INTEGRO-DIFFERENTIAL OPERATORS: CONNECTIONS TO DEGENERATE ELLIPTIC EQUATIONS AND SOME FREE BOUNDARY PROBLEMS

By

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ABSTRACT

INTEGRO-DIFFERENTIAL OPERATORS: CONNECTIONS TO DEGENERATE ELLIPTIC EQUATIONS AND SOME FREE BOUNDARY PROBLEMS

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In this dissertation, we study aspects of integro-differential operators, and how they relate to different types of equations. In each case, we use information and results about the operators in a lower dimension to analyse an equation in a higher dimension, and vice-versa. We begin in chapter 1 with an introduction to the operators and equations we will be considering.

In Chapters 2 and 3, we discuss certain integro-differential operators of functions in a relatively smooth space like $C^{1,\alpha}(\mathbb{R}^n)$. However, to understand more about the structure of these operators, particularly about the measure associated with them, we study certain equations in a higher dimension such as degenerate elliptic equations in the upper half space. We analyse the solution of such an equation and its gradient, followed by estimates on its Green's function and Poisson kernel. These estimates then help reveal some properties of the measure associated with the integro-differential operator in the lower dimension. The structure of the degenerate elliptic equations is similar to that of uniformly elliptic equations, but with an additional complexity of a term which involves distance to the boundary. This degeneracy complicates the analysis; as such, the classical techniques of finding pointwise estimates as mentioned above do not work so well anymore. So we provide some revised results for the same. Thus understanding an equation in a higher dimension gives us information about an integro-differential operator in a lower dimension.

In Chapters 4 and 5, we prove some results about the solutions of free boundary problems

in $\mathbb{R}^{n+1} \times [0, T]$, where the free boundary for a fixed time t can be seen as the graph of a function over a sphere. This time, we connect the solution of the free boundary problem to the solution of a parabolic equation on the sphere – that is, in a lower dimension. This parabolic equation involves an integro-differential operator, which has a min-max representation that is consistent with all the results about viscosity solutions of parabolic equations in $\mathbb{R}^n$. We modify these results for parabolic equations on the sphere, which then gives us existence and uniqueness results about the free boundary problem in a higher dimension.

For Amma, the most resilient person I know.

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TABLE OF CONTENTS

LIST OF FIGURES

KEY TO SYMBOLS

- $\mathbb{R}^{n+1}_+$, the half space in dimension $n+1$, i.e. $\mathbb{R}^{n+1}_+ = \{X = (x, y) : x \in \mathbb{R}^n, y \in \mathbb{R}, y > 0\}$. We sometimes use, $N = n + 1$ for brevity.
- $\partial\mathbb{R}^N_+ = \mathbb{R}^n$, the boundary of the half space.
- $\partial_i = \frac{\partial}{\partial x_i}$, the partial derivative with respect to x_i. For a function u, we will also sometimes denote this as u_i or u_{x_i}. Further, $\nabla u = (u_{x_1}, u_{x_2}, \cdots, u_{x_n})$.
- Q_R, a cube centred at 0 in $\mathbb{R}^N_+$ of side-length R.
- $Q^+_R = Q_R \cap R^+_N$.
- $\mathbb{S}^n$, the n-dimensional sphere.
- ∂_ν, the normal derivative. The notation is also used later for the co-normal derivative.
- A, a uniformly elliptic matrix, and $\tilde{A}$, the matrix with weights. In this work, $\tilde{A} = y^a A$, where $a \in (0, 1)$ or $a \in (-1, 1)$, as specified.
- p.v., Principal Value integral, i.e. if x_0 is a point of singularity for an integral $\int_\Omega f dx$, then
$$p.v. \int_\Omega f dx = \lim_{\varepsilon \to 0} \int_{\Omega \backslash B_\varepsilon} f dx$$
- σ, the Lebesgue measure.
- $||f||_p$, the L^p-norm, i.e.
$$||f||_p := \left(\int_\Omega |f(x)|^p dx \right)^{1/p}$$
- $L^p(\Omega) = \{f | f : \Omega \to \mathbb{R},\ ||f||_p < \infty\}$.
- $L^p(\Omega, w)$, the weighted L^p-space.
- $\text{Lip}(\overline{\Omega}) = \{f : \overline{\Omega} \to \mathbb{R}, f \text{ Lipschitz}\}$.
- $H^{1,p}(\Omega)$, the Sobolev space of L^p functions whose weak first derivatives are also in L^p.
- $D(u, v)$, the Dirichlet form, i.e. $\int_\Omega a^{ij}(x)u(x)v(x)dx$.
- $\text{cap}(E)$, the capacity of a set E.
- $[f]_{C^\gamma}$, the γ-th Hölder seminorm of f, i.e. $[f]_{C^\gamma} = \sup\limits_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\gamma}$.

- $C^{1,\gamma}(\Omega) = \{f : \Omega \to \mathbb{R} : ||f||_{L^\infty} + ||\nabla f||_{L^\infty} + [\nabla f]_{C^\gamma} < \infty\}$.
  (equivalently)

$$C^{1,\gamma}(\Omega) = \left\{ f \in L^\infty(\Omega) : \sup_{z\in\Omega} \sup_{r>0} r^{-1-\gamma} \inf_{\substack{P(x)=c+p\cdot x \\ c\in\mathbb{R},\ p\in\mathbb{R}^n}} ||f - P||_{L^\infty(B_r(z))} < \infty \right\}.$$

- For $\gamma \in (0,1), \delta > 0, m > 0$, the convex set $\mathcal{K}(\gamma, \delta, m)$ is defined as
  $\mathcal{K}(\gamma, \delta, m) := \{f \in C^{1,\gamma}(\Omega) : f(x) \geq \delta \ \forall x \in \Omega \text{ and } ||f||_{C^{1,\gamma}(\Omega)} \leq m\}$.

- $\mathcal{K}(\gamma, \delta) = \bigcup_{m>0} \mathcal{K}(\gamma, \delta, m)$.

- $\mathcal{K}(\delta) = \bigcup_{\gamma\in(0,1)} \bigcup_{m>0} \mathcal{K}(\gamma, \delta, m)$.

- For $f : \mathbb{R}^n \to \mathbb{R}$, $\quad D_f = \{(x, y) \in \mathbb{R}^{n+1}, 0 < y < f(x)\}$

- $\Gamma_f = \{(x, y) \in \mathbb{R}^{n+1}, y = f(x)\}$

- Similarly, when $\delta \leq f \leq L$,

$$D_f^+ = \{(x, y) \in \mathbb{R}^{n+1}, 0 < y < f(x)\} \text{ and } D_f^- = \{(x, y) \in \mathbb{R}^{n+1}, f(x) < y < L\}$$

- ν is the inward facing normal to the boundary of the set D_f^+, and

-

$$\partial_\nu^+ u = \lim_{t\to 0^+} \frac{u(X_0 + t\nu(X_0)) - u(X_0)}{t},$$
$$\partial_\nu^- u = -\lim_{t\to 0^+} \frac{u(X_0 - t\nu(X_0)) - u(X_0)}{t}.$$

Chapter 1

Introduction

1.1. Degenerate elliptic equations and integro-differential operators

The first area of my work is in degenerate elliptic equations on a half-space and their Dirichlet to Neumann (D-to-N) maps expressed as integro-differential operators. The D-to-N map appears in wide range of contexts in analysis, probability, Calderón or inverse problems, and mathematical physics. For example, the D-to-N map arises naturally in the study of operators that describe boundary processes of diffusions in a bounded domain. The map is also central to studying free boundary problems, acting as the natural quantity that drives the free boundary. It is a fundamental object in problems relating voltage to current, where the D-to-N map takes the voltage on the boundary and gives the resulting current density on the boundary. When D-to-N maps are expressed as integro-differential operators, I study the properties of the measure associated with this representation; in particular its relationship with the Lebesgue measure and the density in question. To analyse these properties I proved some estimates of the Green's function associated with these equations, as well as the connection between the Green's function and the harmonic measure for the equation.

There is a connection between certain partial differential equations on some "nice" domain (like the half space) of the form

$$\begin{aligned} Lu = -\partial_j(a_{ij}\partial_i u) = 0 \quad \text{in } \Omega \\ u = f \quad \text{on } \partial\Omega \end{aligned} \tag{1.1.1}$$

and operators of the kind

$$\mathcal{I}(f,x) = \int_{\partial\Omega} f(h) - f(x)\mu(x,dh) \tag{1.1.2}$$

for a_{ij} and μ that satisfy certain conditions. For example, one possible connection between the two is that the operator $\mathcal{I}$ in (1.1.2) will often arise as the D-to N map for (1.1.1). These operators have generated considerable interest lately and it is important to understand when the two situations overlap. We can ask the following two questions – for what μ does the representation in (1.1.2) hold? And what can we say about the order of $\mathcal{I}$, i.e. is $\mathcal{I}(u(r\cdot),x) = r^{\alpha}\mathcal{I}(u,rx)$? for some α? Classically, it is known that when L in (1.1.1) is the Laplacian, then the corresponding D-to-N map $\mathcal{I}$ will have the structure in (1.1.2), and it is the 1/2-Laplacian of the boundary data f. Subsequently, it is equally natural to consider the $\alpha/2-$Laplacian for some α instead of the $1/2-$Laplacian. As for μ, it is typically reflected in a weight such as $\mu(x,dh) \approx |h|^{-(n+\alpha)}dh$ in this case, as opposed to $\mu(x,dh) \approx |h|^{-(n+1)}dh$ in case of the 1/2-Laplacian. However, to make the connection between the D-to-N map $\mathcal{I}$ and the equation in (1.1.2), one must study a weighted equation, which consists of $L = \text{div}(y^a\nabla u)$. This has been studied by many authors, for example, Caffarelli and Silvestre [4] and in the context of probability theory, by Song and Vondraček in [36], and also in the book, ‘Bernstein functions. Theory and applications’ [31] by Schilling, Song, and Vondraček.

The main goal of the following chapters (2 and 3) is to generalize these results for operators that are not translation invariant, and thus we present work on the weighted extensions $Lu = \mathrm{div}(y^a A\nabla u)$, where $a \in (-1,1)$, and the matrix $A(X) = (a_{ij}(X))$ satisfies the following conditions:

- *Uniform ellipticity* i.e. $\exists \lambda, \Lambda > 0$ for all $X, \xi \in \mathbb{R}^{n+1}_+$, $\lambda|\xi|^2 \leq a_{ij}(X)\xi_i\xi_j \leq \Lambda|\xi|^2$.
- A modified version of the *Dini-continuity* given in [22, Section 3]; we have for $X = (x,y)$ and $Z = (z,s)$

  $$|y^a A(X) - t^a A(Z)| \leq \omega(|X - Z|),$$

  where ω is a type of Dini-modulus of continuity, particular to our degenerate equation. ω satisfies the properties in [22, Section 3], and additionally,

  $$\int_\Omega y^{-a}\frac{\omega(|X-Z|)}{|X-Z|^N}dX \leq C.$$

Thus, the equation we will study in this work is:

$$\begin{aligned} Lu = -\partial_j(y^a a_{ij}\partial_i u) = 0 \quad &\text{in } \mathbb{R}^{n+1}_+ \\ u = f \quad &\text{on } \partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n \end{aligned} \tag{1.1.3}$$

The uniformly elliptic equation ($a = 0$) has been studied by Guillen, Kitagawa, and Schwab [23]. In this case solutions behave linearly at the boundary, yet the analysis is quite delicate. However, in the weighted setting, we lose uniform ellipticity and the equation degenerates as we approach the boundary. As such, the techniques of [23] no longer apply, the normal derivative can blow up, solutions behave like y^{1-a} instead of being linear, and the analysis becomes more complicated as the proofs to many foundational results need to be adapted to these new challenges. So far we have been able to overcome these technical difficulties to the following result.

Theorem 1.1.1. *In equation* (1.1.2) *above,* μ *satisfies the following conditions*

(a) μ *is absolutely continuous with respect to the surface measure,* σ*, i.e. for all* $X \in \partial\Omega$, $\mu(X, \cdot)$ *has a density,* $\mu(X, dh) = K(X, h)\sigma(dh)$.

(b) There exist universal constants $C_1, C_2 > 0$ *so that* $\forall X, h \in \partial\mathbb{R}^{n+1}_+$, $X \neq h$

$$K(X, h) \leq C_2|X - h|^{-n+a-1}.$$

This extends the uniformly elliptic case treated in [23, Theorem 1.1] to the degenerate elliptic case, for which I have established part (a) and the bound in part (b) for $a \geq 0$. The proof of the result in [23] relies on two key steps, namely the relationship between the harmonic measure and the Green's function found in [3], as well as boundary estimates for the Green's function found in [22]. When $a < 0$, we see that writing analogous proofs to [22, Lemmas 3.1,3.2] seem to fail as the estimates blow up the closer you get to the boundary.

To prove the first condition on μ, we prove two important lemmas. The first of these gives a relationship between the harmonic measure and the Green's function – not unlike the well-known result for the uniformly elliptic equation ($a = 0$) which comes from [3, Lemma 2.2] – but with a change reflecting the term y^a. Namely, for $x \in \mathbb{R}^n$, $y \in \mathbb{R}^{n+1}_+ \setminus B_{sr}$ for some $s > 1$, we have constants c_1, c_2 such that

$$C_1 r^{(n+1)+a-2} G(y, x + r\nu(x)) \leq \omega_y(\partial\Omega \cap B_r(x)) \leq C_2 r^{(n+1)+a-2} G(y, x + r\nu(x)) \quad (1.1.4)$$

where $\nu(x)$ represents the inward normal vector at $x \in \partial\mathbb{R}^{n+1}_+$. The above result for certain classes of degenerate elliptic equations has been established in [20].

The second result is to obtain lower and upper estimates for the Green's function associated with the operator L, especially as we get close to the boundary. This will in turn give us the bounds in condition (b) of theorem 1. Thus, there are constants C_1, C_2 that depend only on $\lambda, \Lambda, n, \omega$ such that

$$C_1 \frac{\delta^p(X)\delta^p(Y)}{|X-Y|^{(n+1)-a}} \leq G(X,Y) \leq C_2 \frac{\delta^p(X)\delta^p(Y)}{|X-Y|^{(n+1)-a}} \tag{1.1.5}$$

for a good choice of p, where $\delta(X) = \text{dist}(X, \partial\mathbb{R}^{n+1}_+)$. For this, I have obtained the upper bounds when $a \geq 0$ with $p = 1-a$ by first proving that for the solutions of the degenerate elliptic equation in a cube with specific boundary conditions, we have $\delta^a(X)|\nabla u(X)| \leq C$, where $C = C(\lambda, \Lambda, n, \omega)$. The proof here follows in a similar fashion to the one given by Grüter and Widman in [22, Lemma 3.2] for uniformly elliptic equations. However, the proof only works when $a \geq 0$, as when $a < 0$ the estimate we get on $\delta^a(X)|\nabla u(X)|$ seems to blow up as you get close to the boundary.

1.2. Parabolic equations and some free boundary problems

We analyse a function $u : \mathbb{R}^{n+1} \times [0,T] \to \mathbb{R}$, which is harmonic on the sets $\{u > 0\}, \{u < 0\} \subset \mathbb{R}^{n+1} \times [0,T]$. These sets have boundary $\partial\{u > 0\}$ which, for a fixed time t, is a hypersurface in $\mathbb{R}^{n+1}$. There is an additional constraint on u which arises naturally from either physical principles or energy minimization, and it concerns the balance of the normal derivatives $\partial_n^+ u^+$ and $\partial_n^- u^-$ in the inward direction along the free boundary, which is given by the boundary velocity. More formally, for our two-phase problem, we seek a function u

that solves

$$\Delta u = 0 \text{ in } \{u > 0\} \cup \{u < 0\} \tag{1.2.1}$$

under the velocity condition

$$\partial\{u > 0\} \text{ moves with normal velocity } \; V = |\nabla u^{+}| - |\nabla u^{-}|. \tag{1.2.2}$$

This model includes a special case of the stationary two phase problem under the condition that u does not depend upon time and the balance condition becomes

$$|\nabla u^{+}| - |\nabla u^{-}| = 1. \tag{1.2.3}$$

It also includes the one phase version that usually carries the name Hele-Shaw, and that corresponds to ignoring $\partial\{u < 0\}$ and setting the velocity condition to be $V = |\nabla u^{+}|$.

I work with these canonical examples, and the goal is to find a solution u that satisfies (1.2.1) above with either of the conditions in (1.2.2) or (1.2.3). Viscosity solutions give a way of handling this when other techniques fail, and there is substantial literature on the variants of these equations in [1, 5, 27, 28]. Recently, the work of Chang-Lara, Guillen, and Schwab [9] gives a new technique which reduces the variants in the family of problems with conditions (1.2.2) and (1.2.3) to an equivalent problem of the integro-differential type by considering the hypersurface $\partial\{u(\cdot, t) > 0\}$ as the graph of a function $f : \mathbb{R}^{n+1} \times [0, T]$. Then, for a sufficiently smooth f, the equation (1.2.1) with the condition in (1.2.2) is equivalent to

$$\begin{aligned} \partial_t f &= H(f, x) \qquad \text{on } \mathbb{R}^n \times [0, T] \\ f(\cdot, 0) &= f_0 \qquad\qquad\quad\;\; \text{on } \mathbb{R}^n \end{aligned} \tag{1.2.4}$$

where $H : C^{1,\gamma} \to C^{\gamma}$, and $H(f) = (|\nabla U_f^+| - |\nabla U_f^-|)\sqrt{1+|\nabla f|^2}$. The main technique used to show the equivalence of solving (1.2.1) under condition (1.2.2) with solving (1.2.4) was given in [9]; the key idea is to use that the operator H has a special structure of the min-max form

$$H(f,x) = \min_i \max_j \{a^{ij} + c^{ij} f(x) + b^{ij} \cdot \nabla f(x) + p.v. \int_{\mathbb{R}^n} f(x+h) - f(x) \mu^{ij}(x,dh)\} \quad (1.2.5)$$

for an appropriate family of a^{ij}, b^{ij}, c^{ij} and μ^{ij}. Equations that admit a similar min-max form are frequently amenable to a large collection of tools from the viscosity solutions context.

The assumption that $\partial\{u > 0\} = \text{graph} f$ is not ideal; we see that it does not appear as a requirement in [1, 5, 11, 12, 27, 28]. However, it gives a natural reduction to explore new techniques, especially with an assumption that $\{u > 0\}$ is a star-shaped domain with respect to $X = 0$. Thus it makes sense to expand the ideas in [9] to the case of the functions u, f defined on the sphere $\mathbb{S}^n$ instead of $\mathbb{R}^{n+1}$.

My work in this area is about expanding this new technique; in particular, when the functions u, f are defined over $\mathbb{S}^n \times [0, T]$. So if time t is fixed, then $f(\cdot, t) : \mathbb{S}^n \to [\delta, \infty)$. As such, any previous assumption about translation invariance of the domain or operators now becomes a matter of rotational invariance.

The advantage of using the approach of (1.2.4) is that it now allows us to generalise these techniques to many more variations in the free boundary problems described above. One important variant of the Hele-Shaw type problem is to have $\Delta u = \rho(u)$ with monotone ρ in $\{u > 0\}$ and the condition $V = k(x,t)|\nabla u^+|$ in (1.2.3); this was used as a semilinear model for tumor growth [30]. Another equation to study is when $\Delta u = g(x)$ in (1.2.1) and the

Batchelor-Prandtl equation is a special case of this with the condition (1.2.2) [9, 16, 17].

The second area of my work is in free boundary problems, which have an additional layer of complexity as their domain and boundary are not fixed; but rather, the domain of the function is an unknown in the equation, it is in fact the set where the solution is positive. Furthermore in the type of problem we study, it may evolve in time. When expressed as a function over the d-dimensional sphere, this boundary is a hypersurface that may not be regular. One physical example of this is the motion of a pressurized fluid through a medium with friction, which is modeled by the Hele-Shaw type equations. In fact, we study a generalized version of Hele-Shaw that allows for two phases of a fluid.

Chapter 2

Background for degenerate elliptic equations

Overall , we are interested in learning more about the connection between partial differential equations on some "nice" domain of the form

$$Lu = -\partial_j(a_{ij}\partial_i u) = 0 \quad \text{in } \Omega$$

$$u = f \quad \text{on } \partial\Omega$$

and operators of the kind

$$\mathcal{I}(f,x) = \int_{\partial\Omega} f(h) - f(x)\mu(x,dh)$$

for a_{ij} and μ that satisfy certain conditions. One of the possible connections we know of is that the operator $\mathcal{I}$ will often arise as the D-to N map for the equation above. Once we know we can express the D-to-N map as an integro-differential operator, we are further interested in studying the properties of the measure μ associated with this representation. We also want to investigate the order of $\mathcal{I}$, i.e. is $\mathcal{I}(u(r\cdot),x) = r^\alpha \mathcal{I}(u,rx)$ for some α?

These questions have already been answered for the case of an equation where the coefficients a^{ij} are uniformly elliptic, and hence the operator L is translation invariant. Here the order is 1, and the operator is the half-Laplacian, $-\Delta^{1/2}$. In this work, we want to show we can create non translation invariant operators of order α, where we have $\alpha = 2s = 1 - a$ for $a \in (-1, 1)$. This is to be done by the process of a weighted D-to-N map for degenerate elliptic equations in the half space with variable coefficients, which is the main goal for chapters 2 and 3. In particular, we will be able to express $\mathcal{I}$ in the degenerate case as an integro-differential operator like in (1.1.2), and establish some properties of this measure μ. In the second part of this background chapter, we will provide some background about equations with weighted coefficients with results mainly from [4], [19], and [20].

But first, in the first part of this chapter, we will provide some background about uniformly elliptic equations, listing the known results about the integro-differential operators of their D-to-N maps. The proofs of these results rely heavily of some results of the Green's function for uniformly elliptic equations, so we start by first providing a list of results from [22] and [3], and then go on to describe what is known about the structure of μ for the uniformly elliptic operators from [23].

2.1. Uniformly elliptic equations

In this section we will provide some background for the uniformly elliptic equation and the corresponding Green's function. Most of the following results are standard and appear in many places, but as [3, 22] was a main reference in this dissertation, we will list the relevant definitions and provide references to results only from [3, 22]. We first define the notions of uniformly elliptic equations and operators.

Definition 2.1.1. *$A : \mathbb{R}^n \to \mathbb{R}^n$ is called uniformly elliptic if $\exists$ positiveand integro- differ-*

ential operators constants λ, Λ *such that*

$$\lambda|\xi|^2 \leq \langle A(x)\xi, \xi\rangle \leq \Lambda|\xi|^2 \quad \forall x, \xi \in \mathbb{R}^n \tag{2.1.1}$$

A partial differential equation of the type

$$\begin{aligned} Lu_\phi = -\operatorname{div}(A\nabla u_\phi) = 0 \quad &\text{in } \Omega \subset \mathbb{R}^n \\ u_\phi = \phi \quad &\text{on } \partial\Omega \end{aligned} \tag{2.1.2}$$

where A is uniformly elliptic; is a uniformly elliptic partial differential equation. One of the classic and simplest examples of a uniformly elliptic equation is when $A = Id$, making $L = -\Delta$, the Laplacian operator.

The set-up for Dirichlet-to-Neumann maps (henceforth D-to-N maps) is as follows.

Definition 2.1.2. *Given the equation* (2.1.2) *as above with* $\phi \in C^{1,\alpha}(\partial\Omega)$*, we define a map*

$$andintegro-differentialoperators\mathcal{I} : C^{1,\alpha}(\partial\Omega) \to C^{\alpha}(\partial\Omega) \quad as \quad \phi \mapsto \partial_\nu u_\phi, \tag{2.1.3}$$

where $\nu(x)$ *is the inward normal vector to* $\partial\Omega$ *at* x*.*

The simplest possible case of the map $\mathcal{I}$ is when $A = Id$ and the domain is $\Omega = \mathbb{R}^{n+1}$, the upper half space. In this case, the inward normal to the boundary is $(0, 0, \cdots, 1)$, we will have $\partial_\nu u = u_y$. Here u is the harmonic extension of ϕ, and it is well known that $\mathcal{I} = (-\Delta)^{1/2}$.

This D-to-N map $\mathcal{I}$ is the one that has a certain inteand integro- differential operatorsgro-differential representation by the results in [14]. To prove our main goal which is to describe properties of the special measure associated with $\mathcal{I}$, we will need another important measure which is used in the context of a partial differential equation, i.e. the harmonic measure. Another key factor in the proofs we will present are estimates on the Green's function of the equation. We define tand integro- differential operatorshese concepts below.

Definition 2.1.3. *Given the operator L as in* (2.1.2) *and $\phi \in C(\partial\Omega)$, there exists a unique u_ϕ that solves* (2.1.2)*. Hence, for a fixed $x \in \Omega$, the mapping $x \mapsto u_\phi(x)$ is well defined. By the comparison principle, this is a non-negative linear functional on $C(\partial\Omega)$ (i.e. the mapping $\phi \mapsto u_\phi(x)$). The unique Borel measure that represents this functional is called the L-harmonic measure, and we denote it by ω_x. In other words, ω_x is uniquely characterized by*

$$\forall \phi \in C(\partial\Omega),\ u_\phi(x) = \int_{\partial\Omega} \phi(z)\omega_x(dz)$$

2.1.1. Green's function

and integro- differential operators The primary tool of all of our analyses is to understand the boundary behaviour of the Green's function. So in what follows, I will give the reader the known results of the Green's function as well as my new results.

Definition 2.1.4. *Given an operator L and a function f (in some appropriate function space), suppose u is the unique solution of the equation*

$$\begin{aligned} Lu &= f && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

Then the Green's function is the unique function such that u can be represented as

$$u(x) = \int_\Omega G(x,z)f(z)dz$$

The equation (2.1.2) above can be rewritten by using $u' = u - \phi$. Then

$$Lu' = L(u - \phi) = Lu - L\phi = L\phi \qquad \text{in } \Omega$$

$$u' = u - \phi = 0 \qquad \text{on } \partial\Omega$$

Thus the Green's function here is the unique function such that u' can be represented as

$$u'(x) = \int_\Omega G(x,z)L\phi(z)dz \implies u(x) = \phi(x) + \int_\Omega G(x,z)L\phi(z)dz$$

One of first results we are interested in and will later modify and use in this work is the relationship between the Green's function and the harmonic measure.

Proposition 2.1.5. *[3, Lemma 2.2] Let* $\{\omega_x\}_{x\in\Omega}$ *be the* $L-$*harmonic measure where* L *is as given in the divergence equation in* (2.1.2)*. Then there are universal constants* r_0, C_1, C_2*, and* $s > 1$ *such that for any* $r \in (0, r_0)$*,* $z \in \partial\Omega$*, and* $x \in \Omega \setminus B_{sr}(z) \subset \mathbb{R}^n$*, the following holds:*

$$C_1 r^{n-2} G(x, z + r\nu(z)) \le \omega_x(\partial\Omega \cap B_r(z)) \le C_2 r^{n-2} G(x, z + r\nu(z))$$

and integro- differential operators

Further, we will be interested in studying the behaviour of the Green's function in the interior of the domain, as well as if an how it can be approximated when you get closer to the boundary of the domain. Of course, the latter will depend on how regular the boundary of the domain is. Therefore in the results that follow, we will discuss the estimates on the Green's function starting with some crude domains that fulfil some minimum requirements (e.g. the cone condition in definition 2.1.6). At first, we will see that we find some very basic Hölder regularity up to the boundary. However, as we improve the conditions on the boundary (e.g. the exterior ball condition (see definition 2.1.7)) we get an improved estimate near the boundary on both the Green's function and it's gradient. Naturally, these results are not special to a Green's function, but to any solution with zero boundary data.

Definition 2.1.6. *[22, Assumption 1.6] A set Ω is said to satisfy the exterior cone condition if $\exists h > 0,\ 0 < \theta < \pi/2$ such that the following is true: for each $z \in \partial\Omega$, $\exists$ a cone $C = C(z, h, \theta)$ such that $\Omega^\circ \cap C = \emptyset$, $\overline{\Omega} \cap \overline{C} = \{z\}$. Here, $C(z, h, \theta)$ denotes a cone with cusp z, height h, and opening angle θ.*

Definition 2.1.7. *A set Ω is said to satisfy the exterior sphere condition if $\exists r > 0$ such that the following is true: for each $z \in \partial\Omega$, there exists a sphere of radius r $B = B(z, r)$ such that $\Omega^\circ \cap B = \emptyset$, $\overline{\Omega} \cap \overline{B} = \{z\}$.*

For a domain that satisfies the exterior cone condition, we can get the following estimates on the solution and the Green's functions as we approach the boundary.

Lemma 2.1.8. *[22, Lemma 1.7] Let $r > 0$ and $D_r = B_r(0) \setminus C(0, r/2, \theta)$. Consider the weak solution u_r of the following equation with the given boundary conditions*

$$\begin{aligned} Lu_r &= 0 && \text{in } D_r \\ u_r &= 0 && \text{in } \partial C(0, r/2, \theta) \\ u_r &= 1 && \text{in } \partial B_r(0) \end{aligned}$$

Then, $\exists\ K(n, \lambda, \Lambda, \theta) > 0,\ \alpha(n, \lambda, \Lambda, \theta) \in (0, 1)$ such that $\forall x \in D_r$

$$u_r(x) \leq K \frac{|x|^\alpha}{r^\alpha}$$

The above lemma zooms into a portion of the domain close to the boundary, by constructing cone at a point x_0 on the boundary and taking a little ball around the point which of course overlaps with a little part of the domain close to the boundary. For convenience, this point is taken to be $x_0 = 0$ and this works for any operator L with bounded, measurable coefficients. We look at $B_r(0)$ because the ellipticity class is invariant by rescaling, i.e. if we have $\tilde{a}^{ij}(x) = a^{ij}(rx)$ or $\hat{a}^{ij}(x) = a^{ij}(x/r)$, then the ellipticity constants of $\tilde{a}^{ij}, \hat{a}^{ij}$ are also λ, Λ. The lemma redefines the boundary values in this little domain D_r which comprises a small neighbourhood around this boundary point. As mentioned earlier, this is a very basic Hölder estimate. This lemma then leads us to a a result about the boundary regularity of the Green's function, as below.

