participants to confirm the effectiveness of the practice (Cook et al., 2014). There is an obvious need for more research focused on algebra and students with high-incidence disabilities in order to validate that these seemingly effective interventions are in fact backed by multiple sources of high-quality evidence. However, the results offer secondary educators options for consideration when teaching algebra, given that no practice could be considered evidence-based for teaching algebra to secondary students with high-incidences disabilities.

Although none of the practices reviewed met evidence standards, interestingly over one-third of the high-quality studies reviewed involved manipulatives either as a stand-alone tool (e.g., Satsangi et al., 2016) or as part of a framework (e.g., Bouck et al., 2019, Maccini & Ruhl, 2000). This aligns with the recommendation of The National Council of Teachers of Mathematics to use manipulatives for teaching mathematics at all levels (NCTM, 2013). Although manipulatives are a vital component of graduated instructional sequences like CRA (Watt et al, 2016), they are not always used as part of a specific sequence of instruction (Satsangi et al., 2016). For the purpose of this review, studies involving manipulatives were categorized and analyzed based on how they were used within instruction resulting in three separate interventions: CRA, manipulatives, and VA. Both CRA and manipulatives were identified as potentially evidence-based, while there was only one study examining VA and thus insufficient evidence to classify the practice.

If researchers analyzed all manipulative studies as one category, overall results of the review may have been different, including an increase of evidence to support the use of manipulatives for secondary students with high-incidence disabilities learning algebra. However, the authors believe it would have been less meaningful to group all of those interventions into one practice as educators would have limited information pertaining to how a specific practice impacted performance (Cook & Cook, 2013). Future evidence-based systematic reviews should consider examining virtual manipulatives and concrete manipulatives as separate practices. While these mathematical tools have similar characteristics, there are benefits and drawbacks to using each and one type – concrete or virtual – may be more effective and appropriate for specific groups of students based on needs and preferences (Satsangi, et al., 2016).

Implications for Practice

The results of this systematic review hold implications for practice. The first is that there are five potentially evidence-based practices teachers can use to teach algebra to students with high-incidence disabilities (e.g., CRA, manipulatives, EAI, SBI, and PALS). When implemented with fidelity, researchers demonstrated these practices support student learning in mathematics (e.g., Maccini & Hughes, 2000; Satsangi et al, 2018a). While they are not guaranteed to work for every student, these mathematic interventions should be a consideration in algebra instruction decision making for secondary students with high-incidence disabilities (Cook & Cook, 2013). Although secondary students with high-incidence disabilities are making gains in mathematics, gaps still exist between those with disabilities and those without (NAEP, 2017), thus it is imperative teachers use practices that are shown to be effective for students with disabilities (NCTM, 2013).

Although students improved algebra performance while the mathematical practice or intervention was being implemented in the original studies, these improvements often dipped during maintenance and generalization phases (e.g., Bouck et al., 2019; Maccini & Hughes, 2000). In order to support students in maintaining the skills they acquire during instruction, teachers may want to consider implementing an intervention package. For example, when implementing graduated instructional sequences like CRA or VA, the teacher could increase criterion levels or include a plan to fade teacher support (Park et al., 2020; Richling et al., 2019). Often criterion levels are set at 80% to move to the next phase, but by increasing this to 100% students would need to demonstrate mastery and the increased criterion may support overlearning (Richling et al., 2019). When fading support, teachers gradually fade their prompts and cues to allow students to initiate independent problem solving (Park et al., 2020). Additional sessions with manipulatives in which the teacher gradually reduces instruction as the intervention progresses is one way to fade support and build fluency. If students are able to maintain the skills they acquire, they will be able to build on these in their new learning which is especially important in the area of mathematics.

A final implication for practice is the emergent of the use of virtual manipulatives to support secondary students with high-incidence disabilities in the area of algebra. The four most recent studies included in this review involved virtual manipulatives in some capacity (e.g. Bouck et al., 2019, Satsangi et al., 2018a; Satsangi et al., 2018b; Satsangi et al., 2016), which suggests virtual manipulatives represent an up-to-date and relevant practice. Accessible through Chromebooks, iPads, or computers, virtual manipulatives do not take up extra space in the classroom, allow for individualized scaffolding within the program, and are often less stigmatizing to older students compared to concrete manipulatives (Satsangi & Miller, 201). Secondary teachers delivering mathematics instruction may want to consider using virtual manipulatives during whole group instruction or for small group interventions targeting specific skill deficits.

Limitations and Future Direction

This evidence-based systematic review is not without its limitations. Although steps were taken to ensure all relevant research was included, there is a chance the inclusion criteria excluded literature that could add to the results and discussion. Only peer-reviewed journal articles published in English were considered, leaving out both dissertations and chapters from books. Additionally, participant inclusion criteria involved middle school and high school-aged students with high-incidence disabilities. Research conducted with students with moderate-to-severe disabilities, although valuable, was not included in this review. Further, even though the included population shared the diagnosis of a high-incidence disability, there are great variations among these disabilities and how they impact specific students. For example, a student with an EBD may struggle with mathematics due to missed instruction or be resistant to working with peers, while a student with a learning disability may need the information delivered in smaller parts and benefit more from a graduated sequence of instruction (Marita & Hord, 2017). In order to better understand how students with various disabilities respond to algebra interventions, future research should limit participants to specific disabilities (e.g., LD, MID, ADHD, EBD) within a single research design. In addition, when more research is available, evidence-based syntheses of mathematics interventions to support students with disabilities learning algebra should be disability specific, such that researchers can say this intervention is an evidence-based practice for students with learning disabilities while that one is for students with mild intellectual disability (Cook & Cook, 2013).

Another limitation was that the interventions examined among studies were sometimes an intervention package (i.e., a combination of instructional components, such as manipulatives plus explicit instruction) making it difficult to credit the effects to a single element (e.g., manipulatives; Watt et al, 2016). In some situations, the intervention was paired with a strategy that has already been established as evidence-based to teach mathematics to secondary students with high-incidence disabilities (e.g., National Center on Intensive Intervention, 2016; Satsangi, et al, 2018a). When paring explicit instruction with other practices, like virtual manipulatives, it is difficult to determine whether the use of the manipulatives or the quality instruction was the source of success. Future research should include direct comparisons of instructional practices in order to help educators in determining the most effective evidence-based practices for teaching algebra to students with disabilities. However, the researchers also acknowledge that intervention packages are more likely to help educators achieve the acquisition, maintenance, and generalization of algebra they seek to achieve for their secondary students with disabilities (Park, 2020).

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Table 2.1

Study characteristics

Study	Participants	Disability	Content	Practice	Design	Results
Bottge et al. (2002)	n= 100 6-12th grade	LD, EBD, CD, SL	pre-algebra	EAI	quasi-experimental	SWD benefited from EAI, and retained a majority of what they learned over time.
Bottge et al. (2007)	n = 42 8 = SWD 7th grade	LD, EBD	pre-algebra	EAI	quasi-experimental	Participants in EAI outperformed TPI group on contextual posttest and transfer. No difference in computation and word problems. All SWD in EAI group had higher scores on the word problems, but 75% had lower computation.
Bouck et al. (2019)	n = 4 middle school	ID, LD, ADHD	linear algebra equations	VA	multiple probe across behaviors replicated across participants	All four participants acquired the algebra skills, but were unable to maintain skills when instruction was not provided directly before completing the probe.
Calhoon & Fuchs (2003)	n = 92 9th-12th grade	LD, EBD, ID	operations & algebraic thinking, measurement, & geometry	PALS	quasi-experimental	PALS + CBM group outperformed the control group on computation scores. Both groups increased comparably on concepts/applications.
Ives, 2007 (a&b)	n = 14 (GO) n =16 (CG)	language related disabilities	solving systems of linear equations	graphic organizer	experimental	Participants who used graphic organizers outperformed those who did not when solving systems of linear equations. In Study 1, participants maintained learning over a couple of weeks. In Study 2, students had more success actually solving the problems.
Jitendra et al. (2002)	n = 4 8^{th} grade	LD	word problems	SBI	multiple probe across participant	All participants improved word problem–solving performance and maintained performance over the duration of the intervention. All participants demonstrated high scores during generalization.

Table 2.1 (cont'd)

Study	Participants	Disability	Content	Practice	Design	Results
Jitendra et al. (1999)	n = 4 6^{th}-7^{th} grade	LD	word problems	SBI	multiple baseline across participants replicated across behaviors	All participants improved from baseline to intervention in using correct operations. Participants generalized strategy. 2 participants maintained, 1 slightly decreased, 1 drastically decreased.
Maccini & Hughes (2000)	n = 6 ages 14-18	LD	problem solving with integers	CRA	multiple probe across participant	Problem-solving skills dramatically improved following instruction at the concrete, semi concrete, and abstract levels. Participants' strategy use increased and they were able to generalize skills to novel situations.
Maccini & Ruhl (2000)	n = 3 8th grade	LD	subtraction of integers	CRA	multiple probe across participant	All participants demonstrated an improvement in strategy use, accuracy on problem representation, and average accuracy on problem solution from baseline to concrete instruction. Participants were able to maintain skills over time, and were able to generalize to near tasks, but have lower transfer skills for far generalization.
Satsangi et al. (2016)	n = 3 11^{th}-12^{th} grade	MLD	linear algebraic equations	manipulatives (virtual & concrete)	alternating treatment	All participants solved more algebraic questions correctly with both types of manipulatives. 2 students learned the material quickest with concrete, and one student learned quickest with virtual. All 3 students had fewer prompts and completed problems quicker using virtual manipulatives suggesting greater independence.

Table 2.1 (cont'd)

Study	Participants	Disability	Content	Practice	Design	Results
Satsangi et al. (2018a)	n = 3 9th grade	MLD	multistep algebraic equations	manipulati ves (virtual)	multiple baseline across participant	Using virtual manipulatives all participants scored above their baseline scores during intervention, maintenance, and generalization. All participants said they benefited and enjoyed using the virtual manipulatives.
Satsangi et al. (2018b)	n = 3 9th grade	MLD	multistep algebraic equations	manipulati ves (virtual)	multiple baseline across participants	All participants improved from baseline. Accuracy scores ranged from 70-100% during intervention and maintenance. Independence scores ranged from 78-100% during intervention and maintenance.
Scheuermann et al. (2009)	n = 14 6th-8th grade	LD	one-variable equations	Explicit Inquiry Routine (EIR)	multiple probe across participant	All participants made substantial progress, all but 1 student reached mastery criterion (80% accuracy) by the final instructional probe. Participants were able to generalize their skills to new problems written in the same format and maintained performance for up to 11 weeks.
Strickland & Maccini (2012)	n = 3 8th-9th grade	LD	multiplying linear expressions	CRA-I + Graphic Organizer	multiple probe across participants	All participants substantially increased overall accuracy from baseline to intervention. 2/3 demonstrated mastery level during maintenance, 1 was improved from baseline, but not mastery. Participants transferred info to novel situations but were not able to transfer to higher level mathematics. Participants reported that they found the intervention beneficial and enjoyable.

Table 2.1 (cont'd)

Study	Participants	Disability	Content	Practice	Design	Results
Strickland & Maccini (2013)	n = 5 high school	LD, MD	quadratic expressions within area word problems	CRA-I + Graphic Organizer	multiple probe across two groups	Participants' accuracy improved and they maintained their skills over time. Participants reported the intervention was beneficial and they would recommend it to peers.
Van Garderen (2007)	n = 3 8th grade	LD	word problems (algebra skills)	Diagrams	multiple probe across participants	Students improved ability to generate diagrams and use them to solve 1 and 2-step word problems. Participants generalized their skills to solve different types of word problems.
Witzel et al. (2003)	n = 358 n = 68 6th-7th grade	LD or at-risk for algebra failure	expressions and equations	CRA	quasi-experimental	Both groups showed improvement but CRA group improved by a greater amount.
Witzel (2005)	n = 231 SWD = 49 middle school	LD	linear algebraic functions	CRA	quasi-experimental	Participants who learned through the CRA method performed significantly better on the posttest than the repeated abstract explicit instruction group. This was true for low, medium, and high achievement groups.
Xin et al. (2005)	n = 22 6th-8th grade	disability or at-risk for math failure	solve for unknown in word problems	SBI	quasi-experimental	Both groups improved but SBI group performed significantly better than control group.

Table 2.2

Cook et al. (2014) Quality Indicators (QI) Applied to Group Design Algebra Studies Involving Students with High-Incidence Disabilities

	1.1	2.1	2.2	3.1	3.2	4.1	4.2	5.1	5.2	5.3	6.1	6.2	6.3	6.4	6.8	6.9	7.1	7.2	7.3	7.4	7.5	7.6	8.1	8.3
Group Design Studies																								
*Bottge et al. (2002)	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
*Bottge et al. (2007)	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
*Calhoon & Fuchs (2009)	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
Ives (2007a)	Y	Y	Y	Y	Y	N	Y	N	N	N	Y	N	Y	N	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
Ives (2007b)	N	Y	Y	N	N	N	Y	N	N	N	Y	N	Y	N	Y	Y	Y	Y	Y	Y	N	Y	Y	Y
Witzel et al. (2003)	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	N	Y	Y	Y
Witzel (2005)	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	N	N	Y	Y
*Xin et al. (2005)	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y

Note: Y = yes, quality indicator present & N = no, quality indicator not present. Quality indicators 6.5, 6.6, 6.7, and 8.2 only applied to single case studies and are not included here. See Cook et al. (2014) for the complete list of quality indicator (e.g., 1.1 – 8.2).
* means the study met all quality indicates

Table 2.3

Cook et al. (2014) Quality Indicators Applied to Single Case Algebra Studies Involving Students with High-Incidence Disabilities

	1.1	2.1	2.2	3.1	3.2	4.1	4.2	5.1	5.2	5.3	6.1	6.2	6.3	6.5	6.6	6.7	7.1	7.2	7.3	7.4	7.5	8.2
Single Case Design Studies																						
*Bouck et al. (2019)	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
*Jitendra et al. (2002)	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
*Jitendra et al. (1999)	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
*Maccini & Hughes (2000)	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
*Maccini & Ruhl (2000)	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
*Satsangi et al. (2016)	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
*Satsangi et al. (2018a)	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
*Satsagni et al.(2018b)	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
Scheuermann et al. (2009)	Y	Y	Y	Y	Y	Y	Y	N	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
Strickland & Maccini (2012)	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	N	Y	Y	Y	Y	Y	N	Y	Y
*Strickland & Maccini (2013)	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
*VanGarderen (2007)	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y

Note: Y = yes, quality indicator present & N = no, quality indicator not present. Quality indicators 6.4, 6.8, 6.9, 7.6, 8.1, and 8.3 only applied to group design studies and are not included here. See Cook et al. (2014) for the complete list of quality indicator (e.g., 1.1 – 8.2).
* means the study met all quality indicators.

Table 2.4

Evidence-Base Categories of Practices

Practice	Group	Single	Participants	Random	Results	Category
Concrete-Representational-Abstract	0	3	14	N/A	Positive	Potentially Evidence-Based
Enhanced Anchor Instruction	2	0	142	Yes	Positive/Neutral	Potentially Evidence-Based
Peer-Assisted Learning Strategies	1	0	92	Yes	Positive	Potentially Evidence-Based
Schema Based Instruction	1	2	27/8	Yes	Positive	Potentially Evidence-Based
Manipulatives	0	3	9	N/A	Positive	Potentially Evidence-Based
Virtual Abstract	0	1	4	N/A	Positive	Insufficient Evidence
Graphic Organizers & Diagrams	0	1	3	N/A	Positive	Insufficient Evidence

CHAPTER 3

Comparing the Effectiveness of Virtual and Concrete Manipulatives to Teach Algebra to Middle School Students with Disabilities

In order to meet the mathematical demands of adult life, students are introduced to basic mathematical principles early with the expectation that their skills will develop each year (Geary, 2013; Jitendra et al., 2018). By the time they enter middle school, students are expected to have a clear understanding of fundamental mathematical concepts, mastery of basic skills, and the necessary procedural and declarative knowledge for higher-level mathematics (Montague & Jitendra, 2018). However, an estimated 7% of school-aged students have a disability that hinders their progress in developing mathematical skills and an additional 5-10% of students exhibit mild but persistent difficulties in mathematics despite average abilities in other academic areas (Geary, 2013). The barriers encountered by these students create gaps in learning that impact performance in basic mathematics classes, and often intensify as students move into more advanced courses like algebra (Jitendra et al., 2018; Watt et al., 2016).

