THE NECESSARY AND SUFFICIENT CONDITIONS IN WEIGHTED INEQUALITIES FOR SINGULAR INTEGRALS AND A LOCAL TB THEOREM WITH AN ENERGY SIDE CONDITION.

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ABSTRACT

THE NECESSARY AND SUFFICIENT CONDITIONS IN WEIGHTED INEQUALITIES FOR SINGULAR INTEGRALS AND A LOCAL TB THEOREM WITH AN ENERGY SIDE CONDITION.

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We provide an essentially complete dictionary of all implications among the basic and fundamental conditions in weighted theory such as the doubling, one weight $A_p(w)$, A_∞ and C_p conditions as well as the two weight $\mathcal{A}_p(\omega,\sigma)$ and the "buffer" Energy and Pivotal conditions. The most notable implication is that in the case of A_∞ weights the two weight $\mathcal{A}_p$ condition implies the $p-$Pivotal condition hence giving an elegant and short proof of the known NTV-conjecture with $p=2$ for A_∞ weights in terms of existing T1 theory. We also provide a quite technical construction inspired by [15] proving that we can have doubling weights satisfying the C_p condition which are not in A_∞.

We obtain a local two weight Tb theorem with an energy side condition for higher dimensional fractional Calderón-Zygmund operators. The proof follows the general outline of the proof for the corresponding one-dimensional Tb theorem in [69], but encountering a number of new challenges, including several arising from the failure in higher dimensions of T. Hytönen's one-dimensional two weight A_2 inequality.

Hytönen used this inequality to deal with estimates for measures living in adjacent intervals. Hytönen's theorem states that the off-testing condition for the Hilbert transform is controlled by the Muckenhoupt's A_2 and A_2^* conditions. So in attempting to extend the two weight T_b theorem to higher dimensions, it is natural to ask if a higher dimensional analogue of Hytönen's theorem holds that permits analogous control of terms involving measures that

live on adjacent cubes.We show that it is not the case even in the presence of the energy conditions used in one dimension [69]. Thus, in order to obtain a local T_b theorem in higher dimensions, it was necessary to find some substantially new arguments to control the notoriously difficult nearby form. More precisely, we show that Hytönen's off-testing condition for the two weight fractional integral and the Riesz transform inequalities is not controlled by Muckenhoupt's A_2^α and $A_2^{\alpha,*}$ conditions and energy conditions.

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Introduction

The goal of this section is to first provide the reader with an exposition to the basic tools needed to work in weighted theory. We start with a few basic facts about harmonic functions and the Poisson kernel which leads to the definition of the Hilbert transform for a special class of functions, the Schwartz functions. Then, in the attempt to extend the Hilbert transform to L^p spaces we talk about the Fourier transform and tempered distributions. Last, we generalize from the Hilbert transform to singular integrals and the Maximal operator.

1.1 Harmonic functions in the upper half plane

Let $f : \mathbb{R} \to \mathbb{R}$ be a real valued *smooth* function, i.e. it has derivatives of all orders. We can extend f to a harmonic function u in the upper half plane $H = \{(x, y) : y > 0, x \in \mathbb{R}\}$ by convolution with the *Poisson kernel*:

$$P_t(x) = \frac{1}{\pi} \frac{t}{t^2 + x^2} \tag{1.1.1}$$

and we get

$$u(x, t) = \int_{\mathbb{R}} P_t(x - y) f(y) dy, t > 0$$

It is simple to prove that u is harmonic since $P_t(x)$ is harmonic (with respect to the variables x, t). Indeed

$$\frac{\partial P_t(x)}{\partial x^2} = \frac{-2t^3 + 6x^2 t}{(x^2+t^2)^3}, \frac{\partial P_t(x)}{\partial t^2} = \frac{2t^3 - 6x^2 t}{(x^2+t^2)^3}$$

and since f is smooth we can move the derivative inside the integral and hence we have

$$u_{xx}(x,t) + u_{tt}(x,t) = \int_{\mathbb{R}} \left(\frac{\partial P_t(x)}{\partial x^2} + \frac{\partial P_t(x)}{\partial t^2} \right) f(y) dy = 0.$$

Using the fact $\int_{\mathbb{R}} P_t(x) dx = 1$ we can also check that

$$\lim_{t \to 0} u(x,t) = f(x)$$

Singular integrals arise when we try to find the harmonic conjugate of u, i.e. the function v such that $g(z) = u(z) + iv(z)$ is analytic.

First we notice that the Poisson kernel $P_t(x)$ is the real part of the analytic function $h(z) = \frac{i}{\pi z}$. Indeed, writing $z = x + it$ we have

$$h(z) = \frac{i}{\pi z} = \frac{i\bar{z}}{\pi |z|^2} = \frac{t + ix}{t^2 + x^2} = P_t(x) + iQ_t(x)$$

where

$$Q_t(x) = \frac{1}{\pi} \frac{x}{t^2 + x^2} \tag{1.1.2}$$

and hence $v(x,t) = \int_{\mathbb{R}} Q_t(x-y) f(y) dy$ is the harmonic conjugate of u. But now it is not easy to define $\lim_{t \to 0} v(x,t)$. We can see there is a problem by taking $\lim_{t \to 0} Q_t(x) = \frac{1}{\pi x}$ since this is not even locally integrable since the integral around 0 does not exist. The singularity at $x = 0$ is the motivation for the name singular integrals. So we cannot just talk about $\int_{\mathbb{R}} \frac{1}{\pi x} f(x)dx$ since this integral is not properly defined so let's see how to interpret that integral.

1.2 The Hilbert transform

It is clear that for $\int_{\mathbb{R}} \frac{1}{\pi x} f(x)dx$ to make sense f cannot be any function (just try any constant function on $\mathbb{R}$). For this reason we define the Schwartz space.

Definition 1.2.1. *Let f be a real valued function. We say $f \in \mathcal{S}$ if and only if f is infinitely differentiable and*

$$\sup_{x\in\mathbb{R}} |x^n f^{(k)}(x)| \leq c_{n,k} = c_{n,k}(f) < \infty, n, k \in \mathbb{N} \tag{1.2.1}$$

where $f^{(k)}$ denotes the $k-th$ derivative of f.

We can now define the principal value of $\frac{1}{x}$.

Definition 1.2.2. *We define the principal value of $\frac{1}{x}$ and we write $p.v.\frac{1}{x}$, by*

$$p.v.\frac{1}{x}(f) = \lim_{\varepsilon\to 0} \int_{|x|>\varepsilon} \frac{f(x)}{x} dx, \quad f \in \mathcal{S} \tag{1.2.2}$$

So $p.v.\frac{1}{x}$ maps Schwartz functions to the real numbers.

To see that this definition makes sense, we have to verify that the limit in the definition actually exists. So let $f \in \mathcal{S}$, we have

$$\begin{aligned} p.v.\frac{1}{x}(f) &= \lim_{\varepsilon\to 0} \int_{|x|>\varepsilon} \frac{f(x)}{x} dx = \lim_{\varepsilon\to 0} \int_{\varepsilon<|x|<1} \frac{f(x)}{x} dx + \int_{|x|\geq 1} \frac{f(x)}{x} dx \\ &= \lim_{\varepsilon\to 0} \int_{\varepsilon<|x|<1} \frac{f(x)-f(0)}{x} dx + \int_{|x|\geq 1} \frac{f(x)}{x} dx \end{aligned}$$

and the last equality holds since $\int_{\varepsilon<|x|<1} \frac{1}{x} dx = 0$. Now since f is a Schwartz function, by (1.2.1) we get

$$\begin{aligned} |p.v.\frac{1}{x}(f)| &\leq \sup_{\varepsilon<|x|<1} |f'(x)| + \sup_{|x|\geq 1} |f(x)| \int_{|x|\geq 1} \frac{c_{1,0}}{x^2} dx \leq ||f'||_\infty + c_{1,0}||f||_\infty \\ &\leq c_{0,1} + c_{0,0}c_{1,0} < \infty \end{aligned}$$

where $||f||_\infty = \sup_{x\in\mathbb{R}} |f(x)|$. So the limit makes sense.

This brings us to the definition of the *Hilbert* transform which is the archetype of *singular integrals*.

Definition 1.2.3. *Let $f \in \mathcal{S}$. We define the Hilbert transform by*

$$Hf(x) = \lim_{t\to 0} Q_t * f(x) = \lim_{\varepsilon\to 0} \frac{1}{\pi} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} dy \tag{1.2.3}$$

The Hilbert transform is extremely important mainly for the reason of mapping the boundary values (f) of a harmonic function (u) to it's harmonic conjugate (v) and it's been extensively studied since then.

It is quite restrictive though that right now we can talk about the Hilbert transform only for the limited class of Schwartz functions. To extend the definition of the Hilbert transform to bigger classes of functions we need to talk about the Fourier transform of a function and tempered distributions.

1.3 The Fourier transform and tempered distributions

In the class of Schwartz functions $\mathcal{S}$ we can define a topology by (1.2.2). We say a sequence $\{f_m\}_{m\in\mathbb{N}}$, where $f \in \mathcal{S}$ converges to 0 if and only if for all $n, k \in \mathbb{N}$

$$\lim_{n,k\to\infty} c_{n,k}(f) = 0. \tag{1.3.1}$$

Definition 1.3.1. *We say a function $f \in L^p(\mathbb{R}^n)$ if and only if*

$$\int_{\mathbb{R}^n} |f(x)|^p dx < \infty$$

It is easy to see using (1.2.2) that $\mathcal{S} \subset L^p(\mathbb{R}^n)$ for all $p > 1$. The Schwartz class is also *dense* in $L^p(\mathbb{R}^n)$ meaning that for any $f \in L^p(\mathbb{R}^n)$ there exist a sequence $\{f_m\}$ such that

$$\lim_{m\to\infty} ||f_m - f||_p = 0.$$

With this metric now in the Schwartz class we can define the space of bounded linear functionals in $\mathcal{S}$, we call it S' the space of *tempered distributions*.

Definition 1.3.2. *We say a map $T : \mathcal{S} \to \mathbb{R}$ (or $\mathcal{S} \to \mathbb{C}$) is in S', the space of tempered distributions, if*

$$\lim_{m\to\infty} f_m = 0 \Rightarrow \lim_{m\to\infty} T(f_m) = 0$$

Remember the limit on the left hand side is in the sense of (1.3.1). *One can easily check that the principal value defined in* (1.2.2) *is a tempered distribution.*

Let's now define the Fourier transform which we will see is a tempered distribution itself. We first define the Fourier transform for L^1 functions.

Definition 1.3.3. *Let $f \in L^1(\mathbb{R}^n)$. The Fourier transform $\mathcal{F}$ of f is given by*

$$\mathcal{F}(f)(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx \tag{1.3.2}$$

where $x \cdot \xi = x_1\xi_1 + x_2\xi_2 + ... + x_n\xi_n$. We will also write $\mathcal{F}(f)(\xi) = \hat{f}(\xi)$.

Here are some properties of the Fourier transform that we need:

1. Linearity: $(\widehat{af + bg}) = a\hat{f} + b\hat{g}$.

2. $||\hat{f}||_\infty \leq ||f||_1$ and hence $\hat{f}$ is continuous.

3. $\widehat{f * g} = \hat{f}\hat{g}$.

We can also define the Fourier transform of tempered distributions.

Definition 1.3.4. *The Fourier transform of $T \in \mathcal{S}'$ is the tempered distribution $\hat{T}$ given by*

$$\hat{T}(f) = T(\hat{f}), f \in \mathcal{S}.$$

The following theorems can be found in [12], pg 13-16.

Theorem 1.3.5. *The Fourier transform is a continuous map from $\mathcal{S}$ to $\mathcal{S}$ such that*

$$\int_{\mathbb{R}^n} \hat{f} g = \int_{\mathbb{R}^n} f \hat{g}$$

and

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi$$

Theorem 1.3.6. *The Fourier transform is a bounded linear bijection from $\mathcal{S}'$ to $\mathcal{S}'$ whose inverse is also bounded.*

Theorem 1.3.7. *The Fourier transform is an isometry on L^2 meaning for $f \in \mathcal{S}$, $\hat{f} \in L^2$ and $||\hat{f}||_2 = ||f||_2$.*

Now we can extend the Fourier transform to functions in L^2. Let $f \in L^2(\mathbb{R}^n)$. By theorem 1.3.7 and using the density of $\mathcal{S}$ in L^2 there exist $\{f_m\}_{m\in\mathbb{N}}$ such that $||f_m||_2 \to ||f||_2$ as $m \to \infty$ hence there is a function g such that $||\hat{f}_m||_2 \to ||g||_2$ as $m \to \infty$. We call then $g = \hat{f}$ the Fourier transform of f and we have equality of norms, i.e. $||f||_2 = ||\hat{f}||_2$.

1.4 The Hilbert transform on L^p spaces

Now we have all the tools we need to extend the Hilbert transform from the space of Schwartz functions to the space of L^p functions.

First, by (1.1.2) and calculating it's Fourier transform we get

$$\hat{Q}_t(\xi) = -isgn(\xi)e^{-2\pi t|\xi|} \tag{1.4.1}$$

Now by (1.2.3) we get for $f \in \mathcal{S}$:

$$\hat{H}f(x) = \lim_{t\to 0} \widehat{Q_t * f}(x) = \lim_{t\to 0} \hat{Q}_t(\xi)\hat{f}(\xi) = -isgn(\xi)\hat{f}(\xi). \tag{1.4.2}$$

The first equality holds by the continuity of the Fourier transform on tempered distributions (theorem 1.3.6), the second equality from property (2) after (1.3.2) and the last equality by (1.4.1).

Now by the density of $\mathcal{S}$ in L^2 and using theorem 1.3.7 we can extend the definition of the Hilbert transform to L^2 functions and we also have

$||Hf||_2 = ||\hat{H}f||_2 = ||f||_2, \quad f \in L^2(\mathbb{R}^n)$

Next, the extension to $L^p, p > 1$ came from Riesz in 1928 with the following theorem.

Theorem 1.4.1 (Riesz 1928)**.** *Let $f \in \mathcal{S}$. The Hilbert transform is strong (p,p), $1 < p < \infty$ i.e.*

$$||Hf||_p \leq C_p ||f||_p$$

Again, like in the case for L^2 functions, using the density of $\mathcal{S}$ in L^p we extend the definition of the Hilbert transform to L^p functions.

1.5 Singular integrals and the maximal operator.

The Hilbert transform is the archetype of a more general class of operators that we want to study the so called *singular integral* operators.

Definition 1.5.1. *A singular integral of convolution type is an operator T defined by convolution with a kernel K that is locally integrable on $\mathbb{R}^n \backslash \{0\}$, in the sense that*

$$T(f)(x) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} K(x-y) f(y) dy$$

Typically the kernel satisfies the following conditions:

1. $||\hat{K}||_\infty \leq C < \infty$

2. $\sup\limits_{y \neq 0} \int_{|x|>2|y|} |K(x-y) - K(x)| dx \leq C$

For $K(x) = \frac{1}{\pi x}$ we get the Hilbert transform.

The first property, usually referred as a *size condition*, on $\hat{K}$ is used to ensure that the *tempered distribution* $p.v.K$ given by the principal value integral

$$p.v.K[\phi] = \lim_{\varepsilon \to 0} \int_{|x|>\varepsilon} \phi(x)K(x)dx$$

is well defined on L^2.

The second property, usually referred as the *Hörmander condition* or *smoothness condition*, is used to ensure the boundedness of the operators in $L^p(\mathbb{R}^n)$ for $p > 1$.

Example 1.5.2. *An important example of singular integrals is the Riesz transforms:*

$$R_j f(x) = c_n \lim_{\varepsilon \to 0} \int_{|y|>\varepsilon} \frac{y_j}{|y|^{n+1}} f(x-y)dy$$

where

$$c_n = \Gamma\left(\frac{n+1}{2}\right)\pi^{-\frac{n+1}{2}}.$$

The Riesz transforms are the extension of the Hilbert transform in higher dimensions. The constant c_n *is so that* $\hat{R}_j f(\xi) = -i\frac{\xi_j}{|\xi|}\hat{f}(\xi)$.

We have the following theorem for singular integrals of convolution type.

Theorem 1.5.3 (Calderon-Zygmund 1952, Hörmander 1960)**.** *Let* $f \in \mathcal{S}$. *Assume* K *is a kernel as in definition 1.5.1. Then the operator* T *associated with* K *is strong* (p,p), $1 < p < \infty$ *i.e.*

$$||Hf||_p \leq C_p ||f||_p$$

These singular integrals are called of convolution type because $Tf = K * f$ (with an abuse of notation since we do not write the limit here). We can extend to operators T that have a kernel $K(x,y)$ and the resulting operator is not of convolution type. First we need to define what a *standard kernel* is.

Definition 1.5.4. *We say $K(x,y): \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a standard kernel when:*

1. $|K(x,y)| \le \frac{C}{|x-y|^n}$
2. $|K(x,y) - K(x',y)| \le \frac{C|x-x'|^\delta}{(|x-y|+|x'-y|)^{n+\delta}}$, *for* $|x-x'| \le \frac{1}{2}\max\left(|x-y|, |x'-y|\right)$
3. $|K(x,y) - K(x,y')| \le \frac{C|y-y'|^\delta}{(|x-y|+|x-y'|)^{n+\delta}}$, *for* $|y-y'| \le \frac{1}{2}\max\left(|x-y|, |x-y'|\right)$

property (1) here is the analog of $||\hat{K}||_\infty \le C < \infty$ and is again a size condition on the kernel. Properties (2) and (3) are the analog of condition (2) and are again called *Hörmander conditions.*

Definition 1.5.5. *A general singular integral is an operator T associated with a standard kernel K, in the sense that*

$$T(f)(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy, x \notin suppf, f \in \mathcal{S} \tag{1.5.1}$$

An important note here is that the operator uniquely defines the Kernel but the kernel does not uniquely define the operator. The classical example on that is the operators $T_b(f)(x) = b(x)f(x)$ where b is a bounded function. All these are general singular integrals with kernel $K(x,y) = 0$. This comes from the fact that the integral formula holds only for $x \notin suppf$ so we have

$$0 = T_b(f)(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy \Rightarrow K(x,y) = 0$$

and the implication comes from setting $f_1(x) = 1_A(x)$, $f_2(x) = 1_B(x)$ where $A = \{K(x,y) > 0\}$ and $B = \{K(x,y) < 0\}$.

Example 1.5.6. *A general singular integral studied a lot is the Cauchy integral. Let's remember the Cauchy integral theorem. Let U be an open subset of $\mathbb{C}$ and $D_{z_0} = \{z : |z - z_0| \leq r\}$ be contained in U. Given a holomorphic function $f : U \to \mathbb{C}$ and any w in the interior of D_{z_0} we have*

$$f(w) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + re^{i\theta})ire^{i\theta}}{z_0 + re^{i\theta} - w} d\theta$$

Inspired by the classic Cauchy integral, we get the Cauchy integral on a Lipschitz curve. Let $L : \mathbb{R} \to \mathbb{R}$ be Lipschitz or equivalently $||L'||_\infty \leq C < \infty$ and let $\Gamma = (t, L(t))$ be a curve on the plane. Let $f \in \mathcal{S}(\mathbb{R})$. The Cauchy integral along Γ is given by

$$C_\Gamma f(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(t)(1 + iL'(t))}{t + iL(t) - z} dt$$

$C_\Gamma f(z)$ is a holomorphic function on $\mathbb{C} \setminus \Gamma$.

Another operator that often appears together with the Hilbert transform and singular integrals, as we will later see, is the Maximal operator of Hardy and Littlewood.

Definition 1.5.7. *The Maximal operator is given by*

$$Mf(x) = \sup_{I \ni x} \frac{1}{|I|}\int_I |f(y)|dy \tag{1.5.2}$$

where the supremum is taken over all cubes containing $x \in \mathbb{R}^n$.

Hardy and Littlewood proved that the maximal operator is bounded in $L^p(\mathbb{R})$, $p > 1$ and Wiener later extended the result to higher dimensions. We have:

Theorem 1.5.8 (Hardy-Littlewood-Wiener 1930,1939)**.** *The Maximal operator is a bounded* *operator on* $L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n), p > 1.$

$$\left(\int_{\mathbb{R}^n} |Mf(x)|^p dx\right)^{\frac{1}{p}} \leq C \left(\int_{\mathbb{R}^n} |f(x)|^p dx\right)^{\frac{1}{p}}.$$

(1.5.2) is not the only way to define the maximal operator. We could also use

$$M'f(x) = \sup_{r>0} \frac{1}{(r)^n} \int_{I_r} |f(y)| dy \qquad (1.5.3)$$

where I_r is the cube centered at x with side length r. We call M' the *centered maximal operator* and M is the *non-centered maximal operator*. One can use either of them since they are pointwise equivalent, i.e. there exist constants c_n, C_n such that

$$c_n Mf(x) \leq M'f(x) \leq C_n Mf(x).$$

1.6 Extending the Lebesgue measure

All the theorems presented until now were using the Lebesgue measure. The work in this monograph is concerned on extending all these results to more general measures. The following definition contains all the different measures that will concern us.

Definition 1.6.1.

1. *The measure μ is called inner regular if for any open set U, $\mu(U) = \sup_{K \subset U} \mu(K)$ where the supremum is taken over all compact subsets of U.*

2. *The measure μ is called outer regular if for any Borel set B, $\mu(B) = \inf_{B \subset U} \mu(U)$ where the infimum is taken over all open sets containing B.*

3. *The measure μ is called locally finite if for every point $x \in \mathbb{R}^n$ there is an open set U containing x such that $\mu(U) < \infty$. Equivalently μ is called locally finite if for every K compact $\mu(K) < \infty$.*

4. *The measure μ is called Radon if it satisfies all the above, i.e. it is inner regular, outer regular and it is locally finite.*

5. *The measure μ is called absolutely continuous if there exists $w : \mathbb{R}^n \to \mathbb{R}^+$ such that for any Borel set B, $\mu(B) = \int_B w(x)dx$. The function $w(x)$ in this case is called a weight.*

6. *The measure μ is called doubling if there exists a constant $C > 0$ such that for any cube I we have*

$$\mu(2I) \leq C\mu(I) \tag{1.6.1}$$

7. *The measure μ on $\mathbb{R}^n$ is called upper doubling if there exists a constant $C > 0$ such that for any ball B_r of radius r we have*

$$\mu(B_r) \leq Mr^n. \tag{1.6.2}$$

8. *We say a measure σ is reverse doubling if there exists $\varepsilon > 0$ depending only on the measure σ such that for all cubes I:*

$$\sigma(2I) \geq (1+\varepsilon)\sigma(I). \tag{1.6.3}$$

Doubling measures satisfy the reverse doubling property as the following lemma from [52] proves.

Lemma 1.6.2. *Let σ be a doubling measure with doubling constant K_σ. Then there exist a constant $\delta_\sigma > 0$ depending only on the doubling constant of σ such that for all cubes I we have $\sigma(2I) \geq (1 + \delta_\sigma)\sigma(I)$.* $\square$

Chapter 2

On weighted theory.

The goal of this section is to provide the reader with the development of the theory from Muckenhoupt's A_p condition for one weight results up to 2020 where we have general singular integrals and two Radon measures σ, ω . Weighted theory tries to answer questions as the following.

Question 1. *Given two locally finite positive Borel measures ω, σ in $\mathbb{R}^n$ and a general singular integral operator T, what are the necessary and sufficient conditions so that the following inequality holds:*

$$||T(fd\sigma)||_{L^p(\omega)} \lesssim ||f||_{L^p(\sigma)}, \quad \forall f \in L^p(\sigma), \ \ p > 1. \tag{2.0.1}$$

2.1 One weight theory

(2.0.1) is a generalization of the one weight inequality for the Hilbert transform where $T = H$, $d\omega(x) = w(x)dx$, $d\sigma(x) = w(x)^{1-p\prime}dx$ and $f \in L^p(w)$

$$||Hf||_{L^p(\omega)} \lesssim ||f||_{L^p(\omega)} \tag{2.1.1}$$

which was shown by Hunt, Muckenhoupt and Wheeden [20] to be equivalent to the finiteness of the Muckenhoupt one weight A_p condition, namely ω has to be absolutely continuous to

Lebesgue measure $d\omega = w(x)dx$ and

$$A_p(w) = \sup_I \frac{1}{|I|} \int_I w(x)dx \left(\frac{1}{|I|} \int_I w(x)^{\frac{1}{1-p}} dx \right)^{p-1} \leq C < \infty \tag{2.1.2}$$

where the supremum is taken uniformly over all cubes in $\mathbb{R}^n$. There has been a huge amount of work in harmonic analysis and boundary value problems around the A_p condition, check Stein [71], Duoandikoetxea [12], Garnett [14] and references there.

Coifman and Fefferman in [7] proved (2.1.1) using the following inequality, which holds for any $w \in A_\infty = \bigcup_{p \geq 1} A_p$,

$$\int_{\mathbb{R}^n} |Tf(x)|^p w(x)dx \leq C \int_{\mathbb{R}^n} |Mf(x)|^p w(x)dx \tag{2.1.3}$$

where T is any singular integral operator and f is bounded and compactly supported. We can extend to any locally integrable f for which the right hand side is finite (since otherwise there is nothing to prove) using the dominated convergence theorem.

Muckenhoupt in [39] proved, for $n = 1$, that a more general class of weights than the A_p weights, namely the C_p weights (see (2.1.4)), are necessary for (2.1.3) to hold. This was generalized in higher dimensions by Sawyer in [54]. Sawyer in [54] also shows that the C_q condition for $q > p$ is sufficient for (2.1.3) to hold. It is still unknown if the C_p condition is sufficient for (2.1.3) to hold.

We say the measure ω satisfies the C_p condition, $1 < p < \infty$ if it is absolutely continuous to the Lebesgue measure, i.e. $d\omega = w(x)dx$, and there exist $C, \epsilon > 0$ such that

$$\frac{|E|_w}{\int_{\mathbb{R}^n} |M\mathbf{1}_I(x)|^p w(x)dx} \leq C \left(\frac{|E|}{|I|} \right)^\epsilon, \text{for } E \text{ compact subset of } I \text{ cube} \tag{2.1.4}$$

with $\int_{\mathbb{R}^n} (M\mathbf{1}_I(x))^p w(x)dx < \infty$, where $|E|_w = \int_E w(x)dx$. Here M denotes the classical Hardy-Littlewood maximal operator. We will call $w(x)$ a C_p weight.

We prove that C_p weights is a strictly larger class than the A_∞ weights. We actually show that there exist even doubling weights (see (1.6.1)) that are also C_p weights that are not in A_∞. Check the diagram at the end of the introduction.

Theorem 2.1.1. *($C_p \cap \mathcal{D} \not\Rightarrow A_\infty$) There exist a weight w that is doubling and satisfies the C_p condition but w is not an A_∞ weight.*

The weight w used in theorem 2.1.1. has a doubling constant $C_w \gtrsim 3^{np}$. We show that this is sharp, i.e. if the doubling constant C_w of the weight w does not satisfy $C_w \geq 3^{np}$ then the C_p condition is equivalent to A_∞.

Theorem 2.1.2. *(C_p+small doubling $\Rightarrow \mathcal{A}_\infty$) Let w be a doubling C_p weight with doubling constant $C_w < 3^{np}$ in $\mathbb{R}^n$. Then $w \in A_\infty$.*

2.2 Two weight theory.

The generalization of the one weight A_p condition was naturally modified to the two weight problem by:

$$\mathcal{A}_p(\omega, \sigma) = \sup_I \left(\frac{\omega(I)}{|I|}\right)^{\frac{1}{p}} \left(\frac{\sigma(I)}{|I|}\right)^{\frac{1}{p'}} < \infty \tag{2.2.1}$$

where the supremum is taken over all cubes in $\mathbb{R}^n$ and the weight w gives its place to two positive locally finite Borel measures. Notice that by setting $d\omega = w(x)^{\frac{1}{1-p}}dx$, $d\sigma = w(x)dx$ we retrieve the one weight A_p condition (2.1.2).

The two weight problem could have applications in a number of problems connected to higher dimensional analogs of the Hilbert transform. For example, questions regarding sub-spaces of the Hardy space invariant under the inverse shift operator (see [75], [46]),questions concerning orthogonal polynomials (see [76], [49], [50]) and some questions in quasiconformal theory for example the conjecture of Iwaniec and Martin (see [25]) or higher dimensional analogues of the Astala conjecture (see [30]).

The classical $\mathcal{A}_p$ condition (2.2.1) is necessary for (2.0.1) to hold but is no longer sufficient, which is an indication that makes two weight theory much more complicated. F. Nazarov in [40] has shown that even the strengthened $\mathcal{A}_p(\omega,\sigma)$ conditions with one or two tails of Nazarov, Treil and Volberg

$$\mathcal{A}_p^{t_1}(\omega,\sigma) = \sup_I \left(\frac{\omega(I)}{|I|}\right)^{\frac{1}{p}} (P(I,\sigma))^{\frac{1}{p'}} < \infty \tag{2.2.2}$$

$$\mathcal{A}_p^{t_2}(\omega,\sigma) = \sup_I (P(I,\omega))^{\frac{1}{p}} (P(I,\sigma))^{\frac{1}{p'}} < \infty \tag{2.2.3}$$

where

$$P(I,\omega) \equiv \int_{\mathbb{R}^n} \left(\frac{|I|^{\frac{1}{n}}}{(|I|^{\frac{1}{n}} + \operatorname{dist}(x,I))^2}\right)^n \omega(dx) \tag{2.2.4}$$

along with their duals $\mathcal{A}_p^{t_1,*}(\omega,\sigma), \mathcal{A}_p^{t_2,*}(\omega,\sigma)$, where the roles of σ and ω are interchanged, are no longer sufficient for (2.0.1) to hold.

When the operator T in (2.0.1) is a fractional operator such as the Cauchy transform or the fractional Riesz transforms then the fractional analogs of (2.2.1), (2.2.2), (2.2.3) are used

$$\mathcal{A}_p^{\alpha}(\omega,\sigma) = \sup_I \left(\frac{\omega(I)}{|I|^{1-\frac{\alpha}{n}}}\right)^{\frac{1}{p}} \left(\frac{\sigma(I)}{|I|^{1-\frac{\alpha}{n}}}\right)^{\frac{1}{p'}} < \infty \tag{2.2.5}$$

$$\mathcal{A}_p^{t_1,\alpha}(\omega,\sigma) = \sup_I \left(\frac{\omega(I)}{|I|^{1-\frac{\alpha}{n}}}\right)^{\frac{1}{p}} (\mathcal{P}^{\alpha}(I,\sigma))^{\frac{1}{p'}} < \infty \tag{2.2.6}$$

$$\mathcal{A}_p^{t_2,\alpha}(\omega,\sigma) = \sup_I \left(\mathcal{P}^\alpha(I,\omega)\right)^{\frac{1}{p}} \left(\mathcal{P}^\alpha(I,\sigma)\right)^{\frac{1}{p'}} < \infty \tag{2.2.7}$$

where $\mathcal{P}^\alpha$ is the *reproducing* Poisson integral and is given by

$$\mathcal{P}^\alpha(I,\omega) \equiv \int_{\mathbb{R}^n} \left(\frac{|I|^{\frac{1}{n}}}{(|I|^{\frac{1}{n}} + \operatorname{dist}(x,I))^2} \right)^{n-\alpha} \omega(dx)$$

The *standard* Poisson integral, is given by

$$P^\alpha(I,\omega) \equiv \int_{\mathbb{R}^n} \frac{|I|^{\frac{1}{n}}}{(|I|^{\frac{1}{n}} + \operatorname{dist}(x,I))^{n+1-\alpha}} \omega(dx)$$

and is used for the definition of the fractional “buffer" conditions. The two Poisson integrals agree for $n=1$, $\alpha=0$. We refer the reader to [64] for more details. All the results that we are proving here for the $\mathcal{A}_p$ conditions hold for their fractional analogs without any modification in the proofs.

We show that the classical $\mathcal{A}_p$ condition is weaker than the tailed conditions, but the two tailed $\mathcal{A}_p$ condition holding is equivalent to both one tailed $\mathcal{A}_p$ conditions holding.

Theorem 2.2.1. *We have the following implications:*

1. *($\mathcal{A}_p \nRightarrow \mathcal{A}_p^{t_1} \cap \mathcal{A}_p^{t_1,*}$) The two weight classical $\mathcal{A}_p$ condition does not imply the one tailed $\mathcal{A}_p$ conditions.*
2. *($\mathcal{A}_p^{t_1} \nRightarrow \mathcal{A}_p^{t_2}$) The one tailed $\mathcal{A}_p^{t_1}$ condition does not imply the two tailed $\mathcal{A}_p^{t_2}$ condition.*
3. *($\mathcal{A}_p^{t_1} \cap \mathcal{A}_p^{t_1,*} \Leftrightarrow \mathcal{A}_p^{t_2}$) The two tailed $\mathcal{A}_p^{t_2}$ condition holding is equivalent to both one tailed $\mathcal{A}_p^{t_1}, \mathcal{A}_p^{t_1,*}$ conditions holding.*

The measures that we use for the proof of theorem 2.2.1. are non doubling and we show that this is the only case. All doubling measures are reverse doubling (see lemma 1.6.2.). So the previous sentence is justified by the following theorem:

Theorem 2.2.2. *($\omega, \sigma \in \mathcal{D}$, $\mathcal{A}_p \Rightarrow \mathcal{A}_p^{t_1} \Rightarrow \mathcal{A}_p^{t_2}$) If ω, σ are reverse doubling measures, then the classical two weight classical $\mathcal{A}_p$ implies the tailed $\mathcal{A}_p$ conditions.*

A proof of theorem 2.2.2 for the one tailed conditions can be found in [70].

2.2.1 The testing conditions.

Since the two weight $\mathcal{A}_p$ conditions are not sufficient for (2.0.1) to hold, some other necessary conditions are required, namely the **1**-testing conditions

$$\begin{aligned} ||T(\mathbf{1}_I d\sigma)||_{L^p(\omega)} &\leq \mathfrak{T}^p |I|_\sigma \\ ||T^*(\mathbf{1}_I d\omega)||_{L^p(\sigma)} &\leq (\mathfrak{T}^*)^p |I|_\omega \end{aligned} \tag{2.2.8}$$

where I runs over all cubes and $\mathfrak{T}, \mathfrak{T}^*$ are the best constants so that (2.2.8) holds.

The testing conditions were first introduced by Sawyer in [56] in 1982 for the boundedness of the maximal operator and they involved two weights, u, v namely Sawyer proved that the Maximal operator is bounded on $L^p(u) \to L^q(w)$ if and only if the *Sawyer testing condition* is satisfied, i.e. if and only if

$$S^{p,q}(w, u^{1-p'}) = \sup_I \left(\int_I u(x)^{1-p'} dx \right)^{\frac{-1}{p}} \left(\int_I \left[M(\mathbf{1}_I u^{1-p'})(x) \right]^q w(x) dx \right)^{\frac{1}{q}} < \infty \tag{2.2.9}$$

where the supremum is taken over all cubes $I \subset \mathbb{R}^n$.

Two years later in 1984, David and Journé in [10] using (2.2.8) for $d\sigma = d\omega = dx$ proved that a general singular integral is bounded in L^p as long as (2.2.8) holds.

The testing conditions alone are trivially not sufficient for (2.0.1) to hold for general measures since as pointed out in [48] for example, the second Riesz transform R_2 of any measure supported on the real line is the zero element in $L^p(\omega)$ for any measure ω carried by the upper half plane. On the other hand, such a pair of measures need not satisfy the Muckenhoupt conditions, which are necessary for (2.0.1) to hold.

The famous Nazarov-Treil-Volberg conjecture (NTV conjecture), states that $\mathcal{A}_p(\omega, \sigma)$ and testing conditions are necessary and sufficient for (2.0.1) to hold.

2.2.2 The "buffer" Pivotal and Energy conditions.

Nazarov, Treil and Volberg in a series of very clever papers assumed the pivotal condition, for $p = 2$, and proved (2.0.1) (see [42],[45],[75]).

The Pivotal condition $\mathcal{V}$ is given by

$$\mathcal{V}(\omega, \sigma)^p = \sup_{I_0 = \cup I_r} \frac{1}{\sigma(I_0)} \sum_{r \geq 1} \omega(I_r) P(I_r, 1_{I_0}\sigma)^p < \infty \tag{2.2.10}$$

where the supremum is taken over all possible decompositions of I_0 in disjoint cubes $\{I_r\}_{r \in \mathbb{N}}$ and all cubes I_0 such that $\sigma(I_0) \neq 0$, and its dual $\mathcal{V}^*$ where σ and ω are interchanged.

Lacey, Sawyer and Uriarte-Tuero in [32] proved, again for $p = 2$, that (2.0.1) for the Hilbert transform implies the weaker Energy condition $\mathcal{E}$

$$\mathcal{E}(\omega, \sigma)^p = \sup_{I_0 = \cup I_r} \frac{1}{\sigma(I_0)} \sum_{r \geq 1} \omega(I_r) E(I_r, \omega)^2 P(I_r, 1_{I_0}\sigma)^p < \infty \tag{2.2.11}$$

where the supremum is taken over all possible decompositions of I_0 in disjoint cubes $\{I_r\}_{r \in \mathbb{N}}$

and all cubes I_0 such that $\sigma(I_0) \neq 0$, where

$$E(I,\omega)^2 \equiv \frac{1}{2}\mathbb{E}_I^{\omega(dx)}\mathbb{E}_I^{\omega(dx')}\frac{(x-x')^2}{|I|^2} \tag{2.2.12}$$

and its dual $\mathcal{E}^*$ where σ and ω are interchanged.

In the same paper, Lacey, Sawyer and Uriarte-Tuero proved that a hybrid of the Pivotal and Energy conditions was sufficient but not necessary in the two weight inequality for the Hilbert transform.

Both the energy and the pivotal conditions, sometimes referred to as “buffer conditions", are used to approximate certain forms that appear in the proofs of almost all two weight inequalities. The NTV conjecture states that we can prove (2.0.1) without assuming them.

It is true though that if both ω, σ are individually $\mathcal{A}_\infty$ weights, the classical $\mathcal{A}_p(\omega,\sigma)$ condition implies the Pivotal condition providing a short and elegant proof of the NTV-conjecture for $\mathcal{A}_\infty$ weights assuming existing $T1$ theory. Earlier, Sawyer in [58] gave a proof using different methods for the case of smooth kernels.

Theorem 2.2.3. *($T1$ theorem for $\mathcal{A}_\infty$ weights) Assume ω, σ are in A_∞, T is an α-fractional singular integral and we have the $T1$ testing and the fractional $\mathcal{A}_2^\alpha(\omega,\sigma)$ conditions to hold, along with their duals. Then, T is bounded on $L^2(\mathbb{R}^n)$.*

We can get another proof of this theorem for singular integrals by using theorem 3 in [47] by inserting an A_p weight w between ω, σ and then using the one weight results in [7] or [20].

2.2.3 The relationship between the two weight $\mathcal{A}_p$ and "buffer" conditions.

It is shown in [32] that we can have a pair of measures satisfying the tailed $\mathcal{A}_2$ conditions (2.2.2), (2.2.3) but failing to satisfy the Pivotal condition (2.2.10), hence proving the implication $\mathcal{A}_2^{t_2} \not\Rightarrow \mathcal{V}^2$.

We show here that the Pivotal condition (2.2.10) does not imply the tailed $\mathcal{A}_2$ conditions (2.2.2), (2.2.3).

Theorem 2.2.4. *($\mathcal{V}^p \not\Rightarrow \mathcal{A}_p^{t_1}$) Let $1 < p \leq 2$. The Pivotal condition $\mathcal{V}^p$ does not imply the one tailed $\mathcal{A}_p$ condition $\mathcal{A}_p^{t_1}$.*

Remark 2.2.5. *It is immediate from* (2.2.12) *that the Energy condition* (2.2.11) *is dominated by the Pivotal condition* (2.2.10) *hence we immediately get the following important corollary.*

Corollary 2.2.6. *($\mathcal{E} \not\Rightarrow \mathcal{A}_2^{t_1}$) Let $1 < p \leq 2$. The Energy condition $\mathcal{E}$ does not imply the one tailed $\mathcal{A}_p$ condition $\mathcal{A}_p^{t_1}$.* □

2.2.4 Known cases of the NTV conjecture.

While the general case of the NTV conjecture in $\mathbb{R}^n$ is still not completely understood, several important special cases have been completely solved.

First, in the two part paper by Lacey, Sawyer, Shen and Uriarte-Tuero [34] and Lacey [27] proved the NTV conjecture, namely that $\mathcal{A}_p(\omega, \sigma)$ and testing conditions are necessary and sufficient for (2.0.1) to hold, assuming also that the measures σ and ω had no common point masses, for the Hilbert Transform. Hytönen [22] with his new offset version of $\mathcal{A}_2$

$$\mathcal{A}_2^{\text{offset}}(\omega,\sigma) = \sup_I \frac{\omega(I)}{|I|} \int_{\mathbb{R}^n \setminus I} \left(\frac{|I|^{\frac{1}{n}}}{(|I|^{\frac{1}{n}} + \operatorname{dist}(x,I))^2} \right)^n \sigma(dx) < \infty \tag{2.2.13}$$

removed the restriction of common point masses on σ, ω. An alternate approach using "punctured" versions of $\mathcal{A}_2$ appears in [67].

Other important cases include first Sawyer, Shen, Uriarte-Tuero [64] for α-fractional singular integrals, Lacey-Wick [37] for the Riesz transforms, Lacey, Sawyer, Shen, Uriarte-Tuero and Wick [35] for the Cauchy transform and Sawyer, Shen, Uriarte-Tuero [65] for the Riesz tranform when a measure is supported on a curve in $\mathbb{R}^n$ and recently [58] for general Calderon-Zygmund operators and doubling measures that also satisfy the fractional $\mathcal{A}_\infty^\alpha$ condition, check (3.1.16). The NTV conjecture is yet to be proven for a general operator T.

2.3 Tb theory

The original $T1$ theorem of David and Journé [10], which characterized boundedness of a singular integral operator by testing over indicators $\mathbf{1}_Q$ of cubes Q, was quickly extended to a Tb theorem by David, Journé and Semmes [11], in which the indicators $\mathbf{1}_Q$ were replaced by testing functions $b\mathbf{1}_Q$ for an accretive function b, i.e. $0 < c \leq Reb \leq |b| \leq C < \infty$. Here the accretive function b could be chosen to adapt well to the operator at hand, resulting in almost immediate verification of the b-testing conditions, despite difficulty in verifying the 1-testing conditions. One motivating example of this phenomenon is the boundedness of the Cauchy integral on Lipschitz curves, easily obtained from the above Tb theorem[1]. See e.g.

[1] The problem reduces to boundedness on $L^2(\mathbb{R})$ of the singular integral operator C_A with kernel $K_A(x,y) \equiv \frac{1}{x-y+i(A(x)-A(y))}$, where the curve has graph $\{x+iA(x) : x \in \mathbb{R}\}$. Now $b(x) \equiv 1+iA'(x)$ is

[72, pages 310-316].

Subsequently, M. Christ [5] obtained a far more robust *local* Tb theorem in the setting of homogeneous spaces, in which the testing functions could be further specialized to $b_Q \mathbf{1}_Q$, where now the accretive functions b_Q can be *chosen by the reader to differ* for *each* cube Q. Applications of the local Tb theorem included boundedness of layer potentials, see e.g. [2] and references there; and the Kato problem, see [19], [18] and [3]: and many authors, including G. David [8]; Nazarov, Treil and Volberg [43], [42]; Auscher, Hofmann, Muscalu, Tao and Thiele [4], Hytönen and Martikainen [24], and more recently Lacey and Martikainen [29], set about proving extensions of the local Tb theorem, for example to include a single upper doubling weight together with weaker upper bounds on the function b. But these extensions were modelled on the 'nondoubling' methods that arose in connection with upper doubling measures in the analytic capacity problem, see Mattila, Melnikov and Verdera [38], G. David [8], [9], X. Tolsa [74], and alsoVolberg [75], and were thus constrained to a single weight - a setting in which both the Muckenhoupt and energy conditions follow from the upper doubling condition.

More recently, in a precursor to the present result, [69] obtained a general two weight Tb theorem for the Hilbert transform on the real line. Here, we extend this precursor to higher dimensions. As in [69], we adapt methods from the theory of two weight $T1$ theorems, which arose from [44], [75], [34], [27], [64] and [66], and were used in [24] as well, to prove a two weight local Tb theorem. These methods involve the 'testing' perspective toward characterizing two weight norm inequalities for an operator T. As suggested by work originating in [10] and [55], it is plausible to conjecture that a given operator T is

accretive and the b-testing condition $\int_I \left|C_A\left(\mathbf{1}_I b\right)(x)\right|^2 dx \leq \mathfrak{T}_H^b |I|$ follows from $\left|C_A\left(\mathbf{1}_I b\right)(x)\right|^2 \approx \ln \frac{x-\alpha}{\beta-x}$, for $x \in I = [\alpha, \beta]$. In the case of a $C^{1,\delta}$ curve, the kernel K_A is $C^{1,\delta}$ and a Tb theorem applies with $T = C_A$ and $\sigma = \omega = dx$, to show that C_A is bounded on $L^2(\mathbb{R})$.

bounded from one weighted space to another if and only if both it and its dual are bounded when tested over a suitable family of functions related geometrically to T, e.g. testing over indicators of intervals for fractional integrals T as in [55].

The main two weight local Tb theorem: Here is a brief statement of our main theorem.

Theorem 2.3.1 (local Tb in higher dimensions)**.** *Let T^{α} denote a Calderón-Zygmund operator on $\mathbb{R}^n$, and let σ and ω be locally finite positive Borel measures on $\mathbb{R}^n$ that satisfy the energy and Muckenhoupt buffer conditions. Then T^{α}_{σ}, where $T^{\alpha}_{\sigma} f \equiv T^{\alpha}(f\sigma)$, is bounded from $L^2(\sigma)$ to $L^2(\omega)$* if and only if *the* $\mathbf{b}$*-testing and* $\mathbf{b}^*$*-testing conditions*

$$\int_I \left|T^{\alpha}_{\sigma} b_I\right|^2 d\omega \leq \left(\mathfrak{T}^{\mathbf{b}}_{T^{\alpha}}\right)^2 |I|_{\sigma} \text{ and } \int_J \left|T^{\alpha,*}_{\omega} b^*_J\right|^2 d\sigma \leq \left(\mathfrak{T}^{\mathbf{b}^*,*}_{T^{\alpha}}\right)^2 |J|_{\omega} , \tag{2.3.1}$$

taken over two families of test functions $\{b_I\}_{I\in\mathcal{P}}$ and $\{b^*_J\}_{J\in\mathcal{P}}$, where b_I and b^*_J are only required to be nondegenerate in an average sense, and to be just slightly better than L^2 functions themselves, namely L^p for some $p > 2$.

2.3.1 Challenges in higher dimensional two weight Tb theory and a counterexample to Hytönen's off testing condition

A number of difficulties arise in generalizing to higher dimensions the work that was done in [69] for dimension $n = 1$. The main difficulty lies in the strictly-one dimensional nature of a fundamental inequality of Hytönen, namely that local testing, i.e. testing the integral of $\left|T_{\sigma}\mathbf{1}_Q\right|^2$ over the cube Q, together with the A_2 condition, imply full testing, meaning that $\left|T_{\sigma}\mathbf{1}_Q\right|^2$ is integrated over the entire space $\mathbb{R}^n$. For the proof of full testing, Hytönen uses an inequality for the Hardy operator that is true only in dimension $n = 1$ - in fact we prove that this property of the Hardy operator is not available in higher dimensions. Before stating the theorem we need to define the fractional energy and the off testing conditions.

Definition 2.3.2. *We say that the pair (σ,ω) satisfies the energy (resp. dual energy) condition if*

$$(\mathcal{E}_2^{\alpha})^2 \equiv \sup_{Q=\dot{\cup}Q_r} \frac{1}{\sigma(Q)} \sum_{r=1}^{\infty} \left(\frac{\mathrm{P}^{\alpha}\left(Q_r, \mathbf{1}_Q\sigma\right)}{|Q_r|^{\frac{1}{n}}} \right)^2 \left\| x - m_{Q_r}^{\omega} \right\|^2_{L^2\left(\mathbf{1}_{Q_r}\omega\right)} < \infty$$

$$\left(\mathcal{E}_2^{\alpha,*}\right)^2 \equiv \sup_{Q=\dot{\cup}Q_r} \frac{1}{\omega(Q)} \sum_{r=1}^{\infty} \left(\frac{\mathrm{P}^{\alpha}\left(Q_r, \mathbf{1}_Q\omega\right)}{|Q_r|^{\frac{1}{n}}} \right)^2 \left\| x - m_{Q_r}^{\sigma} \right\|^2_{L^2\left(\mathbf{1}_{Q_r}\sigma\right)} < \infty$$

where the supremum is taken over arbitrary decompositions of a cube Q using a pairwise disjoint union of subcubes Q_r, where $\mathrm{P}^{\alpha}(Q,\mu)$ is the standard Poisson integral and

$$m_I^{\mu} \equiv \frac{1}{\mu(I)} \int x d\mu(x) = \left\langle \frac{1}{|I|_{\mu}} \int x_1 d\mu(x), ..., \frac{1}{|I|_{\mu}} \int x_n d\mu(x) \right\rangle.$$

Definition 2.3.3. *The off-testing constants $\mathcal{T}_{\mathit{off},\alpha}$ and $\mathcal{R}_{j,\mathit{off},\alpha}$ in $\mathbb{R}^2$ by*

$$\mathcal{T}^2_{\mathit{off},\alpha} = \sup_Q \frac{1}{\omega(Q)} \int_{\mathbb{R}^2\setminus Q} \left(\int_Q \frac{1}{|x-y|^{2-\alpha}} d\omega(y) \right)^2 d\sigma(x)$$

$$\mathcal{R}^2_{m,\mathit{off},\alpha} = \sup_Q \frac{1}{\omega(Q)} \int_{\mathbb{R}^2\setminus Q} \left(\int_Q \frac{t_m - x_m}{|x-t|^{3-\alpha}} d\omega(t) \right)^2 d\sigma(x), \quad 1 \le m \le 2$$

for all cubes $Q \subset \mathbb{R}^2$ whose sides are parallel to the axes.

Theorem 2.3.4. *For $0 \le \alpha < 2$, there exists a pair of locally finite Borel measures σ, ω in $\mathbb{R}^2$ such that the fractional Muckenhoupt $\mathcal{A}_2^{\alpha}, \mathcal{A}_2^{\alpha,*}$ and the energy $\mathcal{E}_2^{\alpha}$, $\mathcal{E}_2^{\alpha,*}$ constants are finite but the off-testing constant $\mathcal{T}_{\mathit{off},\alpha}$ is not.*

Theorem 2.3.5. *For $0 \leq \alpha < 2$, there exists a pair of locally finite Borel measures σ, ω in $\mathbb{R}^2$ such that the fractional Muckenhoupt $\mathcal{A}_2^\alpha, \mathcal{A}_2^{\alpha,*}$ and the energy $\mathcal{E}_2^\alpha$, $\mathcal{E}_2^{\alpha,*}$ constants are finite but the off-testing constants $\mathcal{R}_{m,off,\alpha}$ are not.*

With full testing in hand, we obtain a number of properties that greatly simplify matters but we no longer have this tool. Here are the main challenges encountered in passing from the one-dimensional setting to the higher dimensional analog.

1. **The nearby form**. The main difficulty in proving the Tb theorem in dimensions $n > 1$ arises in treating the nearby form in Chapter 5. Full testing is used repeatedly everywhere in this chapter, and a demanding technical approach involving random surgery and averaging, is needed throughout this chapter. In particular, to obtain estimates over adjacent cubes, we decomposed one of the cubes into a smaller rectangle that is separated from the other cube by a halo. The separated part is estimated by the Muckenhoupt's A_2 condition, while the halo is estimated by applying probability over grids. A typical example is the following: Let I be a cube in the grid associated to the function f and J a cube in the grid associated to the function g. Let also b_I, b_J^* be the testing functions used in the theorem for these cubes.

   We would like to estimate $\int T_\sigma^\alpha \left(b_I \mathbf{1}_{I\setminus J}\right) b_J^* \mathbf{1}_J d\omega$. The domains of integration inside the operator and inside the integral are adjacent. In dimension $n = 1$ we could use Hytönen's result. Now we instead argue by splitting the integral as follows:

$$\left|\int T_\sigma^\alpha \left(b_I \mathbf{1}_{I\setminus J}\right) b_J^* \mathbf{1}_J d\omega\right| \leq \left|\int T_\sigma^\alpha \left(b_I \mathbf{1}_{I\setminus(1+\delta)J}\right) b_J^* \mathbf{1}_J d\omega\right| + \left|\int T_\sigma^\alpha \left(b_I \mathbf{1}_{(I\setminus J)\cap(1+\delta)J}\right) b_J^* \mathbf{1}_J d\omega\right|.$$

The first term on the right hand side, where the domains inside the operator and the integral are disjoint with positive distance, is bounded by a constant multiple, depending on δ and n, times the A_2 constant. Using averaging over grids, the second term on the right hand side is bounded by $\delta\mathfrak{N}_T\alpha$ where the small δ gain comes from the fact that $|(I\backslash J)\cap(1+\delta)J|^{\frac{1}{n}} \approx \delta|I|$ where $|\cdot|$ denotes the Lebesque measure of the cube.

2. **Splitting forms**. Here we begin with a pair of smooth compactly supported functions (f, g) and we would like to decompose the functions into their Haar expansions. However, when we select a grid $\mathcal{G}$ for f, the support of f may not be contained in any of the dyadic cubes in the grid $\mathcal{G}$, with a similar problem when selecting a grid $\mathcal{H}$ for g. To deal with this, we follow NTV by adding and subtracting certain averages for these terms, resulting in four integrals to be controlled by our hypotheses. In the one dimensional setting, full testing was used to eliminate three out of the four such integrals that appear after decomposing the functions in sums of martingale differences. Here in this paper, the argument was adjusted to avoid using full testing by averaging over the two grids $\mathcal{G}$ and $\mathcal{H}$ associated with f and g.

3. **Pointwise Lower Bound Property (PLBP)**. In [69] for $n = 1$, the $PLBP$ was used to control terms involving certain 'modified dual martingale differences' in which a factor b_Q had been removed. Moreover, it was proved there that, without loss of generality, the p-weakly accretive families of testing functions b_Q and b_Q^* for $Q \in \mathcal{P}$ could be assumed to satisfy the *pointwise lower bound property*, written $PLBP$:

$$\left|b_Q(x)\right| \geq c_1 > 0 \qquad \text{for } Q \in \mathcal{P} \text{ and } \sigma\text{-a.e. } x \in \mathbb{R},$$

for some positive constant c_1. However, this reduction to assuming $PLBP$ depended heavily on Hytönen's $\mathcal{A}_2$ characterization for supports on disjoint intervals, something that is unavailable in higher dimensions as the following theorem shows:

To circumvent this difficulty we used an observation (that goes back to Hytönen and Martikainen) that under the additional assumption that the *breaking cubes* Q, those for which there is a dyadic child Q' of Q with $b_{Q'} \neq \mathbf{1}_{Q'} b_Q$, satisfy an appropriate Carleson measure condition.

4. **Indented corona**. In chapter 8 (dealing with the stopping form) we construct an 'indented corona'. In dimension $n = 1$ this construction simply reduces to consideration of the 'left and right ends' of the intervals. In the absence of 'right and left ends' in higher dimensions, this simple construction is replaced by a more intricate tower of Carleson cubes.

2.4 Organization of the paper

In chapter 3, section 3.1 we prove theorems 2.1.1 and 2.1.2. In section 3.2 we prove theorem 2.2.1 and theorem 2.2.2. We prove the $T1$ theorem for $\mathcal{A}_\infty$ weights, theorem 2.2.3, in subsection 3.2.3, using the Sawyer testing condition (see (2.2.9) and theorem 3.2.6). In subsection 3.2.4 we prove theorem 2.2.4 and give a partial answer to question 2 in theorem 3.2.11. In chapter 4 we prove theorems 2.3.4 and 2.3.5. In chapter 5 we prove theorem 2.3.1. Check the lattices and the graph in section 2.5 for a summary of the theorems presented in this paper and the history of weighted theory.

2.5 Open problems, lattices and a historic diagram

Here is a list of open questions.

1. The most difficult and important problem in the theory of $T1$ and Tb arises from the fact that, while the Muckenhoupt buffer conditions are necessary for boundedness of a wide range of singular integrals, the *energy* buffer conditions are only necessary for boundedness of the Hilbert transform and some perturbations in dimension $n = 1$, see [57], [68]. What is a reasonable substitute for the energy buffer conditions in a $T1$ or Tb theorem?

2. Does Theorem 2.3.1 remain true in the case $p = 2$, i.e. when $\mathbf{b} = \left\{b_Q\right\}_{Q\in\mathcal{P}}$ is a 2-weakly σ-accretive family of functions, and $\mathbf{b}^* = \left\{b_Q^*\right\}_{Q\in\mathcal{P}}$ is a 2-weakly ω-accretive family of functions?

3. In the special case of the Hilbert transform in dimension $n = 1$, are the energy conditions in Theorem 2.3.1 already implied by the Muckenhoupt, $\mathbf{b}$-testing and dual $\mathbf{b}^*$-testing conditions for a pair of p-weakly accretive families, $p > 2$?

The following lattices provide a summary of the implications among the necessary and sufficient conditions in weighted theory.

One weight conditions

Combining (3.1.3), theorem 2.1.1, *remark 3.1.3*, *remark 3.1.4*, *remark 3.1.5* and theorem 3.1.6 we get, for $p < q$, the following lattice of inclusions for the conditions used in one weighted theory

$$
\begin{array}{ll}
 & \mathcal{A}^{\alpha}_{\infty}(\omega) \cap \mathcal{D}(\omega) \subsetneq \begin{cases} \mathcal{D}(\omega) \\ \\ \mathcal{A}^{\alpha}_{\infty}(\omega) \end{cases} \\
 & \rotatebox{45}{$\subsetneq$} \\
A_1(\omega) \subsetneq A_p(\omega) \subsetneq A_q(\omega) \subsetneq A_\infty(\omega) & \\
 & \rotatebox{-45}{$\subsetneq$} \\
 & C_p(\omega) \cap \mathcal{D}(\omega) \subsetneq \begin{cases} \mathcal{D}(\omega) \\ \\ C_p(\omega) \end{cases}
\end{array}
$$

Two weight conditions

Combining *remark 3.2.1*, theorem 2.2.1, theorem 2.2.2, *remark 3.2.9*, theorem 2.2.4, *remark 2.2.5*, corollary 2.2.6, theorem 3.1.6, theorem 3.2.6, corollary 3.2.7 and the example in [32] we get the following lattice of inclusions for the conditions used in two weighted theory.

For general Radon measures:

Theorem 2.2.1,

remark 3.2.1, *remark 3.2.9*, [32]: $\mathcal{V}(\omega,\sigma)^p \subsetneq \mathcal{A}_p(\omega,\sigma) \subsetneq \mathcal{A}_p^{t_1}(\omega,\sigma) \cup \mathcal{A}_p^{t_1}(\sigma,\omega)$

$$A_p^{t_1}(\omega,\sigma) \cap A_p^{t_1}(\sigma,\omega) = A_p^{t_2}(\omega,\sigma)$$

$$A_p^{t_1}(\omega,\sigma) \subsetneq A_p^{t_2}(\omega,\sigma)$$

Remark 2.2.5,

theorem 2.2.4, corollary 2.2.6: $\mathcal{E}(\omega,\sigma)^p \subsetneq \mathcal{V}(\omega,\sigma)^p \not\Longrightarrow \mathcal{A}_p^{t_1}(\omega,\sigma) \subsetneq \mathcal{A}_p^{t_2}(\omega,\sigma)$

For doubling measures:

Theorem 2.2.2:	$\mathcal{A}_p(\omega,\sigma) = A_p^{t_1}(\omega,\sigma) = \mathcal{A}_p^{t_2}(\omega,\sigma)$
Theorem 3.2.6, corollary 3.2.7:	$\mathcal{A}_p(\omega,\sigma) \cap \mathcal{A}_\infty(\omega) \subsetneq S_d(\omega,\sigma) \subseteq \mathcal{V}(\omega,\sigma)^p$
Theorem 3.2.11:	$\mathcal{A}_p(\omega,\sigma) \cap \mathcal{D}(\sigma) \cap \mathcal{D}(\omega) \subsetneq \mathcal{V}(\omega,\sigma)^p$
	(small doubling constant)

We end the introduction with a diagram detailing the relevant history of two weight theory for this paper. Many important contributions are omitted, such as those dealing with L^p, L^q assumptions in the case of Lebesgue measure, see for example [17] and references there, and results for dyadic operators, see for example [4] and references there.

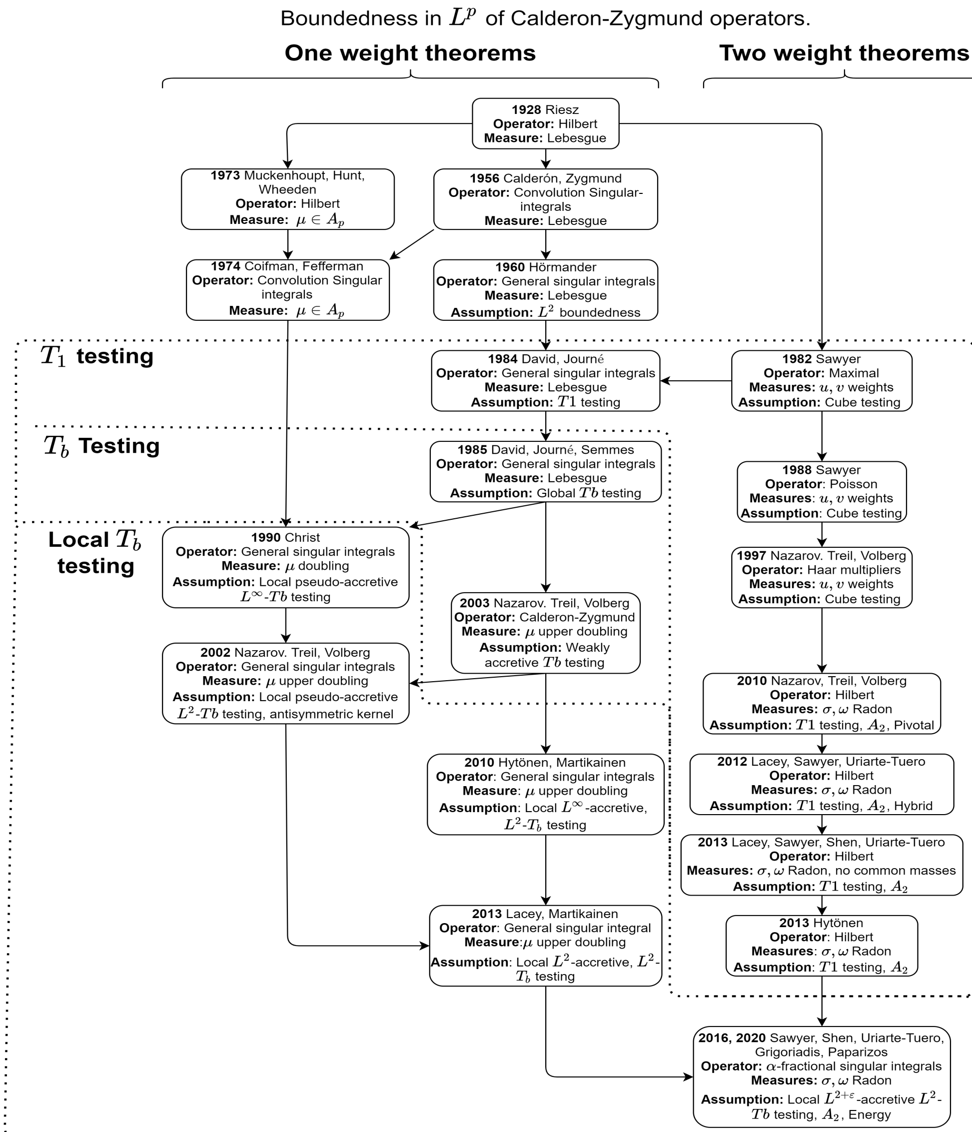

Figure 2.5.1: Theory development diagram

As is evident from the diagram, theorem 2.3.1 (and its precursor for $n = 1$) is the first local Tb theorem for two weights.

Chapter 3

Necessary and sufficient conditions in weighted theory

3.1 One weight conditions

3.1.1 The A_1 and A_∞ conditions.

We say the weight $w(x)$ is an A_1 weight if and only if

$$Mw(x) \leq [w]_{A_1} w(x) \tag{3.1.1}$$

and we call $[w]_{A_1}$ the A_1 constant of w. A_1 is a stronger condition than the A_p condition for $p > 1$.

If we take the union of all the A_p weights for the different p we get the larger class of A_∞ weights, i.e. $A_\infty = \bigcup_{p>1} A_p$ (check [12], chapter 7). Another equivalent and commonly used characterization for A_∞ weights is the following: We say $w \in A_\infty$, if for all $I \subset \mathbb{R}^n$ and $E \subset I$, there exist uniform constants $C, \varepsilon > 0$ such that

$$\frac{w(E)}{w(I)} \leq C \left(\frac{|E|}{|I|} \right)^{\varepsilon}. \tag{3.1.2}$$

Remark 3.1.1. *We have the following linear lattice for $1 < p < q < \infty$:*

$$A_1 \subsetneq A_p \subsetneq A_q \subsetneq A_\infty \tag{3.1.3}$$

The power weights $w(x) = |x|^\alpha$ show that all the inclusions are proper.

In particular we have the following known lemma.

Lemma 3.1.2. *Let $w(x) = |x|^\alpha$, $x \in \mathbb{R}^n$. Then*

$$[w]_{\mathcal{A}_p} \approx \begin{cases} (\alpha+n)^{-1}(-\alpha\frac{p\prime}{p}+n)^{-\frac{p}{p\prime}}, & -n < \alpha < n(p-1) \\ \infty, & otherwise \end{cases}$$

□

3.1.2 A C_p and doubling weight that is not in A_∞

For the rest of this section, we are going to say that a measure ω is doubling if

$$\omega(3I) \leq C\omega(I). \tag{3.1.4}$$

This definition is equivalent to (1.6.1). In this subsection we give the proof for theorem 2.1.1. The construction is a very involved variation of the construction in [15].

Proof of theorem 2.1.1: Let $I_0 = [-\frac{1}{2}, \frac{1}{2}]$ and $I_n = 3I_{n-1} = 3^n I_0$, the intervals centered at 0 with length 3^n. We call $\mathcal{G}$ the triadic grid created by the intervals I_n. Define the measure w as follows: $w(x) = 1$, $x \in I_0$ and $w(I_n) = \frac{1}{\delta_1^n}$, $\frac{1}{3} > \delta_1 > 0$ to be determined later.

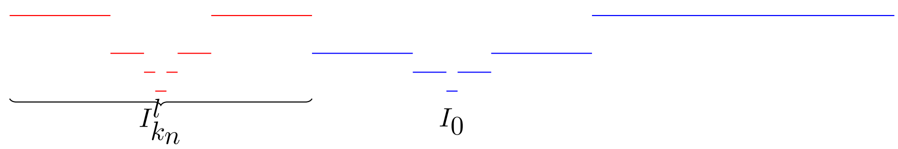

Figure 3.1.1: Building construction

Call I_n^l, I_n^m, I_n^r the left, middle and right third of I_n respectively. Let $w(x) = \frac{3^{-n+1}(1-\delta_1)}{2\delta_1^n}$, $x \in I_n^r$.

Fix $k \in \mathbb{N}$ and $n_k \in \mathbb{N}$ to be determined later. Let $I_{n_k}^{l,m}$ to have the same center as $I_{n_k}^l$ and $|I_{n_k}^{l,0}| = 3^m$, $0 \leq m \leq n_k - 1$. Let $w(I_{n_k}^{l,m}) = \delta_2^{n_k - m} w(I_{n_k}^l)$, where $\frac{1}{3} > \delta_2 > 0$ to be determined later. For $m \geq 2$ let $w(x) = \frac{3(1-\delta_2)}{2|I_{n_k}^{l,m}|} w\left(I_{n_k}^{l,m}\right)$, for all $x \in I_{n_k}^{l,m} \backslash I_{n_k}^{l,m-1}$. This defines w completely outside $3I_{n_k}^{l,0}$. Check the figure below.

Now let $I \subset I_{n_k}^{l,0}$ be any triadic interval such that $|I| \geq 3^{-i_k}$, and $i_k \in \mathbb{N}$ will be determined later. Let

$$w(I) = \begin{cases} \delta_2 w(\pi I) & \text{if } \partial I \cap \partial \pi I = \emptyset \\ \frac{1-\delta_2}{2} w(\pi I) & \text{if } \partial I \cap \partial \pi I \neq \emptyset \end{cases}$$

where πI is the triadic parent of I in the grid $\mathcal{G}$. Let $w(x)$ be constant for any triadic interval $I \subset I_{n_k}^{l,0}$ with $|I| \leq 3^{-i_k}$.

We are left with defining w on $3I_{n_k}^{l,0} \backslash I_{n_k}^{l,0}$. Call $J_{n_k}^l$ the left third of $3I_{n_k}^{l,0}$ and $J_{n_k}^r$ its right third. Let $J_{n_k}^{l,i_k}$ be the right most triadic i_k child of $J_{n_k}^l$ and let $w(x) = \left(\frac{1-\delta_2}{2}\right)^{i_k} w(J_{n_k}^l)$, $x \in J_{n_k}^{l,i_k}$. Now for all triadic I such that $J_{n_k}^{l,i_k} \subset I \subset J_{n_k}^l$, let I^l, I^m, I^r denote the left, middle and right thirds of I and define $w(x) = \frac{3(1-\delta_2)}{2|I|} w(I)$, $x \in I^l$, $w(x) = \frac{3\delta_2}{|I|} w(I)$,

$x \in I^m$ and $w(I^r) = \frac{1-\delta_2}{2} w(I)$. Similarly (but on the left end) we define w on $J^r_{n_k}$. This construction on $J^l_{n_k}$ and $J^r_{n_k}$ is done so that w is doubling.

Indeed, to see that w is doubling, let J_1, J_2 be two triadic intervals of the same length that touch. If they have the same triadic parent then $w(J_1)/w(J_2) \lesssim \frac{1}{\min(\delta_1,\delta_2)}$. If not, we apply the first case to their common ancestor and get again $w(J_1)/w(J_2) \lesssim \frac{1}{\min(\delta_1,\delta_2)}$. For an arbitrary interval I, let $3^m \leq |I| \leq 3^{m+1}$. Then $I \subset J_1 \cup J_2$ triadic intervals with $|J_1| = |J_2| = 3^{m+1}$. Then $w(3I) \lesssim \frac{1}{\min(\delta_1,\delta_2)} w(I)$.

Allowing $i_k \to \infty$ makes w singular to Lebesgue. Check ([15], Lemma 2.2). Choose i_k so that there exists an interval J_{n_k} and $E_{n_k} \subset J_{n_k} \subset 3I^{l,0}_{n_k}$,such that

$$\frac{w\left(E_{n_k}\right)}{w\left(J_{n_k}\right)} \approx \frac{1}{2}, \quad \frac{|E_{n_k}|}{|J_{n_k}|} \approx \frac{1}{2^k} \quad \text{and} \quad \frac{w\,(E)}{w\,(I)} \lesssim 2^k \frac{|E|}{|I|} \tag{3.1.5}$$

for all intervals $I \subset 3I^{l,0}_{n_k}$ and $E \subset I$. This can be done by following in [15] definition 2.1. and lemma 2.2. Note that because we stop at height i_k, (3.1.5) tells us that there is a “worst interval" J_{n_k}.

By letting $k \to \infty$ it is clear that A_∞ fails to hold for w. So we now need to prove that the C_p condition holds.

By the end of the next calculation we will determine δ_1, δ_2. We want to prove w is C_p and for that we need to show that (2.1.4) holds with $\int_{\mathbb{R}} (M\mathbf{1}_I\,(x))^p\, w\,(x)\, dx < \infty$ for any interval I. Let first, $I = I^{l,0}_{n_k}$.

We have

$$\int_{\mathbb{R}} \left(M\mathbf{1}_{I_{n_k}^{l,0}}(x) \right)^p w(x)\,dx = \int_{I_{n_k}^{l,0}} \left(M\mathbf{1}_{I_{n_k}^{l,0}}(x) \right)^p w(x)\,dx + \tag{3.1.6}$$

$$+ \int_{I_{n_k}^{l} \setminus I_{n_k}^{l,0}} \left(M\mathbf{1}_{I_{n_k}^{l,0}}(x) \right)^p w(x)\,dx + \int_{\mathbb{R}\setminus I_{n_k}^{l}} \left(M\mathbf{1}_{I_{n_k}^{l,0}}(x) \right)^p w(x)\,dx$$

$$\equiv A + B + C$$

We have immediately $A = w\left(I_{n_k}^{l,0}\right)$, for B we get

$$\begin{aligned} B &= \int_{I_{n_k}^{l} \setminus I_{n_k}^{l,0}} \left(\frac{|I_{n_k}^{l,0}|}{2|I_{n_k}^{l,0}| + 2dist\left(x, I_{n_k}^{l,0}\right)} \right)^p w(x)\,dx \\ &\approx 2^{-p}(1-\delta_2) \sum_{m=1}^{n_k-1} \frac{3^{-mp}}{\delta_2^m} w\left(I_{n_k}^{l,0}\right) = 2^{-p}(1-\delta_2)\, w\left(I_{n_k}^{l,0}\right) \sum_{m=1}^{n_k-1} \left(\frac{3^{-p}}{\delta_2}\right)^m \end{aligned}$$

Now choose $\delta_2 = \frac{3^{-p}}{2}$ so that the series above diverges (any $\delta_2 \le 3^{-p}$ works here). We also want n_k so that

$$2^{-p}(1-\delta_2)\, w\left(I_{n_k}^{l,0}\right) \sum_{m=1}^{n_k-1} \left(\frac{3^{-p}}{\delta_2}\right)^m \gtrsim 2^k w\left(I_{n_k}^{l,0}\right). \tag{3.1.7}$$

We are only left with calculating term C. We have,

$$\begin{aligned} C &= \int_{\mathbb{R}\setminus I_{n_k}^{l}} \left(\frac{|I_{n_k}^{l,0}|}{2|I_{n_k}^{l,0}| + 2dist\left(x, I_{n_k}^{l,0}\right)} \right)^p w(x)\,dx \\ &\approx 2^{-p}3^{-n_k p}(1-\delta_1) \frac{1-\delta_2}{\delta_2^{n_k-1}} w\left(I_{n_k}^{l,0}\right) \sum_{m=1}^{\infty} \left(\frac{3^{-p}}{\delta_1}\right)^m \end{aligned} \tag{3.1.8}$$

Choose $\delta_1 > 3^{-p}$ so that the infinite series converges. Combining the estimates for A, B and

C we get:

$$\int_{\mathbb{R}} \left(M\mathbf{1}_{I^{l,0}_{n_k}}(x) \right)^p w(x)\,dx < \infty \tag{3.1.9}$$

and

$$\frac{w(E)}{\int_{\mathbb{R}} \left(M\mathbf{1}_{I^{l,0}_{n_k}}(x) \right)^p w(x)\,dx} \leq \frac{w(E)}{2^k w(I^{l,0}_{n_k})} \lesssim \frac{2^k}{2^k} \frac{|E|}{|I^{l,0}_{n_k}|} = \frac{|E|}{|I^{l,0}_{n_k}|}. \tag{3.1.10}$$

for $E \subset I^{l,0}_{n_k}$. We want to extend (3.1.9) and (3.1.10) to all triadic intervals. Note that (3.1.9) holds for any interval I. To see that, choose n big enough so that $I \subset I_n$. Then, following the calculations for estimating C in (3.1.8) we get that

$$\int_{\mathbb{R}\setminus I_n} (M\mathbf{1}_I(x))^p w(x)\,dx < \infty$$

which of course gives us

$$\int_{\mathbb{R}} (M\mathbf{1}_I(x))^p w(x)\,dx < \infty \tag{3.1.11}$$

To get (3.1.10) for any triadic $I \subset 3I^{l,0}_{n_k}$, note that we can follow the same calculations that led to (3.1.7) and just choose n_k big enough so that we get the gain $2^k w(I)$. This is possible since the construction is finite and it stops at some height i_k. For that finite number of intervals, we choose n_k big enough so that all the intervals get the gain 2^k, i.e. $\int_{\mathbb{R}} (M\mathbf{1}_I(x))^p w(x)\,dx \geq 2^k w(I)$. So we have for any $E \subset I$, using (3.1.5),

$$\frac{w(E)}{\int_{\mathbb{R}} (M\mathbf{1}_I(x))^p w(x)\,dx} \leq \frac{w(E)}{2^k w(I)} \lesssim \frac{|E|}{|I|}. \tag{3.1.12}$$

We will use the following calculation for triadic intervals $I \subset I^l_{n_k}$. Let $I = 3I^{l,0}_{n_k}$, following

(3.1.6) and using $\delta_2 = \frac{3-p}{2}$,

$$\int_{\mathbb{R}} (M1_{\mathbf{I}})^p dw \equiv A' + B' + C'.$$

and $A' + B' \approx 3^p(A+B)$ hence

$$\int_{I^l_{n_k}} (M\mathbf{1}_I(x))^p\, w(x)\, dx \approx 3^p \int_{I^l_{n_k}} \left(M\mathbf{1}_{I^{l,0}_{n_k}}(x) \right)^p w(x)\, dx$$

and

$$\frac{w(E)}{\int_{\mathbb{R}} (M\mathbf{1}_I(x))^p\, w(x)\, dx} \lesssim \frac{w(E)}{3^p 2^k w(I^{l,0}_{n_k})} \lesssim 3^{1-p} \frac{|E|}{|I|} \leq \frac{|E|}{|I|} \tag{3.1.13}$$

for any $E \subset 3I^{l,0}_{n_k}$, so we don't lose any of the "gain" necessary for (3.1.12) to hold. We can repeat for all triadic intervals I such that $I^{l,0}_{n_k} \subset I \subset I^l_{n_k}$. Note that for $I = I^l_{n_k}$ $B' = 0$. To extend (3.1.13) to triadic intervals $I \supset I^l_{n_k}$ notice that

$$\frac{w(E)}{w(\pi(I))} \lesssim \delta_1 \frac{w(E)}{w(I)} \lesssim \delta_1 \frac{|E|}{|I|} \leq 3\delta_1 \frac{|E|}{|\pi(I)|} \leq \frac{|E|}{|\pi(I)|} \Longrightarrow \frac{w(E)}{w(\pi(I))} \lesssim \frac{|E|}{|\pi(I)|}$$

for any $E \subset 3I^{l,0}_{n_k}$, where we used $\delta_1 < \frac{1}{3}$.

To get (3.1.12) for an arbitrary triadic interval, let I be a triadic interval not contained in any $I^{l,0}_{n_k}$ and E any subset of I. We write

$$E = \left(\bigcup_{I^{l,0}_{n_k} \subset I} \left(E \cap 3I^{l,0}_{n_k} \right) \right) \bigcup \left(E \setminus \bigcup_{I^{l,0}_{n_k} \subset I} 3I^{l,0}_{n_k} \right) = E_1 \cup E_2$$

Using (3.1.13) we see that

$$w(E_1) = \sum_{I_{n_k}^{l,0} \subset I} w(E \cap 3I_{n_k}^{l,0}) \lesssim \sum_{I_{n_k}^{l,0} \subset I} \frac{|E \cap 3I_{n_k}^{l,0}|}{|I|} \int_{\mathbb{R}} (M1_{\mathbf{I}})^p dw \tag{3.1.14}$$
$$= \frac{|E_1|}{|I|} \int_{\mathbb{R}} (M1_{\mathbf{I}})^p dw$$

To deal with E_2, note that for $x \in I \setminus \bigcup_{I_{n_k}^{l,0} \subset I} 3I_{n_k}^{l,0}$, $w(x) \lesssim \frac{3(1-\delta_2)}{2|I|} w(I)$ so we get

$$\frac{w(E_2)}{w(I)} \approx \frac{|E_2|}{|I|} \tag{3.1.15}$$

combining (3.1.14), (3.1.15) we get (3.1.12) for a triadic interval I.

We are left with extending (3.1.12) to an arbitrary interval I. Let $3^m \leq |I| \leq 3^{m+1}$ and $E \subset I$. Then $I \subset J_1 \cup J_2$, J_1, J_2 triadic intervals such that $|J_1| = |J_2| = 3^{m+1}$. Since

$$M\mathbf{1}_I(x) \approx M\mathbf{1}_{J_1}(x) \approx M\mathbf{1}_{J_2}(x), \text{for all } x \in \mathbb{R}.$$

we get

$$\frac{w(E)}{\int_{\mathbb{R}} (M\mathbf{1}_I(x))^p \, w(x) \, dx} \approx \frac{w(E \cap J_1)}{\int_{\mathbb{R}} \left(M\mathbf{1}_{J_1}(x)\right)^p w(x) \, dx} + \frac{w(E \cap J_2)}{\int_{\mathbb{R}} \left(M\mathbf{1}_{J_2}(x)\right)^p w(x) \, dx}$$
$$\lesssim \frac{|E \cap J_1|}{|J_1|} + \frac{|E \cap J_2|}{|J_2|} \approx \frac{|E|}{|I|}$$

This shows that w satisfies (2.1.4) and hence w is a C_p weight and the proof is complete. $\square$

3.1.3 Doubling C_p weights are in $\mathcal{A}_\infty$ for small doubling constants

Note that the construction in the proof of theorem 2.1.1 we depended heavily on the big doubling constant of the weight w. Here we show that this is the only case by proving theorem 2.1.2.

Proof of theorem 2.1.2: It will be enough to show that

$$\int_{\mathbb{R}^n} |M\mathbf{1}_I|^p w(x)dx \approx w(I)$$

the result then follows immediately from (2.1.4). Let $I_n = 3^m I$ the cubes with same center as I and side length $\ell(I_n) = 3^m \ell(I)$. We write

$$\begin{aligned}\int_{\mathbb{R}^n} |M\mathbf{1}_I|^p w dx &= \sum_{m=0}^{\infty} \int_{I_m \backslash I_{m-1}} |M\mathbf{1}_I|^p w dx \approx \sum_{m=0}^{\infty} \int_{I_m \backslash I_{m-1}} \frac{|I|^p w(x)dx}{(|I|^{\frac{1}{n}} + \operatorname{dist}(x,I))^{np}} \\ &\approx \sum_{m=0}^{\infty} |I|^p \frac{w(I_m)}{|I_m|^p} \lesssim \sum_{m=0}^{\infty} \frac{(C_w)^m w(I)}{(3^{np})^m} \lesssim w(I)\end{aligned}$$

since $C_\sigma < 3^{np}$ by hypothesis and the series converges. □

Remark 3.1.3. *Not all doubling weights are C_p weights. For an example just choose $\delta_1 < 3^{-p}$ in the construction of theorem 2.1.1.*

Remark 3.1.4. *There exist non-doubling C_p weights. For an example choose $\delta_{2,k} = \frac{1}{5k}$ in each I_{n_k} in the construction of theorem 2.1.1. A much simpler example is given by getting the Lebesgue measure in $\mathbb{R}^n$ and setting the measure of the unit ball equal to 0, i.e. define $w(E) = m(E \backslash B(0,1))$ where $B(0,1)$ is the unit ball in $\mathbb{R}^n$.*

3.1.4 The $\mathcal{A}_\infty^\alpha$ condition.

To complete the picture for the one weight conditions we are introducing the fractional $\mathcal{A}_\infty^\alpha$ condition. We are following very closely [58] where $\mathcal{A}_\infty^\alpha$ was introduced.

First we define the $\alpha-$relative capacity of a measure $\mathbf{Cap}_\alpha(E;I)$ of a compact subset E of a cube I by

$$\mathbf{Cap}_\alpha(E;I) = \inf\left\{\int h(x)dx : h \geq 0, Supph \subset 2I \text{ and } I_\alpha h \geq (diam2I)^{\alpha-n} \text{ on } E\right\}$$

Check [1] for more properties on capacity.

We say that a locally finite positive Borel measure ω is an $\mathcal{A}_\infty^\alpha$ measure if

$$\frac{\omega(E)}{\omega(2I)} \leq \eta(\mathbf{Cap}_\alpha(E,I)) \tag{3.1.16}$$

when $\omega(2I) > 0$, for all compact subsets E of a cube I, for some function $\eta : [0,1] \to [0,1]$ with $\lim\limits_{t\to 0} \eta(t) = 0$.

