

STABILITY INVESTIGATIONS OF NON-CONSERVATIVE DYNAMIC SYSTEMS

By

Mahmoud Nabil Abdullatif

A DISSERTATION

Submitted to

Michigan State University

in partial fulfillment of the requirements

for the degree of

Mechanical Engineering – Doctor of Philosophy

2020

ABSTRACT

STABILITY INVESTIGATIONS OF NON-CONSERVATIVE DYNAMIC SYSTEMS

By

Mahmoud Nabil Abdullatif

Elastic stability continues to be a subject of considerable interest due to the increasing demands of lightweight structures. It is well known that structures may lose stability through divergence or flutter and the nature of the instability depends on the type of loading. A structure may lose stability through divergence when the loading is conservative but may lose stability through divergence or flutter when the loading is non-conservative. In this work, we investigate stability transitions in structures due to: damping in the presence of non-conservative loading, terminal dynamic moment with and without intermediate support, and external flow with dynamic moment. The role of damping on non-conservative systems, which was first investigated by Ziegler, is revisited here. It is shown that increasing the level of damping can cause an unstable non-conservative system to become stable, then unstable, then stable again at the same value of the non-conservative load. This sequence of stability transitions, which has not been reported in the literature, is found to exist for several non-conservative systems, including articulated linkages with follower end forces and fluid-conveying pipes. The stability transitions were investigated by applying the Routh-Hurwitz criterion twice: once for the characteristic polynomial and the second time for the polynomial that guarantees the existence of a second-order auxiliary polynomial in the Routh array. Nonconservative loading in elastic structures has previously been limited to follower forces and forces generated by a fluid jet exiting a fluid-conveying pipe. A new type of non-conservative loading is introduced here in the form of a terminal dynamic moment. The terminal moment is dynamic in the sense that it is proportional to the slope or curvature of a point along the structure. Irrespective of whether the moment is slope-dependent or curvature-dependent, stability is lost in a beam through divergence when the constant of

proportionality is positive, and through flutter when the constant of proportionality is negative. Some of the theoretical investigations are supported by experiments with a cantilever beam. In non-conservative systems, the introduction of an intermediate support is known to result in stability transitions between flutter and divergence. This result was confirmed for the new type of non-conservative loading, namely, the terminal dynamic moment. For a cantilever beam with terminal dynamic moment, a rich set of stability transitions were observed for the first time. This includes multiple stability transitions between divergence and flutter and between different modes of flutter. In some cases, the transitions are accompanied by an abrupt change in the critical load, which is commonly referred to as the “jump phenomenon” in the literature. External flow is known to result in non-conservative loading and, therefore, stability transitions are investigated in a hinged beam in external flow. A dynamic moment, proportional to the curvature of the beam at some point along its length, is applied at the hinge. The combined effect of non-conservative loading due to both dynamic moment and external flow is investigated with the objective of developing a mechanism for energy harvesting from flow fields.

Copyright By
MAHMOUD NABIL ABDULLATIF
2020

I dedicate this work to my parents, who have raised me to be the person I am today, and to my beloved wife, Sarah, for nursing me with love and affection and always being by my side through this tremendous journey.

ACKNOWLEDGEMENTS

Thanks to Almighty ALLAH for giving me the strength and ability to learn, understand, and complete this work.

I would like to thank my advisor, Ranjan Mukherjee, for his guidance and support.

I am thankful for Aren Hellum for helping me see new avenues of progress when I got stuck.

A tremendous thank you to my wife, Sarah, for always being by my side: helping me get through the lows and celebrating the highs with me. I could not have done this without you.

I am also thankful for my wonderful colleagues, Amer Allafi, Sanders Aspelund, Connor Boss, Sheryl Chau, Krissy Kamensky, and Nilay Kant, for their constant source of camaraderie and fruitful discussions.

TABLE OF CONTENTS

LIST OF TABLES	ix
--------------------------	----

LIST OF FIGURES	x
---------------------------	---

Chapter 1 Introduction	1
1.1 Background	1
1.2 Literature Review	2
1.2.1 Stability Transitions Induced by Damping	2
1.2.2 Stability Transitions Induced by Intermediate Support	4
1.3 Motivation	5
1.4 Research Contributions	6
1.5 Thesis Outline	7

Chapter 2 Divergence and Flutter Modes of Instability	8
2.1 Discrete System Example	8
2.2 Continuous System Example	12

Chapter 3 Stability Transitions Induced by Damping	15
3.1 Two Degree-Of-Freedom System with a Follower Force	15
3.1.1 Effect of γ_1 on the Stability of the System	18
3.1.2 Effect of γ_2 on the Stability of the System	21
3.2 Two Degree-Of-Freedom Fluid-Conveying Pipe	28
3.2.1 Effect of Relative Damping on the Stability of the System	29
3.2.2 Effect of Absolute Damping on the Stability of the System:	31
3.3 Three Degree-Of-Freedom Systems	32
3.3.1 Three-Link System with a Follower Force	32
3.3.2 Three-Link Fluid-Conveying Pipe	34
3.4 Cantilevered Fluid-Conveying Pipe	37

Chapter 4 Non-conservative Effects of Dynamic Terminal Moment . . .	41
4.1 Problem Formulation	41
4.2 Solution Methods	43
4.2.1 Analytical Solution	43
4.2.2 Galerkin method	45

4.3	Instability Investigation	46
4.3.1	Dynamic Moment Proportional to Positive Slope	46
4.3.2	Dynamic Moment Proportional to Negative Slope	48
4.3.3	Dynamic Moment Proportional to Positive and Negative Curvature	51
4.4	Experimental Investigations	52
4.4.1	Hardware Description	52
4.4.2	Model Verification	54
4.4.3	Flutter Oscillations	54
Chapter 5	Stability Transitions Induced by an Intermediate Support . .	57
5.1	Cantilever Beam with Terminal Dynamic Moment	57
5.1.1	Mathematical Model	57
5.2	Terminal Dynamic Moment Proportional to Slope	59
5.2.1	Model Verification	60
5.2.2	Effect of Intermediate Support on Critical Stability	61
5.3	Terminal Dynamic Moment Proportional to Curvature	65
5.3.1	Effect of Intermediate Support on Critical Stability	67
5.4	Cantilever Beam with a Follower End Load	74
5.4.1	Mathematical Model	74
5.4.2	Analytical Solution	75
5.4.3	Numerical Solution	77
5.4.4	Effect of Intermediate Support on Critical Stability	78
Chapter 6	Non-conservative Behavior of Hinged Beam with a Dynamic Moment	81
6.1	Hinged Beam with Dynamic Moment	81
6.1.1	Mathematical Model	81
6.1.2	Instability Investigation	83
6.2	Effect of External Flow on Flutter Instability	85
6.2.1	Mathematical Model	85
6.2.2	Flutter Instability and Nature of Oscillations	87
6.3	Fluid-Immersed Hinged Beam Affixed to a Rigid Body	90
6.3.1	Mathematical model	90
6.3.2	Method of Analysis	93
6.3.3	Flutter Instability	97
Chapter 7	Conclusion	104
7.1	Divergence and Flutter Modes of Instability	104
7.2	Stability Transitions Induced by Damping	104
7.3	Non-conservative Effects of Dynamic Terminal Moment	106
7.4	Stability Transitions Induced by an Intermediate Support	107
7.5	Non-conservative Behavior of Hinged Beam with a Dynamic Moment	109

BIBLIOGRAPHY	112
---------------------	-----

LIST OF TABLES

Table 4.1:	Natural frequencies of the beam in Fig.4.8: analytical, numerical, and experimental values.	54
Table 4.2:	Analytical, numerical and experimentally observed values of frequency and proportionality constant at the flutter instability point for the experimental setup in Fig.4.8.	55
Table 5.1:	Critical values of $\overline{C}_s$ for $\overline{C}_s > 0$ that results in divergence.	61
Table 5.2:	Critical values of $\overline{C}_s$ for $\overline{C}_s < 0$ that results in flutter.	61
Table 6.1:	Assumed values for the non-dimensional parameters used in the analysis.	95

LIST OF FIGURES

Figure 2.1:	Two-link fluid conveying pipe with torsional joint stiffnesses.	8
Figure 2.2:	The locus of the system roots as the non-dimensional fluid velocity varied.	10
Figure 2.3:	(a) Flutter instability for $u_f = 5.007$. (b) Divergence instability for $u_d = 7.434$	10
Figure 2.4:	Nonlinear simulation of θ for different values of u	11
Figure 2.5:	Cantilevered beam with end loads	12
Figure 2.6:	(a) Cantilevered beam with axial end loading (b) variation of the first two modal frequencies of a cantilevered beam as the axial loading increases (c) locus of the first two modal frequencies illustrating divergence instability.	14
Figure 2.7:	(a) Cantilevered beam with follower end loading (b) variation of the first two modal frequencies of a cantilevered beam as the follower loading increases (c) locus of the first two modal frequencies illustrating flutter instability.	14
Figure 3.1:	A two-link system with a follower force.	16
Figure 3.2:	Absolute instability curve in γ_1 - p space for the two-link system in Fig.3.1 when $\gamma_2 = 1.0$. For $p \in [0, p^*)$, there is only one real root of γ_1 . For $p \geq p^*$, there are three real roots of γ_1 but only one of them is positive.	20
Figure 3.3:	Two-link system in Fig.3.1 with $\gamma_1 = 1.0$: (a) Absolute instability curve in γ_2 - p space showing the S-D-S effect of damping (b) locus of the roots that determine system stability as γ_2 increases for the particular case of $p = 2.08$	22
Figure 3.4:	Contour plot for $\Delta = 0$ in γ_2 - p plane for the two-link system with follower force: regions with positive and negative values of Δ are denoted by (+) and (-), respectively.	24

Figure 3.5:	Flutter instability curves in γ_1 - p space for the two-link system with follower force for: (a) $\gamma_2 = 0.02$ and $p = 1$, (b) $\gamma_2 = 0.02$ and $p = 2.05$, (c) $\gamma_2 = 0.02$ and $p = 5$	24
Figure 3.6:	The locus of the roots that determine system stability as γ_1 increases for the particular cases: (a) $\gamma_2 = 0.02$ and $p = 1$, (b) $\gamma_2 = 0.02$ and $p = 2.05$, (c) $\gamma_2 = 0.02$ and $p = 5$	25
Figure 3.7:	Flutter instability curves in p - γ_1 space for the two-link system with follower force and range of $\gamma_2 \in [0.01, 0.10]$	26
Figure 3.8:	Contour plot for $\Delta = 0$ in γ_1 - p plane for the two-link system with follower force: regions with positive and negative values of Δ are denoted by (+) and (-), respectively.	26
Figure 3.9:	Flutter instability curves in p - γ_2 space for the two-link system with follower force and range of $\gamma_1 \in [0.25, 2.00]$	27
Figure 3.10:	Flutter instability curves in γ_1 - γ_2 space for the two-link system with follower force $p \in [0.5, 2.0]$	27
Figure 3.11:	(a) Relative damping and (b) absolute damping is introduced in the fluid-conveying pipe considered by Benjamin [1] with equal link lengths ℓ and equal torsional joint stiffness k	28
Figure 3.12:	Contour plot for $\Delta = 0$ in β - u plane for the two-link fluid-conveying pipe with relative damping: regions with positive and negative values of Δ are denoted by (+) and (-), respectively.	30
Figure 3.13:	Absolute instability curves in γ - u space for the two-link pipe with relative damping for different values of β ; the S-D-S effect of damping is seen for $\beta = 0.04$ and 0.08	30
Figure 3.14:	Absolute instability curves in γ - u space for the two-link pipe with absolute damping showing D, S-D, and S-D-S effects for (a) $\beta = 1.50$, (b) $\beta = 0.70$, and (c) $\beta = 0.10$, respectively. For (b) and (c), the critical flow velocity u_{cr} for $\gamma = 0$ lies outside the range of u ; these values are 2.245 for $\beta = 0.70$ and 5.007 for $\beta = 0.10$	32
Figure 3.15:	A three-link system with a follower force.	33
Figure 3.16:	Absolute instability curves in γ_3 - p space for the three-link system with follower force showing D, D-S, S-D-S, and S effects of damping for (a) $\gamma_c = 0.01$, (b) $\gamma_c = 2.0$, (c) $\gamma_c = 4.0$, and (d) $\gamma_c = 6.0$, respectively.	34

Figure 3.17:	(a) Flutter instability curves and absolute instability curve in γ - u space for the three-link fluid-conveying pipe in Example 3.6 for $\beta = 0.10$ (b) locus of all roots of the system as u increases for the particular case of $\gamma = 0.2$	35
Figure 3.18:	(a) Flutter instability curves in γ - u space for the three-link fluid-conveying pipe in Example 3.7 for $\beta = 0.10$ (b) A magnified view of the region of intersection of the flutter instability curves is shown along with the absolute instability curve.	36
Figure 3.19:	Absolute instability curves in c - u space for the cantilever fluid-conveying pipe with $\beta = 0.1$ for: (a) $\alpha = 0.86$ (b) $\alpha \in [0.2, 1.0]$. The S-D-S effect of damping is seen for $\alpha = 0.86$	39
Figure 3.20:	Absolute instability curves in c_s - u space for the cantilever fluid-conveying pipe with $\beta = 0.35$ showing the D-S effect of damping.	40
Figure 4.1:	A flexible cantilever beam with a tip mass and a terminal dynamic moment	42
Figure 4.2:	Terminal moment is proportional to the positive slope of the beam: (a) Variation in the first natural frequency with $\overline{C}_s$ ($\overline{C}_s \geq 0$) for four different values of α . Analytical and numerical results are plotted using solid lines and dashed lines, respectively. (b) Analytical results showing variation of the second, third, fourth, and fifth natural frequencies with $\overline{C}_s$ for $\alpha = 0.25$	47
Figure 4.3:	Terminal moment is proportional to the negative slope of the beam: (a) Variation in the first four natural frequencies with $\overline{C}_s$ ($\overline{C}_s \leq 0$) for $\alpha = 0.1$ shows the first flutter instability mode. Analytical and numerical results are plotted using solid lines and dashed lines, respectively. (b) Analytical results showing the first flutter instability mode for five different values of α in range 1: $[0.02, 0.63]$, including the case shown in (a); the critical stability point is shown using the $\bullet$ sign.	48
Figure 4.4:	Terminal moment is proportional to the negative slope of the beam: Analytical results show the (a) second flutter instability mode for range 2: $[0.64, 0.95]$, (b) Switching from first to second flutter instability mode for $\alpha = 0.6394$ and $\overline{C}_s^* = -52.5$. In both figures, the critical stability point is shown using the $\bullet$ sign.	49

Figure 4.5:	Terminal moment is proportional to the negative slope of the beam: Critical stability curve for $\alpha \in [0.02, 0.95]$. The critical stability curve (solid line) was obtained numerically using a ten-mode approximation; the analytical results are shown for discrete values of α using the $\bullet$ sign.	50
Figure 4.6:	Terminal moment is proportional to the positive curvature of the beam: Variation in the first natural frequency with $\bar{C}_c$ ($\bar{C}_c \geq 0$) for three different values of α . Analytical and numerical results are plotted using solid lines and dashed lines, respectively.	51
Figure 4.7:	Terminal moment is proportional to the negative curvature of the beam: Critical stability curve for $\alpha \in [0.01, 0.88]$. The critical stability curve (solid line) was obtained numerically using a ten-mode approximation; the analytical results are shown for discrete values of α using the $\bullet$ sign.	52
Figure 4.8:	Cantilever beam with tip mass: the tip mass is comprised of a motor for application of a dynamic moment and an air bearing to avoid excitation of torsional modes.	52
Figure 4.9:	Analytical and numerical results showing the first flutter instability mode for the Euler-Bernoulli beam corresponding to the experimental setup in Fig.4.8. The analytical results are shown using a solid line; the numerical results are shown using a dashed line but are indistinguishable from the analytical results.	55
Figure 4.10:	A comparison of normalized, non-dimensional mode shapes of the cantilever in Fig.4.8: solid and dashed lines show the mode shapes obtained analytically and from experimental data.	56
Figure 5.1:	A flexible cantilever beam with an intermediate support subjected to a terminal dynamic moment.	58
Figure 5.2:	Critical stability curve for the beam with intermediate support: $\bar{C}_s > 0$ and $\kappa = 0.25$	62
Figure 5.3:	Critical stability curve for the beam with intermediate support: $\bar{C}_s > 0$ and $\kappa = 0.50$	62
Figure 5.4:	Critical stability curve for the beam with intermediate support: $\bar{C}_s > 0$ and $\kappa = 0.75$	63
Figure 5.5:	Critical stability curve for the beam with intermediate support: $\bar{C}_s < 0$ and $\kappa = 0.25$	64

Figure 5.6:	Critical stability curve for the beam with intermediate support: $\bar{C}_s < 0$ and $\kappa = 0.50$	64
Figure 5.7:	Critical stability curve for the beam with intermediate support: $\bar{C}_s < 0$ and $\kappa = 0.75$	65
Figure 5.8:	Terminal moment is proportional to the positive curvature: (a) $\bar{C}_{cr}$ is plotted on a fine mesh grid of κ and α values; divergence instability is shown by dark points on the grid (b) Top view of (a).	68
Figure 5.9:	Case 1: (a) Critical stability curve - divergence instability at $\alpha = 0.1$ and flutter instability for $\alpha \in [0.101, 0.9]$ (b) Second flutter instability mode at $\alpha = 0.513$ showing veering of the locus of the first natural frequency.	69
Figure 5.10:	(a) Case 2: Critical stability curve - divergence instability for $\kappa \in (0.47, 0.54] \cup (0.72, 0.78]$ and flutter instability for other values of κ (b) Case 3: Critical stability curve - divergence instability for $\alpha \in (0.631, 0.642] \cup (0.673, 0.900]$ and flutter instability for other values of α	70
Figure 5.11:	Case 4: Locus of the first six natural frequencies as $\bar{C}_c$ is increased from zero for $\alpha = 0.25$ and (a) $\kappa = 0.249$, (b) $\kappa = 0.251$	71
Figure 5.12:	Case 5: (a) Critical stability curve - divergence instability for $\alpha \in (0.62, 0.65]$, first-mode flutter instability for $\alpha \in (0.65, 0.715]$ and second-mode flutter instability for $\alpha \in (0.715, 0.78]$ (b) Second-mode flutter instability at $\alpha = 0.716$ showing veering of the locus of the first natural frequency.	72
Figure 5.13:	Terminal moment is proportional to the negative curvature: (a) $\bar{C}_{cr}$ is plotted on a fine mesh grid of κ and α values; divergence instability is shown by dark points on the grid (b) Top view of (a).	73
Figure 5.14:	A flexible cantilever beam with an intermediate support subjected to a follower force	74
Figure 5.15:	(a) Variation in the critical follower force σ as the location of the intermediate support κ is varied; both analytical (solid line) and numerical (dots) results are shown (b) Variation in the critical value of σ for 7 different values of κ ; stability is lost through flutter for $\kappa = \{0.40, 0.45, 0.49, 0.50\}$ and through divergence for $\kappa = \{0.51, 0.52, 0.53\}$ (c) Loci of the first two natural frequencies for $\kappa = \{0.5, 0.5001\}$ showing jump and veering phenomena; mode of instability changes from flutter to divergence and the critical value of σ jumps from 71.8 to 80.6 when κ changes from 0.5 to 0.5001.	79

Figure 6.1:	A flexible hinged-beam with dynamic moment	82
Figure 6.2:	Critical stability curve for the hinged beam: $\overline{C}$ versus α for the case where $\overline{k} = 0.5$ and $\overline{C} > 0$	84
Figure 6.3:	Critical stability curve for the hinged beam: $\overline{C}$ versus α for the case where $\overline{k} = 0.5$ and $\overline{C} < 0$	85
Figure 6.4:	The hinged-beam in Fig.6.1 immersed in axial flow	86
Figure 6.5:	Root locus of the first two fundamental frequencies for the hinged beam without external flow: $\overline{C} < 0$ and $\overline{k} = \alpha = 0.5$	88
Figure 6.6:	Argand diagram for $\overline{k} = \alpha = 0.5$, $\overline{U}_e = 0.7$, and $\beta = 0.9$; the critical $\overline{C}$ is denoted by $\overline{C}_{cr}$	88
Figure 6.7:	Root locus of the second mode complex conjugate frequency for the hinged beam subjected to external flow as $\overline{C}$ varies negatively for $\overline{k} = \alpha = 0.5$, $\overline{U}_e = 0.7$, and $\beta = 0.9$	89
Figure 6.8:	Fluid submersible tail mechanism affixed to a rigid body.	90
Figure 6.9:	Free-body diagrams of the tail mechanism and the rigid body.	91
Figure 6.10:	Critical stability curves in $\mathbb{C}-u_e$ space: (a) a range of $\alpha \in (0.10-0.20]$ (b) for $\alpha = 0.20$ the efficiency, η , and the critical frequency, Ω_{cr} , are denoted as (η, Ω_{cr}) for five different points along the critical stability curve.	98
Figure 6.11:	Critical stability curves in $\mathbb{C}-u_e$ space: (a) a range of $\alpha \in [0.22-0.38]$ with an increment of 0.02 (b) for $\alpha = 0.22$ the efficiency, η , and the critical frequency, Ω_{cr} , are denoted as (η, Ω_{cr}) for three different points along the critical stability curve.	99
Figure 6.12:	Critical stability curves in $\mathbb{C}-u_e$ space: (a) critical stability curves are depicted for eight different values of $\alpha \in [0.40, 0.70]$ (b) for $\alpha = 0.58$ the efficiency, η , and the critical frequency, Ω_{cr} , are denoted as (η, Ω_{cr}) for three different points along the critical stability curve.	100
Figure 6.13:	Critical stability curves in $\mathbb{C}-u_e$ space: (a) critical stability curves are depicted for four different values of α , namely, $\alpha = 0.80, 0.84, 0.90, 0.92$ (b) for $\alpha = 0.90$ the efficiency, η , and the critical frequency, Ω_{cr} , are denoted as (η, Ω_{cr}) for three different points along the critical stability curve; The darkness region of the critical stability curve depicts the region of negative thrust.	101

- Figure 6.14: Region 1: critical stability curves in $\mathbb{C} - u_e$ space: (a) critical stability curves are depicted for seven different values of $\alpha \in [0.18, 0.36]$ (b) for $\alpha = 0.30$ the efficiency, η , and the critical frequency, Ω_{cr} , are denoted as (η, Ω_{cr}) for three different points along the critical stability curve. 102
- Figure 6.15: Region 2: critical stability curves in $\mathbb{C} - u_e$ space: (a) critical stability curves are depicted for three different values of α , namely, $\alpha = 0.50, 0.52, 0.58$ (b) for $\alpha = 0.62$ the efficiency, η , and the critical frequency, Ω_{cr} , are denoted as (η, Ω_{cr}) for three different points along the critical stability curve. 102
- Figure 6.16: Region 3: critical stability curves in $\mathbb{C} - u_e$ space: (a) critical stability curves are depicted for five different values of $\alpha \in [0.66, 0.98]$ (b) for $\alpha = 0.98$ the efficiency, η , and the critical frequency, Ω_{cr} , are denoted as (η, Ω_{cr}) for three different points along the critical stability curve. 103

Chapter 1

Introduction

1.1 Background

The dynamics and stability characteristics of elastic structures have been studied extensively since the early work by Prandtl [2] and Reut [3]. Depending on the nature of the boundary loading, elastic systems exhibit divergence and flutter modes of instability. In the absence of damping, a loss of stability through divergence occurs when the lowest frequency of a system passes through the origin and the property of the equilibrium shifts from stable to unstable. In contrast, flutter instability occurs when two successive frequencies approach each other, coexist, and become complex conjugates. In many instances, the system reaches a limit cycle post-flutter and converts the gained energy into a steady periodic oscillation. According to Herrmann and Bungay [4], a flexible system, in the absence of damping, loses its stability solely through divergence when the boundary loading is conservative in nature. On the other hand, the same system may lose stability through divergence or flutter when subjected to a non-conservative boundary load. The critical load for divergence can always be determined through the static Euler method; on the contrary, the critical load for flutter can be determined through the kinetic method alone.

A non-conservative boundary load can be applied to both discrete and continuous systems; the common examples of the discrete system are Ziegler's double pendulum [5] and Benjamin's articulated fluid conveying pipe [6]. The most common example of the continuous system is a cantilevered column subjected to a follower loading. Several variations of the column problem are discussed in the literature: a Beck's column in which a cantilever is subjected to tangential follower boundary loading; a Pfluger's column, which is a Beck's column with a concentrated end mass; a Leipholz's column, which is a beam subjected to uniformly distributed follower loading along the column span; and finally, a Hauger's column,

where a linearly increasing distributed follower loading is subjected to a cantilevered beam. A survey on the dynamic stability of these columns can be found in [7]. Another typical example of a continuous non-conservative system is a fluid conveying pipe, which is extensively reviewed by Paidoussis and Li [8]. Several techniques were addressed in the literature to tackle the stability problems. Early work [3,9,10] focused on the static instability aspects of a column under follower loading. More recently, Kounadis [11] utilized nonlinear dynamic analysis to reexamine the mechanism of stability on the problem considered by Herrmann and Bungay [4]. Qiu and Nemat-Nasser [12] reported experimental results on instability modes of an articulated cantilever subjected an impinging air-jet at its free end. A comprehensive discussion of the literature on non-conservative systems, including both theoretical and experimental investigations, can be found in the review paper by Elishakoff [13].

1.2 Literature Review

In this work we investigate stability transitions in structures due to damping in the presence of non-conservative loading, due to terminal dynamic moment with and without intermediate support, and due to external flow with a dynamic moment. Since the terminal dynamic moment is introduced here for the first time, we review the literature on stability transitions induced by damping and intermediate support.

1.2.1 Stability Transitions Induced by Damping

A counter-intuitive property of some non-conservative systems is that the application of damping can render the system unstable. This property is frequently referred to as Ziegler's paradox [14] after a classical finding that the addition of viscous damping to the joints of a double pendulum subjected to a follower end force destabilizes the system. Since Ziegler's publication in 1952, investigations of the destabilizing effect of damping have continued to the present day, [15–20]. The effect of both internal (structural) and external (viscous) damping has been investigated in systems subjected to follower end forces; this includes

fluid-conveying pipes where the conveyed fluid contributes to an additional damping-like velocity-dependent force in addition to the follower end-force.

Research interest in the general phenomenon of destabilization by damping has been robust. Bottema [21] investigated the conditions for damping to induce instability in Ziegler's work [14] and Bolotin [22] provided some general results for the stability of two-DOF non-conservative systems. Herrmann and Jong [23] established that a relationship exists between the damped and undamped critical loads of a system. Nemat-Nasser and coworkers provided extensions to multi-dof [24] and continuous [25] systems and destabilization by damping in a continuous system were observed experimentally by Gregory and Paidoussis in fluid-conveying pipes [26]. Twenty-one years after Ziegler's finding, the destabilization-by-damping phenomenon was known to be sufficiently common that Done [27] felt compelled to demonstrate mathematically that there exist damping configurations which are never destabilizing. This result was supported by observations published earlier [22], [28], and corroborated in other non-conservative systems such as fluid-conveying pipes with different boundary conditions [29], curved pipes [30], tapered columns [31], panel flutter [32], and fluid-conveying shells [33]. Although the destabilizing effect implies divergent behavior of trajectories in the neighborhood of the equilibrium; Kounadis [34] used nonlinear analysis to show that the trajectories, despite diverging from the equilibrium, may remain bounded.

Several researchers have provided a physical explanation of the destabilizing effect of damping. For example, Semler et al. [15] used energy methods to investigate discrete systems with viscous damping and continuous systems with Kelvin-Voigt (structural) damping. For discrete systems, however, it was shown that the phase difference plays a vital role in the determination of the mode that extracts energy from the follower force and the magnitude of the critical load. For small damping, Gallina [16] used eigenvalue analysis to show that a positive derivative of an eigenvalue with respect to the damping coefficient results in instability. Sugiyama and Langthjem [17] used energy methods to investigate Beck's column and concluded that the follower force could only do work in each cycle if the column vibration

has a traveling wave component; additionally, the wavenumbers at the point of application of the force determines the rate of energy growth. Kirillov and Seyranian [35] used perturbation methods to derive necessary and sufficient conditions for the matrix of velocity-dependent forces to guarantee asymptotic stability.

1.2.2 Stability Transitions Induced by Intermediate Support

The transition between flutter and divergence in non-conservative systems was found to occur due to the introduction of an intermediate support. In some cases, the transition is accompanied by an abrupt variation in the critical non-conservative loading, commonly referred to as the “jump phenomenon”. This phenomenon was reported earlier by Zorii and Chernukha [36] and followed by a series of papers by Kounadis [37–39] in which regions in the parameter space related to divergence and flutter were identified. In related work, Kounadis and Economou [40] spotted that the transition from flutter to divergence and vice versa is associated with the jump phenomenon. Elishakoff and Hollkamp [41] equally report this unforeseen variation in the critical non-conservative loading.

A later work by Elishakoff and Lottati [42], based on an exact solution for a simply supported and clamped columns resting on intermediate support, reported that the jump phenomenon in the earlier work could not be observed. Sundararajan [11], on the other hand, noticed that the addition of elastic support might destabilize the undamped continuous non-conservative system. There has been a surge of interest in examining the effect of intermediate support, taking into account some of the previous stability investigations on various aspects of the flexible systems. Examples are two degree-of-freedom flexible systems subjected to follower loading [43], flexible fluid conveying pipes [29, 44], and rotating shafts with axial loading [45]. Although most of the studies dealt with undamped systems, where stability investigations were carried out using both analytical [42, 46–48] and numerical [41, 44, 49] methods, studies involving damping have also been delineated [50, 51]. The influence of intermediate support on the dynamic stability remains a fascinating topic for researchers,

even to this day, for further exploration. Some of the recent studies include: the transverse vibrations for an axially moving beam supported by rollers [52], the influence of elastic end support on flutter and buckling instability of the Beck's column in the presence of an arbitrary number of weak sections [53], and buckling behavior of beams and columns with additional intermediate flexible supports [54].

1.3 Motivation

The pioneering works on structural stability of non-conservative systems was the motivation behind this work. Among these works, noteworthy is the counterintuitive role of damping, whereby the addition of damping causes destabilization of the structure. A non-conservative system can exhibit multiple stability transitions [15], such that increasing the damping for a given non-conservative force can stabilize an unstable system, which then becomes unstable again as damping is further increased. There has been little or no efforts on investigation of multiple stability transitions and this may be attributed to several factors. Many efforts in the field consider the damping levels to be too small to produce the phenomenon [17, 18, 23, 24, 35, 55]. The appearance of additional stability transitions is also somewhat subtle, depending on the method of analysis. In at least one case [29], the phenomenon was overlooked by the authors, or was simply outside the scope of their interest. This relative lack of investigation stimulated the first part of this work where a procedure is developed in the context of examples, to study multiple stability transitions. Three damping-induced stability transitions has been observed, a phenomenon that has not been reported heretofore. The majority of the results discussed in the literature also pertain to the commonly known non-conservative loads, namely, follower forces and forces induced by the fluid jet in fluid-conveying pipes. A new type of non-conservative loading is introduced in beam-like structures in the form of a terminal dynamic moment. The moment is proportional to the slope or the curvature of a point along the beam length. Thereupon, the stability characteristics of a cantilever beam subjected to the dynamic moment are also investigated. An

extension of this work, inspired by the work done by Lee [56], include the role of intermediate support on structural stability. This investigation reveals a rich set of stability transitions between divergence and flutter and different modes of flutter that have not been reported earlier.

1.4 Research Contributions

This work provides additional insight into the role of system parameters on the stability characteristics of structures. The study begins with the investigation of the role of damping and additional stability transitions are shown to be both possible and common. A non-conservative system that is unstable for zero damping is stabilized, destabilized, and restabilized as the damping is increased. The observation of this phenomenon is made possible by utilizing the Routh-Hurwitz procedure [57] twice: once for the characteristic polynomial and a second time for the auxiliary polynomial. Although other works have employed the Routh procedure to assess the stability of the system [15], it has not been used to determine the number of marginal stability points.

Stability characteristics of a cantilevered beam with a tip mass subjected to a terminal non-conservative moment was also studied. The magnitude of the bending moment varies with the slope or the curvature of the beam at some point along its length. To the best of our knowledge, the stability analysis of such systems has not appeared in the literature, and this work is the first to show both theoretical and experimental investigations with this new type of non-conservative load.

Flutter instability, which is the main topic of investigation in this work, is explored due to its ability to produce a sustained oscillation. Such oscillations are ideal for engineering applications like underwater propulsion [58,59] and energy harvesting [60]. With this in mind, the instability effect of the non-conservative dynamic moment is investigated in the presence of external flow. Galerkin approximation and analytical solutions are used to determine the range of parameters for which flutter oscillations can be obtained.

1.5 Thesis Outline

This work is organized as follows: Chapter 2 provides a short discussion on the two types of structural instabilities using examples of discrete and continuous systems. Chapter 3 provides new results on the role of damping on system stability; the stability transitions of different non-conservative systems are examined through examples. Chapter 4, explores a new way to destabilize a cantilever beam by using a non-conservative dynamic moment. Chapter 5 investigates the role of the intermediate support on stability of a structure subjected to the dynamic moment. In Chapter 6 the instability of a hinged beam subjected to a non-conservative moment is examined in the presence of external flow. Finally, concluding remarks are provided in Chapter 7.

Chapter 2

Divergence and Flutter Modes of Instability

Elastic structures can lose stability through divergence or flutter. In the absence of damping, a loss of stability through divergence occurs when the lowest frequency of the system passes through the origin and the property of the equilibrium shifts from stable to unstable. In contrast, flutter instability occurs when two successive frequencies approach each other, coexist, and become complex conjugates. While the flutter mode of instability can only occur for non-conservative loading, the divergence mode of instability can occur for both conservative and non-conservative loadings. In general, both discrete and continuous systems can lose stability through flutter or divergence and the load at which stability is lost is known as the critical load. In this chapter we consider two examples: an articulated fluid conveying pipe and a cantilever beam subjected to terminal loading, to illustrate the both types of instabilities in a discrete system and a continuous system, respectively.

2.1 Discrete System Example

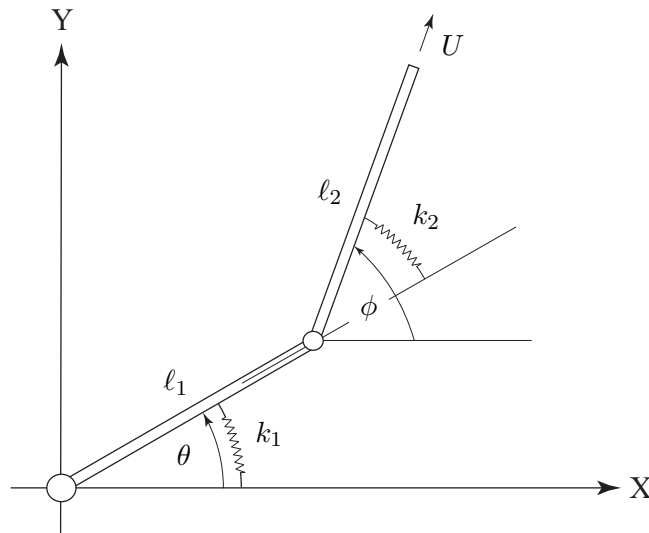


Figure 2.1: Two-link fluid conveying pipe with torsional joint stiffnesses.

Consider the two-link fluid-conveying tube shown in Fig.2.1. The non-dimensional equations of motion in absence of gravity force, following Benjamin's derivation [6], can be expressed as

$$\begin{aligned}
& -u^2\alpha\beta\sin(\theta - \phi) + (1 + \kappa)\theta - \phi + u\alpha^2\beta\dot{\theta} + 2u\alpha\beta\cos(\theta - \phi)\dot{\phi} \\
& + \frac{1}{2}(3\alpha\sin(\theta - \phi)\dot{\phi}^2 + 2\alpha^2(3 + \alpha)\ddot{\theta} + 3\alpha\cos(\theta - \phi)\ddot{\phi}) = 0 \\
& \frac{3}{2}\alpha\cos(\theta - \phi)\ddot{\theta} - \frac{3}{2}\alpha\sin(\theta - \phi)\dot{\phi}^2 + \ddot{\phi} + \beta u\dot{\phi} + \phi - \theta = 0
\end{aligned} \tag{2.1}$$

where

$$\beta \triangleq \frac{3M}{(M + m)}, \quad \alpha \triangleq \frac{\ell_1}{\ell_2}, \quad \kappa \triangleq \frac{k_1}{k_2}$$

Using small angle assumption, the nonlinear equation of motion can be linearized to

$$\begin{aligned}
(\alpha^3 + 3\alpha^2)\ddot{\theta} + \frac{3}{2}\alpha\ddot{\phi} + \beta u(\alpha^2\dot{\theta} + 2\alpha\dot{\phi}) - \alpha\beta u^2(\theta - \phi) + (1 + \kappa)\theta - \phi &= 0 \\
\frac{3}{2}\alpha\ddot{\theta} + \ddot{\phi} + \beta u\dot{\phi} + \phi - \theta &= 0
\end{aligned} \tag{2.2}$$

In matrix form, Eq.(2.2) can be written as

$$\begin{bmatrix} (\alpha^3 + 3\alpha^2) & \frac{3}{2}\alpha \\ \frac{3}{2}\alpha & 1 \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{\phi} \end{bmatrix} + \begin{bmatrix} \beta u\alpha^2 & 2\beta u\alpha \\ 0 & \beta u \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} (1 + \kappa) - \alpha\beta u^2 & \alpha\beta u^2 - 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \theta \\ \phi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2.3}$$

Assuming $k_1 = k_2 = k \Rightarrow \kappa = 1$, $\ell_1 = \ell_2 = \ell \Rightarrow \alpha = 1$, $\beta = 0.1$, and starting from $u = 0$ results in purely imaginary roots - see Fig.2.2. As the non-dimensional fluid velocity u is varied from zero, the purely imaginary roots turn-into complex conjugate roots due to the damping effect that arises from internal fluid flow. By increasing u further, the second mode turns back towards the imaginary axis while the first mode keeps moving away from the axis. At $u_f = 5.007$, the system loses its stability as the second mode crosses the imaginary axis. Since this instability is associated with damping, it is commonly known by single-

mode flutter [61]. At the same time, increasing u further render a second loss of stability at $u_d = 7.434$ where the system exhibits divergence type of instabilities as shown in Fig.2.2.

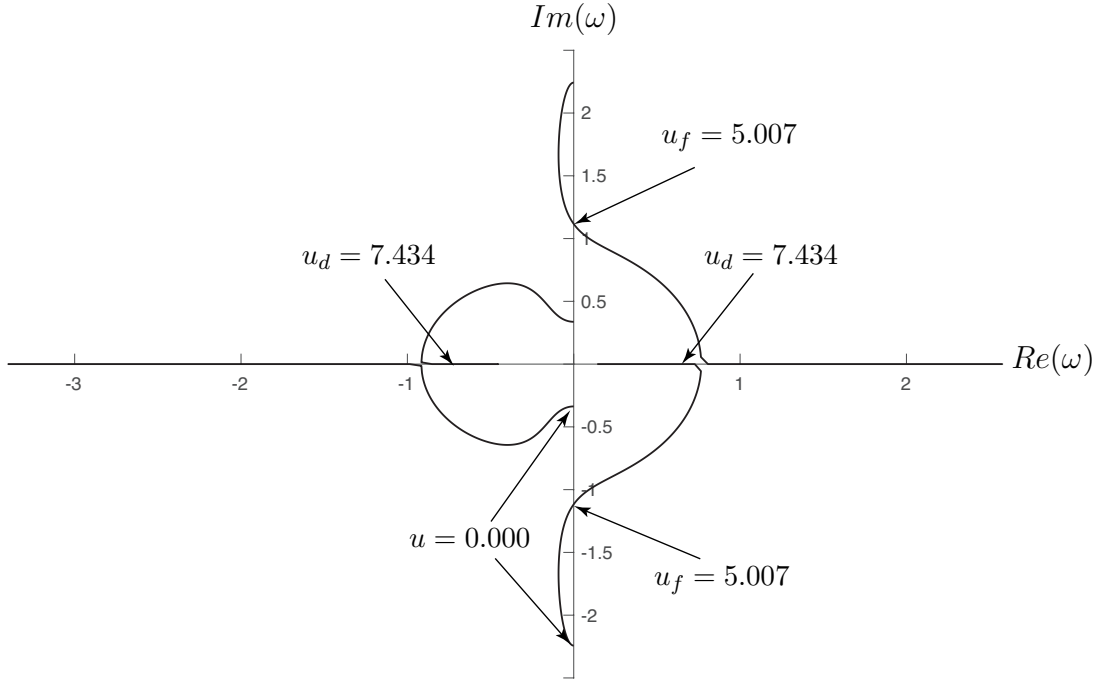


Figure 2.2: The locus of the system roots as the non-dimensional fluid velocity varied.