Algebra is often viewed as a gatekeeper to higher-level mathematics, with proficiency considered essential for all students, but often students with disabilities struggle to develop grade-appropriate algebra skills (Watt at al., 2016). Some students with disabilities have limited or incorrect foundational knowledge while others have deficits in cognitive processes including problems with working memory, information retrieval, and attention regulation (Geary, 2013). Across students with high-incidence disabilities, mathematical struggles can manifest themselves as students with attention problems experiencing difficulties learning the steps of an algebra problem, students with memory deficits knowing the steps but unable to retrieve them, students unable to interpret the meaning of variables represented as letters, and students struggling to transition from concrete mathematical concepts to symbolic algebraic ideas (Geary, 2013). Teachers need to have access to effective and efficient educational tools to help all students achieve success in algebra, including students with disabilities (National Council of Teachers of Mathematics, 2013).

Concrete Manipulatives

The National Council of Teachers of Mathematics recommends teachers use manipulatives to enhance learning at all grade levels (NCTM, 2013). Manipulatives include concrete manipulatives (CM) and virtual manipulatives (VM). CM are physical objects that students can feel and maneuver during mathematical thinking (Bouck & Park, 2018). Common CM include chips, geoboards, base ten blocks, tangrams, and algebra tiles (Carbonneau et al., 2013). CM are effective in building conceptual understanding because they allow students to see and touch objects that represent the mathematical principles students are learning (Carbonneau et al., 2013).

CM can play an important role in developing mathematical understanding of algebra for students with disabilities (Satsangi et al., 2016). With these tools, students with disabilities have the opportunity to construct a better understanding of how abstract concepts relate to novel and authentic situations, a challenging task for students with disabilities (Carbonneau at al., 2013). Maccini and Ruhl (2000) found students with disabilities improved their ability to subtract integers when using algebra tiles as part of the concrete-representational-abstract (CRA) framework, and over time all students generalized their skills to other types of problems. Strickland and Maccini (2013) also studied concrete algebra blocks, as part of a CRA-integration strategy. Students with disabilities were more accurate in their ability to multiply linear algebra expressions as part of area and perimeter problems when they used algebra blocks and graphic organizers. Students were able to maintain their new skills over time, generalize their algebra skills to novel tasks, and agreed the intervention including the use of manipulatives was beneficial (Strickland & Maccini, 2013).

Although researchers established CM as an evidence-based practice for teaching mathematics to students with disabilities (National Center on Intensive Intervention, 2016), disadvantages exist to using physical objects to assist in developing mathematical skills. CM take up physical space in the classroom, often require additional time for setup and cleanup, cannot be easily transported, and could be stigmatizing for older students (Satsangi & Miller, 2017). Even when older students have access to CM, some will evade tools they view as stigmatizing (Satsangi et al., 2016). By choosing not to use CM, students who struggle with mathematics may be unsupported and thus unable to be successful with the higher-order and abstract thinking required in algebra. For some, VM may provide an appropriate non-stigmatizing support to build the bridge between abstract and concrete mathematical concepts (Satsangi & Bouck, 2015).

Virtual Manipulatives

VM are interactive, digital representations of objects displayed on computers, tablets, and handheld devices through app-based purchases, whole programs, or online resources (Bouck, Working, & Bone, 2018). These tools simulate many of the same objects as CM including fraction tiles, base ten blocks, tangrams, number lines, and algebra tiles (Shin et al., 2017). Many VM have settings that can be adjusted to provide additional support and guidance to students struggling to learn specific mathematical concepts (Shin et al, 2017). These settings promote student independence and may include unlimited quantities to use with a problem, the option for hints and extensions, and the availability of immediate feedback (Bouck et al., 2017; Satsangi & Miller, 2017). As a virtual tool, students are able to get the support they need without looking different from their peers (Satsangi & Bouck, 2015). This is especially true in a one-to-one technology school (e.g., chromebook), and becomes more important for students at the secondary level (Satsangi et al., 2016).

Recently, researchers validated the benefits of using VM to support the mathematical learning of students with disabilities (e.g., Bouck et al., 2019; Satsangi & Bouck, 2015; Satsangi, Hammer, & Evmenova, 2018; Satsangi, Hammer, & Hogan, 2018). When using VM to teach concepts of area and perimeter to three secondary students with disabilities, Satsangi and Bouck (2015) found students were able to learn the information, maintain it over time, and generalize the concepts to abstract word problems. Students noted the advantages of having a visual display organized within a device and liked being able to use technology when working with VM. Satsangi and colleagues (2018) conducted two single-case studies examining the effectiveness of VM to teach secondary students with disabilities to solve multi-step algebra equations. In the first study, participants' accuracy was above 90% during both intervention and maintenance phases using virtual balance scales, and 60-90% during the generalization phase without VM (Satsangi, Hammer, & Evmenova, 2018). Satsangi, Hammer, and Hogan (2018) paired VM with explicit instruction and produced similar results. During the intervention and maintenance phases, secondary students with disabilities averaged 88-100% accuracy with an average of 91-100% independence (Satsangi, Hammer, & Hogan, 2018). Most recently, Bouck and colleagues (2019) investigated VM as part of the Virtual-Abstract (VA) instructional framework. All four participants experienced an immediate effect on their accuracy in solving linear equations upon entering the intervention phase, but struggled to maintain their learning over time.

Concrete vs Virtual Manipulatives

Researchers found both CM and VM effective for students with disabilities (Bouck & Park, 2018; Satsangi, et al., 2016). However, a small research base exists in which CM and VM are compared in terms of efficacy and efficiency. To date, however, researchers have yet to determine whether one form is more effective and efficient (i.e., time to complete the task). In their alternating treatment design study, Satsangi et al. (2016) found secondary students with disabilities significantly increased their accuracy in solving area and perimeter problems when they used both CM and VM. However, a majority of students solved more single-variable equations correctly using CM and, when interviewed, students expressed varying opinions regarding preference for either type of manipulative. Bouck, Chamberlain, and Park (2017) also used an alternating treatment design to compare the efficacy of CM and VM. The three middle school students with disabilities were successful subtracting with regrouping using either tool. All three students were more independent in solving problems when using the VM, but one preferred using the CM (Bouck et al., 2017). Finally, Bouck et al. (2018) explored the use of virtual and concrete fraction tiles to support middle school students with disabilities. Students showed improvements adding fractions with unlike denominators regardless of whether they used virtual or concrete fraction tiles. Students reported liking both types of manipulatives, but two students preferred the VM and described certain features unique to VM that made it easier to clear problems and check accuracy.

Overall, more information is needed regarding the effectiveness of CM and VM, including student preference and how each type of manipulative promotes independence and impacts total task completion time. Currently, there is limited research comparing CM and VM that support students in the area of algebra, and no research exists making this comparison with algebra tiles. Building on previous literature, this study will compare the effectiveness of CM to VM to support students' acquisition of algebraic concepts. The research questions addressed in this study included: (a) Are CM and VM effective tools for middle school students with disabilities in terms of accuracy in solving algebraic equations? (b) Are middle school students with disabilities more independent as they solve algebraic equations using CM or VM? (c) Are middle school students with disabilities more efficient solving algebraic equations in terms of task completion time using CM or VM? (d) What are middle school students with disabilities perceptions regarding the use of CM and VM to solve algebraic equations?

Method

Participants

Participants for this study included seventh-grade students previously identified as eligible for specialized services including an Individual Education Program (IEP) with a goal in mathematics. Specific inclusion criteria for participation in the study included: (a) receiving specially designed instruction in the area of mathematics, (b) a pre-assessment score below 50%, (c) non-proficient scores in mathematics during the mid-year administration of Iowa Assessments for a minimum of two years (Iowa Assessments, 2017), and (d) parental consent and student assent to participate. All participants were enrolled in a general education co-taught math class taught by a general education teacher and a special education teacher. In addition to this class, students received specially designed mathematics instruction either one-on-one or in a small group setting (<6 students) a minimum of 20 minutes each week. Following this inclusion criteria, all students participating in the study were suspected of having a disability in the area of mathematics according to the state guidelines.

Alan

Alan was a 13-year-old, white, male who first became eligible for special education services in preschool where he received speech services. His academic goals, including reading and mathematics, were added at the beginning of first grade, with a writing goal added in third grade. Alan earned 195 on the mathematics subtest of Iowa Assessments in 5^{th}-grade and 197 in 6^{th}-grade; both were below the proficiency level. He earned 31% on the pre-assessment where he successfully solved some one-step equations and some two-step equations, but lacked consistency. Alan demonstrated a gap in knowledge of integers, and an inability to apply algebra processes to multistep problems. On his weekly progress monitoring, Alan scored an average of five correct answers on AAIMS algebra probes, and his IEP math goal stated that he would average 15 correct answers by the end of seventh grade.

Cara

Cara was a 12-year-old, white, female who began receiving special education services in first grade due to a learning disability in the area of mathematics. Her last two Iowa Assessment scores placed her just outside of the proficiency range and just under the expected 12-point yearly growth. Cara scored 38% on her pre-assessment. She demonstrated a basic understanding of algebra terminology and simple algebra equations, but often made minor calculation errors or confused integer signs. When given five minutes to work, Cara answered an average of 10 problems correct on the AAIMS algebra probes. Her IEP goal stated that by the end of seventh grade she would correctly answer 15 problems in five minutes.

Jan

Jan was a 13-year old, African American, female who was identified as having a learning disability when she was in first grade with academic goals in reading, writing, and mathematics.

Jan received 40 minutes a week of additional direct instruction in the area of mathematics. Direct instruction occurred in a small group setting where Jan received pre-teaching and re-teaching of skills covered in the general education setting, as well as instruction in deficit areas to fill gaps in mathematical knowledge. Jan earned a score of 25% on her pre-assessment. She was able to identify variables within an equation, had a basic understanding of integers, and solved for the unknown in a one-step algebra problem. She was unfamiliar with some algebra terminology, and was unable to solve two-step and three-step equations, or problems with multiple negative numbers. Jan was non-proficient on the mathematics component of the Iowa Assessments as both a 5th-grade student, and a 6th-grade student. When given 5 minutes to work, Jan scored four answers correct on AAIMS algebra probes, and her IEP goal stated that by the end of seventh grade she would average 14 correct answers.

Setting

This study was conducted in a public school district in a Midwest community with a population of 5,800. The district enrollment at the time of the study was approximately 2,000 students with 95% identified as Caucasian. The district consisted of four schools: lower elementary, upper elementary, middle school, and high school. The middle school consisted of students in grades sixth- through eighth-grade with an enrollment of 485 students. According to school records, 15.5% of students in the middle school were eligible for special education services, with 61% of those students having a mathematics goal as part of their IEP.

The researcher worked one-on-one with each student for all sessions. Data collection occurred during an academic support period which was a scheduled time during the school day for students to work on their specific academic goal areas. Students were given the choice to work in the classroom, in the conference room, or in a study area in the library. Each of these were quiet spaces used regularly by students when working independently or with partners.

Materials

The materials for this study included pencils, data collection sheets, the stopwatch app on a cellphone, task analysis checklist, researcher created pre-assessment, five-problem probes, concrete algebra tiles, and a Chromebook with the Algebra Tiles program (Algebra Tiles by Brainingcamp [2019]). The probes were constructed by a research assistant, under the guidance of the first author who was a doctoral candidate and certified special education teacher, and aligned with the middle school mathematics curriculum and the Common Core State Standards. A minimum of 125 two-step algebra problems consisting of integers ranging from -20 to 20 were dispersed among probes. Problems were presented in four different formats in which the location of the constants and variables differed (e.g., $5 + 2x = 13$, $2x + 5 = 13$, $13 = 5 + 2x$, $13 = 2x + 5$). All probes included a problem of each type as well as a fifth problem randomly selected from the four formats. Each probe was unique and although some problems had the same solution for the variable, no problem was repeated in the same format across the study.

The manipulatives used by participants during the intervention phases were virtual and concrete algebra tiles. Algebra tiles use recognizable pieces to represent variables and constants. Regardless of the concrete or virtual algebra tiles used in this study, positive constants were represented by small yellow squares, negative constants by small red squares, positive x variables by green rectangles, and negative x variable by red rectangles (see Figure 3.1). Although not used for this specific study, a large blue square was available to represent x^2. Whether working with CM or VM, the items were placed on an organizational tool referred to as an algebra mat to represent each side of the equation. The CM were plastic tiles participants could physically pick up and move. Participants accessed the VM algebra tiles from Brainingcamp (2019) on their school-issued Chromebook. The researcher chose the Brainingcamp program because the algebra tiles were the most comparable to the concrete tiles, could be accessed using a device students were comfortable using, and had several other forms of VM that could be used for future learning.

Independent and Dependent Variables

The independent variable for the study was the concrete or virtual algebra tiles. Using the concrete algebra tiles was defined as the participant setting up the two-step algebra problem with the concrete tiles and then physically moving the tiles to solve for the variable. Using virtual algebra tiles was defined as the participant displaying the two-step algebra problem on the Chromebook using the digital algebra tiles and moving the tiles as needed to solve for the unknown. The dependent variables for the study included accuracy, independence and task completion time. Accuracy was defined as the number of two-step algebra problems answered correctly out of five, represented as a percentage. Independence was calculated as the number of steps in the task analysis of solving the two-step algebra problems participants completed without any prompting, represented as a percentage. Task completion time was defined as the time needed to complete the five-problem probe. Event recording was used to measure the effectiveness of manipulatives to solve two-step algebra problems. The researcher summed the problems participants solved correctly out of five to calculate accuracy and reported as a percentage. When assessing answers, no partial credit was awarded. For example, if the correct answer to a problem was 8 and the participant wrote -8, it was recorded as incorrect. The percentage of prompts needed for each student during each phase was recorded to measure independence and the duration of each session was recorded as the task completion time.

When determining independence, the researcher used the system of least prompts in which participants were encouraged to complete each step of the problem independently, but received increasingly intrusive prompts (i.e., gesture, verbal, and modeling) as needed (Ault & Griffen, 2013). In order to correctly solve each problem participants needed to accurately complete all steps described in the task analysis. These steps included: (a) set up left side of equation using manipulatives, (b) set up right side of equation using manipulatives, (c) identify the side of the equation with the variable, (d) place the correct number of inverse squares on the side with the variable, (e) place the same number and color of squares on the opposite side, (f) remove squares from the side with the variable to represent a sum of zero, (g) remove all inverse squares on the opposite side of the equals sign, (h) physically separate the variables with rows, (i) divide the ones among the variable, and (j) record solution.

If participants made an error or did not complete the next task within 10 seconds, a prompt was given and the participant had another opportunity to respond. From least-to-most intrusive, prompts included: gesturing (i.e., pointing to algebra mat or algebra tiles), verbal prompts (i.e., asking "what is the next step?"), and modeling (e.g., showing how to move a digital algebra tile). All problems were two-step; regardless of the problem or condition, there were 10 steps to the task analysis for each problem. Participants could have been prompted up to three times per step and therefore could have received between 0 and 30 prompts on one problem. The researcher calculated the difference between the total number of prompts given and the opportunity for prompts and divided that by the opportunity for prompts (150). The number was then multiplied by 100 to determine independence for each probe. The researcher used the timer on an iPhone to measure task completion time and times were recorded to the nearest second following standard rounding rules. Timing began when the researcher gave the participant the probe with the direction to start working and ended when the participant completed the last problem.

Experimental Design

This study used a single-subject adaptive alternating treatment design to compare the efficacy of concrete and virtual algebraic tiles as instructional tools to support middle school students with disabilities as they solved two-step algebra problems. This study consisted of three phases: baseline, intervention (including extended baseline), and best treatment. Once baseline data were collected, participants alternated between using concrete algebra tiles, virtual algebra tiles, and no manipulatives to complete the five-question probe with no more than two of the same conditions completed consecutively. If there was no clear separation in data between conditions, additional concrete and virtual manipulative sessions were conducted to appropriately identify best treatment for each participant. The researcher, a doctoral candidate and highly-qualified special education teacher, delivered all sessions in a one-on-one format.

