TRANSPORT PROPERTIES OF RANDOM SCHRÖDINGER OPERATORS ON CORRELATED ENVIRONMENTS.

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ABSTRACT

TRANSPORT PROPERTIES OF RANDOM SCHRÖDINGER OPERATORS ON CORRELATED ENVIRONMENTS.

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This Ph.D. thesis presents recent developments in the theory of random Schrödinger operators. Differently from what is often studied in the subject, our main results consider potentials which are not independent at distinct sites but, rather, display some form of long range correlation. These are natural objects to investigate if one wishes to understand the long term behavior of a single particle which evolves in a disordered environment but also interacts with different members of this environment (other particles, spins, etc).

In chapter 2 it is shown that, within the Hartree-Fock approximation for the disordered Hubbard Hamiltonian, weakly interacting Fermions at positive temperature exhibit localization, suitably defined as exponential decay of eigenfunction correlators. Our result holds in any dimension in the regime of large disorder and at any disorder in the one dimensional case. As a consequence of our methods, we are able to show Hölder continuity of the integrated density of states with respect to energy, disorder and interaction using known techniques. This is based on joint work with Jeffrey Schenker [46]

Chapter 3 is based on joint work with Jeffrey Schenker and Rajinder Mavi. There we present simple, physically motivated, examples where small geometric changes on a two-dimensional graph $\mathbb{G}$, combined with high disorder, have a significant impact on the spectral and dynamical properties of the random Schrödinger operator $-A_{\mathbb{G}} + V_\omega$ obtained by adding a random potential to the graph's adjacency operator. Differently from the standard Anderson model, the random potential will be constant along vertical line, hence the models exhibit long range correlations. Moreover, one of the models presented here is a natural example where the transient and recurrent components of the absolutely continuous spec-

trum, introduced by Avron and Simon in [9], coexist and allow us to capture a sharp phase transition present in the system.

To Marsya, Sandra and Cory.

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CHAPTER 1

INTRODUCTION

1.1 A short introduction to Anderson Localization

The effects of disorder on transport properties of quantum systems have drawn a significant amount of attention in the mathematics community since their introduction in 1958 in the celebrated paper [8] by the Physicist Philip Anderson, who later in 1977 became a Nobel prize winner. The efforts to encode Anderson's claim that randomness may prevent diffusion into a rigorous mathematical statement and to obtain a proof for it gave rise to a beautiful theory, incomplete up to these days. Absence of diffusion in the disordered context is now generically referred to as *Anderson Localization* which we shall briefly review in this chapter, which aims at introducing the main concepts and results relevant for the remainder of the thesis. For a more complete historical picture we refer to the survey [58] and the book [6]. The proofs for well-known results given in this introduction are deeply influenced by the presentation in these two sources.

The Anderson model is the family of operators

$$H_\omega = -\Delta + \lambda V_\omega$$

acting on $\ell^2\left(\mathbb{Z}^d\right)$, the space of functions $f : \mathbb{Z}^d \to \mathbb{C}$ which are square summable, where

- Δ is the discrete Laplacian acting on $\varphi \in \ell^2\left(\mathbb{Z}^d\right)$ through

$$\left(\Delta\varphi\right)(n) = \sum_{|m-n|_1=1} \left(\varphi(m) - \varphi(n)\right) \quad , n \in \mathbb{Z}^d, \;\; |n|_1 = |n_1| + ...|n_d|.$$

.

- The on site potential V_ω acts as a multiplication operator on $\ell^2\left(\mathbb{Z}^d\right)$ via

$$\left(V_\omega\varphi\right)(n) = \omega(n)\varphi(n) \;, n \in \mathbb{Z}^d.$$

- $\omega = \{\omega(n)\}_{n \in \mathbb{Z}^d}$ is a list of independent, identically distributed random variables.
- $\lambda > 0$ denotes the disorder strength.

For technical reasons one often includes the following regularity condition on the random variables

$$\mathbb{P}\left(\omega(0) \in I\right) = \int_I \rho(v)\, dv. \tag{1.1.1}$$

A typical assumption on the density ρ is requiring it to be bounded and of compact support, which makes the random variable $\omega(n)$ bounded for every $n \in \mathbb{Z}^d$. Sometimes it is more convenient to work with distributions with unbounded support such as the Gaussian distribution given by $\rho(v) = \frac{1}{\sqrt{\pi}} e^{-v^2}$ and the Cauchy distribution given by $\rho(v) = \frac{1}{\pi(1+v^2)}$.

The lattice structure of $\mathbb{Z}^d$ allow us to make use of the canonical basis $\{\delta_n\}_{nin\mathbb{Z}^d}$ of $\ell^2\left(\mathbb{Z}^d\right)$ defined by $\delta_n(m) = \delta_{mn}$ where δ_{mn} is the Kronecker delta. The probability of finding a particle at position n and time t, given that it started at position 0 is given by

$$|\langle \delta_n, e^{-itH} \delta_0 \rangle|^2.$$

Dynamical Localization is naturally defined as averaged decay of the matrix elements

$$|\langle \delta_n, e^{-itH} \delta_0 \rangle|.$$

which can be explicit in a bound as

$$\mathbb{E}\left(|\langle \delta_n, e^{-itH} \delta_0 \rangle|\right) \leq r(n). \tag{1.1.2}$$

where

$$\sum_{n \in \mathbb{Z}^d} r(n) < \infty.$$

In the best scenario, the function $r(n)$ was shown to have exponential decay. In dimension $d \geq 2$, dynamical localization has been achieved at large disorder, meaning that λ is taken sufficiently large or at weak disorder and extreme energies, see [6, Theorem 10.4] for a precise statement. We shall outline a proof of localization at large disorder in the following sections of this introduction.

If one is only concerned whether diffusion takes place or not in a system, a useful tool is given by the averaged q-moments of the position operator defined as

$$M_T^q(H) = \frac{2}{T}\int_0^\infty e^{\frac{-2t}{T}}\,\mathbb{E}\,\langle 0|e^{itH_\omega}|X|^q e^{-itH_\omega}|0\rangle\;dt.$$

Where the position operator X is formally defined as $(X\varphi)(n) = |n|\varphi(n)$. For instance, whenever

$$\sum_{n\in\mathbb{Z}^d} |n|^q r^2(n) < C < \infty$$

dynamical localization in the above sense implies the bound

$$M_T^q(H) \leq C < \infty$$

The above inequality is a convenient signature of Anderson localization, whereas its counterpart

$$M_T^q(H) \geq CT^\alpha$$

for $\alpha > 0$ indicates diffusion.

Finally, recall that associated to the action of H in the Hilbert space $\mathscr{H} = \ell^2\left(\mathbb{Z}^d\right)$ is the decomposition $\mathscr{H} = \mathscr{H}^{\mathrm{pp}} \oplus \mathscr{H}^{\mathrm{sc}} \oplus \mathscr{H}^{\mathrm{ac}}$ of $\mathscr{H}$ into the pure point, singular continuous and absolutely continuous subspaces and the corresponding splitting of its spectrum into components $\sigma(H) = \sigma_{\mathrm{ac}}(H) \cup \sigma_{\mathrm{sc}}(H) \cup \sigma_{\mathrm{pp}}(H)$. The so called RAGE theorem (after Ruelle, Amrein, Georgescu and Enss.[6, Theorem 2.6]) provides dynamical characterizations for the spectral components. In particular, one of its consequences is the fact that dynamical localization in the form of (1.1.2) implies that H has only pure point spectrum. In other words, dynamical localization implies that the spectral projection of H into the continuous component $\mathscr{H}^c = \mathscr{H}^{\mathrm{sc}} \oplus \mathscr{H}^{\mathrm{ac}}$ vanishes. This means that the spectrum $\sigma(H)$ is the closure of the set of eigenvalues for H. If the associated eigenfunctions decay exponentially, the operator

H is said to exhibit Anderson Localization. A sufficient condition is the exponential decay of the eigenfunction correlators

$$\sup_{|g|\leq 1} \mathbb{E}\left(\left| \langle n|g(H_\omega)|0\rangle \right|\right) \leq Ce^{-\mu|n|} \tag{1.1.3}$$

where the above supremum is taken over all Borel measurable functions $g : \mathbb{R} \to \mathbb{C}$ and $\langle n|g(H_\omega)|0\rangle = \langle \delta_n, g(H_\omega)\delta_0\rangle$ (in this thesis we may alternate between these two notations for the inner product). For proofs for some of the above facts and more precise statements we refer to the next few sections of this introduction.

In the series of works by Jakšić and Molchanov [38], [37] and [39] the authors consider a Schrödinger operator $H = -\Delta + V_\omega$ on $\ell^2\left(\mathbb{Z}^d \times \mathbb{Z}^+\right)$ with randomness being introduced only along the surface $\partial\left(\mathbb{Z}^d \times \mathbb{Z}^+\right) = \mathbb{Z}^d$ through independent, identically distributed random variables $\{\omega(m)\}_{m\in\mathbb{Z}^d}$. More precisely, let

$$V_\omega(n_1, n_2) = \begin{cases} 0 \ \text{ if } n_2 \neq 0. \\ \omega(n_1) \ \text{ if } n_2 = 0. \end{cases}$$

In [37] (see also remark (3) in [39]), for $d = 1$ a sharp dichotomy is shown as

$$\sigma_{\text{pp}}(H) = \overline{\sigma(H) \setminus \sigma(-\Delta)} \ , \sigma_{\text{ac}}(H) = \sigma(-\Delta) \tag{1.1.4}$$

holds almost surely.

An important fact, which will allow us to obtain an analogue of (1.1.4) in the correlated context later in this thesis, is that the absolutely continuous subspace can be further decomposed into its transient and recurrent subspaces $\mathscr{H}^{\text{ac}} = \mathscr{H}^{\text{tac}} \oplus \mathscr{H}^{\text{rac}}$, an idea which was introduced by Avron and Simon in [9]. As explained there, one of the motivations is that in case $\mu_\psi = \chi_C\, dx$, where C is a Cantor-like set of positive Lebesgue measure, the measure μ_ψ resembles a singular measure, despite its absolute continuity. This will be a typical situation in which ψ belongs to the recurrent subspace $\mathscr{H}^{\text{rac}}$. The subspace $\mathscr{H}^{\text{tac}}$ is the closure of the set of all $\psi \in \mathscr{H}^{\text{ac}}$ such that, for all $N \in \mathbb{N}$

$$\left| \langle \psi|e^{-itH}|\psi\rangle \right| = O\left(t^{-N}\right). \tag{1.1.5}$$

Recall that, by the Riemann-Lebesgue lemma, if $\psi \in \mathscr{H}^{\mathrm{ac}}$,

$$\lim_{t\to\infty} \left| \langle \psi | e^{-itH} | \psi \rangle \right| = 0$$

hence, considering the spectral measure μ_ψ associated to ψ in $\mathscr{H}^{\mathrm{tac}}$, we see that its density with respect to Lebesgue measure $\frac{d\mu_\psi}{dE}$ must have additional regularity properties. In fact, according to [9, Proposition 3.1] if ψ is a transient vector then $\mu_\psi = f(x)\,dx$ where f is a smooth function. The converse statement is also true under the additional assumption of f being compactly supported. We will show in theorem 3.7 below that the transient and recurrent spectrum can naturally arise and coexist in a situation of physical relevance.

1.2 Dynamical localization at large disorder

We let $H = -\Delta + \lambda V_\omega$ for the remainder of this chapter, unless otherwise specified. In terms of background for this thesis, the following result is of great relevance. Strictly speaking, the version presented here was proven first by Aizenman in [1] although localization at large disorder was shown earlier by Aizenman and Molchanov [5]. The technical assumptions on the random potential V_ω will be clear when we outline a proof of this result within a few paragraphs.

Theorem 1.1. *There is a $\lambda > 0$ such that for all $\lambda > \lambda_0$*

$$\mathbb{E}\left(\sup_t |\langle \delta_n, e^{-itH}\delta_0\rangle|\right) \leq Ce^{-\nu|n|}.$$

for all $n \in \mathbb{Z}^d$ and some $C < \infty$ and $\nu > 0$.

As mentioned in the above section, theorem 1.1 means that, on average, the probability of finding the wavepacket at a site n at any time is exponentially small. This is the strongest form of single particle localization, known as exponential dynamical localization. We shall present a version of this theorem for the interacting case in chapter 2. A related statement can be achieved via multiscale analysis although with a subexponential rate, see [28]. Some general consequences of dynamical localization will be detailed in section 1.7.

1.3 Green's function decay: A step towards dynamical localization.

An object which is often helpful to the analysis of random operators is the Green's function. It is defined at $z \in \mathbb{C}$ by $G(m,n;z) = \langle \delta_m, (H-z)^{-1}\delta_n \rangle$ when this expression makes sense. In particular, since the operators object of our study are self-adjoint, this will be the case for all $z \in \mathbb{C} \setminus \mathbb{R}$. The first step to show dynamical localization as in theorem 1.1 is obtaining exponential decay for the above quantity as follows.

Theorem 1.2. *Given $s \in (0,1)$ and λ sufficiently large there exist positive constants C_{And} and μ_{And} such that for all $z \in \mathbb{C}^+$*

$$\mathbb{E}\left(|G(m,n;z)|^s\right) \leq C_{\mathrm{And}} e^{-\mu_{\mathrm{And}}|m-n|} \tag{1.3.1}$$

We shall outline two different proofs for the above fact, from which the relationship between the aforementioned constants and the parameters λ and s will be more clear.

1.4 First proof theorem 1.2: the self-avoiding walk expansion

The proposition below is the self-avoiding walk expansion of the Green's function, also called Feenberg's loop-erased expansion (since it seems to date back to Eugene Feenberg in the 1940s) or the locator expansion (as in Anderson's seminal work [8]). It will be useful not only for this chapter but also on chapter 3, when applied to a graph which is different from $\mathbb{Z}^d$. For this reason we keep the general formulation as in [6], i.e, replacing $\mathbb{Z}^d$ by an arbitrary vertex set $\mathbb{G}$ of a graph with finite degree, denoted henceforth by $\deg_{\mathbb{G}}$.

Proposition 1.3. *The Green's function of a self-adjoint operator $H = H_0 + V$ on $\ell^2(\mathbb{G})$ admits the expansion*

$$G(m,n;z) = \sum_{\tau: m\mapsto n}^{\mathrm{SAW}} (-1)^{|\tau|} H_0(\tau(0),\tau(1)) \ldots H_0(\tau(|\tau|-1),\tau(|\tau|)) \times \prod_{k=0}^{|\tau|} G_k(z).$$

On any graph, the above expansion converges whenever $\mathrm{Im} z > \|T\|$. *On finite graphs it is convergent for all $z \in \mathbb{C}^+$.*

Here $G_k(z) = \langle \delta_{\tau(k)}, (H_{\tau,k} - z)^{-1}\delta_{\tau(k)}\rangle$ where $H_{\tau,k}$ denotes the restriction of H to the set $\ell^2\left(\mathbb{G} \setminus \cup_{j<k}\tau(j)\right)$ (in case $k = 0$ we set $H_{\tau,k} = H$). The summation ranges over all self-avoiding walks τ from m to n with a finite number of steps. It is then clear from Feenberg's expansion that in order to control arbitrary matrix elements of $(H - z)^{-1}$ can firstly focus on analysing (meaning obtaining bounds on) the diagonal ones. For that end, fix $n \in \mathbb{Z}^d$ and let P_n be the projection onto the subspace spanned by δ_n. Denote by $\hat{H}$ the operator obtained from H by setting the potential at site n equal to zero, in other words

$$\hat{H} = H - \lambda V(n)P_n. \tag{1.4.1}$$

The first resolvent identity gives

$$(H - z)^{-1} - (\hat{H} - z)^{-1} = -\lambda V(n)(H - z)^{-1}P_n(\hat{H} - z)^{-1}. \tag{1.4.2}$$

In particular,

$$\begin{aligned} G(m, n; z) - \hat{G}(m, n; z) &= \langle \delta_m, ((H - z)^{-1} - (\hat{H} - z)^{-1})\delta_n\rangle \\ &= -\lambda V(n)\langle \delta_m, (H - z)^{-1}P_n(\hat{H} - z)^{-1}\delta_n\rangle \\ &= -\lambda V(n)G(m, n; z)\hat{G}(n, n; z). \end{aligned}$$

Solving for $G(m, n; z)$ we reach

$$G(m, n; z) = \frac{\hat{G}(m, n; z)}{1 + \lambda V(n)\hat{G}(n, n; z)}. \tag{1.4.3}$$

The above formula is the basic ingredient for the following a-priori bound.

Proposition 1.4. *For all $\lambda > 0$, $s \in (0, 1)$ and $z \in \mathbb{C}^+$ we have*

$$\mathbb{E}\left(|G(n, n; z)|^s\right) \le \frac{C_{\mathrm{AP}}(s)}{\lambda^s}. \tag{1.4.4}$$

where $C_{\mathrm{AP}}(s) = \frac{2^s\|\rho\|_\infty^s}{1-s}$.

Proof. Letting $m = n$ in (1.4.3) we find that

$$G(n, n; z) = \frac{1}{\lambda\omega(n) - \Sigma(z)} \tag{1.4.5}$$

where $\Sigma(z) = -\left(\hat{G}(n,n;z)\right)^{-1}$ is a complex number independent of $\omega(n)$. Let $v = \omega(n)$. Under the assumption that $\rho \in L^{\infty}(\mathbb{R})$ we have that for any interval $I \subset \mathbb{R}$

$$\mathbb{P}(v \in I) \leq \|\rho\|_{\infty}|I|. \tag{1.4.6}$$

Combining the above observations with the layer cake representation we discover that

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{|v-\xi|^s}\rho(v)dv &= \int_0^{\infty} \mathbb{P}(\{|v-\xi| \leq t^{-1/s}\})dt \\
&\leq \int_0^{\infty} \min\{1, 2\|\rho\|_{\infty}t^{-1/s}\}dt \\
&= 2^s\|\rho\|_{\infty}^s + 2\|\rho\|_{\infty}\int_{2^s\|\rho\|_{\infty}^s}^{\infty} t^{-1/s}dt \\
&= 2^s\|\rho\|_{\infty}^s + 2\|\rho\|_{\infty}\frac{s}{1-s}((2\|\rho\|_{\infty})^s)^{\frac{s-1}{s}} \\
&= 2^s\|\rho\|_{\infty}^s + 2\|\rho\|_{\infty}\frac{s}{1-s}(2\|\rho\|_{\infty})^{s-1} \\
&= \frac{2^s\|\rho\|_{\infty}^s}{1-s}.
\end{aligned}$$

finishing the proof of the a-priori bound.

□

It readily follows from Feenberg's expansion that whenever $\lambda > C_{\text{AP}} \deg_{\mathbb{G}}$

$$\begin{aligned}
\mathbb{E}\left(|G(m,n;z)|^s\right) &\leq \sum_{k=|m-n|}^{\infty} \deg_{\mathbb{G}}^k \left(\frac{C_{\text{AP}}}{\lambda^s}\right)^{k+1} \\
&= \frac{C_{\text{AP}}}{\lambda^s}\frac{1}{1-\frac{\deg_{\mathbb{G}} C_{\text{AP}}}{\lambda^s}}\left(\frac{\deg_{\mathbb{G}} C_{\text{AP}}}{\lambda^s}\right)^{|m-n|}
\end{aligned}$$

finishing the proof of theorem 1.2.

1.5 Second proof of theorem 1.2: the 'depleted' resolvent identity

Let us now provide another proof of theorem 1.2, based on the so called 'depleted' resolvent identity which, for $m \neq n$, reads

$$G_{\Lambda}(m,n;z) = -G_{\Lambda}(m,m;z)\sum_{n' \neq m} H_0(m,n')G_{\Lambda\setminus\{m\}}(n',n;z).$$

where $\langle m|(H^{\Lambda}-z)^{-1}|n\rangle$ and H^{Λ} denotes the restriction of H to a box of size $|\Lambda|$ centered at the origin. Using the a-priori bound and that $G_{\Lambda\setminus\{m\}}(n',n;z)$ is independent of m we reach

$$\mathbb{E}\left(|G_{\Lambda}(m,n;z)|^{s}\right)\leq\frac{C_{AP}}{\lambda^{s}}\sum_{n'}|H_{0}(m,n')|^{s}\mathbb{E}\left(|G_{\Lambda\setminus\{m\}}(n',n;z)|^{s}\right). \tag{1.5.1}$$

Exponential decay is now achieved whenever

$$\lambda^{s}>C_{AP}\|H_{0}\|_{\infty,\infty} \tag{1.5.2}$$

with

$$\|H_{0}\|_{\infty,\infty}:=\sup_{m}\sum_{n}|H_{0}(m,n)|. \tag{1.5.3}$$

through a suitable iteration of (1.5.1). A Couple of Remarks are in order:

- Note that both proofs make use of independence but that the existence of a conditional density is enough.
- The second of the above proofs also allows H_0 to be of the general form

  $$\sup_{m}\sum_{n}|H_{0}(m,n)|\left(e^{\nu|m-n|}-1\right)<\eta<\infty. \tag{1.5.4}$$

  In case the matrix elements of H_0 only exhibit polynomial decay, the argument given above yields polynomial decay of the Green's function.
- In [47], a refined estimate for the large disorder threshold is presented. Remarkably, it coincides with Anderson's original prediction of the large disorder regime.

Further comments and details may be found on [6, Section 10.1].

1.5.1 Further Aspects of A-priori bounds

The estimates for the Green's function given in this section heavily rely on the rank-one formula (1.4.3). An alternative, and more general, approach, originally due to [2], which is useful in the continuum and also for the operators studied on chapter 3 will be given in section 3.8 of that chapter.

1.6 From Green's function decay to dynamical localization

The next result we shall review explains why, in the context of the Anderson model, it is enough to shown Green's function decay in order to obtain dynamical localization. A proof will not be given here, but related ideas are presented in chapter 3, section 3.4.1. The original argument is due to Aizenman in [1] with a streamlined version being presented in [3, Appendix A]. We will say that a random operator H_ω has uniform Green's function decay on an energy domain $I \subset \mathbb{R}$ when given some $s \in (0,1)$ there exist $C > 0$ and $\mu > 0$ such that

$$\mathbb{E}\left(|G^\Lambda(m,n,E)|^s\right) \leq Ce^{-\mu|m-n|} \tag{1.6.1}$$

holds for all $\Lambda \subset \mathbb{Z}^d$ and every $E \in I$.

Theorem 1.5. *Let $H_\omega = H_0 + V_\omega$ be a random operator with the hopping term H_0 satisfying (1.5.4). Assume that*

- $\mathbb{E}(|V_\omega(n)|^\delta) < \infty$ *holds for some $\delta > 0$.*
- *For every $n_0 \in \mathbb{Z}^d$ the conditional distribution of $V_\omega(n_0) = v$ at specified values of $\{V_\omega(n)\}_{n \neq n_0}$ has a density $\rho_{n_0}(v)$ and*

$$\sup_{\omega \in \Omega} \sup_{n_0 \in \mathbb{Z}^d} \sup_{v \in \mathbb{R}} \rho_{n_0}(v) < \infty.$$

Suppose further that for some $s \in (0,1)$ the uniform Green's function decay (1.6.1) is satisfied in some interval $I \subset \mathbb{R}$. Then, there exist $C' > 0$ and $\mu' > 0$ such that

$$\mathbb{E}\left(\sup_{|f| \leq 1} \langle m|f(H_\omega)P_I(H_\omega)|n\rangle\right) \leq C'e^{-\mu'|m-n|} \tag{1.6.2}$$

where the supremum is taken over all Borel measurable functions $f : \mathbb{R} \to \mathbb{C}$ bounded by one.

1.6.1 Independence as a key tool

When establishing localization by the above arguments, independence of the random variables $\{\omega(n)\}_{n\in\mathbb{Z}^d}$ is important in many steps. Strict independence can, at most, be replaced by an assumption of a weaker form of independence which cannot be removed. This is not simply a technical issue related to fractional moments. For instance, the robust multiscale analysis technique is usually formulated under the assumption of *independence at distance*, meaning that there exists a $\rho > 0$ such that events based on boxes $\Lambda_{L_1}(m)$ are independent of events based on boxes $\Lambda_{L_2}(n)$ if $\mathrm{dist}\left(\Lambda_{L_1}(m), \Lambda_{L_2}(n)\right) > \rho$. Here $\Lambda_L(m) = \{m' \in \mathbb{Z}^d \ : |m - m'|_\infty < \frac{L}{2}\}$, see [41] for a survey in the subject.

1.7 Instructive consequences of dynamical localization

Let us point out some consequences of theorem 1.1. More precisely, the facts presented in this section will follow from dynamical localization as in (1.6.2).

1.7.1 From dynamical localization to finite averaged moments

Let

$$M_T^q(H) = \frac{2}{T}\int_0^\infty e^{\frac{-2t}{T}}\,\mathbb{E}\left(\langle 0|e^{itH_\omega}|X|^q e^{-itH_\omega}|0\rangle\right) dt.$$

be the averaged q-moments of the position operator X, formally defined as $(X\varphi)(n) = |n|\varphi(n)$. The following proposition holds true.

Proposition 1.6. *Assume that dynamical localization in the sense of (1.6.2) holds. Then*

$$M_T^q(H) \leq C$$

for some finite constant C.

Proof. By definition and from (1.6.2), we have that

$$M_T^q(H) = \frac{2}{T}\int_0^\infty e^{\frac{-2t}{T}} \sum_{n\in\mathbb{Z}^d} |n|^q \mathbb{E}\left(|\langle \delta_n, e^{-itH}\delta_0\rangle|^2\right) dt.$$
$$\leq C' \sum_{n\in\mathbb{Z}^d} |n|^q e^{-\mu'|n|}$$

where the above quantity is finite.

∎

1.7.2 From dynamical localization to pure point spectrum

For the convenience of the reader, we shall recall the definition of the spectral types below in section 1.8. The next consequence of dynamical localization we would like to point out is the following.

Proposition 1.7. *Suppose that dynamical localization as described by (1.6.2) holds on an interval $I \subset \mathbb{R}$. Then, almost surely, H_ω has only pure point spectrum in I.*

Proof. By the RAGE theorem (see [6, Theorem 2.6]), given $\varphi \in \ell^2\left(\mathbb{Z}^d\right)$ and denoting by P_I^c the projection onto the continuous component of H within I we have that

$$\|P_I^c\varphi\|^2 = \lim_{L\to\infty}\lim_{T\to\infty}\frac{1}{T}\int_0^T \|\mathbb{1}_{\Lambda_L^c} e^{-itH} P_I(H)\varphi\|^2\, dt. \tag{1.7.1}$$

Let $\varphi = \delta_n$ for some $n \in \mathbb{Z}^d$. Then,

$$\begin{aligned}\|\mathbb{1}_{\Lambda_L^c} e^{-itH} P_I(H)\varphi\|^2 &\leq \|\mathbb{1}_{\Lambda_L^c} e^{-itH} P_I(H)P_n\|^2 \|\delta_n\|^2\\ &\leq \|\mathbb{1}_{\Lambda_L^c} e^{-itH} P_I(H)P_n\|\\ &\leq \sum_{|m|>L} \left|\langle \delta_m, e^{-itH} P_I(H)\delta_n\rangle\right|.\end{aligned}$$

Taking expectations, using Fatou's lemma and Fubini's theorem, we conclude that under the assumption (1.6.2)

$$\|P_I^c\varphi\|^2 \leq C' \lim_{L\to\infty} \sum_{|m|>L} e^{-\mu'|m-n|} = 0. \tag{1.7.2}$$

The general result follows from density of finitely supported functions on $\ell^2\left(\mathbb{Z}^d\right)$. □

1.7.3 Localization centers and Eigenfunction decay

Lastly, let us discuss the phenomenon on engenfunction decay in the regime of dynamical localization.

Definition 1.8. *Given a Borel set $I \subset \mathbb{R}$, the eigenfunction correlator on I is defined by*

$$Q(m,n;I) = \sup_{|f|\leq 1} \langle m|f(H_\omega)P_I(H_\omega)|n\rangle \,. \tag{1.7.3}$$

with the supremum being taken over Borel measurable functions.

If H has only pure point spectrum in I, we have

$$Q(m,n;I) = \sum_{E\in I} |\varphi_{E,m}(m)||\varphi_{E,m}(n)| \tag{1.7.4}$$

where $\varphi_{E,m}$ is the normalized eigenfunction within the cyclic subspace $\mathscr{H}_{H,\delta_m}$ associated to the eigenvalue E. Recall that, in general,

$$\mathscr{H}_{H,v} = \overline{\operatorname{span}\{(H-z)^{-1}v : \; z\in\mathbb{C}\setminus\mathbb{R}\}}. \tag{1.7.5}$$

In case H is a bounded operator, the above set coincides with the closure of the set spanned by vectors of the form $q(H)v$ where q is a polynomial, see [19, lemma 2.4.3]. The expression (1.7.4) is obtained expanding (1.7.3) into the eigenfunction basis.

Definition 1.9. *Given $\varphi \in \ell^2\left(\mathbb{Z}^d\right)$ with $\|\varphi\|_2 = 1$, a point $n_0 \in \mathbb{Z}^d$ is said to be a localization center for φ if*

$$|\varphi(n_0)|^2 \geq \frac{C_d}{(1+|n_0|)^{d+1}} \tag{1.7.6}$$

where $C_d^{-1} := \sum_{n\in\mathbb{Z}^d} \frac{1}{(1+|n_0|)^{d+1}}$.

It is immediate to check that any such φ admits a localization center.

Proposition 1.10. *Assume that for some $\nu > 0$.*

$$\sum_m \frac{C_d}{(1+|m|)^{d+1}} \sum_n e^{\nu|m-n|} \mathbb{E}\left(Q^2(m,n;I)\right) < \infty. \tag{1.7.7}$$

Then, almost surely, for any simple eigenvalue $E \in I$ of H there is a localization center $n_E(\omega)$ such that the normalized eigenfunction $\varphi_{E,\omega}$ satisfies

$$|\varphi_{E,\omega}(n)| \leq C(I,\omega)(1+|n|)^{d+1} e^{-\frac{\nu|n-n_E|}{2}}. \tag{1.7.8}$$

for all $n \in \mathbb{Z}^d$. Moreover, $\omega \mapsto C^2(I,\omega)$ belongs to $L^1(\Omega;\mathbb{P})$.

