# GALOIS MODULE STRUCTURE OF WEAKLY RAMIFIED COVERS OF CURVES

By

Sugil Lee

A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

Mathematics – Doctor of Philosophy

2020

**ABSTRACT**

GALOIS MODULE STRUCTURE OF WEAKLY RAMIFIED COVERS OF CURVES

By

Sugil Lee

The main theme of our study is the obstruction to the existence of a *normal integral basis* for certain Galois modules of geometric origin. When $G$ is a finite group acting on a projective scheme $X$ over Spec $\mathbb{Z}$ and $\mathscr{F}$ is a $G$-equivariant coherent sheaf of $\mathcal{O}_X$-modules, the sheaf cohomology groups $\mathrm{H}^i(X, \mathscr{F})$ are $G$-modules, and one asks if its equivariant Euler characteristic

$$\chi(X, \mathscr{F}) := \sum_i (-1)^i [\mathrm{H}^i(X, \mathscr{F})]$$

can be calculated using a bounded complex of finitely generated *free* modules over $\mathbb{Z}[G]$. Then we say that the cohomology of $\mathscr{F}$ has a *normal integral basis*. The obstruction to the existence of a normal integral basis has been of great interest in the classical case of number fields: As conjectured by Fröhlich and proven by Taylor, when $N/\mathbb{Q}$ is a finite tamely ramified Galois extension with Galois group $G$, the Galois module structure of the ring of integers $\mathcal{O}_N$ is determined (up to stable isomorphism) by the root numbers appearing in the functional equations of Artin $L$-functions associated to symplectic representations of $G$. Chinburg started a generalization of the theory to some schemes with tame group actions by introducing the reduced *projective* Euler characteristic classes $\overline{\chi}^P(X, \mathscr{F})$. These Euler characteristics are elements of the class group $\mathrm{Cl}(\mathbb{Z}[G])$ and give the obstruction to the existence of normal integral basis.

Our aim is to generalize the theory to the ‘‘simplest” kind of wild ramification, namely to *weakly ramified* covers of curves over Spec $\mathbb{Z}$. If $N/\mathbb{Q}$ is wildly ramified, then $\mathcal{O}_N$ is not a free $\mathbb{Z}[G]$-module. Erez showed that when the order $|G|$ is odd, then the different ideal $\mathfrak{D}_{N/\mathbb{Q}}$ is a square, and the square root of the inverse different is a locally free $\mathbb{Z}[G]$-module if and only if $N/\mathbb{Q}$ is weakly ramified. Köck classified all fractional ideals of weakly ramified local rings that have normal integral bases. We generalize both of the results to curves over Spec $\mathbb{Z}$ to construct projective Euler characteristic for certain equivariant sheaves on weakly ramified covers of curves.

To my family.

## ACKNOWLEDGEMENTS

My past seven years of graduate school have been a complete re-discovery of who I am. It was like running a marathon just as many senior colleagues said, and there had been numerous moments that I felt completely tired and lost. However, these wonderful people around me have given me courage and hope, reminded me of how much I wanted to learn more in mathematics. Indeed, nothing would have been possible without their warmth, encouragement, and the endless support.

First and foremost, I'm deeply grateful for my advisor Georgios Pappas. Not only he gave me an identity as a mathematician by leading me to the topic of the thesis, he has always been there being unconditionally supportive to me when I didn't meet his initial expectations and felt completely lost for a long time. With his wisdom, vision, and immense patience, I was able to gain strength to finish my thesis in this form. George, thank you so much for everything. I hope to see you in person again when the pandemic situation is over.

I also thank the rest of the faculty and staff at Michigan State University. Rajesh Kulkarni, my temporary advisor when I joined the graduate program, had given me valuable advices to set realistic goals in the graduate school. He was also in my dissertation committee along with other members: Michael Shapiro and Aaron Levin. My deepest thanks to the committee for the work and time.

Teaching has become an important part of my life as a mathematician, and I want to mention Tsvetanka Sendova, Jane Zimmerman, and Gabriel Nagy for years of great teaching experience.

Another wonderful group to mention is my friends/colleagues that I've met at Michigan State University. Thank you Charlotte for conversations since the first year and helping me preparing for the defense. Thank you Woongbae for our countless conversations in many topics of common interest including mathematics. Thank you Adam, Ákos, Bosu, Dongsoo, Hitesh, Hyoseon, Jihye, Jiwoong, John, Manousos, Paata, Rami, Reshma, Sami, Seonghyeon, Tyler, Wenzhao, Yewon, and all others for a good time.

I also would like to thank my family for the love and the support that I have received since forever. Thank you Sangjae Lee and Myeongsook Kim, my dearest parents in Korea, for the endless love and sacrifices you made for me. Thank you Sumoon Lee and Byunglok Jeon, my only sister and brother-in-law, for encouraging me and bringing lovely smiles of Haeun and Jaeyoon to our family.

## TABLE OF CONTENTS

## LIST OF FIGURES

## KEY TO SYMBOLS

| | |
|---|---|
| $\mathbf{K}_0(G, A)$ | the Grothendieck group of finitely generated $A[G]$-modules |
| $\mathbf{K}_0(A[G])$ | the Grothendieck group of finitely generated projective $A[G]$-modules |
| $\mathbf{K}_0(A[G])^{\mathbf{red}}$ | the class group of finitely generated projective $A[G]$-modules |
| $\mathbf{Cl}(A[G])$ | the class group of finitely generated locally free $A[G]$-modules |
| $\mathbf{H}^i(X, \mathscr{F})$ | the $i$'th sheaf cohomology group |
| $\hat{\mathbf{H}}^i(G, M)$ | the $i$'th Tate cohomology group |
| $\chi(G, X, \mathscr{F})$ | the equivariant Euler characteristic |
| $\chi^P(G, X, \mathscr{F})$ | the projective equivariant Euler characteristic |
| $\mathcal{O}_K$ | the ring of integers of a number field |
| $\mathfrak{p}$ | a prime ideal |
| $\mathfrak{m}_K$ | the maximal ideal of a discrete valuation ring |
| $\kappa$ | the residue field of a discrete valuation ring |
| $\hat{M}_I$ | $I$-adic completion of a module |
| $G_i$ | the $i$'th ramification group |
| $A[G]$ | group ring of $G$ over $A$ |
| $\mathbf{R}\Gamma$ | right derived functor of the global section functor |
| $\mathbb{P}^1_A$ | projective space of dimension 1 over the spectrum of a ring $A$ |
| $\Omega^1_{X/Y}$ | the sheaf of relative differentials of $X/Y$ |
| $\mathfrak{D}_{N/K}$ | the different ideal of $N/K$ |

# CHAPTER 1

# INTRODUCTION

The history of the study of Galois module structure is deeply rooted in the *normal basis theorem* for finite Galois extensions. If $N/K$ is a finite Galois extension of fields with $G = \mathrm{Gal}(N/K)$, the theorem asserts that $N$, as a $K$-vector space, has a basis of the form $\{\sigma(\alpha)\}_{\sigma \in G}$ for some $\alpha \in N$ which is then called a *normal basis*, i.e., $N$ is a free $K[G]$-module of rank 1. To formulate the basic problem, let $N/\mathbb{Q}$ be a finite Galois extension with Galois group $G$. Then the ring of integers $O_N$ has a natural $\mathbb{Z}[G]$-module structure. The analogous question can be asked: Does $O_N$ have a *normal integral basis*, i.e., is there an element $\alpha \in O_N$ such that $\{\sigma(\alpha)\}_{\sigma \in G}$ forms a $\mathbb{Z}$-basis of $O_N$, so that $O_N$ is free over $\mathbb{Z}[G]$? This existence problem and the nature of the global obstructions have been the central theme of the subject.

The first approach to the problem was made via localization with completion. We call a finitely generated $\mathbb{Z}[G]$-module $M$ *locally free* if, for all primes $p$ in $\mathbb{Z}$, the $p$-adic completion $M_p$ is free over $\mathbb{Z}_p[G]$. (This is equivalent to $M$ being *projective*, see [Swa60].) A theorem of Noether [Noe32] states that

**Theorem 1.0.1.** *With $N/\mathbb{Q}$ as above, $O_N$ is locally free over $\mathbb{Z}[G]$ if and only if $N/\mathbb{Q}$ is tame, i.e., at most tamely ramified at every $p$.*

Thus tame ramification is clearly necessary for the existence of a normal integral basis. The natural question is then to ask if it is sufficient, and if not, what are the global obstructions. Explicit examples for $N/\mathbb{Q}$ tame with no normal integral bases can be found in [Mar71].

The question can be rephrased in terms of a calculation of the class $(O_N)_{\mathbb{Z}[G]}$ of $O_N$ in the class group $\mathrm{Cl}(\mathbb{Z}[G])$ of the Grothendieck group $\mathrm{K}_0(\mathbb{Z}[G])$ of finitely generated *projective* modules over $\mathbb{Z}[G]$. The class group $\mathrm{Cl}(\mathbb{Z}[G])$ is a finite abelian group which is defined as the quotient of $\mathrm{K}_0(\mathbb{Z}[G])$ by the subgroup generated by the class of the free module $\mathbb{Z}[G]$. If $M$ is a finitely generated locally free $\mathbb{Z}[G]$-module, then the rank of $M$ and its class $(M)_{\mathbb{Z}[G]} \in \mathrm{Cl}(\mathbb{Z}[G])$ determine

$M$ up to stable isomorphism: If $N$ is another finitely generated projective $\mathbb{Z}[G]$-module, then the classes of $M$, $N$ in $\mathrm{Cl}(\mathbb{Z}[G])$ and their ranks coincide if and only if $M \oplus \mathbb{Z}[G] \cong N \oplus \mathbb{Z}[G]$. Moreover, for most $G$, e.g. of odd order or abelian, a stable isomorphism of locally free modules implies an isomorphism by the Swan-Jacobinski theorem ([Swa86]; see also [Frö83]), i.e., we have ‘‘cancellation’’.

The most important discovery that provides the power, depth, and interest of the theory is the conjecture made by Fröhlich and proved by Taylor in [Tay81]. It states that, for a finite Galois extension of number fields $N/K$, if $N/K$ is tame, then the class $(O_N)_{\mathbb{Z}[G]}$ is equal to another invariant $W_{N/\mathbb{Q}}$ in $\mathrm{Cl}(\mathbb{Z}[G])$ which Cassou-Noguès had defined in [CN78] using the Artin root-numbers of symplectic representations of the Galois group $G$. The root-number $W(\chi)$ of a character $\chi$ of $G$ is the complex constant of absolute value 1 appearing in the functional equation of the extended Artin $L$-function $\tilde{L}(s,\chi) = \tilde{L}(s, N/K, \chi)$ (with Euler factors at infinity, see [Frö83]):

$$\tilde{L}(s,\chi) = \tilde{L}(1-s,\overline{\chi})W(\chi)A(\chi)^{\frac{1}{2}-s}$$

where $\overline{\chi}$ is the complex conjugate representation, $A(\chi)$ is a positive constant. When $\chi$ is real valued, then the root numbers are known to be $\pm 1$. These root numbers influence the existence of non-trivial zeros of the Dedekind zeta functions: $\tilde{L}(s,\chi)A(\chi)^{\frac{s}{2}}$ is either symmetric or asymmetric about $s = \frac{1}{2}$ depending on whether $W(\chi) = 1$ or $= -1$. If $\chi$ is a character of $\mathrm{Gal}(N/K)$, the later case implies in turn that the Dedekind zeta function of $N$ vanishes at $s = \frac{1}{2}$, see [Arm71].

The connection between Galois-structure invariants and Artin $L$-functions has been a basic theme in research on Galois structure since then. For example, as the Artin root numbers of symplectic characters are $\pm 1$, $(O_N)_{\mathbb{Z}[G]} = W_{N/K}$ is 2-torsion by Fröhlich and Taylor [Tay81], and indeed

$$\gcd(2, |G|) \cdot (O_N)_{\mathbb{Z}[G]} = 0. \tag{1.1}$$

Thus when the order $|G|$ of the group is odd, then $O_N$ is stably free; but cancellation holds when $|G|$ is odd, therefore we know that $O_N$ is actually a free $\mathbb{Z}[G]$-module.

The study of the Galois module structure of the ring of integers $O_N$ naturally extends to a geometric situation. Suppose $X$ is a smooth projective variety defined over $\mathbb{Q}$ with an action of a finite group $G$. For any $G$-equivariant coherent sheaf $\mathscr{F}$ of $\mathcal{O}_X$-modules, i.e., a coherent sheaf of $\mathcal{O}_X$-modules equipped with a $G$-action compatible with the $G$-action on $X$, the equivariant Euler characteristic $\chi(X,\mathscr{F}) = \chi(G,X,\mathscr{F})$ is the virtual representation

$$\chi(X,\mathscr{F}) := \sum_i (-1)^i [\mathrm{H}^i(X,\mathscr{F})].$$

It is well-known (see [BK82]) that, if the action is free, $\chi(X,\mathscr{F})$ is a multiple of the regular representation $\mathbb{Q}[G]$, i.e., the complex $\mathrm{R}\Gamma(X,\mathscr{F})$ in the derived category of complexes of $\mathbb{Q}[G]$-modules is isomorphic to a bounded complex of finitely generated free $\mathbb{Q}[G]$-modules. Suppose now that the variety $X$ is extended to a regular projective scheme $X'$ which is flat over Spec $\mathbb{Z}$ and the action of $G$ extended to $X'$ as well. For a $G$-equivariant coherent sheaf of $\mathcal{O}_X$-modules $\mathscr{F}$ on $X'$, we say that

**Definition.** The cohomology of $\mathscr{F}$ has a *normal integral basis* if there exists a bounded complex of finitely generated *free* $\mathbb{Z}[G]$-modules which is isomorphic to $\mathrm{R}\Gamma(X,\mathscr{F})$ in the derived category of complexes of finitely generated $\mathbb{Z}[G]$-modules.

Unlike the previous case over $\mathbb{Q}$, it is not true in general that the cohomology of $\mathscr{F}$ has a normal integral basis. The classical example of $\mathcal{O}_N$ is reconstructed under these settings when $X'$ is the spectrum of $\mathcal{O}_N$ in a Galois extension of number fields $N/\mathbb{Q}$ with the Galois group $G$, and $\mathscr{F} = \mathcal{O}_{X'} = \widetilde{\mathcal{O}_N}$.

From now on, we consider the general setting for geometric Galois module structure as following: For a finite group $G$, we consider a $G$-cover $\pi : X \to Y$ of projective flat schemes over Spec $\mathbb{Z}$. If $\mathscr{F}$ is a $G$-equivariant coherent sheaf of $\mathcal{O}_X$-modules, then the sheaf cohomology groups

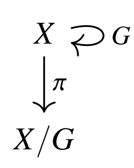



Figure 1.1: Galois cover.

$\mathrm{H}^i(X,\mathscr{F})$ are finitely generated modules over the group ring $\mathbb{Z}[G]$. Thus the equivariant Euler characteristic $\chi(X, \mathscr{F})$ can be defined as earlier in the Grothendieck group $\mathrm{K}_0(G, \mathbb{Z})$ of finitely generated $\mathbb{Z}[G]$-modules. The main objective is to calculate the obstruction to the existence of a normal integral basis of the cohomology of $\mathscr{F}$.

The solution to the problem depends on many geometric invariants of $\pi : X \to Y$ and $\mathscr{F}$, and the foundation of the study has been developed when $\pi$ is assumed to be at most tamely ramified, extending the classical results on $(O_N)_{\mathbb{Z}[G]}$.

Chinburg introduced the general framework ([Chi94]; see also [CE92], [CEPT96]) in the following sense: When $\pi : X \longrightarrow Y = X/G$ is tamely ramified, then for any $G$-equivariant coherent sheaf $\mathscr{F}$ of $O_X$-modules on $X$, the complex $\mathrm{R}\Gamma(X, \mathscr{F})$ is quasi-isomorphic to a bounded complex of finitely generated *projective* $\mathbb{Z}[G]$-modules. In fact, one can take such bounded complex $P^\bullet$ to be of free $\mathbb{Z}[G]$-modules except for the last projective term, thus the class of the last term in $\mathrm{Cl}(\mathbb{Z}[G])$ is the obstruction to the existence of a normal integral basis for the cohomology of $\mathscr{F}$. The Euler characteristic $\sum_i (-1)^i [P^i]$ of such a bounded complex in $\mathrm{K}_0(\mathbb{Z}[G])$ does not depend on the choice of the bounded complex, and is called the *projective* Euler characteristic $\chi^P(X, \mathscr{F})$. This maps to the equivariant Euler characteristic $\chi(X, \mathscr{F})$ in $\mathrm{K}_0(G, \mathbb{Z})$ under the natural forgetful functor

$$\begin{aligned} \mathrm{K}_0(\mathbb{Z}[G]) &\to \mathrm{K}_0(G, \mathbb{Z}) \\ \chi^P(X, \mathscr{F}) &\mapsto \chi(X, \mathscr{F}) \end{aligned}$$

called the Cartan homomorphism, giving us much more information on the Galois module structure of the equivariant Euler characteristic.