The linear system in Eq.(2.3) is simulated for an arbitrary initial conditions. The plot of θ is shown in Fig.2.3(a) for $u_f = 5.007$ and in Fig.2.3(b) for $u_d = 7.434$.

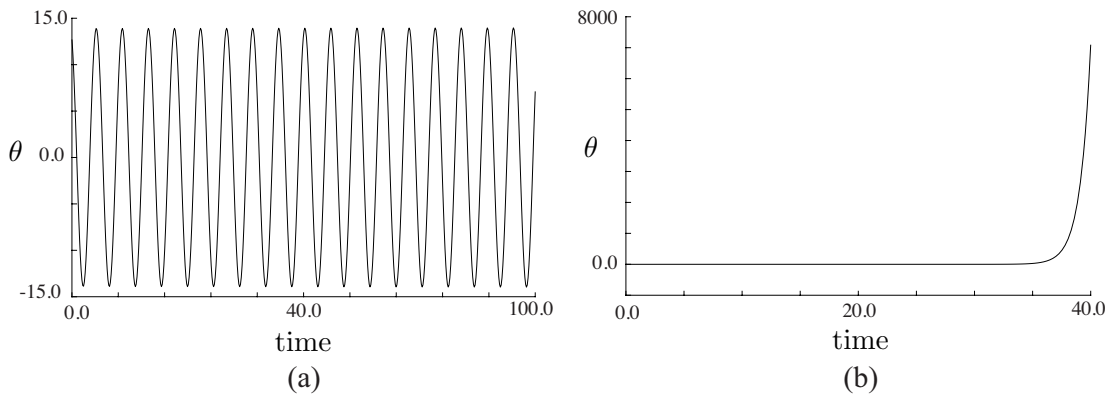


Figure 2.3: (a) Flutter instability for $u_f = 5.007$. (b) Divergence instability for $u_d = 7.434$.

According to Kounadis [11], the nonlinear system will not destabilize globally as the critical parameter exceeds its critical value. This can be seen by simulating the nonlinear

equation of motion in Eq.(2.1). The results, shown in Fig.2.4, indicate that the system exhibits a limit cycle at critical stability point as well as post flutter. The system might exhibit instability at much higher values of u , which is not shown here.

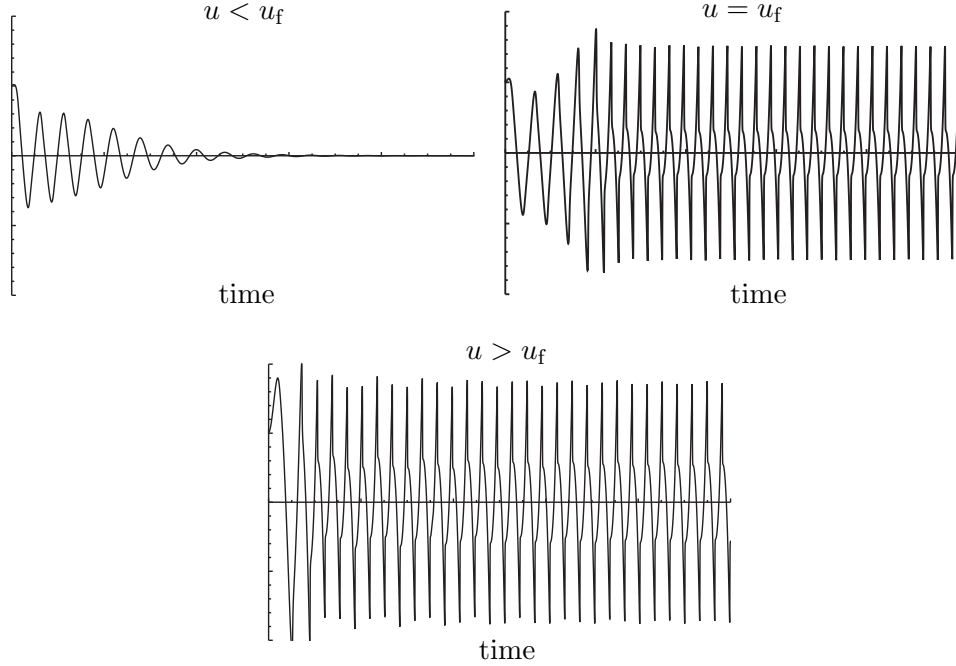


Figure 2.4: Nonlinear simulation of θ for different values of u .

The results presented in Figs.2.2-2.3 reflect two different dynamic stability perspectives, namely, local (linear) and global (nonlinear). It should be noted that the subsequent chapters are mainly going to focus on the local dynamic stability behaviors of slender structures. Ultimately, the main objective is to foretell the flutter instability points of these structures by estimating the critical values of the system parameters.

2.2 Continuous System Example

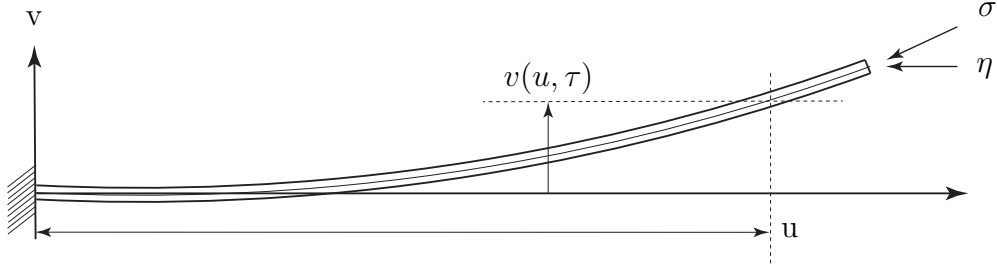


Figure 2.5: Cantilevered beam with end loads

The flutter and divergence instabilities in continuous system can be investigated using a straight cantilever beam subjected to both conservative and non-conservative loadings. These forces are an axial end load η and follower force σ applied to beam free-end, respectively, as shown in Fig.2.5. The motion of the beam is assumed to be planar and the transverse non-dimensional displacement of the beam is denoted by $v(u, \tau)$. Assuming small deformations, the non-dimensional equation of motion can be expressed as [62]:

$$v''''(u, \tau) + (\sigma + \eta)v''(u, \tau) + \ddot{v}(u, \tau) = 0 \quad (2.4)$$

where the geometric and natural boundary conditions correspond to cantilevered beam are

$$v(0, t) = 0, \quad v'(0, \tau) = 0, \quad v''(1, \tau) = 0, \quad v'''(1, \tau) + \eta v'(1, \tau) = 0 \quad (2.5)$$

The dynamic instabilities of the system can be examined through numerical analysis, in particular, using the Galerkin method [63], where the equation of motion Eq.(2.4) is multiplied by a weight function and integrated over the length of the beam. Assuming the solution to be of the form

$$v(u, \tau) = \sum_{i=1}^N a_i(\tau) \phi_i(u) \quad (2.6)$$

where N is the number of terms in the approximation, $\phi_i(u)$, $i = 1, \dots, N$, are functions

that satisfy the boundary conditions in Eq.(2.5), and $a_i(\tau)$, $i = 1, \dots, N$, are coefficients that will be determined. Substituting Eq.(2.6) into Eq.(2.4) and integrating over the length, yields

$$\mathbf{K}a + (\sigma + \eta)\mathbf{G}a + \mathbf{M}\ddot{a} = 0$$

where the (i, j) -th elements of the $\mathbf{K}$, $\mathbf{G}$ and $\mathbf{M}$ matrices are defined as

$$\begin{aligned}\mathbf{K}_{ij} &\triangleq -\eta\phi_i(1)\phi_j'''(1) + \int_0^1 \phi_i''(u)\phi_j''(u) du \\ \mathbf{G}_{ij} &\triangleq \phi_i(1)\phi_j'(1) - \int_0^1 \phi_i'(u)\phi_j'(u) du \\ \mathbf{M}_{ij} &\triangleq \int_0^1 \phi_i(u)\phi_j(u) du\end{aligned}$$

The eigenvalue problem can be performed for a given σ or η , where the first two natural frequencies can be depicted a function of the unknown loading. Figure 2.6(b) shows the first two natural frequencies as a function of the axial end load η for zero follower force, i.e. $\sigma = 0$. The result reveals a decreasing nature in the natural frequencies as η increases, and estimates the end loading at which the beam buckles to be $\eta_{cr} = 2.48$. Figure 2.7(b) shows the same first two natural frequencies as a function of the follower force σ for $\eta = 0$. Note that as σ increases, the first modal frequency increases while the second modal natural frequency decreases. These two frequencies merge, resulting in flutter instability, at a critical value of the follower force given by $\sigma_{cr} = 20.05$. To demonstrate this, the root loci of the system are depicted in Fig.2.6(c) and Fig.2.7(c), respectively. A loss of stability through divergence occurs when the lowest natural frequency of an undamped system passes through the origin as the equilibrium of the system shifts from stable to unstable, as shown in Fig.2.6(c). In contrast, in the absence of damping, flutter instability occurs when two successive natural frequencies approach each other, coexist, and then become complex conjugate – see Fig.2.7(c). According to Paidoussis [61], this instability is known by coupled-mode fluttering via Hamiltonian-Hopf bifurcation.

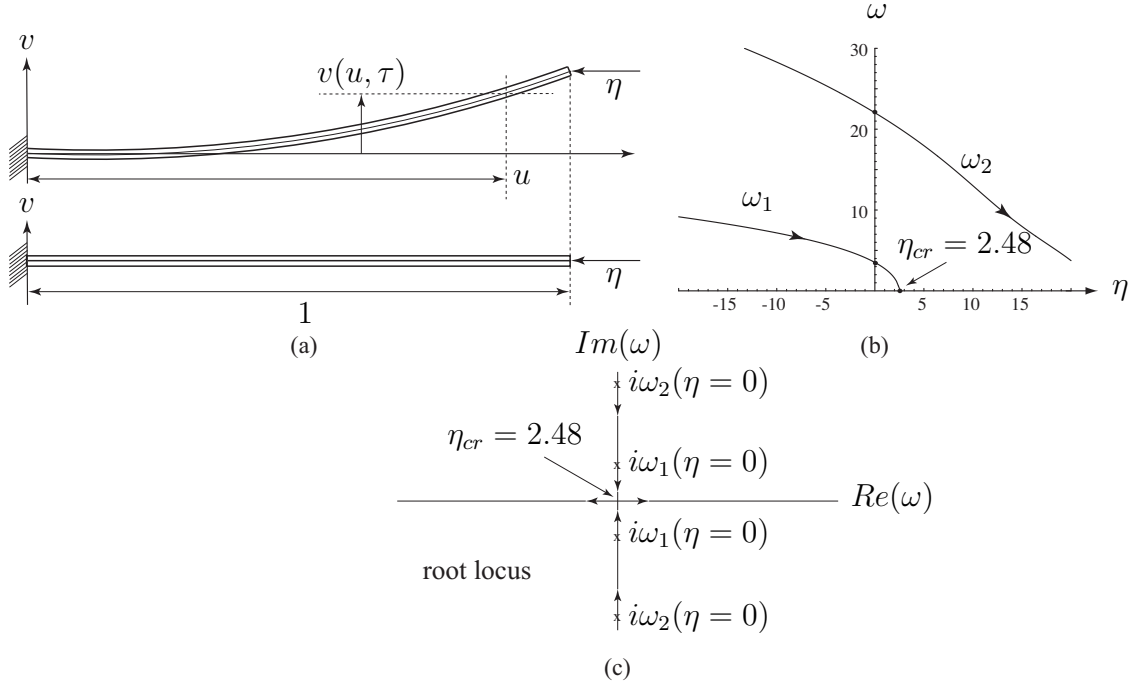


Figure 2.6: (a) Cantilevered beam with axial end loading (b) variation of the first two modal frequencies of a cantilevered beam as the axial loading increases (c) locus of the first two modal frequencies illustrating divergence instability.

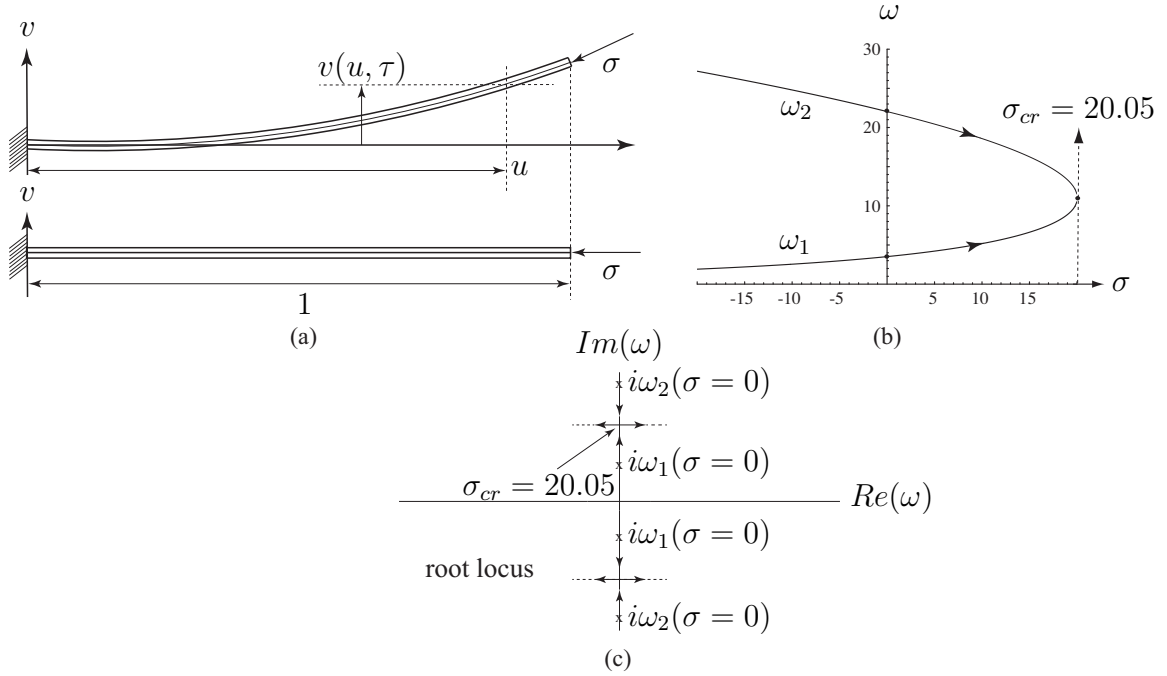


Figure 2.7: (a) Cantilevered beam with follower end loading (b) variation of the first two modal frequencies of a cantilevered beam as the follower loading increases (c) locus of the first two modal frequencies illustrating flutter instability.

Chapter 3

Stability Transitions Induced by Damping

Chapter 2 introduced the two types of instabilities in elastic structures, namely, divergence and flutter. Damping plays an important role in flutter instability as we have seen - double mode flutter - in the absence of damping is replaced by single mode flutter in the presence of damping. A counter-intuitive property of damping is that it can render some non-conservative systems unstable. This property is frequently referred to as Ziegler's paradox after the classical finding that the addition of damping destabilizes a stable double pendulum subjected to a follower end force. Since this finding, investigations of the phenomenon have continued to the present day. For example, It has been shown that some non-conservative systems may exhibit up to two stability transitions when the level of damping is increased while the non-conservative load is held fixed. This chapter focuses on additional stability transitions associated with damping; it is shown that a system which is unstable for zero damping can be stabilized, then destabilized, and finally restabilized as the level of damping is increased. This action is illustrated with the help of examples which show various taxonomies of damping-induced stabilization and destabilization behavior in various non-conservative systems.

3.1 Two Degree-Of-Freedom System with a Follower Force

Consider the two-link system shown in Fig.3.1, which is subjected to the follower force P . Constrained to move in the horizontal plane, the two links are comprised of point masses m_1 and m_2 , connected by massless rods of equal length ℓ . The rotational joints have torsional stiffness k_1 and k_2 and their damping coefficients are c_1 and c_2 . The stability characteristics

of the system can be investigated from the linearized equations of motion¹:

$$\begin{aligned} (m_1 + m_2)\ell^2\ddot{\theta}_1 + m_2\ell^2\ddot{\theta}_2 + (c_1 + c_2)\dot{\theta}_1 - c_2\dot{\theta}_2 + (k_1 + k_2 - P\ell)\theta_1 - (k_2 - P\ell)\theta_2 &= 0 \\ m_2\ell^2\ddot{\theta}_1 + m_2\ell^2\ddot{\theta}_2 - c_2\dot{\theta}_1 + c_2\dot{\theta}_2 - k_2\theta_1 + k_2\theta_2 &= 0 \end{aligned} \quad (3.1)$$

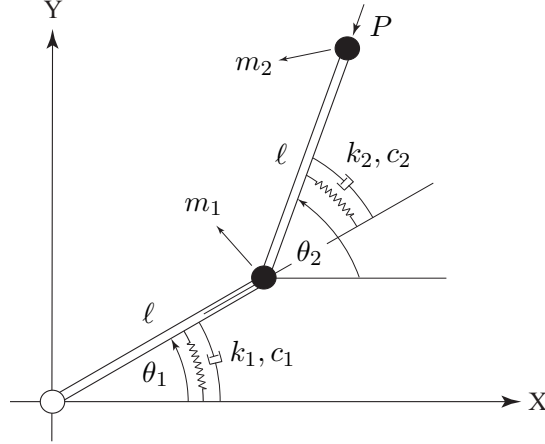


Figure 3.1: A two-link system with a follower force.

For a comparison with the results in [15], we assume $m_1 = 2m_2 = 2m$, $k_1 = k_2 = k$, and scale the time, damping and force variables as follows:

$$\tau = \sqrt{k/m\ell^2} t, \quad \gamma_1 = \sqrt{1/km\ell^2} c_1, \quad \gamma_2 = \sqrt{1/km\ell^2} c_2, \quad p = (\ell/k) P \quad (3.2)$$

to obtain the non-dimensional equations [15]:

$$\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} \gamma_1 + \gamma_2 & -\gamma_2 \\ -\gamma_2 & \gamma_2 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 2 - p & p - 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (3.3)$$

In Eq.(3.3), the $(\dot{})$ and $(\ddot{})$ denote the first and second derivatives with respect to the non-

¹The equations of motion presented here are a special case of the equations presented in [64].

dimensional time variable τ . Assuming the solution to be of the form

$$\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix} e^{s\tau}$$

we get the characteristic polynomial

$$\begin{aligned} \text{Det} \begin{bmatrix} 3s^2 + (\gamma_1 + \gamma_2)s + 2 - p & s^2 - \gamma_2s + p - 1 \\ s^2 - \gamma_2s - 1 & s^2 + \gamma_2s + 1 \end{bmatrix} &= 0 \\ \Rightarrow 2s^4 + (\gamma_1 + 6\gamma_2)s^3 + (7 + \gamma_1\gamma_2 - 2p)s^2 + (\gamma_1 + \gamma_2)s + 1 &= 0 \end{aligned} \quad (3.4)$$

The Routh array [57] can be constructed as follows:

s^4	2	$(7 + \gamma_1\gamma_2 - 2p)$	1
s^3	$(\gamma_1 + 6\gamma_2)$	$(\gamma_1 + \gamma_2)$	
s^2	ϵ_{21}	1	
s^1	ϵ_{11}		
s^0	1		

where ϵ_{21} and ϵ_{11} are given by the expressions

$$\begin{aligned} \epsilon_{21} &= \frac{(\gamma_1 + 6\gamma_2)(7 + \gamma_1\gamma_2 - 2p) - 2(\gamma_1 + \gamma_2)}{\gamma_1 + 6\gamma_2} \\ \epsilon_{11} &= \frac{(\gamma_1 + 6\gamma_2)(7 + \gamma_1\gamma_2 - 2p)(\gamma_1 + \gamma_2) - 2(\gamma_1 + \gamma_2)^2 - (\gamma_1 + 6\gamma_2)^2}{(\gamma_1 + 6\gamma_2)(7 + \gamma_1\gamma_2 - 2p) - 2(\gamma_1 + \gamma_2)} \end{aligned} \quad (3.5)$$

Flutter instability is characterized by a pair of complex conjugate roots with zero real part. The characteristic polynomial in Eq.(3.4) will consequently have a second-order auxilliary polynomial [57] as a factor. This corresponds to the third case of the Routh-Hurwitz criterion

[57], and necessitates

$$\begin{aligned} \epsilon_{11} &= 0 \\ \Rightarrow (\gamma_1 + 6\gamma_2)(7 + \gamma_1\gamma_2 - 2p)(\gamma_1 + \gamma_2) - 2(\gamma_1 + \gamma_2)^2 - (\gamma_1 + 6\gamma_2)^2 &= 0 \end{aligned} \quad (3.6)$$

To ensure that the roots are purely imaginary, we additionally need $\epsilon_{21} > 0$. The critical frequency can then be obtained as follows

$$(\epsilon_{21}s^2 + 1) = 0 \quad \Rightarrow \quad \omega_{\text{cr}} = \sqrt{\frac{1}{\epsilon_{21}}} = \sqrt{\frac{(\gamma_1 + 6\gamma_2)}{(\gamma_1 + 6\gamma_2)(7 + \gamma_1\gamma_2 - 2p) - 2(\gamma_1 + \gamma_2)}} \quad (3.7)$$

Remark 3.1: An auxilliary polynomial, by definition, is symmetric with respect to both the real and imaginary axes [57]. This implies that a second-order auxilliary polynomial must be comprised of either a pair of roots on the imaginary axis located symmetrically with respect to the real axis, or a pair of roots on the real axis located symmetrically with respect to the imaginary axis. The first case is associated with the onset of flutter instability and will be the subject of investigation of this paper. The second case will not be pursued further as it corresponds to a system that is already unstable in divergence, and not a system at the onset of instability.

We now investigate two separate cases that are discussed below.

3.1.1 Effect of γ_1 on the Stability of the System

To this end, we rewrite the left-hand side of Eq.(3.6) as a polynomial in γ_1 :

$$\alpha_1 \gamma_1^3 + \alpha_2 \gamma_1^2 + \alpha_3 \gamma_1 + \alpha_4 = 0 \quad (3.8)$$

$$\alpha_1 \triangleq \gamma_2, \quad \alpha_2 \triangleq 7\gamma_2^2 + 4 - 2p, \quad \alpha_3 \triangleq \gamma_2(6\gamma_2^2 + 33 - 14p), \quad \alpha_4 \triangleq 4\gamma_2^2(1 - 3p) \quad (3.9)$$

The coefficients of the polynomial, α_i , $i = 1, 2, 3, 4$, are functions of p and γ_2 ; here, both of

them are assumed to be positive. The discriminant of the cubic equation in Eq.(3.8), namely

$$\Delta = 18\alpha_1\alpha_2\alpha_3\alpha_4 - 4\alpha_2^3\alpha_4 + \alpha_2^2\alpha_3^2 - 4\alpha_1\alpha_3^3 - 27\alpha_1^2\alpha_4^2 \quad (3.10)$$

determine the number and types of roots of γ_1 [65]. In particular, Eq.(3.8) has

- three distinct real roots if $\Delta > 0$.
- a multiple root and all its roots are real if $\Delta = 0$.
- has one real root and two complex conjugate roots if $\Delta < 0$.

For a real root to be an admissible solution of γ_1 , it has to be positive. On the other hand, the number of roots with positive real parts will be equal to the number of sign changes [57] in the first column of the Routh array constructed using the coefficients of the cubic polynomial in Eq.(3.8), namely

$$\begin{array}{c|cc} \gamma_1^3 & \gamma_2 & \gamma_2(6\gamma_2^2 + 33 - 14p) \\ \gamma_1^2 & (7\gamma_2^2 + 4 - 2p) & 4\gamma_2^2(1 - 3p) \\ \hline \gamma_1^1 & \mu_{11} & \\ \gamma_1^0 & 4\gamma_2^2(1 - 3p) & \end{array} \quad (3.11)$$

where

$$\mu_{11} \triangleq \gamma_2(6\gamma_2^2 + 33 - 14p) - \frac{4\gamma_2^3(1 - 3p)}{(7\gamma_2^2 + 4 - 2p)}$$

The number of positive real roots can therefore be deduced from the sign of Δ in Eq.(3.10) and the number of sign changes in the first column of the Routh array in Eq.(3.11). This is illustrated with the help of a numerical example.

Example 3.1: Consider the case where $\gamma_2 = 1.0$. From Eqs.(3.9) and (3.10), we have

$$\begin{aligned} \alpha_1 &= 1, \quad \alpha_2 = 11 - 2p, \quad \alpha_3 = 39 - 14p, \quad \alpha_4 = 4(1 - 3p) \\ \Rightarrow \quad \Delta &= 25(16p^4 - 64p^3 - 104p^2 + 1008p - 1763) \end{aligned}$$

When p is assumed positive, it can be shown $\Delta = 0$ for $p = p^* \approx 3.245$. For $p = p^*$, we can use Eq.(3.8) to compute $\gamma_1 = \gamma_1^* \approx 2.69$. For $p \in [0, p^*)$, $\Delta < 0$ and therefore there is only one real root of γ_1 . Using the first column of the Routh array in Eq.(3.11), it can be shown that there will be only one sign change for $p \in (0, \bar{p})$. This implies that the critical load $p_{cr} = 0.\bar{3}$ for $\gamma_1 = 0$ and it increases to $p_{cr} = p^*$ as γ_1 increases to γ_1^* . For $p \geq p^*$, $\Delta \geq 0$ and therefore there are three real roots of γ_1 . However, only one of these three roots will be positive; this can be concluded from the single sign change in the first column of the Routh array in Eq.(3.11). These results, which can be verified from the plot of p_{cr} in Fig.3.2, implies that γ_1 has a purely stabilizing (S) effect when $\gamma_2 = 1.0$. The positive sign of ϵ_{21} in Eq.(3.5) can be used to ascertain that the nature of the instability is flutter, the critical frequency can be determined using Eq.(3.7).

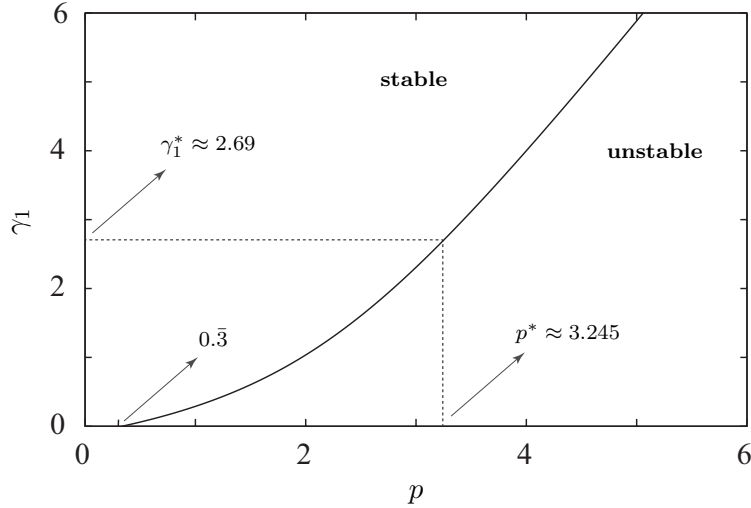


Figure 3.2: Absolute instability curve in γ_1 - p space for the two-link system in Fig.3.1 when $\gamma_2 = 1.0$. For $p \in [0, p^*)$, there is only one real root of γ_1 . For $p \geq p^*$, there are three real roots of γ_1 but only one of them is positive.

We now introduce the following definitions:

Flutter instability curve: The flutter instability curve is the locus of points in the parameter space for which a pair of complex conjugate roots maintain their position on the imaginary axis.

Absolute instability curve: The absolute instability curve is the locus of points in the parameter space for which a pair of complex conjugate roots lie on the imaginary axis while all other roots lie in the open left-half plane.

Remark 3.2: The curve in Fig.3.2 is the flutter instability curve corresponding to the first mode of the system in the γ_1 - p space; it is also the absolute instability curve.

3.1.2 Effect of γ_2 on the Stability of the System

To investigate the effect of γ_2 on the stability of the system, we rewrite the left-hand side of Eq.(3.6) as a polynomial in γ_2

$$\beta_1 \gamma_2^3 + \beta_2 \gamma_2^2 + \beta_3 \gamma_2 + \beta_4 = 0 \quad (3.12)$$

$$\beta_1 \triangleq 6\gamma_1, \quad \beta_2 \triangleq 7\gamma_1^2 + 4(1 - 3p), \quad \beta_3 \triangleq \gamma_1(\gamma_1^2 + 33 - 14p), \quad \beta_4 \triangleq \gamma_1^2(4 - 2p) \quad (3.13)$$

and consider the specific example discussed below.

Example 3.2: Consider the case where $\gamma_1 = 1.0$. Using Eqs.(3.13) we get

$$\beta_1 = 6, \quad \beta_2 = 11 - 12p, \quad \beta_3 = 34 - 14p, \quad \beta_4 = 2(2 - p)$$

The discriminant of the cubic polynomial in Eq.(3.12) and the Routh array are as follows

$$\Delta = 100(144p^4 - 936p^3 + 409p^2 + 5172p - 6787)$$

γ_2^3	6	$(34 - 14p)$	$\nu_{11} \triangleq \frac{2(84p^2 - 275p + 175)}{(11 - 12p)}$
γ_2^2	$(11 - 12p)$	$2(2 - p)$	
γ_2^1	ν_{11}		
γ_2^0	$2(2 - p)$		

When p is assumed positive, it can be shown $\Delta = 0$ for $p = \{p_1^*, p_2^*, p_3^*\} \approx \{1.964, 2.144, 4.75\}$.

For $p = p_1^*$ and p_2^* , we can use Eq.(3.12) to compute the γ_2 values to be ≈ 1.052 and ≈ 0.149 . We focus our attention on the domain $p \in [p_1^*, p_2^*]$ where $\Delta \geq 0$; this implies that all three roots of γ_2 are real. Within this domain, it can be shown that there will be two sign changes in the first column of the Routh array above for $p \in [p_1^*, 2)$ and three sign changes for $p \in (2, p_2^*]$. These results can be verified from the plot of p_{cr} vs γ_2 in Fig.3.3 (a). Based on the notion introduced in [15], γ_2 has a stabilizing effect in the range $[0, 0.149]$; a destabilizing effect in the range $[0.149, 1.052]$; and finally a stabilizing effect for $\gamma_2 > 1.052$. This anomalous property, which is attributed to three positive real roots of γ_2 , has not been reported in the literature to the best of our knowledge. The locus of the roots that determine the stability of the system are plotted in Fig.3.3 (b) for the particular case of $p = 2.08$. It can be seen that the system is unstable for $\gamma_2 \in (0.00, 0.036) \cup (0.382, 1.907)$ and stable for $\gamma_2 \in (0.036, 0.382) \cup (1.907, \infty)$. Similar to Example 3.1, the positive sign of ϵ_{21} in Eq.(3.5) can be used to ascertain that the nature of the instability is flutter, the critical frequency can be determined using Eq.(3.7). The curve in Fig.3.3 is the flutter instability curve corresponding to the second mode of the system in the γ_2 - p space; it is also the absolute instability curve.

Remark 3.3: Example 3.1 showed that γ_1 has a purely stabilizing effect and Example 3.2

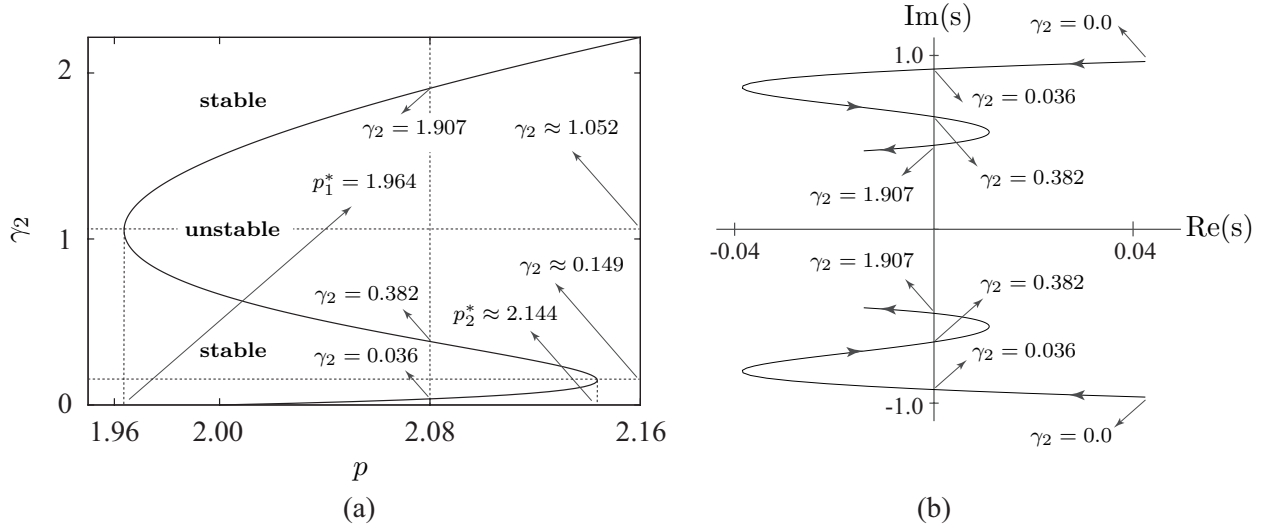


Figure 3.3: Two-link system in Fig.3.1 with $\gamma_1 = 1.0$: (a) Absolute instability curve in γ_2 - p space showing the S-D-S effect of damping (b) locus of the roots that determine system stability as γ_2 increases for the particular case of $p = 2.08$.

showed that γ_2 has a heretofore unobserved “stabilizing-destabilizing-stabilizing” (S-D-S) effect. It should be mentioned that γ_1 can also exhibit the stabilizing-destabilizing-stabilizing (S-D-S) effect for specific values of γ_2 , and γ_2 can have a purely stabilizing effect for specific values of γ_1 .

Remark 3.4: In Example 4.2, the value of γ_1 was assumed to be equal to 1.0. For $\gamma_1 \in \{0.02, 0.05, 0.10\}$, γ_2 exhibits the same stabilizing-destabilizing-stabilizing (S-D-S) effect. For all three cases, there exist a range of p values for which γ_2 has three positive real roots. However, two of the three roots are very close to one another and the third root is distantly placed. For example, for $\gamma_1 = 0.1$ and $p = 2.05$, the roots of γ_2 are 0.0026, 0.0183 and 34.1956. The presence of the third root was overlooked in [15] for all three cases; consequently, the authors reached the conclusion that γ_2 has a stabilizing-destabilizing (S-D) effect as opposed to the stabilizing-destabilizing-stabilizing (S-D-S) effect.

Remark 3.5: The stabilizing and destabilizing effect of damping was defined in a broader sense in [15]. Based on this definition, the damping is said to be stabilizing when the slope of the flutter instability curve is positive, it is said to be destabilizing when the slope is negative. It will be shown later with the help of Example 3.7 that this definition is not suitable when flutter instability curves of different modes intersect each other and the absolute stability curve is comprised of segments of multiple flutter instability curves.

Overall, the role of damping in the joints on the system stability can be investigated using the discriminant of the cubic equations in Eq.(3.8) and Eq.(3.12), respectively. For instance, the contour plot of Δ , Eq.(3.10), in γ_2 - p plane is depicted in Fig.3.4 to show the positive and the negative regions of Δ . A negative value of Δ implies that there can be at most one real root, and therefore at most one admissible (positive and real) solution; this implies that γ_1 will certainly not exhibit the stabilizing-destabilizing-stabilizing effect. A positive value of Δ implies that there exists three real roots of γ_1 . However, all of these roots may not be admissible solutions; therefore, γ_1 may or may not exhibit the stabilizing-destabilizing-stabilizing effect.

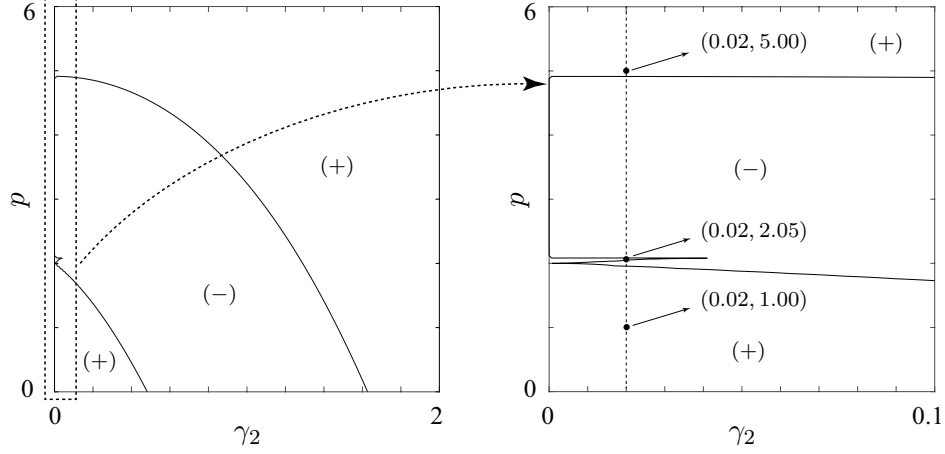


Figure 3.4: Contour plot for $\Delta = 0$ in γ_2 - p plane for the two-link system with follower force: regions with positive and negative values of Δ are denoted by (+) and (-), respectively.

It can be seen from Fig.3.4 that Δ is positive for $\gamma_2 = 0.02$ and $p = 1.00, 2.05$, and 5.00 , respectively. Therefore, it may be possible to obtain three admissible solutions for γ_1 .

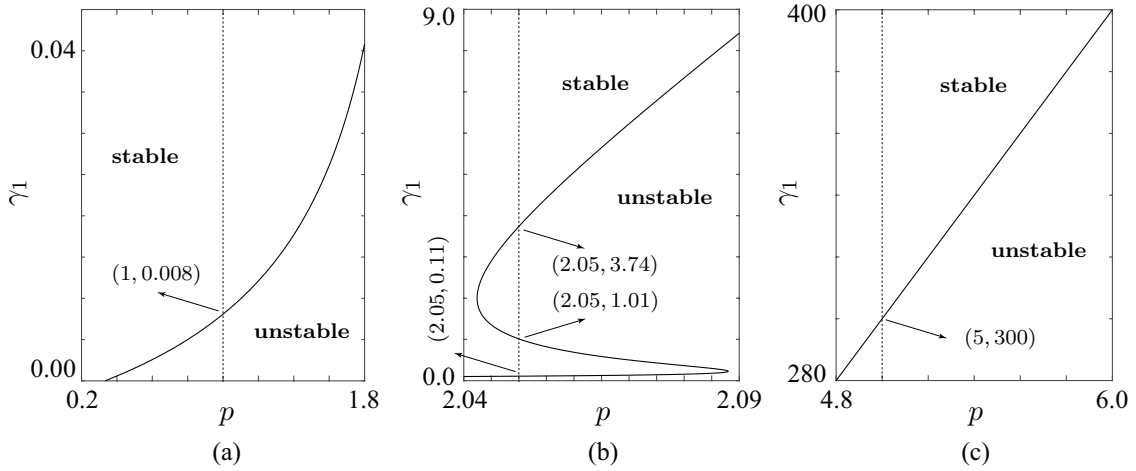


Figure 3.5: Flutter instability curves in γ_1 - p space for the two-link system with follower force for: (a) $\gamma_2 = 0.02$ and $p = 1$, (b) $\gamma_2 = 0.02$ and $p = 2.05$, (c) $\gamma_2 = 0.02$ and $p = 5$.