Proof. Let

$$C(I,\omega) := \left(\sum_m \frac{C_d}{(1+|m|)^{d+1}} \sum_n e^{\nu|m-n|} Q^2(m,n;I)\right)^{1/2}. \tag{1.7.9}$$

By assumption, $C^2(I,\omega)$ belongs to $L^1(\Omega;\mathbb{P})$ hence it is a finite quantity almost surely. By proposition 1.7, the spectrum of H is, almost surely, pure point in I. Moreover, by (1.7.4), any simple eigenfunction of H satisfies

$$|\varphi_{E,m}(m)|^2 |\varphi_{E,m}(n)|^2 \leq C^2(I,\omega) C_d^{-1} (1+|n|)^{d+1} e^{-\nu|n-n_E|}. \tag{1.7.10}$$

recalling the definition of localization center and letting $m = n_E$ we reach

$$|\varphi_{E,m}(n)|^2 \leq C^2(I,\omega) C_d^{-2} (1+|n|)^{2d+2} e^{-\nu|n-n_E|}. \tag{1.7.11}$$

which yields the desired result after taking square roots. □

1.8 Spectral Measures

Finally, we conclude this introduction with a few notions on spectral theory. Let $\mathcal{H}$ be a Hilbert space and H a self-adjoint operator on $\mathcal{H}$. Given $\psi \in \mathcal{H}$, the spectral measure μ_ψ is the unique finite Borel measure on $\mathbb{R}$ satisfying, for all $z \in \mathbb{C}^+$,

$$\langle \psi, (H-z)^{-1}\psi \rangle = \int_{\mathbb{R}} \frac{1}{t-z}\, d\mu_\psi(t).$$

The existence and uniqueness of μ_ψ follows from the representation theorem for Herglotz functions (see [6, Appendix B]) since

$$F(z) = \langle \psi, (H - z)^{-1} \psi \rangle \tag{1.8.1}$$

defines a map $F : \mathbb{C}^+ \to \mathbb{C}^+$ whenever $\psi \neq 0$. By the Radon-Nikodym theorem, we can decompose μ_ψ (with respect to Lebesgue measure) according to its pure point, singular continuous and absolutely continuous components.

$$\mu_\psi = \mu_\psi^{\text{pp}} + \mu_\psi^{\text{sc}} + \mu_\psi^{\text{ac}}$$

Associated to the above decomposition is the decomposition of the Hilbert space

$$\mathscr{H} = \mathscr{H}^{\text{pp}} \oplus \mathscr{H}^{\text{ac}} \oplus \mathscr{H}^{\text{sc}}$$

where

$$\mathscr{H}^{\text{pp}} = \{\psi \; | \mu_\psi = \mu_\psi^{\text{pp}}\} \;, \mathscr{H}^{\text{sc}} = \{\psi \; | \mu_\psi = \mu_\psi^{\text{sc}}\}$$

and

$$\mathscr{H}^{\text{ac}} = \{\psi \; | \mu_\psi = \mu_\psi^{\text{ac}}\}.$$

The spectrum of H, denoted by $\sigma(H)$, is defined as the set of $z \in \mathbb{C}$ such that $H - z$ does not have a bounded inverse and the spectral components $\sigma^{\text{pp}}(H), \sigma^{\text{sc}}(H)$ and $\sigma^{\text{ac}}(H)$ are defined by restricting the operator to the subspaces $\mathscr{H}^{\text{pp}}$,$\mathscr{H}^{\text{sc}}$ and $\mathscr{H}^{\text{ac}}$.

CHAPTER 2

LOCALIZATION AND IDS REGULARITY IN THE HUBBARD MODEL WITH HARTREE-FOCK THEORY

2.1 Chapter Outline

Our main goal in this chapter is to study Anderson localization in the context of infinitely many particles. We shall formulate our results for the disordered Hubbard model within Hartree-Fock theory. However, as the techniques involved are quite flexible, we expect that similar statements can be made in a more general framework, under appropriate modifications of the decorrelation estimates in section 2.7.1. The (deterministic) Hubbard model under Generalized Hartree-Fock Theory has been discussed (at zero and positive temperature) by Lieb, Bach and Solovej in [11] however, to the best of our knowledge, the localization properties of the disordered version of this model remained unexplored, even in the context of restricted Hartree-Fock theory, up to the present work. The main difficulty lies in the addition of a self-consistent effective field, which will be random and non-local by nature, to a random Schrödinger operator. The conclusion of this chapter can be summarized as follows: under technical assumptions, the results on (single-particle) Anderson localization obtained in the non-interacting setting in the regimes of large disorder (in dimension $d \geq 2$) and at any disorder (in dimension $d = 1$), remain valid under the presence of sufficiently weak interactions. More specifically, in the regime of strong disorder this is accomplished in any dimension by theorem 2.2 below. Theorem 2.1 contains the improvement in dimension one, where any disorder strength leads to localization, provided the interaction strength is taken sufficiently small. Our methods contain various bounds in the fluctuations of the effective interaction which are interesting on their own right and potentially useful in different contexts. To exemplify this, we prove Hölder regularity of the integrated density of states (IDS) with respect to various parameters by adapting arguments of [29], which is the content of theorem 2.3.

2.1.1 Discussion of the results and main obstacles

Mathematically, our setting can be understood as an Anderson-type model $H_\omega = H_0 + \lambda U_\omega$ where the values of the random potential U at different sites are correlated in a highly non-local and self-consistent fashion. The correlations are governed by a nonlinear function of H_ω, as explained in section 2.3. In comparison to the recent result on Hartree-Fock theory for lattice fermions of [23], achieved via multiscale analysis, we use the fractional moment method to establish exponential decay of the eigenfunction correlators at large disorder in any dimension but also at any disorder in dimension one. In particular, in the above regimes, we show exponential decay (on expectation) of the matrix elements of the Hamiltonian evolution at all times, which means that, on average, $|\langle m|e^{-itH}|n\rangle|$ decays exponentially on $|m-n|$ for all $t > 0$.

The result of complete localization in dimension one in such interacting context is new and deserves attention on its own. There, the main technical difficulty also lies in the non-local correlations of the potential, which means that standard tools such as Furstenberg's theorem and Kotani theory are not available. Moreover, a large deviation theory for the Green's function is a further obstacle to establishing dynamical localization even if one obtains uniform positivity of the Lyapunov exponent. We overcome these challenges using ideas of [6, Chapter 12], where arguments reminiscent of the proof of the main result in [43] are presented. We then obtain positivity of the Lyapunov exponent at any disorder using uniform positivity for the Lyapunov exponent of the Anderson model, combined with an explicit bound on how this quantity depends on the interaction strength, see theorem 2.19. When it comes to establishing a large deviation theorem, our modification of the argument in [6, Theorem 12.8] relies on quantifying the decorrelations on the effective potential, which is presented on lemma 2.22 in the form of a strong mixing statement. It is worth clarifying that, since our proof is based on fractional moments, we have not established localization in one dimension for rough potentials as in [34]. Moreover, the gap assumption in [23] is replaced by working at positive temperature thus our results do not apply to Hartree-Fock ground states.

2.1.2 Hartree-Fock theory

Hartree-Fock theory has been widely applied in computational physics and chemistry. It also has a rich mathematical literature which goes well beyond the scope of random operators, see for instance [31],[32],[11],[12],[33] and references therein.

2.1.3 Background on Localization for interacting systems

The main results of this chapter lie in between the vast literature on (non-interacting) single particle localization and the recent efforts to study many particle systems, as in the case of an arbitrary, but finite, number of particles in the series of works by Chulaevsky-Suhov [15],[16],[17] and Aizenman-Warzel [7]. In comparison to the later, we only seek for a single-particle localization result but allow for infinitely many interactions, which occur in the form of a mean field. In comparison to the recent developments on spin chains, as the study of the XY spin chain in [30] and the droplet spectrum of the XXZ quantum spin chain in [25] and [13], the notions of localization for a single-particle effective Hamiltonian are more clear and can be displayed from pure point spectrum to exponential decay of eigenfunctions and exponential decay of eigenfunction correlators. The later is agreed to be the strongest form of single particle localization and it is what we end up accomplishing. If fact, as we have explained in the introduction of this thesis, dynamical localization in the form of theorems 2.1 and 2.2 implies pure point spectrum via the RAGE theorem and exponential decay of eigenfunctions, see also [58, Proposition 5.3] and [6, Theorem 7.2 and Theorem 7.4].

2.2 Definitions and Statement of the Main Result

2.2.1 Notation

In what follows, $\mathbb{Z}^d$ will be equipped with the norm $|n| = |n_1| + ... + |n_d|$ for $n = (n_1, ..., n_d)$. Given a subset $\Lambda \subset \mathbb{Z}^d$, we define $\ell^2(\Lambda) := \{\varphi : \Lambda \to \mathbb{C} \,|\, \sum_{n\in\Lambda} |\varphi(n)|^2 < \infty\}$ and, for $\varphi \in \ell^2(\Lambda)$, we let $\|\varphi\|_{\ell^2(\Lambda)} := \left(\sum_{n\in\Lambda} |\varphi(n)|^2 < \infty\}\right)^{1/2}$. Throughout this chapter, η will be a positive constant and $F_{\beta,\kappa}$ will denote the Fermi-Dirac function at inverse temperature $\beta > 0$ and chemical potential κ, meaning that

$$F_{\beta,\kappa}(z) = \frac{1}{1 + e^{\beta(z-\kappa)}}. \tag{2.2.1}$$

We shall omit the dependence of F on the above parameters whenever it is clear from the context. For many of our bounds, the specific form of (2.2.1) is not important and F could denote an arbitrarily chosen function which is analytic in the strip $\mathcal{S} = \{z \in \mathbb{C} : \ |\mathrm{Im} z| < \eta\}$ and continuous up to the boundary of $\mathcal{S}$, in which case we let $\|F\|_\infty := \sup_{z\in\mathcal{S}} |F(z)|$. For the function $F_{\beta,\kappa}$ in (2.2.1) one can take $\eta = \frac{\pi}{2\beta}$. However, to obtain robust results which incorporate delicate fluctuations, further properties of the Fermi-Dirac function are necessary. Namely, in section 2.7.1 we use the the fact that $tF(t)$ is bounded as $t \to \infty$ and that $t(1-F(t))$ is bounded $t \to -\infty$. These properties will also play a role in the decoupling estimates needed in the proof of theorem 2.1 but could be relaxed if one is only interested in the large disorder proof of theorem 2.2 for a specific distribution with heavy tails (for instance, the Cauchy distribution).

Our main goal is to study localization properties of non-local perturbations of the Anderson model $H_{\mathrm{And}} := -\Delta + \lambda V_\omega$ which naturally arise in the context of Hartree-Fock theory for the Hubbard model. The random potential V_ω is the multiplication operator on $\ell^2(\mathbb{Z}^d)$ defined as

$$\left(V_\omega \varphi\right)(n) = \omega_n \varphi(n) \tag{2.2.2}$$

for all $n \in \mathbb{Z}^d$ and $\{\omega_n\}_{n\in\mathbb{Z}^d}$ are independent, identically distributed random variables on which we impose technical assumptions described in the next paragraph. The hopping operator Δ is the discrete Laplacian on $\mathbb{Z}^d$, defined via

$$(\Delta\varphi)(n) = \sum_{|m-n|=1} (\varphi(m) - \varphi(n)) \,. \tag{2.2.3}$$

The proofs of localization via fractional moments usually do not require the hopping to be dictated by the Laplacian and, indeed, we will replace Δ by a more general operator H_0 whose matrix elements decay sufficiently fast away from the diagonal. It is technically useful to formulate some of our results in finite volume, i.e, we will work with restrictions of the operators to $\ell^2(\Lambda)$ but the estimates obtained will be volume independent, meaning that all the constants involved are independent of $\Lambda \subset \mathbb{Z}^d$. We will use $\mathbb{1}_\Lambda$ to denote the characteristic function of Λ as well as the natural projection $P_\Lambda : \ell^2(\mathbb{Z}^d) \to \ell^2(\Lambda)$. With these preliminaries we are ready to define the Schrödinger operators studied in this chapter.

2.2.2 Definition of the operators

Let $H_{\text{And}} = H_0 + \lambda V_\omega$ be the Anderson model on $\ell^2\left(\mathbb{Z}^d\right)$ where:

(A_1)

$$\zeta(\nu) := \sup_m \sum_{n\in\mathbb{Z}^d} |H_0(m,n)| \left(e^{\nu|m-n|} - 1\right) < \frac{\eta}{2}, \quad \text{for some } \nu > 0 \text{ fixed.}$$

(A_2) V_ω is defined as in (2.2.2) and the random variables $\{\omega(n)\}_{n\in\mathbb{Z}^d}$ are independent, identically distributed with a density ρ:

$$\mathbb{P}\left(\omega(0) \in I\right) = \int_I \rho(x)\,dx, \quad \text{for } I \subset \mathbb{R} \text{ a Borel set}\,.$$

(A_3) We also assume that $\operatorname{supp}\rho = \mathbb{R}$ with

$$\frac{\rho(x)}{\rho(x')} \geq e^{-c_1(\rho)|x-x'|(1+c_2(\rho)\max\{\,|x|,|x'|\,\})} \tag{2.2.4}$$

for some $c_1(\rho) > 0$, $c_2(\rho) \geq 0$ and any $x, x' \in \mathbb{R}$.

Before stating the remaining assumptions on ρ, we need to introduce some notation. Assume that ρ satisfies (2.2.4). Let

$$\overline{\rho}(x) = \frac{\rho(x)}{\int_{-\infty}^{\infty} \rho(\alpha) h(x-\alpha)\, d\alpha} \tag{2.2.5}$$

where

$$h(x) = \begin{cases} e^{-\overline{c}_\rho |x|} & \text{if } c_2(\rho) = 0. \\ \\ e^{-\overline{c}_\rho |x|^2} & \text{if } c_2(\rho) > 0. \end{cases} \tag{2.2.6}$$

(A_4) The function $\overline{\rho}$ is bounded for some $\overline{c}_\rho > 0$.

Remark 1. *The technical assumptions* $(A_3)-(A_4)$ *will be needed for the large disorder result of theorem 2.2 below. They include, for instance, the Cauchy distribution, the Gaussian, and the exponential distribution* $\rho(v) = \frac{m}{2} e^{-m|v|}$*. Further details an related comments will be provided on the appendix of this thesis, section A.1.*

The above requirements will suffice to show localization at large disorder on theorem 2.2 below. To show complete localization in dimension one, theorem 2.1 will also require a moment condition on $\overline{\rho}$, which is the following.

(A_5) For some $\varepsilon > 0$ and some $\overline{c}_\rho > 0$, $\int_{-\infty}^{\infty} |x|^\varepsilon \overline{\rho}(x)\, dx < \infty$.

Remark 2. *The assumption* (A_5) *covers, for example, the Gaussian and the exponential distributions but it does not cover the Cauchy or other distribution with heavy tails. It will be necessary for the one dimensional result of theorem 2.1 below. More specifically, this requirement will imply a moment condition which will be needed to relate the Green's function to the Lyapunov exponent, see sections 2.11 and 2.11.5.*

Remark 3. *The specific bound on* $\zeta(\nu)$ *is necessary to ensure that the Combes-Thomas bound* $|G(m, n; E + i\eta)| \leq \frac{2}{\eta} e^{-\nu|m-n|}$ *holds [6, Theorem 10.5], where* $G(m, n; z)$ *denotes* $\langle m|(H-z)^{-1}|n\rangle$*, whenever this quantity is defined.*

Define the operator H_{Hub}, acting on $\ell^2\left(\mathbb{Z}^d\right) \oplus \ell^2\left(\mathbb{Z}^d\right)$ by

$$H_{\text{Hub}} = \begin{pmatrix} H_{\uparrow}(\omega) & 0 \\ 0 & H_{\downarrow}(\omega) \end{pmatrix} := \begin{pmatrix} H_0 + \lambda V_\omega + gV_{\uparrow}(\omega) & 0 \\ 0 & H_0 + \lambda V_\omega + gV_{\downarrow}(\omega) \end{pmatrix} \tag{2.2.7}$$

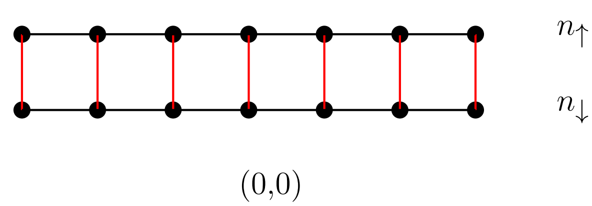
where the operators $H_{\uparrow}(\omega)$ and $H_{\downarrow}(\omega)$ act on $\ell^2\left(\mathbb{Z}^d\right)$ and the so-called effective potentials are defined via

$$\begin{pmatrix} V_{\uparrow}(\omega)(n) \\ V_{\downarrow}(\omega)(n) \end{pmatrix} = \begin{pmatrix} \langle n|F(H_{\downarrow})|n\rangle \\ \langle n|F(H_{\uparrow})|n\rangle \end{pmatrix}. \tag{2.2.8}$$

Note that the above equations only define $H_{\uparrow}(\omega)$ and $H_{\downarrow}(\omega)$ implicitly. Existence and uniqueness of $V_{\uparrow}$ and $V_{\downarrow}$ will be shown in section 2.9.1 via a fixed point argument. The model (2.2.7) is usually referred to as the Hartree approximation, due to the absence of exchange terms. In section 2.3 below we will show that the terminology Hartree-Fock approximation is justified when $g < 0$, which represents a repulsive interaction.

The Hubbard model is schematically represented in the following picture. The black (horizontal) edges represent hopping between sites and the red (vertical) edges represent the effective interaction between the two layers, which are identical copies of $\mathbb{Z}^d$.

Figure 2.1: The Hubbard Model



2.2.3 Main Theorems

Fix an interval $I \subset \mathbb{R}$ and define the eigenfunction correlator through

$$Q_I(m,n) := \sup_{|\varphi|\leq 1} \left(|\langle m|\varphi(H_{\uparrow})|n\rangle| + |\langle m|\varphi(H_{\downarrow})|n\rangle| \right). \tag{2.2.9}$$

The operators $H_\uparrow$ and $H_\downarrow$ are defined as in (2.2.7) and the supremum is taken over Borel measurable functions bounded by one and supported on the interval I. In case $I = \mathbb{R}$ we simply write $Q(m, n)$. Our first result is the following:

Theorem 2.1. *In dimension $d = 1$, let $H_0 = -\Delta$ and assume that the conditions $(A_1)-(A_5)$ hold. For any $\lambda > 0$ and any closed interval $I \subset \mathbb{R}$, there is a constant $g_1 > 0$ such that whenever $|g| < g_1$ we have*

$$\mathbb{E}\left(Q_I(m,n)\right) \leq Ce^{-\mu_1|m-n|}. \tag{2.2.10}$$

for any $m, n \in \mathbb{Z}^d$ and positive constants $\mu_1 = \mu_1(\lambda, \nu, \eta, I)$, $C(\eta, g, \lambda, \|F\|_\infty, I)$.

Theorem 2.2. *Suppose that the conditions $(A_1) - (A_4)$ hold. For any dimension $d \geq 1$, there exists a constant $g_d = g(d, \eta, \|F\|_\infty, \nu)$ such that, whenever $|g| < g_d$, there is a positive constant $\lambda_0(g)$ for which*

$$\mathbb{E}\left(Q(m,n)\right) \leq Ce^{-\mu_d|m-n|}. \tag{2.2.11}$$

holds for $\lambda > \lambda_0(g)$, any $m, n \in \mathbb{Z}^d$ and some positive constants $\mu_d = \mu(d, \lambda, g, \nu, \eta)$, $C(\eta, \nu, d, g, \lambda, \|F\|_\infty)$.

Remark 4. *It will follow from the proof that the constant g_d in theorem 2.2 can be taken proportional to $\frac{\eta\left(1-e^{-\nu}\right)^d}{\|F\|_\infty}$.*

Remark 5. *The constant g_1 in theorem 2.1 can be taken equal to be the minimum among a factor proportional to $\frac{\eta\left(1-e^{-\nu}\right)}{\|F\|_\infty}$ and the upper bound obtained in corollary 2.20, which also depends on the lower bound for the Lyapunov exponent of the Anderson model on $\ell^2(\mathbb{Z})$.*

Before proceeding to the next result, recall the definition of the integrated density of states for an ergodic operator H :

$$N_H(E) = \lim_{|\Lambda|\to\infty} \frac{\mathrm{Tr} P_{(-\infty,E)}(\mathbb{1}_\Lambda H \mathbb{1}_\Lambda)}{|\Lambda|}. \tag{2.2.12}$$

Recall also that H is said to be ergodic (on $\mathbb{Z}^d$) when for every $\omega \in \Omega$ and $j \in \mathbb{Z}^d$

$$H_{T_j\omega} = U_j H_\omega U_j^*. \tag{2.2.13}$$

In the above equality, $T : \Omega \to \Omega$ denotes the measure preserving transformation given by $T_j\omega(n) = \omega(n-j)$ and the unitary maps U_j are defined via

$$\left(U_j\varphi\right)(n) = \varphi(n-j). \tag{2.2.14}$$

In fact, one can relax the above conditions and allow the maps U_j to be of more general form, the interested reader may consult [6, Definition 3.4]. In what follows, we denote by $N_0(E)$ the corresponding quantity for the free operator H_0 defined above, which is assumed to be ergodic for the result of theorem 2.3, where we shall be concerned with the small disorder regime and aim for bounds which do not depend upon λ as $\lambda \to 0$.

Theorem 2.3. *Assume that $(A_1)-(A_2)$ hold with $x^2\rho(x)$ bounded and that $g^2 < \lambda$ and $g'^2 < \lambda'$. Fix a interval I where $E \mapsto N_0(E)$ is α_0-Hölder continuous and a bounded interval $J \subset \mathbb{R}$. The integrated density of states $N_{\lambda,g}(E)$ of H_{Hub} is Hölder continuous with respect to E and with respect to the pair (λ, g). More precisely:*

- *For $E, E' \in I$*

$$|N_{\lambda,g}(E) - N_{\lambda,g}(E')| \leq C(\alpha, I, g)|E-E'|^\alpha \tag{2.2.15}$$

  for $\alpha \in [0, \frac{\alpha_0}{2+\alpha_0}]$ and $C(\alpha, I, g)$ independent of λ.

- *If $\lambda, \lambda' \in J$, we have that, for any $E \in I$, $\alpha \in [0, \frac{\alpha_0}{2+\alpha_0}]$ and $\beta \in [0, \frac{2\alpha}{\alpha+3d+4}]$,*

$$|N_{\lambda,g}(E) - N_{\lambda',g'}(E)| \leq C(\alpha_0.d, I)\left(|\lambda-\lambda'|^\beta + |g-g'|^\beta\right). \tag{2.2.16}$$

2.3 Motivation

We shall explain the motivation for the above choice of the effective potential. We are only going to outline the derivation of the self-consistent equations later in section A.2 as this is a standard topic, see, for instance, [44, Chapter 3].

Let $\Lambda \subset \mathbb{Z}^d$ be a finite set. Following the notation of [11], we use Γ to denote a one particle density matrix, i.e, a 2×2 matrix whose entries are operators on $\ell^2(\Lambda)$ and which satisfies $0 \leq \Gamma \leq \mathbb{1}$. We then write

$$\Gamma := \begin{pmatrix} \Gamma_{\uparrow} & \Gamma_{\uparrow\downarrow} \\ \Gamma_{\downarrow\uparrow} & \Gamma_{\uparrow} \end{pmatrix}$$

where $\Gamma_{\downarrow\uparrow} = \Gamma_{\uparrow\downarrow}^{\dagger}$.

As in [11, Equation 3a.8], the pressure functional $\mathscr{P}(\Gamma)$ is defined as

$$-\mathscr{P}(\Gamma) = \mathscr{E}(\Gamma) - \beta^{-1}\mathscr{S}(\Gamma). \tag{2.3.1}$$

The energy functional is

$$\mathscr{E}(\Gamma) = \mathrm{Tr}\left(H_0 - \kappa + \lambda V_\omega\right)\Gamma + g\sum_n \langle n|\Gamma_{\uparrow}|n\rangle\ \langle n|\Gamma_{\downarrow}|n\rangle\,, \tag{2.3.2}$$

where we have identified $H_0 - \kappa + \lambda V_\omega$ with $\begin{pmatrix} H_0 - \kappa + \lambda V_\omega & 0 \\ 0 & H_0 - \kappa + \lambda V_\omega \end{pmatrix}$.

The entropy is given by

$$\mathscr{S}(\Gamma) = -\mathrm{Tr}\left(\Gamma\log\Gamma + (1-\Gamma)\log(1-\Gamma)\right). \tag{2.3.3}$$

Generally, the choice of energy functional (2.3.2) is referred to as Hartree approximation as exchange terms are neglected. However, in the case of a repulsive interaction among the particles, it is easy to prove that such exchange terms do not affect the choice of minimizer for $-\mathscr{P}(\Gamma)$ and the process may be referred to as the Hartree-Fock approximation. Indeed, the Hartree-Fock energy for the repulsive interaction would incorporate the term $-g|\langle n|\Gamma_{\uparrow\downarrow}|n\rangle|^2$, which is non-negative when $g < 0$, inside the summation. Thus, for repusive interactions, off-diagonal terms can be disregarded for minimization purposes, see the analogue discussion in [11, Section 4a]. The minimizer Γ of $-\mathscr{P}(\Gamma)$ exists since Λ is a finite set. Moreover, it satisfies

$$\langle n|\Gamma_{\uparrow}|n\rangle = \langle n|\frac{1}{1 + e^{\beta\left(H_0 - \kappa + \lambda V_\omega + \mathrm{Diag}\left(\Gamma_{\downarrow}\right)\right)}}|n\rangle\,. \tag{2.3.4}$$

$$\langle n|\Gamma_{\downarrow}|n\rangle = \langle n|\frac{1}{1+e^{\beta(H_0-\kappa+\lambda V_\omega+\mathrm{Diag}\left(\Gamma_{\uparrow}\right))}}|n\rangle\,. \tag{2.3.5}$$

Thus, the effective Hamiltonian on $\ell^2(\Lambda)\oplus\ell^2(\Lambda)$ is determined by

$$H_\omega^\Lambda := \begin{pmatrix} H_0+\lambda\omega(n)+gV_\uparrow^\Lambda(n) & 0 \\ 0 & H_0+\lambda\omega(n)+gV_\downarrow^\Lambda(n) \end{pmatrix}$$

$$V_\uparrow^\Lambda(\omega)(n) := \langle n|\frac{1}{1+e^{\beta(H_0-\kappa+\lambda\omega+gV_\downarrow)}}|n\rangle \tag{2.3.6}$$

$$V_\downarrow^\Lambda(\omega)(n) := \langle n|\frac{1}{1+e^{\beta(H_0-\kappa+\lambda\omega+gV_\uparrow)}}|n\rangle\,. \tag{2.3.7}$$

It will follow from arguments given below that if Λ_R is an increasing sequence with $\cup_{R\in\mathbb{N}}\Lambda_R=\mathbb{Z}^d$ then, for fixed $m\in\mathbb{Z}^d$,

$$\lim_{R\to\infty} V_{\mathrm{eff}}^{\Lambda_R}(m) = V_{\mathrm{eff}}(m) \tag{2.3.8}$$

and this fact ensures that, for localization purposes in the Hubbard model, it suffices to study H_{Hub} and its finite volume restrictions.

2.4 Outline of the Proof of theorem 2.2

We now want to outline the proof of the theorem 2.2 in the related model where H_{Hub} is replaced by the operator

$$H = H_0+\lambda\omega(n)+gV_{\mathrm{eff}}(n) \tag{2.4.1}$$

acting on $\ell^2\left(\mathbb{Z}^d\right)$ with

$$V_{\mathrm{eff}}(n) = \langle n|\frac{1}{1+e^{\beta(H_0+\lambda\omega+gV_{\mathrm{eff}})}}|n\rangle\,. \tag{2.4.2}$$

In this case, the eigenfunction correlator is defined as

$$Q_I(m,n) := \sup_{|\varphi|\le 1} |\langle m|\varphi(H)|n\rangle|. \tag{2.4.3}$$

where the supremum is taken over functions φ which are Borel measurable and supported on I.

The above operator exhibits the main mathematical features of the Hubbard model, namely: the effective potential is defined self-consistently as a non-local and non-linear function of H. Thus, it is natural to first illustrate our methods here. For now let's assume the existence and uniqueness of V_{eff} are proven, as well as its regularity with respect to $\{\omega(n)\}_{n\in\mathbb{Z}^d}$. Combined with estimates on the derivatives of V_{eff}, the above facts form a significant portion of the proof which is developed in sections 2.5 and 2.6. The, somewhat straightforward, extension of the proof to H_{Hub} will be explained in section 2.9. A feature which theorem 2.1 and theorem 2.2 have in common is that the eigenfunction correlator decay will be achieved via analysis of the Green's function of $H^\Lambda = \mathbb{1}_\Lambda H \mathbb{1}_\Lambda$, which is H restricted to a finite set $\Lambda \subset \mathbb{Z}^d$. Let

$$G^\Lambda(m,n,z) = \langle m|(H^\Lambda - z)^{-1}|n\rangle\,. \tag{2.4.4}$$

Using the basics of the fractional moment method, which dates back to [5] and [1], we aim at showing that, for some $s \in (0,1)$,

$$\mathbb{E}\left(\left|G^\Lambda(m,n;z)\right|^s\right) \le Ce^{-\mu_d|m-n|} \tag{2.4.5}$$

holds uniformly in $z \in \mathbb{C}^+$, with positive constants $C = C$ and μ depending on the parameters $(d,s,g,\lambda,\nu,\eta,\|F\|_\infty)$ but independent of the volume $|\Lambda|$. In this context, the Green's function decay expressed by equation (2.4.5) implies

$$\mathbb{E}\left(Q(m,n)\right) \le C'e^{-\mu_d'|m-n|} \tag{2.4.6}$$

for some exponent $\mu_d' = \mu'(d,s,g,\lambda,\nu,\eta,\|F\|_\infty) > 0$ and $C' = C'(\eta,\nu,d,g,\lambda,s,\|F\|_\infty)$. This is well known and explained in the introduction of this thesis.