Note that omitting the factor 2 in $\omega(2I)$ above makes the condition more restrictive in general, but remains equivalent for doubling measures. It is shown in [58] that $\omega \in \mathcal{A}_\infty^\alpha$ implies the Wheeden-Muckenhoupt inequality

$$\int |I_\alpha f|^p \, d\omega \leq \int |M_\alpha f|^p \, d\omega \tag{3.1.17}$$

for all f positive Borel measures.

Remark 3.1.5. $\mathcal{A}_\infty^\alpha$ *measures are not necessarily doubling. Take for example the Lebesgue* *measure in $\mathbb{R}^n$ and set the measure of the unit ball equal to 0, i.e. define $\omega(E) = m(E \backslash B(0,1))$ where $B(0,1)$ is the unit ball in $\mathbb{R}^n$. This measure is clearly non-doubling and hence not in A_∞ but it is an $\mathcal{A}_\infty^\alpha$ measure.*

There exist also doubling fractional A_∞ measures that are not in A_∞. The example we use for that is exactly the one used in [15] but here we have to calculate the relative capacities of the sets used.

Theorem 3.1.6. *($\mathcal{A}_\infty^\alpha \cap \mathcal{D} \not\Rightarrow A_\infty$) There exist a measure μ singular to the Lebesgue measure that is doubling and satisfies the $\mathcal{A}_\infty^\alpha$ condition with $\eta(t) = t$ but μ is not an A_∞ weight.*

Proof. Let $\mu([0,1]) = 1$, $0 < \delta < 3^{-1}$ to be determined later, and for any triadic $I \subset [0,1]$ let

$$\mu(I) = \begin{cases} \delta\mu(\pi I) & \text{if } \partial I \cap \partial\pi I = \emptyset \\ \frac{1-\delta}{2}\mu(\pi I) & \text{if } \partial I \cap \partial\pi I \neq \emptyset \end{cases}$$

It was shown in [15] that μ is a doubling measure. It is also shown that it is singular to the Lebesgue measure hence it does not satisfy the A_∞ condition.

To show that it satisfies the $\mathcal{A}_\infty^\alpha$ condition, let $I \subset [0,1]$ be a triadic interval and $E \subset I$ be compact.

We claim that $||I_\alpha \mu_I||_{L^\infty(I)} = C_{\alpha,\delta}\mu(I)|I|^{\alpha-1}$, where μ_I is the restriction of μ on the set I and the constant $C_{\alpha,\delta}$ is independent of I. For any $x \in I$ we have

$$I_\alpha\mu_I(x) = \int_I |x-y|^{\alpha-1} d\mu(y) \lesssim \mu(I)|I|^{\alpha-1} \sum_{k=0}^{\infty} 3^{k(1-\alpha)} \left(\frac{1-\delta}{2}\right)^k = C_{\alpha,\delta}\mu(I)|I|^{\alpha-1}$$

as long as $3^{1-\alpha}\frac{1-\delta}{2} < 1 \Rightarrow \alpha > 1 - \frac{\ln(\frac{2}{1-\delta})}{\ln 3}$.

Now for any $f \geq 0$, $Suppf \subset 2I$ and $I_\alpha f \geq |2I|^{\alpha-1}$ on E, using Fubini's theorem we have

$$\mu(E) = \int_I \mathbf{1}_E d\mu \leq \int_I |2I|^{1-\alpha} I_\alpha f(x) \mathbf{1}_E(x) d\mu(x) = \int |2I|^{1-\alpha} I_\alpha \mu_E(x) f(x) dx$$
$$\leq ||f||_1 ||I_\alpha \mu_E||_\infty |2I|^{1-\alpha} \leq ||f||_1 ||I_\alpha \mu_I||_\infty |2I|^{1-\alpha} \lesssim ||f||_1 \mu(I)$$

So $Cap_\alpha(E, I) \gtrsim \frac{\mu(E)}{\mu(I)}$ hence $\mathcal{A}_\infty^\alpha$ holds with $\eta(t) = t$ and the proof is complete. $\square$

3.2 Two weight conditions

We start this section with the proofs of theorems 2.2.1 and 2.2.2.

3.2.1 Non doubling $\mathcal{A}_p$ examples

Remark 3.2.1. *Note first that we have the following simple implications $\mathcal{A}_p^{t_2} \Rightarrow \mathcal{A}_p^{t_1} \Rightarrow \mathcal{A}_p$. Indeed it is easy to see:*

$$\begin{aligned} P(I, \sigma) &= \int_I \frac{|I|}{(|I| + \text{dist}(x, I))^2} \sigma(dx) + \int_{\mathbb{R}/I} \frac{|I|}{(|I| + \text{dist}(x, I))^2} \sigma(dx) \\ &= \frac{\sigma(I)}{|I|} + \int_{\mathbb{R}/I} \frac{|I|}{(|I| + \text{dist}(x, I))^2} \sigma(dx) \geq \frac{\sigma(I)}{|I|} \end{aligned}$$

and so immediately from the definitions (2.2.1), (2.2.2) *and* (2.2.3) *we get*

$$\mathcal{A}_p(\omega, \sigma) \subseteq \mathcal{A}_p^{t_1}(\omega, \sigma) \subseteq \mathcal{A}_p^{t_2}(\omega, \sigma).$$

Remark 3.2.2. *We work with $p = 2$ for simplicity. The examples we use work with trivial modifications for any $p > 1$.*

Proof of theorem 2.2.1:

(1) We want to construct two measures ω, σ such that the two weight classical $\mathcal{A}_2$ condition holds but both one tailed $\mathcal{A}_2$ conditions fail. First, we construct measures u_k and v_k^n that satisfy

$$\frac{u_k(I)v_k^n(I)}{|I|^2} \leq M,$$

$$\frac{u_k(I)}{|I|}P(I, v_k^n) \gtrsim n$$

where the constant M does not depend on k, n. Then we will combine the measures u_k and v_k^n to create ω, σ such that the two weight classical $\mathcal{A}_2$ condition holds and both one tailed $\mathcal{A}_2$ conditions fail for the pair ω, σ. Let

$$u_k(E) = m(E \cap [k, k+1]), \quad v_k^n(E) = \sum_{i=0}^{n} 2^i m\left(E \cap \left[k + 2^i, k + 2^{i+1}\right]\right)$$

where m is the classic Lebesgue measure on $\mathbb{R}$. Let $I = (a, b)$, $a < k+1$ and $k + 2^{i-1} \leq b < k + 2^i$, for some $i \geq 0$ (of course if the interval does not intersect $[k, k+1]$ then $u_k(I) = 0$). Then

$$\frac{u_k(I)v_k^n(I)}{|I|^2} \leq \frac{4^{i+1} - 1}{(4-1)(2^{i-1} - 2)^2} = M \tag{3.2.1}$$

which is bounded for $i > 2$ (the cases $i = 0, 1, 2$ can be seen directly). Now let $I = [k, k+1]$. We have then:

$$\frac{u_k(I)}{|I|}P(I, v_k^n) = \int_{\mathbb{R}} \frac{v_k^n(dx)}{(1 + \mathrm{dist}(x, [k, k+1]))^2} = \sum_{i=0}^{n} \int_{I_i} \frac{v_k^n(dx)}{(1 + \mathrm{dist}(x, [k, k+1]))^2}$$

where $I_i = [k+2^i, k+2^{i+1}]$. We get:

$$\sum_{i=0}^{k} \int_{I_i} \frac{v_k^n(dx)}{(1+\operatorname{dist}(x,[1,2]))^2} \geq 1 + \sum_{i=1}^{n} \int_{I_i} \frac{v_k^n(dx)}{\left(1+2^i-2\right)^2} \tag{3.2.2}$$
$$= 1 + \sum_{i=1}^{n} \frac{v_k^n(I_i)}{\left(2^i-1\right)^2} = 1 + \sum_{i=1}^{n} \frac{2^{2i}}{\left(2^i-1\right)^2} \approx n.$$

Now we define ω, σ as follows:

$$\omega(E) = \sum_{k=1}^{\infty} u_{100^k}(E) + \sum_{k=1}^{\infty} v^k_{-100^k}(E)$$

$$\sigma(E) = \sum_{k=1}^{\infty} u_{-100^k}(E) + \sum_{k=1}^{\infty} v^k_{100^k}(E)$$

It is easy to see that with $I = I_k = [100^k, 100^k+1]$ both one tailed $\mathcal{A}_2$ conditions fail using (3.2.2).

To see that the classical $\mathcal{A}_2$ condition holds, let $I = (a,b)$ be any interval. It is simple to check that if I is big enough such that $|a| \approx 100^k, |b| \approx 100^n$ for $k \neq n$ then

$$\frac{\omega(I)\sigma(I)}{|I|^2} \leq 1.$$

While if $|a| \approx |b| \approx 100^k$ for some k then using (3.2.1) we get

$$\frac{\omega(I)\sigma(I)}{|I|^2} \leq 2M$$

hence the classical two weight $\mathcal{A}_2$ condition holds but both one tailed $\mathcal{A}_2$ conditions fail.

(2) Now we turn to proving $\mathcal{A}_2^{t_1} \nRightarrow \mathcal{A}_2^{t_2}$. Let the new measures be:

$$\begin{aligned}
\omega(E) &= \sum_{n=1}^{\infty} 2^n m\left(E \cap \left[2^n, 2^{n+1}\right]\right) \\
\sigma(E) &= m(E \cap [0,1])
\end{aligned}$$

From the construction above we can see that with $I = [0,1]$ we get:

$$P(I,\omega)P(I,\sigma) = \int_{\mathbb{R}} \frac{\omega(dx)}{(1+\operatorname{dist}(x,[0,1]))^2} \int_{\mathbb{R}} \frac{\sigma(dx)}{(1+\operatorname{dist}(x,[0,1]))^2}$$
$$= \int_{\mathbb{R}} \frac{\omega(dx)}{(1+\operatorname{dist}(x,[0,1]))^2} \int_0^1 \frac{dx}{(1+\operatorname{dist}(x,[0,1]))^2} = \int_{\mathbb{R}} \frac{\omega(dx)}{(1+\operatorname{dist}(x,[0,1]))^2}$$

Now from the definition of ω the last expression is equal to:

$$\sum_{n=1}^{\infty} \int_{2^n}^{2^{n+1}} \frac{2^n}{(1+\operatorname{dist}(x,[0,1]))^2} dx \geq \sum_{n=1}^{\infty} \frac{2^{2n}}{2^{2n}} = \infty$$

To prove that $\mathcal{A}_2^{t_1}$ hold let I be an interval such that $2^n \leq |I| < 2^{n+1}$ and $2^k - 1 \leq \operatorname{dist}(I,[0,1]) < 2^{k+1} - 1$ with $k \geq 0$. We have two cases:

(i) $n \geq k$.

$$\frac{\omega(I)}{|I|} P(I,\sigma) \leq \frac{\sum_{l=1}^{n+1} 2^l m\left(I \cap \left[2^l, 2^{l+1}\right]\right)}{|I|^2} \leq \frac{\sum_{l=1}^{n+1} 2^{2l}}{2^{2n}} = \frac{2^{2(n+2)} - 1}{(4-1)2^{2n}} < M < \infty$$

where the first inequality uses the fact that the interval cannot intersect any point in $[2^{n+2}, \infty)$ otherwise $n \geq k$ would not be satisfied.

(ii) $n < k$. If $k = 0$ then $I \cap [2, \infty) = \emptyset$ and there is nothing to prove. So assume $k > 0$.

$$\frac{\omega(I)}{|I|} P(I, \sigma) \leq \frac{\sum_{l=k}^{k+1} 2^l m \left(I \cap \left[2^l, 2^{l+1} \right] \right)}{2^{2k}} \leq \frac{2^{2k} + 2^{2(k+1)}}{2^{2k}} = 5 < \infty$$

where the first inequality now holds because I cannot contain neither any point in $(0, 2^k)$ for otherwise $\mathrm{dist}(I, (0, 1)) < 2^k - 1$ nor any point in $[2^{k+1}, \infty)$ because $n < k$ would not be satisfied and the proof is complete.

(3) Last, for the equivalence of the two tailed $\mathcal{A}_2$ condition to both one tailed $\mathcal{A}_2$ conditions let $I \in \mathbb{R}^n$ be a cube. We have:

$$P(I, \sigma) \approx \frac{\sigma(I)}{|I|} + \sum_{k=1}^{\infty} \sum_{m=1}^{3^n - 1} \frac{\sigma(I_m^k)}{3^{kn} |I_m^k|}, \quad P(I, \omega) \approx \frac{\omega(I)}{|I|} + \sum_{k=1}^{\infty} \sum_{m=1}^{3^n - 1} \frac{\omega(I_m^k)}{3^{kn} |I_m^k|}$$

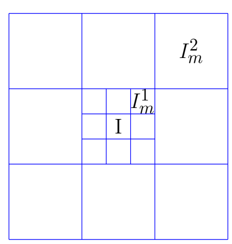

Figure 3.2.1: Splitting for $n = 2$

where $|I_m^k|^{\frac{1}{n}} = 3^k |I|^{\frac{1}{n}}$ and $d(I_m^k, I) \approx 3^k$, and all the implied constants depend only on the dimension, check Figure 3.2.1. There exist $k_1, k_2 \geq 0$ such that

$$P(I, \sigma) \approx 2 \left(\frac{\sigma(I)}{|I|} + \sum_{k=1}^{k_1} \sum_{m=1}^{3^n - 1} \frac{\sigma(I_m^k)}{3^{kn} |I_m^k|} \right) \approx 2 \sum_{k=k_1}^{\infty} \sum_{m=1}^{3^n - 1} \frac{\sigma(I_m^k)}{3^{kn} |I_m^k|}$$

$$P(I,\omega) \approx 2\left(\frac{\omega(I)}{|I|} + \sum_{k=1}^{k_2}\sum_{m=1}^{3^n-1}\frac{\omega(I_m^k)}{3^{kn}|I_m^k|}\right) \approx 2\sum_{k=k_2}^{\infty}\sum_{m=1}^{3^n-1}\frac{\omega(I_m^k)}{3^{kn}|I_m^k|}$$

We can assume without loss of generality that $k_1 \leq k_2$.

Let $J = I \cup \left(\bigcup_{k=1}^{k_1}\bigcup_{m=1}^{3^n-1} I_m^k\right)$, hence $|J|^{\frac{1}{n}} \approx 3^{k_1}|I|^{\frac{1}{n}}$ where again the implied constant depends only on dimension. We calculate

$$\begin{aligned}
\frac{\sigma(J)}{|J|}P(J,\omega) &\approx \frac{1}{|J|}\left(\sigma(I) + \sum_{k=1}^{k_1}\sum_{m=1}^{3^n-1}\sigma(I_m^k)\right)\left(\frac{\omega(J)}{|J|} + \sum_{k=1}^{\infty}\sum_{m=1}^{3^n-1}\frac{\omega(J_m^k)}{3^{kn}|J_m^k|}\right) \\
&\approx \frac{1}{3^{k_1 n}|I|}\left(\sigma(I) + \sum_{k=1}^{k_1}\sum_{m=1}^{3^n-1}\sigma(I_m^k)\right)\left(\frac{\omega(J)}{|J|} + \sum_{k=1}^{\infty}\sum_{m=1}^{3^n-1}\frac{\omega(I_m^{k+k_1})}{3^{kn}|I_m^{k+k_1}|}\right) \\
&\approx \frac{1}{3^{k_1 n}|I|}\left(\sigma(I) + \sum_{k=1}^{k_1}\sum_{m=1}^{3^n-1}\sigma(I_m^k)\right)\left(\frac{\omega(J)}{|J|} + \sum_{k=k_1}^{\infty}\sum_{m=1}^{3^n-1}\frac{3^{k_1 n}\omega(I_m^k)}{3^{kn}|I_m^k|}\right) \\
&\gtrsim \frac{1}{|I|}\left(\sigma(I) + \sum_{k=1}^{k_1}\sum_{m=1}^{3^n-1}\sigma(I_m^k)\right)\left(\frac{\omega(J)}{|J|} + \sum_{k=k_2}^{\infty}\sum_{m=1}^{3^n-1}\frac{\omega(I_m^k)}{3^{kn}|I_m^k|}\right) \\
&\gtrsim \left(\frac{\sigma(I)}{|I|} + \sum_{k=1}^{k_1}\sum_{m=1}^{3^n-1}\frac{\sigma(I_m^k)}{3^{kn}|I_m^k|}\right)\left(\frac{\omega(J)}{|J|} + \sum_{k=k_2}^{\infty}\sum_{m=1}^{3^n-1}\frac{\omega(I_m^k)}{3^{kn}|I_m^k|}\right) \\
&\approx P(I,\sigma)P(I,\omega)
\end{aligned}$$

hence showing that the one tailed $\mathcal{A}_p$ conditions bound the two tailed $\mathcal{A}_p$ condition and the proof is complete.

□

Remark 3.2.3. *From the above construction we see that the same measures could work to prove the same implications for* $\mathcal{A}_p^{offset}$ (2.2.13)*, and it's two tailed analogue since it's exactly the nature of the tail that we take advantage of in the construction.*

3.2.2 Two weight $\mathcal{A}_p$ equivalence for doubling measures

Remark 3.2.4. *We are going to use $p = 2$ in the proof for simplicity. The general case follows immediately since $\frac{1}{p}, \frac{1}{p'} < 1$ and hence*

$$P(I,\omega)^{\frac{1}{p}} \approx \left(\sum_{k=1}^{\infty} \sum_{j=1}^{3^n-1} \frac{\omega(I_i^k)}{3^{2kn}|I|} \right)^{\frac{1}{p}} \leq \sum_{k=1}^{\infty} \sum_{j=1}^{3^n-1} \left(\frac{\omega(I_i^k)}{3^{2kn}|I|} \right)^{\frac{1}{p}}$$

and from here the proof follows the same way as for $p = 2$.

Proof of theorem 2.2.2: Let ω, σ be reverse doubling measures with reverse doubling constants $1 + \delta_\omega$ and $1 + \delta_\sigma$ respectively. It is enough to prove that we can bound the two tailed $\mathcal{A}_p^{t2}(\omega, \sigma)$ from the classical $\mathcal{A}_p(\omega, \sigma)$. Let I be a cube. We then have,

$$\begin{aligned} P(I,\omega)P(I,\sigma) \lesssim & \frac{\omega(I)\sigma(I)}{|I|^2} + \frac{\omega(I)}{|I|} \sum_{m=1}^{\infty} \sum_{i=1}^{3^n-1} \frac{\sigma(I_i^m)}{3^{2mn}|I|} + \frac{\sigma(I)}{|I|} \sum_{k=1}^{\infty} \sum_{j=1}^{3^n-1} \frac{\omega(I_i^k)}{3^{2kn}|I|} \\ & + \sum_{m=1}^{\infty} \sum_{i=1}^{3^n-1} \frac{\sigma(I_i^m)}{3^{2mn}|I|} \sum_{k=1}^{\infty} \sum_{j=1}^{3^n-1} \frac{\omega(I_j^k)}{3^{2kn}|I|} \equiv A + B + C + D \end{aligned}$$

where $|I_j^m| = 3^{mn}|I|$, $\text{dist}(I_j^m, I) \approx 3^m |I|^{\frac{1}{n}}$, $\bigcup_{m\in\mathbb{N}} \bigcup_{j=1}^{3^n-1} I_j^m = \mathbb{R}^n \backslash I$ and the implied constant depends only on dimension. A is bounded immediately by $\mathcal{A}_2(\omega, \sigma)$. For B we have:

$$B = \sum_{m=1}^{\infty} \sum_{i=1}^{3^n-1} \frac{\omega(I)\sigma(I_i^m)}{3^{2mn}|I|^2} \lesssim \sum_{m=1}^{\infty} (1+\delta_\omega)^{-m} \sum_{i=1}^{3^n-1} \frac{\omega(I^m)\sigma(I^m)}{|I^m|^2} \lesssim \mathcal{A}_2(\omega,\sigma) < \infty$$

where $I^m = I \cup \left(\bigcup_{\ell=1}^{m} \bigcup_{j=1}^{3^n-1} I_j^\ell \right)$ and the implied constant again depends only on dimension and the reverse doubling constant of ω. The bound for C is similar to B.

For D we have:

$$\begin{aligned} D &= \sum_{m=1}^{\infty}\sum_{k=1}^{m}\sum_{i=1}^{3^n-1}\sum_{j=1}^{3^n-1}\frac{\sigma(I_i^m)\omega(I_j^k)}{3^{2mn}|I|3^{2kn}|I|}+\sum_{k=1}^{\infty}\sum_{m=1}^{k-1}\sum_{j=1}^{3^n-1}\sum_{i=1}^{3^n-1}\frac{\sigma(I_i^m)\omega(I_j^k)}{3^{2mn}|I|3^{2kn}|I|} \\ &\equiv \mathbf{I}+\mathbf{II} \end{aligned}$$

We will get the bound for **I**, the calculations for **II** are identical.

$$\begin{aligned} \mathbf{I} &\lesssim \sum_{m=1}^{\infty}\sum_{k=1}^{m}\sum_{i=1}^{3^n-1}\sum_{j=1}^{3^n-1}(1+\delta_\omega)^{k-m}\frac{\sigma(I^m)\omega(I^m)}{3^{2kn}|I^m|^2} \lesssim \\ &\lesssim \mathcal{A}_2(\omega,\sigma)\sum_{m=1}^{\infty}(1+\delta_\omega)^{-m}\sum_{k=1}^{m}\frac{(1+\delta_\omega)^k}{3^{2kn}} \le C_{n,\sigma}\mathcal{A}_2(\omega,\sigma)<\infty \end{aligned}$$

Combining all the above bounds and getting supremum over the cubes I we get

$$\mathcal{A}_2^{t_2}(\omega,\sigma) \le C_{n,\omega,\sigma}\mathcal{A}_2(\omega,\sigma)$$

which completes the proof of the theorem. $\square$

Remark 3.2.5. *The same proof works for the fractional $\mathcal{A}_p(\omega,\sigma)$ conditions as defined in [64].*

3.2.3 The T_1 theorem for A_∞ weights.

The goal of this subsection is to prove theorem 2.2.3. For that we are going to use the Sawyer testing condition.

3.2.3.1 The Sawyer testing condition.

Sawyer in [56] proved that the Maximal operator is bounded on $L^p(u) \to L^q(w)$ if and only if 2.2.9 holds, i.e. if and only if

$$S^{p,q}(w, u^{1-p'}) = \sup_I \left(\int_I u(x)^{1-p'} dx \right)^{\frac{-1}{p}} \left(\int_I \left[M(\mathbf{1}_I u^{1-p'})(x) \right]^q w(x) dx \right)^{\frac{1}{q}} < \infty \quad (3.2.3)$$

where the supremum is taken over all cubes $I \subset \mathbb{R}^n$. Replacing the weights w, u with the measures ω, σ we call $S_d^{p,q}(\omega, \sigma)$ the dyadic Sawyer testing condition where the Maximal operator in (3.2.3) is replaced by the dyadic Maximal operator M_d where the supremum in the operator is taken over only dyadic cubes.

Theorem 3.2.6. *($\sigma \in A_\infty$, $\mathcal{A}_p(\omega, \sigma) \Rightarrow S_d^{p,p}(\omega, \sigma)$) Let ω, σ be Radon measures in $\mathbb{R}^n$ such that $\sigma \in A_\infty$. If ω, σ satisfy the $\mathcal{A}_p(\omega, \sigma)$ condition then the dyadic Sawyer testing condition $S_d^{p,p}(\omega, \sigma)$ holds.*

Proof. Let I be a cube in $\mathbb{R}^n$. Let $\Omega_m = \{x \in I : (M_d \mathbf{1}_I \sigma)(x) > K^m\} = \dot{\bigcup} I_j^m$, where K is a constant to be determined later and I_j^m are the maximal, disjoint dyadic cubes such that $\frac{\sigma(I_j^m)}{|I_j^m|} > K^m$. We have

$$\begin{aligned}
\int_I (M_d \mathbf{1}_I \sigma)^p (x) d\omega(x) &\lesssim \sum_{m,j} \left(\frac{\sigma(I_j^m)}{|I_j^m|} \right)^p \omega(I_j^m) \\
&= \sum_{m,j} \left(\frac{\sigma(I_j^m)^{\frac{1}{p'}} \omega(I_j^m)^{\frac{1}{p}}}{|I_j^m|} \right)^p \sigma(I_j^m) \leq \mathcal{A}_p(\omega, \sigma) \sum_{m,j} \sigma(I_j^m)
\end{aligned}$$

Call $A_t^m = \bigcup_{I_j^{m+1} \subset I_t^m} I_j^{m+1}$. Since $\sigma \in A_\infty$ we get

$$\sigma\left(A_t^m\right) \leq C \left(\frac{|A_t^m|}{|I_t^m|}\right)^\varepsilon \sigma(I_t^m)$$

for some C positive and ε like in (3.1.2). From the maximality of I_j^m we obtain

$$|A_t^m| = \sum_{I_j^{m+1} \subset I_t^m} \left|I_j^{m+1}\right| \leq \frac{1}{K^{m+1}} \sigma\left(A_t^m\right) \leq \frac{2^n}{K} |I_t^m|$$

Choose K big enough that $C \left(\frac{|A_t^m|}{|I_t^m|}\right)^\varepsilon \leq \frac{1}{2}$. Fix $m \in \mathbb{N}$, $k \geq -m$, then

$$\sum_j \sigma(I_j^k) \leq \left(\frac{1}{2}\right)^{m+k} \sum_j \sigma(I_j^{-m}) \leq \left(\frac{1}{2}\right)^{m+k} \sigma(I)$$

$$\sum_{k=-m}^{\infty} \sum_j \sigma(I_j^k) \leq \sum_{k=-m}^{\infty} 2^{-m-k} \sigma(I) \leq 2\sigma(I)$$

and by taking $m \to \infty$ we get

$$\sum_{k,j} \sigma(I_j^k) = \lim_{m \to \infty} \sum_{k=-m}^{\infty} \sum_j \sigma(I_j^k) \leq 2\sigma(I)$$

and this completes the proof of the theorem. $\square$

With theorem 3.2.6. at hand we get the following corollary.

Corollary 3.2.7. *($\omega \in A_\infty$, $\mathcal{A}_p(\omega, \sigma) \Rightarrow \mathcal{V}(\omega, \sigma)^p$) Let ω, σ be Radon measures in $\mathbb{R}^n$ such that $\sigma \in A_\infty$. Then the $\mathcal{A}_p(\omega, \sigma)$ condition implies the pivotal condition $\mathcal{V}(\omega, \sigma)^p$.*

Proof. Let I be a cube in $\mathbb{R}^n$.

$$\begin{aligned} \mathrm{P}(I,\sigma) &= \int \frac{|I|}{\left(|I|^{\frac{1}{n}} + |x - x_I|\right)^{2n}} d\sigma(x) \lesssim \sum_{m=0}^{\infty} \frac{\sigma\big((2^m+1)I\big)}{2^m|2^m I|} \\ &\lesssim \sum_{m=0}^{\infty} \inf_{x\in I} M_d\sigma(x) 2^{-m} \lesssim \inf_{x\in I} M_d\sigma(x) \end{aligned} \tag{3.2.4}$$

where M_d denotes the dyadic maximal function.

Let I_0 be a cube in $\mathbb{R}^n$. Let $I_0 = \bigcup_{r\geq 1} I_r$ be a decomposition of I_0 in disjoint cubes. Using (3.2.4) we get

$$\sum_{r\geq 1} \omega(I_r)\mathrm{P}^p(I_r, \mathbf{1}_{I_0}\sigma) \leq \sum_{r\geq 1} \omega(I_r) \inf_{x\in I_r} \left(M_d \mathbf{1}_{I_0}\sigma\right)^p (x) \leq \int_{I_0} \left(M_d \mathbf{1}_{I_0}\sigma\right)^p (x) d\omega(x)$$

and using theorem 3.2.6. the last expression is bounded by a constant multiple of $\sigma(I_0)$. So we have

$$\sum_{r\geq 1} \omega(I_r)\mathrm{P}^p(I_r, \mathbf{1}_{I_0}\sigma) \leq K\sigma(I_0)$$

and that completes the proof of the corollary. □

Question 2. *A_∞ is a special class of doubling measures. Is it true that for ω doubling measure, $\mathcal{A}_p(\omega,\sigma) \Rightarrow \mathcal{V}(\omega,\sigma)^p$?*

Question 3. *In corollary 3.2.7 we prove that for $\omega \in A_\infty$ dyadic Sawyer testing implies pivotal. Is it true that $S_d^{p,p}(\omega,\sigma) = \mathcal{V}(\omega,\sigma)^p$?*

Remark 3.2.8. *The proof of corollary 3.2.7, holds also for the fractional $\mathcal{A}_p(\omega,\sigma)$ and pivotal conditions as stated in [64] (stated for $p = 2$ but extends immediately to any $p > 1$).*

Proof of theorem 2.2.3: If both the measures ω, σ are in the one weight A_∞, then by corollary 3.2.7, the the two weight $\mathcal{A}_2(\omega, \sigma)$ condition implies both the pivotal conditions $\mathcal{V}(\omega, \sigma)^2$ ((2.2.10) and it's dual) and we can apply the main theorem from [64] (or the one in [37]) to get the result. $\square$

3.2.4 The "buffer" conditions do not imply the tailed $\mathcal{A}_p$ conditions.

The goal of this subsection is to give a proof of theorem 2.2.4. First we make the following simple remark.

Remark 3.2.9. *It is immediate to see that the Pivotal condition implies the classical $\mathcal{A}_p$ condition. Just let the decomposition in* (2.2.10) *be just a single cube.*

Remark 3.2.10. *We are going to use $p = 2$ in the proof for simplicity. The proof works for $1 < p \leq 2$, without any modifications.*

Proof of theorem 2.2.4: We construct measures ω and σ so that the pivotal condition $\mathcal{V}^2$ (2.2.10) holds, but $\mathcal{A}_2^{t_1}$ (2.2.2) does not. Let

$$\omega(E) = \delta_0, \quad \sigma(E) = \sum_{n=2}^{\infty} n\delta_n(E)$$

where δ_n denotes the point mass at $x = n$. First we check $\mathcal{A}_2^{t_1}$ does not hold. Let $I = [0, 1]$. Then

$$\frac{\omega(I)}{|I|}P(I, \sigma) = \int_{\mathbb{R}} \frac{1}{(1 + dist(x, [0, 1]))^2}\sigma(dx) = \sum_{n=2}^{\infty} \frac{n}{(n)^2} = \infty$$

To show the pivotal condition holds, let $I_0 = (a, b)$ where $a < 0$ and $n \leq b < n+1, n \geq 2$ (we need I_0 to contain some masses from σ and $0 \in I_0$ for otherwise there is nothing to prove). Decomposing $I_0 = \dot{\cup} I_r$, only the I_r such that $0 \in I_r$ contributes to the pivotal condition. Call that cube I_1. We consider the cases:

(i) $|I_1| \leq 1$. We calculate:

$$\frac{\omega(I_1)P(I_1, I_0\sigma)^2}{\sigma(I_0)} = \frac{\left(\int_{I_0} \frac{|I_1|\sigma(dx)}{(|I_1| + \mathrm{dist}(x, I_1))^2}\right)^2}{\sigma(I_0)} \leq |I_1|^2 \left(\sum_{k=2}^{n} \frac{k}{k^2}\right)^2 \Big/ \sum_{k=2}^{n} k$$
$$\leq M < \infty$$

where the constant M does not depend on n.

(ii) $|I_1| \geq n$. We get:

$$\frac{\omega(I_1)P(I_1, I_0\sigma)^2}{\sigma(I_0)} = \frac{\left(\int_{I_0} \frac{|I_1|\sigma(dx)}{(|I_1| + \mathrm{dist}(x, I_1))^2}\right)^2}{\sigma(I_0)} \leq |I_1|^2 \left(\sum_{k=2}^{n} \frac{k}{|I_1|^2}\right)^2 \Big/ \sum_{k=2}^{n} k$$
$$\leq \frac{n^2}{|I_1|^2} \leq 1$$

(iii) $1 \leq |I_1| \leq n$. We have:

$$\frac{\omega(I_1)P(I_1, I_0\sigma)^2}{\sigma(I_0)} \lesssim \frac{|I_1|^2}{n^2}\left(\sum_{k=2}^{|I_1|} \frac{k}{|I_1|^2} + \sum_{k=|I_1|}^{n} \frac{1}{k}\right)^2 \lesssim \frac{|I_1|^2}{n^2}\left(1 + \log\left(\frac{n}{|I_1|}\right)\right)^2$$
$$\lesssim \frac{|I_1|^2}{n^2} + \frac{|I_1|^2}{n^2}\log^2\left(\frac{n}{|I_1|}\right)$$

Now, on the last expression setting $x = \frac{n}{|I_1|}$ we get the function $f(x) = \frac{\log^2 x}{x^2}$, $x \geq 1$ which is bounded independent of n. Combining all three cases we see that the pivotal condition is bounded. $\square$

Question 4. *In the above example, one can check that the dual pivotal condition does not hold. Is it true that $\mathcal{V}(\sigma,\omega)^p \cap \mathcal{V}(\omega,\sigma)^p \Rightarrow \mathcal{A}_p^{t_1}(\omega,\sigma)$?*

3.2.4.1 Doubling measures and the Pivotal condition.

The result in this subsection is essentially in [58], equation (4.4), but we include it for completeness. We partially answer positively **question 2**.

If the measures ω, σ are doubling but not in $\mathcal{A}_\infty$ then we do not in general know if the Pivotal condition can be controlled by the $\mathcal{A}_p(\omega,\sigma)$ condition. For measures with small doubling constant though the $\mathcal{A}_p(\omega,\sigma)$ condition implies $\mathcal{V}(\omega,\sigma)^p$.

Theorem 3.2.11. *(Small doubling+$\mathcal{A}_p(\omega,\sigma) \Rightarrow \mathcal{V}(\omega,\sigma)^p$) Let ω, σ be doubling measures in $\mathbb{R}^n$ with doubling constants K_ω, K_σ and reverse doubling constants $1+\delta_\omega, 1+\delta_\sigma$ respectively. If $K_\sigma < 2^p(1+\delta_\omega)$ then the $\mathcal{A}_p(\omega,\sigma)$ condition implies the pivotal condition $\mathcal{V}(\omega,\sigma)^p$.*

Proof. Let I_0 be a cube in $\mathbb{R}^n$ and $I_0 = \cup_{r\geq 1} I_r$ be a decomposition of I_0 in disjoint cubes.

$$\begin{aligned}
\sum_{r\geq 1} \omega(I_r) P(I_r, I_0\sigma)^p &\approx \sum_{r\geq 1} \omega(I_r) \left(\sum_{m=1}^{m_r} \frac{\sigma(I_r^m)}{2^m |I_r^m|} \right)^p \\
\leq\ & \sum_{r\geq 1} \left(\sum_{m=1}^{m_r} (1+\delta_\omega)^{-\frac{m}{p}} \frac{\omega^{\frac{1}{p}}(I_r^m) \sigma^{\frac{1}{p'}}(I_r^m)}{2^m |I_r^m|} \sigma^{\frac{1}{p}}(I_r^m) \right)^p \\
\leq\ & \mathcal{A}_p(\omega,\sigma) \sum_{r\geq 1} \sigma(I_r) \left(\sum_{m=1}^{m_r} \left(\frac{K_\sigma}{2^p(1+\delta_\omega)} \right)^{\frac{m}{p}} \right)^p \lesssim \mathcal{A}_p(\omega,\sigma)\sigma(I_0)
\end{aligned}$$

where $m_r = \log_2 \left(\frac{|I_0|}{|I_r|}\right)^{\frac{1}{n}}$, I_r^m is the cube with same center as that of I_r and $|I_r^m|^{\frac{1}{n}} = 2^m |I_r|^{\frac{1}{n}}$. where the implied constant depends only on the doubling constant of σ and the reverse doubling constant of ω. This completes the proof of the theorem. $\square$

Remark 3.2.12. *For a doubling measure ω and a cube $I \subset \mathbb{R}^n$ we have that in* (2.2.12) $E(I,\omega)^2 \geq c_\omega > 0$ *since $\omega(I_1) \approx \omega(I_2)$ where $|I_1| = |I_2| = 2^{-n}|I|$ and I_1 is in the top left corner of I, I_2 in the bottom right corner of I. Hence for ω doubling the Pivotal condition $\mathcal{V}(\omega,\sigma)^p$ is equivalent to the Energy condition $\mathcal{E}(\omega,\sigma)^p$.*

Chapter 4

Counterexample to Hytönen's off-testing condition in two dimensions

We begin with the proof of Theorem 2.3.4. The proof of Theorem 2.3.5 will be very similar and we will only have to deal with the cancellation occurring in the kernel with Lemma 4.3.1 being useful.

Proof of Theorem 2.3.4. First we build two measures in $\mathbb{R}$, generalizing the work done in [32], and then they will be used for our two dimensional construction.

4.1 The One-Dimensional Construction

Given $0 \leq \alpha < 2$, choose $\frac{1}{3} \leq b < 1$ such that $\frac{1}{9} \leq \left(\frac{1-b}{2}\right)^{2-\alpha} \leq \frac{1}{3}$. Let $s_0^{-1} = \left(\frac{1-b}{2}\right)^{2-\alpha}$. Recall the middle-$b$ Cantor set E_b and the Cantor measure $\ddot{\omega}$ on the closed interval $I_1^0 = [0,1]$. At the kth generation in the construction, there is a collection $\{I_j^k\}_{j=1}^{2^k}$ of 2^k pairwise disjoint closed intervals of length $|I_j^k| = \left(\frac{1-b}{2}\right)^k$. The Cantor set is defined by $\mathrm{E}_b = \bigcap_{k=1}^{\infty} \bigcup_{j=1}^{2^k} I_j^k$ and the Cantor measure $\ddot{\omega}$ is the unique probability measure supported in E with the property that is equidistributed among the intervals $\{I_j^k\}_{j=1}^{2^k}$ at each scale k, i.e

$$\ddot{\omega}(I_j^k) = 2^{-k}, \qquad k \geq 0, 1 \leq j \leq 2^k.$$

We denote the removed open middle bth of I_j^k by G_j^k and by $\ddot{z}_j^k$ its center. Following closely [32], we define

$$\ddot{\sigma} = \sum_{k,j} s_j^k \delta_{\ddot{z}_j^k}$$

where the sequence of positive numbers s_j^k is chosen to satisfy $\dfrac{s_j^k \ddot{\omega}(I_j^k)}{|I_j^k|^{4-2\alpha}} = 1$, i.e.

$$s_j^k = \left(\frac{2}{s_0^2}\right)^k, \quad k \geq 0,\ 1 \leq j \leq 2^k.$$

4.1.1 The Testing Constant is Unbounded

. Consider the following operator

$$\ddot{T} f(x) = \int_{\mathbb{R}} \frac{f(y)}{|x-y|^{2-\alpha}} dy$$

Note that

$$\ddot{T}\ddot{\omega}(\ddot{z}_1^k) = \int_{I_1^0} \frac{d\ddot{\omega}(y)}{|\ddot{z}_1^k - y|^{2-\alpha}} \geq \int_{I_1^k} \frac{d\ddot{\omega}(y)}{|\ddot{z}_1^k - y|^{2-\alpha}} \geq \frac{\ddot{\omega}(I_1^k)}{\left(\frac{1}{2}\left(\frac{1-b}{2}\right)^k\right)^{2-\alpha}} \approx \left(\frac{s_0}{2}\right)^k$$

since $|\ddot{z}_1^k - y| \leq |\ddot{z}_1^k|$ for $y \in I_1^k$ and $\ddot{z}_1^k = \frac{1}{2}(\frac{1-b}{2})^k$. Similar inequalities hold for the rest of $\ddot{z}_j^k$. This implies that the following testing condition fails:

$$\int_{I_1^0} \left(\ddot{T}(\mathbf{1}_{I_1^0}\ddot{\omega})(x)\right)^2 d\ddot{\sigma}(y) \gtrsim \sum_{k=1}^{\infty} \sum_{j=1}^{2^k} s_j^k \cdot \left(\frac{s_0}{2}\right)^{2k} = \sum_{k=1}^{\infty} \sum_{j=1}^{2^k} \frac{1}{2^k} = \infty \tag{4.1.1}$$

4.1.2 The $\ddot{\mathcal{A}}_2$ Condition

. Let us now define

$$\ddot{\mathcal{P}}(I,\mu) = \int_{\mathbb{R}} \left(\frac{|I|}{(|I| + |x - x_I|)^2} \right)^{2-\alpha} d\mu(x)$$

and the following variant of the A_2^α condition:

$$\ddot{\mathcal{A}}_2^\alpha(\ddot{\sigma},\ddot{\omega}) = \sup_I \ddot{\mathcal{P}}(I,\ddot{\sigma}) \cdot \ddot{\mathcal{P}}(I,\ddot{\omega})$$

where the supremum is taken over all intervals in $\mathbb{R}$. We verify that $\ddot{\mathcal{A}}_2^\alpha$ is finite for the pair $(\ddot{\sigma},\ddot{\omega})$. The starting point is the estimate

$$\ddot{\sigma}(I_r^\ell) = \sum_{(k,j):\ddot{z}_j^k \in I_r^\ell} s_j^k = \sum_{k=l}^{\infty} 2^{k-\ell} s_j^k = 2^{-\ell} \sum_{k=l}^{\infty} \left(\frac{4}{s_0^2} \right)^k \approx \left(\frac{2}{s_0^2} \right)^\ell = s_r^\ell$$

and from this, it immediately follows,

$$\frac{\ddot{\sigma}(I_j^\ell)\ddot{\omega}(I_j^\ell)}{|I_j^\ell|^{4-2\alpha}} \approx \frac{s_j^\ell \ddot{\omega}(I_j^\ell)}{|I_j^\ell|^{4-2\alpha}} = 1, \text{ for } \ell \geq 0,\ 1 \leq j \leq 2^\ell. \tag{4.1.2}$$

Now from the definition of $\ddot{\sigma}$ we get,

$$\begin{aligned}
\ddot{\mathcal{P}}(I_r^\ell,\ddot{\sigma}) &\leq \frac{\ddot{\sigma}(I_r^\ell)}{|I_r^\ell|^{2-\alpha}} + \int_{I_1^0\setminus I_r^\ell}\left(\frac{|I_r^\ell|}{\left(|I_r^\ell|+|x-x_{I_r^\ell}|\right)^2}\right)^{2-\alpha} d\ddot{\sigma}(x) \qquad (4.1.3)\\
&\leq \frac{\ddot{\sigma}(I_r^\ell)}{|I_r^\ell|^{2-\alpha}} + \sum_{m=0}^{\ell}\sum_{k=m}^{\infty}\frac{2^{k-m}s_j^k\;|I_r^\ell|^{2-\alpha}}{\left(|I_r^\ell|+b\left(\frac{1-b}{2}\right)^m\right)^{4-2\alpha}}\\
&\lesssim \frac{\ddot{\sigma}(I_r^\ell)}{|I_r^\ell|^{2-\alpha}} + \sum_{m=0}^{\ell}\frac{2^{-m}|I_r^\ell|^{2-\alpha}\left(\frac{4}{s_0^2}\right)^m}{\left(b\left(\frac{1-b}{2}\right)^{m-\ell}|I_r^\ell|\right)^{4-2\alpha}}\\
&= \frac{\ddot{\sigma}(I_r^\ell)}{|I_r^\ell|^{2-\alpha}} + \frac{b^{2\alpha-4}}{|I_r^\ell|^{2-\alpha}}\left(\frac{1}{s_0^2}\right)^\ell\sum_{m=0}^{\ell}2^m\\
&\lesssim \frac{\ddot{\sigma}(I_r^\ell)}{|I_r^\ell|^{2-\alpha}} + \frac{s_r^\ell}{|I_r^\ell|^{2-\alpha}} \approx \frac{\ddot{\sigma}(I_r^\ell)}{|I_r^\ell|^{2-\alpha}}
\end{aligned}$$

and using the uniformity of $\ddot{\omega}$,

$$\begin{aligned}
\ddot{\mathcal{P}}(I_r^\ell,\ddot{\omega}) &\leq \frac{\ddot{\omega}(I_r^\ell)}{|I_r^\ell|^{2-\alpha}} + \int_{I_1^0\setminus I_r^\ell}\left(\frac{|I_r^\ell|}{\left(|I_r^\ell|+|x-x_{I_r^\ell}|\right)^2}\right)^{2-\alpha} d\ddot{\omega}(x) \qquad (4.1.4)\\
&\leq \frac{\ddot{\omega}(I_r^\ell)}{|I_r^\ell|^{2-\alpha}} + \sum_{k=1}^{\ell}\frac{|I_r^\ell|^{2-\alpha}\;\ddot{\omega}(I_{j_k}^k)}{\left(|I_r^\ell|+b\left(\frac{1-b}{2}\right)^{k-1}\right)^{4-2\alpha}}\\
&\leq \frac{\ddot{\omega}(I_r^\ell)}{|I_r^\ell|^{2-\alpha}} + \sum_{k=1}^{\ell}\frac{|I_r^\ell|^{2-\alpha}\;\ddot{\omega}(I_{j_k}^k)}{\left(b\left(\frac{1-b}{2}\right)^{k-1-\ell}|I_r^\ell|\right)^{4-2\alpha}}
\end{aligned}$$

$$\lesssim \frac{\ddot{\omega}(I_r^\ell)}{|I_r^\ell|^{2-\alpha}} + \frac{2^{-\ell}}{|I_r^\ell|^{2-\alpha}} = 2\frac{\ddot{\omega}(I_r^\ell)}{|I_r^\ell|^{2-\alpha}},$$

where $I_{j_k}^k \subset I_t^{k-1}$, $I_r^\ell \subset I_t^{k-1}$ and $I_{j_k}^k \cap I_r^\ell = \emptyset$, and where all the implied constants in the above calculations depend only on α. From (4.1.3), (4.1.4) and (4.1.2), we see that

$$\ddot{\mathcal{P}}(I_r^\ell,\ddot{\sigma})\ddot{\mathcal{P}}(I_r^\ell,\ddot{\omega})\lesssim 1.$$

Let us now consider an interval $I\subset I_1^0$ and let $A>1$ be fixed. Then, let k be the smallest integer such that $\ddot{z}_j^k\in AI$; if there is no such k, then $AI\subsetneq G_j^\ell$, for some ℓ. We have the following cases:

Case 1. Assume that $I\subset AI\subsetneq G_j^k\subset I_j^k$. If $|x_I-\ddot{z}_j^k|\le \operatorname{dist}(x_I,\partial G_j^k)$ then,

$$\begin{aligned}\ddot{\mathcal{P}}(I,\ddot{\sigma})\ddot{\mathcal{P}}(I,\ddot{\omega}) &= |I|^{4-2\alpha}\int_{I_1^0}\frac{d\ddot{\sigma}(x)}{(|I|+|x-x_I|)^{4-2\alpha}}\int_{I_1^0}\frac{d\ddot{\omega}(x)}{(|I|+|x-x_I|)^{4-2\alpha}} \quad (4.1.5)\\ &\lesssim |I|^{4-2\alpha}\left(\frac{s_j^k}{|I|^{4-2\alpha}}+\frac{1}{|I_j^k|^{2-\alpha}}\int_{I_1^0\setminus G_j^k}\frac{|I_j^k|^{2-\alpha}d\ddot{\sigma}(x)}{(|I_j^k|+|x-x_{I_j^k}|)^{4-2\alpha}}\right)\frac{\ddot{\mathcal{P}}(I_j^k,\ddot{\omega})}{|I_j^k|^{2-\alpha}}\\ &\lesssim \frac{|I|^{4-2\alpha}}{|I_j^k|^{2-\alpha}}\left(\frac{s_j^k}{|I|^{4-2\alpha}}+\frac{\ddot{\sigma}(I_j^k)}{|I_j^k|^{4-2\alpha}}\right)\frac{\ddot{\omega}(I_j^k)}{|I_j^k|^{2-\alpha}}\lesssim\frac{\ddot{\sigma}(I_j^k)\ddot{\omega}(I_j^k)}{|I_j^k|^{4-2\alpha}}\approx 1\end{aligned}$$

where in the first inequality we used the fact that $|x-x_I|\approx|x-\ddot{z}_j^k|\gtrsim|I_j^k|$ when $x\notin G_j^k$, since x_I is "close" to the center of G_j^k, and for the second inequality we used (4.1.3) and (4.1.4).

If $|x_I-\ddot{z}_j^k|>\operatorname{dist}(x_I,\partial G_j^k)$, we can assume $b\left(\frac{1-b}{2}\right)^{m-1}\le|I|\le b\left(\frac{1-b}{2}\right)^m$ for some $m>k$, since for $m=k$ we have $|I|\approx|I_j^k|$, $|x-x_I|\gtrsim|x-x_{I_j^k}|$ for $x\notin G_j^k$ and we can repeat the proof of (4.1.5). Now let I_t^m be the m-th generation interval that is closer to I that touches the boundary of G_j^k. We have, using $|x_{I_t^m}-\ddot{z}_j^\ell|\lesssim|x_I-\ddot{z}_j^\ell|$, for all $\ell\ge1, 1\le j\le 2^\ell$, $\ddot{\mathcal{P}}(I,\ddot{\sigma})\lesssim\ddot{\mathcal{P}}(I_t^m,\ddot{\sigma})$ and $\ddot{\mathcal{P}}(I,\ddot{\omega})\lesssim\ddot{\mathcal{P}}(I_t^m,\ddot{\omega})$, which imply

$$\ddot{\mathcal{P}}(I,\ddot{\sigma})\ddot{\mathcal{P}}(I,\ddot{\omega})\lesssim 1.$$

Case 2. Now assume $G_j^k \subset AI$. If $I_j^k \cap I = \emptyset$, then, using the minimality of k, $I \subset G_t^m$ for some $m < k$ and we can repeat the proof of (4.1.5). If $I_j^k \cap I \neq \emptyset$ then $|I| \lesssim |I_j^k|$ since otherwise AI would contain $\ddot{z}_t^{k-1}$, contradicting the minimality of k if we fix A big enough depending only on α. Hence we have:

$$|G_j^k| + |x - \ddot{z}_j^k| \leq |G_j^k| + |x_I - \ddot{z}_j^k| + |x - x_I| \leq \left(A + \frac{A}{2}\right)|I| + |x - x_I|$$

which implies that

$$\ddot{\mathcal{P}}(I,\ddot{\sigma}) \lesssim \int_{I_1^0} \frac{|I|^{2-\alpha}}{\left(|G_j^k| + |x - \ddot{z}_j^k|\right)^{4-2\alpha}} d\ddot{\sigma}(x) \lesssim \frac{|I|^{2-\alpha}}{|I_j^k|^{2-\alpha}} \int_{I_1^0} \frac{|I_j^k|^{2-\alpha}}{\left(|I_j^k| + |x - \ddot{z}_j^k|\right)^{4-2\alpha}} d\ddot{\sigma}(x)$$

and similarly

$$\ddot{\mathcal{P}}(I,\ddot{\omega}) \lesssim \frac{|I|^{2-\alpha}}{|I_t^k|^{2-\alpha}} \ddot{\mathcal{P}}(I_j^k,\ddot{\omega}) \leq \ddot{\mathcal{P}}(I_j^k,\ddot{\omega}).$$

which implies

$$\ddot{\mathcal{P}}(I,\ddot{\sigma})\ddot{\mathcal{P}}(I,\ddot{\omega}) \lesssim 1$$

Case 3. If neither $G_j^k \cap AI \neq G_j^k$ nor $G_j^k \cap AI \neq AI$, note that $G_j^k \subset 3AI$ and we repeat again the proof of Case 2.

Thus, for any interval $I \subset I_1^0$, we have shown that $\ddot{\mathcal{P}}(I,\ddot{\sigma})\ddot{\mathcal{P}}(I,\ddot{\omega}) \lesssim 1$, which implies

$$\ddot{\mathcal{A}}_2^{\alpha}(\ddot{\sigma},\ddot{\omega}) < \infty \tag{4.1.6}$$

4.1.3 The Energy Constants $\ddot{\mathcal{E}}$ and $\ddot{\mathcal{E}}^*$

. Now define the following variant of the energy constants

$$\begin{aligned}\ddot{\mathcal{E}} &= \sup_{I=\dot{\bigcup} I_r} \frac{1}{\ddot{\sigma}(I)} \sum_{r\geq 1} \ddot{\omega}(I_r) E(I_r,\ddot{\omega})^2 \ddot{\mathrm{P}}(I_r, \mathbf{1}_I \ddot{\sigma})^2 \\ \ddot{\mathcal{E}}^* &= \sup_{I=\dot{\bigcup} I_r} \frac{1}{\ddot{\omega}(I)} \sum_{r\geq 1} \ddot{\sigma}(I_r) E(I_r,\ddot{\sigma})^2 \ddot{\mathrm{P}}(I_r, \mathbf{1}_I \ddot{\omega})^2\end{aligned}$$

where the supremum is taken over the different intervals I and all the different decompositions of $I = \dot{\bigcup}_{r\geq 1} I_r$, and

$$\ddot{\mathrm{P}}(I,\mu) = \int_{\mathbb{R}} \frac{|I|}{(|I| + |x - x_I|)^{3-\alpha}} d\mu(x),$$

$$E(I,\mu)^2 = \frac{1}{2}\frac{1}{\mu(I)^2} \int_I \int_I \frac{(x-x')^2}{|I|^2} d\mu(x') d\mu(x) = \frac{1}{\mu(I)} \cdot \left\| x - m_I^\mu \right\|^2_{L^2(\mathbf{1}_I \mu)} \leq 1.$$

We first show that $\ddot{\mathcal{E}}$ is bounded. We have

$$\begin{aligned}\ddot{\mathrm{P}}(I,\ddot{\sigma}) = \int \frac{|I|}{(|I| + |x - x_I|)^{3-\alpha}} d\ddot{\sigma}(x) &\lesssim \sum_{n=0}^{\infty} \frac{\ddot{\sigma}\big((2^n+1)I\big)}{(2^n)|2^n I|^{2-\alpha}} \\ &\leq \sum_{n=0}^{\infty} \inf_{x\in I} M^\alpha \ddot{\sigma}(x) 2^{-n} \lesssim \inf_{x\in I} M^\alpha \ddot{\sigma}(x)\end{aligned}$$

where $M^\alpha \mu(x) = \sup_{I\ni x} \frac{1}{|I|^{2-\alpha}} \int_I d\mu$ and the implied constants depend only on α. Thus, given an interval $I = \dot{\cup}_{r\geq 1} I_r$, we have:

$$\sum_{r\geq 1} \ddot{\omega}(I_r) \ddot{\mathrm{P}}^2(I_r, \mathbf{1}_I \ddot{\sigma}) \leq \sum_{r\geq 1} \ddot{\omega}(I_r) \inf_{x\in I} \left(M^\alpha \mathbf{1}_I \ddot{\sigma}\right)^2(x) \leq \int_I \left(M^\alpha \mathbf{1}_I \ddot{\sigma}\right)^2(x) d\ddot{\omega}(x)$$

and so we are left with estimating the right hand term of the above inequality. We will prove the inequality

$$\int_{I_r^\ell} \left(M^\alpha \mathbf{1}_{I_r^l} \ddot{\sigma}\right)^2 (x) d\ddot{\omega}(x) \leq C\ddot{\sigma}(I_r^\ell). \tag{4.1.7}$$

where the constant C depends only on α. This will be enough, since for an interval I containing a point mass $\ddot{z}_r^\ell$ but no masses $\ddot{z}_j^k$ for $k < \ell$, we have

$$\begin{aligned}\int_I (M^\alpha \ddot{\sigma})^2 (x) d\ddot{\omega}(x) = \int_{I\cap I_r^\ell} \left(M^\alpha \mathbf{1}_{I\cap I_r^\ell} \ddot{\sigma}\right)^2 (x) d\ddot{\omega}(x) \;&\leq\; \int_{I_r^\ell} \left(M^\alpha \mathbf{1}_{I_r^\ell} \ddot{\sigma}\right)^2 (x) d\ddot{\omega}(x) \\ &\leq\; \ddot{\sigma}(I_r^\ell) \approx \ddot{\sigma}(I)\end{aligned}$$

Since the measure $\ddot{\omega}$ is supported in the Cantor set E_b, we can use the fact that for $x \in I_r^\ell \cap \mathrm{E}_b$,

$$M^\alpha(\mathbf{1}_{I_r^\ell}\ddot{\sigma})(x) \lesssim \sup_{(k,j):x\in I_j^k} \frac{1}{\left|I_j^k\right|^{2-\alpha}} \int_{I_j^k\cap I_r^\ell} d\ddot{\sigma} \approx \sup_{(k,j):x\in I_j^k} \frac{{s_0}^{-2(k\vee\ell)} 2^{k\vee\ell}}{{s_0}^{-k}} \approx \frac{\ddot{\sigma}(I_r^\ell)}{|I_r^\ell|^{2-\alpha}} \approx \left(\frac{2}{s_0}\right)^\ell$$

Fix m and let the approximations $\ddot{\omega}^{(m)}$ and $\ddot{\sigma}^{(m)}$ to the measures ω and $\ddot{\sigma}$ given by

$$d\ddot{\omega}^{(m)}(x) = \sum_{i=1}^{2^m} 2^{-m} \frac{1}{\left|I_i^m\right|} \mathbf{1}_{I_i^m}(x)\, dx \quad \text{and} \quad \ddot{\sigma}^{(m)} = \sum_{k<m} \sum_{j=1}^{2^k} s_j^k \delta_{z_j^k}.$$

For these approximations we have in the same way the estimate for $x \in \bigcup_{i=1}^{2^m} I_i^m$,

$$M^\alpha \left(\mathbf{1}_{I_r^\ell} \ddot{\sigma}^{(m)}\right)(x) \lesssim \sup_{(k,j):x\in I_j^k} \frac{1}{\left|I_j^k\right|^{2-\alpha}} \int_{I_j^k\cap I_r^\ell} d\ddot{\sigma} \approx \sup_{(k,j):x\in I_j^k} \frac{\left(\frac{1}{s_0}\right)^{k\vee\ell} \left(\frac{2}{s_0}\right)^{k\vee\ell}}{\left(\frac{1}{s_0}\right)^k} \leq C \left(\frac{2}{s_0}\right)^\ell$$

Thus for each $m \geq n \geq \ell$ we have

$$\int_{I_r^\ell} M^\alpha \left(\mathbf{1}_{I_r^\ell}\ddot{\sigma}^{(n)}\right)^2 d\ddot{\omega}^{(m)} \leq C \sum_{i:I_i^m \subset I_r^\ell} \left(\frac{2}{s_0}\right)^{2\ell} 2^{-m} = C2^{m-\ell}\left(\frac{2}{s_0}\right)^{2\ell} 2^{-m} = Cs_r^\ell \approx C \int_{I_r^\ell} d\ddot{\sigma}$$

Now since $\ddot{\omega}^m$ converges weakly to $\ddot{\omega}$ and using the fact that M^α is lower semi-continuous we get:

$$\int_{I_r^\ell} M^\alpha \left(\mathbf{1}_{I_r^\ell}\ddot{\sigma}^{(n)}\right)^2 d\ddot{\omega} \leq \liminf_{m\to\infty} \int_{I_r^\ell} M^\alpha \left(\mathbf{1}_{I_r^\ell}\ddot{\sigma}^{(n)}\right)^2 d\ddot{\omega}^{(m)} \leq C\ddot{\sigma}(I_r^\ell)$$

Now, taking $n \to \infty$, by monotone convergence we get (4.1.7). This proves

$$\sum_{r\geq 1} \ddot{\omega}(I_r)\ddot{\mathrm{P}}^2(I_r, \mathbf{1}_I\ddot{\sigma}) \leq C\ddot{\sigma}(I) \tag{4.1.8}$$

which in turn implies $\ddot{\mathcal{E}} < \infty$ as $E(I_r, \ddot{\omega}) \leq 1$.

Finally, we show that the dual energy constant $\ddot{\mathcal{E}}^*$ is finite. Let us show that for $I \subset I_1^0$

$$\ddot{\sigma}(I)E(I, \ddot{\sigma})^2\ddot{\mathrm{P}}(I, \ddot{\omega})^2 \lesssim \ddot{\omega}(I). \tag{4.1.9}$$

as if we let $\{I_r \ : \ r \geq 1\}$ be any partition of I, (4.1.9) gives

$$\sum_{r\geq 1} \ddot{\sigma}(I_r)E(I_r, \ddot{\sigma})^2\ddot{\mathrm{P}}(I_r, \ddot{\omega})^2 \lesssim \sum_{r\geq 1} \ddot{\omega}(I_r) = \ddot{\omega}(I)\ .$$

Now let us establish (4.1.9). We can assume that $E(I, \ddot{\sigma}) \neq 0$. Let k be the smallest integer for which there is a r with $\ddot{z}_r^k \in I$. And let n be the smallest integer so that for some

s we have $\ddot{z}_s^{k+n} \in I$ and $\ddot{z}_s^{k+n} \neq \ddot{z}_r^k$. We have that

$$\begin{aligned}
E(I,\ddot{\sigma})^2 &= \frac{1}{2}\frac{1}{\ddot{\sigma}(I)^2}\int_I\int_I \frac{|x-x'|^2}{|I|^2}d\ddot{\sigma}(x)d\ddot{\sigma}(x') \\
&= \frac{1}{\ddot{\sigma}(I)^2}\left[\ddot{\sigma}(\ddot{z}_r^k)\int_I \frac{|x-\ddot{z}_r^k|^2}{|I|^2}d\ddot{\sigma}(x) + \int_I\int_{I\backslash\{\ddot{z}_r^k\}}\frac{|x-x'|^2}{|I|^2}d\ddot{\sigma}(x)d\ddot{\sigma}(x')\right] \\
&\lesssim \frac{\ddot{\sigma}(\ddot{z}_r^k)\ddot{\sigma}(I\backslash\{\ddot{z}_r^k\})}{\ddot{\sigma}(I)^2} + \frac{\ddot{\sigma}(I\backslash\{\ddot{z}_r^k\})}{\ddot{\sigma}(I)} \lesssim \left(\frac{2}{s_0^2}\right)^n
\end{aligned}$$

Finally, $\ddot{\sigma}(I) \approx \left(\frac{2}{s_0^2}\right)^k$, $\ddot{\omega}(I) \approx 2^{-k-n}$, and $\ddot{\mathrm{P}}(I,\ddot{\omega}) \approx \left(\frac{s_0}{2}\right)^k$, which proves (4.1.9).

4.2 The Two Dimensional Construction

It is time now to define the two dimensional measures that prove the statement of Theorem 2.3.4. For any set $E \subset \mathbb{R}^2$ let

$$\omega(E) = \sum_{n=0}^{\infty}\ddot{\omega}_n(E)$$

where $\ddot{\omega}_0(E) = \ddot{\omega}(E_x \cap I_1^0)$, E_x the projection of E on the x-axis, and $\ddot{\omega}_n$ are copies of $\ddot{\omega}_0$ at the intervals $[a_n, a_n+1] \times \{0\}$ with $k_n = a_{n+1} - (a_n+1)$ to be determined later. In the same way, let

$$\sigma(E) = \sum_{n=0}^{\infty}\ddot{\sigma}_n(E)$$

where $\ddot{\sigma}_0(E) = \ddot{\sigma}([E \cap (I_1^0 \times \{\gamma_0\})]_x)$, and $\ddot{\sigma}_n$ are copies of $\ddot{\sigma}_0$ at the intervals $[a_n, a_n+1] \times \{\gamma_n\}$, where the height γ_n will be determined later. Check Figure 4.2.1.

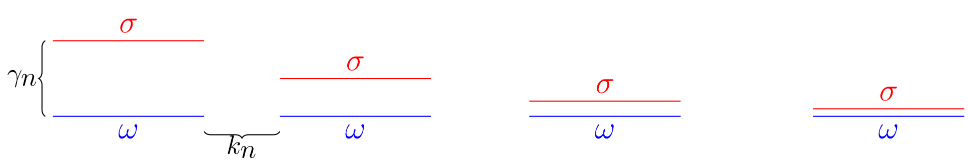

Figure 4.2.1: Positioning of measures

4.2.1 The $\mathcal{A}_2$ conditions.

We will now prove that both $\mathcal{A}_2^\alpha$ and $\mathcal{A}_2^{\alpha,*}$ constants are bounded. Let Q be a cube in $\mathbb{R}^2$, $J_0^n = [a_n, a_n+1] \times \{0\}$ and $J_{\gamma_n}^n = [a_n, a_n+1] \times \{\gamma_n\}$. We take cases for Q. If Q intersects only one of the intervals J_0^n, say J_0^0 for convenience, and $(Q \cap J_0^0)_x =: J_0$ we have:

$$\begin{aligned}
\mathcal{P}^\alpha(Q, \mathbf{1}_{Q^c}\sigma)\frac{\omega(Q)}{|Q|^{1-\frac{\alpha}{2}}} &\lesssim \ddot{\mathcal{P}}(J_0, \ddot{\sigma})\frac{\ddot{\omega}(J_0)}{|J_0|^{2-\alpha}} + \mathcal{P}^\alpha(Q, \mathbf{1}_{(J_{\gamma_1}^1)^c}\sigma)\frac{\ddot{\omega}(I_1^0)}{|Q|^{1-\frac{\alpha}{2}}} \\
&\leq \ddot{\mathcal{A}}_2^\alpha(\ddot{\sigma}, \ddot{\omega}) + C < \infty
\end{aligned}$$

using (4.1.6) and taking k_n large enough so that the second summand is bounded independently of the interval ($k_n = 4^{2n \cdot \max\{(2-\alpha)^{-1}, 1\}}$ would do here). If Q intersects more than one of the intervals J_0^n, it is easy to see, using that Q is very big (since it intersects more than one of the intervals) and that k_n is also large, that:

$$\mathcal{P}^\alpha(Q, \mathbf{1}_{Q^c}\sigma)\frac{\omega(Q)}{|Q|^{1-\frac{\alpha}{2}}} \lesssim 1$$

which of course shows that $\mathcal{A}_2^\alpha$ is bounded. Essentially using the same calculations we see that $\mathcal{A}_2^{\alpha,*}$ is bounded as well.

4.2.2 Off-Testing Constant

. Let us now check that the off-testing constant is not bounded. Choose the cube $Q_n = [a_n, a_n+1] \times [0,-1]$. Then,

$$\frac{1}{\omega(Q_n)}\int_{Q_n^c}\left[\int_{Q_n}\frac{d\omega(y)}{|x-y|^{2-\alpha}}\right]^2 d\sigma(x) \geq \frac{1}{\ddot{\omega}(I_1^0)}\int_{I_1^0}\left[\int_{I_1^0}\frac{d\ddot{\omega}(y_1)}{\sqrt{(x_1-y_1)^2+\gamma_n^2}^{2-\alpha}}\right]^2 d\ddot{\sigma}(x_1)$$

for $x=(x_1,x_2)$ and $y=(y_1,y_2)$. Taking γ_n such that the last expression on the display above equals n (note that this is feasible, since for $\gamma_n = 0$, (4.1.1) gives infinity in the latter expression above) we have

$$\mathcal{T}_{\text{off},\alpha}^2 \geq \frac{1}{\omega(Q_n)}\int_{Q_n^c}\left[\int_{Q_n}\frac{d\omega(y)}{|x-y|^{2-\alpha}}\right]^2 d\sigma(x) \geq n$$

and by letting $n \to \infty$ we obtain that the off-testing constant is not bounded.

4.2.3 The Energy Conditions

. For the energy condition $\mathcal{E}_2^\alpha$ first, let Q be a cube and $Q = \dot{\cup} Q_r$, where $\{Q_r\}_{r=1}^\infty$ is a decomposition of Q. Then we have

$$\frac{1}{\sigma(Q)}\sum_{r=1}^{\infty}\left(\frac{\mathrm{P}^\alpha\left(Q_r, \mathbf{1}_Q\sigma\right)}{|Q_r|^{\frac{1}{2}}}\right)^2 \left\|x - m_{Q_r}^\omega\right\|_{L^2\left(\mathbf{1}_{Q_r}\omega\right)}^2 \leq \frac{2}{\sigma(Q)}\sum_{r=1}^{\infty}\omega(Q_r)\left(\mathrm{P}^\alpha\left(Q_r, \mathbf{1}_Q\sigma\right)\right)^2$$

Assume that Q intersects m intervals of the form J_0^n. Then we have $m-2 \lesssim \sigma(Q) \lesssim m$. The case $m=1$ is exactly the same as the one dimensional analog for $\ddot{\mathcal{E}}$. Assume $m=2$.

Now we need to take cases for Q_r:

(i) Let Q^1 be the set of cubes Q_r that intersect only one of the intervals J_0^n. Then we have, following the proof of (4.1.8), that

$$\sum_{Q_r \in Q^1} \omega(Q_r) \left(\mathrm{P}^\alpha\left(Q_r, \mathbf{1}_Q \sigma\right)\right)^2 \leq C\sigma(Q)$$

(ii) If Q_r intersects both of the intervals J_0^n then this Q_r is unique since the family $\{Q_r\}_{r\in\mathbb{N}}$ forms a decomposition of Q. Therefore we have:

$$\omega(Q_r) \left(\mathrm{P}^\alpha\left(Q_r, \mathbf{1}_Q \sigma\right)\right)^2 \lesssim \frac{\omega(Q_r)\sigma(Q)}{|Q_r|^{2-\alpha}}\sigma(Q) \lesssim \sigma(Q)$$

using the fact that $|Q_r| \gtrsim 4^2$ since it intersect two of the intervals J_0^n and $\omega(Q_r) \lesssim 2, \sigma(Q) \lesssim 2$.

For $m \geq 3$, again we take cases for Q_r:

(i) If Q_r intersects only one J_0^n we again have, following the proof of (4.1.8), that

$$\sum_{Q_r \in Q^1} \omega(Q_r) \left(\mathrm{P}^\alpha\left(Q_r, \mathbf{1}_Q \sigma\right)\right)^2 \leq C\sigma(Q)$$

(ii) If Q_r intersects more than one of the intervals J_0^n, the last one being J_0^{n0} we have

$$\omega(Q_r) \left(\mathrm{P}^\alpha\left(Q_r, \mathbf{1}_Q \sigma\right)\right)^2 \lesssim \frac{\omega(Q_r)\sigma(Q_r^-)^2}{|Q_r|^{2-\alpha}} + \omega(Q_r)\sum_{k=1}^{m} \frac{1}{4^{2k}|Q_r|^{2-\alpha}} \lesssim 2$$

where Q_r^- contains all the intervals J_0^n such that $n \leq n_0$. Again in the last inequality we use the fact that Q_r is very big since it intersects at least two intervals J_0^n. Now since Q_r form a decomposition of Q we can have at most $m-1$ of these.

Combining the above cases, we obtain

$$\sum_{r=1}^{\infty} \omega(Q_r)\big(\mathrm{P}^{\alpha}\left(Q_r, \mathbf{1}_Q\sigma\right)\big)^2 \leq C\sigma(Q) + 2m - 2 \leq 2C\sigma(Q)$$

and that proves the energy condition is bounded.

The dual energy $\mathcal{E}_2^{\alpha,*}$ can also be proved bounded with the same calculations as in the energy condition following the proof of (4.1.9) instead of (4.1.8) as in the first case above. This completes the proof of the Theorem 2.3.4. □

4.3 The Riesz transform lemma

To obtain the same result for the Riesz transforms, we need to deal with the fact that the kernel is not positive. This prevents us from placing the masses for $\ddot{\sigma}$ at the center of the intervals G_j^k, as we did in the proof of Theorem 2.3.4. Since otherwise, if the point-mass $\ddot{\sigma}$ is located at the center of G_j^k, it would result in the cancellation of much of the mass not letting us deduce that the off testing condition for the Riesz transform is unbounded. The following lemma, whose proof follows closely the work in [32] but with a two dimensional twist, helps us overcome this problem, showing that, while not being able to place the point masses in the middle of G_j^k, we can place them far from the boundary. This enables us to show that the $\ddot{A}_2$ condition is bounded, like in the proof of Theorem 2.3.4. First we need to define the operator

$$\ddot{R}f(x) = \int_{\mathbb{R}} \frac{(x-y)f(y)}{|x-y|^{3-\alpha}}dy$$

Lemma 4.3.1. *For $k \geq 1$, $1 \leq j \leq 2^k$, write $G_j^k = (a_j^k, b_j^k)$. Then there exists $0 < c < 1$*

that depends only on α such that

$$\ddot{R}\ddot{\omega}\left(a_j^k+c\left(\frac{1-b}{2}\right)^k b\right) \approx \left(\frac{s_0}{2}\right)^k$$

where $\ddot{\omega}$ is the measure defined above.

Proof. Fix k. We have

$$\ddot{R}\ddot{\omega}\left(a_1^k+c\left(\frac{1-b}{2}\right)^k b\right) \leq \ddot{R}\ddot{\omega}\left(a_j^k+c\left(\frac{1-b}{2}\right)^k b\right) \leq \ddot{R}\ddot{\omega}\left(a_{2^k}^k+c\left(\frac{1-b}{2}\right)^k b\right)$$

from monotonicity. So it is enough to prove the following:

$$\left(\frac{s_0}{2}\right)^k \lesssim \ddot{R}\ddot{\omega}\left(a_1^k+c\left(\frac{1-b}{2}\right)^k b\right) \leq \ddot{R}\ddot{\omega}\left(a_{2^k}^k+c\left(\frac{1-b}{2}\right)^k b\right) \lesssim \left(\frac{s_0}{2}\right)^k$$

We start with right hand inequality. Following the definitions of $\ddot{R}, \ddot{\omega}$ we get

$$\begin{aligned}
\ddot{R}\ddot{\omega}\left(a_{2^k}^k+c\left(\frac{1-b}{2}\right)^k b\right) &\leq \int_{[0,a_{2^k}^k]} \frac{d\ddot{\omega}(y)}{\left(a_{2^k}^k+c\left(\frac{1-b}{2}\right)^k b-y\right)^{2-\alpha}} \\
&\leq \sum_{\ell=1}^{k} \frac{2^{-\ell}}{\left(a_{2^k}^k+c\left(\frac{1-b}{2}\right)^k b-\left[1-\left(\frac{1-b}{2}\right)^{\ell-1}\left(\frac{1+b}{2}\right)\right]\right)^{2-\alpha}} \\
&\approx \frac{2^{-k}}{c^{2-\alpha}s_0^{-k}}+\sum_{\ell=1}^{k-1} \frac{2^{-\ell}}{s_0^{-\ell}\left[\frac{1+b}{2}+\left(\frac{1-b}{2}\right)^{k-\ell+1}\left[cb-\frac{1+b}{2}\right]\right]^{2-\alpha}} \\
&\leq \frac{2^{-k}}{c^{2-\alpha}s_0^{-k}}+\sum_{\ell=1}^{k-1} \frac{2^{-\ell}}{s_0^{-\ell}\left[\frac{1+b}{2}-\frac{1+b}{2}\left(\frac{1-b}{2}\right)^{k-\ell+1}\right]^{2-\alpha}}
\end{aligned}$$

since $a_{2^k}^k = 1-\left(\frac{1+b}{2}\right)\left(\frac{1-b}{2}\right)^k$. The square bracket inside the last fraction is minimized for

$\ell = k-1$ and we get the inequality

$$\ddot{R}\ddot{\omega}\left(a_{2^k}^k + c\left(\frac{1-b}{2}\right)^k b\right) \lesssim \frac{2^{-k}}{c^{2-\alpha}s_0^{-k}} + \sum_{\ell=1}^{k-1}\left(\frac{s_0}{2}\right)^\ell \lesssim \frac{1}{c^{2-\alpha}}\left(\frac{s_0}{2}\right)^k$$

where the implied constants depend again only on α. We should note here that the summand with $\ell = k$ is the dominant one in the above inequality.

Now we consider the left hand inequality. We have that $\ddot{R}\ddot{\omega}\left(a_1^k + c\left(\frac{1-b}{2}\right)^k b\right)$ equals

$$\ddot{R}\ddot{\omega}\mathbf{1}_{I_1^{k+1}}\left(a_1^k + c\left(\frac{1-b}{2}\right)^k b\right) + \sum_{\ell=1}^{k+1}\ddot{R}\ddot{\omega}\mathbf{1}_{I_2^\ell}\left(a_1^k + c\left(\frac{1-b}{2}\right)^k b\right) \tag{4.3.1}$$

and following the argument for the previous inequality we see that

$$\left|\sum_{\ell=1}^{k+1}\ddot{R}\ddot{\omega}\mathbf{1}_{I_2^\ell}\left(a_1^k + c\left(\frac{1-b}{2}\right)^k b\right)\right| \le A\left(\frac{s_0}{2}\right)^k$$

where A depends only on α but not on c. The first summand of (4.3.1) gives

$$\begin{aligned}
\int_{I_1^{k+1}} \frac{d\ddot{\omega}(y)}{\left(a_1^k + c\left(\frac{1-b}{2}\right)^k b - y\right)^{2-\alpha}} \quad &\ge \quad \sum_{\ell=k+1}^{\infty} \frac{2^{-\ell-1}}{\left(\left(\frac{1-b}{2}\right)^\ell + c\left(\frac{1-b}{2}\right)^k b\right)^{2-\alpha}} \\
&\approx \quad \frac{s_0^k}{2^k}\sum_{\ell=k+1}^{\infty}\frac{2^{-\ell+k-1}}{\left(\left(\frac{1-b}{2}\right)^{\ell-k} + cb\right)^{2-\alpha}} \\
&= \quad \frac{s_0^k}{2^k}\sum_{\ell=1}^{\infty}\frac{2^{-\ell-1}}{\left(\left(\frac{1-b}{2}\right)^{\ell} + cb\right)^{2-\alpha}}.
\end{aligned}$$

Choosing c small enough not depending on k (since the last sum does not depend on k), we obtain

$$\int_{I_1^{k+1}} \frac{d\ddot{\omega}(y)}{\left(a_1^k+c\left(\frac{1-b}{2}\right)^k b-y\right)^{2-\alpha}} \geq C_1\left(\frac{s_0}{2}\right)^k$$

with $C_1 > 2A$ and we conclude our lemma. $\square$

Proof of Theorem 2.3.5. Set $\dot{z}_j^k = a_j^k+cb\left(\frac{1-b}{2}\right)^k$ and define the measure $\dot{\sigma} = \sum_{k,j} s_j^k \delta_{\dot{z}_j^k}$ where $s_j^k = \left(\frac{2}{s_0^2}\right)^k$ as before. Following verbatim the calculations of Theorem 2.3.4, one can show that $\ddot{A}_2(\dot{\sigma},\ddot{\omega}) < \infty$. Now define the measures ω and σ, as before, for any measurable set $E \subset \mathbb{R}^2$ by

$$\omega(E) = \sum_{n=0}^{\infty} \ddot{\omega}_n(E) \quad \text{and} \quad \sigma(E) = \sum_{n=0}^{\infty} \dot{\sigma}_n(E)$$

where $\dot{\sigma}_0(E) = \dot{\sigma}([E\cap(I_1^0\times\{\gamma_0\})]_x)$, and $\dot{\sigma}_n$ are copies of $\dot{\sigma}_0$ at the intervals $[a_n, a_n{+}1]\times\{\gamma_n\}$, and where the height γ_n will be determined later. Again, as before, it is easy to see that both $\mathcal{A}_2^{\alpha}$ and $\mathcal{A}_2^{\alpha,*}$ and both $\mathcal{E}_2^{\alpha}$ and $\mathcal{E}_2^{\alpha,*}$ are bounded. Let us now finish the proof by showing that the off-testing constant for the Riesz transforms are unbounded. From Lemma 4.3.1 we have $\ddot{R}\ddot{\omega}(\dot{z}_j^k) \gtrsim \left(\frac{s_0}{2}\right)^k$ which implies

$$\int_{I_1^0} \left(\ddot{R}(\mathbf{1}_{I_1^0}\ddot{\omega})(x)\right)^2 d\dot{\sigma}(y) \gtrsim \sum_{k=1}^{\infty}\sum_{j=1}^{2^k} s_j^k \cdot \left(\frac{s_0}{2}\right)^{2k} = \sum_{k=1}^{\infty}\sum_{j=1}^{2^k} \frac{1}{2^k} = \infty. \tag{4.3.2}$$

Now choose the cube $Q_n = [a_n, a_n + 1] \times [0, -1]$. Then,

$$\begin{aligned}
\mathcal{R}_{1,\mathit{off},\alpha}^2 &\geq \frac{1}{\omega(Q_n)} \int_{Q_n^c} \left[\int_{Q_n} \frac{(x_1-y_1)d\omega(y)}{|x-y|^{3-\alpha}}\right]^2 d\sigma(x) \\
&\geq \frac{1}{\omega(Q_n)} \int_{I_1^0} \left[\int_{I_1^0} \frac{(x_1-y_1)d\ddot{\omega}(y_1)}{\sqrt{(x_1-y_1)^2+\gamma_n^2}^{3-\alpha}}\right]^2 d\dot{\sigma}(x_1) = \frac{n}{\omega(Q_n)}
\end{aligned}$$

by choosing the height γ_n so that $\int_{I_1^0}\left[\int_{I_1^0} \frac{(x_1-y_1)d\ddot{\omega}(y_1)}{\sqrt{(x_1-y_1)^2+\gamma_n^2}^{3-\alpha}}\right]^2 d\dot{\sigma}(x_1) = n$ by (4.3.2). Letting $n \to \infty$, we see that the off-testing constant is unbounded. $\square$

Chapter 5

A two weight local Tb theorem for n-dimensional Fractional Integrals

5.1 The local Tb theorem and proof preliminaries

5.1.1 Standard fractional singular integrals

Let $0 \leq \alpha < n$. We define a standard α-fractional CZ kernel $K^{\alpha}(x,y)$ to be a real-valued function defined on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying the following fractional size and smoothness conditions of order $1+\delta$ for some $\delta > 0$: For $x \neq y$,

$$\begin{aligned}
\left|K^{\alpha}(x,y)\right| &\leq C_{CZ}\left|x-y\right|^{\alpha-n} \qquad (5.1.1)\\
\left|\nabla K^{\alpha}(x,y)\right| &\leq C_{CZ}\left|x-y\right|^{\alpha-n-1}\\
\left|\nabla K^{\alpha}(x,y)-\nabla K^{\alpha}\left(x',y\right)\right| &\leq C_{CZ}\left(\frac{\left|x-x'\right|}{\left|x-y\right|}\right)^{\delta}\left|x-y\right|^{\alpha-n-1}, \qquad \frac{\left|x-x'\right|}{\left|x-y\right|} \leq \frac{1}{2},
\end{aligned}$$

and the last inequality also holds for the adjoint kernel in which x and y are interchanged. We note that a more general definition of kernel has only order of smoothness $\delta > 0$, rather than $1+\delta$, but the use of the Monotonicity and Energy Lemmas in arguments below involves first order Taylor approximations to the kernel functions $K^{\alpha}(\cdot,y)$.

5.1.1.1 Defining the norm inequality

We now turn to a precise definition of the weighted norm inequality

$$\left\| T_{\sigma}^{\alpha} f \right\|_{L^{2}(\omega)} \leq \mathfrak{N}_{T_{\sigma}^{\alpha}} \left\| f \right\|_{L^{2}(\sigma)}, \qquad f \in L^{2}(\sigma). \tag{5.1.2}$$

For this we introduce a family $\left\{ \eta_{\delta,R}^{\alpha} \right\}_{0<\delta<R<\infty}$ of nonnegative functions on $[0,\infty)$ so that the truncated kernels $K_{\delta,R}^{\alpha}(x,y) = \eta_{\delta,R}^{\alpha}(|x-y|) K^{\alpha}(x,y)$ are bounded with compact support for fixed x or y. Then the truncated operators

$$T_{\sigma,\delta,R}^{\alpha} f(x) \equiv \int_{\mathbb{R}^{n}} K_{\delta,R}^{\alpha}(x,y) f(y)\, d\sigma(y), \qquad x \in \mathbb{R}^{n}, \tag{5.1.3}$$

are pointwise well-defined, and we will refer to the pair $\left(K^{\alpha}, \left\{ \eta_{\delta,R}^{\alpha} \right\}_{0<\delta<R<\infty} \right)$ as an α-fractional singular integral operator, which we typically denote by T^{α}, suppressing the dependence on the truncations.

Definition 5.1.1. *We say that an α-fractional singular integral operator T^{α} satisfies the norm inequality (5.1.2) provided*

$$\left\| T_{\sigma,\delta,R}^{\alpha} f \right\|_{L^{2}(\omega)} \leq \mathfrak{N}_{T_{\sigma}^{\alpha}} \left\| f \right\|_{L^{2}(\sigma)}, \qquad f \in L^{2}(\sigma), 0 < \delta < R < \infty.$$

It turns out that, in the presence of the Muckenhoupt conditions (5.1.7) below, the norm inequality (5.1.2) is essentially independent of the choice of truncations used, and this is explained in some detail in [67]. Thus, as in [67], we are free to use the tangent line truncations described there throughout the proofs of our results.

