COMPUTATIONS IN TOPOLOGICAL COHOCHSCHILD HOMOLOGY

By

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ABSTRACT

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In recent work, Hess and Shipley defined an invariant of coalgebra spectra called topological coHochschild homology (coTHH). In 2018, Bohmann-Gerhardt-Høgenhaven-Shipley-Ziegenhagen developed a coBökstedt spectral sequence to compute the homology of coTHH for coalgebras over the sphere spectrum. However, examples of coalgebras over the sphere spectrum are limited, and one would like to have computational tools to study coalgebras over other ring spectra. In this thesis, we construct a relative coBökstedt spectral sequence to study the topological coHochschild homology of more general coalgebra spectra. We consider $H\mathbb{F}_p \wedge_{H\mathbb{Z}} H\mathbb{F}_p$ and $H\mathbb{F}_p \wedge_{BP<n>} H\mathbb{F}_p$ for certain values of n as $H\mathbb{F}_p$-coalgebras and compute the E_2-term of the spectral sequence in these cases. Further, we show that this spectral sequence has additional algebraic structure, and exploit this structure to complete coTHH calculations. Finally, we show that coHochschild homology is a bicategorical shadow, in the sense of Ponto.

To my grandparents, who instilled in me a love of learning and who have intentionally invested in my education at every step along the way.

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Chapter 1

Introduction

Hochschild homology, which we will denote as HH, is a classical algebraic invariant of rings that can be extended topologically to give an invariant of ring spectra called topological Hochschild homology (THH). In the 1970's, Doi [14] studied a construction dual to Hochschild homology for coalgebras, called coHochschild homology (coHH). Recent work of Hess and Shipley [20] defines the topological analog of Doi's work, topological coHochschild homology (coTHH), to study *co*algebra spectra.

Work of Malkiewich [25] and Hess-Shipley [20] shows coTHH of suspension spectra is related to THH for simply connected spaces X via

$$\mathrm{coTHH}(\Sigma^\infty_+ X) \simeq \mathrm{THH}(\Sigma^\infty_+(\Omega X)) \simeq \Sigma^\infty_+ \mathscr{L}X,$$

where the last equivalence comes from work of Bökstedt and Waldhausen. Thus coTHH is relevant for studying the homology of free loop spaces, $\mathscr{L}X$, the main topic of the field of string topology [12, 13]. Further, because THH is directly related to algebraic K-theory via trace methods, coTHH also has applications for algebraic K-theory of spaces.

In this paper we will develop computational tools for studying topological coHochschild homology. The primary tools used to compute topological (co)Hochschild homology are spectral sequences. In the late 1980's, Bökstedt identified the E^2-term of the spectral sequence induced from the skeletal filtration of the simplicial object $\mathrm{THH}(A)_\bullet$ to be the familiar algebraic theory of Hochschild homology:

$$E^2_{*,*} = \mathrm{HH}_*(H_*(A;k)) \implies H_*(\mathrm{THH}(A);k).$$

This spectral sequence is referred to as the *Bökstedt spectral sequence* for a ring spectrum A.

In 2018, Bohmann-Gerhardt-Høgenhaven-Shipley-Ziegenhagen showed that in the dual situation there is a *coBökstedt spectral sequence*. This is the Bousfield-Kan spectral sequence for the cosimplicial spectrum $\mathrm{coTHH}(C)^\bullet$, for C a coalgebra spectrum over the sphere spectrum [4]. As we would hope, this spectral sequence has classical coHochschild homology as its E_2-term, and in cases when it does indeed converge we have:

$$E_2^{*,*} = \mathrm{coHH}_*(H_*(C;k)) \implies H_*(\mathrm{coTHH}_*(C);k).$$

In their work however, these tools are only set up to study coalgebra spectra over the sphere spectrum $\mathbb{S}$, i.e. for C with comultiplication map $C \to C \wedge_{\mathbb{S}} C$. Examples of this sort are closely related to suspension spectra of spaces and are fairly limited as shown by recent work of Péroux-Shipley [30]. In this paper, we broaden these tools to apply to coTHH for coalgebras over any commutative ring spectrum.

In order to motivate the need for a relative coBökstedt spectral sequence, we examine a variety of examples of coalgebras over spectra other than $\mathbb{S}$. For example, the following proposition gives a way of generating coalgebra spectra over a commutative ring spectrum B.

Proposition 1.0.1

A map of commutative ring spectra $A \to B$ induces a B-coalgebra structure on the spectrum $B \wedge_A B$.

We call the spectral sequence that allows us to study the topological coHochschild homology of coalgebras over an arbitrary commutative ring spectrum R the *relative coBökstedt spectral sequence*:

Theorem 1.0.2

Let E and R be commutative ring spectra, C an R-coalgebra spectrum that is cofibrant as an R-module, and N a (C, C)-bicomodule spectrum. If $E_*(C)$ is flat over $E_*(R)$, then there exists a Bousfield-Kan spectral sequence for the cosimplicial spectrum $\mathrm{coTHH}^R(N, C)^\bullet$ that abuts to $E_{t-s}(\mathrm{coTHH}^R(N, C))$ with E_2-page

$$E_2^{s,t} = \mathrm{coHH}_{s,t}^{E_*(R)}(E_*(N), E_*(C))$$

given by the classical coHochschild homology of $E_*(C)$ with coefficients in $E_*(N)$.

Note in particular that this holds for any generalized homology theory E in addition to being over the more general ring spectrum R. Further, we identify conditions for convergence of this spectral sequence. In particular, for the case when $E = \mathbb{S}$, if for every s there exists some r so that $E_r^{s,s+i} = E_\infty^{s,s+i}$ then the relative coBökstedt spectral sequence converges completely to $\pi_*(\mathrm{coTHH}^R(N, C))$.

A first question is whether the E_2-term of this spectral sequence is computable. By the above proposition, $H\mathbb{F}_p \wedge_{H\mathbb{Z}} H\mathbb{F}_p$ and $H\mathbb{F}_p \wedge_{BP<n>} H\mathbb{F}_p$ for $n = 0, 1$ and for $n = 2$ at the primes $p = 2, 3$ are $H\mathbb{F}_p$-coalgebras. In these cases for example, the E_2-term is indeed computable:

Proposition 1.0.3

For the $H\mathbb{F}_p$-coalgebra $H\mathbb{F}_p \wedge_{H\mathbb{Z}} H\mathbb{F}_p$, the E_2-page of the spectral sequence calculating $\pi_{t-s}(\mathrm{coTHH}^{H\mathbb{F}_p}(H\mathbb{F}_p \wedge_{H\mathbb{Z}} H\mathbb{F}_p))$ is

$$E_2^{s,t} = \mathrm{coHH}^{\mathbb{F}_p}_{s,t}(\pi_*(H\mathbb{F}_p \wedge_{H\mathbb{Z}} H\mathbb{F}_p)) \cong \Lambda_{\mathbb{F}_p}(\tau) \otimes \mathbb{F}_p[\omega]$$

for $||\tau|| = (0,1), ||\omega|| = (1,1)$.

Proposition 1.0.4

For the $H\mathbb{F}_p$-coalgebra $H\mathbb{F}_p \wedge_{BP<n>} H\mathbb{F}_p$ for $n = 0,1$ and for $n = 2$ at the primes $p = 2,3$, the E_2-page of the spectral sequence calculating $\pi_{t-s}(\mathrm{coTHH}^{H\mathbb{F}_p}(H\mathbb{F}_p \wedge_{BP<n>} H\mathbb{F}_p))$ is

$$E_2^{s,t} = \mathrm{coHH}^{\mathbb{F}_p}_{s,t}(\pi_*(H\mathbb{F}_p \wedge_{BP<n>} H\mathbb{F}_p)) \cong \Lambda_{\mathbb{F}_p}(\tau_0, \dots \tau_n) \otimes \mathbb{F}_p[\omega_0, \dots \omega_n]$$

for $||\tau_i|| = (0, 2p^i - 1), ||\omega_i|| = (1, 2p^i - 1)$.

Because computations with this relative coBökstedt spectral sequence are quite complicated, any additional structure on the spectral sequence can help in these calculations. By work of Angeltveit-Rognes, the classical Bökstedt spectral sequence for a commutative ring spectrum has the structure of a spectral sequence of Hopf algebras under some flatness conditions [1]. Bohmann-Gerhardt-Shipley show that under appropriate coflatness conditions, the coBökstedt spectral sequence for a cocommutative coalgebra spectrum has what is called a $\square$-*Hopf algebra structure*, an analog of a Hopf algebra structure for working over a coalgebra [5]. The following proposition follows from Bohmann-Gerhardt-Shipley's work:

Theorem 1.0.5

For C a cocommutative coalgebra spectrum, if for $r \geq 2$ each $E_r^{*,*}(C)$ is coflat over $\pi_*(C)$, then the relative coBökstedt spectral sequence is a spectral sequence of $\square_{\pi_*(C)}$-Hopf algebras.

This additional $\square$-Hopf algebra structure is very computationally useful. For instance, we can extend the work of Bohmann-Gerhardt-Shipley [5] to show the following:

Theorem 1.0.6

For a field k, let C be a cocommutative Hk-coalgebra spectrum such that $\mathrm{coHH}_*(\pi_*(C))$ is coflat over $\pi_*(C)$ and the graded coalgebra $\pi_*(C)$ is connected. Then the E_2-term of the relative coBökstedt spectral sequence calculating $\pi_*(\mathrm{coTHH}^{Hk}(C))$,

$$E_2^{*,*}(C) = \mathrm{coHH}_*^k(\pi_*(C)),$$

is a $\square_{\pi_*(C)}$-bialgebra, and the shortest non-zero differential $d_r^{s,t}$ in lowest total degree $s+t$ maps from a $\square_{\pi_*(C)}$-algebra indecomposable to a $\square_{\pi_*(C)}$-coalgebra primitive.

This algebraic structure proves very useful for explicit computations with the coBökstedt spectral sequence. We use the relative coBökstedt spectral sequence to show:

Theorem 1.0.7

For a field k, let C be a cocommutative Hk-coalgebra spectrum that is cofibrant as an Hk-module with $\pi_*(C) \cong \Lambda_k(y)$ for $|y|$ odd and greater than 1. Then the relative coBökstedt spectral sequence collapses and

$$\pi_*(\mathrm{coTHH}^{Hk}(C)) \cong \Lambda_k(y) \otimes k[w]$$

as graded k-modules for $|w| = |y| - 1$.

Theorem 1.0.8

Let k be a field and let $p = char(k)$, including 0. For C a cocommutative Hk-coalgebra spectrum that is cofibrant as an Hk-module with $\pi_*(C) \cong \Lambda_k(y_1, y_2)$ for $|y_1|, |y_2|$ both odd and greater than 1, if p^m is not equal to $\frac{|y_2|-1}{|y_1|-1}$ or $\frac{|y_2|-1}{|y_1|-1} + 1$ for all $m \geq 0$, then the relative coBökstedt spectral sequence collapses and

$$\pi_*(\mathrm{coTHH}^{Hk}(C)) \cong \Lambda_k(y_1, y_2) \otimes k[w_1, w_2],$$

as graded k-modules for $|w_i| = |y_i| - 1$.

Further, in a result analogous to the work of Bohmann-Gerhardt-Høgenhaven-Shipley-Ziegenhagen [4] we have

Theorem 1.0.9

Let k be a field and let C be a cocommutative coassociative Hk-coalgebra spectrum that is cofibrant as an Hk-module spectrum, and whose homotopy coalgebra is

$$\pi_*(C) = \Gamma_k[x_1, x_2, \ldots],$$

where the x_i are in non-negative even degrees and there are only finitely many of them in each degree. Then the relative coBökstedt spectral sequence calculating the homotopy groups of the topological coHochschild homology of C collapses at E_2, and

$$\pi_*(\mathrm{coTHH}^{Hk}(C)) \cong \Gamma_k[x_1, x_2, \ldots] \otimes \Lambda_k(z_1, z_2, \ldots)$$

as k-modules, with z_i in degree $|x_i| - 1$.

Finally, we will show that coHochschild homology (coHH) is a bicategorical shadow. Ponto [31] and Ponto-Shulman [32] developed the original framework for shadows and traces in bicategories. More recently, work of Campbell-Ponto [11] used this fundamental framework to show that THH is an example of a shadow. In particular, the shadow structure formally provides many of the desirable properties of THH, including Morita invariance. Therefore in the same vein, we will show that coHochschild homology (coHH) is also a shadow for the appropriately defined bicategory.

This thesis is organized as follows. Chapter 2 introduces coalgebras in spectra and topological coHochschild homology. In Chapter 3 we construct the relative coBökstedt spectral sequence. In Chapter 4 we study the algebraic structures of this spectral sequence, and Chapter 5 discusses some explicit topological coHochschild homology calculations. Finally in Chapter 6 we delve into the theoretical framework of coHochschild homology and show that it is a shadow.

Chapter 2

Background

2.1 Hochschild Homology

To begin, we recall the definition of Hochschild homology for associative k-algebras.

Definition 2.1.1

Let k be a commutative ring, A an associative algebra over k, and M an (A,A)-bimodule. Recall that this algebraic structure gives us a multiplication map $\mu : A \otimes_k A \to A$ and a unit map $\eta : k \to A$. Let $\mathrm{HH}(A,M)_\bullet$ denote the simplicial k-module with r-simplices $\mathrm{HH}(A,M)_r := M \otimes A^{\otimes r}$. The face maps are given by

$$d_i(m \otimes a_1 \otimes \ldots \otimes a_r) = \begin{cases} ma_1 \otimes \ldots \otimes a_r & i = 0 \\ m \otimes a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_r & 1 \le i < r \\ a_r m \otimes a_1 \otimes \ldots \otimes a_{r-1} & i = r, \end{cases}$$

and the degeneracy maps insert the unit map. In particular,

$$s_i(m \otimes a_1 \otimes \cdots \otimes a_r) = \begin{cases} m \otimes 1 \otimes a_1 \otimes \cdots \otimes a_r & i = 0 \\ m \otimes a_1 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_r & 1 \le i \le r \end{cases}$$

This simplicial object has the form:

$$\begin{array}{c} \vdots \\ M \otimes_k A \otimes_k A \\ \downarrow\uparrow \downarrow\uparrow\downarrow \\ M \otimes_k A \\ \downarrow \uparrow\downarrow \\ M \end{array}$$

Let $C_*(A, M)$ denote the associated chain complex with boundary map $d = \Sigma_i (-1)^i d_i$. Then the q^{th} **Hochschild homology** of A with coefficients in M is:

$$\mathrm{HH}_q(A, M) := H_q(C_*(A, M)).$$

The Dold-Kan correspondence gives us the equivalent definition

$$\mathrm{HH}_q(A, M) := \pi_q(|\mathrm{HH}(A, M)_\bullet|)$$

for the geometric realization of the simplicial k-module $\mathrm{HH}(A, M)_\bullet$. This latter formulation makes it clearer how we would extend this definition to create an analogous topological theory.

Remark 2.1.2

Hochschild homology is defined here with coefficients in an (A, A)-bimodule M. When $M = A$, considered as a bimodule over itself, this is denoted by $\mathrm{HH}_q(A)$.

Remark 2.1.3

If $A = M$, then together with the extra structure of a cyclic operator

$$t_n(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = a_n \otimes a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1},$$

these face and degeneracy maps determine a cyclic object, $\mathrm{HH}(A)_\bullet$. This complex is called the **cyclic bar construction**.

One may also define the Hochschild homology of a graded algebra:

Definition 2.1.4

For a differential graded algebra (A, δ) with trivial derivation δ, let $C_*(A, \delta)$ be the cyclic chain complex given by

$$[n] \mapsto (A, \delta)^{\otimes(n+1)}$$

with face maps

$$d_i(a_0 \otimes \ldots \otimes a_n) = \begin{cases} a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n & 0 \leq i < n \\ (-1)^{|a_n|(|a_0|+|a_1|+\ldots+|a_{n-1}|)} a_n a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1} & i = n, \end{cases}$$

degeneracy maps that insert the unit

$$s_i(a_0 \otimes \ldots \otimes a_n) = a_0 \otimes \ldots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \ldots \otimes a_n,$$

and cyclic operator

$$t_n(a_0 \otimes \ldots \otimes a_n) = (-1)^{|a_n|(|a_0|+|a_1|+\ldots+|a_{n-1}|)+n}(a_n \otimes a_0 \otimes \ldots \otimes a_{n-1}).$$

Then the Hochschild complex associated to $C_*(A, \delta)$ is a complex of complexes with boundary map d given by an alternating sum of the face maps

$$d = \sum_{i=0}^{n} (-1)^i d_i : C_*(A, \delta)^{\otimes(n+1)} \longrightarrow C_*(A, \delta)^{\otimes n}.$$

This bicomplex below has trivial horizontal maps:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\downarrow & & \downarrow & & \downarrow & & \\
(A\otimes A\otimes A)_0 & \xleftarrow[\delta]{} & (A\otimes A\otimes A)_1 & \xleftarrow[\delta]{} & (A\otimes A\otimes A)_2 & \longleftarrow & \cdots \\
\downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \\
(A\otimes A)_0 & \xleftarrow[\delta]{} & (A\otimes A)_1 & \xleftarrow[\delta]{} & (A\otimes A)_2 & \longleftarrow & \cdots \\
\downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \\
A_0 & \xleftarrow[\delta]{} & A_1 & \xleftarrow[\delta]{} & A_2 & \longleftarrow & \cdots
\end{array}$$

and the weight parts of the tensor product are given by:

$$(A^{\otimes(n+1)})_k = \bigoplus_{i_0+\ldots+i_n=k} A_{i_0}\otimes\ldots\otimes A_{i_n}.$$

Then taking the homology of the total complex of this bicomplex with boundary $d+0$ (since δ is trivial) gives the **Hochschild homology** of this differential graded algebra, denoted by $\mathrm{HH}_*(A,\delta)$.

2.2 Topological Hochschild Homology

Motivated by applications to algebraic K-theory, in the late 1980's Bökstedt constructed a topological version of Hochschild homology [6, 7]. Topological Hochschild homology is an invariant of ring spectra and can be defined as follows:

Definition 2.2.1

For a commutative ring spectrum R, an R-algebra A, and an (A,A)-bimodule spectrum M, we have a multiplication map $\mu : A\wedge_R A \to A$ and a unit map $\eta : R \to A$ along with left and right actions of A on M

$$\psi : A \wedge_R M \to M \qquad \gamma : M \wedge_R A \to M.$$

Let $\mathrm{THH}^R(A,M)_\bullet$ be the simplicial R-module spectrum with r-simplices $\mathrm{THH}^R(A,M)_r :=$ $M \wedge_R A^{\wedge_R r}$ and face maps

$$d_i = \begin{cases} \gamma \wedge \mathrm{Id}^{\wedge(r-1)} & i = 0 \\ \mathrm{Id}^{\wedge i} \wedge \mu \wedge \mathrm{Id}^{\wedge(r-i-1)} & 1 \le i < r \\ (\psi \wedge \mathrm{Id}^{\wedge(r-1)}) \circ t & i = r \end{cases}$$

where t is the cyclic permutation bringing the last factor around to the front. The degeneracy maps will again insert the unit map appropriately. Then the **topological Hochschild homology** relative to R of A with coefficients in M is the geometric realization

$$\mathrm{THH}^R(A,M) := |\mathrm{THH}^R(A,M)_\bullet|$$

Remark 2.2.2

As before, when $M = A$ we then eliminate it from the notation and write

$$\mathrm{THH}^R(A) := |\mathrm{THH}^R(A,A)_\bullet|$$

In summary, we have now introduced the algebraic definition of Hochschild homology and its topological analog, topological Hochschild homology. In the following sections we will introduce analogous theories for coalgebras, coHochschild homology (coHH) and topological coHochschild homology (coTHH).

$$\begin{array}{ccc} & \underline{\textit{Algebra}} & \underline{\textit{Topology}} \\ \underline{\textit{Algebras:}} & \mathrm{HH}(A) & \longrightarrow & \mathrm{THH}(A) \\ & \downarrow & & \downarrow \\ \underline{\textit{Coalgebras:}} & \mathrm{coHH}(C) & \longrightarrow & \mathrm{coTHH}(C) \end{array}$$

2.3 Coalgebras

First we will recall the definitions of coalgebras and comodules in order to introduce the classical coHochschild homology of Doi [14] and coHH of a graded coalgebra over a graded ring, since we will need such structure for the spectral sequence. Then we will introduce coalgebras in spectra and look at examples.

Definition 2.3.1

Let R be a commutative ring. Then a (coassociative, counital) **coalgebra** C over R is an R-module with R linear maps that are the comultiplication $\Delta : C \to C \otimes C$ that is coassociative and counit $\epsilon : C \to R$ that is counital, i.e. the following coassociativity and counitality diagrams commute:

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle \mathrm{Id} \otimes \Delta} \\ C \otimes C & \xrightarrow{\Delta \otimes \mathrm{Id}} & C \otimes C \otimes C \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ {\scriptstyle\Delta}\downarrow & \searrow^{\mathrm{Id}} & \downarrow{\scriptstyle \mathrm{Id} \otimes \epsilon} \\ C \otimes C & \xrightarrow{\epsilon \otimes \mathrm{Id}} & R \otimes C \cong C \cong C \otimes R \end{array}$$

Example 2.3.2

For a field k, the polynomial coalgebra $k[w_1, w_2, \ldots]$ for w_i in even degree is the vector space with basis given by $\{w_i^j\}$ for $j \geq 0$ and $i \geq 1$. It has coproduct

$$\Delta(w_i^j) = \sum_k \binom{j}{k} w_i^k \otimes w_i^{j-k}$$

and counit

$$\epsilon(w_i^j) = \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{if } j \neq 0. \end{cases}$$

Example 2.3.3

For a field k, the exterior coalgebra $\Lambda_k(y_1, y_2, \ldots)$ for y_i in odd degrees is the vector space with basis given by $\{1, y_i\}$ for $i \geq 1$, which has coproduct

$$\begin{aligned} \Delta(y_i) &= 1 \otimes y_i + y_i \otimes 1 \\ \Delta(1) &= 1 \otimes 1 \end{aligned}$$

and counit

$$\begin{aligned} \epsilon(y_i) &= 0 \\ \epsilon(1) &= 1 \end{aligned}$$

Example 2.3.4

For a field k, the divided power coalgebra $\Gamma_k[x_1, x_2, \ldots]$ with x_i in even degrees is the vector space with basis given by $\{\gamma_j(x_i)\}$ for $j \geq 0$ and $i \geq 1$. It has coproduct

$$\Delta(\gamma_j(x_i)) = \sum_{a+b=j} \gamma_a(x_i) \otimes \gamma_b(x_i)$$

where $\gamma_0(x_i) = 1, \gamma_1(x_i) = x_i$, and counit

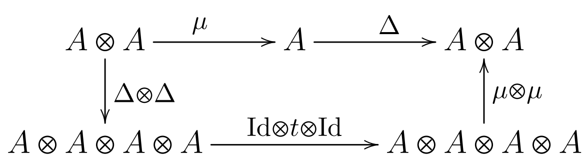
$$\epsilon(\gamma_j(x_i)) = \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{if } j \neq 0. \end{cases}$$

Understanding Hopf algebras will also be essential for this work, so we recall the definition here. First we introduce bialgebras.

Definition 2.3.5

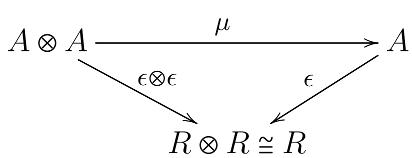
A **bialgebra** A over a commutative ring R is a unital associative R-algebra with multiplication $\mu : A \otimes_R A \to A$ and unit $\eta : R \to A$ along with comultiplication $\Delta : A \to A \otimes_R A$ and counit $\epsilon : A \to R$ such that A is also a counital coassociative R-coalgebra satisfying the following commutative diagrams, where all $\otimes$ below are over R:

1.

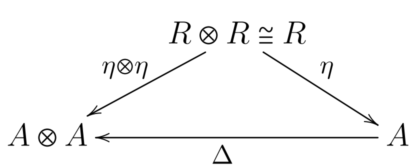


where t swaps the two components of $A \otimes A$.

2.



3.