Procedures

Pre-assessment

Prior to baseline, participants were given a pre-assessment to determine their current ability to solve basic algebra problems. The pre-assessment was given in two parts to reduce the effect of fatigue and consisted of 16 total problems. The first three problems asked participants to identify key algebra terminology, and the remaining 13 problems required participants to solve for an unknown variable. The difficulty level of each problem was influenced by the number and types of steps required to solve the problem and the presence or absence of negative numbers. Students who answered more than eight questions correctly were not eligible for the study as they likely did not need the support of manipulatives and would reach a ceiling before the conclusion of the study.

Baseline

The baseline phase consisted of a minimum of five total sessions for each participant, with one probe administered each session. Participants did not have access to manipulatives when completing probes during baseline. No instructional training on how to solve problems was offered, and no prompting was provided. Once a minimum of five sessions occurred and baseline data were stable (80/25 rule, Gast & Spriggs, 2014) with a zero-celerating or de-celerating trend, each student underwent a training on how to use the manipulatives.

Training

Prior to the intervention phase, the researcher used explicit instruction to train participants on how to use the manipulatives as tools to assist in solving two-step algebra problems. Students were trained first on CM, followed by VM. General instruction included explanation of the colored tiles and representation, setup procedures, review of inverse principles, and modeling of correct use of the manipulatives. For VM, additional instructions included selecting the program from the bookmarked list on the Chromebook, using the cursor to move tiles, highlighting and selecting groups of tiles, reversing steps, canceling out tiles, and clearing the entire board.

The researcher used explicit instruction to teach the process of solving algebra problems with manipulatives. After the researcher modeled two problems, inclusive of using a verbal narration (i.e., think aloud), the researcher guided (i.e., provided prompts or cues as needed) participants as they solved two problems independently. Participants were then asked to solve five problems independently without any prompts or support. Once a participant correctly solved

80% of the problems using each of the respective manipulatives, he or she moved to the intervention phase. In the event a participant did not meet this criterion, the researcher repeated the training process during the next session with the same type of manipulative. Participants were not allowed to move into the intervention phase until they were able to successfully complete training on both CM and VM.

Intervention

The researcher randomly alternated the two intervention conditions – CM and VM – as well as the extended baseline of no manipulative, with no more than two consecutive sessions of the same condition. Condition order was determined using a random number generator. Participants completed each of the conditions during the intervention phase at least five times, resulting in a minimum of 15 sessions during the intervention phase. The independent variable for each session and the five-problem probe were randomly selected for each participant. The researcher served as the interventionist and implemented the system of least prompts when participants did not initiate a step of the task analysis within 10 seconds during CM and VM conditions; no prompts were used during extended baseline sessions during the intervention phase. Prompts became more intrusive if the participant failed to respond to a previous prompt and included gesturing, verbal prompts, and modeling (Ault & Griffen, 2013).

Concrete manipulatives. In the CM condition, participants were given a pencil, the five-problem probe, an algebra mat with two sides separated by an equal sign, 50 reversible square algebra tiles to represent constants, and 15 reversible rectangle algebra tiles to represent variables (refer to Figure 3.1). The reversible squares could be alternated between yellow and red as necessary to represent positive and negative numbers, and the rectangles from green and red for positive and negative variables. The participant displayed the written equation in a visual format by physically picking up the plastic algebra tiles and placing them with the correct color (i.e., yellow, red, or green) facing upward. For example, when working with $-2 + 3x = 4$, the participant first placed two red squares to represent -2 and three green rectangles to represent 3x on the left side of the algebra mat, and then four yellow squares to represent + 4 on the right side of the algebra mat.

Once the problem was correctly displayed, the participant read the problem aloud and identified the side of the equation with the variable. The participant began to solve the equation by placing the correct number of inverse (opposite) square tiles on the side with the variable and then placed the same number and color of squares on the opposite side. In this example, the participant placed two yellow squares on the left side to represent the additive inverse of -2, and then two yellow squares on the right side to represent a balanced distribution of values to both sides of the equal sign. The participant then simplified the equation by summing the numerical values. As the additive inverse, the sum of -2 and 2 is zero and the participant illustrated this by removing all four squares on the left side leaving three green rectangles representing the variables. The participant then summed the squares as necessary on the right side to determine the value of the variable. In the example $-2 + 3x = 4$, the two yellow tiles would be added to the original four yellow tiles and would remain on the algebra mat. The left side of the algebra mat would now have three green rectangles representing the variable and the right side of the algebra mat would have six yellow squares representing the ones (3x = 6). The second step of solving the algebra problem requires division. To do this with support of the manipulatives, the participant separated the three green rectangles and equally dispersed, or divided, the six yellow squares. In the example, each variable represented by a rectangle piece would be paired with two yellow squares indicating a positive solution (e.g., x = 2). The participant recorded the answer on the worksheet before moving onto the next problem. This procedure continued for all five problems listed on the probe.

Virtual manipulatives. For the VM condition, participants were given a pencil, the five-problem probe, and were directed to use the algebra tiles manipulative (Brainingcamp, 2019) bookmarked on their school-issued Chromebook. As part of the program, participants had access to an unlimited number of squares (ones) and rectangles (*x*) in yellow, green, and red in order to display each algebra problem listed on the assessment. Upon accessing the program, participants were directed to select the equation background to prepare for solving the two-step equations. Like in the concrete condition, participants began by displaying the problem in a visual format. Instead of physically picking up pieces, the participant dragged squares and rectangles on the appropriate side of the equals sign to represent the written algebra problem. For example, when solving the problem $5x + 2 = -8$ the student began by dragging a green rectangle to the left side and then clicking on the tile until five green rectangles were present to represent 5x. The student then dragged a yellow square to the left side of the equation and clicked once so two yellow squares were present representing the value +2. Finally, the student dragged a red square to the right side of the equals sign and clicked on the tile until eight red squares appeared to symbolize -8.

Once the equation was correctly displayed the steps necessary to solve the equation with the assistance of the virtual algebra tiles were the same as when using CM with three exceptions. First, when moving pieces, participants were able to highlight and select groups of tiles to move them instead of picking each up individually. Second, when removing pieces from the virtual algebra mat, participants could place the opposite colors on top of one another and both pieces disappeared, or summed to zero. In the example $5x + 2 = -8$ the student began by placing two red squares on each side of the equal sign to represent the inverse of 2. When placing the red squares on the left side of the equal sign the student placed them on top of the two yellow squares representing the positive constants. When the red squares were placed on the yellow squares, the tiles summed to zero and disappeared. When the student placed the two red squares on the right side of the equals sign they were added to the eight red squares already being displayed. At this time the student was left with five green rectangles and ten red squares representing $5x = -10$. Finally, when dividing using VM, participants selected the number of rows based on the coefficient and then dragged the variable tiles to the correct position. Using the same example, the student selected five rows and dragged each green rectangle to its own row on the left side of the equal sign. The red squares on the right were then divided equally among each of the rows. When the problem was solved, the participant looked at one portioned section which contained a green rectangle representing the variable on one side of the equals sign, and a sole numerical value as represented by squares on the other side. In the current example $x = -2$. The participant recorded the answer on the probe before moving onto the next problem and continued the procedure for all five probe problems.

No-manipulative condition. Procedures during the no-manipulative sessions were the same as the procedures during the baseline sessions. Participants were given the five-problem probe and a pencil and asked to solve the algebra problems. Participants did not receive cues or prompting while they worked. Participants completed a minimum of five sessions without access to manipulatives.

Best treatment

Students engaged in three sessions solving two-step algebra problems using the treatment condition (i.e., CM or VM) determined to be most effective for that particular student in terms of accuracy. When identifying best treatment, the no-manipulative condition was not considered as it was not an intervention. Each session during the best treatment phase involved the same type of five-problem probe as used during the baseline and intervention phases. The researcher followed the same procedure in terms of giving participants materials, using the system of least prompts, and calculating task completion time.

To determine the most effective treatment, the researcher calculated the percentage of non-overlapping data (PND) for accuracy. PND is a commonly used method to compare the data from one condition to another within alternating treatment designs (Gast & Spriggs, 2014). In order to calculate PND, the researcher found the number of sessions one condition (e.g., VM) was more effective than the other (e.g., CM), and then divided the summed number by five and multiplied by 100 (Wolery, Gast, & Ledford, 2014). If no difference between conditions for accuracy existed, the researcher conducted one additional session of each intervention before calculating PND for independence to determine the condition considered best for each participant.

Social Validity

Before baseline and then after intervention, the researcher conducted interviews with each participant. Participants were asked six open-ended questions focused on confidence in abilities, attitudes and personal preference toward past learning experiences, accommodations, technology use, and mathematics instructions. These social validity interviews helped to determine participants' preferences regarding CM and VM.

Interobserver Agreement and Treatment Fidelity

Interobserver agreement (IOA) data were collected for accuracy for all participants during baseline, intervention, and best treatment phases. IOA data for independence were collected for all participants during intervention and best treatment phases, but not during baseline as no prompts were offered during this condition. IOA data were recorded for each participant for (a) two sessions (40%) during baseline, (b) two intervention sessions (40%) for each condition, and (c) one session (33%) of best treatment. A middle school mathematics teacher examined student work to determine IOA for accuracy. IOA was calculated by dividing the number of agreements between the teacher and the researcher by five. IOA for all participants for accuracy was 100%. A special education teacher observing the intervention sessions collected and recorded the number of prompts given for each step of the problem to determine IOA for independence. IOA was calculated by summing the number of agreement for each session, and dividing that by the total opportunities for agreement. IOA scores for independence were as follows: Alan 99.86%, Cara 99.74% and Jan 100%.

Treatment fidelity data for two intervention sessions (40%) for each condition and one best treatment session (33%) for each participant were collected using a checklist. Treatment fidelity included participants being provided a pencil, assessment, and the appropriate manipulative (concrete or virtual), participants using the manipulative to help them solve the problems, and implementation of the system of least prompts taking place within 10 seconds of students not proceeding to the next task on the list. Treatment fidelity was 100% for all participants for this study.

Data Analysis

Data analysis consisted of visual analysis as well as calculating level, trend, and effect size for the three main dependent variables (i.e., accuracy, independence, and task completion time; Gast & Spriggs, 2014). To calculate level, the researcher determined the stability of the intervention conditions. The data were considered stable if 80% of the data fell within 25% of the median (Gast & Spriggs, 2014). The researcher used the split middle method to calculate trend. The researcher divided the intervention data in half for each condition, drew a line between calculated mid-rate and mid-date, and the determined whether each trend was accelerating, decelerating, or zero-celerating (White & Haring, 1980). Finally, the researcher used the Tau-U web-based calculator to calculate effect size for accuracy (Vannest et al., 2011). Tau-U scores less than .65 are considered a small effect, between .66-.92 a medium effect, and greater than .93 a large effect (Parker et al., 2009).

Results

Regardless of the type of manipulative, all three participants were more accurate using algebra tiles as compared to baseline and extended baseline (See Figure 3.2). A functional relation existed between accuracy and use of manipulatives with participants' accuracy at or above 60% on all probes when using manipulatives. One student scored better using concrete manipulatives, one student scored better using virtual manipulatives, and one student was equally successful in terms of accuracy.

Alan

Alan correctly solved an average of 13% of algebra problems across six baseline sessions. Alan was unable to solve any problems correct on his first two probes, and then solved one problem correct on the next three probes. The researcher issued an additional baseline probe to ensure Alan's accuracy data were stable and zero-celerating before moving to the intervention phase. Alan spent an average of 14:38 (range 9:49-18:32) completing the probes during baseline. His task completion time data were variable with a decelerating trend.

Accuracy

Alan averaged 92% accuracy (range 80-100%) using CM to solve 2-step algebra problems, and averaged 84% accuracy (range 80-100%) using VM. His accuracy data were stable with a strong effect (Tau-U = 1.0) with the support of either type of manipulative, with a zero-celerating trend using CM and a decelerating trend using VM. When Alan did not use manipulatives during extended baseline, he averaged 32% accuracy (range 20-80%). Alan answered one question correctly on each probe, except during the second extended baseline session when he correctly solved 80% of the 2-step algebra problems; his extended baseline data were stable with a zero-celerating trend. According to PND, Alan was more accurate using CM on 60% of the probes, more accurate using VM on 20% of the probes, and equally accurate regardless of the type of manipulative on 20% of the probes. Based on accuracy data, using CM was considered best treatment for Alan. Alan's best treatment data were stable with a zero-celerating trend and he averaged 93% accuracy (range 80-100%).

Independence

Alan solved problems with an average 91% independence (range 83.3-97.3%) when using CM. When using VM, Alan worked through the probes with an average of 93% independence (range 79-97%). During the best treatment phase, Alan used CM and solved problems with 99% independence (range 97-100%). All of Alan's independence data were variable, but with an accelerating trend indicating that as he became more comfortable working with the manipulatives he was able to complete more steps without prompts from the researcher.

Time

Alan averaged 12:18 for task completion time on each probe (range 6:35-19:15) using CM and 15:03 (range 8:57-22:01) using VM. Alan spent an average of 9:56 (range 5:27-17:04) completing probes during the extended baseline phase. During his best treatment phase, Alan used CM and averaged 8:58 for task completion time (range 7:54-9:42). Alan's task completion time data were all variable with a decelerating trend, except during best treatment where data were stable.

Cara

During the baseline phase, Cara was unable to correctly solve any two-step algebra problems during the five sessions. Her task completion time average was 9:41(range 7:56-11:13). Both accuracy and task completion time data were stable with a zero-celerating trend.

Accuracy

When Cara had access to manipulatives, either concrete or virtual, she averaged 96% accuracy (range 80-100%) over five sessions with a strong effect (Tau-U = 1.0). Without manipulatives, Cara's extended baseline scores improved over her initial baseline; she averaged 52% accuracy (range 40-60%). Her accuracy data were stable for all intervention phases with a zero-celerating trend. When comparing accuracy data using PND, Cara was equally successful regardless of the type of manipulative on 60% of the probes. She was more accurate using CM on 20% of probes and VM on 20% of probes. The researcher administered two more probes, one with CM and one with VM. Cara correctly solved 100% of the problems during both the virtual and concrete sessions. The researcher then compared independence data to determine Cara's best treatment was VM. During the three best treatment sessions, Cara averaged 100% accuracy using VM. Her data were stable with a zero-celerating trend.

Independence

Cara solved problems with 95% independence (range 83-97%) when using CM and 94% independence (range 79-100%) when using VM. After calculating PND, the researcher determined VM was the best treatment with Cara showing greater independence solving problems with VM compared to CM on 67% percent of the probes. During the best treatment phase, Cara used VM to help her work through each probe with 100% independence. Independence data during each intervention phase were variable with an accelerating trend, and stable with a zero-celerating trend during best treatment.

Time

Cara averaged 12:58 for task completion time (range 8:56-19:19) using CM and 15:20 (range 8:57-21:29) using VM during the intervention phase. Cara averaged 8:11 for task completion time (range 7:45-8:31) using VM as part of the best treatment phase. During the extended baseline phase where Cara did not use manipulatives, average task completion time was 4:48 (range 2:05-12:55). Cara's task completion time data were variable with a decelerating trend during intervention and extended baseline. As she completed sessions, her task completion time decreased; by the best treatment phase her time was stable with a zero-celerating trend.

Jan

Jan attempted all problems during her baseline phase, but was only able to correctly solve one problem. Her average task completion time was 9:26 (range 7:17-10:32) and she stated she was frustrated by the lack of instruction. Jan's accuracy data were stable with a zero-celerating trend, and her task completion time data were stable with a decelerating trend.

Accuracy

Using CM had a small effect size (Tau-U = .64) on Jan's ability to correctly solve two-step algebra problems where she averaged 84% accuracy (range 60-100). VM had a medium effect (Tau-U = .84) and her average accuracy was 92% (range 80-100%). Jan's accuracy data were stable with a zero-celerating trend when working with CM and decelerating trend when working with VM. When Jan worked without manipulatives, during the extended baseline sessions, she averaged 52% accuracy (range 20-80%). These data were variable with a zero-celerating trend. During her best treatment phase, Jan used VM, where she scored 100% accuracy during all three sessions.