Another aspect which is shared by the proofs of theorems 2.1 and 2.2 is that the starting point to obtain (2.4.5) will be the following *a-priori* bound.

Lemma 2.4. *Given a finite set $\Lambda \subset \mathbb{Z}^d$, there exist a constant C_{AP} depending on the parameters $(\eta,\nu,d,g,\lambda,s,\|F\|_\infty)$, but independent of Λ, such that*

$$\mathbb{E}\left(\left|G^\Lambda(m,n;z)\right|^s\right) \le C_{\text{AP}} \tag{2.4.7}$$

holds for any $m, n \in \Lambda$.

The proof of lemma 2.4 will follow from lemma 2.5 below. Let

$$U_\omega(n) = \omega(n) + \frac{g}{\lambda} V_{\text{eff}}(n, \omega). \tag{2.4.8}$$

be the "full" potential at site n. From now on, to keep the notation simple, we drop the dependence on ω in the new variables $\{U(n)\}_{n \in \Lambda}$. Note that $U(n)$ and $U(m)$ are correlated for all values of m and n. The strategy is to show that, for g sufficiently small, they still behave as if they were independent in the following sense:

Lemma 2.5. *Fix* $\Lambda \subset \mathbb{Z}^d$ *finite and* $n_0 \in \Lambda$. *The conditional distribution of* $U(n_0) = u$ *at specified values of* $\{U(n)\}_{n \in \Lambda \setminus \{n_0\}}$ *has density* $\rho^\Lambda_{n_0}$. *Moreover, under assumptions* $(A_1) - (A_4)$ *we have that*

$$\sup_\Lambda \sup_{n_0 \in \Lambda} \sup_{u \in \mathbb{R}} \rho^\Lambda_{n_0}(u) < \infty. \tag{2.4.9}$$

If, additionally, assumption (A_5) *holds then* $\rho^\Lambda_{n_0}(u) \in L^1(\mathbb{R}, |x|^\varepsilon dx)$.

The proof of the above result is detailed in section 2.8; it requires exponential decay of $|\frac{\partial V_{\text{eff}}(n)}{\partial \omega(m)}|$ and $|\frac{\partial^2 V_{\text{eff}}(n)}{\partial \omega(m) \omega(l)}|$ with respect to $|m - n|$ and $|m - n| + |l - n|$, respectively. The need for this decay is the main reason to require $\beta > 0$ or, in other words, to require analiticity of F in a strip. The intuitive explanation for lemma 2.5 is that the random variables $U(n)$ and $U(n_0)$ decorrelate in a strong fashion as $|n - n_0|$ becomes large. As explained in the introduction of this manuscript, lemma 2.5 implies (2.4.5) for any $0 < s < 1$ as long as λ is taken sufficiently large, see also [6, Theorem 10.2].

The proof of theorem 2.1 will require additional efforts involving tools which are specific to one dimension, which we shall comment on below.

2.5 Existence of the effective potential

To justify the definition of the effective potential in (2.4.1), let $\Phi(V) : \ell^\infty(\mathbb{Z}^d) \to \ell^\infty(\mathbb{Z}^d)$ be given by $\Phi(V)(n) := \langle n | F(T + \lambda V_\omega + gV) | n \rangle$. Recall that F is analytic, bounded in the strip $S = \{|\mathrm{Im} z| < \eta\}$ and continuous up to the boundary of S. Our goal is to check that Φ is a contraction in $\ell^\infty\left(\mathbb{Z}^d\right)$, meaning that

$$\|\Phi(V) - \Phi(W)\|_{\ell^\infty(\mathbb{Z}^d)} < c\|V - W\|_{\ell^\infty(\mathbb{Z}^d)} \tag{2.5.1}$$

holds for some $c < 1$ and all $V, W \in \ell^\infty(\mathbb{Z}^d)$. Let $R(z, H) = (H - z)^{-1}$ and T be a self-adjoint operator. Using the analiticity of F we have the following representation for $F(T)$ [4, Equation (D.2)]

$$F(T) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \left(R(t + i\eta, T) - R(t - i\eta, T)\right) f(t)\, dt \tag{2.5.2}$$

for all $V \in \ell^\infty(\mathbb{Z}^d)$, where $f = F_+ + F_- - D * F$ for $F_\pm(u) = F(u \pm i\eta)$ and $D(u) = \frac{\eta}{\pi\left(\eta^2+u^2\right)}$ is the Poisson kernel. As explained in [4], the above representation is a consequence of the identities

$$D_+ + D_- = \delta + D * D \tag{2.5.3}$$

(with δ denoting the Dirac delta) and

$$(D_+ + D_-) * F = D * (F_+ + F_-). \tag{2.5.4}$$

It follows immediately that $\|f\|_\infty \le 3\|F\|_\infty$. This is a prelude for the following fixed point argument, where the operator T will be assumed to satisfy

$$\sup_n \sum_m |T(m, n)| \left(e^{\nu|m-n|} - 1\right) < \frac{\eta}{2}. \tag{2.5.5}$$

Proposition 2.6. *The following contraction estimates hold true.*

- *For any self-adjoint operator T on $\ell^2\left(\mathbb{Z}^d\right)$ satisfying (2.5.5) and bounded potentials V, W, we have, for any $\nu' \in (0, \nu)$, that*

$$\left|\, \langle m|(F(T + V) - F(T + W))|n\rangle \,\right| \le \frac{72\sqrt{2}e^{-\nu'|m-n|}}{\eta\left(1 - e^{\nu'-\nu}\right)^d}\|F\|_\infty\|V - W\|_\infty. \tag{2.5.6}$$

- *For any self-adjoint operator T on $\ell^2\left(\mathbb{Z}^d\right) \oplus \ell^2\left(\mathbb{Z}^d\right)$ satisfying (2.5.5) and bounded potentials V, W on $\ell^2\left(\mathbb{Z}^d\right) \oplus \ell^2\left(\mathbb{Z}^d\right)$ we have, for any $\nu' \in (0,\nu)$, that*

$$\Big| \langle m|(F(T+V) - F(T+W))|n\rangle \Big| \leq \frac{144\sqrt{2}e^{-\nu'|m-n|}}{\eta\left(1-e^{\nu'-\nu}\right)^d} \|F\|_\infty \|V-W\|_\infty \tag{2.5.7}$$

- *For any $m, n, j \in \mathbb{Z}^d$, the matrix elements $\langle m|F(T+gV)|n\rangle$ are differentiable with respect to $V(j)$ and*

$$\Big|\frac{\partial \langle m|F(T+gV)|n\rangle}{\partial V(j)}\Big| \leq |g| \frac{72\sqrt{2}e^{-\nu(|m-j|+|n-j|)}}{\eta} \|F\|_\infty \|V\|_\infty. \tag{2.5.8}$$

Proof. The resolvent identity gives

$$\begin{aligned}
&\langle m|\frac{1}{T+V-t-i\eta} - \frac{1}{T+W-t-i\eta}|n\rangle + \langle m|\frac{1}{T+W-t+i\eta} - \frac{1}{T+V-t+i\eta}|n\rangle \\
&= \langle m|(\frac{1}{T+V-t-i\eta} - \frac{1}{T+V-t+i\eta})(W-V)\frac{1}{T+W-t-i\eta}|n\rangle \\
&- \langle m|\left(\frac{1}{T+W-t+i\eta} - \frac{1}{T+W-t-i\eta}\right)(W-V)\frac{1}{T+V-t+i\eta}|n\rangle\,.
\end{aligned}$$

Taking absolute values in the first term on the right-hand side we obtain

$$\begin{aligned}
&\Big| \langle m|\left(\frac{1}{T+V-t-i\eta} - \frac{1}{T+V-t+i\eta}\right)(W-V)\frac{1}{T+W-t-i\eta}|n\rangle \Big| \\
&\leq \sum_{l\in\mathbb{Z}^d} |G^V(m,l;t+i\eta) - G^V(m,l;t-i\eta)||(W-V)(l)|G^W(l,n;t+i\eta)| \\
&\leq 24\sum_l |(V-W)(l)|e^{-\nu(|l-n|+|l-m|)}\,\langle l|\frac{1}{(T+V-E)^2+\eta^2/2}|l\rangle^{1/2} \\
&\quad\times\ \langle m|\frac{1}{(T+V-t)^2+\eta^2/2}|m\rangle^{1/2}\,.
\end{aligned}$$

In the last step we made use of the Combes-Thomas bound $|G^W(m,n;t+i\eta)| \leq \frac{2}{\eta}e^{-\nu|m-n|}$ as well as lemma 3 in [4, appendix D] to estimate the difference between the Green functions as

$$
\begin{aligned}
&|G^V(m,l;t+i\eta) - G^V(m,l;t-i\eta)| \\
&\leq 12\eta e^{-\nu|m-l|}\,\langle m|\frac{1}{(T+V-t)^2+\eta^2/2}|m\rangle^{1/2}\ \langle l|\frac{1}{(T+V-E)^2+\eta^2/2}|l\rangle^{1/2}.
\end{aligned}
$$

Integrating over t we conclude, using Cauchy-Schwarz and the spectral measure representation, that

$$
\int_{-\infty}^{\infty}\left|\langle m|\left(\tfrac{1}{T+V-t-i\eta}-\tfrac{1}{T+V-t+i\eta}\right)(W-V)\tfrac{1}{T+W-E-i\eta}|n\rangle\right|dt \tag{2.5.9}
$$

$$
\leq \tfrac{24\sqrt{2}\pi}{\eta}\sum_l |(V-W)(l)|e^{-\nu(|l-n|+|l-m|)}. \tag{2.5.10}
$$

The above implies that

$$
\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\langle m|\left(\frac{1}{T+V-t-i\eta}-\frac{1}{T+V-t+i\eta}\right)(W-V)\frac{1}{T+W-t-i\eta}|n\rangle\right|dt \\
&\leq \frac{12\sqrt{2}}{\eta}\|V-W\|_\infty e^{-\nu'|m-n|}\sum_{l\in\mathbb{Z}^d}e^{(\nu'-\nu)|l-n|} \\
&= \frac{12\sqrt{2}}{\eta}\|V-W\|_\infty e^{-\nu'|m-n|}\frac{1}{\left(1-e^{\nu'-\nu}\right)^d}.
\end{aligned}
$$

As a similar bound holds for the term

$$
\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\langle m|\left(\frac{1}{H_0+W-t+i\eta}-\frac{1}{H_0+W-t-i\eta}\right)(V-W)\frac{1}{H_0+V-t+i\eta}|n\rangle\right|dt
$$

we conclude the proof of the inequality (2.5.6) by recalling that $\|f\|_\infty \leq 3\|F\|_\infty$. The inequality (2.5.7) follows from the same argument with the only difference that one has to sum two geometric series, hence the modification of the upper bound by a factor of two. The bound (2.5.8) is proven similarly: note that $|g||h|\frac{24\sqrt{2}\pi}{\eta}e^{-\nu|j-n|}e^{-\nu|m-j|}$ is an upper bound for the left-hand side of equation (2.5.9) with V replaced by gV and $W = g(V+hP_j)$, where

P_j denotes the projection onto Span$\{\delta_j\}$. We also observe that this time there will be no summation over l, hence the introduction of the ν' is unnecessary. We then conclude that

$$\left|\frac{\langle m|F(T+gV+hP_j)|n\rangle - \langle m|F(T+gV)|n\rangle}{h}\right| \le \frac{|g|72\sqrt{2}}{\eta} e^{-\nu|j-n|} e^{-\nu|m-j|}. \tag{2.5.11}$$

Letting $h \to 0$ finishes the proof. □

Taking $m = n$ we conclude that the contraction mapping condition (2.5.1) holds whenever

$$|g| < \frac{\eta\left(1-e^{-\nu}\right)^d}{72\sqrt{2}\|F\|_\infty}. \tag{2.5.12}$$

This observation yields the following.

Proposition 2.7. *Let* $g_d = \frac{\eta\left(1-e^{-\nu}\right)^d}{72\sqrt{2}\|F\|_\infty}$. *Then, for* $|g| < g_d$*, there is a unique effective potential* $V_{\text{eff}} \in \ell^\infty\left(\mathbb{Z}^d\right)$ *satisfying*

$$V_{\text{eff}}(n) = \langle n|F(H_0 + \lambda\omega + gV_{\text{eff}}|n\rangle\,. \tag{2.5.13}$$

Moreover, for $\Lambda \subset \mathbb{Z}^d$*, there is a unique* V_{eff}^Λ *in* $\ell^2(\Lambda)$ *satisfying (2.5.13) with* H *replaced by* $H^\Lambda = \mathbb{1}_\Lambda H \mathbb{1}_\Lambda$.

Remark 6. *Replacing* H_0 *by* $H_0 - \kappa I$ *we can incorporate a chemical potential in our results. For simplicity, we shall make no further reference to* κ *during the proofs and assume it was already incorporated to* H_0.

2.6 Regularity of the effective potential

Our goal in this section is to conclude that, for a fixed finite subset $\Lambda \subset \mathbb{Z}^d$ with $|\Lambda| = n$, the effective potential V_{eff} is a smooth function of $\{\omega(j)\}_{j\in\Lambda}$. This will be of relevance for several resampling arguments later in the thesis. For that purpose, define a map

$$\xi : \ell^\infty\left(\mathbb{Z}^d\right) \times \mathbb{R}^n \to \ell^\infty\left(\mathbb{Z}^d\right) \tag{2.6.1}$$

by

$$\xi(V,\omega)(j) = V(j) - \langle j|F(H_0 + \lambda\omega + gV)|j\rangle \tag{2.6.2}$$

Then, V_{eff} is the unique solution of $\xi(V,\omega) = 0$. Thus, its regularity can inferred via the implicit function theorem once we check that the derivative $D\xi(\cdot,\omega)$ is non-singular. Note that

$$\frac{\partial \xi(V,\omega)(j)}{\partial V(l)} = \delta_{jl} - \frac{\partial \langle j|F(H_0+\lambda\omega+gV)|j\rangle}{\partial V(l)}. \tag{2.6.3}$$

Using proposition 2.6, we have that

$$\Big|\frac{\partial \langle j|F(H_0+\lambda\omega+gV)|j\rangle}{\partial V(l)}\Big| \le |g|\frac{72\sqrt{2}e^{-2\nu|j-l|}}{\eta}\|F\|_\infty. \tag{2.6.4}$$

In particular, whenever $|g|\frac{72\sqrt{2}\|F\|_\infty}{\eta(1-e^{-2\nu})^d} < 1$ we have that the operator $D\xi(\omega,.) : \ell^\infty(\Lambda) \to \ell^\infty(\Lambda)$ is invertible since it has the form $I+gM$ where gM has operator norm less than one. It is worth observing that the smallness condition on g is independent of $\Lambda \subset \mathbb{Z}^d$. It is a consequence of the implicit function theorem that V is a smooth function of $(\omega(1),...,\omega(n))$.

2.7 Decay estimates for the effective potential

We start this section with the following lemma, which will be useful in order to formulate the decay of correlations between $U(n)$ and $U(m)$ as $|m-n| \to \infty$.

Lemma 2.8. *Whenever* $\frac{72\sqrt{2}|g|\|F\|_\infty}{\eta(1-e^{-\nu})^d} < 1$, *there exist constants* $C_1(d,\lambda,g,\eta,\|F\|_\infty,\nu)$ *and* $C_2(d,\lambda,g,\eta,\|F\|_\infty,\nu)$ *such that*

$$\max\Big\{\sum_m e^{\nu|n-m|}\Big|\frac{\partial V_{\text{eff}}(n)}{\partial\omega(m)}\Big|, \sum_n e^{\nu|n-m|}\Big|\frac{\partial V_{\text{eff}}(n)}{\partial\omega(m)}\Big|\Big\} \le C_1 \tag{2.7.1}$$

$$\sum_{l,m,n} e^{\nu(|l-n|+|n-m|+|l-m|)}\Big|\frac{\partial^2 V_{\text{eff}}(n)}{\partial\omega(m)\partial\omega(l)}\Big| \le C_2. \tag{2.7.2}$$

Moreover C_1 *and* C_2 *can be bounded from above by a constant of the form* $\frac{\lambda D}{1-g\theta}$ *with* D *and* θ *independent of* g *and these constants are explicit in the proof.*

Proof. For convenience of notation we let $V_{\text{eff}} = V$ in this proof. As in section 2.5 we write

$$F(H) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\left(\frac{1}{H-t-i\eta} - \frac{1}{H-t+i\eta}\right) f(t)\,dt$$

where f is bounded by $3\|F\|_\infty$. Thus

$$V(n,\omega) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} K(n,t,\omega) f(t)\,dt \tag{2.7.3}$$

where $K(n,t,\omega) = G(n,n;t-i\eta) - G(n,n;t+i\eta)$. Denote by P_m the projection mapping $\ell^2\left(\mathbb{Z}^d\right)$ onto $\ell^2\left(\mathrm{Span}\{\delta_m\}\right)$. Using difference quotients, it is easy to check

$$\frac{\partial}{\partial\omega(m)}\frac{1}{H-z} + g\frac{1}{H-z}\frac{\partial V}{\partial\omega(m)}\frac{1}{H-z} = -\lambda\frac{1}{H-z}P_m\frac{1}{H-z}. \tag{2.7.4}$$

Taking matrix elements we obtain

$$\frac{\partial K(n,t,\omega)}{\partial\omega(m)} = -g\sum_l \tilde{G}(l,n)\frac{\partial V(l)}{\partial\omega(m)} + \lambda r(m,n).$$

$$\tilde{G}(l,n) := G(l,n;t+i\eta)G(n,l;t+i\eta) - G(l,n;t-i\eta)G(n,l;t-i\eta).$$

$$r(m,n) := G(n,m;t+i\eta)G(m,n;t+i\eta) - G(n,m;t-i\eta)G(m,n;t-i\eta).$$

Note

$$\begin{aligned}\tilde{G}(l,n) &= \left(G(l,n;t+i\eta) - G(l,n;t-i\eta)\right)G(n,l;t+i\eta) \\ &\quad + \left(G(n,l;t+i\eta) - G(n,l;t-i\eta)\right)G(l,n;t-i\eta).\end{aligned} \tag{2.7.5}$$

We now make use of [4, Lemma 3]:

$$\begin{aligned}|G(l,n;t+i\eta) - G(l,n;t-i\eta)| &\le 12\eta e^{-\nu|l-n|}\,\langle n|\tfrac{1}{(H-t)^2+\eta^2/2}|n\rangle^{1/2} \\ &\quad \times \langle l|\tfrac{1}{(H-t)^2+\eta^2/2}|l\rangle^{1/2}.\end{aligned} \tag{2.7.6}$$

This, together with the Combes-Thomas bound $|G(l,n,t\pm i\eta)| \le \frac{2}{\eta}e^{-\nu|l-n|}$ and (2.7.5) implies

$$|\tilde{G}(l,n)| \le 48e^{-2\nu|l-n|}\,\langle n|\frac{1}{(H-t)^2+\eta^2/2}|n\rangle^{1/2}\,\langle l|\frac{1}{(H-t)^2+\eta^2/2}|l\rangle^{1/2}.$$

$$|r(m,n)| \le 48e^{-2\nu|m-n|}\,\langle m|\frac{1}{(H-t)^2+\eta^2/2}|m\rangle^{1/2}\,\langle n|\frac{1}{(H-t)^2+\eta^2/2}|n\rangle^{1/2}.$$

Thus

$$\tilde{K}(l,n) := \int_{-\infty}^{\infty} |\tilde{G}(l,n)|\, dt \leq \frac{48\sqrt{2}\pi}{\eta} e^{-2\nu|l-n|}$$

$$\tilde{r}(m,n) := \int_{-\infty}^{\infty} |r(m,n)|\, dt \leq \frac{48\sqrt{2}\pi}{\eta} e^{-2\nu|m-n|}.$$

To summarize, we have shown the following inequality

$$\Big|\frac{\partial V(n)}{\partial \omega(m)}\Big| \leq \frac{3\|F\|_\infty}{2\pi}\left(|g|\sum_l \tilde{K}(l,n)\Big|\frac{\partial V(l)}{\partial \omega(m)}\Big| + \lambda \tilde{r}(m,n)\right).$$

Whenever $\frac{72\sqrt{2}|g|}{\eta\left(1-e^{-2\nu}\right)^d} < 1$ we have that

$$|g|\|\tilde{K}\|_{\infty,\infty} < 1 \tag{2.7.7}$$

where

$$\|\tilde{K}\|_{\infty,\infty} = \sup_l \sum_m \tilde{K}(l,m). \tag{2.7.8}$$

Considering the weight $W(n) := e^{\nu|m-n|}$ we let

$$\theta := \frac{3\|F\|_\infty}{2\pi} \sup_n \sum_l \frac{W(n)}{W(l)} \tilde{K}(n,l). \tag{2.7.9}$$

By the triangle inequality,

$$\begin{aligned}\theta &\leq \frac{3\|F\|_\infty}{2\pi} \sup_n \sum_l e^{\nu|n-l|}\tilde{K}(n,l) \\ &\leq \frac{72\sqrt{2}\|F\|_\infty}{\eta\left(1-e^{-\nu}\right)^d}.\end{aligned}$$

hence, whenever $\frac{72\sqrt{2}|g|}{\eta(1-e^{-\nu})^d} < 1$, we have that

$$|g|\theta < 1. \tag{2.7.10}$$

Moreover, with the choice

$$D_1 := \sum_n W(n)\tilde{r}(m,n) \tag{2.7.11}$$

we have

$$D_1 \leq \frac{72\sqrt{2}\|F\|_\infty}{\eta\left(1-e^{-\nu}\right)^d}. \tag{2.7.12}$$

After conditions (2.7.7), (2.7.10) and (2.7.12) have been verified, the general result [6, Theorem 9.2] applies, yielding

$$\sum_m e^{\nu|n-m|}\left|\frac{\partial V(n)}{\partial\omega(m)}\right| < \frac{\lambda D_1}{1-g\theta} := C_1(d,\|F\|_\infty,\lambda,g,\eta,\nu). \tag{2.7.13}$$

Differentiating (2.7.4) with respect to $\omega(l)$,

$$\begin{aligned}
&\frac{\partial^2}{\partial\omega(m)\partial\omega(l)}\frac{1}{H-z} + g\left(\frac{\partial}{\partial\omega(l)}\frac{1}{H-z}\right)\frac{\partial V}{\partial\omega(m)}\frac{1}{H-z}\\
&+g\frac{1}{H-z}\frac{\partial V}{\partial\omega(m)}\left(\frac{\partial}{\partial\omega(l)}\frac{1}{H-z}\right) + g\frac{1}{H-z}\frac{\partial^2 V}{\partial\omega(m)\partial\omega(l)}\frac{1}{H-z}\\
&= -\lambda\left(\frac{\partial}{\partial\omega(l)}\frac{1}{H-z}\right)P_m\frac{1}{H-z} - \lambda\frac{1}{H-z}P_m\left(\frac{\partial}{\partial\omega(l)}\frac{1}{H-z}\right)
\end{aligned}$$

Repeating the previous argument and using the established decay of $\frac{\partial V(n)}{\partial\omega(m)}$ we reach (2.7.2), finishing the proof. □

Given a finite set $\Lambda \subset \mathbb{Z}^d$, let us define $\mathcal{T} : \mathbb{R}^{|\Lambda|} \to \mathbb{R}^{|\Lambda|}$ by

$$\left(\mathcal{T}\omega\right)(n) = \omega(n) + \frac{g}{\lambda}V_{\text{eff}}(n). \tag{2.7.14}$$

Let $U(n) := \left(\mathcal{T}\omega\right)(n)$ be the new coordinates in the probability space. The bound (2.7.1) implies that, for $|g|$ sufficiently small, $\mathcal{T}$ is a differentiable perturbation of the identity by an operator with norm less than one hence $\mathcal{T}^{-1}$ is well defined.

Fix $n_0 \in \Lambda$ and denote by $U_\alpha = U + \left(\alpha - U(n_0)\right)\delta_{n_0}$ the new potential obtained from U by changing its value at n_0 to $\alpha \in \mathbb{R}$. Let $\omega_\alpha(n) = \left(\mathcal{T}^{-1}U_\alpha\right)(n)$. The variables $\omega_\alpha(n)$ correspond to the change in $\omega(n)$ when a resampling argument is applied to the new probability space at the point n_0. Intuitively, the exponential decay shown above guarantees that this change is not too large if n and n_0 are far away. This is the content of the result below.

Lemma 2.9. *Given $\alpha \in \mathbb{R}$ and $|g| < \lambda C_1^{-1}$, we have*

$$\sum_{n \neq n_0} e^{\nu|n-n_0|}\big|\omega_\alpha(n) - \omega(n)\big| \leq \frac{C_1|g|}{\lambda}\frac{\left(|\alpha - U(n_0)| + 2\frac{|g|\|F\|_\infty}{\lambda}\right)}{\left(1 - \frac{|g|}{\lambda}C_1\right)}.$$

where C_1 is the upper bound in equation (2.7.1).

Proof. Using the given definitions and the mean value inequality we obtain, for $n \neq n_0$,

$$\begin{aligned}
|\omega(n) - \omega_\alpha(n)| &\leq \frac{|g|}{\lambda}|V(n,\omega) - V(n,\omega_\alpha)| \\
&\leq \frac{|g|}{\lambda}\sum_{l \in \mathbb{Z}^d}\Big|\frac{\partial V_{\text{eff}}(n,\hat{\omega}_\alpha)}{\partial \omega(l)}\Big|\Big|\omega_\alpha(l) - \omega(l)\Big| \\
&\leq \frac{|g|}{\lambda}\Big|\frac{\partial V_{\text{eff}}(n,\hat{\omega}_\alpha)}{\partial \omega(n_0)}\Big|\left(|\alpha - U(n_0)| + 2\frac{|g|\|F\|_\infty}{\lambda}\right) + \\
&\frac{|g|}{\lambda}\sum_{l \neq n_0}\Big|\frac{\partial V_{\text{eff}}(n,\hat{\omega}_\alpha)}{\partial \omega(l)}\Big|\Big|\omega_\alpha(l) - \omega(l)\Big|.
\end{aligned}$$

Where $\hat{\omega}_\alpha$ denotes some configuration in the probability space with $\hat{\omega}_\alpha(l)$ in the interval connecting $\omega(l)$ to $\omega_\alpha(l)$. Let $W(n) = e^{\nu|n-n_0|}$. According to (2.7.1),

$$\begin{aligned}
\sup_n \sum_l \frac{W(n)}{W(l)}\Big|\frac{\partial V_{\text{eff}}(n,\hat{\omega}_\alpha)}{\partial \omega(l)}\Big| &\leq \sup_n \sum_l e^{\nu|n-l|}\Big|\frac{\partial V_{\text{eff}}(n,\hat{\omega}_\alpha)}{\partial \omega(l)}\Big| \\
&\leq C_1.
\end{aligned}$$

Once again, the conditions of [6, Theorem 9.2] are satisfied for $|g| < \lambda C_1^{-1}$ therefore

$$\sum_{n \neq n_0} e^{\nu|n-n_0|}\big|\omega_\alpha(n) - \omega(n)\big| \leq \frac{C_1|g|}{\lambda}\frac{\left(|\alpha - U(n_0)| + 2\frac{|g|\|F\|_\infty}{\lambda}\right)}{\left(1 - \frac{|g|}{\lambda}C_1\right)}.$$

□

Since another application of the mean value theorem gives, after a possible correction on $\hat{\omega}_\alpha$ that

$$\Big|\frac{\partial V_{\text{eff}}(n,\omega)}{\partial \omega(m)} - \frac{\partial V_{\text{eff}}(n,\omega_\alpha)}{\partial \omega(m)}\Big| \leq \sum_{l \in \mathbb{Z}^d}\Big|\frac{\partial^2 V_{\text{eff}}(n,\hat{\omega}_\alpha)}{\partial \omega(l)\partial \omega(m)}\Big|\Big|\omega(l) - \omega_\alpha(l)\Big|$$

we obtain that, for any $\nu' \in (0,\nu)$, the difference $\left|\frac{\partial V_{\text{eff}}(n,\omega)}{\partial \omega(m)} - \frac{\partial V_{\text{eff}}(n,\omega_\alpha)}{\partial \omega(m)}\right|$ is bounded from above by

$$\frac{C_2|g|}{\lambda}\left(C_1\frac{\left(|\alpha - U(n_0)| + 2\frac{|g|\|F\|_\infty}{\lambda}\right)}{\left(1-\frac{|g|}{\lambda}C_1\right)(1-e^{\nu'-\nu})^d} + 2\|F\|_\infty\right)e^{-\nu'(|m-n|+|n-n_0|+|m-n_0|)}.$$

where C_2 is the constant in (2.7.2).

In particular, letting $\nu' = \nu/2$ and with the choices

$$A = \frac{g}{\lambda}\left(\frac{\partial V_{\text{eff}}(n_i,\omega_\alpha)}{\partial \omega(n_j)}\right)_{|\Lambda|\times|\Lambda|}, \quad B = \frac{g}{\lambda}\left(\frac{\partial V_{\text{eff}}(n_i,\omega)}{\partial \omega(n_j)}\right)_{|\Lambda|\times|\Lambda|}$$

we have

$$\sum_{(m,n)\in\Lambda\times\Lambda}|(A-B)_{m,n}| \leq \frac{C_2|g|^2}{\lambda^2}\left(C_1\frac{\left(|\alpha - U(n_0)| + 2\frac{|g|\|F\|_\infty}{\lambda}\right)}{\left(1-\frac{|g|}{\lambda}C_1\right)(1-e^{-\nu/2})^{3d}} + \frac{2\|F\|_\infty}{(1-e^{-\nu/2})^{2d}}\right). \tag{2.7.15}$$

We summarize the above observation as a lemma.