Understanding these projective Euler characteristics has been the main problem of the theory of geometric Galois structure. In [Chi94] and [CEPT96], under some additional hypotheses, the projective Euler characteristic of a version of the de Rham complex of sheaves on $X$ (which generalizes the classical obstruction $(O_N)_{\mathbb{Z}[G]}$ if $X = \mathrm{Spec}\, O_N$) was calculated using $\epsilon$-factors of Hasse-Weil-Artin $L$-functions for the cover $X \to Y$. In [Pap98], Pappas considered the general $\mathscr{F}$ on the ‘‘relative curve” case over Spec $\mathbb{Z}$ with a free action of $G$ by introducing the technique of cubic structures. He generalizes the equation (1.1) by showing

$$\gcd(2, |G|) \cdot \overline{\chi}^P(X, \mathscr{F}) = 0 \tag{1.2}$$

when all the Sylow subgroups of $G$ are abelian. Here, $\overline{\chi}^P(X, \mathscr{F})$ is the class of $\chi^P(X, \mathscr{F})$ in $\mathrm{Cl}(\mathbb{Z}[G])$ called the *reduced* projective Euler characteristic. Later the abelian restriction on (1.2) was overcome with a stronger result with the introduction of Adams operations in [Pap15]. This proves a case of a conjecture on unramified covers that there are integers $n$ (that depends only on the relative dimension of $X$ over $\mathbb{Z}$) and $\delta$ (that depends only on the order of $G$) such that

$$\gcd(n, |G|)^{\delta} \cdot \overline{\chi}^P(X, \mathscr{F}) = 0,$$

see [Pap08] and [CPT09] for more.

Unlike the rich progress in non-ramified and tamely ramified covers, little is known about the simplest kind of wild ramification called *weakly ramified*. The Galois group $G$ of a finite Galois extension of local fields $L/K$ with respect to a discrete valuation $v_L$ has a finite chain of normal subgroups

$$G = G_{-1} \trianglerighteq G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \cdots$$

given by

$$G_n = \{\sigma \in G \mid v_L(\sigma(x) - x) \geq n + 1 \;\; \forall x \in O_L\}$$

called the $n$'th ramification group. We say $L/K$ is *weakly ramified* (resp. *tamely ramified*, *unramified*) if $G_n = \{1\}$ for $n = 2$ (resp. $n = 1$, $n = 0$) and the corresponding residue field extensions are separable. A Galois extension of number fields $N/K$ is *weakly ramified* if its localizations at all primes in $N$ are weakly ramified. The main result of this thesis generalizes some aspects of the theory of geometric Galois module structure to weakly ramified covers of curves.

Turning back to the classical example, Noether's theorem asserts that when the ramification of $N/\mathbb{Q}$ is wild, then $O_N$ does not possess a normal integral basis. However, there could be other ambiguous ideals, i.e., fractional ideals that are also stable under $G$, that are free over $\mathbb{Z}[G]$. Regardless of the ramification type, Ullom in [Ull70] showed that if an ambiguous ideal in $N$

is locally free over $\mathbb{Z}[G]$, then all the second ramification groups are trivial. Thus the classical examples of normal integral bases are all bound to weakly ramified cases. The first general result on locally free ambiguous ideals over $\mathbb{Z}[G]$ other than the ring of integers itself was given by Erez in [Ere91]. When the order of the Galois group $G$ is odd, then by Hilbert's valuation formula on the different ideal $\mathfrak{D}(N/\mathbb{Q})$, there exists a canonical ambiguous ideal $A(N/\mathbb{Q})$ called the *square root of the inverse different* whose square is the inverse of $\mathfrak{D}(N/\mathbb{Q})$. Erez's theorem states that

**Theorem 1.0.2.** *$A(N/\mathbb{Q})$ is locally free over $\mathbb{Z}[G]$ if and only if $N/\mathbb{Q}$ is weakly ramified.*

Note that, since the order $|G|$ is assumed to be odd, $\mathbb{Q}[G]$ satisfies the Eichler condition in the Swan-Jacobinski theorem, thus the class $(A(N/\mathbb{Q}))_{\mathbb{Z}[G]}$ determines the Galois structure of $A(N/\mathbb{Q})$ up to $\mathbb{Z}[G]$-isomorphism.

In [Köc04], Köck classified all ambiguous ideals of weakly ramified extensions that are locally free over $\mathbb{Z}[G]$. Suppose $L/K$ is a finite Galois extension of local fields with Galois group $G$, and $\mathfrak{m}_L$ denote the maximal ideal of the ring of integers of $L$. Then the maximal tamely ramified subfield $L^t$ of $L$ over $K$ determines the wild inertia group $G_1 = \mathrm{Gal}(L/L^t)$, and

**Theorem 1.0.3.** *The fractional ideal $\mathfrak{m}_L^b$ for some $b \in \mathbb{Z}$ is free over $\mathcal{O}_K[G]$ if and only if $L/K$ is weakly ramified and $b \equiv 1 \mod |G_1|$.*

In the same paper, this result is then generalized to the geometric Galois module case of dimension 1, namely weakly ramified cover of curves $\pi : X \to X/G$ over an algebraically closed field $k$ with $G$-equivariant invertible sheaves $\mathcal{O}_X(D)$. These sheaves admit projective Euler characteristics defined in $\mathrm{K}_0(k[G])$.

Our main objective is to study this Galois module structure problem when $\pi$ is a weakly ramified cover of curves over Spec $\mathbb{Z}$, the first higher dimensional case. Once the meaning of ‘‘weak ramification” is clarified on finite covers of schemes of dimension 2 (which is discussed in Chapter 4), the natural question is whether we can study the cohomologies of $G$-equivariant sheaves using an obstruction well-defined in $\mathrm{Cl}(\mathbb{Z}[G])$. The equivariant Euler characteristic $\chi(X, \mathscr{F})$ is in $\mathrm{K}_0(G, \mathbb{Z})$, and the Cartan homomorphism $\mathrm{K}_0(\mathbb{Z}[G]) \to \mathrm{K}_0(G, \mathbb{Z})$ is not surjective in general.

Our first result is to define the projective Euler characteristic $\chi^p$ for a class of $G$-equivariant invertible sheaves on $X$ using Chinburg's criterion given in [Chi94]. This generalizes Köck's work to schemes of dimension 2 (The exact statement of the theorem is given in Chapter 4).

**Theorem 1.0.4.** *Let $\pi : X \to X/G$ be a weakly ramified cover of curves over $\mathbb{Z}$ with a finite action of $G$. Suppose $\mathscr{F} = \mathcal{O}_X(D)$ is an invertible sheaf of $\mathcal{O}_X$-$G$-modules on $X$ corresponding to a horizontal divisor $D$. Suppose the restriction $D \cap X_p$ to the special fibre $X_p$ over each prime divisor $p$ of the order of $G$ is given by a $G$-equivariant Weil divisor $\sum_{x \in X_p} n_x \cdot [x]$ where $n_x \equiv -1 \mod |G_{x,1}|$. Then the derived complex $\mathrm{R}\Gamma(X, \mathscr{F})$ is isomorphic in the derived category to a bounded complex $P^\bullet$ of finitely generated projective $\mathbb{Z}[G]$-modules. Its Euler characteristic $\sum_i (-1)^i [P]^i$ in $K_0(\mathbb{Z}[G])$ is independent of choices and defines the projective Euler characteristic $\chi^P(X, \mathscr{F})$.*

The second main result is that the condition imposed on the $G$-equivariant invertible sheaves in the theorem is not too strong, that there canonically exists a sheaf satisfying the condition when $X \to Y$ is weakly ramified relative cover of curves over Spec $\mathbb{Z}$. The canonical existence is a generalization of the square root of the inverse different $A(N/\mathbb{Q})$ introduced by Erez (This is discussed in Chapter 5).

**Theorem 1.0.5.** *There exists an invertible sheaf $\mathscr{F}$ on $X$ such that $\mathscr{F}^{\otimes -2}$ is the torsion-free part of the quotient sheaf $\mathcal{O}_X/\mathrm{Ann}(\Omega^1_{X/Y})$ by the annihilator of the sheaf of relative differentials $\Omega^1_{X/Y}$ and which satisfies the assumptions of the previous theorem, so that the projective Euler characteristic $\chi^P(X, \mathscr{F})$ exists.*

The thesis is organized as follows. In Chapter 2, we review the background mathematics that will be used throughout the thesis. For example, we discuss some facts about ramified extensions of discrete valuation rings, Grothendieck groups, projectivity and cohomological triviality, quotient scheme by a finite group, and $G$-equivariant sheaves. Chapter 3 is an extension to Chapter 2, discussing more on theorems of Chinburg, Köck, and Erez that will be used directly in our main results. In Chapter 4, we introduce our first main result which gives a sufficient condition for $G$-equivariant invertible sheaves on weakly ramified relative curves to have projective Euler characteristic. We also demonstrate an example which naturally leads to the topic of the next chapter. Chapter 5 discusses the canonical existence of sheaves that meet the hypothesis of the theorem in Chapter 4, generalizing the square root of the inverse different that Erez introduced.

# CHAPTER 2

# BACKGROUND

This chapter provides a succinct review of mathematics that will be used throughout this thesis. We first recall a basic classification of ramification of discrete valuation rings using higher ramification groups. The second part reviews group cohomology and the Grothendieck group $\mathrm{K}_0(\mathbb{Z}[G])$ of finitely generated projective $\mathbb{Z}[G]$-modules. In the last section, we review group actions on schemes and the Euler characteristic of equivariant sheaves.

Throughout the text, $G$ always denotes a finite group, and all rings are assumed to be commutative and have identity.

## 2.1 Discrete Valuation Rings and Dedekind Domains

*Dedekind domains* are fundamental objects in algebraic number theory and smooth curves. This section heavily relies on [Ser79], and proofs of basic results are mostly omitted.

**Discrete Valuation Rings.** Let $K$ be a field, $K^\times$ the multiplicative group of non-zero elements of $K$. A *discrete valuation* of $K$ is a surjective homomorphism $v : K^\times \to \mathbb{Z}$ such that

$$v(x+y) \geq \mathrm{Inf}(v(x), v(y)) \quad \text{for } x, y \in K^\times.$$

Here $v$ is extended to $K$ by setting $v(0) = +\infty$.

The set $\mathcal{O}$ of elements $x \in K$ such that $v(x) \geq 0$ is a subring of $K$ called the *valuation ring* of $v$. It has a unique maximal ideal, namely the set $\mathfrak{m}$ of all $x \in K$ such that $v(x) > 1$. An element $\pi \in \mathfrak{m}$ with $v(\pi) = 1$ generates the maximal ideal $\mathfrak{m}$, and such an element is called a *uniformizer* of $\mathcal{O}$ (or of $v$). The field $\kappa = \mathcal{O}/\mathfrak{m}$ is called the residue field of $\mathcal{O}$ (or of $v$).

A *discrete valuation ring* $(\mathcal{O}, \mathfrak{m})$ (or simply $\mathcal{O}$) is a principal ideal domain $\mathcal{O}$ with exactly one non-zero maximal ideal $\mathfrak{m}$. Then the field of quotients $K$ of $\mathcal{O}$ is equipped with a discrete valuation $v$ derived by $\mathfrak{m}$: For $x \in K^\times$, $v(x) = n$ where $x \in \mathfrak{m}^n$ and $x \notin \mathfrak{m}^{n+1}$.

A discrete valuation ring is characterized by the following proposition:

**Proposition 2.1.1.** *Let $A$ be a noetherian integral domain. Then $A$ is a discrete valuation ring if and only if the two following conditions are met:*

*i) $A$ is integrally closed.*

*ii) $A$ has a unique non-zero prime ideal.*

**Dedekind Domains.** Let $\mathcal{O}$ be a noetherian integral domain.

**Proposition 2.1.2.** *For a noetherian integral domain $\mathcal{O}$, the following are equivalent:*

*i) For every non-zero prime ideal $\mathfrak{p} \in Spec\ \mathcal{O}$, the localization $\mathcal{O}_{\mathfrak{p}}$ is a discrete valuation ring.*

*ii) $\mathcal{O}$ is integrally closed and of dimension $\leq 1$.*

If $\mathcal{O}$ satisfies these equivalent conditions and is of dimension $1$, it is called a *Dedeking domain*.

If $A$ is an integral domain with the field of fractions $K$, a *fractional ideal $I$* of $A$ is a sub-$A$-module of $K$ finitely generated over $A$. If $J$ is another fractional ideal of $A$, the product ideal $IJ$ is generated by products of elements of $I$ and $J$. One says $I$ is *invertible* if there exists a fractional ideal $J$ such that $IJ = A$.

**Proposition 2.1.3.** *In a Dedekind domain, every fractional ideal is invertible.*

The non-zero fractional ideals of a Dedekind domain form a group under multiplication called the *ideal group* of the ring.

Let $\mathcal{O}$ be a Dedekind domain. For each non-zero prime ideal $\mathfrak{p}$ of $\mathcal{O}$, the localization $\mathcal{O}_{\mathfrak{p}}$ defines a discrete valuation of the field of fractions $K$ denoted by $v_{\mathfrak{p}}$.

**Proposition 2.1.4.** *If $x \in \mathcal{O}$, $x \neq 0$, then only finitely many prime ideals contain $x$.*

**Corollary 2.1.4.1.** *For every $x \in K^{\times}$, the numbers $v_{\mathfrak{p}}(x)$ are almost all zero, i.e., zero except for a finite number.*

If $I$ is a fractional ideal and $\mathfrak{p}$ is a non-zero prime ideal of $O$, then the image $I_\mathfrak{p}$ of $I$ in $O_\mathfrak{p}$ has the form $I_\mathfrak{p} = (\mathfrak{p}O_\mathfrak{p})^{v_\mathfrak{p}(I)}$ defining the valuation $v_\mathfrak{p}(I)$ of the ideal $I$ at $\mathfrak{p}$.

**Proposition 2.1.5.** *Every fractional ideal $I$ of $O$ can be written uniquely in the form:*

$$I = \prod_{\mathfrak{p}} \mathfrak{p}^{v_\mathfrak{p}(I)},$$

*where $v_\mathfrak{p}(I)$ are integers almost all zero.*

**Extensions of Dedekind Domains.** Let $O_K$ be a Dedekind domain with its field of fractions $K$. Let $N$ be a separable extension of $K$ and denote by $O_N$ the *integral closure* of $O_K$ in $N$, i.e., the set of elements of $N$ that are integral over $O_K$.

**Proposition 2.1.6.** *The ring $O_N$ is a finitely generated $O_K$-module and Dedekind.*

If $\mathfrak{p}$ is a non-zero prime ideal of $O_K$, and $\mathfrak{q}$ is a prime ideal of $O_N$ such that $\mathfrak{q} \cap O_K = \mathfrak{p}$, we say $\mathfrak{q}$ *is over* $\mathfrak{p}$ and write $\mathfrak{q}|\mathfrak{p}$. The fractional ideal $\mathfrak{p}O_N$ decomposes into prime ideals of $O_N$:

$$\mathfrak{p}O_N = \prod_{\mathfrak{q}|\mathfrak{p}} \mathfrak{q}^{e_\mathfrak{q}}$$

where $e_\mathfrak{q} = v_\mathfrak{q}(\mathfrak{p}O_N) \geq 0$. The integer $e_\mathfrak{q}$ is called the *ramification index* of $\mathfrak{q}$ in the extension $N/K$.

When $\mathfrak{q}$ is over $\mathfrak{p}$, the residue field $\lambda = O_N/\mathfrak{q}$ is a finite field extension of the residue field $\kappa = O_K/\mathfrak{p}$. The degree $f_\mathfrak{q}$ of extension $\lambda/\kappa$ is called the *residue degree* of $\mathfrak{q}$.