Independence

Jan demonstrated 97% independence (range 95-99%) when completing probes using CM and 97% independence (range 95-100%) when working with VM. During the best treatment phase, Jan solved problems with an average 98% independence (range 96-100%). Jan was able to successfully work through the probes with 100% independence on two separate occasions, both using VM. All independence data were stable with an accelerating trend.

Time

When solving problems using CM, Jan's average task completion time was 13:12 (range 6:39-23:20). These data were variable with a decelerating trend; her task completion time decreased every session. Using VM, Jan averaged 12:54 for task completion time (range 8:41-19:12); these were variable with a decelerating trend. During extended baseline sessions, Jan's task completion time average was 7:46 (range 4:28-9:51). During the best treatment phase with VM, Jan averaged 7:11 for task completion time (range 6 :15-7:57). Task completion time data during extended baseline and best treatment were stable with a decelerating trend.

Social Validity

Before the intervention participants stated they had experience using multiplication charts and notes from class to help them work through new or difficult math content. They found benefit in having notes because it helped them remember the steps for more complex problems, but they weren't allowed to use these on tests or quizzes so they tried not to use them too often.

Students did not mention using manipulatives, but when asked they were familiar with concrete manipulatives and had used them before entering middle school. After explaining the difference between CM and VM and asking which they thought they would prefer to use, Alan stated neither and that he would rather do the work in his head, Cara said VM to help her stay organized, and Jan said CM because she prefers hands on materials and doesn’t enjoy doing work on a computer.

After completing the intervention, all three participants stated using manipulatives to learn how to solve the algebra problems was helpful and that they preferred VM. Cara again emphasized how important it was for her to stay organized, and that she felt like the CM were too scattered. Alan said he thought completing the problems in his head would be faster, but when he used manipulatives he had to really think about what he was doing and it helped him catch his mistakes. He preferred the VM because it was easier for him to backtrack when he noticed he did something wrong. Jan also preferred the VM stating it was difficult to pick up the small pieces when using the CM, and that it would be much easier to use VM at home compared to transferring the CM back and forth from school. All students agreed they would use the manipulatives to help them with their assignments, but Cara also noted she probably would only need to use the manipulatives for multi-step problems with a lot of negatives because the tiles have helped her learn how to solve the problems. Participants also stated it was easy to access and operate the VM because they were familiar with how to use their Chromebooks and using the VM felt the same as working through other computer-based school activities. Participants also noted that their peers would likely use the VM on their Chromebook if they needed them, but probably wouldn’t use the CM because they were messy and difficult to organize.

Discussion

This study explored students' use of virtual and concrete algebra tiles to solve two-step algebra problems. The researcher used an alternating treatment single case research design to compare the effectiveness of each type of manipulative (concrete and virtual) in terms of students' accuracy, independence, and task completion time when using the tools. Results indicate both types of manipulatives were effective in terms of accuracy and no notable differences when comparing independence or task completion time. When reviewing social validity interview data, all three students with disabilities preferred the virtual algebra tiles.

Students were equally successful solving algebra equations regardless of the type of manipulative. These findings are similar to previous research and support the idea VM are equally as effective as CM in supporting students with disabilities as the learn new math skills (Bouck et al., 2017; Satsangi & Bouck, 2015; Satsangi et al., 2016). Regardless of what type of manipulative they were using, two participants scored 80% accuracy or greater on all probes, and the third participant scored 60% accuracy or greater on all probes (see Figure 3.2). This was a significant improvement from when students did not use any manipulative (i.e., answering 0 or 1 correct), and consistent with previous research comparing the two types of manipulatives (e.g., Bouck et al., 2018; Bouck et al., 2014; Satsangi et al., 2016). Consistent with research involving manipulatives, all participants experienced an immediate improvement in accuracy moving from baseline to intervention (Bouck et al., 2019; Bouck et al., 2017; Satsangi & Bouck, 2015).

Also consistent with previous research, participants needed less support as they gained experience using the manipulatives and learned the algebraic process (Bouck et al., 2018; Bouck et al., 2014). Participants demonstrated 90% or greater independence when working with manipulatives and all independence data had an accelerating trend. Like others who have conducted studies using manipulatives (Bouck et al., 2017), researchers hypothesize additional training and practice using manipulatives prior to intervention would have increased initial independence. Unlike previous research (Satsangi et al., 2016), participants were not significantly more independent using one type of manipulative over the other. Although the virtual algebra tiles have certain features available to support students and allow them to be more independent in their learning (Bouck et al., 2017; Shin et al., 2017), these features were turned off during the study to ensure the comparison between the two types of manipulatives were equivalent. Having these features available to students may have an impact on the number and types of prompts needed to solve problems, and thus could lead students to be more independent using VM over CM.

Two participants averaged a shorter task completion time using CM, and the third participant's task completion time was comparable regardless of the type of manipulative. Task completion time data were decelerating for all participants during intervention for both types of manipulatives. One hypothesis is that task completion time decreased as students became more comfortable with the algebraic process and using the manipulatives. In most situations, participants physically picked up multiple tiles when setting up the algebra problem using CM but only moved one virtual piece at a time, which took more time. There are features available when using VM that would allow participants to copy and paste tiles and select groups of tiles to move them. Once learned, the use of these features may have decreased the task completion time (e.g., Satsangi et al., 2016). Similar to previous research exploring efficiency and mathematics manipulatives, task completion time began when students were given the probe and ended when students finished their last problem (Bouck et al., 2017). In this study, students spent a lot of their total task completion time setting up the problem and certain problems (i.e., those with double digit constants and variables) took considerably longer to set up. This was controlled for by assigning probes and condition (CM, VM, and no manipulative) randomly for each participant. However, more specific efficiency data could be collected in the future by analyzing set up time and solving time individually.

Implications for Practice

This study holds implications for practice as it suggests manipulatives, whether they be concrete or virtual, are an option for supporting secondary students with disabilities as they develop algebraic skills. While manipulatives are considered an evidence-based practice (National Center on Intensive Intervention, 2016; Watt et al., 2016), there remains questions as to whether this included virtual tools and if one type of manipulative was more effective than the other (Satsangi et al., 2016). In this study, two of three students had a clear best practice where they solved more problems correct using one type of manipulative over the other. One student performed better using CM, one performed better using VM, and one performed equally well regardless of the type of manipulative. As VM proved just as effective as CM, this study suggests the two may be interchangeable. Thus, teachers who implement the CRA instructional framework, a graduated progression in which students first learn to solve problems using manipulatives, before moving to a representational phase, and eventually an abstract phase (Maccini & Ruhl, 200) could choose to use VM. Specifically, teachers would alter the first phase by using VM to support the acquisition of algebra skills in place of CM.

All three participants preferred using VM and averaged greater than 90% independence when using the virtual tools. Students felt comfortable navigating the program using their school-issued Chromebook, and noted that using VM felt the same as completing other types of educational tasks online. These findings help support the use of VM in a classroom where older students might otherwise feel stigmatized (Satsangi & Miller, 2017). Since both types of manipulatives have been shown to be effective, student preference supports the use of VM over CM. As more schools move to one-to-one technology, VM in particular can be used in a large group setting without using too much space or requiring additional setup time (Bouck, Working & Bone, 2018). In a classroom where students are at various skill levels, teachers could choose to utilize support features within the virtual tool to individualize scaffolding or encourage independent practice (Bouck, Working & Bone, 2018; Shin, et al, 2017). For example, by turning on the “eye” feature students can see the written form of the equation they are displaying to ensure it is accurate before they begin solving the equation, or they can use the pen feature to show work and label steps.

Another implication for practice is that teachers may need to spend more time using explicit instruction, when teaching mathematics to students with disabilities. In this study, manipulatives were paired with explicit instruction to ensure students received specific and systematic instruction on the mathematical concepts and procedures as well as how to use the manipulatives to support learning. While accuracy was far better during the intervention phase, two participants also showed a significant improvement in accuracy from initial baseline to extended baseline. The use of explicit instruction, a practice shown to be effective by multiple, methodologically rigorous studies to teach mathematics to students with disabilities (Jitendra et al., 2018), includes a clear explanation of the algebraic concepts and procedures with modeling and the opportunity for guided practice to ensure authentic learning (Satsangi, Hammer, & Hogan, 2018). When working with manipulatives, using explicit instruction helps to ensure students have a clear understanding of why manipulatives are used to support learning, what the manipulatives represent, and how the manipulatives relate to the equations. Thus, creating more opportunities for teachers to use explicit instruction paired with manipulatives to teach algebra related concepts to secondary students with and without disabilities may lead to even greater success for all.

Limitations and Future Direction

This study, like most research, is not without limitations. One limitation is that this study did not include a generalization phase. Students learned how to solve 2-step algebra problems with positive and negative numbers, but without a true generalization phase the researcher cannot say with certainty that participants developed a deep understanding of the algebraic-concepts that they would be able to apply to more advanced problems. This study also did not include a maintenance phase, which is usually used to demonstrate the learned skills have been retained over time. The researchers chose not to include a maintenance phase because students were moving into a review unit that would include multiple days of instruction devoted to equations. It would be difficult to determine whether the skills displayed during the maintenance phase were a result of the manipulatives or the review instruction received in class.

While there is a growing body of research involving students with disabilities and mathematics instructions, more information is needed on how technology can help students at the secondary level develop their mathematics skills. Currently, studies exist that explore how technology can support students as they develop basic algebra and geometry skills (e.g., Satsangi, Hammer & Evmenova, 2018), but little is known about how virtual manipulatives and other types of technology can support higher order thinking skills that can be applied to even more advanced mathematics. There is also a need for a better understanding of how specific characteristics of virtual and concrete manipulatives benefits students with specific disabilities. For example, students with spatial issues may prefer and experience more support from VM because the technology restricts where pieces can be placed. Other students may benefit from physically touching the manipulatives which is only possible using CM. Finally, this study explored how algebra tiles supported students with disabilities, but there are other types of manipulatives that may assist students as they develop their algebra skills such as virtual balance scales and virtual graphing programs.

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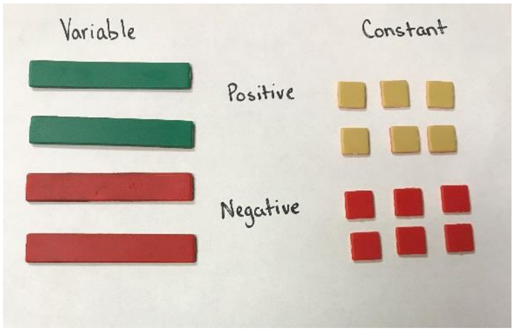
Figure 3.1. Algebra Tiles

Concrete algebra tiles



Virtual algebra tiles

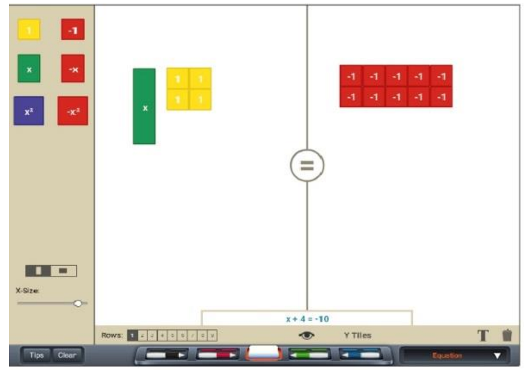
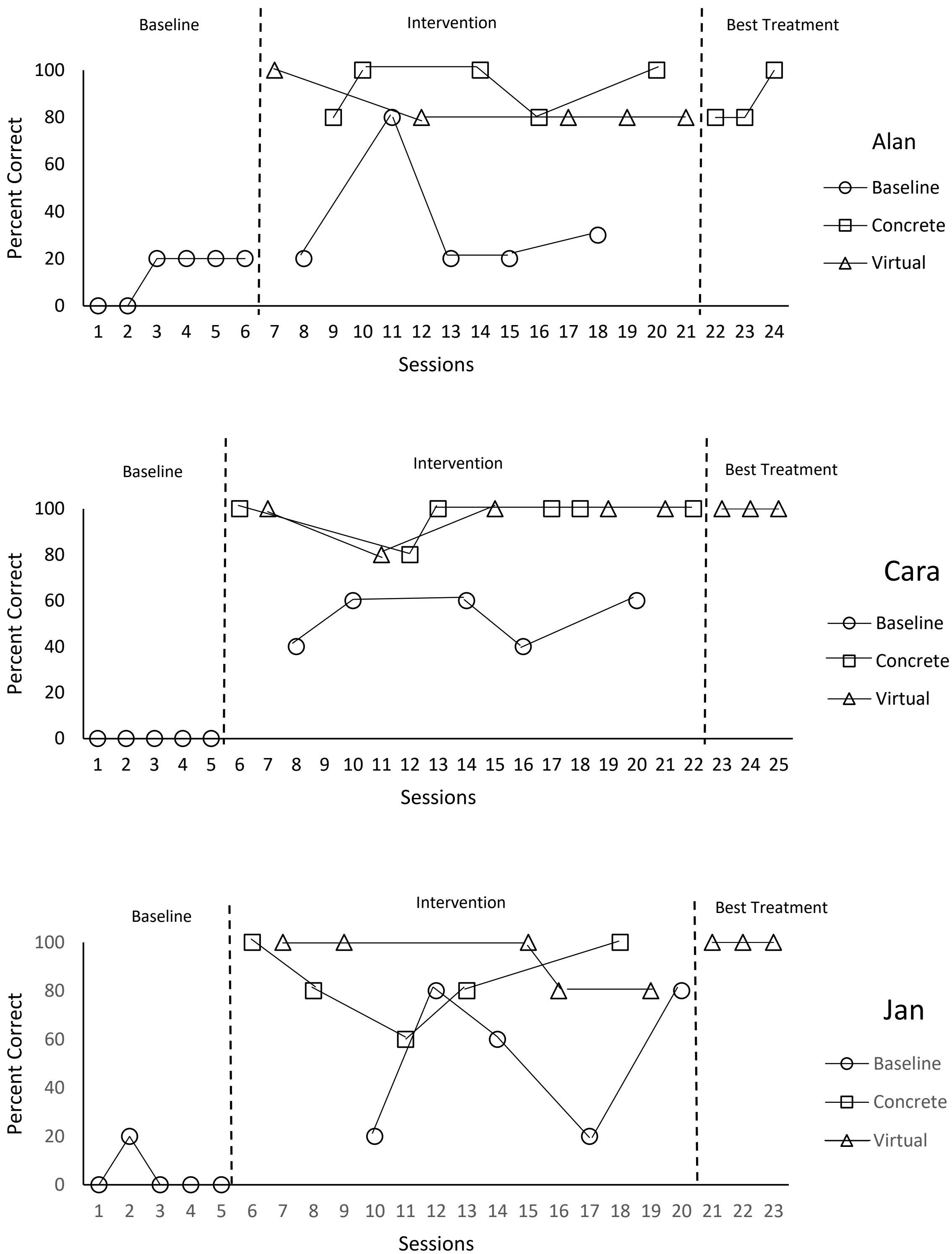


Figure 3.2. *Percent of accuracy on probes for all participants*



CHAPTER 4

Using the VA Framework to Teach Algebra to Middle School Students with Disabilities

Developing algebraic reasoning is important for all students, as algebra is considered by many to be a gateway to higher-level mathematics and eventual economic success (Watt, et al., 2016). As such, the foundations of algebra are introduced in early elementary where students are taught to recognize patterns, form generalizations about numbers, and identify equivalency (Ketterlin-Geller et al., 2019; Watt et al., 2016). Students are challenged to build on these foundational skills every year, and, by middle school, those with established algebraic reasoning skills are able to identify and depict various mathematical structures, describe algebraic principles, and write and solve problems involving variables (Ketterlin-Geller et al., 2019). More specifically, students proficient in mathematics are expected to reason about and solve one-variable equations by the end of sixth grade, use tools strategically to solve multi-step mathematical problems by the end of seventh grade, and analyze and solve linear equations by the end of eighth grade (Common Core State Standards, 2010).