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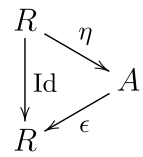
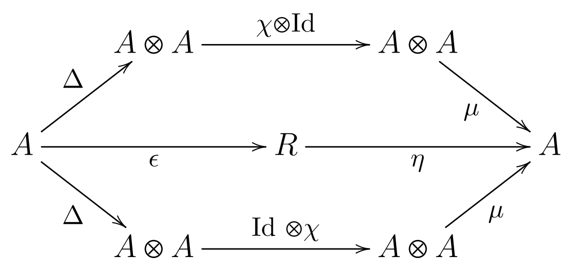


Definition 2.3.6

A **Hopf algebra** A is a bialgebra over a commutative ring R together with a map of R-modules $\chi : A \to A$ called the *antipode* such that the following diagram commutes:

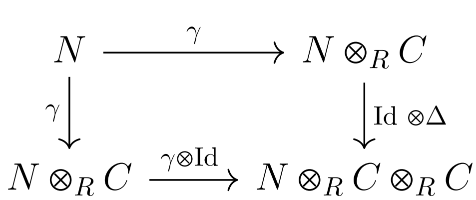


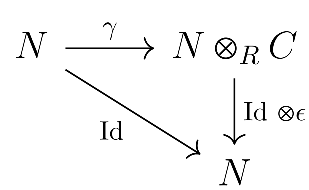
where μ is the multiplication, Δ is the comultiplication, η is the unit, and ϵ is the counit.

It will also be useful to have the "dual" notion of modules on which C *coacts*:

Definition 2.3.7

Let R be a commutative ring and C an R-coalgebra. Then N is a **right C-comodule** if it is an R-module together with an R-linear map $\gamma : N \to N \otimes_R C$ that is coassociative and counital, i.e. that makes the following diagrams commute:





γ is referred to as a *right C-coaction*.

Similarly, a **left C-comodule** is an R-module together with an R-linear map $\psi : N \to C \otimes_R N$

that is coassociative and counital, i.e. that makes the following diagrams commute:

$$\begin{array}{ccc} N & \xrightarrow{\psi} & C \otimes_R N \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle \Delta\otimes \mathrm{Id}} \\ C \otimes_R N & \xrightarrow{\mathrm{Id}\,\otimes\psi} & C \otimes_R C \otimes_R N \end{array} \qquad \begin{array}{ccc} N & \xrightarrow{\psi} & C \otimes_R N \\ & {\scriptstyle \mathrm{Id}}\searrow & \downarrow{\scriptstyle \epsilon\otimes \mathrm{Id}} \\ & & N \end{array}$$

ψ is referred to as a *left C-coaction.*

Similar to the way in which right and left module structures together may create a bimodule structure, the analogous definition holds for comodules:

Definition 2.3.8

For R-coalgebras C, D, a (C, D)-**bicomodule** N is a left C-comodule with left coaction $\psi : N \to C \otimes_R N$ and right D-comodule with right coaction $\gamma : N \to N \otimes_R D$ that satisfies the following commutative diagram:

$$\begin{array}{ccc} N & \xrightarrow{\gamma} & N \otimes_R D \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle \psi\otimes \mathrm{Id}} \\ C \otimes_R N & \xrightarrow{\mathrm{Id}\,\otimes\gamma} & C \otimes_R N \otimes_R D \end{array}$$

2.4 CoHochschild Homology

In [14], Doi defines coHochschild homology as an invariant of coalgebras:

Definition 2.4.1

Let k be a commutative ring, C a (coassociative, counital) k-coalgebra, and N a (C, C)-bicomodule. Then C comes equipped with a coassociative comultiplication $\Delta : C \to C \otimes C$

and counit $\epsilon : C \to k$. We build the cochain complex $C^*(N, C)$:

$$\cdots \longleftarrow N \otimes C \otimes C \longleftarrow N \otimes C \longleftarrow N \longleftarrow 0$$

with coboundary map $\delta = \Sigma_i (-1)^i \delta_i$ for δ_i given by

$$\delta_i = \begin{cases} \gamma \otimes \mathrm{Id}^{\otimes r} & i = 0 \\ \mathrm{Id}^{\otimes i} \otimes \Delta \otimes \mathrm{Id}^{\otimes (r-i)} & 1 \le i \le r \\ \tilde{t} \circ (\psi \otimes \mathrm{Id}^{\otimes r}) & i = r+1 \end{cases}$$

where γ denotes the right coaction, ψ denotes the left coaction, and $\tilde{t}$ is the map that twists the first factor to the last. Then the qth **coHochschild homology** of C with coefficients in N is

$$\mathrm{coHH}_q(N, C) := H^q(C^*(N, C)).$$

Remark 2.4.2

We denote coHochschild homology with coefficients in C by $\mathrm{coHH}_q(C)$ when C is viewed as a (C, C)-bicomodule over itself.

Remark 2.4.3

Work of Hess-Parent-Scott [19] shows that the coHochschild homology of a differential graded coalgebra over a graded ring follows as above with the addition of signs that follow from the Koszul rule. In the spectral sequence of the next chapter we will use the following definition of coHH of a graded coalgebra over a graded ring based on their definition with trivial differential:

Definition 2.4.4

For a differential graded coalgebra (D, ∂) with trivial coderivation ∂, let $C^*(D, \partial)$ be the cyclic cochain complex given by

$$[n] \mapsto (D, \partial)^{\otimes(n+1)}$$

with coface maps

$$\delta_i = \begin{cases} \mathrm{Id}^{\otimes i} \otimes \Delta \otimes \mathrm{Id}^{\otimes(r-i)} & 0 \le i \le r \\ (-1)^{|d_0|(|d_1|+|d_2|+\ldots+|d_n|)} \tilde{t} \circ (\Delta \otimes \mathrm{Id}^{\otimes r}) & i = r+1, \end{cases}$$

where $\tilde{t}$ is the map that twists the first factor to the last, codegeneracy maps that insert the counit

$$\sigma_i = \mathrm{Id}^{\otimes(i+1)} \otimes \epsilon \otimes \mathrm{Id}^{r-i},$$

and cocyclic operator

$$\bar{t}_n(d_0 \otimes \ldots \otimes d_n) = (-1)^{|d_0|(|d_1|+|d_2|+\ldots+|d_n|)+n}(d_1 \otimes \ldots \otimes d_n \otimes d_0).$$

Then the coHochschild complex associated to $C^*(D, \partial)$ has coboundary map given by an alternating sum of the coface maps

$$\delta = \sum_{i=0}^{n} (-1)^i \delta_i : C^*(D, \partial)^{\otimes n} \longrightarrow C^*(D, \partial)^{\otimes(n+1)},$$

and the bicomplex below has trivial horizontal maps:

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \\
(D\otimes D\otimes D)_0 & \xrightarrow{\partial} & (D\otimes D\otimes D)_1 & \xrightarrow{\partial} & (D\otimes D\otimes D)_2 & \longrightarrow & \cdots \\
\delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \\
(D\otimes D)_0 & \xrightarrow{\partial} & (D\otimes D)_1 & \xrightarrow{\partial} & (D\otimes D)_2 & \longrightarrow & \cdots \\
\delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \\
D_0 & \xrightarrow{\partial} & D_1 & \xrightarrow{\partial} & D_2 & \longrightarrow & \cdots
\end{array}
$$

where the weight parts of the tensor product are given by:

$$(D^{\otimes(n+1)})_k = \bigoplus_{i_0+\ldots+i_n=k} D_{i_0} \otimes \ldots \otimes D_{i_n}.$$

Taking the cohomology of the total complex of this bicomplex with coboundary $\delta+0$ gives the **coHochschild homology** of this differential graded coalgebra, denoted by $\mathrm{coHH}_*(D,\partial)$.

2.5 Coalgebras in Spectra

Now we will introduce coalgebras in spectra. First we state the definition of a coalgebra for the general setting of a symmetric monoidal category.

Definition 2.5.1

Let $(\mathscr{D},\otimes,1)$ be a symmetric monoidal category. Then a (coassociative, counital) **coalgebra** $C\in\mathscr{D}$ has a comultiplication $\Delta : C \to C\otimes C$ that is coassociative and a counit morphism $\epsilon : C \to 1$ that is counital, i.e. the following coassociativity and counitality diagrams commute:

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}\,\otimes\Delta} \\ C \otimes C & \xrightarrow{\Delta\otimes\mathrm{Id}} & C \otimes C \otimes C \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ {\scriptstyle \Delta}\downarrow & \overset{\mathrm{Id}}{\searrow} & \downarrow{\scriptstyle \mathrm{Id}\,\otimes\epsilon} \\ C \otimes C & \xrightarrow{\epsilon\otimes\mathrm{Id}} & 1 \otimes C \cong C \cong C \otimes 1 \end{array}$$

Definition 2.5.2

A **coalgebra spectrum** is a coalgebra in one of the symmetric monoidal categories of spectra.

Definition 2.5.3

For a commutative ring spectrum R, an R**-coalgebra spectrum** C is a coalgebra in the symmetric monoidal category of R-modules. It has comultiplication $\Delta : C \to C \wedge_R C$ and counit $\epsilon : C \to R$, satisfying the coassociativity and counitality conditions.

Example 2.5.4

For a space X, the diagonal map $X \to X \wedge X$ on topological spaces induces a comultiplication map on the suspension spectrum $\Sigma^\infty(X)$:

$$\Delta : \Sigma^\infty(X) \to \Sigma^\infty(X \wedge X) \simeq \Sigma^\infty(X) \wedge \Sigma^\infty(X),$$

making $\Sigma^\infty(X)$ into a coalgebra spectrum.

However, it should be noted that most spectra do not have diagonal maps and thus examples of this form are quite limited, as shown in work of Péroux-Shipley [30]. In particular, they prove that all coalgebra spectra over $\mathbb{S}$ are cocommutative in strict monoidal categories of spectra. As we saw in the above example, some coalgebras over the sphere spectrum come from suspension spectra, but Péroux-Shipley further show that in model categories all

$\mathbb{S}$-coalgebras are closely related to suspension spectra. This rigid structure of $\mathbb{S}$-coalgebras therefore provides motivation for studying other kinds of coalgebra spectra.

Because examples of coalgebras in spectra over $\mathbb{S}$ are limited, a primary goal of this work is to develop tools to study coalgebras over other commutative ring spectra. One source of such coalgebra spectra is the following general construction:

Proposition 2.5.5

A map of commutative ring spectra $\phi : A \to B$ induces a B-coalgebra structure on the spectrum $B \wedge_A B$.

Proof.

To have a coalgebra structure, we first want a comultiplication map

$$\begin{array}{rl} \Delta : B \wedge_A B \to & (B \wedge_A B) \wedge_B (B \wedge_A B) \\ & \downarrow \cong \\ & B \wedge_A B \wedge_A B \end{array}$$

But we have the equivalence $i_A \wedge \mathrm{Id} : B \wedge_A B \to B \wedge_A A \wedge_A B$. The map i_A inserts an extra copy of A:

$$i_A : \qquad \begin{array}{ccc} B & \longrightarrow & B \wedge_A A \\ \cong\big\downarrow & & \big\uparrow \\ B \wedge \mathbb{S} & \xrightarrow{\mathrm{Id} \wedge \eta_A} & B \wedge A \end{array}$$

for unit map η_A. Applying ϕ yields

$$B \wedge_A B \cong B \wedge_A A \wedge_A B \xrightarrow{\mathrm{Id} \wedge \phi \wedge \mathrm{Id}} B \wedge_A B \wedge_A B$$

which induces our desired comultiplication map.

We further need a counit map $\epsilon : B \wedge_A B \to B$. By definition, $B \wedge_A B$ is the coequalizer of

$$B \wedge A \wedge B \underset{\gamma\wedge\mathrm{Id}}{\overset{\mathrm{Id}\wedge\psi}{\rightrightarrows}} B \wedge B \to B \wedge_A B$$

for module actions

$$\begin{array}{ccc} A \wedge B & \xrightarrow{\psi} & B \\ {\scriptstyle \phi\wedge\mathrm{Id}}\downarrow & \nearrow{\scriptstyle m} & \\ B \wedge B & & \end{array}$$

$$\begin{array}{ccc} B \wedge A & \xrightarrow{\gamma} & B \\ {\scriptstyle \mathrm{Id}\wedge\phi}\downarrow & \nearrow{\scriptstyle m} & \\ B \wedge B & & \end{array}$$

where the ring map $m : B \wedge B \to B$ is the multiplication in the commutative ring spectrum B. Consider mapping to B in this diagram:

$$\begin{array}{ccccc} B \wedge A \wedge B & \underset{\gamma\wedge\mathrm{Id}}{\overset{\mathrm{Id}\wedge\psi}{\rightrightarrows}} & B \wedge B & \longrightarrow & B \wedge_A B \\ & & \downarrow{\scriptstyle m} & & \\ & & B & & \end{array}$$

Diagram 2.1

If we consider the two composites in Diagram 2.1,

$$\begin{array}{ccccc} B \wedge A \wedge B & \xrightarrow{\mathrm{Id}\wedge\psi} & B \wedge B & \xrightarrow{m} & B \\ {\scriptstyle \mathrm{Id}\wedge\phi\wedge\mathrm{Id}}\downarrow & \nearrow{\scriptstyle \mathrm{Id}\wedge m} & & & \\ B \wedge B \wedge B & & & & \end{array}$$

$$\begin{array}{ccccc} B \wedge A \wedge B & \xrightarrow{\gamma\wedge\mathrm{Id}} & B \wedge B & \xrightarrow{m} & B \\ {\scriptstyle \mathrm{Id}\wedge\phi\wedge\mathrm{Id}}\downarrow & \nearrow{\scriptstyle m\wedge\mathrm{Id}} & & & \\ B \wedge B \wedge B & & & & \end{array}$$

we see that they agree since B is in particular an associative ring spectrum, so we have a map from $B \wedge A \wedge B \to B$, making the diagram 2.1 commute. By the universal property of coequalizers, there exists a unique map $\epsilon : B \wedge_A B \to B$ in this diagram

$$\begin{array}{ccccc} B \wedge A \wedge B & \underset{\gamma\wedge\mathrm{Id}}{\overset{\mathrm{Id}\wedge\psi}{\rightrightarrows}} & B \wedge B & \longrightarrow & B \wedge_A B \\ & \searrow & \downarrow{\scriptstyle m} & \overset{\epsilon}{\swarrow} & \\ & & B & & \end{array}$$

and this map ϵ gives us the desired counit map. Now we must check that this is indeed a coalgebra by confirming that it satisfies the necessary coassociativity and counitality diagrams.

First, we need to check that we satisfy coassociativity of the comultiplication, that is $(\mathrm{Id} \wedge \Delta)\Delta = (\Delta \wedge \mathrm{Id})\Delta$. This property can be shown by proving the following diagram commutes:

$$\begin{array}{ccc} B \wedge_A B & \xrightarrow{\qquad\Delta\qquad} & (B \wedge_A B) \wedge_B (B \wedge_A B) \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}\,\wedge\Delta} \\ (B \wedge_A B) \wedge_B (B \wedge_A B) & \xrightarrow{\Delta\wedge\mathrm{Id}} & (B \wedge_A B) \wedge_B (B \wedge_A B) \wedge_B (B \wedge_A B) \end{array}$$

This result follows from a diagram chase through the definition of the comultiplication. Recall that i_A inserts $- \wedge_A A$ and similarly i_B will insert $- \wedge_B B$, so that the map Δ in this setting is the composition:

$$\Delta : B \wedge_A B \xrightarrow{i_A\wedge\mathrm{Id}} B \wedge_A A \wedge_A B \xrightarrow{\mathrm{Id}\wedge\phi\wedge\mathrm{Id}} B \wedge_A B \wedge_A B \xrightarrow{\mathrm{Id}\wedge i_B\wedge\mathrm{Id}} (B \wedge_A B) \wedge_B (B \wedge_A B).$$

So now expanding the above diagram with this decomposition of the comultiplication yields Diagram 1 given in the Appendix. Commutativity of each of the squares in that diagram follows because each of the vertical (and horizontal) maps are equivalent at each level with additional copies of the identity inserted as appropriate. Thus the coassociativity axiom is satisfied.

Counitality follows from showing $(\epsilon \wedge \mathrm{Id})\Delta = \mathrm{Id} = (\mathrm{Id} \wedge \epsilon)\Delta$. Because B is commutative, the left and right module actions of B on $B \wedge_A B$ coincide:

$$\begin{array}{ccccc}(B\wedge_A B)\wedge_B(B\wedge_A B) & \xrightarrow{\mathrm{Id}\wedge\epsilon} & (B\wedge_A B)\wedge_B B\cong B\wedge_A B\cong B\wedge_B(B\wedge_A B) & \xleftarrow{\epsilon\wedge\mathrm{Id}} & (B\wedge_A B)\wedge_B(B\wedge_A B)\\ & \nwarrow^{\Delta} & \| & \nearrow^{\Delta} & \\ & & B\wedge_A B & & \end{array}$$

so it suffices to show why one of these triangles commutes. The right-hand composition can be broken down as

$$B\wedge_A B\xrightarrow{i_A\wedge\mathrm{Id}} B\wedge_A A\wedge_A B\xrightarrow{\mathrm{Id}\wedge\phi\wedge\mathrm{Id}} B\wedge_A B\wedge_A B\xrightarrow{\mathrm{Id}\wedge i_B\wedge\mathrm{Id}} B\wedge_A B\wedge_B B\wedge_A B\xrightarrow{\epsilon\wedge\mathrm{Id}} B\wedge_B(B\wedge_A B)$$
$$\downarrow\cong$$
$$B\wedge_A B.$$

Observe that the last factor of B is unchanged by this map, and so this composition map comes from

$$B\cong B\wedge\mathbb{S}\xrightarrow{\mathrm{Id}\wedge\eta_A} B\wedge A\xrightarrow{\mathrm{Id}\wedge\phi} B\wedge B\xrightarrow{m} B,$$
$$\left(B\wedge\mathbb{S}\xrightarrow{\mathrm{Id}\wedge\eta_B} B\wedge B\right)$$

where the first part of this composite must be $\mathrm{Id}\wedge\eta_B$ by the definition of map of ring spectra. However, unitality of B implies that $m(\mathrm{Id}\wedge\eta_B)=\mathrm{Id}_B$, and therefore the right triangle in our diagram is equivalent to the identity on $B\wedge_A B$. A similar justification can be used to show the left triangle commutes as well, so we have shown that $B\wedge_A B$ coming from the map of commutative ring spectra $\phi: A\to B$ is a coassociative, counital B-coalgebra spectrum. $\square$

Examples of these kinds of coalgebra spectra include $H\mathbb{F}_p$-coalgebras such as $H\mathbb{F}_p\wedge H\mathbb{F}_p$, $H\mathbb{F}_p\wedge_{H\mathbb{Z}} H\mathbb{F}_p$, and $H\mathbb{F}_p\wedge_{BP<n>} H\mathbb{F}_p$ for $n=0,1$ and for $n=2$ at the primes $p=2,3$, some of which we will examine later on in further detail.

2.6 Topological CoHochschild Homology

We will define (topological) coHochschild homology for any general symmetric monoidal category as in Bohmann-Gerhardt-Høgenhaven-Shipley-Ziegenhagen [4]. Thereafter we will be primarily using the definition as it applies to spectra, although classical coHochschild homology as defined by Doi [14] can be recovered from the following more general definition by considering the category of coalgebras over a field.

Definition 2.6.1

Let $(\mathscr{D}, \otimes, 1)$ be a symmetric monoidal model category and let $C \in \mathscr{D}$ be a coalgebra with coassociative comultiplication $\Delta : C \to C \otimes C$ and counit $\epsilon : C \to 1$. Further, let N be a (C, C)-bicomodule with left and right coactions $\psi : N \to C \otimes N$ and $\gamma : N \to N \otimes C$. Then let $\mathrm{coTHH}(N, C)^{\bullet}$ be the cosimplicial object with r-simplices $\mathrm{coTHH}(N, C)^{r} := N \otimes C^{\otimes r}$, with coface maps

$$\delta_i = \begin{cases} \gamma \otimes \mathrm{Id}^{\otimes r} & i = 0 \\ \mathrm{Id}^{\otimes i} \otimes \Delta \otimes \mathrm{Id}^{\otimes (r-i)} & 0 < i \leq r \\ \tilde{t} \circ (\psi \otimes \mathrm{Id}^{\otimes r}) & i = r + 1 \end{cases}$$

where $\tilde{t}$ is the map that twists the first factor to the last, and with codegeneracy maps $\sigma_i : N \otimes C^{\otimes (r+1)} \to N \otimes C^{\otimes r}$ for $0 \leq i \leq r$

$$\sigma_i = \mathrm{Id}^{\otimes (i+1)} \otimes \epsilon \otimes \mathrm{Id}^{\otimes r-i}.$$

This gives a cosimplicial object of the form

$$\begin{array}{c} \vdots \\ N \otimes C \otimes C \\ \uparrow\downarrow\ \uparrow\downarrow\uparrow \\ N \otimes C \\ \uparrow\ \downarrow\uparrow \\ N \end{array}$$

Then the **topological coHochschild homology** of the coalgebra C with coefficients in N is given by

$$\mathrm{coTHH}(N, C) := \mathrm{Tot}(\mathscr{R}\mathrm{coTHH}(N, C)^{\bullet})$$

where $\mathscr{R}$ is the Reedy fibrant replacement and Tot represents the totalization.

Remark 2.6.2

We defined topological coHochschild homology with coefficients in a (C, C)-bicomodule N, but when we consider C as a bicomodule over itself, we write $\mathrm{coTHH}(C)$ for coefficients in C. As with topological Hochschild homology, we will further decorate the notation with $\mathrm{coTHH}^R(C)$ when we consider the topological coHochschild homology of C relative to R, i.e. coTHH of an R-coalgebra C for R a commutative ring spectrum.

Remark 2.6.3

Recall that for rings and ring spectra A, $\mathrm{HH}(A)_{\bullet}$ and $\mathrm{THH}(A)_{\bullet}$ are examples of *cyclic bar constructions*. Similarly for a coalgebra C, the coHochschild complex for C together with a cyclic operator

$$\tilde{t}_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = a_1 \otimes \cdots \otimes a_{n-1} \otimes a_n \otimes a_0.$$

is called the **cyclic cobar construction**, and both $\mathrm{coHH}(C)^{\bullet}$ and $\mathrm{coTHH}(C)^{\bullet}$ are examples.

As mentioned above, working in the category of coalgebras over a field recovers the classical coHH of Doi [14]. Our general convention will be to specifically refer to this construction as *topological* coHochschild homology when we are considering as input some coalgebra *spectrum*. For instance, we will refer to the work of Hess-Parent-Scott [19] as studying *coHochschild homology* of differential graded coalgebras (dg-coalgebras) over a field.

2.7 The Classical Bökstedt Spectral Sequence

Before constructing the coBökstedt spectral sequence for coTHH, we first recall the classical Bökstedt spectral sequence for THH, due to Bökstedt in [6]. Recall that for a ring spectrum A (i.e. an $\mathbb{S}$-algebra with multiplication $\mu : A \wedge A \to A$ and unit $\eta : A \to \mathbb{S}$), we can build the simplicial spectrum $\mathrm{THH}(A)_\bullet$ via

$$\begin{array}{c} \vdots \\ A \wedge A \wedge A \\ \downarrow\uparrow \downarrow\uparrow\downarrow \\ A \wedge A \\ \downarrow \uparrow\downarrow \\ A \end{array}$$

where the face maps $d_i : A^{\wedge(r+1)} \to A^{\wedge r}$ are given by

$$d_i = \begin{cases} \mathrm{Id}^{\wedge i} \wedge \mu \wedge \mathrm{Id}^{\wedge(r-i-1)} & 0 \leq i < r \\ (\mu \wedge \mathrm{Id}^{\wedge i}) \circ t & i = r \end{cases}$$

for t that cyclically permutes the last element to the front, and where the degeneracy maps

$s_i : A^{\wedge(r+1)} \to A^{\wedge(r+2)}$ insert the unit map for $0 \le i \le r$:

$$s_i = \mathrm{Id}^{\wedge(i+1)} \wedge \eta \wedge \mathrm{Id}^{\wedge(r-i)}$$

Then for this simplicial spectrum $\mathrm{THH}(A)_\bullet$, one can consider the spectral sequence that comes from its skeletal filtration and converges to $H_*(\mathrm{THH}(A); k)$, where k is a field. For this spectral sequence, $E^1_{*,q}$ is isomorphic to the normalized chain complex of $H_q(\mathrm{THH}(A)_\bullet)$. So we consider the homology applied to each simplicial level of $\mathrm{THH}(A)_\bullet$:

$$\begin{array}{c} \vdots \\ H_*(A \wedge A \wedge A; k) \\ \downarrow\uparrow\downarrow\uparrow\downarrow \\ H_*(A \wedge A; k) \\ \downarrow\uparrow\downarrow \\ H_*(A; k). \end{array}$$

Note then that

$$\begin{aligned} H_*(A \wedge A \wedge \cdots \wedge A; k) &\cong \pi_*(A \wedge A \wedge \cdots \wedge A \wedge Hk) \\ &\cong \pi_*((A \wedge Hk) \wedge_{Hk} (A \wedge Hk) \wedge_{Hk} \cdots \wedge_{Hk} (A \wedge Hk)) \\ &\cong \pi_*(A \wedge Hk) \otimes_{\pi_* Hk} \pi_*(A \wedge Hk) \otimes_{\pi_* Hk} \cdots \otimes_{\pi_* Hk} \pi_*(A \wedge Hk) \end{aligned}$$

where the last isomorphism follows from the Künneth spectral sequence because $\pi_*(A \wedge Hk)$ is flat over $\pi_* Hk \cong k$. Thus we can rewrite each level to get:

$$H_*(A \wedge A \wedge \cdots \wedge A; k) \cong H_*(A; k) \otimes_k H_*(A; k) \otimes_k \cdots \otimes_k H_*(A; k)$$

Further, the d_1-differential of the spectral sequence under this identification agrees with the differential of the complex computing $\mathrm{HH}_*(H_*(A;k))$. Therefore the E^2-term of this spectral sequence is $\mathrm{HH}_*(H_*(A;k))$, the classical Hochschild homology of $H_*(A;k)$.