Figure 3.5 (a) and (c) show that γ_1 has purely stabilizing effects since they have a single stability transition. Figure 3.5 (b) reveals a “stabilizing-destabilizing-stabilizing” (S-D-S) effect due to three stability transitions. These results can also be verified from the number of sign changes in the first column of the Routh array in Eq.(3.11). For (a) and (c), a single sign change is obtained. However, three sign changes are observed in (b). The locus of the eigenfrequencies of the system are plotted in Fig.3.6: γ_1 is increased from zero for the cases

shown in Fig.3.5.

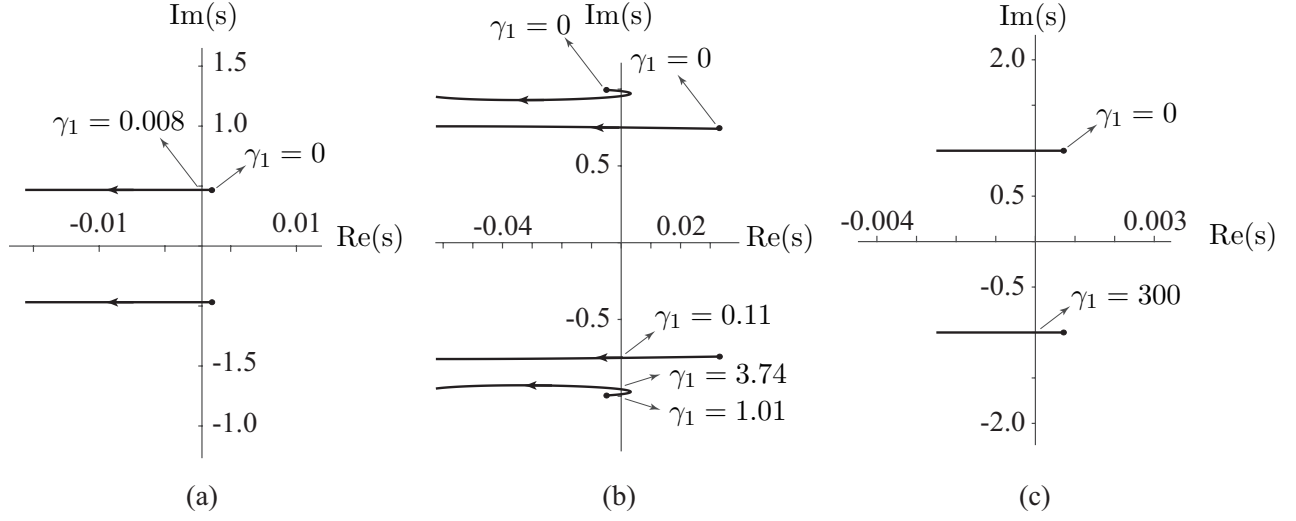


Figure 3.6: The locus of the roots that determine system stability as γ_1 increases for the particular cases: (a) $\gamma_2 = 0.02$ and $p = 1$, (b) $\gamma_2 = 0.02$ and $p = 2.05$, (c) $\gamma_2 = 0.02$ and $p = 5$.

It is worth noting that for the case where the system exhibited the (S-D-S) effect of damping ($\gamma_2 = 0.02$, $p = 2.05$), the system is originally unstable. As γ_1 is increased, the system is stabilized when the fundamental complex conjugate pair of roots cross the imaginary axis. When γ_1 is increased further, the system is destabilized by the second pair of complex conjugate roots which cross the imaginary axis to move from the left-half plane to the right-half plane. For even higher values of γ_1 the system regains stability at $\gamma_1 = 3.74$. This interesting behavior, where the S-D-S effect is produced by two pairs of complex conjugate roots, is not discussed in the literature.

To complete the investigation, the flutter instability curve in p - γ_1 space is plotted for a range of γ_2 values in the range $[0.01, 0.10]$ - see Fig.3.7. The figure discloses that at higher values γ_2 ($\gamma_2 > 0.06$), γ_1 loses its ability to produce the three stability transitions (stabilizing-destabilizing-stabilizing). On contrary, for lower values of γ_2 the system does exhibit those three stability transitions.

Similarly, the role of damping in the second link on the system stability is investigated through the contour plot of the discriminant - see Fig.3.8. Consider the horizontal dotted

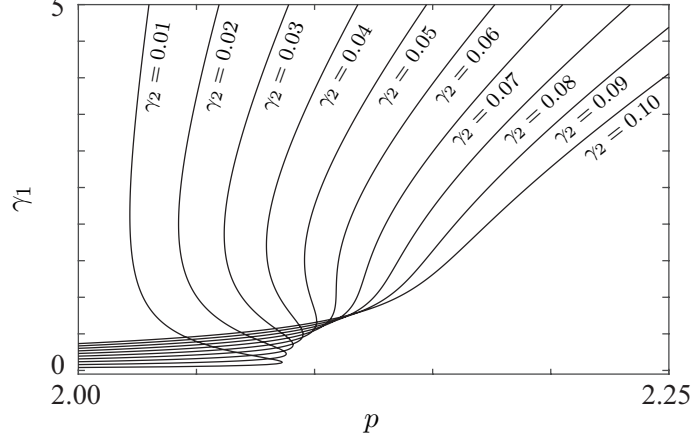


Figure 3.7: Flutter instability curves in p - γ_1 space for the two-link system with follower force and range of $\gamma_2 \in [0.01, 0.10]$

line in Fig.3.8 where $p = 2$. For $\gamma_1 \in [0.25, 2.00]$, the discriminant Δ is positive for four values of γ_1 and negative for the other four values. A plot of the flutter instability curves in p - γ_2 space for $\gamma_1 \in [0.25, 2.00]$, show that for $\gamma_1 > 1$ the discriminant $\Delta < 0$. Consequently, Eq.(3.12) has one admissible solution, which also can be verified from a single sign change in the first column of Routh array. At the same time, for $\gamma_1 < 1$, the discriminant $\Delta > 0$, which may result in three admissible solution for (3.12). The first column of the array shows three sign changes and accordingly it is possible to obtain three admissible solutions for γ_2 . Figure 3.9 shows both cases where Eq.(3.12) has single and three stability transitions for a

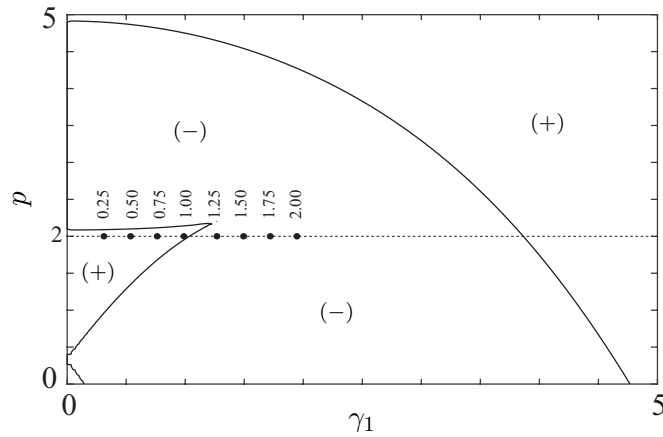


Figure 3.8: Contour plot for $\Delta = 0$ in γ_1 - p plane for the two-link system with follower force: regions with positive and negative values of Δ are denoted by (+) and (-), respectively.

range of $\gamma_1 \in [0.25, 2.00]$.

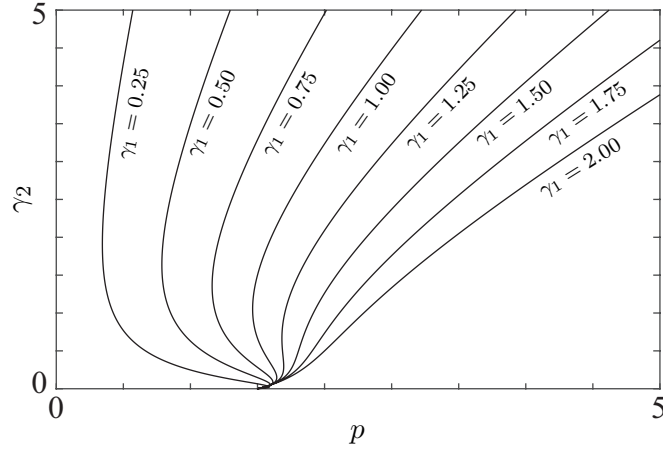


Figure 3.9: Flutter instability curves in p - γ_2 space for the two-link system with follower force and range of $\gamma_1 \in [0.25, 2.00]$.

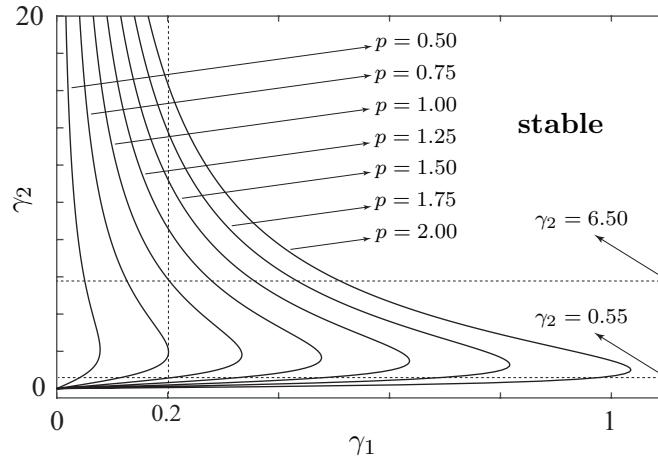


Figure 3.10: Flutter instability curves in γ_1 - γ_2 space for the two-link system with follower force $p \in [0.5, 2.0]$.

As a final analysis, the effect of γ_1 and γ_2 on the stability of the system is examined for a range of p values in $[0.5, 2.0]$; this is shown with the help of Fig.3.10. The system is initially unstable for $\gamma_1 = \gamma_2 = 0$ and $p \in [0.5, 2.0]$. Increasing γ_1 in the range $[0, 1.1]$, the system gains stability based upon the value of p . In the meantime, increasing γ_2 caused the system losing its stability. However, Increasing γ_2 beyond a certain value (6.5 for example) for the range of p the two-link system regains stability. To this end, this work extensively studied the two-link articulated rod subjected to follower end force and highlighted the condition on

which stabilizing-destabilizing-stabilizing transition in the system stability occurs due to the application of damping.

3.2 Two Degree-Of-Freedom Fluid-Conveying Pipe

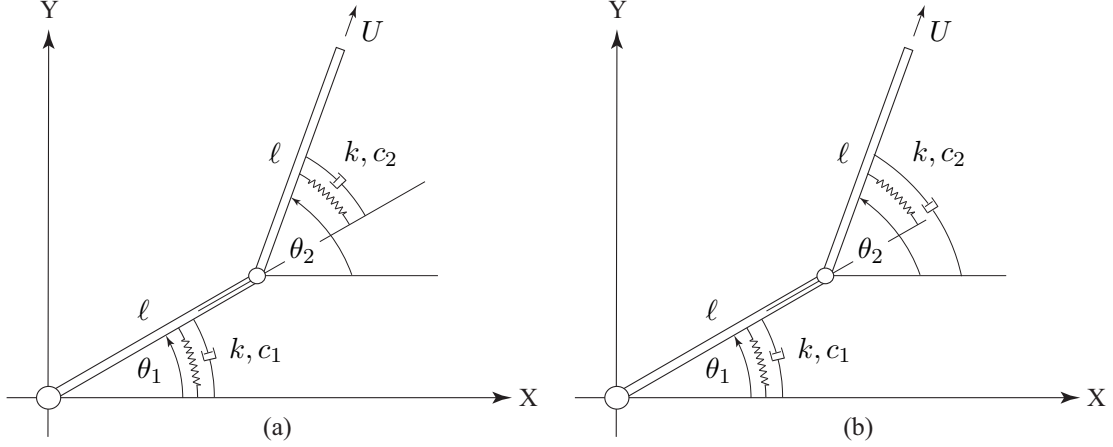


Figure 3.11: (a) Relative damping and (b) absolute damping is introduced in the fluid-conveying pipe considered by Benjamin [1] with equal link lengths ℓ and equal torsional joint stiffness k .

The two-link articulated fluid conveying pipe studied by Benjamin [1] is investigated here under the assumptions that the link lengths are equal, the torsional stiffness of the joints are the same, and the motion of the links are restricted to the horizontal plane. Two different models of damping, namely, relative damping and absolute damping, are introduced in the system; this is illustrated with the help of Figs.3.11 (a) and (b). We use the same scaling of variables in [1] together with the following scaling for the damping variable

$$\gamma_1 = c_1 \sqrt{\frac{3}{(M+m)k\ell^3}}, \quad \gamma_2 = c_2 \sqrt{\frac{3}{(M+m)k\ell^3}}$$

where M and m denote the mass per unit length of the fluid and pipe, respectively. We investigate the effect of damping on the stability of the system for the two damping models separately.

3.2.1 Effect of Relative Damping on the Stability of the System

Example 3.3: Consider the case where $\gamma_1 = 0$ and $\gamma_2 = \gamma$. The non-dimensional equations of motion are given as follows:

$$\begin{bmatrix} 4 & 3/2 \\ 3/2 & 1 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} \beta u + \gamma & 2\beta u - \gamma \\ -\gamma & \beta u + \gamma \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 2 - \beta u^2 & \beta u^2 - 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (3.14)$$

where θ_1 and θ_2 denote the absolute joint angles of the two links as shown in Fig.3.11, u denotes the non-dimensional fluid velocity, and β is the mass fraction

$$\beta \triangleq 3M/(M + m)$$

Using the procedure outlined in Section 3.1, the characteristic polynomial can be obtained as follows:

$$\begin{aligned} &1.75s^4 + (8\gamma + 2\beta u)s^3 + (9 + 4\gamma\beta u + \beta^2 u^2 \\ &- 2.5\beta u^2)s^2 + (\gamma + 5\beta u - \beta^2 u^3)s + 1 = 0 \end{aligned} \quad (3.15)$$

Constructing the Routh array from Eq.(3.15), the conditions for flutter instability can be expressed by the following cubic equation in γ :

$$\begin{aligned} &\alpha_1 \gamma^3 + \alpha_2 \gamma^2 + \alpha_3 \gamma + \alpha_4 = 0 \\ &\alpha_1 \triangleq 128\beta u \\ &\alpha_2 \triangleq \beta u^2(-128\beta^2 u^2 + 704\beta - 80) + 25 \\ &\alpha_3 \triangleq \beta u(-64\beta^3 u^4 + 80\beta^2 u^4 + 328\beta^2 u^2 - 694\beta u^2 + 1314) \\ &\alpha_4 \triangleq \beta^2 u^2(-8\beta^3 u^4 + 13\beta^2 u^4 + 40\beta^2 u^2 - 102\beta u^2 + 169) \end{aligned} \quad (3.16)$$

The discriminant of the cubic polynomial Δ can be evaluated using Eq.(3.10). The contour plot in Fig.3.12 shows the regions where Δ is positive and negative. A negative value of

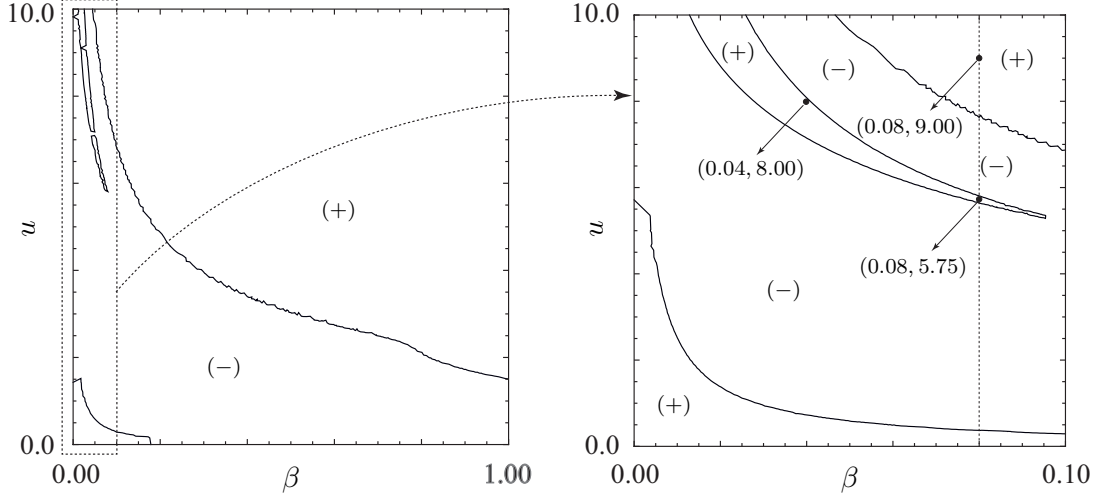


Figure 3.12: Contour plot for $\Delta = 0$ in β - u plane for the two-link fluid-conveying pipe with relative damping: regions with positive and negative values of Δ are denoted by (+) and (-), respectively.

Δ implies that there can be at most one real root, and therefore at most one admissible (positive and real) solution; this implies that γ will certainly not exhibit the stabilizing-destabilizing-stabilizing effect. A positive value of Δ implies that there exists three real roots of γ . However, all of these roots may not be admissible solutions; therefore, γ may or may not exhibit the stabilizing-destabilizing-stabilizing effect. It can be seen from Fig.3.12 that Δ is positive for $\beta = 0.08$ and $u = 5.75$; therefore, it not surprising that γ has three admissible solutions - see Fig.3.13.

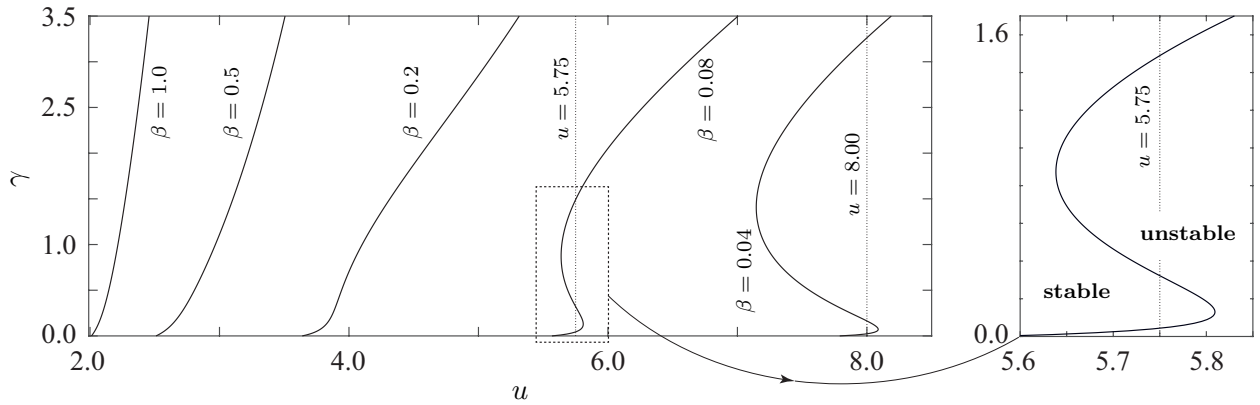


Figure 3.13: Absolute instability curves in γ - u space for the two-link pipe with relative damping for different values of β ; the S-D-S effect of damping is seen for $\beta = 0.04$ and 0.08 .

The flutter instability curves for different values of β are shown in Fig.3.13. Among the different β values considered in Fig.3.13, γ has a stabilizing-destabilizing-stabilizing effect only for $\beta = 0.04$ and 0.08 . This can be verified from the three intersections of the critical stability curve for $\beta = 0.04$ with the vertical line $u = 8.00$ and the critical stability curve for $\beta = 0.08$ with the vertical line $u = 5.75$. From Fig.3.12 it can be verified that the points $(\beta, u) = (0.04, 8.00)$ and $(0.08, 5.75)$ lie in the region where $\Delta > 0$.

Remark 3.6: From the results in Fig.3.13 it can be inferred that the damping parameter γ has a stabilizing-destabilizing-stabilizing (S-D-S) effect for small values of β .

3.2.2 Effect of Absolute Damping on the Stability of the System:

Example 3.4: Consider the case where $\gamma_1 = \gamma_2 = \gamma$. The non-dimensional equations of motion are given as follows:

$$\begin{bmatrix} 4 & 3/2 \\ 3/2 & 1 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} \beta u + \gamma & 2\beta u \\ 0 & \beta u + \gamma \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 2 - \beta u^2 & \beta u^2 - 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (3.17)$$

The characteristic polynomial is given by the expression:

$$\begin{aligned} 1.75s^4 + (5\gamma + 2\beta u)s^3 + (9 + \gamma^2 + 2\gamma\beta u + \beta^2 u^2 - 2.5\beta u^2)s^2 \\ + (3\gamma + 5\beta u - \gamma\beta u^2 - \beta^2 u^3)s + 1 = 0 \end{aligned} \quad (3.18)$$

Constructing the Routh array from Eq.(3.18), the condition for flutter instability can be described by a quartic equation of γ ; this implies that γ may have up to four admissible solutions. However, a plot of the critical stability curves for different values of β and u indicate that γ has one, two, or three admissible (positive and real) solutions. Three specific cases are shown in Fig.3.14; it can be seen that γ has a purely destabilizing (S) effect when the quartic equation has one admissible solution - see Fig.3.14(a), a stabilizing-destabilizing (S-D) effect when the quartic equation has two admissible solutions - see Fig.3.14(b), and

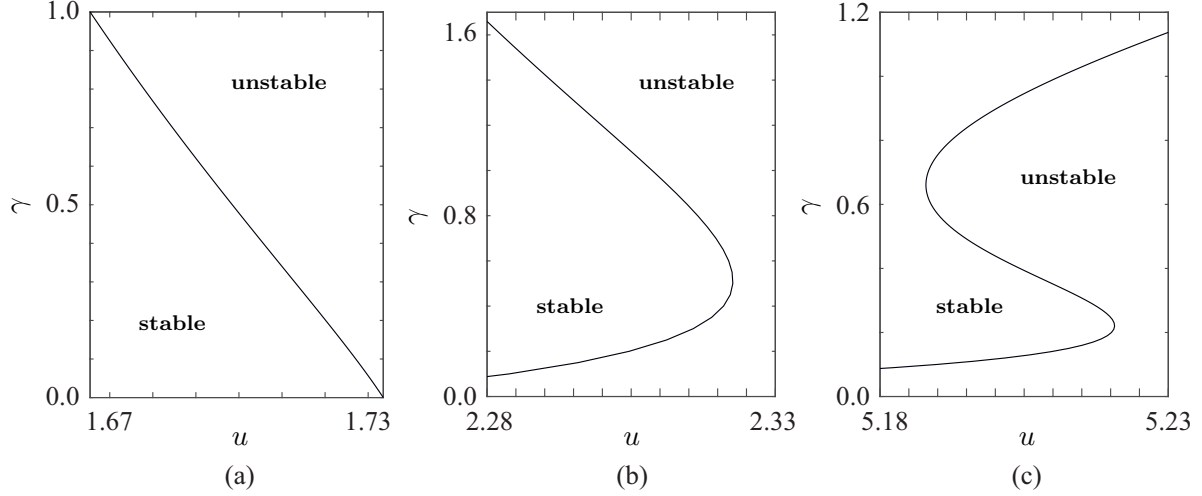


Figure 3.14: Absolute instability curves in γ - u space for the two-link pipe with absolute damping showing D, S-D, and S-D-S effects for (a) $\beta = 1.50$, (b) $\beta = 0.70$, and (c) $\beta = 0.10$, respectively. For (b) and (c), the critical flow velocity u_{cr} for $\gamma = 0$ lies outside the range of u ; these values are 2.245 for $\beta = 0.70$ and 5.007 for $\beta = 0.10$.

a stabilizing-destabilizing-stabilizing (S-D-S) effect when the quartic equation has three admissible solutions - see Fig.3.14(c).

3.3 Three Degree-Of-Freedom Systems

3.3.1 Three-Link System with a Follower Force

Consider the three-link system with the follower force P , shown in Fig.3.15. Each link is comprised of a point mass m and a massless rod of length ℓ and is constrained to move in the horizontal plane. The rotational joints have the same torsional stiffness k but their damping coefficients are different, equal to c_1 , c_2 and c_3 , respectively. The stability characteristics of the system can be investigated from the non-dimensional linearized equations of motion:

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{bmatrix} + \begin{bmatrix} \gamma_1 + \gamma_2 & -\gamma_2 & 0 \\ -\gamma_2 & \gamma_2 + \gamma_3 & -\gamma_3 \\ 0 & -\gamma_3 & \gamma_3 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} + \begin{bmatrix} 2 - p & -1 - p & 2p \\ -1 & 2 - p & -1 + p \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (3.19)$$

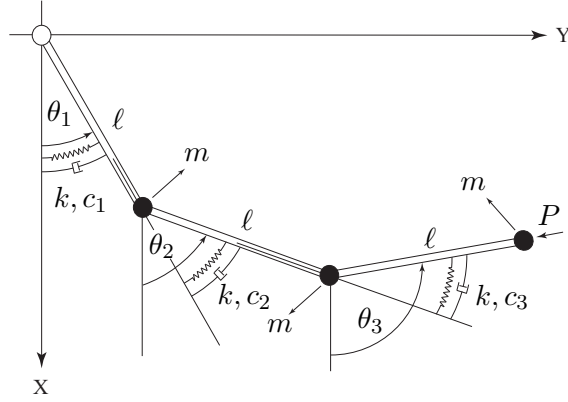


Figure 3.15: A three-link system with a follower force.

where p is the non-dimensional follower force, γ_i , $i = 1, 2, 3$, are the non-dimensional damping coefficients, and τ is the non-dimensional time variable; the non-dimensional variables are obtained using the scaling expressions provided in Eq.(3.2). Assuming the solution to be of the form $\theta_i = \Theta_i e^{s\tau}$, $i = 1, 2, 3$, we get a sixth-order characteristic polynomial in s . We now consider the following example.

Example 3.5: Consider the case where $\gamma_1 = \gamma_2 \triangleq \gamma_c = \text{constant}$. Constructing the Routh array and equating the coefficient of s^1 equal to zero, we get a quintic polynomial in γ_3 that describes the flutter instability curve; the coefficients of the polynomial are functions of γ_c and p . The quintic polynomial has five roots but numerical solutions based on different values of γ_c and p yield one, two, or three admissible (positive and real) solutions. Four specific cases are shown in Fig.3.16; it can be seen that γ_3 can have a purely destabilizing (D) or a purely stabilizing (S) effect when the quintic equation has one admissible solution - see Fig.3.16(a) and Fig.3.16(d), a destabilizing-stabilizing (D-S) effect when the quintic equation has two admissible solutions - see Fig.3.16(b), and a stabilizing-destabilizing-stabilizing (S-D-S) effect when the quintic equation has three admissible solutions - see Fig.3.16(c).

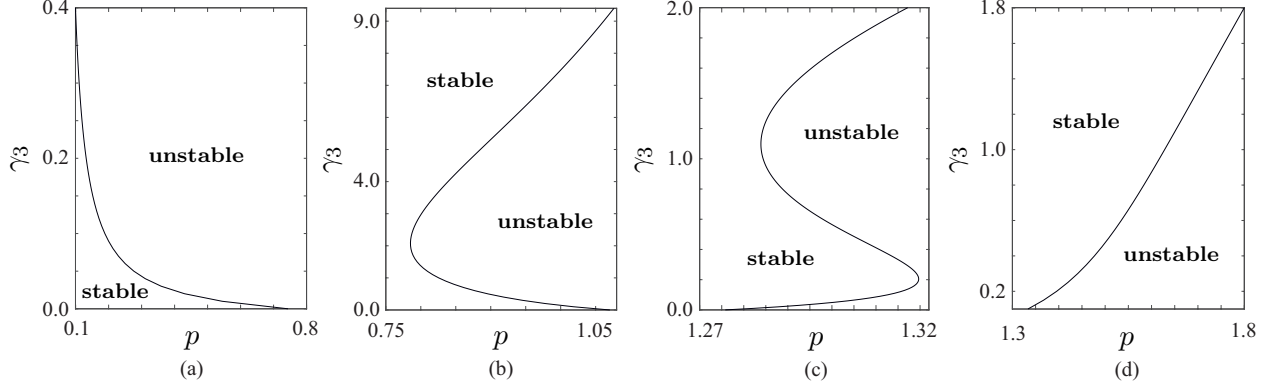


Figure 3.16: Absolute instability curves in γ_3 - p space for the three-link system with follower force showing D, D-S, S-D-S, and S effects of damping for (a) $\gamma_c = 0.01$, (b) $\gamma_c = 2.0$, (c) $\gamma_c = 4.0$, and (d) $\gamma_c = 6.0$, respectively.

3.3.2 Three-Link Fluid-Conveying Pipe

The three-link articulated fluid conveying pipe studied by Benjamin [1] is investigated here for the case where the link lengths are equal, the torsional stiffness of the joints are the same, and the motion of the links are restricted to the horizontal plane. Similar to the two-link pipe shown in Fig.3.11 (a), the configuration of the system is described using absolute joint angles θ_i , $i = 1, 2, 3$. Relative damping is introduced in the system and the non-dimensional damping coefficients of the three joints are assumed to be γ_i , $i = 1, 2, 3$. The non-dimensional equations are:

$$\begin{aligned}
 & \begin{bmatrix} 7 & 4.5 & 1.5 \\ 4.5 & 4 & 1.5 \\ 1.5 & 1.5 & 1 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{bmatrix} + \begin{bmatrix} \beta u + \gamma_1 + \gamma_2 & 2\beta u - \gamma_2 & 2\beta u \\ -\gamma_2 & \beta u + \gamma_2 + \gamma_3 & 2\beta u - \gamma_3 \\ 0 & -\gamma_3 & \beta u + \gamma_3 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} \\
 & + \begin{bmatrix} -\beta u^2 + 2 & -1 & \beta u^2 \\ -\beta u^2 - 1 & 2 & \beta u^2 - 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (3.20)
 \end{aligned}$$

Example 3.6: Consider the case where $\gamma_1 = \gamma_2 = 0.10$ and $\gamma_3 = \gamma$. The condition for flutter instability is a quintic polynomial in γ whose coefficients are functions of β and u . The quintic polynomial has five roots but numerical solutions based on $\beta = 0.10$ yielded a maximum of three admissible (positive and real) solutions. There are two flutter instability curves - see Fig.3.17 (a); the one to the left is also the absolute instability curve. It can be seen from this figure that γ has three admissible solutions for both $u = 12.0$ and 16.5 . For $u = 12.0$, two of the three roots ($\gamma = 4.63, 5.78$) lie on the absolute instability curve. The system is unstable for $\gamma \in (0, 4.63)$, stable for $\gamma \in (4.63, 5.78)$, and unstable for $\gamma > 5.78$; consequently, the system exhibits a stabilizing-destabilizing (S-D) effect. For $u = 16.5$, none of the roots lie on the absolute instability curve; therefore, the system is unstable and damping has no effect on the stability property of the system.

The locus of the roots are plotted in Fig.3.17 (b) for $\gamma = 0.20$. Consistent with Fig.3.17 (a), the locus is shown for $u \in [4, 20]$. At the starting point, where $u = 4.0$, there are three pairs of complex conjugate roots in the left-half plane. At $u = 4.49$, one pair of roots crosses the imaginary axis into the right-half plane; this identifies the absolute instability curve, which is also a flutter instability curve. When u is increased to 12.44 , a second pair of roots

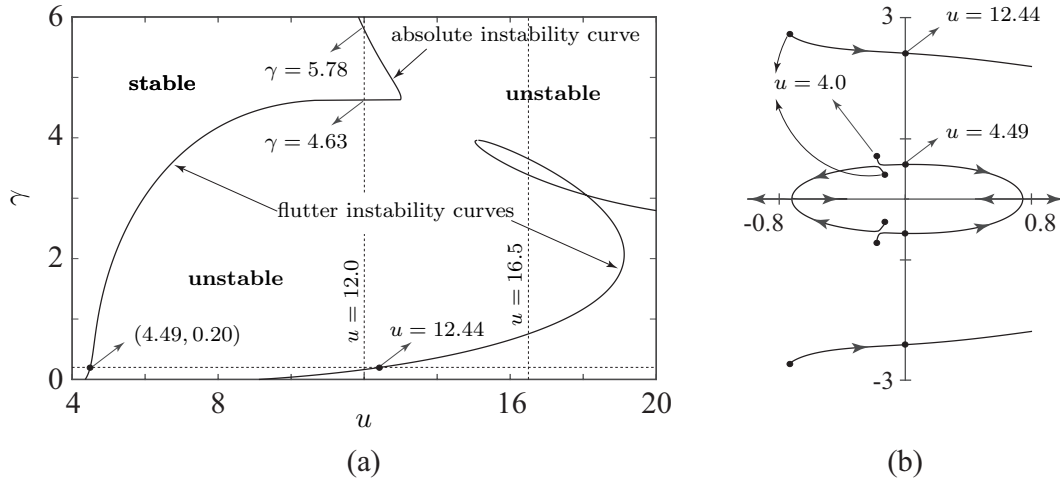


Figure 3.17: (a) Flutter instability curves and absolute instability curve in γ - u space for the three-link fluid-conveying pipe in Example 3.6 for $\beta = 0.10$ (b) locus of all roots of the system as u increases for the particular case of $\gamma = 0.2$.

cross the imaginary axis into the right-half plane; this identifies the second flutter instability curve that intersects the $\gamma = 0.20$ line at $u = 12.44$.

Example 3.7: Consider the case where $\gamma_2 = 0.0$, $\gamma_1 = 0.5$ and $\gamma_3 = \gamma$. The conditions for flutter instability results is an octic polynomial in γ whose coefficients are functions of β and u . The octic polynomial has eight roots but numerical solutions with $\beta = 0.10$ yielded a maximum of four admissible solutions. The flutter instability curves are shown in Fig.3.18(a); it can be seen from this figure that γ has four roots for $u = 12.0$. Similar to Example 3.6, there are two flutter instability curves which correspond to the second and third modes of the system. Unlike Example 3.6, however, the two flutter instability curves

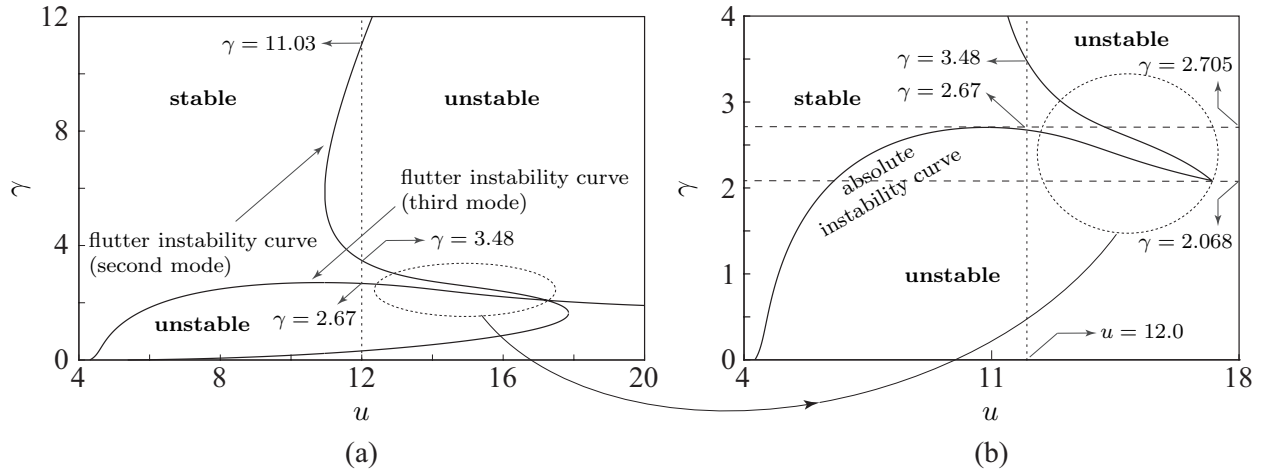


Figure 3.18: (a) Flutter instability curves in γ - u space for the three-link fluid-conveying pipe in Example 3.7 for $\beta = 0.10$ (b) A magnified view of the region of intersection of the flutter instability curves is shown along with the absolute instability curve.

intersect each other; consequently, the absolute instability curve is piecewise continuous and comprised of segments of both flutter instability curves - see Fig.3.18(b). Similar results have been reported in the literature earlier [66] where nonlinear analysis was used to show the intersection of two curves in the bifurcation diagram.

For $u = 12.0$, three of the four roots ($\gamma = 2.67$, 3.48 and 11.03) lie on the absolute instability curve - see Figs.3.18 (a) and (b). The system is unstable for $\gamma \in (0, 2.67)$, stable for $\gamma \in (2.67, 3.48)$, unstable for $\gamma \in (3.48, 11.03)$ and stable for $\gamma \in (11.03, \infty)$; consequently, the system exhibits a stabilizing-destabilizing-stabilizing (S-D-S) effect.

Remark 3.7: The absolute instability curve in Example 3.7 has a positive slope and two negative slopes in the range $\gamma \in [2.068, 2.705]$ - see Fig.3.18 (b). This is at odds with the definition of “destabilizing” and “stabilizing” damping proposed in [15]. We instead propose the following definition.

Destabilizing (Stabilizing) damping: The addition of damping is “destabilizing” (“stabilizing”) if it causes the system to become unstable (stable) at some value of the non-conservative forcing variable (u , in Example 3.7). This definition may be overly-reductive but the emphasis on the non-conservative forcing variable rather than the local slope is needed to cover all observed cases of stability transitions.

Remark 3.8: The destabilizing effect of damping was first described by Ziegler [14] relative to an undamped system. The above definition is valid for changes in stability properties between any two non-negative levels of damping.

3.4 Cantilevered Fluid-Conveying Pipe

For the sake of completeness, we consider a continuous system, namely, a cantilever fluid-conveying pipe with a damping force applied at a fixed point along the length of the pipe. In the absence of gravity, the non-dimensional equation of motion of the pipe and its boundary conditions are:

$$\frac{\partial^4 y(x, t)}{\partial x^4} + u^2 \frac{\partial^2 y(x, t)}{\partial x^2} + 2u\sqrt{\beta} \frac{\partial^2 y(x, t)}{\partial x \partial t} + c_s \frac{\partial^5 y(x, t)}{\partial^4 x \partial t} + c \frac{\partial y(\alpha, t)}{\partial t} + \frac{\partial^2 y(x, t)}{\partial t^2} = 0 \quad (3.21)$$

$$y(0, t) = 0, \quad \frac{\partial y}{\partial x}(0, t) = 0, \quad \frac{\partial^2 y}{\partial x^2}(1, t) = 0, \quad \frac{\partial^3 y}{\partial x^3}(1, t) = 0$$

where u is the non-dimensional fluid velocity, β is the ratio of the mass of the fluid to the mass of the fluid and pipe, c_s is the non-dimensional structural damping coefficient, c is the non-dimensional external damping coefficient, and $\alpha \in [0, 1]$ defines the point of application of the external damping force. Using Galerkin approximation [63], we assume the solution

to be of the form

$$y(x, t) = \sum_{i=1}^N a_i(t) \phi_i(x) \quad (3.22)$$

where N is the number of terms in the approximation, $\phi_i(x)$, $i = 1, \dots, N$, are functions that satisfy the boundary conditions, and $a_i(t)$, $i = 1, \dots, N$, are coefficients that will be determined. Substituting Eq.(3.22) into Eq.(3.21) and integrating over the length of the pipe, we get

$$M\ddot{a} + C\dot{a} + Ka = 0$$

where the (i, j) -th elements of the M , C and K matrices are defined as

$$\begin{aligned} M_{ij} &\triangleq \int_0^1 \phi_i \phi_j dx \\ C_{ij} &\triangleq 2u\sqrt{\beta} \int_0^1 \phi_i \phi_j' dx + c_s \int_0^1 \phi_i'' \phi_j'' dx + c \phi_i(\alpha) \phi_j(\alpha) \\ K_{ij} &\triangleq \int_0^1 \phi_i'' \phi_j'' dx + u^2 \left[\phi_i(1) \phi_j'(1) - \int_0^1 \phi_i' \phi_j' dx \right] \end{aligned}$$

and $(.)'$ denotes the first spatial derivative of $(.)$.