Lemma 2.10. *Let* $A = \frac{g}{\lambda}\left(\frac{\partial V_{\text{eff}}(n_i,\omega_\alpha)}{\partial \omega(n_j)}\right)_{|\Lambda|\times|\Lambda|}$ *and* $B = \frac{g}{\lambda}\left(\frac{\partial V_{\text{eff}}(n_i,\omega)}{\partial \omega(n_j)}\right)_{|\Lambda|\times|\Lambda|}$. *Whenever* $\frac{|g|}{\lambda}C_1 < 1$ *we have*

$$\sum_{(m,n)\in\Lambda\times\Lambda}|(A-B)_{m,n}| \leq |g|^2\left(C_3|\alpha - U(n_0)| + C_4\right). \tag{2.7.16}$$

Moreover, the constant C_3 *can be chosen independent of* λ *and* C_4 *is proportional to* $\frac{1}{\lambda}$.

Finally, we analyze how the effective potential varies with respect to disorder and interaction. This will be relevant to the Integrated Density of States regularity with respect to disorder and interaction strengths developed in chapter four.

Lemma 2.11. *For a fixed* $\omega \in \Omega$

$$|V_{\lambda,g}(n) - V_{\lambda',g'}(n)| \leq \frac{C_5(d,\|F\|_\infty,g,\eta,\nu,\omega)}{1-gC_6(d,\|F\|_\infty,g,\eta,\nu)}|\lambda-\lambda'| + C_7(d,\|F\|_\infty,g,\eta,\nu)|g-g'|. \tag{2.7.17}$$

Note when $\lambda \neq \lambda'$ the bound depends on ω through the constant C_5.

Proof. Let $R_{\lambda,g}(z) = \frac{1}{H_0+\lambda\omega+gV_{\lambda,g}-z}$ and $R_{\lambda',g'}(z) = \frac{1}{H_0+\lambda'\omega+g'V_{\lambda',g'}-z}$ for $z = t + i\eta$. Similarly as in the above proofs, it is immediate to check that

$$\begin{aligned} R_{\lambda,g}(z) - R_{\lambda',g'}(z) =& (\lambda' - \lambda) R_{\lambda,g}(z) V_\omega R_{\lambda',g'}(z) + (g' - g) R_{\lambda,g}(z) V_{\lambda',g'} R_{\lambda',g'}(z) \\ &+ g R_{\lambda,g}(z) \left(V_{\lambda,g} - V_{\lambda',g'} \right) R_{\lambda',g'}(z). \end{aligned}$$

Replacing z by $\bar{z}$ and subtracting the resulting equations:

$$\begin{aligned} &\left(R_{\lambda,g}(z) - R_{\lambda,g}(\bar{z})\right) - \left(R_{\lambda',g'}(z) - R_{\lambda',g'}(\bar{z})\right) = \\ &\left(R_{\lambda,g}(z) - R_{\lambda,g}(\bar{z})\right) \left((\lambda' - \lambda)V_\omega + (g' - g)V_{\lambda',g'}\right) R_{\lambda',g'}(z) \\ &+ R_{\lambda,g}(\bar{z}) \left((\lambda' - \lambda)V_\omega + (g' - g)V_{\lambda',g'}\right) \left(R_{\lambda',g'}(z) - R_{\lambda',g'}(\bar{z})\right) \\ &+ g \left(R_{\lambda,g}(z) - R_{\lambda,g}(\bar{z})\right) \left(V_{\lambda,g} - V_{\lambda',g'}\right) R_{\lambda',g'}(z) \\ &+ g R_{\lambda,g}(z) \left(V_{\lambda,g} - V_{\lambda',g'}\right) \left(R_{\lambda',g'}(z) - R_{\lambda',g'}(\bar{z})\right). \end{aligned}$$

Taking matrix elements, multiplying by $f(t)$, integrating with respect to t and taking absolute values we can read from the representation (2.7.3) that, denoting

$$K_{\lambda,g}(n,l) = |G_{\lambda,g}(n,l;z) - G_{\lambda,g}(n,l;\bar{z})|, \tag{2.7.18}$$

$$\begin{aligned} &|V_{\lambda,g}(n) - V_{\lambda',g'}(n)| \leq \\ &\frac{3\|F\|_\infty}{2\pi} |\lambda - \lambda'| \sum_{l\in\mathbb{Z}^d} |\omega(l)| \int_{-\infty}^{\infty} \left(K_{\lambda,g}(n,l) K_{\lambda',g'}(l,n) + |G_{\lambda,g}(n,l)| \tilde{K}_{\lambda',g'}(l,n)\right) dt \\ &+ \frac{3\|F\|_\infty^2}{2\pi} |g - g'| \sum_{l\in\mathbb{Z}^d} \int_{-\infty}^{\infty} \left(|G_{\lambda,g}(n,l)| K_{\lambda',g'}(l,n) + G_{\lambda',g'}(n,l)| K_{\lambda,g}(l,n)\right) dt. \\ &+ 2g \sum_{l\in\mathbb{Z}^d} \int_{-\infty}^{\infty} |G_{\lambda,g}(n,l)| |V_{\lambda,g}(l) - V_{\lambda',g'}(l)| K_{\lambda',g'}(l,n)\, dt. \end{aligned}$$

Using equation (2.7.6) together with [6, Theorem 9.2] we conclude the proof. □

2.7.1 Improvements

We will now improve upon the previous bounds. More specifically, we need robust estimates which also reflect the decay of the derivatives of $V_{\text{eff}}(n)$ when the local potential $\omega(n)$ is large. The content of this section will be important for a general fluctuation analysis on section 2.8 and for complete localization in the one dimensional setting. Before stating the main result of the section we start with the following deterministic estimate, which incorporates ideas from [4, Lemma 3].

Lemma 2.12.

$$|G(m,l;t+i\eta)| \leq \sqrt{2}\,\langle l|\frac{1}{(H-t)^2+\eta^2}|l\rangle^{1/2}\, e^{-\nu|m-l|} \tag{2.7.19}$$

Proof. To keep the notation simple, we set $t=0$ without loss of generality. Let $H_f = e^f H e^{-f}$ where $f(n) = \nu\min\{|n-l|, R\}$ for a fixed $l \in \mathbb{Z}^d$ and $R > 0$. By choosing R sufficiently large, we may assume that $|m-l| < R$. We then have

$$e^{\nu|m-l|}G(m,l;i\eta) = \langle m|(H_f - i\eta)^{-1}|l\rangle\,.$$

We claim that

$$||(H_f - i\eta)^{-1}(H^2+\eta^2)^{1/2}|| \leq \sqrt{2}. \tag{2.7.20}$$

Indeed,

$$\begin{aligned}||(H_f - i\eta)^{-1}(H^2+\frac{\eta^2}{2})^{1/2}||^2 &= ||(H^2+\frac{\eta^2}{2})^{1/2}(H_f^*+i\eta)^{-1}(H_f-i\eta)^{-1}(H^2+\frac{\eta^2}{2})^{1/2}|| \\ &= ||(H^2+\frac{\eta^2}{2})^{1/2}\frac{1}{(H_f-i\eta)(H_f^*+i\eta)}(H^2+\frac{\eta^2}{2})^{1/2}||\end{aligned}$$

where by [4, Eq D.9] (with f replaced by $-f$) we have

$$(H_f-i\eta)(H_f^*+i\eta) \geq \frac{1}{2}\left(H^2+\frac{\eta^2}{2}\right)$$

showing the claim in (2.7.20). The inequality (2.7.19) will now follow from

$$\begin{aligned}|\,\langle m|(H_f-i\eta)^{-1}|l\rangle\,| &\leq \|(H_f-i\eta)^{-1}(H^2+\frac{\eta^2}{2})^{1/2}\|\,|(H^2+\frac{\eta^2}{2})^{-1/2}\delta_l| \\ &\leq \sqrt{2}\,\langle l|(H^2+\frac{\eta^2}{2})^{-1}|l\rangle^{1/2}\,.\square\end{aligned}$$

Lemma 2.13. *There exists $C_7(\lambda, \eta, d, g, \|F\|_\infty, \nu) > 0$ such that, for $m \neq n$,*

$$\max\{\,|\omega(n)|, |\omega(m)|\,\}\left|\frac{\partial V(n)}{\partial \omega(m)}\right| \leq C_7 e^{-2\nu|m-n|} \tag{2.7.21}$$

and, for $n \neq n_0$,

$$\left|\omega(n)(\omega_\alpha(n) - \omega(n))\right| \leq \frac{C_7|g|}{\lambda - |g|C_1}\left(|\alpha - U(n_0)| + 2\frac{|g|\|F\|_\infty}{\lambda} + \frac{1}{(1-e^{-\nu})^d}\right) e^{-\nu|n-n_0|}. \tag{2.7.22}$$

Moreover, whenever $\frac{|g|}{\lambda}C_1 < 1$, C_7 can be chosen to be uniformly bounded as a function of the parameters λ and g.

Proof. Recall that $U(n) = \omega(n) + \frac{g}{\lambda}V_{\text{eff}}(n)$ denotes the "full" potential at site n. We divide the proof into two cases.

- Case one: $U(n) \geq 0$.

  Let us start by noting that lemma 2.7.19 implies that for $n, l \in \mathbb{Z}^d$

  $$\int_{-\infty}^{\infty} |G(n,l;t+i\eta)G(l,n;t+i\eta)|\,dt \leq \frac{2\sqrt{2}\pi}{\eta}e^{-2\nu|n-l|}.$$

  From the previous section we already know that

  $$\frac{\partial V(n)}{\partial \omega(m)} = \int_{-\infty}^{\infty}\left(-g\sum_l r(n,l)\frac{\partial V(l)}{\partial \omega(m)} + \lambda r(m,n)\right) f(t)\,dt \tag{2.7.23}$$

  where $f(t) = F_+(t+i\eta) + F_-(t-i\eta) - D * F(t)$ and

  $$r(m,n) = G(n,m;t+i\eta)G(m,n;t+i\eta) - G(n,m;t-i\eta)G(m,n;t-i\eta).$$

  Observe that, for $z = t + i\eta$ and $n \neq m$,

  $$\begin{aligned}
  &\lambda|U(n)G(n,m;z)G(m,n;z)| \\
  &= \frac{\lambda|U(n)|}{|\lambda U(n) - z|}\sum_l |H_0(n,l)G(l,m;z)G(m,n,z)| \\
  &\leq \left(1 + \frac{|z|}{|\lambda U(n) - z|}\right)\sum_l |H_0(n,l)G(l,m;z)G(m,n,z)|
  \end{aligned}$$

where we made use of the identity

$$(\lambda U(n) - z)G(n,m;z) = \delta_{mn} - \sum_l H_0(n,l)G(l,m;z). \tag{2.7.24}$$

Note that if $U(n) \geq 0$ and $t = \text{Re}z < 0$, then

$$\frac{|z|}{|\lambda U(n) - z|} \leq 1. \tag{2.7.25}$$

Using the fact that $tf(t)$ goes to zero as $t \to \infty$ we conclude that

$$\int_{-\infty}^{\infty} |U(n)G(n,m;t+i\eta)G(m,n;t+i\eta)||f(t)|\,dt \leq \frac{C(\nu,\eta,\|F\|_\infty)}{\lambda} e^{-2\nu|m-n|}. \tag{2.7.26}$$

Since a similar equation holds with m replaced by l, we can proceed as in the previous section and, using the exponential decay of $\frac{\partial V(n)}{\partial \omega(m)}$, conclude the proof.

- Case two: $U(n) < 0$.

In this case, the argument given above must be modified to take into account that the inequality

$$\frac{|z|}{|\lambda U(n) - z|} \leq 1. \tag{2.7.27}$$

is satisfied when $t = \text{Re} > 0$. In this case, the immediate use of (2.7.23) would result in a problem as $tf(t)$ is unbounded as $t \to -\infty$. This can be addressed by observing that the Fermi-Dirac function $F(z)$ has the following symmetry

$$\frac{1}{1 + e^{\beta(z-\mu)}} = 1 - \frac{1}{1 + e^{\beta(-z+\mu)}}. \tag{2.7.28}$$

Hence we can make use of the representation (2.7.23) corresponding to

$$-\frac{1}{1 + e^{\beta(-z+\mu)}} := F_\mu^*(z) \tag{2.7.29}$$

since, for $m \neq n$, the constant term does not affect the calculation of $\frac{\partial V(n)}{\partial \omega(m)}$. Denoting by

$$f^*(t) = F_+^*(t + i\eta) + F_-^*(t - i\eta) - D * F^*(t)$$

we reach

$$\frac{\partial V(n)}{\partial \omega(m)} = \int_{-\infty}^{\infty} \left(-g \sum_{l} r(n,l) \frac{\partial V(l)}{\partial \omega(m)} + r(m,n) \right) f^*(t)\, dt \tag{2.7.30}$$

where now $t f^*(t) \to 0$ as $t \to -\infty$. Proceeding as in the first case the proof is finished, showing (2.7.21). Following the proof of lemma 2.9 and using (2.7.21) we conclude (2.7.22) □

Lemma 2.14. *Let Λ_1 and Λ_2 be subsets of $\mathbb{Z}^d$ with $dist(\Lambda_1, \Lambda_2) \geq r$. Let V_2^c be the effective potential defined by*

$$V_2^c(n) = \langle n | F(H^{\Lambda_2^c}) | n \rangle \quad \text{for } n \in \mathbb{Z}^d$$

where $H^{\Lambda_2^c}$ denotes the restriction of H to the complement of Λ_2.

Then, for any $n \in \Lambda_1$

$$\left| \frac{\partial V(n)}{\partial \omega(m)} - \frac{\partial V_2^c(n)}{\partial \omega(m)} \right| \leq C(\eta, d, \lambda, g, \|F\|_\infty, \nu) e^{-\nu(|m-n|+r)} \tag{2.7.31}$$

Proof. The proof follows the same steps as in the previous results. The only modification which is required comes when comparing the quantities $r(m,n)$ and $r^{\Lambda_2^c}(m,n)$ given by

$$r(m,n) = G(n,m;t+i\eta)G(m,n;t+i\eta) - G(n,m;t-i\eta)G(m,n;t-i\eta)$$

$$r^{\Lambda_2^c}(m,n) = G^{\Lambda_2^c}(n,m;t+i\eta)G^{\Lambda_2^c}(m,n;t+i\eta) - G^{\Lambda_2^c}(n,m;t-i\eta)G^{\Lambda_2^c}(m,n;t-i\eta).$$

We observe that

$$G(n,m;z)G(m,n;z) - G^{\Lambda_2^c}(n,m;z)G^{\Lambda_2^c}(m,n;z) =$$
$$G(n,m;z)\left(G(m,n;z) - G^{\Lambda_2^c}(m,n;z)\right) + \left(G(n,m;z) - G^{\Lambda_2^c}(n,m;z)\right) G^{\Lambda_2^c}(m,n;z).$$

Moreover,

$$G(m,n;z) - G^{\Lambda_2^c}(m,n;z) = -\lambda \sum_{l \in \Lambda_2} G(m,l;z)\omega(l) G^{\Lambda_2^c}(l,n;z)$$
$$+ \frac{g}{\lambda} \sum_{l} G(m,l;z)\left(V_{\text{eff}}(l) - V_2^c(l)\right) G^{\Lambda_2^c}(l,n;z).$$

Since $V_{\text{eff}}(l) - V_2^c(l)$ can be expressed in terms of Green's functions of H and $H^{\Lambda_2^c}$ the proof is now finished using arguments identical to the proof of lemma 2.8 and the improvement on lemma 2.13.

□

2.8 Proof of lemma 2.5

We now verify the existence of the density $\rho_{n_0}^{\Lambda}$. Fix $\Lambda \subset \mathbb{Z}^d$ finite. Recall that we have defined

$$U(n, \omega) = \omega(n) + \frac{g}{\lambda} V_{\text{eff}}(n, \omega). \tag{2.8.1}$$

Until the end of this section we suppress the ω dependence on $U(n)$ and V_{eff}. Note that, for $m, n \in \Lambda$,

$$\frac{\partial U(m)}{\partial \omega(n)} = \delta_{mn} + \frac{g}{\lambda} \frac{\partial V_{\text{eff}}(m)}{\partial \omega(n)}. \tag{2.8.2}$$

We have denoted the above change of variables by $\mathcal{T} : \mathbb{R}^{|\Lambda|} \to \mathbb{R}^{|\Lambda|}$, which reads

$$\mathcal{T}(\omega(n_1), ..., \omega(n_{|\Lambda|})) = (U(n_1), ..., U(n_{|\Lambda|})). \tag{2.8.3}$$

It is now simple compute the joint distribution of the variables $\{U(n)\}_{n \in \mathbb{Z}^d}$. Using the fact that the random variables $\{\omega(n)\}_{n \in \mathbb{Z}^d}$ have a common density ρ we conclude that for all Borel sets $I_1, ..., I_N$ in $\mathbb{R}$:

$$\mathbb{P}\left(U(n_1) \in I_1, \ldots, U(n_{|\Lambda|}) \in I_{|\Lambda|}\right) =$$

$$\int_{\mathcal{T}^{-1}\left(I_1 \times \ldots \times I_{|\Lambda|}\right)} \prod_{k=1}^{|\Lambda|} \rho(\omega(n_k))\, d\omega(n_k)$$

$$= \int_{I_1 \times \ldots \times I_{|\Lambda|}} |\det J_{\mathcal{T}^{-1}}| \prod_{k=1}^{|\Lambda|} \rho\left(\mathcal{T}^{-1} U(n_k)\right)\, dU(n_1) \ldots dU(n_{|\Lambda|})$$

$$= \int_{I_1 \times \ldots \times I_{|\Lambda|}} \left|\det\left(I + \frac{g}{\lambda} \frac{\partial V_{\text{eff}}(n_i, \mathcal{T}^{-1} U)}{\partial U(n_j)}\right)\right| \prod_{k=1}^{|\Lambda|} \rho\left(U(n_k) - \frac{g}{\lambda} V_{\text{eff}}(n_k, \mathcal{T}^{-1} U)\right)\, dU(n_k).$$

Therefore the joint distribution of $\{U(n_k)\}_{k=1}^{|\Lambda|}$ is given by the measure

$$\Big|\det\left(I+\frac{g}{\lambda}\frac{\partial V_{\text{eff}}(n_i,\mathcal{T}^{-1}U)}{\partial U(n_j)}\right)\Big|\prod_{k=1}^{|\Lambda|}\rho\left(U(n_k)-\frac{g}{\lambda}V_{\text{eff}}(n_k,\mathcal{T}^{-1}U)\right)\,dU(n_1)...dU(n_{|\Lambda|}). \tag{2.8.4}$$

It follows that for each $n_0\in\Lambda$ the conditional distribution of $U(n_0)$ at specified values of $\{U(n)\}_{n\neq n_0}$ has a density given by

$$\rho_{n_0}^{\Lambda}=\frac{\big|\det\left(I+\frac{g}{\lambda}\frac{\partial V_{\text{eff}}(n_i,\mathcal{T}^{-1}U)}{\partial U(n_j)}\right)\big|\prod_{k=1}^{|\Lambda|}\rho\left(U(n_k)-\frac{g}{\lambda}V_{\text{eff}}(n_k,\mathcal{T}^{-1}U)\right)}{\int_{-\infty}^{\infty}\big|\det\left(I+\frac{g}{\lambda}\frac{\partial V_{\text{eff}}(n_i,\mathcal{T}^{-1}U^{\alpha})}{\partial U(n_j)}\right)\big|\prod_{k=1}^{|\Lambda|}\rho\left(U^{\alpha}(n_k)-\frac{g}{\lambda}V_{\text{eff}}(n_k,\mathcal{T}^{-1}U^{\alpha})\right)\,d\alpha} \tag{2.8.5}$$

Where $U^{\alpha}(n):=U(n)+(\alpha-U(n_0))\,\delta_{n=n_0}$. This strategy naturally leads to the analysis of ratios of determinants. A sufficient condition for finding an upper bound to the right-hand side of (2.8.5) is to obtain positive constants $C=C_{\text{fluct}}(U(n_0))$ and $D=D(\alpha)$ which are independent of $|\Lambda|$ and such that the following estimates hold true

$$\frac{\big|\det\left(I+\frac{g}{\lambda}\frac{\partial V_{\text{eff}}(n_i,\mathcal{T}^{-1}U^{\alpha})}{\partial U(n_j)}\right)\big|}{\big|\det\left(I+\frac{g}{\lambda}\frac{\partial V_{\text{eff}}(n_i,\mathcal{T}^{-1}U)}{\partial U(n_j)}\right)\big|}\geq D(\alpha). \tag{2.8.6}$$

$$\int_{-\infty}^{\infty}D(\alpha)\rho\left(\alpha-\frac{g}{\lambda}V_{\text{eff}}(n_0,\mathcal{T}^{-1}U^{\alpha})\right)d\pi^{\alpha}\geq C_{\text{fluct}}(U(n_0)) \tag{2.8.7}$$

where

$$d\pi^{\alpha}:=\prod_{n\in|\Lambda|\setminus\{n_0\}}\frac{\rho\left(U^{\alpha}(n)-\frac{g}{\lambda}V_{\text{eff}}(n,\mathcal{T}^{-1}U^{\alpha})\right)}{\rho\left(U(n)-\frac{g}{\lambda}V_{\text{eff}}(n,\mathcal{T}^{-1}U)\right)}\,d\alpha. \tag{2.8.8}$$

The bounds (2.8.7) and (2.8.6) readily imply that letting $U(n_0)=u$

$$\rho_{n_0}^{\Lambda}(u)\leq\frac{\rho\left(u-\frac{g}{\lambda}V_{\text{eff}}(n_0,\mathcal{T}^{-1}U)\right)}{C_{\text{fluct}}(u)}. \tag{2.8.9}$$

Lemma 2.5 will follow from a precise control of the right-hand side of equation (2.8.9).

We now execute the strategy which was outlined above. The ratio of determinants can be controlled through the following bound, where $\|M\|_1$ denotes the trace norm of a matrix M.

Lemma 2.15. *Let A, B be square matrices with $I + B$ invertible. Then,*

$$\left|\frac{\det(I+A)}{\det(I+B)}\right| \leq e^{\|(A-B)(I+B)^{-1}\|_1}. \tag{2.8.10}$$

Proof. We make use of the elementary identities

$$\frac{\det(I+A)}{\det(I+B)} = \det(I+A)(I+B)^{-1} \tag{2.8.11}$$

and

$$(I+A)(1+B)^{-1} = I + (A-B)(I+B)^{-1}. \tag{2.8.12}$$

The proof is now finished with the inequality

$$|\det(1+M)| \leq e^{\|M\|_1}$$

which holds in the general setting of trace class operators, see [49, Lemma 3.3] □

The triangle inequality for the trace norm implies the following.

Corollary 2.16. *Under the above conditions*

$$\left|\frac{\det(I+B)}{\det(I+A)}\right| \geq e^{-\sum_{m,n}\left|\left((A-B)(I+B)^{-1}\right)_{mn}\right|} \tag{2.8.13}$$

Letting $A = \frac{g}{\lambda}\left(\frac{\partial V(n_i,\omega)}{\partial U(n_j)}\right)_{|\Lambda|\times|\Lambda|}$, $B = \frac{g}{\lambda}\left(\frac{\partial V(n_i,\omega_\alpha)}{\partial U(n_j)}\right)_{|\Lambda|\times|\Lambda|}$ and using lemma 2.8 we see that, for $|g| < \lambda C_1^{-1}$, $(1+B)^{-1}$ has uniformly bounded operator norm. Using lemma 2.10 and corollary 2.16 we conclude that (2.8.6) holds with $D(\alpha) = e^{-|g|^2 C_3(|\alpha - U(n_0)| + C_4)}$.

We now check that equation (2.8.7) holds when ρ satisfies the fluctuation bound (2.2.4). We divide the proof in two cases:

- Suppose that $c_2(\rho) > 0$.

  Let $c_\rho = \max\{c_1(\rho), c_2(\rho)\}$. The left-hand side of (2.8.7) is bounded from below by

$$\int_{-\infty}^{\infty} D(\alpha)\rho\left(\alpha - \frac{g}{\lambda}V_{\text{eff}}(n_0, \mathcal{T}^{-1}U^\alpha)\right) \prod_{n\in\Lambda\setminus\{n_0\}} e^{-c_\rho|\omega(n)-\omega_\alpha(n)|(1+|\omega(n)|+|\omega_\alpha(n)|)}\, d\alpha$$

which equals

$$\int_{-\infty}^{\infty} D(\alpha)\rho\left(\alpha - \frac{g}{\lambda}V_{\text{eff}}(n_0, \mathcal{T}^{-1}U^{\alpha})\right) e^{S(n)}\, d\alpha$$

for

$$S(n) = \sum_{n\in\Lambda\setminus\{n_0\}} -c_\rho|\omega(n) - \omega_\alpha(n)|(1 + |\omega(n)| + |\omega_\alpha(n)|\,). \tag{2.8.14}$$

Due to the triangle inequality and lemmas 2.9 and 2.13, we conclude that there is a positive constant $C = C(d, \|F\|_\infty, g, \eta, \nu)$ with $\lim_{g\to 0} C(d, \|F\|_\infty, g, \eta, \nu) < \infty$ such that for $n \neq n_0$

$$\begin{aligned}
&- c_\rho|\omega(n) - \omega_\alpha(n)|\left(1 + |\omega(n)| + |\omega_\alpha(n)|\right) \geq \\
&- c_\rho|\omega(n) - \omega_\alpha(n)|\left(1 + 2|\omega(n)| + |\omega_\alpha(n) - \omega(n)|\right) \\
&\geq -|g|c_\rho e^{-\nu|n-n_0|}\left(C^2|\alpha - U(n_0)|^2 + 2C|\alpha - U(n_0)|\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&- c_\rho \sum_{n\in\Lambda\setminus\{n_0\}} |\omega(n) - \omega_\alpha(n)|\left(1 + |\omega(n)| + |\omega_\alpha(n)|\right) \geq \\
&- |g|\frac{2c_\rho}{(1-e^{-\nu})^d}\left(C^2|\alpha - U(n_0)|^2 + 2C|\alpha - U(n_0)|\right)).
\end{aligned}$$

Thus, by choosing $|g|$ sufficiently small so that $|g|\frac{4c_\rho}{(1-e^{-\nu})^d}C^2 < \bar{c}_\rho$ and using the assumption (A_4) we obtain that the integral below is finite and bounded from below by a positive constant independent of Λ and n_0.

$$\int_{-\infty}^{\infty} D(\alpha)\frac{\rho\left(\alpha - \frac{g}{\lambda}V_{\text{eff}}(n_0, \mathcal{T}^{-1}U^{\alpha})\right)}{\rho\left(U(n_0) - \frac{g}{\lambda}V_{\text{eff}}(n_0, \mathcal{T}^{-1}U)\right)}e^{S(n)}\, d\alpha.$$

where $S(n)$ was defined in (2.8.14). This, together with (2.8.9), verifies lemma 2.5 when $c_2(\rho) > 0$. If ρ satisfies the assumption (A_5) the above argument yields $\rho_{n_0}^{\Lambda}(u) \in L^1(\mathbb{R}, |u|^\varepsilon du)$.

- Assume that $c_2(\rho) = 0$:

  Similarly to the above argument, the left-hand side of (2.8.7) is bounded from below by

$$\int_{-\infty}^{\infty} D(\alpha) \frac{\rho\left(\alpha - \frac{g}{\lambda} V_{\text{eff}}(n_0, \mathcal{T}^{-1}U^{\alpha})\right)}{\rho\left(U(n_0) - \frac{g}{\lambda} V_{\text{eff}}(n_0, \mathcal{T}^{-1}U)\right)} e^{-|g| \frac{c_1(\rho)}{(1-e^{-\nu})^d} C|\alpha - U(n_0)|}. \tag{2.8.15}$$

  Where, from (2.2.4)

$$\frac{\rho\left(\alpha - \frac{g}{\lambda} V_{\text{eff}}(n_0, \mathcal{T}^{-1}U^{\alpha})\right)}{\rho\left(U(n_0) - \frac{g}{\lambda} V_{\text{eff}}(n_0, \mathcal{T}^{-1}U)\right)} \geq e^{-c_1(\rho)\left(|\alpha - U(n_0)| + 2\frac{|g|}{\lambda}\right)}. \tag{2.8.16}$$

  Again, choosing $|g|$ sufficiently small we conclude that

$$0 < \int_{-\infty}^{\infty} D(\alpha) \frac{\rho\left(\alpha - \frac{g}{\lambda} V_{\text{eff}}(n_0, \mathcal{T}^{-1}U^{\alpha})\right)}{\rho\left(U(n_0) - \frac{g}{\lambda} V_{\text{eff}}(n_0, \mathcal{T}^{-1}U)\right)} \prod_{n \in \Lambda \setminus \{n_0\}} e^{-c_1(\rho)|\omega(n) - \omega_\alpha(n)|} \, d\alpha < \infty.$$

  finishing the proof.

2.9 The Hartree approximation for the Hubbard model

Let us now explain how the techniques from the previous sections can adapted to the Hubbard model. Recall that H_{Hub} is defined as

$$\begin{pmatrix} H_{\uparrow}(\omega) & 0 \\ 0 & H_{\downarrow}(\omega) \end{pmatrix} := \begin{pmatrix} H_0 + \lambda V_\omega + g V_{\uparrow}(\omega) & 0 \\ 0 & H_0 + \lambda V_\omega + g V_{\downarrow}(\omega) \end{pmatrix}$$

acting on $\ell^2\left(\mathbb{Z}^d\right) \oplus \ell^2\left(\mathbb{Z}^d\right)$. The operators H_0 and V_ω are defined as before, i.e; $H_0 + \lambda V_\omega$ is the standard Anderson model acting on $\ell^2\left(\mathbb{Z}^d\right)$. The effective potentials are given by

$$\begin{pmatrix} V_{\uparrow}(\omega)(n) \\ V_{\downarrow}(\omega)(n) \end{pmatrix} = \begin{pmatrix} \langle n|F(H_{\downarrow})|n\rangle \\ \langle n|F(H_{\uparrow})|n\rangle \end{pmatrix}. \tag{2.9.1}$$

Mathematically, the treatment of the above model is very similar to the one explained above, therefore most details are skipped and we just indicate the required modifications.