Theorem 2.1.9. *[22, Theorem 1.8] There are constants* $K(n, \lambda, \Lambda, \theta, \operatorname{diam}\Omega, \partial\Omega) > 0$, $\alpha(n, \lambda, \Lambda, \theta) \in (0, 1)$ *such that* $\forall x, z \in \Omega$

$$G(x, z) \leq K\delta^{\alpha}(z)|x - z|^{2-n-\alpha}$$

where $\delta(z) = \operatorname{dist}(z, \partial\Omega)$.

If we improve the boundary of our domain to one that satisfies the exterior sphere condition, we will get better estimates for the solutions and the Green's function. But first, we also require some special assumptions on the coefficients a^{ij} [22, Section 3].

Definition 2.1.10. *The coefficient matrix A is called Dini-continuous if it satisfies,*

$$|a^{ij}(x) - a^{ij}(z)| \leq \omega(|x - z|) \tag{2.1.4}$$

where $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ *is non-decreasing and it satisfies the doubling condition,*

$$\omega(2t) \leq K\omega(t) \quad \textit{for some } K > 0 \quad \textit{and all } t > 0 \tag{2.1.5}$$

$$\textit{and} \quad \int_0^1 \frac{\omega(t)}{t} dt < \infty. \tag{2.1.6}$$

With coefficients as given above, we have the following two lemmas from [22] regarding the size of the gradient of the solution. These estimates are then used to study the behaviour of the Green's function close to the boundary.

Lemma 2.1.11. *[22, Lemma 3.1] Suppose* u *is a bounded solution of* $Lu = 0$ *in* Ω*. with* L *as in* (2.1.2) *and Dini-continuous coefficients* a^{ij} *as described above. Then* $\exists\ K = (n, \lambda, \Lambda, \omega, \Omega) > 0$ *such that for any* $x \in \Omega$

$$|\nabla u(x)| \leq K\delta^{-1}(x) \sup_{\Omega} |u|,$$

where $\delta^{-1}(x) := \operatorname{dist}(x, \partial\Omega)$*.*

In the above result we assume no special conditions about the domain Ω. As such, we note that when you get closer to $\partial\Omega$, it is quite possible for the gradient to blow up. The following lemma describes a bound of the gradient of the solution on a much nicer and integro- differential operatorsdomain, i.e. an annulus.

Lemma 2.1.12. *[22, Lemma 3.2] Let* u *be the solution of the Dirichlet problem (with Dini* *continuous coefficients a^{ij})*

$$\begin{aligned} Lu &= 0 && \text{in } D_r = B_{2r} \setminus \overline{B}_r \\ u &= 0 && \text{in } \partial B_r \\ u &= 1 && \text{in } \partial B_{2r} \end{aligned}$$

Then $\exists\ K = (n, \lambda, \Lambda, \omega) > 0$ such that for any $x \in D_r,\ r \in (0, 1]$,

$$|\nabla u(x)| \leq \frac{K}{r}.$$

The above lemmas are used powerfully in proving the following theorem about the estimates of the Green's function. If we have a domain that satisfies the exterior sphere condition, we can construct an annulus of a suitable radius at every point on the boundary of the domain, with the inner sphere touching the boundary on the outside. We then employ the same techniques used in the proof of 2.1.9 to prove the boundary regularity of the Green's function.

Theorem 2.1.13. *[22, Theorem 3.3] Let Ω be a domain that satisfies the exterior sphere condition and let L satisfy* (2.1.1)–(2.1.6). *Let G be the corresponding Green's function. Then the following inequalities are true for any $x, z \in \Omega$*

(a) $G(x, z) \leq K|x - z|^{2-n}, \quad K = K(n, \lambda, \Lambda)$

(b) $G(x, z)\delta(x) \leq K|x - z|^{1-n}, \quad K = K(n, \lambda, \Lambda, \omega, \Omega)$

(c) $G(x, z) \leq K\delta(x)\delta(z)|x - z|^{-n},$

(d) $|\nabla_x G(x, z)| \leq K|x - z|^{1-n},$

(e) $|\nabla_x G(x,z)| \leq K\delta(z)|x-z|^{-n}$,

(f) $|\nabla_x \nabla_z G(x,z)| \leq K|x-z|^{-n}$,

2.1.2. Dirichlet-to-Neumann maps

We seek to study the relationship between the equation (1.1.1) and the operator in (1.1.2) in a degenerate elliptic setting. However, we already have a lot of information about this $\mathcal{I}$, which arises as the Dirichlet-to-Neumann map in the case of the uniformly elliptic equation (see definition 2.1.2). In this section, we provide some background and results about $\mathcal{I}$ as a D-to-N map, its integro-differential representation, and properties of the associated measure. To begin with, this map $\mathcal{I}$ is not only well defined from $C^{1,\alpha}(\partial\Omega)$ to $C^{\alpha}(\partial\Omega)$, but it also satisfies the global comparison property as defined below.

Definition 2.1.14. *The global comparison property (GCP) for* $I : C^{1,\alpha}(X) \to C^0(X)$ *requires that for all* $f, g \in C^{1,\alpha}(X)$ *such that* $f(x) \leq g(x)$ *for all* $x \in X$ *and* $f(x_0) = g(x_0)$ *for some* $x_0 \in X$*, then the operator* I *satisfies* $I(f, x_0) \leq I(g, x_0)$*. That is to say that* I *preserves the ordering on the functions* f, g *on* X *at any points where the graphs of* f, g *touch.*

In the case of the divergence equations, L, and hence also $\mathcal{I}$ are linear operators. It was proved in the 1960s, by Courrège [14] and Bony-Courrège-Priouret [2], through linearity and the global comparison property, that $\mathcal{I}$ must be an integro-differential operator of the form

$$\mathcal{I}(\phi, x) = b(x) \cdot \nabla\phi(x) - p.v. \int_{\partial\Omega} \phi(x+h) - \phi(x)\mu(x, dh).$$

Following is the main theorem from [23]. We will prove the analogous result for degenerate elliptic equations in the next chapter.

Theorem 2.1.15. *[23, Theorem 1.1] Suppose L is the divergence operator as in* (2.1.2) *with* $\Omega = \mathbb{R}^{n+1}_+$ *and $\mathcal{I}$ is as defined via* (2.1.2) *and* (2.1.3)*, then there exists a vector field, b, and a family of measures parametrised by $x, \mu(x, dh)$ such that $\forall \phi \in C^{1,\alpha}(\mathbb{R}^n)$,*

$$\mathcal{I}(\phi, x) = \langle b(x), \nabla\phi(x)\rangle - \int_{\mathbb{R}^n} \phi(x+h) - \phi(x) - \mathbb{1}_{B_1(x)}(h)\,\langle \nabla\phi(x), h - x\rangle\,\mu(x, dh).$$

Further, μ satisfies

(i) For all $x \in \mathbb{R}^n$, $\mu(x, \cdot)$ has density $\mu(x, dh) = K(x, h)\sigma(dh)$,

(ii) There exists universal $c_1 > 0, c_2 \geq c_1$ such that for all $x, h \in \mathbb{R}^n$, $x \neq h$,

$$c_1|x - h|^{-n-1} \leq K(x, h) \leq c_2|x - h|^{-n-1}$$

In this dissertation, we explore the answers to these questions in a setting where the coefficients are not uniformly elliptic; in fact they come attached with variable coefficients and weights making them degenerate as we get closer to the boundary of the domain. As a result, we are dealing with operators that are no longer translation invariant. This means many of the classically known results, particularly about the Green's function may no longer be valid or have modified versions to suit the new conditions. In the next section, we discuss some degenerate elliptic equations with weights similar to the ones in our work, and well known results about the Green's function of these equations.

2.2. Degenerate Elliptic Equations

Consider the following type of degenerate elliptic equation in divergence form

$$\begin{aligned} Lu = -\partial_i(\tilde{a}^{ij}u_j) = 0 \quad &\text{in } \mathbb{R}^{n+1}_+ \\ u = f \quad &\text{on } \mathbb{R}^n \end{aligned} \tag{2.2.1}$$

The coefficients $\tilde{a}^{ij}$ will be real-valued, measurable, symmetric, and satisfy

$$\lambda|Z|^2 w(X) \leq \tilde{a}^{ij}(X)Z_iZ_j \leq \Lambda|Z|^2 w(X) \tag{2.2.2}$$

for all $X = (x, y), Z = (z, s) \in \mathbb{R}^{n+1}_+$. The weight $w(X)$ will be a non-negative, measurable function satisfying Muckenhoupt's condition, or the A_2 condition, which is

$$\sup_B \left(\frac{1}{|B|}\int_B w(X)dX\right)\left(\frac{1}{|B|}\int_B \frac{1}{w(X)}dX\right) \leq C. \tag{2.2.3}$$

Here the supremum is taken over all Euclidean balls B and $\int_B 1dX = |B|$. Further, for a set E, we denote $w(E) = \int_E w(X)dX$. Another way of writing the equation above would be to denote $\tilde{A}(X) = w(X)A(X)$, where A is uniformly elliptic.

Now if $w(X)$ satisfies the A_2 condition (2.2.3), then following are two well known facts about the measure $w(X)dX$:

(a) $w(X)dX$ and $\sigma(dX)$ are mutually absolutely continuous.

(b) $w(B(X, 2r)) \leq cw(B(X, r))$ (doubling condition.

The above description of the degenerate elliptic equation is taken from [19]. In this section, we will list many of the results in [19], many of which we will adapt later in chapter

3 to our particular version of the degenerate elliptic equation given in (2.2.1). First, we start by listing some commonly known function spaces with their definitions which will show up in multiple results in this section and the next chapter.

Definition 2.2.1. *(Some function spaces):*

(a) $L^p(\Omega)$ *is the set of all functions* f *such that their* L^p*-norm,* $(\int_\Omega |f(x)|^p dx)^{1/p}$*, is finite.*

(b) $L^p(\Omega, w)$ *is the Lebesgue class with the norm* $(\int_\Omega |f(x)|^p w(x) dx)^{1/p}$*, or the weighted* L^p*-space.*

(c) $Lip(\overline{\Omega})$ *is the set of all functions* f *on* $\overline{\Omega}$ *that satisfy the Lipschitz condition, i.e.* $|f(x) - f(y)| \leq M|x - y|$ *for some* M*.*

(d) If we consider the inclusion from $Lip(\overline{\Omega}) \to [L^p(\Omega, w)]^{n+1}$ *given by the mapping* $f \mapsto (f, \nabla f) = (f, f_{x_1}, f_{x_2}, \cdots, f_{x_n})$*, then* $H^{1,p}(\Omega)$ *denotes the closure of the image of* $Lip(\overline{\Omega})$ *in* $[L^p(\Omega, w)]^{n+1}$*. Essentially,* $H^{1,p}(\Omega)$ *is the space of* L^p *functions on* Ω *whose weak first derivatives are also in* $L^p(\Omega)$*.*

(e) $H_0^{1,p}(\Omega)$ *denotes the closure of compactly supported functions in the image of* $Lip(\Omega)$ *in* $[L^p(\Omega, w)]^{n+1}$*. i.e.* $f \in H_0^{1,p}(\Omega)$ *if and only if there exist functions* $f_m \in Lip(\Omega)$ *such that* f_m *are all compactly supported in* Ω *and* $f_m \to f$ *in* $H^{1,p}(\Omega)$*.*

We further describe a notion of convergence in these Sobolev spaces, and the capacity of a set, which later appears in the interior estimates of the Green's function.

Definition 2.2.2. *Let* $K \subset \overline{\Omega}$*. We say that* $u \geq c$ *on* K *in the* $H^{1,2}(\Omega)$ *sense if* $\exists\ \varphi_j \in Lip(\overline{\Omega})$ *such that* $\varphi_j \geq c\ \forall x \in K$ *and* $\varphi_j \longrightarrow u$ *in* $H^{1,2}(\Omega)$*.*

Definition 2.2.3. *The Dirichlet form* $D : H^{1,2}(\Omega) \times H^{1,2}(\Omega) \to \mathbb{R}$ *is defined by*

$$D(u, v) := \int_\Omega \tilde{a}^{ij} u(x)v(x)dx.$$

Definition 2.2.4. *Let* $K \subset \Omega$ *be compact. The capacity of* K *in* Ω *is given by*

$$\text{cap}(K) := \inf\{D(u, u) : u \in H_0^{1,2}(\Omega),\ u \geq 1 \text{ on } K \text{ in the } H^{1,2}(\Omega) \text{ sense.}\}$$

For an open set Θ *in* Ω, $\text{cap}(\Theta) := \sup\{\text{cap}(K) : K \text{ compact},\ K \subset \Theta\}$.

2.2.1. Green's function

Definition 2.2.5. *The Green's function for the degenerate equation is defined in a similar manner as the elliptic case. Given the equation* $Lu = f$ *in* Ω *with* L *as in* (2.2.1)*, the Green's function is the unique function such that* u *can be represented as*

$$u(x) = \int_\Omega G(x, z)f(z)dz$$

As before, we can rewrite this equation $u' = u - \phi$ *and get*

$$u(x) = \phi(x) + \int_\Omega G(x, z)L\phi(z)dz$$

Remark 2.2.6. *Another way of looking at the Green's function is that is we fix* $z \in \Omega$*, then the Green's function denoted by* $G(x, z)$ *is the weak solution of* $L_x u(x) = \delta_z(x)$ *as a function* *of x, where δ_z is the unit mass at z. Thus, we claim that $L_x G(x,z) = \delta_z(x)$. This is because*

$$
\begin{aligned}
\textit{first,} \quad u(z) &= \int_\Omega \delta_z(x) u(x) dx. && \text{(by the definition of } \delta_z) \\
\textit{Next,} \quad u(z) &= \int_\Omega G(x,z) f(x) dx && \text{(by the definition of } G(x,z)) \\
&= \int_\Omega G(x,z) L_x u(x) dx && \text{(since } L_x u(x) = f(x)) \\
&= \int_\Omega L_x G(x,z) u(x) dx && \text{(since } L \text{ is self-adjoint)} \\
\implies \int_\Omega \delta_z(x) u(x) dx &= \int_\Omega L_x G(x,z) u(x) dx
\end{aligned}
$$

Thus we have, as claimed, $L_x G(x,z) = \delta_z(x)$.

Now we list results from [19, Section 3] about the size of the Green's function for the degenerate elliptic equation described in (2.2.1), which will be useful for finding the estimates of the Green's function to the case of the specific degenerate elliptic equation we will look at in the next chapter.

Lemma 2.2.7. *[19, Lemma 3.1] If $B(x, 2r) \in \Omega$ and $z \in \partial B(x, r)$, then*

$$G(x,z) \cong 1/\operatorname{cap}(B(x,r)).$$

Lemma 2.2.8. *[19, Lemma 3.2] If $x \in \Omega$ and $\frac{3}{2} r \leq \operatorname{dist}(x, \partial\Omega) \leq 8r$, then $\operatorname{cap}(B(x,r)) \simeq w(B(x,r))/r^2$.*

Using these lemmas, we get an estimate for the Green's function in the interior of the domain.

Theorem 2.2.9. *[19, Theorem 3.3] Let $B_R \subset \Omega$ and $x, z \in B_{R/4}$. Denote $r = |x-z|$. Then*

$$G(x,z) \simeq \int_r^R \frac{s^2}{w(B(x,s))} \frac{ds}{s}. \tag{2.2.4}$$

The above result gives interior estimates for the Green's function, and we will still need results about the behaviour of the Green's function as we approach the boundary, which will take up the entire first section in chapter 3.

Additionally, we also need to describe some relationship between the Green's function and the harmonic measure (same as defined in definition 2.1.3) for the equation (2.2.1). There is such a result in [20] which concerns nontangentially accessible (NTA) domains. One of the conditions in the definition of an NTA domain Ω is that $\exists A > 1, r_0 > 0$ such that $\forall r,\ 0 < r < r_0,\ \forall z \in \partial\Omega,\ \exists z' \in \Omega$ such that $|z - z'| < Ar$ and $B(z', r/A) \subset \Omega$. In a sense, this is an interior ball condition analogous to the exterior sphere condition described in definition 2.1.7.

Lemma 2.2.10. *[20, Lemma 3] Let $z \in \partial\Omega, z' \in \Omega$ be as above in the definition of an NTA domain, and let $\{\omega_x\}_{x\in\Omega}$ be the L-harmonic measure where L is as given in (2.2.1). If $x \in \Omega \setminus B(z, 4Ar)$ then*

$$G(z', x) \simeq \omega_x(\partial\Omega \cap B(z,r)) \frac{r^2}{w(B(x,r))}$$

This is of course analogous to proposition 2.1.5, by considering $z' = z + r\nu(z)$.

2.2.2. Dirichlet-to-Neumann maps

Now that we are equipped with sufficient background about degenerate elliptic equations, we will look at the D-to-N maps of these equations. A canonical result of this is the fractional Laplacian which we mentioned in Chapter 1. Let us consider a fairly simple type of degenerate elliptic equation in the half-space in $\mathbb{R}^{n+1}$. For $X \in \mathbb{R}^{n+1}_+$, we denote $X = (x, y)$, where $x \in \mathbb{R}^n$, and $y > 0$. Also note that $\partial\mathbb{R}^{n+1} = \mathbb{R}^n$.

$$\begin{aligned} Lu = -\operatorname{div}(y^a \nabla u) = 0 \qquad & \text{in } \mathbb{R}^{n+1}_+ \\ u = f \qquad & \text{in } \mathbb{R}^n \end{aligned} \tag{2.2.5}$$

This is an equation that looks like (2.2.1), and the weights as given in (2.2.2) are y^a. where $a \in (-1, 1)$. Thus, $\tilde{a}^{ij}(X) = a^{ij}(X)w(X) = y^a Id$. Thus $w(X) = w(x, y) = y^a$ and $A(x) = Id$, which is uniformly elliptic. Among the many authors to study this equation in the context of the fractional Laplacian are Caffarelli and Silvestre [4].

Definition 2.2.11. *The fractional Laplacian of a function $f : \mathbb{R}^n \to \mathbb{R}$ is expressed by*

$$(-\Delta)^s f(x) = C_{n,s} \int_{\mathbb{R}^n} \frac{f(x) - f(z)}{|x - z|^{n+2s}} dz$$

where the parameter $s \in (0, 1)$ and $C_{n,s}$ is some normalization constant.

One of the main results of [4] is the connection between the fractional Laplacian and the D-to-N map for the equation (2.2.5). The D-to-N map is given by the co-normal derivative of the solution of the equation.

Definition 2.2.12. *The co-normal derivative for an equation is defined with the help of Green's identity. In particular, it is the term that appears in the integral on the boundary.*

For the above equation (2.1.2)*, using, integration by parts, we have*

$$\int_{\mathbb{R}^{n+1}_+} -\operatorname{div}(y^a \nabla u) \cdot \varphi dX = \int_{\mathbb{R}^{n+1}_+} y^a \nabla u \cdot \nabla \varphi dX - \int_{\mathbb{R}^n} y^a \nabla u \cdot \varphi \cdot \nu dS$$

Since the inward facing normal to the upper half space is $(0, 0, \ldots, y)$*, the last term above is* $\int_{\mathbb{R}^n} \varphi \cdot (y^a u_y) dS$*. The co-normal derivative is the term multiplied to the test function, i.e.* $-y^a u_y$*.*

Theorem 2.2.13. *[4, Section 3] For u, the solution of* (2.2.4)*, we have up to a constant factor, that the co-normal derivative is the fractional Laplacian of order* $s = (1-a)/2$*, i.e.*

$$\lim_{y \to 0^+} y^a u_y = (-\Delta)^s f(x)$$

Note that $s \in (0,1) \implies a \in (-1,1)$*.*

$$\lim_{y \to 0^+} y^a u_y = (-\Delta)^{\frac{1-a}{2}} f(x) = C_{n,s} \int_{\mathbb{R}^n} \frac{f(x) - f(z)}{|x-z|^{n+1-a}} dz$$

This also fits with our earlier mention of the D-to-N map in the case of the Laplacian, i.e. when $a = 0$, we have $\mathcal{I} = (-\Delta)^{1/2}$.

Now for this equation, say we would like to develop a Poisson formula P to explicitly solve (2.2.5) i.e. we want P that satisfies for $X = (x, y) \in \mathbb{R}^{n+1}_+, Z \in \partial\mathbb{R}^{n+1}_+ = \mathbb{R}^n$

$$u(X) = \int_{\mathbb{R}^n} P(X, Z) f(Z) dZ$$

We will also denote this Poisson Kernel P as $P(X, Z) = P_X(Z)$. Another notable comment about the Poisson kernel P is that it is the co-normal derivative of the Green's function.

Chapter 3

Connection to Dirichlet-to-Neumann (D-to-N) maps

3.1. Introduction to our equation and coefficients

In this work, we will be proving many of the above results for a special case of the degenerate elliptic equation set in the upper half-space. For some $f \in C^{1,\alpha}(\mathbb{R}^{n+1}_+)$, let u solve the following equation:

$$\begin{aligned} Lu = -\operatorname{div}(y^a a^{ij} u_i) = 0 \qquad &\text{in } \mathbb{R}^{n+1}_+ \\ u = f \qquad &\text{in } \mathbb{R}^n \end{aligned} \tag{3.1.1}$$

where $[a^{ij}] = A$ is measurable, symmetric, and uniformly elliptic. At times, we will use the notation $\tilde{a}^{ij}$ to denote $y^a a^{ij}$, or $\tilde{A}$ to denote $y^a A$. Thus, $\tilde{a}^{ij}$ are like the weighted coefficients in (2.2.2) with weights $w(X) = y^a$. Notice that when $a = 0$, we get the uniformly elliptic case that was discussed in chapter 2. However, if $a \neq 0$, then this equation degenerates as you get closer to the boundary, i.e. $\mathbb{R}^n$, where $y = 0$.

Now, we will consider a special version of Dini-continuity for our coefficients $\tilde{a}^{ij}$, namely for any bounded subset $\Omega \subset \mathbb{R}^{n+1}_+$ we have

$$\int_\Omega y^{-a} \frac{|\tilde{a}^{ij}(X) - \tilde{a}^{ij}(Z)|}{|X - Z|^N} dX \le C \tag{3.1.2}$$

where $X, Z \in \mathbb{R}^{n+1}_+$, $X = (x, y)$, $Z = (z, s)$, $a \in (0, 1)$. The reason for imposing this extra condition on the coefficients will become evident in the proofs of the results about the Green's function to this equation, particularly lemma 3.2.2.

Before we look at the D-to-N map for this equation, we will need a definition for the co-normal derivative in this setting.

Definition 3.1.1. *The co-normal derivative for an equation is defined with the help of Green's identity, just as in the definition 2.2.12. In particular, it is the term that appears in the integral on the boundary. For the above equation* (3.1.1)*, using, integration by parts, we have*

$$\int_{\mathbb{R}^{n+1}_+} -\operatorname{div}(y^a a^{ij} u_i) \cdot \varphi dX = \int_{\mathbb{R}^{n+1}_+} y^a a^{ij} u_i \cdot \varphi_j dX - \int_{\mathbb{R}^n} y^a a^{ij} u_i \cdot \varphi \cdot \nu dS$$

Since the inward facing normal to the upper half space is $(0, 0, \ldots, e_{n+1})$*, the last term above is* $\int_{\mathbb{R}^n} \varphi \cdot (y^a \sum_j a^{n+1,j} u_j) dS$*. The co-normal derivative is the term multiplied to the test function, i.e.* $-y^a \sum_j a^{n+1,j} u_j$*.*

Remark 3.1.2. *There are, in essence, two choices for the "normal" derivative. This is because for the divergence equation, there are two possible normal vectors we can consider. One of them is the natural inward normal vector we see geometrically, which in the case of the half-space is just* $(0, \cdots, 0, 1)$*. We can see that this is the normal vector taken into consideration for the definition 2.2.12 of the co-normal derivative for the equation 2.2.5. The other choice of normal vector comes from the equation, i.e.* $(a^{n+1,1}, \cdots, a^{n+1,n+1})$ *as in the definition 3.1.1 above. This is still an inward normal vector, but it is weighted by the coefficients* $y^a a^{ij}$*. Since* a^{ij} *are uniformly elliptic, the distance between this normal vector and the conventional geometric normal vector are comparable up to a constant.*

The Dirichlet-to-Neumann map for this equation (3.1.1) is given by this weighted normal derivative, i.e. $\mathcal{I}(f,x) = -y^a \partial_\nu u = -y^a u_y$. This map $\mathcal{I}$ is not only well defined from $C^{1,\alpha}(\partial\Omega)$ to $C^{\alpha}(\partial\Omega)$, but it also satisfies the global comparison property as defined in definition 2.2.1. Again, here in (3.1.1), L is a linear operator and hence so is $\mathcal{I}$. By a result of Bony-Courrège-Priouret [2] in the 1960s using linearity and the global comparison property, it is known that $\mathcal{I}$ can be expressed as an integro-differential operator of the form

$$\mathcal{I}(\phi,x) = b(x)\cdot\nabla\phi(x) - p.v.\int_{\partial\Omega} \phi(x+h) - \phi(x)\mu(x,dh).$$

The main result in this chapter is a modification of the theorem 2.1.15 from [23] for this particular elliptic equation which degenerates as you get close to the boundary.

Theorem 3.1.3. *Suppose L is the divergence operator as in (3.1.1) with $\Omega = \mathbb{R}^{n+1}_+$ and $a \in (0,1)$. If $\mathcal{I}$ is as defined via (2.1.2) and (2.1.3), then we know there exists a vector field, b, and a family of measures parametrised by $x, \mu(x,dh)$ such that $\forall \phi \in C^{1,\alpha}(\mathbb{R}^n)$,*

$$\mathcal{I}(\phi,x) = \langle b(x), \nabla\phi(x)\rangle - \int_{\mathbb{R}^n} \phi(x+h) - \phi(x) - \mathbb{1}_{B_1(x)}(h)\langle \nabla\phi(x), h-x\rangle \mu(x,dh).$$

Further, μ satisfies

(i) μ is absolutely continuous with respect to the surface measure, σ, i.e. for all $x \in \mathbb{R}^n$, $\mu(x,\cdot)$ has density $\mu(x,dh) = K(x,h)(dh)$,

(ii) There exists universal $C > 0$ such that for all $x, h \in \mathbb{R}^n$, $x \neq h$,

$$K(x,h) \leq C|x-h|^{-n+a-1}.$$

The proof of this theorem relies on many properties and results about the Green's function associated with (3.1.1), which we will outline in the next section.

3.2. Estimates for the Green's function

We start by finding estimates for the Green's function of (3.1.1) as defined in definition 2.1.4, first in the interior of our domain, and then closer to the boundary.

3.2.1. Interior estimates

Since we have the the weighted coefficients, i.e. $y^a a^{ij}$, we know that the A_2 weights for (3.1.1) are $w(X) = y^a$. The following result for the interior estimate for the Green's function are a direct result of applying the results in [19, Section 3]. Since our domain is a half-space, we will consider cubes that are contained in the domain instead of the spheres/balls that appear in lemmas 2.2.7, 2.2.8 for the proofs. Following is the modified version of theorem 2.2.9, which is [19, Theorem 3.3].

Theorem 3.2.1. *Let $X, Z \in Q_R^{1/4} = \{(m, n+3/4) : M = (m,n) \in Q_R\}$ and let $r = |X-Z|$. Then*

$$G(X,Z) \simeq \int_r^R \frac{t^2}{w(B(X,t))} \frac{dt}{t}$$

The proof of this theorem relies on the two aforementioned lemmas 2.2.7, 2.2.8. Using this theorem and $w(X) = w(x,y) = y^a$ we can approximate the size of the Green's function for our degenerate equation if we let $P = (p,q) \in B(X,t)$, then

$$w(B(X,t)) = \int_{B(X,t)} w(P)dP = \int_{B(X,t)} q^a dP \simeq Ct^{n+1+a}$$

Then $\displaystyle\int_r^R \frac{t^2}{w(B(X,t))}\frac{dt}{t} = \int_{|X-Z|}^R \frac{t^2}{Ct^{n+1+a}}\frac{dt}{t} = \int_{|X-Z|}^R Ct^{-n-a}dt \simeq C|X-Z|^{-n-a+1}$

But $1-n-a = 2-(n+1)-a = 2-N-a$ and thus $\exists\, C = C(R,n)$

$$G(X,Z) \simeq C|X-Z|^{2-N-a}$$

One can also write an alternate lengthier proof using the techniques in [29].

3.2.2. Boundary estimates

In this section, we will modify some of the results in [22] to suit the degenerate elliptic equation in (3.1.1). Recall from section 2.1.1 that first, in order to find estimates on the Green's function, it is helpful to get some estimates on the gradient of the solution. We saw what with a domain that satisfies the exterior cone condition defined in definition 2.1.6, we get some crude Hölder-type estimates on the solution of the equation, and subsequently the Green's function, as seem in lemma 2.1.8 and 2.1.9. After modifying the coefficients, when we have no assumptions on the boundary, we see in lemma 2.1.11 that the gradient of the solution can possibly blow up as you get closer to the boundary. Finally, we considered a domain with the exterior sphere condition; by first proving some estimates for the uniformly elliptic equation with Dini-continuous coefficients on an annulus (lemma 2.1.12), we are then able to get estimates for the Green's function and it's gradient up to the boundary in theorem 2.1.13.

Since the equation we are dealing with is set in the half-space $\mathbb{R}^{n+1}_+$, the boundary of our domain i.e. $\mathbb{R}^n$, is incredibly nice. It satisfies both the interior and exterior cone and sphere conditions. However, the roadblock in our work is due to the degeneracy of the equation as you get closer to the boundary. To be able to uniformly study points close to the boundary, we make use of cubes in $\mathbb{R}^{n+1}$. Particularly, we look at the intersection of cubes centred at the origin with the half-space. Of course, the operator in (3.1.1) is not translation invariant in general, but for any fixed distance from the boundary, we do have translation invariance. This is why it is acceptable to look at cubes centred around 0 alone, as we can "slide" the cubes "horizontally" along the boundary $\mathbb{R}^n$ without changing the equation.