5.1.2 Weakly accretive functions

Denote by $\mathcal{P}$ the collection of cubes in $\mathbb{R}^n$. Note that we include an L^p upper bound in our definition of 'p-weakly accretive family' of functions.

Definition 5.1.2. *Let $p \geq 2$ and let μ be a locally finite positive Borel measure on $\mathbb{R}^n$. We say that a family $\mathbf{b} = \left\{b_Q\right\}_{Q\in\mathcal{P}}$ of functions indexed by $\mathcal{P}$ is a p-*weakly μ-accretive* family of functions on $\mathbb{R}^n$ if for $Q \in \mathcal{P}$,*

$$\begin{aligned} &\operatorname{supp} b_Q \subset Q \\ 0 < c_{\mathbf{b}} \;\; &\leq \;\; \frac{1}{|Q|_\mu}\int_Q b_Q d\mu \leq \left(\frac{1}{|Q|_\mu}\int_Q \left|b_Q\right|^p d\mu\right)^{\frac{1}{p}} \leq C_{\mathbf{b}} < \infty. \end{aligned} \tag{5.1.4}$$

5.1.3 b-testing conditions

Suppose σ and ω are locally finite positive Borel measures on $\mathbb{R}^n$. The $\mathbf{b}$-testing conditions for T^α and $\mathbf{b}^*$-testing conditions for the dual $T^{\alpha,*}$ are given by

$$\begin{aligned} \int_Q \left|T_\sigma^\alpha b_Q\right|^2 d\omega \;\; &\leq \;\; \left(\mathfrak{T}_{T^\alpha}^{\mathbf{b}}\right)^2 |Q|_\sigma \;, \qquad \text{for all cubes } Q, \\ \int_Q \left|T_\omega^{\alpha,*} b_Q^*\right|^2 d\sigma \;\; &\leq \;\; \left(\mathfrak{T}_{T^{\alpha,*}}^{\mathbf{b}^*}\right)^2 |Q|_\omega \;, \qquad \text{for all cubes } Q. \end{aligned} \tag{5.1.5}$$

5.1.4 Poisson integrals and the Muckenhoupt conditions

Let μ be a locally finite positive Borel measure on $\mathbb{R}^n$, and suppose Q is a cube in $\mathbb{R}^n$. Recall that $|Q|^{\frac{1}{n}} = \ell(Q)$ for a cube Q. The two α-fractional Poisson integrals of μ on a cube Q are given by the following expressions:

$$\begin{aligned}
\mathrm{P}^{\alpha}(Q,\mu) &\equiv \int_{\mathbb{R}^n} \frac{|Q|^{\frac{1}{n}}}{\left(|Q|^{\frac{1}{n}}+\left|x-x_Q\right|\right)^{n+1-\alpha}} d\mu(x), \\
\mathcal{P}^{\alpha}(Q,\mu) &\equiv \int_{\mathbb{R}^n} \left(\frac{|Q|^{\frac{1}{n}}}{\left(|Q|^{\frac{1}{n}}+\left|x-x_Q\right|\right)^{2}}\right)^{n-\alpha} d\mu(x),
\end{aligned}$$

where $\left|x-x_Q\right|$ denotes distance between x and the center x_Q of Q and $|Q|$ denotes the Lebesgue measure of the cube Q. We refer to P^{α} as the *standard* Poisson integral and to $\mathcal{P}^{\alpha}$ as the *reproducing* Poisson integral. Note that these two kernels satisfy for all cubes Q and positive measures μ,

$$\begin{aligned}
0 &\leq \mathrm{P}^{\alpha}(Q,\mu) \leq C\mathcal{P}^{\alpha}(Q,\mu), \quad n-1 \leq \alpha < n, \\
0 &\leq \mathcal{P}^{\alpha}(Q,\mu) \leq C\mathrm{P}^{\alpha}(Q,\mu), \quad 0 \leq \alpha < n-1.
\end{aligned}$$

We now define the *one-tailed constant with holes* $\mathcal{A}_2^{\alpha}$ using the reproducing Poisson kernel $\mathcal{P}^{\alpha}$. On the other hand, the standard Poisson integral P^{α} arises naturally throughout the proof of the Tb theorem in estimating oscillation of the fractional singular integral T^{α}, and in the definition of the energy conditions below.

Definition 5.1.3. *Suppose σ and ω are locally finite positive Borel measures on $\mathbb{R}^n$. The one-tailed constants $\mathcal{A}_2^{\alpha}$ and $\mathcal{A}_2^{\alpha,*}$ with holes for the weight pair (σ,ω) are given by*

$$\begin{aligned}
\mathcal{A}_2^{\alpha} &\equiv \sup_{Q\in\mathcal{P}} \mathcal{P}^{\alpha}\left(Q,\mathbf{1}_{Q^c}\sigma\right)\frac{|Q|_{\omega}}{|Q|^{1-\frac{\alpha}{n}}} < \infty, \\
\mathcal{A}_2^{\alpha,*} &\equiv \sup_{Q\in\mathcal{P}} \mathcal{P}^{\alpha}\left(Q,\mathbf{1}_{Q^c}\omega\right)\frac{|Q|_{\sigma}}{|Q|^{1-\frac{\alpha}{n}}} < \infty.
\end{aligned}$$

Note that these definitions are the conditions with 'holes' introduced by Hytönen [22] - the supports of the measures $\mathbf{1}_{Q^c}\sigma$ and $\mathbf{1}_{Q^c}\omega$ in the definition of $\mathcal{A}_2^\alpha$ are disjoint, and so any common point masses of σ and ω do not appear simultaneously in the factors of any of the products $\mathcal{P}^\alpha\left(Q,\mathbf{1}_{Q^c}\sigma\right)\frac{|Q|_\omega}{|Q|^{1-\frac{\alpha}{n}}}$. Recall, the definition of the classical Muckenhoupt condition

$$A_2^\alpha = \sup_{Q\in\mathcal{P}} \frac{|Q|_\omega}{|Q|^{1-\frac{\alpha}{n}}}\frac{|Q|_\sigma}{|Q|^{1-\frac{\alpha}{n}}}$$

but it will find no use in the two weight setting with common point masses permitted.

Initially, these definitions of Muckenhoupt type were given in the following 'one weight' case, $d\omega(x) = w(x)\,dx$ and $d\sigma(x) = \frac{1}{w(x)}dx$, where $\mathcal{A}_2^\alpha\left(\lambda w, (\lambda w)^{-1}\right) = \mathcal{A}_2^\alpha\left(w, w^{-1}\right)$ is homogeneous of degree 0. Of course the two weight version is homogeneous of degree 2 in the weight pair, $\mathcal{A}_2^\alpha(\lambda\sigma,\lambda\omega) = \lambda^2\mathcal{A}_2^\alpha(\sigma,\omega)$, while all of the other conditions we consider in connection with two weight norm inequalities, including the operator norm $\mathfrak{N}_{T^\alpha}(\sigma,\omega)$ itself, are homogeneous of degree 1 in the weight pair. This awkwardness regarding the homogeneity of Muckenhoupt conditions could be rectified by simply taking the square root of $\mathcal{A}_2^\alpha$ and renaming it, but the current definition is so entrenched in the literature, in particular in connection with the A_2 conjecture, that we will leave it as is.

5.1.4.1 Punctured A_2^α conditions

The *classical* A_2^α characteristic fails to be finite when the measures σ and ω have a common point mass - simply let Q in the sup above shrink to a common mass point. But there is a substitute that is quite similar in character that is motivated by the fact that for large cubes Q, the sup above is problematic only if just *one* of the measures is *mostly* a point mass when restricted to Q.

Given an at most countable set $\mathfrak{P} = \{p_k\}_{k=1}^{\infty}$ in $\mathbb{R}^n$, a cube $Q \in \mathcal{P}$, and a positive locally finite Borel measure μ, define

$$\mu\left(Q,\mathfrak{P}\right) \equiv \left|Q\right|_{\mu} - \sup\left\{\mu\left(p_k\right) : p_k \in Q \cap \mathfrak{P}\right\}, \tag{5.1.6}$$

where the supremum is actually achieved since $\sum_{p_k \in Q\cap\mathfrak{P}} \mu\left(p_k\right) < \infty$ as μ is locally finite. The quantity $\mu\left(Q,\mathfrak{P}\right)$ is simply the $\widetilde{\mu}$ measure of Q where $\widetilde{\mu}$ is the measure μ with its largest point mass from $\mathfrak{P}$ in Q removed. Given a locally finite positive measure pair (σ,ω), let $\mathfrak{P}_{(\sigma,\omega)} = \{p_k\}_{k=1}^{\infty}$ be the at most countable set of common point masses of σ and ω. Then the weighted norm inequality (5.1.2) typically implies finiteness of the following *punctured* Muckenhoupt conditions:

$$\begin{aligned} A_2^{\alpha,punct}\left(\sigma,\omega\right) &\equiv \sup_{Q\in\mathcal{P}} \frac{\omega\left(Q,\mathfrak{P}_{(\sigma,\omega)}\right)}{\left|Q\right|^{1-\frac{\alpha}{n}}} \frac{\left|Q\right|_{\sigma}}{\left|Q\right|^{1-\frac{\alpha}{n}}}, \\ A_2^{\alpha,*,punct}\left(\sigma,\omega\right) &\equiv \sup_{Q\in\mathcal{P}} \frac{\left|Q\right|_{\omega}}{\left|Q\right|^{1-\frac{\alpha}{n}}} \frac{\sigma\left(Q,\mathfrak{P}_{(\sigma,\omega)}\right)}{\left|Q\right|^{1-\frac{\alpha}{n}}}. \end{aligned}$$

In particular, all of the above Muckenhoupt conditions $\mathcal{A}_2^{\alpha}$, $\mathcal{A}_2^{\alpha,*}$, $A_2^{\alpha,punct}$ and $A_2^{\alpha,*,punct}$ are necessary for boundedness of an elliptic α-fractional singular integral T_{σ}^{α} from $L^2\left(\sigma\right)$ to $L^2\left(\omega\right)$. It is convenient to define

$$\mathfrak{A}_2^{\alpha} \equiv \mathcal{A}_2^{\alpha} + \mathcal{A}_2^{\alpha,*} + A_2^{\alpha,punct} + A_2^{\alpha,*,punct} . \tag{5.1.7}$$

5.1.5 Energy Conditions

Here is the definition of the strong energy conditions, which we sometimes refer to simply as the energy conditions. Let

$$m_I^\mu \equiv \frac{1}{|I|_\mu}\int x d\mu(x) = \left\langle \frac{1}{|I|_\mu}\int x_1 d\mu(x), ..., \frac{1}{|I|_\mu}\int x_n d\mu(x)\right\rangle$$

be the average of x with respect to the measure μ, which we often abbreviate to m_I when the measure μ is understood.

Definition 5.1.4. *Let $0 \leq \alpha < n$. Suppose σ and ω are locally finite positive Borel measures on $\mathbb{R}^n$. Then the* strong *energy constant $\mathcal{E}_2^\alpha$ is defined by*

$$(\mathcal{E}_2^\alpha)^2 \equiv \sup_{I=\dot{\cup} I_r} \frac{1}{|I|_\sigma}\sum_{r=1}^{\infty}\left(\frac{\mathrm{P}^\alpha\left(I_r, \mathbf{1}_I\sigma\right)}{|I_r|^{\frac{1}{n}}}\right)^2 \left\|x - m_{I_r}^\omega\right\|_{L^2\left(\mathbf{1}_{I_r}\omega\right)}^2 , \tag{5.1.8}$$

where the supremum is taken over arbitrary decompositions of a cube I using a pairwise disjoint union of subcubes I_r. Similarly, we define the dual strong *energy constant $\mathcal{E}_2^{\alpha,*}$ by switching the roles of σ and ω:*

$$\left(\mathcal{E}_2^{\alpha,*}\right)^2 \equiv \sup_{I=\dot{\cup} I_r} \frac{1}{|I|_\omega}\sum_{r=1}^{\infty}\left(\frac{\mathrm{P}^\alpha\left(I_r, \mathbf{1}_I\omega\right)}{|I_r|^{\frac{1}{n}}}\right)^2 \left\|x - m_{I_r}^\sigma\right\|_{L^2\left(\mathbf{1}_{I_r}\sigma\right)}^2 . \tag{5.1.9}$$

These energy conditions are necessary for boundedness of elliptic and gradient elliptic operators, including the Hilbert transform (but not for for certain elliptic singular operators that fail to be gradient elliptic) - see [68] and [69]. It is convenient to define

$$\mathfrak{E}_2^\alpha \equiv \mathcal{E}_2^\alpha + \mathcal{E}_2^{\alpha,*}$$

as well as

$$\mathcal{NTV}_\alpha \equiv \mathfrak{T}_{T^\alpha}^{\mathbf{b}} + \mathfrak{T}_{T^\alpha,*}^{\mathbf{b}^*} + \sqrt{\mathfrak{A}_2^\alpha} + \mathfrak{E}_2^\alpha\ . \tag{5.1.10}$$

5.1.6 The two weight local Tb Theorem

Here we derive a local Tb theorem based in part on the *proof* of the $T1$ theorem in [63], and in part on the *proof* of a one weight Tb theorem in Hytönen and Martikainen [24]. Recall from [68] that an α-fractional singular integral T^α with kernel K^α is said to be *elliptic* if $|K^\alpha(x,y)| \geq c\,|x-y|^{\alpha-1}$ and *gradient elliptic* if the kernel $K^\alpha(x,y)$ satisfies

$$|\nabla K^\alpha(x,y)| \geq c\,|x-y|^{\alpha-n-1}\,. \tag{5.1.11}$$

The Hilbert transform kernel $K(x,y) = \frac{1}{y-x}$ satisfies (5.1.11) with $\alpha = 0$, $n = 1$. In dimension $n = 1$ the Muckenhoupt conditions are necessary for norm boundedness of elliptic operators by results in [32], [22] and [66], and the energy conditions are necessary for norm boundedness of gradient elliptic operators by results in [68]. Moreover, in dimension $n = 1$, Hytönen [22, Corollary 3.10] proves that full testing is controlled by testing and the Muckenhoupt conditions for the Hilbert transform, and this is easily extended to $0 \leq \alpha < 1$:

$$\mathfrak{F}\mathfrak{T}_{T^\alpha}^{\mathbf{b}} \lesssim \mathfrak{T}_{T^\alpha}^{\mathbf{b}} + \sqrt{\mathcal{A}_2^\alpha} + \sqrt{\mathcal{A}_2^{\alpha,*}} \text{ and } \mathfrak{F}\mathfrak{T}_{T^\alpha,*}^{\mathbf{b}^*} \lesssim \mathfrak{T}_{T^\alpha,*}^{\mathbf{b}^*} + \sqrt{\mathcal{A}_2^\alpha} + \sqrt{\mathcal{A}_2^{\alpha,*}}.$$

Theorem 5.1.5. *Suppose that σ and ω are locally finite positive Borel measures on Euclidean space $\mathbb{R}^n$. Suppose that T^α is a standard α-fractional singular integral operator on $\mathbb{R}^n$, and set $T_\sigma^\alpha f = T^\alpha(f\sigma)$ for any smooth truncation of T_σ^α, so that T_σ^α is* apriori *bounded from $L^2(\sigma)$ to $L^2(\omega)$. Assume the Muckenhoupt and energy conditions hold, i.e.*

$\mathcal{A}_2^\alpha, \mathcal{A}_2^{\alpha,*}, A_2^{\alpha,punct}, A_2^{\alpha,*,punct}, \mathcal{E}_2^\alpha, \mathcal{E}_2^{\alpha,*} < \infty$. *Finally, let* $p > 2$ *and let* $\mathbf{b} = \{b_Q\}_{Q\in\mathcal{P}}$ *be a* p*-weakly* σ*-accretive family of functions on* $\mathbb{R}^n$*, and let* $\mathbf{b}^* = \left\{b_Q^*\right\}_{Q\in\mathcal{P}}$ *be a* p*-weakly* ω*-accretive family of functions on* $\mathbb{R}^n$*. Then for* $0 \leq \alpha < n$*, the operator* T_σ^α *is bounded from* $L^2(\sigma)$ *to* $L^2(\omega)$ *with operator norm* $\mathfrak{N}_{T_\sigma^\alpha}$*, i.e.*

$$\|T_\sigma^\alpha f\|_{L^2(\omega)} \leq \mathfrak{N}_{T_\sigma^\alpha} \|f\|_{L^2(\sigma)}, \qquad f \in L^2(\sigma),$$

uniformly *in smooth truncations of* T^α if and only if *the* $\mathbf{b}$*-testing conditions for* T^α *and the* $\mathbf{b}^*$*-testing conditions for the dual* $T^{\alpha,*}$ *both hold. Moreover, we have*

$$\mathfrak{N}_{T^\alpha} \lesssim \mathfrak{T}_{T^\alpha}^{\mathbf{b}} + \mathfrak{T}_{T^\alpha}^{\mathbf{b}^*} + \sqrt{\mathfrak{A}_2^\alpha} + \mathfrak{E}_2^\alpha \ .$$

Remark 5.1.6. *In the special case that* $\sigma = \omega = \mu$*, the classical Muckenhoupt* A_2^α *condition is*

$$\sup_{Q\in\mathcal{P}} \frac{|Q|_\mu}{|Q|^{1-\frac{\alpha}{n}}} \frac{|Q|_\mu}{|Q|^{1-\frac{\alpha}{n}}} < \infty,$$

which is the upper doubling measure condition with exponent $n - \alpha$*, i.e.*

$$|Q|_\mu \leq C\ell(Q)^{n-\alpha}, \quad \text{for all cubes } Q,$$

which of course prohibits point masses in μ. Both Poisson integrals are then bounded,

$$\mathrm{P}^{\alpha}(Q,\mu)\lesssim\sum_{k=0}^{\infty}\frac{|Q|^{\frac{1}{n}}}{\left(2^{k}|Q|^{\frac{1}{n}}\right)^{n+1-\alpha}}\left|2^{k}Q\right|_{\mu}\lesssim\sum_{k=0}^{\infty}\frac{|Q|^{\frac{1}{n}}}{\left(2^{k}|Q|^{\frac{1}{n}}\right)^{n+1-\alpha}}\left(2^{k}\ell(Q)\right)^{n-\alpha}=2$$

$$\mathcal{P}^{\alpha}(Q,\mu)\lesssim\sum_{k=0}^{\infty}\left(\frac{|Q|^{\frac{1}{n}}}{\left(2^{k}|Q|^{\frac{1}{n}}\right)^{2}}\right)^{n-\alpha}\left|2^{k}Q\right|_{\mu}\lesssim\sum_{k=0}^{\infty}\left(\frac{|Q|^{\frac{1}{n}}}{\left(2^{k}|Q|^{\frac{1}{n}}\right)^{2}}\right)^{n-\alpha}\left(2^{k}\ell(Q)\right)^{n-\alpha}=C_{\alpha}$$

and it follows easily that the equal weight pair (μ,μ) satisfies not only the Muckenhoupt $\mathfrak{A}_2^{\alpha}$ condition, but also the strong energy condition $\mathfrak{E}_2^{\alpha}$:

$$\begin{aligned}\sum_{r=1}^{\infty}\left(\frac{\mathrm{P}^{\alpha}(I_r,\mathbf{1}_I\sigma)}{|I_r|}\right)^{2}\left\|x-m_{I_r}^{\omega}\right\|_{L^2(\omega)}^{2} &\leq C\sum_{r=1}^{\infty}\left\|\frac{x-m_{I_r}^{\omega}}{|I_r|}\right\|_{L^2(\omega)}^{2}\\ &\leq C\sum_{r=1}^{\infty}|I_r|_{\omega}\leq C|I|_{\omega}=C|I|_{\sigma}\ ,\end{aligned}$$

since $\omega=\sigma$. Thus Theorem 5.1.5, when restricted to a single weight $\sigma=\omega$, recovers a slightly weaker, due to our assumption that $p>2$, version of the one weight theorem of Lacey and Martikainen [29, Theorem 1.1] for dimension $n=1$. On the other hand, the possibility of a two weight theorem for a 2-weakly μ-accretive family is highly problematic, as one of the key proof strategies used in [29] in the one weight case is a reduction to testing over f and g with controlled L^{∞} norm, a strategy that appears to be unavailable in the two weight setting.

In order to prove Theorem 5.1.5, it is convenient to establish some improved properties for our p-weakly μ-accretive family, and also necessary to establish some improved energy conditions related to the families of testing functions $\mathbf{b}$ and $\mathbf{b}^{*}$. We turn to these matters in the next two subsections.

5.1.7 Reduction to real bounded accretive families

We begin by noting that if b_Q satisfies (5.1.4) with $\mu = \sigma$, and satisfies a given $\mathbf{b}$-testing condition for a weight pair (σ, ω), then Reb_Q satisfies

$$\left(\frac{1}{|Q|_\mu} \int_Q |Reb_Q|^p \, d\mu \right)^{\frac{1}{p}} \leq C_{\mathbf{b}}(p)$$

and the given $\mathbf{b}$-testing condition for (σ, ω) with Reb_Q in place of b_Q.

Thus we may assume throughout the proof of Theorem 5.1.5 that our p-weakly μ-accretive families $\mathbf{b} \equiv \{b_Q\}_{Q \in \mathcal{D}}$ and $\mathbf{b}^* \equiv \left\{b_Q^*\right\}_{Q \in \mathcal{G}}$ consist of **real-valued** functions.

Next we show that the assumption of testing conditions for a fractional integral T^α and p-weakly μ-accretive testing functions $\mathbf{b} = \{b_Q\}_{Q \in \mathcal{P}}$ and $\mathbf{b}^* = \left\{b_Q^*\right\}_{Q \in \mathcal{P}}$ with $p > 2$ can always be replaced with real-valued ∞-weakly μ-accretive testing functions, thus reducing the Tb theorem for the case $p > 2$ to the case when $p = \infty$. We now proceed to develop a precise statement. We extend (5.1.4) to $2 < p \leq \infty$ by

$$\operatorname{supp} b_Q \subset Q \ , \qquad Q \in \mathcal{P}, \tag{5.1.12}$$

$$1 \ \leq \ \frac{1}{|Q|_\mu} \int_Q b_Q d\mu \leq \begin{cases} \left(\frac{1}{|Q|_\mu} \int_Q |b_Q|^p \, d\mu \right)^{\frac{1}{p}} \leq C_{\mathbf{b}}(p) < \infty & \text{for } 2 < p < \infty \\ \|b_Q\|_{L^\infty(\mu)} \leq C_{\mathbf{b}}(\infty) < \infty & \text{for } p = \infty \end{cases}$$

Proposition 5.1.7. *Let $0 \leq \alpha < 1$, and let σ and ω be locally finite positive Borel measures on $\mathbb{R}^n$, and let T^α be a standard α-fractional elliptic and gradient elliptic singular integral operator on $\mathbb{R}^n$. Set $T_\sigma^\alpha f = T^\alpha (f\sigma)$ for any smooth truncation of T_σ^α, so that T_σ^α is* apriori *bounded from $L^2(\sigma)$ to $L^2(\omega)$. Finally, define the sequence of positive extended real numbers*

$$\{p_m\}_{m=0}^{\infty} = \left\{ \frac{2}{1-\left(\frac{2}{3}\right)^m} \right\}_{m=0}^{\infty} = \left\{ \infty, 6, \frac{18}{5}, \frac{162}{65}, ... \right\}.$$

Suppose that the following statement is true:

$(\mathcal{S}_\infty)$ *If $\mathbf{b} = \{b_Q\}_{Q\in\mathcal{P}}$ is an ∞-weakly σ-accretive family of functions on $\mathbb{R}^n$ and if $\mathbf{b}^* = \{b_Q^*\}_{Q\in\mathcal{P}}$ is an ∞-weakly ω-accretive family of functions on $\mathbb{R}^n$, then the operator norm $\mathfrak{N}_{T_\sigma^\alpha}$ of T_σ^α from $L^2(\sigma)$ to $L^2(\omega)$, i.e. the best constant in*

$$\|T_\sigma^\alpha f\|_{L^2(\omega)} \leq \mathfrak{N}_{T_\sigma^\alpha} \|f\|_{L^2(\sigma)}, \qquad f \in L^2(\sigma),$$

***uniformly** in smooth truncations of T^α, satisfies*

$$\mathfrak{N}_{T^\alpha} \lesssim \left(C_{\mathbf{b}}(\infty) + C_{\mathbf{b}^*}(\infty)\right)\left(\mathfrak{T}_{T^\alpha}^{\mathbf{b}} + \mathfrak{T}_{T^\alpha}^{\mathbf{b}^*} + \sqrt{\mathfrak{A}_2^\alpha} + \mathfrak{E}_2^\alpha\right),$$

where $C_{\mathbf{b}}(\infty), C_{\mathbf{b}^*}(\infty)$ are the accretivity constants in (5.1.12), and the constants implied by $\lesssim$ depend on α and the constant C_{CZ} in (5.1.1).

Then for each $m \geq 0$, the following statements hold:

$(\mathcal{S}_m)$ *Let $p \in (p_{m+1}, p_m]$. If $\mathbf{b} = \{b_Q\}_{Q\in\mathcal{P}}$ is a p-weakly σ-accretive family of functions on $\mathbb{R}^n$, and if $\mathbf{b}^* = \{b_Q^*\}_{Q\in\mathcal{P}}$ is a p-weakly ω-accretive family of functions on $\mathbb{R}^n$, then the operator norm $\mathfrak{N}_{T_\sigma^\alpha}$ of T_σ^α from $L^2(\sigma)$ to $L^2(\omega)$, **uniformly** in smooth truncations of T^α, satisfies*

$$\mathfrak{N}_{T^\alpha} \lesssim \left(C_{\mathbf{b}}(p) + C_{\mathbf{b}^*}(p)\right)^{3^{m+1}}\left(\mathfrak{T}_{T^\alpha}^{\mathbf{b}} + \mathfrak{T}_{T^\alpha}^{\mathbf{b}^*} + \sqrt{\mathfrak{A}_2^\alpha} + \mathfrak{E}_2^\alpha\right),$$

where $C_{\mathbf{b}}(p), C_{\mathbf{b}^*}(p)$ are the accretivity constants in (5.1.4), and the constants implied by $\lesssim$ depend on p, α, and the constant C_{CZ} in (5.1.1).

Proof of Proposition 5.1.7. We will prove it by induction. We first prove $(\mathcal{S}_0)$. So fix $p \in (p_1, p_0) = (6, \infty)$, and let $\mathbf{b} = \{b_Q\}_{Q\in\mathcal{P}}$ be a p-weakly σ-accretive family of functions on $\mathbb{R}^n$, and let $\mathbf{b}^* = \left\{b_Q^*\right\}_{Q\in\mathcal{P}}$ be a p-weakly ω-accretive family of functions on $\mathbb{R}^n$. Let $0 < \varepsilon < 1$ (to be chosen differently at various points in the argument below) and define

$$\lambda = \lambda(\varepsilon) = \left(\frac{p}{p-2} C_{\mathbf{b}}(p)^p \frac{1}{\varepsilon}\right)^{\frac{1}{p-2}} \tag{5.1.13}$$

and a new collection of test functions,

$$\widehat{b}_Q \equiv 2b_Q \left(\mathbf{1}_{\left\{\left|b_Q\right|\leq\lambda\right\}} + \frac{\lambda}{\left|b_Q\right|}\mathbf{1}_{\left\{\left|b_Q\right|>\lambda\right\}}\right), \qquad Q \in \mathcal{P}, \tag{5.1.14}$$

We compute

$$\begin{aligned}
\int_{\left\{\left|b_Q\right|>\lambda\right\}} \left|b_Q\right|^2 d\sigma &= \int_{\left\{\left|b_Q\right|>\lambda\right\}} \left[\int_0^{\left|b_Q\right|} 2tdt\right] d\sigma \\
&= \int\int_{\left\{(x,t)\in\mathbb{R}^n\times(0,\infty):\max\{t,\lambda\}<\left|b_Q(x)\right|\right\}} 2tdtd\sigma(x) \\
&= \int_0^\lambda \int_{\left\{x\in\mathbb{R}^n:\lambda<\left|b_Q(x)\right|\right\}} d\sigma(x)\, 2tdt + \int_\lambda^\infty \int_{\left\{x\in\mathbb{R}^n:t<\left|b_Q(x)\right|\right\}} d\sigma(x)\, 2tdt \\
&= \lambda^2 \left|\left\{\left|b_Q\right| > \lambda\right\}\right|_\sigma + \int_\lambda^\infty \left|\left\{\left|b_Q\right| > t\right\}\right|_\sigma 2tdt,
\end{aligned}$$

and hence

$$\int_{\left\{\left|b_Q\right|>\lambda\right\}} \left|b_Q\right|^2 d\sigma \leq \lambda^2 \frac{1}{\lambda^p} \left(\int \left|b_Q\right|^p d\sigma \right) + \int_\lambda^\infty \frac{1}{t^p} \left(\int \left|b_Q\right|^p d\sigma \right) 2tdt \tag{5.1.15}$$

$$\leq \left\{ \lambda^{2-p} + \int_\lambda^\infty 2t^{1-p} dt \right\} C_{\mathbf{b}}(p)^p \left|Q\right|_\sigma$$

$$= \frac{p}{p-2} \lambda^{2-p} C_{\mathbf{b}}(p)^p \left|Q\right|_\sigma = \varepsilon \left|Q\right|_\sigma ,$$

by (5.1.13). Thus we have the lower bound,

$$\left| \frac{1}{|Q|_\sigma} \int_Q \widehat{b}_Q d\sigma \right| = 2 \left| \frac{1}{|Q|_\sigma} \int_Q b_Q d\sigma - \frac{1}{|Q|_\sigma} \int_Q b_Q \left(1 - \frac{\lambda}{|b_Q|} \right) \mathbf{1}_{\left\{\left|b_Q\right|>\lambda\right\}} d\sigma \right| \tag{5.1.16}$$

$$\geq 2 \left| \frac{1}{|Q|_\sigma} \int_Q b_Q d\sigma \right| - 2 \left(\frac{1}{|Q|_\sigma} \int_Q \left|b_Q\right|^2 \mathbf{1}_{\left\{\left|b_Q\right|>\lambda\right\}} d\sigma \right)^{\frac{1}{2}}$$

$$\geq 2 - 2 \left(\frac{1}{|Q|_\sigma} \varepsilon \left|Q\right|_\sigma \right)^{\frac{1}{2}} = 2 - 2\sqrt{\varepsilon} \geq 1 > 0, \qquad Q \in \mathcal{P}.$$

For an upper bound we have

$$\left\| \widehat{b}_Q \right\|_{L^\infty(\sigma)} \leq 2\lambda = 2\lambda(\varepsilon) = 2 \left(\frac{p}{p-2} C_{\mathbf{b}}(p)^p \frac{1}{\varepsilon} \right)^{\frac{1}{p-2}} ,$$

which altogether shows that

$$C_{\widehat{\mathbf{b}}}(\infty) \leq 2 \left(\frac{p}{p-2} C_{\mathbf{b}}(p)^p \frac{1}{\varepsilon} \right)^{\frac{1}{p-2}} = 2 \left(\frac{p}{p-2} \right)^{\frac{1}{p-2}} C_{\mathbf{b}}(p)^{\frac{p}{p-2}} \varepsilon^{-\frac{1}{p-2}} \tag{5.1.17}$$

if we choose $0 < \varepsilon \leq \frac{1}{4}$. Similarly we have

$$C_{\widehat{\mathbf{b}}^*}(\infty) \;\leq\; 2\left(\frac{p}{p-2}C_{\mathbf{b}^*}(p)^p \frac{1}{\varepsilon^*}\right)^{\frac{1}{p-2}} = 2\left(\frac{p}{p-2}\right)^{\frac{1}{p-2}} C_{\mathbf{b}^*}(p)^{\frac{p}{p-2}} (\varepsilon^*)^{-\frac{1}{p-2}}$$

for $0 < \varepsilon^* \leq \frac{1}{4}$. Moreover, we also have, using (5.1.15),

$$\begin{aligned}
\sqrt{\int_Q \left|T_\sigma^\alpha \widehat{b}_Q\right|^2 d\omega} &\leq 2\sqrt{\int_Q \left|T_\sigma^\alpha b_Q\right|^2 d\omega} + 2\sqrt{\int_Q \left|T_\sigma^\alpha \mathbf{1}_{\left\{\left|b_Q\right|>\lambda\right\}}\left(\frac{\lambda}{\left|b_Q\right|}-1\right) b_Q\right|^2 d\omega} \\
&\leq \; 2\mathfrak{T}_{T^\alpha}^{\mathbf{b}}\sqrt{|Q|_\sigma} + 2\mathfrak{N}_{T^\alpha}\sqrt{\int_{\left\{\left|b_Q\right|>\lambda\right\}} \left|b_Q\right|^2 d\sigma} \\
&\leq \; 2\left\{\mathfrak{T}_{T^\alpha}^{\mathbf{b}} + \sqrt{\varepsilon}\mathfrak{N}_{T^\alpha}\right\}\sqrt{|Q|_\sigma}\,, \qquad \text{for all cubes } Q,
\end{aligned}$$

which shows that

$$\mathfrak{T}_{T^\alpha}^{\widehat{\mathbf{b}}} \leq 2\mathfrak{T}_{T^\alpha}^{\mathbf{b}} + 2\sqrt{\varepsilon}\mathfrak{N}_{T^\alpha}\,. \tag{5.1.18}$$

Now we apply the fact that $(\mathcal{S}_\infty)$ holds to obtain

$$\mathfrak{N}_{T^\alpha} \lesssim \left(C_{\widehat{\mathbf{b}}}(\infty) + C_{\widehat{\mathbf{b}}^*}(\infty)\right)\left\{\mathfrak{T}_{T^\alpha}^{\widehat{\mathbf{b}}} + \mathfrak{T}_{T^\alpha,*}^{\widehat{\mathbf{b}}^*} + \sqrt{\mathfrak{A}_2^\alpha + \mathfrak{E}_2^\alpha}\right\}$$

and take $\varepsilon = \varepsilon^*$ to conclude, using (5.1.17) and (5.1.18), that

$$\begin{aligned}
\mathfrak{N}_{T^\alpha} \lesssim C_{implied}\left(C_{\mathbf{b}}(p) + C_{\mathbf{b}^*}(p)\right)^{\frac{p}{p-2}} \varepsilon^{-\frac{1}{p-2}}\left\{\mathfrak{T}_{T^\alpha}^{\mathbf{b}} + \mathfrak{T}_{T^\alpha,*}^{\mathbf{b}^*} + \sqrt{\mathfrak{A}_2^\alpha + \mathfrak{E}_2^\alpha}\right\} \quad (5.1.19)\\
+C_{implied}\left(C_{\mathbf{b}}(p) + C_{\mathbf{b}^*}(p)\right)^{\frac{p}{p-2}} \varepsilon^{\frac{1}{2}-\frac{1}{p-2}}\mathfrak{N}_{T^\alpha}
\end{aligned}$$

Now we choose

$$\varepsilon = \frac{1}{\Gamma}\left(C_{\mathbf{b}}(p) + C_{\mathbf{b}^*}(p)\right)^{-\frac{\frac{p}{p-2}}{\frac{1}{2}-\frac{1}{p-2}}}$$

with $\Gamma = \left(2C_{implied}\right)^4$, which satisfies $\Gamma \geq 1$, so that the final term on the right satisfies

$$C_{implied}\left(C_{\mathbf{b}}(p) + C_{\mathbf{b}^*}(p)\right)^{\frac{p}{p-2}} \varepsilon^{\frac{1}{2}-\frac{1}{p-2}} \mathfrak{N}_{T^\alpha} \leq C_{implied}\left(\frac{1}{\Gamma}\right)^{\frac{1}{4}} \mathfrak{N}_{T^\alpha} = \frac{1}{2}\mathfrak{N}_{T^\alpha}$$

where we have used $\frac{1}{2} - \frac{1}{p-2} \geq \frac{1}{4}$ for $p > 6$. This term can then be absorbed into the left hand side of (5.1.19) to obtain

$$\mathfrak{N}_{T^\alpha} \lesssim \left(C_{\mathbf{b}}(p) + C_{\mathbf{b}^*}(p)\right)^{\frac{p}{p-2}\left\{1+\frac{\frac{1}{p-2}}{\frac{1}{2}-\frac{1}{p-2}}\right\}} \left\{\mathfrak{T}_{T^\alpha}^{\mathbf{b}} + \mathfrak{T}_{T^{\alpha,*}}^{\mathbf{b}^*} + \sqrt{\mathfrak{A}_2^\alpha} + \mathfrak{E}_2^\alpha\right\}$$

Since

$$\frac{p}{p-2}\left\{1 + \frac{\frac{1}{p-2}}{\frac{1}{2} - \frac{1}{p-2}}\right\} = \left(1 + \frac{2}{p-2}\right)\left(1 + \frac{2}{p-4}\right) \leq 3 \text{ for } p > 6,$$

we get

$$\mathfrak{N}_{T^\alpha} \lesssim \left(C_{\mathbf{b}}(p) + C_{\mathbf{b}^*}(p)\right)^3 \left\{\mathfrak{T}_{T^\alpha}^{\mathbf{b}} + \mathfrak{T}_{T^{\alpha,*}}^{\mathbf{b}^*} + \sqrt{\mathfrak{A}_2^\alpha} + \mathfrak{E}_2^\alpha\right\},$$

which completes the proof of $(\mathcal{S}_0)$.

We now show that $\left(\mathcal{S}_p\right)$ holds for all $p \in (p_{m+1}, p_m]$. So fix $m \geq 1$, $p \in (p_{m+1}, p_m]$, and suppose that $\mathbf{b} = \left\{b_Q\right\}_{Q\in\mathcal{P}}$ is a p-weakly σ-accretive family of functions on $\mathbb{R}^n$ and that $\mathbf{b}^* = \left\{b_Q^*\right\}_{Q\in\mathcal{P}}$ is a p-weakly ω-accretive family of functions on $\mathbb{R}^n$. Note that the sequence $\{p_m\}_{m=0}^\infty = \left\{\frac{2}{1-\left(\frac{2}{3}\right)^m}\right\}_{m=0}^\infty$ satisfies the recursion relation

$$p_{m+1} = \frac{6}{1 + \frac{4}{p_m}}, \text{ equivalently, } p_m = \frac{4}{\frac{6}{p_{m+1}} - 1}, \quad m \geq 0.$$

Choose $q \in (p_m, p_{m-1}]$ so that

$$p > \frac{6}{1+\frac{4}{q}} = \frac{6q}{q+4}, \text{ i.e. } q < \frac{4}{\frac{6}{p}-1} = \frac{4p}{6-p}, \tag{5.1.20}$$

which can be done since $p > p_{m+1} = \frac{2}{1-\left(\frac{2}{3}\right)^{m+1}}$ is equivalent to $p_m = \frac{2}{1-\left(\frac{2}{3}\right)^{m}} < \frac{4}{\frac{6}{p}-1}$, which leaves room to choose q satisfying $p_m < q < \frac{4}{\frac{6}{p}-1}$.

Now let $0 < \varepsilon < 1$ (to be fixed later), define $\lambda = \lambda(\varepsilon)$ as in (5.1.13), and define $\widehat{b}_Q$ as in (5.1.14). Recall from (5.1.15) and (5.1.16) that we then have

$$\int_{\left\{\left|b_Q\right|>\lambda\right\}} \left|b_Q\right|^2 d\sigma \leq \varepsilon \left|Q\right|_\sigma \text{ and } \left|\frac{1}{\left|Q\right|_\sigma}\int_Q \widehat{b}_Q d\sigma\right| \geq 1, \qquad Q \in \mathcal{P} ,$$

if we choose $0 < \varepsilon \leq \frac{1}{4}$. We of course have the previous upper bound

$$\left\|\widehat{b}_Q\right\|_{L^\infty(\sigma)} \leq 2\lambda = 2\lambda(\varepsilon) = 2\left(\frac{p}{p-2}C_{\mathbf{b}}(p)^p \frac{1}{\varepsilon}\right)^{\frac{1}{p-2}}$$

and while this turned out to be sufficient in the case $m = 0$, we must do better than $O\left(\frac{1}{\varepsilon}\right)^{\frac{1}{p-2}}$ in the case $m \geq 1$. In fact we compute the L^q norm instead, recalling that $q > p$

and using Chebysev's inequality,

$$
\begin{aligned}
\left(\frac{1}{|Q|_\mu}\int_Q\left|\widehat{b}_Q\right|^q d\mu\right)^{\frac{1}{q}} &= 2\left(\frac{1}{|Q|_\mu}\int_Q\left|b_Q\left(\mathbf{1}_{\left\{\left|b_Q\right|\le\lambda\right\}}+\frac{\lambda}{\left|b_Q\right|}\mathbf{1}_{\left\{\left|b_Q\right|>\lambda\right\}}\right)\right|^q d\mu\right)^{\frac{1}{q}} \\
&= 2\left(\frac{1}{|Q|_\mu}\int_{\left\{\left|b_Q\right|\le\lambda\right\}}\left[\int_0^{\left|b_Q\right|} qt^{q-1}dt\right]d\sigma+\frac{\lambda^q\left|\left\{\left|b_Q\right|>\lambda\right\}\right|_\mu}{|Q|_\mu}\right)^{\frac{1}{q}} \\
&\le 2\left(\frac{1}{|Q|_\mu}\int_0^\lambda\left[\int_{\left\{t<\left|b_Q\right|\le\lambda\right\}}d\sigma\right]qt^{q-1}dt+C_{\mathbf{b}}(p)^p\lambda^{q-p}\right)^{\frac{1}{q}} \\
&\le 2\left(\frac{1}{|Q|_\mu}\int_0^\lambda\left[\frac{1}{t^p}\int\left|b_Q\right|^p d\sigma\right]qt^{q-1}dt+C_{\mathbf{b}}(p)^p\lambda^{q-p}\right)^{\frac{1}{q}} \\
&\le 2C_{\mathbf{b}}(p)^{\frac{p}{q}}\left(\int_0^\lambda qt^{q-p-1}dt+\lambda^{q-p}\right)^{\frac{1}{q}} \\
&= 2C_{\mathbf{b}}(p)^{\frac{p}{q}}\left(\frac{2q-p}{q-p}\lambda^{q-p}\right)^{\frac{1}{q}}
\end{aligned}
$$

which shows that $C_{\widehat{\mathbf{b}}}(q)$ satisfies the estimate

$$
\begin{aligned}
C_{\widehat{\mathbf{b}}}(q) &\le 2C_{\mathbf{b}}(p)^{\frac{p}{q}}\left(\frac{2q-p}{q-p}\right)^{\frac{1}{q}}\left[\left(\frac{p}{p-2}C_{\mathbf{b}}(p)^p\frac{1}{\varepsilon}\right)^{\frac{1}{p-2}}\right]^{1-\frac{p}{q}} \\
&\lesssim C_{\mathbf{b}}(p)^{\frac{p}{q}\left(\frac{q-2}{p-2}\right)}\varepsilon^{-\frac{1-\frac{p}{q}}{p-2}} \lesssim C_{\mathbf{b}}(p)^{\frac{3}{2}}\varepsilon^{-\frac{1-\frac{p}{q}}{p-2}},
\end{aligned}
$$

a significant improvement over the bound $O\left(\varepsilon^{-\frac{1}{p-2}}\right)$. Here we have used that if $p>\frac{6q}{q+4}$, then

$$
\frac{p}{q}\left(\frac{q-2}{p-2}\right)<\frac{\frac{6q}{q-4}}{\frac{6q}{q-4}-2}\frac{q-2}{q}<\frac{3}{2}
$$

as the function $x \mapsto \frac{x}{x-2}$ is decreasing when $x > 2$. Moreover, from (5.1.18) we also have

$$\mathfrak{T}_{T^\alpha}^{\widehat{\mathbf{b}}} \leq 2\mathfrak{T}_{T^\alpha}^{\mathbf{b}} + 2\sqrt{\varepsilon}\mathfrak{N}_{T^\alpha} \ .$$

We can do the same for the dual testing functions $\mathbf{b}^* = \left\{b_Q^*\right\}_{Q\in\mathcal{P}}$ and then altogether, provided $0 < \varepsilon \leq \frac{1}{4}$, we have both

$$\begin{aligned} 1 \ \leq \ \left|\frac{1}{|Q|_\sigma}\int_Q \widehat{b}_Q d\sigma\right| \leq \left\|\widehat{b}_Q\right\|_{L^q(\sigma)} \leq C_{\mathbf{b}}\left(p\right)^{\frac{3}{2}} \varepsilon^{-\frac{1-\frac{p}{q}}{p-2}}, \qquad Q \in \mathcal{P} \ ,\\ \mathfrak{T}_{T^\alpha}^{\widehat{\mathbf{b}}} \leq 2\mathfrak{T}_{T^\alpha}^{\mathbf{b}} + 2\sqrt{\varepsilon}\mathfrak{N}_{T^\alpha} \ , \end{aligned}$$

as well as

$$\begin{aligned} 1 \ \leq \ \left|\frac{1}{|Q|_\omega}\int_Q \widehat{b^*}_Q d\omega\right| \leq \left\|\widehat{b^*}_Q\right\|_{L^q(\omega)} \leq C_{\mathbf{b}^*}\left(p\right)^{\frac{3}{2}} \varepsilon^{-\frac{1-\frac{p}{q}}{p-2}}, \qquad Q \in \mathcal{P} \ ,\\ \mathfrak{T}_{T^\alpha}^{\widehat{\mathbf{b}^*}} \leq 2\mathfrak{T}_{T^\alpha}^{\mathbf{b}^*} + 2\sqrt{\varepsilon}\mathfrak{N}_{T^\alpha} \end{aligned}$$

We now use these estimates, together with the fact that $(\mathcal{S}_{m-1})$ holds, to obtain

$$\begin{aligned} \mathfrak{N}_{T^\alpha} &\lesssim \left(C_{\widehat{\mathbf{b}}}\left(q\right) + C_{\widehat{\mathbf{b}}^*}\left(q\right)\right)^{3^n} \left\{\mathfrak{T}_{T^\alpha}^{\widehat{\mathbf{b}}} + \mathfrak{T}_{T^\alpha,*}^{\widehat{\mathbf{b}}^*} + \sqrt{\mathfrak{A}_2^\alpha} + \mathfrak{E}_2^\alpha\right\} \\ &\lesssim \left(C_{\mathbf{b}}\left(p\right) + C_{\mathbf{b}^*}\left(p\right)\right)^{\frac{3}{2}3^n} \varepsilon^{-\frac{1-\frac{p}{q}}{p-2}} \left\{\left[\mathfrak{T}_{T^\alpha}^{\mathbf{b}} + \sqrt{\varepsilon}\mathfrak{N}_{T^\alpha}\right] + \left[\mathfrak{T}_{T^\alpha,*}^{\mathbf{b}^*} + \sqrt{\varepsilon}\mathfrak{N}_{T^\alpha}\right] + \sqrt{\mathfrak{A}_2^\alpha} + \mathfrak{E}_2^\alpha\right\} \\ &\lesssim \left(C_{\mathbf{b}}\left(p\right) + C_{\mathbf{b}^*}\left(p\right)\right)^{\frac{3}{2}3^n} \left(\varepsilon^{-\frac{1-\frac{p}{q}}{p-2}} \left\{\mathfrak{T}_{T^\alpha}^{\mathbf{b}} + \mathfrak{T}_{T^\alpha,*}^{\mathbf{b}^*} + \sqrt{\mathfrak{A}_2^\alpha} + \mathfrak{E}_2^\alpha\right\} + \sqrt{\varepsilon}\varepsilon^{-\frac{1-\frac{p}{q}}{p-2}} \mathfrak{N}_{T^\alpha}\right) \end{aligned}$$

We can absorb the term $\left(C_{\mathbf{b}}\left(p\right) + C_{\mathbf{b}^*}\left(p\right)\right)^{\frac{3}{2}3^n} \sqrt{\varepsilon}\varepsilon^{-\frac{1-\frac{p}{q}}{p-2}} \mathfrak{N}_{T^\alpha}$ into the left hand side as before, by choosing

$$\varepsilon = \frac{1}{\Gamma}\left(C_{\mathbf{b}}(p) + C_{\mathbf{b}^*}(p)\right)^{\left(\frac{\frac{3}{2}3^n}{\frac{1-\frac{p}{q}}{p-2}-\frac{1}{2}}\right)}$$

with Γ sufficiently large, depending only on the implied constant, since (5.1.20) gives $\frac{\frac{6}{p}-1}{2} < \frac{2}{q}$, and hence

$$\frac{1}{2} - \frac{1-\frac{p}{q}}{p-2} = \frac{p\left(1+\frac{2}{q}\right)-4}{2p-4} > \frac{p\left(1+\frac{\frac{6}{p}-1}{2}\right)-4}{2p-4} = \frac{1}{4}. \tag{5.1.21}$$

Thus,

$$\mathfrak{N}_{T^\alpha} \lesssim \left(C_{\mathbf{b}}(p) + C_{\mathbf{b}^*}(p)\right)^{\frac{3}{2}3^n(1+1)}\left\{\mathfrak{T}_{T^\alpha}^{\mathbf{b}} + \mathfrak{T}_{T^\alpha,*}^{\mathbf{b}^*} + \sqrt{\mathfrak{A}_2^\alpha} + \mathfrak{E}_2^\alpha\right\}.$$

Here we have used that (5.1.21) implies $\frac{\frac{1-\frac{p}{q}}{p-2}}{\frac{1}{2}-\frac{1-\frac{p}{q}}{p-2}} < 4\frac{1-\frac{p}{q}}{p-2} \leq 1$. So we finally have

$$\mathfrak{N}_{T^\alpha} \lesssim \left(C_{\mathbf{b}}(p) + C_{\mathbf{b}^*}(p)\right)^{3^{n+1}}\left\{\mathfrak{T}_{T^\alpha}^{\mathbf{b}} + \mathfrak{T}_{T^\alpha,*}^{\mathbf{b}^*} + \sqrt{\mathfrak{A}_2^\alpha} + \mathfrak{E}_2^\alpha\right\},$$

which completes the proof of Proposition 5.1.7. $\square$

Thus we may assume for the proof of Theorem 5.1.5 given below that $p = \infty$ and that the testing functions are real-valued and satisfy

$$\operatorname{supp} b_Q \subset Q\ , \qquad Q \in \mathcal{P}, \tag{5.1.22}$$

$$1 \leq \frac{1}{|Q|_\mu}\int_Q b_Q d\mu \leq \|b_Q\|_{L^\infty(\mu)} \leq C_{\mathbf{b}}(\infty) < \infty, \qquad Q \in \mathcal{P}\ .$$

5.1.8 Reverse Hölder control of children

Here we begin to further reduce the proof of Theorem 5.1.5 to the case of bounded real testing functions $\mathbf{b}=\left\{b_Q\right\}_{Q\in\mathcal{P}}$ having reverse Hölder control

$$\left|\frac{1}{\left|Q'\right|_\sigma}\int_{Q'} b_Q d\sigma\right| \geq c\left\|\mathbf{1}_{Q'} b_Q\right\|_{L^\infty(\sigma)} > 0, \tag{5.1.23}$$

for all children $Q'\in\mathfrak{C}(Q)$ with $\left|Q'\right|_\sigma > 0$ and $Q\in\mathcal{P}$.

5.1.8.1 Control of averages over children

Lemma 5.1.8. *Suppose that σ and ω are locally finite positive Borel measures on $\mathbb{R}^n$. Assume that T^α is a standard α-fractional elliptic and gradient elliptic singular integral operator on $\mathbb{R}^n$, and set $T^\alpha_\sigma f = T^\alpha(f\sigma)$ for any smooth truncation of T^α_σ, so that T^α_σ is* apriori *bounded from $L^2(\sigma)$ to $L^2(\omega)$. Let $Q\in\mathcal{P}$ and let $\mathfrak{N}_{T^\alpha}(Q)$ be the best constant in the local inequality*

$$\sqrt{\int_{Q'}\left|T^\alpha_\sigma\left(\mathbf{1}_Q f\right)\right|^2 d\omega} \leq \mathfrak{N}_{T^\alpha}(Q)\sqrt{\int_Q |f|^2 d\sigma}\ , \qquad f\in L^2\left(\mathbf{1}_Q\sigma\right).$$

Suppose that b_Q is a real-valued function supported in Q such that

$$1 \leq \frac{1}{|Q|_\sigma}\int_Q b_Q d\sigma \leq \left\|\mathbf{1}_Q b_Q\right\|_{L^\infty(\sigma)} \leq C_{\mathbf{b}}\ ,$$
$$\sqrt{\int_Q\left|T^\alpha_\sigma b_Q\right|^2 d\omega} \leq \mathfrak{T}^{b_Q}_{T^\alpha}(Q)\sqrt{|Q|_\sigma}\ .$$

Then for every $0<\delta<\frac{1}{2^{n+1}C^3_{\mathbf{b}}}$, there exists a real-valued function $\widetilde{b}_Q$ supported in Q such that

(1). $1 \le \frac{1}{|Q|_\sigma} \int_Q \widetilde{b}_Q d\sigma \le \left\| \mathbf{1}_Q \widetilde{b}_Q \right\|_{L^\infty(\sigma)} \le 2 \left(1 + \sqrt{C_\mathbf{b}}\right) C_\mathbf{b}$,

(2). $\sqrt{\int_Q \left| T_\sigma^\alpha \widetilde{b}_Q \right|^2 d\omega} \le \left[\mathfrak{T}_{T^\alpha}^{b_Q}(Q) + 2C_\mathbf{b}^{\frac{3}{4}} \delta^{\frac{1}{4}} \mathfrak{N}_{T^\alpha}(Q) \right] \sqrt{|Q|_\sigma}$,

(3). $0 < \left\| \mathbf{1}_{Q_i} \widetilde{b}_Q \right\|_{L^\infty(\sigma)} \le \frac{16 C_\mathbf{b}}{\delta} \left| \frac{1}{|Q_i|_\sigma} \int_{Q_i} \widetilde{b}_Q d\sigma \right| , \qquad Q_i \in \mathfrak{C}(Q)$.

Proof. Let $0 < \delta < 1$ and fix $Q \in \mathcal{P}$. By assumption we have

$$1 \le \frac{1}{|Q|_\sigma} \int_Q b_Q d\sigma \le \left\| \mathbf{1}_Q b_Q \right\|_{L^\infty(\sigma)} \le C_\mathbf{b}.$$

Let Q_i be the children of Q. We now define $\widetilde{b}_Q$. First we note that the inequality

$$\left| \frac{1}{|Q_i|_\sigma} \int_{Q_i} b_Q d\sigma \right| < \frac{\delta}{C_\mathbf{b}} \left\| \mathbf{1}_{Q_i} b_Q \right\|_{L^\infty(\sigma)} \tag{5.1.24}$$

cannot hold for *all* Q_i, since otherwise we obtain the contradiction

$$\begin{aligned} \left| \int_Q b_Q d\sigma \right| &\le \sum_{i=1}^{2^n} \left| \int_{Q_i} b_Q d\sigma \right| < \frac{\delta}{C_\mathbf{b}} \sum_{i=1}^{2^n} |Q_i|_\sigma \left\| \mathbf{1}_{Q_i} b_Q \right\|_{L^\infty(\sigma)} \\ &\le \frac{\delta}{C_\mathbf{b}} |Q|_\sigma \left\| \mathbf{1}_Q b_Q \right\|_{L^\infty(\sigma)} \le \delta \left| \int_Q b_Q d\sigma \right| < \left| \int_Q b_Q d\sigma \right|. \end{aligned}$$

If (5.1.24) holds for none of the Q_i, then we simply define $\widetilde{b}_Q = b_Q$, and trivially all the conclusions of the Lemma 5.1.8 hold. If (5.1.24) holds for at least one of the children, say Q_{i_0}, then we define $\widetilde{b}_Q$ differently according to how large the $L^1(\sigma)$-average $\frac{1}{\left|Q_{i_0}\right|_\sigma} \int_{Q_{i_0}} \left| b_Q \right| d\sigma$ is. In this case, define $\tilde{G}$ to be the set of indices for which (5.1.24) holds and G the set of indices for which (5.1.24) fails. We define

$$
\begin{aligned}
\widetilde{b}_Q &\equiv \sum_{i\in G} b_Q \mathbf{1}_Q + \sum_{i\in G_0} \delta \mathbf{1}_{Q_i} + \sum_{i\in G_+} \left(\frac{1}{|Q_i|_\sigma} \int_{Q_i} \left|b_Q\right| d\sigma \right) \mathbf{1}_{Q_i} \\
&\quad + \sum_{i\in B_-} \left(p_i - n_i \left(1 + \sqrt{C_{\mathbf{b}}\delta} \right) \right) \mathbf{1}_{Q_i} + \sum_{i\in B_+} \left(\left(1 + \sqrt{C_{\mathbf{b}}\delta} \right) p_i - n_i \right) \mathbf{1}_{Q_i}
\end{aligned}
$$

where

$$
\begin{aligned}
G_0 &\equiv \left\{ i \in \tilde{G} : \frac{1}{|Q_i|_\sigma} \int_{Q_i} \left|b_Q\right| d\sigma = 0 \right\} \\
G_+ &\equiv \left\{ i \in \tilde{G} : 0 < \frac{1}{|Q_i|_\sigma} \int_{Q_i} \left|b_Q\right| d\sigma \leq \sqrt{C_{\mathbf{b}}\delta} \right\}, \\
B_- &\equiv \left\{ i \in \tilde{G} : \frac{1}{|Q_i|_\sigma} \int_{Q_i} \left|b_Q\right| d\sigma > \sqrt{C_{\mathbf{b}}\delta} \text{ and } \int_{Q_i} n_i d\sigma > \int_{Q_i} p_i d\sigma \right\}, \\
B_+ &\equiv \left\{ i \in \tilde{G} : \frac{1}{|Q_i|_\sigma} \int_{Q_i} \left|b_Q\right| d\sigma > \sqrt{C_{\mathbf{b}}\delta} \text{ and } \int_{Q_i} p_i d\sigma \geq \int_{Q_i} n_i d\sigma \right\}.
\end{aligned}
$$

and p_i, n_i are the positive and negative parts of b_Q respectively on Q_i, i.e.

$$
\begin{aligned}
\mathbf{1}_{Q_i}(x)\, b_Q(x) &= p_i(x) - n_i(x), \\
\mathbf{1}_{Q_i}(x) \left|b_Q(x)\right| &= p_i(x) + n_i(x),
\end{aligned}
$$

Now let us check the conclusions of the Lemma 5.1.8. For *(1)* we have

$$
\begin{aligned}
1 &\leq \frac{1}{|Q|_\sigma} \int_Q b_Q d\sigma \\
&\leq \frac{1}{|Q|_\sigma} \int_Q \widetilde{b}_Q d\sigma + \frac{1}{|Q|_\sigma} \sum_{i\in B_-} \int_{Q_i} n_i \sqrt{C_b\delta} d\sigma - \frac{1}{|Q|_\sigma} \sum_{i\in B_+} \int_{Q_i} p_i \sqrt{C_b\delta} d\sigma \\
&\leq \frac{1}{|Q|_\sigma} \int_Q \widetilde{b}_Q d\sigma + \sqrt{C_b\delta} C_b \frac{1}{|Q|_\sigma} \sum_{i\in B_-} |Q_i|_\sigma \leq \frac{1}{|Q|_\sigma} \int_Q \widetilde{b}_Q d\sigma + C_b^{\frac{3}{2}} \sqrt{\delta}
\end{aligned}
$$

and choosing δ small enough we get

$$\frac{1}{2} \leq \frac{1}{|Q|_\sigma} \int_Q \widetilde{b}_Q d\sigma \leq \left\| \mathbf{1}_Q \widetilde{b}_Q \right\|_{L^\infty(\sigma)},$$

which in turn is bounded by

$$\sup_{Q_i \in \mathfrak{C}(Q)} \left\| \mathbf{1}_{Q_i} \widetilde{b}_{Q_i} \right\|_{L^\infty(\sigma)} \leq 2 \left(1 + \sqrt{C_{\mathbf{b}}} \right) C_{\mathbf{b}}$$

by taking the different cases on Q_i:

(a) For $i \in G_0$, $\left\| \mathbf{1}_{Q_i} \widetilde{b}_{Q_i} \right\|_{L^\infty} \leq \delta$,

(b) For $i \in G_+$, $\left\| \mathbf{1}_{Q_i} \widetilde{b}_{Q_i} \right\|_{L^\infty} \leq C_{\mathbf{b}}$,

(c) For $i \in B_- \cup B_+$, $\left\| \mathbf{1}_{Q_i} \widetilde{b}_{Q_i} \right\|_{L^\infty} \leq 2(1 + \sqrt{C_{\mathbf{b}}}) C_{\mathbf{b}}$.

This completes the proof for *(1)*.

For *(2)*, we have from Minkowski's inequality

$$\begin{aligned}
\sqrt{\frac{1}{|Q|_\sigma} \int_Q \left| T_\sigma^\alpha \widetilde{b}_Q \right|^2 d\omega} &\leq \sqrt{\frac{1}{|Q|_\sigma} \int_Q \left| T_\sigma^\alpha b_Q \right|^2 d\omega} + \sqrt{\frac{1}{|Q|_\sigma} \int_Q \left| T_\sigma^\alpha \left(\widetilde{b}_Q - b_Q \right) \right|^2 d\omega} \\
&\leq \mathfrak{T}_{T^\alpha}^{b_Q}(Q) + \mathfrak{N}_{T^\alpha}(Q) \sqrt{\frac{1}{|Q|_\sigma} \int_Q \left| \widetilde{b}_Q - b_Q \right|^2 d\sigma} \\
&= \mathfrak{T}_{T^\alpha}^{b_Q}(Q) + \mathfrak{N}_{T^\alpha}(Q) \sqrt{\frac{1}{|Q|_\sigma} \sum_{Q_i \in \mathfrak{C}(Q)} \int_{Q_i} \left| \widetilde{b}_Q - b_Q \right|^2 d\sigma}
\end{aligned}$$

and this last term is bounded by:

$$\left(\sum_{i \in G} + \sum_{i \in G_0} + \sum_{i \in G_+} + \sum_{i \in B_-} + \sum_{i \in B_+} \right) \sqrt{\frac{1}{|Q|_\sigma} \int_{Q_i} \left| \widetilde{b}_Q - b_Q \right|^2 d\sigma}$$

and since we have:

(a) for $i \in G$,

$$\frac{1}{|Q|_\sigma} \int_{Q_i} \left|\widetilde{b}_Q - b_Q\right|^2 d\sigma = 0$$

(b) for $i \in G_0$,

$$\begin{aligned}
\frac{1}{|Q|_\sigma} \int_{Q_i} \left|\widetilde{b}_Q - b_Q\right|^2 d\sigma &\leq \frac{1}{|Q|_\sigma} \left(\int_{Q_i} \delta^2 d\sigma + \int_{Q_i} |b_Q|^2 d\sigma \right) \\
&\leq \frac{1}{|Q|_\sigma} \left(\delta^2 |Q_i|_\sigma + C_{\mathbf{b}} \int_{Q_i} |b_Q| d\sigma \right) = \delta^2 \frac{|Q_i|_\sigma}{|Q|_\sigma}
\end{aligned}$$

by the accretivity of b_Q and the definition of G_0.

(c) for $i \in G_+$,

$$\begin{aligned}
\frac{1}{|Q|_\sigma} \int_{Q_i} \left|\widetilde{b}_Q - b_Q\right|^2 d\omega &= \frac{1}{|Q|_\sigma} \int_{Q_i} \left| \left(\frac{1}{|Q_i|_\sigma} \int_{Q_i} |b_Q|\, d\sigma \right) - b_Q \right|^2 d\sigma \\
&\leq \frac{1}{|Q|_\sigma} \left(\int_{Q_i} \left| \frac{1}{|Q_i|_\sigma} \int_{Q_i} |b_Q|\, d\sigma \right|^2 d\sigma + \int_{Q_i} \left|b_Q\right|^2 d\sigma \right) \\
&\leq \frac{1}{|Q|_\sigma} \left(\int_{Q_i} C_{\mathbf{b}} \delta d\sigma + C_{\mathbf{b}} \int_{Q_i} \left|b_Q\right| d\sigma \right) \\
&\leq \left(C_{\mathbf{b}} \delta + C_{\mathbf{b}} \sqrt{C_{\mathbf{b}} \delta} \right) \frac{|Q_i|_\sigma}{|Q|_\sigma} \leq 2 C_{\mathbf{b}}^{\frac{3}{2}} \delta^{\frac{1}{2}} \frac{|Q_i|_\sigma}{|Q|_\sigma}.
\end{aligned}$$

(d) for $i \in B_-$,

$$\begin{aligned}
\frac{1}{|Q|_\sigma} \int_{Q_i} \left|\widetilde{b}_Q - b_Q\right|^2 d\sigma = \frac{1}{|Q|_\sigma} \int_{Q_i} \left|C_{\mathbf{b}} \delta n_i\right|^2 d\sigma &= C_{\mathbf{b}} \delta \frac{1}{|Q|_\sigma} \int_{Q_i} |n_i|^2 d\sigma \\
&\leq C_{\mathbf{b}}^3 \delta \frac{|Q_i|_\sigma}{|Q|_\sigma}.
\end{aligned}$$

(e) and for $i \in B_+$, the same estimate as in the previous case,

we obtain

$$\sqrt{\frac{1}{|Q|_\sigma}\int_Q \left|T_\sigma^\alpha \widetilde{b}_Q\right|^2 d\omega} \leq \mathfrak{T}_{T^\alpha}^{b_Q}(Q) + 2\cdot 2^n C_{\mathbf{b}}^{\frac{3}{4}} \delta^{\frac{1}{4}} \mathfrak{N}_{T^\alpha}(Q) \ .$$

where the dimensional constant comes from

$$\frac{1}{\sqrt{|Q|_\sigma}}\sum_{i=1}^{2^n}\sqrt{|Q_i|_\sigma} \leq 2^n.$$

Now we are left with verifying *(3)*. Note that

(a) for $i \in G$, the inequality (5.1.24) does not hold and as $\widetilde{b}_Q = b_Q$ there, immediately we obtain

$$\left\|\mathbf{1}_{Q_i}\widetilde{b}_Q\right\|_{L^\infty(\sigma)} \leq \left|\frac{C_{\mathbf{b}}}{\delta}\int_{Q_i}\widetilde{b}_Q d\sigma\right|$$

(b) for $i \in G_0 \cup G_+$,

$$\frac{\left\|\mathbf{1}_{Q_i}\widetilde{b}_Q\right\|_{L^\infty(\sigma)}}{\left|\frac{1}{|Q_i|_\sigma}\int_{Q_i}\widetilde{b}_Q d\sigma\right|} = 1 < \frac{C_{\mathbf{b}}}{\delta}$$

(c) for $i \in B_-$,

$$\begin{aligned}
\frac{\left\|\mathbf{1}_{Q_i}\widetilde{b}_Q\right\|_{L^\infty(\sigma)}}{\left|\frac{1}{|Q_i|_\sigma}\int_{Q_i}\widetilde{b}_Q d\sigma\right|} &\leq \frac{\left(1+\sqrt{C_{\mathbf{b}}\delta}\right)C_{\mathbf{b}}}{\left|\frac{1}{|Q_i|_\sigma}\int_{Q_i}\left[p_i - n_i\left(1+\sqrt{C_{\mathbf{b}}\delta}\right)\right]d\sigma\right|} \\
&\leq \frac{\left(1+\sqrt{C_{\mathbf{b}}\delta}\right)C_{\mathbf{b}}}{\left|\sqrt{C_{\mathbf{b}}\delta}\frac{1}{|Q_i|_\sigma}\int_{Q_i} n_i d\sigma\right|} \\
&\leq \frac{2(1+\sqrt{C_{\mathbf{b}}\delta})C_{\mathbf{b}}}{\sqrt{C_{\mathbf{b}}\delta}\frac{1}{|Q_i|_\sigma}\int_{Q_i}\left|b_Q\right| d\sigma} \\
&\leq \frac{4C_{\mathbf{b}}}{C_{\mathbf{b}}\delta} = \frac{4}{\delta}\ ,
\end{aligned}$$

as, by taking $0 < \delta < \frac{1}{4C_{\mathbf{b}}^3}$, we have $1 + \sqrt{C_{\mathbf{b}}\delta} < 2$.

(d) and for $i \in B_+$ similarly as in the previous case.

In order to obtain the inequalities for $\widetilde{b}_Q$ in the conclusion of Lemma 5.1.8, we simply multiply the above function $\widetilde{b}_Q$ by a factor of 2.

Finally, if $\left|b_Q\right| \geq c_1 > 0$, we easily see that $\left|\widetilde{b}_Q\right| \geq \left|b_Q\right| \geq c_1 > 0$ as well. This completes the proof of Lemma 5.1.8. $\square$

5.1.8.2 Control of averages in coronas

Let $\mathcal{D}_Q$ be the grid of dyadic subcubes of Q. In the construction of the triple corona below, we will need to repeat the construction in the previous subsubsection for a subdecomposition $\{Q_i\}_{i=1}^{\infty}$ of dyadic subcubes $Q_i \in \mathcal{D}_Q$ of a cube Q. Define the corona corresponding to the subdecomposition $\{Q_i\}_{i=1}^{\infty}$ by

$$\mathcal{C}_Q \equiv \mathcal{D}_Q \setminus \bigcup_{i=1}^{\infty} \mathcal{D}_{Q_i} \ .$$

Lemma 5.1.9. *Suppose that σ and ω are locally finite positive Borel measures on $\mathbb{R}^n$. Assume that T^α is a standard α-fractional elliptic and gradient elliptic singular integral operator on $\mathbb{R}^n$, and set $T_\sigma^\alpha f = T^\alpha(f\sigma)$ for any smooth truncation of T_σ^α, so that T_σ^α is* apriori *bounded from $L^2(\sigma)$ to $L^2(\omega)$. Let $Q \in \mathcal{P}$ and let $\mathfrak{N}_{T^\alpha}(Q)$ be the best constant in the local inequality*

$$\sqrt{\int_Q \left|T_\sigma^\alpha\left(\mathbf{1}_Q f\right)\right|^2 d\omega} \leq \mathfrak{N}_{T^\alpha}(Q) \sqrt{\int_Q |f|^2 d\sigma} \ , \qquad f \in L^2\left(\mathbf{1}_Q \sigma\right).$$

Let $\{Q_i\}_{i=1}^{\infty} \subset \mathcal{D}_Q$ be a collection of pairwise disjoint dyadic subcubes of Q. Suppose that

b_Q is a real-valued function supported in Q such that

$$1 \leq \frac{1}{|Q'|_\sigma} \int_{Q'} b_Q d\sigma \leq \left\| \mathbf{1}_{Q'} b_Q \right\|_{L^\infty(\sigma)} \leq C_{\mathbf{b}} \ , \qquad Q' \in \mathcal{C}_Q \ ,$$
$$\sqrt{\int_Q \left| T_\sigma^\alpha b_Q \right|^2 d\omega} \leq \mathfrak{T}_{T^\alpha}^{b_Q}(Q) \sqrt{|Q|_\sigma} \ .$$

Then for every $0 < \delta < \frac{1}{4C_{\mathbf{b}}^3}$, there exists a real-valued function $\widetilde{b}_Q$ supported in Q such that

$$1 \leq \frac{1}{|Q'|_\sigma} \int_{Q'} \widetilde{b}_Q d\sigma \leq \left\| \mathbf{1}_{Q'} \widetilde{b}_Q \right\|_{L^\infty(\sigma)} \leq 2\left(1 + \sqrt{C_{\mathbf{b}}}\right) C_{\mathbf{b}} \ , \qquad Q' \in \mathcal{C}_Q \ ,$$
$$\sqrt{\int_Q \left| T_\sigma^\alpha \widetilde{b}_Q \right|^2 d\omega} \leq \left[2\mathfrak{T}_{T^\alpha}^{b_Q}(Q) + 4C_{\mathbf{b}}^{\frac{3}{2}} \delta^{\frac{1}{4}} \mathfrak{N}_{T^\alpha}(Q) \right] \sqrt{|Q|_\sigma} \ ,$$
$$0 < \left\| \mathbf{1}_{Q_i} \widetilde{b}_Q \right\|_{L^\infty(\sigma)} \leq \frac{16 C_{\mathbf{b}}}{\delta} \left| \frac{1}{|Q_i|_\sigma} \int_{Q_i} \widetilde{b}_Q d\sigma \right| \ , \qquad 1 \leq i < \infty.$$

Moreover, if $|b_Q| \geq c_1 > 0$, then we may take $\left|\widetilde{b}_Q\right| \geq c_1$ as well.

The additional gain in the lemma is in the final line that controls the degeneracy of $\widetilde{b}_Q$ at the 'bottom' of the corona $\mathcal{C}_Q$ by establishing a reverse Hölder control. Note that if we combine this control with the accretivity control in the corona $\mathcal{C}_Q$, namely

$$\left\| \mathbf{1}_{Q'} \widetilde{b}_Q \right\|_{L^\infty(\sigma)} \leq 2\left(1 + \sqrt{C_{\mathbf{b}}}\right) C_{\mathbf{b}} \leq 2\left(1 + \sqrt{C_{\mathbf{b}}}\right) C_{\mathbf{b}} \frac{1}{|Q'|_\sigma} \int_{Q'} \widetilde{b}_Q d\sigma,$$

we obtain reverse Hölder control throughout the entire collection $\mathcal{C}_Q \cup \{Q_i\}_{i=1}^\infty$:

$$\left\| \mathbf{1}_I \widetilde{b}_{Q'} \right\|_{L^\infty(\sigma)} \leq C_{\delta,\mathbf{b}} \left| \frac{1}{|I|_\sigma} \int_I \widetilde{b}_{Q'} d\sigma \right|, \qquad I \in \mathfrak{C}(Q') , Q' \in \mathcal{C}_Q \ .$$

This has the crucial consequence that the martingale and dual martingale differences $\triangle_{Q'}^{\sigma,\mathbf{b}}$

and $\square^{\sigma,\mathbf{b}}_{Q'}$ associated with these functions as defined in (5.1.38), satisfy

$$\left|\triangle^{\sigma,\mathbf{b}}_{Q'}h\right|,\left|\square^{\sigma,\mathbf{b}}_{Q'}h\right|\leq C_{\delta,\mathbf{b}}\sum_{I\in\mathfrak{C}(Q')}\left(\frac{1}{|I|_{\sigma}}\int_{I}|h|\,d\sigma+\frac{1}{|Q'|_{\sigma}}\int_{Q'}|h|\,d\sigma\right)\mathbf{1}_{I}\ . \tag{5.1.25}$$

However, the defect in this lemma is that we lose the weak testing condition for $\widetilde{b}_Q$ in the corona even if we had assumed it at the outset for b_Q.

Proof. The proof of Lemma 5.1.9 is similar to that of the Lemma 5.1.8. Indeed, we define

$$\begin{aligned}\widetilde{b}_Q\ \equiv\ &\sum_{i\in G_0}\delta\mathbf{1}_{Q_i}+\sum_{i\in G_+}\left(\frac{1}{|Q_i|_{\sigma}}\int_{Q_i}\left|b_Q\right|d\sigma\right)\mathbf{1}_{Q_i}\\&+\sum_{i\in B_-}\left(\frac{1}{|Q_i|_{\sigma}}\int_{Q_i}\left[p_i-n_i\left(1+\sqrt{C_{\mathbf{b}}\delta}\right)\right]d\sigma\right)\mathbf{1}_{Q_i}\\&+\sum_{i\in B_+}\left(\frac{1}{|Q_i|_{\sigma}}\int_{Q_i}\left[\left(1+\sqrt{C_{\mathbf{b}}\delta}\right)p_i-n_i\right]d\sigma\right)\mathbf{1}_{Q_i}\\&+b_Q\mathbf{1}_{Q\setminus\cup_{i=1}^{\infty}Q_i}\ ,\end{aligned}$$

where

$$\begin{aligned}G_0\ &\equiv\ \left\{i:\frac{1}{|Q_i|_{\sigma}}\int_{Q_i}\left|b_Q\right|d\sigma=0\right\},\\G_+\ &\equiv\ \left\{i:0<\frac{1}{|Q_i|_{\sigma}}\int_{Q_i}\left|b_Q\right|d\sigma\leq\sqrt{C_{\mathbf{b}}\delta}\right\},\\B_-\ &\equiv\ \left\{i:\frac{1}{|Q_i|_{\sigma}}\int_{Q_i}\left|b_Q\right|d\sigma>\sqrt{C_{\mathbf{b}}\delta}\text{ and }\int_{Q_i}n_i d\sigma>\int_{Q_i}p_i d\sigma\right\},\\B_+\ &\equiv\ \left\{i:\frac{1}{|Q_i|_{\sigma}}\int_{Q_i}\left|b_Q\right|d\sigma>\sqrt{C_{\mathbf{b}}\delta}\text{ and }\int_{Q_i}p_i d\sigma\geq\int_{Q_i}n_i d\sigma\right\}.\end{aligned}$$

and p_i,n_i the positive and negative parts of b_Q on each Q_i. The proof of Lemma 5.1.8 can be applied verbatim. We emphasise only that when estimating the testing condition, we need the bound

$$\int_Q \left|\widetilde{b}_Q - b_Q\right|^2 d\sigma \leq C\left(C_{\mathbf{b}}\right)\delta^{\frac{1}{4}} \sum_{i=1}^{\infty} |Q_i|_\sigma \leq C\left(C_{\mathbf{b}}\right)\delta^{\frac{1}{4}} |Q|_\sigma .$$

□

Remark 5.1.10. *The estimate* $\int_Q \left|\widetilde{b}_Q - b_Q\right|^2 d\sigma \leq C\left(C_{\mathbf{b}}\right)\delta^{\frac{1}{4}} \sum_{i=1}^{\infty} |Q_i|_\sigma$ *in the last line of the above proof is of course too large in general to be dominated by a fixed multiple of* $\left|Q'\right|_\sigma$ *for* $Q' \in \mathcal{C}_Q$*, and this is the reason we have no control of weak testing for* $\widetilde{b}_Q$ *in the rest of the corona even if we assume weak testing for* b_Q *in the corona* $\mathcal{C}_Q$*. This defect is addressed in the next subsection below.*

5.1.9 Three corona decompositions

We will use multiple corona constructions, namely a Calderón-Zygmund decomposition, an accretive/testing decomposition, and an energy decomposition, in order to reduce matters to the stopping form, which is treated in Section 5.6 by adapting the bottom/up stopping time and recursion of M. Lacey in [27]. We will then iterate these corona decompositions into a single corona decomposition, which we refer to as the *triple corona*. More precisely, we iterate the first generation of common stopping times with an infusion of the reverse Hölder condition on children, followed by another iteration of the first generation of weak testing stopping times. Recall that we must show the bilinear inequality

$$\left|\int \left(T_\sigma^\alpha f\right) g d\omega\right| \leq \mathfrak{N}_{T^\alpha} \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)}, \qquad f \in L^2(\sigma) \text{ and } g \in L^2(\omega).$$

5.1.9.1 The Calderón-Zygmund corona decomposition

In this section, we introduce the Calderón-Zygmund stopping times $\mathcal{F}$ for a function $\phi \in L^2(\mu)$ relative to a cube S_0 and a positive constant $C_0 \geq 4$. Let $\mathcal{F} = \{F\}_{F \in \mathcal{F}}$ be the collection of Calderón-Zygmund stopping cubes for ϕ defined so that $F \subset S_0$, $S_0 \in \mathcal{F}$, and for all $F \in \mathcal{F}$ with $F \subsetneqq S_0$ we have

$$\begin{aligned} \frac{1}{|F|_\mu} \int_F |\phi| \, d\mu &> C_0 \frac{1}{|\pi_{\mathcal{F}} F|_\mu} \int_F |\phi| \, d\mu; \\ \frac{1}{|F'|_\mu} \int_{F'} |\phi| \, d\mu &\leq C_0 \frac{1}{|\pi_{\mathcal{F}} F|_\mu} \int_F |\phi| \, d\mu \qquad \text{for } F \subsetneqq F' \subset \pi_{\mathcal{F}} F. \end{aligned}$$

We denote by $\pi_{\mathcal{F}} F$ be the smallest member of $\mathcal{F}$ that *strictly* contains F. For a cube $I \in \mathcal{D}$ let $\pi_{\mathcal{D}} I$ be the $\mathcal{D}$-parent of I in the grid $\mathcal{D}$. For $F, F' \in \mathcal{F}$, we say that F' is an $\mathcal{F}$-child of F if $\pi_{\mathcal{F}}(F') = F$ (it could be that $F = \pi_{\mathcal{D}} F'$), and we denote by $\mathfrak{C}_{\mathcal{F}}(F)$ the set of $\mathcal{F}$-children of F. We call $\pi_{\mathcal{F}}(F')$ the $\mathcal{F}$-parent of $F' \in \mathcal{F}$.

To achieve the construction above we use the following definition.

Definition 5.1.11. *Let $C_0 \geq 4$. Given a dyadic grid $\mathcal{D}$ and a cube $S_0 \in \mathcal{D}$, define $\mathcal{S}(S_0)$ to be the* maximal *$\mathcal{D}$-subcubes $I \subset S_0$ such that*

$$\frac{1}{|I|_\mu} \int_I |\phi| \, d\mu > C_0 \frac{1}{|S_0|_\mu} \int_{S_0} |\phi| \, d\mu \ ,$$

and then define the Calderón-Zygmund stopping cubes of S_0 to be the collection

$$\mathcal{F} = \{S_0\} \cup \bigcup_{m=0}^{\infty} \mathcal{S}_m$$

where $\mathcal{S}_0 = \mathcal{S}(S_0)$ and $\mathcal{S}_{m+1} = \bigcup_{S \in \mathcal{S}_m} \mathcal{S}(S)$ for $m \geq 0$.

Define the corona of F by

$$\mathcal{C}_F \equiv \left\{F' \in \mathcal{D} : F \supset F' \supsetneqq H \text{ for some } H \in \mathfrak{C}_{\mathcal{F}}(F)\right\}.$$

The stopping cubes $\mathcal{F}$ above satisfy a Carleson condition:

$$\sum_{F\in\mathcal{F}:\ F\subset\Omega} |F|_\mu \leq C\,|\Omega|_\mu\ , \qquad \text{for all open sets } \Omega.$$

Indeed,

$$\sum_{F'\in\mathfrak{C}_{\mathcal{F}}(F)} |F'|_\mu \leq \sum_{F'\in\mathfrak{C}_{\mathcal{F}}(F)} \frac{\int_{F'} |\phi|\, d\mu}{C_0 \frac{1}{|F|_\mu}\int_F |\phi|\, d\mu} \leq \frac{1}{C_0}|F|,$$

and standard arguments now complete the proof of the Carleson condition.

We emphasize that accretive functions b play no role in the Calderón-Zygmund corona decomposition.

5.1.9.2 The accretive/testing corona decomposition

We use a corona construction modelled after that of Hytönen and Martikainen [24], that delivers a *weak corona testing condition* that coincides with the testing condition itself **only** at the tops of the coronas. This corona decomposition is developed to optimize the choice of a new family of real valued testing functions $\left\{\widehat{b}_Q\right\}_{Q\in\mathcal{D}}$ taken from the vector $\mathbf{b} \equiv \left\{b_Q\right\}_{Q\in\mathcal{D}}$ so that we have

1. the telescoping property at our disposal in each accretive corona,
2. a weak corona testing condition remains in force for the new testing functions $\widehat{b}_Q$ that coincides with the testing condition at the tops of the coronas,

3. the tops of the coronas, i.e. the stopping cubes, enjoy a Carleson condition.

We will henceforth refer to the old family as the *original* family, and denote it by $\left\{b_Q^{orig}\right\}_{Q\in\mathcal{D}}$. The original family will reappear later in helping to estimate the nearby form.