**Proposition 2.1.7.** *Let $\mathfrak{p}$ be a non-zero prime ideal of $O_K$. Then the ring $O_N/\mathfrak{p}O_N$ is an $O_K/\mathfrak{p}$-algebra of degree $n = [N : K]$ isomorphic to the product*

$$\prod_{\mathfrak{q}|\mathfrak{p}} O_N/\mathfrak{q}^{e_\mathfrak{q}}.$$

*We have the formula*

$$n = \sum_{\mathfrak{q}|\mathfrak{p}} e_\mathfrak{q} f_\mathfrak{q}.$$

If $\mathfrak{q}|\mathfrak{p}$ and $v_{\mathfrak{q}}$ is the valuation induced by $\mathfrak{q}$, then for $x \in K$, $v_{\mathfrak{q}}(x) = e_{\mathfrak{q}} v_{\mathfrak{p}}(x)$. We say the valuation $v_{\mathfrak{q}}$ extends $v_{\mathfrak{p}}$ with index $e_{\mathfrak{q}}$. Conversely:

**Proposition 2.1.8.** *Let $w$ be a discrete valuation of $N$ extending $v_{\mathfrak{p}}$ for some $\mathfrak{p} \in \text{Spec}\, O_K$ with index $e$. Then there is $\mathfrak{q} \in \text{Spec}\, O_N$, $w = v_{\mathfrak{q}}$ with $e_{\mathfrak{q}} = e$.*

**Galois Extensions.** Let $N/K$ be as before and assume further that the extension is Galois.

**Proposition 2.1.9.** *If $N/K$ is Galois and $\mathfrak{q}|\mathfrak{p}$, the integers $e_{\mathfrak{q}}, f_{\mathfrak{q}}$ depend only on $\mathfrak{p}$. If we denote them by $e_{\mathfrak{p}}, f_{\mathfrak{p}}$, and if $r_{\mathfrak{p}}$ denotes the number of prime ideals $\mathfrak{q}$ over $\mathfrak{p}$, then*

$$n = e_{\mathfrak{p}} f_{\mathfrak{p}} r_{\mathfrak{p}}.$$

Let $G = \text{Gal}(N/K)$. For a non-zero prime ideal $\mathfrak{q} \in \text{Spec}\, O_N$, the *decomposition group* $G_{\mathfrak{q}}$ is the subgroup of $G$ fixing $\mathfrak{q}$:

$$G_{\mathfrak{q}} := \{\sigma \in G \mid \sigma(\mathfrak{q}) = \mathfrak{q}\}.$$

For $\mathfrak{q}|\mathfrak{p}$, denote $e_{\mathfrak{p}}, f_{\mathfrak{p}}, r_{\mathfrak{p}}$ by $e, f, r$ respectively. Then we have extensions of fields:

$$\begin{array}{c} N \\ \big|\, {\scriptstyle e \cdot f} \\ N^{G_{\mathfrak{q}}} \\ \big|\, {\scriptstyle r} \\ K \end{array}$$

where $N^{G_{\mathfrak{q}}}$ is the fixed group of $N$ by the decomposition group $G_{\mathfrak{q}}$.

Now consider the residue field extension $\lambda/\kappa$ at $\mathfrak{q}$. There is a natural homomorphism

$$G_{\mathfrak{q}} \to \text{Gal}(\lambda/\kappa),$$

and the kernel $I_{\mathfrak{q}}$ of this morphism is called the *inertia group* of $\mathfrak{q}$. This gives extensions:

$$\begin{array}{c} N \\ \Big| \\ N^{I_{\mathfrak{q}}} \\ \Big| \\ N^{G_{\mathfrak{q}}} \\ \Big|\, r \\ K \end{array}$$

where $N^{I_{\mathfrak{q}}}/N^{G_{\mathfrak{q}}}$ is a Galois extension with Galois group $G_{\mathfrak{q}}/I_{\mathfrak{q}}$.

**Proposition 2.1.10.** *The residue extension $\lambda/\kappa$ is normal, and the natural homomorphism*

$$G_{\mathfrak{q}} \to \mathrm{Gal}(\lambda/\kappa)$$

*is surjective.*

The defining short exact sequence of the inertia group $I_{\mathfrak{q}}$ is thus given as

$$0 \to I_{\mathfrak{q}} \to G_{\mathfrak{q}} \to \mathrm{Gal}(\lambda/\kappa) \to 0.$$

**Completion.** Suppose $A$ is a commutative ring with identity, and $I$ is an ideal of $A$. We denote by $I^n$ the $n$'th power of the ideal $I$. Then there are natural homomorphisms

$$A/I \leftarrow A/I^2 \leftarrow A/I^3 \leftarrow \cdots.$$

This makes $(A/I^n)$ an inverse system of rings, and its inverse limit ring $\hat{A}^I = \varprojlim A/I^n$ is called the *completion of $A$ with respect to $I$* or the *$I$-adic completion of $A$*. For each $n$, we have a natural map $A \to A/I^n$, and by the universal property of the inverse limit, we obtain a homomorphism $A \to \hat{A}^I$.

Similarly, if $M$ is an $A$-module, we define $\hat{M}^I = \varprojlim M/I^n M$, and call it the *$I$-adic completion of $M$*. It has a natural structure of $\hat{A}^I$-module. When no confusion arises, we would skip $I$ and denote $\hat{A}^I$, $\hat{M}^I$ by $\hat{A}$, $\hat{M}$, respectively.

We state some important properties of $I$-adic completion without proofs (see II. §9 in [Har77]).

**Proposition 2.1.11.** *Let $A$ be a noetherian ring, and $I$ an ideal of $A$. Then:*

*i) $I$-adic completion commutes with finite direct sums (this holds true without noetherian assumption on $A$);*

*ii) $\hat{I} = \varprojlim I/I^n$ is an ideal of $\hat{A}$. For any $n$, the power $\hat{I}^n = I^n\hat{A}$, and $\hat{A}/\hat{I}^n \cong A/I^n$;*

*iii) if $M$ is a finitely generated $A$-module, then $\hat{M} \cong M \otimes_A \hat{A}$;*

*iv) the functor $M \mapsto \hat{M}$ is an exact functor on the category of finitely generated $A$-modules;*

*v) $\hat{A}$ is a flat $A$-module;*

*vi) $\hat{A}$ is a noetherian ring;*

*vii) if $(M_n)$ is an inverse system where each $M_n$ is a finitely generated $A/I^n$-module, and for all $n < n'$,*

$$0 \to I^n M_{n'} \to M_{n'} \to M_n \to 0$$

*is exact, then $M = \varprojlim M_n$ is a finitely generated $\hat{A}$-module, and for each $n$, $M_n \cong M/I^n M$.*

In most of cases of our study, we consider the $\mathfrak{m}$-adic completion for a discrete valuation ring $(A, I) = (\mathcal{O}, \mathfrak{m})$.

**Complete Fields.** Consider a field $K$ equipped with a discrete valuation ring $(\mathcal{O}_K, \mathfrak{m}_K)$. The $\mathfrak{m}_K$-adic completion of $\mathcal{O}_K$ is a discrete valuation ring $\hat{\mathcal{O}}_K$ with the maximal ideal $\hat{\mathfrak{m}}_K$. Its field of fractions $\hat{K}$ is the completion of $K$ with respect to an ultrametric absolute value on $K$ defined by, for some real number $0 < a < 1$,

$$||x|| = a^{v(x)} \quad \text{for} \quad x \neq 0,$$
$$||0|| = 0.$$

The topology determined by this metric does not depend on the choice of $a$. The field $K$ is *complete* if $\hat{K} = K$.

A complete field $K$ is a *local field* if its residue field $\kappa$ is finite. A local field of characteristic zero is either $\mathbb{Q}_p$, the completion of $\mathbb{Q}$ for the topology defined by the $p$-adic valuation, or the Laurent power series in a formal variable. A local field with finite characteristic $p > 0$ is isomorphic to the field $F((T))$ of formal Laurent power series for some finite field $F$. When $K$ is a local field, a canonical way to choose the number $a$ is to take $a = q^{-1}$, where $q$ is the cardinality of the finite residue field $\kappa$.

**Extension of a Complete Field.** Let $K$ be a field equipped with a discrete valuation ring $O_K$. Suppose $K$ is complete in the topology determined by $O_K$. Let $L/K$ be a finite extension of fields, and $O_L$ the integral closure of $O_K$.

**Proposition 2.1.12.** *Under the above assumptions,*

*i)* $O_L$ *is a discrete valuation ring;*

*ii)* $O_L$ *is a free* $O_K$*-module of rank* $n = [L : K]$*;*

*iii)* $L$ *is complete in the topology determined by the maximal ideal of* $O_L$*.*

**Corollary 2.1.12.1.** *If* $L/K$ *is separable, then* $O_L$ *is a finitely generated free* $O_K$*-module.*

There is a unique valuation $v_L$ of $L$ extending the valuation $v_K$ of $K$.

**Corollary 2.1.12.2.** *For every* $x \in L$, $v_L(x) = \frac{1}{f} \cdot v_K(N_{L/K}(x))$ *where* $f$ *is the residue degree.*

**Extension and Completion.** Suppose a field $K$ is the fraction field of a discrete valuation ring $(O_K, \mathfrak{m}_K)$ which is not assumed to be complete, and there is a finite separable extension $N/K$ with $O_N$ the integral closure of $O_K$ in $N$. Then $\mathfrak{m}_K$ decomposes into a product of primes $\mathfrak{q}_i$ of $O_N$ as

$$\mathfrak{m}_K O_N = \prod_{i=1}^{r} \mathfrak{q}_i^{e_i}.$$

Each $\mathfrak{q}_i|\mathfrak{m}_K$ has a residue degree $f_i = [O_N/\mathfrak{q}_i : O_K/\mathfrak{m}_K]$. Let $\hat{N}_i$, $\hat{K}$ be the completions of $N$, $K$ in the topologies induced by $\mathfrak{q}_i$, $\mathfrak{m}_K$, respectively.

$$\begin{array}{ccc} N & \longrightarrow & \hat{N}_i \\ \Big| & & \Big| \\ K & \longrightarrow & \hat{K} \end{array}$$

Figure 2.1: Extension and then completion.

**Proposition 2.1.13.** *For $\hat{N}_i$, $\hat{K}$ as given above,*

*i)* $[\hat{N}_i : K] = n_i = e_i f_i$;

*ii) The ramification index of $\hat{N}_i/\hat{K}$ is $e_i$, and the residue degree of $\hat{N}_i/\hat{K}$ is $f_i$;*

*iii) the canonical homomorphism $N \otimes_K \hat{K} \to \prod_i \hat{N}_i$ is an isomorphism.*

**Corollary 2.1.13.1.** *If $N/K$ is Galois with Galois group $G$, and if $G_{\mathfrak{q}_i}$ denotes the decomposition group of $\mathfrak{q}_i|\mathfrak{m}_K$, the extension $\hat{N}_i/\hat{K}$ is Galois with Galois group $G_{\mathfrak{q}_i}$.*

Thus $N \otimes_K \hat{K}$ is a $n = [N : K]$-dimensional vector space over $\hat{K}$. The following proposition shows the similar for $O_N \otimes_{O_K} \hat{O}_K$.

**Proposition 2.1.14.** *With the same hypotheses and notation as above, let $O_{N_i}$ be the ring of valuation with respect to $\mathfrak{q}_i|\mathfrak{m}_K$. Then the canonical homomorphism*

$$O_N \otimes_{O_K} \hat{O}_K \to \prod_i \hat{O}_{N_i}$$

*is an isomorphism.*

**Corollary 2.1.14.1.** *Let $M$ be a finitely generated $O_N$-module. Consider $M$ as an $O_K$-module via the natural inclusion $O_K \to O_N$. Let $\hat{M}$ denote the $\mathfrak{m}_K$-adic completion of $M$. Then*

$$\hat{M} \cong \prod_i \hat{M}_{\mathfrak{q}_i}.$$

**Higher Ramification Groups.** Let $L/K$ be a finite Galois extension of fields with compatible non-trivial discrete valuations. Let $G = \mathrm{Gal}(L/K)$, and assume that the residue field extension is *separable* with characteristic $p > 0$. Denote the corresponding rings of integers by $\mathcal{O}_L/\mathcal{O}_K$, maximal ideals by $\mathfrak{m}_L/\mathfrak{m}_K$, and residue fields $\lambda/\kappa$, respectively.

For integers $n \geq -1$, the $n$'th ramification group $G_n$ is defined by

$$G_n := \{\sigma \in G \mid \text{for all } x \in \mathcal{O}_L,\ \sigma(x) - x \in \mathfrak{m}_L^{n+1}\}.$$

Thus $G_{-1} = G$; $G_0 = I$ is the *inertia subgroup*, the kernel of the natural quotient

$$0 \to I \to G \to \mathrm{Gal}(\lambda/\kappa) \to 0.$$

Recall that the *wild inertia subgroup* in $I$ is $\mathrm{Gal}(L/L^t)$ where $L^t$ is the maximal tamely ramified subextension in $L/K$. Indeed, $G_1$ is the wild inertia group which is also a Sylow $p$-subgroup of $G_0$, where $G_0/G_1$ is finite cyclic of order $e_t$ prime to $p$ by the canonical injection into $k^\times$ given the choice of a uniformizer of $\mathcal{O}_L$.

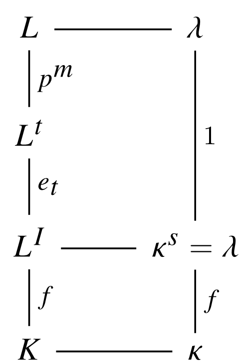



Figure 2.2: Subextensions with respect to higher ramification groups.

The ramification groups form a chain

$$G \trianglerighteq G_0 \trianglerighteq G_1 \trianglerighteq G_2 \trianglerighteq \cdots$$

of normal subgroups of $G$. For sufficiently big $n$, $G_n$ is trivial. The extension $L/K$ is called *weakly ramified* (*tamely ramified*, *unramified*), if $G_n$ is trivial for $n = 2$ ($n = 1$, $n = 0$, respectively) and $\lambda/\kappa$ is separable.

Let $O_N/O_K$ be an extension of Dedekind domains with a finite Galois extension $N/K$. For a non-zero prime ideal $\mathfrak{q}$ of $O_N$ and $\mathfrak{p} = \mathfrak{q} \cap O_K$, let $\hat{N}_{\mathfrak{q}}$, $\hat{K}_{\mathfrak{p}}$ denote the completions with respect to corresponding discrete valuations of $N$, $K$, respectively. We say $N/K$ is *weakly ramified* (*tamely ramified*, *unramified*) at $\mathfrak{q}$ if $\hat{N}_{\mathfrak{q}}/\hat{K}_{\mathfrak{p}}$ is *weakly ramified* (*tamely ramified*, *unramified*). If $N/K$ is *weakly ramified* (*tamely ramified*, *unramified*) at all non-zero prime ideal $\mathfrak{q} \in \operatorname{Spec} O_L$, then $N/K$ is called *weakly ramified* (*tamely ramified*, *unramified*).

**Hilbert's Formula.** Let $K$ be a complete discrete valued field under $v_L$ and $L/K$ finite Galois with $G = \operatorname{Gal}(L/K)$. Then the different ideal $\mathfrak{D}(L/K)$ can be determined by the ramification groups (cf. [Ser79], IV.1 Proposition 4).

**Proposition 2.1.15.** *If $\mathfrak{D}(L/K)$ denotes the different of $L/K$, then*

$$v_L \mathfrak{D}(L/K) = \sum_{i \geq 0} (|G_i| - 1).$$

**Normal Integral Basis.** Let $O_N/O_K$ be an extension of Dedekind domains with the extension of fields of fractions $N/K$ as before. Suppose $N/K$ is Galois with $G = \operatorname{Gal}(N/K)$. We saw that $O_N$ is a finitely generated $O_K$-module. If there is an element $\alpha \in O_N$ such that the set of its conjugates $\{\sigma(\alpha)\}_{\sigma \in G}$ forms a $O_K$-basis of $O_N$, we say $O_N$ has a *normal integral basis*. Thus having a normal integral basis means $O_N$ is a free $O_K[G]$-module. For a prime $\mathfrak{p} \in \operatorname{Spec} O_K$, we say $O_N$ is *free* at $\mathfrak{p}$ if the $\mathfrak{p}$-adic completion $\hat{O}_{N,\mathfrak{p}}$ is free over $\hat{O}_{K,\mathfrak{p}}[G]$. We say $O_N$ is *locally free* if it is free at all primes $\mathfrak{p}$ of $O_K$ simultaneously.

Assume further that $N/K$ is an extension of number fields, i.e., finite extensions of $\mathbb{Q}$. To be a free $O_K[G]$-module, it is certainly necessary that the module should be locally free. Noether's theorem (cf. [Noe32]) part of which goes back to Speicer (cf. [Spe16]) relates the existence of normal integral basis to ramification.

**Theorem 2.1.1** (Noether)**.** *$O_N$ is locally free over $O_K[G]$ if and only if $N/K$ is tame.*

## 2.2 Galois Modules

The term *Galois module* is used as a synonym for $G$-module, i.e., $\mathbb{Z}[G]$-module. Let $A$ be a ring and $M$ a finitely generated module over the group ring $A[G]$ for some finite group $G$. We say $M$ has a *normal integral basis* over $A$ if $M$ has elements $\alpha_1, ..., \alpha_n$ such that the set $\{\sigma(\alpha_1), ..., \sigma(\alpha_n)\}_{\sigma \in G}$ is an $A$-basis of $M$, i.e., $M$ is free over the group ring $A[G]$.