Our goal is to be able to prove the estimates for the Green's function as given in theorem 3.2.7. For this, we will need some specific estimates on the solution of (3.1.1) and its gradient. In the following results, we make some discoveries regarding how the gradient of the solution and the solution itself behaves close to the boundary. In lemma 3.2.2 and corollary 3.2.6, we see that when we restrict the values of a to $(0,1)$, we can get the estimates we want which lead us to the estimates for the Green's function in the following theorem 3.2.7.

Lemma 3.2.2. *Suppose u solves the following equation in $Q^+_{4R} = Q_{4R} \cap \mathbb{R}^{n+1}_+$ (with L as given in (3.1.1))*

$$\begin{aligned} Lu &= 0 \quad in \quad Q^+_{4R} \\ u &= 0 \quad on \quad Q^+_{2R} \cap \partial\mathbb{R}^{n+1}_+ \\ u &= 1 \quad on \quad \partial Q^+_{4R} \setminus \partial\mathbb{R}^{n+1}_+ \\ 0 \leq u &\leq 1 \quad on \quad (Q^+_{4R} \setminus Q^+_{2R}) \cap \partial\mathbb{R}^{n+1}_+ \end{aligned} \tag{3.2.1}$$

Then $\exists K = K(n, \lambda, \Lambda, \omega)$ such that for any $X = (x, y) \in Q^+_R$,

$$|\nabla u(X)| \leq K \frac{y^{-a}}{R^{1-a}} \tag{3.2.2}$$

Here, $Lu = -\operatorname{div}(y^a A \nabla u)$, with A uniformly elliptic, and $a \in (0,1)$. (We will sometimes denote $\tilde{A} = y^a A$.)

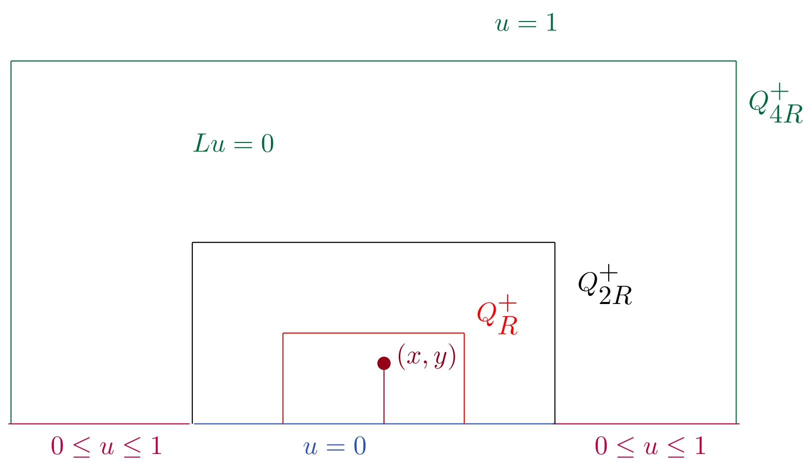

Figure 3.1: Boundary conditions for lemma 3.2.2

Remark 3.2.3. *We shed light on some things we may take for granted in this proof, which we will elaborate upon later.*

*(a) We will first consider the case where $R = 1$, and see later that we can re-scale to get the result **(see Remark 3.2.4)**.*

(b) We do not yet know if Schauder theory applies to (3.2.1) *with the given coefficients, but we can fix this with a limiting argument **(see Remark 3.2.5)**. For now, we shall assume that $u \in C^1_{loc}(Q_4^+)$ and proceed with the proof.*

Proof.

$$\text{Let} \quad \sup_{X \in Q_1^+} y^a |\nabla u(X)| =: M < \infty$$

Let $X_0 = (x_0, t) \in Q_1^+$ be such that $y^a|\nabla u(X_0)| \geq \frac{1}{2}M$, i.e. X_0 is a point where this value is particularly large.

Consider a ball of radius d around X_0, where $d \leq 1/2$ will be determined later. We choose a cut-off function $\eta \in C_c^\infty(B_d(X_0))$ such that $\eta \equiv 1$ on $B_{d/2}(X_0)$, $|\nabla \eta| \leq k_1 d^{-1}$ and

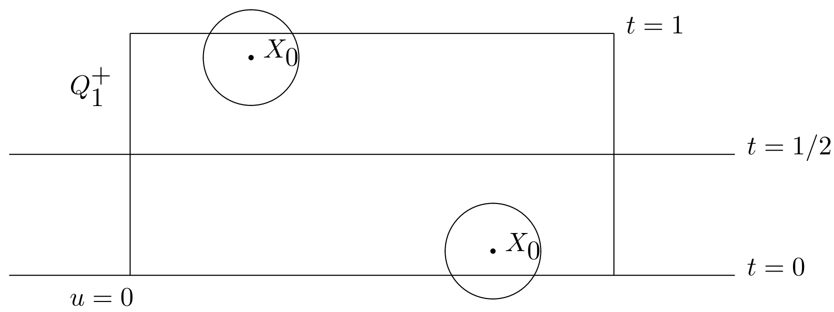

Figure 3.2: Different locations for $B_d(X_0)$

$|\nabla^2\eta| \leq k_2 d^{-2}$. ($k_1, k_2$ are universal constants.)

Note that we also have $u\eta \equiv 0$ on ∂Q_4^+. Indeed, if for $X_0 = (x_0, t)$ we have $t > 1/2$ then $B_d(X_0)$ is safely contained inside Q_4^+ and since $\eta = 0$ outside $B_d(X_0)$, $u\eta = 0$ on ∂Q_4^+. On the other hand, if $t < 1/2$ then part of $B_d(X_0)$ may be outside Q_4^+, but the only part of the boundary it intersects is where $Q_2^+ \cap \partial\mathbb{R}_+^{n+1}$, where $u = 0$. The figure above shows the different places that $B_d(X_0)$ could be with $X_0 \in Q_1^+$ and given $d < 1/2$.

Now suppose F is the Green's function in Q_4^+ for the equation in (3.2.1) corresponding to the operator with the constant coefficients $\tilde{a}_0^{ij}(X) = \tilde{a}^{ij}(X_0) = t^a a^{ij}(X_0)$. i.e we have $L_0 = -\operatorname{div}(t^a a_0^{ij}\nabla)$, and $L_0 F = 0$ in Q_4^+, $F \equiv 0$ on ∂Q_4^+. Hence, ηF is a valid test function. Then for $Z = (z, s) \in Q_4^+$ we use $\eta(\cdot)F(\cdot, Z)$ as a test function to get

$$0 = \int_{Q_4^+} \tilde{a}^{ij}(X) u_{x_i}(X)[\eta(X)F(X,Z)]_{x_j} dX = \int_{Q_4^+} \tilde{a}^{ij} u_i (\eta F)_j$$

We will use the second notation for brevity.

$$
\begin{aligned}
0 &= \int_{Q_4^+} \tilde{a}^{ij} u_i (\eta F)_j \\
&= \int_{Q_4^+} \tilde{a}_0^{ij} u_i (\eta F)_j + \int_{Q_4^+} (\tilde{a}^{ij} - \tilde{a}_0^{ij}) u_i (\eta F)_j \\
&= \int_{Q_4^+} \tilde{a}_0^{ij} u_i (\eta_j F + \eta F_j) + \int_{Q_4^+} (\tilde{a}^{ij} - \tilde{a}_0^{ij}) u_i (\eta F)_j
\end{aligned}
$$

We will use integration by parts and note that $u\eta \equiv 0$ on ∂Q_4^+, as is F.

$$
0 = -\int_{Q_4^+} \tilde{a}_0^{ij} u \eta_j F_i - \int_{Q_4^+} \tilde{a}_0^{ij} u \eta_{ji} F - \int_{Q_4^+} \tilde{a}_0^{ij} u \eta_i F_j - \int_{Q_4^+} \tilde{a}_0^{ij} u \eta F_{ji} + \int_{Q_4^+} (\tilde{a}^{ij} - \tilde{a}_0^{ij}) u_i (\eta F)_j
$$

Since $-a_0^{ij} F_{ji}(\cdot, Z) = \delta_Z$, we have

$$
\begin{aligned}
u(Z)\eta(Z) = &\int_{Q_4^+} \tilde{a}_0^{ij} u \eta_j F_i + \int_{Q_4^+} \tilde{a}_0^{ij} u \eta_{ji} F + \int_{Q_4^+} \tilde{a}_0^{ij} u \eta_i F_j \\
&+ \int_{Q_4^+} (\tilde{a}_0^{ij} - \tilde{a}^{ij}) u_i (\eta F)_j
\end{aligned}
\tag{3.2.3}
$$

Now we differentiate (take gradient) (3.2.3) with respect to Z and set $Z = X_0$. Note that $\eta(X_0) = 1$ and $\nabla \eta(X_0) = 0$ since $\eta \equiv 1$ on $B_{d/2}(X_0)$.

$$
\begin{aligned}
\nabla u(X_0) = &\int_{Q_4^+} t^a a_0^{ij} u \eta_j \nabla_Z F_i(\cdot, X_0) + \int_{Q_4^+} t^a a_0^{ij} u \eta_{ji} \nabla_Z F(\cdot, X_0) \\
&+ \int_{Q_4^+} t^a a_0^{ij} u \eta_i \nabla_Z F_j(\cdot, X_0) \\
&+ \int_{Q_4^+} (t^a a_0^{ij} - y^a a^{ij}) u_i [\eta_j \nabla_Z F(\cdot, X_0) + \eta \nabla_Z F_j(\cdot, X_0)] \\
&= I_1 + I_2 + I_3 + I_4
\end{aligned}
\tag{3.2.4}
$$

Since a_0^{ij} is just a constant coefficient and t^a is a fixed constant multiplied to it, the estimates we have on the Green's function F from [37] are as follows:

$$|\nabla_Z F(X, X_0)| \leq t^{-a} C_1 |X - X_0|^{1-N}, \quad \text{and} \quad \nabla_Z F_{x_i}(X, X_0) \leq t^{-a} C_2 |X - X_0|^{-N}$$

For I_1 and I_3, since $\eta_j = \eta_i = 0$ on $B_{d/2}(X_0)$ as well as outside $B_d(X_0)$, we can integrate over the annulus $B_d \setminus B_{d/2} \cap Q_4^+$. Also using the fact that $u \leq 1$, in this case we have the following estimates

$$\begin{aligned} |I_1| &\leq \int_{B_d \setminus B_{d/2}} t^a \Lambda k_1 d^{-1} C_2 t^{-a} |X - X_0|^{-N} dX \\ &= \Lambda k_1 C_2 d^{-1} \int_{d/2}^{d} r^{-N} r^{N-1} dr = \Lambda k_1 C_2 d^{-1} \ln(2) = C d^{-1} \end{aligned}$$

Similarly, we also have $|I_3| \leq C d^{-1}$

For I_2, we will just integrate over B_d by using the fact that $\eta \equiv 0$ outside B_d and hence $\eta_{ij} \equiv 0$ too.

$$\begin{aligned} |I_2| &\leq \int_{B_d} t^a \Lambda k_2 d^{-2} C_1 t^{-a} |X - X_0|^{1-N} dX \\ &= \Lambda k_2 C_1 d^{-2} \int_0^d r^{1-N} r^{N-1} dr = C d^{-1} \end{aligned}$$

As for I_4, we know that $u_i(X) \leq M y^{-a}$ by the assumption and recall we have the condition

$$\int_{\mathbb{R}^{n+1}_+ \cap B_d} y^{-a} \frac{|\tilde{a}^{ij}(X) - \tilde{a}^{ij}(Z)|}{|X - Z|^N} dX \leq C$$

Therefore,

$$\begin{aligned}
|I_4| &\leq \int_{B_d} (\tilde{A}(X_0) - \tilde{A}(X))u_i[\eta_j \nabla_Z F(\cdot, X_0) + \eta \nabla_Z F_j(\cdot, X_0)] \\
&\leq \int_{B_d} (\tilde{A}(X_0) - \tilde{A}(X))My^{-a}\left[k_1 d^{-1} C_1 t^{-a}|X_0 - X|^{1-N} + t^{-a}C_2|X_0 - X|)^{-N}\right] dX \\
&\leq C' M t^{-a} \int_{B_d} y^{-a} \frac{\tilde{A}(X_0) - \tilde{A}(X)}{|X_0 - X|^N} dX \qquad \text{(since } d \geq |X_0 - X|\text{)}
\end{aligned}$$

Putting all the above estimates together, we have

$$\begin{aligned}
|\nabla u(X_0)| &\leq Cd^{-1} + C' M t^{-a} \int_{B_d} y^{-a} \frac{\tilde{A}(X_0) - \tilde{A}(X)}{|X_0 - X|^N} dX \\
\implies \frac{1}{2}M \leq t^a |\nabla u(X_0)| &\leq C\frac{t^a}{d} + C' M \int_{B_d} y^{-a} \frac{\tilde{A}(X_0) - \tilde{A}(X)}{|X_0 - X|^N} dX
\end{aligned}$$

Let $d_0 = \sup\left\{ d : C' \int_{B_d} y^{-a} \frac{\tilde{A}(X_0) - \tilde{A}(X)}{|X_0 - X|^N} dX \leq 1/4 \right\}$, choose $d = \min\{d_0, 1/2\}$.

Then,

$$\begin{aligned}
\frac{1}{2}M &\leq C\frac{t^a}{d} + \frac{1}{4}M \\
M &\leq C\frac{t^a}{d} \leq C \sup\{2, 1/d_0\} = K \qquad \text{(since } X_0 \in Q_1^+,\ t^a < C\text{)}
\end{aligned}$$

Note that in order to be controlled by this universal constant, it is important that $a \geq 0$, hence our assumption that $a \in (0, 1)$. Finally, this gives us

$$\begin{aligned}
M = \sup_{X \in Q_1} y^a |\nabla u(X)| &\leq K \\
\implies |\nabla u(X)| &\leq K y^{-a}. \qquad \text{(for any } X = (x, y) \in Q_1^+\text{)}
\end{aligned}$$

□

Remark 3.2.4. ***Scaling***

u solves the equation $Lu(X) = -\operatorname{div}(y^a A(X)\nabla u(X)) = 0$ in Q_{4R}^+ with all the boundary values as in (3.1.1). Now if instead we look at Q_4^+, then $X \in Q_{4R}^+ \implies \frac{X}{R} = Z \in Q_4^+$. We make a change of variables as follows:

$$\text{Let } \; u(RZ) = \hat{u}(Z) \implies u(X) = \hat{u}\left(\frac{X}{R}\right)$$

$$\begin{aligned} \text{Then, we will have } \; \hat{u} &= 0 \quad \text{on} \quad Q_2^+ \cap \partial\mathbb{R}_+^{n+1} \\ \hat{u} &= 1 \quad \text{on} \quad \partial Q_4^+ \setminus \partial\mathbb{R}_+^{n+1} \\ 0 \leq \hat{u} &\leq 1 \quad \text{on} \quad (Q_4^+ \setminus Q_2^+) \cap \partial\mathbb{R}_+^{n+1} \end{aligned}$$

Since we have the same boundary conditions, we must ask - what equation does $\hat{u}$ solve in Q_4^+ if u solves (3.1.1) in Q_{4R}^+? Note that $\nabla_X u = \frac{1}{R}\nabla_Z \hat{u}$, and since $X = (x, y) = RZ = (Rz, Rs)$, we have $y^a = R^a s^a$. If we also denote $a^{ij}(X) = a^{ij}(RZ) = \hat{a}^{ij}(Z)$, and we know that $\hat{A}$ will satisfy the same conditions as A, then $\hat{u}$ solves the following equation in Q_4^+

$$-\frac{1}{R^{1-a}}(s^a \hat{a}^{ij}(Z)\nabla_Z \hat{u}(Z)) = 0$$

Repeat all the calculations in the proof to see that the factor $\frac{1}{R^{1-a}}$ remains throughout, thus giving us the result in lemma 3.2.2.

Remark 3.2.5. ***Limit argument***

Now we don't really know whether $u \in C^1_{loc}(Q_4^+)$. However, we can consider $\forall m \in \mathbb{N}$ the coefficients $\tilde{A}_m(X) = \tilde{A}_m(x, y) = \max\{\frac{1}{m}, y^a\} \cdot A(x, y)$, where A is uniformly elliptic. Note *that $\tilde{A}_m \to \tilde{A}$ in the L^∞-norm. Also, for every m,*

$$\tilde{A}_m(X) \geq \frac{1}{m} A(X) \implies \tilde{a}^{ij}_m(X)\xi_i\xi_j \geq \frac{1}{m}\lambda|\xi|^2 \quad \forall X, \xi \in \mathbb{R}^{n+1}_+$$

and as such the equation $L_m u_m = -(\tilde{a}^{ij}_m (u_m)_i)_j = 0$ with the same boundary values given in (3.1.1) is non-degenerate. If fact, each $\tilde{A}_m$ is uniformly elliptic, and hence we know that each $u_m \in C^1(Q_4^+)$ by Schauder. Also, $u_m \in H^1(Q_4^+)$ as it solves the elliptic PDE. On the other hand, we also know that u which solves (3.1.1) is in the weighted Sobolev space $H^1(Q_4^+, w)$, where $w(X) = y^a$ is the weight. We could also say that $u_m \in H^1(Q_4^+, w)$ as $H^1(Q_4^+, w) \subset H^1(Q_4^+)$.

If we proceed as in the proof above, then we see that we have $|\nabla u_m| \leq K y^{-a}$, which is a uniform bound independent of m, and we also know that for all m, $0 \leq u_m \leq 1$. Thus the sequence u_m is uniformly bounded in $H^1(Q_4^+, w)$, a Hilbert space, which means there is a $u^\star \in H^1(Q_4^+, w)$ such that a subsequence u_{m_k} converges weakly to $u^\star$ in $H^1(Q_4^+, w)$. On the other hand, we also know that $H^1(Q_4^+, w) \Subset L^2(Q_4^+, w)$. Thus, not only does $u_{m_k} \to u^\star$ weakly in $L^2(Q_4^+)$, there is also a $v \in L^2(Q_4^+)$ such that $u_{m_k} \to v$ strongly in $L^2(Q_4^+)$. Therefore, we must have $u^\star = v$, and $u_{m_k} \to u^\star$ strongly in $H^1(Q_4^+, w)$. Hence the uniform bound on $|\nabla u_m|$ also applies to its limit, $|\nabla u^\star|$.

Finally, what guarantees that this $u^\star$ which is the limit of u_m is the same function u that solves the equation (3.1.1)? Using φ as a test function

$$
\begin{aligned}
&\lim_{m\to\infty}\int_{Q_4^+}[\tilde{a}_m^{ij}(u_m)_i\varphi_j-\tilde{a}^{ij}u_i^\star\varphi_j]\\
&=\lim_{m\to\infty}\int_{Q_4^+}\tilde{a}_m^{ij}(u_m)_i\varphi_j-\tilde{a}^{ij}(u_m)_i\varphi_j+\tilde{a}^{ij}(u_m)_i\varphi_j-\tilde{a}^{ij}u_i^\star\varphi_j\\
&=\lim_{m\to\infty}\int_{Q_4^+}(\tilde{a}_m^{ij}-\tilde{a}^{ij})(u_m)_i\varphi_j+\lim_{m\to\infty}\int_{Q_4^+}\tilde{a}^{ij}((u_m)_i-u_i^\star)\varphi_j\\
&\le\lim_{m\to\infty}||\tilde{A}_m-\tilde{A}||_{L^\infty}\int_{Q_4^+}|\nabla u_m||\nabla\varphi|\\
&\quad+\lim_{m\to\infty}\int_{Q_4^+}y^a\Lambda|\nabla u_m-\nabla u^\star||\nabla\varphi|\\
&\le\lim_{m\to\infty}||\tilde{A}_m-\tilde{A}||_{L^\infty}||u_m||_{H^1(Q_4^+,w)}||\nabla\varphi||_{L^\infty}|Q_4^+|\\
&\quad+\lim_{m\to\infty}||u_m-u^\star||_{H^1(Q_4^+,w)}C_N\Lambda||\nabla\varphi||_{L^\infty}\\
&=0
\end{aligned}
$$

Thus, we have $\quad 0=\lim_{m\to\infty}\int_{Q_4^+}\tilde{a}_m^{ij}(u_m)_i\varphi_j=\int_{Q_4^+}a^{ij}u_i^\star\varphi_j$

Thus, $u^\star$ *solves the equation* (3.1.1) *weakly. But* u *solves* (3.1.1)*, and the solution is unique, which means* $u=u^\star$.

As for the argument with freezing the coefficients, we have

$$
\tilde{A}_{0,m}=\tilde{A}_m(X_0)=\max\left\{\frac{1}{m},t^a\right\}\cdot A(x_0,t)
$$

which means we either have $\tilde{a}_{0,m}^{ij}(X)=t^aA(x_0,t)$ *or* $\tilde{a}_{0,m}^{ij}(Z)=\frac{1}{m}A(x_0,t).$

In the proof above, we only considered the former case, however, it is easy to see that if we consider the latter, we would have the Laplacian multiplied by a constant which gives us the *same estimates for the Green's function, up to a universal constant.*

Corollary 3.2.6. *Let u solve the same equation above* (3.1.1). *Then $\exists K = K(n, \lambda, \Lambda, \omega)$ such that $\forall X = (x, y) \in Q_R^+$,*

$$u(X) \leq K \frac{y^{1-a}}{R^{1-a}}$$

Proof. For $X \in Q_R^+$, let $X^\star \in \partial\mathbb{R}_+^{n+1}$ be such that $|X - X^\star| = \delta(X) = y$. If $[X, X^\star]$ is the line joining X and $X^\star$,

$$\frac{|u(X) - u(X^\star)|}{|X - X^\star|} \leq \sup_{Z \in [X, X^\star]} |\nabla u(Z)| \leq K \frac{y^{-a}}{R^{1-a}} \qquad \text{(using the lemma above)}$$

Since we know that $X^\star \in Q_{2R}^+ \cap \partial\mathbb{R}_+^{n+1}$, we have $u(X^\star) = 0$, which gives

$$u(X) \leq K \frac{y^{-a}}{R^{1-a}} |X - X^\star| = K \frac{y^{1-a}}{R^{1-a}}.$$

□

Theorem 3.2.7. *Let G be the Green's function for* (3.1.1) *with $a \in (0, 1)$. Then $\exists K_1, K_2$ that depend only on n, μ, λ, ω such that $\forall\ X, Z \in \mathbb{R}_+^N$*

(i) $G(X, Z) \leq K_1 \delta(X)^{1-a} |X - Z|^{1-N}$

(ii) $G(X, Z) \leq K_2 \delta(X)^{1-a} \delta(Z)^{1-a} |X - Z|^{-N+a}$

Proof. (i) Let X be fixed and look at G as a function of Z alone, i.e. $G(\cdot) = G(X, \cdot)$. Let $R = |X - Z|/5$. Consider the following 2 cases:

(a) $\delta(Z) \geq R$. This means X, Z are safely inside the domain. So we can apply the interior estimates to get

$$\begin{aligned} G(X,Z) &= K|X-Z|^{2-N-a}\frac{R^{1-a}}{R^{1-a}} \\ &\le K\delta(Z)^{1-a}\frac{|X-Z|^{2-N-a}}{\left(\frac{X-Z}{5}\right)^{1-a}} \\ &\le K_1\delta(Z)^{1-a}|X-Z|^{1-N} \end{aligned}$$

(b) $\delta(Z) \le R$. In this case, choose $Z^\star \in \partial\mathbb{R}^N_+$ such that $|Z - Z^\star| = \delta(Z)$. With the point $Z^\star$ as the center, consider $Q^+_{4R}(Z^\star)$. Let u_R be as in lemma 3.1.

Now, for any point $P \in \partial Q^+_{4R} \setminus \partial\mathbb{R}^N_+$, we have $G(P) \le K|X-P|^{2-N-a}$ since we have interior estimates. Also, since $u_R = 1$ here, we have $G(P) \le K|X - P|^{2-N-a}u_R(P)$. We also know that

$$\begin{aligned} &|X-Z| \le |X-P| + |P-Z| \le |X-P| + 4R \implies |X-Z| \le 5|X-P| \\ &\implies |X-P|^{2-N-a} \le |X-Z|^{2-N-a} \end{aligned}$$

Thus, we have $G(\cdot) \le K|X-Z|^{2-N-a}u_R(\cdot)$ for any point on $\partial Q^+_{4R} \setminus \partial\mathbb{R}^N_+$. On the other hand, on the bottom boundary, i.e., on $Q^+_{4R} \cap \partial\mathbb{R}^N_+$, we have $G \equiv 0$, and $u_R \ge 0$, and thus we have $G(\cdot) \le K|X-Z|^{2-N-a}u_R(\cdot)$.

Thus, not only are $G(\cdot)$ and $K|X-Z|^{2-N-a}u_R(\cdot)$ both solutions of $Lu = 0$ in $Q^+_{4R}(Z^\star)$, we also have $G(\cdot) \le K|X-Z|^{2-N-a}u_R(\cdot)$ on $\partial Q^+_{4R}(Z^\star)$. By comparison principle, we get

$$G(P) \leq K|X-Z|^{2-N-a} u_R(P) \quad \forall P \in Q^+_{4R}(Z^\star)$$

In particular, this statement is true for $Z \in Q^+_{4R}(Z^\star)$.

From lemma 3.1, we get

$$\begin{aligned} G(X,Z) &\leq K|X-Z|^{2-N-a} u_R(Z) \\ &\leq K|X-Z|^{2-N-a}|u_R(Z) - u_R(Z^\star| \\ &\leq K|X-Z|^{2-N-a}|\nabla(Z)|\delta(Z) \\ &\leq K|X-Z|^{2-N-a}\frac{\delta(Z)^{1-a}}{R^{1-a}} = K_1\delta(Z)^{1-a}|X-Z|^{1-N} \end{aligned}$$

(ii) The proof of this is obtained from (i) the same way in which we prove (i) from the interior estimates.

□

Remark 3.2.8. *Recall that the Poisson kernel P is one that helps explicitly solve the equation, which is this case is* (3.1.1) *i.e. we want P that satisfies for $X = (x,y) \in \mathbb{R}^{n+1}_+$, $Z = (z,0) \in \mathbb{R}^n$*

$$u(X) = \int_{\mathbb{R}^n} P(X,Z) f(Z) dZ$$

We will use the notation $P_X(Z) = P(X,Z)$ at times. This Poisson Kernel can be recognized as the co-normal derivative of the Green's function. Since from the above theorem, we have an estimate for the Green's function as $G(X,Z) \leq Cy^{1-a}s^{1-a}|X-Z|^{-N+a}$. Using *this estimate we will directly compute the co-normal derivative of the Green's function from the definition 3.1.1. We will use the notation* $a^{n+1,j} = \vec{a}^{n+1}, \; X = (x,y), \; Z = (z,s)$

$$
\begin{aligned}
P_X(Z) &= y^a \sum_j a^{n+1,j} G_{x_j}(X,Z) \\
&= y^a \lim_{y \to 0} \frac{G(X + y\vec{a}^{n+1}, Z) - G(X,Z)}{y} \\
&\leq \lim_{y \to 0} y^{a-1}[C y^{1-a} s^{1-a} |X - Z|^{-N+a}] \\
&\leq C s^{1-a} |X - Z|^{-N+a} \qquad (3.2.5)
\end{aligned}
$$

3.2.3. Relationship with harmonic measure

As seen in the case of the uniformly elliptic equation, one of the other results about the Green's function which is crucial to the proof of our main result is the relationship between the Green's function and harmonic measure. We recall from the previous chapter (stated as lemma 2.2.10) that we know what this relationship is directly from [20, lemma 3].

However, since we do not need the assumption of an NTA domain with our domain being very smooth and regular, we will adapt this result to the half-space to establish the connection between the Green's function and harmonic measure that we will call upon later. Recall that in section 3.2, we worked out that $w(B(X,r)) = Cr^{n+1+a} = r^{N+a}$. Thus applying [20, lemma 3] to (3.1.1), we get

Lemma 3.2.9. *[20, lemma 3] Let* $Z = (z,s) \in \mathbb{R}^n = \partial\mathbb{R}^{n+1}_+$, $r > 0$, *and* $\{\omega_x\}_{x\in\Omega}$ *be the L-harmonic measure where L is as given in* (3.1.1)*. Also let* $\nu(Z)$ *be the inward normal vector at Z (i.e.* $\nu(Z) = (0, \cdots, 0, 1)$*), and let* $Z'_r = Z + r\nu(Z)$*. Then, if* $X \in \mathbb{R}^{n+1}_+ \setminus B(Z, 4r)$*, we* *have*

$$G(X, Z_r') \simeq \omega_X(\partial\Omega \cap B(Z,r))\frac{r^2}{w(B(x,r))} = r^{2-N-a}\omega_X(\partial\Omega \cap B(Z,r))$$

In other words, we can also say that $\exists C_1, C_2 > 0$ *such that*

$$C_1 r^{2-N-a}\omega_X(\partial\Omega \cap B(Z,r)) \leq G(X, Z + r\nu(Z)) \leq C_2 r^{2-N-a}\omega_X(\partial\Omega \cap B(Z,r))$$

which is exactly the analogous result to [3, Lemma 2.2] that we need (stated in this work as 2.1.5).