Since this structure follows from the general spectral sequence that arises from the skeletal filtration of the simplicial spectrum $\mathrm{THH}(A)_\bullet$, this can be extended to any homology theory, which is formally stated in the context of $\mathbb{S}$-modules in the following theorems of Elmendorf-Kriz-Mandell-May [15]:

Theorem ([15] Thm IX.2.9)

Let E be a commutative ring spectrum, A a ring spectrum, and M a cell (A,A)-bimodule. If $E_*(A)$ is E_*-flat, then there is a spectral sequence of the form

$$E^2_{p,q} = \mathrm{HH}^{E_*}_{p,q}(E_*(A), E_*(M)) \implies E_{p+q}(\mathrm{THH}^{\mathbb{S}}(A,M))$$

Here E_* is the homology theory associated to the commutative ring spectrum E, i.e. $E_*(A) = \pi_*(E \wedge A)$. Thus the above result comes from the spectral sequence derived from the simplicial filtration of $\mathrm{THH}^R(A,M)$ (for the case $R = \mathbb{S}$) as given in:

Theorem ([15] Thm X.2.9)

Let E and R be ring spectra and K_* be a proper simplicial R-module spectrum. Then there is a natural homological spectral sequence $\{E^r_{p,q}K_*\}$ such that

$$E^2_{p,q}K_* = H_p(E_q(K_*))$$

and $\{E^r_{p,q}K_*\}$ converges strongly to $E_*(|K_*|)$.

Note that this Theorem X.2.9 from [15] gives a more general statement of the Bökstedt spectral sequence for topological Hochschild homology of an R-algebra, which we state here for clarity.

Theorem 2.7.1

Suppose E and R are commutative ring spectra, A is an R-algebra, and M is a cell (A, A)-bimodule. Then if $E_*(A)$ is flat over $E_*(R)$, then there exists a spectral sequence

$$E^2_{p,q} = \mathrm{HH}^{E_*(R)}_{p,q}(E_*(A), E_*(M)) \implies E_{p+q}(\mathrm{THH}^R(A, M))$$

We quickly verify that this spectral sequence has the indicated E^2-term. For this spectral sequence, $E^1_{*,q}$ is isomorphic to the normalized chain complex of $E_q(\mathrm{THH}^R(A, M)_\bullet)$. So we consider the E-homology applied to each simplicial level of $\mathrm{THH}^R(A, M)_\bullet$:

$$\begin{aligned}
E_*(M \wedge_R A \wedge_R \cdots \wedge_R A) &\cong \pi_*(E \wedge M \wedge_R A \wedge_R \cdots \wedge_R A) \\
&\cong \pi_*(E \wedge M \wedge_{E\wedge R} E \wedge R \wedge_R A \wedge_{E\wedge R} E \wedge R \wedge_R \cdots \wedge_{E\wedge R} E \wedge R \wedge_R A) \\
&\cong \pi_*(E \wedge M \wedge_{E\wedge R} E \wedge A \wedge_{E\wedge R} E \wedge A \cdots \wedge_{E\wedge R} E \wedge A) \\
&\cong \pi_*(E \wedge M) \otimes_{\pi_*(E\wedge R)} \pi_*(E \wedge A) \otimes_{\pi_*(E\wedge R)} \cdots \otimes_{\pi_*(E\wedge R)} \pi_*(E \wedge A) \\
&\qquad\qquad \text{(since } \pi_*(E \wedge A) \text{ is flat over } \pi_*(E \wedge R) \text{ by hypothesis)} \\
&\cong E_*(M) \otimes_{E_*(R)} E_*(A) \otimes_{E_*(R)} \cdots \otimes_{E_*(R)} E_*(A)
\end{aligned}$$

The d_1-differential of the spectral sequence under this identification agrees with the differential of the complex computing $\mathrm{HH}^{E_*(R)}(E_*(A), E_*(M))$, so we identify the E^2-term:

$$E^2_{p,q} = \mathrm{HH}^{E_*(R)}_{p,q}(E_*(A), E_*(M)) \implies E_{p+q}(\mathrm{THH}^R(A, M))$$

Now that we're equipped with the relative statement of the theorem for the theories of HH and THH, we want to investigate what is currently known about the dual situation for coTHH, and then see how we can extend that work in order to have a tool for general R-coalgebra spectra.

Chapter 3

Construction of a relative coBökstedt spectral sequence

Associated to a cosimplicial spectrum is a Bousfield-Kan spectral sequence [8]. Applied to the cosimplicial spectrum $\mathrm{coTHH}(C)^\bullet$ this yields a spectral sequence whose E_2-term was identified in Bohmann-Gerhardt-Høgenhaven-Shipley-Ziegenhagen [4] as the classical coHochschild homology of coalgebras in the sense of Doi [14]. Specifically, they show:

Theorem ([4] Thm 4.1)

Let k be a field and C a coalgebra spectrum that is cofibrant as a spectrum. Then the Bousfield-Kan spectral sequence for the cosimplicial spectrum $\mathrm{coTHH}(C)^\bullet$ gives a coBökstedt spectral sequence for calculating $H_{t-s}(\mathrm{coTHH}(C);k)$ with E_2-page

$$E_2^{s,t} = \mathrm{coHH}^k_{s,t}(H_*(C;k))$$

given by the classical coHochschild homology of $H_*(C;k)$ as a graded k-module.

Bear in mind that the Bousfield-Kan spectral sequence **does not always converge**, although the authors do specify conditions under which the coBökstedt spectral sequence will converge completely.[1] Note that this spectral sequence applies to the ordinary homology of $\mathrm{coTHH}(C)$ where C is an $\mathbb{S}$-coalgebra. Now we want to create a relative version of this theorem for R-coalgebra spectra that would give the Bousfield-Kan spectral sequence computing the homology of $\mathrm{coTHH}^R(C)$. As we saw in the THH setting, we would expect that some flatness conditions must be satisfied.

For commutative ring spectra E and R, an R-coalgebra spectrum C, and a (C, C)-bicomodule N, we will see that if $E_*(C)$ is flat over $E_*(R)$, then the Bousfield-Kan spectral sequence for the cosimplicial spectrum $\mathrm{coTHH}^R(N, C)^\bullet$ can be used in calculating $E_{t-s}(\mathrm{coTHH}^R(N, C))$ with E_2-page

$$E_2^{s,t} = \mathrm{coHH}_{s,t}^{E_*(R)}(E_*(N), E_*(C)).$$

We will refer to this spectral sequence as the *relative coBökstedt spectral sequence*. Note in particular that this holds for any generalized homology theory in addition to being over the more general ring spectrum R.

We will first formally state and prove that this relative coBökstedt spectral sequence exists, and then identify a corollary that will be useful for later computations. Further, we will describe conditions for convergence of this spectral sequence.

Theorem 3.0.1

Let E and R be commutative ring spectra, C an R-coalgebra spectrum that is cofibrant as an R-module, and N a (C, C)-bicomodule spectrum. If $E_*(C)$ is flat over $E_*(R)$, then there exists a Bousfield-Kan spectral sequence for the cosimplicial R-module $\mathrm{coTHH}^R(N, C)^\bullet$ that

[1] For instance, the coBökstedt spectral sequence converges when C is a suspension spectrum $\Sigma_+^\infty X$ for simply connected X [4].

abuts to $E_{t-s}(\mathrm{coTHH}^R(N,C))$ with E_2-page

$$E_2^{s,t} = \mathrm{coHH}_{s,t}^{E_*(R)}(E_*(N), E_*(C))$$

given by the classical coHochschild homology of $E_*(C)$ with coefficients in $E_*(N)$.

Proof. To begin, we will recall the general construction of the Bousfield-Kan spectral sequence [8] for a general Reedy fibrant cosimplicial R-module $X^\bullet$.

Let Δ be the cosimplicial space with the standard n-simplex Δ^n as its n^{th} level. The category of R-modules is cotensored over pointed spaces (see e.g. [2]), and the notation D^K will be used for the cotensor of an R-module D with a simplicial space K. So for our Reedy fibrant cosimplicial R-module $X^\bullet$ the totalization of $X^\bullet$ is given by:

$$\mathrm{Tot}(X^\bullet) = eq\Big(\prod_{n\geq 0} (X^n)^{\Delta^n} \rightrightarrows \prod_{\alpha\in\Delta([a],[b])} (X^b)^{\Delta^a} \Big).$$

Let $sk_s\Delta \subset \Delta$ be the cosimplicial subspace with n^{th} level $sk_s\Delta^n$ that is the s-skeleton of the n-simplex. Then one can define

$$\mathrm{Tot}_s(X^\bullet) \coloneqq eq\Big(\prod_{n\geq 0} (X^n)^{sk_s\Delta^n} \rightrightarrows \prod_{\alpha\in\Delta([a],[b])} (X^b)^{sk_s\Delta^a} \Big).$$

The inclusions $sk_s\Delta \hookrightarrow sk_{s+1}\Delta$ induce a tower of fibrations

$$\begin{array}{ccccccc}
\cdots \to & Tot_s(X^\bullet) \xrightarrow{p_s} & Tot_{s-1}(X^\bullet) \xrightarrow{p_{s-1}} & Tot_{s-2}(X^\bullet) \to & \cdots \xrightarrow{p_1} & Tot_0(X^\bullet) \cong X^0. \\
& \uparrow i_s & \uparrow i_{s-1} & \uparrow i_{s-2} & & \uparrow i_0 \\
& F_s & F_{s-1} & F_{s-2} & & F_0
\end{array}$$

We then have an associated exact couple

$$\begin{array}{ccc} \pi_*(\mathrm{Tot}_*(X^\bullet)) & \xrightarrow{p_*} & \pi_*(\mathrm{Tot}_*(X^\bullet)) \\ & {\scriptstyle i_*}\nwarrow \quad \swarrow{\scriptstyle \partial} & \\ & \pi_*(F_*) & \end{array}$$

that yields a half plane cohomological spectral sequence $\{E_r, d_r\}$ with differentials

$$d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}.$$

We now want to identify the fiber F_s. Recall that the normalized cochain complex $N^s X^\bullet$ is defined to be:

$$N^s X^\bullet = \bigcap_{i=0}^{s-1} \ker(\sigma_i : X^s \to X^{s-1})$$

for codegeneracy maps σ_i as given by the cosimplicial structure.

Then each fiber F_s can be identified with

$$F_s := \Omega^s(N^s X^\bullet).$$

The E_1-term of the spectral sequence can thus be identified:

$$\begin{aligned} E_1^{s,t} &= \pi_{t-s}(F_s) \\ &\cong \pi_{t-s}(\Omega^s(N^s X^\bullet)) \\ &\cong \pi_t(N^s X^\bullet) \\ &\cong N^s \pi_t(X^\bullet) \end{aligned}$$

with differential $d_1 : N^s \pi_t(X^\bullet) \to N^{s+1}\pi_t(X^\bullet)$. This map can then be identified with

$\Sigma(-1)^i \pi_t(\delta^i)$, and we have

$$H^*(N^s \pi_t(X^\bullet)) \cong H^s(\pi_t(X^\bullet))$$
$$\Longrightarrow E_2^{s,t} \cong H^s(\pi_t(X^\bullet), \Sigma(-1)^i \pi_t(\delta^i))$$

Here we care about the specific case when $X^\bullet = \mathscr{R}(E \wedge \mathrm{coTHH}^R(N,C)^\bullet)$, where $\mathscr{R}$ indicates the Reedy fibrant replacement, and so we get

$$\pi_*(X^\bullet) = \pi_*(\mathscr{R}(E \wedge \mathrm{coTHH}^R(N,C)^\bullet) \cong \pi_*(E \wedge \mathrm{coTHH}^R(N,C)^\bullet).$$

Recall that $\mathrm{coTHH}^R(N,C)$ has cosimplicial structure:

$$\begin{array}{c} \vdots \\ N \wedge_R C \wedge_R C \\ \uparrow\downarrow\ \uparrow\downarrow\uparrow \\ N \wedge_R C \\ \uparrow\ \downarrow\uparrow \\ N \end{array}$$

so when we take $\pi_*(E \wedge -)$, at the n^{th} level we see that:

$$\begin{aligned} \pi_*(E \wedge N \wedge_R C \wedge_R \cdots \wedge_R C) &\cong \pi_*(E \wedge N \wedge_{E\wedge R} E \wedge R \wedge_R C \wedge_{E\wedge R} E \wedge R \wedge_R \cdots \wedge_{E\wedge R} E \wedge R \wedge_R C) \\ &\cong \pi_*(E \wedge N \wedge_{E\wedge R} E \wedge C \wedge_{E\wedge R} E \wedge \cdots \wedge_{E\wedge R} E \wedge C) \\ &\cong \pi_*(E \wedge N) \otimes_{\pi_*(E\wedge R)} \pi_*(E \wedge C) \otimes_{\pi_*(E\wedge R)} \cdots \otimes_{\pi_*(E\wedge R)} \pi_*(E \wedge C) \\ &\qquad \text{(since } \pi_*(E \wedge C) \text{ is flat over } \pi_*(E \wedge R) \text{ by hypothesis)} \\ &\cong E_*(N) \otimes_{E_*(R)} E_*(C) \otimes_{E_*(R)} \cdots \otimes_{E_*(R)} E_*(C) \end{aligned}$$

and so we get

$$\begin{aligned}\pi_*\mathscr{R}(E \wedge \mathrm{coTHH}^R(N,C)^n) &\cong \pi_*(E \wedge \mathrm{coTHH}^R(N,C)^n)\\ &\cong E_*(N) \otimes_{E_*(R)} E_*(C)^{\otimes_{E_*(R)} n}.\end{aligned}$$

Then $\Sigma(-1)^i \pi_*(\delta^i)$ gives the coHochschild differential under this identification, and thus we get the coHochschild complex:

$$\begin{aligned}E_2^{s,t} &\cong H^s(\pi_t(X^\bullet), \Sigma(-1)^i \pi_t(\delta^i))\\ &\cong H^s(E_t(N) \otimes_{E_*(R)} E_t(C)^{\otimes_{E_*(R)} n}, \Sigma(-1)^i \pi_t(\delta^i))\\ &\cong \mathrm{coHH}_{s,t}^{E_*(R)}(E_*(N), E_*(C))\end{aligned}$$

Therefore the result is the Bousfield-Kan spectral sequence

$$E_2^{s,t} = \mathrm{coHH}_{s,t}^{E_*(R)}(E_*(N), E_*(C)) \overset{??}{\Longrightarrow} E_{t-s}(\mathrm{coTHH}^R(N,C))$$

where we use ?? as a reminder that this sequence does not converge in general. □

Because it will be particularly useful in future examples, we state the following special case when $E = \mathbb{S}$ as a corollary:

Corollary 3.0.2

Let R be a commutative ring spectrum and C an R-coalgebra spectrum. If $\pi_*(C)$ is flat over $\pi_*(R)$, then there exists a Bousfield-Kan spectral sequence that abuts to $\pi_{t-s}(\mathrm{coTHH}^R(C))$ with E_2-page

$$E_2^{s,t} = \mathrm{coHH}_{s,t}^{\pi_*(R)}(\pi_*(C))$$

given by the classical coHochschild homology of $\pi_*(C)$.

Now we want to see the conditions we require for convergence. Based on work of Bousfield-

Kan [8] and Bohmann-Gerhardt-Høgenhaven-Shipley-Ziegenhagen [4], we have the following convergence result.

Proposition 3.0.3

If for every s there exists some r so that $E_r^{s,s+i} = E_\infty^{s,s+i}$, then the relative coBökstedt spectral sequence for $\mathrm{coTHH}^R(C)$ converges completely to

$$\pi_* \mathrm{Tot}\mathscr{R}(E \wedge \mathrm{coTHH}^R(C)^\bullet)$$

Conditions for complete convergence can be found in Goerss-Jardine [18]. Further, from the natural construction of a map $\mathrm{Hom}(X,Y) \wedge Z \to \mathrm{Hom}(X, Y \wedge Z)$ we get a natural map

$$P : E \wedge \mathrm{Tot}(\mathscr{R}\mathrm{coTHH}^R(C)^\bullet) \to \mathrm{Tot}\mathscr{R}(E \wedge \mathrm{coTHH}^R(C)^\bullet)$$

Applying homotopy to the above gives us the following corollary.

Corollary 3.0.4

If for every s there exists some r so that $E_r^{s,s+i} = E_\infty^{s,s+i}$ and $P : E \wedge \mathrm{Tot}(\mathscr{R}\mathrm{coTHH}^R(C)^\bullet) \to \mathrm{Tot}\mathscr{R}(E \wedge \mathrm{coTHH}^R(C)^\bullet)$ induces an isomorphism in homotopy, then the relative coBökstedt spectral sequence for $\mathrm{coTHH}^R(C)$ converges completely to $E_*(\mathrm{coTHH}^R(C))$.

For the examples in which we will compute topological coHochschild homology in this thesis, we are taking $E = \mathbb{S}$ and so the condition on the map P is satisfied. We formally state this specific case here for easy reference:

Corollary 3.0.5

When considering $E = \mathbb{S}$, if for every s there exists some r so that $E_r^{s,s+i} = E_\infty^{s,s+i}$ then the relative coBökstedt spectral sequence converges completely to $\pi_*(\mathrm{coTHH}^R(C))$.

Chapter 4

Algebraic structures in the (relative) (co)Bökstedt spectral sequence

Understanding additional algebraic structure in a spectral sequence can help facilitate calculations. In this section we study the structure of the relative coBökstedt spectral sequence. By work of Angeltveit-Rognes, the classical Bökstedt spectral sequence for a commutative ring spectrum has the structure of a spectral sequence of Hopf algebras under some flatness conditions [1]. Bohmann-Gerhardt-Shipley show that under appropriate coflatness conditions the coBökstedt spectral sequence for a cocommutative coalgebra spectrum has what is called a $\square$-*Hopf algebra structure*, an analog of a Hopf algebra structure working over a coalgebra [5]. It follows from Bohmann-Gerhardt-Shipley's work that the relative coBökstedt spectral sequence also has this type of $\square$-Hopf algebra structure, and this additional algebraic structure is computationally useful. For instance, with this structure the shortest nonzero differential maps from an algebra indecomposable to a coalgebra primitive.

In this section, we will begin by examining the structure of the Bökstedt spectral sequence and the implications of that algebraic structure as in Angeltveit-Rognes [1]. We will then introduce the necessary definitions and theorems for the $\square$-Hopf algebra structure and state the result of Bohmann-Gerhardt-Shipley [5] that the coBökstedt spectral sequence has this structure. Finally we will see how the relative coBökstedt spectral sequence by extension also has this $\square$-Hopf algebra structure.

4.1 Hopf algebra structure in the Bökstedt spectral sequence

In order to show that the Bökstedt spectral sequence is a spectral sequence of Hopf algebras, Angeltveit-Rognes [1] show as in Elmendorf-Kriz-Mandell-May [15] that for a commutative ring spectrum A, $\mathrm{THH}(A)$ is a Hopf algebra over A itself in the homotopy category. One can identify $\mathrm{THH}(A)$ with $A \otimes S^1$ as commutative $\mathbb{S}$-algebras [26], which induces maps on THH that give the Hopf structure in the homotopy category. In particular, $\mathrm{THH}(A)$ is a commutative Hopf algebra, but it is not cocommutative in general because the coproduct is induced from the pinch map, which is not homotopy cocommutative. Angeltveit-Rognes also study an analogous Hopf structure on the entire Bökstedt spectral sequence, under flatness conditions [1].

Definition 4.1.1

The standard **simplicial circle** $S^1_\bullet$ is given by $\Delta^1/\partial\Delta^1$ for the one simplex Δ^1. Then Δ^1_r has elements $\{x_0, \ldots, x_{r+1}\}$ for $x_j : [r] \to [1]$ that sends j terms to 0. Identifying $x_0 \sim x_{r+1}$ creates the quotient S^1_r with face maps

$$d_i(x_r) = \begin{cases} x_r & r \leq i \\ x_{r-1} & r > i \end{cases}$$

and degeneracy maps

$$s_i(x_r) = \begin{cases} x_r & r \le i \\ x_{r+1} & r > i. \end{cases}$$

In the case of $\mathrm{THH}(A)_\bullet \cong A \otimes S^1_\bullet$, the Hopf algebra structure over A on $\mathrm{THH}(A)$ is induced by simplicial maps on $S^1_\bullet$:

- The inclusion of the basepoint $\eta : * \to S^1_\bullet$ induces the unit map $\eta : A \to \mathrm{THH}(A)_\bullet$.
- The retraction $\epsilon : S^1_\bullet \to *$ induces the counit map $\epsilon : \mathrm{THH}(A)_\bullet \to A$.
- The fold map $\phi : S^1_\bullet \vee S^1_\bullet \to S^1_\bullet$ induces the multiplication map

$$\phi : \mathrm{THH}(A)_\bullet \wedge_A \mathrm{THH}(A)_\bullet \to \mathrm{THH}(A)_\bullet.$$

There is no simplicial pinch map $S^1_\bullet \to S^1_\bullet \vee S^1_\bullet$ for this simplicial model of of S^1, so [1] needed to also employ a double circle model. The double circle $dS^1_\bullet$ is the quotient of the double 1-simplex given by

$$dS^1_\bullet = (\Delta^1 \coprod \Delta^1) \coprod_{\partial\Delta^1 \coprod \partial\Delta^1} \partial\Delta^1.$$

They then use the double circle model $dS^1_\bullet$ to define the coproduct. There is a simplicial pinch map $\psi : dS^1_\bullet \to S^1_\bullet \vee S^1_\bullet$ taking the double circle to the wedge of two circles, and a simplicial reflection map $\chi : dS^1_\bullet \to dS^1_\bullet$, which interchanges the two copies of Δ^1. These maps induce maps on a corresponding "double model" of THH, which [1] shows is weakly equivalent to ordinary THH. Therefore, one gets the following coproduct and antipode maps in the homotopy category that make THH into an A-Hopf algebra in the homotopy category:

$$\begin{aligned}\psi &: \mathrm{THH}(A) \to \mathrm{THH}(A) \wedge_A \mathrm{THH}(A)\\ \chi &: \mathrm{THH}(A) \to \mathrm{THH}(A).\end{aligned}$$

Angeltveit-Rognes further prove that these simplicial maps on the circle yield structure in the Bökstedt spectral sequence as well.

Theorem 4.1.2 ([1] 4.5)

If A is a commutative ring spectrum, then:

1. If $H_*(\mathrm{THH}(A);\mathbb{F}_p)$ is flat over $H_*(A;\mathbb{F}_p)$, then there is a coproduct

$$\psi : H_*(\mathrm{THH}(A);\mathbb{F}_p) \to H_*(\mathrm{THH}(A);\mathbb{F}_p) \otimes_{H_*(A;\mathbb{F}_p)} H_*(\mathrm{THH}(A);\mathbb{F}_p)$$

   and $H_*(\mathrm{THH}(A);\mathbb{F}_p)$ is an $\mathscr{A}_*$-comodule $H_*(A;\mathbb{F}_p)$-Hopf algebra, where $\mathscr{A}_*$ is the dual Steenrod algebra.