We say $M$ is *free* at $\mathfrak{p} \in \text{Spec } A$ if $M_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}[G]$-module. We say $M$ is *locally free* over $A$ if it is free at all $\mathfrak{p} \in \text{Spec } A$. When $A = \mathbb{Z}$, we simply say $M$ has a normal integral basis or is locally free without referring to $\mathbb{Z}$.

Local-global methods originated in number theory (Hasse principle) are extended to Galois module theory, and one of the most important results is Swan's theorem: Suppose $A$ is a Dedekind domain whose field of fractions has characteristic zero. Let $G$ be a finite group such that no rational prime dividing the order of $G$ is a unit in $A$. Then finitely generated projective $A[G]$-modules are locally free over $A[G]$ and also locally free after completion, i.e., for all non-zero prime $\mathfrak{p} \in \text{Spec } A$, $\hat{M}_{\mathfrak{p}}$ is is a free $\hat{A}_{\mathfrak{p}}[G]$-module ([Swa60]). The converse follows readily from Theorem 7.3.29 in [BK00]. Thus the terms ‘‘projective" and ‘‘locally free" will be used interchangeably throughout the rest of the text.

An important corollary to Swan's theorem (Corollary 6.4 in [Swa60]) is as following:

**Proposition 2.2.1.** *Let $G$ be a finite group, $A$ a local integral domain, $K$ its field of fractions. Let $P$, $Q$ be finitely generated projective modules over $A[G]$. If $P \otimes K \cong Q \otimes K$ as $K[G]$-modules, then $P \cong Q$.*

**Tate Cohomology Groups.** We first recall the basic definitions and results on the homology and cohomology of groups. A $G$-module $M$ is *projective* if the functor $Hom_G(M, \cdot)$ is exact, *injective* if the functor $Hom_G(\cdot, M)$ is exact.

The $G$-module $M$ is *induced* if it has the form $\mathbb{Z}[G] \otimes_{\mathbb{Z}} X$ for some abelian group $X$. Every

$G$-module is a quotient of an induced module, canonically given by the surjection

$$\mathbb{Z}[G] \otimes M_0 \to M$$

where $M_0$ is just $M$ with its abelian group structure. Dually, a $G$-module $M$ is called *co-induced* if it has the form $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], X)$ for some abelian group $X$. Each $G$-module $M$ embeds canonically in the co-induced $G$-module $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], M_0)$. When the group $G$ is assumed to be finite, the notions of induced and co-induced modules coincide, and this is our case.

Let $M^G$ be the submodule of $M$ consisting of the elements fixed by $G$; it is the largest submodule of $M$ on which $G$ acts trivially. If $f : M \to M'$ is a morphism of $G$-modules, then $f$ maps $M^G$ to $M'^G$, thus we can speak of the functor $M^G$. It is a left exact additive functor. By definition, the *right derived functors* of the functor $M^G$ are the cohomology groups of $G$ with coefficients in $M$, denoted by $\mathrm{H}^i(G, M)$, $i \geq 0$. Note that $M^G = \mathrm{Hom}_G(\mathbb{Z}, M)$ where $\mathbb{Z}$ is considered as a $G$-module with trivial action. Then $\mathrm{H}^i(G, M) = \mathrm{Ext}^i_{\mathbb{Z}[G]}(\mathbb{Z}, M)$ since $\mathrm{Ext}^i_{\mathbb{Z}[G]}(\mathbb{Z}, \cdot)$ are the derived functors of the functor $\mathrm{Hom}_G(\mathbb{Z}, \cdot) = \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, \cdot)$.

Let $DM$ be the subgroup of $M$ generated by $\sigma(x) - x$, $x \in M$, $\sigma \in G$. The quotient $M/DM$ will be denoted by $M_G$; it is the largest quotient module of $M$ on which $G$ acts trivially. The functor $M_G$ is a right exact additive functor. By definition, its *left derived functors* are the homology groups of $G$ with coefficients in $M$, denoted by $\mathrm{H}_i(G, M)$ for $i \geq 0$. We have $M_G = \mathbb{Z} \otimes_{\mathbb{Z}[G]} M$, hence $\mathrm{H}_i(G, M) = \mathrm{Tor}_i^{\mathbb{Z}[G]}(\mathbb{Z}, M)$ as the derived functors.

In the group algebra $\mathbb{Z}[G]$, the element $\sum_{\sigma \in G} \sigma$ will be called the *norm* and be denoted $N$. For every $G$-module $M$, $N$ defines an endomorphism of $M$ by the formula

$$N(x) = \sum_{\sigma \in G} \sigma(x).$$

Let $I_G$ denote the *augmentation ideal* of $\mathbb{Z}[G]$, the set of linear combinations of the $\sigma - 1$, $\sigma \in G$. Then obviously

$$I_G M \subset \mathrm{Ker}(N) \quad \text{and} \quad \mathrm{Im}(N) \subset M^G.$$

Since $\mathrm{H}_0(G,M) = M/I_G M$ and $\mathrm{H}^0(G,M) = M^G$, it follows that $N$ defines an induced homomorphism

$$N^* : \mathrm{H}_0(G,M) \to \mathrm{H}^0(G,M).$$

Suppose that $0 \to A \to B \to C \to 0$ is an exact sequence of $G$-modules. Then the diagram

$$\begin{array}{ccccccccc}
\mathrm{H}_1(G,C) & \longrightarrow & \mathrm{H}_0(G,A) & \longrightarrow & \mathrm{H}_0(G,B) & \longrightarrow & \mathrm{H}_0(G,C) & \longrightarrow & 0 \\
\downarrow & & \downarrow N_A^* & & \downarrow N_B^* & & \downarrow N_C^* & & \downarrow \\
0 & \longrightarrow & \mathrm{H}^0(G,A) & \longrightarrow & \mathrm{H}^0(G,B) & \longrightarrow & \mathrm{H}^0(G,C) & \longrightarrow & \mathrm{H}^1(G,A)
\end{array}$$

is commutative, and there is a canonical homomorphism

$$\mathrm{Ker}(N_C^*) \to \mathrm{CoKer}(N_A^*)$$

by the snake lemma. Moreover (cf. [CE56], V. 10), the above diagram gives a long exact sequence

$$\begin{aligned}
\cdots \to \mathrm{H}_1(G,C) \to \mathrm{Ker}(N_A^*) \to \mathrm{Ker}(N_B^*) \to \mathrm{Ker}(N_C^*) \\
\to \mathrm{CoKer}(N_A^*) \to \mathrm{CoKer}(N_B^*) \to \mathrm{CoKer}(N_C^*) \to \mathrm{H}^1(G,A) \to \cdots .
\end{aligned}$$

This leads to the Tate cohomology groups $\hat{\mathrm{H}}^i(G,M)$ which are defined as

$$\begin{aligned}
\hat{\mathrm{H}}^i(G,M) &= \mathrm{H}^i(G,M) \quad \text{if } i \geq 1 \\
\hat{\mathrm{H}}^0(G,M) &= \mathrm{Coker}(N^*) \\
\hat{\mathrm{H}}^{-1}(G,M) &= \mathrm{Ker}(N^*) \\
\hat{\mathrm{H}}^{-i}(G,M) &= \mathrm{H}_{i-1}(G,M) \quad \text{if } i \geq 2.
\end{aligned}$$

**Restriction and Corestriction.** If $H \leqslant G$ is a subgroup, a $G$-module $M$ inherits the natural $H$-module structure by restricting the group action to $H$. We write this $H$-module as $\mathrm{Res}_H^G(M)$. Clearly $M^G \subset M^H$, and this induces *restriction* homomorphisms

$$\mathrm{Res} : \mathrm{H}^i(G,M) \to \mathrm{H}^i(H,M).$$

Also this induced action of $H$ yields $M_H \to M_G$ which leads to *corestriction* homomorphisms

$$\mathrm{Cor} : \mathrm{H}_i(H,M) \to \mathrm{H}_i(G,M).$$

Now given a transversal of the left cosets $G/H$ in $G$, an element $x$ of $M^H$ can be mapped to $M^G$ by taking the *norm* of $x$,

$$\sum_{\sigma\in G/H} \sigma(x).$$

This induces *corestriction* homomorphisms

$$\mathrm{Cor} : \mathrm{H}^i(H, M) \to \mathrm{H}^i(G, M)$$

and *restriction* homomorphisms

$$\mathrm{Res} : \mathrm{H}_i(G, M) \to \mathrm{H}_i(H, M).$$

These naturally extend to the Tate cohomology groups as *restriction* homomorphisms

$$\mathrm{Res} : \hat{\mathrm{H}}^i(G, M) \to \hat{\mathrm{H}}^i(H, M)$$

and *corestriction* homomorphisms

$$\mathrm{Cor} : \hat{\mathrm{H}}^i(H, M) \to \hat{\mathrm{H}}^i(G, M).$$

We have the following restriction-corestriction formula (cf. [Ser79], VIII. 2. Proposition 4).

**Proposition 2.2.2.** *If $n = [G : H]$, then Cor $\circ$ Res $= n$.*

**Corollary 2.2.2.1.** *If $n$ is the order of $G$, then all the groups $\hat{\mathrm{H}}^i(G, M)$ are annihilated by $n$.*

**Shapiro's Lemma.** Let $M$ be an $H$-module, $H$ a subgroup of $G$. Then the *induced* and *coinduced* $G$-modules are defined by

$$\mathrm{Ind}_H^G(M) = \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M,$$
$$\mathrm{Coind}_H^G(M) = \mathrm{Hom}_H(\mathbb{Z}[G], M).$$

A fundamental tool in calculation is Shapiro's lemma (cf. [Wei94], Lemma 6.3.2).

**Proposition 2.2.3** (Shapiro)**.** $\mathrm{H}_i(G, \mathrm{Ind}_H^G(M)) \cong \mathrm{H}_i(H, M)$*; and* $\mathrm{H}^i(G, \mathrm{Coind}_H^G(M)) \cong \mathrm{H}^i(H, M)$.

**Proposition 2.2.4.** *If the index* $[G : H]$ *is finite,* $\mathrm{Ind}_H^G(M) \cong \mathrm{Coind}_H^G(M)$.

**Corollary 2.2.4.1.** *If* $M$ *is a projective* $G$*-module, then all Tate cohomology groups* $\hat{\mathrm{H}}^i(G, M)$ *vanish.*

**Double Coset Formula.** Another useful tool is Mackey's double coset formula (cf. [Ser77], 7.3 Proposition 22). Let $H_1$, $H_2$ be two subgroups of $G$. If $W$ is a representation of $H_1$, we set $V = \mathrm{Ind}_{H_1}^G(W)$. The double coset formula determines the restriction $\mathrm{Res}_{H_2}^G(V)$. Choose a set of representatives $S$ for the $(H_1, H_2)$ double cosets of $G$, i.e., $G$ is the disjoint union of $H_1 s H_2$, for $s \in S$. For $s \in S$, let $H_{1,s} = sH_1s^{-1} \cap H_2 \leqslant H_2$. Let $W_s$ be the $H_{1,s}$-module with the underlying set $W$ and the action of $x \in H_{1,s}$ given by $s^{-1}xs \in H_1$.

**Proposition 2.2.5** (Mackey)**.** $\mathrm{Res}_{H_2}^G \mathrm{Ind}_{H_1}^G(W) \cong \bigoplus_{s\in S} \mathrm{Ind}_{H_{1,s}}^{H_2}(W_s)$

**Projectivity and Cohomological Triviality** A $G$-module $M$ is called *cohomologically trivial* if, for every subgroup $H$ of $G$ and every $n \in \mathbb{Z}$, $\hat{\mathrm{H}}^n(H, M) = 0$. By Corollary 2.2.4.1, projective $\mathbb{Z}[G]$-modules are cohomologically trivial. A refined statement of the converse is given as Theorem 7 in [Ser79], IX. 5 (see also Proposition 1.3 in [Köc04]):

**Proposition 2.2.6.** *Let* $A$ *be a Dedekind domain and* $M$ *be a* $A[G]$*-module. Then* $M$ *is projective over* $A[G]$ *if and only if* $M$ *is projective over* $A$ *and cohomologically trivial.*

As a result, cohomological triviality and projectivity would often be equivalent in what follows. Checking cohomological triviality will be our main tool in finding locally free $\mathbb{Z}[G]$-modules.

Here are some general statements describing locally free Galois modules in terms of group cohomology. First we recall the notion of the projective dimension of a module. Given a module $M$, a *projective resolution* of $M$ is an infinite exact sequence of modules

$$\cdots \to P_n \to \cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

with all the $P_i$ projective. Every module possesses a projective resolution made out of free modules, but the resolution could be infinite. The length of a finite resolution is the subscript $n$ such that $P_n$ is non-zero and $P_i = 0$ for $i > n$. If $M$ admits a finite projective resolution, the *projective dimension* of $M$ is the minimal length among all finite projective resolutions of $M$.

**Proposition 2.2.7** ([Chi94]). *Suppose $A$ is a ring and $M$ is an $A[G]$-module.*

*i) If $A$ is a field, then $M$ is cohomologically trivial for $G$ if and only if $M$ is projective for $A[G]$.*

*ii) Suppose $A$ is a Dedekind domain. Then $M$ is cohomologically trivial for $G$ if and only if $M$ has projective dimension at most one as an $A[G]$-module.*

*iii) Suppose $A$ is a Dedekind domain and that the image of the natural morphism* $\operatorname{Spec} A \to \operatorname{Spec} \mathbb{Z}$ *contains the prime divisors of the order of $G$. A finitely generated $A[G]$-module $M$ is projective if and only if it is locally free.*

**Grothendieck Groups of $\mathbb{Z}[G]$-modules.** Our problems are often stated in terms of *Grothendieck groups*. Let $R$ be a ring and $\mathcal{C}$ a category of left $R$-modules. The *Grothendieck group* of $\mathcal{C}$ is the abelian group defined by generators and relations as follows: A generator $[A]$ associated with each $A \in \mathcal{C}$, and the relation $[B] = [A] + [C]$ is associated with each exact sequence

$$0 \to A \to B \to C \to 0 \quad \text{where } A, B, C \in \mathcal{C}.$$

The two most common examples are the Grothendieck group $\mathrm{K}_0(G, \mathbb{Z})$ of finitely generated $\mathbb{Z}[G]$-modules and the Grothendieck group $\mathrm{K}_0(\mathbb{Z}[G])$ of finitely generated *projective* $\mathbb{Z}[G]$-modules. When working with a category of $G$-modules, elements of Grothendieck groups are also called *virtual representations*. For example, given an exact sequence of modules

$$0 \to M_1 \to M_2 \to M_2 \to M_3 \to M_4 \to \cdots \to M_n \to 0$$

in any of the above examples of Grothendieck groups, we have a corresponding virtual representation given by an alternating sum

$$[M_1] = [M_2] - [M_3] + [M_4] - [M_5] + \cdots + (-1)^n [M_n].$$

An equality in $\mathrm{K}_0(\mathbb{Z}[G])$ gives a stable isomorphism: If $[M] = [M']$ in $\mathrm{K}_0(\mathbb{Z}[G])$, there is $n \geq 0$, $M \oplus \mathbb{Z}[G]^{\oplus n} \cong M' \oplus \mathbb{Z}[G]^{\oplus n}$. The Swan-Jacobinski theorem states that whenever the Eichler condition is satisfied (cf. p. 178 in [Swa86]; see also p. 50 in [Frö83]), stable isomorphism implies isomorphism, i.e., ‘‘cancellation” holds. This is the case whenever none of the simple components of $\mathbb{Q}[G]$ are totally definite quaternion division algebras over a totally real field, e.g., for $G$ of odd order or $G$ abelian. There are more groups with the cancellation law, but it does not hold for all groups (e.g. $G = H_{32}$, see [Swa79]). Nevertheless, cancellation often holds, and knowing that $[M]$ is equal to a free $\mathbb{Z}[G]$-module in $\mathrm{K}_0(\mathbb{Z}[G])$ is a strong approximation to the existence of a normal integral basis.