3.3. D-to-N maps of the weighted equation (main result and proof)

The main goal of this chapter and the previous one had been to modify the theorem 2.1.15 to include the degenerate elliptic equations with weights. The work of [14] already confirms that the D-to-N map $\mathcal{I}$ associated with the equation (3.1.1) can be represented as the integro-differential operator

$$\mathcal{I}(\phi, x) = \langle b(x), \nabla\phi(x)\rangle - \int_{\mathbb{R}^n} \phi(x+h) - \phi(x) - \mathbb{1}_{B_1(x)}(h)\,\langle \nabla\phi(x), h - x\rangle\,\mu(x, dh). \quad (3.3.1)$$

We aim to prove the properties of the measure μ in the following main result:

Theorem 3.3.1. *Suppose* L *is the divergence operator as in* (3.1.1) *with* $\Omega = \mathbb{R}^{n+1}_+$ *and* $\mathcal{I}$ *is as defined via* (2.1.2) *and* (2.1.3), *then it is known that there exists a vector field,* b, *and a family of measures parametrised by* $x, \mu(x, dh)$ *such that* $\forall\phi \in C^{1,\alpha}(\mathbb{R}^n)$, *we have the representation in* (3.3.1). *The measure* μ *in this expression satisfies* *(a) μ is absolutely continuous with respect to the Lebesgue measure, i.e. for all $X = (x, 0) \in \mathbb{R}^n$, $\mu(x, \cdot)$ has a density, $\mu(x, dh) = K(x, h)(dh)$. (We will use the notation $(x, 0) = x$ since we are talking about points on the boundary, $\mathbb{R}^n$.)*

(b) There exist universal constants $C_1, C_2 > 0$ so that $\forall x, h \in \mathbb{R}^n,\ x \neq h$

$$K(x, h) \leq C_2 |x - h|^{-n+a-1}.$$

Proof. First, we will want to know if μ is comparable to the Lebesgue measure on $\mathbb{R}^n$. To do this, fix $X = (x, 0) = x \in \mathbb{R}^n$, show that $\mu(x, \cdot)$ is absolutely continuous with respect to dh on $\mathbb{R}^n \setminus \{x\}$. We will do this by showing absolute continuity on the set $\mathbb{R}^n \setminus \overline{B_r(x)}$ for any arbitrary $r > 0$, and then we can exhaust $\mathbb{R}^n \setminus \{x\}$ by a union of such sets. Thus, fix $r > 0$ and any set $E \subset \mathbb{R}^n \setminus \overline{B_r(x)}$ with $|E| = 0$. Our goal is to show that $\mu(E) = 0$. We fix $\delta > 0$, and find a countable cover $\{B(x_j, r_j)\}_{j=1}^{\infty}$ of E by open geodesic balls such that $\sum_{j=1}^{\infty} r_j^n < \delta$, and we will write $B_j = B(x_j, r_j)$ for brevity. Note that all of these balls are in $\mathbb{R}^n$. Now let $\phi \in C^2(\mathbb{R}^n)$ be such that $0 \leq \phi \leq \mathbb{1}_{\cup_{j=1}^{\infty} B_j}$.

If δ is sufficiently small, then we will have $\phi \equiv 0$ on $\overline{B_{r/2}(x)}$. This is because since $E \subset \mathbb{R}^n \setminus \overline{B_r(x)}$, the worst case scenario is that the balls cover all of $\mathbb{R}^n \setminus \overline{B_r(x)}$, and maybe some part of $\overline{B_r(x)}$. But if δ is small, then the radii of the balls are really small, and so they won't cross over too much, and we can have B_j in such a way that none of them intersect $\overline{B_{r/2}(x)}$. Since $0 \leq \phi \leq \mathbb{1}_{\cup_{j=1}^{\infty} B_j} = 0$ on $\overline{B_{r/2}(x)}$, we have $\phi \equiv 0$ there. Thus in (3.3.1), $\phi(X) = 0,\ \nabla\phi(x) = 0$. Also, $\phi(x) = 0$ when $x \in \overline{B_{r/2}(x)}$ and so we have

$$\mathcal{I}(\phi, x) = \int_{\mathbb{R}^n \setminus \overline{B_{r/2}(x)}} \phi(z)\mu(z, dz)$$
$$= \int_{\mathbb{R}^n} \phi(z)\mu(z, dz) \tag{3.3.2}$$

Let $\{\omega_X\}_{X \in \mathbb{R}^{n+1}_+}$ be the L-harmonic measure for L given by (3.1.1), and recall for any $X \in \mathbb{R}^{n+1}_+$,

$$U_\phi(X) = \int_{\partial \mathbb{R}^n} \phi(z)\omega_X(dz)$$

We are now going to employ lemma 3.2.9, and we choose any $t > 0$ as for the $X = x \in \mathbb{R}^n$, we have $X' = X + t\nu(X) \in \mathbb{R}^{n+1}_+$ always. Then we have

$$\omega_{X+t\nu(X)}(\partial\Omega \cap B_{r_j}(x_j)) \leq C r_j^{N+a-2} G(X + t\nu(X), X_j + r_j\nu(X_j)) \tag{3.3.3}$$

Where G is the Green's function and C only depends on the ellipticity of the equation. This works because x_j are far away enough from x, and so $X_j + r_j\nu(X_j)$ will be far away enough from $X + r\nu(X)$. Thus we have the estimate

$$
\begin{aligned}
U_\phi(X+t\nu(X)) &= \int_{\mathbb{R}^n} \phi(z)\omega_{X+t\nu(X)}(dz) \\
&\le \sum_{j=1}^{\infty} \omega_{X+t\nu(X)}(\mathbb{R}^n \cap B_{r_j}(z_j)) && (\phi \le \mathbb{1}_{\cup B_j}) \\
&\le C\sum_{j=1}^{\infty} r_j^{N-2+a} G(X+t\nu(X), X_j + r_j\nu(X_j)) && (\text{from } (3.3.3)) \\
&\le C\sum_{j=1}^{\infty} r_j^{N-2+a} \frac{t^{1-a}\cdot r_j^{1-a}}{|X+t\nu(X)-(X_j+r_j\nu(X_j))|^{N-2+a}} \\
& && (\text{from theorem 3.2.7 (ii)}) \\
&\le C_r t^{1-a}\delta
\end{aligned}
$$

(the distance between $X_j + r_j\nu(X_j)$ and $X + r\nu(X)$ is some constant that only depends on r)

From the definition, we have $\mathcal{I}$ is the weighted normal derivative, i.e. $\mathcal{I}(\phi, X) = \partial_\nu U_\phi$

$$
\mathcal{I}(\phi, X) = t^a \lim_{t\to 0} \frac{U_\phi(X+t\nu(X)) - U_\phi(X)}{t} = \lim_{t\to 0} \frac{C_r t^{1-a}\delta - 0}{t^{1-a}} = C_r\delta
$$

But B_j covers E, so we can take a sequence of functions $\phi_k \in C^2(\mathbb{R}^n)$ such that $\mathbb{1}_E \le \phi_k \le \mathbb{1}_{\cup B_j}$ and that they decrease pointwise to $\mathbb{1}_E$, so we have

$$
\begin{aligned}
\mathcal{I}(\phi_k, X) &= \int_{\mathbb{R}^n\setminus\overline{B_{r/2}(X)}} \phi_k(Y)\mu(X, dY) \\
&= \int_{\mathbb{R}^n} \phi_k(Y)\mu(X, dY) && (\phi_k \equiv 0 \text{ on } \overline{B_{r/2}(X)}) \\
&\ge \int_{R^n} \mathbb{1}_E \mu(X, dY) = \mu(X, E) \\
\implies \mu(x, E) &\le \mathcal{I}(\phi_k, X) \le C_r\delta. && (\text{from earlier})
\end{aligned}
$$

This shows μ is absolutely continuous with respect to the Lebesgue measure, this establishing part (a) of the theorem.

Now, in order to prove part (b), we use the Poisson kernel. From 3.2.8, we expect

$$U_\phi(X) = \int_{\mathbb{R}^n} P(X,Z)\phi(Z)dZ$$

and we also have an analogous estimate to the Poisson kernel similar to [4, Section 2.4] from (3.2.5), which is

$$P_X(Z) \le C\frac{s^{1-a}}{|X-Z|^{n+1-a}}$$

In the following calculation, we denote $X + t\nu(X) = X' \in \mathbb{R}^{n+1}_+$ for $X = (x,0) \in \mathbb{R}^n$, and thus for $X' = (x,t)$ and $\xi = (\xi, 0) \in \mathbb{R}^n$ we have

$$P_{X'}(\xi) \le Ct^{1-a}|X'-\xi|^{-N+a} = Ct^{1-a}\left(|x-\xi|^2 + t^2\right)^{\frac{-n-1+a}{2}}$$

$$\begin{aligned}
\text{Thus }\ \mathcal{I}(\phi,x) &= \lim_{t\to 0} t^{a-1}\left(U_\phi(X+t\nu(X)) - U_\phi(X)\right) \\
&= \lim_{t\to 0} t^{a-1}\left(\int_{\mathbb{R}^n} P_{X'}(Z)\phi(Z)dZ\right) \\
&\qquad \text{(Since } U_\phi(X) = \phi(X) = 0 \text{ and using the definition of the Poisson kernel)} \\
&\le \lim_{t\to 0} t^{a-1}\int_{\mathbb{R}^n} t^{1-a}|X'-Z|^{-n-1+a}dZ \\
&\le C\int_{\mathbb{R}^n} |X-Z|^{-n-1+a}dZ \\
&\qquad \text{(Since } X' = X+\nu(X),\ |X-Z| \text{ and } |X'-Z| \text{ are comparable)}
\end{aligned}$$

Also, since $X = (x,0),\ Z = (z,0) \in \mathbb{R}^n$, we can say $|X-Z| = |x-z|$. Now from (3.3.2),

$$\mathcal{I}(\phi,x) = \int_{\mathbb{R}^n} \phi(z)\mu(x,dz) \le C\int_{\mathbb{R}^n} |x-z|^{-n-1+a}\phi(z)dz$$

The above statement is true for every ϕ on $\mathbb{R}^n$ such that $\phi \equiv 0$ on $B_{r/2}(x)$. Thus, for $E \subset \mathbb{R}^n \setminus B_{r/2}(x)$, we have

$$\mu(E) \leq \int_{\mathbb{R}^n} \mathbb{1}_E \mu(x,z) = \int_{\mathbb{R}^n} \mathbb{1}_E |x-z|^{-n-1+a} dz = \int_E |x-z|^{-n-1+a} dz$$

Since this is true for every set $E \subset \mathbb{R}^n \setminus B_{r/2}(x)$, and we can exhaust $\mathbb{R}^n$ by the union over r of such sets $\mathbb{R}^n \setminus B_{r/2}(x)$, we have proved part (b) of the theorem. □

Remark 3.3.2. *The above "proof" is comprised of two different proofs of the same result. We note here that using the estimates on the Poisson Kernel in the second half of the proof would be enough to shove both parts (a) and (b) of the theorem. However, we also include an alternate proof of why $\mu \ll \sigma$ in the first half of the above proof.*

Remark 3.3.3. *In the proof of the above theorem, we have used in the definition of the weighted normal derivative the inward normal vector which is the natural geometric candidate as explained in 3.1.2. However, we reiterate here that this theorem and it's proof also works identically using the co-normal derivative defined with the help of the equation in definition 3.1.1. Since the operators we use fulfil the global comparison property, the canonical derivative to investigate is always the one that comes from the natural geometric normal.*

Chapter 4

Background on free boundary problems

4.1. Introduction and background

The second project in this dissertation studies some two-phase free boundary problems that are similar to Hele-Shaw equations. To do this, we will be generalising some recent work that used methods from integro-differential parabolic equations. We shall now briefly describe what these equations are.

The simplest type of such an equation is as follows. Consider a function $U : \mathbb{R}^{n+1} \times [0,T] \to \mathbb{R}$, which is harmonic (i.e $\Delta U = 0$) on the sets $\{U > 0\}, \{U < 0\} \subset \mathbb{R}^{n+1} \times [0,T]$. These sets have boundary $\partial\{U > 0\}$ which, for a fixed time t, is a hypersurface in $\mathbb{R}^{n+1}$, and may not be regular. Thus these equations have an additional layer of complexity as their domain and boundary are not fixed; but rather, the domain of the function is an unknown in the equation and changes with time. There is an additional constraint on u which arises naturally from either physical principles or energy minimization, and it concerns the balance of the normal derivatives $\partial_n^+ U^+$ and $\partial_n^- U^-$ in the inward direction along the free boundary, which is given by the boundary velocity. More formally, for our two-phase problem, we seek a function u that solves

$$\Delta U = 0 \text{ in } \{U > 0\} \cup \{U < 0\} \tag{4.1.1}$$

under the velocity condition

$$\partial\{U > 0\} \text{ moves with normal velocity } \; V = |\nabla U^+| - |\nabla U^-|. \tag{4.1.2}$$

This model includes a special case of the stationary two phase problem under the condition that U does not depend upon time and the balance condition becomes

$$|\nabla U^+| - |\nabla U^-| = 1. \tag{4.1.3}$$

It also includes the one phase version that usually carries the name Hele-Shaw, and that corresponds to ignoring $\partial\{U < 0\}$ and setting the velocity condition to be $V = |\nabla U^+|$.

I work with these canonical examples, and the goal is to find a solution u that satisfies (4.1.1) above with the condition in (4.1.2). The work presented in this dissertation applies to more general equations in which the stationary equation for U us given by

$$\begin{aligned} F_1(D^2U) &= 0 && \text{in } \{U(\cdot, t) > 0\} \\ F_2(D^2U) &= 0 && \text{in } \{U(\cdot, t) < 0\} \\ V &= G(\partial_n^+ U, \partial_n^- U) && \text{in } \partial\{U > 0\} \end{aligned} \tag{4.1.4}$$

where F_1, F_2 are uniformly elliptic rotationally invariant nonlinear operators (see the section 4.2 for precise definitions). The example stated before in equations (4.1.1)-(4.1.3) is a special case of the same, where both $F_1(D^2U) = F_2(D^2U) = \Delta U$.

Numerous works have studied these types of equations and their solutions in the case that $U : \mathbb{R}^{n+1} \to \mathbb{R}$ and some of them include [9, 11, 18, 27]. Viscosity solutions give a way of handling these equations when other techniques fail, and there is substantial literature on the variants of these equations in [1, 5, 27, 28].

Recently, the work of Chang-Lara, Guillen, and Schwab [9] gives a new technique which reduces the variants in the family of problems with the condition (4.1.2) to an equivalent problem of the integro-differential type by considering the hypersurface $\partial\{U(\cdot, t) > 0\}$ as the graph of a function $f : \mathbb{R}^d \times [0, T] \to \mathbb{R}$. Then, for a sufficiently smooth f, the equation (4.1.1) with the condition in (4.1.2) is equivalent to

$$\begin{aligned} \partial_t f &= H(f, x) \qquad \text{on } \mathbb{R}^n \times [0, T] \\ f(\cdot, 0) &= f_0 \qquad\qquad\quad \text{on } \mathbb{R}^n \end{aligned} \tag{4.1.5}$$

where $H : C^{1,\gamma} \to C^{\gamma}$. In our example of (4.1.2), we will have

$$H = (|\nabla U_f^+| - |\nabla U_f^-|)\sqrt{1 + |\nabla f|^2}.$$

This particular H is a form of the Hamilton-Jacobi equation, however, before the methods of viscosity solutions are applicable to our case, we will need more information about the structure of $|\nabla U_f^+|, |\nabla U_f^-|$ as operators on f.

In chapters 4 and 5 of this dissertation, we will be looking at the case where the free boundary set $\partial\{U(\cdot, t) > 0\}$ is given again by the graph of a function, but that the function is now a function on $\mathbb{S}^n \times [0, T]$ instead of $f : \mathbb{R}^n \times [0, T]$, which was the assumption for the graph in the previous works. This graph assumption is a technical restriction for the methods herein. In this regard, the point of the work in this part is to extend the work of [9] to the case of above by expanding this new technique, with the free boundary assumed to be the graph of a function over the sphere $\mathbb{S}^n$ instead of $\mathbb{R}^n$. Thus we have that U, f are functions defined over $\mathbb{S}^n \times [0,T]$, and when time t is fixed, then $f(\cdot,t) : \mathbb{S}^n \to [\delta,\infty)$. As such, any previous assumption about translation invariance of the domain or operators now becomes a matter of rotational invariance.

(4.1.5) is a parabolic equation, but now we shall describe a little more explicitly the examples of parabolic equations that play a role for the free boundary analysis. In the earlier chapters regarding degenerate elliptic equations, we considered the canonical operator to be the 1/2-Laplacian, which can also be considered the canonical integro-differential operator in this context. Recall that

$$-(-\Delta)^{1/2} f(x) = c_n \int_{\mathbb{R}^n} \left(f(x+h) - f(x) - \mathbb{1}_{B_1}(h)|h|^{-n-1} \right) dh.$$

The canonical parabolic equation in our context is the 1/2-heat equation given by

$$\partial_t f + (-\Delta)^{1/2} f = g$$

Just like in the earlier chapters, the Dirichlet to Neumann operator in $\mathbb{R}^{n+1}$ is given by the linear operator $-(-\Delta)^{1/2}$. Define an operator that is suitable for (4.1.2) using the normal derivative i.e. $f \mapsto \mathcal{I}(f) = \partial_\nu U_f$. Here U_f is the unique function that solves a one-phase problem, namely

$$\begin{aligned} \Delta U_f = 0 \quad & \text{in } \{0 < y < f(x)\} \\ U_f = 0 \quad & \text{on } \{y = f(x)\} \\ U_f = 0 \quad & \text{on } \{y = 0\} \end{aligned} \tag{4.1.6}$$

The condition $U_f = 1$ on $\{y = 0\}$ can simply be interpreted as there being some background pressure in the system. (As earlier, we denote $X \in \mathbb{R}^{n+1}$ as $X = (x, y)$ where $x \in \mathbb{R}^n, y \in \mathbb{R}$.

The main technique used to show the equivalence of solving (4.1.1) under condition (4.1.2) with solving (4.1.5) was given in [9]. It turns out that the set up above is in some sense a non-linear version of the D-to-N mapping, and the key idea is to use that the operator H has a special structure of the min-max form

$$H(f,x) = \min_i \max_j \{a^{ij} + c^{ij} f(x) + b^{ij} \cdot \nabla f(x) + p.v. \int_{\mathbb{R}^n} f(x+h) - f(x) \mu^{ij}(x, dh)\} \quad (4.1.7)$$

for an appropriate family of a^{ij}, b^{ij}, c^{ij} and μ^{ij}. Equations that admit a similar min-max form are frequently amenable to a large collection of tools from the viscosity solutions context. Hence, in the next chapter, we will utilize the solutions of

$$\partial_t f - \min_i \max_j \{a^{ij} + c^{ij} f(x) + b^{ij} \cdot \nabla f(x) + p.v. \int_{\mathbb{R}^n} f(x+h) - f(x) \mu^{ij}(x, dh)\} = g$$

to deduce existence, uniqueness, and some low regularity results for the solutions of the free boundary problems described above.

The assumption that $\partial\{U > 0\} = \operatorname{graph} f$ is not ideal; we see that it does not appear as a requirement in [1, 5, 11, 12, 27, 28]. However, it gives a natural reduction to explore new techniques, especially with an assumption that $\{U > 0\}$ is a star-shaped domain with respect to $X = 0$. Thus it makes sense to expand the ideas in [9] to the case of the functions u, f defined on the sphere $\mathbb{S}^n$ instead of $\mathbb{R}^{n+1}$.

The advantage of using the approach of (4.1.5) is that it now allows us to generalise these techniques to many more variations in the free boundary problems described above. One important variant of the Hele-Shaw type problem is to have $\Delta U = \rho(U)$ with monotone ρ

in $\{U > 0\}$ and the condition $V = k(x,t)|\nabla U^+|$ in (4.1.3); this was used as a semilinear model for tumor growth [30]. Another equation to study is when $\Delta U = g(x)$ in (4.1.1) and the Batchelor-Prandtl equation is a special case of this with the condition (4.1.2) [9, 16, 17].

4.2. Definitions and examples

Since the main goal of this project is to extend results owing to the connections between certain parabolic equations and free boundary problems using integro-differential operators, we use this section to provide definitions of the different equations and operators, as well as detailed descriptions of the existing results in the field. There is a tremendous amount of work on integro-differential parabolic equations [6, 7, 8, 10, 32, 33, 34, 35]. For this project, the main works that contain most of the techniques which we generalize to the context of rotationally invariant equations on $\mathbb{S}^n$ is that of Silvestre [34] and [9].

4.2.1. General Definitions

In this section, we provide a general list of definitions of various concepts, operators, and their properties which are present in this chapter and the next one. First, we start with the definitions of some of the properties of the integro-differential operators which appear in the parabolic equations we study.

Definition 4.2.1. *As a function, $R_x : \mathbb{S}^n \to \mathbb{S}^n$ is a rotation such that $R_x(0) = x$. The rotation operator R_x, which acts on functions on $\mathbb{S}^n$ (say $U : \mathbb{S}^n \to \mathbb{R}$), is defined for a fixed x as*

$$R_x U(\cdot) := U \circ R_x(\cdot)$$

Definition 4.2.2. *We say that an operator* $J : C^{1,\gamma}(\mathbb{S}^n) \to C^0(\mathbb{S}^n)$ *is rotationally invariant if* $\forall f \in C^{1,\gamma}(\mathbb{S}^n)$ *we have* $J(R_x f, z) = J(f, R_x z)$.

Definition 4.2.3. *(GCP). We say that* $J : C^{1,\gamma}(\mathbb{S}^n) \to C^0(\mathbb{S}^n)$ *has the global comparison property (GCP) if* $\forall\ f, g \in C^{1,\gamma}(\mathbb{S}^n)$ *such that* $f(x) \leq g(x)$ $\forall x \in \mathbb{S}^n$, *and for some* x_0, $f(x_0) = g(x_0)$, *the operator* J *satisfies* $J(f, x_0) \leq J(g, x_0)$. *In other words,* J *preserves the ordering of functions on* $\mathbb{S}^n$ *wherever their graphs touch. We also say that* J *has GCP at* x_0 *if the above property only holds for one fixed* x_0, *instead of all possible* x_0.

Recall that we are generally interested in equations that involve uniformly elliptic operators, and hence we shall define what a uniformly elliptic operator means in this context, for which we first need to define Pucci operators.

Definition 4.2.4. *(Extremal Operators) For a function* u *that is second-differentiable at* X, *the second order* (λ, Λ) *Pucci operators are defined as* $\mathcal{M}^+$, $\mathcal{M}^-$ *via*

$$\mathcal{M}^-(D^2U, X) = \min_{\lambda Id \leq B \leq \Lambda Id} \operatorname{tr}(BD^2U(X)) = \Lambda \sum_{e_i \leq 0} e_i + \lambda \sum_{e_i > 0} e_i$$

$$\mathcal{M}^+(D^2U, X) = \max_{\lambda Id \leq B \leq \Lambda Id} \operatorname{tr}(BD^2U(X)) = \lambda \sum_{e_i \leq 0} e_i + \Lambda \sum_{e_i > 0} e_i$$

Where D^2U *is the Hessian matrix of all the second partial derivatives of* u, $\{e_i\}_{i=1,2,\cdots d+1}$ *are the eigenvalues of* $D^2U(X)$.

Definition 4.2.5. *(Uniformly Elliptic) We define uniform ellipticity for* F *that is either linear of non-linear in the following ways:*

- *When* F *is linear, i.e. when* $F(D^2U, \nabla U) = \operatorname{tr}(AD^2U) + B \cdot \nabla U$, *then we need* $||B||_{L^\infty} \leq \Lambda$ *and* $\lambda Id \leq A \leq \Lambda Id$ *to say* F *is uniformly elliptic, and*

- *when F is non-linear, if for all $U, V \in C^2$,*

$$\mathcal{M}^-(D^2U - D^2V) - \Lambda|\nabla U - \nabla V| \leq F(D^2U, \nabla V) - F(D^2V, \nabla V)$$
$$\leq \mathcal{M}^+(D^2U - D^2V) + \Lambda|\nabla U - \nabla V|$$

The integro-differential form with which we can represent our operators consists of a linear operator on $C^{1,\gamma}(\mathbb{S}^n)$, but we can later restrict this operator to some special subsets of $C^{1,\gamma}(\mathbb{S}^n)$ to suit the parabolic equations we work with. We shall now list some definitions for these special subsets.

Definition 4.2.6. *($C^{1,\gamma}(\Omega)$) The $\gamma-th$ Hölder semi-norm of $f : \Omega \to \mathbb{R}$ (where $\Omega \subset \mathbb{S}^n$ is given by*

$$[f]_\gamma := \sup_{\substack{x,y \in \Omega \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\gamma}$$

The $C^{1,\gamma}$-norm of f is given by

$$||f||_{C^{1,\gamma}(\Omega)} := ||f||_{L^\infty} + ||\nabla f||_{L^\infty} + [\nabla f]_{C^\gamma}$$

The set $C^{1,\gamma}(\Omega)$ consists of all functions $f : \Omega \to \mathbb{R}$ whose $C^{1,\gamma}$-norm is finite. Equivalently, we can also say that

$$C^{1,\gamma}(\Omega) = \left\{ f \in L^\infty(\Omega) : \sup_{z \in \Omega} \sup_{r > 0} r^{-1-\gamma} \inf_{\substack{P(x) = c + p \cdot x \\ c \in \mathbb{R},\ p \in \mathbb{R}^n}} ||f - P||_{L^\infty(B_r(z))} < \infty \right\}.$$

Definition 4.2.7. *For $\gamma \in (0,1), \delta > 0, m > 0$, the convex set $\mathcal{K}(\gamma,\delta,m)$ is defined as*

$$\mathcal{K}(\gamma,\delta,m) := \{f \in C^{1,\gamma}(\Omega) : f(x) \geq \delta \ \forall x \in \Omega \text{ and } ||f||_{C^{1,\gamma}(\Omega)} \leq m\}$$

$$\text{Further,} \quad \mathcal{K}(\gamma,\delta) = \bigcup_{m>0} \mathcal{K}(\gamma,\delta,m) \quad \text{and}$$

$$\mathcal{K}(\delta) = \bigcup_{\gamma\in(0,1)} \bigcup_{m>0} \mathcal{K}(\gamma,\delta,m).$$

Definition 4.2.8. *(Upper Gradient). Suppose $\mathcal{K} \in C^{1,\gamma}(\mathbb{S}^n)$ is an open convex subset and $\phi : \mathcal{K} \to \mathbb{R}$ is Lipschitz. Then the upper gradient of ϕ at $f \in \mathcal{K}$ in the direction of $g \in C^{1,\gamma}(\mathbb{S}^n)$ is defined as*

$$\phi^0(f;g) := \limsup_{t\searrow 0} \frac{\phi(f+tg)-\phi(f)}{t}$$

This can be viewed as a function $\phi^0 : \mathcal{K} \times C^{1,\gamma}(\mathbb{S}^n) \to \mathbb{R}$

Definition 4.2.9. *(Subdifferential). Let ϕ be as in definition 4.2.3. The Clarke differential (or the generalized gradient) of ϕ at $f \in \mathcal{K}$ is a subset of $\left(C^{1,\gamma}(\mathbb{S}^n)\right)^\star$ given by*

$$\partial\phi(f) := \left\{\ell \in \left(C^{1,\gamma}(\mathbb{S}^n)\right)^\star |\forall\psi \in C^{1,\gamma}(\mathbb{S}^n),\ \phi^0 \geq \langle \ell,\psi\rangle\right\}$$

$$\text{and} \quad [\partial\phi]_{\mathcal{K}} := \text{hull}\left(\bigcup_{f\in\mathcal{K}} \partial\phi(f)\right) \qquad \text{(the convex hull)}$$

Definition 4.2.10. *($C^{1,\gamma}$-semi-concave) Given $\gamma \in (0,1]$, $m > 0$, a Lipschitz function $f : \mathbb{S}^n \to \mathbb{R}$ is said to be $C^{1,\gamma}$-semi-concave with constant m if there is a real-valued function, $r : \mathbb{S}^n \to \mathbb{R}$, such that*

$$f(x) = \inf_{y\in\mathbb{S}^n}\{r(y) + m|x-y|^{1+\gamma}\}$$

f is said to be $C^{1,\gamma}$-semi-convex with constant m if $(-f)$ is $C^{1,\gamma}$-semi-concave with constant

m.

Remark 4.2.11. *The original definition above is for functions from $\mathbb{R}^n \to \mathbb{R}$, but we can see that we can do an easy replacement of $\mathbb{R}^n$ by $\mathbb{S}^n$ and the definition still holds. Indeed, since $\mathbb{S}^n$ is a manifold, we have charts from all open subsets $U \in \mathbb{S}^n$ to $B_1 \in \mathbb{R}^n$. Let us call this map $\Phi_U : U(\subset \mathbb{S}^n) \to B_1 \subset \mathbb{R}^n$ for an arbitrary open set. We will use $x, y \in \mathbb{R}^n$ and $p, q \in \mathbb{S}^n$ and let $\Phi_U(p) = x,\ \Phi_U(q) = y$. Then we can say that $f : \mathbb{S}^n \to \mathbb{R}$ is $C^{1,\gamma}$-semi-concave if $f \circ \Phi_U^{-1} : \mathbb{R}^n \to \mathbb{R}$ is $C^{1,\gamma}$-semi-concave by the usual definition. Thus, $\exists r' : \mathbb{R}^n \to \mathbb{R}$, a real-valued function such that*

$$\begin{aligned} f(p) = f \circ \Phi_U^{-1}(x) &= \inf_{y \in \mathbb{R}^n} \{r'(y) + m|x-y|^{1+\gamma}\} \\ &= \inf_{q \in \mathbb{S}^n} \left\{ r'(\Phi_U(q)) + m|\Phi_U(p) - \Phi_U(q)|^{1+\gamma} \right\} \\ &= \inf_{q \in \mathbb{S}^n} \left\{ (r' \circ \Phi_U)(q) + m \cdot C|p-q|^{1+\gamma} \right\} \end{aligned}$$

Thus, we now have a real-valued function $r = r' \circ \Phi_U : \mathbb{S}^n \to \mathbb{R}$ that satisfies the condition in definition 4.2.5, with the constant $m \cdot C > 0$, where C depends on the Lipschitz constant of Φ_U.

Definition 4.2.12. *(Pointwise or punctually $C^{1,\gamma}$). Let $\gamma \in (0,1]$ and $m > 0$ be fixed. We say that f is pointwise $m - C^{1,\gamma}$ at x_0 (denoted $f \in m - C^{1,\gamma}(x_0)$) if $\nabla f(x_0)$ exists, $|f(x_0)| \leq m,\ |\nabla f(x_0)| \leq m$, and $\exists r > 0$ such that*

$$\forall x \in B_r(x_0), |f(x) - f(x_0) - \nabla f(x_0) \cdot (x - x_0)| \leq m|x - x_0|^{1+\gamma}$$

4.2.2. Some examples of free boundary problems

To support the motivation behind studying the two-phase free boundary problem on the sphere, we shall give some examples of free boundary problems, their D-to-N maps, and the parabolic equations associated with them.