Even when gradually and systematically introducing concepts, the abstract nature of algebra creates unique challenges for some students, including students with disabilities (Ketterlin-Geller et al., 2019; Star et al., 2015; Watt at al., 2016). Transitioning from concrete arithmetic (e.g., $3\ cats + 7\ cats =$) to the symbolic algebra (e.g., $\frac{2}{3}m + 5 = 9$) is especially difficult for students with disabilities (Star et al., 2015). In fact, when examining only questions pertaining to algebra (i.e., understanding of patterns, using variables, algebraic representation, and functions) students with disabilities scored 46 points lower as compared to their same-aged peers without disabilities on the National Assessment of Educational Progress (NAEP, 2017). In order to ensure all students, including those with disabilities, can be successful in algebra, educators need to use effective and efficient instructional methods and interventions supported by evidence (Watt et al., 2016).

Evidence-Based Practices

Evidence-based practices (EBP) are instructional supports and teaching methods shown by high-quality, methodological-sound research studies to have a positive impact on student learning (Cook & Cook, 2013). Teachers provide high-quality instruction to students when they implement EBP with fidelity to address standards-based curriculum (Cook & Cook, 2013). One such evidence-based mathematics intervention for students with disabilities is the concrete-representational-abstract (CRA) framework (Agrawal & Morin, 2016; Bouck, Satsangi et al., 2018). The CRA framework focuses on conceptual understanding and students' ability to perform mathematical procedures across lessons and is considered an effective intervention for students with disabilities across a variety of mathematical content areas including place value, basic operations, fractions, and algebra (Bouck & Park, 2018).

The CRA framework is a graduated instructional sequence in which students make connections with mathematical concepts at a concrete, representational, and abstract level, with instruction provided via explicit instruction (Agrawal & Morin, 2016; Bouck, Satsangi et al., 2018). At the concrete level, students use concrete manipulatives to aid in solving mathematical problems. Common concrete manipulatives include geoboards, base ten blocks, fraction strips, and algebra tiles (Bouck & Park, 2018). Once a set criterion level is met, students move to the representational phase where they create their own visual (i.e., picture, drawing) to make a connection to the abstract. At the abstract level of mathematical understanding, students are able to reason with numerical strategies without concrete or representational support (Agrawal & Morin, 2016). At each phase, teachers employ explicit instruction through a sequence of instruction where they first model and then provide guided practice. Finally, students are able to perform the various steps independent of teacher support (Agrawal & Morin, 2016).

CRA and Algebra

Researchers demonstrated the effectiveness of the CRA framework to teach algebraic concepts to students with disabilities (Maccini & Ruhl 2000; Strickland & Maccini 2013; Witzel, 2005; Witzel et al., 2003). Maccini and Ruhl (2000) found students with disabilities improved their ability to subtract integers when using algebra tiles as part of the CRA framework, and over time all students were able to generalize their skills to other types of problems. When comparing the CRA framework to traditional algebra instruction, Witzel et al. (2003) found the CRA framework to be more effective for middle school students with disabilities or at risk of algebra failure. Students who received the CRA instruction outperformed their matched pair and committed fewer procedural errors when solving for variables. Witzel (2005) compared the CRA framework to an approach with repeated explicit instruction without manipulatives and found sixth- and seventh-grade students with disabilities were better able to transform linear equations when they learned the algebraic concept through CRA. Most recently, Strickland and Maccini (2013) examined the Concrete-Representational-Abstract Integration strategy (CRA-I), which modified the sequence of CRA by using each phase simultaneously, on teaching multiplication of linear expressions to secondary students with disabilities. The CRA-I strategy was effective in improving students’ conceptual understanding and procedural fluency.

Mathematic Manipulatives

A vital component of the CRA framework is the effective use of manipulatives (Agrawal & Morin, 2016). Manipulatives are a recommended tool for teaching mathematics at all levels (NCTM, 2013), considered an effective instructional practice for students with disabilities

(Strickland & Maccini, 2013), and shown to improve performance of secondary students with disabilities on algebra related content (Satsangi et al., 2016). Despite research supporting the benefits of using concrete manipulatives, there is a decrease in the use of manipulatives during mathematics instruction from kindergarten to middle school (Swan & Marshall, 2010). Often middle school and high school-aged students view these tools as stigmatizing and find them inappropriate for their age (Satsangi & Bouck, 2015). As schools focus more time and resources on technology, the use of virtual manipulatives rather than concrete manipulatives may be more appropriate for older students (Bouck, Working, et al., 2018; Satsangi & Miller 2017).

Virtual manipulatives are dynamic visual representations of concrete objects that can be accessed through online applications and programs (Bouck, Working, et al., 2018). Teachers can use virtual manipulatives in comparable ways as concrete manipulatives, like introducing mathematical ideas, developing understanding through visual representation, and scaffolding students as they actively engage in learning (Bouck, Working, et al., 2018). Virtual manipulatives help support students with learning disabilities as they work with more advanced mathematical concepts (Satsangi & Bouck, 2015; Satsangi, et al., 2016), and these tools can be customized to offer appropriate support to a variety of learners without the stigmatizing effects of being different from peers (Satsangi & Miller, 2017). When learning algebra-related concepts, for example, older students with disabilities can use virtual manipulatives as part of the virtual-representational-abstract framework.

Virtual–Representational-Abstract Framework

The virtual-representational-abstract (VRA) framework uses the same systematic approach to teaching mathematical strategies as the CRA but uses virtual manipulatives rather than concrete manipulatives (Bouck & Sprick, 2018). Students begin in the virtual phase where they are provided access to virtual manipulatives (i.e., virtual algebra tiles, virtual base-10 blocks) as the teacher gives them explicit instruction on how to use the tool to support their acquisition of the mathematical behavior (Bouck, Bassette, et al., 2017). Following the virtual phase, students move into the representational phase where they draw pictures and figures to represent the mathematical concept, and then to the abstract phase where they rely on the numerical strategies (Bouck & Sprick, 2018).

In their multiple probe across participant study, Bouck, Bassette, et al. (2017) found middle school students with disabilities were able to solve equivalent fraction problems successfully using the VRA framework. Similar to studies conducted using the CRA framework, student performance improved from baseline throughout the VRA intervention. Yet, researchers found it was challenging for students to draw the fraction representations (Bouck, Bassette, et al., 2017). The representational phase may have added unnecessary stress for the student. Previous researchers also found the representational phase—within the CRA framework—may not be a crucial component for all students or mathematical content (Cass, et al., 2003). An instructional approach using virtual manipulatives without the representational phase may be a solution.

Virtual-Abstract Framework

The virtual-abstract (VA) framework is an adaptation of the VRA framework, where the representations phase is removed and students move directly to the abstract phase from the virtual. Although limited, literature indicates students with disabilities benefit from the VA framework (Bouck, et al., 2019; Bouck, Park et al., 2017). Bouck, Park, et al. (2017) found three of the four students in their study improved performance in solving addition of fractions with unlike denominators using the VA framework. All participants experienced an immediate effect moving from baseline to intervention, but two students needed additional sessions during the abstract phase in order to meet criterion. Overall the lack of the representational phase did not seem to impact student success and eliminating it from the framework may be beneficial especially for students with spatial issues. Most recently, Bouck and colleagues (2019) examined the effects of the VA framework of the acquisition of algebra skills for middle school students with disabilities. In their study, all four students experienced an immediate effect when moving from baseline to intervention and only two students needed to repeat sessions. However, students were not able to consistently demonstrate their learning during maintenance and the researchers highlighted the need for more research on the framework.

Although there is an obvious need for a better understanding of the impact of the VA framework, even more can be gained from further examination of how to support students with disabilities in the development of algebra skills. The current study aims to build on the limited understanding of how virtual manipulatives support students with disabilities by investigating their use within the VA framework. This study is a systematic replication of the recent VA research conducted by Bouck et al. (2019) in that the research design and focus on solving algebra equations is the same, but a variation between participants exists in terms of specific disabilities. The research questions include: (a) Using the VA framework, to what extent does the performance of seventh-grade students with high-incidence disabilities on solving grade-level algebra problems improve? (b) To what extent do seventh-grade students with high-incidence disabilities maintain their performance solving grade-level algebra problems when no instruction is given? (c) What is the perception held by seventh-grade students with high-incidence disabilities regarding the VA framework?

Method

Participants

This study involved seventh-grade students previously identified as eligible for special education services due to a disability negatively impacting their acquisition of mathematics skills. Specific inclusion criteria for participants included: (a) an Individual Education Plan (IEP) with a math goal, (b) an Algebra Assessment Instruction: Meeting Standards (AAIMS) Basic Algebra score at or below 10, (c) limited or inadequate progress with the general education mathematics curriculum (i.e., failure to demonstrate growth toward meeting essential standards), (d) considered non-proficient in the area of mathematics according to the Iowa Statewide Assessment of Student Progress (ISASP), (e) identified as at-risk in mathematics based on the aMath fall screener, and (f) parental consent and student assent to participate.

Emily

Emily was a 12-year-old, White, female in the 7th grade. Emily had an IEP due to a learning disability and received specially designed instruction in the areas of mathematics and reading. Emily received mathematics instruction from a special education teacher for 45 minutes daily in a small group setting. Her goal in the area of mathematics stated that in 36-school weeks given 5 minutes to work, Emily would be able to answer 14 problems correct on an AAIMS Basic Algebra probe three out of four consecutive trials. At the time of the study, Emily scored a median of 6 problems correct on AAIMS Basic Algebra probes. According to the ISASP assessment administered in the spring of 6th grade, Emily was not yet proficient in mathematics. In the area of equations and expressions, she answered 23% of the questions correctly. Emily scored a 206 on her aMath screener administered in the fall of 7th grade, placing her at the 8%ile nationally and flagging her as high risk in mathematics. According to this assessment, Emily had not yet mastered any skills under the umbrella of equations and expressions.

Sara

Sara was a 12-year-old Latino female in the 7th grade. Using state guidelines, she was identified as having a learning disability in the area of mathematics in 5th grade and began receiving special education services at that time. Sara was enrolled in a co-taught grade-level mathematics class in which the general education teacher and special education teacher delivered instruction using the co-teaching model. She received additional specially designed instruction in mathematics from the special education teacher in a small group setting (i.e., < 6 students) for 60 minutes weekly. Her IEP math goal stated in 36-school weeks when given 5 minutes to work, Sara will be able to answer 10 problems correct on AAIMS Basic Algebra probe on three out of four trials. Her baseline for this goal was 3 correct answers. According to the ISASP assessment administered during the spring of 6th grade, Sara was not yet proficient in mathematics. She successfully answered 54% of the questions in the equations and expressions section of the assessment. Sara scored 210 on her aMath screener administered in the fall of 7th grade, placing her at the 17%ile nationally and flagging her as some risk in mathematics. While the test results indicated Sara was developing the skills needed to use variables to represent numbers and write expressions, she had not yet mastered any skills in the area of expressions and equations.

Paul

Paul was a 12-year-old, White, male in the 7th grade. Paul was found eligible for special education services in 3rd grade and currently has goals in the areas of mathematics and reading. Paul receives mathematics instruction from a special education teacher for 45 minutes daily in a small group setting (i.e., < 6 students). Paul's math goal states in 36-school weeks when given five minutes to work he will correctly answer 12 problems on the AAIMS Basic Algebra probe on three out of four trials. Paul's baseline score for this goal was 5 correct answers. Paul was not yet proficient in mathematics according to the ISASP assessment administered in the spring of 6^{th} grade. Paul struggled significantly in the area of equations and expressions where he only answered 15% of the questions correctly. Paul scored a 206 on his aMath screener administered in the fall of 7^{th} grade. This score placed him at the 8%ile nationally and he was flagged as high risk in mathematics. Paul had not mastered any skills under the expressions and equations category, but was developing the skills needed to use variables to represent numbers and write expressions.

Setting

The study took place in a small Midwest community in Iowa where 95% of the population identified as Caucasian. The community's school district served 2,000 students across four buildings: lower elementary school, upper elementary school, middle school, and high school. The middle school had 485 students in grades 6-8, and, according to school records, 15.5% of students in the middle school were eligible for special education services. Of those receiving services, 61% had a goal in the area of mathematics as part of their IEP. Data collection occurred at a rectangular table in an open classroom. This was a quiet space that students commonly used to work one-on-one with a teacher, with a peer, or independently when an alternate setting was needed. Data were collected during an intervention period, math skills period, or academic support period. Academic support was a scheduled time during the school day for students to work on their specific academic goal areas.

Materials

The main materials for this study were the algebra probes, the teacher model and student practice problems, a pencil, and the virtual manipulatives (Algebra Tiles by Brainingcamp [2019]). The probes were constructed by the researchers and aligned with the middle school mathematics curriculum and the Common Core State Standards (2010). Each probe consisted of five algebra problems presented in four different formats in which the location of the constants and variables differed (e.g. $3 + 4x = 11$, $4x + 3 = 11$, $11 = 3 + 4x$, $11 = 4x + 3$). All probes included a problem in each of the noted formats with the fifth problem randomly selected from the four formats. Each probe contained a unique set of problems and no probe was repeated.

Probes evaluated one specific algebraic behavior based on each individual participant's mathematical level. Mathematical behaviors were determined on an individual basis based on teacher-input and data collected from the pre-assessment. However, based on this information, all participants qualified and were assessed using the same mathematical behaviors. Behaviors varied based on the number and types of steps needed to solve for the variable, as well as the inclusion and absence of negative numbers. Mathematical behaviors based on grade-level curriculum included: one-step equations with positive and negative numbers (e.g., $x + (-4) = 7$), two-step equations with positive numbers (e.g., $3x + 6 = 18$), and two-step equations with positive and negative numbers (e.g., $-2x + 4 = 14$).

Participants used the virtual Algebra Tiles from Brainingcamp LLC (2019), accessed on their school-issued Chromebook (see Figure 4.1). The app involved an equation background, which helped participants organize the algebra problem by separating the two sides of the equation with the equal sign. Participants had access to an unlimited number of algebra tiles that were used to represent constants and variables. Positive constants were represented by small yellow squares, negative constants by small red squares, positive variables (*x*) by green rectangles, and negative variables by red rectangles. A large blue square was also available to represent x^2, but was not used for the purpose of this study.

Independent and Dependent Variables

The independent variable for the study was the VA framework for each of the three behaviors examined across participants (e.g., one-step equations with positive and negative numbers, two-step equations with positive numbers, two-step equations with positive and negative numbers). The dependent variable was accuracy, defined as the number of algebra problems answered correctly out of five. The researcher summed the problems participants solved correctly (no partial credit) and divided the number by five to calculate percent accuracy. Researchers recorded accuracy using event recording.

Experimental Design

This study employed a multiple probe across behaviors, replicated across participants, design to examine the effectiveness of the VA framework to teach and support middle school students with disabilities as they solved grade-level algebra problems (Gast, Lloyd, & Ledford, 2014). With the VA framework, students begin by solving mathematical problems (i.e., two-step algebra problems) with virtual manipulatives (i.e., virtual algebra tiles) before solving problems abstractly without the support of manipulatives (Bouck, Park, et al., 2019; Bouck, Park, et al., 2017). The multiple probe across behaviors design can effectively evaluate an intervention intended to accelerate the frequency of a non-reversible behavior, which for this study was solving algebra problems correctly (Gast et al., 2014). The researcher and first author, a doctoral candidate and highly-qualified special education teacher, delivered all sessions in a one-on-one format. Sessions occurred two-to-three days a week for 10-12 weeks depending on the participant. No participant completed more than two sessions per day.

Pre-assessment

Prior to baseline, participants completed a pre-assessment to determine their current ability to solve basic algebra problems. The pre-assessment included eight problems examining four levels of behaviors: one-step addition equations with positive and negative numbers, two-step equations with positive numbers, two-step equations with positive and negative numbers, three-step equations with positive numbers and variables on both side of the equals sign, and two questions regarding basic algebra principles. The researcher scored the pre-assessment and used the data paired with current classroom progress and input from the special education teacher to determine eligibility for the study as well as the mathematical behaviors appropriate for each student.