Let σ and ω be locally finite Borel measures on $\mathbb{R}^n$. We assume that the vector of 'testing functions' $\mathbf{b}\equiv\left\{b_Q\right\}_{Q\in\mathcal{D}}$ is a ∞-weakly σ-accretive family, i.e. for $Q\in\mathcal{D}$

$$\operatorname{supp} b_Q\subset Q,$$
$$0<c_{\mathbf{b}}\leq\frac{1}{|Q|_\mu}\int_Q b_Q d\sigma\leq||b_Q||_{L^\infty(\sigma)}\leq C_{\mathbf{b}}<\infty$$

and also that $\mathbf{b}^*\equiv\left\{b_Q\right\}_{Q\in\mathcal{D}}$ is a ∞-weakly ω-accretive family, and we assume in addition the testing conditions

$$\begin{aligned}\int_Q\left|T_\sigma^\alpha\left(\mathbf{1}_Q b_Q\right)\right|^2 d\omega &\leq \left(\mathfrak{T}_{T^\alpha}^{\mathbf{b}}\right)^2|Q|_\sigma\ , \qquad \text{for all cubes } Q,\\ \int_Q\left|T_\omega^{\alpha,*}\left(\mathbf{1}_Q b_Q^*\right)\right|^2 d\sigma &\leq \left(\mathfrak{T}_{T^{\alpha,*}}^{\mathbf{b}^*}\right)^2|Q|_\omega\ , \qquad \text{for all cubes } Q.\end{aligned}$$

Definition 5.1.12. *Given a cube S_0, define $\mathcal{S}(S_0)$ to be the* maximal *subcubes $I\subset S_0$ such that satisfy one of the following*

(a). $\left|\frac{1}{|I|_\mu}\int_I b_{S_0}d\sigma\right|<\gamma$, *or*

(b). $\int_I\left|T_\sigma^\alpha\left(b_{S_0}\right)\right|^2 d\omega>\Gamma\left(\mathfrak{T}_{T^\alpha}^{\mathbf{b}}\right)^2|I|_\sigma$

where the positive constants γ,Γ satisfy $0<\gamma<1<\Gamma<\infty$. Then define the $\mathbf{b}$-accretive stopping cubes of S_0 to be the collection

$$\mathcal{F}=\{S_0\}\cup\bigcup_{m=0}^{\infty}\mathcal{S}_m$$

where $\mathcal{S}_0 = \mathcal{S}(S_0)$ *and* $\mathcal{S}_{m+1} = \bigcup_{S \in \mathcal{S}_m} \mathcal{S}(S)$ *for* $m \geq 0$.

For $\varepsilon > 0$ chosen small enough depending on $p > 2$, the $\mathbf{b}$-accretive stopping cubes satisfy a σ-Carleson condition relative to the measure σ, and the new testing functions $\left\{\widetilde{b}_Q\right\}_{Q \in \mathcal{D}}$, defined by $\widetilde{b_S} = \mathbf{1}_S b_{S_0}$ for $S \in \mathcal{C}_{S_0}$, satisfy *weak* testing inequalities. The following lemma is essentially in [24], but we include a proof for completeness.

Lemma 5.1.13. *For* γ *small enough and* Γ *large enough, we have the following:*

(1). For every open set Ω *we have we have the inequality,*

$$\sum_{S \in \mathcal{F}:\ S \subset \Omega} |S|_\sigma \leq C\, |\Omega|_\sigma \ . \tag{5.1.26}$$

(2). For every cube $S \in \mathcal{C}_{S_0}$ *we have the weak corona testing inequality,*

$$\int_S \left| T_\sigma^\alpha b_{S_0} \right|^2 d\omega \leq C \left(\mathfrak{T}_{T^\alpha}^{\mathbf{b}} \right)^2 |S|_\sigma \ . \tag{5.1.27}$$

Proof. Inequality (5.1.27) is immediate from the definition of $\mathcal{F}$ in the definition 5.1.12. We now address the Carleson condition (5.1.26). A standard argument reduces matters to the case where Ω is a cube $Q \in \mathcal{F}$ with $|Q|_\sigma > 0$. It suffices to consider each of the two stopping criteria separately. We first address the stopping condition $\left| \frac{1}{|I|_\sigma} \int_I b_{S_0} d\sigma \right| < \gamma$. Throughout this proof we will denote the union of these children $\mathcal{S}(Q)$ of Q by $E(Q) \equiv \bigcup_{S \in \mathcal{S}(Q)} S$. Then we have

$$\left| \int_{E(Q)} b_Q d\sigma \right| \leq \sum_{S \in \mathcal{S}(Q)} \left| \int_S b_Q d\sigma \right| < \gamma \sum_{S \in \mathcal{S}(Q)} |S|_\sigma \leq \gamma\, |Q|_\sigma \ ,$$

which together with our hypotheses on b_Q gives

$$\begin{aligned}
|Q|_\sigma &\leq \left|\int_Q b_Q d\sigma\right| \leq \left|\int_{E(Q)} b_Q d\sigma\right| + \left|\int_{Q\setminus E(Q)} b_Q d\sigma\right| \\
&\leq \gamma\,|Q|_\sigma + \sqrt{\int_{Q\setminus E(Q)} \left|b_Q\right|^2 d\sigma}\sqrt{|Q\setminus E\,(Q)|_\sigma} \\
&\leq \gamma\,|Q|_\sigma + C_{\mathbf{b}}\sqrt{|Q|_\sigma}\sqrt{|Q\setminus E\,(Q)|_\sigma}.
\end{aligned}$$

Rearranging the inequality yields

$$(1-\gamma)\,|Q|_\sigma \leq C_{\mathbf{b}}\sqrt{|Q|_\sigma}\sqrt{|Q\setminus E\,(Q)|_\sigma}$$

or

$$\frac{(1-\gamma)^2}{C_{\mathbf{b}}^2}\,|Q|_\sigma \leq |Q\setminus E\,(Q)|_\sigma \ ,$$

which in turn gives

$$\begin{aligned}
\sum_{S\in\mathcal{S}(Q)} |S|_\sigma &= |E(Q)| = |Q|_\sigma - |Q\setminus E\,(Q)|_\sigma \\
&\leq |Q|_\sigma - \frac{(1-\gamma)^2}{C_{\mathbf{b}}^2}\,|Q|_\sigma = \left(1-\frac{(1-\gamma)^2}{C_{\mathbf{b}}^2}\right)|Q|_\sigma \equiv \beta\,|Q|_\sigma \ ,
\end{aligned}$$

where $0<\beta<1$ since $1\leq C_{\mathbf{b}}$. If we now iterate this inequality, we obtain for each $k\geq 1$,

$$\begin{aligned}
\sum_{\substack{S\in\mathcal{F}:\ S\subset Q\\ \pi_{\mathcal{F}}^{(k)}(S)=Q}}|S|_{\sigma} &= \sum_{\substack{S\in\mathcal{F}:\ S\subset Q\\ \pi_{\mathcal{F}}^{(k-1)}(S)=Q}}\ \sum_{S'\in\mathcal{S}(S)}\left|S'\right|_{\sigma}\leq \sum_{\substack{S\in\mathcal{F}:\ S\subset Q\\ \pi_{\mathcal{F}}^{(k-1)}(S)=Q}}\beta\left|S\right|_{\sigma}\\
&\vdots\\
&\leq \sum_{\substack{S\in\mathcal{F}:\ S\subset Q\\ \pi_{\mathcal{F}}^{(1)}(S)=Q}}\beta^{k-1}\left|S\right|_{\sigma}\leq\beta^{k}\left|Q\right|_{\sigma}\ .
\end{aligned}$$

Finally then

$$\sum_{S\in\mathcal{F}:\ S\subset Q}|S|_{\sigma}\leq\sum_{k=0}^{\infty}\sum_{\substack{S\in\mathcal{F}:\ S\subset Q\\ \pi_{\mathcal{F}}^{(k)}(S)=Q}}|S|_{\sigma}\leq\sum_{k=0}^{\infty}\beta^{k}\left|Q\right|_{\sigma}=\frac{1}{1-\beta}\left|Q\right|_{\sigma}=\frac{C_{\mathbf{b}}^{2}}{(1-\gamma)^{2}}\left|Q\right|_{\sigma}.$$

Now we turn to the second stopping criterion $\int_I\left|T_{\sigma}^{\alpha}\left(b_{S_0}\right)\right|^2d\omega>\Gamma\left(\mathfrak{T}_{T^{\alpha}}^{\mathbf{b}}\right)^2|I|_{\sigma}$. We have

$$\begin{aligned}
\sum_{S\in\mathfrak{C}_{\mathcal{F}}(S_0)}|S|_{\sigma} &\leq \frac{1}{\Gamma\left(\mathfrak{T}_{T^{\alpha}}^{\mathbf{b}}\right)^{2}}\sum_{S\in\mathfrak{C}_{\mathcal{F}}(S_0)}\int_{S}\left|T_{\sigma}^{\alpha}\left(b_{S_0}\right)\right|^{2}d\omega\\
&\leq \frac{1}{\Gamma\left(\mathfrak{T}_{T^{\alpha}}^{\mathbf{b}}\right)^{2}}\int_{S_0}\left|T_{\sigma}^{\alpha}\left(b_{S_0}\right)\right|^{2}d\omega\leq\frac{1}{\Gamma}\left|S_0\right|_{\sigma}.
\end{aligned}$$

Iterating this inequality gives

$$\sum_{\substack{S\in\mathcal{F}\\ S\subset S_0}}|S|_{\sigma}\leq\sum_{k=0}^{\infty}\frac{1}{\Gamma^{k}}\left|S_0\right|_{\sigma}=\frac{\Gamma}{\Gamma-1}\left|S_0\right|_{\sigma},$$

and then

$$\sum_{\substack{S\in\mathcal{F}\\ S\subset\Omega}}|S|_{\sigma}=\sum_{\substack{\text{maximal } S_0\in\mathcal{F}\\ S_0\subset\Omega}}\ \sum_{\substack{S\in\mathcal{F}\\ S\subset S_0}}|S|_{\sigma}\leq\frac{\Gamma}{\Gamma-1}\sum_{\substack{\text{maximal } S_0\in\mathcal{F}\\ S_0\subset\Omega}}|S_0|_{\sigma}=\frac{\Gamma}{\Gamma-1}|\Omega|_{\sigma}\ .$$

This completes the proof of Lemma 5.1.13. □

5.1.9.3 The energy corona decompositions

Given a weight pair (σ,ω), we construct an energy corona decomposition for σ and an energy corona decomposition for ω, that uniformize estimates (c.f. [43], [32], [63] and [64]). In order to define these constructions, we recall that the energy condition constant $\mathcal{E}_2^{\alpha}$ is given by

$$(\mathcal{E}_2^{\alpha})^2\equiv\sup_{\substack{Q\in\mathcal{P}\\ Q=\dot{\cup}J_r}}\frac{1}{|Q|_{\sigma}}\sum_{r=1}^{\infty}\left(\frac{\mathrm{P}^{\alpha}\left(J_r,\mathbf{1}_Q\sigma\right)}{|J_r|^{\frac{1}{n}}}\right)^2\left\|x-m_{J_r}\right\|_{L^2\left(\mathbf{1}_{J_r}\omega\right)}^2\ ,$$

where $\dot{\cup}J_r$ is an arbitrary subdecomposition of Q into cubes $J_r\in\mathcal{P}$ and interchanging the roles of σ and ω we have the constant $\mathcal{E}_2^{\alpha,*}$. Also recall that $\mathfrak{E}_2^{\alpha}=\mathcal{E}_2^{\alpha}+\mathcal{E}_2^{\alpha,*}$. In the next definition we restrict the cubes Q to a dyadic grid $\mathcal{D}$, but keep the subcubes J_r unrestricted.

Definition 5.1.14. *Given a dyadic grid $\mathcal{D}$ and a cube $S_0\in\mathcal{D}$, define $\mathcal{S}(S_0)$ to be the* maximal *$\mathcal{D}$-subcubes $I\subset S_0$ such that*

$$\sup_{I\supset\dot{\cup}J_r}\sum_{r=1}^{\infty}\left(\frac{\mathrm{P}^{\alpha}\left(J_r,\mathbf{1}_I\sigma\right)}{|J_r|^{\frac{1}{n}}}\right)^2\left\|x-m_{J_r}\right\|_{L^2\left(\mathbf{1}_{J_r}\omega\right)}^2\geq C_{en}\left[(\mathfrak{E}_2^{\alpha})^2+\mathfrak{A}_2^{\alpha}\right]\ |I|_{\sigma}\,,\qquad(5.1.28)$$

where the cubes $J_r\in\mathcal{P}$ are pairwise disjoint in I, $\mathfrak{E}_2^{\alpha}$ is the energy condition constant, and C_{en} is a sufficiently large positive constant depending only on α. Then define the σ-energy *stopping cubes of S_0 to be the collection*

$$\mathcal{F}=\{S_0\}\cup\bigcup_{m=0}^{\infty}\mathcal{S}_m$$

where $\mathcal{S}_0=\mathcal{S}(S_0)$ and $\mathcal{S}_{m+1}=\bigcup_{S\in\mathcal{S}_m}\mathcal{S}(S)$ for $m\geq 0$.

We now claim that from the energy condition $\mathfrak{E}_2^{\alpha}<\infty$, we obtain the σ-Carleson estimate,

$$\sum_{S\in\mathcal{S}:\ S\subset I}|S|_{\sigma}\leq 2|I|_{\sigma},\qquad I\in\mathcal{D}. \tag{5.1.29}$$

Indeed, for any $S_1\in\mathcal{F}$ we have

$$\begin{aligned}\sum_{S\in\mathfrak{C}_{\mathcal{F}}(S_1)}|S|_{\sigma}&\leq\frac{1}{C_{en}\left(\mathfrak{A}_2^{\alpha}+\left(\mathcal{E}_2^{\alpha}\right)^2\right)}\sum_{S\in\mathfrak{C}_{\mathcal{F}}(S_1)}\sup_{S\supset\dot{\cup}J_r}\sum_{r=1}^{\infty}\left(\frac{\mathrm{P}^{\alpha}(J_r,\mathbf{1}_S\sigma)}{|J_r|^{\frac{1}{n}}}\right)^2\left\|x-m_{J_r}\right\|_{L^2(\mathbf{1}_{J_r}\omega)}^2\\&\leq\frac{1}{C_{en}\left(\mathcal{E}_2^{\alpha}\right)^2}\left(\mathcal{E}_2^{\alpha}\right)^2|S_1|_{\sigma}=\frac{1}{C_{en}}|S_1|_{\sigma}\ ,\end{aligned}$$

upon noting that the union of the subdecompositions $\dot{\cup}J_r\subset S$ over $S\in\mathfrak{C}_{\mathcal{F}}(S_1)$ is a subdecomposition of S_1, and the proof of the Carleson estimate is now finished by iteration in the standard way.

Finally, we record the reason for introducing energy stopping times. If

$$\mathbf{X}_{\alpha}(\mathcal{C}_S)^2\equiv\sup_{I\in\mathcal{C}_S}\frac{1}{|I|_{\sigma}}\sup_{I\supset\dot{\cup}J_r}\sum_{r=1}^{\infty}\left(\frac{\mathrm{P}^{\alpha}(J_r,\mathbf{1}_I\sigma)}{|J_r|^{\frac{1}{n}}}\right)^2\left\|x-m_{J_r}\right\|_{L^2\left(\mathbf{1}_{J_r}\omega\right)}^2 \tag{5.1.30}$$

is (the square of) the α-*stopping energy* of the weight pair (σ,ω) with respect to the corona

$\mathcal{C}_S$, then we have the *stopping energy bounds*

$$\mathbf{X}_\alpha\left(\mathcal{C}_S\right) \leq \sqrt{C_{en}}\sqrt{\left(\mathfrak{E}_2^\alpha\right)^2 + \mathfrak{A}_2^\alpha}, \qquad S \in \mathcal{F}, \tag{5.1.31}$$

where $\mathfrak{A}_2^\alpha$ and the energy constant $\mathfrak{E}_2^\alpha$ are controlled by the assumptions in Theorem 5.1.5.

5.1.10 Iterated coronas and general stopping data

We will use a construction that permits *iteration* of the above three corona decompositions by combining Definitions 5.1.11, 5.1.12 and 5.1.14 into a single stopping condition. However, there is one remaining difficulty with the triple corona constructed in this way, namely if a stopping cube $I \in \mathcal{A}$ is a child of a cube Q in the corona $\mathcal{C}_A$, then the modulus of the average $\left|\frac{1}{|I|_\sigma}\int_I b_Q d\sigma\right|$ of b_Q on I may be far smaller than the sup norm of $\left|b_Q\right|$ on the child I, indeed it may be that $\frac{1}{|I|_\sigma}\int_I b_Q d\sigma = 0$. This of course destroys any reasonable estimation of the martingale and dual martingale differences $\triangle_Q^{\sigma,\mathbf{b}} f$ and $\square_Q^{\sigma,\mathbf{b}} f$ used in the proof of Theorem 5.1.5, and so we will use Lemma 5.1.9 on the function b_A to obtain a new function $\widetilde{b}_A$ for which this problem is circumvented at the 'bottom' of the corona, i.e. for those $A' \in \mathfrak{C}_{\mathcal{A}}(A)$. We then refer to the stopping times $A' \in \mathfrak{C}_{\mathcal{A}}(A)$ as 'shadow' stopping times since we have lost control of the weak testing condition relative to the new function $\widetilde{b}_A$. Thus we must redo the weak testing stopping times for the new function $\widetilde{b}_A$, but also stopping if we hit one of the shadow stopping times. Here are the details.

Definition 5.1.15. *Let $C_0 \geq 4$, $0 < \gamma < 1$ and $1 < \Gamma < \infty$. Suppose that $\mathbf{b} = \left\{b_Q\right\}_{Q\in\mathcal{P}}$ is an ∞-weakly σ-accretive family on $\mathbb{R}^n$. Given a dyadic grid $\mathcal{D}$ and a cube $Q \in \mathcal{D}$, define the collection of 'shadow' stopping times $\mathcal{S}_{shadow}(Q)$ to be the* maximal *$\mathcal{D}$-subcubes $I \subset Q$ such that one of the following holds:*

(a).

$$\frac{1}{|I|_\sigma}\int_I |f|\,d\sigma > C_0\frac{1}{|Q|_\sigma}\int_Q |f|\,d\sigma \ ,$$

(b).

$$\left|\frac{1}{|I|_\mu}\int_I b_Q d\sigma\right| < \gamma \ \text{or} \ \int_I \left|T_\sigma^\alpha\left(b_Q\right)\right|^2 d\omega > \Gamma\left(\mathfrak{T}_{T^\alpha}^{\mathbf{b}}\right)^2 |I|_\sigma \ ,$$

(c).

$$\sup_{I\supset\dot{\cup}J_r}\sum_{r=1}^\infty\left(\frac{\mathrm{P}^\alpha\left(J_r,\sigma\right)}{|J_r|^{\frac{1}{n}}}\right)^2\left\|x-m_{J_r}\right\|_{L^2\left(\mathbf{1}_{J_r}\omega\right)}^2 \geq C_{en}\left[(\mathfrak{E}_2^\alpha)^2+\mathfrak{A}_2^\alpha\right] \ |I|_\sigma \ .$$

Now we apply Lemma 5.1.9 to the function b_Q with $\mathcal{S}_{shadow}\left(Q\right)\equiv\{Q_i\}_{i=1}^\infty$ to obtain a new function $\widetilde{b}_Q$ satisfying the properties

$$\begin{aligned}
&\operatorname{supp}\widetilde{b}_Q \subset Q \ , && (5.1.32)\\
&1\leq\frac{1}{|Q'|_\sigma}\int_{Q'}\widetilde{b}_Q d\sigma \leq \left\|\mathbf{1}_{Q'}\widetilde{b}_Q\right\|_{L^\infty(\sigma)} \leq 2\left(1+\sqrt{C_{\mathbf{b}}}\right)C_{\mathbf{b}} \ , \qquad Q'\in\mathcal{C}_Q \ ,\\
&\sqrt{\int_Q\left|T_\sigma^\alpha b_Q\right|^2 d\omega} \leq \left[2\mathfrak{T}_{T^\alpha}^{\mathbf{b}}\left(Q\right)+4C_{\mathbf{b}}^{\frac{3}{2}}\delta^{\frac{1}{4}}\mathfrak{N}_{T^\alpha}\left(Q\right)\right]\sqrt{|Q|_\sigma} \ ,\\
&\left\|\mathbf{1}_{Q_i}\widetilde{b}_Q\right\|_{L^\infty(\sigma)} \leq \frac{16C_{\mathbf{b}}}{\delta}\left|\frac{1}{|Q_i|_\sigma}\int_{Q_i}\widetilde{b}_Q d\sigma\right| \ , \qquad 1\leq i<\infty.
\end{aligned}$$

Note that each of the functions $\widetilde{b}_{Q'}\equiv\mathbf{1}_{Q'}\widetilde{b}_Q$, for $Q'\in\mathcal{C}_Q$, now satisfies the crucial reverse Hölder property

$$\left\|\mathbf{1}_I\widetilde{b}_{Q'}\right\|_{L^\infty(\sigma)} \leq C_{\delta,\mathbf{b}}\left|\frac{1}{|I|_\sigma}\int_I\widetilde{b}_{Q'}d\sigma\right| \ , \qquad \text{for all } I\in\mathfrak{C}\left(Q'\right), \ Q'\in\mathcal{C}_Q.$$

Indeed, if I equals one of the Q_i then the reverse Hölder condition in the last line of (5.1.32) applies, while if $I\in\mathcal{C}_Q$ then the accretivity in the second line of (5.1.32) applies.

Since we have lost the weak testing condition in the corona for this new function $\widetilde{b}_Q$, the next step is to run again the weak testing construction of stopping times, but this time starting with the new function $\widetilde{b}_Q$, and also stopping if we hit one of the 'shadow' stopping times Q_i. Here is the new stopping criterion.

Definition 5.1.16. *Let $C_0 \geq 4$ and $1 < \Gamma < \infty$. Let $\mathcal{S}_{shadow}(Q) \equiv \{Q_i\}_{i=1}^{\infty}$ be as in Definition 5.1.15. Define $\mathcal{S}_{iterated}(Q)$ to be the* maximal *$\mathcal{D}$-subcubes $I \subset Q$ such that either*

$$\int_I \left| T_\sigma^\alpha \left(\widetilde{b}_Q \right) \right|^2 d\omega > \Gamma \left(\mathfrak{T}_{T^\alpha}^{\widetilde{\mathbf{b}}} \right)^2 |I|_\sigma \ ,$$

or

$$I = Q_i \text{ for some } 1 \leq i < \infty.$$

Thus for each cube Q we have now constructed *iterated stopping children* $\mathcal{S}_{iterated}(Q)$ by first constructing shadow stopping times $\mathcal{S}_{shadow}(Q)$ using one step of the triple corona construction, then modifying the testing function to have reverse Hölder controlled children, and finally running again the weak testing stopping time construction to get $\mathcal{S}_{iterated}(Q)$. These iterated stopping times $\mathcal{S}_{iterated}(Q)$ have control of CZ averages of f and energy control of σ and ω, simply because these controls were achieved in the shadow construction, and were unaffected by either the application of Lemma 5.1.9 or the rerunning of the weak testing stopping criterion for $\widetilde{b}_Q$. And of course we now have weak testing within the corona determined by Q and $\mathcal{S}_{iterated}(Q)$, and we also have the crucial reverse Hölder condition on all the children of cubes in the corona. With all of this in hand, here then is the definition of the construction of iterated coronas.

Definition 5.1.17. *Let $C_0 \geq 4$, $0 < \gamma < 1$ and $1 < \Gamma < \infty$. Suppose that $\mathbf{b} = \left\{ b_Q \right\}_{Q \in \mathcal{P}}$ is* *an ∞-weakly σ-accretive family on $\mathbb{R}^n$. Given a dyadic grid $\mathcal{D}$ and a cube S_0 in $\mathcal{D}$, define the iterated stopping cubes of S_0 to be the collection*

$$\mathcal{F} = \{S_0\} \cup \bigcup_{m=0}^{\infty} \mathcal{S}_m$$

where $\mathcal{S}_0 = \mathcal{S}_{iterated}(S_0)$ and $\mathcal{S}_{m+1} = \bigcup_{S \in \mathcal{S}_m} \mathcal{S}_{iterated}(S)$ for $m \geq 0$, and where $\mathcal{S}_{iterated}(Q)$ is defined in Definition 5.1.16.

It is useful to append to the notion of stopping times $\mathcal{S}$ in the above σ-iterated corona decomposition a positive constant A_0 and an additional structure $\alpha_{\mathcal{S}}$ called stopping bounds for a function f. We will refer to the resulting triple $(A_0, \mathcal{F}, \alpha_{\mathcal{F}})$ as constituting stopping data for f. If $\mathcal{F}$ is a grid, we define $F' \prec F$ if $F' \subsetneqq F$ and $F', F \in \mathcal{F}$. Recall that $\pi_{\mathcal{F}} F'$ is the smallest $F \in \mathcal{F}$ such that $F' \prec F$.

Suppose we are given a positive constant $A_0 \geq 4$, a subset $\mathcal{F}$ of the dyadic grid $\mathcal{D}$ (called the stopping times), and a corresponding sequence $\alpha_{\mathcal{F}} \equiv \{\alpha_{\mathcal{F}}(F)\}_{F \in \mathcal{F}}$ of nonnegative numbers $\alpha_{\mathcal{F}}(F) \geq 0$ (called the stopping bounds). Let $(\mathcal{F}, \prec, \pi_{\mathcal{F}})$ be the tree structure on $\mathcal{F}$ inherited from $\mathcal{D}$, and for each $F \in \mathcal{F}$ denote by $\mathcal{C}_F = \{I \in \mathcal{D} : \pi_{\mathcal{F}} I = F\}$ the corona associated with F:

$$\mathcal{C}_F = \left\{ I \in \mathcal{D} : I \subset F \text{ and } I \not\subset F' \text{ for any } F' \prec F \right\}.$$

Definition 5.1.18. *We say the triple $(A_0, \mathcal{F}, \alpha_{\mathcal{F}})$ constitutes* stopping data *for a function $f \in L^1_{loc}(\sigma)$ if*

(1). $E_I^\sigma |f| \leq \alpha_{\mathcal{F}}(F)$ for all $I \in \mathcal{C}_F$ and $F \in \mathcal{F}$,

(2). $\sum_{F' \preceq F} |F'|_\sigma \leq A_0 |F|_\sigma$ for all $F \in \mathcal{F}$,

(3). $\sum_{F\in\mathcal{F}}\alpha_{\mathcal{F}}(F)^2|F|_\sigma\leq A_0^2\|f\|^2_{L^2(\sigma)}$,

(4). $\alpha_{\mathcal{F}}(F)\leq\alpha_{\mathcal{F}}(F')$ *whenever* $F',F\in\mathcal{F}$ *with* $F'\subset F$.

Property *(1)* says that $\alpha_{\mathcal{F}}(F)$ bounds the averages of f in the corona $\mathcal{C}_F$, and property *(2)* says that the cubes at the tops of the coronas satisfy a Carleson condition relative to the weight σ. Note that a standard 'maximal cube' argument extends the Carleson condition in property *(2)* to the inequality

$$\sum_{F'\in\mathcal{F}:\ F'\subset A}|F'|_\sigma\leq A_0|A|_\sigma \text{ for all open sets } A\subset\mathbb{R}^n. \tag{5.1.33}$$

Property *(3)* is the quasi-orthogonality condition that says the sequence of functions $\{\alpha_{\mathcal{F}}(F)\mathbf{1}_F\}_{F\in\mathcal{F}}$ is in the vector-valued space $L^2(\ell^2;\sigma)$ with control and is often referred to as a Carleson embedding theorem, and property *(4)* says that the control on stopping data is nondecreasing on the stopping tree $\mathcal{F}$. We emphasize that we are *not* assuming in this definition the stronger property that there is $C>1$ such that $\alpha_{\mathcal{F}}(F')>C\alpha_{\mathcal{F}}(F)$ whenever $F',F\in\mathcal{F}$ with $F'\subsetneqq F$. Instead, the properties *(2)* and *(3)* substitute for this lack. Of course the stronger property *does* hold for the familiar *Calderón-Zygmund* stopping data determined by the following requirements for $C>1$,

$$E^\sigma_{F'}|f|>CE^\sigma_F|f| \text{ whenever } F',F\in\mathcal{F} \text{ with } F'\subsetneqq F,$$

$$E^\sigma_I|f|\leq CE^\sigma_F|f| \text{ for } I\in\mathcal{C}_F,$$

which are themselves sufficiently strong to automatically force properties *(2)* and *(3)* with $\alpha_{\mathcal{F}}(F)=\mathbb{E}^\sigma_F|f|$.

We have the following useful consequence of *(2)* and *(3)* that says the sequence $\{\alpha_{\mathcal{F}}(F)\mathbf{1}_F\}_{F\in\mathcal{F}}$ has a *quasi-orthogonal* property relative to f with a constant C_0' depending only on C_0:

$$\left\|\sum_{F\in\mathcal{F}}\alpha_{\mathcal{F}}(F)\mathbf{1}_F\right\|_{L^2(\sigma)}^2\leq C_0'\|f\|_{L^2(\sigma)}^2. \tag{5.1.34}$$

Proposition 5.1.19. *Let $f\in L^2(\sigma)$, let $\mathcal{F}$ be as in Definition 5.1.17, and define stopping data $\alpha_{\mathcal{F}}$ by $\alpha_F=\frac{1}{|F|_\sigma}\int_F|f|\,d\sigma$. Then there is $A_0\geq 4$, depending only on the constant C_0 in Definition 5.1.11 , such that the triple $(A_0,\mathcal{F},\alpha_{\mathcal{F}})$ constitutes* stopping data *for the function f.*

Proof. This is an easy exercise using (5.1.26) and (5.1.29), and is left for the reader. □

5.1.11 Reduction to good functions

We begin with a specification of the various parameters that will arise during the proof, as well as the extension of goodness introduced in [24].

Definition 5.1.20. *The parameters $\mathbf{r}$, τ and ρ will be fixed below to satisfy*

$$\tau>\mathbf{r}\ \textit{and}\ \rho>\mathbf{r}+\tau,$$

where $\mathbf{r}$ is the goodness parameter fixed in (5.2.16).

Let $0<\varepsilon<1$ to be chosen later. Define J to be $\varepsilon-$*good* in a cube K if

$$d(J,skelK)>2|J|^{\varepsilon}|K|^{1-\varepsilon},$$

where the skeleton $skelK \equiv \bigcup_{K' \in \mathfrak{C}(K)} \partial K'$ of a cube K consists of the boundaries of all the children K' of K. Define $\mathcal{G}^{\mathcal{D}}_{(k,\varepsilon)-good}$ to consist of those $J \in \mathcal{G}$ such that J is good in every supercube $K \in \mathcal{D}$ that lies at least k levels above J. We also define J to be $\varepsilon - good$ in a cube K and beyond if $J \in \mathcal{G}^{\mathcal{D}}_{(k,\varepsilon)-good}$ where $k = \log_2 \frac{\ell(K)}{\ell(J)}$. We can now say that $J \in \mathcal{G}^{\mathcal{D}}_{(k,\varepsilon)-good}$ if and only if J is $\varepsilon - good$ in $\pi^k J$ and beyond. As the goodness parameter ε will eventually be fixed throughout the proof, we sometimes suppress it, and simply say "J is *good* in a cube K and beyond" instead of "J is $\varepsilon - good$ in a cube K and beyond".

As pointed out on page 14 of [24] by Hytönen and Martikainen, there are subtle difficulties associated in using dual martingale decompositions of functions which depend on the entire dyadic grid, rather than on just the local cube in the grid. We will proceed at first in the spirit of [24]. The goodness that we will infuse below into the main 'below' form $\mathsf{B}_{\Subset_\rho}(f,g)$ will be the Hytönen-Martikainen 'weak' goodness: every pair $(I,J) \in \mathcal{D} \times \mathcal{G}$ that arises in the form $\mathsf{B}_{\Subset_\rho}(f,g)$ will satisfy $J \in \mathcal{G}^{\mathcal{D}}_{(k,\varepsilon)-good}$ where $\ell(I) = 2^k \ell(J)$.

It is important to use *two* independent random grids, one for each function f and g simultaneously, as this is necessary in order to apply probabilistic methods to the dual martingale averages $\square^{\mu,\mathbf{b}}_I$ that depend, not only on I, but also on the underlying grid in which I lives. The proof methods for functional energy from [64] and [63] relied heavily on the use of a single grid, and this must now be modified to accomodate two independent grids.

5.1.11.1 Parameterizations of dyadic grids

It is important to use two independent grids, one for each function f and g simultaneously, as it is necessary in order to apply probabilistic methods to the dual martingale averages $\square^{\mu,\mathbf{b}}_I$ that depend not only on I but also on the underlying grid in which I lives.

Now we recall the construction from the paper [67]. We momentarily fix a large positive integer $M \in \mathbb{N}$, and consider the tiling of $\mathbb{R}^n$ by the family of cubes $\mathbb{D}_M \equiv \left\{ I_\alpha^M \right\}_{\alpha \in \mathbb{Z}}$ having side length 2^{-M} and given by $I_\alpha^M \equiv I_0^M + \alpha \cdot 2^{-M}$ where $I_0^M = \left[0, 2^{-M}\right)$. A *dyadic grid* $\mathcal{D}$ built on $\mathbb{D}_M$ is defined to be a family of cubes $\mathcal{D}$ satisfying:

1. Each $I \in \mathcal{D}$ has side length $2^{-\ell}$ for some $\ell \in \mathbb{Z}$ with $\ell \leq M$, and I is a union of $2^{n(M-\ell)}$ cubes from the tiling $\mathbb{D}_M$,
2. For $\ell \leq M$, the collection $\mathcal{D}_\ell$ of cubes in $\mathcal{D}$ having side length $2^{-\ell}$ forms a pairwise disjoint decomposition of the space $\mathbb{R}^n$,
3. Given $I \in \mathcal{D}_i$ and $J \in \mathcal{D}_j$ with $j \leq i \leq M$, it is the case that either $I \cap J = \emptyset$ or $I \subset J$.

We now momentarily fix a *negative* integer $N \in -\mathbb{N}$, and restrict the above grids to cubes of side length at most 2^{-N}:

$$\mathcal{D}^N \equiv \left\{ I \in \mathcal{D} : \text{side length of } I \text{ is at most } 2^{-N} \right\}.$$

We refer to such grids $\mathcal{D}^N$ as a (truncated) dyadic grid $\mathcal{D}$ built on $\mathbb{D}_M$ of size 2^{-N}. There are now two traditional means of constructing probability measures on collections of such dyadic grids, namely parameterization by choice of parent, and parameterization by translation.

Construction #1: Consider first the special case of dimension $n = 1$. For any

$$\beta = \{\beta_i\}_{i \in \mathbb{Z}_M^N} \in \omega_m^N \equiv \{0,1\}^{\mathbb{Z}_M^N},$$

where $\mathbb{Z}_M^N \equiv \{\ell \in \mathbb{Z} : N \leq \ell \leq M\}$, define the dyadic grid $\mathcal{D}_\beta$ built on $\mathbb{D}_m$ of size 2^{-N} by

$$\mathcal{D}_\beta = \left\{ 2^{-\ell} \left([0,1) + k + \sum_{i:\ \ell < i \leq M} 2^{-i+\ell} \beta_i \right) \right\}_{N \leq \ell \leq M,\, k \in \mathbb{Z}} \tag{5.1.35}$$

Place the uniform probability measure ρ_M^N on the finite index space $\omega_M^N = \{0,1\}^{\mathbb{Z}_M^N}$, namely that which charges each $\beta \in \omega_M^N$ equally. This construction is then extended to Euclidean space $\mathbb{R}^n$ by taking products in the usual way and using the product index space $\Omega_M^N \equiv (\omega_M^N)^n$ and the uniform product probability measure $\mu_M^N = \rho_M^N \times ... \times \rho_M^N$.

Construction #2: Momentarily fix a (truncated) dyadic grid $\mathcal{D}$ built on $\mathbb{D}_M$ of size 2^{-N}. For any

$$\gamma \in \Gamma_M^N \equiv \left\{2^{-M}\mathbb{Z}_+^n : |\gamma_i| < 2^{-N}\right\},$$

where $\mathbb{Z}_+^n = (\mathbb{N} \cup \{0\})^n$, define the dyadic grid $\mathcal{D}^\gamma$ built on $\mathbb{D}_m$ of size 2^{-N} by

$$\mathcal{D}^\gamma \equiv \mathcal{D} + \gamma.$$

Place the uniform probability measure ν_M^N on the finite index set Γ_M^N, namely that which charges each multiindex γ in Γ_M^N equally.

The two probability spaces $\left(\left\{\mathcal{D}_\beta\right\}_{\beta\in\Omega_M^N}, \mu_M^N\right)$ and $\left(\{\mathcal{D}^\gamma\}_{\gamma\in\Gamma_M^N}, \nu_M^N\right)$ are isomorphic since both collections $\left\{\mathcal{D}_\beta\right\}_{\beta\in\Omega_M^N}$ and $\{\mathcal{D}^\gamma\}_{\gamma\in\Gamma_M^N}$ describe the set $\boldsymbol{A}_M^N$ of **all** (truncated) dyadic grids $\mathcal{D}^\gamma$ built on $\mathbb{D}_m$ of size 2^{-N}, and since both measures μ_M^N and ν_M^N are the uniform measure on this space. The first construction may be thought of as being *parameterized by scales* - each component β_i in $\beta = \{\beta_i\}_{i\in\mathbb{Z}_M^N} \in \omega_M^N$ amounting to a choice of the two possible tilings at level i that respect the choice of tiling at the level below - and since any grid in $\boldsymbol{A}_M^N$ is determined by a choice of scales , we see that $\left\{\mathcal{D}_\beta\right\}_{\beta\in\Omega_M^N} = \boldsymbol{A}_M^N$. The second construction may be thought of as being *parameterized by translation* - each $\gamma \in \Gamma_M^N$ amounting to a choice of translation of the grid $\mathcal{D}$ fixed in construction #2 - and since any grid in $\boldsymbol{A}_M^N$ is determined by any of the cubes at the top level, i.e. with side length 2^{-N}, we see that $\{\mathcal{D}^\gamma\}_{\gamma\in\Gamma_M^N} = \boldsymbol{A}_M^N$ as well, since every cube at the top level in $\boldsymbol{A}_M^N$ has the form $Q+\gamma$ for some $\gamma \in \Gamma_M^N$ and $Q \in \mathcal{D}$ at the top level in $\boldsymbol{A}_M^N$ (i.e. every cube at the top level in $\boldsymbol{A}_M^N$ is a union of small cubes in $\mathbb{D}_m$, and so must be a translate of some $Q \in \mathcal{D}$ by an amount 2^{-M} times an element of $\mathbb{Z}_+$). Note also that in all dimensions, $\#\Omega_M^N = \#\Gamma_M^N = 2^{n(M-N)}$. We will use $\boldsymbol{E}_{\Omega_M^N}$ to denote expectation with respect to this common probability measure on $\boldsymbol{A}_M^N$.

Notation 5.1.21. *For purposes of notation and clarity, we now suppress all reference to M and N in our families of grids, and in the notations Ω and Γ for the parameter sets, and we use $\boldsymbol{P}_\Omega$ and $\boldsymbol{E}_\Omega$ to denote probability and expectation with respect to families of grids, and instead proceed as if all grids considered are unrestricted. The careful reader can supply the modifications necessary to handle the assumptions made above on the grids $\mathcal{D}$ and the functions f and g regarding M and N.*

5.1.12 Formulas

We need the following formulas defined on Appendix A of [69].

$$\begin{aligned}
\mathbb{E}_Q^{\mu,\mathbf{b}} f(x) &\equiv \mathbf{1}_Q(x) \frac{1}{\int_Q b_Q d\mu} \int_Q f b_Q d\mu, \qquad Q \in \mathcal{P}, \\
\mathbb{F}_Q^{\mu,\mathbf{b}} f(x) &\equiv \mathbf{1}_Q(x)\, b_Q(x) \frac{1}{\int_Q b_Q d\mu} \int_Q f d\mu, \qquad Q \in \mathcal{P},
\end{aligned} \tag{5.1.36}$$

$$\widehat{\mathbb{F}}_Q^{\mu,\mathbf{b}} f(x) \equiv \mathbf{1}_Q(x) \frac{1}{\int_Q b_Q d\mu} \int_Q f d\mu, \qquad Q \in \mathcal{P}. \tag{5.1.37}$$

and

$$\triangle_Q^{\mu,\mathbf{b}} f(x) \equiv \left(\sum_{Q' \in \mathfrak{C}(Q)} \mathbb{E}_{Q'}^{\mu,\mathbf{b}} f(x) \right) - \mathbb{E}_Q^{\mu,\mathbf{b}} f(x) = \sum_{Q' \in \mathfrak{C}(Q)} \mathbf{1}_{Q'}(x) \left(\mathbb{E}_{Q'}^{\mu,\mathbf{b}} f(x) - \mathbb{E}_Q^{\mu,\mathbf{b}} f(x) \right) \quad (5.1.38)$$

$$\square_Q^{\mu,\mathbf{b}} f(x) \equiv \left(\sum_{Q' \in \mathfrak{C}(Q)} \mathbb{F}_{Q'}^{\mu,\mathbf{b}} f(x) \right) - \mathbb{F}_Q^{\mu,\mathbf{b}} f(x) = \sum_{Q' \in \mathfrak{C}(Q)} \mathbf{1}_{Q'}(x) \left(\mathbb{F}_{Q'}^{\mu,\mathbf{b}} f(x) - \mathbb{F}_Q^{\mu,\mathbf{b}} f(x) \right)$$

We also need

$$\bigtriangledown_Q^{\mu} f \equiv \sum_{Q' \in \mathfrak{C}_{brok}(Q)} \left(\frac{1}{|Q'|_\mu} \int_{Q'} |f| \, d\mu \right) \mathbf{1}_{Q'}, \quad (5.1.39)$$

$$\widehat{\bigtriangledown}_Q^{\mu} f \equiv \sum_{Q' \in \mathfrak{C}_{brok}(Q)} \left(\frac{1}{|Q'|_\mu} \int_{Q'} |f| \, d\mu + \frac{1}{|Q|_\mu} \int_{Q} |f| \, d\mu \right) \mathbf{1}_{Q'},$$

$$\sum_{Q \in \mathcal{D}} \left\| \widehat{\bigtriangledown}_Q^{\mu} f \right\|_{L^2(\mu)}^2 \lesssim \|f\|_{L^2(\mu)}^2 \ . \quad (5.1.40)$$

and

$$\square_Q^{\mu,\pi,\mathbf{b}} f = \left[\sum_{Q' \in \mathfrak{C}(Q)} \mathbb{F}_{Q'}^{\mu,\pi,\mathbf{b}} f \right] - \mathbb{F}_Q^{\mu,\mathbf{b}} f = \sum_{Q' \in \mathfrak{C}(Q)} \mathbb{F}_{Q'}^{\mu,b_Q} f - \mathbb{F}_Q^{\mu,b_Q} f, \quad (5.1.41)$$

$$\mathbb{F}_Q^{\mu,\pi,\mathbf{b}} f = \mathbf{1}_Q \frac{b_{\pi Q}}{\int_Q b_{\pi Q} d\mu} \int_Q f d\mu, \quad (5.1.42)$$

$$\square_Q^{\mu,\mathbf{b}} = \square_Q^{\mu,\pi,\mathbf{b}} \square_Q^{\mu,\pi,\mathbf{b}} + \square_{Q,brok}^{\mu,\mathbf{b}} \text{ and } \square_Q^{\mu,\mathbf{b}} = \square_Q^{\mu,\pi,\mathbf{b}} + \square_{Q,brok}^{\mu,\pi,\mathbf{b}} \quad (5.1.43)$$

$$\square_{Q,brok}^{\mu,\mathbf{b}} f = \sum_{Q' \in \mathfrak{C}_{brok}(Q)} \mathbb{F}_{Q'}^{\mu,b_{Q'}} f - \mathbb{F}_{Q'}^{\mu,b_Q} f,$$

$$\left| \square_{Q,brok}^{\mu,\pi,\mathbf{b}} f \right| \lesssim \left| \widehat{\bigtriangledown}_Q^{\mu} f \right|, \quad (5.1.44)$$

with similar equalities and inequalities for $\triangle$ and $\mathbb{E}$. Here $\mathfrak{C}_{brok}(Q)$ denotes the set of broken children, i.e. those $Q' \in \mathfrak{C}(Q)$ for which $b_{Q'} \neq \mathbf{1}_{Q'} b_Q$, and more generally and typically,

$\mathfrak{C}_{brok}(Q) = \mathfrak{C}(Q) \cap \mathcal{A}$ where $\mathcal{A}$ is a collection of stopping cubes that includes the broken children and satisfies a σ-Carleson condition and πQ is the dyadic father of Q.

Define another modified dual martingale difference by

$$\square_I^{\sigma,\flat,\mathbf{b}} f \equiv \square_I^{\sigma,\mathbf{b}} f - \sum_{I' \in \mathfrak{C}_{brok}(I)} \mathbb{F}_{I'}^{\sigma,\mathbf{b}} f = \left(\sum_{I' \in \mathfrak{C}_{nat}(I)} \mathbb{F}_{I'}^{\sigma,\mathbf{b}} f \right) - \mathbb{F}_I^{\sigma,\mathbf{b}} f, \tag{5.1.45}$$

where we have removed the averages over broken children from $\square_I^{\sigma,\mathbf{b}} f$, but left the average over I intact. On any child I' of I, the function $\square_I^{\sigma,\flat,\mathbf{b}} f$ is thus a constant multiple of b_I, and so we have

$$\begin{aligned} \square_I^{\sigma,\flat,\mathbf{b}} f &= b_I \sum_{I' \in \mathfrak{C}(I)} \mathbf{1}_{I'} E_{I'}^{\sigma} \left(\frac{1}{b_I} \square_I^{\sigma,\flat,\mathbf{b}} f \right) = b_I \sum_{I' \in \mathfrak{C}(I)} \mathbf{1}_{I'} E_{I'}^{\sigma} \left(\widehat{\square}_I^{\sigma,\flat,\mathbf{b}} f \right); \qquad (5.1.46) \\ \widehat{\square}_I^{\sigma,\flat,\mathbf{b}} f &\equiv \sum_{I' \in \mathfrak{C}(I)} \mathbf{1}_{I'}\, E_{I'}^{\sigma} \left(\frac{1}{b_I} \square_I^{\sigma,\flat,\mathbf{b}} f \right), \\ &= \sum_{I' \in \mathfrak{C}_{nat}(I)} \mathbf{1}_{I'} \left[\frac{1}{\int_{I'} b_I d\mu} \int_{I'} f d\mu - \frac{1}{\int_I b_I d\mu} \int_I f d\mu \right] - \sum_{I' \in \mathfrak{C}_{brok}(I)} \mathbf{1}_{I'} \left[\frac{1}{\int_I b_I d\mu} \int_I f d\mu \right] \end{aligned}$$

Thus for $I \in \mathcal{C}_A$ we have

$$\square_I^{\sigma,\flat,\mathbf{b}} f = b_A \sum_{I' \in \mathfrak{C}(I)} \mathbf{1}_{I'} E_{I'}^{\sigma} \left(\widehat{\square}_I^{\sigma,\flat,\mathbf{b}} f \right) = b_A \widehat{\square}_I^{\sigma,\flat,\mathbf{b}} f, \tag{5.1.47}$$

where the averages $E_{I'}^{\sigma} \left(\widehat{\square}_I^{\sigma,\flat,\mathbf{b}} f \right)$ satisfy the following telescoping property for all $K \in (\mathcal{C}_A \setminus \{A\}) \cup \left(\bigcup_{A' \in \mathfrak{C}_{\mathcal{A}}(A)} A' \right)$ and $L \in \mathcal{C}_A$ with $K \subset L$:

$$\sum_{I:\ \pi K \subset I \subset L} E_{I_K}^{\sigma} \left(\widehat{\square}_I^{\sigma,\flat,\mathbf{b}} f \right) = \begin{cases} -E_L^{\sigma} \widehat{\mathbb{F}}_L^{\sigma} f & \text{if} \quad K \in \mathfrak{C}_{\mathcal{A}}(A) \\ E_K^{\sigma} \widehat{\mathbb{F}}_K^{\sigma} f - E_L^{\sigma} \widehat{\mathbb{F}}_L^{\sigma} f & \text{if} \quad K \in \mathcal{C}_A \end{cases}, \tag{5.1.48}$$

where $\widehat{\mathbb{F}}_K^{\sigma}$ is defined in (5.1.37) above.

Finally, in analogy with the broken differences $\triangle_{Q,brok}^{\mu,\pi,\mathbf{b}}$ and $\square_{Q,brok}^{\mu,\pi,\mathbf{b}}$ introduced above, we define

$$\triangle_{I,brok}^{\mu,\flat,\mathbf{b}} f \equiv \sum_{I'\in\mathfrak{C}_{brok}(I)} \mathbb{E}_{I'}^{\sigma,\mathbf{b}} f \quad \text{and} \quad \square_{I,brok}^{\mu,\flat,\mathbf{b}} f \equiv \sum_{I'\in\mathfrak{C}_{brok}(I)} \mathbb{F}_{I'}^{\sigma,\mathbf{b}} f \ , \tag{5.1.49}$$

so that

$$\triangle_I^{\mu,\mathbf{b}} = \triangle_I^{\mu,\flat,\mathbf{b}} + \triangle_{I,brok}^{\mu,\flat,\mathbf{b}} \quad \text{and} \quad \square_I^{\mu,\mathbf{b}} = \square_I^{\mu,\flat,\mathbf{b}} + \square_{I,brok}^{\mu,\flat,\mathbf{b}} \ . \tag{5.1.50}$$

These modified differences and the identities (5.1.47) and (5.1.48) play a useful role in the analysis of the nearby and paraproduct forms.

Lemma 5.1.22. *For dyadic cubes R and Q we have*

$$\triangle_R^{\mu,b} \triangle_Q^{\mu,b} = \begin{cases} \triangle_Q^{\mu,b} & \text{if} \quad R = Q \\ 0 & \text{if} \quad R \neq Q \end{cases} .$$

For the reader's convenience we now collect the various martingale and probability estimates that will be used in the proof that follows. First we summarize the martingale identities and estimates that we will use in our proof. Suppose μ is a positive locally finite Borel measure, and that $\mathbf{b}$ is a ∞-weakly μ-controlled accretive family. Then,

Martingale identities: Both of the following identities hold pointwise μ-almost everywhere, as well as in the sense of strong convergence in $L^2(\mu)$:

$$\begin{aligned} f &= \sum_{I\in\mathcal{D}:\ I\subset I_\infty,\ \ell(I)\geq 2^{-N}} \square_I^{\sigma,\mathbf{b}} f + \mathbb{F}_{I_\infty}^{\sigma,\mathbf{b}} f, \\ f &= \sum_{I\in\mathcal{D}:\ I\subset I_\infty,\ \ell(I)\geq 2^{-N}} \triangle_I^{\sigma,\mathbf{b}} f + \mathbb{E}_{I_\infty}^{\sigma,\mathbf{b}} f. \end{aligned}$$

Frame estimates: Both of the following frame estimates hold:

$$\begin{aligned} \|f\|_{L^2(\mu)}^2 &\approx \sum_{Q\in\mathcal{D}} \left\{ \left\|\square_Q^{\mu,\mathbf{b}} f\right\|_{L^2(\mu)}^2 + \left\|\nabla_Q^{\mu,\mathbf{b}} f\right\|_{L^2(\mu)}^2 \right\} \\ &\approx \sum_{Q\in\mathcal{D}} \left\{ \left\|\triangle_Q^{\mu,\mathbf{b}} f\right\|_{L^2(\mu)}^2 + \left\|\nabla_Q^{\mu,\mathbf{b}} f\right\|_{L^2(\mu)}^2 \right\} . \end{aligned} \tag{5.1.51}$$

Weak upper Riesz estimates: Define the pseudoprojections,

$$\begin{aligned} \Psi_{\mathcal{B}}^{\mu,\mathbf{b}} f &\equiv \sum_{I\in\mathcal{B}} \square_I^{\mu,\mathbf{b}} f, \\ \left(\Psi_{\mathcal{B}}^{\mu,\mathbf{b}}\right)^* f &\equiv \sum_{I\in\mathcal{B}} \left(\square_I^{\mu,\mathbf{b}}\right)^* f = \sum_{I\in\mathcal{B}} \triangle_I^{\mu,\mathbf{b}} f. \end{aligned} \tag{5.1.52}$$

We have the 'upper Riesz' inequalities for pseudoprojections $\Psi_{\mathcal{B}}^{\mu,\mathbf{b}}$ and $\left(\Psi_{\mathcal{B}}^{\mu,\mathbf{b}}\right)^*$:

$$\begin{aligned} \left\|\Psi_{\mathcal{B}}^{\mu,\mathbf{b}} f\right\|_{L^2(\mu)}^2 &\leq C\sum_{I\in\mathcal{B}} \left\|\square_I^{\mu,\mathbf{b}} f\right\|_{L^2(\mu)}^2 + \sum_{I\in\mathcal{B}} \left\|\widehat{\nabla}_I^{\mu,\mathbf{b}} f\right\|_{L^2(\mu)}^2, \\ \left\|\left(\Psi_{\mathcal{B}}^{\mu,\mathbf{b}}\right)^* f\right\|_{L^2(\mu)}^2 &\leq C\sum_{I\in\mathcal{B}} \left\|\triangle_I^{\mu,\mathbf{b}} f\right\|_{L^2(\mu)}^2 + \sum_{I\in\mathcal{B}} \left\|\left(\widehat{\nabla}_I^{\mu,\mathbf{b}}\right)^* f\right\|_{L^2(\mu)}^2, \end{aligned} \tag{5.1.53}$$

for all $f \in L^2(\mu)$ and all subsets $\mathcal{B}$ of the grid $\mathcal{D}$. Here the positive constant C and depends only on the accretivity constants, and is *independent* of the subset $\mathcal{B}$ and the testing family $\mathbf{b}$. The Haar martingale differences $\triangle_Q^{\mu,\mathbf{b}}$ are independent of both the testing families and the grid, while the Carleson averaging operators $\triangledown_Q^\mu$ depend on the grid only through the choice of broken children of Q.

5.1.13 Monotonicity Lemma

As in virtually all proofs of a two weight $T1$ theorem (see e.g. [27], [34] , [64] and/or [63]), the key to starting an estimate for any of the forms we consider below, is the Monotonicity Lemma and the Energy Lemma, to which we now turn. In dimension $n = 1$ ([34], [27]) the Haar functions have opposite sign on their children, and this was exploited in a simple but powerful monotonicity argument. In higher dimensions, this simple argument no longer holds and that Monotonicity Lemma is replaced with the Lacey-Wick formulation of the Monotonicity Lemma (see [37], and also [63]) involving the smaller Poisson operator. As the martingale differences with test functions b_Q here are no longer of one sign on children, we will adapt the Lacey-Wick formulation of the Monotonicity Lemma to the operator T^α and the dual martingale differences $\left\{\square_J^{\omega,\mathbf{b}^*}\right\}_{J\in\mathcal{G}}$, bearing in mind that the operators $\square_J^{\omega,\mathbf{b}^*}$ are no longer projections, which results in only a one-sided estimate with additional terms on the right hand side. It is here that we need the crucial property that the Range of $\square_J^{\omega,\mathbf{b}^*}$ is orthogonal to constants, $\int\left(\square_J^{\omega,\mathbf{b}^*}\Psi\right)d\sigma=\int\left(\triangle_J^{\sigma,\mathbf{b}^*}1\right)\Psi d\omega=\int(0)\,\Psi d\omega=0$.

We will also need the smaller Poisson integral used in the Lacey-Wick formulation of the Monotonicity Lemma,

$$\mathrm{P}_{1+\delta}^{\alpha}(J,\mu)\equiv\int\frac{|J|^{\frac{1+\delta}{n}}}{\left(|J|+|y-c_J|\right)^{n+1+\delta-\alpha}}d\mu(y),$$

which is discussed in more detail below.

Lemma 5.1.23 (Monotonicity Lemma)**.** *Suppose that I and J are cubes in $\mathbb{R}^n$ such that*

$J \subset \gamma J \subset I$ for some $\gamma > 1$, and that μ is a signed measure on $\mathbb{R}^n$ supported outside I. Let $0 < \delta < 1$ and let $\Psi \in L^2(\omega)$. Finally suppose that T^α is a standard fractional singular integral on $\mathbb{R}^n$ with $0 \leq \alpha < 1$, and suppose that $\mathbf{b}^*$ is an ∞-weakly μ-controlled accretive family on $\mathbb{R}^n$. Then we have the estimate

$$\left|\left\langle T^\alpha \mu, \Box_J^{\omega,\mathbf{b}^*}\Psi\right\rangle_\omega\right| \lesssim C_{\mathbf{b}^*}C_{CZ}\ \Phi^\alpha(J,|\mu|)\ \left\|\Box_J^{\omega,\mathbf{b}^*}\Psi\right\|_{L^2(\omega)}^{\bigstar}, \tag{5.1.54}$$

where

$$\begin{aligned}
\Phi^\alpha(J,|\mu|) &\equiv \frac{\mathrm{P}^\alpha(J,|\mu|)}{|J|}\left\|\triangle_J^{\omega,\mathbf{b}^*}x\right\|_{L^2(\omega)}^{\spadesuit} + \frac{\mathrm{P}_{1+\delta}^\alpha(J,|\mu|)}{|J|}\|x-m_J\|_{L^2(\mathbf{1}_J\omega)}, \\
\left\|\triangle_J^{\omega,\mathbf{b}^*}x\right\|_{L^2(\omega)}^{\spadesuit 2} &\equiv \left\|\triangle_J^{\omega,\mathbf{b}^*}x\right\|_{L^2(\omega)}^2 + \inf_{z\in\mathbb{R}}\sum_{J'\in\mathfrak{C}_{brok}(J)}|J'|_\omega\left(E_{J'}^\omega|x-z|\right)^2, \\
\left\|\Box_J^{\omega,\mathbf{b}^*}\Psi\right\|_{L^2(\mu)}^{\bigstar 2} &\equiv \left\|\Box_J^{\omega,\mathbf{b}^*}\Psi\right\|_{L^2(\mu)}^2 + \sum_{J'\in\mathfrak{C}_{brok}(J)}|J'|_\omega\left[E_{J'}^\omega|\Psi|\right]^2.
\end{aligned}$$

All of the implied constants above depend only on $\gamma > 1$, $0 < \delta < 1$ and $0 < \alpha < 1$.

Using $\triangledown_J^\omega h = \sum_{J'\in\mathfrak{C}_{brok}(J)}\left(E_{J'}^\omega|h|\right)\mathbf{1}_{J'}$ defined in (5.1.39), we can rewrite the expressions $\left\|\triangle_J^{\omega,\mathbf{b}^*}x\right\|_{L^2(\omega)}^{\spadesuit 2}$ and $\left\|\Box_J^{\omega,\mathbf{b}^*}\Psi\right\|_{L^2(\mu)}^{\bigstar 2}$ as

$$\begin{aligned}
\left\|\triangle_J^{\omega,\mathbf{b}^*}x\right\|_{L^2(\omega)}^{\spadesuit 2} &\equiv \left\|\triangle_J^{\omega,\mathbf{b}^*}x\right\|_{L^2(\omega)}^2 + \inf_{z\in\mathbb{R}}\left\|\triangledown_J^\omega(x-z)\right\|_{L^2(\omega)}^2, \\
\left\|\Box_J^{\omega,\mathbf{b}^*}\Psi\right\|_{L^2(\mu)}^{\bigstar 2} &\equiv \left\|\Box_J^{\omega,\mathbf{b}^*}\Psi\right\|_{L^2(\mu)}^2 + \left\|\triangledown_J^\omega\Psi\right\|_{L^2(\omega)}^2.
\end{aligned}$$

Proof. Using $\square_J^{\omega,\mathbf{b}^*} = \square_J^{\omega,\pi,\mathbf{b}^*}\square_J^{\omega,\pi,\mathbf{b}^*} + \square_{J,brok}^{\omega,\pi,\mathbf{b}^*}$, we write

$$\begin{aligned}
\left|\left\langle T^\alpha\mu, \square_J^{\omega,\mathbf{b}^*}\Psi\right\rangle_\omega\right| &= \left|\left\langle T^\alpha\mu, \left(\square_J^{\omega,\pi,\mathbf{b}^*}\square_J^{\omega,\pi,\mathbf{b}^*} + \square_{J,brok}^{\omega,\pi,\mathbf{b}^*}\right)\Psi\right\rangle_\omega\right| \\
&\leq \left|\left\langle T^\alpha\mu, \square_J^{\omega,\pi,\mathbf{b}^*}\square_J^{\omega,\pi,\mathbf{b}^*}\Psi\right\rangle_\omega\right| + \left|\left\langle T^\alpha\mu, \square_{J,brok}^{\omega,\pi,\mathbf{b}^*}\Psi\right\rangle_\omega\right| \\
&\equiv \mathrm{I} + \mathrm{II}.
\end{aligned}$$

Since $\left\langle 1, \square_J^{\omega,\pi,\mathbf{b}^*} h\right\rangle_\omega = 0$, we use $m_J = \frac{1}{|J|_\omega}\int_J x d\omega(x)$ to obtain

$$\begin{aligned}
T^\alpha\mu(x) - T^\alpha\mu(m_J) &= \int \left[(K^\alpha)(x,y) - (K^\alpha)(m_J,y)\right] d\mu(y) \\
&= \int \left[\nabla (K^\alpha)^T(\theta(x,m_J), y)\cdot(x-m_J)\right] d\mu(y)
\end{aligned}$$

for some $\theta(x,m_J) \in J$ to obtain

$$\begin{aligned}
\mathrm{I} &= \left|\int \left[T^\alpha\mu(x) - T^\alpha\mu(m_J)\right]\ \square_J^{\omega,\pi,\mathbf{b}^*}\square_J^{\omega,\pi,\mathbf{b}^*}\Psi(x)\,d\omega(x)\right| \\
&= \left|\int\left\{\int \nabla(K^\alpha)^T(\theta(x,m_J))\,d\mu(y)\right\}\cdot(x-m_J)\ \square_J^{\omega,\pi,\mathbf{b}^*}\square_J^{\omega,\pi,\mathbf{b}^*}\Psi(x)\,d\omega(x)\right| \\
&\leq \left|\int\left\{\int \nabla(K^\alpha)^T(m_J,y)\,d\mu(y)\right\}\cdot\ (x-m_J)\ \square_J^{\omega,\pi,\mathbf{b}^*}\square_J^{\omega,\pi,\mathbf{b}^*}\Psi(x)\,d\omega(x)\right| \\
&\quad +\left|\int\left\{\int\left[\nabla(K^\alpha)^T(\theta(x,m_J),y) - \nabla(K^\alpha)^T(m_J,y)\right]d\mu(y)\right\}\right. \\
&\quad \left.\cdot(x-m_J)\ \square_J^{\omega,\pi,\mathbf{b}^*}\square_J^{\omega,\pi,\mathbf{b}^*}\Psi(x)\,d\omega(x)\right| \\
&\equiv \mathrm{I}_1 + \mathrm{I}_2
\end{aligned}$$

Now we estimate

$$\begin{aligned}
\mathrm{I}_1 &= \left|\left[\int \nabla(K^\alpha)(m_J,y)\,d\mu(y)\right]^T \cdot \int (x-m_J)\ \square_J^{\omega,\pi,\mathbf{b}^*}\square_J^{\omega,\pi,\mathbf{b}^*}\Psi(x)\,d\omega(x)\right| \\
&\le n\int\int |\nabla(K^\alpha)(m_J,y)|\,d|\mu|(y)\ \left|\triangle_J^{\omega,\pi,\mathbf{b}^*}x\right|\ \left|\square_J^{\omega,\pi,\mathbf{b}^*}\Psi(x)\right|\ d\omega(x) \\
&\lesssim n\cdot C_{CZ}\frac{\mathrm{P}^\alpha(J,|\mu|)}{|J|^{\frac{1}{n}}}\ \left\|\triangle_J^{\omega,\pi,\mathbf{b}^*}x\right\|_{L^2(\omega)}\left\|\square_J^{\omega,\pi,\mathbf{b}^*}\Psi\right\|_{L^2(\omega)}
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{I}_2 &\lesssim C_{CZ}\frac{\mathrm{P}^\alpha_{1+\delta}(J,|\mu|)}{|J|^{\frac{1}{n}}}\int |x-m_J|\left|\square_J^{\omega,\pi,\mathbf{b}^*}\square_J^{\omega,\pi,\mathbf{b}^*}\Psi(x)\right|d\omega(x) \\
&\lesssim C_{CZ}\frac{\mathrm{P}^\alpha_{1+\delta}(J,|\mu|)}{|J|}\sqrt{\int_J |x-m_J|^2\,d\omega(x)}\left\|\square_J^{\omega,\pi,\mathbf{b}^*}\square_J^{\omega,\pi,\mathbf{b}^*}\Psi\right\|_{L^2(\omega)} \\
&\lesssim C_{CZ}\frac{\mathrm{P}^\alpha_{1+\delta}(J,|\mu|)}{|J|}\|x-m_J\|_{L^2(\mathbf{1}_J\omega)}\left\|\square_J^{\omega,\pi,\mathbf{b}^*}\Psi\right\|_{L^2(\omega)}\ .
\end{aligned}$$

For term II we fix $z\in\overline{J}$ for the moment. Then since

$$\left\langle 1,\square_{J,brok}^{\omega,\mathbf{b}^*}h\right\rangle_\omega=\left\langle 1,\square_J^{\omega,\mathbf{b}^*}h-\square_J^{\omega,\pi,\mathbf{b}^*}h\right\rangle_\omega=0$$

we have

$$\begin{aligned}
\mathrm{II} &= \left|\left\langle T^\alpha\mu,\square_{J,brok}^{\omega,\mathbf{b}^*}\Psi\right\rangle_\omega\right| \\
&= \left|\int\left\{\int \nabla(K^\alpha)^T(\theta(x,z),y)\,d\mu(y)\right\}\cdot(x-z)\ \square_{J,brok}^{\omega,\pi,\mathbf{b}^*}\Psi(x)\,d\omega(x)\right| \\
&\le C_{CZ}\frac{\mathrm{P}^\alpha(J,|\mu|)}{|J|^{\frac{1}{n}}}\int |x-z|\cdot\left|\square_{J,brok}^{\omega,\pi,\mathbf{b}^*}\Psi(x)\right|d\omega(x) \\
&\le C_{CZ}\frac{\mathrm{P}^\alpha(J,|\mu|)}{|J|^{\frac{1}{n}}}\sum_{J'\in\mathfrak{C}_{brok(J)}}\int_{J'}|x-z|\cdot\mathbf{1}_{J'}E^\omega_{J'}|\Psi|\,d\omega(x)
\end{aligned}$$

having used the reverse Hölder control of children (5.1.23) to obtain

$$\left|\square_{J,brok}^{\omega,\mathbf{b}^*}\Psi\right| = \left|\sum_{J'\in\mathfrak{C}_{brok}(JQ)}\left(\mathbb{F}_{J'}^{\omega,b_{J'}} - \mathbb{F}_{J'}^{\omega,b_J}\right)\Psi\right| \lesssim \sum_{J'\in\mathfrak{C}_{brok}(J)} \mathbf{1}_{J'} E_{J'}^{\omega}|\Psi|,$$

and since

$$\int_{J'} |x-z|\cdot\mathbf{1}_{J'} E_{J'}^{\omega}|\Psi|\,d\omega(x) = \int_{J'} \frac{|x-z|}{\sqrt{|J'|_{\omega}}} \frac{\mathbf{1}_{J'}\int_{J'}|\Psi|d\omega(x)}{\sqrt{|J'|_{\omega}}} d\omega(x)$$

we get

$$\mathrm{II} \leq C_{CZ}\frac{\mathrm{P}^{\alpha}(J,|\mu|)}{|J|^{\frac{1}{n}}}\sqrt{\sum_{J'\in\mathfrak{C}_{brok}(J)}|J'|_{\omega}\left(E_{J'}^{\omega}|x-z|\right)^2}\sqrt{\sum_{J'\in\mathfrak{C}_{brok}(J)}|J'|_{\omega}\left[E_{J'}^{\omega}|\Psi|\right]^2}.$$

Combining the estimates for terms I and II, we obtain

$$\begin{aligned}
&\left|\left\langle T^{\alpha}\mu,\square_J^{\omega,\mathbf{b}^*}\Psi\right\rangle_{\omega}\right| \\
\lesssim\;& C_{CZ}\frac{\mathrm{P}^{\alpha}(J,|\mu|)}{|J|^{\frac{1}{n}}}\left\|\triangle_J^{\omega,\pi,\mathbf{b}^*}x\right\|_{L^2(\omega)}\left\|\square_J^{\omega,\pi,\mathbf{b}^*}\Psi\right\|_{L^2(\omega)} \\
+\;& C_{CZ}\frac{\mathrm{P}^{\alpha}_{1+\delta}(J,|\mu|)}{|J|^{\frac{1}{n}}}\left\|x-m_J\right\|_{L^2(\mathbf{1}_J\omega)}\left\|\square_J^{\omega,\pi,\mathbf{b}^*}\Psi\right\|_{L^2(\omega)} \\
+\;& C_{CZ}\frac{\mathrm{P}^{\alpha}(J,|\mu|)}{|J|^{\frac{1}{n}}}\inf_{z\in\overline{J}}\sqrt{\sum_{J'\in\mathfrak{C}_{brok}(J)}|J'|_{\omega}\left(E_{J'}^{\omega}|x-z_2|\right)^2}\sqrt{\sum_{J'\in\mathfrak{C}_{brok}(J)}|J'|_{\omega}\left[E_{J'}^{\omega}|\Psi|\right]^2}
\end{aligned}$$

and then noting that the infimum over $z\in\mathbb{R}$ is achieved for $z\in\overline{J}$, and using the triangle inequality on $\square_J^{\omega,\pi,\mathbf{b}^*} = \square_J^{\omega,\mathbf{b}^*} - \square_{J,brok}^{\omega,\pi,\mathbf{b}^*}$ we get (5.1.54). $\square$

The right hand side of (5.1.54) in the Monotonicity Lemma will be typically estimated in what follows using the frame inequalities for any cube K,

$$\begin{aligned} \sum_{J\subset K}\left\Vert \square_J^{\omega,\mathbf{b}^*}\Psi\right\Vert_{L^2(\omega)}^{\bigstar 2} &\lesssim \left\Vert \Psi\right\Vert_{L^2(\omega)}^2 \ , \\ \sum_{J\subset K}\left\Vert \triangle_J^{\omega,\mathbf{b}^*}x\right\Vert_{L^2(\omega)}^{\spadesuit 2} &\lesssim \int_K \left\vert x-m_K\right\vert^2 d\omega\left(x\right) \ , \end{aligned}$$

together with these inequalities for the square function expressions. To see the last one, write $x=(x_1,\dots,x_n)$ and note that for $J\subset K$,

$$\begin{aligned} \left\Vert \triangle_J^{\omega,\mathbf{b}^*}x\right\Vert_{L^2(\omega)}^{\spadesuit 2}=\int_J \left\vert \triangle_J^{\omega,\mathbf{b}^*}x\right\vert^2 d\omega &= \int_J \sum_{i=1}^n \left\vert \triangle_J^{\omega,\mathbf{b}^*}x_i\right\vert^2 d\omega \\ &\leq \sum_{i=1}^n \int_K \left\vert x_i-m_{Ki}\right\vert^2 d\omega = \left\vert\left\vert x-m_k\right\vert\right\vert_{L^2(\mathbf{1}_K\omega)}^2 \end{aligned}$$

using the one-variable result from [69].

Lemma 5.1.24. *For any cube K we have*

$$\begin{aligned} \sum_{J\subset K}\sum_{J'\in\mathfrak{C}_{brok}(J)}\left\vert J'\right\vert_\omega \left[E_{J'}^\omega\left\vert\Psi\right\vert(x)\right]^2 &\lesssim \int_K \left\vert\Psi\left(x\right)\right\vert^2 d\omega\left(x\right), \qquad (5.1.55) \\ \textit{and} \ \sum_{J\subset K}\inf_{z\in\mathbb{R}}\sum_{J'\in\mathfrak{C}_{brok}(J)}\left\vert J'\right\vert_\omega \left(E_{J'}^\omega\left\vert x-z\right\vert\right)^2 &\lesssim \int_K \left\vert x-m_K\right\vert^2 d\omega\left(x\right). \end{aligned}$$

Proof. The first inequality in (5.1.55) is just the Carleson embedding theorem since the cubes $\left\{J'\in\mathfrak{C}_{brok}\left(J\right):J\subset K\right\}$ satisfy an ω-Carleson condition, and the second inequality in (5.1.55) follows by choosing $z=m_K$ to obtain

$$\inf_{z\in\mathbb{R}}\sum_{J'\in\mathfrak{C}_{brok}(J)}\left\vert J'\right\vert_\omega\left(E_{J'}^\omega\left\vert x-z\right\vert\right)^2\leq\sum_{J'\in\mathfrak{C}_{brok}(J)}\left\vert J'\right\vert_\omega\left(E_{J'}^\omega\left\vert x-m_K\right\vert\right)^2,$$

and then applying the Carleson embedding theorem again:

$$\sum_{J\subset K}\sum_{J'\in\mathfrak{C}_{brok}(J)}\left|J'\right|_{\omega}\left(E_{J'}^{\omega}\left|x-m_K\right|\right)^2\lesssim\int_K\left|x-m_K\right|^2 d\omega\left(x\right).$$

$\square$

5.1.13.1 The smaller Poisson integral

The expressions

$$\inf_{z\in\mathbb{R}}\frac{\mathrm{P}_{1+\delta}^{\alpha}\left(J,\left|\mu\right|\right)}{\left|J\right|}\left\|x-z\right\|_{L^2(\mathbf{1}_J\omega)}\left\|\square_J^{\omega,\mathbf{b}^*}\Psi\right\|_{L^2(\omega)}^{\bigstar}$$

are typically easier to sum due to the small Poisson operator $\mathrm{P}_{1+\delta}^{\alpha}\left(J,\left|\mu\right|\right)$. To illlustrate, we show here one way in which we can exploit the additional decay in the Poisson integral $\mathrm{P}_{1+\delta}^{\alpha}$. Suppose that J is good in I with $\ell\left(J\right)=2^{-s}\ell\left(I\right)$ (see Definition 5.2.5 below for 'goodness'). We then compute

$$\begin{aligned}\frac{\mathrm{P}_{1+\delta}^{\alpha}\left(J,\mathbf{1}_{A\setminus I}\sigma\right)}{\left|J\right|^{\frac{1}{n}}}&\approx\int_{A\setminus I}\frac{\left|J\right|^{\frac{\delta}{n}}}{\left|y-c_J\right|^{n+1+\delta-\alpha}}d\sigma\left(y\right)\\&\leq\int_{A\setminus I}\left(\frac{\left|J\right|^{\frac{1}{n}}}{\mathrm{dist}\left(c_J,I^c\right)}\right)^{\delta}\frac{1}{\left|y-c_J\right|^{n+1-\alpha}}d\sigma\left(y\right)\\&\lesssim\left(\frac{\left|J\right|^{\frac{1}{n}}}{\mathrm{dist}\left(c_J,I^c\right)}\right)^{\delta}\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{A\setminus I}\sigma\right)}{\left|J\right|^{\frac{1}{n}}},\end{aligned}$$

and use the goodness inequality,

$$\mathrm{dist}\left(c_J,I^c\right)\geq 2\ell\left(I\right)^{1-\varepsilon}\ell\left(J\right)^{\varepsilon}\geq 2\cdot 2^{s(1-\varepsilon)}\ell\left(J\right),$$

to conclude that

$$\left(\frac{\mathrm{P}^{\alpha}_{1+\delta}\left(J,\mathbf{1}_{A\setminus I}\sigma\right)}{|J|^{\frac{1}{n}}}\right)\lesssim 2^{-s\delta(1-\varepsilon)}\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{A\setminus I}\sigma\right)}{|J|^{\frac{1}{n}}} \tag{5.1.56}$$

Now we can estimate

$$\begin{aligned}
&\sum_{J\subset K:\ J\text{ good in }K}\inf_{z\in\mathbb{R}}\frac{\mathrm{P}^{\alpha}_{1+\delta}\left(J,\mathbf{1}_{K^c}|\mu|\right)}{|J|^{\frac{1}{n}}}\|x-z\|_{L^2(\mathbf{1}_J\omega)}\left\|\square^{\omega,\mathbf{b}^*}_J\Psi\right\|^{\bigstar}_{L^2(\omega)}\\
\leq&\sqrt{\sum_{\substack{J\subset K\\ J\text{ good in }K}}\left(\frac{\mathrm{P}^{\alpha}_{1+\delta}\left(J,\mathbf{1}_{K^c}|\mu|\right)}{|J|^{\frac{1}{n}}}\right)^2\inf_{z\in\mathbb{R}}\|x-z\|^2_{L^2(\mathbf{1}_J\omega)}}\sqrt{\sum_{\substack{J\subset K\\ J\text{ good in }K}}\left\|\square^{\omega,\mathbf{b}^*}_J\Psi\right\|^{\bigstar 2}_{L^2(\omega)}}
\end{aligned}$$

where

$$\begin{aligned}
&\sum_{J\subset K:\ J\text{ good in }K}\left(\frac{\mathrm{P}^{\alpha}_{1+\delta}\left(J,\mathbf{1}_{K^c}|\mu|\right)}{|J|}\right)^2\inf_{z\in\mathbb{R}}\|x-z\|^2_{L^2(\mathbf{1}_J\omega)}\\
=&\sum_{s=0}^{\infty}\sum_{\substack{J\subset K:\ J\text{ good in }K\\ \ell(J)=2^{-s}\ell(I)}}\left(\frac{\mathrm{P}^{\alpha}_{1+\delta}\left(J,\mathbf{1}_{K^c}|\mu|\right)}{|J|}\right)^2\inf_{z\in\mathbb{R}}\|x-z\|^2_{L^2(\mathbf{1}_J\omega)}\\
\leq&\sum_{s=0}^{\infty}\sum_{\substack{J\subset K:\ J\text{ good in }K\\ \ell(J)=2^{-s}\ell(I)}}\left(2^{-s\delta(1-\varepsilon)}\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{K^c}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2\inf_{z\in\mathbb{R}}\|x-z\|^2_{L^2(\mathbf{1}_J\omega)}\\
\leq&\left(\frac{\mathrm{P}^{\alpha}\left(K,\mathbf{1}_{K^c}\sigma\right)}{|K|^{\frac{1}{n}}}\right)^2\sum_{s=0}^{\infty}\sum_{\substack{J\subset K:\ J\text{ good in }K\\ \ell(J)=2^{-s}\ell(I)}}2^{-2s\delta(1-\varepsilon)}\inf_{z\in\mathbb{R}}\|x-z\|^2_{L^2(\mathbf{1}_K\omega)}\\
\lesssim&\left(\frac{\mathrm{P}^{\alpha}\left(K,\mathbf{1}_{K^c}\sigma\right)}{|K|^{\frac{1}{n}}}\right)^2\inf_{z\in\mathbb{R}}\|x-z\|^2_{L^2(\mathbf{1}_K\omega)}\ ,
\end{aligned}$$

and where we have used (5.5.10), which gives in particular

$$\mathrm{P}^{\alpha}(J,\mu\mathbf{1}_{I^c})\lesssim\left(\frac{\ell(J)}{\ell(I)}\right)^{1-\varepsilon(n+1-\alpha)}\mathrm{P}^{\alpha}(I,\mu\mathbf{1}_{I^c}).$$

for $J \subset I$ and $d(J,\partial I) > 2\ell(J)^{\varepsilon}\ell(I)^{1-\varepsilon}$. We will use such arguments repeatedly in the sequel.

Armed with the Monotonicity Lemma and the lower frame inequality

$$\sum_{I\in\mathcal{D}} \left\|\square_I^{\omega,\mathbf{b}^*} g\right\|_{L^2(\mu)}^{\bigstar 2} \lesssim \|g\|_{L^2(\omega)}^2 \ ,$$

we can obtain a $\mathbf{b}^*$-analogue of the Energy Lemma as in [64] and/or [63].

5.1.13.2 The Energy Lemma

Suppose now we are given a subset $\mathcal{H}$ of the dyadic grid $\mathcal{G}$. Due to the failure of both martingale and dual martingale pseudoprojections $\mathsf{Q}_{\mathcal{H}}^{\omega,\mathbf{b}^*} x$ and $\mathsf{P}_{\mathcal{H}}^{\omega,\mathbf{b}^*} g$ (see below for definition) to satisfy inequalities of the form $\left\|\mathsf{P}_{\mathcal{H}}^{\omega,\mathbf{b}^*} g\right\|_{L^2(\omega)} \lesssim \|g\|_{L^2(\omega)}$ when the children 'break', it is convenient to define the 'square function norms' $\left\|\mathsf{Q}_{\mathcal{H}}^{\omega,\mathbf{b}^*} x\right\|_{L^2(\omega)}^{\spadesuit}$ and $\left\|\mathsf{P}_{\mathcal{H}}^{\omega,\mathbf{b}^*} g\right\|_{L^2(\omega)}^{\bigstar}$ of the pseudoprojections

$$\mathsf{Q}_{\mathcal{H}}^{\omega,\mathbf{b}^*} x = \sum_{J\in\mathcal{H}} \triangle_J^{\omega,\mathbf{b}^*} x \text{ and } \mathsf{P}_{\mathcal{H}}^{\omega,\mathbf{b}^*} g = \sum_{J\in\mathcal{H}} \square_J^{\omega,\mathbf{b}^*} g \ ,$$

by

$$\begin{aligned}
\left\|\mathsf{Q}_{\mathcal{H}}^{\omega,\mathbf{b}^*} x\right\|_{L^2(\omega)}^{\spadesuit 2} &\equiv \sum_{J\in\mathcal{H}} \left\|\triangle_J^{\omega,\mathbf{b}^*} x\right\|_{L^2(\omega)}^{\spadesuit 2} \\
&= \sum_{J\in\mathcal{H}} \left\|\triangle_J^{\omega,\mathbf{b}^*} x\right\|_{L^2(\omega)}^2 + \sum_{J\in\mathcal{H}} \inf_{z\in\mathbb{R}} \sum_{J'\in\mathfrak{C}_{brok}(J)} \left|J'\right|_\omega \left(E_{J'}^\omega |x-z|\right)^2 \\
\left\|\mathsf{P}_{\mathcal{H}}^{\omega,\mathbf{b}^*} g\right\|_{L^2(\omega)}^{\bigstar 2} &\equiv \sum_{J\in\mathcal{H}} \left\|\square_J^{\omega,\mathbf{b}^*} g\right\|_{L^2(\omega)}^{\bigstar 2} \\
&= \sum_{J\in\mathcal{H}} \left\|\square_J^{\omega,\mathbf{b}^*} g\right\|_{L^2(\omega)}^2 + \sum_{J\in\mathcal{H}} \sum_{J'\in\mathfrak{C}_{brok}(J)} \left|J'\right|_\omega \left[E_{J'}^\omega |g|\right]^2
\end{aligned}$$

for any subset $\mathcal{H} \subset \mathcal{G}$. The average $E_J^{\omega} |x-z|$ above is taken with respect to the variable x, i.e. $E_J^{\omega} |x-z| = \frac{1}{|J|_{\omega}} \int |x-z| \, d\omega(x)$, and it is important that the infimum $\inf_{z\in\mathbb{R}}$ is taken *inside* the sum $\sum_{J\in\mathcal{H}}$.

Note that we are defining here square function expressions related to pseudoprojections, which depend not only on the functions $\mathsf{Q}_{\mathcal{H}}^{\omega,\mathbf{b}^*} x$ and $\mathsf{P}_{\mathcal{H}}^{\omega,\mathbf{b}^*} g$, but also on the particular representations $\sum_{J\in\mathcal{H}} \triangle_J^{\omega,\mathbf{b}^*} x$ and $\sum_{J\in\mathcal{H}} \square_J^{\omega,\mathbf{b}^*} g$. This slight abuse of notation should not cause confusion, and it provides a useful way of bookkeeping the sums of squares of norms of martingale and dual martingale differences $\left\| \triangle_J^{\omega,\mathbf{b}^*} x \right\|_{L^2(\omega)}^2$ and $\left\| \square_J^{\omega,\mathbf{b}^*} g \right\|_{L^2(\omega)}^2$, along with the norms of the associated Carleson square function expressions

$$
\begin{aligned}
\sum_{J\in\mathcal{H}} \inf_{z\in\mathbb{R}} \left\| \nabla_J^{\omega} (x-z) \right\|_{L^2(\omega)}^2 &= \sum_{J\in\mathcal{H}} \inf_{z\in\mathbb{R}} \sum_{J'\in\mathfrak{C}_{brok}(J)} |J'|_{\omega} \left(E_{J'}^{\omega} |x-z| \right)^2 \\
\sum_{J\in\mathcal{H}} \left\| \nabla_J^{\omega} \Psi \right\|_{L^2(\omega)}^2 &= \sum_{J\in\mathcal{H}} \sum_{J'\in\mathfrak{C}_{brok}(J)} |J'|_{\omega} \left[E_{J'}^{\omega} |\Psi| \right]^2 .
\end{aligned}
$$

Note also that the upper weak Riesz inequalities yield the inequalities

$$
\begin{aligned}
\left\| \mathsf{Q}_{\mathcal{H}}^{\omega,\mathbf{b}^*} x \right\|_{L^2(\omega)}^2 &\lesssim \sum_{J\in\mathcal{H}} \left\| \triangle_J^{\omega,\mathbf{b}^*} x \right\|_{L^2(\omega)}^2 \leq \left\| \mathsf{Q}_{\mathcal{H}}^{\omega,\mathbf{b}^*} x \right\|_{L^2(\omega)}^{\spadesuit 2} \\
\left\| \mathsf{P}_{\mathcal{H}}^{\omega,\mathbf{b}^*} g \right\|_{L^2(\omega)}^2 &\lesssim \sum_{J\in\mathcal{H}} \left\| \square_J^{\omega,\mathbf{b}^*} g \right\|_{L^2(\omega)}^2 \leq \left\| \mathsf{P}_{\mathcal{H}}^{\omega,\mathbf{b}^*} g \right\|_{L^2(\omega)}^{\bigstar 2}
\end{aligned}
$$

We will exclusively use $\left\| \mathsf{Q}_{\mathcal{H}}^{\omega,\mathbf{b}^*} x \right\|_{L^2(\omega)}^{\spadesuit 2}$ in connection with energy terms, and use $\left\| \mathsf{P}_{\mathcal{H}}^{\sigma,\mathbf{b}^*} f \right\|_{L^2(\sigma)}^{\bigstar 2}$ and $\left\| \mathsf{P}_{\mathcal{H}}^{\omega,\mathbf{b}^*} g \right\|_{L^2(\omega)}^{\bigstar 2}$ in connection with functions $f \in L^2(\sigma)$ and $g \in L^2(\omega)$. Finally, note that $\mathsf{Q}_{\mathcal{H}}^{\omega,\mathbf{b}^*} x = \mathsf{Q}_{\mathcal{H}}^{\omega,\mathbf{b}^*} (x-m)$ for any constant m.