(a) One phase Hele-Shaw on the half-space

As before, we will use the notation $X = (x, y)$ for any X in the half space $\mathbb{R}^{n+1}_{+}$, with $x \in \mathbb{R}^n, y > 0$. We consider functions $f : \mathbb{R}^n \to \mathbb{R}$ which are continuous, non-negative, bounded and bounded away from zero. To any such function, we can associate a domain

$$D_f = \{(x, y) \in \mathbb{R}^{n+1}_{+}, 0 < y < f(x)\}$$

We can also define the hypersurface Γ_f, which is the graph of f as

$$\Gamma_f = \{(x, y) \in \mathbb{R}^{n+1}_{+}, y = f(x)\}$$

Now let $U_f : D_f \to \mathbb{R}$ be the unique bounded solution to the Dirichlet problem

$$\begin{aligned} \Delta U_f &= 0 && \text{in } D_f \\ U_f &= 1 && \text{on } \{y = 0\},\ i.e.\ \mathbb{R}^n \\ U_f &= 0 && \text{on } \Gamma_f \end{aligned} \tag{4.2.1}$$

Now, if ν is the inward facing unit normal to Γ_f (i.e. it points towards D_f), then we

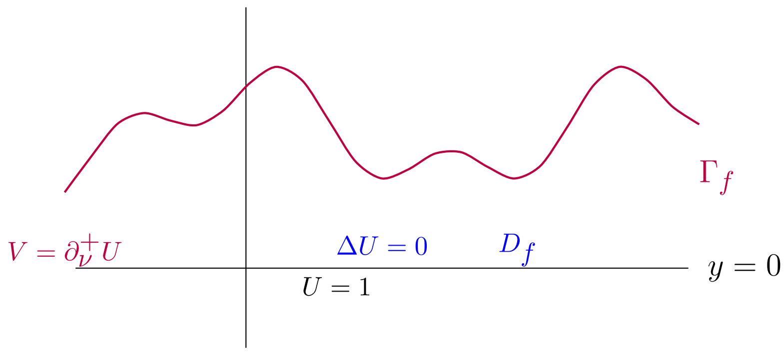

Figure 4.1: One phase Hele-Shaw on the half-space

define the normal derivative $\partial_\nu^+ U$ as follows for $X_0 \in \Gamma_f$,

$$\partial_\nu^+ U = \lim_{t \to 0^+} \frac{U(X_0 + t\nu(X_0)) - U(X_0)}{t}, \quad \text{and} \quad \mathcal{I}(f, x) := \partial_\nu^+ U_f(x, f(x)). \tag{4.2.2}$$

Now, we can put a time evolution into the setup to make a free boundary problem, i.e. consider $U : \mathbb{R}^{n+1} \times [0, T] \to \mathbb{R}$, which solves

$$\begin{aligned} \Delta U = 0 & \qquad \text{in} \;\; \{u > 0\} \\ U = 1 & \qquad \text{on} \;\; \{y = 0\}, \; i.e. \; \mathbb{R}^n \\ V = \partial_\nu^+ U & \qquad \text{on} \;\; \partial\{u > 0\} \end{aligned} \tag{4.2.3}$$

(4.2.3) is the one-phase Hele-shaw problem on the upper half space.

For a sufficiently smooth f, $U = U_f$ is a solution of (4.2.2) if and only if f is a solution to the parabolic equation

$$\partial_t f := \mathcal{I}(f, x)\sqrt{1 + |\nabla f|^2}, \quad \text{on} \;\; \mathbb{R}^n \times [0, T] \tag{4.2.4}$$

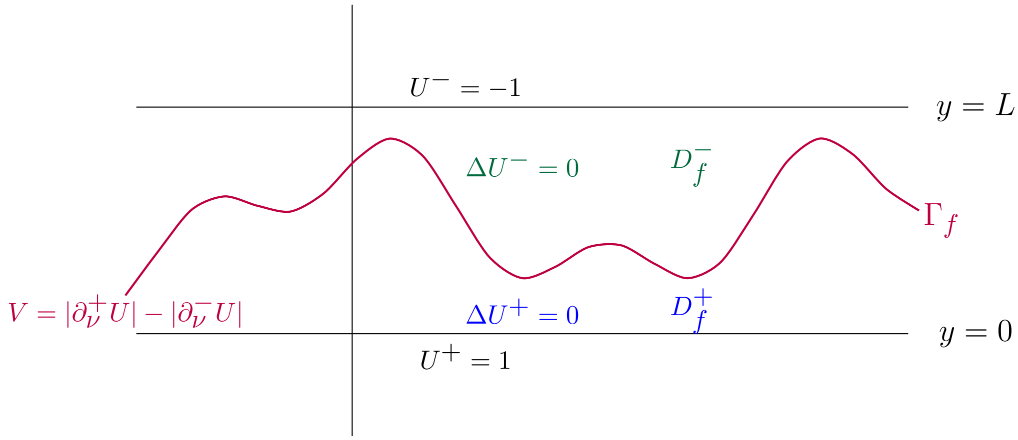

Figure 4.2: Two phase Hele-Shaw on an infinite strip

where $\mathcal{I}$ is as given in (4.2.2), and it is is translation invariant, non-linear, non-local, and it enjoys GCP.

(b) Two phase Hele-Shaw on an infinite strip

We first fix an upper boundary $L > 0$ in $\mathbb{R}^{n+1}_+$. Now for every $f : \mathbb{R}^n \to \mathbb{R}$ such that f is continuous, non-negative, and bounded away from 0 and L, i.e. $0 < \delta \leq f \leq L-\delta < L$. We then define two domains for $\{U > 0\}$ and $\{U < 0\}$

$$D_f^+ = \{(x, y) \in \mathbb{R}^{n+1}_+, 0 < y < f(x)\},$$

$$D_f^- = \{(x, y) \in \mathbb{R}^{n+1}_+, f(x) < y < L\},$$

$$\text{and} \quad \Gamma_f = \{(x, y) \in \mathbb{R}^{n+1}_+, y = f(x)\}$$

Now we suppose $U^+ : D_f^+ \to \mathbb{R}$ and $U^- : D_f^- \to \mathbb{R}$ are respectively the unique bounded solution to the Dirichlet problem

$$\begin{aligned} \Delta U^{\pm} &= 0 && \text{in } D_f^{\pm} \\ U^{\pm} &= \pm 1 \quad \text{on } \{y=0\} \text{ and } \{y=L\} && \text{respectively} \\ U^{\pm} &= 0 && \text{on } \Gamma_f \end{aligned} \tag{4.2.5}$$

Then a Lipschitz function $G : (0,\infty)^2 \to \mathbb{R}$ which satisfies the monotonicity condition

$$\lambda_0 \leq \frac{\partial}{\partial a} G(a,b), \frac{\partial}{\partial b} G(a,b) \leq \Lambda_0$$

will give the normal velocity of the flow depending on $\partial_\nu^{\pm} U$ along the boundary $\partial\{U > 0\}$. $\partial_\nu^{+} U$ is as given above in (4.2.2), and $\partial_\nu^{-} U$ is given by

$$\partial_\nu^{-} U = - \lim_{t \to 0^+} \frac{U(X_0 - t\nu(X_0)) - U(X_0)}{t} \tag{4.2.6}$$

Now, we can talk about the two-phase free boundary problem along an infinite strip as follows:

$$\begin{aligned} \Delta U &= 0 && \text{in } \{U \neq 0\} \\ U &= 1 && \text{on } \{y=0\},\ i.e.\ \mathbb{R}^n \\ U &= -1 && \text{on } \{y=L\} \\ V &= G(\partial_\nu^{+} U, \partial_\nu^{-} U) && \text{on } \partial\{U>0\} \end{aligned} \tag{4.2.7}$$

If we define $\mathcal{H}(f,x)$ as $H(f,x) = G(\partial_\nu^{+} U(x, f(x)), \partial_\nu^{-} U(x, f(x))$, then similarly as in the one-phase problem, a sufficiently smooth f solving the parabolic equation

$$\partial_t f := \mathcal{H}(f,x)\sqrt{1 + |\nabla f|^2}, \quad \text{on } \mathbb{R}^n \times [0,T] \tag{4.2.8}$$

is equivalent to $U = U_f$ solving the free boundary problem (4.2.7).

(c) Prandtl-Batchelor as a non-linear integro-differential equation on the sphere.

Finally, we come to an example where the free boundary is not given by the graph of a function over $\mathbb{R}^n$ but by the graph of a function over a sphere. The Prandtl-Batchelor is a two dimensional model in fluid mechanics that models a vortex patch occupying a convex region, the patch being surroundedby a steady flow. The interface of the vortex patch is what plays the role of the free boundary. We denote the stream function of the flow by u, and it solves the following equation

$$\begin{aligned} \Delta U = 0 \qquad & \text{in } \{U > 0\} \\ \Delta U = 1 \qquad & \text{in } \{U < 0\} \\ |\partial_\nu^+ U|^2 - |\partial_\nu^- U|^2 = 1 \qquad & \text{on } \partial\{U > 0\} \end{aligned} \tag{4.2.9}$$

Now consider any function $f : \mathbb{S}^n \to \mathbb{R}$, which is non-negative, bounded, and bounded away from zero, and satisfies $0 < f(x) \leq \overline{f}(x)$ for a given positive and continuous function $\overline{f}$. Thus we can define the sets

$$D_f^- := \{x \in \mathbb{R}^n + 1 : x = re,\ e \in \mathbb{S}^n,\ 0 \leq r < f(x)\}$$

$$D_f^+ := D_{\overline{f}}^- \setminus D_f^-$$

Suppose $U_f^{\pm} : D_f^{\pm} \to \mathbb{R}$ give the unique solution to the Dirichlet problem

$$\begin{aligned} \Delta U^- &= 1 \quad \text{in } D_f^- \\ U^- &= 0 \quad \text{on } \partial D_f^- \\ \Delta U^+ &= 0 \quad \text{in } D_f^+ \\ U^+ &= 0 \quad \text{on } \partial D_f^- \\ U^+ &= 1 \quad \text{on } \partial D_{\overline{f}}^- \end{aligned} \tag{4.2.10}$$

and we define an operator $\mathcal{P} : C^2(\mathbb{S}^n) \to C^0(\mathbb{S}^n)$ as $\mathcal{P}(f, x) := |\partial_\nu^+ U(x, f(x))|^2 - |\partial_\nu^- U(x, f(x))|^2$. $\mathcal{P}$ admits GCP, and if there is a sufficiently smooth $f : \mathbb{S}^n \to \mathbb{R}$ solving $\mathcal{P}(f, x) = 1$ in $x \in \mathbb{S}^n$, then the radial graph of f is the boundary of a vortex patch in a Batchelor-Prandtl flow, and $U_f^{\pm}$ gives the positive and negative phases of the stream function. Since the Laplacian operator and the boundary conditions on the domain are both rotationally invariant, the operator $\mathcal{P}$ will also remain invariant under the action of rotations on $C^2(\mathbb{S}^n)$.

4.3. Results to be modified

Here we will state some of the main results from [9], which are about the equation (4.1.4). There is a notion of weak solutions for both the free boundary problems and the parabolic equations and these are called viscosity solutions. We will define these notions in detail later, in sections 5.4.1 and 5.3.1 respectively.

Theorem 4.3.1. *[9, Theorem 1.1] If F_1, F_2 are uniformly elliptic and rotationally invariant in the Hessian variable, G is Lipschitz and monotone, and $f : \mathbb{R}^d \times [0, T] \to [\delta, \infty)$, U_f :*

$\mathbb{R}^{d+1} \times [0,T] \to \mathbb{R}$ *are such that*

$$\forall t \in [0,T],\ \ \Gamma(t) = \partial\{U_f(\cdot,t) > 0\} = \operatorname{graph} f(\cdot,t),$$

then,

(a) U_f *is a viscosity solution of the free boundary evolution,* (4.1.4) *if and only if* f *is a viscosity solution of the fractional parabolic equation* (4.1.5).

(b) If additionally, $\operatorname{graph} f(\cdot,t) = \partial\{U_f(\cdot,t)\}$ *enjoys a modulus of continuity, i.e.* $|f(x,0) - f(y,0)| \leq \omega(|x-y|)$, *then* $\forall t \in [0,T]$ *this modulus is preserved for* $f(\cdot,t)$ *and hence for* $\partial\{U_f(\cdot,t)$.

(c) If $\operatorname{graph} f(\cdot,t) = \partial\{U_f(\cdot,t)\}$ *enjoys a modulus of continuity, i.e.* $|f(x,0) - f(y,0)| \leq \omega(|x-y|)$, *then there exists a unique viscosity solution to both* (4.1.4) *and* (4.1.5).

We refer the reader to [9] for the proof of the above theorem. The main tool used in the analysis of the above results is what could be called a non-linear version of the D-to-N operator $\mathcal{I}$, defined via the inward normal derivative of U_f which is given by

$$\mathcal{I}(f,x) := \partial_\nu U_f(x, f(x)) \tag{4.3.1}$$

for the one-phase problem. For the two phase problem, we define the D-to-N operators as

$$\mathcal{I}^+(f,x) := \partial_\nu^+ U_f(x,f(x)),\ \ \mathcal{I}^-(f,x) := \partial_\nu^- U_f(x,f(x))$$

One of the results that leads to the proof of theorem 4.3.1 is the property of $\mathcal{I}$ which we list in the following result.

Theorem 4.3.2. *[9, Theorem 1.4] If F is uniformly elliptic and rotationally invariant in the Hessian variable, $\mathcal{I}$ is as in* (4.3.1)*, and $\gamma \in (0,1)$ is fixed, then $\exists \gamma'$ with $0 < \gamma' < \gamma$ so that*

$$\mathcal{I} : \mathcal{K}(\gamma, \delta, m) \to C^{\gamma'}(\mathbb{R}^d)$$

and $\mathcal{I}$ is locally Lipschitz. Also, $\mathcal{I}$ enjoys the following representation for $a^{ij}, b^{ij}, c^{uj}, \mu^{ij}(dh)$ which are all independent of x:

$$\begin{aligned}\mathcal{I}(f,x) = \min_i \max_j \Big\{ & a^{ij} + c^{ij} f(x) + b^{ij} \cdot \nabla f(x) \\ & + \int_{\mathbb{R}^d} \left(f(x+h) - f(x) - \mathbb{1}_{B_1}(h) \nabla f(x) \cdot h \right) \mu^{ij}(dh) \Big\}\end{aligned}$$

The bounds of $a^{ij}, b^{ij}, c^{uj}, \mu^{ij}(dh)$ all depend on the bounded set

$$\{ f : f \geq \delta, ||f||_{C^{1,\gamma}(\mathbb{R}^d)} \leq m \}.$$

Furthermore, $\exists C > 0$ such that $|a^{ij}|, |b^{ij}| \leq C$, $-C \leq c^{ij} \leq 0$, and

$$\int_{\mathbb{R}^d} \min \left\{ |h|^{1+\gamma}, 1 \right\} \mu^{ij}(dh) \leq C \quad \text{and} \quad \int_{\mathbb{R}^d} \mu^{ij}(dh) \leq C$$

The proof of the above result can also be found in [9]. Inorder to prove these properties about $\mathcal{I}$ and subsequently H as given in (4.1.5), one can study a general class of operators that satisfies certain conditions, which we outline in the assumption below.

Assumption 4.3.3. *(i) $0 < \gamma < 1$ is fixed.*

(ii) $J : \left(\bigcup_{\delta>0} \bigcup_{m>\delta} \mathcal{K}(\gamma, \delta, m) \right) \to C^0(\mathbb{R}^d)$.

(iii) For each δ *and* m*,* J *is a Lipschitz mapping on the sets* $\mathcal{K}(\gamma, \delta, m)$*, whose Lipschitz constant,* $C(\delta, m)$ *increases as* δ *decreases or* m *increases.*

(iv) J *satisfies GCP.*

(v) J *is translation invariant.*

(vi) If $f \in \mathcal{K}(\gamma, \delta, m)$ *and* $c > 0$ *is a constant, then*

$$\forall x \in \mathbb{S}^n, \quad J(f + c, x) \leq J(f, x).$$

(vii) J *enjoys the operator splitting property:* $\exists C = C(\gamma, \delta, m)$*, and a modulus of continuity* ω *with* $\omega(R) \to 0$ *as* $R \to \infty$ *such that for all* $f, g \in \mathcal{K}(\gamma, \delta, m)$*, for* $R > 1$*,*

$$||J(f, \cdot) - J(g, \cdot)||_{L^\infty(B_R)} \leq C \left(||f - g||_{C^{1,\gamma}(B_{2R})} + \omega(R)||f - g||_{L^\infty(\mathbb{R}^d)} + \omega(R) \right)$$

This operator J will enjoy a special min-max structure, and we will state the relevant result below. However, first, the proof for deriving this structure relies on the following the results from [25].

Theorem 4.3.4. *[25, Theorem 1.10] If* $I : C^2_b(\mathbb{R}^d) \to C^0_b(\mathbb{R}^d)$ *(bounded functions in* C^2*,* C^0*, respectively) satisfies the conditions (iii), (iv), and (v) of Assumption 4.3.3, then there exists a family* $\{f_{ab}, L_{ab}\}_{a,b \in \mathcal{K}(I)}$*, that depends only on* I*, where for all* a, b*,* $f_{a,b}$ *are constants, and*

L_{ab} are linear translation invariant operators mapping $I : C^2_b(\mathbb{R}^d) \to C^0_b(\mathbb{R}^d)$ of the form

$$L(u, x) = Cu(x) + B \cdot \nabla u(x) + \mathrm{tr}(AD^2u(x)) \\ + \int_{\mathbb{R}^d} u(x+h) - u(x) - \mathbb{1}_{B_1(0)}(h)\nabla u(x) \cdot h\mu(dh)$$

i.e. with constant coefficients.

Further, $\forall u \in C^2_b(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ we have

$$I(u, x) = \min_a \max_b \{f_{ab} + L_{ab}(u, x)\}.$$

Finally, for a universal C, and for all f_{ab} and L_{ab}, we have

$$|f_{ab}| + |A_{ab}| + |B_{ab}| + |C_{ab}| + \int_{\mathbb{R}^d} \min\{1, |h|^2\}\mu_{ab}(dh) \leq C||I||_{\mathrm{Lip}, C^2_b \to C^0_b}. \tag{4.3.2}$$

The above theorem relies on the following fact from [25], which elaborates on the characterization of linear functionals (whose codomain is $\mathbb{R}$) which have the GCP.

Lemma 4.3.5. *[25, Lemma 3.9] Assume that $\beta \in [0, 3)$. let $\ell : C^\beta_b(\mathbb{R}^d) \to \mathbb{R}$ be a bounded linear functional which has the GCP with respect to 0. For $\beta \geq 2$ and any $u \in C^\beta_b(\mathbb{R}^d) \cap C^2(0)$, we have the representation*

$$\langle l, u\rangle = C_\ell u(0) + (B_\ell, \nabla u(0)) + \mathrm{tr}(A_\ell D^2u(0)) \\ + \int_{\mathbb{R}^d} u(h) - u(0) - \chi_{B_1(0)} \langle \nabla u(0), h\rangle \mu_\ell(dh)$$

This representation is unique, which means that if there were $\tilde{C}, \tilde{B}, \tilde{A}$, and $\tilde{\mu}$ a measure in

$\mathbb{R}^d \setminus \{0\}$ *all such that*

$$\begin{aligned}\langle l, u\rangle =& \tilde{C}u(0) + (\tilde{B}, \nabla u(0)) + \mathrm{tr}(\tilde{A}D^2u(0)) \\ &+ \int_{\mathbb{R}^d} u(h) - u(0) - \chi_{B_1(0)} \langle \nabla u(0), h\rangle \, \tilde{\mu}(dh)\end{aligned}$$

for all u, then $\tilde{C} = C_\ell, \tilde{B} = B_\ell, \tilde{A} = A_\ell$, and $\tilde{\mu} = \mu_\ell$. Furthermore, if $\beta < 2$ and $u \in C^\beta(\mathbb{R}^d) \cap C^1(0)$, then $A_\ell = 0$, and if $\beta < 1$ then $B_\ell = 0$ and the integrand on the right can be replaced by just $u(h) - u(0)$.

The proof of the theorem 4.3.4 above employs a lot of information from [25, Lemma 3.6] and its proof, particularly the part which says that for $\beta \in [1, 2)$ (as it is in the case of $C^{1,\gamma}$, which is the case relevant to us) and $u \in C^\beta(\mathbb{R}^d) \cap C^1(0)$,

$$\langle l, u\rangle = C_\ell u(0) + (B_{\ell,\phi}, \nabla u(0)) + \int_{\mathbb{R}^d} u(h) - \mathcal{P}_{\phi,\eta,u}\mu_\ell(dh), \tag{4.3.3}$$

where $\mathcal{P}_{\phi,\eta,u}$ is given by

$$\mathcal{P}_{\phi,\eta,u,x}(\cdot) = u(x) + \phi(\cdot - x)(\nabla u(x), \cdot - x) \quad \text{if } \beta \in [1, 2). \tag{4.3.4}$$

Finally, we state the result from [9] about the structure of the operator J that satisfies the assumption 4.3.3.

Theorem 4.3.6. *(Translation invariant min-max) If J follows all the conditions in the assumption 4.3.3, then J admits a min-max representation as follows:*

$$\forall f \in \mathcal{K}(\gamma, \delta, m), \quad J(f, x) = \min_{g \in \mathcal{K}(\gamma,\delta,m)} \max_{L \in \mathcal{L}(\mathcal{K}(\gamma,\delta,m))} \{J(g, x) + L(f - g, x)\} \tag{4.3.5}$$

in which each L is also translation invariant. This means $L \in \mathcal{L}_{inv}$ which is the class that contains all linear operators of the form

$$\begin{aligned} L(f,x) =& cf(x) + b\cdot\nabla f(x) \\ &+ \int_{\mathbb{R}^d} f(x+h) - f(x) - \mathbb{1}_{B_1}(h)\nabla f(x)\cdot h\mu(dh), \end{aligned} \tag{4.3.6}$$

where each of c, b, μ are independent of x.

Furthermore, given a $C_1 > 0$, $\exists C_2 > 0$ such that for all J with a Lipschitz norm bounded by C_1, all such c, b, μ resulting from an $L \in \mathcal{L}_{inv}$, we have $|c|, |b| < C_2$, and

$$\int_{\mathbb{R}^d} \max\{|h|^{1+\gamma}, 1\}\mu(dh) \leq C_2, \quad \textit{and} \quad \int_{\mathbb{R}^d\setminus B_R} \mu(dh) \leq C_2\omega(R). \tag{4.3.7}$$

The class of operators, $\mathcal{L}_{inv}$ in (4.3.6) and the min-max in (4.3.5) both depend on γ, δ, m via $\mathcal{K}(\gamma, \delta, m)$.

Chapter 5

Parabolic equations and free boundary problems

5.1. Introduction to the problem and results

The goal of this chapter is to reproduce the results from [9] as mentioned above in section 4.3; particularly theorems 4.3.1 and 4.3.2, but this time for the specific equation given in (4.1.4) with the following details:

(a) $F_1 = F_2 = \Delta$,

(b) $G = |\nabla U_f^+| - |\nabla U_f^-|$,

(c) and for functions f defined over the sphere $\mathbb{S}^n$ instead of $\mathbb{R}^d$.

Thus, consider the following free boundary problem for $U : \mathbb{R}^{n+1} \to \mathbb{R}$

$$\begin{aligned} \Delta U &= 0 && \text{in } \{U(\cdot,t)>0\} \text{ and } \{U(\cdot,t)<0\} \\ V &= |\nabla U^+| - |\nabla U^-| && \text{in } \partial\{U>0\} \end{aligned} \tag{5.1.1}$$

Once again, we consider the hypersurface $\partial\{U(\cdot,t)>0\}$ as the graph of a function f, but this time $f : \mathbb{S}^n \times [0,T]$. Then, for a sufficiently smooth f, the equation (5.1.1) is equivalent to

$$
\begin{aligned}
\partial_t f = H(f,x) = (|\nabla U_f^+| - |\nabla U_f^+|)\sqrt{1+|\nabla f|^2} \quad &\text{on } \mathbb{S}^n \times [0,T] \\
f(\cdot,0) = f_0 \quad &\text{on } \mathbb{S}^n
\end{aligned}
\tag{5.1.2}
$$

where $H : C^{1,\gamma}(\mathbb{S}^n) \to C^\gamma(\mathbb{S}^n)$.

5.1.1. Main results

Theorem 5.1.1. *For the equations given above, if we have* $f : \mathbb{S}^n \times [0,T] \to [\delta,\infty)$, $U_f : \mathbb{S}^n \times [0,T] \to \mathbb{R}$ *are such that*

$$\forall t \in [0,T],\ \Gamma(t) = \partial\{U_f(\cdot,t) > 0\} = \operatorname{graph} f(\cdot,t),$$

then,

(a) U_f *is a viscosity solution of the free boundary evolution,* (5.1.1) *if and only if* f *is a viscosity solution of the fractional parabolic equation* (5.1.2)*.*

(b) If additionally, $\operatorname{graph} f(\cdot,t) = \partial\{U_f(\cdot,t)\}$ *enjoys a modulus of continuity, i.e.* $|f(x,0) - f(y,0)| \leq \omega(|x-y|)$*, then* $\forall t \in [0,T]$ *this modulus is preserved for* $f(\cdot,t)$ *and hence for* $\partial\{U_f(\cdot,t)\}$*.*

(c) If $\operatorname{graph} f(\cdot,t) = \partial\{U_f(\cdot,t)\}$ *enjoys a modulus of continuity, i.e.* $|f(x,0) - f(y,0)| \leq \omega(|x-y|)$*, then there exists a unique viscosity solution to both* (5.1.1) *and* (5.1.2)*.*

Once again, the precise definitions of the notions of viscosity solutions that are mentioned in these results will be given later, in sections 5.4.1 and 5.3.1. Now we can define the D-to-N operator $\mathcal{I}$, via the inward normal derivative of U_f which is given by

$$\mathcal{I}^+(f,x) := \partial_\nu^+ U_f(x, f(x)), \quad \mathcal{I}^-(f,x) := \partial_\nu^- U_f(x, f(x))$$

for the two-phase problem. Here $\partial_\nu^+ U_f, \partial_\nu^- U_f$ are as defined in (4.2.2), (4.2.6) respectively. Since the gradient of U_f at a point is orthogonal to the level curve of U_f at the same point, we have that the normal to the curve at point is given by the gradient. i.e. we have

$$\nu = \pm \frac{\nabla U_f}{|\nabla U_f|},$$

and since $\partial_\nu U_f = \nabla U_f \cdot \nu$, we have $\partial_\nu U_f = |\nabla U_f|$.

$$\text{Similarly, } \; \partial_\nu^+ U_f = |\nabla U_f^+|, \; \text{ and } \; \partial_\nu^- U_f = |\nabla U_f^-|.$$

This of course, establishes the connection between $\mathcal{I}$ and H, and hence one of the results that leads to the proof of theorem 5.1.1 is the property of $\mathcal{I}$ which we list in the following result.

Theorem 5.1.2. *If $\mathcal{I}$ is as given above, and $\gamma \in (0,1)$ is fixed, then $\exists \gamma'$ with $0 < \gamma' < \gamma$ so that*

$$\mathcal{I} : \mathcal{K}(\gamma, \delta, m) \to C^{\gamma'}(\mathbb{S}^n)$$

and $\mathcal{I}$ is locally Lipschitz. Also, $\mathcal{I}$ enjoys the following representation for $a^{ij}, b^{ij}, c^{uj}, \mu^{ij(dh)}$ which are all independent of x:

$$\mathcal{I}(f,x) = \min_i \max_j \Big\{ a^{ij} + c^{ij} f(x) + b^{ij} \cdot \nabla f(x)$$
$$+ \int_{\mathbb{S}^n} \Big(f(x+h) - f(x) - \mathbb{1}_{B_1}(h) \left\langle \nabla f(x), \exp_x^{-1}(h) \right\rangle \Big) \mu^{ij}(dh) \Big\}$$

The bounds of $a^{ij}, b^{ij}, c^{uj}, \mu^{ij(dh)}$ *all depend on the bounded set* $\{f : f \geq \delta, ||f||_{C^{1,\gamma}(\mathbb{S}^n)} \leq m\}$. *Furthermore,* $\exists C > 0$ *such that* $|a^{ij}|, |b^{ij}| \leq C,\ -C \leq c^{ij} \leq 0,$ *and*

$$\int_{\mathbb{S}^n} \min\left\{ |h|^{1+\gamma}, 1 \right\} \mu^{ij}(dh) \leq C \quad \text{and} \quad \int_{\mathbb{S}^n} \mu^{ij}(dh) \leq C$$

We shall give the proofs of these results in section 5.4. However, we will give a brief outline of the steps involved in the proof.

(1) We will first study a general class of operators acting on the functions on the sphere (with some reasonable level of smoothness). The connection of these operators to the problem at hand is that the H described in (5.1.2) will belong to this class of operators. Moreover, this H depends on the D-to-N map, $\mathcal{I}$. We will analyse these operators in terms of their enjoyment of the min-max structure, which is the result given by [9, Theorem 1.4], modified here as 5.1.1.

(2) Next, we will study the existence and uniqueness properties of viscosity solutions to parabolic equations on the sphere. These equations will involve the operators mentioned in the previous step, as given by (5.1.2).

(3) We will go on to show how that the viscosity solutions of the two different equations (free boundary problem, i.e. (5.1.1) and the parabolic equation, i.e. (5.1.2)) produce

an equivalent outcome, accounting for the correspondence between the free boundary in (5.1.1) and the graph of the function f from (5.1.2). This is the equivalence of solutions listed in [9, Theorem 1.1], modified here as 5.1.2.

(4) Finally, we will prove that a modulus of continuity for the initial data will be propagated for all time. This will lead us to part (c) of the theorem 5.1.1

The work in this chapter will focus primarily on steps (1) and (2), and the detailed explanations of steps (3) and (4) can be directly modified from [9], although we given a brief explanation in section 5.4.