Motivated by this, we consider the *reduced* Grothendieck group $\mathrm{K}_0(\mathbb{Z}[G])^{\mathrm{red}}$ which is the quotient of $\mathrm{K}_0(\mathbb{Z}[G])$ by the subgroup generated by $[\mathbb{Z}[G]]$. By Proposition 2.2.7, $\mathrm{K}_0(\mathbb{Z}[G])^{\mathrm{red}}$ is identified with the *locally free class group* $\mathrm{Cl}(\mathbb{Z}[G])$ of $\mathbb{Z}[G]$ which is obtained as following: Let $N$ be a number field and $\mathfrak{A}$ an order in $N[G]$, i.e., a subring of $N[G]$ with $1 \in \mathfrak{A}$. A *locally free* $\mathfrak{A}$-module $M$ is a finitely generated $\mathfrak{A}$-module so that, for all prime divisors $\mathfrak{p}$ of $N$, the $\mathfrak{A}_\mathfrak{p}$-module $M_\mathfrak{p}$ is free. Its rank $r(M)$ is defined as the rank of the free $N[G]$-module $M \otimes_{O_N} N$ spanned by $M$. This rank is finite, and is also the rank of $M_\mathfrak{p}$ over $O_\mathfrak{p}[G]$ for all $\mathfrak{p}$.

The Grothendieck group $\mathfrak{K}_0(\mathfrak{A})$ of locally free $\mathfrak{A}$-modules is the abelian group with generators $[M]$ corresponding to the $\mathfrak{A}$-isomorphism classes of locally free $\mathfrak{A}$-modules $M$ and with relations

$$[M_1 \oplus M_2] = [M_1] + [M_2].$$

The map $\mathbb{N} \to \mathfrak{K}_0(\mathfrak{A})$ which takes $n$ into the class $[\mathfrak{A}^{\oplus n}]$ extends to a homomorphism $\mathbb{Z} \to \mathfrak{K}_0(\mathfrak{A})$, and we define the locally free class group $\mathrm{Cl}(\mathfrak{A})$ to be its cokernel. For an alternative description of $\mathrm{Cl}(\mathfrak{A})$ called "Hom description", see I.2, [Frö83].

The class group is known to be a finite abelian group, and we will denote the image of $[M] \in \mathrm{K}_0(\mathbb{Z}[G])$ in $\mathrm{Cl}(\mathbb{Z}[G])$ by $(M)_{\mathbb{Z}[G]}$.

The natural forgetful functor

$$\mathrm{K}_0(\mathbb{Z}[G]) \to \mathrm{K}_0(G, \mathbb{Z})$$

called the Cartan homomorphism. This morphism is neither injective nor surjective in general. When studying an arbitrary finitely generated $G$-module $M$, knowing that $[M] \in \mathrm{K}_0(G, \mathbb{Z})$ is in the image of the Cartan homomorphism is a huge advantage in analyzing its Galois module structure.

Given a ring $A$, the Grothendieck groups $\mathrm{K}_0(A[G])$ and $\mathrm{K}_0(G, A)$ are defined similarly. We also mention that $\mathrm{CT}(A[G])$ denotes the Grothendieck group of finitely generated $A[G]$-modules which are cohomologically trivial. When $A$ is a Dedekind domain, the forgetful homomorphism $\mathrm{K}_0(A[G]) \to \mathrm{CT}(A[G])$ is an isomorphism by Proposition 2.2.7.

**More on Cohomological Triviality.** Here we list a couple more theorems on cohomology for a finite group which provide background to Proposition 2.2.6 and 2.2.7 (see [Ser79] for proofs and more).

**Proposition 2.2.8.** *Let $G$ be a $p$-group and let $M$ be a $G$-module without $p$-torsion. The following conditions are equivalent:*

*i) $M$ is cohomologically trivial for $G$,*

*ii) $\hat{\mathrm{H}}^i(G, M) = 0$ for two consecutive values of $i$,*

*iii) the $\mathbb{F}_p[G]$-module $M/pM$ is free.*

**Proposition 2.2.9.** *Let $G$ be a finite group, $M$ a $\mathbb{Z}$-free $G$-module, and $G_p$ a Sylow $p$-subgroup of $G$ for each prime number $p$. The following conditions are equivalent:*

*i) $M$ is $\mathbb{Z}[G]$-projective,*

*ii) For every prime number $p$, the $G_p$-module $M$ satisfies the equivalent conditions of Proposition 2.2.8.*

## 2.3 Algebraic Geometry

We introduce basic notions in algebraic geometry required for describing geometric Galois modules and Euler characteristics of equivariant sheaves.

**Quotient Scheme by a Finite Group.** Let $X$ be a scheme and $G \subseteq \mathrm{Aut}(X)$ a finite group acting on $X$. We are interested in when we can form the quotient scheme $X/G$. The following statement can be found in [Mum70].

**Proposition 2.3.1.** *Suppose the orbit of any point is contained in an affine open subset of $X$. Then there is a pair $(Y, \pi)$ where $Y$ is a scheme and $\pi : X \to Y$ is a morphism such that*

*(i) As a topological space, $(Y, \pi)$ is the quotient of $X$ for the action of $G$;*

*(ii) the morphism $\pi : X \to Y$ is $G$-invariant, and if $(\pi_* \mathcal{O}_X)^G$ denotes the subsheaf of $\pi_* \mathcal{O}_X$ of $G$-invariant functions, the natural homomorphism $\mathcal{O}_Y \to (\pi_* \mathcal{O}_X)^G$ is an isomorphism.*

*The pair $(Y, \pi)$ is uniquely determined up to isomorphism by these conditions. The morphism $\pi$ is finite and surjective. We denote $Y$ by $X/G$, and it has the functorial property: for any $G$-invariant morphism $f : X \to Z$, there is a unique morphism $g : Y \to Z$ such that $f = g \circ \pi$.*

In particular, if $X \to \operatorname{Spec} \mathbb{Z}$ is projective, then the quotient scheme $X/G$ exists. Throughout the text, we assume that a scheme $X$ with a finite group action $G$ admits a quotient scheme $\pi : X \to Y$.

We list a few corollaries on properties of $\pi$ and $X/G$.

**Corollary 2.3.1.1.** *The quotient map $\pi : X \to Y$ is open.*

*Proof.* Suppose $U \subseteq X$ is open. Then $\pi(U)$ is open if and only if $\pi^{-1}(\pi(U))$ is open. The later is just the union of the orbits of $U$ under $G$. □

**Corollary 2.3.1.2.** *If $X$ is of finite type over $\mathbb{Z}$ and $Y = X/G$ exists, then both $X$ and $Y$ are noetherian.*

*Proof.* Since the structure morphism $X \to \operatorname{Spec} \mathbb{Z}$ is of finite type and $\operatorname{Spec} \mathbb{Z}$ is noetherian, $X$ is also noetherian. We check that $Y$ is quasi-compact and locally noetherian. Suppose $\{V_i\}_{i \in I}$ is an open covering of $Y$. This gives an open covering $\{U_i = \pi^{-1}(V_i)\}_{i \in I}$ of $X$, and since $X$ is quasi-compact, we can choose a finite subcovering $\{U_i\}_{i \in J}$. Since $\pi$ is open and surjective with

$V_i = \pi(U_i)$, $\{V_i\}_{i\in J}$ is then a finite subcovering of $\{V_i\}_{i\in I}$. Also for each $y \in Y$, there is an open affine neighborhood Spec $B$ such that $\pi^{-1}(\text{Spec } B) = \text{Spec } A$ is noetherian and $B = A^G \subset A$. Since $\pi$ is finite, by the theorem of Eakin-Nagata (Theorem 3.7(i) in [Mat87]), $B$ is also noetherian, hence $Y$ is locally noetherian. □

**Corollary 2.3.1.3.** *Suppose $X$ is of finite type over $\mathbb{Z}$ and $Y = X/G$ exists. If $\mathscr{F}$ is a coherent sheaf of $\mathcal{O}_X$-modules, then $\pi_*\mathscr{F}$ is a coherent sheaf of $\mathcal{O}_Y$-modules.*

*Proof.* This follows from that $X$, $Y$ are both noetherian and $\pi$ is a finite morphism (see II. Ex. 5.5 in [Har77]). □

**Purity of Branch Locus.** Suppose $x \in X$ maps to $y \in Y = X/G$ under the canonical projection $\pi : X \to Y$. Denote the corresponding extension of local rings by $\mathcal{O}_{X,x}/\mathcal{O}_{Y,y}$, maximal ideals by $\mathfrak{m}_x/\mathfrak{m}_y$. Throughout the text, we assume that all residue field extensions are separable. The extension of local rings $\mathcal{O}_{X,x}/\mathcal{O}_{Y,y}$ is *unramified* if $\mathfrak{m}_y\mathcal{O}_{X,x} = \mathfrak{m}_x$ in $\mathcal{O}_{X,x}$. This coincides with the previously defined notion of unramified extension if $\mathcal{O}_{X,x}/\mathcal{O}_{Y,y}$ are discrete valuation rings. The projection $\pi$ is *unramified* at $x$ if $\mathcal{O}_{X,x}/\mathcal{O}_{Y,y}$ is unramified.

Let $U$ be the largest open subscheme of $X$ such that the restriction $\pi|_U$ is *étale*, i.e., $\pi$ is flat and unramified at every point in $U$. The complement $X - U$ is naturally equipped with the closed subscheme structure defined by the annihilator of the sheaf $\Omega^1_{X/Y}$ of relative differentials, and we call this subscheme $R_\pi$ the *ramification locus of* $\pi$. Since $\pi$ is finite, the image $\pi(R_\pi)$ is closed, and its unique reduced induced closed subscheme structure is called $B_\pi$ the *branch locus of* $\pi$. If both $X$ and $Y$ are regular, then $\pi$ is flat (Remark 3.11 in [Liu02]).

Often, the irreducible components of $R_\pi$ (resp. $B_\pi$) are all of codimension 1. This property is called *purity of ramification* (resp. *branch*) locus.

**Proposition 2.3.2** (Purity of branch locus)**.** *Let $f : X \to Y$ be a morphism of locally noetherian schemes. Let $x \in X$ and set $y = f(x)$. Suppose*

*i) $\mathcal{O}_{X,x}$ is normal;*

*ii)* $\mathcal{O}_{Y,y}$ *is regular;*

*iii)* $f$ *is quasi-finite at* $x$*;*

*iv)* $dim(\mathcal{O}_{X,x}) = dim(\mathcal{O}_{Y,y}) \geq 1$*;*

*v) for all* $x'$ *that specialize to* $x$ *with* $dim(\mathcal{O}_{X,x'}) = 1$*,* $f$ *is unramified at* $x'$*.*

*Then* $f$ *is étale at* $x$*.*

**Equivariant Euler Characteristic.** Let $X$ be a scheme over a noetherian ring $A$ with $G \subset \mathrm{Aut}_A(X)$ a finite group. An $\mathcal{O}_X$-$G$-*module* $\mathcal{F}$ on $X$ (or $G$-*equivariant* sheaf) is a sheaf of $\mathcal{O}_X$-modules having an action of $G$ which is compatible with the action of $G$ on $\mathcal{O}_X$ in the following sense: Suppose $x \in X$ and $\sigma \in G$. Let $\sigma(x)$ be the image of $x$ under $\sigma$. The action of $\sigma$ on $\mathcal{O}_X$ and $\mathcal{F}$ gives homomorphisms of stalks $\mathcal{O}_{X,\sigma(x)} \to \mathcal{O}_{X,x}$ and $\mathcal{F}_{\sigma(x)} \to \mathcal{F}_x$; both of these homomorphisms will also be denoted by $\sigma$, and $\sigma(a \cdot m) = \sigma(a) \cdot \sigma(m)$ for all $a \in \mathcal{O}_{X,\sigma(x)}$ and $m \in \mathcal{F}_{\sigma(x)}$. If such $\mathcal{F}$ is a quasi-coherent (resp. coherent, locally free) $\mathcal{O}_X$-module sheaf, then $\mathcal{F}$ is called a quasi-coherent (resp. coherent, locally free) $\mathcal{O}_X$-$G$-module. An example of a locally free $\mathcal{O}_X$-$G$-module of rank 1 on $X$ can be given as $\mathcal{O}_X(D)$ where $D = \sum n_Z \cdot Z$ is a $G$-equivariant divisor on $X$, i.e., $n_{\sigma(Z)} = n_Z$ for all $\sigma \in G$ and $Z \in D$.

If $\mathcal{F}$ is an $\mathcal{O}_X$-$G$-module on $X$, then the sheaf cohomology groups $\mathrm{H}^i(X, \mathcal{F})$, $i \geq 0$, are $\mathbb{Z}[G]$-modules in a natural way. Suppose each cohomology group is a finitely generated $\mathbb{Z}[G]$-module. This holds if, for instance, $\mathcal{F}$ is a coherent $\mathcal{O}_X$-$G$-module on $X$ and $X$ is projective over Spec $A$ (III. Theorem 5.2 in [Har77]). We define the *equivariant Euler characteristic* $\chi(X, \mathcal{F}) = \chi(G, X, \mathcal{F})$ to be the virtual representation

$$\chi(X, \mathcal{F}) := \sum (-1)^i [\mathrm{H}^i(X, \mathcal{F})]$$

in the Grothendieck group $\mathrm{K}_0(G, A)$ of all finitely generated modules over the group ring $A[G]$.

Now assume that $X$ is projective over Spec $A$. As seen before, the quotient $\pi : X \to Y$ then exists, $Y = X/G$. We will call such quotient a *Galois cover*. An $\mathcal{O}_Y[G]$-*module* $\mathcal{G}$ on $Y$ is just a $\mathcal{O}_Y$-$G$-module on $Y$ where $G$ acts trivially on $Y$. In most of our applications, $\mathscr{G} = \pi_*\mathscr{F}$ for a coherent $\mathcal{O}_X$-$G$-module $\mathscr{F}$. In this case, since $\pi$ is finite and thus affine, $\mathrm{H}^i(X, \mathscr{F}) = \mathrm{H}^i(Y, \pi_*\mathscr{F})$ for all $i \in \mathbb{Z}$, and $\pi_*\mathscr{F}$ is coherent by Corollary 5, therefore $\chi(X, \mathscr{F}) = \chi(Y, \pi_*\mathscr{F})$ in $\mathrm{K}_0(G, \mathbb{Z})$.

By an $A[G]$-*module on* $Y$ we mean a sheaf of $A[G]$-modules on $Y$. The category of $A[G]$-modules on $Y$ has enough injectives (cf. [Chi94]). We say the cohomology of $\mathscr{F}$ has a *normal integral basis* over $A$ if there exists a bounded complex of finitely generated free $A[G]$-modules that is isomorphic to $\mathrm{R}\Gamma(X, \mathscr{F})$ in the derived category of complexes of $A[G]$-modules. An equality between $\chi(X, \mathscr{F})$ and a multiple of $[A[G]]$ in $\mathrm{K}_0(G, A)$ is often too weak to study the existence of normal integral basis, thus one hopes to have a better approximation in the Grothendieck group $\mathrm{K}_0(A[G])$ of finitely generated projective $A[G]$-modules. As noted in 2.2, the Cartan homomorphism

$$\mathrm{K}_0(A[G]) \to \mathrm{K}_0(G, A)$$

is not surjective in general, and the foundation of our study is based on characterizing $\mathcal{O}_X$-$G$-modules $\mathscr{F}$ that admit a *projective Euler characteristic* $\chi^P(X, \mathscr{F})$ in $\mathrm{K}_0(A[G])$ which will be introduced in the next chapter.

# CHAPTER 3

# IMPORTANT THEOREMS

In this chapter, we recall two important results, Chinburg's criterion for the existence of projective Euler characteristics and Köck's classification of projective fractional ideals in weakly ramified extension of discrete valuation rings.

## 3.1 Projective Euler Characteristic

Let $\pi : X \to Y$ be a Galois cover over a noetherian ring $A$ with a finite group $G$ as in 2.3.

$$\begin{array}{l} X \circlearrowleft G \\ \downarrow \pi \\ Y \circlearrowleft G \text{ trivial} \\ \downarrow \\ \operatorname{Spec} A \end{array}$$

Figure 3.1: Galois cover.

Let $\mathrm{K}^+(Y, A, G)$ (resp. $\mathrm{K}^+(A, G)$) be the category of complexes of $A[G]$-modules on $Y$ (resp. $A[G]$-modules) which are bounded below. Morphisms in these categories are homotopy classes of morphisms of complexes. A morphism is a quasi-isomorphism if it induces isomorphisms in cohomology. The *derived categories* $\mathrm{D}^+(Y, A, G)$ and $\mathrm{D}^+(A, G)$ are the localizations of $\mathrm{K}^+(Y, A, G)$ and $\mathrm{K}^+(A, G)$, respectively, with respect to the multiplicative systems of quasi-isomorphisms in these categories. There are enough injectives in the category of sheaves of $A[G]$-modules on $Y$ (III. Proposition 2.2 in [Har77]; also [Chi94]). Hence the global section functor $\Gamma$ has a right-derived functor $\mathrm{R}\Gamma^+ : \mathrm{D}^+(Y, \mathbb{Z}, G) \to \mathrm{D}^+(\mathbb{Z}, G)$.