2. If each term $E^r_{*,*}(A)$ for $r \geq 2$ is flat over $H_*(A;\mathbb{F}_p)$, then there is a coproduct

$$\psi : E^r_{*,*}(A) \to E^r_{*,*}(A) \otimes_{H_*(A;\mathbb{F}_p)} E^r_{*,*}(A)$$

   and $E^r_{*,*}(A)$ is an $\mathscr{A}_*$-comodule $H_*(A;\mathbb{F}_p)$-Hopf algebra spectral sequence. In particular, the differentials d^r respect the coproduct ψ.

As mentioned above, we are using the notation $\mathscr{A}_*$ for the dual Steenrod algebra, which is the $\mathbb{F}_p$-Hopf algebra:

$$\mathscr{A}_* = \begin{cases} \mathbb{F}_p[\xi_1, \xi_2, \ldots] \otimes \Lambda_{\mathbb{F}_p}(\tau_0, \tau_1, \ldots) & p \text{ odd} \\ \mathbb{F}_p[\xi_1, \xi_2, \ldots] & p = 2 \end{cases}$$

for $|\xi_i| = 2(p^i - 1)$ (or $2^i - 1$ if $p = 2$), $|\tau_i| = 2p^i - 1$ [28].

In order to understand this spectral sequence structure in the setting of Angeltveit-Rognes [1], we recall a few definitions:

Definition 4.1.3

For an augmented algebra A over a commutative ring R with augmentation $\epsilon : A \to R$, the **indecomposable elements** of A, denoted by the R-module QA, are given by the short exact sequence

$$IA \otimes IA \xrightarrow{\mu} IA \longrightarrow QA \longrightarrow 0$$

for multiplication map μ and $IA := \ker(\epsilon)$.

Example 4.1.4

Indecomposable elements in the polynomial algebra $k[w_1, w_2, \ldots]$ are classes of the form w_i. The augmentation in this case is

$$\begin{aligned} \epsilon : k[w_1, w_2, \ldots] &\to k \\ w_i &\mapsto 0 \end{aligned}$$

so $IA = \ker(\epsilon) = (w_1, w_2, \ldots)$. So then the image of the product on IA will be terms of the form $w_i^{m_i} \ldots w_j^{m_j}$ for $\Sigma m_k > 1$, which means QA is given by elements of the form w_i.

Example 4.1.5

Similarly, in the exterior algebra $\Lambda_k(y_1, y_2, \ldots)$, indecomposable elements are classes of the form y_i. The augmentation is given by

$$\begin{aligned} \epsilon : \Lambda_k(y_1, y_2, \ldots) &\to k \\ y_i &\mapsto 0 \end{aligned}$$

so $IA = \ker(\epsilon) = (y_1, y_2, \ldots)$. The image of the product on IA will be terms of the form $y_{i_1}, y_{i_2} \ldots y_{i_n}$ for $n > 1$, which means QA is given by elements of the form y_i.

Definition 4.1.6

For a coaugmented coalgebra C over a commutative ring R with coaugmentation $\eta : R \to C$ and counit $\epsilon : C \to R$, the **primitive elements** of C, denoted by the R-module PC, are given by the short exact sequence

$$0 \longrightarrow PC \longrightarrow JC \xrightarrow{\Delta} JC \otimes JC$$

for comultiplication map Δ and $JC := \text{coker}(\eta)$. An element $x \in \ker(\epsilon)$ is primitive if its image under the quotient by $Im(\eta)$ in JC is in PC.

Remark 4.1.7

In a coaugmented coalgebra C, x is primitive if $\Delta(x) = 1 \otimes x + x \otimes 1$. Note that this formulation is equivalent to the above definition because the coproduct on $x \in IC = \ker(\epsilon)$ is given by

$$\Delta(x) = 1 \otimes x + x \otimes 1 + \Sigma_i x_i' \otimes x_i''.$$

Since C is coaugmented, it splits as $R \oplus IC$, which means that

$$C \otimes C = (R \otimes R) \oplus (IC \otimes R) \oplus (R \otimes IC) \oplus (IC \otimes IC).$$

Because C is counital,

$$\mathrm{Id} = (\epsilon \otimes \mathrm{Id}) \circ \Delta = (\mathrm{Id} \otimes \epsilon) \circ \Delta,$$

so $\Sigma_i x_i' \otimes x_i'' \in IC \otimes IC$. But

$$\begin{aligned} \eta : R &\longrightarrow C \cong R \oplus IC \\ r &\mapsto (r, 0) \end{aligned}$$

has cokernel $JC \cong IC$, so for primitive $x \in IC \cong JC$,

$$\begin{array}{c} 0 \longrightarrow PC \longrightarrow JC \xrightarrow{\Delta} JC \otimes JC \\ x \mapsto x \mapsto 0 \end{array}$$

means that $\Sigma_i x_i' \otimes x_i'' \in JC \otimes JC$ must be zero, and so $\Delta(x) = 1 \otimes x + x \otimes 1$ as desired.

Example 4.1.8

Primitive elements in the polynomial coalgebra $k[w_1, w_2, \ldots]$ are classes of the form $w_i^{p^m}$ for $p = char(k)$. The coaugmentation

$$\begin{aligned} \eta : k &\longrightarrow k[w_1, w_2, \ldots] \\ 1 &\mapsto 1 \end{aligned}$$

has cokernel JC with basis $\{w_1^j, w_2^j, \ldots\}$ for all $j \geq 1$. Recall the comultiplication is given by

$$\Delta(w_i^j) = \sum_k \binom{j}{k} w_i^k \otimes w_i^{j-k}$$

Since p is the characteristic of k,

$$\Delta(w_i^{p^m}) = 1 \otimes w_i^{p^m} + w_i^{p^m} \otimes 1$$

so $w_i^{p^m}$ is primitive. The other w_i^n are not primitive because $\Delta(w_i^n) \neq 1 \otimes w_i^n + w_i^n \otimes 1$ since those binomial coefficients do not vanish.

Example 4.1.9

In the exterior coalgebra $\Lambda_k(y_1, y_2, \ldots)$, primitive elements are classes of the form y_i. Recall that the coproduct on $\Lambda_k(y_1, y_2, \ldots)$ is given by $\Delta(y_i) = 1 \otimes y_i + y_i \otimes 1$ and therefore the those terms are primitive.

Example 4.1.10

Primitive elements in the divided power coalgebra $\Gamma_k[x_1, x_2, \ldots]$ are classes of the form x_i. Recall that the divided power coalgebra has comultiplication

$$\Delta(\gamma_j(x_i)) = \sum_{a+b=j} \gamma_a(x_i) \otimes \gamma_b(x_i)$$

So since $\gamma_0(x_i) = 1$ and $\gamma_1(x_i) = x_i$, we have

$$\Delta(x_i) = 1 \otimes x_i + x_i \otimes 1.$$

The other $\gamma_j(x_i)$ for $j > 1$ are not primitive because their image under Δ will have additional $\gamma_a(x_i) \otimes \gamma_b(x_i)$ terms.

Studying primitive and indecomposable elements can be particularly useful because of results like the following from Angeltveit and Rognes:

Theorem (Prop 4.8 [1])

Let A be a commutative $\mathbb{S}$-algebra with $H_*(A;k)$ connected and such that $\mathrm{HH}_*(H_*(A;k))$ is flat over $H_*(A;k)$. Then the E^2-term of the Bökstedt spectral sequence

$$E^2_{*,*}(A) = \mathrm{HH}_*(H_*(A;k))$$

is an $H_*(A;k)$-Hopf algebra, and a shortest non-zero differential $d^r_{s,t}$ in lowest total degree $s+t$, if one exists, must map from an algebra indecomposable to a coalgebra primitive in $\mathrm{HH}_*(H_*(A;k))$.

Proof. We will go through the proof as presented by Angeltveit-Rognes [1] since the proof of the analogous result for the coBökstedt spectral sequence will be similar.

First we justify why $E^2_{*,*}(A)$ is an $H_*(A;k)$-Hopf algebra. Recall that the Hopf algebra structure includes a comultiplication map, multiplication map, counit map, unit map, and antipode. As we saw above, THH has an A-Hopf algebra structure in the homotopy category that comes from identifying topological Hochschild homology with the simplicial tensor $\mathrm{THH}(A) \cong A \otimes S^1$. Then this simplicial structure gives a filtration that induces a spectral sequence. The E^2-term in this case is Hochschild homology of $H_*(A;k)$, and the product and coproduct descend to E^2.

Now suppose $d^2, ..., d^{r-1}$ are all zero. Since E^2 is an $H_*(A;k)$-Hopf algebra, this means $E^2_{*,*}(A) = E^r_{*,*}(A)$ is still an $H_*(A;k)$-Hopf algebra. If the class xy is decomposable for classes x, y with positive degree such that $d^r(xy) \neq 0$, then applying the Leibniz Rule yields:

$$d^r(xy) = d^r(x)y \pm x d^r(y),$$

which implies that $d^r(x) \neq 0$ or $d^r(y) \neq 0$. Therefore xy cannot be in the lowest possible total degree for the source of the differential, and thus the lowest total degree nonzero differential must map from an algebra indecomposable instead.

On the other hand, if we assume $d^r(z)$ is not a coalgebra primitive, then the co-Leibniz Rule says

$$\begin{aligned}
\Delta \circ d^r(z) &= (d^r \otimes 1 \pm 1 \otimes d^r)\Delta(z) \\
&= (d^r \otimes 1 \pm 1 \otimes d^r)(z \otimes 1 + 1 \otimes z + \Sigma_i z_i' \otimes z_i'') \\
&= (d^r(z) \otimes 1 + d^r(1) \otimes z + \Sigma_i d^r(z_i') \otimes z_i'') \pm (z \otimes d^r(1) + 1 \otimes d^r(z) + \Sigma_i z_i' \otimes d^r(z_i'')) \\
&= (d^r(z) \otimes 1 + \Sigma_i d^r(z_i') \otimes z_i'') \pm (1 \otimes d^r(z) + \Sigma_i z_i' \otimes d^r(z_i'')) \qquad \text{(since } d^r(1) = 0)
\end{aligned}$$

where the tensor products are over $H_*(A;k)$. So this implies that $d^r(z_i') \neq 0$ or $d^r(z_i'') \neq 0$ for some i, because if they're all zero then

$$\Delta(d^r(z)) = d^r(z) \otimes 1 \pm 1 \otimes d^r(z),$$

which by definition that says that $d^r(z)$ is primitive, contradicting our initial assumption. So since either $d^r(z_i') \neq 0$ or $d^r(z_i'') \neq 0$ and the coproduct preserves degree (i.e. $deg(z_i') + deg(z_i'') = deg(z)$ for z_i' and z_i'' in positive degree), $deg(z_i') < deg(z)$ and $deg(z_i'') < deg(z)$. But then z_i' and z_i'' are in lower total degree than z, so the shortest non-zero differential in lowest total degree has to hit a coalgebra primitive. □

Remark 4.1.11

The result in Angeltveit-Rognes [1] further shows that when $k = \mathbb{F}_p$ there is an $\mathscr{A}_*$-comodule structure, but since we don't use an analogous structure in the coalgebra setting we chose not to include it in the above discussion.

4.2 $\square$-Hopf algebra structure in the coBökstedt spectral sequence

We have now examined certain properties that were known about the Bökstedt spectral sequence and computational implications of that structure, and so we now want to address the analogous coBökstedt spectral sequence setting.

As we saw in the previous section, under appropriate flatness conditions the Bökstedt spectral sequence for a commutative ring spectrum A is a spectral sequence of Hopf algebras over the commutative ring $H_*(A;k)$. However, in this dual setting we would then want to show that we have a Hopf algebra over the coalgebra $H_*(C;k)$. However, this requires a notion of a Hopf algebra over a coalgebra, which Bohmann-Gerhardt-Shipley [5] call a $\square$-Hopf algebra. To start, we will thus recall background information about the cotensor product $\square$ before stating that the coBökstedt spectral sequence has a $\square$-Hopf algebra structure.

For an R-coalgebra C, a right C-comodule M with $\gamma : M \to M \otimes C$, and a left C-comodule N with $\psi : N \to C \otimes N$, the cotensor of M and N over C is defined to be the equalizer in R-modules:

$$M \square_C N := eq\left((M \otimes_R N) \underset{\mathrm{Id}_M \otimes \psi}{\overset{\gamma \otimes \mathrm{Id}_N}{\rightrightarrows}} M \otimes_R C \otimes_R N \right).$$

Note that the cotensor does not always yield a C-comodule, but under some conditions it does. In particular, if C is a coalgebra over a field and M and N are C-bicomodules, then $M \square_C N$ is a C-bicomodule.

In order to give the definition of a $\square_C$-Hopf algebra for a coalgebra C over a field k, we first need the definitions of a $\square_C$-coalgebra and a $\square_C$-bialgebra.

Definition 4.2.1 ([5])

Let C be a coalgebra over a field. A $\square_C$**-coalgebra** D is a C-bicomodule along with a comultiplication map $\Delta : D \to D \,\square_C\, D$ and a counit map $\epsilon : D \to C$ that are coassociative and counital maps of C-comodules.

Definition 4.2.2 ([5])

Let C be a coalgebra over a field. A **$\square_C$-bialgebra** D is a $\square_C$-coalgebra that is also equipped with a multiplication map $\mu : D \square_C D \to D$ and a unit map $\eta : D \to C$ that satisfy associativity and unitality. The multiplication must also be compatible with the $\square_C$-coalgebra structure. A **$\square_C$-Hopf algebra** D is a $\square_C$-bialgebra along with an antipode $\chi : D \to D$ that is a C-comodule map satisfying the corresponding hexagonal antipode diagram.

See [5] for more details on the diagrams for coassociativity and counitality and those specifying the interactions between the algebra and coalgebra structures.

As we saw above, in the commutative case we can identify $\mathrm{THH}(A)_\bullet$ as the tensor of A with $S^1_\bullet$. Similarly, $\mathrm{coTHH}(C)^\bullet$ can be viewed as a *co*tensor with $S^1_\bullet$. We also discussed how [1] shows that the simplicial pinch and fold maps induce maps on THH and the Bökstedt spectral sequence. In the same way, work of Bohmann-Gerhardt-Shipley shows that simplicial pinch and fold maps in the dual setting induce maps on the coBökstedt spectral sequence. In [5], they describe topological coHochschild homology as the following cotensor with S^1

$$\mathrm{coTHH}^\bullet(C) \cong C^{S^1_\bullet},$$

so that the simplicial maps on S^1 ultimately induce a $\square$-Hopf structure on the coBökstedt spectral sequence. In order to state this result, we require the following definition:

Definition 4.2.3

For a coalgebra C over a field k, a right comodule M over C is called **coflat** if $M \,\square_C\, -$ is exact as a functor from left C-comodules to k-modules.

We can now state the analog of [1, Theorem 4.5] for coTHH.

Theorem 4.2.4 ([5])

For C a cocommutative coalgebra spectrum, if for $r \geq 2$ each $E_r^{*,*}(C)$ is coflat over $H_*(C;k)$, then the coBökstedt spectral sequence is a spectral sequence of $\square_{H_*(C;k)}$-bialgebras.

They further show that this bialgebra structure makes the coBökstedt spectral sequence into a spectral sequence of $\square_{H_*(C;k)}$-Hopf algebras by defining the appropriate antipode map.

4.3 $\square$-Hopf algebra structure in the relative coBökstedt spectral sequence

We now want to see how the $\square$-Hopf algebra structure extends to the relative coBökstedt spectral sequence. We will consider coalgebras over Hk, for k a field. Adapting the notation from Bohmann-Gerhardt-Shipley's work [5], we first define the cotensor in the quasicategory of cocommutative Hk-coalgebras, denoted by $CoCAlg_{Hk}$. Note that [5] uses the notation $CoCAlg(Mod_{Hk})$ based on the symmetric monoidal quasicategory of Hk-modules, Mod_{Hk}, so this is minor condensing of notation.

Definition 4.3.1

Given $C \in CoCAlg_{Hk}$ and a simplicial set $X_\bullet$, we write $C^{X_\bullet}$ for the **cotensor** of C with the simplicial set $X_\bullet$. On the n^{th} cosimplicial level this is:

$$(C^{X_\bullet})^n = \prod_{x \in X_n} C.$$

So using this notion of cotensor in $CoCAlg_{Hk}$ and the simplicial circle $S^1_\bullet$ from the last section, $\text{coTHH}^{Hk}(C)^\bullet \cong C^{S^1_\bullet}$.

Remark 4.3.2

Note that since the relative coBökstedt spectral sequence was stated in generality for any homology theory E, we now need to restrict ourselves to the appropriate conditions for the $\square$-coalgebra structure. In this thesis we are specifically interested in such examples where $E = \mathbb{S}$ that are Hk-coalgebras.

Theorem 4.3.3 ([5])

For C a cocommutative coalgebra spectrum, if for $r \geq 2$ each $E_r^{*,*}(C)$ is coflat over $\pi_*(C)$, then the relative coBökstedt spectral sequence is a spectral sequence of $\square_{\pi_*(C)}$-Hopf algebras.

The proof follows as in [5], in which cotensoring with the simplicial fold map $S^1_\bullet \vee S^1_\bullet \to S^1_\bullet$ induces the comultiplication, and the multiplication further comes from the simplicial pinch map on the double circle simplicial model $dS^1_\bullet \to S^1_\bullet \vee S^1_\bullet$. As we saw in the Bökstedt spectral sequence, we now want to use the additional algebraic structure to understand differentials in the spectral sequence. However, in order to make sense of these ideas in the dual setting, we also need the following definitions and results regarding indecomposable and primitive elements.

Definition 4.3.4

A unital $\square_C$-algebra A with multiplication $\mu : A \square_C A \to A$ and unit $\eta : C \to A$ is **augmented** if there exists an augmentation map $\epsilon : A \to C$ such that $\epsilon\mu = \epsilon \square \epsilon$ and $\epsilon\eta = \text{Id}$.

Definition 4.3.5

A counital $\square_C$-coalgebra D with comultiplication $\Delta : D \to D \square_C D$ and counit $\epsilon : D \to C$ is

coaugmented if there exists a coaugmentation map $\eta : C \to D$ such that $\Delta\eta = \eta \,\square\, \eta$ and $\epsilon\eta = \mathrm{Id}$.

Definition 4.3.6 ([5])

Given a coaugmented $\square_C$-coalgebra D, let PD be defined by the short exact sequence

$$0 \longrightarrow PD \longrightarrow JD \xrightarrow{\Delta} JD \,\square_C\, JD,$$

where $JD = \mathrm{coker}(\eta)$. An element in $\ker(\epsilon)$ is **primitive** if its image in JD is in PD.

Definition 4.3.7 ([5])

For an augmented $\square_C$-algebra A, the **indecomposables** of A, denoted by QA, are defined by the short exact sequence

$$IA \,\square_C\, IA \xrightarrow{\mu} IA \longrightarrow QA \longrightarrow 0,$$

where $IA = \ker(\epsilon)$.

Since the theorem regarding the Hopf structure in the relative coBökstedt spectral sequence will require $\pi_*(C)$ to be connected, we define that term here:

Definition 4.3.8 ([5])

A graded k-coalgebra D_* is connected if $D_* = 0$ when $* < 0$, and the counit map $\epsilon : D_* \to k$ is an isomorphism in degree zero.

Theorem 4.3.9

For a field k, let C be a cocommutative Hk-coalgebra spectrum such that $\mathrm{coHH}_*(\pi_*(C))$ is coflat over $\pi_*(C)$ and the graded coalgebra $\pi_*(C)$ is connected. Then the E_2-term of the relative coBökstedt spectral sequence calculating $\pi_*(\mathrm{coTHH}^{Hk}(C))$,

$$E_2^{*,*}(C) = \mathrm{coHH}_*^k(\pi_*(C)),$$

is a $\square_{\pi_*(C)}$-bialgebra, and the shortest non-zero differential $d_r^{s,t}$ in lowest total degree $s+t$ maps from a $\square_{\pi_*(C)}$-algebra indecomposable to a $\square_{\pi_*(C)}$-coalgebra primitive.

Proof. The proof follows as in the non-relative version in [5]. Note that the requirement that $\mathrm{coHH}(\pi_*(C))$ is coflat over $\pi_*(C)$ is really a condition on E_2. However since we can do this argument page by page, no differentials on the E_2-page implies that $E_2 \cong E_3$ and so the same coflatness condition will hold for that page. Thus, since we're just concerned about the first of the non-zero differentials, the only condition we have to satisfy is the one we need for the E_2-page. $\square$

Chapter 5

Explicit calculations

A natural question that comes up when studying coTHH is to ask what kinds of coalgebra spectra exist, and for those that exist, is the E_2-page of the relative coBökstedt spectral sequence computable? Although Bohmann-Gerhardt-Høgenhaven-Shipley-Ziegenhagen [4] demonstrate that the coBökstedt spectral sequence can input examples of the form $\Sigma^\infty_+ X$ for simply connected X, Péroux-Shipley show that examples of $\mathbb{S}$-coalgebras are still quite limited [30]. So now that we have a way of generating new R-coalgebra spectra of the form $B \wedge_A B$ and studying them via the relative coBökstedt spectral sequence, we will go through a few specific examples. In particular, we will examine the coalgebras $H\mathbb{F}_p \wedge_{H\mathbb{Z}} H\mathbb{F}_p$ and $H\mathbb{F}_p \wedge_{BP<n>} H\mathbb{F}_p$ (for those n and p such that $BP < n >$ is commutative). In this chapter, we will start by computing the E_2-term of the relative coBökstedt spectral sequence computing $\pi_*(\mathrm{coTHH}(C))$ for these examples.

These results confirm that we can indeed compute the E_2-pages of the relative coBökstedt spectral sequence calculating relative topological coHochschild homology of some examples of, in this case, $H\mathbb{F}_p$-coalgebras. As discussed in the last chapter, in order to use the $\square$-Hopf algebra structure to find the E_∞-page and complete the computation of the homotopy groups of coTHH, we need our coalgebra spectra to be cocommutative. As a result, these examples cannot be simplified using the $\square$-Hopf algebra techniques because they are not cocommutative. However, we will consider cocommutative coalgebras with homotopy that is similar to the above E_2-page examples later in this chapter.

5.1 E_2-page Examples

Proposition 5.1.1

For the $H\mathbb{F}_p$-coalgebra $H\mathbb{F}_p \wedge_{H\mathbb{Z}} H\mathbb{F}_p$, the E_2-page of the spectral sequence calculating $\pi_{t-s}(\mathrm{coTHH}^{H\mathbb{F}_p}(H\mathbb{F}_p \wedge_{H\mathbb{Z}} H\mathbb{F}_p))$ is

$$E_2^{s,t} = \mathrm{coHH}_{s,t}^{\mathbb{F}_p}(\pi_*(H\mathbb{F}_p \wedge_{H\mathbb{Z}} H\mathbb{F}_p)) \cong \Lambda_{\mathbb{F}_p}(\tau) \otimes_{\mathbb{F}_p} \mathbb{F}_p[\omega]$$

for $\|\tau\| = (0,1), \|\omega\| = (1,1)$.

Proof. By Proposition 2.5.5, $H\mathbb{F}_p \wedge_{H\mathbb{Z}} H\mathbb{F}_p$ is an $H\mathbb{F}_p$-coalgebra coming from the map $\phi : H\mathbb{Z} \to H\mathbb{F}_p$, which is induced by $\mathbb{Z} \xrightarrow{\text{mod } p} \mathbb{F}_p$.