Baseline

During the baseline phase, students worked independently to complete a minimum of five probes for each behavior (i.e., one-step equation with positives and negative numbers, two-step equations with positive number, two-step equations with positive and negative numbers). Students did not receive specific instruction on how to solve problems, and did not have access to manipulatives. Once data stability was evident and the baseline had a zero-celerating or de-celerating trend for the first behavior students moved into the intervention phase for that behavior.

Intervention

The VA framework was conducted during intervention consistent with the CRA and VRA frameworks (Bouck et al., 2019). Participants received at least six intervention sessions (i.e., three in the virtual manipulative phase and three in the abstract phase) per mathematical behavior. The researcher used explicit instruction during each intervention sessions, including the modeling of two problems, before the participant completed two problems with cues and prompts from the researcher as needed. Prompts included gestures (e.g., pointing to a location on the screen) or verbal cues (e.g., “what’s your next step?”). After the lesson, the participant completed the five-problem probe using virtual algebra tiles without assistance from the researcher. Once the participant correctly answered 80% or more during three intervention sessions using the virtual manipulatives, they entered the abstract phase. In the event a participant failed to achieve 80% accuracy in a lesson, they repeated that same lesson the next session. After each participant entered into the abstract phase for the first mathematical behavior, they completed a probe with no support (baseline) and then began the intervention phase for the next mathematical behavior (i.e., two-step with positive numbers and then two-step with positive and negative numbers). This continued until each participant successfully completed both phases for all three behaviors.

Virtual phase. The first three lessons for each behavior involved the researcher using explicit instruction to teach participants how to access and use the Algebra Tiles by Brainingcamp (2019) to solve the problem. Each lesson began with the researcher demonstrating the steps necessary to set-up and solve the algebra problem. During this process, the researcher verbalized her mathematical approach, referred to as a think-aloud. For example, in the problem $5x + (-3) = 12$ the researcher began by reading the problem aloud (e.g., “5*x* plus negative 3 equals twelve”) and then identifying the location of the variable (e.g., “the variable is on the left side of the equals sign”). The think-aloud process continued as the researcher demonstrated each step necessary to solve the equation. In this example, the researcher placed three yellow squares on top of the red squares to represent the additive inverse of -3 (e.g., “the sum of -3 and 3 is zero”), and then three yellow squares on the right side to represent a balanced distribution of values to both sides of the equal sign (e.g., “to keep the equation balanced I have to add three to the right side too”). At this time, the researcher would be left with 5 green rectangles on the left side and 15 yellow squares on the right side ($5x = 15$).

The next step required division so the researcher selected the number of rows based on the coefficient (e.g., “since there are five rectangles, I need five rows”) to appropriately separate or divide the pieces. The researcher then dragged each rectangle to its own row on the left, and then divided the yellow squares on the right equally among the rows (e.g., “each row should have the same number of squares”). When the problem was solved, the researcher looked at one portioned section that contained a green rectangle representing the variable on one side of the equals sign, and a sole numerical value as represented by squares on the other side (e.g., “now I can see that one green rectangle equals three yellow squares”; $x = 3$). Once the researcher demonstrated two problems, the participant used the virtual algebra tiles to solve two new problems. During this time, the researcher provided cues and prompts to the participant as needed (e.g., “Remember if you add 3 to the left side of the equals sign, you must…”). The participant then used the virtual algebra tiles to independently complete the five-problem probe. During this time, the researcher did not provide support to the participant.

Abstract phase. During the final three lessons for each mathematical behavior, participants solved the algebra problems abstractly by applying mathematical principles without the support of the virtual manipulatives. The researcher modeled how to solve two problems using the think-aloud strategy so participants could both see the steps and hear the reasoning behind each step. The researcher focused on the mathematical processes and how they applied to each step of the algebra problem. Once the researcher demonstrated two problems, the participant completed two problems with support from the researcher in the form of cues and prompts (e.g., "Check your addition one more time, what's your next step, etc."). Finally, each participant completed the five-problem probe independently and abstractly for each behavior (i.e., no teacher support and no manipulatives).

Maintenance

All participants completed two maintenance probes per behavior two weeks after their last abstract session for that behavior. The maintenance phase followed the same procedures as baseline and the independent portion of each lesson. In this way, participants did not receive cues, prompts, or feedback from the researcher nor did participants have access to manipulatives while completing probes during maintenance.

Social Validity

The researcher conducted interviews with each participant prior to data collection and at the conclusion of the study. Participants were asked to discuss their attitudes toward mathematics, and after completing the study how they felt about the VA framework. Specifically, participants were asked whether the framework helped them learn how to solve various types of algebra problems, whether they would use this process given the choice, and if they believed virtual manipulatives were tools they would use in the future to help support their skill development and conceptual understanding in the area of mathematics.

Interobserver Agreement and Treatment Fidelity

A middle school mathematics teacher examined student work to determine interobserver agreement (IOA) for accuracy for all participants. IOA was calculated for each probe by dividing the number of agreements between the mathematics teacher and the researcher by five (total problems on probe). IOA data were recorded for each student for at least 33% of baseline sessions, 40% for each intervention condition, and 50% of maintenance sessions. IOA for accuracy was 100% for all participants. The researcher used a checklist to collect treatment fidelity data for a minimum of two intervention sessions for each student. Treatment fidelity included participants being provided appropriate materials: a pencil, access to virtual manipulatives when appropriate (i.e., virtual phase), and the correct type of probe, and the researcher implementing explicit instruction, participants using the virtual manipulatives when appropriate (i.e. virtual phase), and the researcher ensuring participants did not receive prompting during the independent portion of the lesson (i.e. completing assessment). Treatment fidelity for all participants was 100%.

Data Analysis

The researchers used visual analysis and effect size calculations to assess level, trend, and effect size for problem accuracy for each student (Gast & Spriggs, 2014). To calculate level, the researchers determined the stability using the 80/25 rule. When 80% of the data fell within 25% of the median, the data were considered stable for that particular intervention condition (Gast & Spriggs, 2014). The researcher calculated trend and determined if data were accelerating, decelerating, or zero-celerating by using the split middle method in which the researcher divides the data into quarters to determine mid-rate and mid-date (White & Haring, 1980). To determine effect size, the research used Tau-U which combines non-overlapping data between phases with trend from within the intervention phase. The researcher used an online calculator to establish the Tau-U with results being reported using numerical scores between 0 and 1 (Parker, Vannest, Davis, & Sauber, 2011). Scores between 0.93 to 1 were considered a large effect, 0.66 to .92 a medium effect, and 0 to 0.65 a small effect (Parker, Vannest, & Brown, 2009).

Results

Researchers found a functional relation between the VA framework and students' ability to successfully solve grade-level algebra problems (see Figures 4.2-4.4). All participants demonstrated an increase in the number of problems they were able to solve correctly using the VA framework compared to their accuracy during baseline. All participants also solved more algebra problems correctly during the maintenance phase as compared to baseline.

Emily

Emily's accuracy for her first behavior during baseline was 0 for each session (see Figure 4.2); her data were stable, with a zero-celeration trend. Emily completed eight sessions during intervention including five during the virtual phase and three during the abstract phase. Her accuracy range during intervention was 20–100% (μ = 77.5%). Her overall intervention data were variable with an accelerating trend. The Tau-U was 1.0, indicating a large effect. Her accuracy was 80% for both maintenance sessions.

Baseline data were collected intermittently for Emily's second mathematical behavior where her accuracy range was 0-20% (μ =8%) across five sessions. Data were stable with a decelerating tend. Emily completed eight sessions during intervention including four during the virtual phase and four during the abstract phase. Her scores ranged from 60-100% accuracy (μ = 80%). Intervention data were stable with an accelerating trend and the Tau-U was 1.0, indicating a large effect. During the maintenance phase conducted two weeks after the final abstract session, Emily scored 60% accuracy on both assessments.

Emily's baseline for her third behavior ranged from 0 to 20% (μ = 10%) across six sessions. Her last three baseline probes were 20, 0, and 0 indicating data were stable with a decelerating trend. Emily completed eight sessions during intervention including four during the virtual phase and four during the abstract phase. Her accuracy ranged from 60-100% (μ = 77.5%). Data were stable with an accelerating trend and the 1.0 Tau-U indicated a large effect size. Emily's scores on her maintenance assessments were 40% and 60% accuracy.

Sara

During her first behavior Sara scored 40% accuracy on all but one of her five baseline sessions (μ = 36%; see Figure 4.3). The data were stable before she moved into the intervention phase. Sara experienced an immediate effect; her accuracy was 100% during all intervention phases and she did not repeat any sessions. Data were stable with a zero-celerating trend with a Tau-U score of 1.0, indicating a large effect size. Sara also scored 100% accuracy on both of her maintenance assessments.

Sara's baseline accuracy scores for her second behavior were stable and ranged from 0-40% accuracy (μ = 20%) across five sessions. Sara experienced an immediate effect, as her accuracy was 100% for the first virtual intervention session. Her overall intervention accuracy ranged from 80-100% (μ = 96.7%); the data were stable with a zero-celerating trend, and she did not repeat any sessions. The Tau-U for her accuracy was 1.0 indicating a large effect. Sara's accuracy stayed consistent during the maintenance where she scored 80% and 100%.

.On her final mathematical behavior, Sara completed baseline assessments with 20-40% accuracy (μ = 26.7%) across six sessions. Consistent with the previous behaviors, Sara experienced an immediate effect upon entering the intervention phase where she solved 100% on her first assessment during the virtual phase. Her overall intervention accuracy ranged from 80-100% (μ = 96.7%); the data were stable with a zero-celerating trend, and she did not repeat any sessions. The Tau-U for her accuracy was 1.0 indicating a large effect size. Sara's accuracy scores were 100% and 80% during her two maintenance sessions.

Paul

Paul's accuracy for his first behavior ranged from 0-20% accuracy (μ = 16%); data were stable and zero-celerating (see Figure 4.4). Paul required eight intervention sessions including five virtual sessions and three abstract sessions. His intervention accuracy ranged from 40-100% (μ = 80%); data were variable with an accelerating trend and Tau-U was 1.0, indicating a large effect size. Paul maintained his accuracy by scoring 80% during two maintenance sessions.

During baseline of his second mathematical behavior Paul solved 0-20% of the problems correctly (μ = 8%) across five sessions. Data were stable with a zero-celerating trend. Paul completed eight sessions during intervention: four during the virtual phase and four during the abstract phase. His scores ranged from 60-100% accuracy (μ = 85%); data were variable with an accelerating trend. Tau-U was 1.0 indicating a large effect size. During maintenance, Paul solved the equations with 100% and 80% accuracy.

Paul's baseline scores ranged from 0-40% accuracy (μ = 20%) during his third mathematical behavior across six sessions. Data were stable with a zero-celerating trend. Paul's accuracy scores experienced an immediate effect upon entering intervention and he only required three sessions for each phase to reach criterion. He solved problems with 80-100% accuracy (μ = 90%); data were stable with a zero-celerating trend and Tau-U was 1.0, indicating a large effect size. During his maintenance sessions, Paul solved equations with 60% and 80% accuracy.

Social Validity

Before the intervention, all participants stated algebra and algebra-related tasks were difficult. Two participants said knowing how to do basic operations made algebra easier, while one participant stated the need to know operations was an obstacle. Participants also shared they were easily confused by variables and algebra procedures. All participants knew they had access to and experience using multiplication charts, classroom notes, and calculators, but prior to the intervention no participants mentioned using manipulatives.

At the conclusion of the study, both Emily and Paul stated they preferred using the virtual manipulatives over the abstract phase where no manipulatives were available. Emily indicated using the virtual manipulatives helped keep her thoughts in order which was beneficial since she was still learning “how to do algebra”. Paul liked being able to hit clear to get a clean display when he made a mistake, “sometimes I get all jumbled and just need to start over, and when I do this on paper I just scribble it out and then I run out of space and all my work is all over and confusing.” Sara preferred the abstract phase because it didn’t take as long to solve the problems. She said learning how to do the problems with the virtual manipulatives was helpful, but she didn’t need to do it so many times and would rather just “do the math.” All three participants agreed the virtual manipulatives would be helpful for teaching the algebra concepts to peers, and said their teachers would probably like them since they could use the Chromebook.

Discussion

Algebra is considered by many to be a gateway to higher-level mathematics and eventual economic success (Watt et al., 2016), yet students with and without disabilities often struggle to develop the skills necessary to demonstrate algebraic thinking (Star et al., 2015). This study explored whether the VA framework supported the acquisition of three algebra behaviors (e.g., one-step equations with positive and negative numbers, two-step equations with positive numbers, and two-step equations with positive and negative numbers) for seventh-grade students with disabilities. Researchers found a functional relation between the VA framework and student algebraic learning. All three participants improved their performance on each of the three algebra behaviors during intervention, and all participants maintained their accuracy after intervention, as compared to baseline to maintenance, with two maintaining their skills at 60% or greater.

Consistent with previous research exploring graduated instructional sequence interventions (e.g., CRA, VRA, VA; Bouck et al., 2019; Bouck, Bassette, et al, 2017; Strickland & Maccini, 2013), Sara experienced an immediate effect for all behaviors, and the effects from baseline to intervention were immediate for Emily and Paul’s second and third behaviors. In this study, students were able to successfully move directly from the virtual phase to the abstract phase, as intended in the VA framework. Consistent with previous research, the lack of a representational phase did not seem to negatively impact students’ algebraic understanding or performance (Bouck et al., 2019; Bouck, Park at al., 2017; Cass et al., 2003). Students’ ability to score at or above 60% during all abstract phases is consistent with previous research indicating the representational phase may not be necessary for some students with disabilities and certain mathematical content (Cass et al., 2003). The success, despite the lack of a representational phase, lend support to previous research suggesting for certain mathematical behaviors this form of representation (i.e., pictures) might not be necessary, resulting in a more efficient intervention.

Student performance during the maintenance phase demonstrated all students performed better as compared to baseline scores and two of the three students were able to sustain the skills learned over time (e.g., Paul > 60% and Sara > 80%). The ability for students to maintain skills over time is consistent with the findings of Satsangi et al. (2016) and Satsangi et al. (2018), in which students with learning disabilities also demonstrated the acquisition of algebra skills after being taught using virtual manipulatives. However, one major difference between the current study and Satsangi et al.’s research was the absence of manipulatives during the maintenance phase of the current study. While students were able to solve the problems without additional instruction in the aforementioned studies, they also had access to manipulatives to assist in solving the algebra problems. In this study, Paul and Sara solving problems with 60% accuracy or greater without manipulatives demonstrates the effectiveness of the VA framework for acquiring and maintaining algebra skills in the absence of tools.

Consistent with research where the maintenance sessions are conducted without manipulatives (e.g., Bouck et al., 2019; Bouck, Park, et al., 2017), Emily scored better during maintenance as compared to baseline but struggled to maintain a consistent high level of performance when explicit instruction and representation did not proceed the independent attempt. After Emily's first behavior, she never scored above 60% during maintenance. Some students with disabilities may benefit from additional sessions during intervention to allow for more opportunities to practice the skills (e.g., five sessions instead of three; Bouck et al., 2020). Future research should consider targeting maintenance as part of the intervention, as sustaining skills overtime is an essential part of learning (Park et al., 2020; Collins, 2012). Researchers who targeted maintenance as part of a graduated sequence of instruction, while for different mathematical areas (e.g., subtraction with regrouping, multiplication, and division) found positive impacts (Bouck et al., 2020; Park et al., 2020).

Implications for Practice

All three students in this study successfully acquired three linear algebra behaviors with six-to-eight sessions using the VA framework with session length ranging from 15-25 minutes. Overall, the results suggest the VA framework is an effective and efficient intervention for middle school students learning algebra, when working with students one-on-one. At the secondary level, virtual manipulatives could be used for whole group instruction, during station teaching, or even in small groups (e.g., intervention groups), especially as more students gain access to one-to-one technology (e.g., Chromebooks, iPads; Satsangi et al., 2018). Teachers using the VA framework during instruction in a large class could use a projector or smartboard to display their work during the explicit instruction, and then students could work from their devices as the teacher walked around and provided feedback, before students completed the mathematics independently (Bouck, Mathews & Peltier, 2019). However, it is important to remember that part of the effectiveness of the VA intervention relies on the student's ability to move at his or her own pace with some students needing additional sessions (Bouck & Sprick, 2018), and thus whole class implementation may become more challenging.