Example 3.8: Consider the fluid-conveying pipe with external damping only, i.e., $c_s = 0, c \neq 0$. A four-mode approximation ($N = 4$) of the fluid-conveying pipe is obtained using the first four mode shapes of the cantilever beam [67]. The M , C and K matrices have dimension four and the characteristic polynomial is of order eight. Using the procedure outlined in Section 3.1, the conditions for flutter instability can be described by a polynomial equation in c ; the order of the polynomial equation is forty-one and the coefficients are functions of α , β and u . For $\alpha = 0.86$ and $\beta = 0.1$, the polynomial equation has three admissible solutions over a small range of u values - see Fig.3.19(a). Thus the fluid-conveying cantilever also exhibits the stabilizing-destabilizing-stabilizing (S-D-S) effect of damping; for $u = 5.05$, for example. The qualitative behavior of the system does not change when the number of terms in the Galerlin approximation is increased to five ($N = 5$); this can be seen from the plot in Fig.3.19(a).

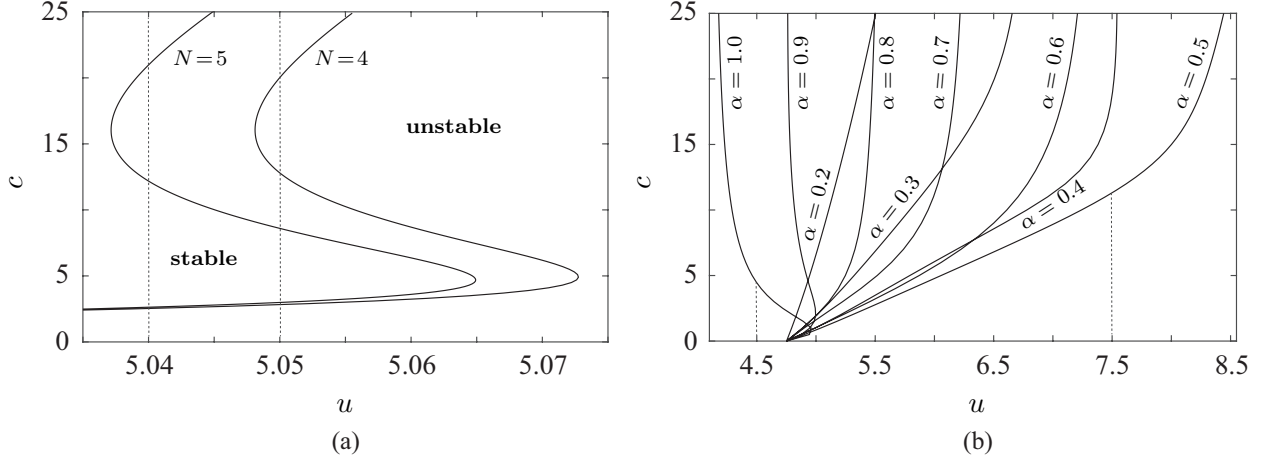


Figure 3.19: Absolute instability curves in c - u space for the cantilever fluid-conveying pipe with $\beta = 0.1$ for: (a) $\alpha = 0.86$ (b) $\alpha \in [0.2, 1.0]$. The S-D-S effect of damping is seen for $\alpha = 0.86$.

We complete this example with some observations on how the stabilizing and destabilizing effects of damping depend on the location where the external damping force is applied. The absolute instability curves for different values of α are shown in Fig.3.19 (b) for $\beta = 0.1$. It can be seen that the curves move towards the right as the point of application of the force changes from the free end of the pipe ($\alpha = 1.0$) to the mid-point ($\alpha = 0.5$), implying that a decrease in α in the range $[0.5, 1.0]$ has a stabilizing (S) effect. However, further decrease in the value of α moves the absolute instability curves back towards the left, i.e., decrease in α in the range $[0.2, 0.5]$ has a destabilizing (D) effect. Overall, the pipe has the best stability characteristics when $\alpha = 0.5$ and the worst stability characteristics when $\alpha = 1.0$. This is also supported by the observation that increasing damping has a stabilizing (S) effect for $\alpha = 0.5$ ($u = 7.5$, for example) whereas it has a destabilizing (D) effect for $\alpha = 1.0$ ($u = 4.5$, for example).

Example 3.9: Consider the fluid-conveying pipe with structural damping only, i.e., $c = 0, c_s \neq 0$. Similar to the previous example, a four-mode approximation ($N = 4$) is used. The conditions for flutter instability can be described by a polynomial equation in c_s ; the order of the polynomial equation is sixty-four and the coefficients are functions of β and u . For $\beta = 0.35$, the polynomial equation has two admissible solutions for $c_s \in [0.0, 0.1]$ - see

Fig.3.20. This figure indicates that the fluid-conveying cantilever exhibits a destabilizing-stabilizing (D-S) effect for structural damping; for $u = 7.7$, for example. This problem was also investigated in [15], where it was claimed that structural damping has a destabilizing effect for $\beta > 0.29$. While this is a correct statement for very small values of damping (approx. $c_s < 0.27$), slightly higher values of damping have a stabilizing effect.

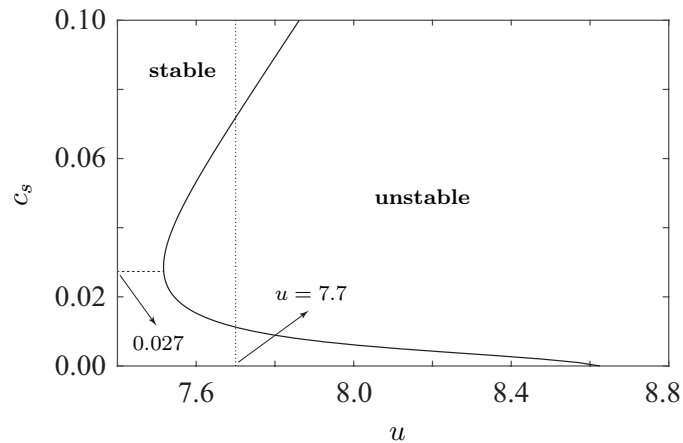


Figure 3.20: Absolute instability curves in c_s - u space for the cantilever fluid-conveying pipe with $\beta = 0.35$ showing the D-S effect of damping.

Chapter 4

Non-conservative Effects of Dynamic Terminal Moment

In this chapter, we introduce a new type of non-conservative load in the form of a terminal dynamic moment. For a flexible cantilevered beam the dynamic moment is applied at the free end and its magnitude varies with the slope or the curvature of the beam at some points along its length. The stability analysis of such a system has not appeared in the literature and the non-conservative effects of this type of loading is shown here using both theory and experiment. This chapter is organized as follows. In Section 1, we provide a problem formulation where the equation of motion of a cantilevered beam with terminal dynamic moment and a tip mass is presented. Section 2 highlights the procedure to solve the problem both analytically and numerically and introduces the characteristic equation as a function of the non-conservative load parameter. The stability investigations are introduced in Section 3 in the context of various loading frameworks and results for an experimental are presented in Section 4.

4.1 Problem Formulation

Consider the cantilevered Euler-Bernoulli beam of length L shown in Fig.4.1; it is assumed to have a uniform cross-sectional area A . The free end of the cantilever is subjected to a dynamic bending moment M . This bending moment is the reaction of a torque produced by an actuator mounted at the free end. The mass and mass moment of inertia of the actuator about the free end is assumed to be m and J , respectively. The transverse displacement of the beam is denoted by $y(x, t)$. For small deformation, the equation of motion of the beam and its boundary conditions are:

$$EI y'''' + \rho A \ddot{y} = 0 \tag{4.1}$$

$$y(0, t) = 0, \quad y'(0, t) = 0, \quad EIy''(L, t) + J\ddot{y}'(L, t) = M, \quad EIy'''(L, t) = m\ddot{y}(L, t) \quad (4.2)$$

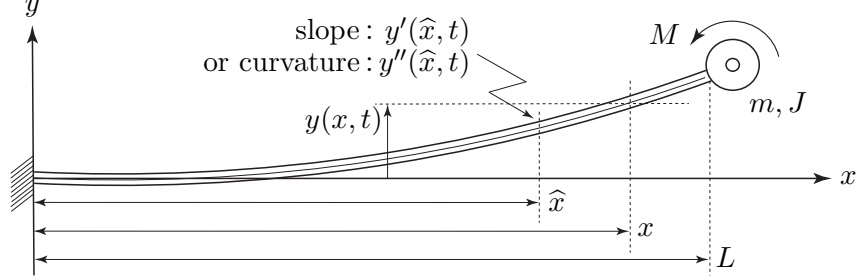


Figure 4.1: A flexible cantilever beam with a tip mass and a terminal dynamic moment

where E , I , and ρ are the Young's modulus of elasticity, cross-sectional area moment of inertia, and the mass per unit volume of the beam, and y' and $\dot{y}$ denote the spatial and time derivatives of $y(x, t)$. Using the following change of variables

$$v = \frac{y}{L}, \quad u = \frac{x}{L}, \quad \tau = t \sqrt{\frac{EI}{\rho AL^4}} \quad (4.3)$$

the non-dimensional equation of motion and boundary conditions are obtained as

$$v''''(u, \tau) + \ddot{v}(u, \tau) = 0 \quad (4.4)$$

$$v(0, \tau) = 0, \quad v'(0, \tau) = 0, \quad v''(1, \tau) + \nu \ddot{v}'(1, \tau) = \overline{M}, \quad v'''(1, \tau) = \mu \ddot{v}(1, \tau) \quad (4.5)$$

where v' and $\dot{v}$ denote the partial derivatives of $v(u, \tau)$ with respect to u and τ , respectively, and

$$\overline{M} \triangleq \frac{ML}{EI}, \quad \mu \triangleq \frac{m}{\rho AL}, \quad \nu \triangleq \frac{J}{\rho AL^3} \quad (4.6)$$

We consider two cases where the dynamic moment M is assumed to be proportional to:

- (a) the slope of the beam at $x = \hat{x}$, and
- (b) the curvature of the beam at $x = \hat{x}$, *i.e.*

$$\begin{aligned} M = M_s &\triangleq C_s y'(\hat{x}, t) & : & \text{ slope-dependent} \\ M = M_c &\triangleq C_c y''(\hat{x}, t) & : & \text{ curvature-dependent} \end{aligned} \tag{4.7}$$

These relationships can be expressed using the following non-dimensional variables

$$\begin{aligned} \overline{M} = \overline{M}_s &\triangleq \overline{C}_s v'(\alpha, \tau), & \overline{C}_s &\triangleq C_s L / EI & : & \text{ slope-dependent} \\ \overline{M} = \overline{M}_c &\triangleq \overline{C}_c v''(\alpha, \tau), & \overline{C}_c &\triangleq C_c / EI & : & \text{ curvature-dependent} \end{aligned}, \quad \alpha \triangleq \frac{\hat{x}}{L} \tag{4.8}$$

where $\alpha \in (0, 1]$ denotes the location of the point on the beam from where the slope or the curvature is measured.

4.2 Solution Methods

We solve Eqs.(4.4) and (4.5) both analytically and numerically. In addition to providing confidence in our analytical results, the numerical method allows us to examine the structure of the stiffness matrix and determine conditions under which the matrix loses its symmetric property, and thereby infer the nature of the instability due to the dynamic moment. The numerical model is also useful for investigating the effect of damping, when present, on the stability characteristics of the system [68].

4.2.1 Analytical Solution

Using variable separation, we substitute $v(u, \tau) = U(u)T(\tau)$ to get solutions of the form

$$\begin{aligned} T(\tau) &= A \cos \overline{\omega} \tau + B \sin \overline{\omega} \tau, \\ U(u) &= P_1 e^{\beta u} + P_2 e^{-\beta u} + P_3 e^{i\beta u} + P_4 e^{-i\beta u} \end{aligned} \quad \overline{\omega} \triangleq \beta^2 \tag{4.9}$$

where A and B are constants that can be obtained from initial conditions, and P_1 through P_4 are constants that can be obtained from the boundary conditions in Eq.(4.5). The boundary conditions result in the following transcendental characteristic equations:

$$\begin{aligned} & [\beta\mu(\cos\beta - \cosh\beta) + \sin\beta - \sinh\beta] [\beta(\sin\beta + \sinh\beta) + \overline{C}_s(\cos\alpha\beta - \cosh\alpha\beta) \\ & + \beta^4\nu(\cos\beta - \cosh\beta)] - [\cos\beta + \cosh\beta - \beta\mu(\sin\beta - \sinh\beta)] [\overline{C}_s(\sin\alpha\beta + \sinh\alpha\beta) \\ & - \beta(\cos\beta + \cosh\beta) + \beta^4\nu(\sin\beta + \sinh\beta)] = 0 \end{aligned} \quad (4.10a)$$

$$\begin{aligned} & [\beta\mu(\sin\beta - \sinh\beta) - (\cos\beta + \cosh\beta)] [\overline{C}_c(\cos\alpha\beta + \cosh\alpha\beta) - (\cos\beta + \cosh\beta) \\ & + \beta^3\nu(\sin\beta + \sinh\beta)] - [\sin\beta - \sinh\beta + \beta\mu(\cos\beta - \cosh\beta)] [\overline{C}_c(\sin\alpha\beta + \sinh\alpha\beta) \\ & - (\sin\beta + \sinh\beta) - \beta^3\nu(\cos\beta - \cosh\beta)] = 0 \end{aligned} \quad (4.10b)$$

where Eqs.(4.10a) and (4.10b) correspond to the two cases where the terminal dynamic moment is slope-dependent and curvature-dependent, respectively. By solving Eq.(4.10), we get the non-dimensional natural frequencies $\overline{\omega} = \beta^2$. The constants P_1 through P_4 are obtained from the boundary conditions as follows:

$$P_1 = 0, \quad P_2 = 1.0, \quad P_3 = 0, \quad P_4 = \begin{cases} \gamma_s & : \quad \overline{M} = \overline{M}_s \text{ is slope-dependent} \\ \gamma_c & : \quad \overline{M} = \overline{M}_c \text{ is curvature-dependent} \end{cases}$$

where

$$\gamma_s \triangleq \frac{\overline{C}_s(\sin\alpha\beta + \sinh\alpha\beta) - \beta[\cos\beta + \cosh\beta - \beta^3\nu(\sin\beta + \sinh\beta)]}{\overline{C}_s(\cos\alpha\beta - \cosh\alpha\beta) + \beta[\sin\beta + \sinh\beta + \beta^3\nu(\cos\beta - \cosh\beta)]} \quad (4.11a)$$

$$\gamma_c \triangleq -\frac{\overline{C}_c(\cos\alpha\beta + \cosh\alpha\beta) - (\cos\beta + \cosh\beta) + \beta^3\nu(\sin\beta + \sinh\beta)}{\overline{C}_c(\sin\alpha\beta + \sinh\alpha\beta) - (\sin\beta + \sinh\beta) - \beta^3\nu(\cos\beta - \cosh\beta)} \quad (4.11b)$$

Each solution of β obtained from Eq.(4.10) results in a unique mode shape given by the expression

$$U(u) = \begin{cases} (\cos \beta u - \cosh \beta u) + \gamma_s(\sin \beta u - \sinh \beta u) & : \quad \overline{M} = \overline{M}_s \text{ is slope-dependent} \\ (\cos \beta u - \cosh \beta u) + \gamma_c(\sin \beta u - \sinh \beta u) & : \quad \overline{M} = \overline{M}_c \text{ is curvature-dependent} \end{cases} \quad (4.12)$$

It is important to note that the terminal moment, independent of whether it is slope- or curvature-dependent, is a boundary condition. Therefore, a boundary-value problem is solved to obtain the characteristic equation and determine critical stability points.

4.2.2 Galerkin method

To obtain a numerical solution, the Galerkin method [63] is used; the solution to Eq.(4.4) is assumed to be of the form

$$v(u, \tau) = \sum_{i=1}^N a_i(\tau) \phi_i(u) \quad (4.13)$$

where N is the number of terms in the approximation, $\phi_i(u)$, $i = 1, \dots, N$, are assumed modes that satisfy the geometric boundary conditions, and $a_i(\tau)$, $i = 1, \dots, N$, are the modal amplitudes. Substituting Eq.(4.13) into Eq.(4.4) and integrating over the length, we get

$$\mathbb{M}\ddot{a} + \mathbb{K}a = 0 \quad (4.14)$$

where the (i, j) -th elements of the $\mathbb{M}$ and $\mathbb{K}$ matrices are defined as

$$\begin{aligned} \mathbb{M}_{ij} &\triangleq \int_0^1 \phi_i \phi_j du + \mu \phi_i(1) \phi_j(1) + \nu \phi_i'(1) \phi_j'(1) \\ \mathbb{K}_{ij} &\triangleq \begin{cases} \int_0^1 \phi_i'' \phi_j'' du - \overline{C}_s \phi_i'(1) \phi_j'(\alpha) & : \quad \overline{M} = \overline{M}_s \text{ is slope-dependent} \\ \int_0^1 \phi_i'' \phi_j'' du - \overline{C}_c \phi_i'(1) \phi_j''(\alpha) & : \quad \overline{M} = \overline{M}_c \text{ is curvature-dependent} \end{cases} \end{aligned} \quad (4.15)$$

and where ϕ' denotes the derivative of ϕ with respect to u . The second and third terms in the expression of $\mathbb{M}_{ij}$ denote the contribution of the mass and rotor inertia of the actuator to the

mass matrix. The second term in the expression of $\mathbb{K}_{ij}$ denotes the geometric stiffness associated with the terminal dynamic moment. When the dynamic moment is slope-dependent, it is clear that the stiffness matrix $\mathbb{K}$ will be symmetric if $\alpha = 1$ and asymmetric otherwise. When the dynamic moment is curvature-dependent, the stiffness matrix $\mathbb{K}$ will be asymmetric. A nonconservative system is associated with an asymmetric stiffness matrix [56], and therefore the cantilever beam may lose stability through flutter for both cases where the terminal moment is slope- or curvature-dependent.

4.3 Instability Investigation

The analytical and numerical methods are both used to investigate the instabilities of the cantilever beam due to the terminal dynamic moment. For the analytical method, the non-dimensional natural frequencies $\bar{\omega}_i = \beta_i^2$, $i = 1, 2, \dots$, are computed from Eq.(4.10) and plotted as a function of $\bar{C}_s$ or $\bar{C}_c$ (depending on whether the moment is slope- or curvature-dependent), for discrete values of α . For the numerical method, the natural frequencies are obtained by solving the eigenvalue problem associated with Eq.(4.14). The numerical method is based on a ten-mode approximation ($N = 10$) and the behavior of the first few natural frequencies is investigated. The values of the mass and inertia ratios, defined in Eq.(4.6), are chosen as $\mu = 7.03$, $\nu = 0.038$ ¹. We investigate the two cases where the dynamic moment is slope-dependent and curvature-dependent, and consider positive and negative values of the proportionality constant for each of these cases.

4.3.1 Dynamic Moment Proportional to Positive Slope

The results for positive values of $\bar{C}_s$ are shown in Fig.4.2 (a). The first natural frequency of the beam with a mass at the tip ($\mu = 7.03$, $\nu = 0.038$) is equal to 0.639 in the absence of the terminal dynamic moment ($\bar{C}_s = 0$). When $\bar{C}_s$ is positive, the system loses stability through divergence: as the value of $\bar{C}_s$ is increased, the first natural frequency reduces to zero. The

¹These values were chosen to be identical to those in our experimental setup, described in Section 4.4.

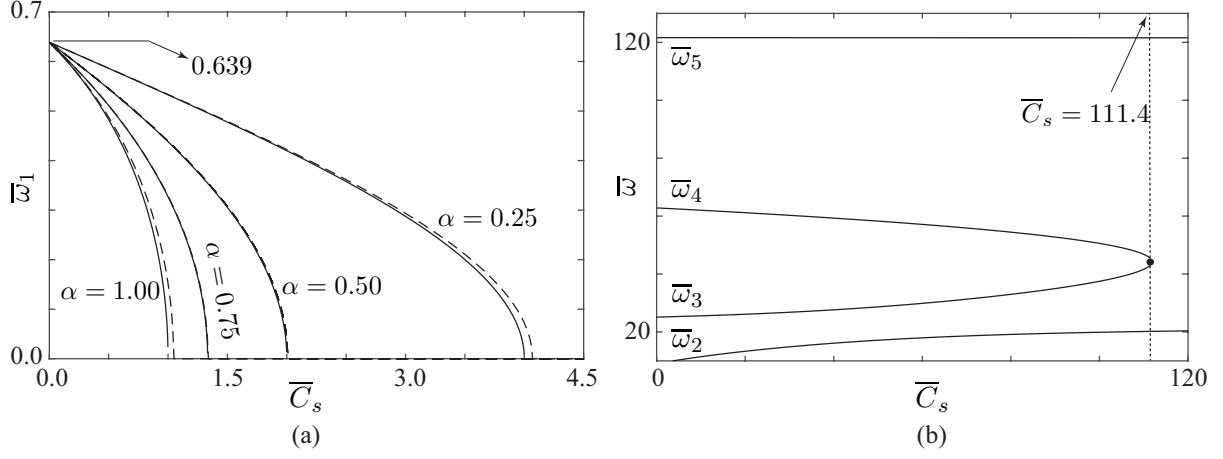


Figure 4.2: Terminal moment is proportional to the positive slope of the beam: (a) Variation in the first natural frequency with $\bar{C}_s$ ($\bar{C}_s \geq 0$) for four different values of α . Analytical and numerical results are plotted using solid lines and dashed lines, respectively. (b) Analytical results showing variation of the second, third, fourth, and fifth natural frequencies with $\bar{C}_s$ for $\alpha = 0.25$.

critical value of $\bar{C}_s$, denoted by $\bar{C}_s^*$, is different for different values of α but the overall trend is the same. The analytical and numerical methods provide similar results, which are shown for four different values of α , $\alpha = \{0.25, 0.50, 0.75, 1.00\}$. It should be noted that the K matrix is asymmetric when $\alpha \neq 1$ and the system loses stability through divergence. If $\bar{C}_s$ is increased beyond its first critical value, some of the higher natural frequencies become complex for $\alpha \neq 1$. Plots of some of the higher natural frequencies are shown in Fig.4.2 (b) for $\alpha = 0.25$; it can be seen that $\bar{\omega}_3$ and $\bar{\omega}_4$ become equal for $\bar{C}_s = 111.4$, they assume complex values (not shown) for higher values of $\bar{C}_s$.

In general, if an undamped system loses stability through flutter due to variation of a system parameter, two consecutive natural frequencies approach each other, assume real equal values at the critical value of the parameter, and assume complex conjugate values for higher values of the parameter. This is described as coupled-mode flutter via a Hamiltonian-Hopf bifurcation [69] in the literature. This motivates us to adopt the following definition for the ease of discussion in the next section.

n-th Flutter Instability Mode: The system loses stability through the n -th flutter instability mode if the n -th and $(n + 1)$ -th natural frequencies of the system are equal at the

lowest critical value of a system parameter, *i.e.*, the n -th and $(n + 1)$ -th natural frequencies exhibit coupled-mode flutter via a Hamiltonian-Hopf bifurcation.

4.3.2 Dynamic Moment Proportional to Negative Slope

When $\overline{C}_s$ is negative, the system loses stability through flutter. The results are discussed for two ranges of α , described below:

range 1: $\alpha \in [0.02, 0.63]$ first flutter instability mode

range 2: $\alpha \in [0.64, 0.95]$ second flutter instability mode

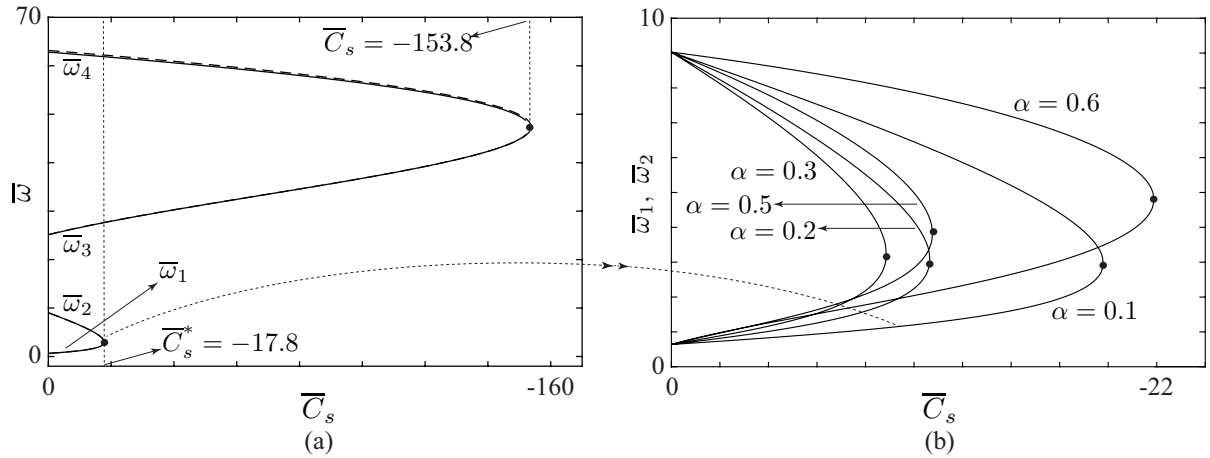


Figure 4.3: Terminal moment is proportional to the negative slope of the beam: (a) Variation in the first four natural frequencies with $\overline{C}_s$ ($\overline{C}_s \leq 0$) for $\alpha = 0.1$ shows the first flutter instability mode. Analytical and numerical results are plotted using solid lines and dashed lines, respectively. (b) Analytical results showing the first flutter instability mode for five different values of α in range 1: $[0.02, 0.63]$, including the case shown in (a); the critical stability point is shown using the $\bullet$ sign.

The first flutter instability mode is observed for range 1: $\alpha \in [0.02, 0.63]$ and the results are shown in Fig.4.3. It was not possible to find a critical value of $\overline{C}_s$ for $\alpha < 0.02$. For a specific value of α in range 1, namely $\alpha = 0.1$, the first four natural frequencies are plotted in Fig.4.3 (a) as the value of $\overline{C}_s$ is decreased from zero. It is seen that the system loses stability for $\overline{C}_s = \overline{C}_s^* = -17.8$: the first two natural frequencies become equal at this value of $\overline{C}_s$, they assume complex values for more negative values of $\overline{C}_s$ and are not shown. The analytical and numerical results provide almost identical plots for all four natural frequencies. Analytical

results for variation of the first two natural frequencies as a function of $\overline{C}_s$ are provided in Fig.4.3 (b) for five different values of α in range 1: $[0.02, 0.63]$; each plot is terminated when the two natural frequencies become equal (shown by the $\bullet$ sign) at the point of flutter instability. It is seen that $\overline{C}_s^*$ decreases as α is increased from 0.1 to 0.3 but increases when α is increased further. This can be viewed as a “destabilizing-stabilizing” effect of the location of the point from where the slope of the beam is measured; further discussion on this topic appears later in this section.

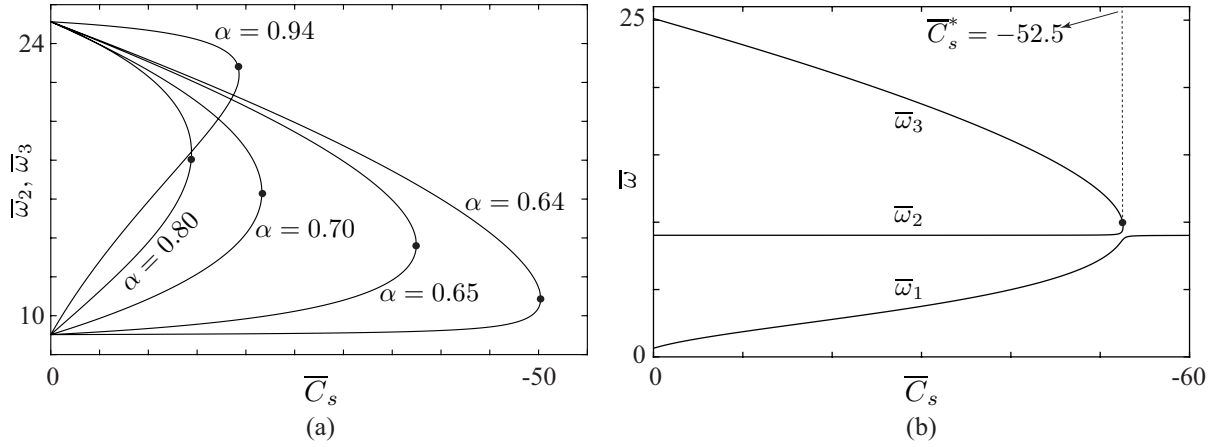


Figure 4.4: Terminal moment is proportional to the negative slope of the beam: Analytical results show the (a) second flutter instability mode for range 2: $[0.64, 0.95]$, (b) Switching from first to second flutter instability mode for $\alpha = 0.6394$ and $\overline{C}_s^* = -52.5$. In both figures, the critical stability point is shown using the $\bullet$ sign.

The second flutter instability mode is observed for higher values of α ; the results are provided in Fig.4.4 (a). Variation of the natural frequencies $\overline{\omega}_2$, $\overline{\omega}_3$ as a function of $\overline{C}_s$ are provided in Fig.4.4 (a) for five different values of α in range 2: $[0.64, 0.95]$; each plot is terminated when the frequencies become equal (shown by the $\bullet$ sign) at the point of flutter instability. Similar to the observation in Fig.4.3 (b), the “destabilizing-stabilizing” effect of α can be observed in Fig.4.4 (a): the value of $\overline{C}_s^*$ decreases as α is increased from 0.64 to 0.80 but increases when α is increased further. For $\alpha > 0.95$, it becomes increasingly difficult to determine the mode of flutter instability. For $\alpha = 0.6394$, which lies in between ranges 1 and 2, the mode of flutter instability switches from the first to the second as $\overline{C}_s$ is decreased. For $\overline{C}_s^* = -52.5$, $\overline{\omega}_2 = \overline{\omega}_3$ and $\overline{\omega}_1$ veers off resulting in the second flutter instability mode -

see Fig.4.4 (b). This veering phenomenon has been reported earlier in the literature [43].

To summarize the results presented in this section and to further investigate the “destabilizing-stabilizing” effect of α , we plot the critical stability curve in Fig.4.5 for $\alpha \in [0.02, 0.95]$, which spans ranges 1 and 2. The numerical results (solid line) match well with the analytical results. The critical stability curve is comprised of two curves for the two ranges of α , each having a “catenary-like” shape. It is possible to find a value of $\overline{C}_s$ (horizontal line) which intersects both “catenaries” twice. For example, the horizontal line $\overline{C}_s = -11$ intersects the catenaries in ranges 1 and 2 twice. This implies that for $\overline{C}_s = -11$, increasing the value of α results in a “destabilizing-stabilizing-destabilizing-stabilizing” effect², where the first destabilization occurs in the first flutter instability mode and the second destabilization occurs in the second flutter instability mode.

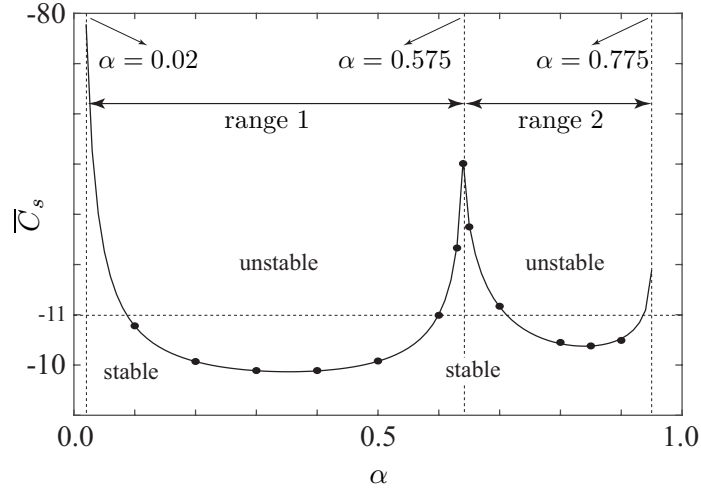


Figure 4.5: Terminal moment is proportional to the negative slope of the beam: Critical stability curve for $\alpha \in [0.02, 0.95]$. The critical stability curve (solid line) was obtained numerically using a ten-mode approximation; the analytical results are shown for discrete values of α using the $\bullet$ sign.

²Similar multiple stability transitions have been observed in non-conservative systems for variation in the level of damping starting with the work by Semler et al. [15]; a survey of the literature along with some new results can be found in [68].

4.3.3 Dynamic Moment Proportional to Positive and Negative Curvature

In this section, we discuss the case where the terminal moment is proportional to the curvature of the beam. For positive values of $\overline{\mathcal{C}}_c$, both analytical and numerical results are shown in Fig.4.6 for three different values of α , $\alpha = \{0.25, 0.50, 0.75\}$. Similar to Fig.4.2 (a), the

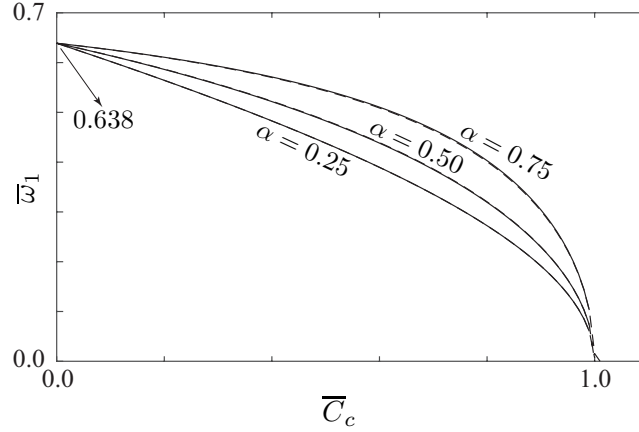


Figure 4.6: Terminal moment is proportional to the positive curvature of the beam: Variation in the first natural frequency with $\overline{\mathcal{C}}_c$ ($\overline{\mathcal{C}}_c \geq 0$) for three different values of α . Analytical and numerical results are plotted using solid lines and dashed lines, respectively.

first natural frequency of the beam with a tip mass ($\mu = 7.03$, $\nu = 0.038$) is equal to 0.638 in the absence of the terminal moment ($\overline{\mathcal{C}}_c = 0$). Also, similar to the results in Section 4.3.1 (moment is proportional to the positive slope), the system loses stability through divergence. However, unlike the results in Section 4.3.1, the critical value of $\overline{\mathcal{C}}_c$ is almost identical, equal to unity, for different values of α . For negative values of $\overline{\mathcal{C}}_c$, the system loses stability through flutter. Similar to the results in Section 4.3.2 (moment is proportional to the negative slope), higher values of α are associated with higher modes of flutter instability but there are four ranges, as described below, and shown in Fig.4.7:

- range 1: $\alpha \in [0.01, 0.29]$ first flutter instability mode
- range 2: $\alpha \in [0.30, 0.74]$ second flutter instability mode
- range 3: $\alpha \in [0.75, 0.84]$ third flutter instability mode
- range 4: $\alpha \in [0.85, 0.88]$ fourth flutter instability mode

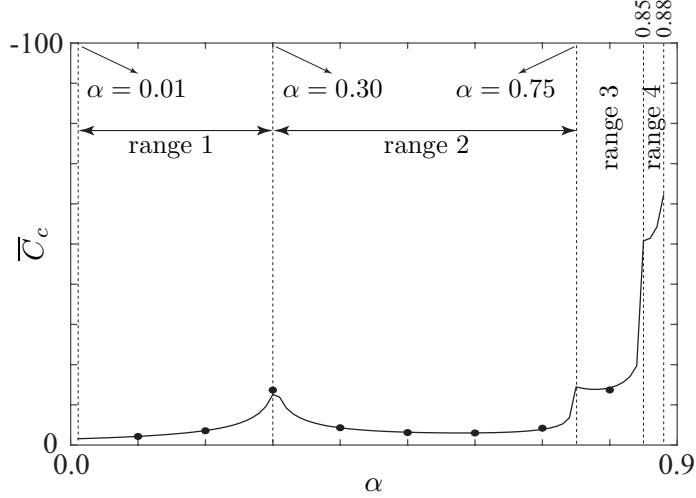


Figure 4.7: Terminal moment is proportional to the negative curvature of the beam: Critical stability curve for $\alpha \in [0.01, 0.88]$. The critical stability curve (solid line) was obtained numerically using a ten-mode approximation; the analytical results are shown for discrete values of α using the $\bullet$ sign.

4.4 Experimental Investigations

4.4.1 Hardware Description

Laboratory experiments were performed using a steel ($E = 200$ GPa, $\rho = 8000$ kg/m³) cantilever beam with tip mass; the experimental setup is shown in Fig.4.8. The beam has a uniform rectangular cross-section with height $h = 0.05$ m and thickness $t = 5 \times 10^{-4}$ m.

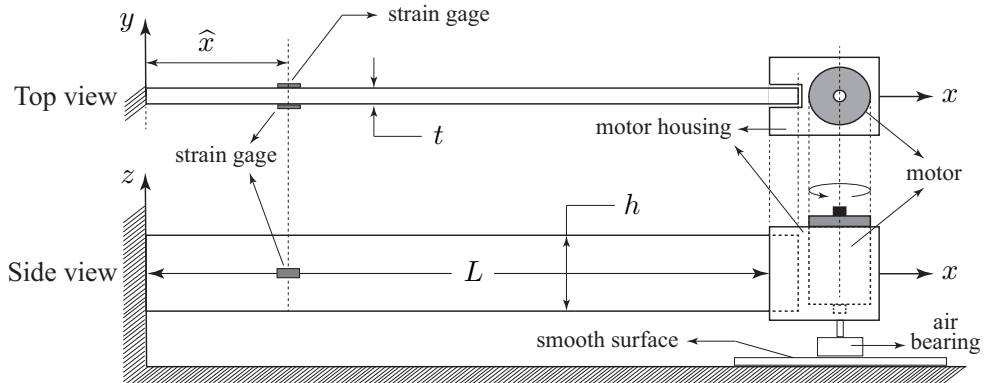


Figure 4.8: Cantilever beam with tip mass: the tip mass is comprised of a motor for application of a dynamic moment and an air bearing to avoid excitation of torsional modes.

The length and cross-sectional area of the beam are

$$L = 0.56 \text{ m}, \quad A = h t = 2.5 \times 10^{-5} \text{ m}^2$$

The tip mass houses a motor³ that is used for applying the dynamic moment at the free end of the beam. The relatively large mass of the motor can cause torsional deformation and/or excite the torsional modes of vibration of the thin beam; a pair of air bearings is therefore used to support the tip mass and allow it to slide freely over a smooth surface during bending motion of the beam. The tip mass, which includes the motor, motor housing, and air bearings, and the ratio of the tip mass to the mass of the beam are given below

$$m = 0.787 \text{ kg}, \quad \mu \triangleq \frac{m}{\rho A L} = 7.03$$

The mass moment of inertia of the motor, motor housing, and air bearings, and the corresponding non-dimensional constant are given below

$$J = 0.00135 \text{ kgm}^2, \quad \nu \triangleq \frac{J}{\rho A L^3} = 0.038$$

In our experiments, the terminal moment was proportional to the curvature of the beam, which was measured using a pair of strain gages located at a distance of

$$\hat{x} = 0.078 \text{ m} \quad \Rightarrow \quad \alpha = \frac{\hat{x}}{L} = 0.14$$

A P3 strain gage measurement unit⁴ was used to condition the strain gage signals and output a voltage proportional to the beam curvature. A dSpace DS1104 board and the Matlab/Simulink environment was used for closed-loop excitation of the beam using motor

³The motor used in our experiment is a product of Micromo, model number 3863H0124CR. It was controlled using an analog servo drive; the servo drive is a product of Advanced Motion Controls, model number 12A8.