We want to find $E_*(\mathrm{coTHH}^R(C))$ for $R = H\mathbb{F}_p$, $E = \mathbb{S}$, and $C = H\mathbb{F}_p \wedge_{H\mathbb{Z}} H\mathbb{F}_p$ as in the Corollary 3.0.2. Note that we satisfy the flatness condition that $\pi_*(H\mathbb{F}_p \wedge_{H\mathbb{Z}} H\mathbb{F}_p)$ is flat over $\pi_*(H\mathbb{F}_p) \cong \mathbb{F}_p$, because modules are flat over fields. Corollary 3.0.2 states that the relative coBökstedt spectral sequence has the form:

$$E_2^{s,t} = \mathrm{coHH}_{s,t}^{\mathbb{F}_p}(\pi_*(H\mathbb{F}_p \wedge_{H\mathbb{Z}} H\mathbb{F}_p)) \overset{??}{\Longrightarrow} \pi_{t-s}(\mathrm{coTHH}^{H\mathbb{F}_p}(H\mathbb{F}_p \wedge_{H\mathbb{Z}} H\mathbb{F}_p))$$

where the ?? serve as a reminder that convergence for this Bousfield-Kan spectral sequence cannot be automatically assumed. We can use the Künneth spectral sequence to calculate $\pi_*(H\mathbb{F}_p \wedge_{H\mathbb{Z}} H\mathbb{F}_p)$:

$$\mathrm{Tor}_{p,q}^{E_*(R)}(E_*(M), E_*(N)) \Rightarrow E_{p+q}(M \wedge_R N)$$

which exists if $E_*(R)$ is a flat right R_*-module [15, Theorem IV.6.2]. Here we have $E = \mathbb{S}$ and $R = H\mathbb{Z}$, so since $E_*(R) = \pi_*(H\mathbb{Z}) \cong \mathbb{Z}$ is indeed flat over $R_* = \pi_*(H\mathbb{Z}) \cong \mathbb{Z}$, we may apply the Künneth spectral sequence to get:

$$\mathrm{Tor}_{p,q}^{\pi_*(H\mathbb{Z})}(\pi_*(H\mathbb{F}_p), \pi_*(H\mathbb{F}_p)) \cong \mathrm{Tor}_{p,q}^{\mathbb{Z}}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow \pi_{p+q}(H\mathbb{F}_p \wedge_{H\mathbb{Z}} H\mathbb{F}_p)$$

To compute this E_2-term, we will need to create a projective resolution of $\mathbb{F}_p$ as a $\mathbb{Z}$-module:

$$\mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \xrightarrow{\text{mod } p} \mathbb{F}_p \longrightarrow 0.$$

Then we can truncate and $- \otimes_{\mathbb{Z}} \mathbb{F}_p$ to get

$$\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{F}_p \xrightarrow{\times p} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{F}_p \longrightarrow 0$$

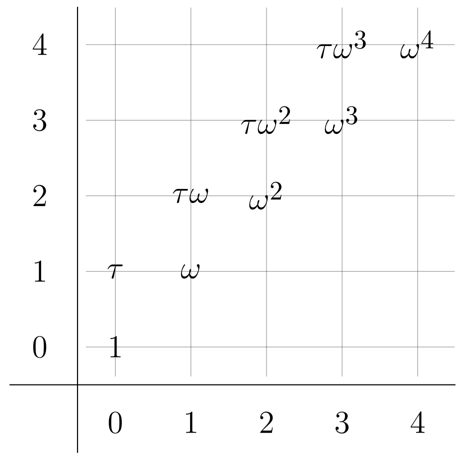
which simplifies to

$$\mathbb{F}_p \xrightarrow[0]{\times p} \mathbb{F}_p \longrightarrow 0$$

Thus we have $\mathbb{F}_p$ in degree 0 and 1. As a coalgebra this is the exterior coalgebra with a single generator in degree 1. Now that we know that $\pi_*(H\mathbb{F}_p \wedge_{H\mathbb{Z}} H\mathbb{F}_p)$ is an exterior coalgebra over $\mathbb{F}_p$, the E_2-page looks like:

$$
\begin{aligned}
E_2^{s,t} &= \mathrm{coHH}_{s,t}^{\mathbb{F}_p}(\pi_*(H\mathbb{F}_p \wedge_{H\mathbb{Z}} H\mathbb{F}_p)) \\
&\cong \mathrm{coHH}_{s,t}^{\mathbb{F}_p}(\Lambda_{\mathbb{F}_p}(\tau)) && (|\tau| = 1) \\
&\cong \Lambda_{\mathbb{F}_p}(\tau) \otimes \mathbb{F}_p[\omega] && \text{(by Proposition 5.1 in [4])}
\end{aligned}
$$

with bidegrees $\|\tau\| = (0,1)$ and $\|\omega\| = (1,1)$. Thus this E_2-page looks like:



Therefore we have shown the desired result. □

Now that we've seen one calculation of an E_2-term for the relative coBökstedt spectral sequence, let's consider a similar $H\mathbb{F}_p$-coalgebra example, $H\mathbb{F}_p \wedge_{BP<n>} H\mathbb{F}_p$. First we introduce BP based on the complex cobordism spectrum MU as in [33].

Definition 5.1.2 ([9])

The spectrum BP, called the **Brown-Peterson spectrum**, is named because Brown and Peterson showed that MU localized at a prime can be split into a wedge product of suspensions of BP. In particular it is characterized by

$$\pi_*(BP) = \mathbb{Z}_{(p)}[v_1, v_2, \ldots],$$

for $|v_i| = 2(p^i - 1)$. There are also truncated Brown-Peterson spectra $BP < n >$, with

$$\pi_*(BP < n >) = \mathbb{Z}_{(p)}[v_1, \dots, v_n]$$

as shown by [22].

Remark 5.1.3

Note that in order to consider $B \wedge_A B$ as a B-coalgebra, we need A to be a commutative ring spectrum. Because $BP < 0 >= H\mathbb{Z}_{(p)}$ is an Eilenberg-Mac Lane spectrum of a commutative ring, it will also be commutative. Similarly, [27] show that $BP < 1 >= \ell$ is commutative since it is equivalent to the algebraic K-theory of a commutative ring spectrum. However for $n = 2$, $BP < 2 >$ is only known to be commutative for $p = 2$ [23] and $p = 3$ [21], and so we limit our examples to these cases.

Proposition 5.1.4

For the $H\mathbb{F}_p$-coalgebra $H\mathbb{F}_p \wedge_{BP<n>} H\mathbb{F}_p$ for $n = 0, 1$ and for $n = 2$ at the primes $p = 2, 3$, the E_2-page of the spectral sequence calculating $\pi_{t-s}(\mathrm{coTHH}^{H\mathbb{F}_p}(H\mathbb{F}_p \wedge_{BP<n>} H\mathbb{F}_p))$ is

$$E_2^{s,t} = \mathrm{coHH}_{s,t}^{\mathbb{F}_p}(\pi_*(H\mathbb{F}_p \wedge_{BP<n>} H\mathbb{F}_p)) \cong \Lambda_{\mathbb{F}_p}(\tau_0, \dots \tau_n) \otimes \mathbb{F}_p[\omega_0, \dots \omega_n]$$

for $||\tau_i|| = (0, 2p^i - 1), ||\omega_i|| = (1, 2p^i - 1)$.

Proof. $H\mathbb{F}_p \wedge_{BP<n>} H\mathbb{F}_p$ is an $H\mathbb{F}_p$-coalgebra because based on the definition for $BP < n >$ we have

$$\pi_* BP < n > \cong \mathbb{Z}_{(p)}[v_1, v_2, ..., v_n]$$

for $|v_i| = 2p^i - 2$, so there is a map of ring spectra

$$BP<n> \to H\mathbb{Z}_{(p)} \to H\mathbb{F}_p,$$

given by mapping to the Eilenberg-Mac Lane spectrum on π_0. The composition gives a coalgebra structure by Proposition 2.5.5 for $n = 0, 1$ and for $n = 2$ at the primes $p = 2, 3$.

We apply the relative coBökstedt spectral sequence of Corollary 3.0.2 with $R = H\mathbb{F}_p$, $E = \mathbb{S}$, and $C = H\mathbb{F}_p \wedge_{BP<n>} H\mathbb{F}_p$, giving us:

$$E_2^{s,t} = \mathrm{coHH}_{s,t}^{\mathbb{F}_p}(\pi_*(H\mathbb{F}_p \wedge_{BP<n>} H\mathbb{F}_p)) \overset{??}{\Longrightarrow} \pi_{t-s}(\mathrm{coTHH}^{H\mathbb{F}_p}(H\mathbb{F}_p \wedge_{BP<n>} H\mathbb{F}_p))$$

Now to compute $\pi_*(H\mathbb{F}_p \wedge_{BP<n>} H\mathbb{F}_p)$, we may again use the Künneth spectral sequence:

$$\mathrm{Tor}_{p,q}^{E_*(R)}(E_*(M), E_*(N)) \Rightarrow E_{p+q}(M \wedge_R N)$$

which exists if $E_*(R)$ is a flat right R_*-module. So here we have

$$\mathrm{Tor}_{p,q}^{\pi_* BP<n>}(\pi_* H\mathbb{F}_p, \pi_* H\mathbb{F}_p) \cong \mathrm{Tor}_{p,q}^{\mathbb{Z}_{(p)}[v_1, v_2, ..., v_n]}(\mathbb{F}_p, \mathbb{F}_p) \quad \Rightarrow \pi_{p+q}(H\mathbb{F}_p \wedge_{BP<n>} H\mathbb{F}_p)$$

To compute this, we will need to create a projective resolution of $\mathbb{F}_p$ as a $\mathbb{Z}_{(p)}[v_1, v_2, ..., v_n]$-module. We may use the Koszul complex, which we will include for $n = 2$ since that is the largest case we will consider:

$$\Sigma^{|v_1v_2|}\mathbb{Z}_{(p)}[v_1,v_2] \xrightarrow{\alpha} \Sigma^{|v_1|}\mathbb{Z}_{(p)}[v_1,v_2] \oplus \Sigma^{|v_2|}\mathbb{Z}_{(p)}[v_1,v_2] \oplus \Sigma^{|v_1v_2|}\mathbb{Z}_{(p)}[v_1,v_2]$$
$$1 \mapsto (v_2, -v_1, p)$$
$$\xrightarrow{\beta} \mathbb{Z}_{(p)}[v_1,v_2] \oplus \Sigma^{|v_1|}\mathbb{Z}_{(p)}[v_1,v_2] \oplus \Sigma^{|v_2|}\mathbb{Z}_{(p)}[v_1,v_2] \xrightarrow{\gamma} \mathbb{Z}_{(p)}[v_1,v_2] \xrightarrow{\psi} \mathbb{F}_p \longrightarrow 0.$$
$$(1,0,0) \mapsto (v_1, -p, 0) \qquad (1,0,0) \mapsto p$$
$$(0,1,0) \mapsto (v_2, 0, -p) \qquad (0,1,0) \mapsto v_1$$
$$(0,0,1) \mapsto (0, v_2, -v_1) \qquad (0,0,1) \mapsto v_2$$

where ψ is the module map $\psi : \mathbb{Z}_{(p)}[v_1,v_2] \xrightarrow{\text{mod } p} \mathbb{F}_p$ that sends v_1 and v_2 to 0. We then truncate and $- \otimes_{\mathbb{Z}_{(p)}[v_1,v_2]} \mathbb{F}_p$ to get the resulting complex:

$$\Sigma^{|v_1v_2|}\mathbb{F}_p \xrightarrow{0} \Sigma^{|v_1|}\mathbb{F}_p \oplus \Sigma^{|v_2|}\mathbb{F}_p \oplus \Sigma^{|v_1v_2|}\mathbb{F}_p \xrightarrow{0} \mathbb{F}_p \oplus \Sigma^{|v_1|}\mathbb{F}_p \oplus \Sigma^{|v_2|}\mathbb{F}_p \xrightarrow{0} \mathbb{F}_p \longrightarrow 0$$

Since coassociative comultiplication preserves total degree, the resulting $\mathbb{F}_p \oplus \Sigma\mathbb{F}_p \oplus \Sigma^{|v_1|+1}\mathbb{F}_p \oplus \Sigma^{|v_2|+1}\mathbb{F}_p$ is an exterior coalgebra over $\mathbb{F}_p$ generated by what we will call τ_0, τ_1, τ_2 in degrees $|\tau_0| = 1$, $|\tau_1| = |v_1| + 1$, and $|\tau_2| = |v_2| + 1$.[1] As in Tilson [35], the spectral sequence computing $\pi_*(H\mathbb{F}_p \wedge_{BP<n>} H\mathbb{F}_p)$ then collapses at this E_2-page, and we recover the calculation:

$$\pi_*(H\mathbb{F}_p \wedge_{BP<n>} H\mathbb{F}_p) \cong \Lambda_{\mathbb{F}_p}(\tau_0, \tau_1, ..., \tau_n)$$

for $|\tau_i| = |v_i| + 1 = 2p^i - 1$, and our computation amounts to the familiar:

$$\begin{aligned} E_2^{s,t} &= \mathrm{coHH}_{s,t}^{\mathbb{F}_p}(\pi_*(H\mathbb{F}_p \wedge_{BP<n>} H\mathbb{F}_p)) \\ &\cong \mathrm{coHH}_{s,t}^{\mathbb{F}_p}(\Lambda_{\mathbb{F}_p}(\tau_0, \tau_1, ..., \tau_n)) && (|\tau| = 2p^i - 1) \\ &\cong \Lambda_{\mathbb{F}_p}(\tau_0, ..., \tau_n) \otimes_{\mathbb{F}_p} \mathbb{F}_p[\omega_0, ..., \omega_n] && \text{(by Lemma 5.1 in [4])} \end{aligned}$$

[1] In [35, Prop 5.6] that specifically examines the case where $n = 2$ and $p = 2$, these generators are called $\bar{2}, \bar{v_1}$, and $\bar{v_2}$.

with bidegrees $||\tau_i|| = (0, 2p^i - 1)$ and $||\omega_i|| = (1, 2p^i - 1)$. □

Remark 5.1.5

Thanks to a conversation with Mike Hill, we also have the following quotients of the dual Steenrod algebra for $p = 2$ that emerge as the homotopy groups of $H\mathbb{F}_2$-coalgebras of the form $B \wedge_A B$ for $|\xi_i| = 2^i - 1$:

	π_*
$H\mathbb{F}_2 \wedge_{H\mathbb{F}_2} H\mathbb{F}_2$	$\mathbb{F}_2$
$H\mathbb{F}_2 \wedge_{H\mathbb{Z}} H\mathbb{F}_2$	$\Lambda(\xi_1)$
$H\mathbb{F}_2 \wedge_{ku} H\mathbb{F}_2$	$\Lambda(\xi_1, \xi_2)$
$H\mathbb{F}_2 \wedge_{ko} H\mathbb{F}_2$	$\mathbb{F}_2[\xi_1, \xi_2]/\xi_1^4, \xi_2^2$
$H\mathbb{F}_2 \wedge_{tmf_1(3)} H\mathbb{F}_2$	$\Lambda(\xi_1, \xi_2, \xi_3)$
$H\mathbb{F}_2 \wedge_{tmf} H\mathbb{F}_2$	$\mathbb{F}_2[\xi_1, \xi_2, \xi_3]/\xi_1^8, \xi_2^4, \xi_3^2$

So by Lemma 5.1 in [4] as in the examples that we saw above, we could similarly find the E_2-pages:

$$E_2^{s,t} = \mathrm{coHH}_{s,t}^{\mathbb{F}_2}(\pi_*(H\mathbb{F}_2 \wedge_{ku} H\mathbb{F}_2)) \cong \Lambda_{\mathbb{F}_2}(\xi_1, \xi_2) \otimes \mathbb{F}_2[\omega_1, \omega_2]$$
$$E_2^{s,t} = \mathrm{coHH}_{s,t}^{\mathbb{F}_2}(\pi_*(H\mathbb{F}_2 \wedge_{tmf_1(3)} H\mathbb{F}_2)) \cong \Lambda_{\mathbb{F}_2}(\xi_1, \xi_2, \xi_3) \otimes \mathbb{F}_2[\omega_1, \omega_2, \omega_3]$$

5.2 Computational Tools

We are going to use the □-Hopf structure of the last chapter to give further computational tools. Recall from the previous chapter that the shortest nonzero differential must go from a □-Hopf algebra indecomposable to a □-coalgebra primitive. We first study the primitives of $\square_C$-coalgebras of the form $C \otimes D$.

Proposition 5.2.1 ([5])

For coaugmented k-coalgebras C and D, $C \otimes D$ is a $\square_C$-coalgebra and an element $c \otimes d \in C \otimes D$ is primitive as an element of the $\square_C$-coalgebra $C \otimes D$ if and only if d is primitive in the k-coalgebra D.

Bohmann-Gerhardt-Shipley prove that if coHH(D) is coflat over D then coHH(D) is a $\square_D$-algebra [5]. We will further need to identify the indecomposable elements, but in that case we will restrict to the specific computational setting we will need.

The last tool we introduce here is that for Hk-coalgebras the relative coBökstedt spectral sequence is itself a spectral sequence of k-coalgebras, which will then allow us to restrict differentials even further to targets that are k-coalgebra primitives. Bohmann-Gerhardt-Høgenhaven-Shipley-Ziegenhagen [4] showed the following result for the coBökstedt spectral sequence, and the relative case follows from their work.

Theorem 5.2.2

If C is a connected cocommutative Hk-coalgebra that is cofibrant as an Hk-module, then the relative coBökstedt spectral sequence for $E = \mathbb{S}$ is a spectral sequence of k-coalgebras. In particular, for every $r > 1$ there is a coproduct

$$\psi : E_r^{*,*} \longrightarrow E_r^{*,*} \otimes_k E_r^{*,*},$$

and the differentials d_r respect the coproduct.

Proof. This proof follows as in [4] since for $E = \mathbb{S}$ we are already in the setting of cosimplicial Hk-modules. □

5.3 Exterior Inputs

The goal of this section is to compute the homotopy groups of the topological coHochschild homology of coalgebra spectra with an exterior homotopy coalgebra. Now because we proved in the previous chapter that our spectral sequence has a $\square$-Hopf algebra structure, we will use that the shortest nonzero differential goes from an algebra indecomposable to a coalgebra primitive.

Theorem 5.3.1

For a field k, let C be a cocommutative Hk-coalgebra spectrum that is cofibrant as an Hk-module with $\pi_*(C) \cong \Lambda_k(y)$ for $|y|$ odd and greater than 1. Then the relative coBökstedt spectral sequence collapses and

$$\pi_*(\mathrm{coTHH}^{Hk}(C)) \cong \Lambda_k(y) \otimes k[w]$$

as graded k-modules for $|w| = |y| - 1$.

Proof. Recall that the flatness condition of the relative coBökstedt spectral sequence is satisfied because we're taking $E = \mathbb{S}$ and $R = Hk$, so the E_2-page is

$$E_2^{s,t} = \mathrm{coHH}^k_{s,t}(\Lambda_k(y)) \cong \Lambda_k(y) \otimes k[w]$$

by Proposition 5.1 in [4]. But now because the degree of y is both odd and greater than 1, we will show that the spectral sequence is sparse enough that all differentials will be zero.

By Proposition 4.3.9 we know that the shortest nontrivial differential in lowest total degree must map from a $\square_{\Lambda_k(y)}$-algebra indecomposable to a $\square_{\Lambda_k(y)}$-coalgebra primitive. Since the E_2-page is given by $\Lambda_k(y) \otimes k[w]$, Proposition 5.2.1 implies that elements in this

$\square_{\Lambda_k(y)}$-coalgebra will be primitive if and only if the component from $k[w]$ is primitive in the k-coalgebra $k[w]$. Recall that primitives in the k-coalgebra $k[w_1, w_2, \dots]$ more generally are of the form $w_i^{p^m}$ for $p = char(k)$, so here

$$\Lambda_k(y) \otimes \big(\text{primitives in } k[w]\big) \cong \Lambda_k(y) \otimes w^{p^m}$$

Therefore the only terms in the spectral sequence that are possible targets of differentials are w^{p^m} and yw^{p^m} for $m \geq 0$ and prime p.

Bohmann-Gerhardt-Shipley also identify the indecomposable elements for the $\square_{\Lambda_k(y)}$-algebra $\Lambda_k(y) \otimes k[w]$ as those of the form $\Lambda_k(y) \otimes w$ since the indecomposable elements of $k[w_1, w_2, \dots]$ more generally are w_i [5]. Thus the only terms in the spectral sequence that are possible sources of differentials are y and yw, since yw^j is decomposable for $j > 1$. Because the possible targets are of the form w^{p^m} and yw^{p^m}, and the yw^{p^m} appear in the same diagonal as both y and yw, those elements cannot be hit by any $(r, r-1)$-bidegree differential. Thus we need only justify why differentials from y and yw cannot hit terms of the form w^{p^m}.

Note that the elements we are considering live in the following bidegrees ($\|-\|$) and $(t-s)$ total degrees ($|-|$) for $m, n \geq 1$:

$$\begin{aligned}
\|y\| &= (0, 2n+1) && \text{(since } |y| = 2n+1 \text{ is odd and } > 1)\\
\|w^{p^m}\| &= (p^m, p^m(2n+1)) = (p^m, 2np^m + p^m) && \text{(since } \|w\| = (1, 2n+1))\\
\|yw\| &= (1, 4n+2)\\
\|d_r(y)\| &= (r, |y| + r - 1) = (r, 2n+1+r-1) = (r, 2n+r)\\
\|d_r(yw)\| &= (1+r, 4n+2+r-1) = (1+r, 4n+r+1)
\end{aligned}$$

First, we will justify that $\|d_r(y)\| \neq \|w^{p^m}\|$ for any $m \geq 1$. Suppose by contradiction that these terms were in the same bidegrees. Then the first coordinate tells us that $r = p^m$, so we have from the second coordinate:

$$\begin{aligned} 2n + p^m &= 2np^m + p^m \\ 2n &= 2np^m \\ 1 &= p^m && \text{(since } n \geq 1) \end{aligned}$$

but p is prime and $m \geq 1$, so we have a contradiction.

Second we justify that $\|d_r(yw)\| \neq \|w^{p^m}\|$ for any $m \geq 1$. Suppose by contradiction that these terms were in the same bidegrees. Then the first coordinate tells us that $r + 1 = p^m$, so we have from the second coordinate:

$$\begin{aligned} 4n + p^m &= 2np^m + p^m \\ 4n &= 2np^m \\ 2 &= p^m, \end{aligned}$$

which is true only when $m = 1$ and $p = 2$. However, if $m = 1$ then $r + 1 = p^m = 2^1$ implies that $r = 1$ and we are already considering the E_2-page, so no such differential exists.

Now we want to make sure the convergence conditions of Corollary 3.0.5 hold; that is, if for every s there exists some r so that $E_r^{s,s+i} = E_\infty^{s,s+i}$ then the relative coBökstedt spectral sequence converges completely to $\pi_*(\mathrm{coTHH}^{Hk}(C))$. However, because the differentials starting at the E_2-page must be trivial, we satisfy this condition for convergence, which yields:

$$E_2 \cong E_\infty \cong \Lambda_k(y) \otimes k[w]$$

and so we have an isomorphism with $\pi_*(\mathrm{coTHH}^{Hk}(C))$ as graded k-modules.

□

Next we consider the computation when we increase the number of cogenerators. Bohmann-Gerhardt-Shipley identifies the indecomposable elements for this setting more generally:

Proposition 5.3.2 ([5])

The indecomposable elements in the $\square_{\Lambda_k(y_1,y_2,\ldots,y_n)}$-algebra

$$\mathrm{coHH}(\Lambda_k(y_1,y_2,\ldots,y_n)) \cong \Lambda_k(y_1,y_2,\ldots,y_n) \otimes k[w_1,w_2,\ldots,w_n]$$

are given by $\Lambda_k(y_1,y_2,\ldots,y_n) \otimes w_i$.

Theorem 5.3.3

Let k be a field and let $p = char(k)$, including 0. For C a cocommutative Hk-coalgebra spectrum that is cofibrant as an Hk-module with $\pi_*(C) \cong \Lambda_k(y_1,y_2)$ for $|y_1|,|y_2|$ both odd and greater than 1, if p^m is not equal to $\frac{|y_2|-1}{|y_1|-1}$ or $\frac{|y_2|-1}{|y_1|-1}+1$ for all $m \geq 0$, then the relative coBökstedt spectral sequence collapses and

$$\pi_*(\mathrm{coTHH}^{Hk}(C)) \cong \Lambda_k(y_1,y_2) \otimes k[w_1,w_2],$$

as graded k-modules for $|w_i| = |y_i| - 1$.

Proof. Suppose $|y_1| = a$ and $|y_2| = b$ so that on the E_2-page of the spectral sequence y_1 appears in bidegree $(0,a)$, and y_2 appears in bidegree $(0,b)$, which implies $||w_1|| = (1,a), ||w_2|| = (1,b)$. Then we assume WLOG that $b \geq a$ and we will determine if there is the possibility for differentials by examining the degrees of the terms in the spectral sequence. We will refer to our assumptions that p^m is not equal to $\frac{|y_2|-1}{|y_1|-1}$ as **condition 1** and not equal to $\frac{|y_2|-1}{|y_1|-1}+1$ as **condition 2**.