2.9.1 Existence of the Effective potential

Let $\Phi(X,Y) : \ell^\infty\left(\mathbb{Z}^d\right) \oplus \ell^\infty\left(\mathbb{Z}^d\right) \to \ell^\infty\left(\mathbb{Z}^d\right) \oplus \ell^\infty\left(\mathbb{Z}^d\right)$ be given by

$$\Phi(X,Y)(m,n) := \left(\langle n|F(H_0 + V_\omega + gY)|n\rangle \, , \, \langle m|F(H_0 + V_\omega + gX)|m\rangle \right).$$

using proposition 2.6, we immediately reach

$$\|\Phi(X_1,Y_1) - \Phi(X_2,Y_2)\|_{\ell^\infty\left(\mathbb{Z}^d\right)\oplus\ell^\infty\left(\mathbb{Z}^d\right)} \leq$$
$$|g|\frac{72\sqrt{2}}{\eta\left(1-e^{\nu'-\nu}\right)^d}\|F\|_\infty\left(\|X_1 - X_2\|_{\ell^\infty\left(\mathbb{Z}^d\right)} + \|Y_1 - Y_2\|_{\ell^\infty\left(\mathbb{Z}^d\right)}\right).$$

Therefore, if $|g|\frac{72\sqrt{2}}{\eta\left(1-e^{\nu'-\nu}\right)^d}\|F\|_\infty < 1$ we conclude Φ has a unique fixed point

$$V_{\text{eff}} = \left(V_\uparrow, V_\downarrow\right) \tag{2.9.2}$$

belonging to $\ell^\infty\left(\mathbb{Z}^d\right) \oplus \ell^\infty\left(\mathbb{Z}^d\right)$.

2.9.2 Regularity of the effective potential

Fix $\Lambda \subset \mathbb{Z}^d$ finite and define functions $\xi : \left(\ell^\infty(\Lambda) \oplus \ell^\infty(\Lambda)\right) \times \mathbb{R}^n \to \ell^\infty(\Lambda) \oplus \ell^\infty(\Lambda)$ through

$$\xi^\uparrow(V,\omega)(j) = V^\uparrow(j) - \langle j|F(H_0 + \lambda\omega + gV_\downarrow)|j\rangle. \tag{2.9.3}$$

$$\xi^\downarrow(V,\omega)(j) = V^\downarrow(j) - \langle j|F(H_0 + \lambda\omega + gV_\uparrow)|j\rangle. \tag{2.9.4}$$

Our goal is to conclude $V^\uparrow$,$V^\downarrow$ are smooth functions of an arbitrary, but finite, list $\{\omega(j)\}_{j\in\Lambda}$. Again, this can be done via implicit function theorem once we check that the derivative

$$\frac{\partial\xi(\omega,V)(j)}{\partial V(l)} = \delta_{jl} - \frac{\partial\langle j|F(H_0 + \lambda\omega + gV)|j\rangle}{\partial V(l)}. \tag{2.9.5}$$

is non-singular. Using lemma 2.6, we have that for $\sharp \in \{\uparrow,\downarrow\}$

$$\left|\frac{\partial\langle j|F(H_0 + \lambda\omega + gV_\sharp)|j\rangle}{\partial V(l)}\right| \leq |g|\frac{72\sqrt{2}e^{-2\nu|j-l|}}{\eta}\|F\|_\infty. \tag{2.9.6}$$

In particular, whenever $|g|\frac{144\sqrt{2}\|F\|_\infty}{\eta(1-e^{-2\nu})^d} < 1$ we have that the operator $D\xi(\cdot,\omega) : \ell^\infty(\Lambda)\oplus\ell^\infty(\Lambda) \to \ell^\infty(\Lambda)\oplus\ell^\infty(\Lambda)$ has an inverse. From the implicit function theorem it follows that V is a smooth function of $(\omega(1),...,\omega(n))$ for $n=|\Lambda|$.

2.9.3 Decay estimates

The decay rate in the case of the Hubbard model is dictated by

$$\Big|\frac{\partial V_\uparrow(n)}{\partial\omega(m)}\Big| \leq 3|g|\|F\|_\infty \sum_l \tilde{K_\downarrow}(l,m)\Big|\frac{\partial V_\downarrow(l)}{\partial\omega(m)}\Big| + \tilde{r_\downarrow}(n). \tag{2.9.7}$$

$$\Big|\frac{\partial V_\downarrow(n)}{\partial\omega(m)}\Big| \leq 3|g|\|F\|_\infty \sum_l \tilde{K_\uparrow}(l,m)\Big|\frac{\partial V_\uparrow(l)}{\partial\omega(m)}\Big| + \tilde{r_\uparrow}(n). \tag{2.9.8}$$

where, for $\sharp\in\{\uparrow,\downarrow\}$

$$\tilde{G_\sharp}(l,m) := G_\sharp(l,n;t+i\eta)G_\sharp(n,l;t+i\eta) - G_\sharp(l,n;t-i\eta)G_\sharp(n,l;t-i\eta).$$

$$r_\sharp(m,n) := G_\sharp(n,m;t+i\eta)G_\sharp(m,n;t+i\eta) - G_\sharp(n,m;t-i\eta)G_\sharp(m,n;t-i\eta).$$

$$\tilde{K_\sharp}(l,m) := \int_{-\infty}^{\infty} |\tilde{G_\sharp}(l,m)|\,dt.$$
$$\tilde{r_\sharp}(n) := \int_{-\infty}^{\infty} |r_\sharp(n)|\,dt.$$

In particular,

$$\Big|\frac{\partial V_\uparrow(n)}{\partial\omega(m)}\Big| + \Big|\frac{\partial V_\downarrow(n)}{\partial\omega(m)}\Big| \leq 3|g|\|F\|_\infty \sum_l \left(\tilde{K_\uparrow}(l,m)+\tilde{K_\downarrow}(l,m)\right)\left(\Big|\frac{\partial V_\uparrow(l)}{\partial\omega(m)}\Big| + \Big|\frac{\partial V_\downarrow(l)}{\partial\omega(m)}\Big|\right)$$
$$+ \left(\tilde{r_\uparrow}(n,m)+\tilde{r_\downarrow}(n,m)\right).$$

The analysis from the previous sections applies and we obtain lemmas 2.8,2.9,2.10 and 2.13 with $|.|$ being replaced by the matrix norm $|M| = |M_{11}|+|M_{21}|$ for $M = \begin{pmatrix} M_{11} \\ M_{21}\end{pmatrix}$. The effective potential and its derivatives are to be interpreted as follows:

$$V_{\text{eff}}(n) = \begin{pmatrix} V^{\uparrow}_{\text{eff}}(n) \\ V^{\downarrow}_{\text{eff}}(n) \end{pmatrix}, \quad \frac{\partial V_{\text{eff}}(n)}{\partial \omega(m)} = \begin{pmatrix} \frac{\partial V^{\uparrow}_{\text{eff}}(n)}{\partial \omega(m)} \\ \frac{\partial V^{\downarrow}_{\text{eff}}(n)}{\partial \omega(m)} \end{pmatrix} \text{ and } \frac{\partial^2 V_{\text{eff}}(n)}{\partial \omega(m) \partial \omega(l)} = \begin{pmatrix} \frac{\partial^2 V^{\uparrow}_{\text{eff}}(n)}{\partial \omega(m) \omega(l)} \\ \frac{\partial^2 V^{\downarrow}_{\text{eff}}(n)}{\partial \omega(m) \omega(l)} \end{pmatrix}.$$

2.10 One dimensional Aspects:proof of theorem 2.1

In this section we will prove theorem 2.1. We let $H_+ = H_{[0,\infty)\cap\mathbb{Z}}$ and denote by $G^+(m,n;z)$ the Green's function of H^+. Recall the definition of the Lyapunov exponent:

$$\mathcal{L}(z) = -\mathbb{E}\left(\ln |G^+(0,0;z)|\right) \tag{2.10.1}$$

$$\mathcal{L}_{\text{And}}(z) = -\mathbb{E}\left(\ln |G^+_{\text{And}}(0,0;z)|\right). \tag{2.10.2}$$

Recall in this case $H_0 = -\Delta$ hence, we define H_{Hub} acting on $\left(\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})\right)$ by

$$H_{\text{Hub}} = \begin{pmatrix} H_{\uparrow}(\omega) & 0 \\ 0 & H_{\downarrow}(\omega) \end{pmatrix} \tag{2.10.3}$$

where, denoting by H_{And} the standard Anderson model $-\Delta + V_\omega$ on $\ell^2(\mathbb{Z})$,

$$\begin{pmatrix} H_{\uparrow}(\omega) & 0 \\ 0 & H_{\downarrow}(\omega) \end{pmatrix} := \begin{pmatrix} H_{\text{And}} + gV_{\uparrow}(\omega) & 0 \\ 0 & H_{\text{And}} + gV_{\downarrow}(\omega) \end{pmatrix}. \tag{2.10.4}$$

The effective potentials are defined as (2.2.8). In the theorem below, we will use an abbreviation and $\mathcal{L}(z)$ will refer to the Lyapunov exponent of either $H_\uparrow$ or $H_\downarrow$ whereas $\mathcal{L}_{\text{And}}(z)$ will denote the Lyapunov exponent of the Anderson model on $\ell^2(\mathbb{Z})$.

2.11 One dimensional aspects: strategy of the proof of theorem 2.1

The argument for proving theorem 2.1 follows closely the approach in the proof of theorem 12.11 in [6], which we now recall.

2.11.1 Main ideas in the i.i.d case

In the reference [6, Chapter 12] the decay rate for the Green's function is described in terms of the moment generating function, defined by

$$\varphi(s,z) = \lim_{|n|\to\infty} \frac{\ln \mathbb{E}(|G(0,n;z)|^s)}{|n|}. \tag{2.11.1}$$

The existence of the above quantity for all $z \in \mathbb{C}^+$ and $s \in (0,1)$ and its relationship to the Lyapunov exponent are a consequence of Fekete's lemma:

Lemma 2.17 (Fekete)**.** *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that, for every pair (m,n) of natural numbers,*

$$a_{n+m} \le a_n + a_m. \tag{2.11.2}$$

Then, $\alpha = \lim_{n\to\infty} \frac{a_n}{n}$ exists and equals $\inf_{n\in\mathbb{N}} \frac{a_n}{n}$.

It is an elementary observation that if, instead, the sequence $\{a_n\}_{n\in\mathbb{N}}$ satisfies the inequality $a_{n+m} \le a_n + a_m + C$ then the above result applies to $b_n := a_n + C$ and that an analogue statement holds for superadditive sequences, i.e, sequences which satisfy (2.11.2) but with the reversed inequality sign. In the i.i.d. context, the sequence $a_n = \ln \mathbb{E}\left(|G(0,n;z)|^s\right)$ is shown to be both subbaditive and superadditive, meaning that there exist constants $C_-(s,z)$ and $C_+(s,z)$ for which

$$a_n + a_m + C_- \le a_{n+m} \le a_n + a_m + C_+. \tag{2.11.3}$$

holds for all $m,n \in \mathbb{N}$, see [6, Lemma 12.10]. A consequence of this fact, together with a precise control of the arising constants, is stated below where we operate under the assumption that the random variables $\{\omega(n)\}_{n\in\mathbb{Z}}$ have a density ρ which satisfies $\rho \in L^1\left(\mathbb{R}; |u|^\varepsilon du\right)$ with $\varepsilon \in (0,1)$.

Lemma 2.18. *[6, Theorem 12.8] For any $z \in \mathbb{C}^+$, there are $c_s(z), C_s(z) \in (0,\infty)$ such that for all $n \in \mathbb{Z}$*

$$c_s^{-1}(z) e^{\varphi(s,z)|n|} \le \mathbb{E}\left(|G_{\text{And}}(0,n;z)|^s\right) \le C_s(z) e^{\varphi(s,z)|n|}. \tag{2.11.4}$$

Moreover, for any compact set $K \subset \mathbb{R}$ and $S \subset [-\varepsilon, 1)$, we have the local uniform bound

$$\sup_{s\in S} \sup_{z\in K+i(0,1]} \max\{c_s(z), C_s(z)\} < \infty \tag{2.11.5}$$

and the same result holds with z replaced by its boundary value $E + i0$ for Lebesgue almost every E.

On the other hand, for fixed $z \in \mathbb{C}^+$, $\varphi(s,z)$ is shown to be a convex function of s and non-increasing in $[-\varepsilon, +\infty)$, with its derivative at $s = 0$ satisfying $\frac{\partial\varphi(0,z)}{\partial s} = -\mathcal{L}(z)$. It is a consequence of these facts that for almost every $E \in \mathbb{R}$ there exists a value $s = s(E) \in (0,1)$ such that

$$\varphi(s,E) \leq -\frac{s}{2}\mathcal{L}(E). \tag{2.11.6}$$

The above is the content of [6, Equation (12.86)]. Dynamical localization is shown to hold locally as a consequence of the inequality (2.11.3) along with lemma 2.18, the inequality (2.11.6) and Kotani theory, which establishes that $\mathcal{L}(E)$ is positive for almost every $E \in \mathbb{R}$.

2.11.2 Modifications

In this section we will outline the proof theorem 2.1 with H_{Hub} again replaced by the operator H on $\ell^2\left(\mathbb{Z}^d\right)$ defined in (2.4.1). For simplicity we set $\lambda = 1$ since the disorder strenght does not play an important role in theorem 2.1. Let $H^+ = H_{[0,\infty)\cap\mathbb{Z}}$ be the restriction of H to $\ell^2\left(\mathbb{Z}^+\right)$ and denote by $G^+(m,n;z)$ the Green's function of H^+. Recall the definition of the Lyapunov exponent: initially, for $z \in \mathbb{C}^+$, we let

$$\mathcal{L}(z) = -\mathbb{E}\left(\ln|G^+(0,0;z)|\right). \tag{2.11.7}$$

By Herglotz theory (see, for instance, [6, Appendix B] and references therein) it is seen that, for Lebesgue almost every $E \in \mathbb{R}$, $\mathcal{L}(E)$ is well defined as $\lim_{\delta\to 0^+} \mathcal{L}(E + i\delta)$. Finally, recall the uniform positivity of the Lyapunov exponent for the Anderson model on $\ell^2(\mathbb{Z})$:

$$\operatorname{ess}\inf_{E\in\mathbb{R}} \mathcal{L}_{\text{And}}(E) > \mathcal{L}_{\text{And}} \tag{2.11.8}$$

for some $\mathscr{L}_{\text{And}} > 0$. The first step towards Green's function decay (2.4.5) will be showing uniform positivity of $\mathscr{L}(E)$, which is accomplished by the following.

Theorem 2.19. *There exists a constant* $C_{\text{Lyap}}(s, \eta, g, \|F\|_\infty) > 0$ *such that*

$$|\mathscr{L}(z) - \mathscr{L}_{\text{And}}(z)| \le C_{\text{Lyap}} |g|^s \tag{2.11.9}$$

for all $z \in \mathbb{C}^+$.

Proof. From the resolvent identity we obtain

$$\frac{|G^+(0,0;z)|}{|G^+_{\text{And}}(0,0;z)|} \le 1 + |g| \|F\|_\infty \sum_n |G^+(0,n;z)| \frac{|G^+_{\text{And}}(n,0;z)|}{|G^+_{\text{And}}(0,0;z)|} \tag{2.11.10}$$

$$\frac{|G^+_{\text{And}}(0,0;z)|}{|G^+(0,0;z)|} \le 1 + |g| \|F\|_\infty \sum_n |G^+_{\text{And}}(0,n;z)| \frac{|G^+(n,0;z)|}{|G^+(0,0;z)|} \tag{2.11.11}$$

Using the bound $\ln(1+x) \le \frac{x^s}{s}$ for $0 < s < 1$ and $x > 0$ we reach, for $0 < s < 1/2$,

$$\ln \left(\frac{|G^+(0,0;z)|}{|G^+_{\text{And}}(0,0;z)|} \right) \le \frac{|g|^s}{s} \|F\|_\infty^s \sum_n |G^+(0,n;z)|^s \frac{|G^+_{\text{And}}(n,0;z)|^s}{|G^+_{\text{And}}(0,0;z)|^s}. \tag{2.11.12}$$

Taking expectations, using the definition of the Lyapunov exponents and the Cauchy-Schwarz inequality

$$\begin{aligned} \mathscr{L}_{\text{And}}(z) - \mathscr{L}(z) \le \quad & \tfrac{|g|^s}{s} \|F\|_\infty^s \sup\nolimits_n \mathbb{E}\left(|G^+(0,n;z)|^{2s}\right)^{1/2} \textstyle\sum_n \mathbb{E}\left(\frac{|G^+_{\text{And}}(n,0;z)|^{2s}}{|G^+_{\text{And}}(0,0;z)|^{2s}} \right)^{1/2} \\ & := C_{\text{Lyap}}(s, \eta, \nu, \|F\|_\infty) |g|^s. \end{aligned} \tag{2.11.13}$$

The fact that C_{Lyap} is a finite quantity follows from a couple of remarks. Firstly, by Feenberg's expansion [6, Theorem 6.2] we have the identity

$$|G^+_{\text{And}}(n,0;z)| = |G^+_{\text{And}}(0,0;z)| |G^+_{\text{And}}(1,n;z)| \tag{2.11.14}$$

where $G^+_{\text{And}}(1,n;z)$ denotes the Green's function of H_{And} restricted to $\ell^2(\mathbb{Z}) \cap [1, \infty)$. From the *a-priori* fractional moment bound on lemma 2.4 combined with the Green's function decay for dimensional Anderson model

$$\mathbb{E}\left(|G^+_{\text{And}}(1,n;z)|^{2s}\right) < C(s) e^{-\mu_{\text{And}}|n|} \tag{2.11.15}$$

we conclude that that $C_{\text{Lyap}} < \infty$. The estimate for $\mathscr{L}(z) - \mathscr{L}_{\text{And}}(z)$ is similar. ☐

In principle one might worry that the pre-factor C_{Lyap} on the above bound will depend on g. However, it is easy to see from the arguments in the proof of lemma 2.4, that C_{AP} converges to a finite quantity as $g \to 0$, thus we shall disregard its dependence on g.

Corollary 2.20. *Whenever* $|g| < \left(\frac{\mathcal{L}_{\text{And}}}{C_{\text{Lyap}}}\right)^{1/s}$ *holds for some* $s \in (0, 1/2)$*, we have*

$$\mathcal{L}_0 := \operatorname{ess}\inf_{E\in\mathbb{R}} \mathcal{L}(E) > 0. \tag{2.11.16}$$

We can now proceed to the second step of the proof of theorem 2.1, which consists of establishing Green's function decay from corollary 2.20. For that purpose, an important detail to keep in mind is that, in the correlated context, if we choose $a_n = \log \mathbb{E}\left(|G(0, n; z)|^s\right)$, the condition (2.11.3) will not be fulfilled for all pairs (m, n) due to the lack of independence between the potentials. This means that Fekete's lemma is not applicable. Moreover, its well-studied modifications (for instance by P. Erdös and N. G. de Bruijn [24]) do not seem to suffice either.

To the best of our knowledge the result given below is new. Its formulation takes into account the strong decorrelation between the potentials in the Hubbard model and introduces a notion of approximate subbaditivity.

Lemma 2.21 (Fekete-type lemma for approximately subbaditive sequences)**.** *Let* $\delta > 0$ *be given and* $\{a_n\}_{n\in\mathbb{N}}$ *be a sequence of real numbers such that, for every triplet* m, n, r *of natural numbers with* $r \geq \delta \max\{\log m, \log n\}$*, the inequality*

$$a_{n+m+r} \leq a_n + a_m + C \tag{2.11.17}$$

holds with a constant C *independent of* m, n *and* r*. Then,*

$$\alpha = \lim_{n\to\infty} \frac{a_n}{n} \tag{2.11.18}$$

exists and equals $\inf_{n\in\mathbb{N}} \frac{a_n+C}{n}$ *. Moreover,* $\alpha \in [-\infty, 0]$*.*

Note that, as a consequence, we have

$$a_n \geq n\alpha - C \tag{2.11.19}$$

for all $n \in \mathbb{N}$, where C is the same constant as in (2.11.17). The following decoupling estimate guarantees the applicability of the above lemma with the choice

$$a_n = \log \mathbb{E}_{[0,n]} \left(|\hat{G}(0,n;z)|^s \right)$$

where $\hat{G}(0,n;z) = \langle 0 | (H_{[0,n]} - z)^{-1} | n \rangle$ is the Green's function of the operator H restricted to $\ell^2 ([0,n] \cap \mathbb{Z})$ and $\mathbb{E}_{[0,n]}$ denotes the expectation with respect to $U(0), ..., U(n)$.

Lemma 2.22. *[Strong mixing decoupling] There exist constants* $C_{\mathrm{Dec}}(s, \nu, \eta, g, \|F\|_\infty)$ *and* $\delta = \delta(\eta, \nu, g, \|F\|_\infty)$ *such that the inequality*

$$\mathbb{E}_{[0,n+m+r]} \left(|\hat{G}(0,n+m+r;z)|^s \right) \leq C_{\mathrm{Dec}} \mathbb{E}_{[0,n]} \left(|\hat{G}(0,n;z)|^s \right) \mathbb{E}_{[0,m]} \left(|\hat{G}(0,m;z)|^s \right) \tag{2.11.20}$$

holds whenever $r \geq \delta \log \max\{m, n\}$.

A combination of lemmas 2.22, 2.21 and equation (2.11.19) yields the lower bound

$$C_{\mathrm{Dec}}^{-1} e^{\varphi(s,z)n} \leq \mathbb{E}_{[0,n]} \left(|\hat{G}(0,n;z)|^s \right) \quad \text{for all } n \in \mathbb{N}. \tag{2.11.21}$$

As we shall see in section 2.11.5 below, an application of the lower bound (2.11.21) in combination with the superadditive version of lemma 2.21 applied to the sequence $b_n = -\varphi(s,E)n + \log \mathbb{E}_{[0,n]} \left(|\hat{G}(0,n;z)|^s \right)$ is enough to establish an upper bound

$$\mathbb{E} \left(|\hat{G}(0,n;z)|^s \right) \leq C(s,z) e^{\varphi(s,z)n} \quad \text{for all } n \in \mathbb{N}. \tag{2.11.22}$$

where the constant $C(s,z)$ is locally uniform in $(s,z) \in (0,1) \times \mathbb{C}^+$.

After obtaining an analogue of lemma 2.18, the final step will be to relate the moment-generating function to the Lyapunov exponent through an inequality of the type

$$\varphi(s,E) \leq -\frac{s}{2} \mathcal{L}_0. \tag{2.11.23}$$

In reference [6], the bound (2.11.23) is stated with $\mathcal{L}_0$ replaced by $\mathcal{L}(E)$ and with s depending on E. However, it is easy to see from the from the arguments given there that s can be chosen locally uniformly in E, see [6, Equations (12.79) and (12.80)]. Moreover, by making use uniform positivity of the Lyapunov exponent obtained in corollary 2.20 we reach the inequality (2.11.23). The Green's function decay follows from the bounds (2.11.22) and (2.11.23).

2.11.3 Proof of Lemma 2.21

For simplicity we set $C = 0$. The general statement will follow by considering the related sequence $b_n := a_n + C$. Given integers L and ℓ with $L >> \ell$, our goal is to bound $\frac{a_L}{L}$ from above in terms of $\frac{a_\ell}{\ell}$. As a initial step, observe that by (2.11.17) we have

$$a_L \leq a_{L-\delta \log L - \ell} + a_\ell.$$

Iterating the above procedure $k+1$ times for

$$k = k_{\ell,L} := \lfloor \frac{L - 2\ell - \delta \log L}{\delta \log L + \ell} \rfloor \tag{2.11.24}$$

we obtain

$$a_L \leq (k+2)\, a_\ell$$

In the above iteration we have made use of the fact that in the assumption (2.11.17) the remainder r can be adjusted as long as it satisfies the inequality given there. Thus,

$$\frac{a_L}{L} \leq \frac{\ell(k+2)}{L} \frac{a_\ell}{\ell} \tag{2.11.25}$$

Before proceeding with the proof, a few remarks are in order. Firstly, nothing is achieved by holding ℓ fixed and letting $L \to \infty$ directly on equation (2.11.25) since this only yields the upper bound of zero. A second attempt would be showing that letting $\ell \to \infty$ (hence, $L \to \infty$ as well) implies that the ratio $\frac{k\ell}{L}$ converges to one. However, as

$$q_{\ell,L} - \frac{\ell}{L} \leq \frac{k\ell}{L} \leq q_{\ell,L} \tag{2.11.26}$$

for the choice

$$q_{\ell,L} = \frac{1 - 2\frac{\ell}{L} - \delta \frac{\log L}{L}}{1 + \frac{\delta \log L}{\ell}} \tag{2.11.27}$$

we see that $\frac{k\ell}{L}$ converges to one as $\ell \to \infty$ only along a subsequence where

$$\frac{\ell}{L} \to 0 \quad \text{and} \quad \frac{\log L}{\ell} \to 0. \tag{2.11.28}$$

Taking this into account, let $\varepsilon > 0$ be given and ℓ_1 be an initial scale to be determined. Let $L >> \ell_1$ be a positive integer to be interpreted as a larger scale. Iterating equation (2.11.25) throughout a sequence of scales

$$\ell_1 < \ell_2 < ... < \ell_{N_L} \leq L < \ell_{N_L+1} < ... \tag{2.11.29}$$

satisfying, for some $p > 0$,

$$p \log\left(\ell_j\right) \leq \log \ell_{j+1} < p^2 \log\left(\ell_j\right). \tag{2.11.30}$$

and

$$\sum_{j=1}^{\infty} \frac{\ell_j}{\ell_{j+1}} < \infty \tag{2.11.31}$$

we reach, for $q_{\ell,L}$ defined in (2.11.27),

$$\frac{a_L}{L} \leq \left(q_{\ell_{N_L},L} + \frac{2\ell_{N_L}}{L}\right) \prod_{j=1}^{N_L-1} \left(q_{\ell_j,\ell_{j+1}} + \frac{2\ell_j}{\ell_{j+1}}\right) \frac{a_{\ell_1}}{\ell_1}. \tag{2.11.32}$$

Since $q_{\ell_j,\ell_{j+1}} \to 1$ as $j \to \infty$, due to (2.11.31), we conclude that the value of ℓ_1 can be chosen (independently of L) so that

$$\sum_{j=1}^{N_L-1} \log\left(q_{\ell_j,\ell_{j+1}} + \frac{2\ell_j}{\ell_{j+1}}\right) + \log\left(q_{\ell_{N_L},L} + \frac{2\ell_{N_L}}{L}\right) < \varepsilon. \tag{2.11.33}$$

Thus

$$\frac{a_L}{L} \leq e^{\varepsilon} \frac{a_{\ell_1}}{\ell_1}. \tag{2.11.34}$$

Moreover, the above conclusion holds for any integer ℓ_1 sufficiently large, as long as $L >> \ell_1$. In particular, we can also require that

$$\frac{a_{\ell_1}}{\ell_1} \leq \inf_n \frac{a_n}{n} + \varepsilon. \tag{2.11.35}$$

Combining equations (2.11.34) and (2.11.35) the proof is finished letting $\varepsilon \to 0$.

2.11.4 Proof of lemma 2.22

We will show that the following inequality holds

$$\mathbb{E}_{[0,n+m+r]}\left(|\hat{G}(0,n+m+r;z)|^s\right) \le C_{\text{Dec}}\mathbb{E}_{[0,n]}\left(|\hat{G}(0,n;z)|^s\right)\mathbb{E}_{[0,m]}\left(|\hat{G}(0,m;z)|^s\right) \tag{2.11.36}$$

where $C_{\text{Dec}} = C_{\text{AP}}e^{C(\eta,g,\|F\|_\infty)e^{-\nu' r}(m^2+n^2)}$, $\mathbb{E}_{[0,n]}$ denotes the expectation with respect to the variables $U(0),...,U(n)$ and C_{AP} is, up to a multiplication by a constant independent of m,n and r, the constant obtained on the *a priori* from lemma 2.4. Denote

$$\hat{\omega} = (\omega(1),...,\omega(n+1),\omega(n+r),...,\omega(n+r+m)) \tag{2.11.37}$$

and let us change variables according to

$$\hat{\omega} \mapsto \hat{U}. \tag{2.11.38}$$

We remark that the variables $\omega(n+2),...,\omega(n+r-1)$ are fixed in this process.