⁴The P3 is a product of Vishay Precision Group.

torque as the input and curvature measurement as the output.

4.4.2 Model Verification

The first two natural frequencies of the beam were evaluated experimentally. A sinusoidal torque was applied to the beam using the motor and the frequency was gradually increased. The resonant behavior of the beam was used to identify the natural frequencies; they are listed as ω_1 and ω_2 in Table 4.1 to distinguish them from the non-dimensional natural frequencies $\bar{\omega}$. The experimentally determined frequencies match well with those obtained analytically and numerically (ten-mode Galerkin approximation).

Table 4.1: Natural frequencies of the beam in Fig.4.8: analytical, numerical, and experimental values.

	Natural Frequencies	
	ω_1 (rad/s)	ω_2 rad/s
Analytical	1.470	20.685
Numerical	1.470	20.688
Experimental	1.6	21.5

4.4.3 Flutter Oscillations

For our experimental setup with $\alpha = 0.14$, the cantilever beam with tip mass was found to lose stability through the first flutter instability mode for $C_c < 0$. The experiments were conducted in the following manner. Initially, the beam was at rest in its undeformed configuration. The value of C was reduced below zero (made more negative), incrementally. The value of C was held constant for some time before it was reduced again. As the value of C approached the critical value, the beam could be seen vibrating with a very small amplitude. This amplitude would neither grow nor decay. At the critical value, the amplitude of vibration increased suddenly and the beam started undergoing limit cycle oscillations. Although the post-flutter behavior is not the subject of investigation of this work, it is well known that system nonlinearities lead to limit cycle oscillations at and beyond the point of flutter instability - see [34, 70, 71], for example.

The experimental results pertain to nonlinear limit cycle oscillations at or slightly beyond the point of instability; nevertheless, they show reasonable match with analytical results obtained using a linear model. The analytical and numerical (ten-mode Galerkin approximation) results based on the Euler-Bernoulli beam model are shown in Fig.4.9; they indicate that the critical frequency is $\omega^* = 7.34$ rad/s or $f^* = 1.167$ Hz and the critical value of C_c is $C_c^* = -0.263^5$. These values match reasonably well with the values obtained from experiments - see Table 4.2. The experimental flutter frequency was obtained from the strain gage voltage signal, which is not presented here.

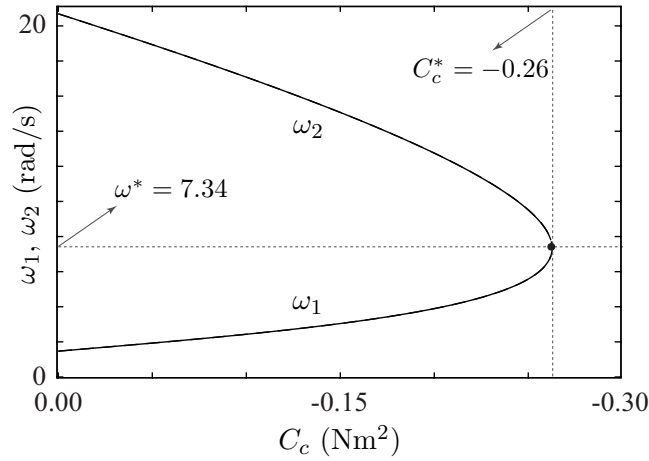


Figure 4.9: Analytical and numerical results showing the first flutter instability mode for the Euler-Bernoulli beam corresponding to the experimental setup in Fig.4.8. The analytical results are shown using a solid line; the numerical results are shown using a dashed line but are indistinguishable from the analytical results.

Table 4.2: Analytical, numerical and experimentally observed values of frequency and proportionality constant at the flutter instability point for the experimental setup in Fig.4.8.

	f^* (Hz)	C_c^* (Nm ²)
Analytical	1.168	-0.262
Numerical	1.167	-0.263
Experimental	1.33	-0.21

A still image of the maximum deflection of the beam was used to determine the non-

⁵The symbols ω^* , f^* and C_c^* should be distinguished from their counterparts $\bar{\omega}^*$, $\bar{f}^*$ and $\bar{C}_c^*$, which are non-dimensional quantities.

dimensional mode shape⁶ at or slightly above the instability point. Although this mode shape pertains to the nonlinear oscillation of the beam, it is found to match reasonably well with the linear mode shape computed analytically using Eq.(4.12) - see Fig.4.10. In Fig.4.10, both mode shapes were normalized by their infinity-norm.

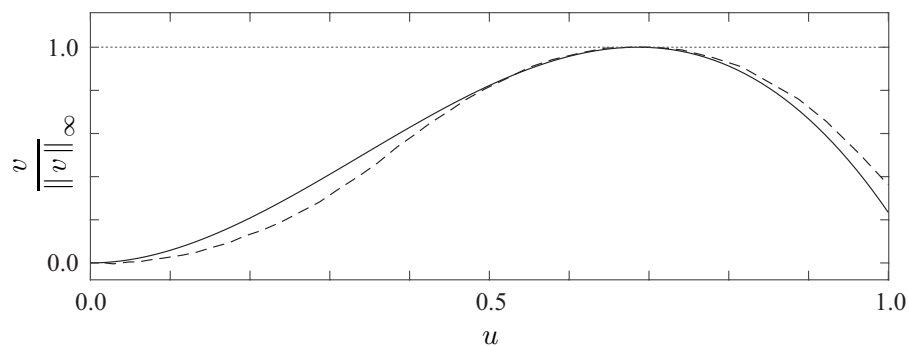


Figure 4.10: A comparison of normalized, non-dimensional mode shapes of the cantilever in Fig.4.8: solid and dashed lines show the mode shapes obtained analytically and from experimental data.

⁶A large number of points were chosen along the length of the deflected beam and the Matlab function “grabit” was used to extract the data from the image file.

Chapter 5

Stability Transitions Induced by an Intermediate Support

The stability investigation in this chapter is heavily inspired by the work done by Lee [72], where multiple stability transitions were reported due to the introduction of an intermediate elastic support. The main objective, however, is to highlight the nature of instability transitions, in a cantilevered beam subjected to a terminal dynamic moment and with an intermediate support. The investigation reveals a rich set of stability transitions not observed heretofore; these include multiple stability transitions between divergence and flutter and between different modes of flutter. These transitions usually involving jumps in the critical non-conservative load and sometimes occurs with the veering phenomenon [43]. In this chapter, the investigation is comprised of three parts; the first two parts consider the non-conservative dynamic moment to be proportional to the slope and the curvature at a point on the structure, respectively. In the last part, the problem of the cantilever beam with a follower force is revisited to draw attention to the jump phenomenon that has been overlooked in the literature [42].

5.1 Cantilever Beam with Terminal Dynamic Moment

5.1.1 Mathematical Model

Consider the Euler-Bernoulli beam of length L and cross-sectional area A , shown in Fig.5.1. The transverse displacement of a point on the beam at a distance x from the fixed end is denoted by $y(x, t)$. The beam has an intermediate pinned support at $x = \ell$ and is subjected to a terminal dynamic moment M that is proportional to the slope or the curvature of the

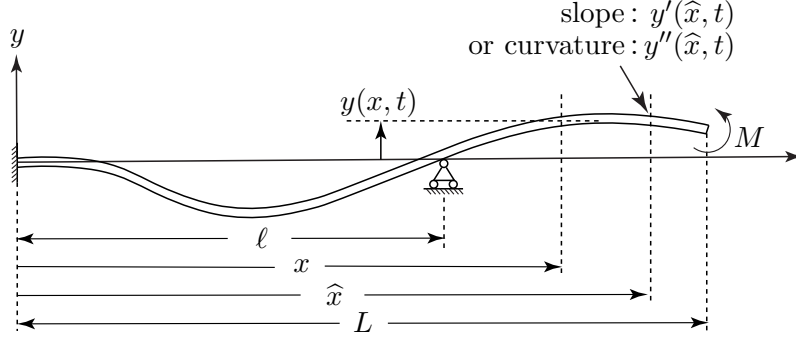


Figure 5.1: A flexible cantilever beam with an intermediate support subjected to a terminal dynamic moment.

beam at $x = \hat{x}$. We introduce the following change of variables

$$u = \frac{x}{L}, \quad v = \frac{y}{L}, \quad \tau = t \sqrt{\frac{EI}{\rho AL^4}}, \quad \kappa = \frac{\ell}{L}, \quad \alpha = \frac{\hat{x}}{L}, \quad \overline{M} = \frac{ML}{EI}$$

where E , I , and ρ are the Young's modulus of elasticity, area moment of inertia, and the mass per unit volume of the beam. The non-dimensional equation of motion is

$$v''''(u, \tau) + \ddot{v}(u, \tau) = 0 \quad (5.1)$$

The dynamic moment is assumed to be proportional to either the slope or the curvature of the beam at $x = \hat{x}$ or $u = \alpha$; therefore

$$M = \begin{cases} C_s \frac{dy(\hat{x}, t)}{dx} & \Rightarrow \quad \overline{M}_s = \overline{C}_s v'(\alpha, \tau) \\ C_c \frac{d^2 y(\hat{x}, t)}{dx^2} & \Rightarrow \quad \overline{M}_c = \overline{C}_c v''(\alpha, \tau) \end{cases}, \quad \overline{C}_s \triangleq \frac{C_s L}{EI}, \quad \overline{C}_c \triangleq \frac{C_c}{EI}$$

To simplify the analysis, the beam is studied over two domains; the displacement of the beam over these domains is defined as:

$$\begin{aligned} v_L(u, \tau) &= v(u, \tau) & \text{if } u \in [0, \kappa] \\ v_R(u, \tau) &= v(1 - u, \tau) & \text{if } u \in [0, 1 - \kappa] \end{aligned} \quad (5.2)$$

The boundary conditions are

$$v_L(0, \tau) = 0, \quad v'_L(0, \tau) = 0, \quad v''_R(1, \tau) = \overline{M} = \begin{cases} \overline{M}_s = \begin{cases} \overline{C}_s v'_L(\alpha, \tau) & : \text{ if } \alpha < \kappa \\ \overline{C}_s v'_R(1 - \alpha, \tau) & : \text{ if } \alpha > \kappa \end{cases} \\ \overline{M}_c = \begin{cases} \overline{C}_c v''_L(\alpha, \tau) & : \text{ if } \alpha < \kappa \\ \overline{C}_c v''_R(1 - \alpha, \tau) & : \text{ if } \alpha > \kappa \end{cases} \end{cases}, \quad v'''_R(1, \tau) = 0 \quad (5.3)$$

and continuity requires that we impose the constraints:

$$v_L(\kappa, \tau) = v_R(1 - \kappa, \tau) = 0, \quad v'_L(\kappa, \tau) = -v'_R(1 - \kappa, \tau), \quad v''_L(\kappa, \tau) = v''_R(1 - \kappa, \tau) \quad (5.4)$$

It has already been shown [73] that for both $\overline{C}_s \neq 0$ and $\overline{C}_c \neq 0$, the system is non-conservative and the cantilever beam may lose stability through divergence or flutter.

5.2 Terminal Dynamic Moment Proportional to Slope

To solve Eq.(5.1) together with the boundary conditions in Eq.(5.3), Galerkin approximation [63] is used where the solution is assumed to have the following form

$$v(u, \tau) = \sum_{i=1}^N a_i(\tau) \phi_i(u) \quad (5.5)$$

where N is the number of terms in the approximation, $\phi_i(u)$, $i = 1, \dots, N$, are assumed modes that satisfy the geometric boundary conditions, and $a_i(\tau)$, $i = 1, \dots, N$, are the modal amplitudes. For the problem with intermediate support, we substitute Eq.(5.5) into Eq.(5.1) and integrate over the length of the beam. After substituting the boundary conditions in Eq.(5.3), we get

$$\mathbb{M}\ddot{a} + \mathbb{K}a = 0 \quad (5.6)$$

where the (i, j) -th elements of the $\mathbb{M}$ and $\mathbb{K}$ matrices are obtained as follows

$$\begin{aligned}\mathbb{M}_{ij} &\triangleq \int_0^\kappa \phi_{Li} \phi_{Lj} du + \int_0^{1-\kappa} \phi_{Ri} \phi_{Rj} du \\ \mathbb{K}_{ij} &\triangleq \int_0^\kappa \phi_{Li}'' \phi_{Lj}'' du + \int_0^{1-\kappa} \phi_{Ri}'' \phi_{Rj}'' du - \begin{cases} \bar{C}_s \phi_{Ri}'(0) \phi_{Lj}'(\alpha) & : \text{ if } \alpha < \kappa \\ \bar{C}_s \phi_{Ri}'(0) \phi_{Rj}'(\alpha) & : \text{ if } \alpha > \kappa \end{cases} \quad (5.7)\end{aligned}$$

ϕ_{Li} and ϕ_{Ri} , however, have the following form

$$\begin{aligned}\phi_{Li}(u_L) &= \cosh(q_i u_L) - \cos(q_i u_L) + \lambda_i [\sin(q_i u_L) - \sinh(q_i u_L)] \\ \lambda_i &\triangleq \frac{\cos(\kappa q_i) - \cosh(\kappa q_i)}{\sin(\kappa q_i) - \sinh(\kappa q_i)}\end{aligned}$$

$$\begin{aligned}\phi_{Ri}(u_R) &= \eta_i [\cosh(q_i u_R) + \cos(q_i u_R)] - \mu_i [\sin(q_i u_R) + \sinh(q_i u_R)] \\ \eta_i &\triangleq \frac{[\cos(\kappa q_i) \cosh(\kappa q_i) - 1]}{[\sin(\kappa q_i) - \sinh(\kappa q_i)]} \times \frac{[\sin([1 - \kappa]q_i) + \sinh([1 - \kappa]q_i)]}{[\cos([1 - \kappa]q_i) \cosh([1 - \kappa]q_i) + 1]} \\ \mu_i &\triangleq \frac{\cos([1 - \kappa]q_i) + \cosh([1 - \kappa]q_i)}{\sin([1 - \kappa]q_i) + \sinh([1 - \kappa]q_i)}\end{aligned}$$

The q_i 's in the above equations are obtained by solving

$$\begin{aligned}& [\cos(\kappa q) \sinh(\kappa q) - \cosh(\kappa q) \sin(\kappa q)] \times [\cos([1 - \kappa]q) \cosh([1 - \kappa]q) + 1] + \\ & [\cos([1 - \kappa]q) \sinh([1 - \kappa]q) - \cosh([1 - \kappa]q) \sin([1 - \kappa]q)] [\cos(\kappa q) \cosh([1 - \kappa]q) - 1] = 0\end{aligned}$$

5.2.1 Model Verification

Previous equations are used to investigate the accuracy of the Galerkin approximation by comparing results obtained using the analytical solution for the beam without intermediate support. These results are used to gain confidence in our Galerkin approximation for the beam with intermediate support, for which analytical results are not readily available. For the beam without intermediate support, the accuracy of the approximation is verified for $N = 4, 5, 6, 7, 8$, by comparing the critical value of the proportionality constant $\bar{C}_s$ for $\alpha =$

0.25, 0.5, 0.75. The results are shown in Tables 5.1 and 5.2 for positive and negative values of the proportionality constant $\bar{C}_s$. It should be noted that $\bar{C}_s > 0$ results in divergence and $\bar{C}_s < 0$ results in flutter.

Table 5.1: Critical values of $\bar{C}_s$ for $\bar{C}_s > 0$ that results in divergence.

α	<i>Analytical</i>	$N = 4$	$N = 5$	$N = 6$	$N = 7$	$N = 8$
0.25	4.05	3.89	4.1	3.96	4	4.03
0.5	2	2.04	1.99	1.99	2.01	2.01
0.75	1.35	1.31	1.33	1.35	1.35	1.35

Table 5.2: Critical values of $\bar{C}_s$ for $\bar{C}_s < 0$ that results in flutter.

α	<i>Analytical</i>	$N = 4$	$N = 5$	$N = 6$	$N = 7$	$N = 8$
0.25	-7.15	-7.51	-6.82	-7.26	-7.13	-7.03
0.5	-13.5	-11.89	-14.38	-14.38	-13.1	-13.1
0.75	-14.65	-21.67	-15.95	-13.83	-13.33	-13.7

The results in Tables 5.1 and 5.2 reveal that $N = 6$ provides a sufficiently accurate solution. For this reason, a six-mode based approximation ($N = 6$) is used in the subsequent analysis.

5.2.2 Effect of Intermediate Support on Critical Stability

In this section we investigate instability of the beam with intermediate support at three different locations along the length of the beam, namely, $\kappa = 0.25, 0.50, 0.75$. For each of these cases, the instability is investigated for positive and negative values of $\bar{C}_s$. We rely on a six-mode Galerkin approximation of the system described by Eqs.(5.6) and (5.7).

Dynamic Moment Proportional to Positive Slope

We first study the case where the moment is proportional to the positive slope, *i.e.*, $\bar{C}_s > 0$. The results of the first case, $\kappa = 0.25$, are presented in Fig.5.2. Unlike the beam

with no intermediate support where stability is lost through divergence alone, here stability is lost through both divergence and flutter. For $\alpha \in [0.1, 0.13] \cup [0.17, 0.25] \cup [0.83, 1.0]$, stability is lost through flutter; divergence instability occurs for the intermediate range of $\alpha \in (0.25, 0.83)$. The six-mode approximation fails to capture loss of stability for the ranges of $\alpha \in (0.13, 0.17) \cup (0.8, 0.83)$.

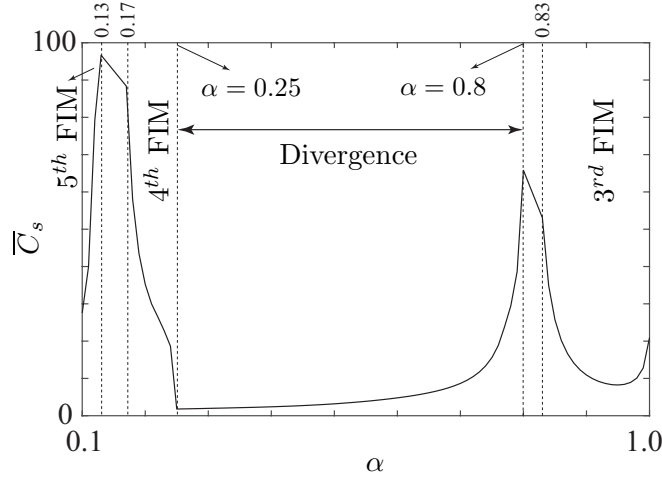


Figure 5.2: Critical stability curve for the beam with intermediate support: $\bar{C}_s > 0$ and $\kappa = 0.25$.

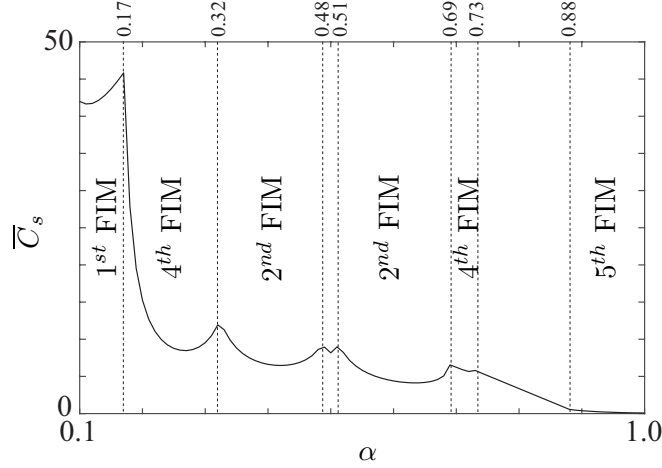


Figure 5.3: Critical stability curve for the beam with intermediate support: $\bar{C}_s > 0$ and $\kappa = 0.50$.

For $\kappa = 0.50$, the results are provided in Fig.5.3. Similar to the results shown in Fig.5.2 for $\kappa = 0.25$, stability is lost through both flutter and divergence. However, divergence

instability occurs for a very narrow intermediate range of $\alpha \in (0.48, 0.51)$. For $\alpha \in [0.1, 0.48] \cup [0.51, 0.73] \cup [0.88, 1.0]$, stability is lost through flutter and the mode of flutter instability switches randomly as α changes. The six-mode approximation, again, fails to capture loss of stability for the range of $\alpha \in (0.73, 0.88)$.

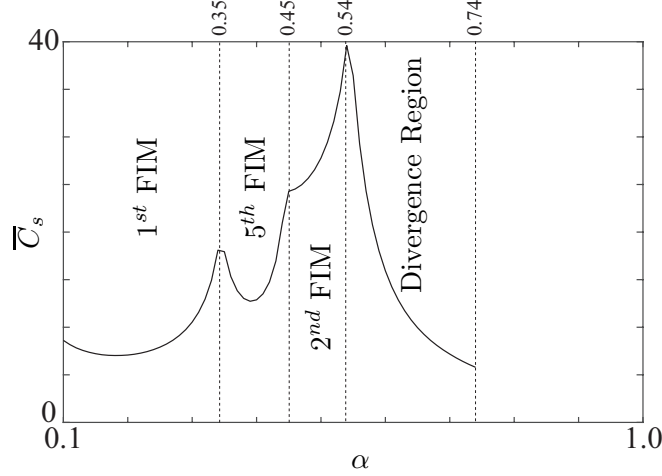


Figure 5.4: Critical stability curve for the beam with intermediate support: $\bar{C}_s > 0$ and $\kappa = 0.75$.

For $\kappa = 0.75$, stability investigations indicated that the loss of stability through flutter for $\alpha \in [0.1, 0.54]$ and through divergence for $\alpha \in [0.54, 0.74]$. The six-mode approximation failed to capture loss of stability for higher values of α .

A comparison of the results presented in Figs. 5.2, 5.3 and 5.4 with those for the beam with no intermediate support, Section 4.3.1, indicates that stability is lost through both flutter and divergence when $\bar{C}_s > 0$. Furthermore, the range of α values that lead to flutter and divergence instability and the mode of flutter instability strongly depend on the location of the intermediate support.

Dynamic Moment Proportional to Negative Slope

We complete this section by investigating the case where the moment is proportional to the negative slope, *i.e.*, $\bar{C}_s < 0$. For α varying between 0.1 and 1.0, we again consider the three cases where the intermediate support locations are $\kappa = 0.25, 0.50, 0.75$. Similar to the cases

with $\bar{C}_s > 0$, stability is lost through both divergence and flutter. For $\kappa = 0.25$, stability is primarily lost through different flutter instability modes as shown in Fig.5.5; divergence instability occurs for a very small region, namely, $\alpha \in [0.11, 0.14]$. For $\kappa = 0.50$, stability is also lost primarily through different flutter instability modes as shown in Fig.5.6; divergence instability occurs for a very small region, namely, $\alpha \in [0.25, 0.29]$. For $\kappa = 0.25$ and 0.50 ,

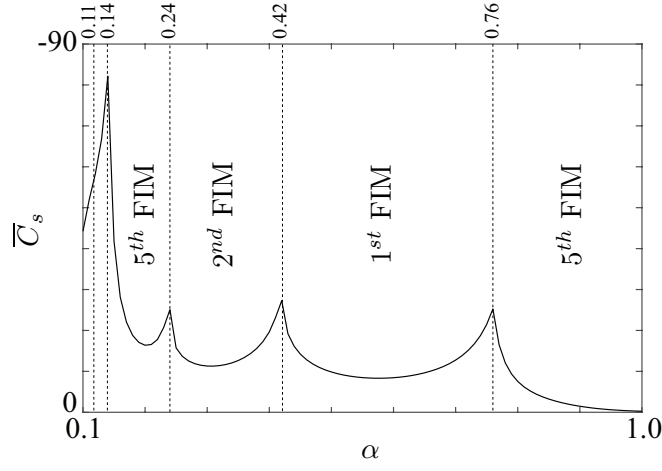


Figure 5.5: Critical stability curve for the beam with intermediate support: $\bar{C}_s < 0$ and $\kappa = 0.25$.

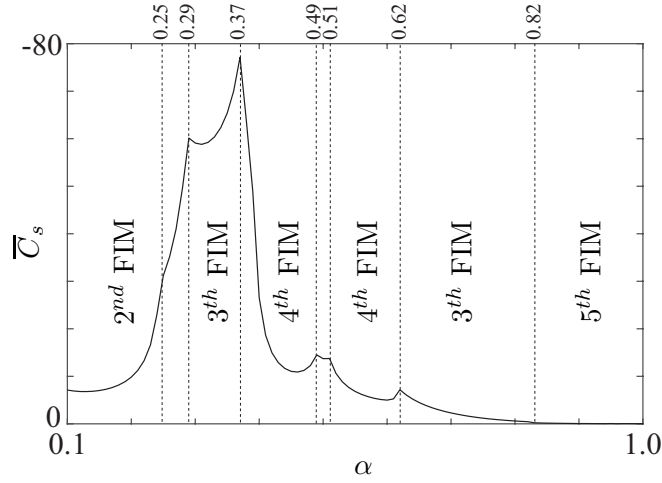


Figure 5.6: Critical stability curve for the beam with intermediate support: $\bar{C}_s < 0$ and $\kappa = 0.50$.

the structure exhibits flutter instability for very small values of α ; as α is increased, the nature of instability switches to divergence and then switches back to flutter. In the region

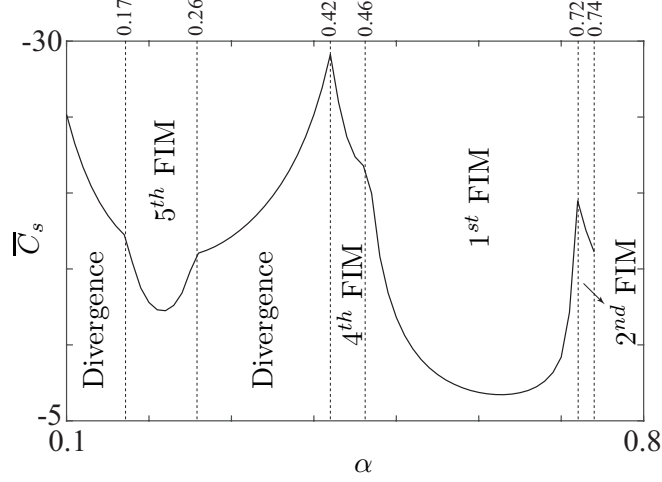


Figure 5.7: Critical stability curve for the beam with intermediate support: $\bar{C}_s < 0$ and $\kappa = 0.75$.

of flutter instability, the flutter instability modes switch randomly. In contrast, for $\kappa = 0.75$, the structure loses stability alternately through divergence and flutter as the value of α increases - see Fig.5.7.

5.3 Terminal Dynamic Moment Proportional to Curvature

In this section, we use the method of variable separation to solve Eq.(5.1). We substitute $v(u, \tau) = U(u)T(\tau)$ into Eq.(5.1) to get

$$\frac{U''''}{U} = \frac{\ddot{T}}{T} = \omega^2 \quad (5.8)$$

The complete solution can then be obtained as

$$U(u) = C_1 e^{\beta u} + C_2 e^{-\beta u} + C_3 e^{i\beta u} + C_4 e^{-i\beta u}, \quad T(\tau) = A \cos \omega \tau + B \sin \omega \tau, \quad \beta \triangleq \sqrt{\omega} \quad (5.9)$$

where C_1 through C_4 , and A and B are constants. Following the notation in Eq.(5.2), the shape function in Eq.(5.9) is defined over the two domains that lie to the left and right of the intermediate support:

$$\begin{aligned}
U_L(u) &= C_1 e^{\beta u} + C_2 e^{-\beta u} + C_3 e^{i\beta u} + C_4 e^{-i\beta u}, & \text{if } u \in [0, \kappa] \\
U_R(u) &= C_5 e^{\beta u} + C_6 e^{-\beta u} + C_7 e^{i\beta u} + C_8 e^{-i\beta u}, & \text{if } u \in [0, 1 - \kappa]
\end{aligned} \tag{5.10}$$

where C_5 through C_8 are four additional constants that are used to describe the shape of the beam to the right of the intermediate support. Substitution of Eq.(5.10) into Eqs.(5.3) and (5.4) results in eight simultaneous homogeneous algebraic equations in the unknown constants C_1 through C_8 . By equating the determinant of the coefficient matrix to zero, we get the following characteristic equations for $\alpha \leq \kappa$:

$$\begin{aligned}
&\overline{C}_c \cos \mu_1 \sin(\mu_2 - \mu_1) - \overline{C}_c \sinh(\alpha\beta) \cos \mu_1 \cos(\beta\kappa) \\
&- \overline{C}_c \cos(\alpha\beta) \cos \mu_1 \sinh(\beta\kappa) + \overline{C}_c \sin(\alpha\beta) \cosh \mu_1 \cosh(\beta\kappa) \\
&+ \overline{C}_c \cosh \mu_1 \sin(\mu_2 - \mu_1) + \overline{C}_c \cosh(\alpha\beta) \sin(\beta\kappa) \cosh \mu_1 \\
&+ \overline{C}_c \sinh(\alpha\beta) \cosh \mu_1 \cosh(\beta\kappa) - \overline{C}_c \cosh(\alpha\beta) \sinh(\beta\kappa) \cosh \mu_1 \\
&+ \overline{C}_c \sin(\alpha\beta) \cos \mu_1 \cosh(\beta\kappa) + \overline{C}_c \cosh(\alpha\beta) \sin(\beta\kappa) \cos \mu_1 \\
&- \overline{C}_c \sinh(\alpha\beta) \cos(\beta\kappa) \cosh \mu_1 + \overline{C}_c \sinh(\alpha\beta) \cos \mu_1 \cosh(\beta\kappa) \\
&- \overline{C}_c \cosh(\alpha\beta) \cos \mu_1 \sinh(\beta\kappa) - \overline{C}_c \cos(\alpha\beta) \sinh(\beta\kappa) \cosh \mu_1 \\
&- 2 \cos \mu_1 \sinh \mu_1 + 2 \cos(\beta\kappa) \sinh(\beta\kappa) - 2 \sin(\beta) \cosh \mu_1 \cosh(\beta\kappa) \\
&+ 2 \sin \mu_1 \cosh \mu_1 - 2 \sin(\beta\kappa) \cosh(\beta\kappa) + 2 \cos \mu_1 \cos(\beta\kappa) \sinh \mu_1 \cosh(\beta\kappa) \\
&+ 2 \cos \mu_1 \cos(\beta\kappa) \sinh(\beta\kappa) \cosh \mu_1 = 0
\end{aligned} \tag{5.11}$$

and $\alpha > \kappa$:

$$\begin{aligned}
&\overline{C}_c \sinh \mu_2 \sin(\beta\kappa) \cos \mu_1 (\cosh \mu_3 + \cosh \beta) \\
&- 2 \overline{C}_c \cos(\beta\kappa) \sinh(\beta\kappa) (\sin \mu_2 + \sinh \mu_2) \\
&+ \cosh \mu_1 [\cosh(\beta\kappa) [\overline{C}_c \cos \mu_4 + 3 \overline{C}_c \cos(\alpha\beta) - 4\beta \sin \beta] \\
&- 4 \overline{C}_c \cos(\mu_2 - \mu_1) + 4\beta \sin \mu_1 + 2 \cos(\beta\kappa) \sinh(\beta\kappa) [\overline{C}_c \sin(\mu_1 - \mu_2) \\
&+ \cos \mu_1 (2\beta - \overline{C}_c \sinh \mu_2)] + \overline{C}_c \cosh \mu_2 [-4 \cos \mu_1 \\
&+ (\cos \mu_3 + 3 \cos \beta) \cosh(\beta\kappa) + 2 \sin \mu_1 \cos(\beta\kappa) \sinh(\beta\kappa)]] \\
&+ 2 \sinh \mu_1 [\overline{C}_c \cos(\beta\kappa) \sinh(\beta\kappa) \{\cos(\mu_2 - \mu_1) \\
&- \sinh \mu_2 \sin \mu_1 + \cosh \mu_2 \cos \mu_1\} - 2 \cos \mu_1 (\beta - \overline{C}_c \sinh \mu_2)] \\
&+ \cosh(\beta\kappa) [2 \sin(\beta\kappa) [\overline{C}_c \sin \mu_2 + \overline{C}_c \sinh \mu_2 - 2\beta] \\
&+ \sinh \mu_1 [2\beta (\cos \mu_3 + \cos \beta) - 2 \overline{C}_c \sin(\beta\kappa) [\cos(\mu_2 - \mu_1) + \cosh \mu_2 \cos \mu_1] \\
&- \overline{C}_c \sinh \mu_2 (\cos \mu_3 + 3 \cos \beta)]] + 4\beta \cos(\beta\kappa) \sinh(\beta\kappa) = 0
\end{aligned} \tag{5.12}$$

where

$$\mu_1 \triangleq \beta(1 - \kappa), \quad \mu_2 \triangleq \beta(1 - \alpha), \quad \mu_3 \triangleq \beta(1 - 2\kappa), \quad \mu_4 \triangleq \beta(\alpha - 2\kappa)$$

Equations (5.11) and (5.12) are functions of ω ($\omega = \beta^2$), κ , α , and $\overline{\mathcal{C}}_c$, and they can be solved for critical stability for different combination of κ and α values. To this end, we start with $\overline{\mathcal{C}}_c = 0$ and compute the first twelve natural frequencies ω for $\kappa \in [0.1, 0.9]$ and $\alpha \in [0.1, 0.9]$. Then, we trace the change in these natural frequencies by gradually increasing the value of $\overline{\mathcal{C}}_c$ up to the point $\overline{\mathcal{C}}_c = \overline{\mathcal{C}}_{cr}$ where the system loses its stability through either divergence or flutter¹. Since there is no damping in the system, stability is lost through divergence when the fundamental frequency become equal to zero and through coupled-mode flutter [69] when the roots associated with two consecutive natural frequencies approach each other and become equal. The proportionality constant $\overline{\mathcal{C}}_c$ can be positive or negative, therefore we study these two cases separately.

The analytical method is used to investigate critical stability; the numerical method requires the computation of mode shapes and this process becomes cumbersome as the location of the intermediate support is changed incrementally with a small step size. In the previous section, the numerical method was used to study critical stability for the case where the terminal moment is proportional to the slope of the beam; however, the study was restricted to three values of κ , namely, $\kappa = 0.25, 0.50, 0.75$.

5.3.1 Effect of Intermediate Support on Critical Stability

Moment is Proportional to Positive Curvature

We first consider the case where $\overline{\mathcal{C}}_c > 0$, *i.e.*, the terminal moment is proportional to the positive curvature. In the absence of intermediate support, stability is always lost through divergence [73]; here it will be shown that introduction of an intermediate support significantly alters the stability characteristics. The critical value of $\overline{\mathcal{C}}_c$ depends on both κ and α ; Eq.(5.11) is used to compute $\overline{\mathcal{C}}_{cr}$ for $\alpha \leq \kappa$ and Eq.(5.12) is used to compute $\overline{\mathcal{C}}_{cr}$ for $\alpha > \kappa$. The critical stability surface is obtained by computing $\overline{\mathcal{C}}_{cr}$ on a fine mesh grid of $\kappa \in [0.1, 0.9]$

¹We restricted ourselves to the locus of the first ten natural frequencies; therefore, the results reported in the next section are based on the implicit assumption that the system does not lose stability through modes higher than ten.

and $\alpha \in [0.1, 0.9]$ with a step size of 0.01; the results are shown in Fig.5.8. This figure shows that the system loses stability through both flutter and divergence; to distinguish between the two types of instability, divergence is shown by dark points on the grid.

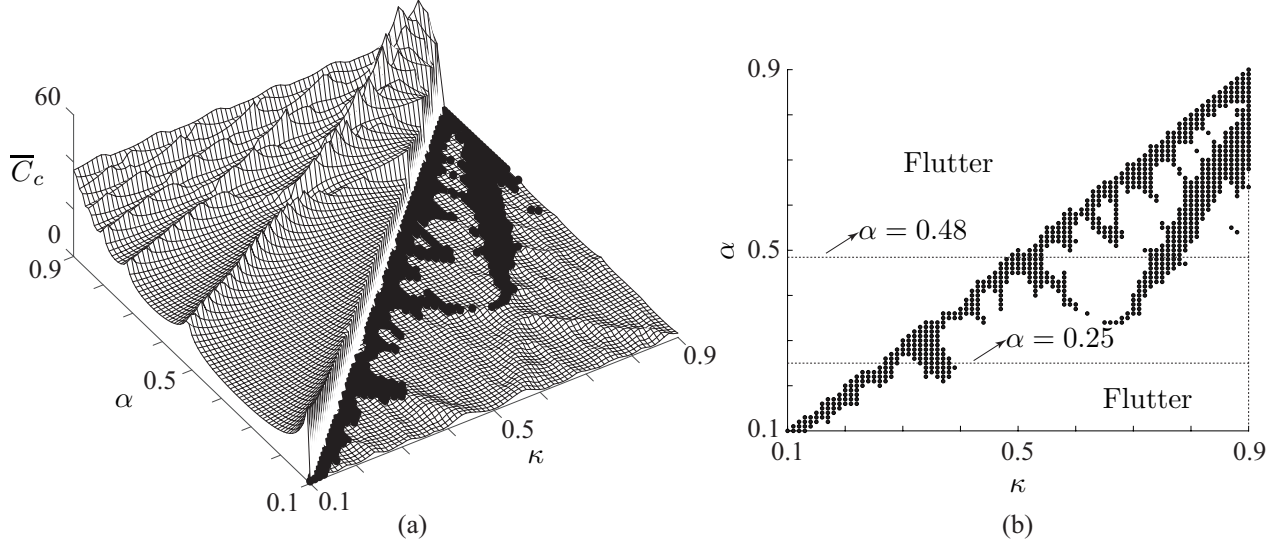


Figure 5.8: Terminal moment is proportional to the positive curvature: (a) $\overline{C}_{cr}$ is plotted on a fine mesh grid of κ and α values; divergence instability is shown by dark points on the grid (b) Top view of (a).

The critical stability surface in Fig.5.8 reveals a rich set of stability transitions between different modes of flutter² and between flutter and divergence. To illustrate these stability transitions, we take a closer look at the following five cases, which are marked on Fig.5.8 (b):

Case 1: $\kappa = 0.1, \alpha \in [0.1, 0.9]$

Case 2: $\alpha = 0.48, \kappa \in [0.1, 0.9]$

Case 3: $\kappa = 0.9, \alpha \in [0.1, 0.9]$

Case 4: $\alpha = 0.25, \kappa \in \{0.249, 0.251\}$

Case 5: $\kappa = 0.65, \alpha \in \{0.62, 0.78\}$

²In our earlier work [73], we defined the term “flutter instability mode” as follows: The system loses stability through the n -th flutter instability mode if two successive natural frequencies of the system, numbered n and $(n+1)$, are equal at the lowest critical value of a system parameter. This is usually referred to as coupled-mode flutter via a Hamiltonian-Hopf bifurcation [69].

For Case 1, the system exhibits divergence instability at $\alpha = 0.1$ and transitions immediately to flutter instability as α is increased by 0.001; the value of $\overline{C}_{cr}$ jumps from 1.00 to 41.87 - see Fig.5.9 (a). In contrast to the cantilever beam with follower force, discussed in section 5.4, no veering phenomenon [43] is associated with this jump. For $\alpha > 0.100$, the system loses stability through different modes of flutter as listed below:

- range 1: $\alpha \in [0.101, 0.513)$ first flutter instability mode
- range 2: $\alpha \in [0.513, 0.717)$ second flutter instability mode
- range 3: $\alpha \in [0.717, 0.798)$ third flutter instability mode
- range 4: $\alpha \in [0.798, 0.843)$ fourth flutter instability mode
- range 5: $\alpha \in [0.843, 0.872)$ fifth flutter instability mode
- range 6: $\alpha \in [0.872, 0.892)$ sixth flutter instability mode
- range 7: $\alpha \in [0.892, 0.900]$ seventh flutter instability mode

It should be noted that an increase in α is associated with an ascending order in the mode of flutter instability. Furthermore, while there is no jump in the value of $\overline{C}_{cr}$ during the flutter-to-flutter instability transitions, each transition is associated with the veering phenomenon.