Note that because of the $\square$-coalgebra structure from Proposition 4.3.9 the shortest nontrivial differential has to hit a coalgebra primitive. If $char(k) = p$ a prime, then by Proposition 5.2.1 coalgebra primitives will be of the form

$$\Lambda_k(y_1, y_2) \otimes w_i^{p^m}$$

since the primitives in $k[w_1, w_2]$ are of the form $w_1^{p^m}$ or $w_2^{p^n}$. However, by Theorem 5.2.2 the relative coBökstedt spectral sequence in this setting also has a coalgebra structure over k. Therefore the first nontrivial differential has to hit a k-coalgebra primitive, that is only classes of the form y_i or $w_i^{p^m}$ (and not any of their tensored combinations). Since the y_is appear in the zero column, they cannot be hit by any differentials, so our only possible targets are classes $w_1^{p^m}$ or $w_2^{p^n}$. Similarly, if $char(k) = 0$ then the only primitives in $k[w_1, w_2]$ are w_1 and w_2.

Further, the source of the shortest nontrivial differential must be a $\square$-algebra indecomposable, which by Proposition 5.3.2 will be of the form $\Lambda_k(y_1, y_2) \otimes w_i$. Thus we only consider differentials from the following sources that land in bidegrees:

$$\begin{aligned}
||d_r(y_1)|| &= (r, a + r - 1)\\
||d_r(y_2)|| &= (r, b + r - 1)\\
||d_r(y_1 y_2)|| &= (r, a + b + r - 1)\\
||d_r(w_1)|| &= (1 + r, a + r - 1)\\
||d_r(w_2)|| &= (1 + r, b + r - 1)\\
||d_r(y_1 w_1)|| &= (1 + r, 2a + r - 1)\\
||d_r(y_2 w_1)|| &= (1 + r, a + b + r - 1)\\
||d_r(y_1 w_2)|| &= (1 + r, a + b + r - 1)\\
||d_r(y_2 w_2)|| &= (1 + r, 2b + r - 1)\\
||d_r(y_1 y_2 w_1)|| &= (1 + r, 2a + b + r - 1)\\
||d_r(y_1 y_2 w_2)|| &= (1 + r, a + 2b + r - 1)
\end{aligned}$$

The primitive elements that could serve as possible targets live in bidegrees:

$$||w_1^{p^m}|| = (p^m, ap^m)$$
$$||w_2^{p^m}|| = (p^m, bp^m)$$

Note that if there is a nonzero differential hitting one of these classes, comparing the degree of the first coordinate implies information about either r or $1+r$. In the $char(k) = 0$ case, $|w_1| = (1, a)$ and $|w_2| = (1, b)$ imply that no nontrivial differentials exist since we are already on the E_2-page. Thus we assume $char(k) = p$ is prime so that the first coordinate implies $r = p^m$ or $1 + r = p^m$, which we will use to simplify the second coordinate of the bidegree.

Suppose $d_r(y_1)$ hits a class $w_1^{p^m}$. Then by comparing degrees:

$$a + p^m - 1 = ap^m.$$

This gives either that $a = 1$ (except we're assuming $|y_i| > 1$) or that $m = 0$ (then a could be anything), but in that case $r = p^m = 1$, and we're already on the E_2-page. Thus there is no such possible differential. A similar argument can be used to justify why $d_r(y_2) \neq w_2^{p^m}$.

Suppose $d_r(y_1)$ hits a class $w_2^{p^m}$. Then by comparing degrees:

$$a + p^m - 1 = bp^m.$$

So $\frac{a-1}{b-1} = p^m$, but we assumed that $b \geq a$, so this equality only holds if $p^m = r = 1$. But we are considering the E_2-page, so no such differential exists. A similar justification regarding

r determines that $d_r(y_1w_1) \neq w_1^{p^m}$ and $d_r(y_2w_2) \neq w_2^{p^m}$.

Suppose $d_r(y_2)$ hits a class $w_1^{p^m}$. Then by comparing degrees:

$$b + p^m - 1 = ap^m$$

so $p^m = \frac{b-1}{a-1}$. Now we assumed in **condition 1** that $\frac{|y_2|-1}{|y_1|-1} \neq p^m$, so no such differential exists.

Suppose $d_r(y_1y_2)$ hits a class $w_1^{p^m}$. Then

$$a + b + p^m - 1 = ap^m,$$

so $b = (p^m - 1)(a - 1)$, but b is odd and $a - 1$ is even and we can't have equality due to the parity issue, so there are no such possible differentials. Similar parity issues arise to show $d_r(y_1y_2) \neq w_2^{p^m}$, as well as for $d_r(w_1) \neq w_1^{p^m}$ or $w_2^{p^m}$, $d_r(w_2) \neq w_1^{p^m}$ or $w_2^{p^m}$, $d_r(y_1y_2w_1) \neq w_1^{p^m}$ or $w_2^{p^m}$, and $d_r(y_1y_2w_2) \neq w_1^{p^m}$ or $w_2^{p^m}$.

Now suppose $d_r(y_1w_1)$ hits a class $w_2^{p^m}$. Then the first coordinate implies that $r+1 = p^m$, so the second coordinate gives:

$$2a + r - 1 = b(1 + r),$$

so $2\frac{a-1}{b-1} = r + 1$. But $a \leq b$ so $2\frac{a-1}{b-1} \leq 2(1) < 3 \leq r + 1$ since $r \geq 2$ and so no such differential exists. Similar justifications based on the assumption that $a \leq b$ allow us to conclude that $d_r(y_2w_1) \neq w_2^{p^m}$ and $d_r(y_1w_2) \neq w_2^{p^m}$.

Suppose $d_r(y_2w_1)$ hits a class $w_1^{p^m}$. Then

$$a + b + p^m - 2 = ap^m,$$

so $\frac{b-1}{a-1} = p^m - 1$. However we assumed in **condition 2** that p^m cannot be equal to $\frac{|y_2|-1}{|y_1|-1} + 1$, so no such differential exists. This condition also arises in the case $d_r(y_1 w_2) \neq w_1^{p^m}$.

Finally suppose $d_r(y_2 w_2)$ hits a class $w_1^{p^m}$. Then

$$2b + p^m - 2 = ap^m,$$

so $2\frac{b-1}{a-1} = p^m$. However, we claim that the assumption $p^m \neq 2\frac{|y_2|-1}{|y_1|-1}$ is already eliminated by the existing conditions. First, if $m = 0$ then $p^m = 1$ and this assumption does not apply since we assumed above that $|y_2| \geq |y_1|$. Therefore, we need only justify that $p^m \neq 2\frac{|y_2|-1}{|y_1|-1}$ for $m \geq 1$. If p is odd, an odd prime p to any power will still be odd and so $p^m \neq 2\frac{|y_2|-1}{|y_1|-1}$ due to parity.

If $p = 2$, consider the case where $\frac{|y_2|-1}{|y_1|-1}$ is odd. Then $2\frac{|y_2|-1}{|y_1|-1}$ will only be equal to a power of $p = 2$ if the power is 1. But $m = 1$ would imply here that $r = 1$ and we are already on the E_2-page. If $\frac{|y_2|-1}{|y_1|-1}$ is even, then verifying that $p^m \neq 2\frac{|y_2|-1}{|y_1|-1}$ for $m \geq 1$ is equivalent to checking that $p^n \neq \frac{|y_2|-1}{|y_1|-1}$ for $n \geq 0$, i.e. **condition 1**. So, no such differential from $y_2 w_2$ to $w_1^{p^m}$ exists if **condition 1** is satisfied.

We have now justified via combinatorics why all possible differentials can be eliminated, whether that is for parity reasons, because we're already on the E_2-page, or because we restricted values of p^m based on the conditions listed in the hypotheses. Thus the spectral sequence collapses, and the convergence conditions of Corollary 3.0.5 hold so we have the desired result. □

Remark 5.3.4

Note that the conditions on p^m allow us to avoid cases like $|y_1| = 3, |y_2| = 5$, which has a possible d_2 differential from y_2 to w_1^2 for the prime $p = 2$ (which is in this case is eliminated by **condition 1**).

5.4 Divided Power Input

Along with proving the existence of the coBökstedt spectral sequence in Bohmann-Gerhardt-Høgenhaven-Shipley-Ziegenhagen [4], they show the following significant computational result:

Theorem 5.4.1 ([4] 5.4)

Let C be a cocommutative coassociative coalgebra spectrum that is cofibrant as a spectrum, and whose homology coalgebra is

$$H_*(C;k) = \Gamma_k[x_1, x_2, \ldots],$$

where the x_i are cogenerators in non-negative even degrees and there are only finitely many cogenerators in each degree. Then the coBökstedt spectral sequence for C collapses at E_2, and

$$E_2 \cong E_\infty \cong \Gamma_k[x_1, x_2 \ldots] \otimes \Lambda_k(z_1, z_2, \ldots)$$

with x_i in degree $(0, |x_i|)$ and z_i in degree $(1, |x_i|)$.

Now we would like to have an analogous result for the relative coBökstedt spectral sequence for the case when $E = \mathbb{S}$ and $R = Hk$ for a field k. Recall that these restrictions allow us to use the $\square$-Hopf algebra structure of the spectral sequence to eliminate certain possible differentials.

Theorem 5.4.2

Let C be a cocommutative coassociative Hk-coalgebra spectrum that is cofibrant as an

Hk-module spectrum, and whose homotopy coalgebra is

$$\pi_*(C) = \Gamma_k[x_1, x_2, \dots],$$

where the x_i are in non-negative even degrees and there are only finitely many of them in each degree. Then the relative coBökstedt spectral sequence calculating the homotopy groups of the topological coHochschild homology of C collapses at E_2, and

$$\pi_*(\mathrm{coTHH}^{Hk}(C)) \cong \Gamma_k[x_1, x_2, \dots] \otimes \Lambda_k(z_1, z_2, \dots)$$

as k-modules, with z_i in degree $|x_i| - 1$.

Proof. Since $E_*(C) = \pi_*(C) = \Gamma_k[x_1, x_2, \dots]$ is flat over $E_*(R) = \pi_*(Hk) \cong k$, the relative coBökstedt spectral sequence that abuts to $\pi_{t-s}(\mathrm{coTHH}^{Hk}(C))$ has E_2-page

$$E_2^{s,t} = \mathrm{coHH}^k_{s,t}(\Gamma_k[x_1, x_2, \dots])$$

By Proposition 5.1 in [4],

$$\mathrm{coHH}^k_{*,*}(\Gamma_k[x_1, x_2, \dots]) \cong \Gamma_k[x_1, x_2, \dots] \otimes \Lambda_k(z_1, z_2, \dots),$$

where $||z_i|| = (1, |x_i|)$. Now we want to examine the differentials on this E_2-page of our spectral sequence. In particular, Theorem 4.3.9 says that the coalgebra structure implies that the shortest nonzero differential has to hit a $\square$-coalgebra primitive. Since $\mathrm{coHH}_*(\pi_*(C))$ is a $\square$-coalgebra over $\pi_*(C) = \Gamma_k[x_1, x_2, \dots]$, we know by Proposition 5.2.1 that the primitive elements will be of the form

$$\Gamma_k[x_1, x_2, \ldots] \otimes \big(\text{primitives in } \Lambda_k(z_1, z_2, \ldots)\big),$$

where the primitives in $\Lambda_k(z_1, z_2, \ldots)$ viewed as a k-coalgebra are of the form z_i.

Note that since all of the x_i's appear in degree $(0, |x_i|)$, all x_i's and all the divided powers will stay in the zero column. Similarly, the exterior cogenerator z_i is in degree $(1, |x_i|)$, and so all possible targets, i.e. combinations of x_i's with a single z_i, will be in the first column. Because we are on the E_2-page, the differentials of bidegree $(2, 1)$ will be mapping outside of these two columns, as will all possible d_r differentials on later E_r-pages. Thus beyond the zero and first columns, the only elements that may be hit by differentials are those that include at least $z_i z_j$. However, as we said above, such elements are not primitive, and the shortest non-zero differential $d_r^{s,t}$ in lowest total degree $s + t$ has to hit a $\square_{\pi_*(C)}$-coalgebra primitive. Therefore, our spectral sequence collapses at E_2.

Now we want to make sure the convergence conditions of Corollary 3.0.5 hold; that is, if for every s there exists some r so that $E_r^{s,s+i} = E_\infty^{s,s+i}$ then the relative coBökstedt spectral sequence converges completely to $\pi_*(\mathrm{coTHH}^{Hk}(C))$. However, because the differentials starting at the E_2-page must be trivial, we satisfy this condition for convergence, and so we have the following isomorphism of k-modules:

$$\pi_*(\mathrm{coTHH}^{Hk}(C)) \cong \Gamma_k[x_1, x_2, \ldots] \otimes \Lambda_k(z_1, z_2, \ldots).$$

. $\square$

Chapter 6

Shadows

In a symmetric monoidal category $(\mathscr{C}, \otimes, 1)$, an object C is called *dualizable* with dual $D \in \mathscr{C}$ if there is a coevaluation map $\eta : 1 \to C \otimes D$ and evaluation map $\epsilon : D \otimes C \to 1$ that satisfy the triangle identities:

$$(\mathrm{Id}_C \otimes \epsilon) \circ (\eta \otimes \mathrm{Id}_C) : C \to C \otimes D \otimes C \to C = \mathrm{Id}_C$$
$$(\epsilon \otimes \mathrm{Id}_D) \circ (\mathrm{Id}_D \otimes \eta) : D \to D \otimes C \otimes D \to D = \mathrm{Id}_D$$

Using this structure, one can define the trace of a map $f : C \to C$ as

$$1 \xrightarrow{\eta} C \otimes D \xrightarrow{f \otimes \mathrm{Id}} C \otimes D \cong D \otimes C \xrightarrow{\epsilon} 1.$$

Observe that the symmetric monoidal setting critically provides the symmetry isomorphism $C \otimes D \cong D \otimes C$. One might want to extend the notion of trace to bicategories. For two objects C and D in a bicategory, there is a horizontal composition $C \odot D$. However, one would not expect to have a symmetry isomorphism relating $C \odot D$ and $D \odot C$. Indeed, $C \odot D$ and $D \odot C$ may not even live in the same category.

Work of Ponto [31] and Ponto-Shulman [32] develops a notion of a bicategorical shadow to address this issue. More recently, work of Campbell-Ponto [11] used this framework to show that THH is a shadow. In this chapter we will show that coHochschild homology (coHH) is also a shadow. Note in particular that once we have the structure of a shadow, other properties such as Morita invariance follow as a consequence. Some of these properties were already shown via other methods, but the framework of shadows gives us another perspective.

6.1 (Co)Bar Constructions

Because we will need bar and cobar constructions to give examples of shadows in this chapter, we state those definitions here.

Definition 6.1.1

Let k be a commutative ring, A a k-algebra, M a right A-module, and N a left A-module. Define a simplicial k-module

$$\begin{array}{c} \vdots \\ M \otimes_k A \otimes_k A \otimes_k N \\ \downarrow\uparrow\downarrow\uparrow\downarrow \\ M \otimes_k A \otimes_k N \\ \downarrow\uparrow\downarrow \\ M \otimes_k N \end{array}$$

with face maps given by

$$d_i(m \otimes a_1 \otimes \ldots \otimes a_r \otimes n) = \begin{cases} ma_1 \otimes \ldots \otimes a_r \otimes n & i = 0 \\ m \otimes a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_r \otimes n & 1 \leq i < r \\ m \otimes a_1 \otimes \ldots \otimes a_{r-1} \otimes a_r n & i = r, \end{cases}$$

and degeneracy maps that insert the unit map:

$$s_i(m \otimes a_1 \otimes \cdots \otimes a_r \otimes n) = \begin{cases} m \otimes 1 \otimes a_1 \otimes \cdots \otimes a_r \otimes n & i = 0 \\ m \otimes a_1 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_r \otimes n & 1 \leq i \leq r. \end{cases}$$

Then the **two-sided bar complex** $Bar_\bullet(M, A, N)$ is given by the simplicial k-module defined above. Further one can form a chain complex of k-modules via the boundary map $d = \Sigma_i(-1)^i d_i$ to create the **two-sided bar construction** $Bar(M, A, N)$.

Definition 6.1.2

Let R be a commutative ring spectrum, A an R-algebra, M a right A-module with structure map $\gamma : M \wedge A \to M$, and N a left A-module with $\psi : A \wedge N \to N$, where all $\wedge$ in this definition are over R. Define a simplicial R-module

$$\begin{array}{c} \vdots \\ M \wedge A \wedge A \wedge N \\ \downarrow\uparrow\downarrow\uparrow\downarrow \\ M \wedge A \wedge N \\ \downarrow\uparrow\downarrow \\ M \wedge N \end{array}$$

with face maps given by

$$d_i = \begin{cases} \gamma \wedge \mathrm{Id}^{\wedge r} & i = 0 \\ \mathrm{Id}^{\wedge i} \wedge \mu \wedge \mathrm{Id}^{\wedge r-i} & 1 \leq i < r \\ \mathrm{Id}^{\wedge r} \wedge \psi & i = r, \end{cases}$$

and degeneracy maps that insert the unit map:

$$s_i = \mathrm{Id}^{i+1} \wedge \eta \wedge \mathrm{Id}^{r-i+1},$$

for $0 \leq i \leq r$. Then the **two-sided bar complex** $Bar_\bullet(M, A, N)$ is given by the simplicial

R-module defined above, which we can then geometrically realize to get the **two-sided bar construction** $Bar(M, A, N)$.

Definition 6.1.3

Let k be a commutative ring, C a k-coalgebra, M a right C-comodule with right coaction $\gamma : M \to M \otimes_k C$, and N a left C-comodule with left coaction $\psi : N \to C \otimes_k N$. Define a cosimplicial k-comodule

$$\begin{array}{c} \vdots \\ M \otimes_k C \otimes_k C \otimes_k N \\ \uparrow\downarrow \uparrow\downarrow\uparrow \\ M \otimes_k C \otimes_k N \\ \uparrow \downarrow\uparrow \\ M \otimes_k N \end{array}$$

with coface maps given by

$$\delta_i = \begin{cases} \gamma \otimes \mathrm{Id}^{\otimes r+1} & i = 0 \\ \mathrm{Id}^{\otimes i} \otimes \Delta \otimes \mathrm{Id}^{\otimes (r-i+1)} & 1 \leq i \leq r \\ \mathrm{Id}^{\otimes r+1} \otimes \psi & i = r+1 \end{cases}$$

and codegeneracy maps that insert the counit map for $0 \leq i \leq r-1$:

$$\sigma_i = \mathrm{Id}^{\otimes (i+1)} \otimes \epsilon \otimes \mathrm{Id}^{\otimes r-i}.$$

We denote the cosimplicial **two-sided cobar complex** by $coBar^{\bullet}(M, C, N)$. One can form a cochain complex of k-comodules via the boundary map $\delta = \Sigma_i (-1)^i \delta_i$ to create the **two-sided cobar construction** $coBar(M, C, N)$.

6.2 Shadow Background

We first recall some basic definitions.

Definition 6.2.1 ([32, 11])

A **bicategory** $\mathscr{B}$ consists of

- a collection objects, $ob(\mathscr{B})$, called 0-cells
- categories $\mathscr{B}(R,T)$ for each pair $R,T \in ob(\mathscr{B})$. The objects in these categories are referred to as 1-cells and the morphisms as 2-cells.
- unit functors $U_R \in ob(\mathscr{B}(R,R))$ for all $R \in ob(\mathscr{B})$
- horizontal composition functors for $R,T,V \in ob(\mathscr{B})$

$$\odot : \mathscr{B}(R,T) \times \mathscr{B}(T,V) \to \mathscr{B}(R,V)$$

  which are not required to be strictly associative or unital.

- natural isomorphisms for $M \in ob(\mathscr{B}(R,T))$, $N \in ob(\mathscr{B}(T,V))$, and $P \in ob(\mathscr{B}(V,W))$, $Q \in ob(\mathscr{B}(W,X))$ for $R,T,V,W,X \in ob(\mathscr{B})$

$$\begin{aligned} a : (M \odot N) \odot P &\xrightarrow{\cong} M \odot (N \odot P) \\ l : U_R \odot M &\xrightarrow{\cong} M \\ r : M \odot U_T &\xrightarrow{\cong} M \end{aligned}$$

  that satisfy the monoidal category coherence axioms (triangle identity and pentagon identity):

$$
\begin{array}{ccc}
(M \odot U_T) \odot N & \xrightarrow{\quad a \quad} & M \odot (U_T \odot N) \\
& {\scriptstyle r\odot \mathrm{Id}} \searrow \quad \swarrow {\scriptstyle \mathrm{Id}\odot l} & \\
& M \odot N &
\end{array}
$$

$$
\begin{array}{ccccc}
& & (M \odot N) \odot (P \odot Q) & & \\
& \overset{a}{\nearrow} & & \overset{a}{\searrow} & \\
((M \odot N) \odot P) \odot Q & & & & M \odot (N \odot (P \odot Q)) \\
\downarrow {\scriptstyle a\odot \mathrm{Id}} & & & & {\scriptstyle \mathrm{Id}\odot a} \uparrow \\
(M \odot (N \odot P)) \odot Q & & \xrightarrow{\qquad a \qquad} & & M \odot ((N \odot P) \odot Q)
\end{array}
$$

Example 6.2.2

The bicategory $Mod/Ring$ whose 0-cells are rings, and $Mod/Ring(R,T) =_R Mod_T$ is the category of (R,T)-bimodules for rings R,T. The unit U_R is the (R,R)-bimodule R, and horizontal composition is given by the tensor product of bimodules

$$
\begin{aligned}
\odot : \mathscr{B}(R,T) \times \mathscr{B}(T,V) &\to \mathscr{B}(R,V) \\
(M,N) &\mapsto M \odot N := M \otimes_T N
\end{aligned}
$$

Example 6.2.3

The bicategory $\mathscr{D}(Ch/Ring)$ has 0-cells that are rings and $\mathscr{D}(Ch/Ring)(R,T) = \mathscr{D}({}_R Mod_T)$ is the derived category of (R,T)-bimodules. The unit U_R is the (R,R)-bimodule R viewed as a chain complex, and horizontal composition is given by the derived tensor product $\otimes^L$:

$$
\begin{aligned}
\odot : \mathscr{B}(R,T) \times \mathscr{B}(T,V) &\to \mathscr{B}(R,V) \\
(M,N) &\mapsto M \odot N := M \otimes_T^L N
\end{aligned}
$$

Note that $Bar_\bullet(M,T,N) \simeq M \otimes_T^L N$ viewed as a trivial simplicial object via the isomorphism that in degree j multiplies together all the factors of T. Thus we may also consider this horizontal composition as the two-sided bar construction.

Example 6.2.4

Let $\mathscr{D}(Mod/Ring\ Spectra)$ denote the bicategory whose 0-cells are ring spectra, and for ring spectra R,T $\mathscr{D}(Mod/Ring\ Spectra)(R,T)$ is the homotopy category of (R,T)-bimodules. The unit U_R is the (R,R)-bimodule spectrum R, and horizontal composition is given by the derived smash product $\wedge^L$ of spectra

$$\begin{aligned} \odot : \mathscr{B}(R,T) \times \mathscr{B}(T,V) &\to \mathscr{B}(R,V) \\ (M,N) &\mapsto M \odot N := M \wedge_T^L N \end{aligned}$$

Note that as in the previous example $M \wedge_T^L N \simeq Bar(M,T,N)$ [15, Prop IV.7.5], and so we may also consider this horizontal composition as the two-sided bar construction.