Recall that

$$\Phi^{\alpha}(J,\nu) \equiv \frac{\mathrm{P}^{\alpha}(J,\nu)}{|J|^{\frac{1}{n}}} \left\| \triangle_{J}^{\omega,\mathbf{b}^{*}} x \right\|_{L^{2}(\omega)}^{\spadesuit} + \frac{\mathrm{P}_{1+\delta}^{\alpha}(J,\nu)}{|J|^{\frac{1}{n}}} \|x - m_{J}\|_{L^{2}(\mathbf{1}_{J}\omega)} .$$

Lemma 5.1.25 (Energy Lemma). *Let J be a cube in $\mathcal{G}$. Let Ψ_J be an $L^2(\omega)$ function supported in J with vanishing ω-mean, and let $\mathcal{H} \subset \mathcal{G}$ be such that $J' \subset J$ for every $J' \in \mathcal{H}$. Let ν be a positive measure supported in $\mathbb{R} \backslash \gamma J$ with $\gamma > 1$, and for each $J' \in \mathcal{H}$, let $d\nu_{J'} = \varphi_{J'} d\nu$ with $|\varphi_{J'}| \leq 1$. Suppose that $\mathbf{b}^*$ is an ∞-weakly μ-controlled accretive family on $\mathbb{R}^n$. Let T^α be a standard α-fractional singular integral operator with $0 \leq \alpha < 1$. Then we have*

$$\begin{aligned}
&\left| \sum_{J' \in \mathcal{H}} \left\langle T^{\alpha}(\nu_{J'}), \square_{J'}^{\omega,\mathbf{b}^{*}} \Psi_{J} \right\rangle_{\omega} \right| \lesssim C_{\gamma} \sum_{J' \in \mathcal{H}} \Phi^{\alpha}(J',\nu) \left\| \square_{J'}^{\omega,\mathbf{b}^{*}} \Psi_{J} \right\|_{L^{2}(\mu)}^{\bigstar} \\
\lesssim\ & C_{\gamma} \sqrt{\sum_{J' \in \mathcal{H}} \Phi^{\alpha}(J',\nu)^{2}} \sqrt{\sum_{J' \in \mathcal{H}} \left\| \square_{J'}^{\omega,\mathbf{b}^{*}} \Psi_{J} \right\|_{L^{2}(\mu)}^{\bigstar 2}} \\
\lesssim\ & \left(\frac{\mathrm{P}^{\alpha}(J,\nu)}{|J|} \left\| \mathsf{Q}_{\mathcal{H}}^{\omega,\mathbf{b}^{*}} x \right\|_{L^{2}(\omega)}^{\spadesuit} + \frac{\mathrm{P}_{1+\delta}^{\alpha}(J,\nu)}{|J|^{\frac{1}{n}}} \|x - m_{J}\|_{L^{2}(\mathbf{1}_{J}\omega)} \right) \left\| \mathsf{P}_{\mathcal{H}}^{\omega,\mathbf{b}^{*}} \Psi_{J} \right\|_{L^{2}(\mu)}^{\bigstar}
\end{aligned}$$

and in particular the 'energy' estimate

$$\begin{aligned}
&|\langle T^{\alpha} \varphi\nu, \Psi_{J} \rangle_{\omega}| \\
\leq & C_{\gamma} \left(\frac{\mathrm{P}^{\alpha}(J,\nu)}{|J|^{\frac{1}{n}}} \left\| \mathsf{Q}_{J}^{\omega,\mathbf{b}^{*}} x \right\|_{L^{2}(\omega)}^{\spadesuit} + \frac{\mathrm{P}_{1+\delta}^{\alpha}(J,\nu)}{|J|^{\frac{1}{n}}} \|x - m_{J}\|_{L^{2}(\mathbf{1}_{J}\omega)} \right) \left\| \sum_{J' \subset J} \square_{J'}^{\omega,\mathbf{b}^{*}} \Psi_{J} \right\|_{L^{2}(\mu)}^{\bigstar}
\end{aligned}$$

where $\left\| \sum_{J' \subset J} \square_{J'}^{\omega,\mathbf{b}^*} \Psi_J \right\|_{L^2(\mu)}^{\bigstar} \lesssim \|\Psi_J\|_{L^2(\mu)}$, *and the 'pivotal' bound*

$$\left|\left\langle T^{\alpha}\left(\varphi\nu\right), \Psi_J\right\rangle_{\omega}\right| \lesssim C_{\gamma} \mathrm{P}^{\alpha}\left(J, |\nu|\right) \sqrt{|J|_{\omega}} \|\Psi_J\|_{L^2(\omega)} \ ,$$

for any function φ *with* $|\varphi| \leq 1$.

Proof. Using the Monotonicity Lemma 5.1.23, followed by $\left|\nu_{J'}\right| \leq \nu$, the Poisson equivalence

$$\frac{\mathrm{P}^{\alpha}\left(J', \nu\right)}{|J'|^{\frac{1}{n}}} \approx \frac{\mathrm{P}^{\alpha}\left(J, \nu\right)}{|J|^{\frac{1}{n}}}, \qquad J' \subset J \subset \gamma J, \quad \mathrm{supp}\nu \cap \gamma J = \emptyset, \tag{5.1.57}$$

and the weak frame inequalities for dual martingale differences, we have

$$\begin{aligned}
&\left|\sum_{J' \in \mathcal{H}} \left\langle T^{\alpha}\left(\nu_{J'}\right), \square_{J'}^{\omega,\mathbf{b}^*} \Psi_J \right\rangle_{\omega}\right| \lesssim \sum_{J' \in \mathcal{H}} \Phi^{\alpha}\left(J', |\mu|\right) \left\|\square_{J'}^{\omega,\mathbf{b}^*} \Psi_J\right\|_{L^2(\mu)}^{\bigstar} \\
\lesssim \ & \left(\sum_{J' \in \mathcal{H}} \left(\frac{\mathrm{P}^{\alpha}\left(J', \nu\right)}{|J'|^{\frac{1}{n}}}\right)^2 \left\|\triangle_{J'}^{\omega,\mathbf{b}^*} x\right\|_{L^2(\omega)}^{\spadesuit 2}\right)^{\frac{1}{2}} \left(\sum_{J' \in \mathcal{H}} \left\|\square_{J'}^{\omega,\mathbf{b}^*} \Psi_J\right\|_{L^2(\omega)}^{\bigstar 2}\right)^{\frac{1}{2}} \\
& + \left(\sum_{J' \in \mathcal{H}} \left(\frac{\mathrm{P}_{1+\delta}^{\alpha}\left(J', |\mu|\right)}{|J'|^{\frac{1}{n}}}\right)^2 \left\|x - m_{J'}\right\|_{L^2\left(\mathbf{1}_{J'}\omega\right)}^2\right)^{\frac{1}{2}} \left(\sum_{J' \in \mathcal{H}} \left\|\square_{J'}^{\omega,\mathbf{b}^*} \Psi_J\right\|_{L^2(\omega)}^{\bigstar 2}\right)^{\frac{1}{2}} \\
\lesssim & \frac{\mathrm{P}^{\alpha}\left(J, \nu\right)}{|J|^{\frac{1}{n}}} \left\|\mathsf{Q}_{\mathcal{H}}^{\omega,\mathbf{b}^*} x\right\|_{L^2(\omega)}^{\spadesuit} \|\Psi_J\|_{L^2(\omega)} + \frac{1}{\gamma^{\delta'}} \frac{\mathrm{P}_{1+\delta'}^{\alpha}\left(J, \nu\right)}{|J|^{\frac{1}{n}}} \|x - m_J\|_{L^2\left(\mathbf{1}_J\omega\right)} \|\Psi_J\|_{L^2(\omega)} \ .
\end{aligned}$$

The last inequality follows from the following calculation using Haar projections $\triangle_K^\omega$:

$$\sum_{J'\in\mathcal{H}}\left(\frac{\mathrm{P}_{1+\delta}^{\alpha}\left(J',\nu\right)}{\left|J'\right|^{\frac{1}{n}}}\right)^{2}\left\|x-m_{J'}\right\|_{L^{2}\left(\mathbf{1}_{J'}\omega\right)}^{2} \tag{5.1.58}$$
$$\begin{aligned}
&= \sum_{J'\in\mathcal{H}}\left(\frac{\mathrm{P}_{1+\delta}^{\alpha}\left(J',\nu\right)}{\left|J'\right|^{\frac{1}{n}}}\right)^{2}\sum_{J''\subset J'}\left\|\triangle_{J''}^{\omega}x\right\|_{L^{2}(\omega)}^{2}\\
&= \sum_{J''\subset J}\left\{\sum_{J':\ J''\subset J'\subset J}\left(\frac{\mathrm{P}_{1+\delta}^{\alpha}\left(J',\nu\right)}{\left|J'\right|^{\frac{1}{n}}}\right)^{2}\right\}\left\|\triangle_{J''}^{\omega}x\right\|_{L^{2}(\omega)}^{2}\\
&\lesssim \frac{1}{\gamma^{2\delta'}}\sum_{J''\subset J}\left(\frac{\mathrm{P}_{1+\delta'}^{\alpha}\left(J'',\nu\right)}{\left|J''\right|^{\frac{1}{n}}}\right)^{2}\left\|\triangle_{J''}^{\omega}x\right\|_{L^{2}(\omega)}^{2}\\
&\leq \frac{1}{\gamma^{2\delta'}}\left(\frac{\mathrm{P}_{1+\delta'}^{\alpha}\left(J,\nu\right)}{\left|J\right|^{\frac{1}{n}}}\right)^{2}\sum_{J''\subset J}\left\|\triangle_{J''}^{\omega}x\right\|_{L^{2}(\omega)}^{2},
\end{aligned}$$

which in turn follows from (recalling $\delta=2\delta'$ and $\left|J'\right|^{\frac{1}{n}}+\left|y-c_{J'}\right|\approx\left|J\right|^{\frac{1}{n}}+\left|y-c_{J}\right|$ and $\frac{|J|}{|J|+\left|y-c_J\right|}\leq\frac{1}{\gamma}$ for $y\in\mathbb{R}^n\setminus\gamma J$)

$$\begin{aligned}
&\sum_{J':\ J''\subset J'\subset J}\left(\frac{\mathrm{P}_{1+\delta}^{\alpha}\left(J',\nu\right)}{\left|J'\right|^{\frac{1}{n}}}\right)^{2}=\\
&\sum_{J':\ J''\subset J'\subset J}\left|J'\right|^{\frac{2\delta}{n}}\left(\int_{\mathbb{R}^n\setminus\gamma J}\frac{1}{\left(\left|J'\right|^{\frac{1}{n}}+\left|y-c_{J'}\right|\right)^{n+1+\delta-\alpha}}d\nu\left(y\right)\right)^{2}\\
\lesssim&\sum_{J':\ J''\subset J'\subset J}\frac{1}{\gamma^{2\delta'}}\frac{\left|J'\right|^{\frac{2\delta}{n}}}{\left|J\right|^{\frac{2\delta}{n}}}\left(\int_{\mathbb{R}^n\setminus\gamma J}\frac{\left|J\right|^{\frac{\delta'}{n}}}{\left(\left|J\right|^{\frac{1}{n}}+\left|y-c_{J}\right|\right)^{n+1+\delta'-\alpha}}d\nu\left(y\right)\right)^{2}\\
=&\ \frac{1}{\gamma^{2\delta'}}\left(\sum_{J':\ J''\subset J'\subset J}\frac{\left|J'\right|^{\frac{2\delta}{n}}}{\left|J\right|^{\frac{2\delta}{n}}}\right)\left(\frac{\mathrm{P}_{1+\delta'}^{\alpha}\left(J,\nu\right)}{\left|J\right|^{\frac{1}{n}}}\right)^{2}\lesssim\frac{1}{\gamma^{2\delta'}}\left(\frac{\mathrm{P}_{1+\delta'}^{\alpha}\left(J,\nu\right)}{\left|J\right|^{\frac{1}{n}}}\right)^{2}.
\end{aligned}$$

Finally we obtain the 'energy' estimate from the equality

$$\Psi_J = \sum_{J' \subset J} \square_{J'}^{\omega,\mathbf{b}^*} \Psi_J \ , \qquad \text{(since } \Psi_J \text{ has vanishing } \omega\text{-mean)},$$

and we obtain the 'pivotal' bound from the inequality

$$\sum_{J'' \subset J} \left\| \triangle_{J''}^{\omega,\mathbf{b}^*} x \right\|_{L^2(\omega)}^{\spadesuit 2} \lesssim \left\| (x - m_J) \right\|_{L^2(\mathbf{1}_J \omega)}^2 \leq |J|^2 \, |J|_\omega \ .$$

$\square$

5.1.14 Organization of the proof

We adapt the proof of the main theorem in [66], but beginning instead with the decomposition of Hytönen and Martikainen [24], to obtain the norm inequality

$$\mathfrak{N}_{T^\alpha} \lesssim \mathfrak{T}_{T^\alpha}^{\mathbf{b}} + \mathfrak{T}_{T^\alpha}^{\mathbf{b}^*} + \sqrt{\mathfrak{A}_2^\alpha} + \mathfrak{E}_2^\alpha$$

under the *apriori* assumption $\mathfrak{N}_{T^\alpha} < \infty$, which is achieved by considering one of the truncations $T_{\sigma,\delta,R}^\alpha$ defined in (5.1.3) above. This will be carried out in the next four sections of this paper. In the next section we consider the various form splittings and reduce matters to the *disjoint* form, the *nearby* form and the *main below* form. Then these latter three forms are taken up in the subsequent three sections, using material from the appendices.

A major source of difficulty will arise in the infusion of goodness for the cubes J into the below form where the sum is taken over all pairs (I, J) such that $\ell(J) \leq \ell(I)$. We will infuse goodness in a weak way pioneered by Hytönen and Martikainen in a one weight setting. This weak form of goodness is then exploited in all subsequent constructions by typically replacing J by $J^{\maltese}$ in defining relations, where $J^{\maltese}$ is the smallest cube K for which J is good w.r.t. K and beyond.

Another source of difficulty arises in the treatment of the nearby form in the setting of two weights. The one weight proofs in [24] and [29] relied strongly on a property peculiar to the one weight setting - namely the fact already pointed out in Remark 5.1.6 above that both of the Poisson integrals are bounded, namely $\mathrm{P}^{\alpha}(Q,\mu) \lesssim 1$ and $\mathcal{P}^{\alpha}(Q,\mu) \lesssim 1$. We will circumvent this difficulty by combining a recursive energy argument with the full testing conditions assumed for the *original* testing functions b_Q^{orig}, before these conditions were suppressed by corona constructions that delivered only weak testing conditions for the new testing functions b_Q.

Of particular importance will be a result proved in Appendices A that follow from known work with some new twists. We show that the functional energy for an arbitrary pair of grids is controlled by the Muckenhoupt and energy side conditions. The somewhat lengthy proof of this latter assertion is similar to the corresponding proof in the $T1$ setting - see e.g. [66] - but requires a different decomposition of the stopping cubes into 'Whitney cubes' in order to accomodate the weaker notion of goodness used here.

5.2 Form splittings

Notation 5.2.1. *Fix grids $\mathcal{D}$ and $\mathcal{G}$. We will use $\mathcal{D}$ to denote the grid associated with $f \in L^2(\sigma)$, and we will use $\mathcal{G}$ to denote the grid associated with $g \in L^2(\omega)$.*

Now we turn to the probability estimates for martingale differences and halos that we will use. Recall that given $\overrightarrow{\lambda} = (\lambda_1, ..., \lambda_n)$, $0 < \lambda_i < \frac{1}{2}$ for all $1 \leq i \leq n$, the λ-halo of J is defined to be

$$\partial_{\overrightarrow{\lambda}} J \equiv \left(1+\overrightarrow{\lambda}\right) J \setminus \left(1-\overrightarrow{\lambda}\right) J.$$

Suppose μ is a positive locally finite Borel measure, and that $\mathbf{b}$ is a p-weakly μ-controlled accretive family for some $p>2$. Then the following probability estimate holds.

Bad cube probability estimates. Suppose that $\mathcal{D}$ and $\mathcal{G}$ are independent random dyadic grids. With $\Psi^{\mu,\mathbf{b}^*}_{\mathcal{G}^{\mathcal{D}}_{k-bad}} g \equiv \sum_{J\in\mathcal{G}^{\mathcal{D}}_{k-bad}} \square^{\mu,\mathbf{b}^*}_J g$ equal to the pseudoprojection of g onto k-bad $\mathcal{G}$-cubes, we have

$$\begin{aligned} \boldsymbol{E}^{\mathcal{D}}_{\Omega}\left(\left\|\Psi^{\mu,\mathbf{b}^*}_{\mathcal{G}^{\mathcal{D}}_{k-bad}} g\right\|^2_{L^2(\mu)}\right) &\lesssim \boldsymbol{E}^{\mathcal{D}}_{\Omega}\left(\sum_{J\in\mathcal{G}^{\mathcal{D}}_{k-bad}}\left[\left\|\square^{\mu,\mathbf{b}^*}_{J,\mathcal{G}} g\right\|^2_{L^2(\mu)}+\left\|\nabla^{\mu}_{J,\mathcal{G}} g\right\|^2_{L^2(\mu)}\right]\right) \\ &\leq Ce^{-k\varepsilon}\left\|g\right\|^2_{L^2(\mu)}, \end{aligned} \tag{5.2.1}$$

where the first inequality is the 'weak upper half Riesz' inequality from Appendix A of [69] for the pseudoprojection $\Psi^{\mu,\mathbf{b}^*}_{\mathcal{G}^{\mathcal{D}}_{k-bad}}$, and the second inequality is proved using the frame inequality in (5.2.10) below.

Halo probability estimates. Suppose that $\mathcal{D}$ and $\mathcal{G}$ are independent random grids. Using the *parameterization by translations* of grids and taking the average over certain translates $\tau+\mathcal{D}$ of the grid $\mathcal{D}$ we have

$$\begin{aligned} \boldsymbol{E}^{\mathcal{D}}_{\Omega}\sum_{I'\in\mathcal{D}:\ \ell(I')\approx\ell(J')}\int_{J'\cap\partial_\delta I'} d\omega &\lesssim \delta\int_{J'} d\omega, \qquad J'\in\mathfrak{C}(J), J\in\mathcal{G}, \\ \boldsymbol{E}^{\mathcal{G}}_{\Omega}\sum_{J'\in\mathcal{G}:\ \ell(J')\approx\ell(I')}\int_{I'\cap\partial_\delta J'} d\sigma &\lesssim \delta\int_{I'} d\sigma, \qquad I'\in\mathfrak{C}(I), I\in\mathcal{D}, \end{aligned} \tag{5.2.2}$$

and where the expectations $\boldsymbol{E}_{\Omega}^{\mathcal{D}}$ and $\boldsymbol{E}_{\Omega}^{\mathcal{G}}$ are taken over grids $\mathcal{D}$ and $\mathcal{G}$ respectively. Indeed, it is geometrically evident that for any fixed pair of side lengths $\ell_1 \approx \ell_2$, the average of the measure $\left|J' \cap \partial_\delta I'\right|_\omega$ of the set $J' \cap \partial_\delta I'$, as a cube $I' \in \mathcal{D}$ with side length $\ell\left(I'\right) = \ell_1$ is translated across a cube $J' \in \mathcal{G}$ of side length $\ell\left(J'\right) = \ell_2$, is at most $C\left|J'\right|_\omega$. Using this observation it is now easy to see that (5.2.2) holds.

In the σ-iterated corona construction we redefined the family $\mathbf{b} = \left\{b_Q\right\}_{Q\in\mathcal{D}}$ so that the new functions b_Q^{new} are given in terms of the original functions b_Q^{orig} by $b_Q^{new} = \mathbf{1}_Q b_A^{orig}$ for $Q \in \mathcal{C}_A^\sigma$, and of course we then dropped the superscript *new*. We continue to refer to the triple stopping cubes A as 'breaking' cubes even if b_A happens to equal $\mathbf{1}_A b_{\pi A}$. The results of Appendix A of [69] apply with this more inclusive definition of 'breaking' cubes, and the associated definition of 'broken' children, since only the Carleson condition on stopping cubes is relevant here.

This and Proposition 5.1.19 give us the *triple corona decomposition* of $f = \sum_{A\in\mathcal{A}} \mathsf{P}_{\mathcal{C}_A}^\sigma f$, where the pseudoprojection $\mathsf{P}_{\mathcal{C}_A}^\sigma$ is defined as:

$$\mathsf{P}_{\mathcal{C}_A}^\sigma f = \sum_{I\in\mathcal{C}_A} \square_I^{\mu,\mathbf{b}} f$$

We now record the main facts proved above for the triple corona.

Lemma 5.2.2. *Let $f \in L^2(\sigma)$. We have*

$$f = \sum_{A\in\mathcal{A}} \mathsf{P}_{\mathcal{C}_A}^\sigma f$$

both in the sense of norm convergence in $L^2(\sigma)$ and pointwise σ-a.e. The corona tops $\mathcal{A}$ and stopping bounds $\{\alpha_{\mathcal{A}}(A)\}_{A\in\mathcal{A}}$ satisfy properties (1), (2), (3) and (4) in Definition 5.1.18, *hence constitute stopping data for f. Moreover, $\mathbf{b} = \{b_I\}_{I\in\mathcal{D}}$ is a ∞-weakly σ-controlled accretive family on $\mathcal{D}$ with corona tops $\mathcal{A} \subset \mathcal{D}$, where $b_I = \mathbf{1}_I b_A$ for all $I \in \mathcal{C}_A$, and the weak corona forward testing condition holds uniformly in coronas, i.e.*

$$\frac{1}{|I|_\sigma}\int_I \left|T_\sigma^\alpha b_A\right|^2 d\sigma \leq C, \qquad I \in \mathcal{C}_A^\sigma \ .$$

Similar statements hold for $g \in L^2(\omega)$.

We have defined corona decompositions of f and g in the σ-iterated triple corona construction above, but in order to start these corona decompositions for f and g respectively within the dyadic grids $\mathcal{D}$ and $\mathcal{G}$, we need to first restrict f and g to be supported in a large common cube Q_∞. Then we cover Q_∞ with 2^n pairwise disjoint cubes $I_\infty \in \mathcal{D}$ with $\ell(I_\infty) = \ell(Q_\infty)$, and similarly cover Q_∞ with 2^n pairwise disjoint cubes $J_\infty \in \mathcal{G}$ with $\ell(J_\infty) = \ell(Q_\infty)$. We can now use the broken martingale decompositions, together with random surgery, to reduce matters to consideration of the four forms

$$\sum_{I\in\mathcal{D}:\ I\subset I_\infty}\ \sum_{J\in\mathcal{G}:\ J\subset J_\infty} \int \left(T_\sigma^\alpha \square_I^{\sigma,\mathbf{b}} f\right) \square_J^{\omega,\mathbf{b}^*} g d\omega,$$

with I_∞ and J_∞ as above, and where we can then use the cubes I_∞ and J_∞ as the starting cubes in our corona constructions below. Indeed, the identities in [24, Lemma 3.5]), give

$$\begin{aligned} f &= \sum_{I\in\mathcal{D}:\ I\subset I_\infty,\ \ell(I)\geq 2^{-N}} \square_I^{\sigma,\mathbf{b}} f + \mathbb{F}_{I_\infty}^{\sigma,\mathbf{b}} f, \\ g &= \sum_{J\in\mathcal{G}:\ J\subset J_\infty,\ \ell(J)\geq 2^{-N}} \square_J^{\omega,\mathbf{b}^*} g + \mathbb{F}_{J_\infty}^{\omega,\mathbf{b}^*} g, \end{aligned}$$

which can then be used to write the bilinear form $\int (T_\sigma f)\, g d\omega$ as a sum of the forms

$$\sum_{\substack{2^{n+1}\text{pairs} \\ (I_\infty, J_\infty)}} \left\{ \sum_{\substack{I\in\mathcal{D} \\ I\subset I_\infty}} \sum_{\substack{J\in\mathcal{G} \\ J\subset J_\infty}} \int \left(T_\sigma^\alpha \square_I^{\sigma,\mathbf{b}} f \right) \square_J^{\omega,\mathbf{b}^*} g d\omega + \sum_{\substack{I\in\mathcal{D} \\ I\subset I_\infty}} \int \left(T_\sigma^\alpha \square_I^{\sigma,\mathbf{b}} f \right) \mathbb{F}_{J_\infty}^{\omega,\mathbf{b}^*} g d\omega \right.$$
$$\left. + \sum_{J\in\mathcal{G}:\ J\subset J_\infty} \int \left(T_\sigma^\alpha \mathbb{F}_{I_\infty}^{\sigma,\mathbf{b}} f \right) \square_J^{\omega,\mathbf{b}^*} g d\omega + \int \left(T_\sigma^\alpha \mathbb{F}_{I_\infty}^{\sigma,\mathbf{b}} f \right) \mathbb{F}_{J_\infty}^{\omega,\mathbf{b}^*} g d\omega \right\} \quad (5.2.3)$$

taken over the 2^{n+1} pairs of cubes (I_∞, J_∞) above. The second, third and fourth sums in (5.2.3) can be controlled using testing and random surgery. For example, for the second sum we have

$$\begin{aligned}
\left| \sum_{I\in\mathcal{D}:\ I\subset I_\infty} \int \left(T_\sigma^\alpha \square_I^{\sigma,\mathbf{b}} f \right) \mathbb{F}_{J_\infty}^{\omega,\mathbf{b}^*} g d\omega \right| &\leq \left| \int_{I_\infty \cap J_\infty} \left(\sum_{I\in\mathcal{D}:\ I\subset I_\infty} \square_I^{\sigma,\mathbf{b}} f \right) T_\omega^{\alpha,*} \left(\mathbb{F}_{J_\infty}^{\omega,\mathbf{b}^*} g \right) d\sigma \right| \\
&+ \left| \int_{I_\infty \cap ((1+\delta)J_\infty \setminus J_\infty)} \left(\sum_{I\in\mathcal{D}:\ I\subset I_\infty} \square_I^{\sigma,\mathbf{b}} f \right) T_\omega^{\alpha,*} \left(\mathbb{F}_{J_\infty}^{\omega,\mathbf{b}^*} g \right) d\sigma \right| \\
&+ \left| \int_{I_\infty \setminus (1+\delta)J_\infty} \left(\sum_{I\in\mathcal{D}:\ I\subset I_\infty} \square_I^{\sigma,\mathbf{b}} f \right) T_\omega^{\alpha,*} \left(\mathbb{F}_{J_\infty}^{\omega,\mathbf{b}^*} g \right) d\sigma \right| \\
&\equiv A_1 + A_2 + A_3
\end{aligned}$$

So we are left with bounding A_1, A_2, A_3. We have

$$A_1 \leq \left(\int_{I_\infty} \left| \sum_{I\in\mathcal{D}:\ I\subset I_\infty} \square_I^{\sigma,\mathbf{b}} f \right|^2 d\sigma \right)^{\frac{1}{2}} \left(\int_{J_\infty} \left| T_\omega^{\alpha,*} \left(\mathbb{F}_{J_\infty}^{\omega,\mathbf{b}^*} g \right) \right|^2 d\sigma \right)^{\frac{1}{2}}$$

and since $\mathbb{F}_{J_\infty}^{\omega,\mathbf{b}^*} g = b_{J_\infty}^* \frac{E_{J_\infty}^{\omega} g}{E_{J_\infty}^{\omega} b_{J_\infty}^*}$ is $b_{J_\infty}^*$ times an 'accretive' average of g on J_∞, we get

$$\begin{aligned} A_1 &\leq \left\| \sum_{I\in\mathcal{D}:\ I\subset I_\infty} \square_I^{\sigma,\mathbf{b}} f \right\|_{L^2(\sigma)} \left(\int_{J_\infty} \left| T_\omega^{\alpha,*}(\mathbf{1}_{J_\infty} b_{J_\infty}^*) \right|^2 d\sigma \right)^{\frac{1}{2}} |E_{J_\infty}^{\omega} g| \cdot \frac{1}{c_{\mathbf{b}^*} |J_\infty|_\omega} \\ &\lesssim \mathfrak{T}_{T^{\alpha,*}}^{\mathbf{b}^*} \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)} \end{aligned}$$

where in the last inequality we used the frame estimates (5.1.51) and the dual testing condition on $b_{J_\infty}^*$.

For A_2 we use expectation on the grid $\mathcal{G}$.

$$\boldsymbol{E}^{\mathcal{G}} A_2 \leq \boldsymbol{E}^{\mathcal{G}} \int_{I_\infty\cap[(1+\delta)J_\infty\setminus J_\infty]} \left| \sum_{I\in\mathcal{D}:\ I\subset I_\infty} \square_I^{\sigma,\mathbf{b}} f \right| \left| T_\omega^{\alpha,*}\left(\mathbb{F}_{J_\infty}^{\omega,\mathbf{b}^*} g\right) \right| d\sigma$$

$$\leq \boldsymbol{E}^{\mathcal{G}} \left(\int_{I_\infty\cap[(1+\delta)J_\infty\setminus J_\infty]} \left| \sum_{I\in\mathcal{D}:\ I\subset I_\infty} \square_I^{\sigma,\mathbf{b}} f \right|^2 d\sigma \right)^{\frac{1}{2}} \left(\int \left| T_\omega^{\alpha,*}\left(\mathbb{F}_{J_\infty}^{\omega,\mathbf{b}^*} g\right) \right|^2 d\sigma \right)^{\frac{1}{2}}$$

$$\leq \left(\boldsymbol{E}^{\mathcal{G}} \int_{I_\infty\cap[(1+\delta)J_\infty\setminus J_\infty]} \left| \sum_{I\in\mathcal{D}:\ I\subset I_\infty} \square_I^{\sigma,\mathbf{b}} f \right|^2 d\sigma \right)^{\frac{1}{2}} \left(\mathfrak{N}_{T^\alpha} \int |g|^2 d\omega \right)^{\frac{1}{2}}$$

$$\leq \left(C\delta \int_{I_\infty} \left| \sum_{I\in\mathcal{D}:\ I\subset I_\infty} \square_I^{\sigma,\mathbf{b}} f \right|^2 d\sigma \right)^{\frac{1}{2}} \left(\mathfrak{N}_{T^\alpha} \int |g|^2 d\omega \right)^{\frac{1}{2}}$$

$$\leq \sqrt{C\delta \mathfrak{N}_{T^\alpha}} \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)}$$

Finally for A_3 we use lemma 5.4.3 since $\mathrm{dist}(I_\infty\setminus(1+\delta)J_\infty, J_\infty) \approx \delta\ell(J_\infty)$ to get

$$A_3 \lesssim \sqrt{\mathfrak{A}_2^\alpha} \delta^{\alpha-n} \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)}.$$

Altogether we get

$$\boldsymbol{E}_{\Omega}^{\mathcal{G}}\left|\sum_{\substack{I\in\mathcal{D}\\ I\subset I_{\infty}}}\int\left(T_{\sigma}^{\alpha}\square_{I}^{\sigma,\mathbf{b}}f\right)\mathbb{F}_{J_{\infty}}^{\omega,\mathbf{b}^{*}}gd\omega\right|\lesssim\left(\mathfrak{T}_{T^{\alpha}}^{\mathbf{b}}+\sqrt{\mathfrak{A}_{2}^{\alpha}}\delta^{\alpha-n}+\delta\mathfrak{N}_{T^{\alpha}}\right)\|f\|_{L^{2}(\sigma)}\|g\|_{L^{2}(\omega)}$$

Similarly we deal with the third and fourth sum of (5.2.3). We are left to deal with the first sum in (5.2.3).

5.2.1 The Hytönen-Martikainen decomposition and weak goodness

Now we turn to the various splittings of forms, beginning with the two weight analogue of the decomposition of Hytönen and Martikainen [24]. Let $\mathbf{b}$ (respectively $\mathbf{b}^{*}$) be a ∞-weakly σ-controlled (respectively ω-controlled) accretive family. Fix the stopping data $\mathcal{A}$ and $\{\alpha_{\mathcal{A}}(A)\}_{A\in\mathcal{A}}$ and dual martingale differences $\square_{I}^{\sigma,\mathbf{b}}$ constructed above with the triple iterated coronas, as well as the corresponding data for g. We are left with the estimation of the bilinear form $\int(T_{\sigma}f)\,gd\omega$ to that of the sum

$$\sum_{I\in\mathcal{D}}\sum_{J\in\mathcal{G}}\int\left(T_{\sigma}^{\alpha}\square_{I}^{\sigma,\mathbf{b}}f\right)\square_{J}^{\omega,\mathbf{b}^{*}}gd\omega,$$

We split the form $\langle T_{\sigma}^{\alpha}f,g\rangle_{\omega}$ into the sum of two essentially symmetric forms by cube size,

$$\begin{aligned}\int(T_{\sigma}f)\,gd\omega &= \left\{\sum_{\substack{I\in\mathcal{D}:\ J\in\mathcal{G}\\ \ell(J)\leq\ell(I)}}+\sum_{\substack{I\in\mathcal{D}:\ J\in\mathcal{G}\\ \ell(J)>\ell(I)}}\right\}\int\left(T_{\sigma}^{\alpha}\square_{I}^{\sigma,\mathbf{b}}f\right)\square_{J}^{\omega,\mathbf{b}^{*}}gd\omega, \qquad (5.2.4)\\ &\equiv \Theta(f,g)+\Theta^{*}(f,g)\end{aligned}$$

and focus on the first sum,

$$\Theta(f,g)=\sum_{I\in\mathcal{D}\text{ and }J\in\mathcal{G}:\ \ell(J)\leq\ell(I)}\left\langle T_{\sigma}^{\alpha}\square_{I}^{\sigma,\mathbf{b}}f,\square_{J}^{\omega,\mathbf{b}^{*}}\right\rangle_{\omega},$$

since the second sum is handled dually, but is easier due to the missing diagonal. Before introducing goodness into the sum, we follow [24] and split the form $\Theta(f,g)$ into 3 pieces:

$$\begin{aligned}&\sum_{I\in\mathcal{D}}\left\{\sum_{\substack{J\in\mathcal{G}:\ \ell(J)\leq\ell(I)\\ d(J,I)>2\ell(J)^{\varepsilon}\ell(I)^{1-\varepsilon}}}+\sum_{\substack{J\in\mathcal{G}:\ \ell(J)\leq 2^{-\mathbf{r}}\ell(I)\\ d(J,I)\leq 2\ell(J)^{\varepsilon}\ell(I)^{1-\varepsilon}}}+\sum_{\substack{J\in\mathcal{G}:\ 2^{-\mathbf{r}}\ell(I)<\ell(J)\leq\ell(I)\\ d(J,I)\leq 2\ell(J)^{\varepsilon}\ell(I)^{1-\varepsilon}}}\right\}\left\langle T_{\sigma}^{\alpha}\square_{I}^{\sigma,\mathbf{b}}f,\square_{J}^{\omega,\mathbf{b}^{*}}\right\rangle_{\omega}\\ &\quad\equiv\ \Theta_{1}(f,g)+\Theta_{2}(f,g)+\Theta_{3}(f,g)\ ,\end{aligned}$$

where $\varepsilon>0$ will be chosen to satisfy $0<\varepsilon<\frac{1}{n+1-\alpha}$ later. Now the disjoint form $\Theta_1(f,g)$ can be handled by 'long-range' and 'short-range' arguments which we give in a section below, and the nearby form $\Theta_3(f,g)$ will be handled using surgery methods and a new recursive argument involving energy conditions and the 'original' testing functions discarded in the corona construction. The remaining form $\Theta_2(f,g)$ will be treated further in this section after introducing weak goodness.

5.2.1.1 Good cubes with 'body'

. We begin with the weaker extension of goodness introduced in [24], except that we will make it a bit stronger by replacing the skeleton '$skelK$' of a cube K, as used in [24], by a larger collection of points '$bodyK$', which we call the dyadic body of K. This modification will prove useful in establishing the Straddling Lemma in the treatment of the stopping form in Section 5.6 below. Let $\mathcal{P}$ denote the collection of all cubes in $\mathbb{R}^n$. The content of the next four definitions is inspired by, or sometimes identical with, that already appearing in the work of Nazarov, Treil and Volberg in [41] and [43].

Definition 5.2.3. *Given a dyadic cube $K \in \mathbb{R}^n$, we define $W(K)$ to be the Whitney cubes in K. Namely, $S \in W(K)$ if:*

- $3S \subset K$.
- $S' \cap S \neq \emptyset$ *and* $3S' \subset K$ *imply* $S' \subset S$.

Definition 5.2.4. *We define the dyadic body 'bodyK' of a dyadic cube $K \in \mathbb{R}^n$ by*

$$bodyK = \bigcup_{S \in W(K)} \partial S$$

where ∂S is the boundary of S.

Definition 5.2.5. *Let $0 < \epsilon < 1$. For dyadic cubes $J, K \in \mathbb{R}^n$ with $\ell(J) \leq \ell(K)$ we define J to be $\epsilon-$good in K if*

$$\operatorname{dist}(J, bodyK) > 2\ell(J)^{\epsilon}\ell(K)^{1-\epsilon} \tag{5.2.5}$$

and we say it is $\epsilon-$bad in K if (5.2.5) fails.

Definition 5.2.6. *Let $\mathcal{D}$ and $\mathcal{G}$ be two dyadic grids in $\mathbb{R}^n$. Define $\mathcal{G}^{\mathcal{D}}_{(k,\epsilon)-good}$ to consist of those cubes $J \in \mathcal{G}$ such that J is $\epsilon-$good inside every cube $K \in \mathcal{D}$ with $K \cap J \neq \emptyset$ and $\ell(K) \geq 2^k \ell(J)$.*

5.2.1.2 Grid probability

As pointed out on page 14 of [24] by Hytönen and Martikainen, there are subtle difficulties associated in using dual martingale decompositions of functions which depend on the entire dyadic grid, rather than on just the local cube in the grid. We will proceed at first in the spirit of [24], and the goodness that we will infuse below into the main 'below' form $\mathsf{B}_{\Subset_{\mathbf{r}}}(f,g)$ will be the Hytönen-Martikainen 'weak' version of NTV goodness, but using the body '*bodyI*' of a cube rather than its skeleton '*skelI*': every pair $(I,J)\in\mathcal{D}\times\mathcal{G}$ that arises in the form $\mathsf{B}_{\Subset_{\mathbf{r}}}(f,g)$ will satisfy $J\in\mathcal{G}^{\mathcal{D}}_{(k,\varepsilon)-good}$ where $\ell(I)=2^k\ell(J)$.

Now we return to the martingale differences $\square_I^{\sigma,\mathbf{b}}$ and $\square_J^{\omega,\mathbf{b}^*}$ with controlled families $\mathbf{b}$ and $\mathbf{b}^*$ in $\mathbb{R}^n$. When we want to emphasize that the grid in use is $\mathcal{D}$ or $\mathcal{G}$, we will denote the martingale difference by $\square_{I,\mathcal{D}}^{\sigma,\mathbf{b}}$, and similarly for $\square_{J,\mathcal{G}}^{\omega,\mathbf{b}^*}$. Recall Definition 5.2.5 for the meaning of when an cube J is ε-*bad* with respect to another cube K.

Definition 5.2.7. *We say that $J\in\mathcal{P}$ is k-bad in a grid $\mathcal{D}$ if there is a cube $K\in\mathcal{D}$ with $\ell(K)=2^k\ell(J)$ such that J is ε-bad with respect to K (context should eliminate any ambiguity between the different use of k-bad when $k\in\mathbb{N}$ and ε-bad when $0<\varepsilon<\frac{1}{2}$).*

Following [69] we know that in one dimension for an interval J and grids $\mathcal{D}_0$

$$\boldsymbol{P}_{\Omega}^{\mathcal{D}_0}(\mathcal{D}_0: J \text{ is } k\text{-}bad \text{ in } \mathcal{D}_0)\equiv\int_{\Omega}\mathbf{1}_{\{\mathcal{D}_0:\ J \text{ is } k\text{-}bad \text{ in } \mathcal{D}_0\}}d\mu_{\Omega}(\mathcal{D}_0)\le C\varepsilon k2^{-\varepsilon k}. \tag{5.2.6}$$

Thus we conclude:

$$\boldsymbol{P}_{\Omega}^{\mathcal{D}_0}(\mathcal{D}_0: J \text{ is } k\text{-}good \text{ in } \mathcal{D}_0)\ge 1-C\varepsilon k2^{-\varepsilon k}. \tag{5.2.7}$$

Now for a cube J to be good in our n-dimensional setting, it needs to be good in each side. So, we conclude that

$$\boldsymbol{P}_{\Omega}^{\mathcal{D}}(\mathcal{D}: J \text{ is } k\text{-}good \text{ in } \mathcal{D})\ge (1-C\varepsilon k2^{-\varepsilon k})^n. \tag{5.2.8}$$

and therefore a cube is bad with probability bounded by:

$$\boldsymbol{P}_{\Omega}^{\mathcal{D}}\left(\mathcal{D}: J \text{ is } k\text{-}bad \text{ in } \mathcal{D}\right) \leq 1-(1-C\varepsilon k 2^{-\varepsilon k})^{n}. \tag{5.2.9}$$

Then we obtain from (5.2.9), using the lower frame inequality, the expectation estimate

$$\begin{aligned}
&\int_{\Omega}\sum_{J\in\mathcal{G}_{k-bad}^{\mathcal{D}}}\left[\left\|\square_{J,\mathcal{G}}^{\omega,\mathbf{b}^{*}}g\right\|_{L^{2}(\omega)}^{2}+\left\|\triangledown_{J,\mathcal{G}}^{\omega}g\right\|_{L^{2}(\omega)}^{2}\right]d\mu_{\Omega}\left(\mathcal{D}\right)\\
=\ &\sum_{J\in\mathcal{G}}\left[\left\|\square_{J,\mathcal{G}}^{\omega,\mathbf{b}^{*}}g\right\|_{L^{2}(\omega)}^{2}+\left\|\triangledown_{J,\mathcal{G}}^{\omega}g\right\|_{L^{2}(\omega)}^{2}\right]\int_{\Omega}\mathbf{1}_{\{\mathcal{D}:\ J\text{ is }k\text{-}bad\text{ in }\mathcal{D}\}}d\mu_{\Omega}\left(\mathcal{D}\right)\\
\leq\ &(1-(1-C\varepsilon k2^{-\varepsilon k})^{n})\sum_{J\in\mathcal{G}}\left[\left\|\square_{J,\mathcal{G}}^{\omega,\mathbf{b}^{*}}g\right\|_{L^{2}(\omega)}^{2}+\left\|\triangledown_{J,\mathcal{G}}^{\omega}g\right\|_{L^{2}(\omega)}^{2}\right]\\
\leq\ &(1-(1-C\varepsilon k2^{-\varepsilon k})^{n})\left\|g\right\|_{L^{2}(\omega)}^{2}\ ,
\end{aligned}$$

where $\triangledown_{J,\mathcal{G}}^{\omega}$ denotes the 'broken' Carleson averaging operator in (5.1.39) that depends on the broken children in the grid $\mathcal{G}$. Altogether then it follows easily that

$$\boldsymbol{E}_{\Omega}^{\mathcal{D}}\left(\sum_{J\in\bigcup_{\ell=k}^{\infty}\mathcal{G}_{\ell-bad}^{\mathcal{D}}}\left[\left\|\square_{J,\mathcal{G}}^{\omega,\mathbf{b}^{*}}g\right\|_{L^{2}(\omega)}^{2}+\left\|\triangledown_{J,\mathcal{G}}^{\omega}g\right\|_{L^{2}(\omega)}^{2}\right]\right)\leq(1-(1-C\varepsilon k2^{-\varepsilon k})^{n})\|g\|_{L^{2}(\omega)}^{2} \tag{5.2.10}$$

for some large positive constant C.

From such inequalities summed for $k\geq\mathbf{r}$, it can be concluded as in [43] that there is an absolute choice of $\mathbf{r}$ depending on $0<\varepsilon<\frac{1}{2}$ so that the following holds. Let $T\ :\ L^{2}(\sigma)\rightarrow L^{2}(\omega)$ be a bounded linear operator. We then have the following traditional inequality for two random grids in the case that $\mathbf{b}$ is an ∞-weakly μ-controlled accretive family:

$$\|T\| \leq 2 \sup_{\|f\|_{L^2(\sigma)}=1} \sup_{\|g\|_{L^2(\omega)}=1} \boldsymbol{E}_{\Omega}\boldsymbol{E}_{\Omega'} \left| \left\langle \sum_{I,J\in\mathcal{D}^{\mathcal{G}}_{\mathbf{r}-good}} T\left(\square^{\sigma,\mathbf{b}}_{I,\mathcal{D}} f\right) f, \square^{\omega,\mathbf{b}^*}_{J,\mathcal{D}} g \right\rangle_{\omega} \right| . \quad (5.2.11)$$

However, this traditional method of introducing goodness is flawed here in the general setting of dual martingale differences, since these differences are no longer orthogonal projections, and as emphasized in [24], we cannot simply add back in bad cubes whenever we want telescoping identities to hold - but these are needed in order to control the right hand side of (5.2.11). In fact, in the analysis of the form $\Theta(f,g)$ above, it is necessary to have goodness for the cubes J and telescoping for the cubes I. On the other hand, in the analysis of the form $\Theta^*(f,g)$ above, it is necessary to have just the opposite - namely goodness for the cubes I and telescoping for the cubes J.

Thus, because in this unfortunate set of circumstances we can no longer 'add back in' bad cubes to achieve telescoping, we are prevented from introducing goodness in the *full* sum (5.2.4) over all I and J, prior to splitting according to side lengths of I and J. Thus the infusion of goodness must come *after* the splitting by side length, but one must work much harder to introduce goodness directly into the form $\Theta(f,g)$ *after* we have restricted the sum to cubes J that have smaller side length than I. This is accomplished in the next subsubsection using the *weaker form of NTV goodness* introduced by Hytönen and Martikainen in [24] (that permits certain additional pairs (I,J) in the good forms where $\ell(J) \leq 2^{-\mathbf{r}}\ell(I)$ and yet J is *bad* in the traditional sense), and that will prevail later in the treatment of the far below forms $\mathsf{T}^1_{farbelow}(f,g)$, and of the local forms $\mathsf{B}^A_{\Subset_{\mathbf{r}}}(f,g)$ (see Subsection 5.7) where the need for using the 'body' of a cube will become apparent in dealing with the stopping form, and also in the treatment of the functional energy in Appendix .

5.2.1.3 Weak goodness

Let $\mathcal{D}$ and $\mathcal{G}$ be dyadic grids. It remains to estimate the form $\Theta_2(f,g)$ which, following [24], we will split into a 'bad' part and a 'good' part. For this we introduce our main definition associated with the above modification of the weak goodness of Hytönen and Martikainen, namely the definition of the cube $R^{\maltese}$ in a grid $\mathcal{D}$, given an arbitrary cube $R \in \mathcal{P}$.

Definition 5.2.8. *Let $\mathcal{D}$ be a dyadic grid. Given $R \in \mathcal{P}$, let $R^{\maltese}$ be the smallest (if any such exist) $\mathcal{D}$-dyadic supercube Q of R such that R is good inside* ***all*** *$\mathcal{D}$-dyadic supercubes K of Q. Of course $R^{\maltese}$ will not exist if there is no $\mathcal{D}$-dyadic cube Q containing R in which R is good. For cubes $R, Q \in \mathcal{P}$ let $\kappa(Q,R) = \log_2 \frac{\ell(Q)}{\ell(R)}$. For $R \in \mathcal{P}$ for which $R^{\maltese}$ exists, let $\kappa(R) \equiv \kappa\left(R^{\maltese}, R\right)$.*

Note that we typically suppress the dependence of $R^{\maltese}$ on the grid $\mathcal{D}$, since the grid is usually understood from context. If $R^{\maltese}$ exists, we thus have that R is good inside all $\mathcal{D}$-dyadic supercubes K of R with $\ell(K) \geq \ell\left(R^{\maltese}\right)$. Note in particular the monotonicity property for $J', J \in \mathcal{P}$:

$$J' \subset J \Longrightarrow (J')^{\maltese} \subset J^{\maltese}.$$

Here now is the decomposition:

$$\begin{aligned}
\Theta_2(f,g) &= \sum_{I\in\mathcal{D}} \sum_{\substack{J\in\mathcal{G}:\ J^{\maltese}\subsetneqq I,\ \ell(J)\leq 2^{-\mathbf{r}}\ell(I) \\ d(J,I)\leq 2\ell(J)^{\varepsilon}\ell(I)^{1-\varepsilon}}} \int \left(T_\sigma^\alpha \square_I^{\sigma,\mathbf{b}} f\right) \square_J^{\omega,\mathbf{b}^*} g d\omega \\
&\quad + \sum_{I\in\mathcal{D}} \sum_{\substack{J\in\mathcal{G}:\ J^{\maltese}\subsetneqq I,\ \ell(J)\leq 2^{-\mathbf{r}}\ell(I) \\ d(J,I)\leq 2\ell(J)^{\varepsilon}\ell(I)^{1-\varepsilon}}} \int \left(T_\sigma^\alpha \square_I^{\sigma,\mathbf{b}} f\right) \square_J^{\omega,\mathbf{b}^*} g d\omega \\
&\equiv \Theta_2^{bad}(f,g) + \Theta_2^{good}(f,g)\ ,
\end{aligned}$$

and where if $J^{\maltese}$ fails to exist, we assume by convention that $J^{\maltese} \subsetneqq I$, i.e. $J^{\maltese}$ is *not* strictly contained in I, so that the pair (I,J) is then included in the bad form $\Theta_2^{bad}(f,g)$. We will in fact estimate a larger quantity corresponding to the bad form, namely

$$\Theta_2^{bad\natural}(f,g) \equiv \sum_{I\in\mathcal{D}} \sum_{\substack{J\in\mathcal{G}:\ J^{\maltese}\subsetneqq I,\ \ell(J)\leq 2^{-\mathbf{r}}\ell(I) \\ d(J,I)\leq 2\ell(J)^{\varepsilon}\ell(I)^{1-\varepsilon}}} \left| \int \left(T_\sigma^\alpha \square_I^{\sigma,\mathbf{b}} f\right) \square_J^{\omega,\mathbf{b}^*} g d\omega \right| \tag{5.2.12}$$

with absolute value signs *inside* the sum.

Remark 5.2.9. *We now make some general comments on where we now stand and where we are going.*

1. *In the first sum $\Theta_2^{bad}(f,g)$ above, we are roughly keeping the pairs of cubes (I,J) such that J is bad with respect to some 'nearby' cube having side length larger than that of I.*

2. *We have defined energy and dual energy conditions that are independent of the testing families (because the definition of $\mathsf{E}(J,\omega) = \mathbb{E}_J^{\omega,x}\mathbb{E}_J^{\omega,x'}\left(\left|\frac{x-x'}{\ell(J)}\right|^2\right)$ does not involve*

pseudoprojections $\square^{\omega,\mathbf{b}^*}_{J,\mathcal{D}}$*), but the functional energy condition defined below* does *involve the dual martingale pseudoprojections* $\square^{\omega,\mathbf{b}^*}_{J,\mathcal{D}}$.

3. *Using the notion of weak goodness above, we will be able to eliminate all pairs of cubes with* J *bad in* I*, which then permits control of the short range form in Section 5.3 and the neighbour form in Section 5.5 provided* $0 < \varepsilon < \frac{1}{n+1-\alpha}$*. Defining shifted coronas in terms of* $J^{\maltese}$ *will then allow existing arguments to prove the Intertwining Proposition and obtain control of the functional energy in Appendix , as well as permitting control of the stopping form in Section 5.6, but all of this with some new twists, for example the introduction of a top/down 'indented corona' in the analysis of the stopping form.*

4. *The nearby form* $\Theta_3(f,g)$ *is handled in Section 5.4 using the energy condition assumption along with the original testing functions* b^{orig}_Q *discarded during the construction of the testing/accretive corona.*

These remarks will become clear in this and the following sections. Recall that we earlier defined in Definition 5.2.6, the set $\mathcal{G}^{\mathcal{D}}_{k-good} = \mathcal{G}^{\mathcal{D}}_{(k,\varepsilon)-good}$ to consist of those $J \in \mathcal{G}$ such that J is $\varepsilon - good$ inside every cube $K \in \mathcal{D}$ with $K \cap J \neq \emptyset$ that lies at least k levels 'above' J, i.e. $\ell(K) \geq 2^k \ell(J)$. We now define an analogous notion of $\mathcal{G}^{\mathcal{D}}_{k-bad}$.

Definition 5.2.10. *Let* $\varepsilon > 0$*. Define the set* $\mathcal{G}^{\mathcal{D}}_{k-bad} = \mathcal{G}^{\mathcal{D}}_{(k,\varepsilon)-bad}$ *to consist of all* $J \in \mathcal{G}$ *such that there is a* $\mathcal{D}$*-cube* K *with sidelength* $\ell(K) = 2^k \ell(J)$ *for which* J *is* $\varepsilon - bad$ *with respect to* K.

Note that for grids $\mathcal{D}$ and $\mathcal{G}$, the complement of $\mathcal{G}^{\mathcal{D}}_{k-good}$ is the union of $\mathcal{G}^{\mathcal{D}}_{\ell-bad}$ for $\ell \geq k$, i.e.

$$\mathcal{G} \setminus \mathcal{G}^{\mathcal{D}}_{k-good} = \bigcup_{\ell \geq k} \mathcal{G}^{\mathcal{D}}_{\ell-bad} .$$

Now assume $\varepsilon > 0$. We then have the following important property, namely for all cubes R, and all $k \geq \mathbf{r}$ (where the goodness parameter $\mathbf{r}$ will be fixed given $\varepsilon > 0$ in (5.2.16) below):

$$\#\left\{Q : \kappa(Q,R) = k \text{ and } d(R,Q) \leq 2\ell(R)^{\varepsilon}\ell(Q)^{1-\varepsilon}\right\} \lesssim 1. \tag{5.2.13}$$

As in [24], set

$$\mathcal{G}_{bad,n}^{\mathcal{D}} \equiv \{J \in \mathcal{G} : J \text{ is } \varepsilon - bad \text{ with respect to some } K \in \mathcal{D} \text{ with } \ell(K) \geq n\}.$$

We will now use the set equality

$$\begin{aligned} &\left\{J \in \mathcal{G} : \ J^{\maltese} \not\subset I,\ \ell(J) \leq 2^{-\mathbf{r}}\ell(I),\ d(J,I) \leq 2\ell(J)^{\varepsilon}\ell(I)^{1-\varepsilon}\right\} \\ = \ &\left\{R \in \mathcal{G}_{bad,\ell(Q)}^{\mathcal{D}} : \ \mathbf{r} \leq \kappa(Q,R) < \kappa(R),\ d(R,Q) \leq 2\ell(R)^{\varepsilon}\ell(Q)^{1-\varepsilon}\right\}, \end{aligned} \tag{5.2.14}$$

which the careful reader can prove by painstakingly verifying both containments.

Assuming only that $\mathbf{b}$ is 2-weakly μ-controlled accretive, and following the proof in [24], we use (5.2.14) to show that for any fixed grids $\mathcal{D}$ and $\mathcal{G}$, and any bounded linear operator T_{σ}^{α} we have the following inequality for the form $\Theta_2^{bad\natural,strict}(f,g)$, defined to be $\Theta_2^{bad\natural}(f,g)$

as in (5.2.12) with the pairs (I,J) removed when $J^{\maltese}=I$. We use $\varepsilon_{Q,R}=\pm 1$ to obtain

$$\Theta_2^{bad\natural,strict}(f,g)=\sum_{Q\in\mathcal{D}}\sum_{\substack{R\in\mathcal{G}_{bad,\ell(Q)}^{\mathcal{D}}:\ \mathbf{r}\leq\kappa(Q,R)<\kappa(R)\\ d(R,Q)\leq 2\ell(R)^{\varepsilon}\ell(Q)^{1-\varepsilon}}}\left|\left\langle T_\sigma^\alpha\left(\square_{Q,\mathcal{D}}^{\sigma,\mathbf{b}}f\right),\square_{R,\mathcal{G}}^{\omega,\mathbf{b}^*}g\right\rangle\right|$$

$$=\sum_{Q\in\mathcal{D}}\sum_{\substack{R\in\mathcal{G}_{bad,\ell(Q)}^{\mathcal{D}}:\ \mathbf{r}\leq\kappa(Q,R)<\kappa(R)\\ d(R,Q)\leq 2\ell(R)^{\varepsilon}\ell(Q)^{1-\varepsilon}}}\varepsilon_{Q,R}\left\langle T_\sigma^\alpha\left(\square_{Q,\mathcal{D}}^{\sigma,\mathbf{b}}f\right),\square_{R,\mathcal{G}}^{\omega,\mathbf{b}^*}g\right\rangle$$

$$\leq\sum_{Q\in\mathcal{D}}\left|\left\langle T_\sigma^\alpha\left(\square_{Q,\mathcal{D}}^{\sigma,\mathbf{b}}f\right),\sum_{\substack{R\in\mathcal{G}_{bad,\ell(Q)}^{\mathcal{D}}:\ \mathbf{r}\leq\kappa(Q,R)<\kappa(R)\\ d(R,Q)\leq 2\ell(R)^{\varepsilon}\ell(Q)^{1-\varepsilon}}}\varepsilon_{Q,R}\square_{R,\mathcal{G}}^{\omega,\mathbf{b}^*}g\right\rangle\right|$$

$$\leq\mathfrak{N}_{T^\alpha}\sum_{Q\in\mathcal{D}}\left\|\square_{Q,\mathcal{D}}^{\sigma,\mathbf{b}}f\right\|_{L^2(\sigma)}\left\|\sum_{\substack{R\in\mathcal{G}_{bad,\ell(Q)}^{\mathcal{D}}:\ \mathbf{r}\leq\kappa(Q,R)<\kappa(R)\\ d(R,Q)\leq 2\ell(R)^{\varepsilon}\ell(Q)^{1-\varepsilon}}}\varepsilon_{Q,R}\square_{R,\mathcal{G}}^{\omega,\mathbf{b}^*}g\right\|_{L^2(\omega)}$$

$$\leq\mathfrak{N}_{T^\alpha}\sum_{Q\in\mathcal{D}}\left\|\square_{Q,\mathcal{D}}^{\sigma,\mathbf{b}}f\right\|_{L^2(\sigma)}\sum_{k=\mathbf{r}}^{\infty}\left\|\sum_{\substack{R\in\mathcal{G}_{bad,\ell(Q)}^{\mathcal{D}}:k=\kappa(Q,R)<\kappa(R)\\ d(R,Q)\leq 2\ell(R)^{\varepsilon}\ell(Q)^{1-\varepsilon}}}\varepsilon_{Q,R}\square_{R,\mathcal{G}}^{\omega,\mathbf{b}^*}g\right\|_{L^2(\omega)},$$

by Minkowski's inequality, and we continue with

$$
\begin{aligned}
&\leq 2\mathfrak{N}_{T^\alpha}\sum_{k=\mathbf{r}}^{\infty}\left(\sum_{Q\in\mathcal{D}}\left\|\square_{Q,\mathcal{D}}^{\sigma,\mathbf{b}}f\right\|_{L^2(\sigma)}^2\right)^{\frac{1}{2}}\cdot \\
&\qquad\left(\sum_{Q\in\mathcal{D}}\sum_{\substack{R\in\mathcal{G}_{bad,\ell(Q)}^{\mathcal{D}}:\ k=\kappa(Q,R)<\kappa(R)\\ d(R,Q)\leq 2\ell(R)^{\varepsilon}\ell(Q)^{1-\varepsilon}}}\left(\left\|\square_{R,\mathcal{G}}^{\omega,\mathbf{b}^*}g\right\|_{L^2(\omega)}^2+\left\|\triangledown_{R,\mathcal{G}}^{\omega}g\right\|_{L^2(\omega)}^2\right)\right)^{\frac{1}{2}} \\
&\lesssim \mathfrak{N}_{T^\alpha}\left\|f\right\|_{L^2(\sigma)}\sum_{k=\mathbf{r}}^{\infty}\left(\sum_{R\in\mathcal{G}_{bad,2^k\ell(R)}^{\mathcal{D}}}\left(\left\|\square_{R,\mathcal{G}}^{\omega,\mathbf{b}^*}g\right\|_{L^2(\omega)}^2+\left\|\triangledown_{R,\mathcal{G}}^{\omega}g\right\|_{L^2(\omega)}^2\right)\right)^{\frac{1}{2}},
\end{aligned}
$$

where $\triangledown_{R,\mathcal{G}}^{\omega}$ denotes the 'broken' Carleson averaging operator in (5.1.39) that depends on the grid $\mathcal{G}$, and

1. the penultimate inequality uses Cauchy-Schwarz in Q and the weak upper Riesz inequalities (5.1.53) for $\sum_{\substack{R\in\mathcal{G}_{bad,\ell(Q)}^{\mathcal{D}}:\ k=\kappa(Q,R)<\kappa(R)\\ d(R,Q)\leq 2\ell(R)^{\varepsilon}\ell(Q)^{1-\varepsilon}}}\varepsilon_{Q,R}\square_{R,\mathcal{G}}^{\omega,\mathbf{b}^*}$, once for the sum when $\varepsilon_{Q,R}=1$, and again for the sum when $\varepsilon_{Q,R}=-1$. However, we note that since the sum in R is pigeonholed by $k=\kappa(Q,R)$, the R's are pairwise disjoint cubes and the pseudoprojections $\square_{R,\mathcal{G}}^{\omega,\mathbf{b}^*}g$ are pairwise orthogonal. Thus we could instead apply Cauchy-Schwarz first in R, and then in Q as was done in [24], but we must still apply weak upper Riesz inequalities as above.

2. and the final inequality uses the frame inequality (5.1.51) together with (5.2.13), namely the fact that there are at most C cubes Q such that $\kappa(Q,R)\geq\mathbf{r}$ is fixed and $d(R,Q)\leq 2\ell(R)^{\varepsilon}\ell(Q)^{1-\varepsilon}$.

Now it is easy to verify that we have the same inequality for the pairs $\left(J^{\maltese},J\right)$ that were removed, and then we take grid expectations and use the probability estimate (5.2.10) to obtain for $\varepsilon'=\frac{1}{2}\varepsilon$ that $\boldsymbol{E}_{\Omega}^{\mathcal{D}}\left(\Theta_2^{bad\natural}\left(f,g\right)\right)$ is bounded by

$$
\begin{aligned}
&\leq \boldsymbol{E}_{\Omega}^{\mathcal{D}}\mathfrak{N}_{T^{\alpha}}\left\|f\right\|_{L^2(\sigma)}\sum_{k=\mathbf{r}}^{\infty}\left(\sum_{R\in\mathcal{G}_{bad,2^k\ell(R)}^{\mathcal{D}}}\left(\left\|\square_{R,\mathcal{G}}^{\omega,\mathbf{b}^*}g\right\|_{L^2(\omega)}^2+\left\|\nabla_{R,\mathcal{G}}^{\omega}g\right\|_{L^2(\omega)}^2\right)\right)^{\frac{1}{2}} \\
&\leq \mathfrak{N}_{T^{\alpha}}\left\|f\right\|_{L^2(\sigma)}\sum_{k=\mathbf{r}}^{\infty}\left(\boldsymbol{E}_{\Omega}^{\mathcal{D}}\sum_{R\in\mathcal{G}_{bad,2^k\ell(R)}^{\mathcal{D}}}\left(\left\|\square_{R,\mathcal{G}}^{\omega,\mathbf{b}^*}g\right\|_{L^2(\omega)}^2+\left\|\nabla_{R,\mathcal{G}}^{\omega}\right\|_{L^2(\omega)}^2\right)\right)^{\frac{1}{2}} \\
&\lesssim 2^{-\frac{1}{2}\varepsilon'\mathbf{r}}\mathfrak{N}_{T^{\alpha}}\left\|f\right\|_{L^2(\sigma)}\sum_{k=\mathbf{r}}^{\infty}\left(\left(1-\left(C_1 2^{-\varepsilon k}\right)^n\right)\left\|g\right\|_{L^2(\omega)}^2\right)^{\frac{1}{2}} \\
&\leq C_{good}2^{-\frac{1}{2}\varepsilon\mathbf{r}}\mathfrak{N}_{T^{\alpha}}\left\|f\right\|_{L^2(\sigma)}\left\|g\right\|_{L^2(\omega)}.
\end{aligned}
\tag{5.2.15}
$$

Clearly we can now fix $\mathbf{r}$ sufficiently large depending on $\varepsilon>0$ so that

$$
C_{good}2^{-\frac{1}{2}\varepsilon\mathbf{r}}<\frac{1}{100}, \tag{5.2.16}
$$

and then the final term above, namely $C_{good}2^{-\frac{1}{2}\varepsilon\mathbf{r}}\mathfrak{N}_{T^{\alpha}}\left\|f\right\|_{L^2(\sigma)}\left\|g\right\|_{L^2(\omega)}$, can be absorbed at the end of the proof in Subsection 5.7. Note that (5.2.16) fixes our choice of the parameter $\mathbf{r}$ for any given $\varepsilon>0$. Later we will choose $0<\varepsilon<\frac{1}{2}\leq\frac{1}{n+1-\alpha}$. It is this type of weak goodness that we will exploit in the local forms $\mathsf{B}_{\Subset\mathbf{r}}^{A}\left(f,g\right)$ treated below in Section 5.5.

We are now left with the following 'good' form to control:

$$\Theta_2^{good}(f,g)=\sum_{I\in\mathcal{D}}\sum_{\substack{J^{\maltese}\subsetneqq I:\ \ell(J)\leq 2^{-\mathbf{r}}\ell(I)\\ d(J,I)\leq 2\ell(J)^{\varepsilon}\ell(I)^{1-\varepsilon}}}\int\left(T_\sigma^\alpha\square_I^{\sigma,\mathbf{b}}f\right)\square_J^{\omega,\mathbf{b}^*}gd\omega.$$

The first thing we observe regarding this form is that the cubes J which arise in the sum for $\Theta_2^{good}(f,g)$ must lie entirely inside I since $J\subset J^{\maltese}\subsetneqq I$. Then in the remainder of the paper, we proceed to analyze

$$\Theta_2^{good}(f,g)=\sum_{I\in\mathcal{D}}\sum_{J^{\maltese}\subsetneqq I:\ \ell(J)\leq 2^{-\mathbf{r}}\ell(I)}\int\left(T_\sigma^\alpha\square_I^{\sigma,\mathbf{b}}f\right)\square_J^{\omega,\mathbf{b}^*}gd\omega, \tag{5.2.17}$$

in the same way we analyzed the below term $\mathsf{B}_{\Subset_{\mathbf{r}}}(f,g)$ in [63]; namely, by implementing the canonical corona splitting and the decomposition into paraproduct, neighbour and stopping forms, but now with an additional broken form. We have (κ,ε)-goodness available for all the cubes $J\in\mathcal{G}$ arising in the form $\Theta_2^{good}(f,g)$, and moreover, the cubes $I\in\mathcal{D}$ arising in the form $\Theta_2^{good}(f,g)$ for a fixed J are tree-connected, so that telescoping identities hold for these cubes I. This will prove decisive in the following three sections of the paper.

The forms $\Theta_1(f,g)$ and $\Theta_3(f,g)$ are analogous to the disjoint and nearby forms $\mathsf{B}_{\cap}(f,g)$ and $\mathsf{B}_{/}(f,g)$ in [63] respectively. In the next two sections, we control the disjoint form $\Theta_1(f,g)$ in essentially the same way that the disjoint form $\mathsf{B}_{\cap}(f,g)$ was treated in [63] and in earlier papers of many authors beginning with Nazarov, Treil and Volberg (see e.g. [75]), and we control the nearby form $\Theta_3(f,g)$ using the probabilistic surgery of Hytönen and Martikainen building on that of NTV, together with a new deterministic surgery involving the energy condition and the original testing functions. But first we recall, in the following subsection, the characterization of boundedness of one-dimensional forms supported on disjoint cubes [22].

5.3 Disjoint form

Here we control the disjoint form $\Theta_1(f,g)$ by further decomposing it as follows:

$$\Theta_1(f,g) = \sum_{I\in\mathcal{D}} \sum_{\substack{J\in\mathcal{G}:\ \ell(J)\leq\ell(I) \\ d(J,I)>2\ell(J)^{\varepsilon}\ell(I)^{1-\varepsilon}}} \int \left(T_\sigma \square_I^{\sigma,\mathbf{b}} f\right) \square_J^{\omega,\mathbf{b}^*} g d\omega$$

which can be rewritten as

$$\begin{aligned}
&\sum_{I\in\mathcal{D}} \left\{ \sum_{\substack{J\in\mathcal{G}:\ \ell(J)\leq\ell(I) \\ d(J,I)>\max(\ell(I),2\ell(J)^{\varepsilon}\ell(I)^{1-\varepsilon})}} + \sum_{\substack{J\in\mathcal{G}:\ \ell(J)\leq\ell(I) \\ \ell(I)\geq d(J,I)>2\ell(J)^{\varepsilon}\ell(I)^{1-\varepsilon}}} \right\} \int \left(T_\sigma \square_I^{\sigma,\mathbf{b}} f\right) \square_J^{\omega,\mathbf{b}^*} g d\omega \\
&\equiv \Theta_1^{long}(f,g) + \Theta_1^{short}(f,g),
\end{aligned}$$

where $\Theta_1^{long}(f,g)$ is a 'long range' form in which J is far from I, and where $\Theta_1^{short}(f,g)$ is a short range form. It should be noted that the goodness plays no role in treating the disjoint form.

5.3.1 Long range form

Lemma 5.3.1. *We have*

$$\sum_{I\in\mathcal{D}} \sum_{\substack{J\in\mathcal{G}:\ \ell(J)\leq\ell(I) \\ d(J,I)>\ell(I)}} \left| \int \left(T_\sigma \square_I^{\sigma,\mathbf{b}} f\right) \square_J^{\omega,\mathbf{b}^*} g d\omega \right| \lesssim \sqrt{\mathfrak{A}_2^\alpha} \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)}$$

Proof. Since J and I are separated by at least $\max\{\ell(J),\ell(I)\}$, we have the inequality

$$\begin{aligned}\mathrm{P}^{\alpha}\left(J,\left|\square_I^{\sigma,\mathbf{b}}f\right|\sigma\right) &\approx \int_I \frac{\ell(J)}{|y-c_J|^{n+1-\alpha}}\left|\square_I^{\sigma,\mathbf{b}}f(y)\right|d\sigma(y)\\ &\lesssim \left\|\square_I^{\sigma,\mathbf{b}}f\right\|_{L^2(\sigma)}\frac{\ell(J)\sqrt{|I|_\sigma}}{d(I,J)^{n+1-\alpha}},\end{aligned}$$

since $\int_I\left|\square_I^{\sigma,\mathbf{b}}f(y)\right|d\sigma(y)\leq\left\|\square_I^{\sigma,\mathbf{b}}f\right\|_{L^2(\sigma)}\sqrt{|I|_\sigma}$. Thus if $A(f,g)$ denotes the left hand side of the conclusion of Lemma 5.3.1, we have using first the Energy Lemma,

$$\begin{aligned}A(f,g) &\lesssim \sum_{I\in\mathcal{D}}\sum_{\substack{J\,:\,\ell(J)\leq\ell(I)\\ d(I,J)\geq\ell(I)}}\left\|\square_I^{\sigma,\mathbf{b}}f\right\|_{L^2(\sigma)}\left\|\square_J^{\omega,\mathbf{b}^*}g\right\|_{L^2(\omega)}\frac{\ell(J)}{d(I,J)^{n+1-\alpha}}\sqrt{|I|_\sigma}\sqrt{|J|_\omega}\\ &\equiv \sum_{(I,J)\in\mathcal{P}}\left\|\square_I^{\sigma,\mathbf{b}}f\right\|_{L^2(\sigma)}\left\|\square_J^{\omega,\mathbf{b}^*}g\right\|_{L^2(\omega)}A(I,J);\\ \text{with } A(I,J) &\equiv \frac{\ell(J)}{d(I,J)^{n+1-\alpha}}\sqrt{|I|_\sigma}\sqrt{|J|_\omega};\\ \text{and } \mathcal{P} &\equiv \{(I,J)\in\mathcal{D}\times\mathcal{G}:\ell(J)\leq\ell(I)\text{ and } d(I,J)\geq\ell(I)\}.\end{aligned}$$

Now let $\mathcal{D}_N\equiv\left\{K\in\mathcal{D}:\ell(K)=2^N\right\}$ for each $N\in\mathbb{Z}$. For $N\in\mathbb{Z}$ and $s\in\mathbb{Z}_+$, we further decompose $A(f,g)$ by pigeonholing the sidelengths of I and J by 2^N and 2^{N-s} respectively:

$$\begin{aligned}A(f,g) &= \sum_{s=0}^{\infty}\sum_{N\in\mathbb{Z}}A_N^s(f,g);\\ A_N^s(f,g) &\equiv \sum_{(I,J)\in\mathcal{P}_N^s}\left\|\square_I^{\sigma,\mathbf{b}}f\right\|_{L^2(\sigma)}\left\|\square_J^{\omega,\mathbf{b}^*}g\right\|_{L^2(\omega)}A(I,J)\\ \text{where } \mathcal{P}_N^s &\equiv \{(I,J)\in\mathcal{D}_N\times\mathcal{G}_{N-s}:d(I,J)\geq\ell(I)\}.\end{aligned}$$

Now let $\mathsf{P}_M^\sigma=\sum_{K\in\mathcal{D}_M}\square_K^{\sigma,\mathbf{b}}$ denote the dual martingale pseudoprojection onto $\mathrm{Span}\left\{\square_K^{\sigma,\mathbf{b}}\right\}_{K\in\mathcal{D}_M}$. Since the cubes K in $\mathcal{D}_M$ are pairwise disjoint, the pseudoprojections

$\square_K^{\sigma,\mathbf{b}}$ are mutually orthogonal, which means that $\left\|\mathsf{P}_M^\sigma f\right\|_{L^2(\sigma)}^2 = \sum_{K\in\mathcal{D}_M} \left\|\square_K^{\sigma,\mathbf{b}} f\right\|_{L^2(\sigma)}^2$. We claim that

$$\left|A_N^s(f,g)\right| \leq C2^{-s}\sqrt{\mathfrak{A}_2^\alpha}\left\|\mathsf{P}_N^\sigma f\right\|_{L^2(\sigma)}^{\bigstar}\left\|\mathsf{P}_{N-s}^\omega g\right\|_{L^2(\omega)}^{\bigstar}, \qquad \text{for } s\geq 0 \text{ and } N\in\mathbb{Z}. \quad (5.3.1)$$

With this proved, we can then obtain

$$\begin{aligned} A(f,g) &= \sum_{s=0}^{\infty}\sum_{N\in\mathbb{Z}} A_N^s(f,g) = \sum_{s=0}^{\infty}\sum_{N\in\mathbb{Z}} A_N^s(f,g) \\ &\leq C\sqrt{\mathfrak{A}_2^\alpha}\sum_{s=0}^{\infty}2^{-s}\sum_{N\in\mathbb{Z}}\left\|\mathsf{P}_N^\sigma f\right\|_{L^2(\sigma)}^{\bigstar}\left\|\mathsf{P}_{N-s}^\omega g\right\|_{L^2(\omega)}^{\bigstar} \\ &\leq C\sqrt{\mathfrak{A}_2^\alpha}\sum_{s=0}^{\infty}2^{-s}\left(\sum_{N\in\mathbb{Z}}\left\|\mathsf{P}_N^\sigma f\right\|_{L^2(\sigma)}^{\bigstar 2}\right)^{\frac{1}{2}}\left(\sum_{N\in\mathbb{Z}}\left\|\mathsf{P}_{N-s}^\omega g\right\|_{L^2(\omega)}^{\bigstar 2}\right)^{\frac{1}{2}} \\ &\leq C\sqrt{\mathfrak{A}_2^\alpha}\sum_{s=0}^{\infty}2^{-s}\left\|f\right\|_{L^2(\sigma)}\left\|g\right\|_{L^2(\omega)} = C\sqrt{\mathfrak{A}_2^\alpha}\left\|f\right\|_{L^2(\sigma)}\left\|g\right\|_{L^2(\omega)}. \end{aligned}$$

To prove (5.3.1), we pigeonhole the distance between I and J:

$$\begin{aligned} A_N^s(f,g) &= \sum_{\ell=0}^{\infty} A_{N,\ell}^s(f,g); \\ A_{N,\ell}^s(f,g) &\equiv \sum_{(I,J)\in\mathcal{P}_{N,\ell}^s}\left\|\square_I^{\sigma,\mathbf{b}} f\right\|_{L^2(\sigma)}\left\|\square_J^{\omega,\mathbf{b}^*} g\right\|_{L^2(\omega)} A(I,J) \\ \text{where } \mathcal{P}_{N,\ell}^s &\equiv \left\{(I,J)\in\mathcal{D}_N\times\mathcal{G}_{N-s} : d(I,J)\approx 2^{N+\ell}\right\}. \end{aligned}$$

If we define $\mathcal{H}\left(A_{N,\ell}^s\right)$ to be the bilinear form on $\ell^2\times\ell^2$ with matrix $\left[A(I,J)\right]_{(I,J)\in\mathcal{P}_{N,\ell}^s}$, then it remains to show that the norm $\left\|\mathcal{H}\left(A_{N,\ell}^s\right)\right\|_{\ell^2\to\ell^2}$ of $\mathcal{H}\left(A_{N,\ell}^s\right)$ on the sequence space ℓ^2 is bounded by $C2^{-s-\ell}\sqrt{\mathfrak{A}_2^\alpha}$. In turn, this is equivalent to showing that the norm $\left\|\mathcal{H}\left(B_{N,\ell}^s\right)\right\|_{\ell^2\to\ell^2}$ of the bilinear form $\mathcal{H}\left(B_{N,\ell}^s\right)\equiv\mathcal{H}\left(A_{N,\ell}^s\right)^{tr}\mathcal{H}\left(A_{N,\ell}^s\right)$ on the sequence space ℓ^2 is bounded by $C^2 2^{-2s-2\ell}\mathfrak{A}_2^\alpha$. Here $\mathcal{H}\left(B_{N,\ell}^s\right)$ is the quadratic form with matrix kernel $\left[B_{N,\ell}^s\left(J,J'\right)\right]_{J,J'\in\mathcal{D}_{N-s}}$ having entries:

$$B_{N,\ell}^s\left(J,J'\right) \equiv \sum_{I\in\mathcal{D}_N:\ d(I,J)\approx d\left(I,J'\right)\approx 2^{N+\ell}} A\left(I,J\right) A\left(I,J'\right), \qquad \text{for } J,J'\in\mathcal{G}_{N-s}.$$

We are reduced to showing the bilinear form inequality,

$$\left\|\mathcal{H}\left(B_{N,\ell}^s\right)\right\|_{\ell^2\to\ell^2} \leq C 2^{-2s-2\ell}\mathfrak{A}_2^\alpha \quad \text{ for } s\geq 0,\ \ell\geq 0 \text{ and } N\in\mathbb{Z}.$$

We begin by computing $B_{N,\ell}^s\left(J,J'\right)$:

$$\begin{aligned} B_{N,\ell}^s\left(J,J'\right) &= \sum_{\substack{I\in\mathcal{D}_N \\ d(I,J)\approx d\left(I,J'\right)\approx 2^{N+\ell}}} \frac{\ell\left(J\right)}{d\left(I,J\right)^{n+1-\alpha}}\sqrt{|I|_\sigma}\sqrt{|J|_\omega}\frac{\ell\left(J'\right)}{d\left(I,J'\right)^{n+1-\alpha}}\sqrt{|I|_\sigma}\sqrt{|J'|_\omega} \\ &= \sum_{\substack{I\in\mathcal{D}_N \\ d(I,J)\approx d\left(I,J'\right)\approx 2^{N+\ell}}} \frac{|I|_\sigma}{d\left(I,J\right)^{n+1-\alpha} d\left(I,J'\right)^{n+1-\alpha}}\cdot \ell\left(J\right)\ell\left(J'\right)\sqrt{|J|_\omega}\sqrt{|J'|_\omega}. \end{aligned}$$

Now we show that

$$\left\|B_{N,\ell}^s\right\|_{\ell^2\to\ell^2} \lesssim 2^{-2s-2\ell}\mathfrak{A}_2^\alpha\ , \tag{5.3.2}$$

by applying the proof of Schur's lemma. Fix $\ell\geq 0$ and $s\geq 0$. Choose the Schur function $\beta\left(K\right)=\frac{1}{\sqrt{|K|_\omega}}$. Fix $J\in\mathcal{D}_{N-s}$. We now group those $I\in\mathcal{D}_N$ with $d\left(I,J\right)\approx 2^{N+\ell}$ into finitely many groups $G_1,...G_C$ for which the union of the I in each group is contained in a cube of side length roughly $\frac{1}{100}2^{N+\ell}$, and we set $I_k^*\equiv\bigcup_{I\in G_k} I$ for $1\leq k\leq C$ (note that I_k^*

is not a cube). We then have

$$
\begin{aligned}
&\sum_{J'\in\mathcal{G}_{N-s}} \frac{\beta(J)}{\beta(J')} B^s_{N,\ell}(J,J') \\
= &\sum_{\substack{J'\in\mathcal{G}_{N-s} \\ d(J',J)\le\frac{1}{100}2^{N+\ell+2}}} \frac{\beta(J)}{\beta(J')} B^s_{N,\ell}(J,J') + \sum_{\substack{J'\in\mathcal{G}_{N-s} \\ d(J',J)>\frac{1}{100}2^{N+\ell+2}}} \frac{\beta(J)}{\beta(J')} B^s_{N,\ell}(J,J') \\
\equiv &\ A+B,
\end{aligned}
$$

where

$$
\begin{aligned}
A &\lesssim \sum_{\substack{J'\in\mathcal{G}_{N-s} \\ d(J,J')\le\frac{1}{100}2^{N+\ell+2}}} \left\{ \sum_{\substack{I\in\mathcal{D}_N \\ d(I,J)\approx 2^{N+\ell}}} |I|_\sigma \right\} \frac{2^{2(N-s)}}{2^{2(\ell+N)(n+1-\alpha)}} |J'|_\omega \\
&= \sum_{\substack{J'\in\mathcal{G}_{N-s} \\ d(J,J')\le\frac{1}{100}2^{N+\ell+2}}} \left\{ \sum_{k=1}^{C} |I_k^*|_\sigma \right\} \frac{2^{2(N-s)}}{2^{2(\ell+N)(n+1-\alpha)}} |J'|_\omega \\
&= \frac{2^{2(N-s)}}{2^{2(\ell+N)(n+1-\alpha)}} \sum_{k=1}^{C} \sum_{\substack{J'\in\mathcal{G}_{N-s} \\ d(J,J')\le\frac{1}{100}2^{N+\ell+2}}} |I_k^*|_\sigma \ |J'|_\omega \\
&\lesssim 2^{-2s-2\ell} \sum_{k=1}^{C} \frac{|I_k^*|_\sigma}{2^{(\ell+N)(n-\alpha)}} \frac{\left|\frac{1}{100}2^{N+\ell+2}J\right|_\omega}{2^{(\ell+N)(n-\alpha)}} \lesssim 2^{-2s-2\ell}\mathfrak{A}_2^\alpha,
\end{aligned}
$$

since I_k^* is contained in a cube $\tilde{I_k^*}$ such that $|I_k^*| \approx |\tilde{I_k^*}|$, with an implied constant depending only on dimension, and $\tilde{I_k^*}$, $\frac{1}{100}2^{N+\ell+2}J$ are well separated. If we let Q_k be the smallest cube containing the set

$$E_k \equiv \bigcup_{\substack{J' \in \mathcal{D}_{N-s}:\ d\left(I_k^*, J'\right) \approx 2^{N+\ell} \\ d\left(J, J'\right) > \frac{1}{100} 2^{N+\ell+2}}} J'$$

we then have

$$\begin{aligned}
B &\lesssim \sum_{\substack{J' \in \mathcal{D}_{N-s} \\ d\left(J, J'\right) > \frac{1}{100} 2^{N+\ell+2}}} \left\{ \sum_{\substack{I \in \mathcal{D}_N \\ d\left(I, J'\right) \approx d(I, J) \approx 2^{N+\ell}}} |I|_\sigma \right\} \frac{2^{2(N-s)}}{2^{2(\ell+N)(n+1-\alpha)}} \left|J'\right|_\omega \\
&\lesssim \sum_{\substack{J' \in \mathcal{D}_{N-s} \\ d\left(J, J'\right) > \frac{1}{100} 2^{N+\ell+2}}} \left\{ \sum_{k:\ d\left(I_k^*, J'\right) \approx 2^{N+\ell}} \left|I_k^*\right|_\sigma \right\} \frac{2^{2(N-s)}}{2^{2(\ell+N)(n+1-\alpha)}} \left|J'\right|_\omega \\
&\lesssim \frac{2^{2(N-s)}}{2^{2(\ell+N)(n+1-\alpha)}} \sum_{k=1}^{C} \left|I_k^*\right|_\sigma \left|E_k\right|_\omega \\
&\lesssim 2^{-2s-2\ell} \sum_{k=1}^{C} \frac{\left|I_k^*\right|_\sigma}{2^{(\ell+N)(n-\alpha)}} \frac{\left|Q_k\right|_\omega}{2^{(\ell+N)(n-\alpha)}} \lesssim 2^{-2s-2\ell} \mathfrak{A}_2^\alpha,
\end{aligned}$$

since I_k^* is contained in a cube $\tilde{I}_k^*$ such that $|I_k^*| \approx |\tilde{I}_k^*|$, with an implied constant depending only on dimension, and $\tilde{I}_k^*$, $\frac{1}{100} 2^{N+\ell+2} J$ are well separated. Thus we can now apply Schur's argument with $\sum_J (a_J)^2 = \sum_{J'} \left(b_{J'}\right)^2 = 1$ to obtain

$$\begin{aligned}
&\sum_{J,J'\in\mathcal{G}_{N-s}} a_J b_{J'} B^s_{N,\ell}\left(J,J'\right) = \sum_{J,J'\in\mathcal{G}_{N-s}} a_J \beta\left(J\right) b_{J'} \beta\left(J'\right) \frac{B^s_{N,\ell}\left(J,J'\right)}{\beta\left(J\right)\beta\left(J'\right)} \\
&\leq \sum_J \left(a_J \beta\left(J\right)\right)^2 \sum_{J'} \frac{B^s_{N,\ell}\left(J,J'\right)}{\beta\left(J\right)\beta\left(J'\right)} + \sum_{J'} \left(b_{J'}\beta\left(J'\right)\right)^2 \sum_J \frac{B^s_{N,\ell}\left(J,J'\right)}{\beta\left(J\right)\beta\left(J'\right)} \\
&= \sum_J \left(a_J\right)^2 \left\{\sum_{J'} \frac{\beta\left(J\right)}{\beta\left(J'\right)} B^s_{N,\ell}\left(J,J'\right)\right\} + \sum_{J'} \left(b_{J'}\right)^2 \left\{\sum_J \frac{\beta\left(J'\right)}{\beta\left(J\right)} B^s_{N,\ell}\left(J,J'\right)\right\} \\
&\lesssim 2^{-2s-2\ell} A^\alpha_2 \left(\sum_J \left(a_J\right)^2 + \sum_{J'} \left(b_{J'}\right)^2\right) = 2^{1-2s-2\ell}\mathfrak{A}^\alpha_2 .
\end{aligned}$$

This completes the proof of (5.3.2). We can now sum in ℓ to get (5.3.1) and we are done.