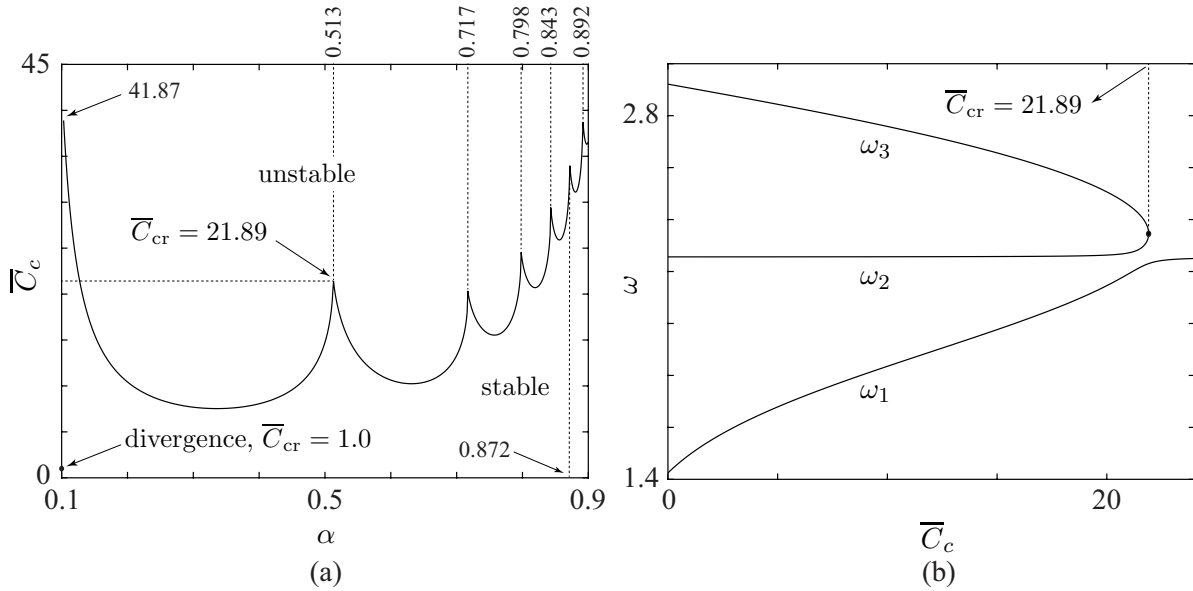


Figure 5.9: Case 1: (a) Critical stability curve - divergence instability at $\alpha = 0.1$ and flutter instability for $\alpha \in [0.101, 0.9]$ (b) Second flutter instability mode at $\alpha = 0.513$ showing veering of the locus of the first natural frequency.

For instance, at $\alpha = 0.513$ the mode of flutter instability switches from the first to the second - see Fig.5.9 (b).

For Case 2, multiple stability transitions occur between flutter and divergence; this is different from Case 1, where a single transition between divergence and flutter is observed. The multiple stability transitions for Case 2, including flutter to flutter transitions, are listed below and shown in Fig.5.10 (a).

- | | | |
|----------|---------------------------|----------------------------------|
| range 1: | $\kappa \in [0.10, 0.47]$ | first flutter instability mode |
| range 2: | $\kappa \in (0.47, 0.54]$ | divergence instability |
| range 3: | $\kappa \in (0.54, 0.58]$ | seventh flutter instability mode |
| range 4: | $\kappa \in (0.58, 0.67]$ | third flutter instability mode |
| range 5: | $\kappa \in (0.67, 0.72]$ | seventh flutter instability mode |
| range 6: | $\kappa \in (0.72, 0.78]$ | divergence instability |
| range 7: | $\kappa \in (0.78, 0.84]$ | seventh flutter instability mode |
| range 8: | $\kappa \in (0.84, 0.90]$ | third flutter instability mode |

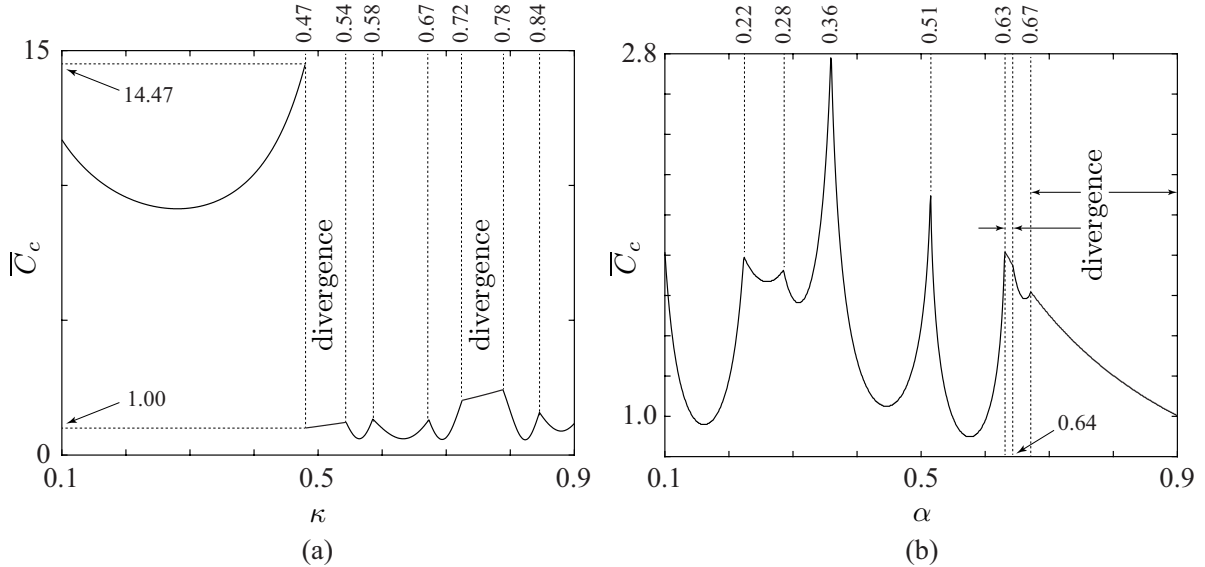


Figure 5.10: (a) Case 2: Critical stability curve - divergence instability for $\kappa \in (0.47, 0.54] \cup (0.72, 0.78]$ and flutter instability for other values of κ (b) Case 3: Critical stability curve - divergence instability for $\alpha \in (0.631, 0.642] \cup (0.673, 0.900]$ and flutter instability for other values of α .

The multiple stability transitions between divergence and flutter occur at $\kappa = 0.47, 0.54, 0.72$, and 0.78 . Among them, the stability transition at $\kappa = 0.47$ is associated with a large jump in the value of $\overline{C}_{cr}$ from 14.47 to 1.00; there is no jump in the value of $\overline{C}_{cr}$ for the other three transitions. Flutter to flutter instability transitions occur in the range of $\kappa \in (0.54, 0.72] \cup (0.78, 0.90]$; unlike Case 1 where the order of flutter instability changes sequentially, these transitions alternate between the third and seventh flutter instability modes. However, similar to Case 1, the critical stability curve has a “catenary-like” shape in the regions where stability is lost through flutter. Finally, there is no veering phenomenon associated with any of the stability transitions for Case 2.

For Case 3, the critical stability curve is shown in Fig.5.10 (b). It can be seen that stability is lost through divergence for $\alpha \in (0.63, 0.64] \cup (0.67, 0.90]$, and through flutter everywhere else. Similar to Cases 1 and 2, the critical stability curve has a “catenary-like” shape in the flutter instability regions. The modes of flutter instability are not specified here as they change randomly between the first five modes. Despite multiple transitions between divergence and flutter, and flutter-to-flutter, unlike Cases 1 and 2 there is no jump in the value of $\overline{C}_{cr}$.

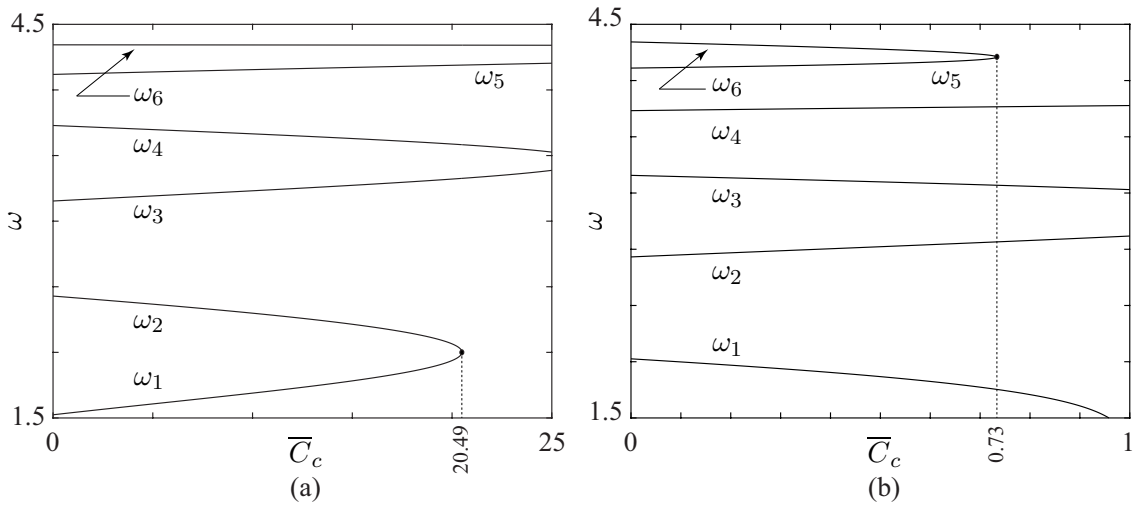


Figure 5.11: Case 4: Locus of the first six natural frequencies as $\overline{C}_c$ is increased from zero for $\alpha = 0.25$ and (a) $\kappa = 0.249$, (b) $\kappa = 0.251$.

For Case 4, the locus of the first six natural frequencies are plotted as $\overline{C}_c$ is increased from zero. These loci, shown in Figs.5.11 (a) and (b), indicate a transition from the first flutter instability mode to the fifth flutter instability mode due to a small change in the value of κ ; importantly, the flutter-to-flutter transition is accompanied with a jump in the value of $\overline{C}_{cr}$ from 20.49 to 0.73.

Remark 5.1: A closer look at Fig.5.8 (b) indicates that the flutter and divergence instability regions are not accurately demarcated by the diagonal line $\alpha = \kappa$. For example, for $\alpha = 0.25$, the two points $\kappa = 0.249$ and $\kappa = 0.251$ lie on opposite sides of the diagonal line, yet stability is lost through flutter for both of these cases. It has been observed that flutter-to-flutter transitions occur with jumps in the critical value of $\overline{C}_{cr}$ wherever the flutter instability region extends from the region $\alpha > \kappa$ to the region $\alpha < \kappa$. The importance of this observation will be better appreciated in the next section where we study the case where the moment is proportional to the negative curvature.

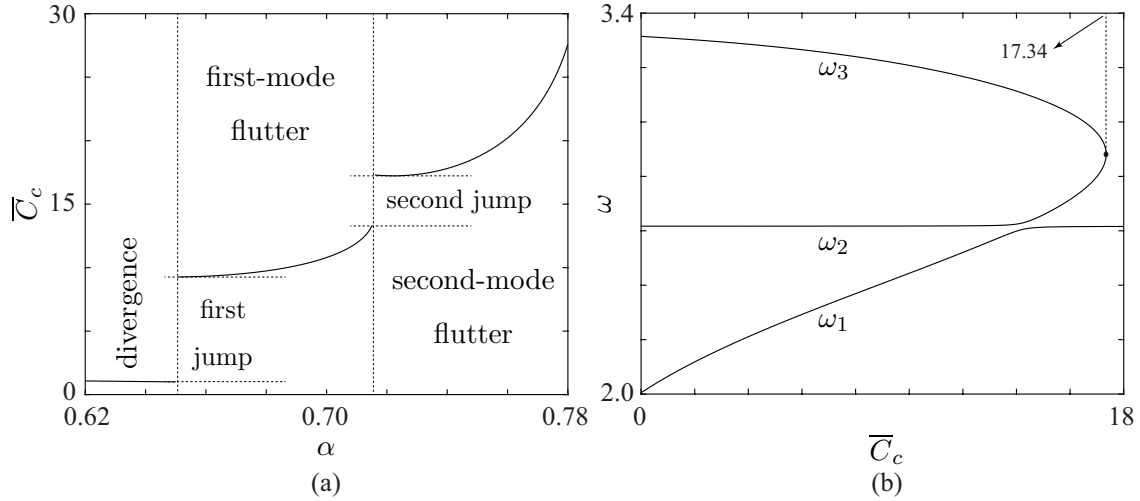


Figure 5.12: Case 5: (a) Critical stability curve - divergence instability for $\alpha \in (0.62, 0.65]$, first-mode flutter instability for $\alpha \in (0.65, 0.715]$ and second-mode flutter instability for $\alpha \in (0.715, 0.78]$ (b) Second-mode flutter instability at $\alpha = 0.716$ showing veering of the locus of the first natural frequency.

For Case 5, the critical stability curve is shown in Fig.5.12 (a). It shows two jumps in the value of $\overline{C}_{cr}$. The value of $\overline{C}_{cr}$ jumps from 1.0 to 9.27 during the transition between

divergence and flutter at $\alpha = 0.65$. The value of $\overline{\mathcal{C}}_{cr}$ jumps from 13.29 to 17.3 during the transition between the first and second flutter instability modes at $\alpha = 0.715$. The second jump results from the first natural frequency veering off; this is illustrated in Fig.5.12 (b).

Moment is Proportional to Negative Curvature

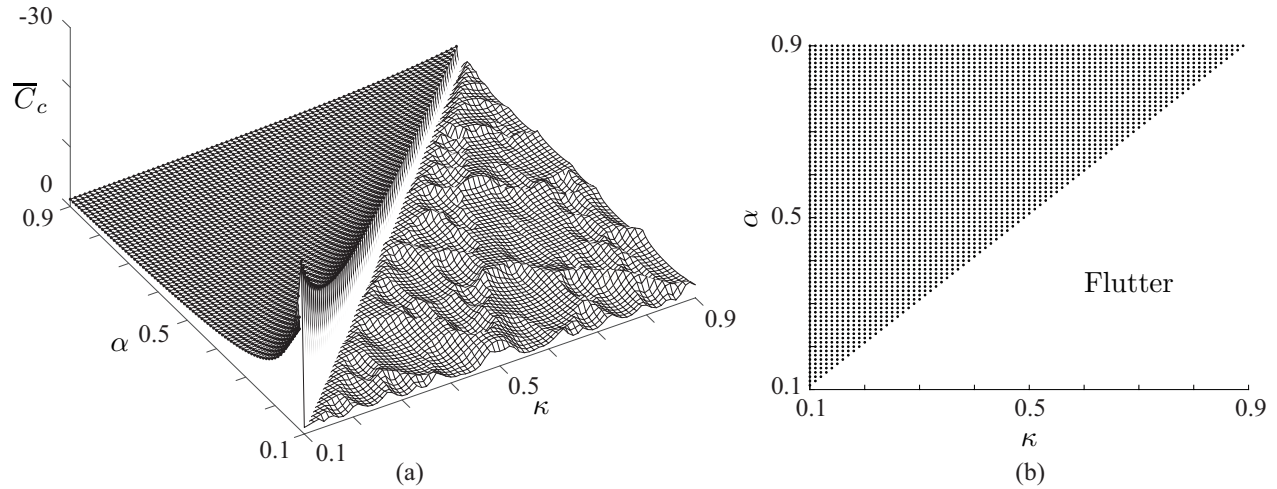


Figure 5.13: Terminal moment is proportional to the negative curvature: (a) $\overline{\mathcal{C}}_{cr}$ is plotted on a fine mesh grid of κ and α values; divergence instability is shown by dark points on the grid (b) Top view of (a).

We now consider the case where $\overline{\mathcal{C}}_c < 0$, *i.e.*, the terminal moment is proportional to the negative curvature. In the absence of intermediate support, stability is always lost through flutter [73]; here it will be shown that introduction of an intermediate support results in both divergence and flutter instabilities. Similar to positive curvature case, the critical stability surface is first obtained; the results are shown in Fig.5.13, where divergence is shown by dark points on the grid to distinguish it from flutter. A comparison of the results in Fig.5.13 with those in Fig.5.8 indicate several similarities and dissimilarities:

- For the case $\overline{\mathcal{C}}_c > 0$, stability is lost through divergence only when $\alpha < \kappa$. In contrast, for the case $\overline{\mathcal{C}}_c < 0$, stability is lost through divergence only when $\alpha > \kappa$.
- For the case $\overline{\mathcal{C}}_c < 0$, the diagonal line $\alpha = \kappa$ clearly demarcates the divergence and flutter instability regions. In contrast, for the case $\overline{\mathcal{C}}_c > 0$, stability can be lost through

flutter on both sides of the diagonal line, *i.e.*, $\alpha > \kappa$ and $\alpha < \kappa$.

- For both $\overline{C}_c > 0$ and $\overline{C}_c < 0$, the diagonal line $\alpha = \kappa$ is associated with a jump in the value of $\overline{C}_{cr}$. For $\overline{C}_c < 0$ this jump is associated with a stability transition between divergence and flutter. In contrast, for $\overline{C}_c > 0$, the jump can be due to a flutter-to-flutter transition or a transition between divergence and flutter.
- For both $\overline{C}_c > 0$ and $\overline{C}_c < 0$, the critical stability surface has an inverted dome-like shape, or the shape of a two-dimensional catenary, when stability is lost through a specific mode of flutter instability - see the region $\alpha > \kappa$ in Fig.5.13 (a), for example. Thus, fixing the value of one parameter (α or κ) results in catenary-like shapes when $\overline{C}_{cr}$ is plotted against the other parameter - see Fig.5.9 (a), Figs.5.10 (a) and (b), and Fig.5.12 (a).

5.4 Cantilever Beam with a Follower End Load

5.4.1 Mathematical Model

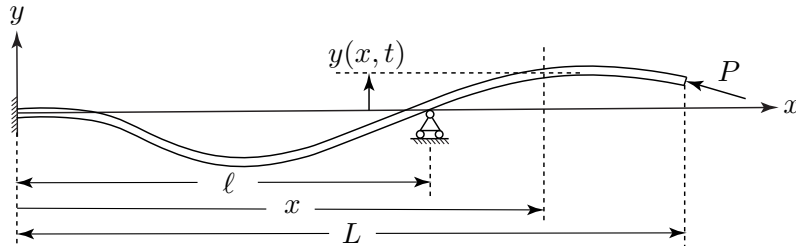


Figure 5.14: A flexible cantilever beam with an intermediate support subjected to a follower force

Consider the Euler-Bernoulli cantilever beam in Fig.5.14, which is subjected to a follower force P . The beam has length L , cross-sectional area A , and is simply supported at an intermediate point $x = \ell$. The transverse displacement of the beam is denoted by $y(x, t)$.

By introducing the change of variables

$$v = \frac{y}{L}, \quad u = \frac{x}{L}, \quad \tau = t \sqrt{\frac{EI}{\rho AL^4}}, \quad \sigma = \frac{PL^2}{EI}, \quad \kappa = \frac{\ell}{L}$$

the non-dimensional equation of motion can be expressed as

$$v''''(u, \tau) + \sigma v''(u, \tau) + \ddot{v}(u, \tau) = 0 \quad (5.13)$$

where E , I , and ρ are the Young's modulus of elasticity, area moment of inertia, and density of the beam, $(.)'$, and $(\dot{})$ denote the partial derivatives of $(.)$ with respect to u and τ , respectively. Similar to Section 5.1, the beam is studied over two domains and the displacement of the beam over these domains is defined as:

$$\begin{aligned} v_L(u, \tau) &= v(u, \tau) & \text{if } u \in [0, \kappa] \\ v_R(u, \tau) &= v(1 - u, \tau) & \text{if } u \in [0, 1 - \kappa] \end{aligned} \quad (5.14)$$

The above equations are identical to those in Eq.(5.2). The boundary conditions are

$$v_L(0, \tau) = v'_L(0, \tau) = 0, \quad v''_R(0, \tau) = v'''_R(0, \tau) = 0 \quad (5.15)$$

and continuity requires that we impose the constraints:

$$v_L(\kappa, \tau) = v_R(1 - \kappa, \tau) = 0, \quad v'_L(\kappa, \tau) = -v'_R(1 - \kappa, \tau), \quad v''_L(\kappa, \tau) = v''_R(1 - \kappa, \tau) \quad (5.16)$$

5.4.2 Analytical Solution

To solve Eq.(5.13) together with the boundary conditions and constraints in Eqs.(5.15) and (5.16), we assume the solution to be of the form

$$v(u, \tau) = U(u) T(\tau), \quad (5.17)$$

Substituting Eq.(5.17) in Eq.(5.13), yields

$$\frac{U''''}{U} + \sigma \frac{U''}{U} = \frac{\ddot{T}}{T} \triangleq \omega^2 \quad (5.18)$$

where ω^2 is constant. The complete solution can be obtained from (5.18) as follows:

$$\begin{aligned} T(\tau) &= A \sin(\omega\tau) + B \cos(\omega\tau) \\ U(u) &= C_1 e^{i\lambda_1 u} + C_2 e^{-i\lambda_1 u} + C_3 e^{\lambda_2 u} + C_4 e^{-\lambda_2 u} \\ \lambda_1 &\triangleq \sqrt{\left(\frac{\sigma}{2}\right) + \sqrt{\left(\frac{\sigma}{2}\right)^2 + \omega^2}}, \quad \lambda_2 \triangleq \sqrt{-\left(\frac{\sigma}{2}\right) + \sqrt{\left(\frac{\sigma}{2}\right)^2 + \omega^2}} \end{aligned} \quad (5.19)$$

where A and B are constants that can be obtained from initial conditions, and C_1 through C_4 are constants that can be obtained separately for each domain from the eight boundary conditions and constraints in Eqs.(5.15) and (5.16). The nontrivial solution of the eight constants result in the following transcendental characteristic equation:

$$\begin{aligned} &\left[\lambda_1 \lambda_2^2 \sin([1 - \kappa]\lambda_1) \cosh([1 - \kappa]\lambda_2) - \lambda_1^2 \lambda_2 \cos([1 - \kappa]\lambda_1) \sinh([1 - \kappa]\lambda_2) \right] \times \\ &\quad \left[-(\lambda_1^2 - \lambda_2^2) \sin(\kappa\lambda_1) \sinh(\kappa\lambda_2) - 2\lambda_1 \lambda_2 \{ \cos(\kappa\lambda_1) \cosh(\kappa\lambda_2) - 1 \} \right] - \\ &\left[\lambda_1 \lambda_2 (\lambda_1^2 - \lambda_2^2) \sin([1 - \kappa]\lambda_1) \sinh([1 - \kappa]\lambda_2) + 2\lambda_2^2 \lambda_1^2 \cos([1 - \kappa]\lambda_1) \cosh([1 - \kappa]\lambda_2) + \lambda_1^4 + \lambda_2^4 \right] \times \\ &\quad \left[\lambda_2 \sin(\kappa\lambda_1) \cosh(\kappa\lambda_2) - \lambda_1 \cos(\kappa\lambda_1) \sinh(\kappa\lambda_2) \right] = 0 \quad (5.20) \end{aligned}$$

Equation (5.20) is a function of ω , κ , and σ and can be solved for the combination of these parameters that result in critical stability. To this end, we start with $\sigma = 0$ and obtain the natural frequencies ω for $\kappa \in [0.1, 1.0]$. Then, we trace the change in the natural frequencies by gradually increasing the value of σ up to the point $\sigma = \sigma_{cr}$ where the system loses its stability through either divergence or flutter.

5.4.3 Numerical Solution

To be consistent with the literature where both analytical and numerical methods have been used [41, 42], we now obtain a numerical solution using a Galerkin approximation for both domains; the solutions are assumed to have the form:

$$v_L(u, \tau) = \sum_{i=1}^N a_i(\tau) \phi_{Li}(u), \quad v_R(u, \tau) = \sum_{i=1}^N a_i(\tau) \phi_{Ri}(u) \quad (5.21)$$

where N is the number of terms in the approximation, $\phi_{Li}(u)$, $\phi_{Ri}(u)$, $i = 1, \dots, N$, are assumed modes that satisfy the boundary conditions and constraints in Eqs.(5.15) and (5.16), and $a_i(\tau)$, $i = 1, \dots, N$, are the modal amplitudes. Substituting Eq.(5.21) into Eq.(5.13) and integrating over the length of the beam, we get

$$\mathbb{M}\ddot{a} + \mathbb{K}a = 0 \quad (5.22)$$

where the (i, j) -th elements of the $\mathbb{M}$ and $\mathbb{K}$ matrices are

$$\begin{aligned} \mathbb{M}_{ij} &\triangleq \int_0^\kappa \phi_{Li} \phi_{Lj} du + \int_0^{1-\kappa} \phi_{Ri} \phi_{Rj} du \\ \mathbb{K}_{ij} &\triangleq \int_0^\kappa \phi_{Li}'' \phi_{Lj}'' du + \int_0^{1-\kappa} \phi_{Ri}'' \phi_{Rj}'' du - \sigma \left[\int_0^\kappa \phi_{Li}' \phi_{Lj}' du + \int_0^{1-\kappa} \phi_{Ri}' \phi_{Rj}' du + \phi_{Ri}(0) \phi_{Rj}'(0) \right] \end{aligned} \quad (5.23)$$

The assumed modes ϕ_{Li} were chosen to be the modes of a freely vibrating cantilever-pinned beam, namely

$$\phi_{Li}(u) = \cosh(q_i u) - \cos(q_i u) + \lambda_i [\sin(q_i u) - \sinh(q_i u)], \quad \lambda_i \triangleq \frac{\cos(\kappa q_i) - \cosh(\kappa q_i)}{\sin(\kappa q_i) - \sinh(\kappa q_i)}$$

Similarly, the assumed modes ϕ_{Ri} were chosen to be the modes of a freely vibrating free-pinned beam, namely

$$\begin{aligned}\phi_{Ri}(u) &= \eta_i [\cosh(q_i u) + \cos(q_i u)] - \mu_i [\sin(q_i u) + \sinh(q_i u)] \\ \eta_i &\triangleq \frac{[\cos(\kappa q_i) \cosh(\kappa q_i) - 1]}{[\sin(\kappa q_i) - \sinh(\kappa q_i)]} \frac{[\sin([1 - \kappa]q_i) + \sinh([1 - \kappa]q_i)]}{[\cos([1 - \kappa]q_i) \cosh([1 - \kappa]q_i) + 1]} \\ \mu_i &\triangleq \frac{\cos([1 - \kappa]q_i) + \cosh([1 - \kappa]q_i)}{\sin([1 - \kappa]q_i) + \sinh([1 - \kappa]q_i)}\end{aligned}$$

In the expressions for the assumed modes above, the q_i 's are obtained by solving the transcendental equation

$$\begin{aligned}& [\cos(\kappa q) \sinh(\kappa q) - \cosh(\kappa q) \sin(\kappa q)][\cos([1 - \kappa]q) \cosh([1 - \kappa]q) + 1] + \\ & [\cos([1 - \kappa]q) \sinh([1 - \kappa]q) - \cosh([1 - \kappa]q) \sin([1 - \kappa]q)][\cos(\kappa q) \cosh([1 - \kappa]q) - 1] = 0\end{aligned}$$

which can be obtained from Eq.(5.20) by substituting $\sigma = 0$. The last three terms in the expression for K_{ij} in Eq.(5.23) represents the geometric stiffness associated with the follower force σ and the last term contributes to asymmetry. Since the stiffness matrix is asymmetric, the system can lose stability through flutter or divergence [72].

5.4.4 Effect of Intermediate Support on Critical Stability

Analytical and the numerical methods were used to investigate the the role of intermediate support on the critical stability of the system. The critical stability curve obtained analytically by solving Eq.(5.20) is shown in Fig.5.15 (a). To independently verify the results, we computed the critical value of σ for select values of κ , $\kappa = \{0.1, 0.2, 0.3, 0.4, 0.6, 0.7, 0.8, 0.9\}$, by solving the eigenvalue problem for the system in Eq.(5.22) using a three-mode ($N = 3$) approximation. These results, shown with the help of $\bullet$ symbols in Fig.5.15 (a), indicate a perfect match with the results obtained analytically. Two observations can be made from the results in Fig.5.15 (a). First, the system loses stability through flutter for $\kappa \leq 0.50$

and through divergence for $\kappa > 0.50$. Second, the transition of stability from flutter to divergence is accompanied by a jump in the critical value of σ : for $\kappa = 0.5000$, stability is lost through flutter with $\sigma_{\text{cr}} = 71.8$; for $\kappa = 0.5001$, stability is lost through divergence with $\sigma_{\text{cr}} = 80.6$. Figures 5.15 (b) and (c) provide a closer look at the neighborhood of the transition point $\kappa = 0.5$. It is clear from these figures that the transition of stability from flutter to divergence is accompanied by what is known as the “veering” phenomenon, first reported in [43]; the loci of the first two natural frequencies approach each other but veer off just before meeting, resulting in divergence instead of flutter. It should be noted that immediately prior to veering off, the loci of the natural frequencies meet at a point where their slopes are unequal ($\kappa = 0.5$, $\sigma = 71.8$); this is different from the “dome-shaped” curves created by the loci for lower values of κ - see Fig.5.15 (b).

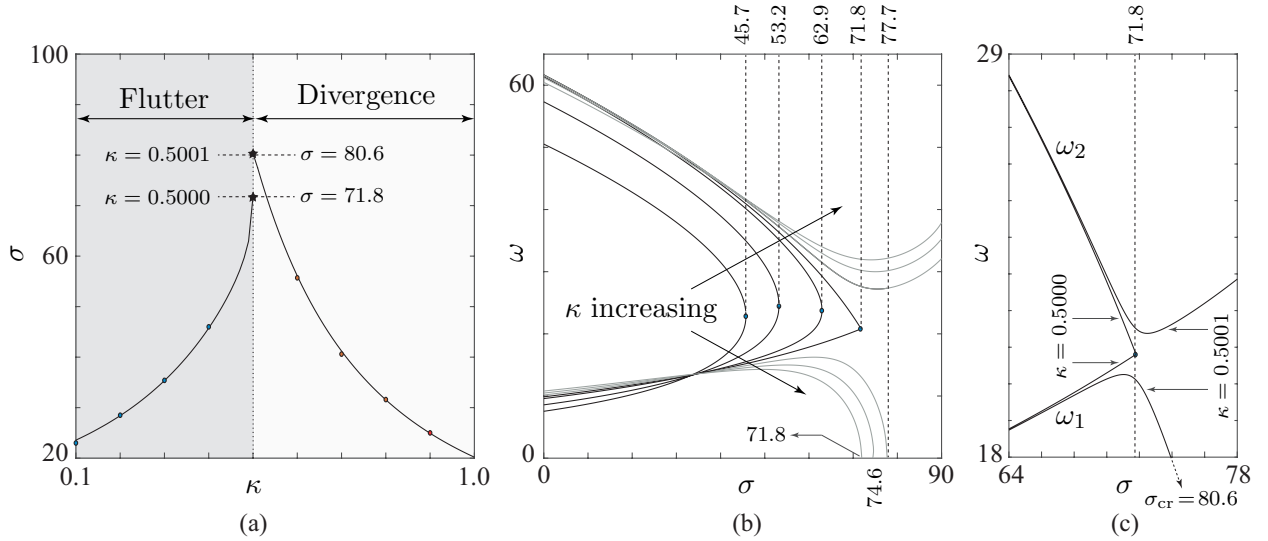


Figure 5.15: (a) Variation in the critical follower force σ as the location of the intermediate support κ is varied; both analytical (solid line) and numerical (dots) results are shown (b) Variation in the critical value of σ for 7 different values of κ ; stability is lost through flutter for $\kappa = \{0.40, 0.45, 0.49, 0.50\}$ and through divergence for $\kappa = \{0.51, 0.52, 0.53\}$ (c) Loci of the first two natural frequencies for $\kappa = \{0.5, 0.5001\}$ showing jump and veering phenomena; mode of instability changes from flutter to divergence and the critical value of σ jumps from 71.8 to 80.6 when κ changes from 0.5 to 0.5001.

Remark 5.2: To the best of our knowledge, the only publication on the transition of critical stability in a cantilever beam with a follower force due to an intermediate support is due to Elishakoff [42]. In this work, the transition of critical stability from flutter to divergence was correctly shown to occur at $\kappa = 0.5$; however, it was claimed that no jump phenomenon was associated with the transition. In this work, we have used analytical methods to verify the existence of the jump phenomenon and shown that the discontinuous jump is due to the veering off by the loci of the natural frequencies.

Chapter 6

Non-conservative Behavior of Hinged Beam with a Dynamic Moment

The stability characteristics of a hinged beam subjected to a non-conservative dynamic moment are investigated in this chapter. The dynamic moment is proportional to the curvature of the beam at some point along its length and the analysis is extended to include interaction of the beam with an external flow. In the absence of external flow, stability is lost through divergence or flutter, depending on the location of the point of measurement of curvature and the sign of the applied moment. In the presence of external flow, the mechanism of flutter instability changes from double-mode flutter to single-mode flutter due to the introduction of a damping-related term in the dynamic model. An extension of the fluid-structure interaction problem is considered by affixing the flexible hinged beam to a rigid body. This results in thrust applied on the rigid body due to the traveling waveform of the flexible-hinged beam at point of flutter instability. The flutter instability conditions are determined and illustrated through examples.

6.1 Hinged Beam with Dynamic Moment

6.1.1 Mathematical Model

Consider the hinged-beam of length L with uniform cross-sectional area A , shown in Fig.6.1. The transverse displacement of the beam at a distance x from the hinged end is denoted by $y(x, t)$. The beam is subjected to a dynamic moment M at the hinged joint, which has torsional stiffness k . The dynamic moment M is proportional to the beam curvature at a distance $\hat{x}$ from the hinged support, *i.e.*, $M = Cy''(\hat{x}, t)$. For small deformation, the non-dimensional equation of motion and boundary conditions are

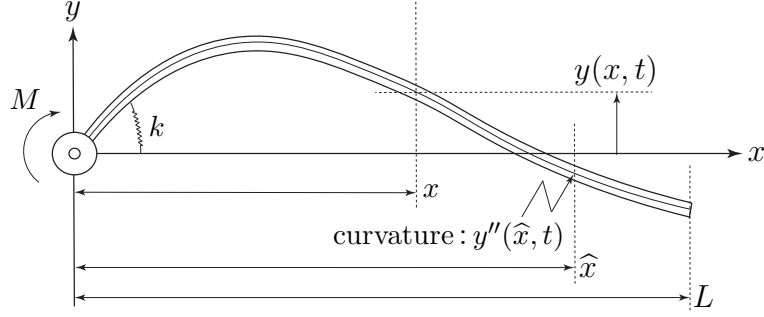


Figure 6.1: A flexible hinged-beam with dynamic moment

$$v''''(u, \tau) + \ddot{v}(u, \tau) = 0 \quad (6.1)$$

$$\begin{aligned} v(0, \tau) &= 0, \quad v''(0, \tau) = \overline{C}v''(\alpha, \tau) - \overline{k}v'(0, \tau) \\ v''(1, \tau) &= 0, \quad v'''(1, \tau) = 0 \end{aligned} \quad (6.2)$$

where $(.)'$ and $(\dot{.})$ denote the partial derivatives of $(.)$ with respect to u and τ , respectively, and the non-dimensional variables are defined as follows:

$$v = \frac{y}{L}, \quad u = \frac{x}{L}, \quad \tau = t \sqrt{\frac{EI}{\rho AL^4}}, \quad \overline{C} \triangleq \frac{C}{EI}, \quad \overline{k} \triangleq \frac{kL}{EI}, \quad \alpha = \frac{\hat{x}}{L}$$

and E , I and ρ denote the Young's modulus, area moment of inertia, and mass per unit volume of the beam. The stability of the system is analyzed using the Galerkin method [63], where the solution to Eq.(6.1) is assumed to be of the form

$$v(u, \tau) = \sum_{i=1}^N a_i(\tau) \phi_i(u) \quad (6.3)$$

where N is the number of terms in the approximation, $a_i(\tau)$ is the i -th modal amplitudes, and $\phi_i(u)$ is the i -th orthogonal eigenfunction of free vibration of the hinged-beam with the spring. The eigenfunctions have the form

$$\phi_i(u) = \frac{\bar{k}}{q_i} [\cos(q_i u) - \cosh(q_i u)] + \sin(q_i u) + \sinh(q_i u) - \gamma [\sin(q_i u) - \sinh(q_i u)] \quad (6.4)$$

$$\gamma \triangleq \frac{\bar{k} [\cosh(q_i) + \cosh(q_i)] + \beta [\sin(q_i) - \sinh(q_i)]}{\beta [\sin(q_i) + \sinh(q_i)]}$$

where the q_i 's are obtained by solving

$$\bar{k} + \cosh(q) [\bar{k} \cos(q) + \beta \sin(q) - \beta (\cos(q) \sinh(q))] = 0$$

Substituting Eq.(6.3) into Eq.(6.1) and integrating over the length of the beam along with satisfying Eq.(6.2), we get

$$\mathbb{M}\ddot{a} + \mathbb{K}a = 0 \quad (6.5)$$

where the (i, j) -th elements of the $\mathbb{M}$ and $\mathbb{K}$ matrices are

$$\mathbb{M}_{ij} \triangleq \int_0^1 \phi_i(u) \phi_j(u) du$$

$$\mathbb{K}_{ij} \triangleq \int_0^1 \phi_i''(u) \phi_j''(u) du + \bar{C} \phi_i'(0) \phi_j''(\alpha) - \bar{k} \phi_i'(0) \phi_j'(0) \quad (6.6)$$

The second term in the expression of $\mathbb{K}_{ij}$ denotes the geometric stiffness associated with the dynamic moment that is applied at the hinge. When $\bar{C} \neq 0$, the stiffness matrix $\mathbb{K}$ will be asymmetric. This implies that the system will be non-conservative [56], and can therefore lose stability through either flutter or divergence [4]. Since Eq.(6.5) represents a mass-spring system, flutter oscillations will not produce traveling waves.

6.1.2 Instability Investigation

The instabilities of the hinged-beam in Eq.(6.5) is investigated using a six-mode approximation ($N = 6$). The point where the curvature is measured, α , was varied along the length of the beam for both cases where $\bar{C}$ is positive and negative. The characteristic equation is

a function of $\overline{C}$, $\overline{k}$ and α ; here, $\overline{k}$ is assumed to be equal to 0.5. By specifying α we search for critical stability points by varying $\overline{C}$. Flutter instability occurs as two successive natural frequencies approach each other and coincide before they turn into a complex conjugate pair; on the other hand, divergence instability happens when the lowest natural frequency passes through the origin as the equilibrium shifts from stable to unstable. The loss of stability through divergence is always associated with the first mode but flutter instability occurs in different flutter instability modes. For ease of discussion we recall the definition of flutter instability mode from Chapter 3:

Flutter Instability Mode (FIM): The system loses stability through the n -th flutter instability mode if two successive natural frequencies of the system, numbered n and $(n+1)$, are equal at the lowest critical value of a system parameter.

For the case where $\overline{C} > 0$ and $\alpha \in [0, 1]$, the results are shown in Fig.6.2. The system loses stability through divergence for $\alpha \in [0, 0.4] \cup [0.45, 0.47]$; it loses stability through flutter for $\alpha \in (0.4, 0.45)$ and for $\alpha > 0.47$. In the region where stability is lost through flutter, the mode of instability changes from fourth to third to second, and then jumps back to fourth as α increases - see Fig.6.2.