Now that we have the underlying bicategorical structure, we will define a shadow on that bicategory:

Definition 6.2.5 ([31, 32])

A **shadow functor** for a bicategory $\mathscr{B}$ consists of functors

$$\langle\langle - \rangle\rangle_C : \mathscr{B}(C,C) \to \mathbf{T}$$

for every $C \in ob(\mathscr{B})$ and some fixed category $\mathbf{T}$ equipped with a natural isomorphism for $M \in \mathscr{B}(C,D)$, $N \in \mathscr{B}(D,C)$

$$\theta : \langle\langle M \odot N \rangle\rangle_C \xrightarrow{\cong} \langle\langle N \odot M \rangle\rangle_D.$$

For $P \in \mathscr{B}(C,C)$, these functors must satisfy the following commutative diagrams (when they make sense):

$$\begin{array}{ccccc}
\langle\langle (M \odot N) \odot P \rangle\rangle_C & \xrightarrow{\theta} & \langle\langle P \odot (M \odot N) \rangle\rangle_C & \xrightarrow{\langle\langle a \rangle\rangle} & \langle\langle (P \odot M) \odot N \rangle\rangle_C \\
\downarrow{\scriptstyle \langle\langle a \rangle\rangle} & & & & \uparrow{\scriptstyle \theta} \\
\langle\langle M \odot (N \odot P) \rangle\rangle_C & \xrightarrow{\theta} & \langle\langle (N \odot P) \odot M \rangle\rangle_D & \xrightarrow{\langle\langle a \rangle\rangle} & \langle\langle N \odot (P \odot M) \rangle\rangle_D
\end{array}$$

$$\begin{array}{ccccc}
\langle\langle P \odot U_C \rangle\rangle_C & \xrightarrow{\theta} & \langle\langle U_C \odot P \rangle\rangle_C & \xrightarrow{\theta} & \langle\langle P \odot U_C \rangle\rangle_C \\
& {\scriptstyle \langle\langle r \rangle\rangle} \searrow & \downarrow{\scriptstyle \langle\langle l \rangle\rangle} & \swarrow {\scriptstyle \langle\langle r \rangle\rangle} & \\
& & \langle\langle P \rangle\rangle_C & &
\end{array}$$

We can now consider shadows for the bicategories that we introduced earlier.

Example 6.2.6

The "underived version" of Hochschild homology (or $\mathrm{HH}_0(R;M)$) is a shadow on the bicategory $Mod/Ring$ [32]. Recall that $Mod/Ring(R,R) = {}_R Mod_R$, so let R be a ring and M be an (R,R)-bimodule to define

$$\begin{aligned}
\langle\langle - \rangle\rangle_R : {}_R Mod_R &\to \mathscr{A}b \\
M &\mapsto R \otimes_{R \otimes R^{op}} M \cong \mathrm{HH}_0(R;M)
\end{aligned}$$

where $\mathscr{A}b$ is the category of abelian groups and the isomorphism above follows since

$$\begin{aligned}
R \otimes_{R \otimes R^{op}} M &= H_0(R \otimes^L_{R \otimes R^{op}} M) \\
&= Tor_0^{R \otimes R^{op}}(R, M) \\
&= \mathrm{HH}_0(R;M)
\end{aligned}$$

Equivalently we could define this shadow of the (R,R)-bimodule M to be the coequalizer of

$$R \otimes M \underset{\psi}{\overset{\gamma}{\rightrightarrows}} M \longrightarrow \langle\langle M \rangle\rangle,$$

where γ and ψ are the right and left module actions respectively.

The main property of shadows that we want to justify is that for an (R,T)-bimodule M

and an (T, R)-bimodule N, there is an isomorphism

$$\theta : \langle\langle M \odot N \rangle\rangle_R \xrightarrow{\cong} \langle\langle N \odot M \rangle\rangle_T$$

Unpacking this we see

$$\begin{aligned}
\langle\langle M \odot N \rangle\rangle_R &:= \langle\langle M \otimes_T N \rangle\rangle_R = R \otimes_{R \otimes R^{op}} (M \otimes_T N) \\
&\cong \mathrm{HH}_0(R; M \otimes_T N) \\
\langle\langle N \odot M \rangle\rangle_T &:= \langle\langle N \otimes_R M \rangle\rangle_T = T \otimes_{T \otimes T^{op}} (N \otimes_R M) \\
&\cong \mathrm{HH}_0(T; N \otimes_R M)
\end{aligned}$$

So justifying that there is such an isomorphism θ comes down to comparing $M \otimes_T N$ quotiented by the action of $R \otimes R^{op}$ and $N \otimes_R M$ quotiented by the action of $T \otimes T^{op}$.

Recall the 0^{th} Hochschild homology of a k-algebra A with coefficients in an (A, A)-bimodule B is given by:

$$\mathrm{HH}_0(A; B) \cong B \big/ < ab - ba > .$$

So to define θ above, we need a map

$$\theta : M \otimes_T N \big/ < rm \otimes n - m \otimes nr > \longrightarrow N \otimes_R M \big/ < tn \otimes m - n \otimes mt > .$$

We would like to define θ as the map that swaps the tensor factors $m \otimes n \mapsto n \otimes m$. However, in order for this map to be a well-defined map, we need to verify that $rm \otimes n - m \otimes nr$ maps to 0. But because of the universal property of $\otimes_R$, we can bring r through the tensor so that θ takes

$$rm \otimes_T n - m \otimes_T nr \longmapsto n \otimes_R rm - nr \otimes_R m = n \otimes_R rm - n \otimes_R rm = 0.$$

Therefore we have defined θ in one direction. A similar argument justifies that its inverse sends $n \otimes m \mapsto m \otimes n$, and together these give the desired isomorphism:

$$\theta : \mathrm{HH}_0(R; M \otimes_T N) \xrightarrow{\cong} \mathrm{HH}_0(T; N \otimes_R M).$$

We have seen above that $\mathrm{HH}_0(R;-)$ is a shadow. Now we will see that Hochschild homology is as well.

Example 6.2.7

Hochschild homology is a shadow on the category $\mathscr{D}(Ch/Ring)$. So let R be a ring and M be a chain complex of (R,R)-bimodules to define

$$\begin{aligned} \langle\langle - \rangle\rangle_R : \mathscr{D}(Ch/Ring)(R,R) &\to \mathscr{D}(Ch_{\mathbb{Z}}) \\ M &\mapsto R \otimes^L_{R\otimes R^{op}} M \cong \mathrm{HH}(R,M) \end{aligned}$$

where $Ch_{\mathbb{Z}}$ is chain complexes of abelian groups and $\mathrm{HH}(R,M)$ denotes the complex whose homology gives Hochschild homology. The isomorphism above follows since

$$\begin{aligned} H_i(R \otimes^L_{R\otimes R^{op}} M) &= Tor_i^{R\otimes R^{op}}(R,M) \\ &= \mathrm{HH}_i(R;M) \end{aligned}$$

Again, the argument amounts to justifying that for M a chain complex of (R,T)-bimodules and N a chain complex of (T,R)-bimodules, there is an isomorphism

$$\theta : \langle\langle M \odot N \rangle\rangle_R \xrightarrow{\cong} \langle\langle N \odot M \rangle\rangle_T$$

Unpacking this and using the fact that $M \otimes^L_T N \simeq Bar(M, T, N)$ we see

$$\begin{aligned}
\langle\langle M \odot N \rangle\rangle_R &:= \langle\langle M \otimes^L_T N \rangle\rangle_R = R \otimes^L_{R \otimes R^{op}} (M \otimes^L_T N) \\
&\cong \mathrm{HH}(R, M \otimes^L_T N) \cong \mathrm{HH}(R, Bar(M, T, N)) \\
\langle\langle N \odot M \rangle\rangle_T &:= \langle\langle N \otimes^L_R M \rangle\rangle_T = T \otimes^L_{T \otimes T^{op}} (N \otimes^L_R M) \\
&\cong \mathrm{HH}(T, N \otimes^L_R M) \cong \mathrm{HH}(T, Bar(N, R, M))
\end{aligned}$$

So justifying that there is such an isomorphism θ amounts to constructing an isomorphism:

$$\theta : \mathrm{HH}(R, Bar(M, T, N)) \xrightarrow{\cong} \mathrm{HH}(T, Bar(N, R, M)).$$

Recall that Hochschild homology is calculated using a cyclic bar construction, and applying the Dold-Kan correspondence between chain complexes and simplicial k-modules allows us to identify both of the above bisimplicial chain complexes, $\mathrm{HH}_\bullet(R; Bar_\bullet(M, T, N))$ and $\mathrm{HH}_\bullet(T; Bar_\bullet(N, R, M))$ with the bisimplicial object $H_{\bullet\bullet}$ that at the (i, j)-spot is given by:

$$\begin{array}{ccccc}
 & & \underbrace{R \otimes R \otimes \ldots \otimes R}_{\text{i}} & & \\
 & \otimes & & \otimes & \\
N & & & & M \\
 & \otimes & & \otimes & \\
 & & \overbrace{T \otimes T \otimes \ldots \otimes T}^{\text{j}} & &
\end{array}$$

where the face maps are given by multiplication of adjacent terms. Then the map θ is given degree-wise by:

$$\begin{aligned}
\theta : M \otimes T \otimes \cdots \otimes T \otimes N \otimes R \otimes \cdots \otimes R &\longrightarrow N \otimes R \otimes \cdots \otimes R \otimes M \otimes T \otimes \cdots \otimes T \\
m \otimes t_1 \otimes \cdots t_j \otimes n \otimes r_1 \otimes \cdots \otimes r_i &\mapsto \pm n \otimes r_1 \otimes \cdots \otimes r_i \otimes m \otimes t_1 \otimes \cdots \otimes t_j,
\end{aligned}$$

where the $\pm$ is determined by the Koszul sign convention. This map is defined because of the (R,T)-bimodule structure on M and the (T,R)-bimodule structure on N. This bisimplicial identification then gives an equivalence of complexes and thus defines the desired isomorphism θ. This approach is often referred to as a Dennis-Waldhausen Morita Argument [36, 3].

Example 6.2.8 ([11])

Topological Hochschild homology is a shadow on the bicategory $\mathscr{D}(Mod/Ring\ Spectra)$. Let R be a ring spectrum and M an (R,R)-bimodule spectrum, and define

$$\begin{aligned} \langle\langle - \rangle\rangle_R : \mathrm{Ho}(Mod_{(R,R)}) &\to \mathrm{Ho}(\mathrm{Sp}) \\ M &\mapsto \mathrm{THH}(R,M) \end{aligned}$$

where Ho(Sp) is the homotopy category of spectra.

Again, the argument amounts to justifying that for an (R,T)-bimodule M and a (T,R)-bimodule N, there is an isomorphism

$$\theta : \langle\langle M \odot N \rangle\rangle_R \xrightarrow{\cong} \langle\langle N \odot M \rangle\rangle_T$$

Unpacking this as before shows that justifying that θ is an isomorphism is equivalent to showing that there is an isomorphism

$$\theta : \mathrm{THH}(R, Bar(M,T,N)) \xrightarrow{\cong} \mathrm{THH}(T, Bar(N,R,M)).$$

As above we may apply the Dennis-Waldhausen Morita Argument to identify both $\mathrm{THH}_\bullet(R, Bar_\bullet(M,T,N))$ and $\mathrm{THH}_\bullet(T, Bar_\bullet(N,R,M))$ with the bisimplicial spectrum that at the (i,j)-spot looks like

$$N \wedge \underbrace{R \wedge R \wedge \ldots \wedge R}_{i} \wedge M.$$
$$N \wedge \overbrace{T \wedge T \wedge \ldots \wedge T}^{j} \wedge M.$$

Then the geometric realization yields the desired equivalence [3].

Now we will see how to extend this work to the dual situation with coalgebra inputs.

6.3 CoHochschild Homology is a Shadow

We want to show that coHochschild homology is also a shadow. We first describe the following bicategory.

Definition 6.3.1

For a field k, the bicategory $CoAlg_k$ has 0-cells that are coalgebras over k, say C, D, and $CoAlg_k(C, D)$ is the category of (C, D)-bicomodules. The unit U_C is the (C, C)-bicomodule C, and horizontal composition is given by the cotensor product $\square$ given by

$$\begin{aligned} \odot : \mathscr{B}(C, D) \times \mathscr{B}(D, E) &\to \mathscr{B}(C, E) \\ (M, N) &\mapsto M \odot N := M \square_D N \end{aligned}$$

For (C, D)-bicomodule M, (D, E)-bicomodule N, and (E, F)-bicomodule P the natural isomorphisms

$$\begin{aligned} a : (M \odot N) \odot P &\xrightarrow{\cong} M \odot (N \odot P) \\ l : U_C \odot M &\xrightarrow{\cong} M \\ r : M \odot U_D &\xrightarrow{\cong} M \end{aligned}$$

follow as in [14] from the natural isomorphisms

$$
\begin{aligned}
(M \,\square_D\, N) \,\square_E\, P &\cong M \,\square_D\, (N \,\square_E\, P) \\
C \,\square_C\, M &\cong M \\
M \,\square_D\, D &\cong M.
\end{aligned}
$$

The maps above are given by:

$$
\begin{aligned}
a : (M \,\square_D\, N) \,\square_E\, P &\cong M \,\square_D\, (N \,\square_E\, P) \\
(m \otimes n) \otimes p &\mapsto m \otimes (n \otimes p)
\end{aligned}
$$

$$
\begin{aligned}
l : C \,\square_C\, M &\cong M \\
\Sigma_i c_i \otimes m_i &\mapsto \Sigma_i \epsilon(c_i) m_i
\end{aligned}
$$

$$
\begin{aligned}
r : M \,\square_D\, D &\cong M \\
\Sigma_i m_i \otimes d_i &\mapsto \Sigma_i m_i \epsilon(d_i)
\end{aligned}
$$

We refer to [10, 11.6] for the proof that the cotensor is associative since the field k ensures flatness.

Theorem 6.3.2

The 0^{th} Hochschild homology, coHH_0, is a shadow on the bicategory $CoAlg_k$. That is, it gives a family of functors

$$
\begin{aligned}
\mathrm{coHH}_0(-, C) : CoMod_{(C,C)} &\to \mathscr{A}b \\
N &\mapsto N \,\square_{C \otimes C^{op}}\, C \cong \mathrm{coHH}_0(N, C)
\end{aligned}
$$

that satisfy the required shadow properties.

Proof. We use the bicategory with 1- and 2-cells from $CoMod_{(C,C)}$, which is the category of (C, C)-bicomodules as defined above. Recall the horizontal composition:

$$\begin{aligned} \odot : \mathscr{B}(C,D) \times \mathscr{B}(D,E) &\to \mathscr{B}(C,E) \\ (M,N) &\mapsto M \odot N := M \,\square_D\, N \end{aligned}$$

Note that since we are working with coalgebras over a field k, the cotensor $M \,\square_D\, N$ is a bicomodule.

We want to define the required functor:

$$\begin{aligned} \langle\langle - \rangle\rangle_C : CoMod_{(C,C)} &\to \mathscr{A}b \\ N &\mapsto N \,\square_{C \otimes C^{op}}\, C \cong \mathrm{coHH}_0(N, C) \end{aligned}$$

where $\mathscr{A}b$ is again the category of abelian groups. The isomorphism above follows from

$$\begin{aligned} N \,\square_{C\otimes C^{op}}\, C &= H^0(N \,\square_{C\otimes C^{op}}\, C) \\ &= CoTor^0_{C\otimes C^{op}}(N, C) \\ &= \mathrm{coHH}_0(N, C) \end{aligned}$$

Now the brunt of what we need to justify to show that coHH_0 is a shadow is that for M a (C,D)-bicomodule and N a (D,C)-bicomodule, we have an isomorphism

$$\theta : \langle\langle M \odot N \rangle\rangle_C \longrightarrow \langle\langle N \odot M \rangle\rangle_D$$

But unpacking our notation gives

$$\begin{aligned} \langle\langle M \odot N \rangle\rangle_C &:= \langle\langle M \,\square_D\, N \rangle\rangle_C = (M \,\square_D\, N) \,\square_{C\otimes C^{op}}\, C \\ &\cong \mathrm{coHH}_0(M \,\square_D\, N, C) \\ \langle\langle N \odot M \rangle\rangle_D &:= \langle\langle N \,\square\, M \rangle\rangle_D = (N \,\square_C\, M) \,\square_{D\otimes D^{op}}\, D \\ &\cong \mathrm{coHH}_0(N \,\square_C\, M, D). \end{aligned}$$

Recall that the 0^{th} coHochschild homology of a k-coalgebra A with coefficients in an (A,A)-bicomodule B with coactions $\gamma : B \to B \otimes A$ and $\psi : B \to A \otimes B$ is given by:

$$\mathrm{coHH}_0(B,A) = \{b \in B \mid \tilde{t}\psi(b) = \gamma(b)\}$$

where $\tilde{t}$ is the twist map [14]. We want to define a map θ

$$\theta : \{m \otimes n \in M \,\square_D\, N \mid \tilde{t}\psi_1(m \otimes n) = \gamma_1(m \otimes n)\} \longrightarrow \{n \otimes m \in N \,\square_C\, M \mid \tilde{t}\psi_2(n \otimes m) = \gamma_2(n \otimes m)\}$$

for the following comodule and structure maps

$$\begin{aligned}
\gamma_1 : M \,\square_D\, N &\to M \,\square_D\, N \otimes C \\
m \otimes n &\mapsto m \otimes \gamma_N(n) \\
\psi_1 : M \,\square_D\, N &\to C \otimes M \,\square_D\, N \\
m \otimes n &\mapsto \psi_M(m) \otimes n \\
\gamma_2 : N \,\square_C\, M &\to N \,\square_C\, M \otimes D \\
n \otimes m &\mapsto n \otimes \gamma_M(m) \\
\psi_2 : N \,\square_C\, M &\to D \otimes N \,\square_C\, M \\
n \otimes m &\mapsto \psi_N(n) \otimes m
\end{aligned}
\qquad\qquad
\begin{aligned}
\gamma_M &: M \to M \otimes D \\
\psi_M &: M \to C \otimes M \\
\gamma_N &: N \to N \otimes C \\
\psi_N &: N \to D \otimes N.
\end{aligned}$$

We would like to define θ as the map that swaps the factors $m \otimes n \mapsto n \otimes m$. In order for this map to be well-defined, we need to verify that $\tilde{t}\psi_2(n \otimes m) = \gamma_2(n \otimes m)$. Note the definition of cotensor

$$M \,\square_D\, N \longrightarrow (M \otimes N) \underset{\mathrm{Id}_M \otimes \psi_N}{\overset{\gamma_M \otimes \mathrm{Id}_N}{\rightrightarrows}} M \otimes D \otimes N$$

implies that for $m \otimes n \in M \,\square_D\, N$, $\gamma_M(m) \otimes n = m \otimes \psi_N(n)$, verifying

$$\tilde{t}\psi_2(n \otimes m) = \tilde{t}(\psi_N(n) \otimes m) = n \otimes \gamma_M(m) = \gamma_2(n \otimes m).$$

A similar argument justifies the map in the other direction as well, so we can define the isomorphism θ by

$$\theta : \mathrm{coHH}_0(M \,\square_D\, N, C) \xrightarrow{\cong} \mathrm{coHH}_0(N \,\square_C\, M, D).$$
$$m \otimes n \mapsto n \otimes m$$

We need to show that for a chain complex of (C, C)-bicomodules P the following diagrams are commutative when they make sense:

$$\begin{array}{ccccc}
\langle\langle (M \odot N) \odot P \rangle\rangle & \xrightarrow{\theta} & \langle\langle P \odot (M \odot N) \rangle\rangle & \xrightarrow{\langle\langle a \rangle\rangle} & \langle\langle (P \odot M) \odot N \rangle\rangle \\
\downarrow{\scriptstyle \langle\langle a \rangle\rangle} & & & & \uparrow{\scriptstyle \theta} \\
\langle\langle M \odot (N \odot P) \rangle\rangle & \xrightarrow{\theta} & \langle\langle (N \odot P) \odot M \rangle\rangle & \xrightarrow{\langle\langle a \rangle\rangle} & \langle\langle N \odot (P \odot M) \rangle\rangle
\end{array}$$

$$\begin{array}{ccccc}
\langle\langle P \odot U_C \rangle\rangle & \xrightarrow{\theta} & \langle\langle U_C \odot P \rangle\rangle & \xrightarrow{\theta} & \langle\langle P \odot U_C \rangle\rangle \\
& {\scriptstyle \langle\langle r \rangle\rangle} \searrow & \downarrow{\scriptstyle \langle\langle l \rangle\rangle} & \swarrow {\scriptstyle \langle\langle r \rangle\rangle} & \\
& & \langle\langle P \rangle\rangle & &
\end{array}$$

But notice that the first diagram above is equivalent to

$$\begin{array}{ccccc}
\mathrm{coHH}_0((M \,\square_D\, N) \,\square_C\, P, C) & \xrightarrow{\theta} & \mathrm{coHH}_0(P \,\square_C\, (M \,\square_D\, N), C) & \xrightarrow{\langle\langle a \rangle\rangle} & \mathrm{coHH}_0((P \,\square_C\, M) \,\square_D\, N, C) \\
\downarrow{\scriptstyle \langle\langle a \rangle\rangle} & & & & \uparrow{\scriptstyle \theta} \\
\mathrm{coHH}_0(M \,\square_D\, (N \,\square_C\, P), C) & \xrightarrow{\theta} & \mathrm{coHH}_0((N \,\square_C\, P) \,\square_C\, M, D) & \xrightarrow{\langle\langle a \rangle\rangle} & \mathrm{coHH}_0(N \,\square_C\, (P \,\square_C\, M), D)
\end{array}$$

So if we apply the definitions of a and θ from above, a tedious check based on the definition of coHH_0 shows that this diagram commutes.

Further by the definition of the shadow, the second diagram is equivalent to:

$$\begin{array}{ccccc}
\mathrm{coHH}_0(P \,\square_C\, C, C) & \xrightarrow{\theta} & \mathrm{coHH}_0(C \,\square_C\, P, C) & \xrightarrow{\theta} & \mathrm{coHH}_0(P \,\square_C\, C, C) \\
& {\scriptstyle \langle\langle r \rangle\rangle} \searrow & \downarrow{\scriptstyle \langle\langle l \rangle\rangle} & \swarrow {\scriptstyle \langle\langle r \rangle\rangle} & \\
& & \mathrm{coHH}_0(P, C) & &
\end{array}$$

This diagram commutes because $\langle\langle r \rangle\rangle$ and $\langle\langle l \rangle\rangle$ just apply the counit ϵ to the copies of C in the coefficients while $\theta : \mathrm{coHH}_0(P \,\square_C\, C, C) \to \mathrm{coHH}_0(C \,\square_C\, P, C)$ by definition just shuffles the P component to the appropriate spot. Therefore the 0^{th} coHochschild homology is a shadow in this bicategorical setting. $\square$

Definition 6.3.3

For a field k, let $\mathscr{D}(CoAlg_k)$ denote the bicategory whose 0-cells are coalgebras over k, and $\mathscr{D}(CoAlg_k)(C,D) = \mathscr{D}(CoMod_{(C,D)})$ is the derived category of (C,D)-bicomodules. The unit U_C is the (C,C)-bicomodule C viewed as a cochain complex, and horizontal composition is given by the derived cotensor product, which we denote by $\widehat{\square}$:

$$\begin{aligned} \odot : \mathscr{B}(C,D) \times \mathscr{B}(D,E) &\to \mathscr{B}(C,E) \\ (M,N) &\mapsto M \odot N := M \widehat{\square}_D N \end{aligned}$$

Note that there is a quasi-isomorphism of cochain complexes $M\widehat{\square}_D N \simeq coBar(M,D,N)$, and so we may also consider this horizontal composition as the two-sided cobar construction.

Remark 6.3.4

Using this equivalence, we define the natural isomorphisms for this setting as:

$$\begin{aligned} a : coBar(coBar(M,D,N),E,P) &\xrightarrow{\cong} coBar(M,D,coBar(N,E,P)) \\ (m \otimes d_1 \otimes \cdots \otimes d_i \otimes n) \otimes e_1 \otimes \cdots \otimes e_j \otimes p &\mapsto m \otimes d_1 \otimes \cdots \otimes d_i \otimes (n \otimes e_1 \otimes \cdots \otimes e_j \otimes p) \\ l : coBar(C,C,M) &\xrightarrow{\cong} M \\ c \otimes c_1 \otimes \cdots \otimes c_i \otimes m &\mapsto \epsilon(c)\epsilon(c_1)\cdots\epsilon(c_i)m \\ r : coBar(M,D,D) &\xrightarrow{\cong} M \\ m \otimes d_1 \otimes \cdots \otimes d_j \otimes d &\mapsto m\epsilon(d_1)\cdots\epsilon(d_j)\epsilon(d). \end{aligned}$$

where i,j denote the number of tensored copies in the cobar construction.

Theorem 6.3.5

CoHochschild homology, coHH, is a shadow on the bicategory $\mathscr{D}(CoAlg_k)$. That is, it gives a family of functors

$$\begin{aligned}\mathrm{coHH}(-,C) : \mathscr{D}(CoMod_{(C,C)}) &\to \mathscr{D}(CoCh_k) \\ N &\mapsto C\widehat{\Box}_{C\otimes C^{op}}N \cong \mathrm{coHH}(N,C)\end{aligned}$$

as a complex that satisfy the required properties, where $CoCh_k$ is the category of cochain complexes.