Note that by lemma 2.4 and a geometric resolvent expansion we have

$$\mathbb{E}\left(\hat{G}(0,n+m+r;z)|^s\right) \le C_{\text{AP}}\mathbb{E}_{\neq n+1,n+r}\left(|\hat{G}(0,n;z)|^s|\hat{G}(n+r+1,n+r+m;z)|^s\right). \tag{2.11.39}$$

where $\mathbb{E}_{\neq n+1,n+r}$ indicates that the variables $U(n+1)$ and $U(n+r)$ were integrated out. Observe that the corresponding Jacobian has the structure

$$\mathcal{J} = \begin{pmatrix} \mathcal{A}_{n\times n} & \mathcal{B}_{n\times(m+r)} \\ \mathcal{C}_{(m+r)\times n} & \mathcal{D}_{(m+r)\times(m+r)} \end{pmatrix}$$

where

$$\mathcal{A}_{jk} = \delta_{jk} + g\frac{\partial V_{\text{eff}}(j)}{\partial U(k)},\ \ \mathcal{B}_{jk} = g\frac{\partial V_{\text{eff}}(j)}{\partial U(n+k)},\ \ \mathcal{C}_{jk} = g\frac{\partial V_{\text{eff}}(n+j)}{\partial U(k)}.$$

Moreover,

$$\mathcal{D} = \begin{pmatrix} \mathcal{I}_{r\times r} & \mathcal{Q}_{r\times m} \\ \mathcal{O}_{m\times r} & \mathcal{P}_{m\times m} \end{pmatrix}$$

where $\mathcal{I}_{r\times r}$ is the identity matrix and

$$\mathcal{P}_{jk} = \delta_{jk} + g\frac{\partial V_{\text{eff}}(n+r+j)}{\partial U(n+r+k)},\ \ \mathcal{Q}_{jk} = g\frac{\partial V_{\text{eff}}(n+j)}{\partial U(n+r+k)},\ \ \mathcal{O}_{jk} = g\frac{\partial V_{\text{eff}}(n+r+j)}{\partial U(n+k)}.$$

By the Schur complement formula

$$\det \mathcal{J} = \det \mathcal{A} \det \mathcal{P} \det \left(\mathcal{I}_{m+r\times m+r} - \mathcal{D}^{-1}\mathcal{C}\mathcal{A}^{-1}\mathcal{B}\right) \det \left(\mathcal{I}_{m\times m} - \mathcal{P}^{-1}\mathcal{O}\mathcal{Q}\right). \quad (2.11.40)$$

where, according to the estimate (2.7.1), the matrices $\mathcal{B}$ and $\mathcal{Q}$ have entries which decay exponentially away from their lower-left corner. Likewise, the entries of $\mathcal{C}$, $\mathcal{O}$ decay exponentially away from their upper-right corner. It readily follows from lemma 2.15 that,

$$\det \left(\mathcal{I}_{m+r\times m+r} - \mathcal{D}^{-1}\mathcal{C}\mathcal{A}^{-1}\mathcal{B}\right) \det \left(\mathcal{I}_{m\times m} - \mathcal{P}^{-1}\mathcal{O}\mathcal{Q}\right) \le C(\eta, \nu, g, \|F\|_\infty).$$

therefore, for C as above,

$$\det \mathcal{J} \le C \det \mathcal{A} \det \mathcal{P}.$$

Let $\rho(l) = \rho\left(U(l) - gV_{\text{eff}}(l)\right)$. We obtain a decoupling estimate by observing that, setting $U(l) = 0$ for $l \ge n+r+1$ would only alter

$$V_{\text{eff}}(j) \text{ and } \frac{\partial V_{\text{eff}}(j)}{\partial U(k)}$$

by at most a factor which decays as $C(\eta, g, \|F\|_\infty)e^{-\nu(r+|j-k|)}$ for $1 \le j, k \le n$. This follows from the exponential decay on lemmas 2.8, 2.13 and 2.14. Similarly, we can set $U(l) = 0$ for $l \le n$ and this only changes

$$V_{\text{eff}}(j) \text{ and } \frac{\partial V_{\text{eff}}(j)}{\partial U(k)}$$

by at most a factor which decays as $Ce^{-\nu(r+|j-k|)}$ for $n+r \le j, k \le m+n+r$. The above process yields two independent measures

$$d\pi^0_{[0,n]} := \det{}_0\left(I + g\frac{\partial V_{\text{eff}}(j)}{\partial U(k)}\right)_{[0,n]} \prod_{0\le l\le n} \rho^0(l)\, dU(l) \quad (2.11.41)$$

and

$$d\pi^0_{[n+r+1,n+m+r]} := \det{}_0\left(I + g\frac{\partial V_{\text{eff}}(j)}{\partial U(k)}\right)_{[n+r+1,n+m+r]} \prod_{n+r+1\le n+r+m} \rho^0(l)\, dU(l). \quad (2.11.42)$$

Using lemma 2.15 we then arrive at an inequality of the type

$$\mathbb{E}_{[0,m+n+r]}\left(|\hat{G}(0,n+m+r;z)|^s\right) \leq C_{\mathrm{AP}} e^{C(\eta,g,\|F\|_\infty)e^{-\nu r}(m^2+n^2)} \int |\hat{G}(0,n;z)|^s \, d\pi^0_{[0,n]} \times \int |G_+(n+r,n+r+m;z)|^s \, d\pi^0_{[n+r+1,n+m+r]}.$$

Rewriting the above conclusion in terms of expectations we obtain

$$\mathbb{E}_{[0,m+n+r]}\left(|\hat{G}(0,n+m+r;z)|^s\right) \leq C_{\mathrm{Dec}} \mathbb{E}_{[0,n]}\left(|\hat{G}(0,n;z)|^s\right)| \times \mathbb{E}_{[n+r+1,n+m+r]}\left(G_+(n+r,n+r+m;z)|^s\right).$$

where $C_{\mathrm{Dec}} = C_{\mathrm{AP}} e^{C(\eta,g,\|F\|_\infty)e^{-\nu r}(m^2+n^2)}$. At the expense of increasing C_{AP} the above expectations can be replaced by expectations over the full probability space, which yields the desired conclusion by translation invariance.

2.11.5 Proof of (2.11.22)

We shall modify the proof of lemma 2.22 to obtain a "super-additive" estimate of the form

$$\mathbb{E}_{[0,n+m+r]}\left(|\hat{G}(0,n+m+r;z)|^s\right) \geq \underline{C}(s,E)e^{\varphi(s,z)r}e^{-C(\eta,g,\|F\|_\infty)e^{-\nu' r}(m^2+n^2)} \times \mathbb{E}_{[0,n]}\left(|\hat{G}(0,n;z)|^s\right)\mathbb{E}_{[0,m]}\left(|\hat{G}(0,m;z)|^s\right)$$

where the constant $\underline{C}(s,z)$ can be chosen locally uniform in z and $s \in (0,1)$. Since the argument is very similar to the one in the proof of (2.11.36), we only explain the key modification which consists in obtaining a lower bound for $\mathbb{E}_{[n+1,n+r]}\left(|\hat{G}(n+1,n+r;z)|^s\right)$ as follows. We start by writing

$$|\hat{G}(n+1,n+r;z)|^s = |\hat{G}(n+1,n+1;z)|^s |\hat{G}(n+2,n+r;z)|^s. \tag{2.11.43}$$

Using Jensen's inequality we have that, for any $\varepsilon \in (0,s)$,

$$\mathbb{E}_{n+1}\left(|\hat{G}(n+1,n+1;z)|^s\right) \geq \mathbb{E}_{n+1}\left(|\hat{G}(n+1,n+1;z)|^{-\varepsilon}\right)^{\frac{-s}{\varepsilon}}. \tag{2.11.44}$$

where $\mathbb{E}_{n+1}$ denotes the conditional expectation with respect to $U(n+1)$. Making use of the discrete Riccati equation [6, Proposition 12.1] we obtain

$$\mathbb{E}_{n+1}\left(|\hat{G}(n+1,n+1;z)|^{-\varepsilon}\right) = \mathbb{E}_{n+1}\left(|U(n+1) - z - \hat{G}(n+2,n+2;z)|^{\varepsilon}\right). \quad (2.11.45)$$

Equations (2.11.43), (2.11.44) and (2.11.45) together with lemma 2.5 yield a lower bound

$$\mathbb{E}_{[n+1,n+r]}\left(|\hat{G}(n+1,n+r;z)|^{s}\right) \geq \underline{C}(z,s)\mathbb{E}_{[n+2,n+r]}\left(|\hat{G}(n+2,n+r;z)|^{s}\right) \quad (2.11.46)$$

which in combination with (2.11.21) implies that, after a suitable adjustment of the constant $\underline{C}(s,z)$,

$$\mathbb{E}_{[n+1,n+r]}\left(|\hat{G}(n+1,n+r;z)|^{s}\right) \geq \underline{C}(z,s)e^{\varphi(s,z)r}. \quad (2.11.47)$$

Equation (2.11.43) follows from the above inequality combined with a decoupling estimate analogous to the one in the proof of (2.11.36). Again, choosing r comparable to $\max\{\log m, \log n\}$ we obtain

$$\mathbb{E}_{[0,n+m+r]}\left(|\hat{G}(0,n+m+r;z)|^{s}\right) \geq \underline{C}(s,z)e^{\varphi(s,z)r} \quad \mathbb{E}_{[0,n]}\left(|\hat{G}(0,n;z)|^{s}\right) \times\mathbb{E}_{[0,m]}\left(|\hat{G}(0,m;z)|^{s}\right)$$

multiplying both sides of the above inequality by $e^{-\varphi(s,z)(m+n)}$ and taking logarithms we conclude that the sequence $b_n = \log e^{-\varphi(s,z)n}\left(\mathbb{E}_{[0,n]}\left(|\hat{G}(0,n;z)|^{s}\right)\right)$ satisfies

$$b_{n+m+r} \geq \log(\underline{C}(s,z)) + b_n + b_m. \quad (2.11.48)$$

The bound (2.11.22) now follows from an application of the supperaditive version of lemma 2.21.

2.12 Hölder Continuity for the integrated density of states at weak interaction

In this section we shall address the problem of Hölder continuity for the integrated density of states for the Hubbard model with respect to energy, disorder and interaction. Our results follow from modifications of the methods in [29] and references therein after we have established the existence of a suitable conditional density as in lemma 2.5.

Let's now prove theorem 2.3, starting from Hölder continuity with respect to energy, equation (2.2.15). We proceed as in [29, Section 2]. For simplicity, we replace H_{Hub} by H defined in (2.4.1). The arguments given below will be applicable to $H_{\uparrow}$ and $H_{\downarrow}$ and, therefore, suffice to show the same result for H_{Hub}.

Fix an energy interval I of length $\varepsilon > 0$ centered at $E \in \mathbb{R}$. The idea is to use the Hölder continuity of N_0 and the resolvent identity to reach the following inequality for $\varepsilon << 1$ and $|I| = \varepsilon$, where we denote by $P_\Lambda(I)$ the spectral projection of H^Λ on the interval I.

$$(1 - \mathrm{o}(\varepsilon))\mathbb{E}\left(\mathrm{Tr} P_\Lambda(I)\right) \leq C(I, \rho)\varepsilon^\alpha |\Lambda|. \tag{2.12.1}$$

Dividing both sides of (2.12.1) by $|\Lambda|$ and letting $|\Lambda| \to \infty$ gives (2.2.15). To obtain (2.12.1) we fix an interval J containing I with $|J|$ to be determined. We then write, with $P_{0,\Lambda}(J) = P\left(H_0^\Lambda\right)(J)$,

$$\mathrm{Tr}(P_\Lambda(I)) = \mathrm{Tr}(P_\Lambda(I) P_{0,\Lambda}(J)) + \mathrm{Tr}(P_\Lambda(I) P_{0,\Lambda}(J^c)). \tag{2.12.2}$$

Note

$$\mathrm{Tr}(P_\Lambda(I) P_{0,\Lambda}(J)) \leq \mathrm{Tr}(P_{0,\Lambda}(J)). \tag{2.12.3}$$

The above inequality combined with to the Hölder continuity of N_0 with respect to $E \in \mathbb{R}$

$$|N_0(E) - N_0(E')| \leq C(I, d)|E - E'|^{\alpha_0}. \tag{2.12.4}$$

yields, for $|\Lambda|$ sufficiently large depending only on J,

$$\mathrm{Tr}(P_\Lambda(I) P_{0,\Lambda}(J)) \leq C(J, d)|J|^{\alpha_0}|\Lambda|. \tag{2.12.5}$$

We now estimate the second term on the left-hand side of equation (2.12.2). By the resolvent identity,

$$\begin{aligned} \mathrm{Tr}\left(P_\Lambda(I) P_{0,\Lambda}(J^c)\right) &= \mathrm{Tr}\left(P_\Lambda(I)(H - E) P_{0,\Lambda}(J^c)(H_{0,\Lambda} - E)^{-1}\right) \\ &\quad -\lambda \mathrm{Tr}\left(P_\Lambda(I) U^\Lambda P_{0,\Lambda}(J^c)(H_{0,\Lambda} - E)^{-1}\right). \end{aligned} \tag{2.12.6}$$

Where we have written $U = V_\omega + \frac{g}{\lambda} V_{\text{eff}}$. Moreover, using using functional calculus and that E is the center of I, we estimate the first term on the left-hand side of equation (2.12.6) by

$$\operatorname{Tr}\left((P_\Lambda(I))(H^\Lambda - E)P_{0,\Lambda}(J^c)(H_{0,\Lambda} - E)^{-1}\right) \leq \frac{|I|}{|J| - |I|}\operatorname{Tr}(P_\Lambda(I)). \tag{2.12.7}$$

Now, the second term in equation (2.12.6) can be controlled by means of

$$-\lambda \operatorname{Tr}\left(P_\Lambda(I)U^\Lambda P_{0,\Lambda}(J^c)(H_{0,\Lambda} - E)^{-1}\right) = A + B$$

for

$$A = -\lambda \operatorname{Tr}\left((H^\Lambda - E)(P_\Lambda(I))U^\Lambda P_{0,\Lambda}(J^c)(H_{0,\Lambda} - E)^{-2}\right) \tag{2.12.8}$$

$$B = \lambda^2 \operatorname{Tr}\left(U^\Lambda (P_\Lambda(I))U^\Lambda P_{0,\Lambda}(J^c)(H_{0,\Lambda} - E)^{-2}\right). \tag{2.12.9}$$

Now, because U^Λ is unbounded, we continue a slight modification of the argument in [29]. The only difference is that we bound term (A) above (which corresponds to [29, (iii) in equation (2.6)] as

$$|\operatorname{Tr}\left((H^\Lambda - E)(P_\Lambda(I))U^\Lambda P_{0,\Lambda}(J^c)(H_{0,\Lambda} - E)^{-2}\right)| \leq \frac{|I|}{(|J| - |I|)^2}|\operatorname{Tr}\left(P_\Lambda(I)U^\Lambda\right)|. \tag{2.12.10}$$

At this point, with an estimate analogous to the one in the proof of Proposition 3.2 in [14] we reach

$$\mathbb{E}\left(|\operatorname{Tr}P_\Lambda(I)V_\omega|\right) \leq \lambda^{-1} \sup_{m\in\mathbb{N}} \left\{\int_{m\varepsilon}^{(m+1)\varepsilon} \omega_j \rho(\omega_j)\, d\omega_j\right\}|\Lambda| \quad \varepsilon = |I|. \tag{2.12.11}$$

Thus, with $M_1(\varepsilon) := \sup_{m\in\mathbb{N}}\left\{\int_{m\varepsilon}^{(m+1)\varepsilon} \omega_j \rho(\omega_j)\, d\omega_j\right\}$,

$$\lambda|\operatorname{Tr}(H_\Lambda - E)P_\Lambda(I)U^\Lambda P_{0,\Lambda}(J^c)(H_0^\Lambda - E)^{-2}| \leq \frac{\lambda|I|}{(|J| - |I|)^2}\left(\frac{M_1(\varepsilon)}{\lambda} + \frac{g\|F\|_\infty}{\lambda}\right)|\Lambda|. \tag{2.12.12}$$

Similarly, with $M_2(\varepsilon) := \sup_{m\in\mathbb{N}}\left\{\int_{m\varepsilon}^{(m+1)\varepsilon} \omega_j^2 \rho(\omega_j)\, d\omega_j\right\}$, we estimate term (B) through

$$\lambda^2|\operatorname{Tr}U^\Lambda(P_\Lambda(I))U^\Lambda P_{0,\Lambda}(J^c)(H_{0,\Lambda} - E)^{-2}| \leq \frac{4\lambda^2}{(|J| - |I|)^2}\left(\frac{M_2(\varepsilon)}{\lambda}|\Lambda| + \frac{g^2}{\lambda^2}\operatorname{Tr}(P_\Lambda(I))\right). \tag{2.12.13}$$

Due lemma 2.5 and the Wegner estimate (see [6, theorem 4.1]) we conclude that

$$\mathrm{Tr}(P_\Lambda(I)) \leq \frac{C}{\lambda}|I||\Lambda|. \tag{2.12.14}$$

Choosing the interval J such that $|J| = \varepsilon^\delta$ for $\delta < 1$, keeping in mind the assumption $g^2 < \lambda$ and combining the bounds (2.12.5), (2.12.7), (2.12.12), (2.12.13), (2.12.14) and optimizing over δ gives $\delta = \frac{1}{2+\alpha_0}$ therefore we reach (2.12.1) for $\alpha \in [0, \frac{\alpha_0}{2+\alpha_0}]$ and (2.2.15) is proven.

To show 2.3 we follow the proof of theorem 1.2 in [29]. We fix $\lambda, \lambda' \in J$, g and g' satisfying the assumptions of theorem 2.3 and $E \in I$. As explained in [29], using Hölder continuity with respect to energy given by equation (2.2.15), trace identities and ergodicity of $H_{\lambda,g}$ and $H_{\lambda',g'}$, it suffices to estimate $\mathbb{E}\left(\mathrm{Tr} P_0\varphi(H_{\lambda,g})(\varphi(H_{\lambda,g}) - \varphi(H_{\lambda',g'}))P_0\right)$ where φ is a smooth function such that

$$\begin{cases} \varphi \equiv 1 \text{ on } (-\infty, E], \\ \varphi \equiv 0 \text{ on } (-\infty, E + |\lambda - \lambda'|^\delta + |g - g'|^\delta)^c, \\ \|\varphi^{(j)}\|_\infty \leq C\left(|\lambda - \lambda'|^\delta + |g - g'|^\delta\right)^{-j}, \ j = 1, 2..., 3d+4 \end{cases} \tag{2.12.15}$$

with $\delta > 0$ to be determined. The need for a high regularity of φ is due to the fact that the random potential V_ω may be unbounded. Let $\tilde{\varphi}$ be an almost analytic extension of φ of order $3 + 3d$. In particular, $\tilde{\varphi}$ is defined in a complex neighborhood of the support of φ and if $z = E + i\eta$ we have that

$$|\partial_{\overline{z}}\tilde{\varphi}(z)| \leq |\eta|^{3d+3}|\varphi^{(3d+4)}(E)|. \tag{2.12.16}$$

By the Helffer-Sjöstrand formula, the quantity

$$\mathrm{Tr}\left(P_0\varphi(H_{\lambda,g})(\varphi(H_{\lambda,g}) - \varphi(H_{\lambda',g'}))P_0\right)$$

equals

$$\frac{1}{\pi}\int_{\mathbb{C}} \partial_{\overline{z}}\tilde{\varphi}\, \mathrm{Tr} P_0 \varphi(H_{\lambda,g}) R_{\lambda,g}(z) \left(\lambda' U_{\lambda',g'} - \lambda U_{\lambda,g}\right) R_{\lambda',g'}(z) P_0 \, d^2 z =$$
$$\frac{(\lambda'-\lambda)}{\pi}\int_{\mathbb{C}} \partial_{\overline{z}}\tilde{\varphi}\, \mathrm{Tr} P_0 \varphi(H_{\lambda,g}) R_{\lambda,g}(z) V_\omega R_{\lambda',g'}(z) P_0 \, d^2 z$$
$$+ \frac{(g'-g)}{\pi}\int_{\mathbb{C}} \partial_{\overline{z}}\tilde{\varphi}\, \mathrm{Tr} P_0 \varphi(H_{\lambda,g}) R_{\lambda,g}(z) V_{\mathrm{eff},\lambda}(g) R_{\lambda',g'}(z) P_0 \, d^2 z$$
$$+ \frac{g'}{\pi}\int_{\mathbb{C}} \partial_{\overline{z}}\tilde{\varphi}\, \mathrm{Tr} P_0 \varphi(H_{\lambda,g}) R_{\lambda,g}(z) \left(V_{\mathrm{eff},\lambda'}(g') - V_{\mathrm{eff},\lambda}(g)\right) R_{\lambda',g'}(z) P_0 \, d^2 z.$$

Since the last two terms enjoy a better modulus of Hölder continuity (since they do not involve V_ω) and can be treated as in [29], we shall only estimate the first of the above integrals. By the resolvent identity,

$$\begin{aligned} R_{\lambda,g}(z) V_\omega R_{\lambda',g'}(z) =& R_{\lambda,g}(z) V_\omega R_{\lambda,g}(z) + (\lambda-\lambda') R_{\lambda,g}(z) V_\omega R_{\lambda',g'}(z) V_\omega R_{\lambda,g}(z) + \\ & (g-g') R_{\lambda,g}(z) V_\omega R_{\lambda',g'}(z) V_{\mathrm{eff},\lambda'}(g') R_{\lambda,g}(z) \\ & - g R_{\lambda,g}(z) V_\omega R_{\lambda',g'}(z) (V_{\mathrm{eff},\lambda'}(g') - V_{\mathrm{eff},\lambda}(g)) R_{\lambda,g}(z). \end{aligned}$$

The above considerations lead to a perturbative expansion of

$$\frac{(\lambda'-\lambda)}{\pi}\int_{\mathbb{C}} \partial_{\overline{z}}\tilde{\varphi}\, \mathrm{Tr} P_0 \varphi(H_{\lambda,g}) R_{\lambda,g}(z) V_\omega R_{\lambda',g'}(z) P_0 \, d^2 z$$

into four terms. We will show below that each of them can be bounded in terms of powers of either $|\lambda-\lambda'|$ or $|g-g'|$.

We start by estimating

$$\mathbb{E}\left(\left|\frac{(\lambda-\lambda')^2}{\pi}\int_{\mathbb{C}} \partial_{\overline{z}}\tilde{\varphi}\, \mathrm{Tr} P_0 \varphi(H_{\lambda,g}) R_{\lambda,g}(z) V_\omega R_{\lambda',g'}(z) V_\omega R_{\lambda,g}(z) P_0 \, d^2 z\right|\right) \tag{2.12.17}$$

with a slight modification of equation (3.15) in [29] since V_ω is unbounded. By the Combes-Thomas bound, equation (2.12.16) and the choice of φ

$$\begin{aligned} &\mathbb{E}\left(\left|\tfrac{(\lambda-\lambda')^2}{\pi}\int_{\mathbb{C}} \partial_{\overline{z}}\tilde{\varphi}\, \mathrm{Tr} P_0 \varphi(H_{\lambda,g}) R_{\lambda,g}(z) V_\omega R_{\lambda',g'}(z) V_\omega R_{\lambda,g}(z) P_0 \, d^2 z\right|\right) \\ &\qquad \le C(d)\left(1+\mathbb{E}^2(|V_\omega|)\right) \frac{|\lambda-\lambda'|^2}{(|\lambda-\lambda'|^\delta + |g-g'|^\delta)^{3d+4}}. \end{aligned}$$

Similarly,

$$\mathbb{E}\left(\Big|\frac{(\lambda-\lambda')(g-g')}{\pi}\int_{\mathbb{C}}\partial_{\overline{z}}\tilde{\varphi}\,\mathrm{Tr}P_0\varphi(H_{\lambda,g})R_{\lambda,g}(z)V_{\mathrm{eff},\lambda'}(g')R_{\lambda',g'}(z)V_\omega R_{\lambda,g}(z)P_0\,d^2z\Big|\right)$$
$$\le C(d)\,(1+\mathbb{E}(|V_\omega|))\,\frac{|\lambda-\lambda'||g-g'|}{(|\lambda-\lambda'|^\delta+|g-g'|^\delta)^{3d+4}}.$$

Moreover, using lemma 2.11 with the the explicit dependence on ω given there, we obtain that the expected value of

$$\Big|\frac{g(\lambda-\lambda')}{\pi}\int_{\mathbb{C}}\partial_{\overline{z}}\tilde{\varphi}\,\mathrm{Tr}P_0\varphi(H_{\lambda,g})R_{\lambda,g}(z)V_\omega R_{\lambda',g'}(z)(V_{\mathrm{eff},\lambda'}(g')-V_{\mathrm{eff},\lambda}(g))R_{\lambda,g}(z)P_0\,d^2z\Big|$$

is bounded from above by

$$C(d)\,(1+\mathbb{E}(|V_\omega|))\,\frac{|g||\lambda-\lambda'|(|g-g'|+|\lambda-\lambda'|)}{(|\lambda-\lambda'|^\delta+|g-g'|^\delta)^{3+3d}}.$$

Using the same arguments as in [29, Equations 3.17 and 3.18] we see that

$$\Big|\frac{(\lambda'-\lambda)}{\pi}\int_{\mathbb{C}}\partial_{\overline{z}}\tilde{\varphi}\,\mathrm{Tr}P_0\varphi(H_{\lambda,g})R_{\lambda,g}(z)V_\omega R_{\lambda,g}(z)P_0\,d^2z\Big|$$

can be bounded from above by

$$|\lambda-\lambda'||\mathbb{E}\left(\mathrm{Tr}(P_0\varphi(H_{\lambda,g})R_{\lambda,g}(z)V_\omega R_{\lambda,g}(z)P_0)\right)|\le\frac{C|\lambda-\lambda'|\mathbb{E}\left(|V_\omega|\right)}{(|\lambda-\lambda'|^\delta+|g-g'|^\delta)}. \qquad (2.12.18)$$

Finally, we conclude that

$$|N_{\lambda,g}(E)-N_{\lambda',g'}(E)|\le C(\alpha_0,d,I)m(|\lambda-\lambda'|)m(|g-g'|) \qquad (2.12.19)$$

where

$$m(x)=x^{\delta\alpha}+x^{2-(3d+4)\delta}+x^{1-\delta}.$$

Choosing $\delta=\frac{2}{\alpha+3d+4}$ we obtain, for any $\beta\in[0,\frac{2\alpha}{\alpha+3d+4}]$,

$$|N_{\lambda,g}(E)-N_{\lambda',g'}(E)|\le C(\alpha_0,I)\left(|\lambda-\lambda'|^\beta+|g-g'|^\beta\right)$$

finishing the proof of theorem 2.3.

CHAPTER 3

SPECTRAL AND DYNAMICAL CONTRAST ON HIGHLY CORRELATED ANDERSON-TYPE MODELS.

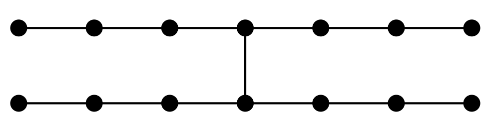
3.1 Introduction and Main results

This chapter aims at presenting and analysing examples of random Schrödinger operators where contrasting dynamical and spectral behaviors can be observed. In comparison to the well established theory of Anderson localization, discussed below in detail, the systems studied here will exhibit some form of long range correlations. Depending on their geometry, the spectral properties of the models can change significantly. More surprisingly, we present a model which exhibits purely absolutely continuous spectrum but where a 'phase transition' can still be observed. Such phenomenon can be captured utilizing the notion of transient and recurrent absolutely continuous spectrum due to Avron and Simon [9], which we shall also review in the subsequent discussion. Before stating our models and main results precisely, we shortly discuss the relevant background on correlated models.

3.1.1 Background on Correlated Models

The effects of strong correlations on localization properties of a given lattice were recently studied in [50] where the authors consider a system consisting of a particle and a 1/2 spin and the particle flips the spin only when it visits the origin.

Figure 3.1: 1/2 spin and single-particle model

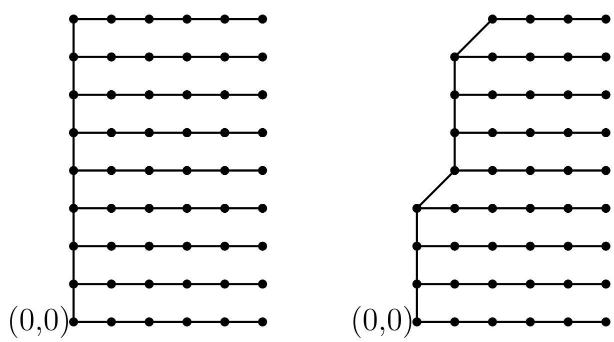


There it is shown that resonant tunneling is compatible with correlated pure point spectrum the since model exhibits Green's function decay on the graph metric, hence pure point spectrum, but has eigenfunctions which are only localized in the particle position [50, Theorems II.2 and II.5].

3.1.2 Main Results

Our goal in this chapter is to study infinite volume analogues of the model in [50]. These will be obtained by connecting infinitely many copies of the Anderson model in two ways. We will present the contrast between two random Schrödinger operators $H_{\text{Sym},\omega} = -A_{\text{Sym},\gamma} + V_\omega$ and $H_{\text{Diag},\omega} = -A_{\text{Diag},\gamma} + V_\omega$, where the random potential only depends on the first coordinate and is defined through $V_\omega(n) = \omega(n_1)$ for $n = (n_1, n_2)$. By $A_{\text{Sym},\gamma}$ and $A_{\text{Diag},\gamma}$ we denote weighted adjacency operators of graphs which are infinite (in the horizontal and vertical directions) subsets of $\mathbb{Z}^2$ obtained by connecting infinitely many copies of $\mathbb{Z}$ along a walk which is either entirely vertical or alternates between vertical components of fixed length $\ell > 0$ and diagonal components.

Figure 3.2: Highly correlated Anderson-type models



The operators $H_{\text{Sym},\omega}$ and $H_{\text{Diag},\omega}$ act in a similar fashion if one fixes $n \in \mathbb{N}$ and considers the horizontal layer $\mathbb{Z} \times \{n\}$ separately, when the models are easily seen to be distinct samples of the Anderson model $-\Delta + V_\omega$ acting on $\ell^2(\mathbb{Z})$. However, as we shall make precise below, by connecting these horizontal components in distinct ways a fundamental difference is introduced. To state the quantum dynamical contrast, let $q > 0$ be given. The averaged q-moments of a self-adjoint operator H are defined as

$$M_T^q(H) = \frac{2}{T}\int_0^\infty e^{\frac{-2t}{T}}\mathbb{E}\left\langle 0|e^{itH_\omega}|X|^q e^{-itH_\omega}|0\right\rangle \, dt \tag{3.1.1}$$

where we recall that the position operator $|X|$ acts as a multiplication operator on $\ell^2\left(\mathbb{Z}^2\right)$ via $\left(|X|^q\phi\right)(n) := |n|^q\phi(n)$ and $|n| = |n_1| + |n_2|$ for $n = (n_1, n_2) \in \mathbb{Z}^2$. Our first result concerns $H_{\mathrm{Sym},\omega}$ where a combination of the symmetry and localization in the horizontal direction induces ballistic transport in the vertical direction, up to a logarithmic correction. For the sake of simplicity, we assume that the random variables $\{\omega(n)\}_{n\in\mathbb{Z}}$ are nonnegative and bounded throughout this chapter so $\omega(n) \in [0, \omega_{\max}]$ for all $n \in \mathbb{Z}^2$.

3.1.3 Dynamical Contrast between H_{Sym} and H_{Diag}

Theorem 3.1. *There exists $T_0 > 0$ for which the averaged moments satisfy*

$$M_T^q\left(H_{\mathrm{Sym}}\right) \geq C_{\delta_0}\left(\frac{T}{\log T}\right)^q \tag{3.1.2}$$

for all times $T \geq T_0$ and some positive constant C_{δ_0} which depends on $\|\rho\|_\infty$.