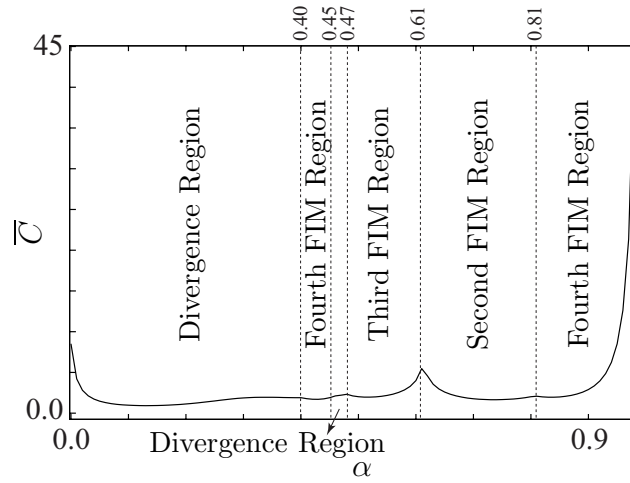


Figure 6.2: Critical stability curve for the hinged beam: $\overline{C}$ versus α for the case where $\overline{k} = 0.5$ and $\overline{C} > 0$.

For $\overline{C} < 0$, the critical stability of the system is investigated for $\alpha \in [0.16, 0.99]$; the results are shown in Fig.6.3. The system always loses its stability through flutter, independent of the value of α . Similar to the case of $\overline{C} > 0$ shown in Fig.6.2, the flutter instability mode switches sequentially downward from the fifth mode to the first mode as α increases from 0.16 to 0.74. For $\alpha > 0.74$, the mode of instability switches back to higher modes. The critical stability curve could not be obtained for $\alpha < 0.16$; this is likely due to the fact that a six-mode approximation was used and the system loses stability in modes greater than or equal to six.

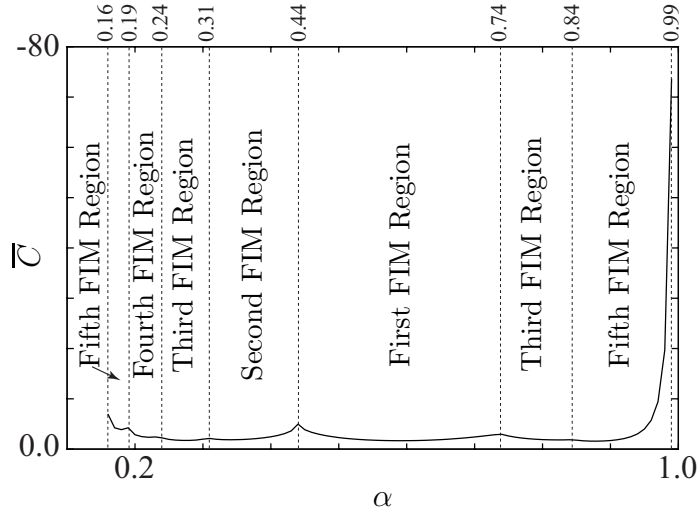


Figure 6.3: Critical stability curve for the hinged beam: $\overline{C}$ versus α for the case where $\overline{k} = 0.5$ and $\overline{C} < 0$.

6.2 Effect of External Flow on Flutter Instability

6.2.1 Mathematical Model

We now consider the case where the hinged-beam, introduced in Section 6.1, is immersed in an inviscid fluid with constant velocity U_e - see Fig.6.4. Using small deformation assumption, the non-dimensional equation of motion and boundary conditions are [58, 69]

$$v''''(u, \tau) + \overline{U}_e^2 v''(u, \tau) + 2\sqrt{\beta} \overline{U}_e \dot{v}'(u, \tau) + \ddot{v}(u, \tau) = 0 \quad (6.7)$$

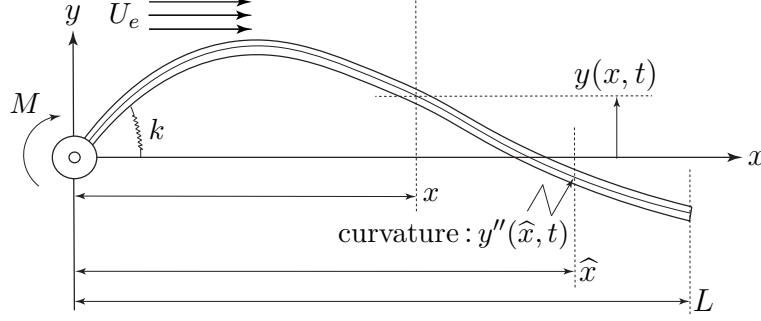


Figure 6.4: The hinged-beam in Fig.6.1 immersed in axial flow

$$\begin{aligned}
 v(0, \tau) &= 0, \quad v''(0, \tau) = \overline{C}v''(\alpha, \tau) - \overline{k}v'(0, \tau) \\
 v''(1, \tau) &= 0, \quad v'''(1, \tau) = 0 \\
 \overline{C} &\triangleq \frac{C}{EI}, \quad \overline{k} \triangleq \frac{KL}{EI}
 \end{aligned} \tag{6.8}$$

A comparison of Eqs.(6.1) and (6.7) that external flow introduces two new terms in the equation of motion. It will be shown later that these terms alter the stiffness and damping characteristics of the system. Equations (6.7) and (6.8) are non-dimensionalized using the following change of variables:

$$u = \frac{x}{L}, \quad v = \frac{y}{L}, \quad \alpha = \frac{\hat{x}}{L}, \quad \tau = \frac{t}{L^2} \left(\frac{EI}{m + M_f} \right)^{1/2} \tag{6.9}$$

We also define the following non-dimensional variables:

$$\overline{U}_e = \left(\frac{M_e}{EI} \right)^{1/2} U_e L, \quad \beta = \frac{M_e}{M_e + m} \tag{6.10}$$

where m , and M_e are the mass per unit length of the beam and mass per unit length of the external fluid that is obtained using the approximation presented in [58]. Following the procedure outlined in Section 6.1.1, we get

$$\mathbb{M}\ddot{a} + \mathbb{D}\dot{a} + \mathbb{K}a = 0 \tag{6.11}$$

where the (i, j) -th elements of the $\mathbb{M}$, $\mathbb{D}$ and $\mathbb{K}$ matrices are

$$\begin{aligned}
\mathbb{M}_{ij} &\triangleq \int_0^1 \phi_i(u) \phi_j(u) du \\
\mathbb{D}_{ij} &\triangleq 2\sqrt{\beta} \bar{U}_e \int_0^1 \phi_i(u) \phi_j'(u) du \\
\mathbb{K}_{ij} &\triangleq \int_0^1 \phi_i''(u) \phi_j''(u) du - \bar{U}_e^2 \int_0^1 \phi_i''(u) \phi_j''(u) du \\
&\quad + \bar{C} \phi_i'(0) \phi_j''(\alpha) - \bar{k} \phi_i'(0) \phi_j'(0) + \bar{U}_e^2 \phi_i(1) \phi_j'(1)
\end{aligned} \tag{6.12}$$

The term $\mathbb{D}_{ij}$ represents the damping due to the external flow. Similarly, the second and the fifth term in the expression of $\mathbb{K}_{ij}$ denote the stiffness effects due to the external flow. The third term in the expression of $\mathbb{K}_{ij}$ denotes the geometric stiffness due to the dynamic moment. Finally, the fourth term corresponds to the torsional stiffness at the rotational joint. The stiffness matrix $\mathbb{K}$ is asymmetric and consequently the system may lose stability through divergence or flutter. The main focus of this section is to highlight the effects of the external flow on the flutter instability modes.

6.2.2 Flutter Instability and Nature of Oscillations

We investigate the effects of external flow on flutter instability. We first consider the example from Section 6.1.2 where $\bar{k} = \alpha = 0.5$ and $\bar{C} < 0$. In this case, the eigenfrequencies ω are purely imaginary in the absence of the external flow and stability is lost through flutter. The root locus is plotted in Fig.6.5; the roots approach each other as $\bar{C}$ decreases from zero, then coexist at $\bar{C} = -2.3$, and finally become complex conjugate pairs. As a result, the structure loses stability through the first flutter instability mode - see Fig.6.3. In the presence of external flow, the eigenfrequencies ω depend on two additional parameters, namely, the flow velocity $\bar{U}_e$ and the mass ratio β . In general, for a specific set of the system parameters, $\bar{k}$, α , β , $\bar{U}_e$, and $\bar{C}$, the ω 's appear as complex conjugate pairs due to the presence of the damping. The locus of these eigenfrequencies in the complex plane is known as the Argand diagram.

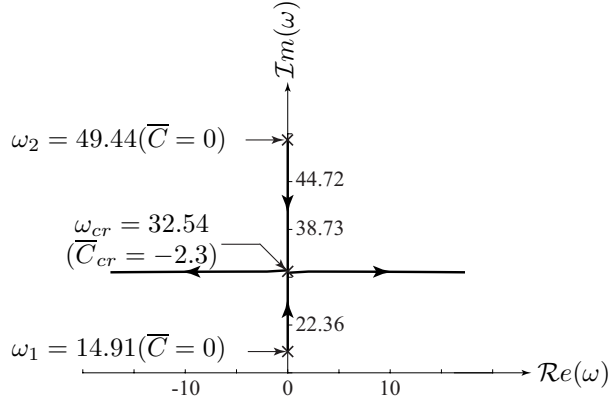


Figure 6.5: Root locus of the first two fundamental frequencies for the hinged beam without external flow: $\overline{C} < 0$ and $\overline{k} = \alpha = 0.5$.

The point on the locus where $\mathcal{Re}(\omega)$ changes sign from positive to negative constitutes the onset of flutter. Accordingly, the structure undergoes a sustained oscillation with a critical frequency $\omega = \omega_{cr}$, which is a purely imaginary. The Argand diagram for $\overline{k} = \alpha = 0.5$, $\beta = 0.9$, $\overline{U}_e = 0.7$, and $\overline{C} < 0$ is shown in Fig.6.6, where the locus of the first three natural frequencies are presented.

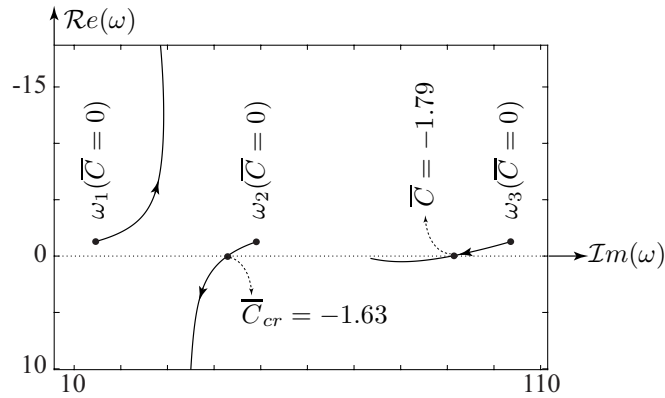


Figure 6.6: Argand diagram for $\overline{k} = \alpha = 0.5$, $\overline{U}_e = 0.7$, and $\beta = 0.9$; the critical $\overline{C}$ is denoted by $\overline{C}_{cr}$.

The results in Fig.6.6 indicate that the structure becomes unstable for $\overline{C} < -1.63$. For the critical value of $\overline{C} = \overline{C}_{cr} = -1.63$, the system undergoes sustained oscillations of the form:

$$v(u, \tau) = \phi_{cr}(u) \sin(\omega_{cr}\tau) \quad (6.13)$$

Substituting Eq.(6.13) into Eq.(6.7), we get

$$[\phi_{cr}''''(u) + U^2 \phi_{cr}''(u) - \omega_{cr}^2 \phi_{cr}(u)] \sin(\omega_{cr} \tau) + [2\sqrt{\beta} U \omega_{cr} \phi_{cr}'(u)] \cos(\omega_{cr} \tau) = 0$$

which can be expressed as

$$\gamma(u) \sin(\omega_{cr} \tau + \eta(u)) = 0 \quad (6.14)$$

$$\gamma(u) = \sqrt{a^2(u) + b^2(u)}, \quad \eta(u) = \tan^{-1} [b(u)/a(u)]$$

$$a(u) = \phi_{cr}''''(u) + U^2 \phi_{cr}''(u) - \omega_{cr}^2 \phi_{cr}(u)$$

$$b(u) = 2\sqrt{\beta} U \omega_{cr} \phi_{cr}'(u)$$

Equation (6.14) represents a traveling wave where the amplitude and wave speed both depend on u , *i.e.*, the location of the point on the beam from the hinged end. These results show that the nature of oscillations transition from standing waves in the absence of external flow to traveling waves in the presence of external flow. Also the mechanism of flutter instability is different. In the absence of external flow, two successive eigenfrequencies becoming identical at the point of instability; this is known as double mode flutter - see Fig.6.5. In contrast, in the presence of external flow, a single complex eigenfrequency crosses the imaginary axis at the point of instability; this is known as single mode flutter - see Fig.6.7.

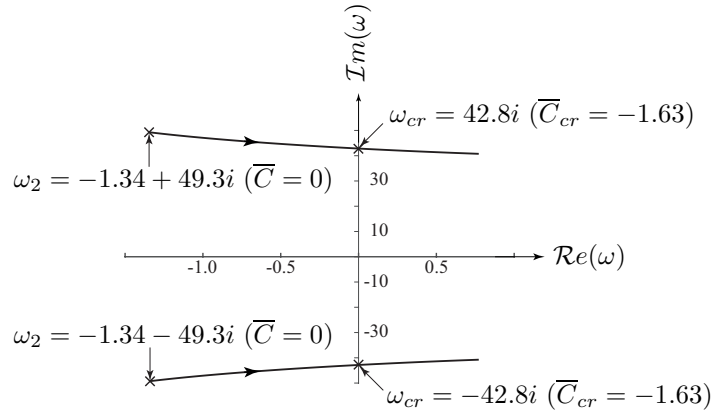


Figure 6.7: Root locus of the second mode complex conjugate frequency for the hinged beam subjected to external flow as $\bar{C}$ varies negatively for $\bar{k} = \alpha = 0.5$, $\bar{U}_e = 0.7$, and $\beta = 0.9$.

The fluid-structure interaction problem is extended by considering the fluid-immersed hinged beam to be affixed to a rigid body.

6.3 Fluid-Immersed Hinged Beam Affixed to a Rigid Body

6.3.1 Mathematical model

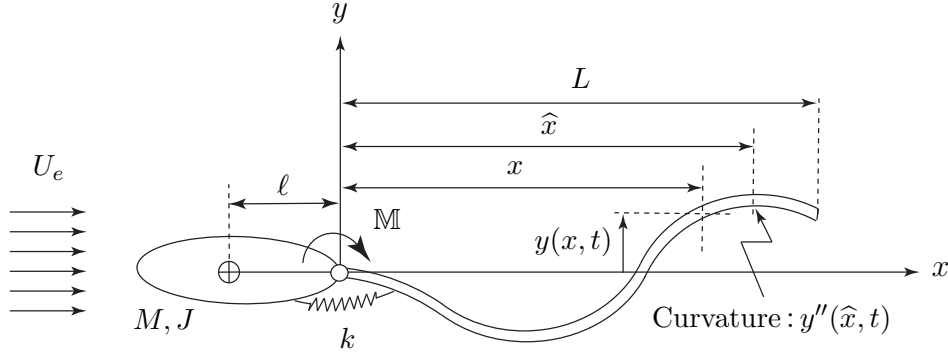


Figure 6.8: Fluid submersible tail mechanism affixed to a rigid body.

Consider the Euler-Bernoulli beam of length L and cross-sectional area A , shown in Fig.6.8, which is hinged to a rigid body of a mass and mass moment of inertia M and J , respectively. The beam is subjected to a dynamic moment $\mathbb{M}$ at the hinged joint, which has a torsional stiffness k , and is submerged in a fluid with a constant relative velocity U_e . The dynamic moment is proportional to the curvature of the beam at $x = \hat{x}$. The transverse displacement of a point on the beam at a distance x from the hinged joint is denoted by $y(x, t)$. For small deformation in absence of the gravity, viscosity, externally imposed tension and pressurization effect, the equation of motion of the beam can be written as follows [69]:

$$EI \frac{\partial^4 y(x, t)}{\partial x^4} + MU_e^2 \frac{\partial^2 y(x, t)}{\partial x^2} + 2MU_e \frac{\partial^2 y(x, t)}{\partial x \partial t} + (M + \rho A) \frac{\partial^2 y(x, t)}{\partial t^2} = 0 \quad (6.15)$$

where E , I , and ρ are the Young's modulus of elasticity, area moment of inertia, and the mass per unit volume of the beam. We introduce the following change of variables

$$u = \frac{x}{L}, \quad v = \frac{y}{L}, \quad \tau = t \sqrt{\frac{EI}{\rho AL^4}}, \quad \alpha = \frac{\hat{x}}{L}, \quad u_e = \sqrt{\frac{M}{EI}} U_e L, \quad \beta = \frac{M}{M + \rho A}$$

The non-dimensional equation of motion is

$$v''''(u, \tau) + u_e^2 v''(u, \tau) + 2u_e^2 \sqrt{\beta} \dot{v}'(u, \tau) + \ddot{v}(u, \tau) = 0 \quad (6.16)$$

where v' and $\dot{v}$ denote the partial derivatives of $v(u, \tau)$ with respect to u and τ , respectively.

We now make the following assumptions:

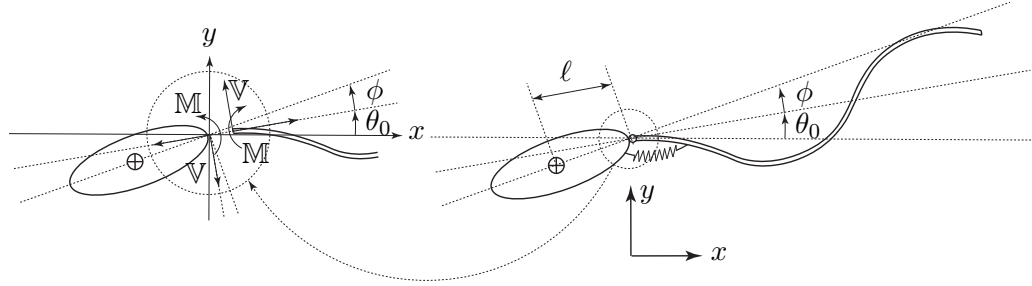


Figure 6.9: Free-body diagrams of the tail mechanism and the rigid body.

- The submersible mechanism is assumed to be traveling with constant forward velocity through the immersing medium along the negative x-axis.
- The rigid body is symmetric about the plane containing the neutral surface of the undeformed beam and has a small angle rotation denoted by ϕ - see Fig.6.9.

The above assumptions yield the following boundary conditions at $x = 0$:

$$EI \frac{\partial^2 y(0, t)}{\partial x^2} + M\ell \frac{\partial^2 y(0, t)}{\partial t^2} - \frac{J + M\ell^2}{J + M\ell^2 - \frac{k}{\omega_n^2}} \left(M\ell \frac{\partial^2 y(0, t)}{\partial t^2} + C \frac{\partial^2 y(\hat{x}, t)}{\partial x^2} + k \frac{\partial y(0, t)}{\partial x} \right) = 0$$

$$EI \frac{\partial^3 y(0, t)}{\partial x^3} + M \frac{\partial^2 y(0, t)}{\partial t^2} - \frac{M\ell}{J + M\ell^2 - \frac{k}{\omega_n^2}} \left(M\ell \frac{\partial^2 y(0, t)}{\partial t^2} + C \frac{\partial^2 y(\hat{x}, t)}{\partial x^2} + k \frac{\partial y(0, t)}{\partial x} \right) = 0$$

where ℓ is the distance from the hinged joint to the center of mass of the rigid body. These two conditions can be derived from the free-body diagram shown in Fig.6.9 as follows:

$$\begin{aligned}\mathbb{V} &= -M\left(\frac{\partial^2 y(0, t)}{\partial t^2} - \ell \frac{\partial^2 \lambda}{\partial t^2}\right) \\ \mathbb{M} &= J \frac{\partial^2 \lambda}{\partial t^2} + \mathbb{V} \ell\end{aligned}$$

where $\lambda = \phi + \theta_0$ and

$$\begin{aligned}\frac{\partial^2 \lambda}{\partial t^2} &= \frac{1}{J + M\ell^2 - \frac{k}{\omega_n^2}} \left(M\ell \frac{\partial^2 y(0, t)}{\partial t^2} + C \frac{\partial^2 y(\hat{x}, t)}{\partial x^2} + k \frac{\partial y(0, t)}{\partial x} \right) \\ \mathbb{M} &= C \frac{\partial^2 y(\hat{x}, t)}{\partial x^2} - k(\lambda - \theta_0)\end{aligned}$$

The boundary conditions at $x = L$ are

$$EI \frac{\partial^2 y(L, t)}{\partial x^2} = EI \frac{\partial^3 y(L, t)}{\partial x^3} = 0$$

Due to the second assumption, the force $\mathbb{F}$ does not affect the boundary conditions. To this end, the non-dimensional form of the boundary conditions can be expressed as

$$\begin{aligned}v''(0, \tau) &= \frac{\mu\psi^2(\gamma^2 + 1)}{\epsilon + \mu\psi^2 - \frac{\kappa}{\Omega_n^2}} \left(\mu\psi\ddot{v}(0, \tau) + \mathbb{C}v''(\alpha, \tau) + \kappa v'(0, \tau) \right) - \mu\psi\ddot{v}(0, \tau) \\ v'''(0, \tau) &= \frac{\mu\psi}{\epsilon + \mu\psi^2 - \frac{\kappa}{\Omega_n^2}} \left(\mu\psi\ddot{v}(0, \tau) + \mathbb{C}v''(\alpha, \tau) + \kappa v'(0, \tau) \right) - \mu\ddot{v}(0, \tau) \\ v''(1, \tau) &= v'''(1, \tau) = 0\end{aligned}\tag{6.17}$$

where Ω is the non-dimensional natural frequencies and

$$\mu = \frac{M}{(\rho A + M_e)L}, \quad \psi = \frac{\ell}{L}, \quad \mathbb{C} = \frac{C}{EI}, \quad \kappa = \frac{KL}{EI}, \quad \epsilon = \frac{J}{(\rho A + M_e)L^3}; \quad M_e = \frac{1}{4}\rho_f \pi S^2$$

In our simulations the external fluid is assumed to be water, *i.e.*, $\rho_f = 1000 \frac{Kg}{m^3}$ and S is the beam height which is assumed to be $0.06m$.

6.3.2 Method of Analysis

To solve Eq.(6.16) along with the boundary conditions in Eq.(6.17), we followed the procedure introduced in [69] and highlighted in [74] by assuming the following separable form of $v(u, \tau)$:

$$v(u, \tau) = f(u) g(\tau); \quad g(\tau) = e^{i\Omega\tau} \quad (6.18)$$

It is worth noting that Eq.(6.16) is not a self-adjoint for $u_e \neq 0$ and therefore the eigenvalues are in general complex values. Upon substitution of Eq.(6.18) into Eqs.(6.16)-(6.17) and dividing through by $e^{i\Omega\tau}$ we get

$$f''''(u) + u_e^2 f''(u) + 2u_e^2 \sqrt{\beta} i \Omega f'(u) - \Omega^2 f(u) = 0 \quad (6.19)$$

$$\begin{aligned} f''(0) &= \frac{\mu\psi^2(\gamma^2 + 1)}{\epsilon + \mu\psi^2 - \frac{\kappa}{\Omega_n^2}} \left(-\mu\psi\Omega^2 f(0) + \mathbb{C}f''(\alpha) + \kappa f'(0) \right) + \mu\psi\Omega^2 f(0) \\ f'''(0) &= \frac{\mu\psi}{\epsilon + \mu\psi^2 - \frac{\kappa}{\Omega_n^2}} \left(-\mu\psi\Omega^2 f(0) + \mathbb{C}f''(\alpha) + \kappa f'(0) \right) + \mu\Omega^2 f(0) \\ f''(1) &= f'''(1) = 0 \end{aligned} \quad (6.20)$$

Equation (6.19) is an ordinary differential equation with constant coefficients, as a result, the solution of $f(u)$ is assumed to be of the form $f(u) = Ae^{zu}$. Accordingly, Eq.(6.19) can

be written as

$$z^4 + u_e^2 z^2 + 2u_e^2 \sqrt{\beta} i \Omega z - \Omega^2 = 0 \quad (6.21)$$

For specific values of u_e and β , the characteristic polynomial, Eq.(6.21), provides four roots of z_n , where $n = 1, 2, 3, 4$. It is also worth noting that z_n is a function of $\alpha, \mu, \psi, \gamma, \epsilon, \mathbb{C}, \kappa$, and Ω in addition to u_e and β . The solution of $f(u)$ is thereupon takes the form

$$f(u) = A_1 e^{z_1 u} + A_2 e^{z_2 u} + A_3 e^{z_3 u} + A_4 e^{z_4 u} \quad (6.22)$$

which requires to satisfy the boundary conditions in Eq.(6.20) and yields to

$$\underbrace{\begin{bmatrix} \delta_1 & \delta_2 & \delta_3 & \delta_4 \\ \zeta_1 & \zeta_2 & \zeta_3 & \zeta_4 \\ z_1^2 e^{z_1} & z_2^2 e^{z_2} & z_3^2 e^{z_3} & z_4^2 e^{z_4} \\ z_1^3 e^{z_1} & z_2^3 e^{z_2} & z_3^3 e^{z_3} & z_4^3 e^{z_4} \end{bmatrix}}_Z \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (6.23)$$

where $\delta_n, \zeta_n, n = 1, 2, 3, 4$, are defined by

$$\delta_n = \frac{\mu \psi \Omega^4 (\epsilon - \gamma^2 \mu \psi^2) + \Omega^2 z_n^2 (\mu \psi^2 (\mathbb{C} (\gamma^2 + 1) e^{\alpha z_n} - 1) - \epsilon) + k (z_n^2 + \mu \psi \Omega^2 ((\gamma^2 + 1) \psi z_n - 1))}{k - \Omega^2 (\epsilon + \mu \psi^2)}$$

$$\zeta_n = \frac{\Omega^2 (\mathbb{C} \mu \psi z_n^2 e^{\alpha z_n} + \epsilon \mu \Omega^2 - z_n^3 (\epsilon + \mu \psi^2)) + k (z_n^3 + \mu \Omega^2 (\psi z_n - 1))}{k - \Omega^2 (\epsilon + \mu \psi^2)}$$

A non-trivial solution of Eq.(6.23), $\det(Z) = 0$, results in a transcendental characteristic equation that has infinite roots in Ω . For different values of $u_e, \alpha, \beta, \mathbb{C}, \mu, \psi, \gamma, \epsilon, \kappa$ the transcendental equation can be solved numerically to get the complex frequencies. Once Ω and $z_n, n = 1, 2, 3, 4$, have been determined, the complete solution of Eq.(6.18) can then be obtained by substituting Eq.(6.22) into Eq.(6.18) as

$$v(u, \tau) = \sum_{n=1}^4 A_n e^{z_n u} e^{i\Omega \tau} = \sum_{n=1}^4 A_n e^{Re[z_n]u} e^{i(Im[z_n]u + Re[\Omega]\tau)} e^{-Im[\Omega]\tau} \quad (6.24)$$

where the coefficients A_n , $n = 1, 2, 3, 4$, can be obtained from the nullspace of the matrix Z in Eq.(6.23). Equation (6.24) indicates that $v(u, \tau)$ is a product of three exponential terms where the first term is bounded since u is bounded. The second term is also bounded due to the imaginary exponent that results in a periodic motion. The last term, however, can grow unbounded with time if $Im[\Omega] < 0$. In this case, the point at which the sign of the imaginary part of the complex frequency changes from positive to negative represents the onset point of flutter instability. The mode at which the beam-like tail, structure, undergoes flutter instability depends on the system parameters, namely, $u_e, \mathbb{C}, \alpha, \beta, \mu, \psi, \gamma, \epsilon$, and κ . In this work, however, we are going to only focus on the effect of first three parameters, that is to say $(u_e, \mathbb{C}, \alpha)$, on the flutter stability; the rest of the parameters are listed in Table 6.1.

Table 6.1: Assumed values for the non-dimensional parameters used in the analysis.

Parameters	Assumed Values
β	0.97
μ	0.27
ψ	0.44
γ	0.36
ϵ	0.01
κ	2.00

The force exerted by the hinged beam-like tail on the surrounding fluid can be determined using the system parameters and Eq.(6.24) evaluated at critical stability point, that is, $(Im[\Omega] = 0)$. At the critical stability point, Eq.(6.24) can be expressed as

$$v(u, \tau) = \sum_{n=1}^4 e^{Re[z_n]u} \left(Re[A_n] \cos(Im[z_n]u + Re[\Omega]\tau) - Im[A_n] \sin(Im[z_n]u + Re[\Omega]\tau) \right)$$

It is important to realize that the sinusoidal terms in the above equation represent a traveling waveform, where the slender beam-like tail passes the wave down along its span with an amplitude increasing over its length. This waveform can produce a positive thrust, according to Lighthill [75], if the phase velocity is greater than the speed of the structure relative to external fluid. In this case

$$\frac{Re[\Omega]}{Im[z_n]} > \frac{u_e}{\sqrt{\beta}} \quad (6.25)$$

Equation (6.25) constitutes the condition under which the waveform generates a positive thrust that produces a forward speed u_e . Following [74–76] derivation, the time-average thrust $\bar{\tau}$ of the harmonic motion is

$$\bar{\tau} = \frac{1}{2}M \left[\left\{ \overline{\left(\frac{\partial y}{\partial t}\right)^2} - U_e^2 \overline{\left(\frac{\partial^2 y}{\partial x^2}\right)^2} \right\}_{x=L} - \left\{ \overline{\left(\frac{\partial y}{\partial t}\right)^2} - U_e^2 \overline{\left(\frac{\partial^2 y}{\partial x^2}\right)^2} \right\}_{x=0} \right] \quad (6.26)$$

and the average power required to provide the displacement $y(x, t)$ is

$$\bar{P} = U_e M \left[\left\{ \overline{\frac{\partial y}{\partial t} \left(\frac{\partial y}{\partial t} + U_e \frac{\partial y}{\partial x} \right)} \right\}_{x=L} - \left\{ \overline{\frac{\partial y}{\partial t} \left(\frac{\partial y}{\partial t} + U_e \frac{\partial y}{\partial x} \right)} \right\}_{x=0} \right] \quad (6.27)$$

It is worth mentioning that Eq.(6.26) suggest a substantial thrust can be obtained if the magnitude of $\frac{\partial y}{\partial t}$ is large. At the same time, a larger magnitude of $\frac{\partial y}{\partial t}$ requires large energy input according to Eq.(6.27). The efficiency introduced in [75] is given by

$$\eta = \frac{\bar{\tau} U_e}{\bar{P}} \quad (6.28)$$

Equations (6.26-6.28) can be non-dimensionalized using the change of variables introduced in Section 6.3.1 as follows:

$$\bar{\tau}^* = \frac{\bar{\tau} L^2}{EI}, \quad \bar{P}^* = \frac{\bar{P} M^{\frac{1}{2}} L^3}{(EI)^{\frac{3}{2}}}, \quad \eta = \frac{\bar{\tau}^* u_e}{\bar{P}^*} \quad (6.29)$$

where the time-average of the non-dimensional thrust $\bar{\tau}^*$ and power $\bar{P}^*$, that are evaluated over one cycle of the harmonic motion, should be sufficient [74]

$$\bar{\tau}^* = \frac{\Omega_{cr}}{4\pi} \int_0^{\frac{2\pi}{\Omega_{cr}}} \left\{ [\beta \dot{v}(u, \tau)^2 - u_e^2 v'(u, \tau)^2]_{u=1} - [\beta \dot{v}(u, \tau)^2 - u_e^2 v'(u, \tau)^2]_{u=0} \right\} d\tau \quad (6.30)$$

$$\begin{aligned} \bar{P}^* = \frac{\Omega_{cr}}{2\pi} \int_0^{\frac{2\pi}{\Omega_{cr}}} \left\{ \left[u_e \beta \dot{v}(u, \tau)^2 - u_e^2 \sqrt{\beta} v'(u, \tau) \dot{v}(u, \tau) \right]_{u=1} \right. \\ \left. - \left[u_e \beta \dot{v}(u, \tau)^2 - u_e^2 \sqrt{\beta} v'(u, \tau) \dot{v}(u, \tau) \right]_{u=0} \right\} d\tau \end{aligned} \quad (6.31)$$

The results in the next section determine the critical stability points, *i.e.*, the value of $\mathbb{C}_{cr}$ at a given u_e that produces flutter instability.

6.3.3 Flutter Instability

We determine the critical stability points, *i.e.*, $Im[\Omega] = 0$ in the $\mathbb{C} - u_e$ space for a range of $\alpha \in [0.1 - 0.9]$. On this account, we start with $\mathbb{C} = 0$ and $u_e = 0.01$ for a specific value of α to acquire the first ten natural frequencies Ω of the beam. These frequencies are used as a first guess to find critical stability for the next higher value of u_e ; the magnitude of $\mathbb{C}$ is gradually increased till one of the Ω 's satisfies the condition $Im[\Omega] = 0$. Then, we trace in the critical stability points to obtain the critical stability curve by gradually varying the value of u_e and compute the corresponding $\mathbb{C}_{cr}$ for the particular value of α . The critical stability curve in the $\mathbb{C} - u_e$ space, is depicted using the automated method proposed by Hellum [74]. Since the proportionality constant $\mathbb{C}$ can be positive or negative, we study these two cases separately.

Positive Proportionality Constant

Consider the case where $\mathbb{C} > 0$, *i.e.*, the dynamic moment $\mathbb{M}$ at the hinge is proportional to the curvature and the constant of proportionality is positive. The critical stability curve is

obtained for each α by calculating $\mathbb{C}_{cr}$ for a range of $u_e \in (0, 20)$ with a step size of 0.01. The objective of this section is to study the effect of the point of measurement of the curvature, α , on the stability. Flutter instability is discussed for four ranges of α in which they are categorized based on the established pattern of behavior. For instance, the critical stability curves exhibit an arch-like shape for $\alpha \in (0.1, 0.38]$. As α increases from 0.12 to 0.20, the curves tend to grow in size and outstretch rightward - see Fig.6.10(a).

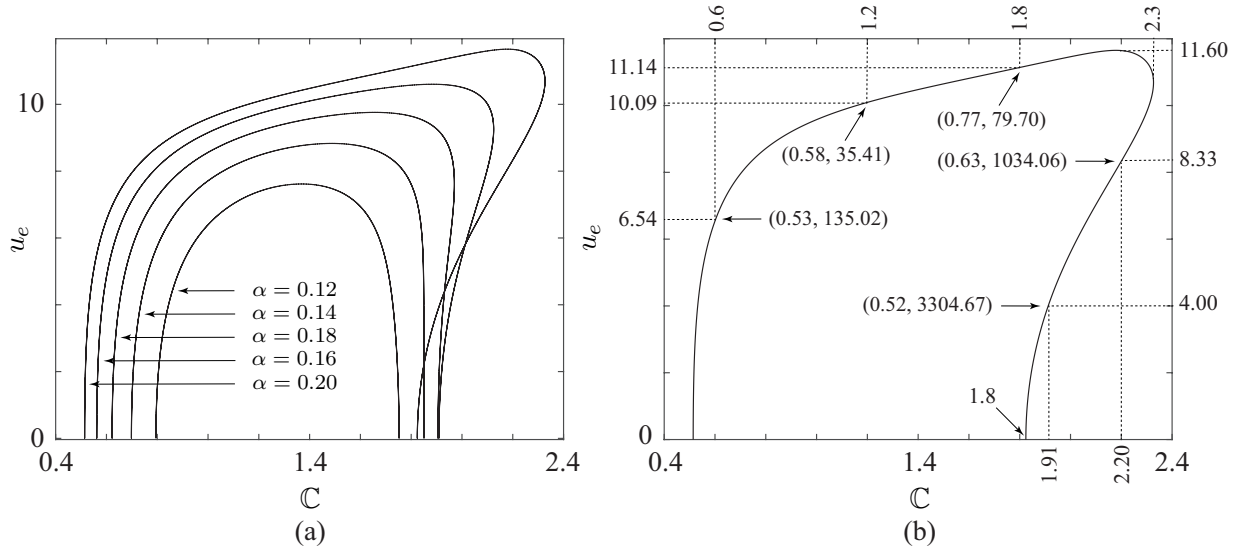


Figure 6.10: Critical stability curves in $\mathbb{C} - u_e$ space: (a) a range of $\alpha \in (0.10 - 0.20]$ (b) for $\alpha = 0.20$ the efficiency, η , and the critical frequency, Ω_{cr} , are denoted as (η, Ω_{cr}) for five different points along the critical stability curve.

The critical stability curve for $\alpha = 0.20$ is detailed in Fig.6.10(b) to include Froude efficiency η and the critical frequency Ω_{cr} . From the figure, we infer that the required value of $\mathbb{C}$ to maintain flutter instability increases as the external fluid velocity is increased for the region of $u_e \in (0.0, 11.6]$. In the same range, the efficiency improves - see the first three chosen points in Fig.6.10(b). Beyond this region, the required value of $\mathbb{C}$ increases to 2.3, with a reduction in the associated u_e . Past this point, $\mathbb{C}$ decreases as the external fluid velocity u_e diminishes and $\mathbb{C} = 1.8$ when u_e reaches zero. During this portion of the curve, the efficiency drops and is coupled with a substantial rise in the critical frequency.

For $\alpha \in [0.22 - 0.38]$ a trend similar to that of $\alpha \in (0.10 - 0.20]$ is observed - see Fig.6.11(a). For $\alpha = 0.22$, the critical stability curve is presented in detail in Fig.6.11(b), mainly to compare with the case $\alpha = 0.20$ which was presented in Fig.6.10(b). These figures indicate that a slight change in the location of α from 0.2 to 0.22 results in a significant change in the flutter instability characteristics. It is observed that the critical stability curve contracts to a dome-like shape with a maximum external fluid velocity $u_e = 3.3$. The value $\mathbb{C}_{cr}$ is reduced, the maximum Froude efficiency is lower, and finally, the frequency flutter instability drops drastically.

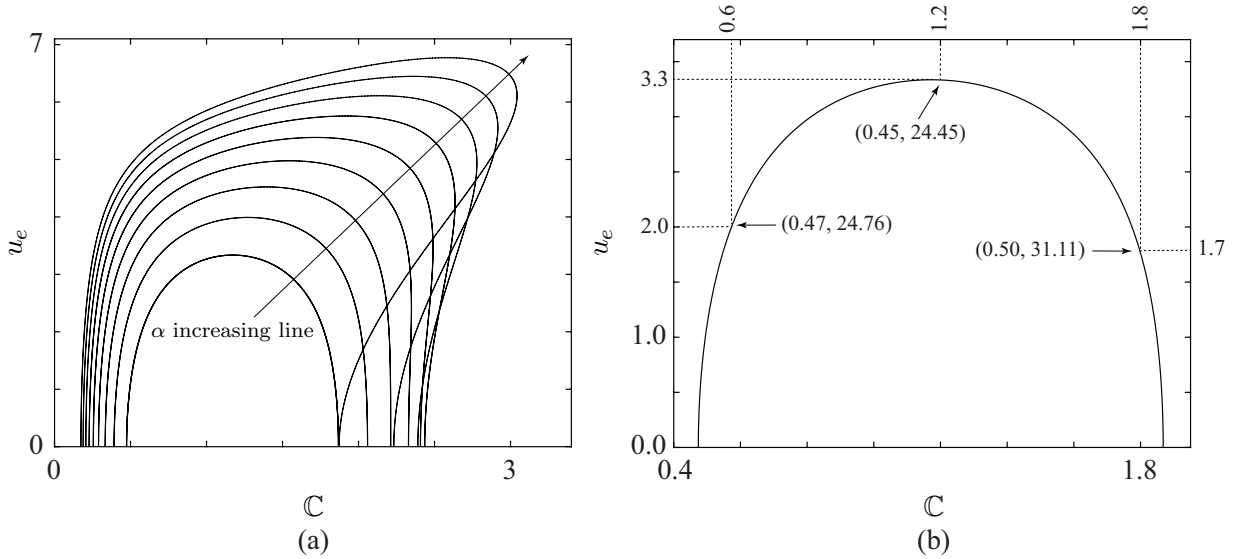


Figure 6.11: Critical stability curves in $\mathbb{C} - u_e$ space: (a) a range of $\alpha \in [0.22 - 0.38]$ with an increment of 0.02 (b) for $\alpha = 0.22$ the efficiency, η , and the critical frequency, Ω_{cr} , are denoted as (η, Ω_{cr}) for three different points along the critical stability curve.