Another implication is the support for the VA framework, which removes the representational phase and transitions students directly from the virtual phase to the abstract phase (Bouck et al., 2019; Bouck et al., 2020). While the representational phase allows an additional opportunity to practice algebraic reasoning as students are challenged to create their own representation of the mathematics (Bouck & Sprick, 2018), it may be redundant at times and cause unnecessary anxiety, especially if the student has difficulties with fine motor skills (Bouck, Bassette, et al., 2017; Cass et al., 2003). Instead the teacher could employ the VA framework and increase the number of sessions for the other phases in order to promote maintenance of the new skills (Bouck et al., 2020). As part of an intervention package, for example, students could complete extra sessions in the virtual phase with the teacher gradually fading support (Park et al., 2020). In this way, students would have access to manipulatives, but teacher prompts and cues would decrease across sessions. Another possibility is to have students complete extra abstract sessions with limited explicit instruction to more closely resemble what students are expected to do outside of the intervention (i.e., maintenance; Bouck et al., 2020). Providing extra opportunities to practice promotes overlearning, which helps students acquire and then maintain their skills over time (Park et al., 2020).

Teachers should regard the VA framework as a possible alternative to the CRA framework to support the acquisition of mathematics skills for students with disabilities (Bouck et al., 2020; Bouck et al., 2019; Bouck et al., 2017). This study, in conjunction with the other VA framework algebra study by Bouck et al. (2019), suggests the efficacy of the VA framework for teaching algebra to students with disabilities. Although limited, there is also literature indicating the VA framework is effective when teaching other mathematical behaviors (e.g. fractions; Bouck et al., 2020; Bouck, Park et al., 2017). In particular, using virtual manipulatives as part of a graduate instruction sequence could be very impactful when working with older students who are either not interested in using concrete manipulatives due to stigma or are simply more interested in using technology (Satsangi & Bouck, 2015; Satsangi & Miller, 2017).

In order to be an effective intervention, teachers must know how to implement it with fidelity (Cook & Cook, 2013). Thus, graduated instructional sequences, like the VA framework, should be taught to teachers during in-service or be a part of teacher preparation courses. This instruction should be ongoing and should include additional coaching when necessary to ensure teachers truly understand implementation procedures. With this knowledge, it is likely teachers will use the intervention more regularly during instruction (Cook & Cook, 2013), whether that be whole group, small group, or one-on-one.

Limitations and Future Direction

This study is not without its limitations. For one, students entered the independent portion of the lesson regardless of the degree or frequency of support given during guided practice. Guided practice should include enough support so the student can experience success solving the problem, but be gradually reduced while the teacher continues to monitor progress (Strickland & Maccini, 2013). Emily and Paul may have benefitted from additional practice sessions and future research should follow recommendations to go back to the modeling stage when students struggled during guided practice (Bouck & Sprick, 2018). It may be advantageous for researchers to determine a set number of prompts as an indicator (e.g., more than five indicate a need for more modeling). To learn more about how students are using virtual manipulatives within the framework, future research should also include data collection on accuracy and independence based on a task analysis during the virtual – or all – phases of the framework.

A second limitation is during her first behavior, Sara moved from baseline to intervention with an accelerating trend which could indicate a threat to internal validity. However, after five sessions, Sara never answered more than two problems correct (40%) and her data were stable. When observing her during the baseline session, Sara voiced frustration throughout the assessment process and self-reported that she was guessing because she did not understand the mathematics. This statement was supported by the time spent completing her last three baseline probes (less than one minute) and her answers (e.g., recording the same answer for all five problems). The researcher chose to move her into the intervention phase because continuing with collecting baseline data without instruction created an aversive experience for the learner, and due to the student's clear frustration with not understanding the mathematical content became ethically questionable.

A third limitation is the three algebra behaviors may have been too closely related. The only difference between the second and third behavior was the inclusion of negative numbers. This behavior was chosen because teachers stated many students required additional lessons or re-teaching before successfully solving equations involving both positive and negative numbers, and the pre-assessment results supported this claim. However, the effectiveness of the VA framework may have been better explored if the behaviors included one-step equations, two-step equations, and three-step equations requiring knowledge of the distributive property, variables on both sides, or combining like terms. These behaviors would have also aligned with the Common Core State Standard for seventh-grade students to use tools strategically to solve multi-step mathematical problems (CCSS, 2010). The over-alignment may have contributed to the high scores during the maintenance phase. These sessions were administered two weeks after the last intervention of each behavior, but while algebra instruction was still occurring. In this way, students were receiving instruction that could easily be applied to the skills being assessed during maintenance. Finally, the assessments were created by the researchers. They followed a set criteria and were reviewed by multiple parties including an algebra instructor, but were not formally evaluated for reliability and validity purposes.

Researchers should continue to investigate the impact of the VA framework to support students with disabilities. Future research should include exploring various types of virtual manipulatives (e.g., virtual balance scale), more advanced algebra concepts (e.g., solving systems of equations), and other mathematics strands taught at the secondary level (e.g., geometry) within the VA framework. Researchers should also consider how students access virtual manipulatives (i.e., web-based vs. app-based), and the implications this may have on accessibility both inside and outside the school setting. Continued exploration of intervention packages that can be used within the VA framework to support maintenance and generalizability is also needed.

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Figure 4.1. Screenshot of virtual algebra tiles (Brainingcamp 2019)

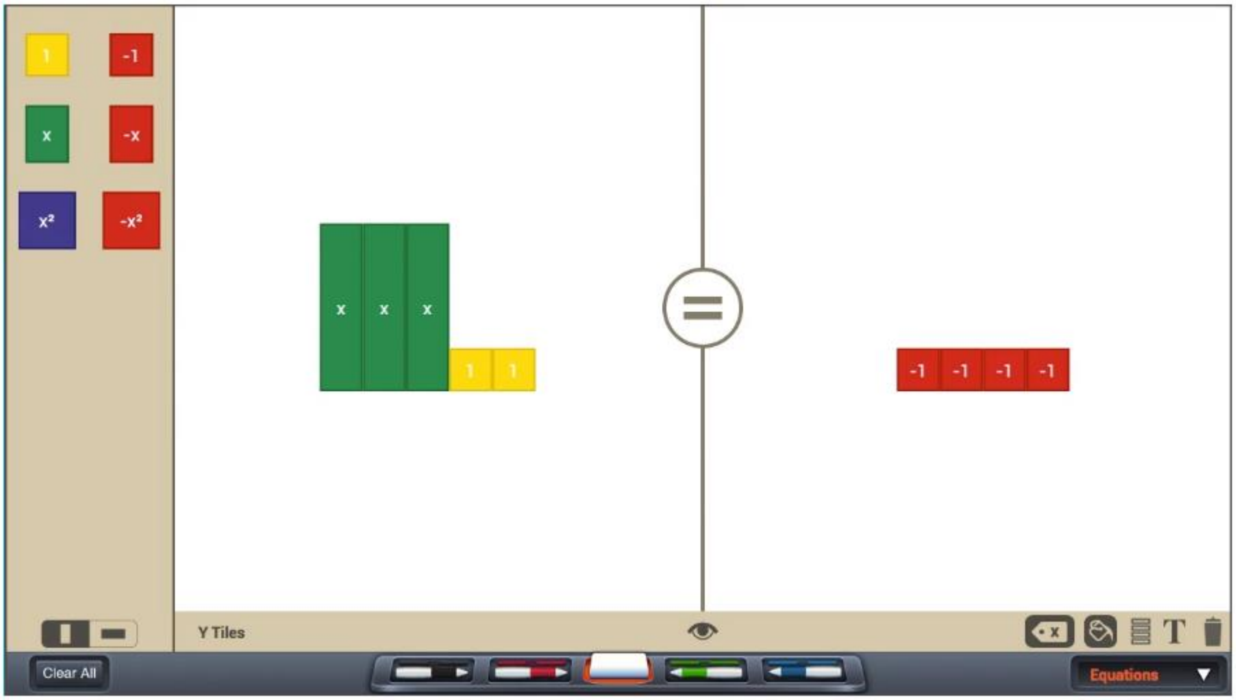


Figure 4.2. *Accuracy and algebra problems for Emily.*

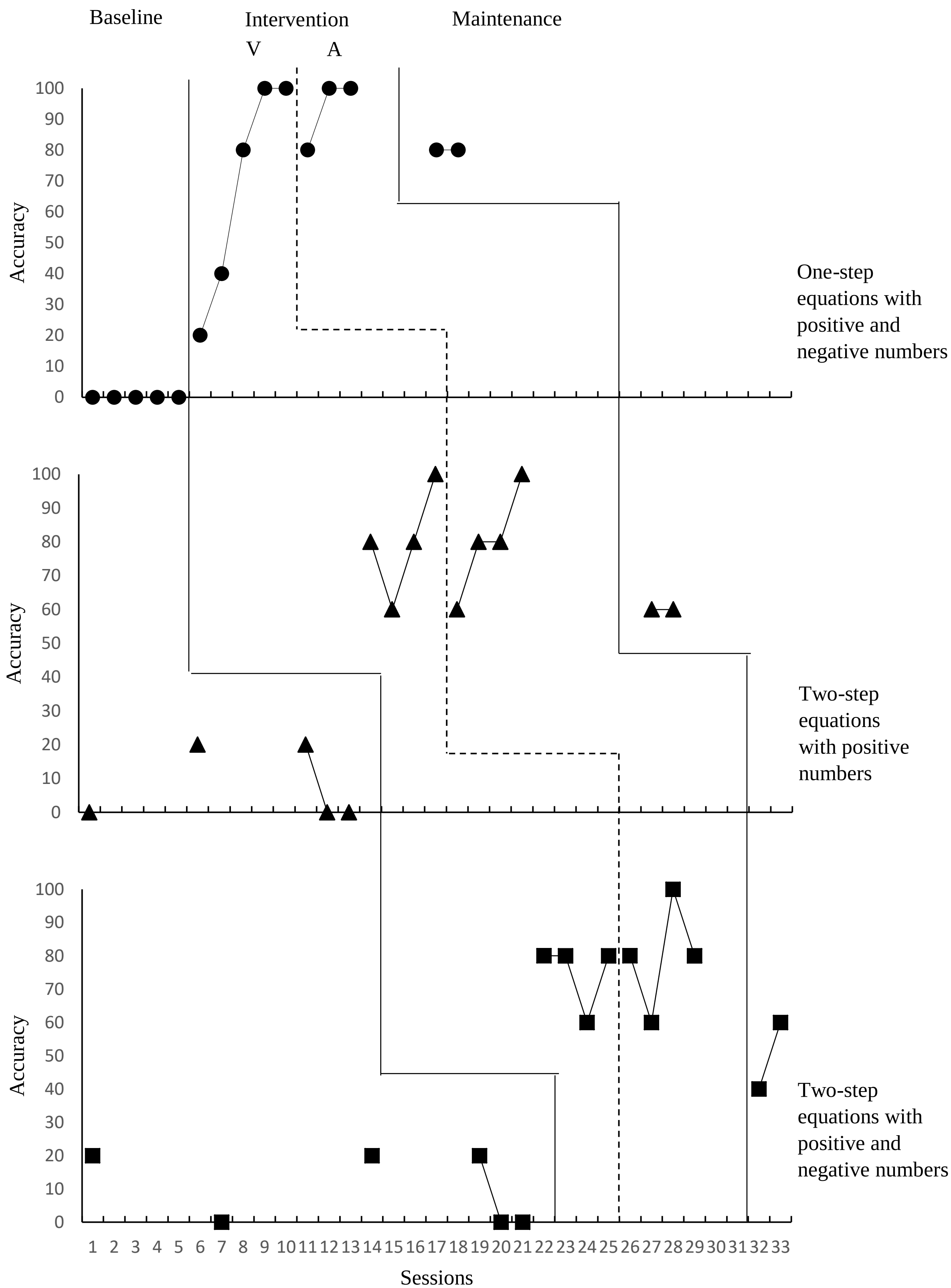


Figure 4.3. Accuracy and algebra problems for Sara.

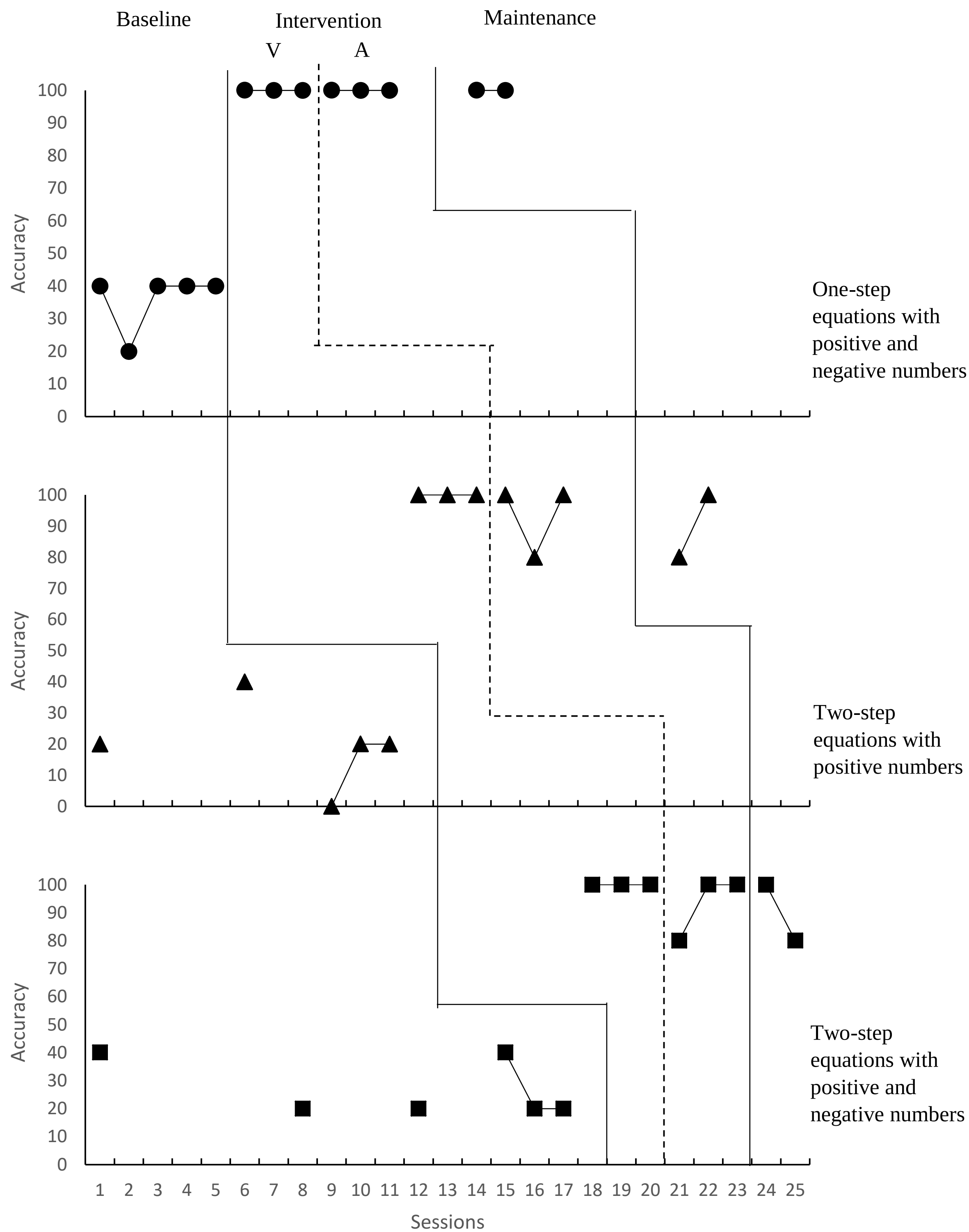
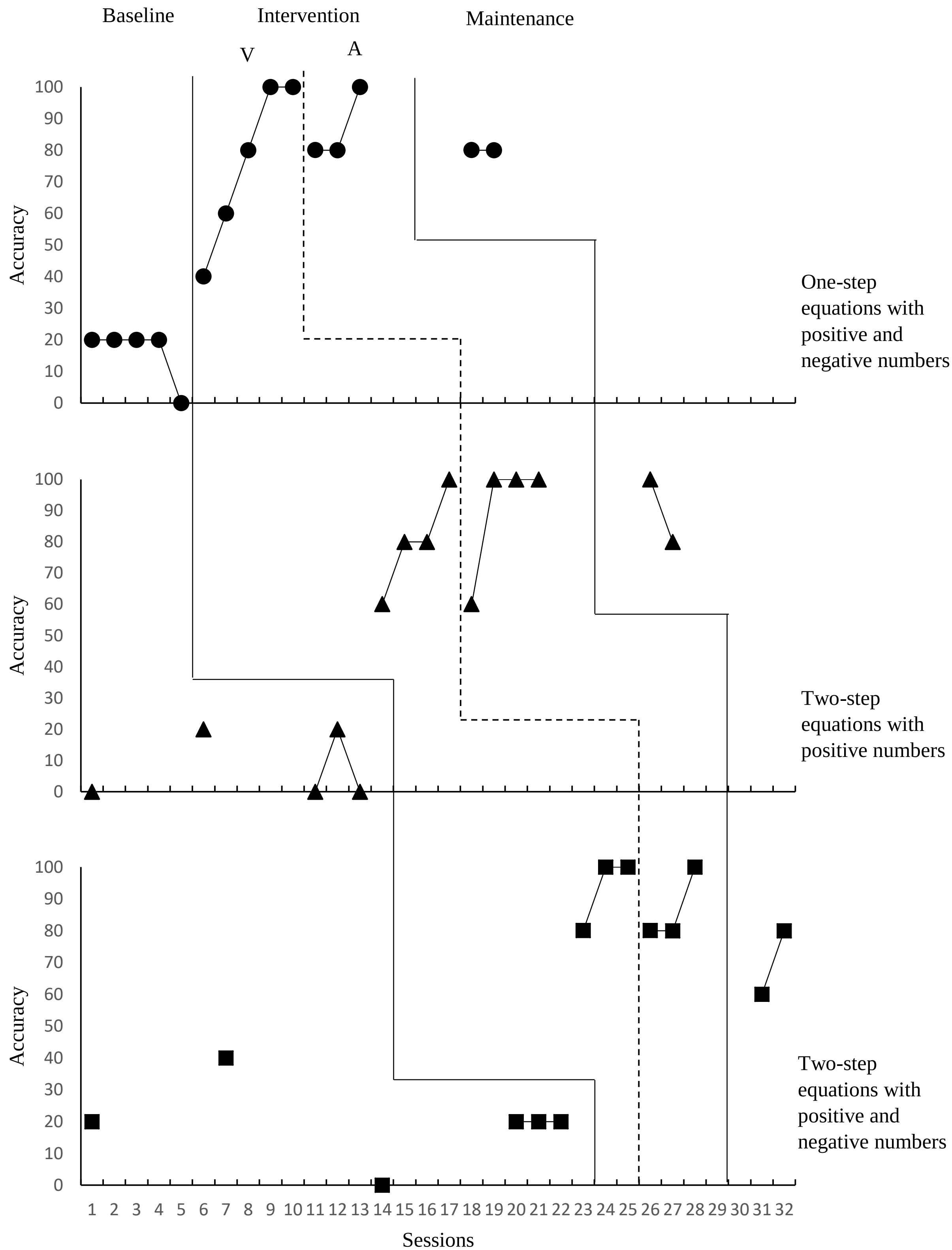