Proof. We use the bicategory $\mathscr{D}(CoMod_{(C,C)})$, which is the derived category of (C,C)-bicomodules as defined above. Recall the horizontal composition:

$$\begin{aligned}\odot : \mathscr{B}(C,D) \times \mathscr{B}(D,E) &\to \mathscr{B}(C,E) \\ (M,N) &\mapsto M \odot N := M\widehat{\Box}_D N \cong coBar(M,D,N)\end{aligned}$$

Note that this definition requires M to be flat over k, which in this case is satisfied because k is a field.

We want to define the required functor:

$$\begin{aligned}\langle\langle - \rangle\rangle_C : \mathscr{D}(CoMod_{(C,C)}) &\to \mathscr{D}(CoCh_k) \\ N &\mapsto C\widehat{\Box}_{C\otimes C^{op}}N \cong \mathrm{coHH}(N,C)\end{aligned}$$

where $CoCh_k$ is cochain complexes of k-modules and $\mathrm{coHH}(N,C)$ denotes the complex whose homology gives coHochschild homology. The isomorphism above follows from

$$\begin{aligned}H^i(C\widehat{\Box}_{C\otimes C^{op}}N) &= CoTor^i_{C\otimes C^{op}}(C,N) \\ &= \mathrm{coHH}_i(N,C)\end{aligned}$$

Now the brunt of what we need to justify to show that coHH is a shadow is that for M a cochain complex of (C,D)-bicomodules and N a cochain complex of (D,C)-bicomodules, we have an isomorphism

$$\theta : \langle\langle M \odot N\rangle\rangle_C \longrightarrow \langle\langle N \odot M\rangle\rangle_D$$

But unpacking our notation and using the fact that $M\widehat{\square}_D N \simeq coBar(M, D, N)$ gives

$$\begin{aligned}
\langle\langle M \odot N\rangle\rangle_C &:= \langle\langle M\widehat{\square}_D N\rangle\rangle_C = C\widehat{\square}_{C\otimes C^{op}}(M\widehat{\square}_D N) \\
&\cong \mathrm{coHH}(M\widehat{\square}_D N, C) \cong \mathrm{coHH}(coBar(M, D, N), C) \\
\langle\langle N \odot M\rangle\rangle_D &:= \langle\langle N\widehat{\square} M\rangle\rangle_D = D\widehat{\square}_{D\otimes D^{op}}(N\widehat{\square}_C M) \\
&\cong \mathrm{coHH}(N\widehat{\square}_C M, D) \cong \mathrm{coHH}(coBar(N, C, M), D)
\end{aligned}$$

and so showing that there exists an isomorphism θ amounts to defining

$$\theta : \mathrm{coHH}(coBar(M, D, N), C) \xrightarrow{\cong} \mathrm{coHH}(coBar(N, C, M), D).$$

Recall that coHochschild homology is calculated using a cyclic cobar construction, and both $\mathrm{coHH}^\bullet(coBar^\bullet(M, D, N), C)$ and $\mathrm{coHH}^\bullet(coBar^\bullet(N, C, M), D)$ can be identified with the bicosimiplicial object $H^{\bullet\bullet}$ that at the (i, j)-spot is given by:

$$\begin{array}{ccccc}
 & & \underbrace{C \otimes C \otimes \ldots \otimes C}_{\text{i}} & & \\
 & \otimes & & \otimes & \\
N & & & & M \\
 & \otimes & & \otimes & \\
 & & \overbrace{D \otimes D \otimes \ldots \otimes D}^{\text{j}} & &
\end{array}$$

where the coface maps are given by comultiplication at the appropriate index. Then the map θ is given degree-wise by:

$$\begin{aligned}
\theta : M \otimes D \otimes \cdots \otimes D \otimes N \otimes C \otimes \cdots \otimes C &\longrightarrow N \otimes C \otimes \cdots \otimes C \otimes M \otimes D \otimes \cdots \otimes D \\
m \otimes d_1 \otimes \cdots d_j \otimes n \otimes c_1 \otimes \cdots \otimes c_i &\mapsto \pm n \otimes c_1 \otimes \cdots \otimes c_i \otimes m \otimes d_1 \otimes \cdots \otimes d_j,
\end{aligned}$$

where the $\pm$ is determined by the Koszul sign. This map behaves well with respect to the coface maps because of the (C,D)-comodule structure on M and the (D,C)-comodule structure on N. This shuffling thus gives an equivalence of cochain complexes and defines the desired isomorphism θ.

We need to show that for a cochain complex of (C,C)-bicomodules P the following diagrams are commutative when they make sense:

$$\begin{array}{ccccc}
\langle\langle (M\odot N)\odot P\rangle\rangle & \xrightarrow{\theta} & \langle\langle P\odot (M\odot N)\rangle\rangle & \xrightarrow{\langle\langle a\rangle\rangle} & \langle\langle (P\odot M)\odot N\rangle\rangle \\
\Big\downarrow{\scriptstyle\langle\langle a\rangle\rangle} & & & & \Big\uparrow{\scriptstyle\theta} \\
\langle\langle M\odot (N\odot P)\rangle\rangle & \xrightarrow{\theta} & \langle\langle (N\odot P)\odot M\rangle\rangle & \xrightarrow{\langle\langle a\rangle\rangle} & \langle\langle N\odot (P\odot M)\rangle\rangle
\end{array}$$

$$\begin{array}{ccccc}
\langle\langle P\odot U_C\rangle\rangle & \xrightarrow{\theta} & \langle\langle U_C\odot P\rangle\rangle & \xrightarrow{\theta} & \langle\langle P\odot U_C\rangle\rangle \\
& {\scriptstyle\langle\langle r\rangle\rangle}\searrow & \Big\downarrow{\scriptstyle\langle\langle l\rangle\rangle} & \swarrow{\scriptstyle\langle\langle r\rangle\rangle} & \\
& & \langle\langle P\rangle\rangle & &
\end{array}$$

But notice that the first diagram above is equivalent to

$$\begin{array}{ccccc}
\mathrm{coHH}((M\widehat{\square}_D N)\widehat{\square}_C P, C) & \xrightarrow{\theta} & \mathrm{coHH}(P\widehat{\square}_C(M\widehat{\square}_D N), C) & \xrightarrow{\langle\langle a\rangle\rangle} & \mathrm{coHH}((P\widehat{\square}_C M)\widehat{\square}_D N, C) \\
\Big\downarrow{\scriptstyle\langle\langle a\rangle\rangle} & & & & \Big\uparrow{\scriptstyle\theta} \\
\mathrm{coHH}(M\widehat{\square}_D(N\widehat{\square}_C P), C) & \xrightarrow{\theta} & \mathrm{coHH}((N\widehat{\square}_C P)\widehat{\square}_C M, D) & \xrightarrow{\langle\langle a\rangle\rangle} & \mathrm{coHH}(N\widehat{\square}_C(P\widehat{\square}_C M), D).
\end{array}$$

So if we apply the definitions of a and θ from above, a tedious check based on the Dennis-Waldhausen Morita Argument shows that this diagram, which is expanded in Diagram 2 of the Appendix, commutes.

Further thanks to the natural isomorphisms of the bicategorical structure, the second diagram is equivalent to:

$$\begin{array}{ccccc}
\mathrm{coHH}(P\widehat{\square}_C C, C) & \xrightarrow{\theta} & \mathrm{coHH}(C\widehat{\square}_C P, C) & \xrightarrow{\theta} & \mathrm{coHH}(P\widehat{\square}_C C, C) \\
& {\scriptstyle\langle\langle r\rangle\rangle}\searrow & \Big\downarrow{\scriptstyle\langle\langle l\rangle\rangle} & \swarrow{\scriptstyle\langle\langle r\rangle\rangle} & \\
& & \mathrm{coHH}(P, C) & &
\end{array}$$

This diagram commutes because $\langle\langle r\rangle\rangle$ and $\langle\langle l\rangle\rangle$ just apply the counit ϵ to the copies of C in the coefficients while $\theta : \mathrm{coHH}(coBar(P,C,C),C) \to \mathrm{coHH}(coBar(C,C,P),C)$ by definition just shuffles the P component to the appropriate spot. Therefore coHH is a shadow. $\square$

6.4 Morita Invariance

Classically, we can define Morita equivalence in the context of rings and bimodules:

Definition 6.4.1 ([29])

Two rings R and T are **Morita equivalent** if there exist bimodules ${}_R M_T$ and ${}_T N_R$ so that

$$M \otimes_T N \cong R \qquad\qquad N \otimes_R M \cong T$$

as R- and T-bimodules respectively.

Hochschild homology is known to be Morita invariant:

Theorem 6.4.2 ([24])

If R and T are Morita equivalent rings, then there is a natural isomorphism

$$\mathrm{HH}_*(R) \cong \mathrm{HH}_*(T).$$

Morita equivalence in the dual setting of coalgebras is often referred to as *Morita-Takeuchi invariance* thanks to work of Takeuchi [34]. This work was further developed by Farinati and Solotar [17], Brezenzinski and Wisbauer [10] and Hess-Shipley [20].

Definition 6.4.3 ([10])

Two coalgebras C and D are **Morita-Takeuchi equivalent** if there exists a (C, D)-bicomodule M and a (D, C)-bicomodule N such that there are bicomodule isomorphisms $M \square_D N \cong C$ and $N \square_C M \cong D$.

This gives an equivalence of categories:

$$N \square_C - : CoMod_C \to CoMod_D$$

CoHochschild homology is similarly known to be Morita-Takeuchi invariant, by work of Farinati-Solotar:

Theorem 6.4.4 ([16])

If C and D are Morita-Takeuchi equivalent coalgebras, then there is a natural isomorphism

$$\mathrm{coHH}_*(C) \cong \mathrm{coHH}_*(D).$$

More generally, Morita equivalence is a natural notion of equivalence in bicategories. We recall the definition of Morita equivalence as presented, for instance, in Campbell-Ponto [11]. In order to define it in the bicategorical setting, we need the following definition.

Definition 6.4.5

For a bicategory $\mathscr{B}$, a 1-cell $M \in \mathscr{B}(C,D)$ is *right dualizable* if there exists a 1-cell $N \in \mathscr{B}(D,C)$, called the *right dual*, along with a coevaluation 2-cell $\eta_{(M,N)} : U_C \to M \odot N$ and an evaluation 2-cell $\epsilon_{(M,N)} : N \odot M \to U_D$ such that they satisfy the triangle identities:

$$\mathrm{Id}_M = (\mathrm{Id}_M \odot \epsilon_{(M,N)}) \circ (\eta_{(M,N)} \odot \mathrm{Id}_M) : M \cong U_C \odot M \to M \odot N \odot M \to M \odot U_D \cong M$$
$$\mathrm{Id}_N = (\epsilon_{(M,N)} \odot \mathrm{Id}_N) \circ (\mathrm{Id}_N \odot \eta_{(M,N)}) : N \cong N \odot U_C \to N \odot M \odot N \to U_D \odot N \cong N$$

The pair (M,N) is called a **dual pair**, and N is *left dualizable* with *left dual* M.

Definition 6.4.6 ([11])

Let $\mathscr{B}$ be a bicategory. Then $C, D \in ob(\mathscr{B})$ are **Morita equivalent** if there exist 1-cells $M \in \mathscr{B}(C,D)$ and $N \in \mathscr{B}(D,C)$ such that (M,N) and (N,M) are dual pairs and the coevaluation maps $\eta_{(M,N)} : U_C \to M \odot N$, $\eta_{(N,M)} : U_D \to N \odot M$ and the evaluation maps $\epsilon_{(M,N)} : N \odot M \to U_D$, $\epsilon_{(N,M)} : M \odot N \to U_C$ are inverses. That is,

$$\eta_{(N,M)} \circ \epsilon_{(M,N)} = \mathrm{Id}_{N \odot M} \qquad \epsilon_{(N,M)} \circ \eta_{(M,N)} = \mathrm{Id}_{U_C}$$
$$\eta_{(M,N)} \circ \epsilon_{(N,M)} = \mathrm{Id}_{M \odot N} \qquad \epsilon_{(M,N)} \circ \eta_{(N,M)} = \mathrm{Id}_{U_D}$$

Example 6.4.7

Using this structure, we see that in the bicategory $Mod/Ring$, R and T are Morita equivalent if there exists an (R,T)-bimodule M and a (T,R)-bimodule N such that (M,N) and (N,M) are dual pairs, giving

$$M \otimes_T N \cong R$$
$$N \otimes_R M \cong T.$$

Note that since HH_0 is a shadow:

$$\langle\langle M \odot N \rangle\rangle_R \cong \langle\langle N \odot M \rangle\rangle_T$$
$$\mathrm{HH}_0(R; M \otimes_T N) \cong \mathrm{HH}_0(T; N \otimes_R M).$$

Morita invariance says that $M \otimes_T N \cong R$ and $N \otimes_R M \cong T$. So then we get

$$\begin{aligned} \mathrm{HH}_0(R;R) &\cong \mathrm{HH}_0(R; M \otimes_T N) && \text{(Morita equivalence)} \\ &\cong \mathrm{HH}_0(T; N \otimes_R M) && \text{(equivalence)} \\ &\cong \mathrm{HH}_0(T;T) && \text{(Morita invariance)} \end{aligned}$$

so our Morita equivalent rings R and T have equivalent "underived" Hochschild homology.

Example 6.4.8

In the bicategory $\mathscr{D}(Ch/Ring)$, R and T are Morita equivalent if there exist chain complexes M of (R,T)-bimodules and N of (T,R)-bimodules such that (M,N) and (N,M) are dual pairs, giving

$$M \otimes_T^L N \cong R$$
$$N \otimes_R^L M \cong T.$$

Again, since HH is a shadow:

$$\langle\langle M \odot N \rangle\rangle_R \cong \langle\langle N \odot M \rangle\rangle_T$$
$$\mathrm{HH}(R, M \otimes_T^L N) \cong \mathrm{HH}(T, N \otimes_R^L M).$$

Morita invariance says that $M \otimes_T^L N \cong R$ and $N \otimes_R^L M \cong T$. So then we get

$$\begin{aligned} \mathrm{HH}(R, R) &\cong \mathrm{HH}(R, M \otimes_T^L N) && \text{(Morita equivalence)} \\ &\cong \mathrm{HH}(T, N \otimes_R^L M) && \text{(shadows)} \\ &\cong \mathrm{HH}(T, T) && \text{(Morita equivalence)} \end{aligned}$$

so our Morita invariant rings R and T have equivalent Hochschild homology as well.

Example 6.4.9

Since [11] further show that THH is a shadow, the same notion of Morita invariance holds for the bicategorical setting $\mathscr{D}(Mod/\textit{Ring Spectra})$. That is, ring spectra R and T are Morita equivalent if there exists an (R, T)-bimodule spectrum M and a (T, R)-bimodule spectrum N such that (M, N) and (N, M) are dual pairs, yielding

$$M \wedge_T^L N \simeq R$$
$$N \wedge_R^L M \simeq T.$$

Then since THH is a shadow,

$$\langle\langle M \odot N \rangle\rangle_R \cong \langle\langle N \odot M \rangle\rangle_T$$
$$\mathrm{THH}(R, M \wedge^L_T N) \simeq \mathrm{THH}(T, N \wedge^L_R M).$$

Since Morita invariance says that $M \wedge^L_T N \simeq R$ and $N \wedge^L_R M \simeq T$,

$$\begin{aligned} \mathrm{THH}(R, R) &\simeq \mathrm{THH}(R, M \wedge^L_T N) && \text{(Morita equivalence)} \\ &\simeq \mathrm{THH}(T, N \wedge^L_R M) && \text{(shadows)} \\ &\simeq \mathrm{THH}(T, T) && \text{(Morita equivalence)} \end{aligned}$$

so our Morita equivalent ring spectra R and T have equivalent topological Hochschild homology as well.

Now we consider which objects are Morita equivalent in the bicategory $CoAlg_k$.

Example 6.4.10

In the bicategory $CoAlg_k$, coalgebras C and D are Morita equivalent if there exists a (C, D)-bicomodule M and a (D, C)-bicomodule N such that (M, N) and (N, M) are dual pairs (i.e. there exist coevaluation and evaluation maps for M and N satisfying the conditions of the definition), yielding

$$M \square_D N \cong C$$
$$N \square_C M \cong D.$$

This recovers the classical notion of Morita-Takeuchi equivalence as in [34]. According to the bicategorical shadow structure we have the following Morita invariance results for coHochschild homology.

Proposition 6.4.11

If C and D are Morita equivalent coalgebras in the bicategory $CoAlg_k$ then

$$\mathrm{coHH}_0(C) \cong \mathrm{coHH}_0(D).$$

Proof. Because coHH_0 is a shadow,

$$\begin{aligned} \langle\langle M \odot N \rangle\rangle_C &\cong \langle\langle N \odot M \rangle\rangle_D \\ \mathrm{coHH}_0(M \,\square_D\, N, C) &\cong \mathrm{coHH}_0(N \,\square_C\, M, D). \end{aligned}$$

Suppose C and D are Morita equivalent. By definition, $M \,\square_D\, N \cong C$ and $N \,\square_C\, M \cong D$ and therefore

$$\begin{aligned} \mathrm{coHH}_0(C, C) &\cong \mathrm{coHH}_0(M \,\square_D\, N, C) && \text{(Morita equivalence)} \\ &\cong \mathrm{coHH}_0(N \,\square_R\, M, D) && \text{(shadows)} \\ &\cong \mathrm{coHH}_0(D, D) && \text{(Morita equivalence)} \end{aligned}$$

Thus C and D have the same 0^{th} coHochschild homology. □

Now we consider Morita equivalent objects in the bicategory $\mathscr{D}(CoAlg_k)$.

Example 6.4.12

In the bicategory $\mathscr{D}(CoAlg_k)$, coalgebras C and D are Morita equivalent if there exists a chain complex of (C, D)-bicomodules M and a chain complex of (D, C)-bicomodules N such that (M, N) and (N, M) are dual pairs (i.e. there exist coevaluation and evaluation maps for M and N satisfying the conditions of the definition), yielding

$$M \widehat{\square}_D N \cong C$$
$$N \widehat{\square}_C M \cong D.$$

Morita equivalent objects in the bicategory $\mathscr{D}(CoAlg_k)$ are classically Morita-Takeuchi equivalent as well.

Proposition 6.4.13

If C and D are Morita equivalent coalgebras in the bicategory $\mathscr{D}(CoAlg_k)$ then

$$\mathrm{coHH}_*(C) \cong \mathrm{coHH}_*(D).$$

Proof. Because coHH is a shadow,

$$\mathrm{coHH}(M \widehat{\square}_D N, C) \cong \mathrm{coHH}(N \widehat{\square}_C M, D).$$

Suppose C and D are Morita equivalent. By definition, $M \widehat{\square}_D N \cong C$ and $N \widehat{\square}_C M \cong D$ and therefore

$$\begin{aligned}
\mathrm{coHH}(C, C) &\cong \mathrm{coHH}(M \widehat{\square}_D N, C) && \text{(Morita equivalence)} \\
&\cong \mathrm{coHH}(N \widehat{\square}_C M, D) && \text{(shadows)} \\
&\cong \mathrm{coHH}(D, D) && \text{(Morita equivalence)}
\end{aligned}$$

Thus Morita equivalent coalgebras C and D have equivalent coHochschild homology. □

This recovers a result of Farinati-Solotar [16], using the perspective of shadows.

APPENDIX

Appendix

The following diagrams are referenced in this thesis.

Diagram 1: Coassociativity from proof of Proposition 2.5.5

$$
\begin{array}{ccccccc}
B \wedge_A B & \xrightarrow{i_A \wedge \mathrm{Id}} & B \wedge_A A \wedge_A B & \xrightarrow{\mathrm{Id} \wedge \phi \wedge \mathrm{Id}} & B \wedge_A B \wedge_A B & \xrightarrow{\mathrm{Id} \wedge i_B \wedge \mathrm{Id}} & (B \wedge_A B) \wedge_B (B \wedge_A B) \\
\downarrow{\scriptstyle i_A \wedge \mathrm{Id}} & & \downarrow{\scriptstyle \mathrm{Id} \wedge i_A \wedge \mathrm{Id}} & & \downarrow{\scriptstyle \mathrm{Id} \wedge i_A \wedge \mathrm{Id}} & & \downarrow{\scriptstyle \mathrm{Id} \wedge \mathrm{Id} \wedge i_A \wedge \mathrm{Id}} \\
B \wedge_A A \wedge_A B & \xrightarrow{i_A \wedge \mathrm{Id} \wedge \mathrm{Id}} & B \wedge_A A \wedge_A A \wedge_A B & \xrightarrow{\mathrm{Id} \wedge \phi \wedge \mathrm{Id} \wedge \mathrm{Id}} & B \wedge_A B \wedge_A A \wedge_A B & \xrightarrow{\mathrm{Id} \wedge i_B \wedge \mathrm{Id} \wedge \mathrm{Id}} & (B \wedge_A B) \wedge_B (B \wedge_A A \wedge_A B) \\
\downarrow{\scriptstyle \mathrm{Id} \wedge \phi \wedge \mathrm{Id}} & & \downarrow{\scriptstyle \mathrm{Id} \wedge \mathrm{Id} \wedge \phi \wedge \mathrm{Id}} & & \downarrow{\scriptstyle \mathrm{Id} \wedge \mathrm{Id} \wedge \phi \wedge \mathrm{Id}} & & \downarrow{\scriptstyle \mathrm{Id} \wedge \mathrm{Id} \wedge \mathrm{Id} \wedge \phi \wedge \mathrm{Id}} \\
B \wedge_A B \wedge_A B & \xrightarrow{i_A \wedge \mathrm{Id} \wedge \mathrm{Id}} & B \wedge_A A \wedge_A B \wedge_A B & \xrightarrow{\mathrm{Id} \wedge \phi \wedge \mathrm{Id} \wedge \mathrm{Id}} & (B \wedge_A B) \wedge_A (B \wedge_A B) & \xrightarrow{\mathrm{Id} \wedge i_B \wedge \mathrm{Id} \wedge \mathrm{Id}} & (B \wedge_A B) \wedge_B (B \wedge_A B \wedge_A B) \\
\downarrow{\scriptstyle \mathrm{Id} \wedge i_B \wedge \mathrm{Id}} & & \downarrow{\scriptstyle \mathrm{Id} \wedge \mathrm{Id} \wedge i_B \wedge \mathrm{Id}} & & \downarrow{\scriptstyle \mathrm{Id} \wedge \mathrm{Id} \wedge i_B \wedge \mathrm{Id}} & & \downarrow{\scriptstyle \mathrm{Id} \wedge \mathrm{Id} \wedge \mathrm{Id} \wedge i_B \wedge \mathrm{Id}} \\
(B \wedge_A B) \wedge_B (B \wedge_A B) & \xrightarrow{i_A \wedge \mathrm{Id} \wedge \mathrm{Id} \wedge \mathrm{Id}} & (B \wedge_A A \wedge_A B) \wedge_B (B \wedge_A B) & \xrightarrow{\mathrm{Id} \wedge \phi \wedge \mathrm{Id} \wedge \mathrm{Id} \wedge \mathrm{Id}} & (B \wedge_A B \wedge_A B) \wedge_B (B \wedge_A B) & \xrightarrow{\mathrm{Id} \wedge i_B \wedge \mathrm{Id} \wedge \mathrm{Id} \wedge \mathrm{Id}} & (B \wedge_A B) \wedge_B (B \wedge_A B) \wedge_B (B \wedge_A B)
\end{array}
$$

Diagram 2: Shadow properties of coHH from proof of Theorem 6.3.5

$$
\begin{array}{ccccc}
\mathrm{coHH}(coBar(coBar(M,D,N),C,P),C) & \xrightarrow{\theta} & \mathrm{coHH}(coBar(P,C,coBar(M,D,N)),C) & \xrightarrow{\langle\langle a^{-1}\rangle\rangle} & \mathrm{coHH}(coBar(coBar(P,C,M),D,N),C) \\
\downarrow{\scriptstyle \langle\langle a\rangle\rangle} & & & & \uparrow{\scriptstyle \theta} \\
\mathrm{coHH}(coBar(M,D,coBar(N,C,P)),C) & \xrightarrow{\theta} & \mathrm{coHH}(coBar(coBar(N,C,P),C,M),D) & \xrightarrow{\langle\langle a\rangle\rangle} & \mathrm{coHH}(coBar(N,C,coBar(P,C,M)),D)
\end{array}
$$

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