For $\alpha \in [0.40, 0.70]$, the critical stability curves exhibit two types of behavior, as shown in Fig.6.12(a). The instability curves have a positive slope as u_e approaches 8. Thereafter, for $\alpha \in [0.40, 0.46]$ the critical stability curves rotate clockwise and eventually reach a zero slope. In contrast, the critical stability curves for $\alpha \geq 0.66$ rotate counterclockwise at a higher value of u_e . For an intermediate value of $\alpha = 0.58$, the result is shown in Fig.6.12(b). The Froude efficiency and the associate flutter frequency are delineated at three different points along the critical stability curve. It is interesting to note that the efficiency increases as both u_e

and $\mathbb{C}$ increase, while the flutter frequency fluctuates. At $u_e = 11.68$ and $\mathbb{C}_{cr} = 7.00$, the efficiency impressively reaches 0.98 at a relatively lower flutter frequency.

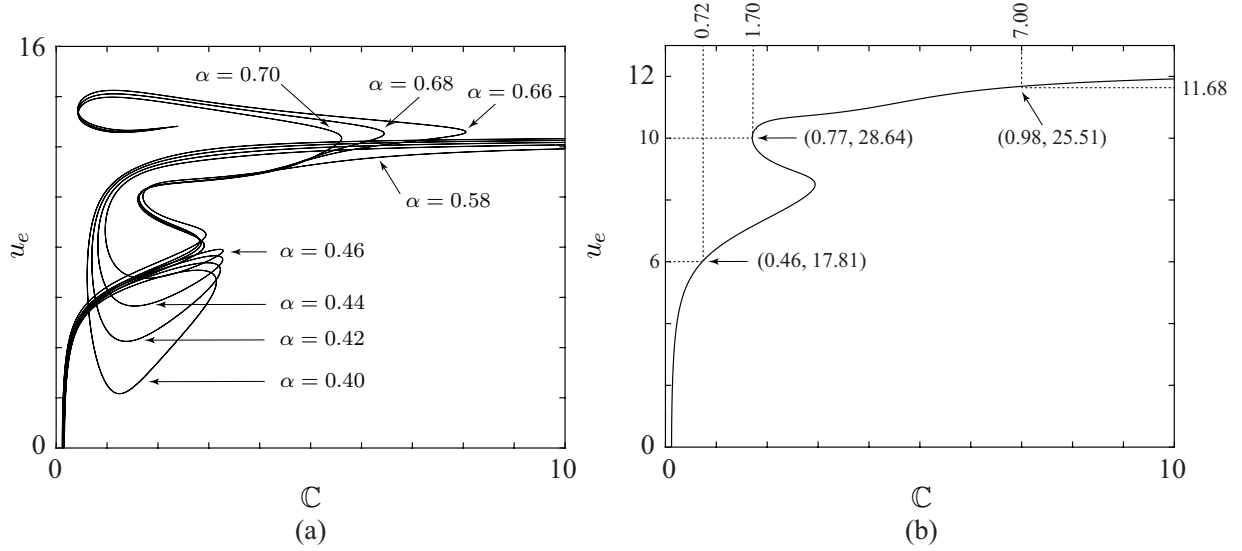


Figure 6.12: Critical stability curves in $\mathbb{C} - u_e$ space: (a) critical stability curves are depicted for eight different values of $\alpha \in [0.40, 0.70]$ (b) for $\alpha = 0.58$ the efficiency, η , and the critical frequency, Ω_{cr} , are denoted as (η, Ω_{cr}) for three different points along the critical stability curve.

The final examined region is $\alpha \in [0.80, 1.00)$, where the critical stability curve exhibits multiple stability transitions¹ - see shown in the Fig.6.13(a). Notably, for $\alpha = 0.90$ and $\mathbb{C} = 1.18$ the beam exhibits up to five stability transitions at different values of u_e - see Fig.6.13(b). These transitions are analogous to the stability transition induced by damping discussed in Chapter 3. For $\mathbb{C} = 1.18$, the beam is unstable for $u_e = 0$. The beam gets stabilized at $u_e = 6.76$, then destabilized at $u_e = 7.55$, stabilized again at $u_e = 13.1$, destabilized again at $u_e = 15.6$, and finally restabilized at $u_e = 18.5$. It should be noted that the generated thrust is mostly positive except in some regions where the curvature is measured closed to the free end of the beam. The negative thrust is illustrated using a darker region on the critical stability curve - see Figs.(6.10 - 6.13)(b).

¹The area toward the left side of the critical stability curve represents the stable region where the beam does not maintain flutter. On the contrary, the area rightward the curve represents flutter instability region.

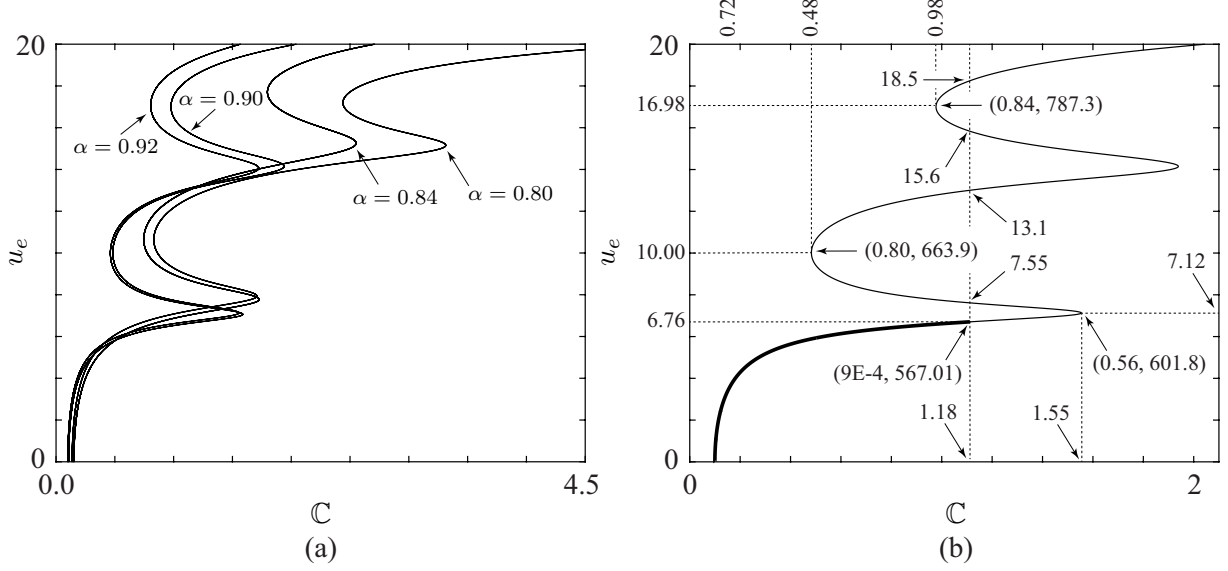


Figure 6.13: Critical stability curves in $\mathbb{C} - u_e$ space: (a) critical stability curves are depicted for four different values of α , namely, $\alpha = 0.80, 0.84, 0.90, 0.92$ (b) for $\alpha = 0.90$ the efficiency, η , and the critical frequency, Ω_{cr} , are denoted as (η, Ω_{cr}) for three different points along the critical stability curve; The darkness region of the critical stability curve depicts the region of negative thrust.

Negative Proportionality Constant

similar to the case where the constant of proportionality is positive, we investigate the case of a negative proportionality constant for three regions of α , described below:

$$\text{region 1: } \alpha \in (0.1, 0.36]$$

$$\text{region 2: } \alpha \in [0.5, 0.58]$$

$$\text{region 3: } \alpha \in [0.66, 0.98]$$

In region 1 the critical stability curve resemble the first two cases when $\mathbb{C} > 0$; however, instead of growing in size, the curves shrink and change shape. For $\alpha = 0.40$, the Froude efficiency and the associated flutter frequency was computed at three different locations along the instability curve. These points are shown in Fig.6.14(b) in which the maximum efficiency η is less than 0.6.

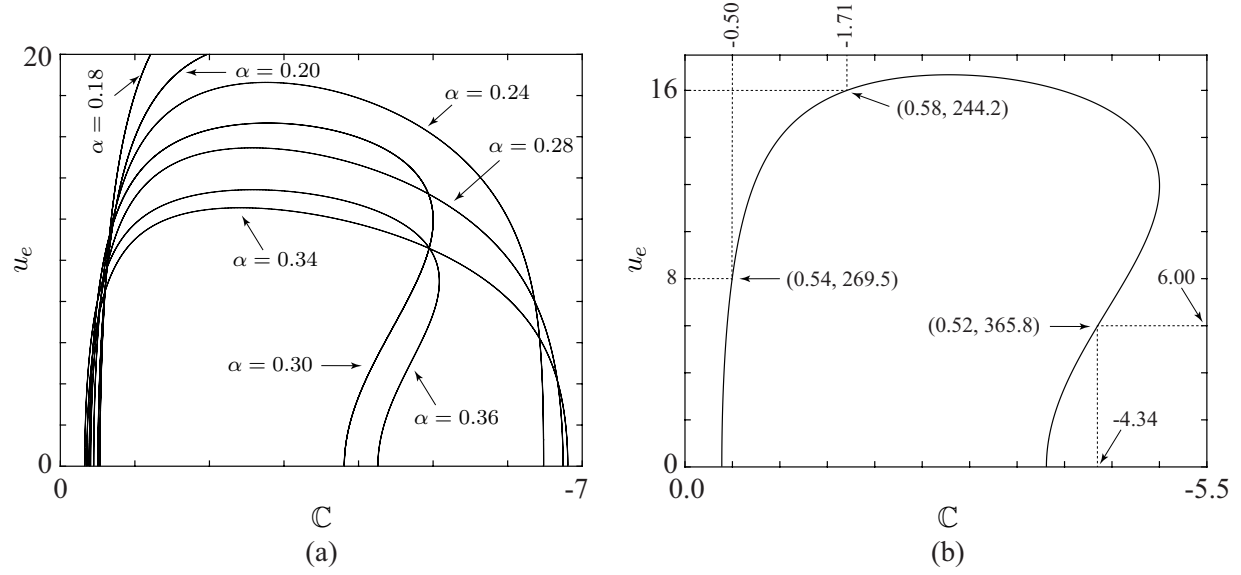


Figure 6.14: Region 1: critical stability curves in $\mathbb{C} - u_e$ space: (a) critical stability curves are depicted for seven different values of $\alpha \in [0.18, 0.36]$ (b) for $\alpha = 0.30$ the efficiency, η , and the critical frequency, Ω_{cr} , are denoted as (η, Ω_{cr}) for three different points along the critical stability curve.

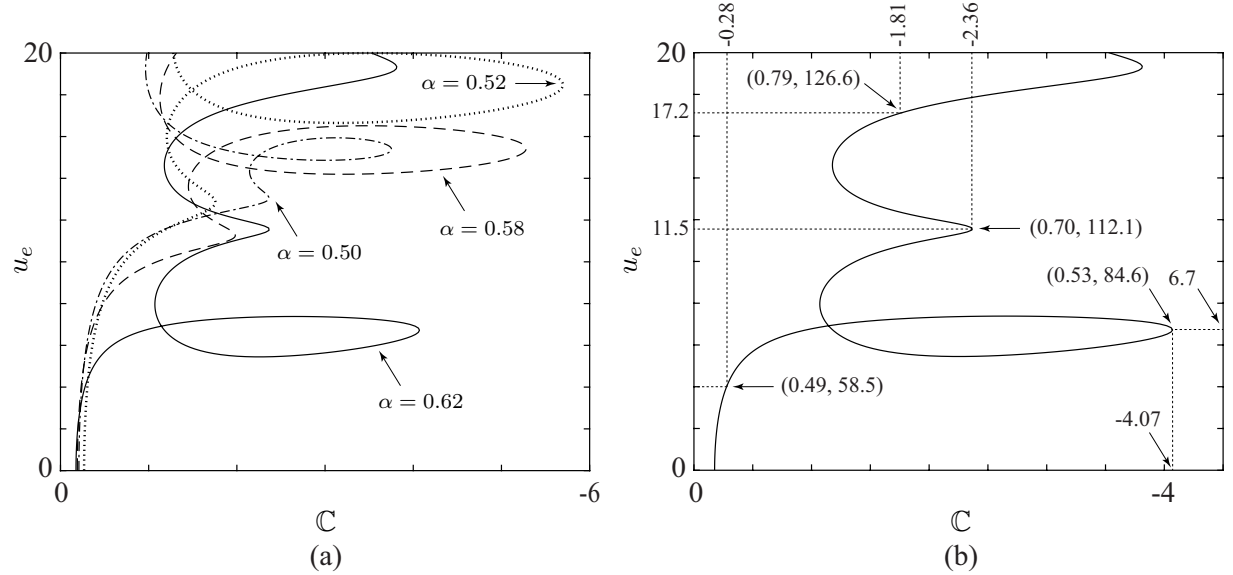


Figure 6.15: Region 2: critical stability curves in $\mathbb{C} - u_e$ space: (a) critical stability curves are depicted for three different values of α , namely, $\alpha = 0.50, 0.52, 0.58$ (b) for $\alpha = 0.62$ the efficiency, η , and the critical frequency, Ω_{cr} , are denoted as (η, Ω_{cr}) for three different points along the critical stability curve.

In region 2 the critical stability curves are similar to both the third and fourth cases with $\mathbb{C} > 0$. It can be seen that the system exhibits a rich set of flutter instability transitions

- see Fig.6.15(a). For $\alpha = 0.62$ the efficiency and the critical frequency is depicted at four locations along the flutter instability curve - see Fig.6.15(b). The figure reveals that an external velocity of 17.2 and $\mathbb{C}_{cr} = -1.81$ results in a positive thrust and an efficiency of 0.8. This indicates that positive thrust can be obtained using both positive and negative proportionality constant.

In region 3, the instability curves exhibit a total of three stability transitions - see Fig.6.16(a). In Fig.6.16(b), we show the special case of $\alpha = 0.98$. The thrust is negative at some locations on the critical stability curve for $\alpha = 0.98$, this is highlighted by the darker region. Although a positive thrust $\bar{\tau}^* > 0$ is needed to produce an external forward velocity and overcome external forces such as the drag, negative thrust can be useful for backward motion. At the two ends of the darker curve, $\bar{\tau}^* = 0$, operating at these points can be useful for hovering motion and quickly changing the direction of motion.

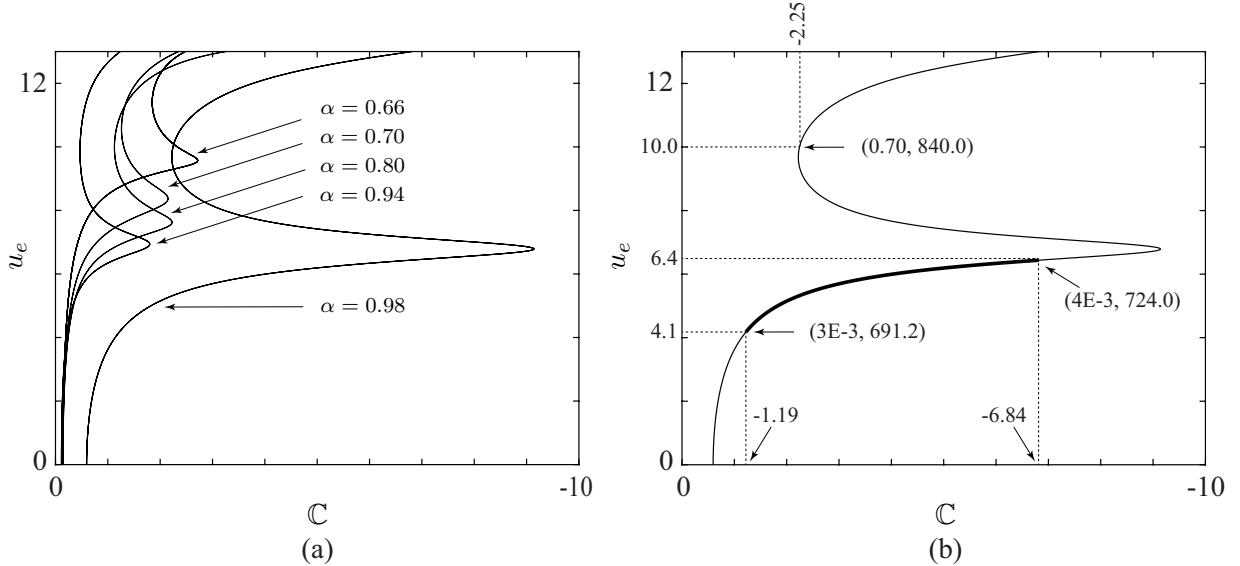


Figure 6.16: Region 3: critical stability curves in $\mathbb{C} - u_e$ space: (a) critical stability curves are depicted for five different values of $\alpha \in [0.66, 0.98]$ (b) for $\alpha = 0.98$ the efficiency, η , and the critical frequency, Ω_{cr} , are denoted as (η, Ω_{cr}) for three different points along the critical stability curve.

Chapter 7

Conclusion

7.1 Divergence and Flutter Modes of Instability

It is well-known that a non-conservative system loses stability through either divergence or flutter as the boundary load exceeds some critical value. At the same time, a conservative system loses its stability only through divergence. With this in mind, the two types of structural instabilities were illustrated using two examples of discrete and continuous systems in Chapter 2. Furthermore, a classification of flutter instability based on the existence of damping was highlighted.

7.2 Stability Transitions Induced by Damping

Chapter 3 assessed the role of damping on the dynamic stability of non-conservative systems. It was found that the initially unstable system, at some fixed value of the non-conservative force, gets stabilized, then destabilized, then restabilized by increasing the amount of damping. This counter-intuitive role of damping - which we refer to as the S-D-S behavior - was observed in every classical non-conservative system considered in this chapter. The non-conservative systems exhibit different types of stability transition behaviors in different regions of their damping parameter space. To the best of our knowledge, this instability behavior has not been described in the literature. This work also highlighted the likelihood for the behavior to exist for most non-conservative systems. The work also showed the possibility for two instability curves to cross, a phenomenon which has multiple implications for the theoretical consideration of stabilization. To accomplish this, it becomes necessary to delineate the stable region of the parameter space using a piecewise continuous absolute instability curve. More critically, the term “stabilizing” must be defined in terms of the

behavior at a fixed value of the non-conservative forcing variable rather than by the local slope of the instability curve as proposed in the literature.

The key to finding the multiple stability transitions, and the S-D-S behavior in particular, was the application of the Routh-Hurwitz criterion twice, back-to-back: the first time for the characteristic polynomial of the system and the second time for the polynomial that guarantees the existence of a second-order auxiliary polynomial in the Routh array. For a given system, the number of stability transitions can be directly determined for a set of system parameters. This method works by determining the number of physically realizable damping coefficients, regardless of their magnitude, at which the system is marginally stable. It is, therefore, possible to demonstrate the existence of the S-D-S behavior in a system that had been investigated earlier in the literature. However, the final restabilization occurs at a damping level higher than the range examined in that work. The overlooked stability transition does not always occur at very high levels of damping. This analysis has been undertaken for each of the S-D-S transitions in this work, and similar observations have been made.

The two degree-of-freedom linkage system subjected to a follower end force was also investigated at greater length by varying the damping level in one of the joints while keeping the damping in the other joint fixed. Through the analysis, it was shown that there could be only one stability transition for some values of the fixed level of damping; for other values of the fixed level of damping, there can be three stability transitions. The study was extended to obtain the critical stability curves in the space of the damping parameters for different values of the non-conservative force. These curves have similar shapes that indicate that an initially unstable system can be made stable by increasing the damping in the first joint, or alternately, by increasing or decreasing the damping in the second joint. It is interesting to note that the maximum number of stability transitions ever found in the investigated systems was three, corresponding to the S-D-S behavior. This work outlined the procedure to determine the number of stability transitions, which is only valid for a particular set of

system parameters. Consequently, the existence of additional transitions cannot be ruled out.

7.3 Non-conservative Effects of Dynamic Terminal Moment

The stability characteristics of a cantilever beam, subjected to a dynamic moment at its free end, was investigated in Chapter 4. The applied moment is assumed to be proportional to the slope or the curvature of the beam, measured from some point along its length. The geometric stiffness associated with the terminal loading imparts a non-symmetric structure to the stiffness matrix when the measurement is taken from any point other than the free end. This suggests that the terminal loading acts as a non-conservative loading for non-terminal measurement. Regardless of whether this non-conservative moment is proportional to the slope or the curvature of the beam, the stability is always lost through divergence when the proportionality constant is positive. On the contrary, for a negative proportionality constant, the system loses stability solely through flutter.

At the divergence point, the critical value of the constant of proportionality drops as the point of measurement shifts from the fixed end to the free end when the terminal moment corresponded to a positive slope. However, for loading proportional to the positive curvature, the location of the point of measurement has little to no effect on the critical values. Flutter instability, on the other hand, is induced at higher modes as the point of measurement shifts from the fixed end to the free end of the beam. For instance, when the terminal moment is proportional to the negative slope, the beam loses stability in the first or second flutter instability mode depending on the location of the point of measurement. Comparably, when the terminal moment is proportional to the negative curvature of the beam: as the point of measurement moves towards the free end, the mode of instability changes sequentially from the first to the fourth flutter instability mode. An interesting observation in both flutter instability cases is that a continuous change in the location of the point of measurement can result in multiple stability transitions and induce different modes of instability. This

behavior is analogous to the multiple instability transitions introduced in Chapter 3.

Experimental validation was a setup where an electric motor was used to provide a terminal moment to a steel cantilever beam. The motor was supported by air-bearings to avoid torsional dynamics. Due to the simplicity of measuring the curvature utilizing a strain gage, the terminal moment was considered to be proportional to the curvature. The critical value of the proportionality constant, frequency, and mode shape at the critical point matched the values predicted using a linear model.

7.4 Stability Transitions Induced by an Intermediate Support

Chapter 5 investigated the critical stability of the non-conservative cantilevered beam presented in Chapter 4, in the presence of intermediate support. The analyses were carried out using both numerical (Galerkin approximation) and analytical methods. In order to examine the effects of the intermediate support, both the locations of the intermediate support and the point of the slope or curvature measurement were varied along the cantilevered beam.

For the case when the terminal loading is proportional to the slope, the critical stability is discussed in three separate locations of the intermediate support using the numerical method. Irrespective of whether the terminal moment is proportional to the positive slope or the negative slope and independent of the location of support, both divergence and flutter modes of instability are exhibited. The regions of the point of measurement of the slope that result in divergence and flutter depend strongly on the location of the intermediate support. For instance, when the intermediate support is close to the fixed end (conversely, free end), divergence occurs when the point of measurement of the slope lies to the right of the support (conversely, left of the support). However, when the intermediate support is in the middle of the beam, divergence occurs when the point of measurement of the slope lies near to the support, on either side. Also, unlike the case without intermediate support presented in Chapter 4, where the flutter instability modes switch sequentially, the flutter instability modes change randomly as the point of measurement shifts from one end to the

other end of the beam.

When the terminal loading is proportional to the positive curvature, five cases are discussed by keeping one parameter fixed and varying the other parameters. The results reveal a rich set of instability transitions between flutter and divergence and flutter-to-flutter when the intermediate support location is shifted from the fixed end to the free end of the beam. The first case involved one stability transition from divergence to flutter that accompanied a jump in the critical proportionally constant values. The second case involves multiple instability transitions between divergence and flutter; however, only the first one is accompanied by a jump in the critical loading. Additionally, transitions between the different flutter instability modes were random, and therefore no veering was observed. The third case involved multiple instability transitions, just like the first two cases, but none of the transitions exhibited jumps in the critical loading. The fourth case illustrated a transition between two random instability modes with no veering but a jump in the critical load. Finally, the last case included two jumps in the critical loading whereby the first one is associated with a transition between divergence and flutter, and the second one involved a transition between two consecutive flutter modes. The flutter to flutter transition results from veering.

For the case where the terminal moment is proportional to the negative curvature, transitions between divergence and flutter occur precisely along the diagonal of the parameter space and are accompanied by the jump phenomenon. Transitions between modes of flutter occur randomly in the demarcated flutter region. Due to the clear demarcation of the regions of divergence and flutter, this case bears a close resemblance to that of the cantilever beam with follower force. Although this problem has been well-studied, it was revisited here in this chapter to highlight the jump phenomenon associated with the instability transitions. This jump phenomenon, which results from veering, was overlooked in the literature.

7.5 Non-conservative Behavior of Hinged Beam with a Dynamic Moment

Chapter 6 presented a framework for investigating the critical stability of a hinged beam with a non-conservative moment both in the presence and absence of external flow. The moment, in this case, is assumed to be proportional to the curvature of the beam, measured at some point along its length. The stability of the hinged beam was first investigated using the Galerkin approximation. The instability analysis was carried out both in the absence and presence of external flow; in both cases the beam dynamics is non-conservative.

In the absence of external flow, the structure lost stability through either divergence or flutter when the moment is proportional to the positive curvature. On the other hand, stability is lost solely through flutter when the moment is considered to be proportional to the negative curvature. The effects of external flow were also investigated using Galerkin approximation whereby the fluid-structure interaction resulted in flow-induced stiffness and damping terms. The damping term results in complex eigenfrequencies that alters the mechanism of flutter instability from that observed in the absence of external flow. Moreover, the standing waves generated at the critical stability point in the absence of external flow converted to traveling waves in the presence of external flow. It is noteworthy that the existence of external flow reduces the magnitude of the critical moment. However, the nature of instability (divergence and flutter) does not change when the external flow is introduced.

The fluid-structure interaction investigation is extended by considering the fluid-immersed hinged beam to be affixed to a rigid body. We focus our attention to exclusively investigating flutter instability. A thrust induced by the traveling waveform is generated at the critical instability point. The critical stability of the structure is determined using the analytical method within a two-parameter space of the applied non-conservative loading and the relative speed of the submersible structure. Since the proportionality constant can be positive or negative, the onset of flutter instability is discussed separately. The critical stability curves in the space of the two considered parameters are depicted; the results showed a similar

trend that can be categorized based on the curvature measurement locations. It should be noted that specific measurement locations can result in a negative thrust, especially when the point is taken close to the free end of the beam. Also, for some other locations, both positive and negative constant of proportionality resulted in positive thrust.

BIBLIOGRAPHY

BIBLIOGRAPHY

- [1] T. B. Benjamin, Dynamics of a system of articulated pipes conveying fluid. i. theory, in: Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences, Vol. 261, The Royal Society, 1961, pp. 457–486.
- [2] L. Prandtl, Kipperscheinungen, Dissertation, Munich.
- [3] V. Reut, About the theory of elastic stability, Proceedings of the Odessa Institute of Civil and Communal Engineering, No. 1.
- [4] G. Herrmann, R. W. Bungay, On the stability of elastic systems subjected to nonconservative forces, Journal of Applied Mechanics, ASME 31 (1964) 435–440.
- [5] H. Ziegler, Die stabilitätskriterien der elastomechanik, Archive of Applied Mechanics 20 (1952) 49–56.
- [6] T. B. Benjamin, Dynamics of a system of articulated pipes conveying fluid - i.theory, Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences 261 (1307) (1962) 457–486.
- [7] M. Langthjem, Y. Sugiyama, Dynamic stability of columns subjected to follower loads: A survey, Journal of Sound and Vibration 238 (5) (2000) 809 – 851.
- [8] M. P. Paidoussis, G. X. Li, Pipes conveying fluid: A model dynamical problem 7 (1993) 137–204.
- [9] E. V. Nikolai, About criterion of stability of elastic systems, Proceedings of the Odessa Institute of Civil and Communal Engineering, No. 1.
- [10] A. Pflüger, Stabilitätsprobleme der elastomechanik, Springer-Verlag, Berlin (1950) P.217.
- [11] A. Kounadis, Some new instability aspects for nonconservative systems under follower loads, International Journal of Mechanical Sciences 33 (4) (1991) 297 – 311.

- [12] Z. Qiu, S. Nemat-Nasser, Instability of an articulated cantilever induced by an impinging airjet, *International Journal of Solids and Structures* 21 (2) (1985) 145 – 154.
- [13] I. Elishakoff, Controversy associated with the so-called “follower forces”: Critical overview, *Applied Mechanics Reviews* 58 (2005) 117.
- [14] H. Ziegler, The stability criteria of elastomechanics, *Archive of Applied Mechanics* 20 (1) (1952) 49–56.
- [15] C. Semler, H. Alighanbari, M. Païdoussis, A physical explanation of the destabilizing effect of damping, *Journal of applied mechanics* 65 (3) (1998) 642–648.
- [16] P. Gallina, Effect of damping on asymmetric systems, *Journal of vibration and acoustics* 125 (3) (2003) 359–364.
- [17] Y. Sugiyama, M. Langthjem, Physical mechanism of the destabilizing effect of damping in continuous non-conservative dissipative systems, *International Journal of Non-Linear Mechanics* 42 (1) (2007) 132–145.
- [18] O. Doaré, Dissipation effect on local and global stability of fluid-conveying pipes, *Journal of Sound and Vibration* 329 (1) (2010) 72–83.
- [19] A. Luongo, F. D’Annibale, On the destabilizing effect of damping on discrete and continuous circulatory systems, *Journal of Sound and Vibration* 333 (24) (2014) 6723–6741.
- [20] M. Tommasini, O. N. Kirillov, D. Misseroni, D. Bigoni, The destabilizing effect of external damping: Singular flutter boundary for the pflüger column with vanishing external dissipation, *Journal of the Mechanics and Physics of Solids* 91 (2016) 204–215.
- [21] O. Bottema, On the stability of the equilibrium of linear mechanical system, *Journal of applied mathematics and physics* 6 (1955) 97–103.
- [22] V. Bolotin, T. Lusher, *Nonconservative Problems of Theory of Elastic Stability*, Elsevier Science & Technology, 1963.
- [23] G. Herrmann, C. Jong, On the destabilizing effect of damping in nonconservative elastic systems, *Journal of Applied Mechanics* 32 (3) (1965) 592–597.

- [24] S. Nemat-Nasser, G. Herrmann, Some general considerations concerning the destabilizing effect in nonconservative systems, *Journal of applied mathematics and physics* 17 (1966) 305–313.
- [25] S. Nemat-Nasser, S. Prasad, G. Hermann, Destabilizing effect of velocity-dependent forces in nonconservative continuous systems., *AIAA Journal* 4 (7) (1966) 1276–1280.
- [26] R. Gregory, M. Paidoussis, Unstable oscillation of tubular cantilevers conveying fluid. i. theory, in: *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, Vol. 293, The Royal Society, 1966, pp. 512–527.
- [27] G. Done, Damping configurations that have a stabilizing influence on nonconservative systems, *International Journal of Solids and Structures* 9 (2) (1973) 203–215.
- [28] M. Paidoussis, Dynamics of tubular cantilevers conveying fluid, *Journal of mechanical engineering sciences* 612 (2) (1970) 85–103.
- [29] I. Lottati, A. Kornecki, The effect of an elastic foundation and of dissipative forces on the stability of fluid-conveying pipes, *Journal of Sound and Vibration* 109 (2) (1986) 327–338.
- [30] R. Aithal, G. Gipson, Instability of internally damped curved pipes, *Journal of engineering mechanics* 116 (1) (1990) 77–90.
- [31] B. N. Rao, G. V. Rao, Stability of tapered cantilever columns subjected to a tip-concentrated follower force with or without damping, *Computers & Structures* 37 (3) (1990) 333–342.
- [32] V. Bolotin, A. Grishko, M. Y. Panov, Effect of damping on the postcritical behaviour of autonomous non-conservative systems, *International journal of non-linear mechanics* 37 (7) (2002) 1163–1179.
- [33] M. Paak, M. Paidoussis, A. Misra, Influence of steady viscous forces on the non-linear behaviour of cantilevered circular cylindrical shells conveying fluid, *International Journal of Non-Linear Mechanics* 58 (2014) 167–183.
- [34] A. N. Kounadis, On the paradox of the destabilizing effect of damping in non-conservative systems, *International journal of non-linear mechanics* 27 (4) (1992) 597–609.

- [35] O. Kirillov, A. Seyranian, Stabilization and destabilization of a circulatory system by small velocity-dependent forces, *Journal of sound and vibration* 283 (3) (2005) 781–800.
- [36] L. Zorii, Y. A. Chernukha, Influence of support conditions on the dynamic stability of elastic column, *Prikl. Mekh* 7 (2) (1971) 134–136.
- [37] A. Kounadis, Static stability analysis of elastically restrained structures under follower forces, *AIAA Journal* 18 (4) (1980) 473–476.
- [38] A. Kounadis, Divergence and flutter instability of elastically restrained structures under follower forces, *International Journal of Engineering Science* 19 (4) (1981) 553–562.
- [39] A. Kounadis, The existence of regions of divergence instability for nonconservative systems under follower forces, *International Journal of Solids and Structures* 19 (8) (1983) 725–733.
- [40] A. Kounadis, A. P. Economou, The effects of the joint stiffness and of the constraints on the type of instability of a frame under a follower force, *Acta Mechanica* 36 (3-4) (1980) 157–168.
- [41] I. Elishakoff, J. Hollkamp, Computerized symbolic solution for a nonconservative system in which instability occurs by flutter in one range of a parameter and by divergence in another, *Computer methods in applied mechanics and engineering* 62 (1) (1987) 27–46.
- [42] I. Elishakoff, I. Lottati, Divergence and flutter of nonconservative systems with intermediate support, *Computer methods in applied mechanics and engineering* 66 (2) (1988) 241–250.
- [43] C. Pierre, Mode localization and eigenvalue loci veering phenomena in disordered structures, *Journal of Sound and Vibration* 126 (3) (1988) 485 – 502.
- [44] Y. Sugiyama, Y. Tanaka, T. Kishi, H. Kawagoe, Effect of a spring support on the stability of pipes conveying fluid, *Journal of Sound and Vibration* 100 (2) (1985) 257–270.
- [45] H. Lee, Divergence of a rotating shaft with an intermediate support and conservative axial loads, *Computer methods in applied mechanics and engineering* 110 (3-4) (1993) 317–324.

- [46] W. Glabisz, Vibration and stability of a beam with elastic supports and concentrated masses under conservative and nonconservative forces, *Computers & structures* 70 (3) (1999) 305–313.
- [47] H.-P. Lin, S. Chang, Free vibration analysis of multi-span beams with intermediate flexible constraints, *Journal of sound and vibration* 281 (1-2) (2005) 155–169.
- [48] H.-Y. Lin, Y.-C. Tsai, Free vibration analysis of a uniform multi-span beam carrying multiple spring–mass systems, *Journal of Sound and Vibration* 302 (3) (2007) 442–456.
- [49] M. De Rosa, C. Franciosi, The influence of an intermediate support on the stability behaviour of cantilever beams subjected to follower forces, *Journal of Sound and Vibration* 137 (1) (1990) 107–115.
- [50] M. De Rosa, Eigenvalues’ behaviour of continuous beams on elastic supports in the presence of damping and follower forces, *International Journal of Mechanical Sciences* 33 (5) (1991) 325–337.
- [51] H. Lee, Effects of damping on the dynamic stability of a rod with an intermediate spring support subjected to follower forces, *Computers & structures* 60 (1) (1996) 31–39.
- [52] S. Park, J. Chung, Dynamic analysis of an axially moving finite-length beam with intermediate spring supports, *Journal of Sound and Vibration* 333 (24) (2014) 6742–6759.
- [53] S. Caddemi, I. Calì, F. Cannizzaro, Influence of an elastic end support on the dynamic stability of beck’s column with multiple weak sections, *International Journal of Non-Linear Mechanics* 69 (2015) 14–28.
- [54] H. Unterweger, A. Taras, S. Loschan, M. Kettler, Influence of imperfections on the stability of beams with intermediate flexible supports, *Journal of Constructional Steel Research* 136 (2017) 140–148.
- [55] V. Bolotin, N. Zhinzher, Effects of damping on stability of elastic systems subjected to nonconservative forces, *International Journal of Solids and Structures* 5 (9) (1969) 965–989.
- [56] H. Lee, Divergence and flutter of a cantilever rod with an intermediate spring support, *International Journal of Solids and Structures* 32 (10) (1995) 1371 – 1382.

- [57] C. L. Phillips, J. M. Parr, Feedback control systems, Pearson, 2011.
- [58] A. Hellum, R. Mukherjee, A. J. Hull, Flutter instability of a fluid-conveying fluid-immersed pipe affixed to a rigid body, *Journal of Fluids and Structures* 27 (7) (2011) 1086 – 1096.
- [59] A. Hellum, R. Mukherjee, A. Bénard, A. J. Hull, Modeling and simulation of the dynamics of a submersible propelled by a fluttering fluid-conveying tail, *Journal of Fluids and Structures* 36 (2013) 83 – 110.
- [60] A. B. Rostami, M. Armandei, Renewable energy harvesting by vortex-induced motions: Review and benchmarking of technologies, *Renewable and Sustainable Energy Reviews* 70 (2017) 193–214.
- [61] M. P. Paidoussis, Fluid-structure interactions: slender structures and axial flow, Vol. 1, Academic press, 1998.
- [62] A. Singh, R. Mukherjee, K. Turner, S. Shaw, Mems implementation of axial and follower end forces, *Journal of Sound and Vibration* 286 (3) (2005) 637 – 644.
- [63] L. Meirovitch, Computational Methods in Structural Dynamics, Sijthoff and Noordhoff, Rockville, MD, 1980.
- [64] I. G. Boruk, L. G. Lobas, On the motion of a reversible double simple pendulum with tracking force, *International Applied Mechanics* 35 (7) (1999) 745–750.
- [65] Z. Wang, D. R. Guo, Special functions, World Scientific, Singapore, 1999.
- [66] A. Di Egidio, A. Luongo, A. Paolone, Linear and non-linear interactions between static and dynamic bifurcations of damped planar beams, *Journal of Non-Linear Mechanics* 42 (1) (2007) 88–98.
- [67] R. R. Craig, Structural Dynamics, John Wiley and Sons, New York, NY, 1981.
- [68] M. Abdullatif, R. Mukherjee, A. Hellum, Stabilizing and destabilizing effects of damping in non-conservative systems: Some new results, *Journal of Sound and Vibration* 413 (2018) 442–455.

- [69] M. Paidoussis, Fluid-Structure Interactions: Slender Structures and Axial Flow, Vol. 2, Academic Press, 2014.
- [70] A. Luongo, F. D’Annibale, M. Ferretti, Hard loss of stability of Ziegler’s column with nonlinear damping, *Meccanica* 51 (11) (2016) 2647–2663.
- [71] V. Zamani, E. Kharazmi, R. Mukherjee, Asymmetric post-flutter oscillations of a cantilever due to a dynamic follower force, *Journal of Sound and Vibration* 340 (2015) 253–266.
- [72] H. Lee, Divergence and flutter of a cantilever rod with an intermediate spring support, *International Journal of Solids and Structures* 32 (10) (1995) 1371 – 1382.
- [73] M. Abdullatif, R. Mukherjee, Divergence and flutter instabilities of a cantilever beam subjected to a terminal dynamic moment, *Journal of Sound and Vibration* 455 (2019) 402–412.
- [74] A. Hellum, R. Mukherjee, A. J. Hull, Flutter instability of a fluid-conveying fluid-immersed pipe affixed to a rigid body, *Journal of Fluids and Structures* 27 (7) (2011) 1086–1096.
- [75] M. Lighthill, Note on the swimming of slender fish, *Journal of fluid Mechanics* 9 (2) (1960) 305–317.
- [76] T. Wu, Hydromechanics of swimming propulsion. part 1. swimming of a two-dimensional flexible plate at variable forward speeds in an inviscid fluid, *Journal of Fluid Mechanics* 46 (2) (1971) 337–355.