Figure 4.4. Accuracy and algebra problems for Paul.



CHAPTER 5

DISCUSSION

This alternative dissertation consisted of three studies that explored mathematical practices to support secondary students with high-incidence disabilities (e.g., learning disabilities, attention deficit hyperactive disorder, mathematics learning disability). The evidence-based systematic review of literature in Chapter 2 identified the practices for teaching algebra to secondary students with high-incidence disabilities and then used the quality indicators and practice standards set by the Council for Exceptional Children to critically analyze each practice (CEC, 2014; Cook et al., 2014). Results of the review suggest insufficient high-quality research exists, as no algebra intervention met the criteria for being an evidence-based practice (i.e., lacked the numbers of studies with positive results). However, five practices did meet standards for potentially evidence-based, suggesting a foundation of literature for future researchers to build upon. Chapter 3 used a single-case experimental design (i.e., adaptive alternating treatment) to compare the effectiveness of concrete manipulatives and virtual manipulatives to support the acquisition of algebra skills for three middle school students with high-incidence disabilities. Both the concrete and virtual algebra tiles were effective in terms of accuracy. There were no significant differences between independence or task completion time when comparing the two types of manipulatives, but all three students preferred the virtual manipulative. Chapter 4 presented a single-case experimental design (i.e., multiple probe across behaviors, replicated across participants) exploring the effectiveness of the VA framework to teach algebra to three middle school students with high-incidence disabilities. Researchers found a functional relation between the VA framework and students' acquisition of algebra skills, and all three students performed better during maintenance than baseline.

The main overall result of this alternative dissertation is that using manipulatives, either as a stand-alone tool or as part of a graduated instructional sequence, is effective in teaching algebra to secondary students with high-incidence disabilities. This aligns with previous recommendations by The National Council of Teachers of Mathematics to use manipulatives to support student learning of mathematics at all stages (NCTM, 2013), as well as literature reviews on manipulatives and the CRA framework for students with disabilities in general or students with learning disabilities in particular (Bouck & Park, 2018; Bouck, Satsangi & Park, 2018; Lafay et al., 2019; Peltier et al., 2019). Results from these three interconnected studies expand on previous research by suggesting manipulatives can be used specifically to support the acquisition of algebra for secondary students with high-incidence disabilities. Researchers can build on the current research by replicating existing studies or work to fill in gaps in the literature by exploring different ways to use manipulatives to support students with high-incidence disabilities as they work with algebra-related content.

From Chapter 2, use of manipulatives as a stand-alone instructional tool as well as manipulatives as part of the CRA framework are both potentially evidence-based practices for the subgroup of students with high-incidence disabilities learning algebra. While previous researchers established these practices as evidence-based for teaching mathematics to students with disabilities in general (Agrawal & Morin, 2016; Bouck, Satsangi & Park, 2018), there is currently lack of research available to verify the effectiveness of the practice for the specific population (i.e., secondary students with high-incidence disabilities) and content area (i.e., algebra). In this way, the current state of research regarding the use of manipulatives to support the acquisition of algebra skills for students with high-incidence disabilities is developing. Over one-third of the studies reviewed in the evidence-based systematic review for algebra and secondary students with high-incidence disabilities involved manipulatives as either a stand-alone tool or as part of a graduated instructional sequence, and all of these studies yielded positive results (e.g., Bouck et al., 2019, Maccini & Ruhl, 2000; Satsangi et al., 2016). This review also indicated a trend toward using virtual manipulatives, with the three most recent studies exploring the impact of virtual manipulatives on student performance solving linear algebraic equations. Additional research on the topic is needed including the consideration of the impact of virtual manipulatives as a stand-alone tool, as part of a framework, and in contrast to concrete manipulatives, as each may have characteristics that are beneficial in various settings.

In Chapter 3, the researchers demonstrated virtual manipulatives were just as effective as concrete manipulatives in teaching algebra content to secondary students with high-incidence disabilities. Consistent with previous research comparing the two types of manipulatives when working to solve basic algebra problems (Satsangi et al., 2016) as well as literature examining the effectiveness of virtual manipulatives to support the acquisition of algebra skills (e.g., Satsangi et al., 2018a; Satsangi et al., 2018b), the performance of three middle school students with high-incidence disabilities improved when they had access to and used either type of manipulative to support their learning. While students in the studies conducted by Satsangi et al. (2016; 2018a; 2018b) used an algebraic balance scale, in the current study students used virtual and concrete algebra tiles, similar to previous work with concrete manipulatives where the focus tended to be on algebra tiles (Maccini & Hughes, 2000; Maccini & Ruhl, 2000; Strickland & Maccini, 2012). Accordingly, this study represented the first comparison on a commonly used and effective concrete manipulative (i.e., algebra tiles) to its virtual counterpart.

Consistent with previous research comparing concrete and virtual manipulatives, students enjoyed using the virtual manipulatives and preferred them to concrete manipulatives (Bouck, Chamberlain & Park, 2017; Bouck, Shurr et al., 2018). Specifically, students in this study found it easy to organize their work on the screen, spoke to how they were able to use the rows to perform division, and could easily move pieces or clear their screen when they realized they made a mistake. There was no notable difference for task time completion, and access to manipulatives, regardless of the type, promoted student independence. Students were able to complete at least 90% of the steps required to solve equations independently when using manipulatives, which is consistent with previous research (Bouck, Shurr et al., 2018; Satsangi et al., 2016) and suggests students will eventually be able to use the tool with limited or no teacher support.

In light of previous research (Bouck, Shurr et al., 2018; Satsangi et al., 2016) as well as the results of Chapter 3 suggesting virtual manipulatives to be as effective in terms of algebra for accuracy, independence, and task time as concrete manipulatives, Chapter 4 explored the VA framework to teach algebra to middle school students with high-incidence disabilities. A functional relation was found between the VA framework and the acquisition of three algebra behaviors (e.g., one-step equations with positive and negative numbers, two-step equations with positive numbers, and two-step equations with positive and negative numbers), consistent with previous research exploring using virtual manipulatives as part of graduated instructional sequence to teach mathematics to students with disabilities (Bouck, Bassette et al., 2017; Bouck, Park et al., 2018). Also consistent with previous literature, student performance and overall algebraic understanding was not negatively impacted by the lack of a representational phase (Bouck, Park et al., 2019; Bouck, Park at al., 2017; Cass et al., 2003). These results add to a developing research base supporting the use of the VA framework to teach mathematics to students with disabilities (Bouck et al., 2020; Bouck, Park et al., 2017), including the acquisition of algebra skills (Bouck, Park et al., 2019). Students enjoyed using the virtual manipulatives and reported feeling comfortable using the support because they accessed it through their school-issued Chromebook just like any other assignment and supports (e.g., Bouck, Mathews & Peltier, 2019).

Implications for Practice

This alternative dissertation offers overall implications for practice. First, is the importance of using effective and efficient practices to support students with disabilities in developing algebra skills. Algebra instruction that is evidence-based or research-based provides greater opportunity for students to improve mathematical performance when implemented with fidelity (Cook & Cook, 2013). High-quality algebra instruction that is effective and efficient provide educators an opportunity to close the achievement gap and help students with disabilities be successful on grade-level curriculum (Cook & Odom, 2013). While additional published, high-quality research is needed regarding algebra instruction for secondary students with high-incidence disabilities, the studies in this dissertation suggest manipulatives offer educators an opportunity to support students and can act as – or part of – a high-quality, effective, and efficient intervention.

Another implication of the overall dissertation is that both concrete and virtual manipulatives are effective to teach algebra (Satsangi et al., 2016). As such, teachers have the flexibility to choose the type that is appropriate in their classroom based on available resources and individual student needs. Some students, for example, benefit more from concrete manipulatives because they are able to physically pick up and interact with pieces (Satsangi et al., 2016). Concrete manipulatives may also be more appropriate for students who struggle with technology or districts who do not have funds as teachers are able to repurpose or create their own at little to no cost (Bouck & Park, 2018). However, as they are just as effective, virtual manipulatives may be more appropriate depending on available resources and student need. Virtual manipulatives are easily accessible on Chromebooks and iPads (Bouck, Mathews & Peltier, 2019; Bouck, Working & Bone, 2018), and secondary students often prefer using virtual manipulatives as they view them as less stigmatizing (Bouck, Shurr et al., 2018; Satsangi & Bouck, 2015; Satsangi & Miller, 2017). This is especially true as more schools move to one-to-one technology and explore virtual learning opportunities. Teachers are also able to turn features on and off within a virtual program to differentiate support based on student need. This is appealing to both teachers and students, as teachers can provide instruction and practice at various skill levels, and students are able to receive appropriate accommodations without looking different from their peers (Satsangi & Miller, 2017).

Related, there are multiple ways manipulatives can be used to support learning. Students have demonstrated success on algebra content using manipulatives as a stand-alone tool (e.g., Satsangi et al., 2018a; 2018b), as well as part of a larger framework (e.g., Witzel, 2005). Although more literature exists demonstrating the effectiveness of a framework with a representational phase, manipulatives can also be part of a two-phase framework like the VA (e.g., Bouck et al., 2019). While all students with disabilities are expected to learn at high levels (ESSA, 2015), students with high-incidence disabilities are often served in the general education setting increasing the need for effective strategies that can be implemented in a variety of way. Flexibility with how manipulatives can be used to support learning can help teachers better fit an effective practice to specific students. For example, teachers may choose to use virtual manipulatives as a part of a graduated instructional sequence when working with students motivated by technology or with those who prefer the virtual option to avoid the stigma associated with using concrete manipulatives (Bouck, Mathews & Peltier, 2019; Satsangi & Bouck, 2015; Satsangi & Miller, 2017). Teachers could also choose to increase the criterion level to move to the next phase while using manipulatives within a framework (e.g., Richling et al., 2019), or increase the number of total sessions, or sessions in a particular phase to promote maintenance of skills (e.g., Bouck et al., 2020; Park et al., 2020). Or teachers may choose to pair manipulatives with other effective practices like explicit instruction and graphic organizers (National Center on Intensive Intervention, 2016). Future research should continue to explore interventions packages and how instructional practices impact specific groups of students (i.e., students with disabilities) to ensure all students have the opportunity to learn at high levels.

A final implication is that manipulatives help to support student independence as they acquire algebra skills. Once taught how to use the manipulatives, students are able to work through the problems with the support of the educational tool instead of relying entirely on teacher support. While a certain level of autonomy is important at all ages, promoting independence is especially important for older students who will increasingly be put into situations where they are challenged to use available resources to problem solve without adult support (Satsangi & Miller, 2017; Satsangi et al., 2018a). Furthermore, if students are able to use the tool with limited adult support, they can complete assignments on their own and practice new skills outside of regular instruction time (Bouck, Mathews & Peltier, 2019). Future research should examine how special features unique to virtual manipulatives, like the ability for teachers to monitor student progress or the immediate and ongoing feedback some programs offer, impacts students' conceptual understanding and overall independence as they work through novel content (Bouck, Working & Bone, 2018; Shin et al, 2017).

Limitations and Future Direction

There are limitations to this dissertation that must be addressed. The first limitation is that it is difficult to generalize findings to all students with disabilities. While inclusion criteria for all three pieces were very similar and resulted in participants sharing many characteristics including their age and having a high-incidence disability leading to difficulty in mathematics, slight variations among students may impact how they respond to specific instructional practices. In this way, there is a need for more research in practices, specifically using manipulatives, to support students with disabilities. Future research should continue to examine specific disabilities (e.g., learning disability, ADHD, mathematics learning disability) in order to learn even more about how specific populations respond to algebra instruction involving manipulatives.

Second, this dissertation focused only on the mathematics strand of algebra. While the evidence-based systematic review included various types of algebra problems (e.g., story problems, quadratic expressions), both experimental studies focused on solving basic equations. Although algebra is considered by many to be a mathematical gatekeeper (Watt, et al., 2016) and solving basic equations is a foundational part of algebra, there is a need for additional research on what practices can support secondary students with disabilities on more advanced algebra as well as different strands of mathematics. Future research should also explore different ways to implement manipulatives into secondary instruction. While the current research explores its impact on student learning in a one-on-one setting (e.g., Bouck, Park et al., 2019), manipulatives may also be able to support students during whole group instruction (Bouck & Cosby, 2017). For students who receive mathematics instruction in the general education setting, the teacher could implement the strategy during an independent work time at the end of a class period, as part of station or alternate teaching in a co-taught setting, or during a shortened intervention period (Bouck, Mathews & Peltier, 2019; Satsangi & Miller, 2017). Finally, this dissertation focused primarily on the acquisition of skills, but failed to address ways to improve maintenance of skills and students' ability to generalize their learning to other types of problems. As these are both essential components of the learning process (Collins, 2012), researchers should examine how the use of manipulatives can be enhanced or combined with other strategies to improve students' maintenance and generalization of skills.

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