This completes our proof of the long range estimate

$$\mathcal{A}\left(f,g\right) \lesssim \sqrt{A^\alpha_2} \left\|f\right\|_{L^2(\sigma)} \left\|g\right\|_{L^2(\omega)} .$$

$\square$

5.3.2 Short range form

The form $\Theta^{short}_1\left(f,g\right)$ is handled by the following lemma.

Lemma 5.3.2. *We have*

$$\sum_{I\in\mathcal{D}} \sum_{\substack{J\in\mathcal{G}:\ \ell(J)\leq 2^{-\rho}\ell(I) \\ \ell(I)\geq d(J,I)>2\ell(J)^\varepsilon \ell(I)^{1-\varepsilon}}} \left|\int \left(T_\sigma \square^{\sigma,\mathbf{b}}_I f\right) \square^{\omega,\mathbf{b}^*}_J g d\omega\right| \lesssim \sqrt{\mathfrak{A}^\alpha_2} \left\|f\right\|_{L^2(\sigma)} \left\|g\right\|_{L^2(\omega)}$$

Proof. The pairs (I,J) that occur in the sum above satisfy $J \subset 4I \backslash I$, so we consider

$$\mathcal{P} \equiv \left\{ (I,J) \in \mathcal{D} \times \mathcal{G} : \ell(J) \leq 2^{-\rho} \ell(I), \ell(I) \geq d(J,I) > 2\ell(J)^{\varepsilon} \ell(I)^{1-\varepsilon}, J \subset 4I \backslash I \right\}$$

For $(I,J) \in \mathcal{P}$, the 'pivotal' estimate from the Energy Lemma 5.1.25 gives

$$\left| \left\langle T_{\sigma}^{\alpha} \left(\Box_I^{\sigma,\mathbf{b}} f \right), \Box_J^{\omega,\mathbf{b}^*} g \right\rangle_{\omega} \right| \lesssim \left\| \Box_J^{\omega,\mathbf{b}^*} g \right\|_{L^2(\omega)} \mathrm{P}^{\alpha} \left(J, \left| \triangle_I^{\sigma} f \right| \sigma \right) \sqrt{|J|_{\omega}}.$$

Now we pigeonhole the lengths of I and J and the distance between them by defining

$$\mathcal{P}_{N,d}^{s} \equiv \left\{ (I,J) \in \mathcal{P} : \ell(I) = 2^N,\ \ell(J) = 2^{N-s},\ 2^{d-1} \leq d(I,J) \leq 2^d,\ J \subset 4I \backslash I \right\}.$$

Note that the closest a cube J can come to I is determined by:

$$2^d \geq 2\ell(I)^{1-\varepsilon} \ell(J)^{\varepsilon} = 2^{1+N(1-\varepsilon)} 2^{(N-s)\varepsilon} = 2^{1+N-\varepsilon s};$$

which implies $N - \varepsilon s + 1 \leq d \leq N$.

Thus we have

$$\begin{aligned}
& \sum_{(I,J) \in \mathcal{P}} \left| \left\langle T_{\sigma}^{\alpha} \left(\Box_I^{\sigma,\mathbf{b}} f \right), \Box_J^{\omega,\mathbf{b}^*} g \right\rangle_{\omega} \right| \\
\lesssim & \sum_{(I,J) \in \mathcal{P}} \left\| \Box_J^{\omega,\mathbf{b}^*} g \right\|_{L^2(\omega)} \mathrm{P}^{\alpha} \left(J, \left| \Box_I^{\sigma,\mathbf{b}} f \right| \sigma \right) \sqrt{|J|_{\omega}} \\
= & \sum_{s=0}^{\infty} \sum_{N \in \mathbb{Z}} \sum_{d=N-\varepsilon s+1}^{N} \sum_{(I,J) \in \mathcal{P}_{N,d}^{s}} \left\| \Box_J^{\omega,\mathbf{b}^*} g \right\|_{L^2(\omega)} \mathrm{P}^{\alpha} \left(J, \left| \Box_I^{\sigma,\mathbf{b}} f \right| \sigma \right) \sqrt{|J|_{\omega}}.
\end{aligned}$$

Now we use

$$\begin{aligned}
\mathrm{P}^{\alpha}\left(J,\left|\square_{I}^{\sigma,\mathbf{b}}f\right|\sigma\right) &= \int_{I}\frac{\ell\left(J\right)}{\left(\ell\left(J\right)+\left|y-c_{J}\right|\right)^{n+1-\alpha}}\left|\square_{I}^{\sigma,\mathbf{b}}f\left(y\right)\right|d\sigma\left(y\right) \\
&\lesssim \frac{2^{N-s}}{2^{d(n+1-\alpha)}}\left\|\square_{I}^{\sigma,\mathbf{b}}f\right\|_{L^{2}(\sigma)}\sqrt{\left|I\right|_{\sigma}}
\end{aligned}$$

and apply Cauchy-Schwarz in J and use $J\subset 4I\setminus I$ to get

$$\begin{aligned}
&\sum_{(I,J)\in\mathcal{P}}\left|\left\langle T_{\sigma}^{\alpha}\left(\square_{I}^{\sigma,\mathbf{b}}f\right),\square_{J}^{\omega,\mathbf{b}^{*}}g\right\rangle_{\omega}\right| \\
&\lesssim \sum_{s=0}^{\infty}\sum_{N\in\mathbb{Z}}\sum_{d=N-\varepsilon s-1}^{N}\sum_{I\in\mathcal{D}_{N}}\frac{2^{N-s}2^{N(n-\alpha)}}{2^{d(n+1-\alpha)}}\left\|\square_{I}^{\sigma,\mathbf{b}}f\right\|_{L^{2}(\sigma)}\frac{\sqrt{\left|I\right|_{\sigma}}\sqrt{\left|4I\setminus I\right|_{\omega}}}{2^{N(n-\alpha)}}\cdot \\
&\qquad\cdot\sqrt{\sum_{\substack{J\in\mathcal{G}_{N-s}\\ J\subset 4I\setminus I\text{ and }d(I,J)\approx 2^{d}}}\left\|\square_{J}^{\omega,\mathbf{b}^{*}}g\right\|_{L^{2}(\omega)}^{2}} \\
&\lesssim (1+\varepsilon s)\sum_{s=0}^{\infty}\sum_{N\in\mathbb{Z}}\frac{2^{N-s}2^{N(n-\alpha)}}{2^{(N-\varepsilon s)(n+1-\alpha)}}\sqrt{\mathfrak{A}_{2}^{\alpha}}\sum_{I\in\mathcal{D}_{N}}\left\|\square_{I}^{\sigma,\mathbf{b}}f\right\|_{L^{2}(\sigma)}\sqrt{\sum_{\substack{J\in\mathcal{G}_{N-s}\\ J\subset 4I\setminus I}}\left\|\square_{J}^{\omega,\mathbf{b}^{*}}g\right\|_{L^{2}(\omega)}^{2}} \\
&\lesssim (1+\varepsilon s)\sum_{s=\rho}^{\infty}2^{-s[1-\varepsilon(n+1-\alpha)]}\sqrt{\mathfrak{A}_{2}^{\alpha}}\left\|f\right\|_{L^{2}(\sigma)}\left\|g\right\|_{L^{2}(\omega)}\lesssim\sqrt{\mathfrak{A}_{2}^{\alpha}}\left\|f\right\|_{L^{2}(\sigma)}\left\|g\right\|_{L^{2}(\omega)}
\end{aligned}$$

where in the third line above we have used $\displaystyle\sum_{d=N-\varepsilon s-1}^{N}1\lesssim 1+\varepsilon s$, and in the last line $\dfrac{2^{N-s}2^{N(n-\alpha)}}{2^{(N-\varepsilon s)(n+1-\alpha)}}=2^{-s[1-\varepsilon(n+1-\alpha)]}$ followed by Cauchy-Schwarz in I and N, using that we have bounded overlap, depending only on dimension and the goodness constant in the quadruples of I for $I\in\mathcal{D}_{N}$. More precisely, if we define $f_{k}\equiv\Psi_{\mathcal{D}_{k}}^{\sigma,\mathbf{b}}f=\sum_{I\in\mathcal{D}_{k}}\square_{I}^{\sigma,\mathbf{b}}f$ and

$g_k \equiv \Psi_{\mathcal{G}_k}^{\sigma,\mathbf{b}^*} g = \sum_{J\in\mathcal{G}_k} \square_J^{\omega,\mathbf{b}^*} g$, then we have the quasi-orthogonality inequality

$$\begin{aligned}\sum_{N\in\mathbb{Z}} \|f_N\|_{L^2(\sigma)} \|g_{N-s}\|_{L^2(\omega)} &\leq \left(\sum_{N\in\mathbb{Z}} \|f_N\|_{L^2(\sigma)}^2\right)^{\frac{1}{2}} \left(\sum_{N\in\mathbb{Z}} \|g_{N-s}\|_{L^2(\omega)}^2\right)^{\frac{1}{2}} \\ &\lesssim \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)} .\end{aligned}$$

We have assumed that

$$0 < \varepsilon < \frac{1}{n+1-\alpha} \tag{5.3.3}$$

in the calculations above, and this completes the proof of Lemma 5.3.2. $\square$

5.4 Nearby form

We dominate the nearby form $\Theta_3(f,g)$ by

$$|\Theta_3(f,g)| \leq \sum_{I\in\mathcal{D}} \sum_{\substack{J\in\mathcal{G}:\ 2^{-\mathbf{r}n}|I|<|J|\leq|I| \\ d(J,I)\leq 2\ell(J)^{\varepsilon}\ell(I)^{1-\varepsilon}}} \left|\int \left(T_\sigma^\alpha \square_I^{\sigma,\mathbf{b}} f\right) \square_J^{\omega,\mathbf{b}^*} g d\omega\right|,$$

and prove the following proposition that controls the expectation, over two independent grids, of the nearby form $\Theta_3(f,g)$. It should be noted that weak goodness plays no role in treating the nearby form. Note also that in various steps we will use a small $\delta > 0$. In all those different instances δ is free of any dependence. Our goal is the following proposition.

Proposition 5.4.1. *Suppose T^α is a standard fractional singular integral with $0 \leq \alpha < n$. Let $\theta \in (0,1)$ be sufficiently small depending only on α, n. Then there is a constant C_θ such that for $f \in L^2(\sigma)$ and $g \in L^2(\omega)$, and dual martingale differences $\square_I^{\sigma,\mathbf{b}}$ and $\square_J^{\omega,\mathbf{b}^*}$ with*

∞*-weakly accretive families of test functions* $\mathbf{b}$ *and* $\mathbf{b}^*$*, we have*

$$\boldsymbol{E}_{\Omega}^{\mathcal{D}}\boldsymbol{E}_{\Omega}^{\mathcal{G}}\sum_{I\in\mathcal{D}}\sum_{\substack{J\in\mathcal{G}:\ 2^{-\mathbf{r}n}|I|<|J|\leq|I| \\ d(J,I)\leq 2\ell(J)^{\varepsilon}\ell(I)^{1-\varepsilon}}}\left|\left\langle T_{\sigma}^{\alpha}\left(\square_I^{\sigma,\mathbf{b}}f\right),\square_J^{\omega,\mathbf{b}^*}g\right\rangle_{\omega}\right| \tag{5.4.1}$$

$$\lesssim \left(C_{\theta}\mathcal{NTV}_{\alpha}+\sqrt{\theta}\mathfrak{N}_{T^{\alpha}}\right)\|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)} .$$

The following diagram is a sketch of the proof of proposition (5.4.1).

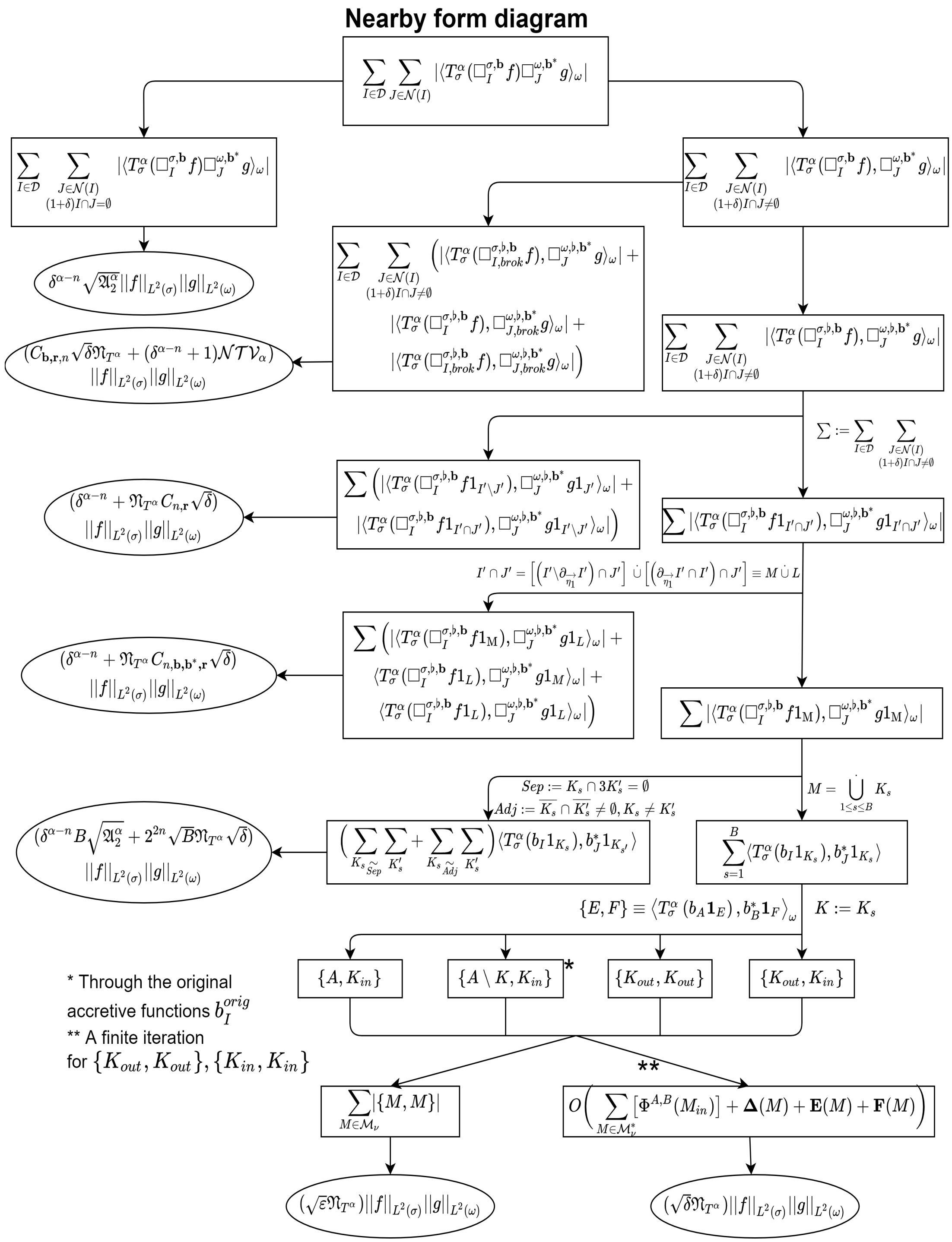

Figure 5.4.1: Nearby form diagram

Before we proceed any further let us mention that we will repeatedly use the inequality

$$\left\|\widehat{\square}_I^{\sigma,\flat,\mathbf{b}} f\right\|_{L^2(\sigma)} \lesssim \left\|\square_I^{\sigma,\mathbf{b}} f\right\|_{L^2(\sigma)}^{\bigstar} \tag{5.4.2}$$

Lemma 5.4.2. *For $f \in L^2(\sigma)$ and $I \in \mathcal{C}_{\mathcal{A}}(A)$ we have $\left\|\widehat{\square}_I^{\sigma,\flat,\mathbf{b}} f\right\|_{L^2(\sigma)} \lesssim \left\|\square_I^{\sigma,\mathbf{b}} f\right\|_{L^2(\sigma)}^{\bigstar}$.*

Proof. Let $I' \in \mathfrak{C}_{\mathcal{D}}(I) \cap \mathcal{C}_{\mathcal{A}}(A)$. Since $I' \in \mathcal{C}_{\mathcal{A}}(A)$, from the corona construction we have

$$\left|\frac{1}{|I'|_\sigma} \int_{I'} b_A d\sigma\right| > \gamma. \tag{5.4.3}$$

Now let $\{I'_j\}_{j\in\mathbb{N}}$ be the collection of maximal subcubes S of I' such that

$$\left|\frac{1}{|S|_\sigma} \int_S b_A d\sigma\right| < \gamma^2.$$

Let $E = \bigcup_j I'_j$. We then have

$$\left|\int_E b_A d\sigma\right| \le \sum_j \left|\int_{I'_j} b_A d\sigma\right| < \gamma^2 \sum_j \left|I'_j\right|_\sigma \le \gamma^2 \left|I'\right|_\sigma$$

which together with (5.4.3) gives

$$\begin{aligned} \gamma \left|I'\right|_\sigma &< \left|\int_{I'} b_A d\sigma\right| = \left|\int_E b_A d\sigma\right| + \left|\int_{I'\setminus E} b_A d\sigma\right| \\ &\le \gamma^2 \left|I'\right|_\sigma + \sqrt{\int_{I'\setminus E} |b_A|^2 d\sigma}\sqrt{|I' \setminus E|_\sigma} \\ &\le \gamma^2 \left|I'\right|_\sigma + C_{\mathbf{b}} \left|I' \setminus E\right|_\sigma, \end{aligned}$$

where in the last inequality we used the ∞-accretivity of b_A. Rearranging the inequality yields successively

$$\begin{aligned}\gamma\left(1-\gamma\right)\left|I'\right|_{\sigma} &\leq C_{\mathbf{b}}\left|I'\setminus E\right|_{\sigma};\\ \frac{\gamma\left(1-\gamma\right)}{C_{\mathbf{b}}}\left|I'\right|_{\sigma} &\leq \left|I'\setminus E\right|_{\sigma}\ ,\end{aligned}$$

which in turn gives

$$\begin{aligned}\sum_j \left|I'_j\right|_{\sigma} &= \left|I'\right|_{\sigma}-\left|I'\setminus E\right|_{\sigma} \\ &\leq \left|I'\right|_{\sigma}-\frac{\gamma\left(1-\gamma\right)}{C_{\mathbf{b}}}\left|I'\right|_{\sigma}=\left(1-\frac{\gamma\left(1-\gamma\right)}{C_{\mathbf{b}}}\right)\left|I'\right|_{\sigma}\equiv\beta\left|I'\right|_{\sigma}\end{aligned} \tag{5.4.4}$$

where $0<\beta<1$ since $1\leq C_{\mathbf{b}}$. This implies

$$\left|I'\right|_{\sigma}\leq\frac{1}{1-\beta}\left|I'\setminus E\right|_{\sigma}$$

Having that in hand and the fact that $\widehat{\square}_I^{\sigma,\flat,\mathbf{b}}f$ is constant on I', say $\mathbf{1}_{I'}\widehat{\square}_I^{\sigma,\flat,\mathbf{b}}f=c_{I'}$ we can now calculate:

$$\begin{aligned}\left\|\mathbf{1}_{I'}\widehat{\square}_I^{\sigma,\flat,\mathbf{b}}f\right\|_{L^2(\sigma)}^2 &= \int_{I'}\left|\widehat{\square}_I^{\sigma,\flat,\mathbf{b}}f\right|^2 d\sigma=\left|I'\right|_{\sigma}\left|c_{I'}\right|^2\\ &\leq \frac{\frac{1}{|I'\setminus E|_{\sigma}}\int_{I'\setminus E}\left|b_A\right|^2 d\sigma}{\gamma^4}\left|I'\right|_{\sigma}\left|c_{I'}\right|^2\\ &= \frac{1}{\gamma^4}\frac{\left|I'\right|_{\sigma}}{|I'\setminus E|_{\sigma}}\int_{I'\setminus E}\left|b_A\right|^2\left|c_{I'}\right|^2 d\sigma\\ &\leq \frac{1}{\gamma^4}\frac{\left|I'\right|_{\sigma}}{|I'\setminus E|_{\sigma}}\int_{I'}\left|b_A\widehat{\square}_I^{\sigma,\flat,\mathbf{b}}f\right|^2 d\sigma\\ &\leq \frac{1}{\gamma^4}\frac{1}{1-\beta}\int_{I'}\left|b_A\widehat{\square}_I^{\sigma,\flat,\mathbf{b}}f\right|^2 d\sigma,\end{aligned}$$

and thus for $I' \in \mathcal{C}_A$ we obtain

$$\int_{I'} \left| \widehat{\square}_I^{\sigma,\flat,\mathbf{b}} f \right|^2 d\sigma \lesssim \int_{I'} \left| b_A \widehat{\square}_I^{\sigma,\flat,\mathbf{b}} f \right|^2 d\sigma,$$

which in turn gives, after summing over all $I' \in \mathfrak{C}_{\mathcal{D}}(I) \cap C_{\mathcal{A}}(A)$,

$$\sum_{I' \in \mathfrak{C}_{\mathcal{D}}(I) \cap C_{\mathcal{A}}(A)} \left\| \mathbf{1}_{I'} \widehat{\square}_I^{\sigma,\flat,\mathbf{b}} f \right\|_{L^2(\sigma)}^2 \lesssim \left\| \mathbf{1}_I b_A \widehat{\square}_I^{\sigma,\flat,\mathbf{b}} f \right\|_{L^2(\sigma)}^2 \leq \left\| b_A \widehat{\square}_I^{\sigma,\flat,\mathbf{b}} f \right\|_{L^2(\sigma)}^2 .$$

Now if $I' \in \mathfrak{C}_{\mathcal{D}}(I) \cap \mathcal{A}$, from the definition of $\widehat{\nabla}_Q^\mu f$ in (5.1.39),

$$\sum_{I' \in \mathfrak{C}_{\mathcal{D}}(I) \cap \mathcal{A}} \left\| \mathbf{1}_{I'} \widehat{\square}_I^{\sigma,\flat,\mathbf{b}} f \right\|_{L^2(\sigma)}^2 \lesssim \left\| \widehat{\nabla}_I^\sigma f \right\|_{L^2(\sigma)}^2 .$$

Now we are ready to prove (5.4.2). As $b_A = b_I$ and

$$\begin{aligned} \left\| \widehat{\square}_I^{\sigma,\flat,\mathbf{b}} f \right\|_{L^2(\sigma)}^2 &= \sum_{I' \in \mathfrak{C}_{\mathcal{D}}(I) \cap C_{\mathcal{A}}(A)} \left\| \mathbf{1}_{I'} \widehat{\square}_I^{\sigma,\flat,\mathbf{b}} f \right\|_{L^2(\sigma)}^2 + \sum_{I' \in \mathfrak{C}_{\mathcal{D}}(I) \cap \mathcal{A}} \left\| \mathbf{1}_{I'} \widehat{\square}_I^{\sigma,\flat,\mathbf{b}} f \right\|_{L^2(\sigma)}^2 \\ &\lesssim \left\| b_I \widehat{\square}_I^{\sigma,\flat,\mathbf{b}} f \right\|_{L^2(\sigma)}^2 + \left\| \widehat{\nabla}_I^\sigma f \right\|_{L^2(\sigma)}^2 \end{aligned}$$

we obtain

$$\begin{aligned} \left\| \widehat{\square}_I^{\sigma,\flat,\mathbf{b}} f \right\|_{L^2(\sigma)} &\lesssim \left\| b_I \widehat{\square}_I^{\sigma,\flat,\mathbf{b}} f \right\|_{L^2(\sigma)} + \left\| \widehat{\nabla}_I^\sigma f \right\|_{L^2(\sigma)} = \left\| \square_I^{\sigma,\flat,\mathbf{b}} f \right\|_{L^2(\sigma)} + \left\| \widehat{\nabla}_I^\sigma f \right\|_{L^2(\sigma)} \\ &\leq \left\| \square_I^{\sigma,\mathbf{b}} f \right\|_{L^2(\sigma)} + \left\| \square_{I,broken}^{\sigma,\flat,\mathbf{b}} f \right\|_{L^2(\sigma)} + \left\| \widehat{\nabla}_I^\sigma f \right\|_{L^2(\sigma)} \lesssim \left\| \square_I^{\sigma,\mathbf{b}} f \right\|_{L^2(\sigma)}^{\bigstar} . \end{aligned}$$

$\square$

Now from quasiorthogonality and (5.4.2) we get,

$$\begin{aligned}\sum_{J\in\mathcal{G}}\sum_{J'\in\mathfrak{C}(J)}\left|J'\right|_{\omega}\left|E^{\omega}_{J'}\left(\widehat{\square}^{\omega,\flat,\mathbf{b}^*}_{J}g\right)\right|^2 &\lesssim \sum_{J\in\mathcal{G}}\left\|\widehat{\square}^{\omega,\flat,\mathbf{b}^*}_{J}g\right\|^2_{L^2(\omega)}\lesssim\sum_{J\in\mathcal{G}}\left\|\square^{\omega,\flat,\mathbf{b}^*}_{J}g\right\|^2_{L^2(\omega)}\\ &\lesssim\sum_{J\in\mathcal{G}}\left(\left\|\square^{\omega,\mathbf{b}^*}_{J}g\right\|^2_{L^2(\omega)}+\left\|\nabla^{\omega}_{J}g\right\|^2_{L^2(\omega)}\right)\lesssim\|g\|^2_{L^2(\omega)}\ .\end{aligned}$$

We also need the following lemma, that controls the above inner product for cubes of positive distance.

Lemma 5.4.3. *Given the ∞-weakly accretive families of test functions $\mathbf{b}$ and $\mathbf{b}^*$ and cubes $Q,R\subset\mathbb{R}^n$, we have*

$$|\langle T^{\alpha}_{\sigma}(b_Q\mathbf{1}_Q),b^*_R\mathbf{1}_{R\setminus(1+\delta)Q}\rangle_{\omega}|\lesssim\delta^{\alpha-n}\sqrt{\mathfrak{A}^{\alpha}_2}\sqrt{|Q|_{\sigma}}\sqrt{|R|_{\omega}}\tag{5.4.5}$$

where the implied constant depends on the accretivity constants of the families $\mathbf{b},\mathbf{b}^*$ and the dimension n.

Proof. We have that $\left|\left\langle T^{\alpha}_{\sigma}\left(b_Q\mathbf{1}_Q\right),b^*_R\mathbf{1}_{R\setminus(1+\delta)Q}\right\rangle_{\omega}\right|$

$$\begin{aligned}&\leq\int_{R\setminus(1+\delta)Q}\left|T^{\alpha}_{\sigma}\left(b_Q\mathbf{1}_Q\right)\right|\left|b^*_R\right|d\omega\\ &\leq\left(\int_{R\setminus(1+\delta)Q}\left|T^{\alpha}_{\sigma}\left(b_Q\mathbf{1}_Q\right)\right|^2d\omega\right)^{\frac{1}{2}}\left(\int_{R\setminus(1+\delta)Q}\left|b^*_R\right|^2d\omega\right)^{\frac{1}{2}}\\ &\lesssim\left(\int_{\mathbb{R}^n\setminus(1+\delta)Q}\left(\int_Q|x-y|^{\alpha-n}\left|b_Q\left(y\right)\right|d\sigma\left(y\right)\right)^2d\omega\left(x\right)\right)^{\frac{1}{2}}\left(\int_R\left|b^*_R\right|^2d\omega\right)^{\frac{1}{2}}\end{aligned}$$

$$
\begin{aligned}
&\lesssim \left(\int_{\mathbb{R}^n\setminus(1+\delta)Q}\left(\int_Q\left(\delta\left|x-c_Q\right|\right)^{\alpha-n}\left|b_Q(y)\right|d\sigma(y)\right)^2 d\omega(x)\right)^{\frac{1}{2}}\sqrt{|R|_\omega}\\
&\lesssim \delta^{\alpha-n}\left(\int_{\mathbb{R}^n\setminus(1+\delta)Q}\left|x-c_Q\right|^{2(\alpha-n)}d\omega(x)\right)^{\frac{1}{2}}\left(\int_Q\left|b_Q(y)\right|d\sigma(y)\right)\sqrt{|R|_\omega}\\
&\lesssim \delta^{\alpha-n}\left(\int_{\mathbb{R}^n\setminus(1+\delta)Q}\left|x-c_Q\right|^{2(\alpha-n)}d\omega(x)\right)^{\frac{1}{2}}|Q|_\sigma\sqrt{|R|_\omega}\\
&\leq \delta^{\alpha-n}\sqrt{\mathfrak{A}_2^\alpha}\sqrt{|Q|_\sigma}\sqrt{|R|_\omega}
\end{aligned}
$$

since

$$
\begin{aligned}
\left(\int_{\mathbb{R}^n\setminus(1+\delta)Q}\left|x-c_Q\right|^{2(\alpha-n)}d\omega(x)\right)|Q|_\sigma &= \left(\int_{\mathbb{R}^n\setminus(1+\delta)Q}\left(\frac{|Q|^{\frac{1}{n}}}{\left|x-c_Q\right|^2}\right)^{n-\alpha}d\omega(x)\right)\frac{|Q|_\sigma}{|Q|^{1-\frac{\alpha}{n}}}\\
&\lesssim \mathcal{P}^\alpha(Q,\omega)\frac{|Q|_\sigma}{|Q|^{1-\frac{\alpha}{n}}}\leq \mathcal{A}_2^{\alpha,*}.
\end{aligned}
$$

$\square$

As usual, we continue to write the independent grids for f and g as $\mathcal{D}$ and $\mathcal{G}$ respectively. Write the dual martingale averages $\square_I^{\sigma,\mathbf{b}}f$ and $\square_J^{\omega,\mathbf{b}^*}g$ as linear combinations

$$
\begin{aligned}
\square_I^{\sigma,\mathbf{b}}f &= b_I\sum_{I'\in\mathfrak{C}_{nat}(I)}\mathbf{1}_{I'}\ E_{I'}^\sigma\left(\widehat{\square}_I^{\sigma,\mathbf{b}}f\right)+\sum_{I'\in\mathfrak{C}_{brok}(I)}b_{I'}\ \mathbf{1}_{I'}\widehat{\mathbb{F}}_{I'}^{\sigma,b_{I'}}f-b_I\sum_{I'\in\mathfrak{C}_{brok}(I)}\mathbf{1}_{I'}\widehat{\mathbb{F}}_I^{\sigma,b_I}f,\\
\square_J^{\omega,\mathbf{b}^*}g &= b_J^*\sum_{J'\in\mathfrak{C}_{nat}(J)}\mathbf{1}_{J'}\ E_{J'}^\omega\left(\widehat{\square}_J^{\omega,\mathbf{b}^*}g\right)+\sum_{J'\in\mathfrak{C}_{brok}(J)}b_{J'}^*\ \mathbf{1}_{J'}\widehat{\mathbb{F}}_{J'}^{\omega,b_{J'}^*}g-b_J^*\sum_{J'\in\mathfrak{C}_{brok}(J)}\mathbf{1}_{J'}\widehat{\mathbb{F}}_J^{\omega,b_J^*}g,
\end{aligned}
$$

of the appropriate function b times the indicators of their children, denoted I' and J' respectively. We will regroup the terms as needed below.

On the natural child I', the expression $\widehat{\square}_I^{\sigma,\mathbf{b}}f=\frac{1}{b_I}\square_I^{\sigma,\mathbf{b}}f$ simply denotes the dual martingale average with b_I removed, so that we need not assume $|b_I|$ is bounded below in order to make sense of $\frac{1}{b_I}\square_I^{\sigma,\mathbf{b}}f$. Similar comments apply to the expressions $\widehat{\mathbb{F}}_{I'}^{\sigma,b_{I'}}f=\frac{1}{b_{I'}}\mathbb{F}_{I'}^{\sigma,b_{I'}}f$ and $\widehat{\mathbb{F}}_{I}^{\sigma,b_{I}}f=\frac{1}{b_{I}}\mathbb{F}_{I}^{\sigma,b_{I}}f$. Now if we set

$$\mathcal{N}(I)=\{J\in\mathcal{G}:\ 2^{-\mathbf{r}n}|I|<|J|\leq|I|,d\left(J,I\right)\leq 2\ell(J)^{\varepsilon}\ell(I)^{1-\varepsilon}\}$$

for the cubes or similar size to I, the left hand side of (5.4.1) is bounded by

$$\begin{aligned}\mathbf{I}+\mathbf{II}\ \equiv\ &\sum_{I\in\mathcal{D}}\sum_{\substack{J\in\mathcal{N}(I)\\(1+\delta)I\cap J=\emptyset}}\left|\left\langle T_\sigma^\alpha\left(\square_I^{\sigma,\mathbf{b}}f\right),\square_J^{\omega,\mathbf{b}^*}g\right\rangle_\omega\right| \\ &+\sum_{I\in\mathcal{D}}\sum_{\substack{J\in\mathcal{N}(I)\\(1+\delta)I\cap J\neq\emptyset}}\left|\left\langle T_\sigma^\alpha\left(\square_I^{\sigma,\mathbf{b}}f\right),\square_J^{\omega,\mathbf{b}^*}g\right\rangle_\omega\right|\end{aligned}\tag{5.4.6}$$

When working in higher dimensions, run the proof pretending you have Hytönen's estimate (which is of course not true due to the result in chapter 4). Then wherever we were supposed to use Hytönen, we use the delta separation trick. The δ-separated part is easily seen to be bounded by the Muckenhoupt conditions, and the δ-close part will get a $\sqrt{\delta}$ estimate. But δ can be chosen at the end, is independent of everything else (it is the Hytönen-delta, not related to anything else in the proof). So, provided the proof only deals with finite estimates and finitely many constructions (like the Cantor set construction, that only does finitely many iterations), those $\sqrt{\delta}$ terms will be absorbable at the end. Here are the details:

5.4.1 The case of δ-separated cubes.

In this subsection we are estimating $\mathbf{I}$ in (5.4.6) by using Lemma 5.4.3.

Definition 5.4.4. *We say that the cubes J and I are* δ-separated*, where $\delta>0$, if $J\cap(1+$*

$\delta)I=\emptyset$.

For the first sum in (5.4.6) we have, following the proof of Lemma 5.4.3, the satisfactory estimate

$$\left|\left\langle T_{\sigma}^{\alpha}\left(\square_{I}^{\sigma,\mathbf{b}}f\right),\square_{J}^{\omega,\mathbf{b}^{*}}g\right\rangle_{\omega}\right|\lesssim\delta^{\alpha-n}\sqrt{\mathfrak{A}_{2}^{\alpha}}\left\|\square_{I}^{\sigma,\mathbf{b}}f\right\|_{L^{2}(\sigma)}\left\|\square_{J}^{\omega,\mathbf{b}^{*}}g\right\|_{L^{2}(\omega)}.$$

Indeed,

$$\begin{aligned}
&\left|\left\langle T_{\sigma}^{\alpha}\left(\square_{I}^{\sigma,\mathbf{b}}f\right),\square_{J}^{\omega,\mathbf{b}^{*}}g\right\rangle_{\omega}\right| \\
\leq\ & \int_{J\setminus(1+\delta)I}\left|T_{\sigma}^{\alpha}\left(\square_{I}^{\sigma,\mathbf{b}}f\right)\right|\left|\square_{J}^{\omega,\mathbf{b}^{*}}g\right|d\omega \\
\leq\ & \left(\int_{J\setminus(1+\delta)I}\left|T_{\sigma}^{\alpha}\left(\square_{I}^{\sigma,\mathbf{b}}f\right)\right|^{2}d\omega\right)^{\frac{1}{2}}\left(\int_{J}\left|\square_{J}^{\omega,\mathbf{b}^{*}}g\right|^{2}d\omega\right)^{\frac{1}{2}} \\
\lesssim\ & \delta^{\alpha-n}\left(\int_{\mathbb{R}^{n}\setminus(1+\delta)I}\left|x-c_{I}\right|^{2(\alpha-n)}d\omega\left(x\right)\right)^{\frac{1}{2}}\left(\int_{I}\left|\square_{I}^{\sigma,\mathbf{b}}f\right|d\sigma\left(y\right)\right)\left|\left|\square_{J}^{\omega,\mathbf{b}^{*}}g\right|\right|_{L^{2}(\omega)} \\
\lesssim\ & \delta^{\alpha-n}\left(\int_{\mathbb{R}^{n}\setminus(1+\delta)I}\left|x-c_{I}\right|^{2(\alpha-n)}d\omega\left(x\right)\right)^{\frac{1}{2}}\sqrt{\left|I\right|_{\sigma}}\left|\left|\square_{I}^{\sigma,\mathbf{b}}f\right|\right|_{L^{2}(\sigma)}\left|\left|\square_{J}^{\omega,\mathbf{b}^{*}}g\right|\right|_{L^{2}(\omega)} \\
\leq\ & \delta^{\alpha-n}\sqrt{\mathfrak{A}_{2}^{\alpha}}\left|\left|\square_{I}^{\sigma,\mathbf{b}}f\right|\right|_{L^{2}(\sigma)}\left|\left|\square_{J}^{\omega,\mathbf{b}^{*}}g\right|\right|_{L^{2}(\omega)}
\end{aligned}$$

So combining all the above we get for the δ-separated cubes that

$$
\begin{aligned}
\mathbf{I} \ \leq\ & \sum_{I\in\mathcal{D}} \sum_{\substack{J\in\mathcal{N}(I)\\(1+\delta)I\cap J=\emptyset}} \delta^{\alpha-n}\sqrt{\mathfrak{A}_2^\alpha}\left\|\square_I^{\sigma,\mathbf{b}}f\right\|_{L^2(\sigma)}\left\|\square_J^{\omega,\mathbf{b}^*}g\right\|_{L^2(\omega)} \\
\leq & \delta^{\alpha-n}\sqrt{\mathfrak{A}_2^\alpha}\left(\sum_{I\in\mathcal{D}} \sum_{\substack{J\in\mathcal{N}(I)\\(1+\delta)I\cap J=\emptyset}}\left\|\square_I^{\sigma,\mathbf{b}}f\right\|_{L^2(\sigma)}^2\right)^{\frac{1}{2}}\left(\sum_{I\in\mathcal{D}} \sum_{\substack{J\in\mathcal{N}(I)\\(1+\delta)I\cap J=\emptyset}}\left\|\square_J^{\omega,\mathbf{b}^*}g\right\|_{L^2(\omega)}^2\right)^{\frac{1}{2}} \\
\lesssim & \delta^{\alpha-n}\sqrt{\mathfrak{A}_2^\alpha}||f||_{L^2(\sigma)}||g||_{L^2(\omega)}
\end{aligned}
\tag{5.4.7}
$$

where the implied constant in the last line depends only on the goodness parameter $\mathbf{r}$ and the finite repetition of I and J in each sum respectively.

5.4.2 The case of δ-close cubes.

Now we turn to the second sum in (5.4.6) which we will bound by using random surgery and expectation.

Definition 5.4.5. *We say that the cubes J and I are δ-close, if $J\cap(1+\delta)I\neq\emptyset$.*

We have

$$
\begin{aligned}
\left\langle T_\sigma^\alpha\left(\square_I^{\sigma,\mathbf{b}}f\right),\square_J^{\omega,\mathbf{b}^*}g\right\rangle_\omega = & \left\langle T_\sigma^\alpha\left(\square_I^{\sigma,\flat,\mathbf{b}}f\right),\square_J^{\omega,\flat,\mathbf{b}^*}g\right\rangle_\omega \\
& + \left\langle T_\sigma^\alpha\left(\square_{I,brok}^{\sigma,\flat,\mathbf{b}}f\right),\square_{J,brok}^{\omega,\flat,\mathbf{b}^*}g\right\rangle_\omega \\
& + \left\langle T_\sigma^\alpha\left(\square_I^{\sigma,\flat,\mathbf{b}}f\right),\square_{J,brok}^{\omega,\flat,\mathbf{b}^*}g\right\rangle_\omega \\
& + \left\langle T_\sigma^\alpha\left(\square_{I,brok}^{\sigma,\flat,\mathbf{b}}f\right),\square_J^{\omega,\flat,\mathbf{b}^*}g\right\rangle_\omega .
\end{aligned}
\tag{5.4.8}
$$

The estimation of the latter three inner products, i.e. those in which a broken operator $\square_{I,brok}^{\sigma,\flat,\mathbf{b}}$ or $\square_{J,brok}^{\omega,\flat,\mathbf{b}^*}$ arises, is simpler, but still requires the use of random surgery in order to avoid the full testing condition that was available in one dimension. Indeed, recall that

$$\begin{aligned}
\square_{I,brok}^{\sigma,\flat,\mathbf{b}} f &= \sum_{I' \in \mathfrak{C}_{brok}(I)} \mathbb{F}_{I'}^{\sigma,\mathbf{b}} f = \sum_{I' \in \mathfrak{C}_{brok}(I)} \left(E_{I'}^{\sigma} \widehat{\mathbb{F}}_{I'}^{\sigma,\mathbf{b}} f \right) b_{I'} \\
\square_{J,brok}^{\omega,\flat,\mathbf{b}^*} g &= \sum_{J' \in \mathfrak{C}_{brok}(J)} \mathbb{F}_{J'}^{\omega,\mathbf{b}^*} g = \sum_{J' \in \mathfrak{C}_{brok}(J)} \left(E_{J'}^{\omega} \widehat{\mathbb{F}}_{J'}^{\omega,\mathbf{b}^*} g \right) b_{J'}^*
\end{aligned}$$

so that if at least one broken difference appears in the inner product, as is the case for the latter three inner products in (5.4.8), we need to use random surgery to get the necessary bound. For example, the fourth term satisfies

$$\left| \left\langle T_\sigma^\alpha \left(\square_{I,brok}^{\sigma,\flat,\mathbf{b}} f \right), \square_J^{\omega,\flat,\mathbf{b}^*} g \right\rangle_\omega \right| = \left| \sum_{I' \in \mathfrak{C}_{brok}(I)} \left(E_{I'}^{\sigma} \widehat{\mathbb{F}}_{I'}^{\sigma,\mathbf{b}} f \right) \left\langle T_\sigma^\alpha b_{I'}, \square_J^{\omega,\flat,\mathbf{b}^*} g \right\rangle_\omega \right|$$

and since

$$\begin{aligned}
\left\langle T_\sigma^\alpha b_{I'}, \square_J^{\omega,\flat,\mathbf{b}^*} g \right\rangle_\omega &= \left\langle \mathbf{1}_{I' \cap J} T_\sigma^\alpha b_{I'}, \square_J^{\omega,\flat,\mathbf{b}^*} g \right\rangle_\omega + \left\langle \mathbf{1}_{J \setminus (1+\delta) I'} T_\sigma^\alpha b_{I'}, \square_J^{\omega,\flat,\mathbf{b}^*} g \right\rangle_\omega \\
&\quad + \left\langle \mathbf{1}_{(J \setminus I') \cap (1+\delta) I'} T_\sigma^\alpha b_{I'}, \square_J^{\omega,\flat,\mathbf{b}^*} g \right\rangle_\omega \\
&\equiv A(f,g) + B(f,g) + C(f,g)
\end{aligned}$$

we have

$$\begin{aligned}
&\left|\sum_{I'\in\mathfrak{C}_{brok}(I)}\left(E^{\sigma}_{I'}\widehat{\mathbb{F}}^{\sigma,\mathbf{b}}_{I'}f\right)A(f,g)\right|\\
\leq\;& C_{\mathbf{b},\mathbf{b}^*}\sum_{I'\in\mathfrak{C}_{brok}(I)}\left|E^{\sigma}_{I'}\widehat{\mathbb{F}}^{\sigma,\mathbf{b}}_{I'}f\right|\mathfrak{T}^{\mathbf{b}}_{T^\alpha}\sqrt{|I'|_\sigma}\left\|\square^{\omega,\flat,\mathbf{b}^*}_{J}g\right\|_{L^2(\omega)}\\
\leq\;& \mathfrak{T}^{\mathbf{b}}_{T^\alpha}\left\|\nabla^{\sigma}_{I}f\right\|_{L^2(\sigma)}\left(\sum_{I'\in\mathfrak{C}_{brok}(I)}\left(\left\|\square^{\omega,\mathbf{b}^*}_{J}g\right\|^2_{L^2(\omega)}+\left\|\square^{\omega,\flat,\mathbf{b}^*}_{J,brok}g\right\|^2_{L^2(\omega)}\right)\right)^{\frac{1}{2}}\\
\lesssim\;& \mathfrak{T}^{\mathbf{b}}_{T^\alpha}\left\|\square^{\sigma,\mathbf{b}}_{I}f\right\|^{\bigstar}_{L^2(\sigma)}\left\|\square^{\omega,\mathbf{b}^*}_{J}g\right\|^{\bigstar}_{L^2(\omega)}
\end{aligned}$$

Next by Lemma 5.4.3,

$$\begin{aligned}
\left|\sum_{I'\in\mathfrak{C}_{brok}(I)}\left(E^{\sigma}_{I'}\widehat{\mathbb{F}}^{\sigma,\mathbf{b}}_{I'}f\right)B(f,g)\right| &\leq \sum_{I'\in\mathfrak{C}_{brok}(I)}\left|E^{\sigma}_{I'}\widehat{\mathbb{F}}^{\sigma,\mathbf{b}}_{I'}f\right|\delta^{\alpha-n}\sqrt{\mathfrak{A}^{\alpha}_{2}}\sqrt{|I'|_\sigma}\left\|\square^{\omega,\flat,\mathbf{b}^*}_{J}g\right\|_{L^2(\omega)}\\
&\leq\; \delta^{\alpha-n}\sqrt{\mathfrak{A}^{\alpha}_{2}}\left\|\square^{\sigma,\mathbf{b}}_{I}f\right\|^{\bigstar}_{L^2(\sigma)}\left\|\square^{\omega,\mathbf{b}^*}_{J}g\right\|^{\bigstar}_{L^2(\omega)}
\end{aligned}$$

Finally, using Cauchy-Schwarz, the norm inequality and accretivity we get

$$\begin{aligned}
&\sum_{I\in\mathcal{D}}\sum_{\substack{J\in\mathcal{N}(I)\\ I\cap J\neq\emptyset}}\left|\sum_{I'\in\mathfrak{C}_{brok}(I)}\left(E^{\sigma}_{I'}\widehat{\mathbb{F}}^{\sigma,\mathbf{b}}_{I'}f\right)C(f,g)\right| \\
\leq\;& C_{\mathbf{b}}\mathfrak{N}_{T^{\alpha}}\sum_{I\in\mathcal{D}}\sum_{\substack{J\in\mathcal{N}(I)\\ I\cap J\neq\emptyset}}\sum_{I'\in\mathfrak{C}_{brok}(I)}\left|E^{\sigma}_{I'}\widehat{\mathbb{F}}^{\sigma,\mathbf{b}}_{I'}f\right|\sqrt{|I'|_{\sigma}}\,\cdot \\
&\cdot\left(\sum_{J'\in\mathfrak{C}(J)}\left[E^{\omega}_{J'}\left(\widehat{\square}^{\omega,\flat,\mathbf{b}^*}_{J}g\right)\right]^2\left|\left((J\backslash I')\cap(1+\delta)I'\right)\cap J'\right|_{\omega}\right)^{\frac{1}{2}} \\
\leq\;& C_{\mathbf{b},\mathbf{r},n}\mathfrak{N}_{T^{\alpha}}||f||_{L^2(\sigma)}\,\cdot \\
&\cdot\left(\sum_{I\in\mathcal{D}}\sum_{\substack{J\in\mathcal{N}(I)\\ I\cap J\neq\emptyset}}\sum_{I'\in\mathfrak{C}_{brok}(I)}\sum_{J'\in\mathfrak{C}(J)}\left[E^{\omega}_{J'}\left(\widehat{\square}^{\omega,\flat,\mathbf{b}^*}_{J}g\right)\right]^2\left|\left((J\backslash I')\cap(1+\delta)I'\right)\cap J'\right|_{\omega}\right)^{\frac{1}{2}}.
\end{aligned}$$

Now, it is geometrically evident that for the Lebesque measure we have

$$\left|\left((J\backslash I')\cap(1+\delta)I'\right)\cap J'\right|\lesssim\delta|J'|.$$

Taking averages over the grid $\mathcal{D}$ we get the same inequality for the ω measure:

$$\boldsymbol{E}^{\mathcal{D}}_{\Omega}\left|\left((J\backslash I')\cap(1+\delta)I'\right)\cap J'\right|_{\omega}\lesssim\delta\left|J'\right|_{\omega}.$$

Thus, if we fix J', there are only finitely many I' involved that contribute (are non-zero), and then the expectation in $\mathcal{D}$ can "go through" the sum in I' to get the estimate

$$\boldsymbol{E}^{\mathcal{D}}_{\Omega}\sum_{I\in\mathcal{D}}\sum_{\substack{J\in\mathcal{N}(I)\\ I\cap J\neq\emptyset}}\left|\sum_{I'\in\mathfrak{C}_{brok}(I)}\left(E^{\sigma}_{I'}\widehat{\mathbb{F}}^{\sigma,\mathbf{b}}_{I'}f\right)C(f,g)\right|\leq C_{\mathbf{b},\mathbf{r},n}\sqrt{\delta}\mathfrak{N}_{T^{\alpha}}||f||_{L^2(\sigma)}||g||_{L^2(\omega)}.$$

The constant $C_{\mathbf{b},\mathbf{r},n}$ depends on the accretivity constant of the family $\mathbf{b}$, the dimension n and the *finite* repetition of the intervals J' appearing in the sum.

The third term in (5.4.8) is handled similarly if we change to $\left\langle \square_I^{\sigma,\flat,\mathbf{b}} f, T_\omega^{\alpha,*}\left(\square_{J,brok}^{\omega,\flat,\mathbf{b}^*} g\right)\right\rangle_\sigma$, the dual operator. For the second term in (5.4.8) the proof is somewhat different: it does not use probability, it is easier because the terms involving g can be estimated as the terms involving f in the proof just done for the fourth term, and then using Carleson estimates. So combining the above we get the following

$$\begin{aligned}
&\boldsymbol{E}_\Omega^{\mathcal{D}} \sum_{I\in\mathcal{D}} \sum_{\substack{J\in\mathcal{N}(I)\\(1+\delta)I\cap J\neq\emptyset}} \left|\left\langle T_\sigma^\alpha\left(\square_I^{\sigma,\mathbf{b}} f\right), \square_J^{\omega,\mathbf{b}^*} g\right\rangle_\omega\right| \qquad (5.4.9)\\
\leq\ & \sum_{I\in\mathcal{D}} \sum_{\substack{J\in\mathcal{N}(I)\\(1+\delta)I\cap J\neq\emptyset}} \left|\left\langle T_\sigma^\alpha\left(\square_I^{\sigma,\flat,\mathbf{b}} f\right), \square_J^{\omega,\flat,\mathbf{b}^*} g\right\rangle_\omega\right| \\
&+\left(C_{\mathbf{b},\mathbf{r},n}\sqrt{\delta}\mathfrak{N}_{T^\alpha}+(\delta^{\alpha-n}+1)\mathcal{NTV}_\alpha\right)||f||_{L^2(\sigma)}||g||_{L^2(\omega)}
\end{aligned}$$

Thus it remains to consider the first inner product $\left\langle T_\sigma^\alpha\left(\square_I^{\sigma,\flat,\mathbf{b}} f\right), \square_J^{\omega,\flat,\mathbf{b}^*} g\right\rangle_\omega$ on the right hand side of (5.4.9), which we call the problematic term, and write it as

$$\begin{aligned}
P(I,J) &\equiv \left\langle T_\sigma^\alpha\left(\square_I^{\sigma,\flat,\mathbf{b}} f\right), \square_J^{\omega,\flat,\mathbf{b}^*} g\right\rangle_\omega \\
&= \sum_{I'\in\mathfrak{C}(I),J'\in\mathfrak{C}(J)} \left\langle T_\sigma^\alpha\left(\mathbf{1}_{I'}\square_I^{\sigma,\flat,\mathbf{b}} f\right), \mathbf{1}_{J'}\square_J^{\omega,\flat,\mathbf{b}^*} g\right\rangle_\omega \\
&= \sum_{I'\in\mathfrak{C}(I),J'\in\mathfrak{C}(J)} E_{I'}^\sigma\left(\widehat{\square}_I^{\sigma,\flat,\mathbf{b}} f\right)\left\langle T_\sigma^\alpha\left(\mathbf{1}_{I'}b_I\right), \mathbf{1}_{J'}b_J^*\right\rangle_\omega\ E_{J'}^\omega\left(\widehat{\square}_J^{\omega,\flat,\mathbf{b}^*} g\right) \qquad (5.4.10)
\end{aligned}$$

It now remains to show that

$$\boldsymbol{E}_\Omega^{\mathcal{D}}\boldsymbol{E}_\Omega^{\mathcal{G}} \sum_{I\in\mathcal{D}} \sum_{J\in\mathcal{N}(I)} |P(I,J)| \lesssim \left(C_\theta\mathcal{NTV}_\alpha+\sqrt{\theta}\mathfrak{N}_{T^\alpha}\right)\|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)}. \qquad (5.4.11)$$

Suppose now that $I \in \mathcal{C}_A$ for $A \in \mathcal{A}$, and that $J \in \mathcal{C}_B$ for $B \in \mathcal{B}$. Then the inner product in the third line of (5.4.10) becomes

$$\left\langle T_\sigma^\alpha \left(b_I \mathbf{1}_{I'} \right), b_J^* \mathbf{1}_{J'} \right\rangle_\omega = \left\langle T_\sigma^\alpha \left(b_A \mathbf{1}_{I'} \right), b_B^* \mathbf{1}_{J'} \right\rangle_\omega \ ,$$

and we will write this inner product in either form, depending on context. We also introduce the following notation:

$$P_{(I,J)}\left(E,F\right) \equiv \left\langle T_\sigma^\alpha \left(b_I \mathbf{1}_E \right), b_J^* \mathbf{1}_F \right\rangle_\omega , \qquad \text{for any sets } E \text{ and } F,$$

so that

$$P\left(I,J\right) = \sum_{I' \in \mathfrak{C}(I) \text{ and } J' \in \mathfrak{C}(J)} E_{I'}^\sigma \left(\widehat{\square}_I^{\sigma,\flat,\mathbf{b}} f \right) \ P_{(I,J)}\left(I',J'\right) \ E_{J'}^\omega \left(\widehat{\square}_J^{\omega,\flat,\mathbf{b}^*} g \right).$$

The first thing we do is reduce matters to showing inequality (5.4.11) in the case that $P_{(I,J)}\left(I',J'\right)$ is replaced by

$$P_{(I,J)}\left(I' \cap J', I' \cap J'\right)$$

in the terms $P\left(I,J\right)$ appearing in (5.4.11). To see this, write $\left\langle T_\sigma^\alpha \left(b_I \mathbf{1}_{I'} \right), b_J^* \mathbf{1}_{J'} \right\rangle_\omega$ as

$$\left\langle T_\sigma^\alpha \left(b_I \mathbf{1}_{I' \setminus J'} \right), b_J^* \mathbf{1}_{J'} \right\rangle_\omega + \left\langle T_\sigma^\alpha \left(b_I \mathbf{1}_{I' \cap J'} \right), b_J^* \mathbf{1}_{J' \setminus I'} \right\rangle_\omega + \left\langle T_\sigma^\alpha \left(b_I \mathbf{1}_{I' \cap J'} \right), b_J^* \mathbf{1}_{I' \cap J'} \right\rangle_\omega$$

Set

$$\mathrm{I} = \left\langle T_\sigma^\alpha \left(b_I \mathbf{1}_{I' \setminus J'} \right), b_J^* \mathbf{1}_{J'} \right\rangle_\omega$$

$$\mathrm{II} = \left\langle T_\sigma^\alpha \left(b_I \mathbf{1}_{I' \cap J'} \right), b_J^* \mathbf{1}_{J' \setminus I'} \right\rangle_\omega \ \text{ and } \ \mathrm{III} = \left\langle T_\sigma^\alpha \left(b_I \mathbf{1}_{I' \cap J'} \right), b_J^* \mathbf{1}_{I' \cap J'} \right\rangle_\omega$$

For the first one, we have

$$\mathrm{I} \le \left|\left\langle T^{\alpha}_{\sigma}\left(b_I \mathbf{1}_{I'\setminus(1+\delta)J'}\right), b^*_J \mathbf{1}_{J'}\right\rangle_{\omega}\right| + \left|\left\langle T^{\alpha}_{\sigma}\left(b_I \mathbf{1}_{(I'\setminus J')\cap(1+\delta)J'}\right), b^*_J \mathbf{1}_{J'}\right\rangle_{\omega}\right| \equiv \mathrm{I}_1 + \mathrm{I}_2$$

Using Lemma 5.4.3, $\mathrm{I}_1 \lesssim \delta^{\alpha-n}\sqrt{\mathfrak{A}^{\alpha}_2}\sqrt{|I'|_{\sigma}}\sqrt{|J'|_{\omega}}$ and for I_2 we need to use random surgery. Summing all the terms for I_2 and using Lemma 5.4.2, we have

$$\begin{aligned}
&\boldsymbol{E}^{\mathcal{G}}_{\Omega}\sum_{I\in\mathcal{D}}\sum_{J\in\mathcal{N}(I)}\sum_{I'\in\mathfrak{C}(I)}\sum_{J'\in\mathfrak{C}(J)} \mathfrak{N}_{T^{\alpha}}\left|E^{\sigma}_{I'}\left(\widehat{\square}^{\sigma,\flat,\mathbf{b}}_{I} f\right)\right|\left(\int_{(I'\setminus J')\cap(1+\delta)J'}|b_I|^2 d\sigma\right)^{\frac{1}{2}}\cdot \qquad (5.4.12)\\
&\quad\cdot\left|\ E^{\omega}_{J'}\left(\widehat{\square}^{\omega,\flat,\mathbf{b}^*}_{J} g\right)\right|\left(\int_{J'}|b_J|^2 d\omega\right)^{\frac{1}{2}}\\
&\lesssim \mathfrak{N}_{T^{\alpha}}\boldsymbol{E}^{\mathcal{G}}_{\Omega}\sum_{I\in\mathcal{D}}\sum_{J\in\mathcal{N}(I)}\sum_{I'\in\mathfrak{C}(I)}\sum_{J'\in\mathfrak{C}(J)}\left|E^{\sigma}_{I'}\left(\widehat{\square}^{\sigma,\flat,\mathbf{b}}_{I} f\right)\right|\cdot\\
&\quad\cdot\left|(I'\setminus J')\cap(1+\delta)J'\right|^{\frac{1}{2}}_{\sigma}\left|\ E^{\omega}_{J'}\left(\widehat{\square}^{\omega,\flat,\mathbf{b}^*}_{J} g\right)\right||J'|^{\frac{1}{2}}_{\omega}\\
&\le \mathfrak{N}_{T^{\alpha}}\boldsymbol{E}^{\mathcal{G}}_{\Omega}\left(\sum\left[E^{\sigma}_{I'}\left(\widehat{\square}^{\sigma,\flat,\mathbf{b}}_{I} f\right)\right]^2\left|(I'\setminus J')\cap(1+\delta)J'\right|_{\sigma}\right)^{\frac{1}{2}}\left(\sum\left[E^{\omega}_{J'}\left(\widehat{\square}^{\omega,\flat,\mathbf{b}^*}_{J} g\right)\right]^2|J'|_{\omega}\right)^{\frac{1}{2}}\\
&\le \mathfrak{N}_{T^{\alpha}}C_{n,\mathbf{r}}||g||_{L^2(\omega)}\left(\sum_{I}\sum_{I'}\left[E^{\sigma}_{I'}\left(\widehat{\square}^{\sigma,\flat,\mathbf{b}}_{I} f\right)\right]^2\boldsymbol{E}^{\mathcal{G}}_{\Omega}\sum_{J}\sum_{J'}\left|(I'\setminus J')\cap(1+\delta)J'\right|_{\sigma}\right)^{\frac{1}{2}}\\
&\le \mathfrak{N}_{T^{\alpha}}C_{n,\mathbf{r}}||g||_{L^2(\omega)}\left(\sum_{I}\sum_{I'}\left[E^{\sigma}_{I'}\left(\widehat{\square}^{\sigma,\flat,\mathbf{b}}_{I} f\right)\right]^2\delta|I'|_{\sigma}\right)^{\frac{1}{2}}\\
&\le \mathfrak{N}_{T^{\alpha}}C_{n,\mathbf{r}}\sqrt{\delta}||f||_{L^2(\sigma)}||g||_{L^2(\omega)}
\end{aligned}$$

Similarly, we get the bound for II.

We are left then with III where we are integrating over $I'\cap J'$. We have to overcome two difficulties at this step. First, $I'\cap J'$ is not necessarily a cube, so we cannot apply any of the testing conditions available. Second, $I'\cap J'$, even if it is a cube, does not need to belong in either of the grids $\mathcal{D}$ or $\mathcal{G}$. We would like to split $I'\cap J'$ in smaller cubes of the grid $\mathcal{G}$.

The problem is that the boundary of $I' \cap J'$ does not necessarily align with the grid $\mathcal{G}$. To deal with this, we cut a slice around $I' \cap J'$ so that what is left inside can be split in cubes of the grid $\mathcal{G}$. This small slice will be bounded using once again random surgery. While for the remaining cubes, we will use a more involved random surgery technique along with the $\mathcal{A}_2$ and testing condition.

Here are the details: Let $\eta_0 = 2^{-m}$ for m large enough. For any cube L we define the $\overrightarrow{\eta_1}$-halo for $\overrightarrow{\eta_1} = (\eta_1^1, \ldots, \eta_1^n)$ by

$$\partial_{\overrightarrow{\eta_1}} L \;\; = \;\; (1 + \overrightarrow{\eta_1})L - (1 - \overrightarrow{\eta_1})L$$

where $(1 + \overrightarrow{\eta_1})L$ means a dilation of each coordinate of L according to the corresponding coordinates of $1 + \overrightarrow{\eta_1}$. Choose the coordinates of $\overrightarrow{\eta_1}$ such that $\frac{\eta_0}{2} \leq \eta_1^i < \eta_0$ for all $1 \leq i \leq n$ and such that if

$$I' \cap J' = \left[\left(I' \backslash \partial_{\overrightarrow{\eta_1}} I' \right) \cap J' \right] \dot{\cup} \left[\left(\partial_{\overrightarrow{\eta_1}} I' \cap I' \right) \cap J' \right] \equiv M \dot{\cup} L \tag{5.4.13}$$

then M consists of $B \lesssim 2^{n \cdot m}$ cubes $K_s \in \mathcal{G}$ with $\ell(K_s) \geq 2^{-m-1} \ell(J')$. Note that either M or L might be empty depending on where J' is located, but this is not a problem. Thus

$$\begin{aligned} \left\langle T_\sigma^\alpha \left(b_I \mathbf{1}_{I' \cap J'} \right), b_J^* \mathbf{1}_{I' \cap J'} \right\rangle_\omega = & \left\langle T_\sigma^\alpha \left(b_I \mathbf{1}_M \right), b_J^* \mathbf{1}_L \right\rangle_\omega + \left\langle T_\sigma^\alpha \left(b_I \mathbf{1}_L \right), b_J^* \mathbf{1}_M \right\rangle_\omega \\ & + \left\langle T_\sigma^\alpha \left(b_I \mathbf{1}_L \right), b_J^* \mathbf{1}_L \right\rangle_\omega + \left\langle T_\sigma^\alpha \left(b_I \mathbf{1}_M \right), b_J^* \mathbf{1}_M \right\rangle_\omega \end{aligned}$$

The first two can be estimated using Lemma 5.4.3 and a random surgery. It is important to mention here that the averages will be taken on the grid $\mathcal{D}$, so that we do not have common intersection among the different translations of the halo. Indeed,

$$\begin{aligned}
\left\langle T_\sigma^\alpha \left(b_I \mathbf{1}_M\right), b_J^* \mathbf{1}_L\right\rangle_\omega &= \left\langle T_\sigma^\alpha \left(b_I \mathbf{1}_M\right), b_J^* \mathbf{1}_{L\setminus(1+\delta)M}\right\rangle_\omega + \left\langle T_\sigma^\alpha \left(b_I \mathbf{1}_M\right), b_J^* \mathbf{1}_{L\cap(1+\delta)M}\right\rangle_\omega \\
&\equiv \mathrm{A}_1 + \mathrm{A}_2
\end{aligned}$$

and

$$\begin{aligned}
\left\langle T_\sigma^\alpha \left(b_I \mathbf{1}_L\right), b_J^* \mathbf{1}_M\right\rangle_\omega &= \left\langle T_\sigma^\alpha \left(b_I \mathbf{1}_L\right), b_J^* \mathbf{1}_{M\setminus(1+\delta)L}\right\rangle_\omega + \left\langle T_\sigma^\alpha \left(b_I \mathbf{1}_L\right), b_J^* \mathbf{1}_{M\cap(1+\delta)L}\right\rangle_\omega \\
&\equiv \mathrm{A}_3 + \mathrm{A}_4
\end{aligned}$$

The first terms on the right hand side of both displays, A_1 and A_3, are bounded, by applying the proof of Lemma 5.4.3 for M and L and using the fact that M consists of $B \lesssim 2^{nm}$ cubes. The bound is a constant multiple of $2^n \delta^{\alpha-n}\sqrt{\mathfrak{A}_2^\alpha}\sqrt{|I'|_\sigma}\sqrt{|J'|_\omega}$, which when plugged into the left hand side of (5.4.11) we get by using Cauchy-Schwarz that

$$\begin{aligned}
&\sum_{I\in\mathcal{D}}\sum_{J\in\mathcal{N}(I)}\sum_{\substack{I'\in\mathfrak{C}(I)\\ J'\in\mathfrak{C}(J)}}\left|E_{I'}^\sigma\left(\widehat{\square}_I^{\sigma,\flat,\mathbf{b}} f\right)\right|\left(\mathrm{A}_1+\mathrm{A}_3\right)\left| E_{J'}^\omega\left(\widehat{\square}_J^{\omega,\flat,\mathbf{b}^*} g\right)\right| \qquad (5.4.14)\\
\lesssim &\sum_{I\in\mathcal{D}}\sum_{J\in\mathcal{N}(I)}\sum_{\substack{I'\in\mathfrak{C}(I)\\ J'\in\mathfrak{C}(J)}}\left|E_{I'}^\sigma\left(\widehat{\square}_I^{\sigma,\flat,\mathbf{b}} f\right)\right|\delta^{\alpha-n}\sqrt{\mathfrak{A}_2^\alpha}\sqrt{|I'|_\sigma}\sqrt{|J'|_\omega}\left|E_{J'}^\omega\left(\widehat{\square}_J^{\omega,\flat,\mathbf{b}^*} g\right)\right|\\
\lesssim &\ \delta^{\alpha-n}\sqrt{\mathfrak{A}_2^\alpha}||f||_{L^2(\sigma)}||g||_{L^2(\omega)}
\end{aligned}$$

For A_2 (and similarly for A_4), we have

$$\begin{aligned}
&\boldsymbol{E}_\Omega^{\mathcal{D}} \sum_{I\in\mathcal{D}} \sum_{J\in\mathcal{N}(I)} \sum_{I'\in\mathfrak{C}(I),J'\in\mathfrak{C}(J)} \left| E_{I'}^{\sigma}\left(\widehat{\square}_I^{\sigma,\flat,\mathbf{b}} f\right)\right| \cdot \left|\left| T_\sigma^\alpha (b_I \mathbf{1}_M)\right|\right|_{L^2(\omega)} \cdot \qquad (5.4.15)\\
&\qquad \cdot \left|\left| b_J^* \mathbf{1}_{L\cap(1+\delta)M}\right|\right|_{L^2(\omega)} \left| E_{J'}^{\omega}\left(\widehat{\square}_J^{\omega,\flat,\mathbf{b}^*} g\right)\right| \\
\leq\ & \mathfrak{N}_{T^\alpha} C_{\mathbf{b}} \boldsymbol{E}_\Omega^{\mathcal{D}} \sum_{\substack{I'\in\mathfrak{C}(I)\&J'\in\mathfrak{C}(J)\\ J\in\mathcal{N}(I)}} \left| E_{I'}^{\sigma}\left(\widehat{\square}_I^{\sigma,\flat,\mathbf{b}} f\right)\right| \left| M\right|_\sigma^{\frac{1}{2}} \left| L\cap(1+\delta)M\right|_\omega^{\frac{1}{2}} \left| E_{J'}^{\omega}\left(\widehat{\square}_J^{\omega,\flat,\mathbf{b}^*} g\right)\right| \\
\leq\ & \mathfrak{N}_{T^\alpha} C_{\mathbf{b},\mathbf{b}^*,\mathbf{r},n} \left(\sum_{\substack{I'\in\mathfrak{C}(I)\&J'\in\mathfrak{C}(J)\\ J\in\mathcal{N}(I)}} \left| E_{I'}^{\sigma}\left(\widehat{\square}_I^{\sigma,\flat,\mathbf{b}} f\right)\right|^2 |M|_\sigma \right)^{\frac{1}{2}} \cdot \\
&\qquad \cdot \left(\boldsymbol{E}_\Omega^{\mathcal{D}} \sum_{\substack{I'\in\mathfrak{C}(I)\&J'\in\mathfrak{C}(J)\\ J\in\mathcal{N}(I)}} \left| E_{J'}^{\omega}\left(\widehat{\square}_J^{\omega,\flat,\mathbf{b}^*} g\right)\right|^2 |L\cap(1+\delta)M|_\omega \right)^{\frac{1}{2}} \\
\leq\ & \mathfrak{N}_{T^\alpha} C_{\mathbf{b},\mathbf{b}^*,\mathbf{r},n} \sqrt{\delta} ||f||_{L^2(\sigma)} ||g||_{L^2(\omega)}
\end{aligned}$$

by noting that $(1+\delta)M\cap L$ is a halo of width δ, much smaller than η_0 (so as to get the estimate by $\sqrt{\delta}$, not $\sqrt{\eta_0}$). Although an estimate of $\sqrt{\eta_0}$ is easy to obtain (as L already has width η_0) and is sufficient for the purposes of this term, the estimate of $\sqrt{\delta}$ will be crucially used later in (5.4.19) to kill the B term. Note also that we can take the averages over all directions, so that we avoid common intersection along the different translations. Notice that L, M are "moving" together. This is not a problem since by "moving" they cover different parts of the cube J'.

Thus we only need to estimate $\left\langle T_\sigma^\alpha\left(b_I\mathbf{1}_L\right), b_J^*\mathbf{1}_L\right\rangle_\omega + \left\langle T_\sigma^\alpha\left(b_I\mathbf{1}_M\right), b_J^*\mathbf{1}_M\right\rangle_\omega$. Applying one more time random surgery to the first term we get that

$$\begin{aligned}
&\boldsymbol{E}_{\Omega}^{\mathcal{D}}\boldsymbol{E}_{\Omega}^{\mathcal{G}}\sum_{I\in\mathcal{D}}\sum_{J\in\mathcal{N}(I)}\sum_{\substack{I'\in\mathfrak{C}(I)\\ J'\in\mathfrak{C}(J)}}\left|E_{I'}^{\sigma}\left(\widehat{\square}_{I}^{\sigma,\flat,\mathbf{b}}f\right)\ \left\langle T_{\sigma}^{\alpha}\left(b_I\mathbf{1}_L\right),b_J^{*}\mathbf{1}_L\right\rangle_{\omega}E_{J'}^{\omega}\left(\widehat{\square}_{J}^{\omega,\flat,\mathbf{b}^*}g\right)\right|\\
\lesssim\ &\boldsymbol{E}_{\Omega}^{\mathcal{G}}\mathfrak{N}_{T^{\alpha}}\left\|f\right\|_{L^2(\sigma)}\boldsymbol{E}_{\Omega}^{\mathcal{D}}\sqrt{\sum_{I\in\mathcal{D}}\sum_{J\in\mathcal{N}(I)}\sum_{\substack{I'\in\mathfrak{C}(I)\\ J'\in\mathfrak{C}(J)}}\left(\int_{\partial_{\eta_1}I'\cap J'}\left|b_J^{*}\right|^2d\omega\right)\left|E_{J'}^{\omega}\left(\widehat{\square}_{J}^{\omega,\flat,\mathbf{b}^*}g\right)\right|^2}
\end{aligned}$$

using (5.4.2) and the frame inequalities again. Then using Cauchy-Schwarz on the expectation $\boldsymbol{E}_{\Omega}^{\mathcal{D}}$, this is dominated by

$$\boldsymbol{E}_{\Omega}^{\mathcal{G}}\mathfrak{N}_{T^{\alpha}}\|f\|_{L^2(\sigma)}\sqrt{\sum_{J\in\mathcal{G}}\sum_{J'\in\mathfrak{C}(J)}\left(\boldsymbol{E}_{\Omega}^{\mathcal{D}}\sum_{\substack{I\in\mathcal{D}:\ 2^{-\mathbf{r}n}|I|<|J|\leq|I|\\ d(J,I)\leq 2\ell(J)^{\varepsilon}\ell(I)^{1-\varepsilon}\\ I'\in\mathfrak{C}(I)}}\left|\partial_{\overrightarrow{\eta_1}}I'\cap J'\right|_{\omega}\right)\left|E_{J'}^{\omega}\left(\widehat{\square}_{J}^{\omega,\flat,\mathbf{b}^*}g\right)\right|^2}$$

$$\begin{aligned}
&\lesssim\boldsymbol{E}_{\Omega}^{\mathcal{G}}\mathfrak{N}_{T^{\alpha}}\left\|f\right\|_{L^2(\sigma)}\sqrt{\sum_{J\in\mathcal{G}}\sum_{J'\in\mathfrak{C}(J)}2^{\mathbf{r}}\left(\boldsymbol{E}_{\Omega}^{\mathcal{D}}\sum_{I'\in\mathcal{D}:|J'|\leq|I'|\leq 2^{\mathbf{r}}|J'|}\left|\partial_{\eta_0}I'\cap J'\right|_{\omega}\right)\left|E_{J'}^{\omega}\left(\widehat{\square}_{J}^{\omega,\flat,\mathbf{b}^*}g\right)\right|^2}\\
&\lesssim\sqrt{\eta_0}\mathfrak{N}_{T^{\alpha}}\left\|f\right\|_{L^2(\sigma)}\left\|g\right\|_{L^2(\omega)}\leq\sqrt{\lambda}\mathfrak{N}_{T^{\alpha}}\left\|f\right\|_{L^2(\sigma)}\left\|g\right\|_{L^2(\omega)}
\end{aligned}$$

where in the last line we have used $\eta_1^i\leq\eta_0$, and then

$$\boldsymbol{E}_{\Omega}^{\mathcal{D}}\sum_{I'\in\mathcal{D}:|J'|\leq|I'|\leq 2^{\mathbf{r}}|J'|}\left|\partial_{\eta_0}I'\cap J'\right|_{\omega}\lesssim\eta_0\left|J'\right|_{\omega}$$

as long as we choose $\eta_0 \ll 2^{-\mathbf{r}}$.

This leaves us to estimate the term $\left\langle T_\sigma^\alpha \left(b_I \mathbf{1}_M\right), b_J^* \mathbf{1}_M \right\rangle_\omega$. It is at this point that we will use the decomposition $M = \dot{\bigcup}_{1 \leq s \leq B} K_s$ constructed above. We have

$$\left\langle T_\sigma^\alpha \left(b_I \mathbf{1}_M\right), b_J^* \mathbf{1}_M \right\rangle_\omega = \sum_{s,s'=1}^{B} \left\langle T_\sigma^\alpha \left(b_I \mathbf{1}_{K_s}\right), b_J^* \mathbf{1}_{K_{s'}} \right\rangle_\omega$$

which can be rewritten as

$$\sum_{s=1}^{B} \left\langle T_\sigma^\alpha \left(b_I \mathbf{1}_{K_s}\right), b_J^* \mathbf{1}_{K_s} \right\rangle_\omega + \left(\sum_{K_s} \sum_{\underset{Sep}{\sim} K_{s'}} + \sum_{K_s} \sum_{\underset{Adj}{\sim} K_{s'}} \right) \left\langle T_\sigma^\alpha \left(b_I \mathbf{1}_{K_s}\right), b_J^* \mathbf{1}_{K_{s'}} \right\rangle_\omega \quad (5.4.16)$$

where we call $K_s \underset{Sep}{\sim} K_{s'}$ the separated cubes, i.e. $3K_s \cap K_{s'} = \emptyset$, while by $K_s \underset{Adj}{\sim} K_{s'}$ are the adjacent cubes, i.e. $K_s \cap K_{s'} = \emptyset$ and $\overline{K_s} \cap \overline{K_{s'}} \neq \emptyset$. The separated terms sum can be estimated directly by $\sqrt{\mathfrak{A}_2^\alpha}$. Indeed, as in the proof of Lemma 5.4.3,

$$\begin{aligned}
\left\langle T_\sigma^\alpha \left(b_I \mathbf{1}_{K_s}\right), b_J^* \mathbf{1}_{K_{s'}} \right\rangle_\omega &\lesssim \left(\int_{K_{s'}} \left(\int_{K_s} |x-y|^{\alpha-n} \left|b_I(y)\right| d\sigma(y) \right)^2 d\omega(x) \right)^{\frac{1}{2}} \sqrt{\left|K_{s'}\right|_\omega} \\
&\lesssim \left(\int_{\mathbb{R}^n \setminus K_s} \left|x - x_{K_s}\right|^{2\alpha-2n} d\omega(x) \right)^{\frac{1}{2}} |K_s|_\sigma \sqrt{\left|K_{s'}\right|_\omega} \\
&\lesssim \sqrt{\mathfrak{A}_2^\alpha} \sqrt{|K_s|_\sigma} \sqrt{|K_{s'}|_\omega}
\end{aligned}$$

thus,

$$\sum_{K_s} \sum_{\underset{Sep}{\sim} K_{s'}} \left\langle T_\sigma^\alpha \left(b_I \mathbf{1}_{K_s}\right), b_J^* \mathbf{1}_{K_{s'}} \right\rangle_\omega \leq C_{\mathbf{b}} \sum_{K_s} \sum_{\underset{Sep}{\sim} K_{s'}} \sqrt{\mathfrak{A}_2^\alpha} \sqrt{|K_s|_\sigma} \sqrt{|K_{s'}|_\omega} \quad (5.4.17)$$

which plugged into (5.4.10) appropriately, we get the bound $B\sqrt{\mathfrak{A}_2^\alpha}\sqrt{|I'|_\sigma}\sqrt{|J'|_\omega}$.