5.1.2. Details of two phase free boundary problem on the sphere

Here we give a few more technical details of the equations in question along with some pictures. Consider any function $f : \mathbb{S}^n \to \mathbb{R}$, which is non-negative, bounded, and bounded away from zero. Thus, we have $r_0, \delta, L > 0$ so that $0 < r_0 < \delta \leq f \leq L - \delta < L$. Now consider the following sets:

$$D_f^+ := \{X \in \mathbb{R}^{n+1} : X = re,\ e \in \mathbb{S}^n,\ 0 \leq r < f(x)\}$$

$$D_f^- := \{X \in \mathbb{R}^{n+1} : X = re,\ e \in \mathbb{S}^n,\ f(x) \leq r < L\}$$

$$\Gamma_f^- := \{X \in \mathbb{R}^{n+1} : X = re,\ e \in \mathbb{S}^n,\ r = f(x)\}$$

Now suppose we have a function $U_f : \mathbb{S}^n \times [0, T] \to \mathbb{R}$ that solves the following free

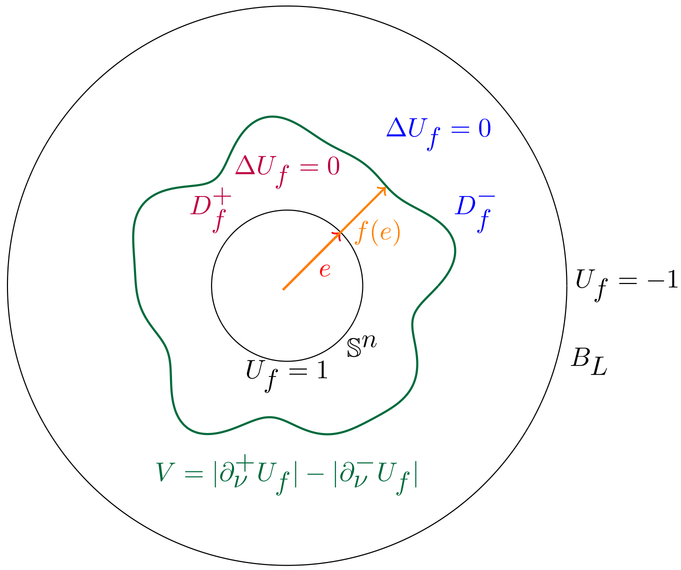

Figure 5.1: Two phase free boundary problem on the sphere

boundary problem

$$
\begin{aligned}
\Delta U_f &= 0 && \text{in } D_f^\pm \\
U_f &= 0 && \text{on } \Gamma_f \\
U_f &= 1 && \text{on } B_\delta(0) \\
U_f &= -1 && \text{on } B_L(0) \\
V &= |\partial_\nu^+ U_f| - |\partial_\nu^- U_f| && \text{on } \Gamma_f
\end{aligned}
\tag{5.1.3}
$$

Now consider the parabolic equation:

$$
\partial_t f = \left(|\partial_\nu^+ U_f| - |\partial_\nu^- U_f|\right)\sqrt{1 + |\nabla f|^2} \text{ on } \mathbb{S}^n \times [0, T] \tag{5.1.4}
$$

The main result adapted from [9] describes the connection between the parabolic equations and the (two-phase) free boundary problem as given in the equations above.

Note that the one-phase problem is given by

$$\begin{aligned} \Delta U_f &= 0 && \text{in } D_f \\ U_f &= 1 && \text{in } B_{r_0} \\ U_f &= 0 && \text{on graph } f \end{aligned} \tag{5.1.5}$$

5.2. Integro-differential representations of certain operators

Before we prove the results for the specific two-phase free boundary problem in this work, we will first consider a broad abstract class of operators that fulfil certain conditions. Such an operator will appear in the parabolic equation as given in (4.1.5). In the next two sections, we will prove a min-max representation for the operators, existence and uniqueness of solutions, as well as the comparison principle for the equations involving these abstract operators. In the final section 5.4, we will elaborate on why the operators in two phase equation we have chosen definitely falls into the category of operators we will describe in this section, and as such it will be evident why all the subsequent results will be applicable to (5.1.1), (5.1.2) as well.

Assumption 5.2.1. *We look at the broad set of operators that act on functions on $\mathbb{S}^n$ and satisfy the following assumptions:*

(i) $0 < \gamma < 1$ *is fixed.*

(ii) $J : \left(\bigcup_{\delta > 0} \bigcup_{m > \delta} \mathcal{K}(\gamma, \delta, m) \right) \to C^0(\mathbb{S}^n)$.

(iii) For each δ and m, J is a Lipschitz mapping on the sets $\mathcal{K}(\gamma, \delta, m)$, whose Lipschitz constant, $C(\delta, m)$ increases as δ decreases or m increases.

(iv) J satisfies GCP.

(v) J is rotational invariant.

(vi) If $f \in \mathcal{K}(\gamma, \delta, m)$ and $c > 0$ is a constant, then

$$\forall x \in \mathbb{S}^n, \quad J(f + c, x) \leq J(f, x).$$

In this section we will first record some of the relevant results about the structure and properties of J, but in the case when J acts on functions in $\mathbb{R}^n$. All of the results below can be found in [25].

Proposition 5.2.2. *If $\ell : \mathcal{K}(\gamma, \delta, m) \to \mathbb{R}$ is a bounded linear function that enjoys GCP at $x_0 = 0$, then $\exists b, c \in \mathbb{R}$ and a measure μ so that $\forall f \in \mathcal{K}(\gamma, \delta, m)$*

$$\ell(f) = cf(0) + b \cdot \nabla f(0) + \int_{\mathbb{S}^n} (f(h) - f(0) - \left\langle \nabla f(0), \exp_0^{-1}(h) \right\rangle \mu(dh),$$

where $b, c,$ and μ satisfy the same bounds as in the above theorem, but none of them depend on x.

This result for functions in $\mathbb{R}^n$ and its proof can both be found in [24, Lemma 3.6]. However, for functions defined on the sphere, the Taylor polynomial given in (4.3.4) will now change to

$$\mathcal{P}_{\phi,\eta,u,x}(\cdot) = u(x) + \phi(R_x(\cdot))(\nabla u(x), \exp_x^{-1}(\cdot))$$

The techniques for the proof in [24, Lemma 3.6] will follow in a straightforward manner. We also note that since $\phi \in C^{1,\gamma}(\mathbb{S}^n)$, the term with the gradient $(\nabla u(x), \exp_x^{-1}(\cdot))$, will decay at the rate of $|x-\cdot|^{1+\gamma}$, which is integrable because we will have

$$\int_{\mathbb{S}^n} \max\{|h|^{1+\gamma}, 1\}\mu(dh) \le C.$$

Theorem 5.2.3. *(Rotationally invariant min-max) If J follows all the assumptions above, then J admits a min-max representation as follows:*

$$\forall f \in \mathcal{K}(\gamma,\delta,m), \quad J(f,x) = \min_{g\in\mathcal{K}(\gamma,\delta,m)} \max_{L\in\mathcal{L}(\mathcal{K}(\gamma,\delta,m))} \{J(g,x) + L(f-g,x)\} \tag{5.2.1}$$

in which each L is also rotationally invariant. This means $L \in \mathcal{L}_{inv}$ which is the class that contains all linear operators of the form

$$\begin{aligned} L(f,x) =& cf(x) + b\cdot\nabla f(x) \\ &+ \int_{\mathbb{S}^n} f(R_x(h) - f(x) - \left\langle \nabla f(x), \exp_x^{-1}(h) \right\rangle \mu(dh), \end{aligned} \tag{5.2.2}$$

where each of c, b, μ are independent of x. Furthermore, given a $C_1 > 0$, $\exists C_2 > 0$ such that for all J with a Lipschitz norm bounded by C_1, all such c, b, μ resulting from an $L \in \mathcal{L}_{inv}$, we have $|c|, |b| < C_2$, and

$$\int_{\mathbb{S}^n} \max\{|h|^{1+\gamma}, 1\}\mu(dh) \le C_2. \tag{5.2.3}$$

The class of operators, $\mathcal{L}_{inv}$ in (5.2.2) *and the min-max in* (5.2.1) *both depend on γ, δ, m via $\mathcal{K}(\gamma,\delta,m)$.*

The consequences of the above theorem are as follows:

- Following as in [9], this identifies a natural class of extremal operators for J as

$$M^+_{inv} = \max_{L \in \mathcal{L}_{inv}} L(f,x), \quad \text{and} \quad M^-_{inv} = \min_{L \in \mathcal{L}_{inv}} L(f,x) \tag{5.2.4}$$

Just by the definition, we can use the min-max expression for J and write

$$\begin{aligned} J(f,x) - J(g,x) &= \min_{g'} \max_{L} \{J(g',x) + L(f-g',x)\} - J(g,x) \\ &\leq \max_{L} L(f-g,x) \qquad\qquad \text{(Taking } g' = g \text{ in the minimum)} \\ \text{and } J(f,x) - J(g,x) &= J(f,x) - \min_{f'} \max_{L} \{J(f',x) + L(g-f',x)\} \\ &\geq -\max_{L} L(g-f,x) \qquad\qquad \text{(Taking } f' = f \text{ in the minimum)} \\ &= \min_{L} L(f-g,x) \end{aligned}$$

These extremal operators are defined specifically to produce, $\forall f, g \in \mathcal{K}(\gamma, \delta, m)$, the following inequalities:

$$M^-_{inv}(f-g,x) \leq J(f,x) - J(g,x) \leq M^+_{inv}(f-g,x) \tag{5.2.5}$$

We note that because of the rotational invariance we have that

$$\forall x \in \mathbb{R}^n, M^{\pm}_{inv}(f,x) = M^{\pm}(f \circ R_x, 0)$$

- Secondly, due to the rotational invariance of J, we see that all of the desired properties of $\mathcal{L}_{inv}$ can be obtained from studying the (possibly non-linear) functional $j : \mathcal{K}(\gamma, \delta, m) \to \mathbb{R}$ with $j(f) := J(f, 0)$.

This functional, j, also satisfies GCP based at $x_0 = 0$. So once J is fixed, the class $\mathcal{L}$ can be further restricted to include only those L that are in the Clarke differential of j (from definition 4.2.4), i.e. those L such that

$$L(f,x) = \ell(f \circ R_x) \text{ for some choice of } \ell \in [\partial j]_{\mathcal{K}(\gamma,\delta,m)}$$

Thus, we can now define a new extremal operator, depending explicitly on J and $\mathcal{K}(\gamma, \delta, m)$ as

$$\begin{aligned} M^{+}_{J,\mathcal{K}(\gamma,\delta,m)}(f,x) &= \max_{\ell \in [\partial j]_{\mathcal{K}(\gamma,\delta,m)}} (\ell(f \circ R_x)) \\ \text{and } \ M^{-}_{J,\mathcal{K}(\gamma,\delta,m)}(f,x) &= \min_{\ell \in [\partial j]_{\mathcal{K}(\gamma,\delta,m)}} (\ell(f \circ R_x)) \end{aligned} \tag{5.2.6}$$

Now for the operator given above, we note that the inequalities in (5.2.4) still hold, they are again, rotation invariant, and they also serve as extremal operators of J, which we talk about in the result below.

Proposition 5.2.4. *If J is fixed, and $M^{\pm}_{J,\mathcal{K}(\gamma,\delta,m)}$ are defined as in* (5.2.6) *above, then J and $M^{\pm}_{J,\mathcal{K}(\gamma,\delta,m)}$ also obey the inequalities in* (5.2.5)*, and $M^{\pm}_{J,\mathcal{K}(\gamma,\delta,m)}$ are Lipschitz functions, as mappings from $C^{1,\gamma}(\mathbb{S}^n) \to C^0(\mathbb{S}^n)$ with a Lipchitz norm bounded by that of J.*

Proof. Firstly, from [13, Proposition 2.1.2], we know that $[\partial j]_{\mathcal{K}(\gamma,\delta,m)} \neq \emptyset$. This is a result of the Hahn-Banach Theorem, which asserts that every positively homogeneous and subadditive functional on a vector space majorises a linear functional on the vector space. In this case, the vector space being $C^{1,\gamma}(\mathbb{S}^n) \times C^{1,\gamma}(\mathbb{S}^n)$ and j^0 is the positively homogeneous and subadditive functional, so there exists some linear functional ℓ so that for all $\psi \in C^{1,\gamma}(\mathbb{S}^n)$,

$j^0(f) \geq \langle \ell, \psi \rangle$, which means $[\partial j]_{\mathcal{K}(\gamma,\delta,m)} \neq \emptyset$ as claimed.

Secondly, j and ∂j enjoy a mean value property due to a theorem by Lebourg [13, Proposition 2.3.7], which says that for any points $x, y \in X$ and f Lipschitz on an open set containing the line segment [x,y], $\exists\ u \in (x, y)$ such that $f(y) - f(x) \in \langle \partial f(u), y - x \rangle$. Thus, for $f, g \in \mathcal{K}(\gamma, \delta, m)$ and j Lipschitz, we can say that there is an element, $\ell \in [\partial j]_{\mathcal{K}(\gamma,\delta,m)}$ such that $j(f) - j(g) = \ell(f - g)$. Now we take a max over $[\partial j]_{\mathcal{K}(\gamma,\delta,m)}$, and again from [13, Proposition 2.1.2], $[\partial j]_{\mathcal{K}(\gamma,\delta,m)}$ is weak-$\star$ compact, thus giving us

$$\forall f, g \in \mathcal{K}(\gamma, \delta, m), \quad j(f) - j(g) \leq \max_{\ell \in [\partial j]_{\mathcal{K}(\gamma,\delta,m)}} \ell(f - g)$$

A similar argument gives us the lower inequality in (5.2.5).

To see why $M^{\pm}$ are Lipschitz, consider

$$\begin{aligned}
& M^{+}_{J,\mathcal{K}(\gamma,\delta,m)}(f_1, x) - M^{+}_{J,\mathcal{K}(\gamma,\delta,m)}(f_2, x) \\
&= \max_{\ell \in [\partial j]_{\mathcal{K}(\gamma,\delta,m)}} (\ell(f_1 \circ R_x)) - \max_{\ell \in [\partial j]_{\mathcal{K}(\gamma,\delta,m)}} (\ell(f_2 \circ R_x)) \\
&\leq \max_{\ell \in [\partial j]_{\mathcal{K}(\gamma,\delta,m)}} \{(\ell(f_1 \circ R_x)) - (\ell(f_2 \circ R_x))\} \\
&= \max_{c,b,\mu} \{c(f_1(x) - f_2(x)) + b \cdot [R_x^{-1} \nabla f_1(x) - R_x^{-1} \nabla f_2(x)] \\
&+ \int_{\mathbb{S}^n} f_1(R_x(h)) - f_2(R_x(h)) - f_1(x) + f_2(x) \\
&- \left\langle R_x^{-1} \nabla (f_1(x)), \exp_0^{-1}(h) \right\rangle + \left\langle R_x^{-1} \nabla (f_1(x)), \exp_0^{-1}(h) \right\rangle \mu(dh)\}
\end{aligned}$$

Owing to the fact that $f_1, f_2 \in \mathcal{K}(\gamma, \delta, m)$ and due to the bound on μ given in (5.2.3), all the terms above are bounded up to a constant by the $C^{1,\gamma}$-norm of $f_1 - f_2$.

□

Proof of theorem 5.2.3:

Proof. From the previous proposition, we can see that

$$\forall f, g \in \mathcal{K}(\gamma,\delta,m),\ j(f) \leq \max_{\ell \in [\partial j]_{\mathcal{K}(\gamma,\delta,m)}} (j(g) + \ell(f-g))$$

$$\text{Thus } \ j(f) \leq \min_{g \in \mathcal{K}(\gamma,\delta,m)} \max_{\ell \in [\partial j]_{\mathcal{K}(\gamma,\delta,m)}} (j(g) + \ell(f-g))$$

(by taking minimum over g)

On the other hand, by letting $g = f$ in the minimum,

$$\min_{g \in \mathcal{K}(\gamma,\delta,m)} \max_{\ell \in [\partial j]_{\mathcal{K}(\gamma,\delta,m)}} (j(g) + \ell(f-g)) \leq j(f)$$

Since J is rotation invariant, we see that $J(f,x) = j(R_x f)$.

Now, any $\ell \in [\partial j]_{\mathcal{K}(\gamma,\delta,m)}$ enjoys the comparison principle, and thus we have that ℓ has a representation as in (5.2.2), and thus the proof follows. □

Corollary 5.2.5. *For each γ, δ, and m fixed, we can extend the operator J (with respect to the set $\mathcal{K}(\gamma,\delta,m)$) to a function on all of $C^{1,\gamma}(\mathbb{S}^n)$. We define*

$$\tilde{J}_{\mathcal{K}(\gamma,\delta,m)} : C^{1,\gamma}(\mathbb{S}^n) \to C^0(\mathbb{S}^n)$$

$$\textit{via} \ \ \tilde{J}_{\mathcal{K}(\gamma,\delta,m)}(f,x) = \min_{g \in \mathcal{K}(\gamma,\delta,m)} \max_{\ell \in [\partial j]_{J,\mathcal{K}(\gamma,\delta,m)}} \{j(g) + \ell((f \circ R_x) - g)\} \qquad (5.2.7)$$

$\tilde{J}$ is Lipschitz, it enjoys GCP, and $\tilde{J} = J$ on $\mathcal{K}(\gamma,\delta,m)$. (It also satisfies all of the other assumptions.)

Proof. We can see that $\tilde{J}$ is Lipschitz because M^+ is Lipschitz and from the following:

$$\begin{aligned}
\tilde{J}(f_1) - \tilde{J}(f_2) &= \min_{h\in\mathcal{K}(\gamma,\delta,m)} \max_{\ell\in[\partial j]_{J,\mathcal{K}(\gamma,\delta,m)}} \{j(h) + \ell((f_1 \circ R_x) - h)\} \\
&- \min_{g\in\mathcal{K}(\gamma,\delta,m)} \max_{k\in[\partial j]_{J,\mathcal{K}(\gamma,\delta,m)}} \{j(g) + k((f_2 \circ R_x) - g)\} \\
&= \min_{h\in\mathcal{K}(\gamma,\delta,m)} \max_{\ell\in[\partial j]_{J,\mathcal{K}(\gamma,\delta,m)}} \{j(h) + \ell((f_1 \circ R_x) - h)\} \\
&- \max_{k\in[\partial j]_{J,\mathcal{K}(\gamma,\delta,m)}} \{j(g^\star) + k((f_2 \circ R_x) - g^\star)\} \\
&\qquad\qquad (\text{Assume } g = g^\star \text{ is the optimizer for the second minimum}) \\
&\le \max_{\ell\in[\partial j]_{J,\mathcal{K}(\gamma,\delta,m)}} \{j(g^\star) + \ell((f_1 \circ R_x) - g^\star)\} \\
&- \max_{k\in[\partial j]_{J,\mathcal{K}(\gamma,\delta,m)}} \{j(g^\star) + k((f_2 \circ R_x) - g^\star)\} \\
&\qquad\qquad (\text{setting } h = g^\star \text{ in the first min}) \\
&\le \max_{\ell\in[\partial j]_{J,\mathcal{K}(\gamma,\delta,m)}} \{j(g^\star) + \ell((f_1 \circ R_x) - g^\star) - j(g^\star) - \ell((f_2 \circ R_x) - g^\star)\} \\
&\le \max_{\ell\in[\partial j]_{J,\mathcal{K}(\gamma,\delta,m)}} (\ell(f_1 \circ R_x) - \ell(f_2 \circ R_x)) \\
&= M^+(f_1 - f_2)
\end{aligned}$$

Further, $\tilde{J}$ satisfies GCP because j, ℓ satisfy GCP at $x_0 = 0$. □

Corollary 5.2.6. *If x is fixed, and γ, δ, m are given, and $f \in m - C^{1,\gamma}(x)$, $\tilde{J}_{\mathcal{K}(\gamma,\delta,m)}(f, x)$ is classically defined via* (5.2.6).

Proof. Since f is $m - C^{1,\gamma}(x)$, then by definition, there are two $C^{1,\gamma}$ functions f^+ and f^- such that they touch f above and below respectively at x. In other words, $f^\pm \in C^{1,\gamma}(\mathbb{R}^n)$, and $\forall y,\ f^-(y) \le f(y) \le f^+(y)$ with $f^-(x) = f(x) = f^+(x)$. By Proposition 5.2.2, we know that if a bounded linear functional ℓ has GCP at a point then $\ell(R_x f) = L(f, x)$. We can work this out explicitly from

$$
\begin{aligned}
\ell(f \circ R_x) =& c(f \circ R_x)(0) + b \cdot \nabla(f \circ R_x)(0) \\
&+ \int_{\mathbb{S}^n} (f \circ R_x)(h) - f \circ R_x(0) - \left\langle \nabla(f \circ R_x)(0), \exp_0^{-1}(h) \right\rangle \mu(dh) \\
=& cf(x) + \left\langle b, R_x^{-1} \nabla f(x) \right\rangle \\
&+ \int_{\mathbb{S}^n} f(R_x h) - f(x) - \left\langle R_x^{-1} \nabla f(x), \exp_0^{-1}(h) \right\rangle \mu(dh) \\
=& cf(x) + \langle R_x \cdot b, \nabla f(x) \rangle \\
&+ \int_{\mathbb{S}^n} f(R_x h) - f(x) - \left\langle \nabla f(x), \exp_x^{-1}(h) \right\rangle \mu(dh) \\
=& L(f, x),
\end{aligned}
$$

with the corresponding $b_L = R_x \cdot b$ and the same c. Thus the formula in (5.2.7) holds classically, i.e. as in theorem 5.2.3 with the estimate on μ that appears in 5.2.3. □

Proposition 5.2.7. *If J is as in Assumption 5.2.1 and $M^+_{J,\mathcal{K}(\gamma,\delta,m)}$ is as defined in* (5.2.5) *then*

$$\forall x \in \mathbb{R}^n, \quad M^+_{J,\mathcal{K}(\gamma,\delta,m)}(1, x) \leq 0$$

As a result, we see that $\forall L \in \mathcal{L}_{inv}$, we have $c \leq 0$ in (5.2.2).

Proof. Note that in (5.2.2), $L(1, 0) = c$. Because of the rotation invariance of $M^+_{J,\mathcal{K}(\gamma,\delta,m)}$, it is enough to show that $M^+_{J,\mathcal{K}(\gamma,\delta,m)}(1, 0) \leq 0$. It can also be observed here that the consequence of the result follows from having

$$c = L(1, 0) = \max_{\ell \in [\partial j]} \ell(R_0 1) = M^+_{J,\mathcal{K}(\gamma,\delta,m)}(1, 0)$$

Suppose we write m_0 to denote the Lipschitz functional

$$m_0(f) = M^+_{J,\mathcal{K}(\gamma,\delta,m)}(f,0) = \max_{\ell\in[\partial j]} \ell(f), \quad \text{where} \;\; j = j(f) = J(f,0).$$

Assumption 5.2.1 says that $J(f+c,x) \leq J(f,x)$ for $c > 0$ and $f \in \mathcal{K}(\gamma,\delta,m)$. Thus

$$j^0(f;1) = \limsup_{x\to 0^+} \frac{j(f+s\cdot 1) - j(f)}{s} \leq 0.$$

$$\text{By definition,} \;\; \partial j(f) := \{\ell \in \left(C^{1,\gamma}(\mathbb{R}^n)\right)^\star | \forall \psi \in C^{1,\gamma}(\mathbb{R}^n), j^0(f;\psi) \geq \langle \ell, \psi\rangle\}$$

$$\text{and} \;\; [\partial j]_{\mathcal{K}} := \text{hull}\left(\bigcup_{f\in\mathcal{K}} \partial j\right)$$

Thus, if $\ell \in [\partial j]_{\mathcal{K}(\gamma,\delta,m)}$, then $\ell(1) \leq 0$. So from the definition of $M^+_{J,\mathcal{K}(\gamma,\delta,m)}$ in (5.2.5), we can say that $M^+_{J,\mathcal{K}(\gamma,\delta,m)}(1,0) = m_0(1) \leq 0$. □

5.3. Comparison theorem and existence for parabolic viscosity solutions on the sphere

5.3.1. Using the assumptions of section 5.2

Now, we look at operators of the type J that satisfy the above assumptions, and for such J, we prove uniqueness for the following equation:

$$\begin{aligned} \partial_t f &= J(f) && \text{in } \mathbb{S}^n \times (0,T] \\ f(\cdot,0) &= f_0 && \text{on } \mathbb{S}^n \end{aligned} \tag{5.3.1}$$

Definition 5.3.1. *We say that f is a viscosity subsolution of* (5.3.1) *if f is upper semi-*

continuous, $\inf f > 0$*, and if* f *has the following property: For all* $(x,t) \in \mathbb{S}^n \times (0,T]$ *for which there exists a function* $\phi \in C^{1,\gamma+\beta}(\mathbb{S}^n)$ *in space and* C^1 *in time (for some* $\beta > 0$ *with* $\gamma+\beta \leq 1$*), and* $\delta \leq \phi \leq L-\delta$ *such that for some* $t > r > 0$*,* $f-\phi$ *attains a global maximum over* $\mathbb{S}^n \times (t-r,t]$*, then* ϕ *must satisfy*

$$\partial_t \phi(x,t) \leq J(\phi, x)$$

On the other hand, we say g *is a viscosity supersolution if* g *is lower semi-continuous,* $\inf g > 0$*, and if we replace the above properties with* $g - \phi$ *attains a minimum and*

$$\partial_t \phi(x,t) \geq J(\phi, x)$$

Of course, f *is a viscosity solution if is satisfies both of the above.*

Lemma 5.3.2. *If* f *is a viscosity subsolution of* (5.3.1)*, and suppose* $(x,t) \in \mathbb{S}^n \times (0,T)$ *is a point such that* $f - \phi$ *attains a maximum at the point* (x,t) *for some* ϕ *that is punctually* $C^{1,\gamma+\beta}(x)$ *in space and* C^1 *in time, with* $\beta > 0$*,* $\gamma+\beta \leq 1$*. Then there exists a choice of* δ_0 *small enough and* m_0 *large enough such that* $\tilde{J}_{\mathcal{K}(\gamma,\delta_0,m_0)}(\phi,x)$ *is classically defined and*

$$\partial_t \phi(x,t) \leq \tilde{J}_{\mathcal{K}(\gamma,\delta_0,m_0)}(\phi, x)$$

Remark 5.3.3. *The analogous result holds for* g *that are supersolutions and for those* ϕ *we also require them to additionally satisfy* $\inf \phi > 0$*. Note that this is not a problem with the subsolution because* $f - \phi \leq 0 \implies \delta \leq f \leq \phi \implies \inf \phi > 0$*.*

Remark 5.3.4. $\tilde{J}$ *is defined in* (5.2.6) *and we need this because* ϕ *is not necessarily in the* *set,* $\mathcal{K}(\gamma, \delta_0, m_0)$*. Also, this result will hold for all* $\delta < \delta_0$ *and* $m > m_0$*.*

Proof. The well defined nature of $\tilde{J}_{\mathcal{K}(\gamma,\delta_0,m_0)}(\phi, x)$ is because of Corollary 5.2.6, from which we can also see that we only need $\phi \in C^{1,\gamma}(x)$ and not $\gamma + \beta$. The extra regularity will come in later.