**Theorem 3.1.1** (Chinburg, Theorem 1.1 in [Chi94])**.** *Suppose $\mathscr{G}^\bullet \in \mathrm{K}^+(Y, A, G)$ has the following properties:*

*i) $\mathscr{G}^\bullet$ is a bounded complex of $A[G]$-modules on $Y$,* *ii) each term of* $\mathscr{G}^\bullet$ *is a quasi-coherent* $O_Y[G]$*-module,*

*iii) each stalk of each term of* $\mathscr{G}^\bullet$ *is a* $G$*-module which is cohomologically trivial for* $G$*, and*

*iv) the cohomology groups of* $\mathrm{R}\Gamma^+(\mathscr{G}^\bullet)$ *are finitely generated* $A[G]$*-modules.*

*Then* $\mathrm{R}\Gamma^+(\mathscr{G}^\bullet)$ *is isomorphic in* $\mathrm{D}^+(A,G)$ *to a bounded complex* $P^\bullet$ *of finitely generated* $A[G]$*-modules which are cohomologically trivial for* $G$*. The Euler characteristic* $\chi(P^\bullet) = \sum(-1)^i[P^i] \in \mathrm{CT}(A[G])$ *depends only on* $\mathscr{G}^\bullet$*, and will be denoted* $\chi\mathrm{R}\Gamma^+(\mathscr{G}^\bullet)$*.*

By Proposition 2.2.7, the natural forgetful homomorphism $\mathrm{K}_0(A[G]) \to \mathrm{CT}(A[G])$ is an isomorphism when $A$ is a Dedekind domain. We call the preimage of $\chi R\Gamma^+(\mathscr{G}^\bullet)$ under the forgetful homomorphism the *projective Euler characteristic* of $\mathscr{G}$ denoted by $\chi^P(Y,\mathscr{G})$. Indeed, modules of a bounded complex $P^\bullet$ of projective finitely generated $A[G]$-modules quasi-isomorphic to $\mathrm{R}\Gamma^+(\mathscr{G}) = \mathrm{R}\Gamma(Y,\mathscr{G})$ can be chosen to be free except the last term. The class of the last module in $\mathrm{Cl}(A[G])$ is $\pm$ the obstruction $\overline{\chi}^P(Y,\mathscr{G})$ to the existence of a normal integral basis, i.e., to $\mathrm{R}\Gamma(Y,\mathscr{G})$ being represented by a perfect complex of $A[G]$-modules. In [CE92], Chinburg and Erez showed that when $\pi : X \to Y$ is tame in the sense that the order of the inertia subgroup at every closed point $x \in X$ is relatively prime to the characteristic of the residue field, then for all $G$-equivariant coherent sheaves $\mathscr{F}$ on $X$, the projective Euler characteristic $\chi^P(X,\mathscr{F}) = \chi^P(Y,\pi_*\mathscr{F})$ is well-defined.

## 3.2 Weakly Ramified Extensions

Recall that when a Galois extension of number fields $N/K$ with Galois group $G$ is tame, then $O_N$ is projective over $O_K[G]$ by Noether's criterion. An addition to that, in [Ull70], Ullom showed that when $N/K$ is tame, all ambiguous ideals in $N$ (fractional ideals of $N$ that are $G$-modules) are indeed projective over $O_K[G]$. The same question can be asked for weakly ramified Galois extensions.

Let $L/K$ be a finite Galois extension of local fields with $G = \mathrm{Gal}(L/K)$ and positive residue characteristic $p$, and write $O_L/O_K$, $\mathfrak{m}_L/\mathfrak{m}_K$, $\lambda/\kappa$ for the corresponding extensions of discrete valuation rings, maximal ideals, and residue fields as in 2.1. In [Köc04], Köck classified all ambiguous ideals in the local field $L$ that are free over $O_K[G]$.

**Theorem 3.2.1** (Köck, Theorem 1.1 in [Köc04]). *Let $b \in \mathbb{Z}$. Then the fractional ideal $\mathfrak{m}_L^b$ of $L$ is free over $O_K[G]$ if and only if $L/K$ is weakly ramified and $b \equiv 1 \mod |G_1|$.*

An important canonical example of a locally free ambiguous ideal in a weakly ramified Galois extension $N/K$ of number fields is the *square root of the inverse different* $A(N/K)$ considered by Erez in [Ere91]. Here we assume $[N : K]$ is odd. By Proposition 2.1.15, the order of the different ideal $\mathfrak{D}(N/K)$ at a prime $\mathfrak{p}$ in $N$ can be calculated in terms of the orders of the higher ramification groups as

$$v_{\mathfrak{p}}(\mathfrak{D}(N/K)) = \sum_{i \geq 0}(|G_{\mathfrak{p},i}| - 1)$$

where $G_{\mathfrak{p}}$ is the decomposition group

$$G_{\mathfrak{p}} = \{\sigma \in G \mid \sigma(\mathfrak{p}) = \mathfrak{p}\}.$$

When $N/K$ is further assumed to be weakly ramified so that all $|G_{\mathfrak{p},2}| = 1$, this shows that there is an ideal $A(N/K)$ in $N$ with

$$A(N/K)^2 = \mathfrak{D}(N/K)^{-1}$$

which justifies the name. Since $\mathfrak{D}(N/K)$ is stable under the action of $G$, so is $A(N/K)$. By the criterion of Köck, $A(N/K)$ is projective over $O_K[G]$.

# CHAPTER 4

# WEAKLY RAMIFIED COVERS OF CURVES

## 4.1 Weakly Ramified Covers

Let $A$ be a Dedekind domain of characteristic 0 with a finite flat structure morphism Spec $A \to$ Spec $\mathbb{Z}$. Let $X$ be a proper flat regular curve over $A$, i.e., $X$ has dimension 2 and all fibres have dimension 1. By Theorem 3.16, § 8.3 in [Liu02], $X$ is projective. Thus, for a finite group action $G \subset \mathrm{Aut}_A(X)$, we have a Galois cover $\pi : X \to Y$ with the quotient scheme $Y = X/G$ in the sense of 2.3. For each $x \in X$ and $y = \pi(x)$, we further assume that the residue field extension $k(x)/k(y)$ is separable. The decomposition group $G_x = \{\sigma \in G \mid \sigma(x) = x\}$ of $x$ acts on $\mathcal{O}_{X,x}$ and on the stack $\mathscr{F}_x$ for any $\mathcal{O}_X$-$G$-module $\mathscr{F}$ on $X$. If $\mathcal{O}_{X,x}/\mathcal{O}_{Y,y}$ is an extension of discrete valuation rings, for each integer $i \geq -1$, the $i$'th ramification group $G_{x,i}$ at $x$ is well-defined for all $i \geq -1$. Recall that an extension $\mathcal{O}_{X,x}/\mathcal{O}_{Y,y}$ is called *weakly ramified* if $G_{x,2}$ is trivial.

We will say that the Galois cover $\pi : X \to Y$ of curves over Spec $A$ is *weakly ramified* if the following extra conditions are met:

i) $Y$ is also regular,

ii) the branch locus $B_\pi$ of $\pi$ is horizontal i.e., each irreducible component of $B_\pi$ surjects onto Spec $A$ via the structure morphism,

iii) for each rational prime $p$ dividing the order of $G$ and each $\mathfrak{p} \in$ Spec $A$ over $p$ via the structure morphism Spec $A \to$ Spec $\mathbb{Z}$, the special fibre $Y_\mathfrak{p}$ intersects each irreducible component of $B_\pi$ transversally and at smooth points, and

iv) at all such smooth points $y \in Y_\mathfrak{p}$ and $x \in X_\mathfrak{p}$ which $\pi(x) = y$, the extension of discrete valuation rings $\mathcal{O}_{X_\mathfrak{p},x}/\mathcal{O}_{Y_\mathfrak{p},y}$ is weakly ramified.

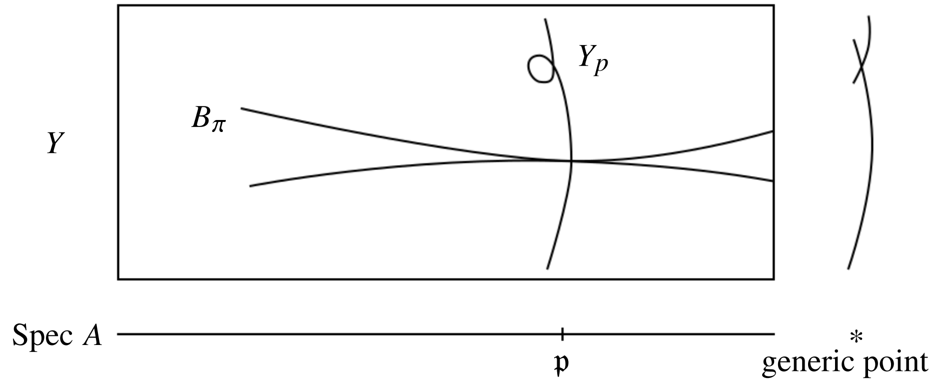



Figure 4.1: Weakly ramified Galois cover of curves.

**Remark.** Our definition of weakly ramified Galois cover is still *tamely ramified* relative to $B_\pi$ in the sense of Definition 2.1 in [Chi94] because of the discrete valuation condition: Since the branch locus $B_\pi$ is assumed to be horizontal and $X, Y$ are regular, for each $y \in B_\pi$ of codimension 1 and $x \in X$ over $y$, $\mathcal{O}_{X,x}/\mathcal{O}_{Y,y}$ is a tamely ramified extension of discrete valuation rings. However, we do not require $B_\pi$ to have normal crossings which is an additional crucial condition for tame covers given in Definition 2.5 in Chinburg's paper.

Our first main result is a sufficient condition for an invertible $\mathcal{O}_X$-$G$-module $\mathscr{F}$ to have a well-defined projective Euler characteristic $\chi^P(X, \mathscr{F})$.

**Theorem 4.1.1** (L, 2020)**.** *Let $\pi : X \to Y$ be a weakly ramified cover of curves over $A$ with a finite action of $G \subset \mathrm{Aut}_A(X)$. Suppose $\mathscr{F} = \mathcal{O}_X(D)$ is an invertible sheaf $\mathcal{O}_X$-$G$-module on $X$ corresponding to a $G$-equivariant horizontal divisor $D$. Assume that, for each $p$ dividing the order of $G$ and $\mathfrak{p} \in$ Spec $A$ over $p$, the pull-back $D \cap X_\mathfrak{p}$ to the special fibre $X_\mathfrak{p}$ is given by a Weil divisor $\sum_{x \in X_\mathfrak{p}} n_x \cdot x$ of $X_\mathfrak{p}$ where $n_x \equiv -1 \mod |G_{x,1}|$. (If $X_\mathfrak{p}$ is not smooth at $x$, take $|G_{x,1}|$ to be $1$.) Then each stalk of the direct image sheaf $\mathscr{G} = \pi_* \mathscr{F}$ is cohomologically trivial for $G$.*

*Proof.* Take $\mathscr{G} = \pi_* \mathscr{F}$ and let $y = \pi(x) \in Y$ for some $x \in X$. If $\pi$ is tamely ramified at $x$, then by Theorem 2.7 in [Chi94], $\mathscr{G}_y$ is cohomologically trivial for $G$. Thus it remains to show cohomological triviality at $y \in B_\pi$ of codimension 2, over $(p) \in \mathrm{Spec}\, \mathbb{Z}$ where $p$ divides the order of $G$. Notice that since $\mathscr{G}_y$ is a flat module over $\mathbb{Z}$ and thus torsion-free, by Proposition 2.2.9, it is enough to show that, for all rational primes $q > 0$, $\mathscr{G}_y$ is cohomologically trivial for the Sylow $q$-subgroup $G_q$ of $G$. For $q$ not dividing the order of $G$, $G_q$ is trivial and cohomological triviality is obvious. For $q \neq p$ dividing the order of $G$, $q$ is a unit in $O_{Y,y}$ inducing an automorphism

$$\hat{\mathrm{H}}^i(H, \mathscr{G}_y) \overset{q}{\cong} \hat{\mathrm{H}}^i(H, \mathscr{G}_y)$$

for all subgroups $H \leq G_q$. Since the order of a finite group annihilates its Tate cohomology groups by Corollary 2.2.2.1, $\mathscr{G}_y$ is cohomologically trivial for $G_q$. The remaining case is for $G_p$, the Sylow $p$-group, so suppose $G = G_p$. We have a short exact sequence

$$0 \longrightarrow \mathscr{G}_y \xrightarrow{p} \mathscr{G}_y \longrightarrow \mathscr{G}_y / p\mathscr{G}_y \longrightarrow 0$$

which induces a long exact sequence of the Tate cohomology groups

$$\cdots \to \hat{\mathrm{H}}^i(H, \mathscr{G}_y) \xrightarrow{p} \hat{\mathrm{H}}^i(H, \mathscr{G}_y) \to \hat{\mathrm{H}}^i(H, \mathscr{G}_y / p\mathscr{G}_y) \to \hat{\mathrm{H}}^{i+1}(H, \mathscr{G}_y) \to \cdots$$

for all subgroups $H \leqslant G$. Since each $\hat{\mathrm{H}}^i(H, \mathscr{G}_y)$ is annihilated by a power of $p$, as in the previous case, it is enough to show that $\mathscr{G}_y / p\mathscr{G}_y$ is cohomologically trivial for $G$. This can be done by restricting $\mathscr{G}_y$ to the special fibre $Y_\mathfrak{p}$. The special fibres $X_\mathfrak{p}$, $Y_\mathfrak{p}$ form a weakly ramified cover of curves $\pi_\mathfrak{p} : X_\mathfrak{p} \to Y_\mathfrak{p}$ with its ramification locus (resp. branch locus) exactly induced by the intersection of $R_\pi$ (resp. $B_\pi$) and $X_\mathfrak{p}$ (resp. $Y_\mathfrak{p}$) in $X$ (resp. $Y$) because $\Omega^1_{X_\mathfrak{p}/Y_\mathfrak{p}} = \Omega^1_{X/Y} \otimes_{O_X} \mathbb{F}_p$. If $x' \in X_\mathfrak{p}$ maps to $x$ and $y' \in Y_\mathfrak{p}$ maps to $y$ via base change, by assumption, $X_\mathfrak{p}$ and $Y_\mathfrak{p}$ are

$$\begin{array}{ccc} X_\mathfrak{p} & \xrightarrow{i} & X \\ \downarrow{\scriptstyle \pi_\mathfrak{p}} & & \downarrow{\scriptstyle \pi} \\ Y_\mathfrak{p} & \xrightarrow{i} & Y \end{array}$$

Figure 4.2: Galois cover of special fibres.

smooth at $x'$, $y'$. Taking the $\mathfrak{p}$-adic completion, we have a weakly ramified extension of discrete valuation rings $\hat{O}_{X_\mathfrak{p},x'} / \hat{O}_{Y_\mathfrak{p},y'}$. Say $\hat{m}_{x'}$ is the maximal ideal of $\hat{O}_{X_\mathfrak{p},x'}$. Again by the assumption,

$i^*\hat{\mathscr{F}}_{x'} = \hat{m}_{x'}^b$ where $b = -n_{x'} \equiv 1 \mod |G_{x',1}|$. Therefore by Theorem 3.2.1, $i^*\hat{\mathscr{F}}_{x'}$ is a free $\hat{O}_{Y_p,y'}[G_{x'}]$-module.