To deal with the adjacent cubes term in (5.4.16), we write

$$\begin{aligned}
&\sum_{K_s}\sum_{\underset{Adj}{\sim}K_{s'}}\left\langle T_\sigma^\alpha\left(b_I\mathbf{1}_{K_s}\right),b_J^*\mathbf{1}_{K_{s'}}\right\rangle_\omega=\sum_{K_s}\sum_{\underset{Adj}{\sim}K_{s'}}\left\langle b_I\mathbf{1}_{K_s},T_\omega^{\alpha,*}\left(b_J^*\mathbf{1}_{K_{s'}}\right)\right\rangle_\sigma\\
= &\sum_{K_s}\sum_{\underset{Adj}{\sim}K_{s'}}\left\langle b_I\mathbf{1}_{K_s\cap(1+\delta)K_{s'}},T_\omega^{\alpha,*}\left(b_J^*\mathbf{1}_{K_{s'}}\right)\right\rangle_\sigma\\
&+\sum_{K_s}\sum_{\underset{Adj}{\sim}K_{s'}}\left\langle b_I\mathbf{1}_{K_s\setminus(1+\delta)K_{s'}},T_\omega^{\alpha,*}\left(b_J^*\mathbf{1}_{K_{s'}}\right)\right\rangle_\sigma\\
\equiv &\ \widetilde{\mathrm{I}}+\widetilde{\mathrm{II}}
\end{aligned}$$

For $\widetilde{\mathrm{II}}$ we use Lemma 5.4.3 to get

$$\begin{aligned}
\widetilde{\mathrm{II}} &\lesssim \delta^{\alpha-n}\sqrt{\mathfrak{A}_2^\alpha}\left(\sum_{s=1}^B|K_s|_\sigma\right)^{\frac{1}{2}}\left(\sum_{s=1}^B\left(\sum_{s'\geq s}\left|K_{s'}\right|_\omega^{\frac{1}{2}}\right)^2\right)^{\frac{1}{2}} \qquad (5.4.18)\\
&\lesssim \delta^{\alpha-n}B\sqrt{\mathfrak{A}_2^\alpha}\sqrt{|I'|_\sigma}\sqrt{|J'|_\omega}
\end{aligned}$$

while summing $\widetilde{\mathrm{I}}$ over

$$\mathcal{T}=\{I\in\mathcal{D},J\in\mathcal{N}(I),I'\in\mathfrak{C}_{nat}(I),J'\in\mathfrak{C}_{nat}(J)\}$$

and using Cauchy-Schwarz, accretivity, taking averages and using Jensen, we get

$$\begin{aligned}
&\boldsymbol{E}_{\Omega}^{\mathcal{G}}\sum_{\mathcal{T}}\left|E_{I'}^{\sigma}\left(\widehat{\square}_{I}^{\sigma,\flat,\mathbf{b}}f\right)E_{J'}^{\omega}\left(\widehat{\square}_{J}^{\omega,\flat,\mathbf{b}^*}g\right)\right|\sum_{K_s}\sum_{\underset{Adj}{\sim}K_{s'}}\left\langle b_I\mathbf{1}_{K_s\cap(1+\delta)K_{s'}},T_{\omega}^{\alpha,*}\left(b_J^*\mathbf{1}_{K_{s'}}\right)\right\rangle_{\sigma} \\
&\lesssim \boldsymbol{E}_{\Omega}^{\mathcal{G}}\sum_{\mathcal{T}}\left|E_{I'}^{\sigma}\left(\widehat{\square}_{I}^{\sigma,\flat,\mathbf{b}}f\right)E_{J'}^{\omega}\left(\widehat{\square}_{J}^{\omega,\flat,\mathbf{b}^*}g\right)\right|\sum_{K_s}\sum_{\underset{Adj}{\sim}K_{s'}}\mathfrak{N}_{T^\alpha}\sqrt{|K_s\cap(1+\delta)K_{s'}|_{\sigma}}\sqrt{|K_{s'}|_{\omega}} \\
&\lesssim \mathfrak{N}_{T^\alpha}\boldsymbol{E}_{\Omega}^{\mathcal{G}}\sum_{\mathcal{T}}\left|E_{I'}^{\sigma}\left(\widehat{\square}_{I}^{\sigma,\flat,\mathbf{b}}f\right)E_{J'}^{\omega}\left(\widehat{\square}_{J}^{\omega,\flat,\mathbf{b}^*}g\right)\right|\Big(\sum_{s=1}^{B}|K_{s'}|_{\omega}\Big)^{\frac12}\cdot \\
&\qquad\cdot\Big(\sum_{s=1}^{B}\Big(\sum_{s\le s'}\sqrt{|K_s\cap(1+\delta)K_{s'}|_{\sigma}}\Big)^2\Big)^{\frac12} \\
&\lesssim \mathfrak{N}_{T^\alpha}\boldsymbol{E}_{\Omega}^{\mathcal{G}}\sum_{\mathcal{T}}\left|E_{I'}^{\sigma}\left(\widehat{\square}_{I}^{\sigma,\flat,\mathbf{b}}f\right)E_{J'}^{\omega}\left(\widehat{\square}_{J}^{\omega,\flat,\mathbf{b}^*}g\right)\right|\sqrt{|J'|_{\omega}}\cdot \\
&\qquad\cdot\Big(\sum_{s=1}^{B}\Big(\sum_{s\le s'}|K_s\cap(1+\delta)K_{s'}|_{\sigma}\cdot\sum_{s\le s'}1\Big)\Big)^{\frac12} \\
&\lesssim \mathfrak{N}_{T^\alpha}\sqrt{B}\,\|g\|_{L^2(\omega)}\left(\sum_{\substack{I\in\mathcal{D}\\ I'\in\mathfrak{C}_{nat}(I)}}\left|E_{I'}^{\sigma}\left(\widehat{\square}_{I}^{\sigma,\flat,\mathbf{b}}f\right)\right|^2\boldsymbol{E}_{\Omega}^{\mathcal{G}}\sum_{\substack{J\in\mathcal{N}(I)\\ J'\in\mathfrak{C}_{nat}(J)}}\sum_{s=1}^{B}\sum_{s\le s'}|K_s\cap(1+\delta)K_{s'}|_{\sigma}\right)^{\frac12} \\
&\lesssim \mathfrak{N}_{T^\alpha}\sqrt{B}\,\|g\|_{L^2(\omega)}\left(\sum_{\mathcal{T}}\left|E_{I'}^{\sigma}\left(\widehat{\square}_{I}^{\sigma,\flat,\mathbf{b}}f\right)\right|^2 2^n\delta|I'|_{\sigma}\right)^{\frac12} \\
&\lesssim \quad \mathfrak{N}_{T^\alpha}2^{2n}\sqrt{B}\sqrt{\delta}\,\|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)}
\end{aligned} \tag{5.4.19}$$

because there are up to 2^n adjacent cubes $K_{s'}$ for a given K_s. The implied constant depends on $\mathbf{r}$ of the nearby form. Note that δ is independent of B or $\mathbf{r}$ and will later be chosen small enough so that the terms containing the norm inequality constant will be absorbed.

Thus now we are left only with the first term of (5.4.16), i.e. we need to estimate

$$\sum_{s=1}^{B} \left\langle T_\sigma^\alpha \left(b_I \mathbf{1}_{K_s}\right), b_J^* \mathbf{1}_{K_s} \right\rangle_\omega$$

Before proceeding further it will prove convenient to introduce some additional notation, namely we will write the energy estimate in the second display of the Energy Lemma as

$$\left|\langle T^\alpha \nu, \Psi_J \rangle_\omega\right| \lesssim C_{\gamma,\delta}\, \mathrm{P}_\delta^\alpha \mathsf{Q}^\omega (J, \upsilon)\ \|\Psi_J\|_{L^2(\mu)} \quad \text{if } \int \Psi_J d\omega = 0 \text{ and } \gamma J \cap \mathrm{supp}\nu = \emptyset \quad (5.4.20)$$

where

$$\mathrm{P}_\delta^\alpha \mathsf{Q}^\omega (J, \upsilon) \equiv \frac{\mathrm{P}^\alpha (J, \nu)}{|J|} \left\| \mathsf{Q}_J^{\omega, \mathbf{b}^*} x \right\|_{L^2(\omega)}^\spadesuit + \frac{\mathrm{P}_{1+\delta}^\alpha (J, \nu)}{|J|} \|x - m_J\|_{L^2(\mathbf{1}_J \omega)} .$$

The use of the compact notation $\mathrm{P}_\delta^\alpha \mathsf{Q}^\omega (J, \upsilon)$ to denote the complicated expression on the right hand side will considerably reduce the size of many subsequent displays.

We now consider the inner product $\left\langle T_\sigma^\alpha (b_A \mathbf{1}_K), b_B^* \mathbf{1}_K \right\rangle_\omega$ and estimate the case when

$$K \in \mathcal{G},\ K \subset I' \cap J',\ I' \in \mathfrak{C}(I),\ J' \in \mathfrak{C}(J),\ I \in \mathcal{C}_A^{\mathcal{A}},\ J \in \mathcal{C}_B^{\mathcal{B}},\ \ell(K) = 2^{-m-1} \ell(J').$$

For subsets $E, F \subset A \cap B$ and cubes $K \subset A \cap B$ we define

$$\{E, F\} \equiv \left\langle T_\sigma^\alpha (b_A \mathbf{1}_E), b_B^* \mathbf{1}_F \right\rangle_\omega , \quad (5.4.21)$$

and K_{in} the 2^n grandchildren of K that do not intersect the boundary of K while K_{out} the rest $4^n - 2^n$ grandchildren of K that intersect its boundary i.e.

$$K_{in} = \left\{ K'' \in \mathfrak{C}^{(2)}(K) : \partial K'' \cap \partial K = \emptyset \right\}$$

$$K_{out} = \left\{ K'' \in \mathfrak{C}^{(2)}(K) : \partial K'' \cap \partial K \neq \emptyset \right\}$$

We can write

$$\{K, K\} = \{A, K_{in}\} - \{A \backslash K, K_{in}\} + \{K_{out}, K_{out}\} + \{K_{in}, K_{out}\}. \tag{5.4.22}$$

Note that the first two terms on the right hand side of (5.4.22) decompose the inner product $\{K, K_{in}\}$, which 'includes' one of the difficult symmetric inner product $\{K_{in}, K_{in}\}$, and where the other difficult symmetric inner products are contained in $\{K_{out}, K_{out}\}$, which can be handled recursively. Thus the difficult symmetric inner products are ultimately controlled by testing on the cube A to handle the 'paraproduct' term $\{A, K_{in}\}$, and by using the energy condition and a trick that resurrects the original testing functions $\left\{ b_J^{*,orig} \right\}_{J \in \mathcal{G}}$, discarded in the corona constructions above, to handle the 'stopping' term $\{A \backslash K, K_{in}\}$. More precisely, these original testing functions $b_J^{*,orig}$ are the testing functions obtained after reducing matters to the case of bounded testing functions.

The first term on the right side of (5.4.22) satisfies

$$\begin{aligned} |\{A, K_{in}\}| &= \left| \int_{K_{in}} (T_\sigma^\alpha b_A)\, b_B^* d\omega \right| \leq \left\| \mathbf{1}_{K_{in}} T_\sigma^\alpha b_A \right\|_{L^2(\omega)} \left\| \mathbf{1}_{K_{in}} b_B^* \right\|_{L^2(\omega)} \\ &\leq \left\| b_B^* \right\|_\infty \left\| \mathbf{1}_{K_{in}} T_\sigma^\alpha b_A \right\|_{L^2(\omega)} \sqrt{|K_{in}|_\omega}\,. \end{aligned} \tag{5.4.23}$$

We now turn to the term $\{A\backslash K, K_{in}\}$. Decompose $\mathbf{1}_{K_{in}} b_B^*$ as

$$\mathbf{1}_{K_{in}} b_B^* = \sum_{\ell=1}^{2^n} \mathbf{1}_{K_{in}^\ell} \left(b_B^* - \frac{1}{\left|K_{in}^\ell\right|_\omega} \int_{K_{in}^\ell} b_B^* d\omega \right) + \sum_{\ell=1}^{2^n} \mathbf{1}_{K_{in}^\ell} \frac{1}{\left|K_{in}^\ell\right|_\omega} \int_{K_{in}^\ell} b_B^* d\omega,$$

and then apply the Energy Lemma to the function

$$k_{K_{in}}^* \equiv \sum_{\ell=1}^{2^n} \mathbf{1}_{K_{in}^\ell} \left(b_B^* - \frac{1}{\left|K_{in}^\ell\right|_\omega} \int_{K_{in}^\ell} b_B^* d\omega \right) \equiv \sum_{j=1}^{2^n} k_{K_{in}}^{*,j}$$

which does indeed satisfy $\square_{K'}^{\omega,\mathbf{b}^*} k_{K_{in}}^* = 0$ unless K' is a dyadic subcube of K that is contained in K_{in}. (Furthermore, we could even replace grandchildren by m-grandchildren in this argument in order that $\square_{K'}^{\omega,\mathbf{b}^*} k_{K_{in}}^* = 0$ unless K' is a dyadic m-grandchild of K that is contained in K_{in}, but we will not need this.) We obtain

$$\begin{aligned} \left\langle T_\sigma^\alpha \left(b_A \mathbf{1}_{A\backslash K} \right), \mathbf{1}_{K_{in}} b_B^* \right\rangle_\omega = & \left\langle T_\sigma^\alpha \left(b_A \mathbf{1}_{A\backslash K} \right), k_{K_{in}}^* \right\rangle_\omega \\ & + \left\langle T_\sigma^\alpha \left(b_A \mathbf{1}_{A\backslash K} \right), \sum_{\ell=1}^{2^n} \mathbf{1}_{K_{in}^\ell} \left(\frac{1}{\left|K_{in}^\ell\right|_\omega} \int_{K_{in}^\ell} b_B^* d\omega \right) \right\rangle_\omega \end{aligned} \tag{5.4.24}$$

and

$$\begin{aligned} \left| \left\langle T_\sigma^\alpha \left(b_A \mathbf{1}_{A\backslash K} \right), k_{K_{in}}^* \right\rangle_\omega \right| & \leq \sum_{\ell=1}^{2^n} \left| \left\langle T_\sigma^\alpha \left(b_A \mathbf{1}_{A\backslash K} \right), k_{K_{in}}^{*,\ell} \right\rangle_\omega \right| \\ & \leq C_{\eta_0,n} \left[\sum_{\ell=1}^{2^n} \mathrm{P}_\delta^\alpha \mathsf{Q}^\omega \left(K_{in}^\ell, \mathbf{1}_{A\backslash K} \sigma \right) \right] \left\| k_{K_{in}}^* \right\|_{L^2(\omega)} \end{aligned}$$

where the constant C_{η_0} depends on the constant C_γ in the statement of the Monotonicity Lemma with $\gamma = \frac{1}{1-\eta_0}$ since $\frac{1}{1-\eta_0} K_{in} \cap (A\backslash K) = \emptyset$, and where we have written $\left\{ K_{in}^\ell \right\}_{\ell=1}^{2^n}$

with K_{in}^{ℓ} denoting the innner grandchildren of K.

Thus we see that $\mathsf{P}_{\mathcal{H}}^{\omega,\mathbf{b}^*}$ and $\mathsf{Q}_{\mathcal{H}}^{\omega,\mathbf{b}^*}$ in the Energy Lemma can be taken to be pseudo-projection onto K_{in}, i.e. $\mathsf{P}_{K_{in}}^{\omega,\mathbf{b}^*} = \sum_{J\in\mathcal{G}:\ J\subset K_{in}} \square_J^{\omega,\mathbf{b}^*}$ and $\mathsf{Q}_{K_{in}}^{\omega,\mathbf{b}^*} = \sum_{J\in\mathcal{G}:\ J\subset K_{in}} \triangle_J^{\omega,\mathbf{b}^*}$, and we will see below that the cubes K_{in} that arise in subsequent arguments will be pairwise disjoint. Furthermore, the energy condition will be used to control these full pseudoprojections $\mathsf{P}_{K_{in}}^{\omega,\mathbf{b}^*}$ when taken over pairwise disjoint decompositions of cubes by subcubes of the form K_{in}.

However, the second line of (5.4.24) remains problematic because we cannot use any type of testing in K_{in}^{ℓ} with b_B^* since K_{in}^{ℓ} does not necessarily belong to $\mathcal{C}_B$, and this is our point in which we exploit the original testing functions $b_{K_{in}^{\ell}}^{*,orig}$.

5.4.2.1 Return to the original testing functions

From the discussion above, we recall the identity (5.4.24) and the estimate (5.4.25). We also have the analogous identity and estimate with $b_{K_{in}^{\ell}}^{*,orig}$ in place of $\mathbf{1}_{K_{in}} b_B^*$:

$$\begin{aligned}
&\left\langle T_\sigma^\alpha \left(b_A \mathbf{1}_{A\setminus K}\right), b_{K_{in}^{\ell}}^{*,orig} \right\rangle_\omega \\
&\qquad = \left\langle T_\sigma^\alpha \left(b_A \mathbf{1}_{A\setminus K}\right), \mathbf{1}_{K_{in}^{\ell}} \left(b_{K_{in}^{\ell}}^{*,orig} - \frac{1}{\left|K_{in}^{\ell}\right|_\omega} \int_{K_{in}^{\ell}} b_{K_{in}^{\ell}}^{*,orig} d\omega \right) \right\rangle_\omega \\
&\qquad\quad + \left\langle T_\sigma^\alpha \left(b_A \mathbf{1}_{A\setminus K}\right), \mathbf{1}_{K_{in}^{\ell}} \left(\frac{1}{\left|K_{in}^{\ell}\right|_\omega} \int_{K_{in}^{\ell}} b_{K_{in}^{\ell}}^{*,orig} d\omega \right) \right\rangle_\omega
\end{aligned} \tag{5.4.25}$$

and

$$\left|\left\langle T_{\sigma}^{\alpha}\left(b_A\mathbf{1}_{A\setminus K}\right),\mathbf{1}_{K_{in}^{\ell}}\left(b_{K_{in}^{\ell}}^{*,orig}-\frac{1}{\left|K_{in}^{\ell}\right|_{\omega}}\int_{K_{in}^{\ell}}b_{K_{in}^{\ell}}^{*,orig}d\omega\right)\right\rangle_{\omega}\right| \tag{5.4.26}$$
$$\lesssim \mathrm{P}_{\delta}^{\alpha}\mathsf{Q}^{\omega}\left(K_{in}^{\ell},\mathbf{1}_{A\setminus K}\sigma\right)\left\|\mathbf{1}_{K_{in}^{\ell}}\left(b_{K_{in}^{\ell}}^{*,orig}-\frac{1}{\left|K_{in}^{\ell}\right|_{\omega}}\int_{K_{in}^{\ell}}b_{K_{in}^{\ell}}^{*,orig}d\omega\right)\right\|_{L^2(\omega)}$$

for $1\le\ell\le 2^n$, where the implied constants depend on L^∞ norms of testing functions and the constant in the Energy Lemma. Using the notation

$$\left\{K_{out},K_{in}^{\ell}\right\}^{orig}\equiv\left\langle T_{\sigma}^{\alpha}b_A\mathbf{1}_{K_{out}},b_{K_{in}^{\ell}}^{*,orig}\right\rangle_{\omega}\text{ for }1\le\ell\le 2^n.$$

note that

$$\begin{aligned}
&\{A\setminus K,K_{in}\}+\sum_{\ell=1}^{2^n}\left(\frac{\frac{1}{|K_{in}^{\ell}|_{\omega}}\int_{K_{in}^{\ell}}b_B^{*}d\omega}{\frac{1}{|K_{in}^{\ell}|_{\omega}}\int_{K_{in}^{\ell}}b_B^{*,orig}d\omega}\right)\left\{K_{out},K_{in}^{\ell}\right\}^{orig}\\
=\ &\{A\setminus K,K_{in}\}-\sum_{\ell=1}^{2^n}\left(\frac{\frac{1}{|K_{in}^{\ell}|_{\omega}}\int_{K_{in}^{\ell}}b_B^{*}d\omega}{\frac{1}{|K_{in}^{\ell}|_{\omega}}\int_{K_{in}^{\ell}}b_B^{*,orig}d\omega}\right)\left\langle T_{\sigma}^{\alpha}\left(b_A\mathbf{1}_{A\setminus K}\right),b_{K_{in}^{\ell}}^{*,orig}\right\rangle_{\omega}\\
&+\sum_{\ell=1}^{2^n}\left(\frac{\frac{1}{|K_{in}^{\ell}|_{\omega}}\int_{K_{in}^{\ell}}b_B^{*}d\omega}{\frac{1}{|K_{in}^{\ell}|_{\omega}}\int_{K_{in}^{\ell}}b_B^{*,orig}d\omega}\right)\left[\left\langle T_{\sigma}^{\alpha}\left(b_A\mathbf{1}_{A}\right),b_{K_{in}^{\ell}}^{*,orig}\right\rangle_{\omega}-\left\langle b_A\mathbf{1}_{K_{in}},T_{\omega}^{\alpha,*}b_{K_{in}^{\ell}}^{*,orig}\right\rangle_{\sigma}\right]\\
\equiv\ &\mathbf{B}+\mathbf{C}
\end{aligned}$$

Now for $\mathbf{B}$, using Energy Lemma to the function

$$\Psi_J^{\ell}=\left(\frac{\frac{1}{|K_{in}^{\ell}|_{\omega}}\int_{K_{in}^{\ell}}b_B^{*}d\omega}{\frac{1}{|K_{in}^{\ell}|_{\omega}}\int_{K_{in}^{\ell}}b_B^{*,orig}d\omega}\right)b_{K_{in}^{\ell}}^{*,orig}-\left(\frac{1}{\left|K_{in}^{\ell}\right|_{\omega}}\int_{K_{in}^{\ell}}b_B^{*}d\omega\right)\mathbf{1}_{K_{in}^{\ell}}$$

for $1 \leq \ell \leq 2^n$ we have

$$\begin{aligned}
|\mathbf{B}| = & \left| \left\langle T_\sigma^\alpha \left(b_A \mathbf{1}_{A \setminus K} \right), \mathbf{1}_{K_{in}} b_B^* \right\rangle_\omega - \sum_{\ell=1}^{2^n} \left(\frac{1}{\left| K_{in}^\ell \right|_\omega} \int_{K_{in}^\ell} b_B^* d\omega \right) \left\langle T_\sigma^\alpha \left(b_A \mathbf{1}_{A \setminus K} \right), \mathbf{1}_{K_{in}^\ell} \right\rangle_\omega \right| \\
& + O \left[\sum_{\ell=1}^{2^n} \left(\frac{\mathrm{P}^\alpha (K_{in}^\ell \mathbf{1}_{A \setminus K} \sigma)}{|K_{in}^\ell|} \left\| \mathsf{Q}_{K_{in}^\ell}^{\omega, \mathbf{b}^*} x \right\|_{L^2(\omega)}^\spadesuit \right) \right] \sqrt{|K_{in}|_\omega} \\
& + O \left[\sum_{\ell=1}^{2^n} \left(\frac{\mathrm{P}_{1+\delta}^\alpha (K_{in}^\ell \mathbf{1}_{A \setminus K} \sigma)}{|K_{in}^\ell|} \left\| x - m_{K_{in}^\ell} \right\|_{L^2(\omega)} \right) \right] \sqrt{|K_{in}|_\omega} \\
\lesssim & \left[\sum_{\ell=1}^{2^n} \mathrm{P}_\delta^\alpha \mathsf{Q}^\omega \left(K_{in}^\ell, \mathbf{1}_{A \setminus K} \sigma \right) \right] \sqrt{|K_{in}|_\omega}
\end{aligned}$$

having used the triangle inequality to get

$$\left\| \Psi_J^\ell \right\|_{L^2(\omega)} \lesssim \left| \frac{\int_{K_{in}^\ell} b_B^* d\omega}{\int_{K_{in}^\ell} b_B^{*,orig} d\omega} \right| \sqrt{|K_{in}^\ell|_\omega} + \sqrt{|K_{in}^\ell|_\omega} \lesssim \sqrt{|K_{in}|_\omega}, \quad 1 \leq \ell \leq 2^n$$

and

$$\begin{aligned}
& \left| \left\langle T_\sigma^\alpha \left(b_A \mathbf{1}_{A \setminus K} \right), \mathbf{1}_{K_{in}} b_B^* \right\rangle_\omega - \sum_{\ell=1}^{2^n} \left(\frac{1}{\left| K_{in}^\ell \right|_\omega} \int_{K_{in}^\ell} b_B^* d\omega \right) \left\langle T_\sigma^\alpha \left(b_A \mathbf{1}_{A \setminus K} \right), \mathbf{1}_{K_{in}^\ell} \right\rangle_\omega \right| \\
\lesssim & \left[\sum_{\ell=1}^{2^n} \mathrm{P}_\delta^\alpha \mathsf{Q}^\omega \left(K_{in}^\ell, \mathbf{1}_{A \setminus K} \sigma \right) \right] \left\| 1_{K_\ell''} \sum_{\ell=1}^{2^n} \left(b_B^* - \frac{1}{|K_l''|_\omega} \int_{k_\ell''} b_B^* d\omega \right) \right\|_{L^2(\omega)} \\
\lesssim & \left[\sum_{\ell=1}^{2^n} \mathrm{P}_\delta^\alpha \mathsf{Q}^\omega \left(K_{in}^\ell, \mathbf{1}_{A \setminus K} \sigma \right) \right] \sqrt{|K_{in}|_\omega}
\end{aligned}$$

where in the last inequality we used accretivity and triangle inequality. We turn our attention in term **C**. We have that

$$\begin{aligned}&\left|\sum_{\ell=1}^{2^n}\left(\frac{\frac{1}{|K_{in}^{\ell}|_\omega}\int_{K_{in}^{\ell}} b_B^* d\omega}{\frac{1}{|K_{in}^{\ell}|_\omega}\int_{K_{in}^{\ell}} b_B^{*,orig} d\omega}\right)\left\langle T_\sigma^\alpha\left(b_A\mathbf{1}_A\right), b^{*,orig}_{K_{in}^{\ell}}\right\rangle_\omega\right|\\ \lesssim\ & \sum_{\ell=1}^{2^n}\sqrt{\int_{K_\ell''}\left|T_\sigma^\alpha b_A\right|^2 d\omega}\sqrt{\int_{K_{in}^{\ell}}\left|b^{*,orig}_{K_\ell''}\right|^2 d\omega}\\ \lesssim\ & \sqrt{\int_{K_{in}}\left|T_\sigma^\alpha b_A\right|^2 d\omega}\sqrt{|K_{in}|_\omega}\end{aligned}$$

Also,

$$\left|\sum_{\ell=1}^{2^n}\left(\frac{\frac{1}{|K_{in}^{\ell}|_\omega}\int_{K_{in}^{\ell}} b_B^* d\omega}{\frac{1}{|K_{in}^{\ell}|_\omega}\int_{K_{in}^{\ell}} b_B^{*,orig} d\omega}\right)\left\langle b_A\mathbf{1}_{K_{in}}, T_\omega^{\alpha,*} b^{*,orig}_{K_{in}^{\ell}}\right\rangle_\sigma\right| \equiv \mathrm{I}+\mathrm{II}+\mathrm{III}$$

where

$$\mathrm{I}=\sum_{\ell=1}^{2^n}\left(\frac{\frac{1}{|K_{in}^{\ell}|_\omega}\int_{K_{in}^{\ell}} b_B^* d\omega}{\frac{1}{|K_{in}^{\ell}|_\omega}\int_{K_{in}^{\ell}} b_B^{*,orig} d\omega}\right)\left\langle b_A\mathbf{1}_{K_{in}}, \mathbf{1}_{K_{in}^{\ell}} T_\omega^{\alpha,*} b^{*,orig}_{K_{in}^{\ell}}\right\rangle_\sigma$$

$$\mathrm{II}=\sum_{\ell=1}^{2^n}\left(\frac{\frac{1}{|K_{in}^{\ell}|_\omega}\int_{K_{in}^{\ell}} b_B^* d\omega}{\frac{1}{|K_{in}^{\ell}|_\omega}\int_{K_{in}^{\ell}} b_B^{*,orig} d\omega}\right)\left\langle b_A\mathbf{1}_{K_{in}\setminus(1+\delta)K_{in}^{\ell}}, T_\omega^{\alpha,*} b^{*,orig}_{K_{in}^{\ell}}\right\rangle_\sigma$$

$$\mathrm{III}=\sum_{\ell=1}^{2^n}\left(\frac{\frac{1}{|K_{in}^{\ell}|_\omega}\int_{K_{in}^{\ell}} b_B^* d\omega}{\frac{1}{|K_{in}^{\ell}|_\omega}\int_{K_{in}^{\ell}} b_B^{*,orig} d\omega}\right)\left\langle b_A\mathbf{1}_{(K_{in}\setminus K_{in}^{\ell})\cap(1+\delta)K_{in}^{\ell}}, T_\omega^{\alpha,*} b^{*,orig}_{K_{in}^{\ell}}\right\rangle_\sigma$$

The first term I is bounded using the dual testing condition. Indeed,

$$\mathrm{I}\leq\left|\left|b_A\mathbf{1}_{K_{in}}\right|\right|_{L^2(\sigma)}\sum_{\ell=1}^{2^n}\mathfrak{T}^* C_{\mathbf{b}^*}\sqrt{|K_{in}^{\ell}|_\omega}\leq 2^n\mathfrak{T}^* C_{\mathbf{b}^*}\left|\left|b_A\mathbf{1}_{K_{in}}\right|\right|_{L^2(\sigma)}\sqrt{|K_{in}|_\omega}$$

The second term II is bounded using Lemma 5.4.3. Indeed,

$$\begin{aligned}
\mathrm{II} &\leq \sum_{\ell=1}^{2^n} \delta^{\alpha-n}\sqrt{\mathfrak{A}_2^\alpha}\sqrt{|K_{in}\backslash(1+\delta)K_{in}^\ell|_\sigma}\sqrt{|K_{in}^\ell|_\omega} \\
&\leq 2^n\delta^{\alpha-n}\sqrt{\mathfrak{A}_2^\alpha}\sqrt{|K_{in}|_\omega}\sqrt{|K_{in}|_\sigma}
\end{aligned}$$

Finally,

$$\begin{aligned}
\mathrm{III} &\leq \sum_{\ell=1}^{2^n}\left|\left|T_\sigma^\alpha\big(b_A\mathbf{1}_{(K_{in}\backslash K_{in}^\ell)\cap(1+\delta)K_{in}^\ell}\big)\right|\right|_{L^2(\omega)}\left|\left|b^{*,orig}_{K_{in}^\ell}\right|\right|_{L^2(\omega)} \\
&\leq \mathfrak{N}_{T^\alpha}\sqrt{C_{\mathbf{b}}C_{\mathbf{b}^*}}\left(\sum_{\ell=1}^{2^n}\left|(K_{in}\backslash K_{in}^\ell)\cap(1+\delta)K_{in}^\ell\right|_\sigma\right)^{\frac{1}{2}}\sqrt{|K_{in}|_\omega} \\
&\equiv \sqrt{C_{\mathbf{b}}C_{\mathbf{b}^*}}\cdot\mathbf{\Delta}(K)
\end{aligned}$$

where we have defined

$$\mathbf{\Delta}(K) = \mathfrak{N}_{T^\alpha}\left(\sum_{\ell=1}^{2^n}\left|(K_{in}\backslash K_{in}^\ell)\cap(1+\delta)K_{in}^\ell\right|_\sigma\right)^{\frac{1}{2}}\sqrt{|K_{in}|_\omega}$$

This last term will be iterated and a final random surgery will give us the desired bound.

5.4.2.2 A finite iteration and a final random surgery.

Letting

$$\begin{aligned}
\Phi^{A,B}(K_{in}) &= \left|\left|\mathbf{1}_{K_{in}}T_\sigma^\alpha(b_A)\right|\right|_{L^2(\omega)}\sqrt{|K_{in}|_\omega} \\
&\quad+\sum_{\ell=1}^{2^n}\mathrm{P}_\delta^\alpha\mathsf{Q}^\omega\left(K_{in}^\ell,\mathbf{1}_{A\backslash K}\sigma\right)\sqrt{|K_{in}|_\omega} \\
&\quad+\left(\mathfrak{T}^\alpha+\mathfrak{T}^{\alpha,*}+\delta^{\alpha-n}\sqrt{\mathfrak{A}_2^\alpha}\right)\sqrt{|K_{in}|_\sigma}\sqrt{|K_{in}|_\omega}
\end{aligned} \tag{5.4.27}$$

and simplifying more our notation

$$\{K_{out},K_{in}\}^{orig}\equiv\sum_{\ell=1}^{2^n}\left(\frac{\frac{1}{|K_{in}^\ell|_\omega}\int_{K_{in}^\ell}b_B^*d\omega}{\frac{1}{|K_{in}^\ell|_\omega}\int_{K_{in}^\ell}b_B^{*,orig}d\omega}\right)\left\{K_{out},K_{in}^\ell\right\}^{orig}$$

we have so far that (5.4.22) is written as

$$\{K,K\}=\{K_{out},K_{in}\}^{orig}+\{K_{out},K_{out}\}+\{K_{in},K_{out}\}+O\big(\Phi^{A,B}(K_{in})+\mathbf{\Delta}(K)\big)$$

Now

$$\{K_{out},K_{out}\}=\sum_\ell\{K_{out}^\ell,K_{out}^\ell\}+\sum_{\substack{m\neq\ell\\ K_{out}^m\cap K_{out}^\ell=\emptyset}}\{K_{out}^\ell,K_{out}^m\}+\sum_{\substack{m\neq\ell\\ K_{out}^m\cap K_{out}^\ell\neq\emptyset}}\{K_{out}^\ell,K_{out}^m\}$$

where $K_{out}^\ell, 1\le\ell\le 4^n-2^n$, are the outer grandchildren of K. For the second sum above, we get

$$\begin{aligned}\Bigg|\sum_{\substack{m\neq\ell\\ K_{out}^m\cap K_{out}^\ell=\emptyset}}\{K_{out}^\ell,K_{out}^m\}\Bigg| &\lesssim \sqrt{\mathfrak{A}_2^\alpha}\sum_\ell\sqrt{|K_{out}^\ell|_\sigma}\sum_{\substack{m\neq\ell\\ K_{out}^m\cap K_{out}^\ell=\emptyset}}\sqrt{|K_{out}^m|_\omega}\\ &\lesssim \sqrt{\mathfrak{A}_2^\alpha}\sqrt{|K_{out}|_\sigma}\sqrt{|K_{out}|_\omega}\end{aligned}$$

where the implied constant depends on dimension and the accretivity of functions involved and since $\operatorname{dist}(K_{out}^\ell,K_{out}^m)\ge\ell(K_{out}^\ell)$ there is no δ. For the third sum, we need to use random surgery again. Using Lemma 5.4.3,

$$
\begin{aligned}
&|\{K^\ell_{out}, K^m_{out}\}| = \left| \left\langle T^\alpha_\sigma \left(b_A \mathbf{1}_{K^\ell_{out}} \right), \mathbf{1}_{K^m_{out}} b^*_B \right\rangle_\omega \right| \\
\leq & \left| \left\langle T^\alpha_\sigma \left(b_A \mathbf{1}_{K^\ell_{out} \setminus (1+\delta) K^m_{out}} \right), \mathbf{1}_{K^m_{out}} b^*_B \right\rangle_\omega \right| + \left| \left\langle T^\alpha_\sigma \left(b_A \mathbf{1}_{K^\ell_{out} \cap (1+\delta) K^m_{out}} \right), \mathbf{1}_{K^m_{out}} b^*_B \right\rangle_\omega \right| \\
\leq & \, \delta^{\alpha-n} \sqrt{\mathfrak{A}^\alpha_2} \sqrt{|K^\ell_{out}|_\sigma} \sqrt{|K^m_{out}|_\omega} + \mathfrak{N}_{T^\alpha} \sqrt{|K^m_{out}|_\omega} \sqrt{|K^\ell_{out} \cap (1+\delta) K^m_{out}|_\sigma}
\end{aligned}
$$

Thus, summing

$$
\begin{aligned}
&\sum_{\ell} \sum_{\substack{m \neq \ell \\ K^m_{out} \cap K^\ell_{out} \neq \emptyset}} |\{K^\ell_{out}, K^m_{out}\}| \qquad (5.4.28) \\
\lesssim & \, \delta^{\alpha-n} \sqrt{\mathfrak{A}^\alpha_2} \sqrt{|K_{out}|_\sigma} \sqrt{|K_{out}|_\omega} + \mathfrak{N}_{T^\alpha} \sum_{\ell} \sum_{\substack{m \neq \ell \\ K^m_{out} \cap K^\ell_{out} \neq \emptyset}} \sqrt{|K^m_{out}|_\omega} \sqrt{|K^\ell_{out} \cap (1+\delta) K^m_{out}|_\sigma} \\
\leq & \, \delta^{\alpha-n} \sqrt{\mathfrak{A}^\alpha_2} \sqrt{|K_{out}|_\sigma} \sqrt{|K_{out}|_\omega} + \mathfrak{N}_{T^\alpha} \sum_{\ell} \left(\sum_{m \neq \ell} |K^\ell_{out} \cap (1+\delta) K^m_{out}|_\sigma \right)^{\frac{1}{2}} \sqrt{|K_{out}|_\omega}
\end{aligned}
$$

Let

$$
\mathbf{E}(K) = \mathfrak{N}_{T^\alpha} \sum_{\ell} \left(\sum_{m \neq \ell} |K^\ell_{out} \cap (1+\delta) K^m_{out}|_\sigma \right)^{\frac{1}{2}} \sqrt{|K_{out}|_\omega}
$$

We will iterate this term below and we will the necessary bound. We now turn to $\{K_{in}, K_{out}\}$ and we have

$$
\begin{aligned}
&|\{K_{in}, K_{out}\}| \\
\leq & \left| \left\langle T^\alpha_\sigma \left(b_A \mathbf{1}_{K_{out} \setminus (1+\delta) K_{in}} \right), \mathbf{1}_{K_{in}} b^*_B \right\rangle_\omega \right| + \left| \left\langle T^\alpha_\sigma \left(b_A \mathbf{1}_{K_{out} \cap (1+\delta) K_{in}} \right), \mathbf{1}_{K_{in}} b^*_B \right\rangle_\omega \right| \\
\lesssim & \, \delta^{\alpha-n} \sqrt{\mathfrak{A}^\alpha_2} \sqrt{|K_{out}|_\sigma} \sqrt{|K_{in}|_\omega} + \mathfrak{N}_{T^\alpha} \sqrt{|K_{in}|_\omega} \sqrt{|K_{out} \cap (1+\delta) K_{in}|_\sigma}
\end{aligned}
$$

and similarly $|\{K_{out}, K_{in}\}^{orig}|$ is bounded by

$$\lesssim \delta^{\alpha-n}\sqrt{\mathfrak{A}_2^\alpha}\sqrt{|K_{out}|_\sigma}\sqrt{|K_{in}|_\omega}+\mathfrak{N}_{T^\alpha}\sqrt{|K_{in}|_\omega}\sqrt{|K_{out}\cap(1+\delta)K_{in}|_\sigma}$$

Let

$$\mathbf{F}(K)=\mathfrak{N}_{T^\alpha}\sqrt{|K_{out}\cap(1+\delta)K_{in}|_\sigma}\sqrt{|K_{in}|_\omega}$$

Using the bounds we found above we have from (5.4.22),

$$\begin{aligned} |\{K,K\}| \lesssim & \sum_{\ell=1}^{4^n-2^n}|\{K_{out}^\ell, K_{out}^\ell\}|+O\big(\Phi^{A,B}(K_{in})\big) \\ & +\mathbf{\Delta}(K)+\mathbf{E}(K)+\mathbf{F}(K)+C_{\delta,\eta_0,\mathbf{b},\mathbf{b}^*}\sqrt{\mathfrak{A}_2^\alpha}\sqrt{|K|_\sigma}\sqrt{|K|_\omega} \end{aligned}$$

Iterating the first term above a *finite* number of times, using again the norm inequality and a final random surgery we get the bound we need. Indeed, for $\nu\in\mathbb{N}$

$$\begin{aligned} |\{K,K\}| \leq & \sum_{M\in\mathcal{M}_\nu}|\{M,M\}|+O\left(\sum_{M\in\mathcal{M}_\nu^*}\left[\Phi^{A,B}(M_{in})\right]+\mathbf{\Delta}(M)+\mathbf{E}(M)+\mathbf{F}(M)\right) \\ & +C_{\delta,\eta_0,\mathbf{b},\mathbf{b}^*}\sqrt{\mathfrak{A}_2^\alpha}\sum_{M\in\mathcal{M}_\nu^*}\sqrt{|M|_\sigma}\sqrt{|M|_\omega} \\ \equiv & A(K)+B(K)+C(K)=A_{(I',J')}(K)+B_{(I',J')}(K)+C_{(I',J')}(K) \end{aligned} \tag{5.4.29}$$

where the collections of cubes $\mathcal{M}_\nu = \mathcal{M}_\nu(K)$ and $\mathcal{M}_\nu^* = \mathcal{M}_\nu^*(K)$ are defined recursively by

$$\begin{aligned}
\mathcal{M}_0 &\equiv \{K\}, \\
\mathcal{M}_{k+1} &\equiv \bigcup_{M\in\mathcal{M}_k} \left\{M_{out}^{\ell}\right\}, \qquad k\geq 0, \\
\mathcal{M}_\nu^* &\equiv \bigcup_{k=0}^{\nu} \mathcal{M}_k .
\end{aligned}$$

We will include the subscript (I',J') in the notation when we want to indicate the pair (I',J') that are defined after (5.4.13). Now the term $C(K)$ can be estimated by

$$C(K) = C_{\delta,\eta_0,\mathbf{b},\mathbf{b}^*}\sqrt{\mathfrak{A}_2^\alpha}\sum_{M\in\mathcal{M}_\nu^*}\sqrt{|M|_\sigma}\sqrt{|M|_\omega} \leq \nu C_{\delta,\eta_0,\mathbf{b},\mathbf{b}^*}\sqrt{\mathfrak{A}_2^\alpha}\sqrt{|K|_\sigma}\sqrt{|K|_\omega} \quad (5.4.30)$$

where ν is chosen below depending on η_0. For the first term $A(K)$, we will apply the norm inequality and use probability, namely

$$\begin{aligned}
|A(K)| &\leq \sqrt{C_{\mathbf{b}}C_{\mathbf{b}^*}}\mathfrak{N}_{T^\alpha}\sum_{M\in\mathcal{M}_\nu}\sqrt{|M|_\sigma}\sqrt{|M|_\omega} \\
&\leq \sqrt{C_{\mathbf{b}}C_{\mathbf{b}^*}}\mathfrak{N}_{T^\alpha}\sqrt{\sum_{M\in\mathcal{M}_\nu}|M|_\sigma}\sqrt{\sum_{M\in\mathcal{M}_\nu}|M|_\omega} \\
&\leq \sqrt{C_{\mathbf{b}}C_{\mathbf{b}^*}}\mathfrak{N}_{T^\alpha}\sqrt{\sum_{M\in\mathcal{M}_\nu}|M|_\sigma}\sqrt{|K|_\omega},
\end{aligned}$$

where $\sqrt{C_{\mathbf{b}}C_{\mathbf{b}^*}}$ is an upper bound for the testing functions involved, followed by

$$\boldsymbol{E}_\Omega^{\mathcal{G}}\left(\sum_{M\in\mathcal{M}_\nu}|M|_\sigma\right) \leq \varepsilon\left|I'\right|_\sigma ,$$

for a sufficiently small $\varepsilon > 0$, where *roughly speaking*, we use the fact that the cubes $M \in \mathcal{M}_\nu$ depend on the grid $\mathcal{G}$ and form a relatively small proportion of I', which captures only a small amount of the total mass $\left|I'\right|_\sigma$ as the grid is translated relative to the grid $\mathcal{D}$ that contains I'.

Here are the details. Recall that the cubes K are taken from the set of consecutive cubes $\{K_i\}_{i=1}^B$ that lie in $I' \cap J'$, that the cubes $M \in \mathcal{M}_\nu(K_i)$ have length $\frac{1}{4^{n\nu}}\ell(K_i)$, and that there are $(4^n - 2^n)^\nu$ such cubes in $\mathcal{M}_\nu(K_i)$ for each i. Thus we have

$$\sum_{M\in\mathcal{M}_\nu(K)} |M| = \sum_{M\in\mathcal{M}_\nu(K)} \frac{1}{4^{n\nu}}|K| = (4^n - 2^n)^\nu \frac{1}{4^{n\nu}}|K|$$

and $\left(\dfrac{4^n-2^n}{4^n}\right)^\nu \to 0$ as $\nu \to \infty$, which implies

$$\boldsymbol{E}_\Omega^{\mathcal{G}}\left(\sum_{i=1}^B \sum_{M\in\mathcal{M}_\nu(K_i)} |M|_\sigma\right) \leq B\left(\frac{4^n-2^n}{4^n}\right)^\nu \left|I'\right|_\sigma \leq \varepsilon\left|I'\right|_\sigma$$

where we have used that the variable B is at most 2^{nm} and where the final inequality holds if ν is chosen large enough such that $B\left(\frac{4^n-2^n}{4^n}\right)^\nu \leq \varepsilon$. Then we have by Cauchy-Schwarz applied first to $\displaystyle\sum_{i=1}^B \sum_{M\in\mathcal{M}_\nu(K_i)}$ and then to $\boldsymbol{E}_\Omega^{\mathcal{G}}$,

$$\boldsymbol{E}_\Omega^{\mathcal{G}}\left(\sum_{i=1}^B |A(K_i)|\right) \leq \boldsymbol{E}_\Omega^{\mathcal{G}}\sqrt{C_{\mathbf{b}}C_{\mathbf{b}^*}}\mathfrak{N}_T\alpha\sqrt{\sum_{i=1}^B \sum_{M\in\mathcal{M}_\nu(K_i)} |M|_\sigma}\sqrt{|J'|_\omega} \tag{5.4.31}$$

$$\leq \sqrt{C_{\mathbf{b}}C_{\mathbf{b}^*}}\mathfrak{N}_T\alpha\sqrt{\boldsymbol{E}_\Omega^{\mathcal{G}}\sum_{i=1}^B \sum_{M\in\mathcal{M}_\nu(K_i)} |M|_\sigma}\sqrt{|J'|_\omega}$$

$$\leq \sqrt{C_{\mathbf{b}}C_{\mathbf{b}^*}}\mathfrak{N}_T\alpha\sqrt{\varepsilon\,|I'|_\sigma}\sqrt{|J'|_\omega} = \sqrt{C_{\mathbf{b}}C_{\mathbf{b}^*}}\sqrt{\varepsilon}\mathfrak{N}_T\alpha\sqrt{|I'|_\sigma}\sqrt{|J'|_\omega},$$

as required.

Now we turn to summing up the remaining terms

$B(K) = C \sum_{M\in\mathcal{M}_\nu^*} \Phi^{A,B}(M_{in}) + \mathbf{\Delta}(M) + \mathbf{E}(M) + \mathbf{F}(M)$ above. In the case when the cube I' is a *natural* child of I, i.e. $I' \in \mathfrak{C}_{nat}(I)$ so that $I' \in \mathcal{C}_A^{\mathcal{A}}$, we have

$$\sum_{M\in\mathcal{M}_\nu^*(K)} \left\|\mathbf{1}_{M_{in}} T_\sigma^\alpha b_A\right\|_{L^2(\omega)}^2 = \sum_{M\in\mathcal{M}_\nu^*(K)} \int_{M_{in}} \left|T_\sigma^\alpha b_A\right|^2 d\omega \leq \int_{I'} \left|T_\sigma^\alpha b_A\right|^2 d\omega \lesssim \left(\mathfrak{T}_{T^\alpha}^{\mathbf{b}}\right)^2 \left|I'\right|_\sigma$$

by the weak testing condition for I' in the corona $\mathcal{C}_A$. Also,

$$\sum_{M\in\mathcal{M}_\nu^*(K)} \left|M_{in}\right|_\omega \leq |K|_\omega \leq \left|J'\right|_\omega$$

because of the crucial fact that the cubes $\{M_{in}\}_{M\in\mathcal{M}_\nu^*(K)}$ form a pairwise disjoint subdecomposition of $K \subset I' \cap J'$ (for any $\nu \geq 1$). Of course, this implies

$$\left(\sum_{M\in\mathcal{M}_\nu^*(K)} \left(\mathfrak{T}_{T^\alpha,*} + \mathfrak{A}_2^\alpha\right)^2 \left|M_{in}\right|_\sigma\right)^{\frac{1}{2}} \left(\sum_{M\in\mathcal{M}_\nu^*(K)} \left|M_{in}\right|_\omega\right)^{\frac{1}{2}} \lesssim \left(\mathfrak{T}_{T^\alpha,*} + \mathfrak{A}_2^\alpha\right) \sqrt{\left|I'\right|_\sigma \left|J'\right|_\omega}$$

and using the definition of $\mathrm{P}_\delta^\alpha \mathsf{Q}^\omega(J, \upsilon)$ in (5.4.2),

$$\begin{aligned}
&\sum_{M\in\mathcal{M}_\nu^*(K)} \sum_{\ell=1}^{2^n} \mathrm{P}_\delta^\alpha \mathsf{Q}^\omega \left(M_{in}^\ell, \mathbf{1}_{A\setminus K}\sigma\right)^2 \\
\lesssim\ & \sum_{M\in\mathcal{M}_\nu^*(K)} \sum_{\ell=1}^{2^n} \left(\frac{\mathrm{P}^\alpha\left(M_{in}^\ell, \mathbf{1}_A\sigma\right)}{\left|M_{in}^\ell\right|}\right)^2 \left\|x - m_{M_{in}^\ell}\right\|_{L^2\left(\mathbf{1}_{M_{in}^\ell}\omega\right)}^2 \\
\lesssim\ & \left(\mathcal{E}_2^\alpha + \mathfrak{A}_2^\alpha\right) \left|I'\right|_\sigma
\end{aligned}$$

upon using the stopping energy condition for I' in the corona $\mathcal{C}_A$, i.e. the failure of (5.1.28), in the corona $\mathcal{C}_A$ with the subdecomposition

$$I' \supset \dot{\bigcup_{M \in \mathcal{M}_{\nu}^{*}(K)}} \bigcup_{\ell=1}^{2^n} M_{in}^{\ell}$$

Combining these four bounds together with the definition of $\Phi^{A,B}$ in (5.4.27), after applying Cauchy-Schwarz, gives

$$\sum_{M \in \mathcal{M}_{\nu}^{*}(K)} \Phi^{A,B}(M_{in}) \lesssim \delta^{\alpha-n} \cdot \mathcal{NTV}_{\alpha} \sqrt{|I'|_{\sigma} |J'|_{\omega}}$$

In particular then, if we now sum over *natural* children I' of $I \in \mathcal{C}_A$ and the associated children J' of $J \in \mathcal{N}(I)$, where

$$\mathcal{N}(I) \equiv \left\{ J \in \mathcal{G} : 2^{-\mathbf{r}} \ell(I) < \ell(J) \leq \ell(I) \text{ and } d(J,I) \leq 2\ell(J)^{\varepsilon} \ell(I)^{1-\varepsilon} \right\}.$$

we obtain the following corona estimate, using the collection of K that is defined after

(5.4.13),

$$\begin{aligned}
&\sum_{\substack{I\in\mathcal{C}_A\\ J\in\mathcal{N}(I)}}\ \sum_{\substack{I'\in\mathfrak{C}_{nat}(I)\&J'\in\mathfrak{C}(J)\\ K\in\mathcal{K}\left(I',J'\right)}}\left|E_{I'}^{\sigma}\left(\widehat{\square}_I^{\sigma,\flat,\mathbf{b}}f\right)\right|\left|B_{(I',J')}(K)\right|\left|E_{J'}^{\omega}\left(\widehat{\square}_J^{\omega,\flat,\mathbf{b}^*}g\right)\right| \qquad (5.4.32)\\
\lesssim\ & \delta^{\alpha-n}\cdot B\cdot\mathcal{NTV}_{\alpha}\sum_{\substack{I\in\mathcal{C}_A\\ J\in\mathcal{N}(I)}}\ \sum_{\substack{I'\in\mathfrak{C}_{nat}(I)\\ J'\in\mathfrak{C}(J)}}\left|E_{I'}^{\sigma}\left(\widehat{\square}_I^{\sigma,\flat,\mathbf{b}}f\right)\right|\ \sqrt{\left|I'\right|_{\sigma}\left|J'\right|_{\omega}}\ \left|E_{J'}^{\omega}\left(\widehat{\square}_J^{\omega,\flat,\mathbf{b}^*}g\right)\right|\\
\lesssim\ & \delta^{\alpha-n}\cdot B\cdot\mathcal{NTV}_{\alpha}\left(\sum_{I\in\mathcal{C}_A}\sum_{I'\in\mathfrak{C}_{nat}(I)}\left|I'\right|_{\sigma}\left|E_{I'}^{\sigma}\left(\widehat{\square}_I^{\sigma,\flat,\mathbf{b}}f\right)\right|^2\right)^{\frac{1}{2}}\cdot\\
&\qquad\cdot\left(\sum_{I\in\mathcal{C}_A}\sum_{J\in\mathcal{N}(I)}\sum_{J'\in\mathfrak{C}(J)}\left|J'\right|_{\omega}\ \left|E_{J'}^{\omega}\left(\widehat{\square}_J^{\omega,\flat,\mathbf{b}^*}g\right)\right|^2\right)^{\frac{1}{2}}\\
\lesssim\ & \delta^{\alpha-n}\cdot B\cdot\mathcal{NTV}_{\alpha}\left\|\mathsf{P}_{\mathcal{C}_A}^{\sigma}f\right\|_{L^2(\sigma)}^{\bigstar}\left\|\mathsf{P}_{\mathcal{C}_A^{\mathcal{G},nearby}}^{\omega}g\right\|_{L^2(\sigma)}^{\bigstar}
\end{aligned}$$

where $\mathcal{C}_A^{\mathcal{G},nearby}=\bigcup_{I\in\mathcal{C}_A}\mathcal{N}(I)$, and the final line uses (5.4.2) to obtain

$$\begin{aligned}
\sum_{I\in\mathcal{C}_A}\sum_{I'\in\mathfrak{C}_{nat}(I)}\left|I'\right|_{\sigma}\left|E_{I'}^{\sigma}\left(\widehat{\square}_I^{\sigma,\flat,\mathbf{b}}f\right)\right|^2 &= \sum_{I\in\mathcal{C}_A}\left\|\widehat{\square}_I^{\sigma,\flat,\mathbf{b}}f\right\|_{L^2(\sigma)}^2\\
&\lesssim \sum_{I\in\mathcal{C}_A}\left\|\square_I^{\sigma,\mathbf{b}}f\right\|_{L^2(\sigma)}^2\leq\left\|\mathsf{P}_{\mathcal{C}_A}^{\sigma}f\right\|_{L^2(\sigma)}^{\bigstar 2}
\end{aligned}$$

and similarly for the sum in J and J', once we note that given $J\in\mathcal{C}_A^{\mathcal{G},nearby}$, there are only boundedly many $I\in\mathcal{C}_A$ for which $J\in\mathcal{N}(I)$.

In order to deal with this sum in the case when the child I' is broken, we must take the estimate one step further and sum over those broken cubes I' whose parents belong to the corona $\mathcal{C}_A$, i.e. $\left\{I'\in\mathcal{D}:I'\in\mathfrak{C}_{brok}(I)\text{ for some }I\in\mathcal{C}_A\right\}$. Of course this collection is precisely the set of $\mathcal{A}$ -children of A, i.e.

$$\left\{I' \in \mathcal{D} : I' \in \mathfrak{C}_{brok}(I) \text{ for some } I \in \mathcal{C}_A\right\} = \mathfrak{C}_{\mathcal{A}}(A). \tag{5.4.33}$$

To obtain the same corona estimate when summing over broken I', we will exploit the fact that the cubes $A' \in \mathfrak{C}_{\mathcal{A}}(A)$ are pairwise disjoint. But first we note that when I' is a broken child, neither weak testing nor stopping energy is available. But if we sum over such broken I', and use (5.4.33) to see that the broken children are pairwise disjoint, we obtain the following estimate where for convenience we use the notation $\widetilde{\mathcal{M}}_\nu \equiv \bigcup_{K\in\mathcal{K}(I',J')} \mathcal{M}^*_\nu(K)$:

$$\begin{aligned}
&\sum_{\substack{I\in\mathcal{C}_A\\ J\in\mathcal{N}(I)}} \sum_{\substack{I'\in\mathfrak{C}_{brok}(I)\&J'\in\mathfrak{C}(J)\\ K\in\mathcal{K}\left(I',J'\right)}} \left|E^\sigma_{I'}\left(\widehat{\square}^{\sigma,\flat,\mathbf{b}}_I f\right)\right| \left|B_{(I',J')}(K)\right| \left|E^\omega_{J'}\left(\widehat{\square}^{\omega,\flat,\mathbf{b}^*}_J g\right)\right| \\
\lesssim\,& \delta^{\alpha-n}\cdot B\cdot \mathcal{NTV}_\alpha \sum_{\substack{I\in\mathcal{C}_A\\ J\in\mathcal{N}(I)}} \sum_{\substack{I'\in\mathfrak{C}_{brok}(I)\\ J'\in\mathfrak{C}(J)}} \left|E^\sigma_{I'}\left(\widehat{\square}^{\sigma,\flat,\mathbf{b}}_I f\right)\right| \sqrt{|J'|_\omega}\ \left|E^\omega_{J'}\left(\widehat{\square}^{\omega,\flat,\mathbf{b}^*}_J g\right)\right| \cdot \\
&\cdot\left(\sum_{M\in\widetilde{\mathcal{M}}_\nu}\left\|\mathbf{1}_{M_{in}} T^\alpha_\sigma b_A\right\|^2_{L^2(\omega)} + \sum_{M\in\widetilde{\mathcal{M}}_\nu}\sum_{\ell=1}^{2^n} \mathrm{P}^\alpha_\delta \mathsf{Q}^\omega\left(M^\ell_{in}, \mathbf{1}_A\sigma\right)^2 + \sum_{M\in\widetilde{\mathcal{M}}_\nu} |M_{in}|_\sigma\right)^{1/2} \\
\lesssim\,& B\delta^{\alpha-n}\mathcal{NTV}_\alpha\Bigg(\sum_{\substack{I\in\mathcal{C}_A\\ I'\in\mathfrak{C}_{brok}(I)}} \sum_{\substack{J\in\mathcal{N}(I)\\ J'\in\mathfrak{C}(J)}} \sum_{M\in\widetilde{\mathcal{M}}_\nu}\Bigg\{\left\|\mathbf{1}_{M_{in}} T^\alpha_\sigma b_A\right\|^2_{L^2(\omega)} + \\
&\sum_{\ell=1} \mathrm{P}^\alpha_\delta \mathsf{Q}^\omega\left(M^\ell_{in}, \mathbf{1}_A\sigma\right)^2 |M_{in}|_\sigma\Bigg\}\Bigg)^{\frac12} \cdot \\
&\cdot\left(\frac{1}{|A|_\sigma}\int_A |f|\,d\sigma\right)\left(\sum_{\substack{J\in\mathcal{C}^{\mathcal{G},nearby}_A\\ J'\in\mathfrak{C}(J)}} \sum_{\substack{I\in\mathcal{C}_A:\ J\in\mathcal{N}(I)\\ I'\in\mathfrak{C}_{brok}(I)}} |J'|_\omega \left|E^\omega_{J'}\left(\widehat{\square}^{\omega,\flat,\mathbf{b}^*}_J g\right)\right|^2\right)^{\frac12}
\end{aligned}$$

which gives that

$$\sum_{\substack{I\in\mathcal{C}_A\\ J\in\mathcal{N}(I)}}\ \sum_{\substack{I'\in\mathfrak{C}_{brok}(I)\&J'\in\mathfrak{C}(J)\\ K\in\mathcal{K}\left(I',J'\right)}} \left|E_{I'}^{\sigma}\left(\widehat{\square}_{I}^{\sigma,\flat,\mathbf{b}}f\right)\right|\ \left|B_{\left(I',J'\right)}\left(K\right)\right|\ \left|E_{J'}^{\omega}\left(\widehat{\square}_{J}^{\omega,\flat,\mathbf{b}^{*}}g\right)\right| \tag{5.4.34}$$

$$\lesssim\ \mathcal{NTV}_{\alpha}\sqrt{\left|A\right|_{\sigma}\left(\frac{1}{\left|A\right|_{\sigma}}\int_{A}\left|f\right|d\sigma\right)^{2}}\left\|\mathsf{P}_{\mathcal{C}_{A}^{\mathcal{G},nearby}}^{\omega}g\right\|_{L^{2}(\sigma)}^{\bigstar}$$

because

$$\left|E_{I'}^{\sigma}\left(\widehat{\square}_{I}^{\sigma,\flat,\mathbf{b}}f\right)\right|=\left|\frac{1}{\int_{I}b_{I}d\sigma}\int_{I}fd\sigma\right|\lesssim\frac{1}{\left|I\right|_{\sigma}}\int_{I}\left|f\right|d\sigma\lesssim\frac{1}{\left|A\right|_{\sigma}}\int_{A}\left|f\right|d\sigma$$

if $I'\in\mathfrak{C}_{brok}\left(I\right)$ and $I\in\mathcal{C}_{A}$, and because

$$\sum_{\substack{I\in\mathcal{C}_A\\ I'\in\mathfrak{C}_{brok}(I)}}\ \sum_{\substack{J\in\mathcal{N}(I)\\ J'\in\mathfrak{C}(J)}}\ \sum_{M\in\widetilde{\mathcal{M}}_{\nu}}\left\{\left\|\mathbf{1}_{M_{in}}T_{\sigma}^{\alpha}b_{A}\right\|_{L^{2}(\omega)}^{2}+\sum_{\ell=1}^{2^{n}}\mathrm{P}_{\delta}^{\alpha}\mathsf{Q}^{\omega}\left(M_{in}^{\ell},\mathbf{1}_{A}\sigma\right)^{2}+\left|M_{in}\right|_{\sigma}\right\} \tag{5.4.35}$$

$$\lesssim\ \left(\mathfrak{T}_{T^{\alpha}}^{\mathbf{b}}+\mathfrak{E}_{2}^{\alpha}+1\right)^{2}\left|A\right|_{\sigma}$$

Indeed, in this last inequality (5.4.35), we have used first the testing condition,

$$\begin{aligned}\sum_{\substack{I\in\mathcal{C}_A\\ I'\in\mathfrak{C}_{brok}(I)}}\ \sum_{\substack{J\in\mathcal{N}(I)\\ J'\in\mathfrak{C}(J)}}\ \sum_{M\in\widetilde{\mathcal{M}}_{\nu}}\left\|\mathbf{1}_{M_{in}}T_{\sigma}^{\alpha}b_{A}\right\|_{L^{2}(\omega)}^{2} &\leq\ \mathfrak{T}_{T^{\alpha}}^{\mathbf{b}}\sum_{\substack{I\in\mathcal{C}_A\\ I'\in\mathfrak{C}_{brok}(I)}}\ \sum_{\substack{J\in\mathcal{N}(I)\\ J'\in\mathfrak{C}(J)}}\left|I\right|_{\sigma}\\ &\lesssim\ \mathfrak{T}_{T^{\alpha}}^{\mathbf{b}}\sum_{\substack{I\in\mathcal{C}_A\\ I'\in\mathfrak{C}_{brok}(I)}}\left|I\right|_{\sigma}\leq\mathfrak{T}_{T^{\alpha}}^{\mathbf{b}}\left|A\right|_{\sigma}\end{aligned}$$

where in the first inequality we used the fact that the M_{in} that appear are all disjoint and form a subdecomposition of $I' \subset I$ and then used testing. On the second inequality we used the bounded overlap of J for any given I, since we are in the case of nearby cubes, and we get the last inequality because the $I \in \mathcal{C}_A$, which have a broken child I', are disjoint and form a subdecomposition of A. The same argument can be applied for the second sum of (5.4.35) upon using the energy condition for all $I \in C_A$ which have a broken child I' and using the finite repetition again since we are in the nearby form.

The inequality (5.4.34) is a suitable estimate since

$$\sum_{A\in\mathcal{A}} \sqrt{|A|_\sigma \left(\frac{1}{|A|_\sigma}\int_A |f|\,d\sigma\right)^2} \left\| \mathsf{P}^\omega_{\mathcal{C}_A^{\mathcal{G},nearby}} g \right\|^\bigstar_{L^2(\sigma)} \lesssim \|f\|_{L^2(\sigma)} \|g\|_{L^2(\sigma)}$$

by quasiorthogonality and the frame inequalities (5.1.40) and (5.1.51), together with the bounded overlap of the 'nearby' coronas $\left\{\mathcal{C}_A^{\mathcal{G},nearby}\right\}_{A\in\mathcal{A}}$. We are left with estimating $\mathbf{\Delta}, \mathbf{E}, \mathbf{F}$ that we get after the iteration.

Let us first deal with $\mathbf{\Delta}$. By $K^j_{i,\ell}$ we mean a grandchild of a cube K^j_i and K^j_i comes from K_i after having iterated j times, so $K^j_{i,\ell}$ is a $(2j+2)$-child of K_i. We have

$$\begin{aligned}
&\sum_{i=1}^{B}\sum_{j=1}^{\nu}\sum_{\ell=1}^{4^n-2^n} \mathbf{\Delta}(K^j_{i,\ell}) \\
\leq\ & \mathfrak{N}_T \alpha C_{\mathbf{b},\mathbf{b}^*,\nu} \sum_{i=1}^{B}\sum_{j=1}^{\nu}\sum_{\ell=1}^{4^n-2^n} \left(\sum_{q=1}^{2^n} \left| (K^j_{i,\ell,in} \backslash K^{j,q}_{i,\ell,in}) \cap (1+\delta) K^{j,q}_{i,\ell,in} \right|_\sigma \right)^{\frac{1}{2}} \sqrt{|K^j_{i,\ell}|_\omega} \\
\leq\ & \mathfrak{N}_T \alpha C_{\mathbf{b},\mathbf{b}^*,\nu} \left(\sum_{i=1}^{B}\sum_{j=1}^{\nu}\sum_{\ell=1}^{4^n-2^n}\sum_{q=1}^{2^n} \left| (K^j_{i,\ell,in} \backslash K^{j,q}_{i,\ell,in}) \cap (1+\delta) K^{j,q}_{i,\ell,in} \right|_\sigma \right)^{\frac{1}{2}} \sqrt{|J'|_\omega}
\end{aligned}$$

where $K^{j,q}_{i,\ell,in}$ is one of the inner grandchildren of $K^j_{i,\ell,in}$. Now fixing $q = q_0$ and taking averages over the grid $\mathcal{G}$ we get

$$\boldsymbol{E}_{\Omega}^{\mathcal{G}} \sum_{i=1}^{B} \sum_{j=1}^{\nu} \sum_{\ell=1}^{4^n-2^n} \left| \left(K_{i,\ell,in}^{j} \backslash K_{i,\ell,in}^{j,q}\right) \cap (1+\delta) K_{i,\ell,in}^{j,q} \right|_{\sigma} \leq C_n \delta |I|_{\sigma}$$

the constant depends on dimension since for the same i, j we can have intersection as ℓ moves. Adding the different q we get finally

$$\boldsymbol{E}_{\Omega}^{\mathcal{G}} \sum_{i=1}^{B} \sum_{j=1}^{\nu} \sum_{\ell=1}^{4^n-2^n} \boldsymbol{\Delta}(K_{i,\ell}^{j}) \leq \mathfrak{N}_T \alpha C_{\mathbf{b},\mathbf{b}^*,\nu,n} \sqrt{\delta} \sqrt{|I'|_{\sigma}} \sqrt{|J'|_{\omega}}. \tag{5.4.36}$$

For $\mathbf{F}$ we get,

$$\sum_{i=1}^{B} \sum_{j=1}^{\nu} \sum_{\ell=1}^{4^n-2^n} \mathbf{F}(K_{i,\ell}^{j}) \leq \mathfrak{N}_T \alpha C_{\mathbf{b},\mathbf{b}^*} \left(\sum_{i=1}^{B} \sum_{j=1}^{\nu} \sum_{\ell=1}^{4^n-2^n} \left| K_{i,\ell,out}^{j} \cap (1+\delta) K_{i,\ell,in}^{j} \right|_{\sigma} \right)^{\frac{1}{2}} \sqrt{|J'|_{\omega}}$$

and again averaging over grids $\mathcal{G}$, we get the bound

$$\boldsymbol{E}_{\Omega}^{\mathcal{G}} \sum_{i=1}^{B} \sum_{j=1}^{\nu} \sum_{\ell=1}^{4^n-2^n} \mathbf{F}(K_{i,\ell}^{j}) \leq \mathfrak{N}_T \alpha C_{\mathbf{b},\mathbf{b}^*} \sqrt{\delta} \sqrt{|I'|_{\sigma}} \sqrt{|J'|_{\omega}} \tag{5.4.37}$$

Note here that upon choosing δ small enough there is no repetition in the different terms that arise. Finally, for $\mathbf{E}$, we have

$$\begin{aligned}
&\sum_{i=1}^{B}\sum_{j=1}^{\nu}\sum_{\ell=1}^{4^n-2^n}\mathbf{E}(K_{i,\ell}^{j}) \qquad (5.4.38)\\
\leq\ & \mathfrak{N}_T\alpha\sum_{i=1}^{B}\sum_{j=1}^{\nu}\sum_{\ell=1}^{4^n-2^n}\sum_{q=1}^{4^n-2^n}\left(\sum_{r>q}\left|K_{i,\ell,out}^{j,q}\cap(1+\delta)K_{i,\ell,out}^{j,r}\right|_{\sigma}\right)^{\frac{1}{2}}\sqrt{\left|K_{i,\ell,out}^{j}\right|_{\omega}}\\
\leq\ & \mathfrak{N}_T\alpha\left(\sum_{i=1}^{B}\sum_{j=1}^{\nu}\sum_{\ell=1}^{4^n-2^n}\sum_{q=1}^{4^n-2^n}\sum_{r>q}\left|K_{i,\ell,out}^{j,q}\cap(1+\delta)K_{i,\ell,out}^{j,r}\right|_{\sigma}\right)^{\frac{1}{2}}\cdot\\
&\qquad\cdot\left(\sum_{i=1}^{B}\sum_{j=1}^{\nu}\sum_{\ell=1}^{4^n-2^n}\sum_{q=1}^{4^n-2^n}\sum_{r>q}\left|K_{i,\ell,out}^{j}\right|_{\omega}\right)^{\frac{1}{2}}\\
\leq\ & \mathfrak{N}_T\alpha\cdot C_{n,\nu}\left(\sum_{i=1}^{B}\sum_{j=1}^{\nu}\sum_{\ell=1}^{4^n-2^n}\sum_{q=1}^{4^n-2^n}\sum_{r>q}\left|K_{i,\ell,out}^{j,q}\cap(1+\delta)K_{i,\ell,out}^{j,r}\right|_{\sigma}\right)^{\frac{1}{2}}\sqrt{|J'|_{\omega}}
\end{aligned}$$

Taking averages,

$$\boldsymbol{E}_{\Omega}^{\mathcal{G}}\sum_{i=1}^{B}\sum_{j=1}^{\nu}\sum_{\ell=1}^{4^n-2^n}\mathbf{E}(K_{i,\ell}^{j})\leq\mathfrak{N}_T\alpha\cdot C_{n,\nu}\sqrt{\delta}\sqrt{|I'|_{\sigma}}\sqrt{|J'|_{\omega}}$$

The constant $C_{n,\nu}$ comes from the intersection of the sets $K_{i,\ell,out}^{j}$.

Recall that after splitting in the cases of δ-seperated and δ-close cubes, we got the bound (5.4.7) in the separated case and after an initial application of random surgery, we reduced the proof of Proposition 5.4.1 to establishing inequality (5.4.11). Then using the bounds in (5.4.12), (5.4.14), (5.4.15), (5.4.16), (5.4.17), (5.4.18) we reduced $P(I,J)$ to getting a bound for $\{K,K\}$ in the notation used in (5.4.21). Then using the estimates in (5.4.30), (5.4.31), (5.4.32) and (5.4.34) together with (5.4.29), (5.4.36), (5.4.37) and (5.4.38) establishes probabilistic control of the sum of all the inner products $\{K,K\}$ taken over appropriate cubes K, yielding (5.4.11) as required if we choose ε, λ, η_0 and δ sufficiently small. And combining all the above bounds we proved proposition 5.4.1, namely we got the bound

$$\boldsymbol{E}_{\Omega}^{\mathcal{D}}\boldsymbol{E}_{\Omega}^{\mathcal{G}}\sum_{I\in\mathcal{D}}\sum_{\substack{J\in\mathcal{G}:\ 2^{-\mathbf{r}n}|I|<|J|\leq|I| \\ d(J,I)\leq 2\ell(J)^{\varepsilon}\ell(I)^{1-\varepsilon}}}\left|\left\langle T_{\sigma}^{\alpha}\left(\square_{I}^{\sigma,\mathbf{b}}f\right),\square_{J}^{\omega,\mathbf{b}^{*}}g\right\rangle_{\omega}\right| \lesssim \left(C_{\theta}\mathcal{NTV}_{\alpha}+\sqrt{\theta}\mathfrak{N}_{T^{\alpha}}\right)\left\|f\right\|_{L^{2}(\sigma)}\left\|g\right\|_{L^{2}(\omega)}$$

5.5 Main below form

Now we turn to controlling the main below form (5.2.17),

$$\Theta_{2}^{good}\left(f,g\right)=\sum_{I\in\mathcal{D}}\sum_{J^{\maltese}\subsetneqq I:\ \ell(J)\leq 2^{-\rho}\ell(I)}\int\left(T_{\sigma}\square_{I}^{\sigma,\mathbf{b}}f\right)\square_{J}^{\omega,\mathbf{b}^{*}}gd\omega.$$

To control $\Theta_{2}^{good}\left(f,g\right)\equiv\mathsf{B}_{\Subset_{\rho}}\left(f,g\right)$ we first perform the *canonical corona splitting* of $\mathsf{B}_{\Subset_{\rho}}\left(f,g\right)$ into a diagonal form and a far below form, namely $\mathsf{T}_{diagonal}\left(f,g\right)$ and $\mathsf{T}_{farbelow}\left(f,g\right)$ as in [63]. This *canonical splitting* of the form $\mathsf{B}_{\Subset_{\rho}}\left(f,g\right)$ involves the corona pseudoprojections $\mathsf{P}_{\mathcal{C}_{A}^{\mathcal{D}}}^{\sigma,\mathbf{b}}$ acting on f and the *shifted* corona pseudoprojections $\mathsf{P}_{\mathcal{C}_{B}^{\mathcal{G},shift}}^{\omega,\mathbf{b}^{*}}$ acting on g, where B is a stopping cube in $\mathcal{A}$. The stopping cubes $\mathcal{B}$ constructed relative to $g\in L^{2}\left(\omega\right)$ play no role in the analysis here, except to guarantee that the frame and weak Riesz inequalities hold for g and $\left\{\square_{J}^{\omega,\mathbf{b}^{*}}g\right\}_{J\in\mathcal{G}}$. Here the shifted corona $\mathcal{C}_{B}^{\mathcal{G},shift}$ is defined to include those cubes $J\in\mathcal{G}$ such $J^{\maltese}\in\mathcal{C}_{B}^{\mathcal{D}}$. Recall that the parameters τ and ρ are fixed to satisfy

$$\tau>\mathbf{r}\text{ and }\rho>\mathbf{r}+\tau,$$

where $\mathbf{r}$ is the goodness parameter already fixed in (5.2.16).

Definition 5.5.1. *For $B \in \mathcal{A}$ we define the shifted $\mathcal{G}$-corona by*

$$\mathcal{C}_B^{\mathcal{G},shift} = \left\{ J \in \mathcal{G} : J^{\maltese} \in \mathcal{C}_B^{\mathcal{D}} \right\}.$$

We will use repeatedly the fact that the shifted coronas $\mathcal{C}_B^{\mathcal{G},shift}$ are pairwise disjoint in B:

$$\sum_{B \in \mathcal{A}} \mathbf{1}_{\mathcal{C}_B^{\mathcal{G},shift}} (J) \leq \mathbf{1}, \qquad J \in \mathcal{D}. \tag{5.5.1}$$

The forms $\mathsf{B}_{\Subset_{\rho,\varepsilon}} (f, g)$ are no longer linear in f and g as the 'cut' is determined by the coronas $\mathcal{C}_A^{\mathcal{D}}$ and $\mathcal{C}_B^{\mathcal{G},shift}$, which depend on f as well as the measures σ and ω. However, if the coronas are held fixed, then the forms can be considered bilinear in f and g. It is convenient at this point to introduce the following shorthand notation:

$$\left\langle T_\sigma^\alpha \left(\mathsf{P}_{\mathcal{C}_A^{\mathcal{D}}}^{\sigma,\mathbf{b}} f \right), \mathsf{P}_{\mathcal{C}_B^{\mathcal{G},shift}}^{\omega,\mathbf{b}^*} g \right\rangle_\omega^{\Subset_{\rho,\varepsilon}} \equiv \sum_{\substack{I \in \mathcal{C}_A^{\mathcal{D}} \text{ and } J \in \mathcal{C}_B^{\mathcal{G},shift} :\ J^{\maltese} \subsetneqq I \\ \ell(J) \leq 2^{-\rho} \ell(I)}} \left\langle T_\sigma^\alpha \left(\square_I^{\sigma,\mathbf{b}} f \right), \square_J^{\omega,\mathbf{b}^*} g \right\rangle_\omega . \tag{5.5.2}$$

Caution One must not assume, from the notation on the left hand side above, that the function $T_\sigma^\alpha \left(\mathsf{P}_{\mathcal{C}_A}^\sigma f \right)$ is simply integrated against the function $\mathsf{P}_{\mathcal{C}_B^{\mathcal{G},shift}}^\omega g$. Indeed, the sum on the right hand side is taken over pairs (I, J) such that $J^{\maltese} \in \mathcal{C}_B^{\mathcal{D}}$ and $J^{\maltese} \subsetneqq I$ and $\ell(J) \leq 2^{-\rho} \ell(I)$.

5.5.1 The canonical splitting and local below forms

We then have the canonical splitting determined by the coronas $\mathcal{C}_A^{\mathcal{D}}$ for $A \in \mathcal{A}$ (the stopping times $\mathcal{B}$ play no explicit role in the canonical splitting of the below form, other than to guarantee the weak Riesz inequalities for the dual martingale pseudoprojections $\square_J^{\omega,\mathbf{b}^*}$)

$$\begin{aligned}
& \mathsf{B}_{\Subset_{\rho,\varepsilon}}(f,g) && (5.5.3)\\
= & \sum_{A,B\in\mathcal{A}} \left\langle T_\sigma^\alpha\left(\mathsf{P}_{\mathcal{C}_A}^{\sigma,\mathbf{b}}f\right), \mathsf{P}_{\mathcal{C}_B^{\mathcal{G},shift}}^{\omega,\mathbf{b}^*}g\right\rangle_\omega^{\Subset_{\rho,\varepsilon}} \\
= & \sum_{A\in\mathcal{A}} \left\langle T_\sigma^\alpha\left(\mathsf{P}_{\mathcal{C}_A}^{\sigma,\mathbf{b}}f\right), \mathsf{P}_{\mathcal{C}_A^{\mathcal{G},shift}}^{\omega,\mathbf{b}^*}g\right\rangle_\omega^{\Subset_{\rho,\varepsilon}} + \sum_{\substack{A,B\in\mathcal{A}\\ B\subsetneqq A}} \left\langle T_\sigma^\alpha\left(\mathsf{P}_{\mathcal{C}_A}^{\sigma,\mathbf{b}}f\right), \mathsf{P}_{\mathcal{C}_B^{\mathcal{G},shift}}^{\omega,\mathbf{b}^*}g\right\rangle_\omega^{\Subset_{\rho,\varepsilon}} \\
& + \sum_{\substack{A,B\in\mathcal{A}\\ B\supsetneqq A}} \left\langle T_\sigma^\alpha\left(\mathsf{P}_{\mathcal{C}_A}^{\sigma,\mathbf{b}}f\right), \mathsf{P}_{\mathcal{C}_B^{\mathcal{G},shift}}^{\omega,\mathbf{b}^*}g\right\rangle_\omega^{\Subset_{\rho,\varepsilon}} + \sum_{\substack{A,B\in\mathcal{A}\\ A\cap B=\emptyset}} \left\langle T_\sigma^\alpha\left(\mathsf{P}_{\mathcal{C}_A}^{\sigma,\mathbf{b}}f\right), \mathsf{P}_{\mathcal{C}_B^{\mathcal{G},shift}}^{\omega,\mathbf{b}^*}g\right\rangle_\omega^{\Subset_{\rho,\varepsilon}} \\
\equiv & \ \mathsf{T}_{diagonal}(f,g) + \mathsf{T}_{farbelow}(f,g) + \mathsf{T}_{farabove}(f,g) + \mathsf{T}_{disjoint}(f,g).
\end{aligned}$$

Now the final two terms $\mathsf{T}_{farabove}(f,g)$ and $\mathsf{T}_{disjoint}(f,g)$ each vanish since there are no pairs $(I,J)\in\mathcal{C}_A^{\mathcal{D}}\times\mathcal{C}_B^{\mathcal{G},shift}$ with both (**i**) $J^{\maltese}\subsetneqq I$ and (**ii**) either $B\subsetneqq A$ or $B\cap A=\emptyset$. The far below form $\mathsf{T}_{farbelow}(f,g)$ requires functional energy, which we discuss in a moment.

Next we follow this splitting by a further decomposition of the diagonal form into local below forms $\mathsf{B}_{\Subset_\rho}^A(f,g)$ given by the individual corona pieces

$$\mathsf{B}_{\Subset_{\rho,\varepsilon}}^A(f,g) \equiv \left\langle T_\sigma^\alpha\left(\mathsf{P}_{\mathcal{C}_A}^{\sigma,\mathbf{b}}f\right), \mathsf{P}_{\mathcal{C}_A^{\mathcal{G},shift}}^{\omega,\mathbf{b}^*}g\right\rangle_\omega^{\Subset_{\rho,\varepsilon}} \tag{5.5.4}$$

and prove the following estimate:

$$\left|\mathsf{B}_{\Subset_{\rho,\varepsilon}}^A(f,g)\right| \lesssim \mathcal{NTV}_\alpha \left(\alpha_{\mathcal{A}}(A)\sqrt{|A|_\sigma} + \left\|\mathsf{P}_{\mathcal{C}_A}^{\sigma,\mathbf{b}}f\right\|_{L^2(\sigma)}^{\bigstar}\right) \left\|\mathsf{P}_{\mathcal{C}_A^{\mathcal{G},shift}}^{\omega,\mathbf{b}^*}g\right\|_{L^2(\omega)}^{\bigstar}.$$

This reduces matters to the local forms since we then have from Cauchy-Schwarz that

$$\begin{aligned}
\sum_{A\in\mathcal{A}}\left|\mathsf{B}^{A}_{\Subset_{\rho,\varepsilon}}(f,g)\right| &\lesssim \mathcal{NTV}_{\alpha}\left(\sum_{A\in\mathcal{A}}\alpha_{\mathcal{A}}(A)^2|A|_{\sigma}+\left\|\mathsf{P}^{\sigma,\mathbf{b}}_{\mathcal{C}^{\mathcal{D}}_{A}}f\right\|^{\bigstar 2}_{L^2(\sigma)}\right)^{\frac{1}{2}}\cdot\\
&\qquad\cdot\left(\sum_{A\in\mathcal{A}}\left\|\mathsf{P}^{\omega,\mathbf{b}^*}_{\mathcal{C}^{\mathcal{G},shift}_{A}}g\right\|^{\bigstar 2}_{L^2(\omega)}\right)^{\frac{1}{2}}\\
&\lesssim \mathcal{NTV}_{\alpha}\ \|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)}\ .
\end{aligned}$$

by the lower frame inequalities

$$\sum_{A\in\mathcal{A}}\left\|\mathsf{P}^{\sigma,\mathbf{b}}_{\mathcal{C}_{A}}f\right\|^{\bigstar 2}_{L^2(\sigma)}\lesssim\|f\|^2_{L^2(\sigma)}\ \text{ and }\ \sum_{A\in\mathcal{A}}\left\|\mathsf{P}^{\omega,\mathbf{b}^*}_{\mathcal{C}^{\mathcal{G},shift}_{A}}g\right\|^{\bigstar 2}_{L^2(\omega)}\lesssim\|g\|^2_{L^2(\omega)}$$

using also quasi-orthogonality $\sum_{A\in\mathcal{A}}\alpha_{\mathcal{A}}(f)^2|A|_{\sigma}\lesssim\|f\|^2_{L^2(\sigma)}$ in the stopping cubes $\mathcal{A}$, and the pairwise disjointedness of the shifted coronas $\mathcal{C}^{\mathcal{G},shift}_{A}$:

$$\sum_{A\in\mathcal{A}}\mathbf{1}_{\mathcal{C}^{\mathcal{G},shift}_{A}}\leq\mathbf{1}_{\mathcal{D}}.$$

From now on we will often write $\mathcal{C}_A$ in place of $\mathcal{C}^{\mathcal{D}}_{A}$ when no confusion is possible.

Finally, the local forms $\mathsf{B}^{A}_{\Subset_{\rho,\varepsilon}}(f,g)$ are decomposed into stopping $\mathsf{B}^{A}_{stop}(f,g)$, paraproduct $\mathsf{B}^{A}_{paraproduct}(f,g)$ and neighbour $\mathsf{B}^{A}_{neighbour}(f,g)$ forms. The paraproduct and neighbour terms are handled as in [63], which in turn follows the treatment originating in [43], and this leaves only the stopping form $\mathsf{B}^{A}_{stop}(f,g)$ to be bounded, which we treat last by adapting the bottom/up stopping time and recursion of M. Lacey in [27].

However, in order to obtain the required bounds of the above forms into which the below form $\mathsf{B}_{\Subset_{\rho}}(f,g)$ was decomposed, we need functional energy. Recall that the vector-valued function $\mathbf{b}$ in the accretive coronas 'breaks' only at a collection of cubes satisfying a Carleson condition. We define $\mathcal{M}_{(\mathbf{r},\varepsilon)-deep}(F)$ to consist of the *maximal* $\mathbf{r}$-deeply embedded dyadic $\mathcal{G}$-subcubes of a $\mathcal{D}$-cube F - see (.0.7) in Appendix for more detail.