Now, since ϕ is punctually $C^{1,\gamma+\beta}$ at x, this means there are two $C^{1,\gamma+\beta}$ functions f^+, f^-, such that

$$\forall y \in \mathbb{R}^n, \quad f^-(y) \leq \phi(y) \leq f^+(y) \ \text{ and } \ f^-(x) = \phi(x) = f^+(x)$$

Now for each $r > 0$, we can define a function ϕ_r such that

$$\phi_r(y) = \begin{cases} f^+(y) & \text{if } y \in B_r(x) \\ \phi(y) & \text{otherwise.} \end{cases}$$

Thus, we also have the ordering $f^- \leq \phi \leq \phi_r \leq f^+$ and
$\nabla f^-(x) = \nabla\phi(x) = \nabla\phi_r(x) = \nabla f^+(x)$ (they all touch at the point x). Finally, we can also assume wlog that

$$\partial_t\phi(x,t) = \partial_t f^+(x,t)$$

$f^+ \in C^{1,\gamma+\beta}(\mathbb{R}^n)$ is a function such that $f - f^+$ attains a maximum at (x,t). Now, $f^+ \geq \phi$ and $\inf\phi > 0$, and also, $f^+ \in C^{1,\gamma+\beta}$. These facts imply that there exists some δ_0 and m_0 such that $f^+ \in \mathcal{K}(\gamma, \delta_0, m_0) = \mathcal{K}$ (we call is so for simplicity). Thus by definition of viscosity subsolution,

$$\partial_t\phi(x,t) = \partial_t f^+(x,t) \le J(f^+,x) = \tilde{J}_{\mathcal{K}}(f^+,x)$$

Now we use Corollaries 5.2.5 and 5.2.6, which tell us that $\tilde{J}$ has many of the properties of J and satisfies many of the same inequalities.

$$\begin{aligned}\tilde{J}_{\mathcal{K}}(f^+,x) &\le \tilde{J}_{\mathcal{K}}(\phi_r,x) + M^+_{J,\mathcal{K}}(f^+-\phi_r,x) && \text{(since } \tilde{J} \text{ also satisfies (5.2.4))}\\ &\le \tilde{J}_{\mathcal{K}}(\phi,x) + M^+_{J,\mathcal{K}}(\phi_r-\phi,x) + M^+_{J,\mathcal{K}}(f^+-\phi_r,x)\end{aligned}$$

All of the above operators are well defined on these functions, as they are all punctually $C^{1,\gamma}(x)$. If we can show that the last two terms $\to 0$ as $r \to 0$, then we complete the proof. This is where we will need the slightly higher regularity. Now following the definitions:

$$M^+_{J,\mathcal{K}}(\phi_r-\phi,x) = \max_{\ell\in[\partial j]_{\mathcal{K}}} \ell((\phi_r-\phi)\circ R_x) \le \max_{L\in\mathcal{L}_{inv}} L(\phi_r-\phi,x)$$

Also, from Theorem 5.2.3, we obtain the following bounds:

$$
\begin{aligned}
L((\phi_r - \phi, x) =& c(\phi_r(x) - \phi(x)) - b \cdot (\nabla\phi_r(x) - \nabla\phi(x)) \\
& \qquad\qquad \text{(at } x\text{, we have } \phi_r = \phi, \nabla\phi_r = \nabla\phi\text{)} \\
& + \int_{\mathbb{S}^n} \phi_r(R_x(h)) - \phi_r(x) - \mathbb{1}_{B_r(x)} \left\langle \nabla\phi_r(x), \exp_x^{-1}(h) \right\rangle \mu(dh) \\
& - \int_{\mathbb{S}^n} \phi(R_x(h)) - \phi(x) - \mathbb{1}_{B_r(x)} \left\langle \nabla\phi(x), \exp_x^{-1}(h) \right\rangle \mu(dh) \\
& \qquad\qquad (B_r(x) \text{ since outside of this set, } \nabla\phi_r - \nabla\phi = 0) \\
\leq & \int_{B_r(x)} C_\phi |h|^{1+\gamma+\beta} \mu(dh) \\
& \qquad\qquad (C_\phi \text{ is the constant depending on } \phi, \phi_r \in C^{1,\gamma+\beta})
\end{aligned}
$$

We can make a similar statement about $f^+ - \phi_r$. Now, for each L, there is a μ and hence $M^+_{J,\mathcal{K}}$ is a max over a family of the measures μ, arising from the set $[J]_{\mathcal{K}}$, and each of these μ satisfy the uniform bound in (5.2.3). Suppose we say

$$
meas(\mathcal{K}) := \{\mu | \exists \ell \in [\partial j]_{\mathcal{K}} \text{ s.t. } \mu \text{ corresponds to } \ell \text{ as in Proposition 5.2.2}\}
$$

Then we can say that we have the following bounds

$$
\begin{aligned}
M^+_{J,\mathcal{K}}(\phi_r - \phi, x) + M^+_{J,\mathcal{K}}(f^+ - \phi, x) &\leq \int_{B_r} C_\phi |h|^{1+\gamma+\beta} \mu(dh) \\
&\leq C_\phi r^\beta \int_{B_r} |h|^{1+\gamma} \mu(dh) \\
&\leq C_\phi r^\beta \sup_{\mu \in meas(\mathcal{K})} \int_{B_r} \min\{|h|^{1+\gamma}, 1\} \mu(dh) \\
&\leq C_2 \cdot C_\phi \cdot r^\beta
\end{aligned}
$$

Hence, as $r \to 0$, we see that $M^+_{J,\mathcal{K}}(\phi_r - \phi, x) + M^+_{J,\mathcal{K}}(f^+ - \phi, x) \to 0$ and thus we have proved the lemma. □

We now propose that the inf and sup convolutions that are appropriate for the parabolic equation on $\mathbb{S}^n \times [0,T]$ are the following:

$$\phi^\varepsilon(x,t) := \sup_{y=R_z x, d(z,0)\le c_0, t' \in [0,T]} \phi(y,s) - \frac{1}{2\varepsilon}\left(||x-y||^2 + |t-t'|^2 \right) \tag{5.3.2}$$

$$\phi_\varepsilon(x,t) := \inf_{y=R_z x, d(z,0)\le c_0, t' \in [0,T]} \phi(y,s) + \frac{1}{2\varepsilon}\left(||x-y||^2 + |t-t'|^2 \right) \tag{5.3.3}$$

Another way to write these (and a notations we will often use) is as follows:

$$\phi^\varepsilon(x,t) := \sup_{d(z,0)\le c_0, s\in[-t,T-t]} \phi(R_z x, t+s) - \frac{1}{2\varepsilon}\left(||R_z||^2 + |s|^2 \right)$$
$$\phi_\varepsilon(x,t) := \inf_{d(z,0)\le c_0, s\in[-t,T-t]} \phi(R_z x, t+s) + \frac{1}{2\varepsilon}\left(||R_z||^2 + |s|^2 \right)$$

Lemma 5.3.5. *We note the following properties of the sup/inf-convolutions:*

(i) The sup-convolution given in (5.3.2) *is semi-convex and the inf-convolution in* (5.3.3) *is semi-concave.*

(ii) the sup and inf convolution are Lipschitz functions.

(iii) Both ϕ^ε and ϕ_ε converge to ϕ pointwise as $\varepsilon \to 0$.

(iv) If u and v are a subsolution and supersolution to the equation $\partial_t u(x,t) = J(u,x)$ in $\mathbb{S}^n$ respectively, then u^ε and v_ε are also a subsolution and supersolution respectively to the same equation.

Proof. (i) It is easy to see from the definition directly that the inf-convolution, ϕ_ε is semi-concave. We can write the sup-convolution as

$$\phi^\varepsilon(x) = -\inf_{d(z,0)\leq c_0, s\in[0,T]} \left\{ -\phi(R_z x, s) + \frac{1}{2\varepsilon}\left(||x - R_z x||^2 + |t-s|^2\right)\right\}$$

Thus, $-\phi^\varepsilon$ is semi-concave by definition, which makes ϕ^ε semi-convex. We can also see this from knowing that $\phi^\varepsilon + |X|^2/2\varepsilon$ is convex (where $X = (x,t)$). Similarly, we can see that ϕ_ε is semi-concave.

(ii) Consider for $X_1 = (x_1, t_1), X_2 = (x_2, t_2) \in \mathbb{S}^n \times [0,T]$, and let $Y_1^\varepsilon = (y_1, s_1) = (R_{z^\varepsilon} x_1, s_1)$ be the point at which the supremum is realized for X_1, or

$$\phi^\varepsilon(x_1, t_1) = \phi(R_{z^\varepsilon} x_1, s_1) - \frac{1}{2\varepsilon}\left(||x_1 - R_{z^\varepsilon} x_1||^2 + |t_1 - s_1|^2\right).$$

On the other hand, we also have

$$\begin{aligned}
\phi^\varepsilon(x_2, t_2) &= \sup_{d(z,0)\leq c_0, s\in[0,T]} \left\{ \phi(R_z x_2, t_2) - \frac{1}{2\varepsilon}\left(||x_2 - R_z x_2||^2 + |t_2 - s|^2\right)\right\} \\
&\geq \phi(R_{z^\varepsilon} x_1, s_1) - \frac{1}{2\varepsilon}\left(||x_2 - R_{z^\varepsilon} x_1||^2 + |t_2 - s_1|^2\right)
\end{aligned}$$

where z_1^ε is such that $R_z x_2 = R_{z_1^\varepsilon} x_1$ for some $z \in B_{c_0}$. Thus, we can write

$$\begin{aligned}
\phi^\varepsilon(x_1, t_1) - \phi^\varepsilon(x_2, t_2) \leq& \cancel{\phi(R_{z^\varepsilon} x_1, s_1)} - \frac{1}{2\varepsilon}\left(||x_1 - R_{z_1^\varepsilon} x_1||^2 + |t_1 - s_1|^2\right) \\
& - \cancel{\phi(R_{z^\varepsilon} x_1, s_1)} + \frac{1}{2\varepsilon}\left(||x_2 - R_{z^\varepsilon} x_1||^2 + |t_2 - s_1|^2\right) \\
\leq& C_1|x_1 - x_2| + C_2|t_1 - t_2| \\
\leq& C||(x_1, t_1) - (x_2, t_2)|| \qquad (C = C(\mathbb{S}^n, T))
\end{aligned}$$

thereby showing that ϕ^ε is Lipschitz. We can similarly show that ϕ_ε is Lipschitz.

(iii) This proof is based on the one given in [15, Lemma A.5].

Assume ψ is an appropriate function that touches ϕ^ε from above at $(x,t) = X$, and that $D\psi(x,t) = \mathbf{q} \in \mathbb{R}^{n+1}$, $D^2\psi(x,t) = M \in \mathcal{S}^{d+1}$, the set of symmetric $(d+1)\times(d+1)$ matrices. We will first show that this means $\exists$ some ψ' that touches ϕ at $X + \mathbf{q}\varepsilon$ and

$$\phi^\varepsilon(X) + \frac{\varepsilon}{2}|\mathbf{q}|^2 = \phi(X + \mathbf{q}\varepsilon)$$

Indeed, suppose $Y^\varepsilon_{(x,t)}$ is the point such that the supremum is achieved in the definition of $\phi^\varepsilon(X)$. We will use the notation $Y^\varepsilon_{(x,t)} = Y^\varepsilon = (y^\varepsilon, t^\varepsilon)$, i.e.

$$\phi^\varepsilon(x,t) = \phi(y^\varepsilon, t^\varepsilon) - \frac{1}{2\varepsilon}(||y^\varepsilon - x||^2 + |t^\varepsilon - t|^2)$$

From similar calculations as in [15], we get $Y^\varepsilon = X + \mathbf{q}\varepsilon$. Thus, we have

$$\begin{aligned}
\phi^\varepsilon(X) + \frac{\varepsilon}{2}|\mathbf{q}|^2 &= \phi(Y^\varepsilon) - \frac{1}{2\varepsilon}|X - Y^\varepsilon|^2 + \frac{\varepsilon}{2}|\mathbf{q}|^2 \\
&= \phi(X + \mathbf{q}\varepsilon) - \frac{\varepsilon}{2}|\mathbf{q}|^2 + \frac{\varepsilon}{2}|\mathbf{q}|^2 \\
&= \phi(X + \mathbf{q}\varepsilon)
\end{aligned}$$

We also know that $|\mathbf{q}|\varepsilon \to 0$ when $\varepsilon \to 0$ because

$$
\begin{aligned}
\phi(X) &\leq \phi^{\varepsilon}(X) \\
&= \phi(Y^{\varepsilon}) - \frac{1}{2\varepsilon}|X - Y^{\varepsilon}|^2 \\
\implies \frac{1}{2\varepsilon}|X - Y^{\varepsilon}|^2 &\leq \phi(Y^{\varepsilon}) - \phi(X) \leq 2||\phi||_{L^\infty} \\
\implies |X - Y^{\varepsilon}| &\leq \sqrt{4\varepsilon||\phi||_{L^\infty}} \to 0 \text{ as } \ \varepsilon \to 0
\end{aligned}
$$

Finally, we see that

$$
\begin{aligned}
\limsup_{\varepsilon\to 0}\left\{\phi^{\varepsilon}(X) + \frac{\varepsilon}{2}|\mathbf{q}|^2\right\} &= \limsup_{\varepsilon\to 0}\phi(X + \mathbf{q}\varepsilon) \\
&\leq \phi(X) && (\phi \text{ is USC}) \\
&\leq \limsup_{\varepsilon\to 0}\phi^{\varepsilon}(X) && (\text{since we have } \phi^{\varepsilon} \geq \phi)
\end{aligned}
$$

This forces $\limsup_{\varepsilon\to 0}\frac{\varepsilon}{2}|\mathbf{q}|^2 = 0$, and eventually, we can see that

$$
\limsup_{\varepsilon\to 0}\phi^{\varepsilon}(X) = \limsup_{\varepsilon\to 0}\phi(X + \mathbf{q}\varepsilon) - \frac{\varepsilon}{2}|\mathbf{q}|^2 = \phi(X)
$$

(iv) Suppose u, v are respectively the subsolution, supersolution of the given equation, i.e., we have in the viscosity sense, that

$$
\partial_t u(x,t) \leq J(u,x), \quad \text{and} \quad \partial_t v(x,t) \geq J(v,x).
$$

We shall use $u\circ R_z$ to denote the function $u\circ R_z(x,t) = u(R_z x, t)$, where $R_z : \mathbb{S}^n \to \mathbb{S}^n$ is the rotation on the sphere that takes 0 to z. Note that we have $J(u\circ R_z, x) = J(u, R_z x)$

since J is a rotation invariant operator. So as u is a subsolution, we should have

$$\partial_t(u \circ R_z)(x,t) = \partial_t u(R_z x, t) \leq J(u, R_z x) = J(u \circ R_z, x). \tag{5.3.4}$$

Now, suppose ϕ is the required function such that $u^{\varepsilon} - \phi$ has its maximum at a point $(x_0, t_0) = X_0$, i.e., we have $u^{\varepsilon} \leq \phi$ and $u^{\varepsilon}(x_0, t_0) = \phi(x_0, t_0)$. Suppose that in the definition of $u^{\varepsilon}(x,t)$, $z^{\varepsilon}, s^{\varepsilon}$ are the parameters so that the supremum is realized for (x,t), and in particular, $z_0^{\varepsilon}, s_0^{\varepsilon}$ are the parameters for the supremum in the definition of $u^{\varepsilon}(x_0, t_0)$. Then we have

$$u^{\varepsilon}(x_0, t_0) = u(R_{z_0^{\varepsilon}} x_0, t_0 + s_0^{\varepsilon}) - \frac{1}{2\varepsilon}\left(||R_{z_0^{\varepsilon}}||^2 + |s_0^{\varepsilon}|^2\right)$$

Further, $\forall (x,t) \in \mathbb{S}^n \times [0,T]$, since u^{ε} is the sup

$$\begin{aligned} \phi(x,t) \geq u^{\varepsilon}(x,t) &\geq u(R_{z_0^{\varepsilon}} x, t + s_0^{\varepsilon}) - \frac{1}{2\varepsilon}\left(||R_{z_0^{\varepsilon}}||^2 + |s_0^{\varepsilon}|^2\right) \\ \text{and} \;\; \phi(x_0,t_0) = u^{\varepsilon}(x_0,t_0) &= u(R_{z_0^{\varepsilon}} x_0, t_0 + s_0^{\varepsilon}) - \frac{1}{2\varepsilon}\left(||R_{z_0^{\varepsilon}}||^2 + |s_0^{\varepsilon}|^2\right). \end{aligned}$$

Let $\phi' = \phi + c_{\varepsilon}$, where $c_{\varepsilon} = 1/2\varepsilon(||R_{z_0^{\varepsilon}}||^2 + |s_0^{\varepsilon}|^2)$.

From earlier, we have $\partial_t u \circ R_{z_0^{\varepsilon}}(x,t) \leq J(u \circ R_{z_0^{\varepsilon}}, x)$.

Note that ϕ' touches $u \circ R_{z_0^\varepsilon}(\cdot + (0, s_0^\varepsilon))$ from above at x_0, i.e. we have

$$\begin{aligned} \phi'(x,t) &\geq u(R_{z_0^\varepsilon} x, t + s_0^\varepsilon) = u \circ R_{z_0^\varepsilon}(x, t + s_0^\varepsilon) \\ \text{and} \quad \phi'(x_0,t_0) &= u \circ R_{z_0^\varepsilon}(x_0, t_0 + s_0^\varepsilon) \end{aligned}$$

Since u is the viscosity subsolution to a rotation invariant equation, we have from (5.3.4) $\partial_t \phi'(x_0, t_0) \leq J(\phi', x_0)$. Also, since c_ε (or s_0^ε) is constant, the function u^ε solves in the viscosity sense

$$\partial_t(u^\varepsilon)(x,t) = \partial_t(u^\varepsilon + c_\varepsilon)(x,t) \leq J(u^\varepsilon + c_\varepsilon, x) \leq J(u^\varepsilon, x)$$

from the properties of J. We have already seen from the proof of part (iii) that $c_\varepsilon \to 0$ as $\varepsilon \to 0$. See previous proof.

□

We also need to know how the family of operators described in Assumption 5.2.1 will operate on rescaled versions of smooth bump functions.

Lemma 5.3.6. *For $x_0 \in \mathbb{S}^n$, if we define smooth functions Φ and Φ_R as*

$$\Phi(x) = \frac{|x - x_0|^2}{1 + |x - x_0|^2}, \quad \textit{and} \quad \Phi_R(x) = \Phi\left(\frac{x}{R}\right)$$

Then, for a fixed J, given and $\delta > 0, m > \delta, \rho > 0, \exists R > 1$, with $R = R(J, \rho, \delta, m)$, so that

$$\sup_{x \in \mathbb{S}^n} M^+_{J,\mathcal{K}(\gamma,\delta,m)}(\Phi_R, x) \leq \rho$$

Proof. First, we will observe the following facts about Φ_R

$$0 \le \Phi_R(x) \le 1 \text{ and } ||\nabla\Phi_R||_{L^\infty} \le \frac{C}{R}$$

$$\left|\Phi_R(R_h x) - \Phi_R(x) - \left\langle \nabla\Phi_R(x), \exp_x^{-1}(h)\right\rangle\right| \le \frac{C|h|^2}{R^2}$$

$$\text{Note that } \nabla\Phi = \frac{2(x-x_0)}{(1+|x-x_0|^2)^2} \implies |\nabla\Phi| \le 1 \implies |\nabla\Phi_R| \le \frac{C}{R}$$

The first fact is owing to the fact that since $|\nabla\Phi| \le 1$, we have $|\nabla\Phi_R| \le C/R$ when we rescale. The second fact follows from the Taylor expansion of Φ_R.

Now, from the estimates in Theorem 5.2.3, we can choose R large enough so that, uniformly across $[\partial J]_{\mathcal{K}(\gamma,\delta,m)}$

$$\int_{\mathbb{S}^n} \left|\Phi_R(R_h x) - \Phi_R(x) - \mathbb{1}_{B_1(x)}(h)\left\langle \nabla\Phi_R(x), \exp_x^{-1}(h)\right\rangle\right| \mu(dh) \le \int_{\mathbb{S}^n} C\frac{|h|^2}{R^2}\mu(dh) \le \frac{\rho}{2}$$

Next, given this t we can choose R large enough so that uniformly across $b \in [\partial j]_{\mathcal{K}(\gamma,\delta,m)}$ (actually, it is $\ell \in [\partial j]_{\mathcal{K}(\gamma,\delta,m)}$, but there is a one to one correspondence between b and ℓ from Proposition 5.2.2), we have $b \cdot \nabla\Phi_R \le \rho/2$. Finally, thanks to Proposition 5.2.7, we see that for any of the constants, c, that appear in (5.2.2), we have $c\Phi_R(x) \le 0$.

Now if we add everything up, we get $\ell(f \circ R_x)$, and if we take the max then by the definition of $M^+_{J,\mathcal{K}(\gamma,\delta,m)}$, we have the required inequality after taking the supremum. □

Lemma 5.3.7. *Given any* $\delta, \rho, C > 0,\ m > \delta,\ \exists R = R(J, \rho, \delta, m)$ *so that*

$$\forall h > 0,\ \forall (x,t) \in \mathbb{S}^n \times [0,T], \quad \Psi(x,t) = C + h\Phi_R(x) + h\rho t$$

is a classical, strict supersolution of

$$\partial_t \Psi > M^+_{J,\mathcal{K}(\gamma,\delta,m)}(\Psi)$$

Proof. It is a direct calculation to show that

$$M^+_{J,\mathcal{K}(\gamma,\delta,m)}(C + h\Phi_R, x) \leq hM^+_{J\mathcal{K}(\gamma,\delta,m)}(\Phi_R)$$

Indeed, $L(C + h\Phi_R)$

$$\begin{aligned}
=& c(C + h\Phi_R) + hb \cdot \nabla\Phi_R \\
&+ \int_{\mathbb{S}^n} h\Phi_R(R_x y) - h\Phi_R(x) - \mathbb{1}_{B_1(x)}(y) h \left\langle \nabla\Phi_R, \exp_x^{-1}(y)\mu(dy) \right\rangle \\
=& c \cdot C + hL(\Phi_R, x) \leq hL(\Phi_R, x)
\end{aligned}$$

(since $c \leq 0$ by Proposition 5.2.7 and $C > 0$ is given)

Now we invoke Lemma 5.3.5 with $\rho/2$ and get

$$M^+_{J,\mathcal{K}(\gamma,\delta,m)}(\Psi) = M^+_{J,\mathcal{K}(\gamma,\delta,m)}(C + h\Phi_R) \leq hM^+_{J,\mathcal{K}(\gamma,\delta,m)}(\Phi_R) \leq h\rho/2 < \partial_t(\Psi)$$

□

5.3.2. Comparison results

Lemma 5.3.8. *If $\delta > 0, m > \delta$ are fixed, and $w : \mathbb{S}^n \times [0,T] \to \mathbb{R}$ is a bounded, upper semi-continuous function such that, in the viscosity sense,*

$$\partial_t w \leq M^+_{J,\mathcal{K}(\gamma,\delta,m)}(w)$$

$$\text{then,} \quad \sup_{\mathbb{S}^n \times [0,T]} w_+(\cdot, t) \leq \sup_{\mathbb{S}^n} w_+(\cdot, 0)$$

Proof. First, we will start the proof by assuming $\sup_{\mathbb{S}^n} w_+(\cdot, 0) = 0$. Then, we need to show that $\sup_{\mathbb{S}^n \times [0,T]} w_+ \leq 0$. Suppose we assume, for the sake of contradiction, that $\sup_{\mathbb{S}^n \times [0,T]} w_+ > 0$. Since for $h, C, \rho > 0$, Ψ as described in Lemma 5.3.6 is strictly above 0, we can choose h and C in a way so that $w_+ - \Psi$ attains a global maximum for some $t > 0$. If we use Ψ as a test function and apply the definition of a viscosity subsolution for w_+, we get a contradiction i.e. $\partial_t \Psi \leq M^+_{J,\mathcal{K}(\gamma,\delta,m)}(\Psi)$, which contradicts what we proved in Lemma 5.3.7.

On the other hand, suppose we do not necessarily assume that $\sup_{\mathbb{S}^n} w_+(\cdot, 0) = 0$. Now we can replace w by the function

$$\tilde{w} = w - \sup_{\mathbb{S}^n} w_+(\cdot, 0)$$

Thus, let $c = \sup_{\mathbb{S}^n} w_+(\cdot, 0) \geq 0$, we see that $w = \tilde{w} + c$, and so in the viscosity sense, $\partial_t w = \partial_t \tilde{w}$. Also, since $c \geq 0$, by Proposition 5.2.7 (which tells us that $c \leq 0$ in the expression for L), just like in the proof of Lemma 5.3.7, we get

$$M^{+}_{J,\mathcal{K}(\gamma,\delta,m)}(w) = M^{+}_{J,\mathcal{K}(\gamma,\delta,m)}(\tilde{w}+c) \leq M^{+}_{J,\mathcal{K}(\gamma,\delta,m)}(\tilde{w})$$

Hence, in the viscosity sense, we have

$$\partial_t \tilde{w} = \partial_t w \leq M^{+}_{J,\mathcal{K}(\gamma,\delta,m)}(w) \leq M^{+}_{J,\mathcal{K}(\gamma,\delta,m)}(\tilde{w})$$

and now, we can apply the first case again since we have $\sup_{\mathbb{S}^n} \tilde{w}_+(\cdot,0) = 0$, which gives us with the assumption above

$$\begin{aligned} & \sup_{\mathbb{S}^n \times [0,T]} \tilde{w}_+(\cdot,t) \leq 0 \\ \implies & \sup_{\mathbb{S}^n \times [0,T]} w_+(\cdot,t) \leq \sup_{\mathbb{S}^n} w_+(\cdot,0) \end{aligned}$$

□

Proposition 5.3.9. *Suppose $f, g : \mathbb{S}^n \times [0,T] \to \mathbb{R}$ are bounded, and for some $\delta > 0$, $f, g \geq \delta$, and they are respectively a subsolution and supersolution in the viscosity sense, of the equation $\partial_t(u) = J(u)$, then*

$$f^{\varepsilon}(x,0) \leq g_{\varepsilon}(x,0) \implies f^{\varepsilon}(x,t) \leq g_{\varepsilon}(x,t) \quad \forall (x,t) \in \mathbb{S}^n \times [0,T], \tag{5.3.5}$$

where $f^{\varepsilon}, g_{\varepsilon}$ are as their definitions given in (5.3.2), (5.3.3).

Proof. We first note from lemma 5.3.5 that f^{ε}, g_{ε} are given in (5.3.2) and (5.3.3), then the following inequalities are satisfied in the viscosity sense:

$$\partial_t f^\varepsilon \leq J(f^\varepsilon), \quad \text{and} \quad \partial_t g_\varepsilon \geq J(g_\varepsilon)$$

Let $\varepsilon \in (0,1)$ be fixed, and we assume $\forall x \in \mathbb{S}^n,\ f^\varepsilon(x,0) - g_\varepsilon(x,0) \leq 0$. We want to show that

$$f^\varepsilon(x,t) - g_\varepsilon(x,t) \leq 0 \quad \forall (x,t) \in \mathbb{S}^n \times [0,T]$$

we will assume for the sake of contradiction that there exists a time $t^\star \in (0,T]$ such that

$$\sup_{(x,t)\in\mathbb{S}^n\times[0,T]} (f^\varepsilon(x,t) - g_\varepsilon(x,t)) = \sup_{x\in\mathbb{S}^n} (f^\varepsilon(x,t^\star) - g_\varepsilon(x,t^\star)) = m_0 > 0$$

Note that this cannot happen at $t = 0$ because of the assumption in this case (when $t = 0$, we have $f^\varepsilon - g_\varepsilon \leq 0$). Since ε is fixed, and m_0 is given, we can invoke Lemma 5.3.6 with $m = c\varepsilon^{-1}$ and $C = cm_0$, i.e. have some $\Psi = cm_0 + h\Phi_R + h\rho t$ (where $R = R(J,\delta,\rho,c\varepsilon^{-1})$) as a strict supersolution. i.e. we know Ψ exists such that the following inequality holds classically for all $(x,t) \in \mathbb{S}^n \times [0,T]$

$$\partial_t \Psi > M^+_{J,\mathcal{K}(\gamma,\delta/2,c\varepsilon^{-1})}(\Psi)$$

Now we can translate Ψ around so it touches $f^\varepsilon - g_\varepsilon$ from above. Since we are assuming that $f^\varepsilon - g_\varepsilon$ attains a positive supremum, and Ψ is always positive, we know they can touch if we translate,i.e. we can choose C so that $\exists (x_0^\varepsilon, t_0^\varepsilon) \in \mathbb{S}^n \times (0,T]$ such that

$$f^\varepsilon - g_\varepsilon \leq \Psi, \quad \text{and} \quad (f^\varepsilon - g_\varepsilon)(x_0^\varepsilon, t_0^\varepsilon) = \Psi(x_0^\varepsilon, t_0^\varepsilon)$$

By definition, for any $\varepsilon > 0$, f^ε and $-g_\varepsilon$ are semi-convex, which means that they both have tangent paraboloids of opening $1/\varepsilon$ touching from below. Thus this is also true for $f^\varepsilon - g_\varepsilon$. Now since a smooth function Ψ touches $f^\varepsilon - g_\varepsilon$ from above at the point $(x_0^\varepsilon, t_0^\varepsilon)$, we have that both f^ε and $-g_\varepsilon$ must be $C^{1,1}$ at the point $(x_0^\varepsilon, t_0^\varepsilon)$. Thus, we can evaluate $\partial_t f^\varepsilon, \partial_t g_\varepsilon, \nabla f^\varepsilon, \nabla g_\varepsilon$ classically at $(x_0^\varepsilon, t_0^\varepsilon)$. Thus, by Lemma (5.3.2), we see that the equations hold classically for f^ε and $-g_\varepsilon$. For f^ε, we use $g_\varepsilon + \Psi$ as the test function, and for g_ε, the test function is $f^\varepsilon - \Psi$. Furthermore, by Proposition 5.2.4 and Lemma 5.3.2, we get classically

$$\begin{aligned}
\partial_t f^\varepsilon - \partial_t g_\varepsilon &\le \tilde{J}_{\mathcal{K}(\gamma,\delta,c\varepsilon^{-1})}(f^\varepsilon, x_0^\varepsilon) - \tilde{J}_{\mathcal{K}(\gamma,\delta,c\varepsilon^{-1})}(g_\varepsilon, x_0^\varepsilon) \\
&\le M^+_{J,\mathcal{K}(\gamma,\delta,c\varepsilon^{-1})}(f^\varepsilon - g_\varepsilon, x_0^\varepsilon) \\
&\le M^+_{J,\mathcal{K}(\gamma,\delta,c\varepsilon^{-1})}(\Psi, x_0^\varepsilon) \qquad \text{(Since } M^+_{J,\mathcal{K}(\gamma,\delta,c\varepsilon^{-1})} \text{ enjoys GCP.)} \\
&< \partial_t \Psi
\end{aligned}$$

Since we know that at $(x_0^\varepsilon, t_0^\varepsilon)$, we have $\partial_t f^\varepsilon - \partial_t g_\varepsilon = \partial_t \Psi$, this gives us a contradiction. So we can conclude that for each ε fixed,

$$\sup_{(x,t)\in\mathbb{S}^n\times[0,T]} (f^\varepsilon(x,t) - g_\varepsilon(x,t)) \le 0$$

$$\text{i.e. } \forall (x,t) \in \mathbb{S}^n \times [0,T],\ f^\varepsilon(x,t) \le g_\varepsilon(x,t)$$

□

Corollary 5.3.10. *As a consequence of the above proposition, we have that if f, g are a subsolution and supersolution respectively of $\partial_t u = J(u)$, and $f^\varepsilon, g_\varepsilon$ are the sup-convolution* *and inf-convolution respectively, then*

$$\sup_{(x,t)\in\mathbb{S}^n\times[0,T]} [f^\varepsilon(x,t) - g_\varepsilon(x,t)] \le \sup_{x\in\mathbb{S}^n} [f^\varepsilon(x,0) - g_\varepsilon(x,0)] \tag{5.3.6}$$

Proof. We assume that $\sup_{x\in\mathbb{S}^n} (f^\varepsilon(x,0) - g_\varepsilon(x,0)) = c^\varepsilon > 0$. Now let

$$\tilde{w}^\varepsilon(x,t) = f^\varepsilon(x,t) - g_\varepsilon(x,t) - c^\varepsilon$$

Since c^ε is the supremum of $f^\varepsilon - g_\varepsilon$ at 0, we obviously have $\tilde{w}^\varepsilon(x,0) \le 0$, and now we are back to the case in the previous proposition. Using the same argument as before leads us to the conclusion that

$$\sup_{(x,t)\in\mathbb{S}^n\times[0,T]} [f^\varepsilon(x,t) - g_\varepsilon(x,t) - c^\varepsilon] \le 0$$

$$\text{i.e.} \quad \sup_{(x,t)\in\mathbb{S}^n\times[0,T]} [f^\varepsilon(x,t) - g_\varepsilon(x,t)] \le \sup_{x\in\mathbb{S}^n} [f^\varepsilon(x,0) - g_\varepsilon(x,0)]$$

□

Remark 5.3.11. *We want to be able to make similar statements like* (5.3.5), (5.3.6) *about* f, g. *However, though we know that* f *and* g *are LSC and USC respectively, we do not know about the regularity of these functions yet. Thus if in* (5.3.6) *we take the limit as* $\varepsilon \to 0$, *the limit commutes with the supremum on the left (since it is the supremum over all* $(x,t) \in \mathbb{S}^n \times [0,T]$*) but not on the right because we are on the boundary. Thus we get*

$$\sup_{(x,t)\in\mathbb{S}^n\times[0,T]} [f(x,t) - g(x,t)] \le \sup_{x\in\mathbb{S}^n} [f^\varepsilon(x,0) - g_\varepsilon(x,0)],$$

when what we really want is

$$\sup_{(x,t)\in\mathbb{S}^n\times[0,T]} [f(x,t)-g(x,t)] \le \sup_{x\in\mathbb{S}^n} [f(x,0)-g(x,0)],$$

In order to take limits on the right of (5.3.6) *we will need the extra assumption in the following proposition.*

Proposition 5.3.12. *Let f and g be a subsolution and supersolution of* (5.3.1) *respectively, and assume $f(-,0) \le g(-,0)$ in a somewhat uniform way, i.e. $\forall \varepsilon > 0, \exists\, \delta > 0$ s.t.*

$$f(x,t) \le g(y,s) + \varepsilon \qquad \forall |x-y| < \delta, |t|, |s| < \delta. \tag{5.3.7}$$

$$\textit{Then} \quad \lim_{\varepsilon\to 0} \sup_{x\in\mathbb{S}^n} f^{\varepsilon}(x,0) - g_{\varepsilon}(x,0) \le 0.$$

Proof. Let $\rho > 0$. Then we will show that

$$\limsup_{\varepsilon\to 0} \sup_{x\in\mathbb{S}^n} f^{\varepsilon}(x,0) - g_{\varepsilon}(x,0) \le \rho$$

By the definitiosn of $f^{\varepsilon}, g_{\varepsilon}$, we know that $\exists\ , x^{\varepsilon}, t^{\varepsilon} x_{\varepsilon}, t_{\varepsilon}$ such that

$$\begin{aligned} f^{\varepsilon}(x,0) &= f(x^{\varepsilon}, t^{\varepsilon}) + \left(\frac{1}{2\varepsilon}|x^{\varepsilon}-x|^2 + \frac{1}{2\varepsilon}|t^{\varepsilon}|^2\right) \\ g_{\varepsilon}(x,0) &= g(x_{\varepsilon}, t_{\varepsilon}) - \left(\frac{1}{2\varepsilon}|x_{\varepsilon}-x|^2 + \frac{1}{2\varepsilon}|t_{\varepsilon}|^2\right) \end{aligned}$$

Since $f^{\varepsilon} \to f, g_{\varepsilon} \to g$ as $\varepsilon \to 0$, this means that entire term in parentheses above $\to 0$. Also, $x^{\varepsilon}, x_{\varepsilon} \to x$ and $t^{\varepsilon}, t_{\varepsilon} \to 0$. Now for $\varepsilon = \rho/3$, $\exists \delta > 0$ so that (5.3.7) holds.