Denote $i^*\mathscr{F}$ and $i^*\mathscr{G}$ by $\mathscr{F}'$, $\mathscr{G}'$, respectively. Since $\widehat{\pi_{\mathfrak{p}*}O_{X'}}_{y'} = \bigoplus_{\pi_{\mathfrak{p}}(x')=y'} \hat{O}_{X',x}$, we have $\mathscr{G}'_{y'} = \mathrm{Ind}_{G_{x'}}^G \mathscr{F}'_{x'} = \mathbb{Z}[G] \otimes_{\mathbb{Z}[G_{x'}]} \mathscr{F}'_{x'}$. Hence for all integers $i$, Shapiro's lemma implies that

$$\hat{\mathrm{H}}^i(G, \mathbb{Z}[G] \otimes_{\mathbb{Z}[G_{x'}]} \mathscr{F}'_{x'}) \cong \hat{\mathrm{H}}^i(G_{x'}, \mathscr{F}'_{x'}).$$

Thus $\hat{\mathrm{H}}^i(G, \mathscr{G}'_{y'}) = 0$ for all $i$. By applying Mackey's double coset formula for all subgroups of $G$, we conclude that $\mathscr{G}'_{y'} = \mathscr{G}_y/p\mathscr{G}_y$ is cohomologically trivial for $G$. □

**Corollary 4.1.1.1.** *Let $\pi : X \to Y$, $\mathscr{F}$ be as in the above theorem. Then there is a well-defined projective Euler characteristic $\chi^P(G, X, \mathscr{F}) \in \mathrm{K}_0(A[G])$ which maps to the equivariant Euler characteristic $\chi(G, X, \mathscr{F}) \in \mathrm{K}_0(G, A)$ via the Cartan homomorphism $\mathrm{K}_0(A[G]) \to \mathrm{K}_0(A, G)$.*

*Proof.* Since $\pi$ is finite and thus affine, we have $\mathrm{H}^i(X, \mathscr{F}) = \mathrm{H}^i(Y, \pi_*\mathscr{F})$ for all $i \in \mathbb{Z}$. Now $\mathscr{G} = \pi_*\mathscr{F}$ and $Y$ satisfy the hypotheses of Theorem 3.1.1: $\mathscr{G}$ is a coherent sheaf on $Y$, and since $Y$ is projective, Theorem 5.2, III. in [Har77] implies that each $\mathrm{H}^i(Y, \mathscr{G})$ is a finitely generated $A[G]$-module. Moreover, Theorem 4.1.1 gives cohomological triviality at each stalk. Therefore, by Theorem 3.1.1, there exists a bounded complex $P^\bullet$ of finitely generated $A[G]$-modules which are cohomologically trivial for $G$, and that its Euler characteristic $\chi(P^\bullet) = \sum(-1)^i[P]^i$ in $\mathrm{CT}(A[G])$ is determined by $\mathscr{F}$ and is mapped to the equivariant Euler characteristic $\chi(G, Y, \mathscr{G}) = \chi(G, X, \mathscr{F})$ via the forgetful functor $\mathrm{CT}(A[G]) \to \mathrm{K}_0(A, G)$.

We reproduce this construction here just to have a self-contained proof: Let $\mathscr{U}$ be a finite open affine cover of $Y$. The Čech complex $C^\bullet(Y, \mathscr{U}, \mathscr{G})$ of $\mathscr{G}$ is isomorphic to $\mathrm{R}\Gamma^+(Y, \mathscr{G})$. Each term of $C^\bullet(Y, \mathscr{U}, \mathscr{G})$ is a direct sum of $A[G]$-modules of the form $\mathscr{G}(U)$ where $U$ is the intersection of finitely many elements of $\mathscr{U}$. Since $Y$ is separated over Spec $A$, $U$ is affine, say $U = \mathrm{Spec}\ B$.

We first show that each $\mathscr{G}(U)$ is cohomologically trivial. We showed that each stalk $\mathscr{G}_y$ at $y \in \mathrm{Spec}\ B$ is cohomologically trivial in the previous theorem. Since the local ring $O_{Y,y} = B_y$ is a flat $B$-module, for all subgroups $H \subset G$,

$$\hat{\mathrm{H}}^i(H, \mathscr{G}_y) = \hat{\mathrm{H}}^i(H, \mathscr{G}(U)) \otimes_B B_y = 0.$$

Since this holds for all $y \in \operatorname{Spec} B$, $\mathscr{G}(U)$ is cohomologically trivial for $G$.

Since $\mathscr{G}$ is coherent, the cohomology groups of $C^\bullet(Y, \mathscr{U}, \mathscr{G})$ are finitely generated $A[G]$-modules. We first construct a complex $P^\bullet$ of finitely generated *free* $A[G]$-modules (possibly not bounded below) together with a quasi-isomorphism of complexes

$$P^\bullet \mapsto C^\bullet(Y, \mathscr{U}, \mathscr{G})$$

by the following usual inductive procedure (cf., Lemma III.12.3, [Har77]). Let

$$0 \to C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \to \cdots \xrightarrow{d^n} C^{n+1} \to \cdots$$

be the Čech complex $C^\bullet(Y, \mathscr{U}, \mathscr{G})$. For a large $N$, we have $C^{n \geq N} = 0$. Take $P^N = 0$. Now suppose the inductive hypothesis: For $i > n$, we have a morphism of complexes

$$\begin{array}{cccccccc} P^\bullet : & & & P^{n+1} & \xrightarrow{\phi^{n+1}} & P^{n+2} & \xrightarrow{\phi^{n+2}} & \cdots \\ \downarrow g & & & \downarrow g^{n+1} & & \downarrow g^{n+2} & & \\ C^\bullet : & \cdots C^n & \xrightarrow{d^n} & C^{n+1} & \longrightarrow & C^{n+2} & \longrightarrow & \cdots \end{array}$$

such that, for $i > n+1$, the induced homology groups are isomorphic

$$h^i(P^\bullet) \cong h^i(C^\bullet)$$

and $\ker \phi^{n+1} \to h^{n+1}(C^\bullet)$ is surjective. We will construct $P^n$ by the following steps: Choose a finite set of generators of $h^i(C^\bullet)$, say $\{\bar{x}_1, ..., \bar{x}_r\}$ from $\{x_1, ..., x_r\} \subset \ker d^n$. Also consider $g^{-1}(\operatorname{im} d^n) \subset P^{n+1}$. Since $P^{n+1}$ is assumed to be finitely generated over the noetherian ring $A$, the submodule $g^{-1}(\operatorname{im} d^n) \subset P^{n+1}$ is also finitely generated. Choose a finite set of generators $y_{r+1}, ..., y_s$ of $g^{-1}(\operatorname{im} d^n) \subset P^{n+1}$. Then their images $g(y_i) \in \operatorname{im} d^n$ can be lifted to $x_i \in C^n$, $i = r+1, ..., s$.

Take $P^n$ to be the free $A[G]$-module of rank $s$ with generators $e_1, ..., e_s$ and define $\phi^n : P^n \to P^{n+1}$ by

$$\{e_1, ..., e_r\} \to \{0\}$$

$$e_{i \geq r+1} \mapsto y_i.$$

Also define $g^n : P^n \to C^n$ by

$$e_i \mapsto x_i$$

for all $i$. Then we have a commutative diagram

$$\begin{array}{ccccccccc}
P^\bullet : & & P^n & \xrightarrow{\phi^n} & P^{n+1} & \xrightarrow{\phi^{n+1}} & P^{n+2} & \xrightarrow{\phi^{n+2}} & \cdots \\
\downarrow g & & \downarrow g^n & & \downarrow g^{n+1} & & \downarrow g^{n+2} & & \\
C^\bullet : & & C^n & \xrightarrow{d^n} & C^{n+1} & \longrightarrow & C^{n+2} & \longrightarrow & \cdots
\end{array}$$

such that $h^{n+1}(P^\bullet) \to h^{n+1}(C^\bullet)$ is an isomorphism, and that $\ker \phi^n \to h^n(C^\bullet)$ is surjective as desired.

The constructed complex $P^\bullet$ of finitely generated free $A[G]$-modules is bounded above but not necessarily below. However, the Čech complex $C^\bullet$ is bounded, say $C^i = \{0\}$ for $i < 0$.

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & P^{-2} & \xrightarrow{\phi^{-2}} & P^{-1} & \xrightarrow{\phi^{-1}} & P^0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & C^0 & \longrightarrow & \cdots
\end{array}$$

Figure 4.3: $P^\bullet$ and $C^\bullet$.

Replace $P^{-1}$ by $P^{-1}/\text{im}\,\phi^{-2}$ and the lower dimensions $P^i$ for $i < -1$ by $\{0\}$. Then we still have a quasi-isomorphism $g : P^\bullet \to C^\bullet$, but the lowest term $P^{-1}$ might not be free anymore. Since $g$ is a quasi-isomorphism, the mapping cylinder $L^\bullet$ of $g$ is an exact bounded complex ([Chi94]). Furthermore, since all of the terms of $C^\bullet$ are cohomologically trivial, at most one term of $L^\bullet$ is not cohomologically trivial for $G$. However, the exactness now forces all of the terms of $L^\bullet$ to be cohomologically trivial for $G$, from which it follows that the same is true for all of the terms of $P^\bullet$.

By Proposition 2.2.7, $\mathrm{CT}(A[G])$ is isomorphic to $\mathrm{K}_0(A[G])$. Thus we have the projective Euler characteristic $\chi^P(G, X, \mathscr{F}) \in \mathrm{K}_0(A[G])$,

$$\chi^P(G, X, \mathscr{F}) \mapsto \chi(P^\bullet)$$

under this isomorphism. □

## 4.2 Example: Cyclic Action

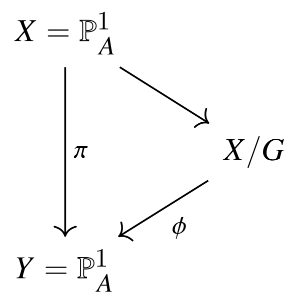
Let $p$ be an odd prime and consider $A = \mathbb{Z}[\zeta]$ for a primitive $p$-th root of unity $\zeta$. Then $p = u \cdot \lambda^{p-1}$ for some unit $u$ where $\lambda = \zeta - 1$ is a uniformizing parameter of the unique prime $\mathfrak{p}$ of $A$ above $p$. Let $X = \mathbb{P}^1_A = \mathrm{Proj}(A[X_0, X_1])$ and $G$ be a cyclic group of order $p$ acting on $X$ generated by

$$\sigma(X_0) = \zeta X_0 + X_1, \quad \sigma(X_1) = X_1.$$

Since $G$ is a finite group, the quotient scheme $X/G$ of the projective space exists. Since $X$ is normal, $X/G$ is also normal. Consider the finite morphism $\pi : X \to Y = \mathbb{P}^1_A$ of degree $p$ given by

$$\pi([X_0 : X_1]) = \left[\frac{(\lambda X_0 + X_1)^p - X_1^p}{\lambda^p} : X_1^p\right].$$

This is invariant under the action of $G$, therefore it factors through the canonical projection




by the universal property of the quotient scheme $X/G$. Since the canonical projection is also a finite morphism of degree $p$, $\phi$ is a finite birational morphism of integral schemes where $X/G$ is normal. This implies that $\phi$ is the normalization of $Y$ which is an isomorphism as $Y$ is already normal.

The ramification locus $R_\pi$ consists of two irreducible components which are the two divisors given by $(\lambda X_0 + X_1)$ and $(X_1)$. For $(\lambda X_0 + X_1) = (\sigma(X_0) - X_0)$,

$$\sigma(\lambda X_0 + X_1) = \lambda(\zeta X_0 + X_1) + X_1 = \zeta(\lambda X_0 + X_1).$$

The branch locus $B_\pi$ is as in the Figure 4.2

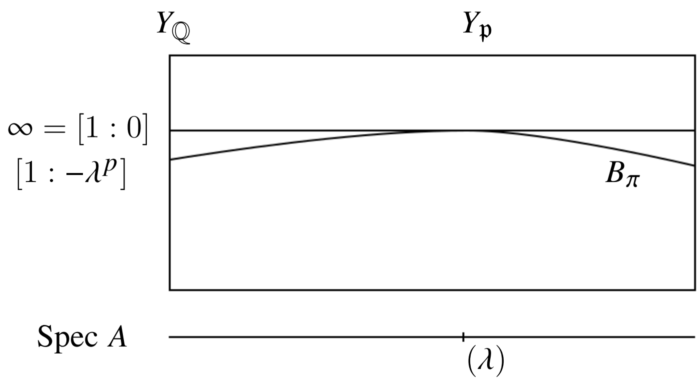



Figure 4.4: The collapse in the example describes the wild ramification from Kummer to Artin-Schreier extension, see [SOS89] for more.

We first see that $\pi$ is weakly ramified: At $\mathfrak{p} = (\lambda)$, the cover of the special fibres $X_{\mathfrak{p}} \to Y_{\mathfrak{p}}$ is ramified only at $(X_1)$ as $\lambda \equiv 0 \mod \mathfrak{p}$. Consider the affine patch where $X_0 = 1$, writing $X_1/X_0 = x_1$. At the local ring $\mathcal{O}_{X_{\mathfrak{p}},x_1}$, we have

$$\sigma(x_1) - x_1 = \frac{x_1}{x_1+1} - x_1 = \frac{-x_1^2}{x_1+1} \not\equiv 0 \mod (x_1)^3.$$

Therefore the second ramification group is trivial at all points of all fibres.

Consider the following $G$-equivariant divisor

$$D = (1-p)\cdot\overline{(X_1)} + (1-p)\cdot\overline{(\lambda X_0 + X_1)}$$

supported on the ramification locus. Since $p$ is assumed to be odd, $-D/2$ is also a well-defined divisor on $X$. At a point $x \in X$ over $(\lambda) \in$ Spec $A$, the two components of $R_\pi$ merge to $\overline{(X_1)}$, and $-D/2$ restricts to

$$-D/2|_{X_{\mathfrak{p}}} = -1\cdot\overline{(X_1)}|_{X_{\mathfrak{p}}}.$$

This shows that the invertible $\mathcal{O}_X$-$G$-module $\mathscr{F} := \mathcal{O}_X(-D/2)$ satisfies the restriction hypothesis given in Theorem 4.1.1. As argued in the proof of the theorem, $\mathscr{G} = \pi_*\mathscr{F}$ has cohomologically trivial stalks.

Let $U_0 =$ Spec $A[x_0]$, $U_1 =$ Spec $A[x_1]$ form an open affine cover $\mathscr{U}$ of $\mathbb{P}^1_A$ where $x_0 = X_0/X_1$,

$x_1 = X_1/X_0$, and consider the Čech complex $C^\bullet$ of $\mathscr{F}$ given by $\mathscr{U}$:

$$\begin{aligned}\mathscr{F}(U_0) &= (\lambda x_0 + 1)^{(1-p)/2} A[x_0],\\ \mathscr{F}(U_1) &= (\lambda + x_1)^{(1-p)/2} \cdot x_1^{(1-p)/2} A[x_1],\\ \mathscr{F}(U_0 \cap U_1) &= (\lambda + x_1)^{(1-p)/2} \cdot x_1^{(1-p)/2} A[x_1, 1/x_1].\end{aligned}$$

A direct computation shows that $\mathrm{H}^0(X, \mathscr{F})$ is a free $A$-module of rank $p$ generated by

$$\frac{1}{(\lambda x_0 + 1)^{(p-1)/2}} \cdot x_0^i$$

for $i = 0, ..., p-1$. The first cohomology group $\mathrm{H}^1(X, \mathscr{F})$ vanishes as $\mathscr{F}(U_0 \cap U_1)$, as an $A$-module, is generated by $(\lambda + x_1)^{(1-p)/2} \cdot x_1^{(1-p)/2} \cdot x_1^n$ for all integers $n$, and

$$(\lambda + x_1)^{(1-p)/2} \cdot x_1^{(1-p)/2} \cdot x_1^n = \begin{cases} (\lambda + x_1)^{(1-p)/2} \cdot x_1^{(1-p)/2} \cdot x_1^n \in \mathscr{F}(U_1), & \text{if } n \geq 0 \\ (\lambda x_0 + 1)^{(1-p)/2} \cdot x_0^{p-1-n} \in \mathscr{F}(U_0), & \text{otherwise.} \end{cases}$$

Thus the equivariant Euler characteristic $\chi(G, X, \mathscr{F})$ is just $[\mathrm{H}^0(X, \mathscr{F})]$ in $\mathrm{K}_0(G, A)$. Since $\mathrm{H}^i(X, \mathscr{F}) \cong \mathrm{H}^i(Y, \mathscr{G})$ for all $i$,

$$\chi(G, X, \mathscr{F}) = [\mathrm{H}^0(Y, \mathscr{G})] \in \mathrm{K}_0(G, A).$$

To find its projective Euler characteristic in $\mathrm{K}_0(A[G])$, first take an open cover $V = \{V_0, V_1\}$ of $Y = \mathbb{P}^1_A$ where each $V_i$ is stable under the action of $G$. For example, take $V_0 = \operatorname{Spec} A[x_0]$ and $V_1 = \operatorname{Spec} A[x_1, \sigma(x_1), ..., \sigma^{p-1}(x_1)]$. The intersection $V_0 \cap V_1$ is also affine, and let $\operatorname{Spec} B$ denote any of $V_0$, $V_1$, or $V_0 \cap V_1$. The corresponding Čech complex used in computing the sheaf cohomology of $\mathscr{G}$ is then

$$C^0 = \mathscr{G}(V_0) \oplus \mathscr{G}(V_1)$$

$$C^1 = \mathscr{G}(V_0 \cap V_1)$$

which are finitely generated $A[G]$-modules. We have a short exact sequence of $A[G]$-modules

$$0 \to \mathrm{H}^0(Y, \mathscr{G}) \to C^0 \to C^1 \to 0.$$

At each $y \in \operatorname{Spec} B$, we saw that $\mathscr{G}_y$ is cohomologically trivial. Since $B_y$ is a flat $B$-module, the Tate cohomology groups over tensor products give

$$\hat{\mathrm{H}}^i(G, \mathscr{G}_y) = \hat{\mathrm{H}}^i(G, \mathscr{G}(\operatorname{Spec} B)) \otimes_B B_y = 0$$

for all $i$. Since this is true for all $y \in \operatorname{Spec} B$, we conclude that $\mathscr{G}(\operatorname{Spec} B)$ is cohomologically trivial for $G$. Thus the terms of $C^\bullet$ are also cohomologically trivial for $G$, and so is $\mathrm{H}^0(Y, \mathscr{G})$. Since $\mathrm{H}^0(Y, \mathscr{G})$ is also a finitely generated free $A$-module, we conclude that the projective Euler characteristic $\chi^P(G, X, \mathscr{F}) \in \mathrm{K}_0(A[G])$ is given by the class $[\mathrm{H}^0(X, \mathscr{F})]$ in the Grothendieck group $\mathrm{K}_0(A[G])$.