Our second result concerns $H_{\mathrm{Diag},\omega,\gamma} = -A_{\mathrm{Diag},\gamma} + V_\omega$. Inspired by the analysis of [40] we show that $M_T^q\left(H_{\mathrm{Diag},\gamma}\right)$ cannot grow faster than a logarithmic power of T, which is an analogue of [40, Theorem 1].

Theorem 3.2. *There are $\gamma_0 > 0$ and $T_0 > 0$ such that whenever $\gamma < \gamma_0$, the averaged moments of the operator $H_{\mathrm{Diag},\omega,\gamma} = -A_{\mathrm{Sym},\gamma} + V_\omega$ satisfy*

$$M_T^q\left(H_{\mathrm{Diag},\gamma}\right) \leq (\log T)^q + C(q) \tag{3.1.3}$$

for all times $T \geq T_0$.

We remark that γ_0 will be explicit in the proof. It depends on ℓ and the decay rate of the Green's function of the one-dimensional Anderson model.

3.1.4 Spectral contrast between $H_{\rm Sym}$ and $H_{\rm Diag}$

The following is a simple consequence of theorem 3.2. Its proof and the relevant definitions of packing measure and dimension will be given in section 3.6.1.

Corollary 3.3. *Whenever $\gamma < \gamma_0$, the packing dimension $\dim_P^+(H_{\rm Diag,\omega})$ vanishes almost surely.*

The spectral contrast between the two models will be evident from the result below.

Theorem 3.4. *For every $\omega \in \Omega$ the spectral measure $\mu_{\delta_0}\left(H_{\rm Sym}\right)$ is purely absolutely continuous and supported on a set of Lebesgue measure 4γ.*

An interesting feature of theorem 3.4 is that, for small values of the parameter γ, the support of $\mu_{\delta_0}\left(H_{\rm Sym}\right)$ has Lebesgue measure much smaller than the spectrum of $H_{\rm Sym}$, since the later contains $[0, 2+\omega_{\max}]$. As we shall see below, this phenomenon is linked to the fact that the portion of $\sigma\left(H_{\rm Sym}\right)$ inside $[0, 2+\omega_{\max}]$ is recurrent. To prove theorem 3.4 we have shown the following abstract result which, to the best of our knowledge, is new.

Proposition 3.5. *Let μ be a finite Borel measure which is purely singular and $F(z) = \int \frac{1}{u-z}\, d\mu(u)$ its Borel transform, defined whenever $z \in \mathbb{C}^+$. Then, the limit $F(E+i0) = \lim_{\delta\to 0^+} F(E+i\delta)$ exists for Lebesgue almost every E and*

$$\left|\{E \in \mathbb{R}\ : \alpha < E + F(E+i0) < \beta\}\right| = \beta - \alpha. \tag{3.1.4}$$

As we could not find a reference in the literature with the exact statement needed, the details of the argument are carried out in section 3.7. Proposition 3.5 is liked to a beautiful equality by Boole [10], although with a slightly different statement. Boole's equality [10] and its extensions have been rediscovered or studied in various contexts by different authors ([45], [57] [20],[21],[36],[22],[52]). For the sake of completeness, we state it below

Proposition 3.6. *Under the above assumptions*

$$\left|\{E \in \mathbb{R}\ : F(E+i0) > t\}\right| = \frac{C}{t}. \tag{3.1.5}$$

Further historical notes on Boole's equality can be found in [55, Chapter 5] and [6, Chapter 8].

3.1.5 Phase transition within $\sigma\left(H_{\text{Sym}}\right)$

Our next result sheds light on theorem 3.4 and provides further information on the dynamics $e^{-itH_{\text{Sym}}}$. Before proceeding with the statement we recall that the notions of transient and recurrent subspaces where given in the introduction of the thesis. The main point of theorem 3.7 below is that the transient and recurrent spectrum can naturally arise and coexist in a situation of physical relevance.

Theorem 3.7. *For all values of γ the spectrum of H_{Sym} is a random set with*

$$[-2, 2+\omega_{\max}] \subset \sigma\left(H_{\text{Sym}}\right) \subset [-2-2\gamma, 2+\omega_{\max}+2\gamma]. \tag{3.1.6}$$

Moreover, $\sigma\left(H_{\text{Sym}}\right) \setminus [-2, 2+\omega_{\max}) = \sigma^{tac}\left(H_{\text{Sym}}\right)$ and $\sigma\left(H_{\text{Sym}}\right) \cap [-2, 2+\omega_{\max}) = \sigma^{rac}\left(H_{\text{Sym}}\right)$.

3.1.6 Definition of the Models

We are going to define the graphs $\mathbb{G}_{\text{Sym}}$ and $\mathbb{G}_{\text{Diag}}$ by specifying the *non-zero* matrix elements of their adjacency operators on the basis $\{\delta_n\}_{n\in\mathbb{Z}^2}$. In both cases the adjacency operators are symmetric. Let $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ and I_X be the indicator function of a set X. The adjacency operator $A_{\text{Sym},\gamma}$ is defined by

$$A_{\text{Sym},\gamma}(x,y) = \begin{cases} \gamma \text{ if } x = (0,n_2) \text{ and } y \in \{(0,n_2-1)I_{\{n_2>0\}}, (0,n_2+1)\} \\ \\ 1 \text{ if } x = (n_1,n_2) \text{ and } y \in \{(n_1-1,n_2)I_{\{n_1>0\}}, (n_1+1,n_2)\} \end{cases} \tag{3.1.7}$$

where $n_1, n_2 \in \mathbb{Z}_+$ in the above definition.

In other words, two vertices x and y satisfying $d(x,y) = 1$ are connected only if their second coordinate is the same or their first coordinate is zero. The constant $\gamma > 0$ is the hopping strength on the vertical direction and it will play a key role later in this chapter. On the other hand, the *non-zero* matrix elements of $A_{\text{Diag},\gamma}$ are defined via

$$A_{\text{Diag},\gamma}(x,y) = \begin{cases} \gamma \text{ if } x = (n_1, (n_1+1)\ell - 1) \text{ and } y = (n_1+1, (n_1+1)\ell) \;, n_1 \in \mathbb{Z}_+ \\ \\ 1 \text{ if } x = (n_1, n_2) \text{ and } y \in \{(n_1-1, n_2)I, (n_1+1, n_2), (n_1, n_2+1)\mathrm{I}_\ell \end{cases} \tag{3.1.8}$$

where $I = I_{\{n_1>0\}}$ and $\mathrm{I}_\ell = \mathrm{I}_{\{n_1\ell \le n_2 < (n_1+1)\ell\}}$ for $n_1, n_2 \in \mathbb{Z}_+$.

Note that $A_{\text{Diag},\gamma}$ is obtained by modifying $A_{\text{Sym},\gamma}$ at points whose second coordinate is a multiple of the fixed length $L > 0$.

3.2 Lower Bound on the Averaged Moments for H_{Sym}: Proof of Theorem 3.1

The proof of Theorem 3.1 consists of rigorously implementing the following ideas. Firstly, the fact that H_{Sym} exhibits some form of transport has its roots on the symmetry of the model. We shall prove that the spectral measure $\mu_{\delta_{(0,0)}}(H_{\text{Sym}})$ is purely absolutely continuous and this implies diffusion by the Guarnieri bound, which we shall revisit below. Secondly, the disordered nature of the system and its connection to the Anderson model allow us to show a horizontal localization result. In other words, transport can only occur in the vertical direction. In particular, this will imply a rate of wavepacket spreading which is faster than what one would expected generically. A careful analysis will then imply the almost ballistic lower bound.

The Guarnieri bound states that if the spectral measure $\mu_\psi(H)$ of a self-adjoint operator H is α-Hölder continuous then

$$M_T^q(H) \ge C_\psi T^{\frac{\alpha q}{2}}. \tag{3.2.1}$$

Recall that μ is said to be α-Hölder continuous if there exists a constant $C < \infty$ such that for all intervals I with $|I| < 1$ we have $\mu(I) \le C|I|^\alpha$. In particular, if μ_ψ is purely absolutely continuous we can directly reach $M_T^q(H) \ge C_\psi T^{\frac{q}{2}}$. However, the proof of (3.2.1) can be adapted to incorporate the improvements due to the disorder which are specific to our context, therefore we reproduce the main arguments below. The starting point is the following estimate on the averaged quantum dynamics.

Theorem 3.8. *(Strichartz-Last) Let H be a self-adjoint operator on a Hilbert space $\mathscr{H}$ and assume the spectral measure of H with respect to ψ is uniformly α-Hölder continuous for some $\alpha \in (0,1)$. Then, there exists a constant $C_\psi < \infty$ such that for all $\phi \in \mathscr{H}$ and all $T > 0$*

$$\frac{1}{T}\int_0^T |\langle \phi | e^{itH} | \psi \rangle|^2 \le \frac{C_\psi \|\phi\|^2}{T^\alpha}. \tag{3.2.2}$$

The above result, which is found in [6, Theorem 2.3], can be used to reach (3.2.1) as follows. For simplicity, let $H_{\text{Sym}} = H$ throughout this section. We firstly rewrite $M_T^q(H)$ as

$$\begin{aligned}
\frac{2}{T}\sum_n |n|^q \int_0^\infty e^{\frac{-2t}{T}} |\langle n | e^{-itH_\omega} | 0 \rangle|^2\, dt &\ge \frac{2e^{-2}}{T} N^q \sum_{|n|>N} \int_0^T |\langle n | e^{-itH_\omega} | 0 \rangle|^2\, dt \\
&= 2e^{-2} N^q \left(1 - \sum_{|n|\le N} \frac{1}{T}\int_0^T |\langle n | e^{-itH_\omega} | 0 \rangle|^2\, dt \right) \\
&\ge 2e^{-2} N^q \left(1 - \frac{N(N+1)}{2}\frac{C_{\delta_0}}{T^\alpha} \right) \\
&\ge C_{\delta_0} T^{\frac{q\alpha}{2}}
\end{aligned}$$

where on the last step we choose N^2 comparable to T^α and the constant C_{δ_0} was adjusted.

In our particular setup, one can improve upon the Guarnieri bound using the localization statement for the horizontal direction in lemma 3.11 which combined with theorem 3.8 yields

$$\frac{1}{T}\sum_{|n|\le N} \int_0^T \mathbb{E} |\langle n | e^{-itH_\omega} | 0 \rangle|^2\, dt \le \sum_{|n|\le N} \min\left\{ \frac{C_{\delta_0}}{T}, Ce^{-\frac{\mu_{\text{And}}}{2-s}|n_1|} \right\}. \tag{3.2.3}$$

where $C = C(\gamma, \omega_{\max}, \|\rho\|_\infty, s) = \frac{2}{\pi}\left((2\gamma)^s \|\rho\|_\infty (4 + 4\gamma + \omega_{\max})\, C_{\text{And}(s)} \right)^{\frac{1}{2-s}}$. Thus, it is natural to split the above sum into two contributions according to $|n_1| \le \frac{2-s}{\mu_{\text{And}}} \log\left(\frac{CT}{C_{\delta_0}} \right)$, in which case $\min\left\{\frac{C_{\delta_0}}{T}, Ce^{-\frac{\mu_{\text{And}}}{2-s}|n_1|}\right\} = \frac{C_{\delta_0}}{T}$, and $|n_1| > \frac{2-s}{\mu_{\text{And}}}\log\left(\frac{CT}{C_{\delta_0}}\right)$ which implies $\min\left\{\frac{C_{\delta_0}}{T}, Ce^{-\frac{\mu_{\text{And}}}{2-s}|n_1|}\right\} = Ce^{-\frac{\mu_{\text{And}}}{2-s}|n_1|}$.

For $T > T_0\left(C, s, C_{\delta_0}, \mu_{\text{And}}\right)$ the first contribution can be estimated from above by

$$\frac{C_{\delta_0}}{T}\sharp\{n \in \mathbb{G}_{\text{Sym}} : |n| < N\,, |n_1| < 2\log T\}. \tag{3.2.4}$$

where N must be chosen bigger than $2\log T$, in which case there are two possibilities for n to be in the above set. The first one is that $n_2 \leq N - 2\log T$, when the condition $|n| < N$ is automatically satisfied whenever $|n_1| < 2\log T$. Secondly, if $\lfloor N - 2\log T + j\rfloor = n_2$ for some $j \in \{1, ..., \lfloor 2\log T\rfloor\}$ then $|n| < N$ is only satisfied when $|n_1| < \lfloor 2\log T\rfloor - j$. This reasoning immediately translates into the bound

$$\begin{aligned}\sharp\{n \in \mathbb{G}_{\text{Sym}} : |n| < N\,, |n_1| < 2\log T\} &\leq 2\log T\,(N - 2\log T) + \sum_{j=1}^{2\log T}(2\log T - j)\\ &= 2\log T\,(N - 2\log T) + \log T(2\log T - 1)\\ &\leq 2\log T\,(N - \log T)\,.\end{aligned}$$

Therefore, the first contribution in the splitting of (3.2.3) is bounded from above by

$$\frac{2C_{\delta_0}\log T\,(N - \log T)}{T}. \tag{3.2.5}$$

The second contribution is estimated by

$$\begin{aligned}\frac{2}{T}\sum_{|n|<N\,,|n_1|>\frac{\log T}{2}} Ce^{-\frac{\mu_{\text{And}}}{2-s}|n_1|} &\leq \frac{2C}{T}\sum_{j=\lfloor\frac{\log T}{2}\rfloor}^{N} e^{-\frac{\mu_{\text{And}}}{2-s}j}(N - \frac{\log T}{2})\\ &= \frac{2C(N - \frac{\log T}{2})}{T}\frac{1 - e^{-\frac{\mu_{\text{And}}}{2-s}\left(N - \lfloor\frac{\log T}{2}\rfloor - 1\right)}}{1 - e^{-1}}.\end{aligned}$$

In particular, the choice $N = \left(\log T + \eta\frac{T}{\log T}\right)$ ensures that, by picking η sufficiently small and eventually changing T_0 (but with the same dependence on $C, s, C_{\delta_0}, \mu_{\text{And}}$), the same argument as in the proof of the Guarnieri bound yields the inequality

$$M_T^q\left(H_{\text{Sym}}\right) \geq C_{\delta_0}\left(\frac{T}{\log T}\right)^q$$

whenever $T > T_0$. Theorem 3.1 is proven.

3.3 The absolute continuity of $\mu_{\delta_{(0,0)}}\left(H_{\text{Sym}}\right)$

We shall explain how the absolute continuity of $\mu_{\delta_{(0,0)}}\left(H_{\text{Sym}}\right)$ follows from recursive relations for the Green's function which are available due to the symmetry of $\mathbb{G}_{\text{Sym}}$. For $z \in \mathbb{C} \setminus \mathbb{R}$, let $\varphi := (H_{\text{Sym}} - z)^{-1}\delta_{(0,0)}$. By definition of H_{Sym} we have

$$-\varphi(1,0) - \gamma\varphi(0,1) + (\omega(0) - z)\varphi(0,0) = 1. \tag{3.3.1}$$

hence

$$-G_{\text{Sym}}\left((0,0),(1,0);z\right) - \gamma G_{\text{Sym}}\left((0,0),(0,1);z\right) + (\omega(0) - z)G_{\text{Sym}}\left((0,0),(0,0);z\right) - 1 = 0. \tag{3.3.2}$$

Thus, denoting by G_{And} the Green's function of the Anderson model on $\ell^2\left(\mathbb{Z}\right)$, it follows from Feenberg's expansion, explained in the introduction of this thesis, that

$$G_{\text{Sym}}\left((0,0),(0,0);z\right)\left(-G^+_{\text{And}}(1,1;z) - \gamma^2 G^+_{\text{Sym}}\left((0,1),(0,1);z\right) + (\omega(0) - z) -\right) - 1 = 0. \tag{3.3.3}$$

Where in a path expansion of the form $G(u,v;z) = G(u,u;z)G^+(v,v;z)$ the term $G^+(v,v;z)$ denotes the Green's function of the operator H_{Sym} restricted to the component of $\mathbb{G}_{\text{Sym}}$ containing v and obtained from $\mathbb{G}_{\text{Sym}}$ by removing the edge which connects the sites u and v. By the vertical symmetry of G_{Sym} we have that

$$G^+_{\text{Sym}}\left((0,1),(0.1);z\right) = G_{\text{Sym}}\left((0,0),(0,0);z\right) \tag{3.3.4}$$

thus we can rewrite equation (3.3.3) as

$$G_{\text{Sym}}\left((0,0),(0,0);z\right)\left(-G^+_{\text{And}}(1,1;z) - \gamma^2 G_{\text{Sym}}\left((0,0),(0,0);z\right) + (\omega(0) - z)\right) - 1 = 0. \tag{3.3.5}$$

Therefore, letting $w = -\omega(0) + z + G^{+}_{\text{And}}(1, 1; z)$ we have

$$2\gamma^2 G_{\text{Sym}}\left((0,0),(0,0);z\right) = -w + \left[w^2 - 4\gamma^2\right]^{1/2} \tag{3.3.6}$$

where we have chosen the branch of the square root onto the upper half-plane and the reason for the positive sign on the second of the above terms is that $G_{\text{Sym}}\left((0,0),(0,0);z\right)$ necessarily has non-negative imaginary part since H_{Sym} is self-adjoint. Equation (3.3.6) implies that

$$2\gamma G_{\text{Sym}}\left((0,0),(0,0);z\right) = -w + w\left(1 - \frac{4\gamma^2}{w^2}\right)^{1/2}. \tag{3.3.7}$$

Therefore, using either equation (3.3.6) or (3.3.7) depending on whether $|w|$ is small or large, we conclude that $G_{\text{Sym}}\left((0,0),(0,0);z\right)$ remains bounded as $\text{Im}z \to 0$ for any $z \in \mathbb{C}_+$ thus the spectral measure $\mu_{\delta_{(0,0)}}\left(H_{\text{Sym}}\right)$ is purely absolutely continuous, see, for instance, proposition B.4 in [6, Appendix B].

3.4 Floquet Theory for H_{Sym}

The vertical symmetry of the graph $\mathbb{G}_{\text{Sym}}$ and the definition of the operator H_{Sym} suggests that the use of a Fourier transform may be helpful when studying the dynamics $e^{-itH_{\text{Sym}}}$. Let us define

$$\mathcal{F} : \ell^2\left(\mathbb{G}_{\text{Sym}}\right) \to \ell^2\left(\mathbb{Z}_+\right) \otimes \mathbb{L}^2\left([0,\pi]\right)$$

through

$$\left(\mathcal{F}\psi\right)(n_1, p) := \sqrt{\frac{2}{\pi}} \sum_{n_2=0}^{\infty} \psi(n_1, n_2) \sin\left(p(n_2+1)\right). \tag{3.4.1}$$

For simplicity of notation we let $\hat{\psi}(n_1, p) = \mathcal{F}\left(\psi\right)(n_1, p)$. It is immediate to check that the following version of Plancherel's identity is satisfied

$$\langle \hat{\varphi}, \hat{\psi} \rangle_{\ell^2(\mathbb{Z}_+)\otimes\mathbb{L}^2([0,\pi])} = \langle \varphi, \psi \rangle_{\ell^2\left(\mathbb{G}_{\text{Sym}}\right)}. \tag{3.4.2}$$

Indeed,

$$\begin{aligned}\langle\hat{\varphi},\hat{\psi}\rangle_{\ell^2(\mathbb{Z}_+)\otimes\mathbb{L}^2([0,\pi])} &= \frac{2}{\pi}\sum_{n_1\in\mathbb{Z}_+}\int_0^\pi \overline{\hat{\varphi}}(n_1,p)\hat{\psi}(n_1,p)\,dp \\ &= \frac{2}{\pi}\sum_{n_1,n_2,n_2'\in\mathbb{Z}_+}\overline{\varphi}(n_1,n_2)\psi(n_1,n_2') \\ &\times\int_0^\pi \sin\left((n_2+1)p\right)\sin\left((n_2'+1)p\right)\,dp \\ &= \sum_{n_1,n_2\in\mathbb{Z}_+}\overline{\varphi}(n_1,n_2)\psi(n_1,n_2) \\ &= \langle\varphi,\psi\rangle_{\ell^2\left(\mathbb{G}_{\mathrm{Sym}}\right)}.\end{aligned}$$

A calculation which is almost identical to the one above gives that, for every $\psi \in \ell^2\left(\mathbb{G}_{\mathrm{Sym}}\right)$

$$\mathcal{F}^{-1}(\hat{\psi}) = \psi.$$

where

$$\mathcal{F}^{-1}(g)(n_1,n_2) = \sqrt{\frac{2}{\pi}}\int_0^\pi g(n_1,p)\sin\left(p(n_2+1)\right)\,dp.$$

Using that $\left\{\sqrt{\frac{2}{\pi}}\sin(np)\ \ : n\in\mathbb{N}\right\}$ is a complete orthonormal system in $[0,\pi]$ one may also check that for $g\in\ell^2\left(\mathbb{Z}_+\right)\otimes\mathbb{L}^2\left([0,\pi]\right)$

$$\mathcal{F}\left(\mathcal{F}^{-1}(g)\right) = g.$$

Therefore, $\mathcal{F}$ is a unitary map with inverse given by

$$\mathcal{F}^{-1}(g)(n_1,n_2) = \sqrt{\frac{2}{\pi}}\int_0^\pi g(n_1,p)\sin\left(p(n_2+1)\right)\,dp. \tag{3.4.3}$$

From the definition of H_{Sym} one readily sees that

$$\widehat{H_{\mathrm{Sym}}\psi}(n_1,p) = -\Delta\hat{\psi}(n_1,p) + \left(\omega(n_1) - 2\gamma\cos p\,\delta_{n_1=0}\right)\hat{\psi}(n_1,p). \tag{3.4.4}$$

where we have committed a slight abuse of notation by letting $\Delta\hat{\psi}(n_1,p) = \hat{\psi}(n_1-1,p) + \hat{\psi}(n_1+1,p)$. In particular, equation (3.4.4) allow us to conclude that H_{Sym} is unitarily equivalent to an operator acting on

$$\ell^2\left(\mathbb{Z}_+\right)\otimes\mathbb{L}^2\left([0,\pi]\right) \tag{3.4.5}$$

with action on each fiber given by $\hbar_p$ where $\hbar_p$ is a rank-one perturbation of the one dimensional Anderson model given by (3.4.4). A consequence of this fact is the following result

Lemma 3.9. *$\delta_{(0,0)}$ is a cyclic vector for H_{Sym}.*

Proof. Firstly, we claim that given $n_1 \in \mathbb{N}$, the vector $\delta_{(n_1,0)}$ belongs to the cyclic subspace generated by $\delta_{(0,0)}$, denoted henceforth by $\mathscr{H}_0$. Indeed, by definition of the Fourier transform (3.4.1),

$$\hat{\delta}_{(n_1,0)} = \sqrt{\frac{2}{\pi}} \sin p \, \delta_{n_1}. \tag{3.4.6}$$

By a result of Simon [54] the vector δ_0 is a cyclic vector for $\hbar_p$ thus we conclude from (3.4.6) that $\hat{\delta}_{(n_1,0)}$ belongs to the cyclic subspace, relative to $\hbar_p$, generated by $\hat{\delta}_{(0,0)}$. By taking the inverse Fourier transform, it follows that $\delta_{(n_1,0)} \in \mathscr{H}_0$. Since

$$H_{\text{Sym}} \delta_{(0,0)} = -\gamma \delta_{(0,1)} + \delta_{(1,0)} + \omega(0) \delta_{(0,0)}$$

we easily conclude that $\delta_{(0,1)} \in \mathscr{H}_0$ as well. Proceeding by induction one shows that $\delta_{(n_1,n_2)} \in \mathscr{H}_0$ for every pair $(n_1, n_2) \in \mathbb{G}_{\text{Sym}}$, finishing the proof. □

Since $\mu_{\delta_{(0,0)}}\left(H_{\text{Sym}}\right)$ is absolutely continuous we readily obtain

Corollary 3.10. *$\sigma\left(H_{\text{Sym}}\right)$ is purely absolutely continuous.*

3.4.1 Localization in the horizontal direction

The above considerations are particularly effective to translate dynamical localization for the one dimensional Anderson model into horizontal localization for H_{Sym}. We start with the following technical result.

Lemma 3.11. *Given $s \in (0,1)$ we have, for all $m, n \in \mathbb{Z}_+$*

$$\mathbb{E}\left(\sup_{|f|\leq 1} |\langle m| f\left(\hbar_p\right)|n\rangle|\right)| \leq (2\gamma)^{\frac{2}{2-s}} \left(\|\rho\|_\infty \left(4 + 4\gamma + \omega_{\max}\right) C_{\text{And}}(s)\right)^{\frac{1}{2-s}} e^{-\frac{\mu_{\text{And}}}{2-s}|m-n|}. \tag{3.4.7}$$

The above supremum is taken over all Borel measurable functions bounded by one.

Proof. It suffices to show that for every $L \in \mathbb{N}$ (3.4.7) holds with $\hbar_p$ replaced by its restriction to $\ell^2(\mathbb{Z}_+ \cap [0, L])$, denoted henceforth by $\hbar_p^L$.

Let $v_p = -2\gamma \cos p$. From rank-one perturbation formulas (see, for instance, [6, Theorem 5.3] and [3, Equation(A.7)]) we find that the spectral measure of $\hbar_p^L$ is given by

$$d\mu_{m,n}^{p,L}(E) = -v_p G_{\mathrm{And}}^L(m, n; E)\, \delta(\Sigma(E) - v_p)\, dE \tag{3.4.8}$$

where

$$\Sigma(E) := -\frac{1}{G_{\mathrm{And}}^L(0, 0; E)} \tag{3.4.9}$$

and $G_{\mathrm{And}}^L(m, n; E)$ denotes the Green's function for the Anderson model on $\ell^2(\mathbb{Z}_+ \cap [0, L])$. Therefore, by general structure of the spectral measures,

$$|v_p|^2 \int \left| G_{\mathrm{And}}^L(m, n; E) \right|^2 \delta(\Sigma(E) - v_p)\, dE \leq 1. \tag{3.4.10}$$

A second observation is that the analogue of equation (3.4.8) for $m = n$ reads

$$d\mu_{m,m}^{p,L}(E) = \delta(\Sigma(E) - v_p)\, dE. \tag{3.4.11}$$

In particular

$$\int_{-\infty}^{\infty} \delta(\Sigma(E) - v_p)\, dE = 1. \tag{3.4.12}$$

Combining equations (3.4.8), (3.4.10) and (3.4.12) with Hölder's inequality (applied to the exponents $(p, q) = (2 - s, \frac{2-s}{1-s})$) and Jensen's inequality for expectations, we conclude that for all intervals $I \subset \mathbb{R}$

$$\mathbb{E}\left(\left| \mu_{m,n}^{p,L} \right| (I) \right) \leq \left[|v_p|^s \mathbb{E}\left(\int_I \left| G_{\mathrm{And}}^L(m, n; E) \right|^s \delta(\Sigma(E) - v_p)\, dE \right) \right]^{\frac{1}{2-s}}. \tag{3.4.13}$$

Thus,

$$\mathbb{E}\left(\left| \mu_{m,n}^{p,L} \right| (I) \right) \leq (2\gamma)^{\frac{s}{2-s}} \|\rho\|_\infty^{\frac{1}{2-s}} \left(\int_I \mathbb{E}\left(|G_{\mathrm{And}}(m, n; E)|^s \right) dE \right)^{\frac{1}{2-s}}. \tag{3.4.14}$$

Since the operator $\hbar_p$ is bounded, with operator norm less or equal than $2 + 2\gamma + \omega_{\max}$, the inequality (3.4.14) suffices to conclude the proof of lemma 3.11. We mention that by introducing an integrable weight, one could also handle the case of unbounded operators. For further details we refer to [3, Equations (A.13)-(A.18)]. □

The localization statement for $\hbar_p$ described on Lemma 3.11 can be immediately translated into a (horizontal) localization statement for H_{Sym} with the use of Plancherel's identity (3.4.2) and the fact that

$$\hat{\delta}_{(m_1,m_2)}(n_1,p) = \sqrt{\frac{2}{\pi}}\delta_{m_1}(n_1)\sin\left(p(m_2+1)\right). \tag{3.4.15}$$

The precise statement reads

Lemma 3.12.

$$\mathbb{E}\left(\sup_{|f|\leq 1}\left|\langle \delta_{m_1,m_2}, f\left(H_{\mathrm{Sym}}\right)\delta_{n_1,n_2}\rangle\right|\right) \leq Ce^{-\frac{\mu_{\mathrm{And}}}{2-s}|m_1-n_1|}. \tag{3.4.16}$$

with $C = \frac{2}{\pi}\left(4\gamma^2\|\rho\|_\infty\left(4+4\gamma+\omega_{\max}\right)C_{\mathrm{And}}(s)\right)^{\frac{1}{2-s}}$.

3.4.2 Transient and recurrent components: proof of theorem 3.7

According to [9, Proposition 3.1] in order for an absolutely continuous measure μ_φ to be transient it is necessary that $\mu_\varphi = f(E)\,dE$ where $f \in C^\infty(\mathbb{R})$. Moreover, if f is also assumed to be compactly supported this condition also sufficient. Since the operators studied here are bounded, this distinction will not be relevant to us. Let

$$S = \mathrm{supp}(\mu_{\delta_{(0,0)}}) \cap [-2, 2+\omega_{\max}).$$

We claim that S cannot be the support of a smooth function and this implies that $\sigma\left(H_{\mathrm{Sym}}\right)\cap [-2, 2+\omega_{\max})$ is recurrent. The claim is a consequence of the lemma below.