Now we can choose ε small, and $|x^\varepsilon - x|,\ |x_\varepsilon - x|,\ |t^\varepsilon|,\ |t_\varepsilon|$ are all $< \delta$. Thus we get

$$\begin{aligned} f^\varepsilon(x,0) &\le f(x^\varepsilon, t^\varepsilon) + \rho/3 && (\text{as } f^\varepsilon \to f) \\ &\le g(x_\varepsilon, t_\varepsilon) + \rho/3 + \rho/3 && (\text{because of (5.3.7)}) \\ &\le g_\varepsilon(x,0) + \rho/3 + \rho/3 + \rho/3. && (\text{as } g_\varepsilon \to g) \end{aligned}$$

Putting together all the above inequalities, we can take the supremum over $x \in \mathbb{S}^n$ and then the limit supremum as $\varepsilon \to 0$ to get

$$\limsup_{\varepsilon \to 0} \sup_{x \in \mathbb{S}^n} [f^\varepsilon(x,0) - g_\varepsilon(x,0)] \le \rho.$$

Since $\rho > 0$ was arbitrary, we have

$$\lim_{\varepsilon \to 0} \sup_{x \in \mathbb{S}^n} f^\varepsilon(x,0) - g_\varepsilon(x,0) \le 0.$$

□

Theorem 5.3.13. *Let f, g be a subsolution and supersolution respectively of* (5.3.1)*, and assume $f(x,0) \le g(x,0)$ on $\mathbb{S}^n$ in a somewhat uniform way, i.e. suppose* (5.3.7) *holds. Then $f(x,t) \le g(x,t) \ \ \forall (x,t) \in \mathbb{S}^n \times [0,T]$.*

The proof of the theorem follows easily if you consider remark 5.3.11, thus by putting together the results from the previous two propositions.

Remark 5.3.14. *If we need to start the previous theorem with just the assumption that $f(x,0) \le g(x,0)\ \forall\ x \in \mathbb{S}^n$, then we need to additionally assume that f, g are bounded and uniformly continuous so that* (5.3.7) *holds.*

5.3.3. Existence (Perron's method)

The proof for existence will follow just like in [34] (which comes from ideas in [26], [15]), with a minor modification just like in [9]. In the Perron method, the solution is given by $u(x) = \sup_{v \in S_\varphi} v(x)$, where S_φ consists of all subsolutions. This u will be a subsolution in the interior, it may not match the values at the boundary, i.e. $u(y) = \varphi(y), \forall y \in \partial\Omega$.

Definition 5.3.15. *If $\exists\ y \in \partial\Omega$ such that there is supersolution w_y and $w_y(y) = 0$ whereas $w_y(x) > 0\ \forall x \neq y$, then such y are called regular points, and the function w_y is called a barrier function. At such points we also get $u(x) \to \varphi(y)$ as $x \to y$.*

We will sketch the proof for existence for (5.3.1). But first, we will list some results and definitions from [26]. Note that we will always have $f, g \in C(\overline{\mathbb{S}^n \times (0, T]})$.

Proposition 5.3.16. *[26, Proposition 2.3] If f is a subsolution and g is a supersolution of (5.3.1), $f \leq g$ on $\mathbb{S}^n \times (0, T]$, then $\exists$ a viscosity solution h such that $f \leq h \leq g$ on $\mathbb{S}^n \times (0, T]$.*

Next, we will give the definitions of the relaxed supremum/infimum of a function.

Definition 5.3.17. *[26, Section 2] Let $\Omega \subset \mathbb{R}^n$ be open. Then for any function $f : \Omega \to \mathbb{R}$ we can define*

$$f^\star(X) = \limsup_{r \downarrow 0} \{f(Z) : Z \in B(X, r)\} \quad \textit{for } X \in \overline{\Omega}$$

$$f_\star(X) = \liminf_{r \downarrow 0} \{f(Z) : Z \in B(X, r)\} \quad \textit{for } X \in \overline{\Omega}$$

These functions are upper semicontinuous and lower semicontinuous respectively, and we can see that $f_\star \leq f \leq f^\star$.

Theorem 5.3.18. *[26, Theorem 3.1] Suppose f, g are respectively viscosity subsolution and supersolution to* (5.3.1)*, and we have $f^\star(X) \leq g_\star(X)$ for $X \in \mathbb{S}^n$. Then we will also have $f^\star \leq g_\star$ on $\mathbb{S}^n \times (0, T]$.*

Remark 5.3.19. *Under the assumptions of this theorem, there is a modulus on continuity, ω_0, such that $f^\star(x) - f_\star(z) \leq \omega_0(|x - z|)$ for $x, z \in \mathbb{S}^n$. We can also now find a modulus of continuity ω so that this is true for $X = (x, t),\ Z = (z, s) \in \mathbb{S}^n \times (0, T]$.*

Theorem 5.3.20. *[26, Theorem 3.2] If f is a viscosity subsolution, and g is a viscosity supersolution to* (5.3.1)*, such that $f \leq g$ in $\mathbb{S}^n \times (0, T]$ and $f = g$ on $\mathbb{S}^n$, then $\exists$ viscosity solution $h \in C(\overline{\mathbb{S}^n \times (0, T]})$ such that $f \leq h \leq g$ in $\overline{\mathbb{S}^n \times (0, T]}$.*

We can now sketch the proof of existence using the following ideas from [34].

Proof. Let f_0 be a uniformly continuous function in $\mathbb{S}^n$. We will prove that there exists a continuous function $f : \mathbb{S}^n \times [0, T] \to \mathbb{R}$ that solves (5.3.1) and $f(-, 0) = f_0$. Perron's method gives us a continuous viscosity solution in the interior by taking a relaxed infimum over the family of supersolutions of the equation. Now we need to show that this infimum is continuous at $t = 0$ and the initial condition $f(-, 0) = f_0$ is satisfied.

Assume $f_0 : \mathbb{S}^n \to \mathbb{R}$ is uniformly continuous and $f_0 \geq \delta$. Let $\mathcal{F}$ be the set of all supersolutions h such that there is some modulus of continuity ω so that for every $x, z \in \mathbb{S}^n$ and $0 < s \leq T$

$$h(z, s) > f_0(x) - \omega(|z - x| + s)$$

Suppose b is a smooth bump function such that $b(0) = 1, b \leq 1$, and $\operatorname{supp} b = B_1$. For an arbitrary $x_0 \in \mathbb{S}^n$, let $b_\delta(x) = b((x - x_0)/\delta)$ Then depending on the modulus of continuity for f_0, $\forall\varepsilon > 0$, $\exists\delta > 0$ so that

$$U_0(x) := b_\delta(x) f_0(x_0) + (1 - b_\delta(x)) \sup_{\mathbb{S}^n} f_0 + \varepsilon \geq f_0(x)$$

$$L_0(x) : b_\delta(x) f_0(x_0) + (1 - b_\delta(x)) \inf_{\mathbb{S}^n} f_0 - \varepsilon \leq f_0(x)$$

Thus, near x_0, f_0 is trapped between U_0 and L_0 with a gap of width 2ε. Since U_0 and L_0 are smooth functions (because b is smooth), $|\nabla U_0|$ and $|\nabla L_0|$ are both bounded by some constant C. Thus we can construct a supersolution and subsolution respectively by

$$U(x,t) := U_0(x) + Ct$$

$$L(x,t) := L_0(x) - Ct$$

Going back to the equation (5.3.1), J in our case will only include terms like $I(f), G(I(f))$, which are Lipschitz. We also have J being Lipschitz regardless from Assumption 5.2.1. Moreover $J(U_0), J(L_0)$ have min-max representations consisting of integro-differential terms which will be controlled by C (from the estimates for μ in Theorem 5.2.3). Thus, we have

$$U_t - J(U) = C - J(U_0) \geq 0 \qquad \text{(we can choose } C \text{ thus)}$$

$$L_t - J(L) = -C - J(L_0) \leq 0.$$

So U and $h \in \mathcal{F}$ are supersolutions and L is a subsolution to (5.3.1), and they are uniformly continuous functions.

$$L(x,t) = L_0(x) - Ct \leq f_0(x) - Ct \leq h(z,s) + \omega(|x-z| + s) - Ct$$

We can choose δ small enough so that $|x-z|<\delta,\ |s|,|t|<\delta \implies \omega(|x-z|+s)-Ct<\varepsilon$. The comparison principle in proposition 5.3.12 gives $L \le h$ in $\mathbb{S}^n \times [0,T]$ for all lower barriers L, which means $\mathcal{F}$ is bounded below. Moreover, $\mathcal{F} \neq \emptyset$ because every upper barrier $U \in \mathcal{F}$. Indeed we have already said why U is a supersolution, but also

$$U(z,s) = U_0(z) + Cs \ge f_0(x) + f_0(z) - f_0(x) + Cs = f_0(x) - \omega(|x-z|+s)$$

$\mathcal{F}$ is a non-empty set that is bounded below, so we can find an infimum. In fact, here we use the relaxed infimum $h_\star$, i.e.

$$h_\star(x,t) = \liminf_{r\to 0} \inf_{\substack{|x-z|<r \\ |t-s|<r}} \inf_{h\in\mathcal{F}} h(z,s)$$

Note that for each $x_0 \in \mathbb{R}^n$ and $\forall\varepsilon > 0$, we can define these functions U, L, and consequently $h_\star$. We have already seen that $\forall h \in \mathcal{F},\ \forall$ lower barriers L

$$\begin{aligned}
L(x,t) &\le h(z,s) + \omega(|x-z|+s) - Ct \\
&= h(z,s) + \omega(|x-z|+|s-t|) \\
\implies L(x,t) &\le \liminf_{r\to 0} \inf_{\substack{|x-z|<r \\ |t-s|<r}} \inf_{h\in\mathcal{F}} h(z,s) + \omega(|x-z|+|s-t|) \\
&= \liminf_{r\to 0} \inf_{\substack{|x-z|<r \\ |t-s|<r}} \inf_{h\in\mathcal{F}} h(z,s) = h_\star(x,t)
\end{aligned}$$

We also have $\inf_{h\in\mathcal{F}} h(z,s) \le U(z,s)$ (since $U \in \mathcal{F}$)

$$= U_0(z) + Cs$$

$$= U_0(x) + Ct + [U_0(z) - U_0(x) + C(s-t)]$$

$$= U(x,t) + \omega(|z-x| + |s-t|)$$

$$\implies h_\star(x,t) \le \liminf_{r\to 0} \inf_{\substack{|x-z|<r \\ |t-s|<r}} U(x,t) + \omega(|z-x| + |s-t|) = U(x,t)$$

Thus, now for each point $x_0 \in \mathbb{S}^n$ and $\forall \varepsilon > 0$, we have trapped a solution $h_\star$ in between a subsolution and a supersolution. Since we can make this ε as small as possible, this means that $h_\star$ is uniformly continuous and we have $h_\star(x_0, 0) = f_0(x_0)$. Since x_0 was arbitrary, we have $h_\star(-, 0) = f_0$. □

This proves part (c) of the main result, i.e. theorem 5.1.1.

5.4. Revisiting the main results

So far, we have proved numerous properties for the broad class of operators J which satisfy the conditions in assumption 5.2.1, and then proved uniqueness for the viscosity solution of the equation (5.1.2), wherein such an operator J makes an appearance. As mentioned in the introduction in chapter 4, the reason for studying these operators and equations is that there is a strong connection between the solutions to the parabolic equations of the type in (4.1.5) and the free boundary problem of the type in (4.1.4), and this connection is explained in [9, Theorems 1.1, 1.4] which were mentioned earlier in section 4.3. The modified versions of these results for the sphere were given in section 5.1.1.

In particular, we are interested to see if these results apply to the case of the free boundary problem (5.1.1) and parabolic equation (5.1.2) introduced in section 5.1.1. Recall that the parabolic equation in question is

$$\partial_t f = \left(|\partial_\nu^+ U| - |\partial_\nu^- U|\right)\sqrt{1+|\nabla f|^2} \text{ on } \mathbb{S}^n \times [0,T]$$

suggesting that the corresponding operator J in this situation for which we proved all the results in sections 5.2 and 5.3 is given by

$$J(f,x) = \left(|\partial_\nu^+ U(x,f(x))| - |\partial_\nu^- U(x,f(x))|\right)\sqrt{1+|\nabla f(x)|^2} \tag{5.4.1}$$

But of course, inorder to apply all of the results in sections 5.2 and 5.3 to this operator J and subsequently to the equations (5.1.1) and (5.1.2), we need to first make sure that the chosen operator J from (5.4.1) satisfies all the properties listed in assumption 5.2.1. We check these properties below.

Indeed, if we fix $0 < \gamma < 1$, and if $f \in \left(\bigcup_{\delta>0}, \bigcup_{m>\delta} \mathcal{K}(\gamma,\delta,m)\right)$, then $f \in C^{1,\gamma}(\mathbb{S}^n)$ and $\delta_0 < f < M$ for some $\delta_0, M > 0$. U and f are at least $C^{1,\gamma}$ ([21, Section 8.11]), therefore their first derivatives are Hölder continuous, and thus $J(f) \in C^0(\mathbb{S}^n)$. Furthermore, in [21, Theorem 8.33] we see that $||U||_{C^{1,\gamma}}$ depends in a linear way on the $C^{1,\gamma}$-norm of the boundary of the domain, which in our case is given by f. This implies that there is a Lipschitz dependence from $f \to \partial_\nu U_f$. This and due to the fact that the mapping $f \mapsto \sqrt{1+|\nabla f(x)|^2}$ is bounded and Lipschitz on each of the sets $\mathcal{K}(\gamma,\delta,m,L)$ (where L is the upper bound for $f : \mathbb{S}^n \to \mathbb{R}$ as described in (5.1.4)), we can see that J is a Lipschitz operator. Further, since all terms in J are rotationally invariant, so is J, and we can also see that the rest of the assumptions work out easily as in [9, Theorem 7.10].

In the following sections, we list some results that follow analogously from the results in [9, Sections 9,10,11], which in turn are useful for proving the main result, theorem 5.1.1. We do not provide any full proofs for the results as they can all be easily adapted from the proofs in [9], but we will mention some ideas behind the proofs and leave some quick remarks where relevant.

5.4.1. Different notions of viscosity solutions

We will elaborate a little more on the topic of the equivalence between solving the parabolic equations of the type in (5.1.2) and the free boundary problem of the type in (5.1.1), but in order to do so, we will need a new notion of viscosity solution. In this notion, we look at test interfaces in lieu of test functions. We shall henceforth denote the set $\Sigma_0 = \mathbb{S}^n \times [\delta, L]$. A test interface S in $[a, b]$ where $0 < a \leq b < T$ will be described as follows.

Definition 5.4.1. *A test interface is a hypersurface $S \subset \Sigma_0 \times [a, b]$, such that each time slice $S(t)$ is a positive distance away from $\partial\Sigma_0$ and it separates Σ_0 into two connected components, say $S(t)^+$ and $S(t)^-$, i.e. we will have $\Sigma_0 \setminus S(t) = S^+ \cup S^-$.*

Recall the definitions of $D_f^{\pm}$ from (5.1.1), and we will also have in the two phase case

$$D_f^+ \subset S(t)^+ \quad \text{and} \quad S(t)^- \subset D_f^-.$$

Now, given a test interface S in $[a, b]$ we will define the function $U_S : \Sigma_0 \times [a, b] \to \mathbb{R}$ as the function which for each fixed time $t \in [a.b]$ is the unique solution to the Dirichlet problem

$$
\begin{aligned}
\Delta U_S = 0 \qquad & \text{in} \quad S^{\pm} \\
U_S = 0 \qquad & \text{on} \quad S \\
U_S = 1 \qquad & \text{on} \quad B_\delta(0) \\
U_S = -1 \qquad & \text{on } B_L(0)
\end{aligned}
\tag{5.4.2}
$$

Below we will give the definitions of the classical subsolution (and supersolution), as well as the original and new notions of viscosity solutions.

Definition 5.4.2. *A function $U : \Sigma_0 \times [a,b] \to \mathbb{R}$ is said to be a classical subsolution (respectively supersolution) of* (5.1.1) *if*

- *The set $\partial\{U > 0\}$ is a differentiable submanifold of $\Sigma_0 \times [a,b]$ with codimension 1, and each time-slice $\partial\{U > 0\} \cap \Sigma_0 \times \{t\}$ $(a \leq t \leq b)$ is a codimension 1 differentiable submanifold of Σ_0. Also, U is twice differentiable in space and differentiable in time in $\Sigma_0 \times [a,b] \setminus \{U \neq 0\}$.*
- *For each fixed t, the function $U(\cdot,t)$ solves, in the viscosity sense*

$$\Delta U \leq 0 \quad \textit{in} \quad \{U > 0\} \textit{ and } \{U < 0\} \quad (\textit{respectively } \geq)$$

and if V denotes the normal velocity of $\partial\{U > 0\}$ (in the outer normal direction) then

$$V \leq |\partial_\nu^+ U| - |\partial_\nu^- U| \quad (\textit{respectively } \geq)$$

U is a classical solution of (5.1.1) *if it is both a supersolution and a subsolution.*

Now we move on to the definition of a viscosity solution.

Definition 5.4.3. *A function U is said to be touched from above at (X_0, t_0) by S if S is a test interface in $[t_0 - \tau, t_0]$ for some $\tau > 0$ such that*

$$\{U > 0\} \cap \{t_0 - \tau \leq t \leq t_0\} \subset \overline{S^+} \quad \text{and} \quad (X_0, t_0) \in \partial\{U > 0\} \cap S$$

Definition 5.4.4. *A viscosity subsolution (respectively supersolution) of the two-phase equation* (5.1.1) *is an upper semicontinuous function (respectively lower semicontinous function) $U : \Sigma_0 \times [a, b] \to \mathbb{R}$, and it is required to have the following properties.*

$$U \leq 1 \quad (\text{respectively } U \geq 1) \text{ on } B_\delta(0), U \leq -1 \quad (\text{respectively } U \geq -1) \text{ on } B_L(0),$$

And it satisfies the following relations in the viscosity sense

$$-\Delta U \geq 0 \quad \text{in } \{U > 0\}^\circ \text{ and } \{U < 0\}^\circ \quad (\text{respectively } \leq),$$

and for any test interface S touching U from above at $(X_0, t_0) \in \partial\{U > 0\}$ we have

$$V_S(X_0, t_0) \leq |\partial_\nu^+ U_s| - |\partial_\nu^- U_s| \quad (\text{respectively } \geq), \tag{5.4.3}$$

where U_s is as given in equation (5.4.2).

We will use the above definitions 5.4.2, 5.4.3, and 5.4.4 for classical and viscosity solutions of (5.1.1), and in [9], it was shown that this is equivalent to the definitions given in [27]

5.4.2. Vertical shifts in the intersurface

It will be necessatry to understand how $\partial U_\nu^\pm U_S$ varies with the vertical shifts of S, and thus we shall introduce some notation for such shifts. Given a test interface S, we define the intersurface S_h, which will result from shifting S in the upward direction by h as follows

$$S_h := \{X = (x, x_{n+1}) | (x, x_{n+1} - h) \in S\},$$

provided $h > 0$; if $h < 0$, then S_h results from shifting S down by $|h|$. The resulting variation in $\partial_\nu^+ U_S$ is recorded below.

Lemma 5.4.5. *Let S be a test interface. Then there is a constant $C = C(S)$ such that for all sufficiently small h and any $(X, t) \in S_h$, we have*

$$|\partial_\nu^+ U_{S_h}(X - he_{n+1}, t) - \partial_\nu^+ U_S(X, t)| \leq C|h|,$$

where both U_S and U_{S_h} are functions given by (5.4.2).

The proof of the above lemma involves looking at a function

$$\tilde{U}(X, t) := U_{S_h}(X - he_{n+1}, t), \quad \text{defined in } S^+ \cap \{(X, t) | X = (x, x_{n+1}), x_{n+1} \geq h\},$$

and then applying the maximum principle comparing $\tilde{U}$ and U_S in $S^+ \setminus \{0 \leq x_{n+1} \leq h\}$. One can use the above estimate for the one-phase problem to then obtain the bound for the two-phase problem relevant to our equation, i.e.

$$\Big| |\nabla U_{S_h}^+| - |\nabla U_{S_h}^-| - |\nabla U_S^+| + |\nabla U_S^-| \Big| \leq C_S |h|.$$

This can be done using a similar approach to the one in [9, Lemma 6.1].

5.4.3. Correspondence between viscosity solutions of the free boundary evolution and viscosity solutions of the parabolic equation

In this subsection, we state the results that ultimately help establish the equivalence between the viscosity solutions of the free boundary evolution in (5.1.1) and viscosity solutions of the parabolic equation given by (5.1.2). These results will show that a viscosity solution to the parabolic equation will yield a viscosity solution to the free boundary problem (in the sense of definition 5.4.4).

Remark 5.4.6. *We also note that we can look at these equations in the context of the level-set equations. In this case, we look at the set $\partial\{U > 0\}$ as the zero level set of the function U, or we can also consider it the domain and the boundary as a function of time, i.r. $\partial\{U(\cdot,t) > 0\} = \partial\Omega(t)$. Assume that we parametrize t by some defining function $\Phi(X,t)$, then we have $\partial\Omega(t) = \{\Phi(\cdot,t) = 0\}$. Now, if the normal velocity is given by V and the outer normal vector is ν, then the level-set equation becomes*

$$\nabla\Phi \cdot (\nu V) + \partial_t\Phi.$$

Further, as Φ is the defining function of a level set, we have

$$\nu = \pm\frac{\nabla\Phi}{|\nabla\Phi|}$$

depending on whether $\Phi > 0$ or $\Phi < 0$. this reduces the flow to $\partial_t\Phi = \pm V|\nabla\Phi|$. To be consistent with the parametrization, we choose $\Phi > 0$, and also $\Phi(X,t) = f(x,t) - x_{n+1}$.

With this and using the normal velocity condition given in (5.1.1)*, we finally obtain* (5.1.2)*, i.e.*

$$\partial_t f = \left(|\partial_\nu^+ U| - |\partial_\nu^- U|\right) \sqrt{1 + |\nabla f|^2} \quad on \quad \mathbb{S}^n \times [0, T]$$

Lemma 5.4.7. *[9, Lemma 9.9] If f is a globally Lipschitz viscosity subsolution (respectively supersolution) of* (5.1.2) *in $\mathbb{S}^n \times [a, b]$, then U_f is a viscosity subsolution (respectively supersolution) of the free boundary evolution*(5.1.1) *in $\Sigma_0 \times [a, b]$ (in the sense of definition 5.4.4).*

For the proof of this lemma (see [9, Lemma 9.9] for full proof), we start with the Γ_f being the graph of f and also the set $\partial\{U_f > 0\}$, and a test interface S touching U_f from above at a point, and we try to find an intermediate test interface between S and Γ_f which is also the graph of some function that lies between U_f and U_S. This is possible because of the Lipschitz nature of f and constructing a function using sup-convolutions. The intermediate function, which touches the graph of f at the same point as the test interface touches the level set of U, will satisfy the equation (5.1.2) by the definition of viscosity solutions, and we get many more inequalities using the comparison principle, which leads us to U_f being the viscosity subsolution (respectively supersolution) of the free boundary evolution (5.1.1).

Lemma 5.4.8. *[9, Lemma 9.10] If f is a viscosity subsolution (respectively supersolution) of* (5.1.2) *in $\mathbb{S}^n \times [a, b]$, then U_f is a viscosity subsolution (respectively supersolution) of the free boundary evolution*(5.1.1) *in $\Sigma_0 \times [a, b]$.*

Note that this lemma does not have the extra assumption of f being globally Lipschitz as the previous lemma 5.4.7. To prove this lemma, we look at the sup-convolution of f, which is in fact a Lipschitz function, and apply lemma 5.4.7. This will tell us that the sup-convolution of U_f is a viscosity subsolution of (5.1.1), and we can now make some limiting arguments to show that U_f is a viscosity subsolution to (5.1.1).

Now the next result will tell us how the test functions for the lower dimensional, non-local parabolic equations will yield to test functions of the free boundary problem.

Proposition 5.4.9. *[9, Proposition 9.11]Let ϕ be an admissible test function touching f from above at some $t_0 \in [a, b]$ and $x_0 \in \mathbb{S}^n$. Then U_ϕ touches U_f from above at (X_0, t_0) where $X_0 = (x_0, f(x_0, t_0))$*

The proof of the above proposition is a straightforward consequence of the comparison principle and can be found in [9, Lemma 9.11].

Lemma 5.4.10. *[9, Lemma 9.12] Let U be a viscosity subsolution (respectively supersolution) of (5.1.1) in $\Sigma_0 \times [a, b]$ whose free boundary is given as the graph of some upper semicontinuous (lower semicontinuous) function f, then f is a viscosity subsolution (respectively supersolution) of (5.1.2) in $\mathbb{S}^n \times [a, b]$.*

Using the previous proposition and the level set equations for (5.1.1) (see remark 5.4.6), we can prove this lemma.

Remark 5.4.11. *All the given results in this section 5.4.3 will give the proof for part (a) of the main result, i.e. theorem 5.1.1.*

5.4.4. Propogation of the modulus of continuity

In this final section, we will give an explanation for the remaining parts of the main result, theorem 5.1.1, which pertain to the modulus of continuity. In particular, we can show that the modulus of continuity of the initial data will be preserved by the fractional parabolic equation. This result follows without difficulty once the comparison theorem for viscosity solutions has been established, which we have done in section 5.3.

Lemma 5.4.12. *Let J be as in assumption 5.2.1, $T > 0$ and $f : \mathbb{S}^n \times [0,T] \to \mathbb{R}$, a continuous viscosity solution of*

$$\partial_t f = J(f) \qquad \text{in } \mathbb{S}^n \times [0,T]$$

$$f(x,0) = f_0(x) \qquad \text{in } \mathbb{S}^n$$

If f_0 is continuous with the modulus of continuity $\omega(\cdot)$, then the same will be true of $f(\cdot,t)$ for all $t \in [0,T]$. In particular, we have the estimate

$$|f(x,t) - f(z,t)| \leq \omega(|x-z|).$$

Proof. Let $z \in \mathbb{S}^n$ be fixed, and consider the function

$$\begin{aligned} w(x,t) &:= (R_z f)(x,t) - f(x,t) \\ &= f(R_z x, t) - f(x,t). \end{aligned}$$

Then by assumption, $w(x,0) \leq \omega(||R_z||), \quad \forall x \in \mathbb{S}^n$

Now, f being a viscosity solution is a subsolution as well as a supersolution, and since J is rotationally invariant, we can also say that $R_z f$ is a viscosity solution. Thus by an argument similar to the one in the proof of proposition 5.3.9, we get

$$\partial_t w \leq M_J^+(w).$$

Now w satisfies all the assumptions of lemma 5.3.8, thus we have

$$\sup_{\mathbb{S}^n \times [0,T]} w_+ \leq \sup_{\mathbb{S}^n} w_+(\cdot, 0)$$

$$\text{i.e. } \ w(x,t) \leq \omega(||R_z||), \quad \forall (x,t) \in \mathbb{S}^n \times [0,T]$$

$$\implies f(R_z x, t) - f(x,t) \leq \omega(||R_z||).$$

Since $z \in \mathbb{S}^n$ was arbitrary, we have proved the lemma. □

Note that the above lemma is the proof of part (b) of the main result, i.e. theorem 5.1.1.

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