Since the rank of $\mathrm{H}^0(X, \mathscr{F})$ over $A$ is $p$, it will be interesting to see if the class of $\chi^P(G, X, \mathscr{F})$, the obstruction to the existence of a normal integral basis of the cohomology of $\mathscr{F}$, is trivial in $\mathrm{Cl}(\mathbb{Z}[G])$.

**Proposition 4.2.1** (L, 2020)**.** $\mathrm{H}^0(X, \mathscr{F})$ *is a free* $A[G]$*-module, so* $\overline{\chi}^P(X, \mathscr{F}) = 0$.

*Proof.* For simplicity, denote $x_0$ by $x$ and replace $\mathrm{H}^0(X, \mathscr{F})$ by the $A$-module generated by $\{1, x, ..., x^{p-1}\}$ which is $A[G]$-isomorphic to $\mathrm{H}^0(X, \mathscr{F})$. We claim that $\mathrm{H}^0(X, \mathscr{F})$ has an $A[G]$-basis given by

$$\alpha = \frac{1}{p} \sum_{i=0}^{p-1} (\lambda x + 1)^i.$$

To see that $\alpha$ belongs to $\mathrm{H}^0(X, \mathscr{F})$, write

$$\sum_{i=0}^{p-1} (\lambda x + 1)^i = \sum_{i=0}^{p-1} \sum_{j=0}^{i} \binom{i}{j} \lambda^j x^j = \sum_{j=0}^{p-1} \sum_{i=j}^{p-1} \binom{i}{j} \lambda^j x^j = \sum_{j=0}^{p-1} \binom{p}{j+1} \lambda^j x^j$$

where the last equality is from

$$\sum_{i=j}^{p-1} \binom{i}{j} = \binom{p}{j+1}.$$

Since the coefficient $\lambda^{p-1}$ of the last term is also divisible by $p$, the whole sum is divisible by $p$, hence $\alpha \in \mathrm{H}^0(X, \mathscr{F})$.

We observe the change of $A$-bases from $\{1, x, ..., x^{p-1}\}$ to $\{\alpha, \sigma(\alpha), ..., \sigma^{p-1}(\alpha)\}$ can be broken into three steps of $A$-linear transformations between $K$-bases, $K$ being the fraction field of

$A$: first from $\{1, x, ..., x^{p-1}\}$ to $\{1, \lambda x + 1, ..., (\lambda x + 1)^{p-1}\}$, then to $\{p\alpha, \sigma(p\alpha), ..., \sigma^{p-1}(p\alpha)\}$, then finally to $\{\alpha, \sigma(\alpha), ..., \sigma^{p-1}(\alpha)\}$. Since

$$(\lambda x + 1)^j = \sum_{i=0}^{j} \binom{j}{i} \lambda^i x^i,$$

the corresponding matrix to the first linear transformation is triangular and the determinant is $\prod_{j=0}^{p-1} \lambda^j = \lambda^{p(p-1)/2}$. The matrix for the second linear transformation is the Vandermonde matrix

$$M = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \zeta & \zeta^2 & \cdots & \zeta^{p-1} \\ 1 & \zeta^2 & (\zeta^2)^2 & \cdots & (\zeta^2)^{p-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \zeta^{p-1} & (\zeta^2)^{p-1} & \cdots & (\zeta^{p-1})^{p-1} \end{bmatrix}$$

where determinant is given by $\prod_{1 \le i < j \le p} (\zeta^{j-1} - \zeta^{i-1}) = u' \cdot \lambda^{p(p-1)/2}$ for some unit $u' \in A^{\times}$. The last linear transformation has determinant $p^{-p} = (u \cdot \lambda^{p-1})^{-p}$. The composition of these three transformations has a unit determinant as all $\lambda$'s cancel out. □

We will see in the next chapter that the choice of $D$ and $-D/2$ in the example is somewhat more canonical than the other possible choices of invertible sheaves satisfying the restriction hypothesis of Theorem 4.1.1.

# CHAPTER 5

## SQUARE ROOT OF THE INVERSE DIFFERENT

Let $\pi : X \to Y = X/G$ be weakly ramified as in the previous chapter with $G$ *of odd order*. In this chapter we discuss the canonical existence of an invertible sheaf on $X$ verifying the conditions of Theorem 4.1.1.

**Lemma 5.0.1.** *Let $A$ be a ring flat over $\mathbb{Z}$, $I$, $J$ ideals of $A$. Suppose that $A/I$, $A/J$ are $\mathbb{Z}$-torsion free. If $I \otimes \mathbb{Q} = J \otimes \mathbb{Q} \subset A \otimes \mathbb{Q}$, then $I = J$.*

*Proof.* We show that one is contained in the other, say $I \subseteq J$. Let $x \in I$. Since $I \otimes \mathbb{Q} = J \otimes \mathbb{Q}$,

$$x \otimes 1 = \sum_i a_j \otimes n_j \in J \otimes \mathbb{Q}.$$

By clearing the denominators of $n_j$, there is an integer $a$ with $ax \in J$. Since $ax \equiv 0$ in $A/J$ and $A/J$ is torsion-free, $x \in J$. □

**Theorem 5.0.2** (L, 2020)**.** *There exists an invertible sheaf $\mathscr{F}$ on $X$ such that $\mathscr{F}^{\otimes -2}$ is the torsion-free part of the quotient sheaf $\mathcal{O}_X/\mathrm{Ann}(\Omega^1_{X/Y})$. Here, $\mathrm{Ann}(\Omega^1_{X/Y})$ is the annihilator of the sheaf of relative differentials $\Omega^1_{X/Y}$. The $\mathcal{O}_Y[G]$-module $\pi_*\mathscr{F}$ has cohomologically trivial stalks, and so the projective Euler characteristic $\chi^P(X, \mathscr{F})$ is well-defined.*

**Remark.** This generalizes *the square root of the inverse different* discussed at the end of 3.2, see [Ere91]. When $N/K$ is an odd degree Galois extension of number fields with Galois group $G$, then by a formula of Hilbert (Proposition 4 on p. 64 in [Ser79]), the order of the different ideal $\mathfrak{D}(N/K)$ at every $\mathfrak{p} \in \mathrm{Spec}\, \mathcal{O}_N$ is always even, hence there exists an ideal whose square is $\mathfrak{D}(N/K)^{-1}$.

*Proof.* Consider the annihilator ideal sheaf $\mathrm{Ann}(\Omega^1_{X/Y}) \subseteq \mathcal{O}_X$ of the sheaf of relative differentials $\Omega^1_{X/Y}$. Then the $\mathbb{Z}$-torsion-free part of $\mathcal{O}_X/\mathrm{Ann}(\Omega^1_{X/Y})$ is again a quotient sheaf of $\mathcal{O}_X$ which determines a closed subscheme $Z_1$ of $X$ flat over $\mathrm{Spec}\, \mathbb{Z}$. We take $\mathscr{I}$ to be the ideal sheaf of $Z_1$. On the other hand, we assume that the ramification locus $R_\pi$ is horizontal, so let $x$ be the generic point of an irreducible component of $R_\pi$ which is of codimension 1. Let $y = \pi(x)$. Since we assume that all residue field extensions are separable, by Proposition 12 on p. 57 in [Ser79], the discrete valuation ring $\mathcal{O}_{X,x}$ can be given by $\mathcal{O}_{Y,y}[T]/(f(T))$ where $f$ is monic. Then $\mathrm{Ann}(\Omega^1_{X/Y,x}) = (f'(t))$ where $t$ is the image of $T$ in $\mathcal{O}_{X,x}$. This is the same as the different ideal of $\mathcal{O}_{X,x}$ over $\mathcal{O}_{Y,y}$ by Proposition 14 on p. 59 in [Ser79], and since the ramification is tame at the horizontal $x$, by Proposition 13 on p. 58 in [Ser79], its order is $e_x - 1$ where $e_x$ is the ramification index at $x$. Using $e_x$ for each irreducible component $\overline{\{x\}} \in R_\pi$, consider the divisor

$$D = \sum_{x \in R_\pi \text{ of codim } 1} (1 - e_x) \cdot \overline{\{x\}}.$$

Let $Z_2$ be the closed subscheme of $X$ with the structure sheaf $\mathcal{O}_{Z_2}$ determined by $\mathcal{O}_X/\mathcal{O}_X(D)$. By construction, $\mathrm{Supp}(\mathcal{O}_X/\mathscr{I}) = \mathrm{Supp}(\mathcal{O}_X/\mathcal{O}_X(D)) = R_\pi$ as a set. Over the generic fibre, $\mathcal{O}_{X_\mathbb{Q}}(D) = \mathscr{I}_\mathbb{Q}$ as $\mathscr{I}_\mathbb{Q} = \mathrm{Ann}(\Omega^1_{X_\mathbb{Q}/Y_\mathbb{Q}})$ is without $\mathbb{Z}$-torsion and we defined $D$ by the closure of the divisor corresponding to $\mathrm{Ann}(\Omega^1_{X_\mathbb{Q}/Y_\mathbb{Q}}) \subseteq \mathcal{O}_{X_\mathbb{Q}}$. Thus the flat closed subschemes $Z_1$, $Z_2$ are identical by Lemma 5.0.1, and we conclude that $\mathscr{I} = \mathcal{O}_X(D)$.

Note that since the order of $G$ is odd, $1 - e_x$ in $D$ is always even, thus $-D/2$ is a well-defined divisor on $X$. Since $D$ is $G$-equivariant, we can take $\mathscr{F} = \mathcal{O}_X(-D/2)$ to be our $\mathcal{O}_X$-$G$-module.

We check that $\mathscr{F}$ satisfies the intersection hypothesis of Theorem 4.1.1. Let $x$ be a point in the intersection of $-D/2$ and $X_\mathfrak{p}$ the special fibre where $\mathfrak{p} \in \operatorname{Spec} A$ is over $p$ and $p$ divides the order of $G$. By assumption, $X_\mathfrak{p}$ is smooth at $x$, and we have an extension of discrete valuation rings $\mathcal{O}_{X_\mathfrak{p},x}/\mathcal{O}_{Y_\mathfrak{p},y}$ where $y$ is the image of $x$ under the canonical projection of special fibres $X_\mathfrak{p}/Y_\mathfrak{p}$. We can compute the valuation of $-D/2$ at $x$ by first computing the valuation of $\mathrm{Ann}(\Omega^1_{X_\mathfrak{p}/Y_\mathfrak{p}})_x$. This can be computed using the higher ramification groups $G_{x,i}$ and a formula of Hilbert on the valuation of the different ideal of local extensions: Since the valuations remain unchanged after taking completion, assume $\mathcal{O}_{X_\mathfrak{p},x}/\mathcal{O}_{Y_\mathfrak{p},y}$ is a weakly ramified extension of complete discrete valuation rings over $\mathbb{F}_p$ (hence $G_{x,2}$ is trivial). Let $\mathfrak{m}_x^a$ be its different ideal for some integer $a$.

Then from Proposition 2.1.15 we have

$$a = \sum_{i=0}^{i=\infty} (|G_{x,i}| - 1) = |G_{x,0}| + |G_{x,1}| - 2.$$

Therefore the valuation $v_x(-D/2)$ of $-D/2$ at $x$ is

$$v_x(-D/2) = \frac{|G_{x,0}| + |G_{x,1}| - 2}{2} \equiv -1 \mod |G_{x,1}|$$

since $G_{x,1} \leqslant G_{x,0}$. Therefore $\mathscr{F}$ satisfies the hypothesis of Theorem 4.1.1, and the direct image sheaf $\pi_*\mathscr{F}$ has cohomologically trivial stalks. Corollary 4.1.1.1 gives the projective Euler characteristic $\chi^P(G, X, \mathscr{F})$ of $\mathscr{F}$. □

# BIBLIOGRAPHY

# BIBLIOGRAPHY


[Arm71] J. V. Armitage. Zeta-functions with zero at s = 1/2. *Inventiones mathematicae*, 15:199--205, 1971.

[BK82] W. Browder and N. M. Katz. Free actions of finite groups on varieties, ii. *Mathematische Annalen*, 260:403412, 1982.

[BK00] A. J. Berrick and M. E. Keating. Categories and modules with K-theory in view. *Cambridge Studies in Advanced Mathematics*, 2000.

[CE56] H. Cartan and S. Eilenberg. Homological algebra. *Princeton University Press*, 1956.

[CE92] T. Chinburg and B. Erez. Equivariant Euler-Poincaré characteristics and tameness. *Astérisque*, 209:179--194, 1992.

[CEPT96] T. Chinburg, B. Erez, G. Pappas, and M. J. Taylor. Tame actions for group schemes: integrals and slices. *Duke Mathematical Journal*, 82(2):269--308, 1996.

[Chi94] T. Chinburg. Galois structure of de Rham cohomology of tame covers of schemes. *Annals of Mathematics*, 139(2):443--490, 1994.

[CN78] Ph. Cassou-Noguès. Quelques theoremes de base normale d'entiers. *Annales de l'Institut Fourier*, 28(3):1--33, 1978.

[CPT09] T. Chinburg, G. Pappas, and M. J. Taylor. Cubic structures, equivariant Euler characteristics and lattices of modular forms. *Annals of Mathematics*, 170:561--608, 2009.

[Ere91] B. Erez. The Galois structure of the square root of the inverse different. *Mathematische Zeitschrift*, 208:239--255, 1991.

[Frö83] A. Fröhlich. Galois module structure of algebraic integers. *Springer*, 1983.

[Har77] R. Hartshorne. Algebraic geometry. *Springer*, 1977.

[Köc04] B. Köck. Galois structure of Zariski cohomology. *American Journal of Mathematics*, 126(5):1085--1107, 2004.

[Liu02] Q. Liu. Algebraic geometry and arithmetic curves. *Oxford Graduate Texts in Mathematics*, 2002.

[Mar71] J. Martinet. Modules sur l'algebra du group quaternionien. *Annales scientifiques de l'École Normale Supérieure*, 4(3):399--408, 1971.

[Mat87] H. Matsumura. Commutative ring theory. *Cambridge University Press*, 1987.

[Mum70] D. Mumford. Abelian varieties. *Oxford University Press*, 1970.

[Noe32] E. Noether. Normalbasis bei körpern ohne höher verzweigung. *J. Reine Angew. Math.*, pages 147--152, 1932.

[Pap98] G. Pappas. Galois modules and the theorem of the cube. *Inventiones mathematicae*, 133:193--225, 1998.

[Pap08] G. Pappas. Galois structure of unramified covers. *Mathematische Annalen*, 341:71--97, 2008.

[Pap15] G. Pappas. Adams operations and galois structure. *Algebra and Number Theory*, 9(6):1477--1514, 2015.

[Ser77] J.-P. Serre. Linear representations of finite groups. *Springer*, 1977.

[Ser79] J.-P. Serre. Local fields. *Springer*, 1979.

[SOS89] T. Sekiguchi, F. Oort, and N. Suwa. On the deformation of Artin-Schreier to Kummer. *Annales scientifiques de l'École Normale Supérieure*, 22(4):345--375, 1989.

[Spe16] A. Speiser. Gruppendeterminante und körpendiskriminante. *Mathematische Annalen*, 77:546--567, 1916.

[Swa60] R. G. Swan. Induced representations and projective modules. *Annals of Mathematics*, 71(3):552--578, May 1960.

[Swa79] R. G. Swan. Strong approximation and locally free modules. *Ring Theory and Algebra III (Proceesings of the Third Conference, University of Oklahoma)*, pages 153--223, 1979.

[Swa86] R. G. Swan. K-theory of finite groups and orders. *Springer*, 1986.

[Tay81] M. J. Taylor. On fröhlich's conjecture for rings of integers of tame extensions. *Inventiones Mathematicae*, 63:41--79, 1981.

[Ull70] S. Ullom. Integral normal bases in Galois extensions of local fields. *Nagoya Mathematical Journal*, 39:141--148, 1970.

[Wei94] C. A. Weibel. An introduction to homological algebra. *Cambridge University Press*, 1994.