Lemma 3.13. *Let* $I \subset [-2, 2+\omega_{\max}]$ *be a closed interval. Then,* $I \cap S^c$ *has positive Lebesgue measure.*

Proof. Let μ_1^+ be the spectral measure of the Anderson model on the half line $[+1, \infty) \cap \mathbb{Z}$, denoted by H_{And}^+, associated to δ_1. Since, almost surely, H_{And}^+ has dense point spectrum in $[-2, 2 + \omega_{\max})$ we have that $\mu_1^+(I) > 0$. It follows from [53, Theorem 1.5] (theorem one in [52], when restated in terms of Borel measures on the real line, would also suffice) that

$$\lim_{t\to\infty} t\left|\{E \in I : |G_{\text{And}}^+(1,1;E)| > t\}\right| > 0. \tag{3.4.17}$$

In particular, by choosing t sufficiently large we may conclude that

$$\left|\{E \in I : |\omega(0) - E - G_{\text{And}}^+(1,1;E)| > 2\gamma\}\right| > 0. \tag{3.4.18}$$

since the above set does not belong to supp $\left(\mu_{\delta_{(0,0)}}\right)$ (this follows from the formula given at the end of section 3.3. An alternative argument is provided by equation (3.4.21) below) the lemma is proven. □

To show the remaining portion of theorem 3.7, having a formula for the spectral measure $\mu_{\delta_{(0,0)}}$ will be useful. Let H_{Sym}^L denote the restriction of H_{Sym} to $\ell^2\left(\{0, ..., L\} \times \mathbb{Z}_+\right)$. As explained in the previous section, for any $L > 0$ we have

$$\begin{aligned}\langle \delta_{(0,0)}, f\left(H_{\text{Sym}}^L\right)\delta_{(0,0)}\rangle &= \frac{2}{\pi}\int_0^\pi \langle \delta_0, f\left(h_p^L\right)\delta_0\rangle \sin^2 p\, dp. \\ &= \frac{2}{\pi}\int_0^\pi \int_{-\infty}^\infty f(E)d\mu^{p,L}(E)\sin^2 p\, dp.\end{aligned}$$

where

$$d\mu^{p,L}(E) = \delta\left(\Sigma_L(E) + 2\gamma\cos p\right)\, dE \tag{3.4.19}$$

and $\Sigma_L(E) = -\frac{1}{G_{\text{And}}^L(0,0;E)}$. In particular,

$$\langle \delta_{(0,0)}, f\left(H_{\text{Sym}}^L\right)\delta_{(0,0)}\rangle = \frac{1}{\pi\gamma}\int_{\{|\Sigma_L(E)|<2\gamma\}} f(E)\sqrt{1 - \frac{\Sigma_L(E)^2}{4\gamma^2}}\, dE. \tag{3.4.20}$$

where we have used the fact that, in this case, $\int_{-1}^1 \delta(\Sigma(E)+v)\, dv$ holds for $E \in \text{supp}(\mu_{\delta_{(0,0)}})$. Letting $L \to \infty$ in the above equation we conclude that

$$d\mu_{\delta_{(0,0)}}(E) = \frac{1}{\pi\gamma}\mathbb{1}_{\{|\Sigma(E)|<2\gamma\}}\sqrt{1 - \frac{\Sigma^2(E)}{4\gamma^2}}\, dE. \tag{3.4.21}$$

Since $\Sigma(E)$ is a smooth function on $[-2, 2+\omega_{\max}]^c$, it follows that $\sigma\left(H_{\text{Sym}}\right) \cap [-2, 2+\omega_{\max}]^c$ is transient, finishing the proof of theorem 3.7.

The above analysis also shows that for any $\omega \in \Omega$ there exist a hopping parameter γ_ω such that $\sigma\left(H_{\text{Sym}}\right) \cap [-2, 2+\omega_{\max}]^c \neq \emptyset$ whenever $\gamma > \gamma_\omega$. Indeed, this follows from the formula (3.4.21) since $\Sigma(E)$ is increasing on $[-2, 2+\omega_{\max}]^c$. The critical values is then $\gamma_\omega = \frac{|\Sigma(E_c)|}{2}$ where $E_c = 2 + \omega_{\max}$. For completeness we provide a proof that this number is non-zero almost surely.

Lemma 3.14. *The Green's function for the half-line Anderson model satisfies*

$$-\Sigma(E) = \omega(0) - E - G^+_{\text{And}}(1,1;E). \tag{3.4.22}$$

Moreover, at the edge $E_c = 2 + \omega_{\max}$ *we have*

$$\Sigma(E_c) \neq 0 \tag{3.4.23}$$

almost surely.

Proof. By definition of the half-line Anderson model H^+_{And}, letting $\varphi = (H^+_{\text{And}} - z)^{-1}\delta_0$ we obtain

$$-\varphi(1) + (\omega(0) - z)\,\varphi(0) = 1.$$

Since $\varphi(0) = G^+_{\text{And}}(0,0;z)$, $\varphi(1) = G^+_{\text{And}}(0,1;z)$ and, by Feenberg's expansion,

$$G^+_{\text{And}}(0,1;z) = G^+_{\text{And}}(0,0;z)G^+_{\text{And}}(1,1;z)$$

and equation (3.4.22) is proven.

For the sake of contradiction, assume that

$$\mathbb{P}\left(\Sigma(E_c) = 0\right) = \delta > 0. \tag{3.4.24}$$

Then, by (3.4.22)

$$\mathbb{P}\left(\omega(0) - E - G^+_{\text{And}}(1,1;E) = 0\right) = \delta > 0$$

Since $G^+_{\text{And}}(1,1;E_c)$ is independent of $\omega(0)$ this would contradict the fact that the distribution of $\omega(0)$ has a density. We conclude that $\mathbb{P}\left(\Sigma(E_c) = 0\right) = 0$.

□

3.5 Absence of Diffusion for $\mathbb{G}_{\text{Diag},\gamma}$: Proof of theorem 3.2

A key step to show absence of diffusion for H_{Diag} is proving that, on expectation, the fractional moments of the Green's function of this operator decay exponentially. We start with a particular case.

Lemma 3.15. *Let $s \in (0,1)$ be given. For all $m \in \mathbb{Z}_+$, $r \in \{0,1,...,\ell\}$ and $z \in \mathbb{C}^+$ we have*

$$\mathbb{E}_{\omega(0),...,\omega(m)}\left(|G_{\text{Diag}}\left((0,0),(m,m(\ell+1)+r);z\right)|^s\right) \leq C_{\text{AP}}\left(C_{\text{AP}}\gamma^s\right)^m. \tag{3.5.1}$$

where C_{AP} is the constant in the a priori bound (3.8.4).

Proof. Fix $r \in \{0,1,...,\ell\}$, $z \in \mathbb{C}^+$ and let $x_{m,r} = (m, m(\ell+1)+r)$. Let us proceed by induction in m. When $m = 0$ the statement reduces to the a priori bound verified in the appendix, equation (3.8.4). Suppose that the desired conclusion holds for some $m \in \mathbb{Z}^+$ and recall the factorization

$$G\left(0, x_{m+1,r}; z\right) = \gamma G\left(0, x_{m,\ell}; z\right) G_+\left(x_{m+1,0}, x_{m+1,r}; z\right). \tag{3.5.2}$$

Taking absolute values on both sides of (3.5.2), raising to the power s and taking expectations we reach

$$\mathbb{E}\left(|G\left(0, x_{m+1,r}; z\right)|^s\right) = \gamma^s \mathbb{E}\left(|G\left(0, x_{m,\ell}; z\right)|^s |G_+\left(x_{m+1,0}, x_{m+1,r}; z\right)|^s\right). \tag{3.5.3}$$

Integrating out the variables $\omega(0),...,\omega(m)$, using the inductive assumption and the fact that $G_+\left(x_{m+1,0}, x_{m+1,r}; z\right)$ only depends on $\omega(j)$ for $j > m$, we obtain from (3.5.3) that

$$\mathbb{E}\left(|G\left(0, x_{m+1,r}; z\right)|^s\right) \leq \left(C_{\text{AP}}\gamma^s\right)^{m+1} \mathbb{E}\left(G_+\left(x_{m+1,0}, x_{m+1,r}; z\right)|^s\right). \tag{3.5.4}$$

Making use of the *a priori* bound (3.8.4) one more time finishes the proof. □

Making use of the Green's function decay for the one-dimensional Anderson model we readily obtain that

$$\mathbb{E}\left(|G\left((0,0),(m+j,m(\ell+1)+r);z\right)|^s\right) \leq C_{\mathrm{AP}}C_{\mathrm{And}}\left(C_{\mathrm{AP}}\gamma^s\right)^m e^{-\mu_{\mathrm{And}}j}. \tag{3.5.5}$$

Pick γ small enough so that

$$C_{\mathrm{AP}}\gamma^s < e^{-\mu_{\mathrm{And}}(\ell+2)}. \tag{3.5.6}$$

Then, (3.5.5) implies

Lemma 3.16. *For any $s \in (0,1)$ there exist a constant C_{Diag} such that for all $n \in \mathbb{G}_{\mathrm{Diag}}$ and $z \in \mathbb{C} \setminus \mathbb{R}$*

$$\mathbb{E}\left(|G_{\mathrm{Diag}}\left(0,n;z\right)|^s\right) \leq C_{\mathrm{Diag}}e^{-\mu_{\mathrm{Diag}}|n|} \tag{3.5.7}$$

with $C_{\mathrm{Diag}} = e^{\mu_{\mathrm{And}}\ell}C_{\mathrm{AP}}C_{\mathrm{And}}$ and $\mu_{\mathrm{Diag}} = \mu_{\mathrm{And}}$.

Proof. The above argument immediately yields the desired conclusion for $z \in \mathbb{C}^+$. By symmetry, reasoning along the same lines we obtain (3.5.7) with $|G_{\mathrm{Diag}}\left(0,n;z\right)|$ replaced by $|G_{\mathrm{Diag}}\left(n,0;z\right)|$ and $z \in \mathbb{C}^+$. Since $G_{\mathrm{Diag}}\left(n,0;z\right) = G_{\mathrm{Diag}}\left(0,n;\overline{z}\right)$ the proof is finished. □

3.6 Upper bounds for quantum dynamics revisited

We are now ready to prove theorem 3.2. Our approach is a combination of the fractional moment method and techniques used by Jitomirskaya and Schulz-Baldes in [40]. Let $q > 0$ and, to simplify the notation, let $H = H_{\mathrm{Diag}}$ throughout this section. Recall that

$$M_T^q = \frac{2}{T}\int_0^\infty e^{\frac{-2t}{T}}\,\mathbb{E}\left\langle 0|e^{itH_\omega}|X|^q e^{-itH_\omega}|0\right\rangle\,dt. \tag{3.6.1}$$

The main advantage of working with time-averaged moments is that they can be directly related to the Green's function via the following formula.

$$M_T^q = \frac{1}{\pi T}\sum_{n\in\mathbb{Z}}|n|^q\int_{\mathbb{R}}\mathbb{E}|G(n,0;E+\frac{i}{T})|^2\,dE. \tag{3.6.2}$$

To verify that (3.6.2) holds, following the proof of lemma 3.2 in [42], one can observe that

$$\int_0^\infty e^{\frac{-2t}{T}}|\left\langle 0|e^{itH_\omega}|n\right\rangle|^2\,dt$$

is the L^2 norm of the function

$$g(t) = e^{-\frac{t}{T}} \int_{\mathbb{R}} e^{iE't}\, d\mu_{0,n}(E')$$

restricted to the positive real line. Let

$$\hat{g}(E) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-iEt} g(t)\, dt \tag{3.6.3}$$

be the Fourier transform of g. It is easy to check that

$$\hat{g}(E) = \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{d\mu_{0,n}(E')}{E' - E + \frac{i}{T}}. \tag{3.6.4}$$

By definition of the spectral measure $\mu_{0,n}$ we discover that

$$\hat{g}(E) = \frac{i}{\sqrt{2\pi}} G(0, n; E - \frac{i}{T}). \tag{3.6.5}$$

Equation (3.6.2) then readily follows from Plancherel's identity.

Returning to the proof theorem 3.2, another useful observation is that, as in to [40, Proposition 1], it suffices to show the desired upper bound for the related quantity

$$M_T^q(E_0, E_1) = \frac{1}{\pi T} \sum_{|n|<T^\alpha} |n|^q \int_{E_0}^{E_1} \mathbb{E}|G(n, 0; E + \frac{i}{T})|^2\, dE$$

whenever $\alpha > 1$ and $\sigma\left(H_{\mathrm{Diag}}\right) \subset (E_0, E_1)$.

To this end, we shall combine the estimate $|G(n, 0; E + \frac{i}{T})| \leq T$ to the fractional moment bound showed in the previous section to reach

$$\mathbb{E}|G(n, 0; E + \frac{i}{T})|^2 \leq C_{\mathrm{Diag}} e^{-\mu_{\mathrm{Diag}}|n|} T^{2-s}.$$

Thus, for any $N \in \mathbb{N}$ we have that

$$\sum_{|n|>N} \mathbb{E}|G(n, 0; E + \frac{i}{T})|^2 \leq C_{\mathrm{Diag}} T^{2-s} \sum_{|n|>N} e^{-\mu_{\mathrm{Diag}}|n|}. \tag{3.6.6}$$

By following exactly the same steps in [40] we can now finish the proof. Namely, we start with the general result

Lemma 3.17. *([40, Lemma 5])*

$$\sum_{|n|\le N} |n|^q \frac{1}{\pi T}\int_{E_0}^{E_1} \mathbb{E}|G(n,0;E+\frac{i}{T})|^2\,dE \le N^q.$$

applied with $N = \lfloor \log T \rfloor$ to obtain

$$\sum_{|n|\le N} |n|^q \frac{1}{\pi T}\int_{E_0}^{E_1} \mathbb{E}|G(n,0;E+\frac{i}{T})|^2\,dE \le (\log T)^q. \tag{3.6.7}$$

On the other hand, using equation (3.6.6) followed by the explicit bound on the exponentially decaying sum [40, Lemma 2] we obtain, with $E_+ := E_1 - E_0$

$$\begin{aligned}\sum_{|n|>\log T} \frac{1}{\pi T}\int_{E_0}^{E_1} \mathbb{E}|G(n,0;E+\frac{i}{T})|^2\,dE &\le C_{\text{Diag}} E_+ T^{1-s} \sum_{|n|>\log T} |n|^q e^{-\mu_{\text{Diag}} N} \\ &\le C_{\text{Diag}} E_+ T^{1-s} e^{-\mu_{\text{Diag}}(\log T)} \\ &\quad \times \frac{2(q+1)!}{\mu_{\text{Diag}}}((\log T)+(\mu_{\text{Diag}})^{-1})^{q+1} \\ &\le C(s,q).\end{aligned}$$

whenever s is chosen so that $1-s<\mu_{\text{Diag}}$ (we remark that this choice is possible since $\mu_{\text{Diag}} = \mu_{\text{And}}$ and the later can be taken proportional to s, see [6, Theorem 12.11 and Equation (12.86)]). Combining the above estimate with (3.6.7) we finish the proof of theorem 3.2

3.6.1 Paking Dimension: Proof of Corollary 3.3

Before proving corollary 3.3, recall that the upper packing dimension of a Borel probability measure μ is defined as

$$\dim_P^+(\mu) := \mu - \operatorname{ess\,sup}_{E\in\mathbb{R}} \overline{d_\mu}(E) \tag{3.6.8}$$

where, for $E \in \operatorname{supp}\mu$,

$$\overline{d_\mu}(E) := \limsup_{\varepsilon\to 0} \frac{\log\left(\mu[E-\varepsilon,E+\varepsilon]\right)}{\log(2\varepsilon)}. \tag{3.6.9}$$

If $E \notin \operatorname{supp}\mu$ we set $\overline{d_\mu}(E) = \infty$. We shall denote by $\dim_P^+(H_{\mathrm{Diag},\omega})$ the packing dimension $\dim_P^+(\mu_{H_{\mathrm{Diag}}})$ where $\mu_{H_{\mathrm{Diag}}}$ is the spectral measure associated to $H_{\mathrm{Diag},\omega}$ and the vector $\delta_{(0,0)}$.

Proof. (of corollary 3.3) From [35, Theorem 1] we have that, for each $\omega \in \Omega$,

$$\dim_P(H_{\mathrm{Diag},\omega}) \leq \limsup_{T\to\infty} \frac{\log\left(M_T^q(H_{\mathrm{Diag},\omega})\right)}{q\log T}.$$

Taking expectations and a subsequence $T_k \to \infty$ which realizes the lim sup we obtain, by Fatou's lemma and Jensen's inequality,

$$\mathbb{E}\left(\dim_P(H_{\mathrm{Diag}})\right) \leq \limsup_{T_k\to\infty} \frac{\log \mathbb{E}\left(M_{T_k}^q(H_{\mathrm{Diag},\omega})\right)}{q\log T_k}$$

where, by theorem 3.2, the limit in the right-hand side equals zero. □

3.7 A Version of Boole's Equality for level sets of Heglotz functions: proof of proposition 3.5

Consider $F(E) = \sum_{n=1}^{N} \frac{p_n}{u_n - E}$ for real numbers $\{u_n\}_{n=1}^N, \{p_n\}_{n=1}^N$. This is the specific form taken by the diagonal elements of the Green's function in finite volume. For a real number α, let Q_α be a polynomial of degree $N+1$ given by

$$Q_\alpha(E) = (\alpha - E - F(E))\prod_{n=1}^{N}(E - u_n).$$

Note that the solutions $v_1, ..., v_{N+1}$ of the equation $E + F(E) = \alpha$ coincide the roots of Q_α. Therefore, the coefficient of E^N in the expansion $Q_\alpha(E) = -\prod_{n=1}^{N+1}(E - v_n)$ equals $\sum_{n=1}^{N+1} v_n$. On the other hand, by definition of Q_α, this coefficient must be $\alpha + \sum_{n=1}^{N+1} u_n$, thus

$$\sum_{n=1}^{N+1} v_n = \alpha + \sum_{n=1}^{N} u_n. \tag{3.7.1}$$

Replacing α by β we conclude the solutions $w_1, ..., w_{N+1}$ of $E + F(E) = \beta$ satisfy

$$\sum_{n=1}^{N+1} w_n = \beta + \sum_{n=1}^{N} u_n. \tag{3.7.2}$$

The set $\{E \in \mathbb{R} \, : \alpha \leq E+F(E) \leq \beta\}$ is a disjoint union of intervals $\cup_{n=1}^{N+1}[v_n, w_n]$ therefore, we may conclude from equations (3.7.1) and (3.7.2) that

$$\begin{aligned}\big|\{E \in \mathbb{R} \, : \alpha \leq E + F(E) \leq \beta\}\big| &= \sum_{n=1}^{N+1} (w_n - v_n) \\ &= \beta - \alpha.\end{aligned}$$

The following argument is inspired by the analysis in [6, Proposition 8.2] and provides a proof which is also valid in the infinite-volume context.

Proof. Since μ is assumed to be purely singular, the boundary values $F(E + i0)$ are real numbers for almost every $E \in \mathbb{R}$ (see [6, Proposition B3]). Note that the indicator function $\mathbb{1}\{E : \alpha < E + F(E + i0) < \beta\}$ can be represented as $\frac{\phi_{\alpha,\beta}(E+F(E+i0))}{\pi}$ for

$$\phi_{\alpha,\beta}(z) = \mathrm{ImLog}(z - \beta) - \mathrm{ImLog}(z - \alpha) \tag{3.7.3}$$

and where Log denotes the principal branch of the logarithm. Let

$$\psi_\eta(E) = \frac{\eta^2}{E^2 + \eta^2}.$$

We then have, by dominated convergence and definition of $\phi_{\alpha,\beta}$,

$$|\{E \in \mathbb{R} \, : \alpha \leq E + F(E + i0) < \beta\}| = \lim_{\eta\to\infty} \int_{-\infty}^{\infty} \pi^{-1}\psi_\eta(E)\phi_{\alpha,\beta}(E + F(E + i0))\, dE.$$

Using dominated convergence one more time and recovering $\phi_{\alpha,\beta}$ from its boundary values,

$$\begin{aligned}\lim_{\eta\to\infty} \int_{-\infty}^{\infty} \pi^{-1}\psi_\eta(E)\phi_{\alpha,\beta}(E + F(E + i0))\, dE &= \lim_{\substack{\eta\to\infty \\ \varepsilon\to 0^+}} \eta \int_{-\infty}^{\infty} \pi^{-1}\psi_\eta(E) \\ &\quad \times \phi_{\alpha,\beta}(E + F(E + i\varepsilon))\, dE \\ &= \lim_{\substack{\eta\to\infty \\ \varepsilon\to 0^+}} \eta\phi_{\alpha,\beta}(i\eta + F(i\eta + i\varepsilon)) \\ &= \lim_{\eta\to\infty} \eta\phi_{\alpha,\beta}(i\eta + F(i\eta)).\end{aligned}$$

On the other hand, by definition of $\phi_{\alpha,\beta}$ we know

$$\phi_{\alpha,\beta}(i\eta + F(i\eta)) = \lim_{\eta\to\infty} \mathrm{Im} \int_\alpha^\beta \frac{1}{E - i\eta - F(i\eta)}\, dE \tag{3.7.4}$$

hence

$$\begin{aligned}
\lim_{\eta\to\infty} \eta\phi_{\alpha,\beta}(i\eta + F(i\eta)) &= \lim_{\eta\to\infty} \mathrm{Im} \int_\alpha^\beta \frac{\eta}{E - i\eta - F(i\eta)}\, dE \\
&= \lim_{\eta\to\infty} \int_\alpha^\beta \frac{\eta^2 + \eta \mathrm{Im} F(i\eta)}{(E - \mathrm{Re}F(i\eta))^2 + (\eta + \mathrm{Im}F(i\eta))^2}\, dE \\
&= \beta - \alpha
\end{aligned}$$

where we have made use of the simple fact that $\lim_{\eta\to\infty} F(i\eta) = 0$ and $\lim_{\eta\to\infty} \eta \mathrm{Im} F(i\eta) = \mu(\mathbb{R})$.

□

3.8 *A priori* bounds on the Green's function

To obtain *a-priori* bounds for the Green's function, we will make use of the weak L_1 bound [2, Lemma 3.1]. This bound is valid for any maximally dissipative operator A on a separable Hilbert space $\mathscr{H}$ and Hilbert-Schmidt operators $M_1 : \mathscr{H} \to \mathscr{H}_1$ and $M_2 : \mathscr{H}_1 \to \mathscr{H}$ where $\mathscr{H}_1$ is another separable Hilbert space. Recall that a densely defined operator A is said to be dissipative if $\mathrm{Im}\langle \varphi, A\varphi \rangle \geq 0$ for every $\varphi \in D(A)$. A is said to be maximally dissipative when it is dissipative and has no proper dissipative extension. Denoting by $|\cdot|$ Lebesgue measure and by $\|\cdot\|_{HS}$ the Hilbert-Schmidt norm, we have

Lemma 3.18.

$$\left| \{ v \; : \|M_1 \frac{1}{A - v + i0} M_2\|_{HS} > t \} \right| \leq C_W \|M_1\|_{HS} \|M_2\|_{HS} \frac{1}{t} \tag{3.8.1}$$

where the constant C_W is independent of A, M_1 and M_2.

For a proof, we refer to [2, Lemma 3.1]. Another fact, which can be found in [2, Proposition 3.2], is

Lemma 3.19. *Let A,M_1 and M_2 be as above and let U_1, U_2 be nonnegative operators.*

$$\left|\{(v_1, v_2) \in [0,1]^2 \ : \|M_1 U_1^{1/2} \frac{1}{A - v + i0} U_2^{1/2} M_2\|_{HS} > t\}\right| \le 2C_W \|M_1\|_{HS} \|M_2\|_{HS} \frac{1}{t} \tag{3.8.2}$$

We also make use of the Birman-Schwinger relation [2, Lemma B.1]

$$P\left(z - H - vP\right)^{-1} P = \Big(P(z-H)^{-1}P - v\Big)^{-1} \tag{3.8.3}$$

for a projection P onto a subspace of $\mathscr{H}$ and the fact that $P(z-H)^{-1}P$ is maximally dissipative on $(\text{Ker}P)^{\perp}$, where the equality in question holds. It immediately follows that

$$\left|\{v \ : \|M_1 P\left(z - H - vP\right)^{-1} P M_2\|_{HS} > t\}\right| \le C_W \|M_1\|_{HS} \|M_2\|_{HS} \frac{1}{t}.$$

equations (3.8.1), (3.8.2) and (3.8.3) together easily imply following apriori bound, for details we refer the reader to [51, appendix A]

$$\mathbb{E}_{m,n}\left(|G_\Lambda^{\text{Diag}}(m,n;z)|^s\right) \le C_{\text{AP}}(s) \tag{3.8.4}$$

and we remark that the constant can be taken as

$$C_{\text{AP}}(s) = \max\left\{\frac{(2C_W \omega_{\max}\|\rho\|_\infty)^s}{1-s}, \frac{\left(4C_W \omega_{\max}^2 \|\rho\|_\infty^2\right)^s}{1-s}\right\}.$$

APPENDIX

APPENDIX

TECHNICAL COMMENTS

A.1 Conditions on the density ρ imposed in chapter 2

Assume that for some $M > 0$ and all $v \in \mathbb{R}$

$$\Big|\frac{d}{dv}\log\rho(v)\Big| \leq M.$$

This condition is satisfied, for instance, by the Cauchy and exponential distributions. Then

$$\begin{aligned}\frac{\rho(x)}{\rho(v)} &= e^{\log\rho(x)-\log\rho(v)} \\ &\geq e^{-M|x-v|}.\end{aligned}$$

Letting $h(v) = e^{-c|v|}$ with $c > 0$ to be determined. The following estimate holds:

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{\rho(x)}{\rho(v)}h(x-v)\,dx &\geq \int_{-\infty}^{\infty} e^{-(M+c)|x-v|}\,dx \\ &= \frac{2}{M+c}.\end{aligned}$$

therefore, letting $\bar{\rho}(v) = \frac{\rho(v)}{\int_{-\infty}^{\infty}\rho(x)h(x-v)\,dx}$, we have shown that $\bar{\rho}(v) \leq \frac{M+c}{2}$ for all $v \in \mathbb{R}$. Let us now verify that $\bar{\rho}$ decays in two cases, starting with when ρ is an exponential. For simplicity, let us ignore normalization factors and let $\rho(v) = e^{-|v|}$. Then, by the triangle inequality,

$$\begin{aligned}\int_{-\infty}^{\infty}\rho(x)h(x-v)\,dx &\geq e^{c|v|}\int_{-\infty}^{\infty} e^{-(1+c)|x|}\,dx \\ &= \frac{2e^{c|v|}}{1+c}.\end{aligned}$$

Thus, in this situation $\bar{\rho}(v) \leq \frac{1+c}{2}e^{-(1+c)|v|}$. Let us now assume that $\rho(v) = e^{-v^2}$ and $h(v) = e^{-cv^2}$ with $c > 0$ to be determined. In this case, we have that

$$\rho(v)h(x-v) \geq e^{-(1+2c)x^2-2cv^2}$$

thus

$$\int_{-\infty}^{\infty} \rho(v)h(x-v)\,dx \geq e^{-2cv^2}\int_{-\infty}^{\infty} e^{-(1+2c)x^2}\,dx$$
$$= \frac{\sqrt{\pi}e^{-2cv^2}}{\sqrt{1+2c}}.$$

In particular, whenever $2c < 1$ we may conclude that $\bar{\rho}$ decays according to

$$\bar{\rho}(v) \lesssim e^{-(1-2c)v^2}.$$

A.2 Derivation of the self-consistent equations

Let $f(\Gamma) = \mathrm{Tr}\,(\Gamma \log \Gamma)$ where $0 < \Gamma < 1$ is a symmetric matrix. Then, given H which is also symmetric with $0 < H < 1$ we have

$$f(\Gamma + H) = f(\Gamma) + \mathrm{Tr}\left(\Gamma \log\left(I + \Gamma^{-1}H\right) + H\log\Gamma(I + \Gamma^{-1}H)\right).$$

Using the power expansion of the logarithm $\log(A) = A - \frac{1}{2}A^2 + ...$ we may conclude that

$$f(\Gamma + H) = f(\Gamma) + \mathrm{Tr}\,(H(I + \log\Gamma) + O(\|H\|^2).$$

therefore, we have the following expression for the Frechét derivative of f

$$f'(\Gamma) = \log\Gamma + I.$$

Similarly, the derivative of the map $\Gamma \mapsto f(1-\Gamma)$ equals $-(I-\Gamma)\log(I-\Gamma) - I$. We may conclude that the entropy $\mathcal{S}(\Gamma) = -\mathrm{Tr}\,(\Gamma\log\Gamma + (1-\Gamma)\log(1-\Gamma))$ satisfies

$$\mathcal{S}'(\Gamma) = -\left(\log\Gamma - \log(I-\Gamma)\right).$$

Moreover, the energy functional

$$\mathcal{E}(\Gamma) = \mathrm{Tr}\,(H_0 - \kappa + \lambda V_\omega)\,\Gamma + g\sum_n \langle n|\Gamma_\uparrow|n\rangle\ \langle n|\Gamma_\downarrow|n\rangle\,,$$

can be more succinctly expressed as

$$\mathcal{E}(\Gamma) = \mathrm{Tr}\left((H_0 + \lambda V_\omega)\,\Gamma + g\mathrm{Diag}(\Gamma_\uparrow)\mathrm{Diag}(\Gamma_\downarrow)\right)$$

from which we see that

$$\mathcal{E}'(\Gamma) = \begin{pmatrix} H_0 + \lambda V_\omega + g\mathrm{Diag}(\Gamma_\downarrow) & 0 \\ 0 & H_0 + \lambda V_\omega + g\mathrm{Diag}(\Gamma_\downarrow) \end{pmatrix}.$$

In particular, we see from the above equations that whenever $\mathcal{E}'(\Gamma) - \beta^{-1}\mathcal{S}'(\Gamma) = 0$ the matrix Γ must be such that

$$\Gamma(I-\Gamma)^{-1} = \begin{pmatrix} \exp\big(-\beta(H_0 + \lambda V_\omega + g\mathrm{Diag}(\Gamma_\downarrow)\big) & 0 \\ 0 & \exp\big(-\beta(H_0 + \lambda V_\omega + g\mathrm{Diag}(\Gamma_\downarrow)\big) \end{pmatrix}.$$

from which we conclude that

$$\Gamma = \begin{pmatrix} \big(1+\exp\big(\beta H_\downarrow\big)\big)^{-1} & 0 \\ 0 & \big(1+\exp\big(\beta H_\uparrow\big)\big)^{-1} \end{pmatrix}$$

where $H_\downarrow = H_0 + \lambda V_\omega + g\mathrm{Diag}(\Gamma_\downarrow)$ and $H_\uparrow = H_0 + \lambda V_\omega + g\mathrm{Diag}(\Gamma_\uparrow)$.

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