Definition 5.5.2. *Let $\mathfrak{F}_\alpha = \mathfrak{F}_\alpha(\mathcal{D},\mathcal{G})$ be the smallest constant in the 'functional energy' inequality below, holding for all $h \in L^2(\sigma)$ and all σ-Carleson collections $\mathcal{F} \subset \mathcal{D}$ with Carleson norm $C_{\mathcal{F}}$ bounded by a fixed constant C:*

$$\sum_{F\in\mathcal{F}} \sum_{M\in\mathcal{M}_{(\mathbf{r},1)-deep,\mathcal{D}}(F)} \left(\frac{\mathrm{P}^\alpha(M,h\sigma)}{|M|^{\frac{1}{n}}}\right)^2 \left\|\mathsf{Q}^{\omega,\mathbf{b}}_{\mathcal{C}_F^{\mathcal{G},shift};M} x\right\|^{\spadesuit 2}_{L^2(\omega)} \leq \mathfrak{F}_\alpha \|h\|_{L^2(\sigma)}, \quad (5.5.5)$$

The main ingredient used in reducing control of the below form $\mathsf{B}_{\Subset_\rho}(f,g)$ to control of the functional energy $\mathfrak{F}_\alpha$ constant and the stopping form $\mathsf{B}^A_{stop}(f,g)$, is the Intertwining Proposition from [63]. The control of the functional energy condition by the energy and Muckenhoupt conditions must also be adapted in light of the p-weakly accretive function $\mathbf{b}$ that only 'breaks' at a collection of cubes satisfying a Carleson condition, but this poses no real difficulties. The fact that the usual Haar bases are orthonormal is here replaced by the weaker condition that the corresponding broken Haar 'bases' are merely frames satisfying certain lower and weak upper Riesz inequalities, but again this poses no real difference in the arguments. Finally, the fact that goodness for J has been replaced with weak goodness, namely $J^{\maltese} \subsetneqq I$, again forces no real change in the arguments.

We then use the paraproduct / neighbour / stopping splitting mentioned above to reduce boundedness of $\mathsf{B}^A_{\Subset_{\rho,\varepsilon}}(f,g)$ to boundedness of the associated stopping form

$$\mathsf{B}^A_{stop}(f,g) \equiv \sum_{I\in\mathcal{C}_A} \sum_{\substack{J\in\mathcal{C}_A^{\mathcal{G},shift}:\ J^{\maltese}\subsetneqq I \\ \ell(J)\leq 2^{-\rho}\ell(I)}} \left(E^\sigma_{I_J}\square^{\sigma,\mathbf{b}}_I f\right) \left\langle T^\alpha_\sigma \mathbf{1}_{A\setminus I_J} b_A, \square^{\omega,\mathbf{b}^*}_J g\right\rangle_\omega \tag{5.5.6}$$

where f is supported in the cube A and its expectations $E^\sigma_I|f|$ are bounded by $\alpha_{\mathcal{A}}(A)$ for $I\in\mathcal{C}^\sigma_A$, the dual martingale support of f is contained in the corona $\mathcal{C}^\sigma_A$, and the dual martingale support of g is contained in $\mathcal{C}_A^{\mathcal{G},shift}$, and where I_J is the $\mathcal{D}$-child of I that contains J.

5.5.2 Diagonal and far below forms

Now we turn to the *diagonal* and the *far below* terms $\mathsf{T}_{diagonal}(f,g)$ and $\mathsf{T}_{farbelow}(f,g)$, where in [63] the far below terms were bounded using the Intertwining Proposition and the control of functional energy condition by the energy conditions, but of course under the restriction there that the cubes J were good. Here we write

$$\begin{aligned}
\mathsf{T}_{farbelow}(f,g) &= \sum_{\substack{A,B\in\mathcal{A} \\ B\subsetneqq A}} \sum_{\substack{I\in\mathcal{C}_A \text{ and } J\in\mathcal{C}_B^{\mathcal{G},shift} \\ J^{\maltese}\subsetneqq I \text{ and } \ell(J)\leq 2^{-\mathbf{r}}\ell(I)}} \left\langle T^\alpha_\sigma\left(\square^{\sigma,\mathbf{b}}_I f\right), \left(\square^{\omega,\mathbf{b}^*}_J g\right)\right\rangle_\omega \\
&= \sum_{B\in\mathcal{A}} \sum_{I\in\mathcal{D}:\ B\subsetneqq I} \left\langle T^\alpha_\sigma\left(\square^{\sigma,\mathbf{b}}_I f\right), \sum_{J\in\mathcal{C}_B^{\mathcal{G},shift}} \square^{\omega,\mathbf{b}^*}_J g\right\rangle_\omega \\
&\quad - \sum_{B\in\mathcal{A}} \sum_{I\in\mathcal{D}:\ B\subsetneqq I} \left\langle T^\alpha_\sigma\left(\square^{\sigma,\mathbf{b}}_I f\right), \sum_{\substack{J\in\mathcal{C}_B^{\mathcal{G},shift} \\ \ell(J)>2^{-\mathbf{r}}\ell(I)}} \square^{\omega,\mathbf{b}^*}_J g\right\rangle_\omega \\
&= \mathsf{T}^1_{farbelow}(f,g) - \mathsf{T}^2_{farbelow}(f,g).
\end{aligned} \tag{5.5.7}$$

since if $I \in \mathcal{C}_A$ and $J \in \mathcal{C}_B^{\mathcal{G},shift}$, with $J^{\maltese} \subsetneqq I$ and $B \subsetneqq A$, then we must have $B \subsetneqq I$. First, we note that expectation of the second sum $\mathsf{T}^2_{farbelow}(f,g)$ is controlled by (5.4.1) in Proposition 5.4.1 , i.e.

$$\begin{aligned}
&\boldsymbol{E}_\Omega^{\mathcal{D}}\boldsymbol{E}_\Omega^{\mathcal{G}}\left|\sum_{B\in\mathcal{A}}\sum_{I\in\mathcal{D}:\ B\subsetneqq I}\left\langle T_\sigma^\alpha\left(\square_I^{\sigma,\mathbf{b}}f\right),\sum_{\substack{J\in\mathcal{C}_B^{\mathcal{G},shift}\\ \ell(J)>2^{-\mathbf{r}}\ell(I)}}\square_J^{\omega,\mathbf{b}^*}g\right\rangle_\omega\right|\\
\lesssim\ &\boldsymbol{E}_\Omega^{\mathcal{D}}\boldsymbol{E}_\Omega^{\mathcal{G}}\sum_{I\in\mathcal{D}}\sum_{\substack{J\in\mathcal{G}:\ 2^{-\mathbf{r}}\ell(I)<\ell(J)\leq\ell(I)\\ d(J,I)\leq 2\ell(J)^\varepsilon\ell(I)^{1-\varepsilon}}}\left|\left\langle T_\sigma^\alpha\left(\square_I^{\sigma,\mathbf{b}}f\right),\square_J^{\omega,\mathbf{b}^*}g\right\rangle_\omega\right|\\
\lesssim\ &\left(C_\theta\mathcal{NTV}_\alpha+\sqrt{\theta}\mathfrak{N}_{T^\alpha}\right)\|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)}\ .
\end{aligned}$$

The form $\mathsf{T}^1_{farbelow}(f,g)$ can be written as

$$\begin{aligned}
\mathsf{T}^1_{farbelow}(f,g) &= \sum_{B\in\mathcal{A}}\sum_{I\in\mathcal{D}:\ B\subsetneqq I}\left\langle T_\sigma^\alpha\left(\square_I^{\sigma,\mathbf{b}}f\right),g_B\right\rangle_\omega;\\
\text{where } g_B &\equiv \sum_{J\in\mathcal{C}_B^{\mathcal{G},shift}}\square_J^{\omega,\mathbf{b}*}g=\mathsf{P}^{\omega,\mathbf{b}^*}_{\mathcal{C}_F^{\mathcal{G},shift}}g
\end{aligned}$$

and the Intertwining Proposition 5.5.7 can now be applied to this latter form to show that it is bounded by $\mathcal{NTV}_\alpha+\mathfrak{F}_\alpha$. Then Proposition .0.1 can be applied to show that $\mathfrak{F}_\alpha\lesssim\mathfrak{A}_2^\alpha+\mathcal{E}_2^\alpha$, which completes the proof that

$$\left|\mathsf{T}_{farbelow}(f,g)\right|\lesssim\mathcal{NTV}_\alpha\ \|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)}\ . \tag{5.5.8}$$

5.5.3 Intertwining Proposition

First we adapt the relevant definitions and theorems from [63].

Definition 5.5.3. *A collection $\mathcal{F}$ of dyadic cubes is σ-*Carleson *if*

$$\sum_{F\in\mathcal{F}:\ F\subset S} |F|_\sigma \leq C_{\mathcal{F}} |S|_\sigma , \qquad S\in\mathcal{F}.$$

The constant $C_{\mathcal{F}}$ is referred to as the Carleson norm of $\mathcal{F}$.

Definition 5.5.4. *Let $\mathcal{F}$ be a collection of dyadic cubes in a grid $\mathcal{D}$. Then for $F\in\mathcal{F}$, we define the shifted corona $\mathcal{C}_F^{\mathcal{G},shift}$ in analogy with Definition 5.5.1 by*

$$\mathcal{C}_F^{\mathcal{G},shift} = \left\{ J\in\mathcal{G} : J^{\maltese}\in\mathcal{C}_F \right\}.$$

Note that the collections $\mathcal{C}_F^{\mathcal{G},shift}$ are pairwise disjoint in F. Let $\mathfrak{C}_{\mathcal{F}}(F)$ denote the set of $\mathcal{F}$-children of F. Given any collection $\mathcal{H}\subset\mathcal{G}$ of cubes, a family $\mathbf{b}^*$ of dual testing functions, and an arbitrary cube $K\in\mathcal{P}$, we define the corresponding dual pseudoprojection $\mathsf{P}_{\mathcal{H}}^{\omega,\mathbf{b}^*}$ and its localization $\mathsf{P}_{\mathcal{H};K}^{\omega,\mathbf{b}^*}$ to K by

$$\mathsf{Q}_{\mathcal{H}}^{\omega,\mathbf{b}^*} = \sum_{H\in\mathcal{H}} \triangle_H^{\omega,\mathbf{b}^*} \text{ and } \mathsf{Q}_{\mathcal{H};K}^{\omega,\mathbf{b}^*} = \sum_{H\in\mathcal{H}:\ H\subset K} \triangle_H^{\omega,\mathbf{b}^*} . \tag{5.5.9}$$

Recall from Definition 5.5.2 that $\mathfrak{F}_\alpha = \mathfrak{F}_\alpha(\mathcal{D},\mathcal{G}) = \mathfrak{F}_\alpha^{\mathbf{b}^*}(\mathcal{D},\mathcal{G})$ is the best constant in (5.5.5), i.e.

$$\sum_{F\in\mathcal{F}} \sum_{M\in\mathcal{M}_{(\mathbf{r},1)-deep,\mathcal{D}}(F)} \left(\frac{\mathrm{P}^\alpha(M,h\sigma)}{|M|^{\frac{1}{n}}}\right)^2 \left\| \mathsf{Q}^{\omega,\mathbf{b}}_{\mathcal{C}_F^{\mathcal{G},shift};M} x \right\|_{L^2(\omega)}^{\spadesuit 2} \leq \mathfrak{F}_\alpha \|h\|_{L^2(\sigma)} .$$

Remark 5.5.5. *If in (5.5.5), we take $h=\mathbf{1}_I$ and $\mathcal{F}$ to be the trivial Carleson collection $\{I_r\}_{r=1}^\infty$ where the cubes I_r are pairwise disjoint in I, then we obtain the deep energy* *condition in Definition .0.4, but with* $\mathsf{P}^{\omega,\mathbf{b}^*}_{\mathcal{C}_F^{\mathcal{G},shift};M}$ *in place of* $\mathsf{P}^{weakgood,\omega}_J$. *However, the pseudoprojection* $\mathsf{P}^{weakgood,\omega}_J$ *is larger than* $\mathsf{P}^{\omega,\mathbf{b}^*}_{\mathcal{C}_F^{\mathcal{G},shift};J}$, *and so we just miss obtaining the deep energy condition as a consequence of the functional energy condition. Nevertheless, this near miss with* $h=\mathbf{1}_I$ *explains the terminology 'functional' energy.*

We will need the following 'indicator' version of the estimates proved above for the disjoint form.

Lemma 5.5.6. *Suppose* T^α *is a standard fractional singular integral with* $0\leq\alpha<1$, *that* $\rho>\mathbf{r}$, *that* $f\in L^2(\sigma)$ *and* $g\in L^2(\omega)$, *that* $\mathcal{F}\subset\mathcal{D}^\sigma$ *and* $\mathcal{G}\subset\mathcal{D}^\omega$ *are* σ*-Carleson and* ω*-Carleson collections respectively, i.e.,*

$$\sum_{F'\in\mathcal{F}:\ F'\subset F}\left|F'\right|_\sigma\lesssim|F|_\sigma\,,\qquad F\in\mathcal{F},\ \text{and}\ \sum_{G'\in\mathcal{G}:\ G'\subset G}\left|G'\right|_\omega\lesssim|G|_\omega\,,\qquad G\in\mathcal{G},$$

that there are numerical sequences $\{\alpha_{\mathcal{F}}(F)\}_{F\in\mathcal{F}}$ *and* $\{\beta_{\mathcal{G}}(G)\}_{G\in\mathcal{G}}$ *such that*

$$\sum_{F\in\mathcal{F}}\alpha_{\mathcal{F}}(F)^2|F|_\sigma\leq\|f\|^2_{L^2(\sigma)}\ \text{and}\ \sum_{G\in\mathcal{G}}\beta_{\mathcal{G}}(G)^2|G|_\sigma\leq\|g\|^2_{L^2(\sigma)}\ ,\tag{5.5.10}$$

Then

$$\begin{aligned}&\sum_{F\in\mathcal{F}}\ \sum_{\substack{J\in\mathcal{G}:\ \ell(J)\leq\ell(F)\\ d(J,F)>2\ell(J)^\varepsilon\ell(F)^{1-\varepsilon}}}\left|\left\langle T^\alpha_\sigma\left(\mathbf{1}_F\alpha_{\mathcal{F}}(F)\right),\square^{\omega,\mathbf{b}^*}_J g\right\rangle_\omega\right|\\&+\sum_{G\in\mathcal{G}}\ \sum_{\substack{I\in\mathcal{D}:\ell(I)\leq\ell(G)\\ d(I,G)>2\ell(I)^\varepsilon\ell(G)^{1-\varepsilon}}}\left|\left\langle T^\alpha_\sigma\left(\square^{\sigma,\mathbf{b}}_I f\right),\mathbf{1}_G\beta_{\mathcal{G}}(G)\right\rangle_\omega\right|\\ \lesssim\ &\sqrt{\mathfrak{A}^\alpha_2}\,\|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)}\,.\end{aligned}\tag{5.5.11}$$

The proof of this lemma is similar to those of Lemmas 5.3.1 and 5.3.2 in Section 5.3 above, using the square function inequalities for $\square_I^{\sigma,\mathbf{b}}$, $\nabla_{I,\mathcal{F}}^{\sigma}$ and $\square_J^{\omega,\mathbf{b}^*}$, $\nabla_{J,\mathcal{G}}^{\omega}$.

Proposition 5.5.7 (The Intertwining Proposition)**.** *Let $\mathcal{D}$ and $\mathcal{G}$ be grids, and suppose that $\mathbf{b}$ and $\mathbf{b}^*$ are ∞-weakly σ-accretive families of cubes in $\mathcal{D}$ and $\mathcal{G}$ respectively. Suppose that $\mathcal{F} \subset \mathcal{D}$ is σ-Carleson and that the $\mathcal{F}$-coronas*

$$\mathcal{C}_F \equiv \left\{ I \in \mathcal{D} : I \subset F \text{ but } I \not\subset F' \text{ for } F' \in \mathfrak{C}_{\mathcal{F}}(F) \right\}$$

satisfy

$$E_I^\sigma |f| \lesssim E_F^\sigma |f| \text{ and } b_I = \mathbf{1}_I b_F, \qquad \text{for all } I \in \mathcal{C}_F \text{ , } F \in \mathcal{F}.$$

Then

$$\boldsymbol{E}_\Omega^{\mathcal{D}} \left| \sum_{F\in\mathcal{F}} \sum_{I:\ I \supsetneqq F} \left\langle T_\sigma^\alpha \square_I^{\sigma,\mathbf{b}} f, \mathsf{P}_{\mathcal{C}_F^{\mathcal{G},shift}}^{\omega,\mathbf{b}^*} g \right\rangle_\omega \right| \lesssim \left(\mathfrak{F}_\alpha + \mathfrak{T}_{T^\alpha}^{\boldsymbol{b}} + \sqrt{\mathfrak{A}_2^\alpha} \delta^{\alpha-n} + \delta \mathfrak{N}_{T^\alpha} \right) \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)} ,$$

where the implied constant depends on the σ-Carleson norm $C_{\mathcal{F}}$ of the family $\mathcal{F}$.

Proof. We write the sum on the left hand side of the display above as

$$\begin{aligned} \sum_{F\in\mathcal{F}} \sum_{I:\ I \supsetneqq F} \left\langle T_\sigma^\alpha \square_I^{\sigma,\mathbf{b}} f, \mathsf{P}_{\mathcal{C}_F^{\mathcal{G},shift}}^{\omega} g \right\rangle_\omega &= \sum_{F\in\mathcal{F}} \left\langle T_\sigma^\alpha \left(\sum_{I:\ I \supsetneqq F} \square_I^{\sigma,\mathbf{b}} f \right), \mathsf{P}_{\mathcal{C}_F^{\mathcal{G},shift}}^{\omega} g \right\rangle_\omega \\ &= \sum_{F\in\mathcal{F}} \left\langle T_\sigma^\alpha \left(f_F^* \right), g_F \right\rangle_\omega ; \end{aligned}$$

where $f_F^* \equiv \sum_{I:\ I \supsetneqq F} \square_I^{\sigma,\mathbf{b}} f$ and $g_F \equiv \mathsf{P}_{\mathcal{C}_F^{\mathcal{G},shift}}^{\omega} g$.

Note that g_F is supported in F. By the telescoping identity for $\square_I^{\sigma,\mathbf{b}}$, the function f_F^*

satisfies

$$\mathbf{1}_F f_F^* = \sum_{I:\ I_\infty \supset I \supsetneqq F} \square_I^{\sigma,\mathbf{b}} f = \mathbb{F}_F^{\sigma,\mathbf{b}} f - \mathbf{1}_F \mathbb{F}_{I_\infty}^{\sigma,\mathbf{b}} f = b_F \frac{E_F^\sigma f}{E_F^\sigma b_F} - \mathbf{1}_F b_{I_\infty} \frac{E_{I_\infty}^\sigma f}{E_{I_\infty}^\sigma b_{I_\infty}} \ .$$

where I_∞ is the starting cube for corona constructions in $\mathcal{D}$. However, we cannot apply the testing condition to the function $\mathbf{1}_F b_{I_\infty}$, and since $E_{I_\infty}^\sigma f$ does not vanish in general, we will instead add and subtract the term $\mathbb{F}_{I_\infty}^{\sigma,\mathbf{b}} f$ to get

$$\begin{aligned} \sum_{F\in\mathcal{F}} \left\langle T_\sigma^\alpha \left(f_F^*\right), g_F\right\rangle_\omega &= \sum_{F\in\mathcal{F}} \left\langle T_\sigma^\alpha \left(\sum_{I:\ I_\infty \supset I \supsetneqq F} \square_I^{\sigma,\mathbf{b}} f \right), \mathsf{P}_{\mathcal{C}_F^{\mathcal{G},shift}}^\omega g \right\rangle_\omega \\ &= \sum_{F\in\mathcal{F}} \left\langle T_\sigma^\alpha \left(\mathbb{F}_{I_\infty}^{\sigma,\mathbf{b}} f + \sum_{I:\ I_\infty \supset I \supsetneqq F} \square_I^{\sigma,\mathbf{b}} f \right), \mathsf{P}_{\mathcal{C}_F^{\mathcal{G},shift}}^\omega g \right\rangle_\omega \\ &\quad - \sum_{F\in\mathcal{F}} \left\langle T_\sigma^\alpha \left(\mathbb{F}_{I_\infty}^{\sigma,\mathbf{b}} f \right), \mathsf{P}_{\mathcal{C}_F^{\mathcal{G},shift}}^\omega g \right\rangle_\omega \ , \end{aligned} \tag{5.5.12}$$

where the second sum on the right hand side of the identity satisfies

$$\boldsymbol{E}_\Omega^{\mathcal{D}} \left| \sum_{F\in\mathcal{F}} \left\langle T_\sigma^\alpha \left(\mathbb{F}_{I_\infty}^{\sigma,\mathbf{b}} f \right), \mathsf{P}_{\mathcal{C}_F^{\mathcal{G},shift}}^\omega g \right\rangle_\omega \right| \lesssim \left(\mathfrak{T}_{T^\alpha}^{\mathbf{b}} + \sqrt{\mathfrak{A}_2^\alpha} \delta^{\alpha-n} + \delta \mathfrak{N}_{T^\alpha} \right) \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)}$$

Indeed, as

$$\begin{aligned} &\sum_{F\in\mathcal{F}} \left\langle T_\sigma^\alpha \left(\mathbb{F}_{I_\infty}^{\sigma,\mathbf{b}} f \right), \mathsf{P}_{\mathcal{C}_F^{\mathcal{G},shift}}^\omega g \right\rangle_\omega \\ = &\left[\int_{I_\infty \cap J_\infty} + \int_{J_\infty \cap ((1+\delta) I_\infty \setminus I_\infty)} + \int_{J_\infty \setminus (1+\delta) I_\infty} \right] \left(\sum_{F\in\mathcal{F}} \mathsf{P}_{\mathcal{C}_F^{\mathcal{G},shift}}^\omega g \right) T_\sigma^\alpha \left(\mathbb{F}_{I_\infty}^{\sigma,\mathbf{b}} f \right) d\omega \\ \equiv &\ A_1 + A_2 + A_3 \end{aligned}$$

by Cauchy-Schwarz and Riesz inequalities, the term A_1 is controlled by testing, the term A_3 by Muckenhoupt's condition using lemma 5.4.3 and finally

$$\begin{aligned} \boldsymbol{E}_{\Omega}^{\mathcal{D}} A_2 &\leq \left(C\delta \int_{I_\infty} \left| \sum_{F\in\mathcal{F}} \mathsf{P}^{\omega}_{\mathcal{C}_F^{\mathcal{G},shift}} g \right|^2 d\omega \right)^{\frac{1}{2}} \left(\mathfrak{N}_{T^\alpha} \int |f|^2 \, d\sigma \right)^{\frac{1}{2}} \\ &\leq \sqrt{C\delta \mathfrak{N}_{T^\alpha}} \, \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)} \, . \end{aligned}$$

The advantage now is that with

$$f_F \equiv \mathbb{F}_{I_\infty}^{\sigma,\mathbf{b}} f + f_F^* = \mathbb{F}_{I_\infty}^{\sigma,\mathbf{b}} f + \sum_{I:\ I_\infty \supset I \supsetneqq F} \square_I^{\sigma,\mathbf{b}} f$$

then in the first term on the right hand side of (5.5.12), the telescoping identity gives

$$\mathbf{1}_F f_F = \mathbf{1}_F \left(\mathbb{F}_{I_\infty}^{\sigma,\mathbf{b}} f + \sum_{I:\ I_\infty \supset I \supsetneqq F} \square_I^{\sigma,\mathbf{b}} f \right) = \mathbb{F}_F^{\sigma,\mathbf{b}} f = b_F \frac{E_F^\sigma f}{E_F^\sigma b_F},$$

which shows that f_F is a controlled constant times b_F on F.

The cubes I occurring in this sum are linearly and consecutively ordered by inclusion, along with the cubes $F' \in \mathcal{F}$ that contain F. More precisely we can write

$$F \equiv F_0 \subsetneqq F_1 \subsetneqq F_2 \subsetneqq \dots \subsetneqq F_n \subsetneqq F_{n+1} \subsetneqq \dots F_N = I_\infty$$

where $F_m = \pi_{\mathcal{F}}^m F$ for all $m \geq 1$. We can also write

$$F = F_0 \equiv I_0 \subsetneqq I_1 \subsetneqq I_2 \subsetneqq \dots \subsetneqq I_k \subsetneqq I_{k+1} \subsetneqq \dots \subsetneqq I_K = F_N = I_\infty$$

where $I_k = \pi_{\mathcal{D}}^k F$ for all $k \geq 1$. There is a (unique) subsequence $\{k_m\}_{m=1}^N$ such that

$$F_m = I_{k_m}, \qquad 1 \leq m \leq N.$$

Then we have

$$f_F(x) \equiv \mathbb{F}_{I_\infty}^{\sigma,\mathbf{b}} f(x) + \sum_{\ell=1}^{K} \square_{I_\ell}^{\sigma,\mathbf{b}} f(x) \quad \text{and} \quad g_F \equiv \sum_{J \in \mathcal{C}_F^{\mathcal{G},shift}} \square_J^{\omega,\mathbf{b}^*} g.$$

Assume now that $k_m \leq k < k_{m+1}$. We denote by $\theta(I)$ the $2^n - 1$ siblings of I, i.e. $\tilde{I} \in \theta(I)$ implies $\tilde{I} \in \mathfrak{C}_{\mathcal{D}}(\pi_{\mathcal{D}} I) \setminus \{I\}$. There are two cases to consider here:

$$\tilde{I}_k \notin \mathcal{F} \text{ and } \tilde{I}_k \in \mathcal{F}.$$

We first note that in either case, using a telescoping sum, we compute that for

$$x \in \tilde{I}_k \subset F_{m+1} \backslash F_m,$$

we have the formula

$$\begin{aligned} f_F(x) &= \mathbb{F}_{I_\infty}^{\sigma,\mathbf{b}} f(x) + \sum_{\ell=k+1}^{K} \square_{I_\ell}^{\sigma,\mathbf{b}} f(x) \\ &= \mathbb{F}_{\tilde{I}_k}^{\sigma,\mathbf{b}} f(x) - \mathbb{F}_{I_{k+1}}^{\sigma,\mathbf{b}} f(x) + \sum_{\ell=k+1}^{K-1} \left(\mathbb{F}_{I_\ell}^{\sigma,\mathbf{b}} f(x) - \mathbb{F}_{I_{\ell+1}}^{\sigma,\mathbf{b}} f(x) \right) + \mathbb{F}_{I_\infty}^{\sigma,\mathbf{b}} f(x) \\ &= \mathbb{F}_{\tilde{I}_k}^{\sigma,\mathbf{b}} f(x) \ . \end{aligned}$$

Now fix $x \in \tilde{I}_k$. If $\tilde{I}_k \notin \mathcal{F}$, then $\tilde{I}_k \in \mathcal{C}_{F_{m+1}}$, and we have

$$\left|f_F(x)\right| = \left|\mathbb{F}_{\tilde{I}_k}^{\sigma,\mathbf{b}} f(x)\right| \lesssim \left|b_{\tilde{I}_k}(x)\right| \ \frac{E_{\tilde{I}_k}^{\sigma}|f|}{\left|E_{\tilde{I}_k}^{\sigma} b_{\theta(I_k)}\right|} \lesssim E_{F_{m+1}}^{\sigma}|f| \ , \tag{5.5.13}$$

since the testing functions $b_{\tilde{I}_k}$ are bounded and accretive, and $E_{\tilde{I}_k}^{\sigma}|f| \lesssim E_{F_{m+1}}^{\sigma}|f|$ by hypothesis. On the other hand, if $\tilde{I}_k \in \mathcal{F}$, then $I_{k+1} \in \mathcal{C}_{F_{m+1}}$ and we have

$$\left|f_F(x)\right| = \left|\mathbb{F}_{\tilde{I}_k}^{\sigma,\mathbf{b}} f(x)\right| \lesssim E_{\tilde{I}_k}^{\sigma}|f| \ .$$

Note that $F^c = \overset{\cdot}{\bigcup_{k\geq 0}} \theta(I_k)$. Now we write

$$\begin{aligned}
f_F &= \varphi_F + \psi_F, \\
\varphi_F \equiv \sum_{k\geq 0} \sum_{\substack{\tilde{I}_k \in \theta(I_k) \\ \tilde{I}_k \in \mathcal{F}}} \mathbb{F}_{\tilde{I}_k}^{\sigma,\mathbf{b}} f \quad &\text{and} \quad \psi_F = f_F - \varphi_F \ ; \\
\sum_{F\in\mathcal{F}} \langle T_\sigma^\alpha f_F, g_F\rangle_\omega &= \sum_{F\in\mathcal{F}} \langle T_\sigma^\alpha \varphi_F, g_F\rangle_\omega + \sum_{F\in\mathcal{F}} \langle T_\sigma^\alpha \psi_F, g_F\rangle_\omega \ ,
\end{aligned}$$

and note that $\varphi_F = 0$ on F, and $\psi_F = b_F \frac{E_F^\sigma f}{E_F^\sigma b_F}$ on F. We can apply the first line in (5.5.11) using $\tilde{I}_k \in \mathcal{F}$ to the first sum above since $J \in \mathcal{C}_F^{\mathcal{G},shift}$ implies $J \subset J^{\maltese} \subset F \subset I_k$, which implies that $d(J,\tilde{I}_k) > 2\ell(J)^{\varepsilon}\,\ell(\tilde{I}_k)^{1-\varepsilon}$. Thus we obtain after substituting F' for $\tilde{I}_k$ below,

$$\begin{aligned}
\left|\sum_{F\in\mathcal{F}} \langle T_\sigma^\alpha \varphi_F, g_F\rangle_\omega\right| &= \left|\sum_{F\in\mathcal{F}} \sum_{J\in\mathcal{C}_F^{\mathcal{G},shift}} \left\langle T_\sigma^\alpha \left(\sum_{k\geq 0} \sum_{\substack{\tilde{I}_k\in\theta(I_k)\\ \tilde{I}_k\in\mathcal{F}}} \mathbb{F}_{\tilde{I}_k}^{\sigma,\mathbf{b}} f\right), \square_J^{\omega,\mathbf{b}^*} g\right\rangle_\omega\right| \\
&\leq \sum_{F\in\mathcal{F}} \sum_{J\in\mathcal{C}_F^{\mathcal{G},shift}} \sum_{k\geq 0} \sum_{\substack{\tilde{I}_k\in\theta(I_k)\\ \tilde{I}_k\in\mathcal{F}}} \left|\left\langle T_\sigma^\alpha \left(\mathbb{F}_{\tilde{I}_k}^{\sigma,\mathbf{b}} f\right), \square_J^{\omega,\mathbf{b}^*} g\right\rangle_\omega\right| \\
&\leq \sum_{F'\in\mathcal{F}} \sum_{\substack{J\in\mathcal{G}:\ \ell(J)\leq\ell\left(F'\right)\\ d\left(J,F'\right)>2\ell(J)^{\varepsilon}\ell\left(F'\right)^{1-\varepsilon}}} \left|\left\langle T_\sigma^\alpha \left(\mathbb{F}_{F'}^{\sigma,\mathbf{b}} f\right), \square_J^{\omega,\mathbf{b}^*} g\right\rangle_\omega\right| \\
&\lesssim \sqrt{\mathfrak{A}_2^\alpha}\,\|f\|_{L^2(\sigma)}\,\|g\|_{L^2(\omega)}\ .
\end{aligned}$$

Turning to the second sum, we note that for $k_m \leq k < k_{m+1}$ and $x\in\tilde{I}_k$ with $\tilde{I}_k\notin\mathcal{F}$, we have

$$|\psi_F(x)| \lesssim \left|b_{\tilde{I}_k}\right|\ E_{\tilde{I}_k}^\sigma |f|\ \mathbf{1}_{\tilde{I}_k}(x) \lesssim \alpha_{\mathcal{F}}(F_{m+1})\,\mathbf{1}_{\tilde{I}_k}(x)$$

Note that for σ-almost all $x\in I_\infty$ there exists a unique $F\in\mathcal{F}$ such that $x\in F\setminus \bigcup_{F'\in\mathfrak{C}_{\mathcal{F}}(F)} F'$ since the family $\mathcal{F}$ is a Carleson family. Also from the stopping criteria we have $\alpha_{\mathcal{F}}(F)\leq \alpha_{\mathcal{F}}(F')$ for $F'\subset F$. Hence we get the following inequality for $x\notin F$,

$$|\psi_F(x)| \lesssim \Phi(x)\ \mathbf{1}_{F^c}(x)\ , \tag{5.5.14}$$

where we have defined

$$\Phi \equiv \sum_{F\in\mathcal{F}} \alpha_{\mathcal{F}}(F)\,\mathbf{1}_{F\setminus\cup\mathfrak{C}_{\mathcal{F}}(F)}\ .$$

Now we write

$$\sum_{F\in\mathcal{F}}\left\langle T_{\sigma}^{\alpha}\psi_{F},g_{F}\right\rangle_{\omega}=\sum_{F\in\mathcal{F}}\left\langle T_{\sigma}^{\alpha}\left(\mathbf{1}_{F}\psi_{F}\right),g_{F}\right\rangle_{\omega}+\sum_{F\in\mathcal{F}}\left\langle T_{\sigma}^{\alpha}\left(\mathbf{1}_{F^{c}}\psi_{F}\right),g_{F}\right\rangle_{\omega}\equiv\mathrm{I}+\mathrm{II}.$$

Then by cube testing,

$$\left|\left\langle T_{\sigma}^{\alpha}\left(b_{F}\mathbf{1}_{F}\right),g_{F}\right\rangle_{\omega}\right|=\left|\left\langle \mathbf{1}_{F}T_{\sigma}^{\alpha}\left(b_{F}\mathbf{1}_{F}\right),g_{F}\right\rangle_{\omega}\right|\lesssim\mathfrak{T}_{T^{\alpha}}\sqrt{\left|F\right|_{\sigma}}\left\Vert g_{F}\right\Vert_{L^{2}(\omega)}^{\bigstar},$$

and so quasi-orthogonality, together with the fact that on F, $\psi_{F}=b_{F}\frac{E_{F}^{\sigma}f}{E_{F}^{\sigma}b_{F}}$ is a constant $c=\frac{E_{F}^{\sigma}f}{E_{F}^{\sigma}b_{F}}$ times b_{F}, where $|c|$ is bounded by $\alpha_{\mathcal{F}}\left(F\right)$, give

$$\begin{aligned}
\left|\mathrm{I}\right|=\left|\sum_{F\in\mathcal{F}}\left\langle T_{\sigma}^{\alpha}\left(\mathbf{1}_{F}cb_{F}\right),g_{F}\right\rangle_{\omega}\right| &\lesssim\sum_{F\in\mathcal{F}}\alpha_{\mathcal{F}}\left(F\right)\left|\left\langle T_{\sigma}^{\alpha}b_{F},g_{F}\right\rangle_{\omega}\right|\\
&\lesssim\sum_{F\in\mathcal{F}}\alpha_{\mathcal{F}}\left(F\right)\mathfrak{T}_{T^{\alpha}}\sqrt{\left|F\right|_{\sigma}}\left\Vert g_{F}\right\Vert_{L^{2}(\omega)}^{\bigstar}\\
&\lesssim\mathfrak{T}_{T^{\alpha}}\left\Vert f\right\Vert_{L^{2}(\sigma)}\left[\sum_{F\in\mathcal{F}}\left\Vert g_{F}\right\Vert_{L^{2}(\omega)}^{\bigstar 2}\right]^{\frac{1}{2}}
\end{aligned}$$

Now $\mathbf{1}_{F^{c}}\psi_{F}$ is supported outside F, and each J in the dual martingale support $\mathcal{C}_{F}^{\mathcal{G},shift}$ of $g_{F}=\mathsf{P}_{\mathcal{C}_{F}^{\mathcal{G},shift}}^{\omega}g$ is in particular *good* in the cube F, and as a consequence, each such cube J as above is contained in some cube M for $M\in\mathcal{W}\left(F\right)$. This containment will be used in the analysis of the term II_{G} below.

In addition, each J in the dual martingale support $\mathcal{C}_{F}^{\mathcal{G},shift}$ of $g_{F}=\mathsf{P}_{\mathcal{C}_{F}^{\mathcal{G},shift}}^{\omega}g$ is $\left(\left[\frac{3}{\varepsilon}\right],\varepsilon\right)$-deeply embedded in F, i.e. $J\Subset_{\left[\frac{3}{\varepsilon}\right],\varepsilon}F$ the definition of $\mathcal{C}_{F}^{\mathcal{G},shift}$. As a consequence, each such cube J as above is contained in some cube M for $M\in\mathcal{M}_{\left(\left[\frac{3}{\varepsilon}\right],\varepsilon\right)-deep,\mathcal{D}}\left(F\right)$. This containment will be used in the analysis of the term II_{B} below.

Notation 5.5.8. *Define* $\rho \equiv \left[\frac{3}{\varepsilon}\right]$, *so that for every* $J \in \mathcal{C}_F^{\mathcal{G},shift}$, *there is* $M \in \mathcal{M}_{(\rho,\varepsilon)-deep,\mathcal{G}}(F)$ *such that* $J \subset M$.

The collections $\mathcal{W}(F)$ and $\mathcal{M}_{(\rho,\varepsilon)-deep,\mathcal{G}}(F)$ used here, and in the display below, are defined in (.0.7) in Appendix. Finally, since the cubes $M \in \mathcal{W}(F)$, as well as the cubes $M \in \mathcal{M}_{\left(\left[\frac{3}{\varepsilon}\right],\varepsilon\right)-deep,\mathcal{G}}(F)$, satisfy $3M \subset F$, we can apply (5.1.54) in the Monotonicity Lemma 5.1.23 using (5.5.14) with $\mu = \mathbf{1}_{F^c}\psi_F$ and J' in place of J there, to obtain

$$
\begin{aligned}
|\mathrm{II}| &= \left|\sum_{F\in\mathcal{F}} \left\langle T_\sigma^\alpha\left(\mathbf{1}_{F^c}\psi_F\right), g_F\right\rangle_\omega\right| = \left|\sum_{F\in\mathcal{F}}\ \sum_{J'\in\mathcal{C}_F^{\mathcal{G},shift}} \left\langle T_\sigma^\alpha\left(\mathbf{1}_{F^c}\psi_F\right), \square_{J'}^{\omega,\mathbf{b}^*} g\right\rangle_\omega\right| \\
&\lesssim \sum_{F\in\mathcal{F}}\ \sum_{J'\in\mathcal{C}_F^{\mathcal{G},shift}} \frac{\mathrm{P}^\alpha\left(J', \mathbf{1}_{F^c}|\psi_F|\sigma\right)}{|J'|^{\frac{1}{n}}} \left\|\triangle_{J'}^{\omega,\mathbf{b}^*} x\right\|_{L^2(\omega)}^{\spadesuit} \left\|\square_{J'}^{\omega,\mathbf{b}^*} g\right\|_{L^2(\omega)}^{\bigstar} \\
&\quad + \sum_{F\in\mathcal{F}}\ \sum_{J'\in\mathcal{C}_F^{\mathcal{G},shift}} \frac{\mathrm{P}_{1+\delta}^\alpha\left(J', \mathbf{1}_{F^c}|\psi_F|\sigma\right)}{|J'|^{\frac{1}{n}}} \left\|x - m_{J'}\right\|_{L^2(\omega)} \left\|\square_{J'}^{\omega,\mathbf{b}^*} g\right\|_{L^2(\omega)}^{\bigstar} \\
&\lesssim \sum_{F\in\mathcal{F}}\ \sum_{M\in\mathcal{W}(F)} \frac{\mathrm{P}^\alpha\left(M, \mathbf{1}_{F^c}\Phi\sigma\right)}{|M|^{\frac{1}{n}}} \left\|\mathsf{Q}_{\mathcal{C}_{F;M}^{\mathcal{G},shift}}^{\omega,\mathbf{b}^*} x\right\|_{L^2(\omega)}^{\spadesuit} \left\|g_{F;M}\right\|_{L^2(\omega)}^{\bigstar} \\
&\quad + \sum_{F\in\mathcal{F}}\ \sum_{J\in\mathcal{M}_{(\rho,\varepsilon),\mathcal{G}}^{deep}(F)}\ \sum_{J'\in\mathcal{C}_{F;J}^{\mathcal{G},shift}} \frac{\mathrm{P}_{1+\delta}^\alpha\left(J', \mathbf{1}_{F^c}|\psi_F|\sigma\right)}{|J'|^{\frac{1}{n}}} \left\|x - m_{J'}\right\|_{L^2\left(\mathbf{1}_{J'}\omega\right)} \left\|\square_{J'}^{\omega,\mathbf{b}^*} g\right\|_{L^2(\omega)}^{\bigstar} \\
&\equiv \mathrm{II}_{\mathrm{G}} + \mathrm{II}_{\mathrm{B}}\ .
\end{aligned}
$$

where $g_{F;M}$ denotes the pseudoprojection $g_{F;M} = \sum_{J'\in\mathcal{C}_F^{\mathcal{G},shift}:\ J'\subset M} \square_{J'}^{\omega,\mathbf{b}^*} g$.

Note: We could also bound II_{G} by using the decomposition $\mathcal{M}_{(\rho,\varepsilon)-deep,\mathcal{G}}(F)$ of F into certain maximal $\mathcal{G}$-cubes, but the 'smaller' choice $\mathcal{W}(F)$ of $\mathcal{D}$-cubes is needed for II_{G} in order to bound it by the corresponding functional energy constant $\mathfrak{F}_\alpha$, which can then be controlled by the energy and Muckenhoupt constants in Appendix .

Then from Cauchy-Schwarz, the functional energy condition, and

$$\left\|\Phi\right\|_{L^2(\sigma)}^2 \leq \sum_{F\in\mathcal{F}} \alpha_{\mathcal{F}}\left(F\right)^2 \left|F\right|_\sigma \lesssim \left\|f\right\|_{L^2(\sigma)}^2 \ ,$$

we obtain

$$\begin{aligned}
\left|\mathrm{II}_{\mathrm{G}}\right| &\leq \left(\sum_{F\in\mathcal{F}}\sum_{M\in\mathcal{W}(F)}\left(\frac{\mathrm{P}^\alpha\left(M,\mathbf{1}_{F^c}\Phi\sigma\right)}{|M|}\right)^2\left\|\mathsf{Q}^{\omega,\mathbf{b}^*}_{\mathcal{C}^{\mathcal{G},shift}_{F;M}}x\right\|^{\spadesuit 2}_{L^2(\omega)}\right)^{\frac{1}{2}}\left(\sum_{F\in\mathcal{F}}\sum_{M\in\mathcal{W}(F)}\left\|g_{F;M}\right\|^{\bigstar 2}_{L^2(\omega)}\right)^{\frac{1}{2}} \\
&\lesssim \ \mathfrak{F}_\alpha\left\|\Phi\right\|_{L^2(\sigma)}\left[\sum_{F\in\mathcal{F}}\left\|g_F\right\|^{\bigstar 2}_{L^2(\omega)}\right]^{\frac{1}{2}} \lesssim \mathfrak{F}_\alpha\left\|f\right\|_{L^2(\sigma)}\left\|g\right\|_{L^2(\omega)},
\end{aligned}$$

by the pairwise disjointedness of the coronas $\mathcal{C}^{\mathcal{G},shift}_{F;M}$ jointly in F and M, which in turn follows from the pairwise disjointedness (5.5.1) of the shifted coronas $\mathcal{C}^{\mathcal{G},shift}_{F}$ in F, together with the pairwise disjointedness of the cubes M. Thus we obtain the pairwise disjointedness of both of the pseudoprojections $\mathsf{P}^{\omega,\mathbf{b}^*}_{\mathcal{C}^{\mathcal{G},shift}_{F;M}}$ and $\mathsf{Q}^{\omega,\mathbf{b}^*}_{\mathcal{C}^{\mathcal{G},shift}_{F;M}}$ jointly in F and M.

In term II_{B} the quantities $\left\|x-m_{J'}\right\|^2_{L^2\left(\mathbf{1}_{J'}\omega\right)}$ are no longer additive except when the cubes J' are pairwise disjoint. As a result we will use (5.1.58) in the form,

$$\begin{aligned}
\sum_{J'\subset J}\left(\frac{\mathrm{P}^\alpha_{1+\delta}\left(J',\nu\right)}{|J'|^{\frac{1}{n}}}\right)^2\left\|x-m_{J'}\right\|^2_{L^2\left(\mathbf{1}_{J'}\right)} \ &\lesssim \ \frac{1}{\gamma^{2\delta'}}\left(\frac{\mathrm{P}^\alpha_{1+\delta'}\left(J,\nu\right)}{|J|^{\frac{1}{n}}}\right)^2\sum_{J''\subset J}\left\|\triangle^\omega_{J''}x\right\|^2_{L^2} \\
&\lesssim \ \left(\frac{\mathrm{P}^\alpha_{1+\delta'}\left(J,\nu\right)}{|J|^{\frac{1}{n}}}\right)^2\left\|x-m_J\right\|^2_{L^2\left(\mathbf{1}_J\right)} \ ,
\end{aligned} \tag{5.5.15}$$

and exploit the decay in the Poisson integral $\mathrm{P}^\alpha_{1+\delta'}$ along with weak goodness of the cubes J. As a consequence we will be able to bound II_{B} *directly* by the strong energy condition

(5.1.8), without having to invoke the more difficult functional energy condition. For the decay we compute that for $J \in \mathcal{M}_{(\rho,\varepsilon)-deep,\mathcal{G}}(F)$

$$\begin{aligned}
\frac{\mathrm{P}^{\alpha}_{1+\delta'}\left(J,\mathbf{1}_{F^c}|\psi_F|\sigma\right)}{|J|^{\frac{1}{n}}} &\approx \int_{F^c} \frac{|J|^{\frac{\delta'}{n}}}{|y-c_J|^{n+1+\delta'-\alpha}} |\psi_F|(y)\,d\sigma \\
&\leq \sum_{t=0}^{\infty} \int_{\pi^{t+1}_{\mathcal{F}}F\setminus\pi^{t}_{\mathcal{F}}F} \left(\frac{|J|^{\frac{1}{n}}}{\operatorname{dist}\left(c_J,\left(\pi^{t}_{\mathcal{F}}F\right)^c\right)}\right)^{\delta'} \frac{|\psi_F|(y)}{|y-c_J|^{n+1-\alpha}} d\sigma \\
&\lesssim \sum_{t=0}^{\infty} \left(\frac{|J|^{\frac{1}{n}}}{\operatorname{dist}\left(c_J,\left(\pi^{t}_{\mathcal{F}}F\right)^c\right)}\right)^{\delta'} \frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{\pi^{t+1}_{\mathcal{F}}F\setminus\pi^{t}_{\mathcal{F}}F}|\psi_F|\sigma\right)}{|J|^{\frac{1}{n}}},
\end{aligned}$$

and then use the weak goodness inequality and the fact that $J \subset F$

$$\operatorname{dist}\left(c_J,\left(\pi^{t}_{\mathcal{F}}F\right)^c\right) \geq 2\ell\left(\pi^{t}_{\mathcal{F}}F\right)^{1-\varepsilon}\ell(J)^{\varepsilon} \geq 2\cdot 2^{t(1-\varepsilon)}\ell(F)^{1-\varepsilon}\ell(J)^{\varepsilon} \geq 2^{t(1-\varepsilon)+1}\ell(J),$$

to conclude that

$$\begin{aligned}
\left(\frac{\mathrm{P}^{\alpha}_{1+\delta'}\left(J,\mathbf{1}_{F^c}|\psi_F|\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2 &\lesssim \left(\sum_{t=0}^{\infty} 2^{-t\delta'(1-\varepsilon)} \frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{\pi^{t+1}_{\mathcal{F}}F\setminus\pi^{t}_{\mathcal{F}}F}|\psi_F|\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2 \qquad (5.5.16) \\
&\lesssim \sum_{t=0}^{\infty} 2^{-t\delta'(1-\varepsilon)} \left(\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{\pi^{t+1}_{\mathcal{F}}F\setminus\pi^{t}_{\mathcal{F}}F}|\psi_F|\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2 .
\end{aligned}$$

where in the last inequality we used the Cauchy-Schwarz inequality. Now we again apply

Cauchy-Schwarz and (5.5.16) to obtain

$$\begin{aligned}
\mathrm{II}_{\mathrm{B}} &= \sum_{F\in\mathcal{F}} \sum_{J\in\mathcal{M}^{deep}_{(\rho,\varepsilon),\mathcal{G}}(F)} \sum_{J'\in\mathcal{C}^{\mathcal{G},shift}_{F;J}} \frac{\mathrm{P}^{\alpha}_{1+\delta}\left(J',\mathbf{1}_{F^c}|\psi_F|\sigma\right)}{|J'|^{\frac{1}{n}}} \left\|x-m_{J'}\right\|_{L^2\left(\mathbf{1}_{J'}\omega\right)} \left\|\square^{\omega,\mathbf{b}^*}_{J'} g\right\|^{\bigstar}_{L^2(\omega)} \\
&\leq \left(\sum_{F\in\mathcal{F}} \sum_{J\in\mathcal{M}^{deep}_{(\rho,\varepsilon),\mathcal{G}}(F)} \sum_{J'\in\mathcal{C}^{\mathcal{G},shift}_{F;J}} \left(\frac{\mathrm{P}^{\alpha}_{1+\delta}\left(J',\mathbf{1}_{F^c}|\psi_F|\sigma\right)}{|J'|^{\frac{1}{n}}}\right)^2 \left\|x-m_{J'}\right\|^2_{L^2\left(\mathbf{1}_{J'}\omega\right)}\right)^{\frac{1}{2}} \\
&\qquad \left[\sum_{F\in\mathcal{F}} \left\|g_F\right\|^{\bigstar 2}_{L^2(\omega)}\right]^{\frac{1}{2}} \\
&\leq \left(\sum_{F\in\mathcal{F}} \sum_{J\in\mathcal{M}_{(\rho,\varepsilon)-deep,\mathcal{G}}(F)} \left(\frac{\mathrm{P}^{\alpha}_{1+\delta'}\left(J,\mathbf{1}_{F^c}|\psi_F|\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2 \left\|x-m_J\right\|^2_{L^2(\mathbf{1}_J\omega)}\right)^{\frac{1}{2}} \|g\|_{L^2(\omega)} \\
&\equiv \sqrt{\mathrm{II}_{\mathrm{energy}}}\, \|g\|_{L^2(\omega)},
\end{aligned}$$

and it remains to estimate $\mathrm{II}_{\mathrm{energy}}$. From (5.5.16) and the strong energy condition (5.1.8),

we have

$$
\begin{aligned}
\mathrm{II}_{\text{energy}} &= \sum_{F\in\mathcal{F}} \sum_{J\in\mathcal{M}_{(\rho,\varepsilon)-deep,\mathcal{G}}(F)} \left(\frac{\mathrm{P}^{\alpha}_{1+\delta'}\left(J, \mathbf{1}_{F^c}|\psi_F|\sigma\right)}{|J|^{\frac{1}{n}}} \right)^2 \|x-m_J\|^2_{L^2(\mathbf{1}_J\omega)} \\
&\leq \sum_{F\in\mathcal{F}} \sum_{J\in\mathcal{M}^{deep}_{(\rho,\varepsilon),\mathcal{G}}(F)} \sum_{t=0}^{\infty} 2^{-t\delta'(1-\varepsilon)} \left(\frac{\mathrm{P}^{\alpha}\left(J, \mathbf{1}_{\pi^{t+1}_{\mathcal{F}}F\setminus\pi^{t}_{\mathcal{F}}F}|\psi_F|\sigma\right)}{|J|^{\frac{1}{n}}} \right)^2 \|x-m_J\|^2_{L^2(\mathbf{1}_J\omega)} \\
&= \sum_{t=0}^{\infty} 2^{-t\delta'(1-\varepsilon)} \sum_{G\in\mathcal{F}} \sum_{F\in\mathfrak{C}^{(t+1)}_{\mathcal{F}}(G)} \sum_{J\in\mathcal{M}^{deep}_{(\rho,\varepsilon),\mathcal{G}}(F)} \left(\frac{\mathrm{P}^{\alpha}\left(J, \mathbf{1}_{G\setminus\pi^{t}_{\mathcal{F}}F}|\psi_F|\sigma\right)}{|J|^{\frac{1}{n}}} \right)^2 \|x-m_J\|^2_{L^2(\mathbf{1}_J\omega)} \\
&\lesssim \sum_{t=0}^{\infty} 2^{-t\delta'(1-\varepsilon)} \sum_{G\in\mathcal{F}} \alpha_{\mathcal{F}}(G)^2 \sum_{F\in\mathfrak{C}^{(t+1)}_{\mathcal{F}}(G)} \sum_{J\in\mathcal{M}^{deep}_{(\rho,\varepsilon)}(F)} \left(\frac{\mathrm{P}^{\alpha}\left(J, \mathbf{1}_{G\setminus\pi^{t}_{\mathcal{F}}F}\sigma\right)}{|J|^{\frac{1}{n}}} \right)^2 \|x-m_J\|^2_{L^2(\mathbf{1}_J\omega)} \\
&\lesssim \sum_{t=0}^{\infty} 2^{-t\delta'(1-\varepsilon)} \sum_{G\in\mathcal{F}} \alpha_{\mathcal{F}}(G)^2 \left(\mathcal{E}^{\alpha}_2\right)^2 |G|_{\sigma} \lesssim \left(\mathcal{E}^{\alpha}_2\right)^2 \|f\|^2_{L^2(\sigma)} .
\end{aligned}
$$

This completes the proof of the Intertwining Proposition 5.5.7. $\square$

The task of controlling functional energy is taken up in Appendix below.

5.5.4 Paraproduct, neighbour and broken forms

In this subsection we reduce boundedness of the local below form $\mathsf{B}^{A}_{\Subset_{\mathbf{r},\varepsilon}}(f,g)$ defined in (5.5.4) to boundedness of the associated stopping form

$$
\mathsf{B}^{A}_{stop}(f,g) \equiv \sum_{\substack{I\in\mathcal{C}^{\mathcal{D}}_{A} \text{ and } J\in\mathcal{C}^{\mathcal{G},shift}_{A} \\ J^{\maltese}\subsetneqq I \text{ and } \ell(J)\leq 2^{-\mathbf{r}}\ell(I)}} \left(E^{\sigma}_{I_J} \widehat{\square}^{\sigma,\flat,\mathbf{b}}_{I} f \right) \left\langle T^{\alpha}_{\sigma}\left(\mathbf{1}_{A\setminus I_J} b_A\right), \square^{\omega,\mathbf{b}^*}_{J} g \right\rangle_{\omega} , \quad (5.5.17)
$$

where the modified difference $\widehat{\square}_I^{\sigma,\flat,\mathbf{b}}$ must be carefully chosen in order to control the corresponding paraproduct form below. Indeed, below we will decompose

$$\mathsf{B}_{\Subset_{\mathbf{r}},\varepsilon}^{A}(f,g)=\mathsf{B}_{paraproduct}^{A}(f,g)-\mathsf{B}_{stop}^{A}(f,g)+\mathsf{B}_{neighbour}^{A}(f,g)+\mathsf{B}_{brok}^{A}(f,g),$$

and we will show that

$$\sum_{A\in\mathcal{A}}\left|\mathsf{B}_{\Subset_{\mathbf{r}},\varepsilon}^{A}(f,g)+\mathsf{B}_{stop}^{A}(f,g)\right|\lesssim\left(\mathfrak{T}_{T^\alpha}^{\mathbf{b}}+\sqrt{\mathfrak{A}_2^\alpha}\right)\|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)}$$

and the bound of $\mathsf{B}_{stop}^{A}(f,g)$ will be the main subject of the next section.

Note that the modified dual martingale differences $\square_I^{\sigma,\flat,\mathbf{b}}$ and $\widehat{\square}_I^{\sigma,\flat,\mathbf{b}}$,

$$\square_I^{\sigma,\flat,\mathbf{b}}f\equiv\square_I^{\sigma,\mathbf{b}}f-\sum_{I'\in\mathfrak{C}_{brok}(I)}\mathbb{F}_{I'}^{\sigma,\mathbf{b}}f=b_A\sum_{I'\in\mathfrak{C}(I)}\mathbf{1}_{I'}E_{I'}^{\sigma}\left(\widehat{\square}_I^{\sigma,\flat,\mathbf{b}}f\right)=b_A\widehat{\square}_I^{\sigma,\flat,\mathbf{b}}f,$$

satisfy the following telescoping property for all $K\in\left(\mathcal{C}_A\setminus\{A\}\right)\cup\left(\bigcup_{A'\in\mathfrak{C}_{\mathcal{A}}(A)}A'\right)$ and $L\in\mathcal{C}_A$ with $K\subset L$:

$$\sum_{I:\ \pi K\subset I\subset L}E_I^{\sigma}\left(\widehat{\square}_I^{\sigma,\flat,\mathbf{b}}f\right)=\begin{cases}-E_L^{\sigma}\widehat{\mathbb{F}}_L^{\sigma,\mathbf{b}}f & \text{if}\quad K\in\mathfrak{C}_{\mathcal{A}}(A)\\ E_K^{\sigma}\widehat{\mathbb{F}}_K^{\sigma,\mathbf{b}}f-E_L^{\sigma}\widehat{\mathbb{F}}_L^{\sigma,\mathbf{b}}f & \text{if}\quad K\in\mathcal{C}_A\end{cases}.$$

Fix $I\in\mathcal{C}_A$ for the moment. We will use

$$\begin{aligned}\mathbf{1}_I&=\mathbf{1}_{I_J}+\sum_{\tilde{I}\in\theta(I_J)}\mathbf{1}_{\tilde{I}}\ ,\\ \mathbf{1}_{I_J}&=\mathbf{1}_A-\mathbf{1}_{A\setminus I_J}\ ,\end{aligned}$$

where $\theta(I_J)$ denotes the 2^n-1 $\mathcal{D}$-children of I other than the child I_J that contains J. We begin with the splitting

$$\begin{aligned}
&\left\langle T_\sigma^\alpha \square_I^{\sigma,\mathbf{b}} f, \square_J^{\omega,\mathbf{b}^*} g\right\rangle_\omega \\
&= \left\langle T_\sigma^\alpha\left(\mathbf{1}_{I_J}\square_I^{\sigma,\mathbf{b}} f\right), \square_J^{\omega,\mathbf{b}^*} g\right\rangle_\omega + \sum_{\tilde{I}\in\theta(I_J)}\left\langle T_\sigma^\alpha\left(\mathbf{1}_{\tilde{I}}\square_I^{\sigma,\mathbf{b}} f\right), \square_J^{\omega,\mathbf{b}^*} g\right\rangle_\omega \\
&= \left\langle T_\sigma^\alpha\left(\mathbf{1}_{I_J}\square_I^{\sigma,\flat,\mathbf{b}} f\right), \square_J^{\omega,\mathbf{b}^*} g\right\rangle_\omega + \left\langle T_\sigma^\alpha\left(\mathbf{1}_{I_J}\sum_{I'\in\mathfrak{C}_{brok}(I)}\mathbb{F}_{I'}^{\sigma,\mathbf{b}} f\right), \square_J^{\omega,\mathbf{b}^*} g\right\rangle_\omega \\
&\qquad + \sum_{\tilde{I}\in\theta(I_J)}\left\langle T_\sigma^\alpha\left(\mathbf{1}_{\tilde{I}}\square_I^{\sigma,\mathbf{b}} f\right), \square_J^{\omega,\mathbf{b}^*} g\right\rangle_\omega \\
&\equiv \mathrm{I}+\mathrm{II}+\mathrm{III}\ .
\end{aligned}$$

From (5.1.47) we have

$$\begin{aligned}
\mathrm{I} &= \left\langle T_\sigma^\alpha\left(\mathbf{1}_{I_J}\square_I^{\sigma,\flat,\mathbf{b}} f\right), \square_J^{\omega,\mathbf{b}^*} g\right\rangle_\omega = \left\langle T_\sigma^\alpha\left[b_A\left(\mathbf{1}_{I_J}\widehat{\square}_I^{\sigma,\flat,\mathbf{b}} f\right)\right], \square_J^{\omega,\mathbf{b}^*} g\right\rangle_\omega \\
&= E_{I_J}^\sigma\left(\widehat{\square}_I^{\sigma,\flat,\mathbf{b}} f\right)\left\langle T_\sigma^\alpha\left(\mathbf{1}_{I_J} b_A\right), \square_J^{\omega,\mathbf{b}^*} g\right\rangle_\omega \\
&= E_{I_J}^\sigma\left(\widehat{\square}_I^{\sigma,\flat,\mathbf{b}} f\right)\left\langle T_\sigma^\alpha b_A, \square_J^{\omega,\mathbf{b}^*} g\right\rangle_\omega - E_{I_J}^\sigma\left(\widehat{\square}_I^{\sigma,\flat,\mathbf{b}} f\right)\left\langle T_\sigma^\alpha\left(\mathbf{1}_{A\setminus I_J} b_A\right), \square_J^{\omega,\mathbf{b}^*} g\right\rangle_\omega
\end{aligned}$$

Since the function $\mathbb{F}_{I_J}^{\sigma,\mathbf{b}} f$ is a constant multiple of b_{I_J} on I_J, we can define $\widehat{\mathbb{F}}_{I_J}^{\sigma,\mathbf{b}} f \equiv \frac{1}{b_{I_J}}\mathbb{F}_{I_J}^{\sigma,\mathbf{b}} f$ and then

$$\mathrm{II} = \left\langle T_\sigma^\alpha\left(\mathbf{1}_{I_J}\sum_{I'\in\mathfrak{C}_{brok}(I)}\mathbb{F}_{I'}^{\sigma,\mathbf{b}} f\right), \square_J^{\omega,\mathbf{b}^*} g\right\rangle_\omega = \mathbf{1}_{\mathfrak{C}_{\mathcal{A}}(A)}(I_J)\, E_{I_J}^\sigma\left(\widehat{\mathbb{F}}_{I_J}^{\sigma,\mathbf{b}} f\right)\left\langle T_\sigma^\alpha b_{I_J}, \square_J^{\omega,\mathbf{b}^*} g\right\rangle_\omega$$

where the presence of the indicator function $\mathbf{1}_{\mathfrak{C}_{\mathcal{A}}(A)}(I_J)$ simply means that term II vanishes unless I_J is an $\mathcal{A}$-child of A. We now write these terms as

$$\begin{aligned}\left\langle T_{\sigma}^{\alpha}\square_{I}^{\sigma,\mathbf{b}}f,\square_{J}^{\omega,\mathbf{b}^{*}}g\right\rangle_{\omega} &= E_{I_J}^{\sigma}\left(\widehat{\square}_{I}^{\sigma,\flat,\mathbf{b}}f\right)\left\langle T_{\sigma}^{\alpha}b_A,\square_{J}^{\omega,\mathbf{b}^{*}}g\right\rangle_{\omega}\\ &\quad -E_{I_J}^{\sigma}\left(\widehat{\square}_{I}^{\sigma,\flat,\mathbf{b}}f\right)\left\langle T_{\sigma}^{\alpha}\left(\mathbf{1}_{A\setminus I_J}b_A\right),\square_{J}^{\omega,\mathbf{b}^{*}}g\right\rangle_{\omega}\\ &\quad +\sum_{\tilde{I}\in\theta(I_J)}\left\langle T_{\sigma}^{\alpha}\left(\mathbf{1}_{\tilde{I}}\square_{I}^{\sigma,\mathbf{b}}f\right),\square_{J}^{\omega,\mathbf{b}^{*}}g\right\rangle_{\omega}\\ &\quad +\mathbf{1}_{\{I_J\in\mathfrak{C}_{\mathcal{A}}(A)\}}\ E_{I_J}^{\sigma}\left(\widehat{\mathbb{F}}_{I_J}^{\sigma,\mathbf{b}}f\right)\ \left\langle T_{\sigma}^{\alpha}b_{I_J},\square_{J}^{\omega,\mathbf{b}^{*}}g\right\rangle_{\omega}\ ,\end{aligned}$$

where the four lines are respectively a paraproduct, stopping, neighbour and broken term.

The corresponding NTV splitting of $\mathsf{B}_{\Subset_{\mathbf{r},\varepsilon}}^{A}(f,g)$ using (5.5.4) and (5.5.2) becomes

$$\begin{aligned}\mathsf{B}_{\Subset_{\mathbf{r},\varepsilon}}^{A}(f,g) &= \left\langle T_{\sigma}^{\alpha}\left(\mathsf{P}_{\mathcal{C}_A}^{\sigma}f\right),\mathsf{P}_{\mathcal{C}_A^{\mathcal{G},shift}}^{\omega}g\right\rangle_{\omega}^{\Subset_{\mathbf{r},\varepsilon}}\\ &= \sum_{\substack{I\in\mathcal{C}_A\text{ and }J\in\mathcal{C}_A^{\mathcal{G},shift}\\ J^{\maltese}\subsetneqq I\text{ and }\ell(J)\leq 2^{-\mathbf{r}}\ell(I)}}\left\langle T_{\sigma}^{\alpha}\left(\square_{I}^{\sigma,\mathbf{b}}f\right),\square_{J}^{\omega,\mathbf{b}^{*}}g\right\rangle_{\omega}\\ &= \mathsf{B}_{paraproduct}^{A}(f,g)-\mathsf{B}_{stop}^{A}(f,g)+\mathsf{B}_{neighbour}^{A}(f,g)+\mathsf{B}_{brok}^{A}(f,g),\end{aligned}$$

where

$$\begin{aligned}\mathsf{B}_{paraproduct}^{A}(f,g) &\equiv \sum_{\substack{I\in\mathcal{C}_A\text{ and }J\in\mathcal{C}_A^{\mathcal{G},shift}\\ J^{\maltese}\subsetneqq I\text{ and }\ell(J)\leq 2^{-\mathbf{r}}\ell(I)}} E_{I_J}^{\sigma}\left(\widehat{\square}_{I}^{\sigma,\flat,\mathbf{b}}f\right)\left\langle T_{\sigma}^{\alpha}b_A,\square_{J}^{\omega,\mathbf{b}^{*}}g\right\rangle_{\omega}\\ \mathsf{B}_{stop}^{A}(f,g) &\equiv \sum_{\substack{I\in\mathcal{C}_A\text{ and }J\in\mathcal{C}_A^{\mathcal{G},shift}\\ J^{\maltese}\subsetneqq I\text{ and }\ell(J)\leq 2^{-\mathbf{r}}\ell(I)}} E_{I_J}^{\sigma}\left(\widehat{\square}_{I}^{\sigma,\flat,\mathbf{b}}f\right)\left\langle T_{\sigma}^{\alpha}\left(\mathbf{1}_{A\setminus I_J}b_A\right),\square_{J}^{\omega,\mathbf{b}^{*}}g\right\rangle_{\omega}\\ \mathsf{B}_{neighbour}^{A}(f,g) &\equiv \sum_{\substack{I\in\mathcal{C}_A\text{ and }J\in\mathcal{C}_A^{\mathcal{G},shift}\\ J^{\maltese}\subsetneqq I\text{ and }\ell(J)\leq 2^{-\mathbf{r}}\ell(I)}}\sum_{\tilde{I}\in\theta(I_J)}\left\langle T_{\sigma}^{\alpha}\left(\mathbf{1}_{\tilde{I}}\square_{I}^{\sigma,\mathbf{b}}f\right),\square_{J}^{\omega,\mathbf{b}^{*}}g\right\rangle_{\omega}\end{aligned}$$

correspond to the three original NTV forms associated with 1-testing, and where

$$\mathsf{B}_{brok}^{A}(f,g) \equiv \sum_{\substack{I\in\mathcal{C}_A \text{ and } J\in\mathcal{C}_A^{\mathcal{G},shift} \\ J^{\maltese}\subsetneqq I \text{ and } \ell(J)\leq 2^{-\mathbf{r}}\ell(I)}} \mathbf{1}_{\{I_J\in\mathfrak{C}_{\mathcal{A}}(A)\}}\, E_{I_J}^{\sigma}\left(\widehat{\mathbb{F}}_{I_J}^{\sigma,\mathbf{b}}f\right)\,\left\langle T_{\sigma}^{\alpha}b_{I_J},\square_J^{\omega,\mathbf{b}^*}g\right\rangle_{\omega} \tag{5.5.18}$$

"vanishes" since $J^{\maltese}\subsetneqq I$ and $I_J\in\mathfrak{C}_{\mathcal{A}}(A)$ imply that $J^{\maltese}\notin\mathcal{C}_A^{\mathcal{G}}$, contradicting $J\in\mathcal{C}_A^{\mathcal{G},shift}$.

Remark 5.5.9. *The inquisitive reader will note that the pairs (I,J) arising in the above sum with $J^{\maltese}\subsetneqq I$ replaced by $J^{\maltese}=I$ are handled in the probabilistic estimate (5.2.15) for the bad form $\Theta_2^{bad\natural}$ defined in (5.2.12).*

5.5.4.1 The paraproduct form

The paraproduct form $\mathsf{B}_{paraproduct}^{A}(f,g)$ is easily controlled by the testing condition for T^{α} together with weak Riesz inequalities for dual martingale differences. Indeed, recalling the telescoping identity (5.1.48), and that the collection $\left\{I\in\mathcal{C}_A\colon \ell(J)\leq 2^{-\mathbf{r}}\ell(I)\right\}$ is tree connected for all $J\in\mathcal{C}_A^{\mathcal{G},shift}$, we have

$$\begin{aligned}
\mathsf{B}_{paraproduct}^{A}(f,g) &= \sum_{\substack{I\in\mathcal{C}_A \text{ and } J\in\mathcal{C}_A^{\mathcal{G},shift} \\ J^{\maltese}\subsetneqq I \text{ and } \ell(J)\leq 2^{-\mathbf{r}}\ell(I)}} E_{I_J}^{\sigma}\left(\widehat{\square}_I^{\sigma,\flat,\mathbf{b}}f\right)\left\langle T_{\sigma}^{\alpha}b_A,\square_J^{\omega,\mathbf{b}^*}g\right\rangle_{\omega} \\
&= \sum_{J\in\mathcal{C}_A^{\mathcal{G},shift}} \left\langle T_{\sigma}^{\alpha}b_A,\square_J^{\omega,\mathbf{b}^*}g\right\rangle_{\omega}\left\{\sum_{I\in\mathcal{C}_A\colon J^{\maltese}\subsetneqq I \text{ and } \ell(J)\leq 2^{-\mathbf{r}}\ell(I)} E_{I_J}^{\sigma}\left(\widehat{\square}_I^{\sigma,\flat,\mathbf{b}}f\right)\right\} \\
&= \sum_{J\in\mathcal{C}_A^{\mathcal{G},shift}} \left\langle T_{\sigma}^{\alpha}b_A,\square_J^{\omega,\mathbf{b}^*}g\right\rangle_{\omega}\left\{\mathbf{1}_{\left\{J:I^{\natural}(J)_J\in\mathcal{C}_A\right\}}E_{I^{\natural}(J)_J}^{\sigma}\widehat{\mathbb{F}}_{I^{\natural}(J)_J}^{\sigma,\mathbf{b}}f - E_A^{\sigma}\widehat{\mathbb{F}}_A^{\sigma,\mathbf{b}}f\right\} \\
&= \left\langle T_{\sigma}^{\alpha}b_A, \sum_{J\in\mathcal{C}_A^{\mathcal{G},shift}}\left\{\mathbf{1}_{\left\{J:I^{\natural}(J)_J\in\mathcal{C}_A\right\}}E_{I^{\natural}(J)_J}^{\sigma}\widehat{\mathbb{F}}_{I^{\natural}(J)_J}^{\sigma,\mathbf{b}}f - E_A^{\sigma}\widehat{\mathbb{F}}_A^{\sigma,\mathbf{b}}f\right\}\square_J^{\omega,\mathbf{b}^*}g\right\rangle_{\omega}
\end{aligned}$$

where $I^{\natural}(J)$ denotes the smallest cube $I \in \mathcal{C}_A$ such that $J^{\maltese} \subsetneqq I$ and $\ell(J) \leq 2^{-\mathbf{r}}\ell(I)$, and of course $I^{\natural}(J)_J$ denotes its child containing J. Note that by construction of the modified difference operator $\square_I^{\sigma,\flat,\mathbf{b}}$, the only time the average $\widehat{\mathbb{F}}_{I^{\natural}(J)_J}^{\sigma} f$ appears in the above sum is when $I^{\natural}(J)_J \in \mathcal{C}_A$, since the case $I^{\natural}(J)_J \in \mathcal{A}$ has been removed to the broken term. This is reflected above with the inclusion of the indicator $\mathbf{1}_{\left\{J: I^{\natural}(J)_J \in \mathcal{C}_A\right\}}$. It follows that we have the bound

$$\left|\mathbf{1}_{\left\{J:I^{\natural}(J)_J\in\mathcal{C}_A\right\}} E_{I^{\natural}(J)_J}^{\sigma} \widehat{\mathbb{F}}_{I^{\natural}(J)_J}^{\sigma,\mathbf{b}} f\right| + \left|E_A^{\sigma}\widehat{\mathbb{F}}_A^{\sigma,\mathbf{b}} f\right| \lesssim E_A^{\sigma}|f| \leq \alpha_{\mathcal{A}}(A)$$

Thus from Cauchy-Schwarz, the upper weak Riesz inequalities for the pseudoprojections $\square_J^{\omega,\mathbf{b}^*} g$ and the bound on the coefficients $\lambda_J \equiv \left(\mathbf{1}_{\left\{J:I^{\natural}(J)_J\in\mathcal{C}_A\right\}} E_{I^{\natural}(J)_J}^{\sigma} \widehat{\mathbb{F}}_{I^{\natural}(J)_J}^{\sigma,\mathbf{b}} f - E_A^{\sigma}\widehat{\mathbb{F}}_A^{\sigma,\mathbf{b}} f\right)$ given by $|\lambda_J| \lesssim \alpha_{\mathcal{A}}(A)$, we have

$$\begin{aligned}
&\left|\mathsf{B}_{paraproduct}^{A}(f,g)\right| = \qquad (5.5.19)\\
&\quad \left|\left\langle T_\sigma^\alpha b_A, \sum_{J\in\mathcal{C}_A^{\mathcal{G},shift}} \left\{\left(\mathbf{1}_{\left\{J:I^{\natural}(J)_J\in\mathcal{C}_A\right\}} E_{I^{\natural}(J)_J}^{\sigma} \widehat{\mathbb{F}}_{I^{\natural}(J)_J}^{\sigma,\mathbf{b}} f - E_A^{\sigma}\widehat{\mathbb{F}}_A^{\sigma,\mathbf{b}} f\right)\right\} \square_J^{\omega,\mathbf{b}^*} g\right\rangle_\omega\right|\\
&\leq \left\|\mathbf{1}_A T_\sigma^\alpha b_A\right\|_{L^2(\omega)} \left\|\sum_{J\in\mathcal{C}_A^{\mathcal{G},shift}} \lambda_J \square_J^{\omega,\mathbf{b}^*} g\right\|_{L^2(\omega)}\\
&\lesssim \alpha_{\mathcal{A}}(A)\ \left\|\mathbf{1}_A T_\sigma^\alpha b_A\right\|_{L^2(\omega)} \sum_{J\in\mathcal{C}_A^{\mathcal{G},shift}} \left\|\square_J^{\omega,\mathbf{b}^*} g\right\|_{L^2(\omega)}\\
&\leq \mathfrak{T}_{T^\alpha}^{\mathbf{b}}\ \alpha_{\mathcal{A}}(A)\ \sqrt{|A|_\sigma}\ \left\|\mathsf{P}_{\mathcal{C}_A^{\mathcal{G},shift}}^{\omega,\mathbf{b}^*} g\right\|_{L^2(\omega)}^{\bigstar}.
\end{aligned}$$

5.5.4.2 The neighbour form

Next, the neighbour form $\mathsf{B}^{A}_{neighbour}(f,g)$ is easily controlled by the $\mathfrak{A}_2^{\alpha}$ condition using the pivotal estimate in Energy Lemma 5.1.25 and the fact that the cubes $J \in \mathcal{C}_A^{\mathcal{G},shift}$ are good in I and beyond when the pair (I,J) occurs in the sum. In particular, the information encoded in the stopping tree $\mathcal{A}$ plays no role here, apart from appearing in the corona projections on the right hand side of (5.5.25) below. We have

$$\mathsf{B}^{A}_{neighbour}(f,g) = \sum_{\substack{I\in\mathcal{C}_A \text{ and } J\in\mathcal{C}_A^{\mathcal{G},shift} \\ J^{\maltese}\subsetneqq I \text{ and } \ell(J)\leq 2^{-\mathbf{r}}\ell(I)}} \sum_{\tilde{I}\in\theta(I_J)} \left\langle T_{\sigma}^{\alpha}\left(\mathbf{1}_{\tilde{I}}\square_I^{\sigma,\mathbf{b}}f\right), \square_J^{\omega,\mathbf{b}^*}g\right\rangle_{\omega} \tag{5.5.20}$$

where we keep in mind that the pairs $(I,J)\in\mathcal{D}\times\mathcal{G}$ that arise in the sum for $\mathsf{B}^{A}_{neighbour}(f,g)$ satisfy the property that $J^{\maltese}\subsetneqq I$, so that J is good with respect to all cubes K of size at least that of $J^{\maltese}$, which includes I. Recall that I_J is the child of I that contains J, and that $\theta(I_J)$ denotes its 2^n-1 siblings in I, i.e. $\theta(I_J)=\mathfrak{C}_{\mathcal{D}}(I)\setminus\{I_J\}$. Fix (I,J) momentarily, and an integer $s\geq\mathbf{r}$. Using $\square_I^{\sigma,\mathbf{b}}=\square_I^{\sigma,\flat,\mathbf{b}}+\square_{I,brok}^{\sigma,\flat,\mathbf{b}}$ and the fact that $\square_I^{\sigma,\flat,\mathbf{b}}f$ is a constant multiple of $b_{\tilde{I}}$ on the cube $\tilde{I}$, we have the estimates

$$\begin{aligned}\left|\mathbf{1}_{\tilde{I}}\square_I^{\sigma,\flat,\mathbf{b}}f\right| &= \left|\left(E_{\tilde{I}}^{\sigma}\widehat{\square}_I^{\sigma,\flat,\mathbf{b}}f\right)b_{\tilde{I}}\right| \leq C_{\mathbf{b}}\left|E_{\tilde{I}}^{\sigma}\widehat{\square}_I^{\sigma,\flat,\mathbf{b}}f\right|,\\ \left|\mathbf{1}_{\tilde{I}}\square_{I,brok}^{\sigma,\flat,\mathbf{b}}f\right| &\leq \mathbf{1}_{\mathfrak{C}_A(A)}(\tilde{I})\ E_{\tilde{I}}^{\sigma}|f|,\end{aligned}$$

and hence

$$\mathbf{1}_{\tilde{I}}\left|\square_I^{\sigma,\mathbf{b}}f\right| \leq C\mathbf{1}_{\tilde{I}}\left(\left|E_{\tilde{I}}^{\sigma}\widehat{\square}_I^{\sigma,\flat,\mathbf{b}}f\right| + \mathbf{1}_{\mathfrak{C}_A(A)}(\tilde{I})\ E_{\tilde{I}}^{\sigma}|f|\right), \tag{5.5.21}$$

which will be used below after an application of the Energy Lemma. We can write $\mathsf{B}_{neighbour}^{A}(f,g)$ as

$$\sum_{\substack{I\in\mathcal{C}_A \& J\in\mathcal{G}^{\mathcal{D}}_{(\kappa(I_J,J),\varepsilon)-good}\cap\mathcal{C}_A^{\mathcal{G},shift} \& J^{\maltese}\subsetneqq I \\ d\left(J,\tilde{I}\right)>2\ell(J)^{\varepsilon}\ell\left(\tilde{I}\right)^{1-\varepsilon} \text{ and } \ell(J)\leq 2^{-\mathbf{r}}\ell(I)}} \sum_{\tilde{I}\in\theta(I_J)} \left\langle T_\sigma^\alpha\left(\mathbf{1}_{\tilde{I}}\square_I^{\sigma,\mathbf{b}}f\right),\square_J^{\omega,\mathbf{b}^*}g\right\rangle_\omega$$

where we have included the conditions

$$J\in\mathcal{G}^{\mathcal{D}}_{(\kappa(I_J,J),\varepsilon)-good} \text{ and } d(J,\tilde{I})>2\ell(J)^{\varepsilon}\,\ell(\tilde{I})^{1-\varepsilon}$$

in the summation since they are already implied the remaining four conditions, and will be used in estimates below.

We will also use the following fractional analogue of the Poisson inequality in [75].

Lemma 5.5.10. *Suppose $0\leq\alpha<1$ and $J\subset I\subset K$ and that $d(J,\partial I)>2\ell(J)^{\varepsilon}\ell(I)^{1-\varepsilon}$ for some $0<\varepsilon<\frac{1}{n+1-\alpha}$. Then for a positive Borel measure μ we have*

$$\mathrm{P}^\alpha(J,\mu\mathbf{1}_{K\setminus I})\lesssim\left(\frac{\ell(J)}{\ell(I)}\right)^{1-\varepsilon(n+1-\alpha)}\mathrm{P}^\alpha(I,\mu\mathbf{1}_{K\setminus I}). \tag{5.5.22}$$

Proof. We have

$$\mathrm{P}^\alpha\left(J,\mu\mathbf{1}_{K\setminus I}\right)\approx\sum_{k=0}^{\infty}2^{-k}\frac{1}{\left|2^kJ\right|^{1-\frac{\alpha}{n}}}\int_{\left(2^kJ\right)\cap(K\setminus I)}d\mu,$$

and $\left(2^kJ\right)\cap(K\setminus I)\neq\emptyset$ requires

$$d(J,K\setminus I)\leq c2^k\ell(J),$$

for some dimensional constant $c>0$. Let k_0 be the smallest such k. By our distance assumption we must then have

$$2\ell(J)^{\varepsilon}\,\ell(I)^{1-\varepsilon}\leq d(J,\partial I)\leq c2^{k_0}\ell(J),$$

or

$$2^{-k_0+1}\leq c\left(\frac{\ell(J)}{\ell(I)}\right)^{1-\varepsilon}.$$

Now let k_1 be defined by $2^{k_1}\equiv\frac{\ell(I)}{\ell(J)}$. Then assuming $k_1>k_0$ (the case $k_1\leq k_0$ is similar) we have

$$\begin{aligned}
\mathrm{P}^{\alpha}\left(J,\mu\mathbf{1}_{K\setminus I}\right) &\approx\left\{\sum_{k=k_0}^{k_1}+\sum_{k=k_1}^{\infty}\right\}2^{-k}\frac{1}{\left|2^kJ\right|^{1-\frac{\alpha}{n}}}\int_{\left(2^kJ\right)\cap(K\setminus I)}d\mu\\
&\lesssim 2^{-k_0}\frac{|I|^{1-\frac{\alpha}{n}}}{\left|2^{k_0}J\right|^{1-\frac{\alpha}{n}}}\left(\frac{1}{|I|^{1-\frac{\alpha}{n}}}\int_{\left(2^{k_1}J\right)\cap(K\setminus I)}d\mu\right)+2^{-k_1}\mathrm{P}^{\alpha}\left(I,\mu\mathbf{1}_{K\setminus I}\right)\\
&\lesssim\left(\frac{\ell(J)}{\ell(I)}\right)^{(1-\varepsilon)(n+1-\alpha)}\left(\frac{\ell(I)}{\ell(J)}\right)^{n-\alpha}\mathrm{P}^{\alpha}\left(I,\mu\mathbf{1}_{K\setminus I}\right)+\frac{\ell(J)}{\ell(I)}\mathrm{P}^{\alpha}\left(I,\mu\mathbf{1}_{K\setminus I}\right),
\end{aligned}$$

which is the inequality (5.5.22). $\square$

Now fix $I_0=I_J, I_\theta\in\theta(I_J)$ and assume that $J\Subset_{\mathbf{r},\varepsilon}I_0$. Let $\frac{\ell(J)}{\ell(I_0)}=2^{-s}$ in the pivotal estimate from Energy Lemma 5.1.25 with $J\subset I_0\subset I$ to obtain

$$\begin{aligned}
&\left|\langle T_\sigma^\alpha\left(\mathbf{1}_{I_\theta}\square_I^{\sigma,\mathbf{b}}f\right),\square_J^{\omega,\mathbf{b}^*}g\rangle_\omega\right|\lesssim\left\|\square_J^{\omega,\mathbf{b}^*}g\right\|_{L^2(\omega)}\sqrt{|J|_\omega}\,\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{I_\theta}\left|\square_I^{\sigma,\mathbf{b}}f\right|\sigma\right)\\
\lesssim&\left\|\square_J^{\omega,\mathbf{b}^*}g\right\|_{L^2(\omega)}\sqrt{|J|_\omega}\cdot2^{-(1-\varepsilon(n+1-\alpha))s}\mathrm{P}^{\alpha}\left(I_0,\mathbf{1}_{I_\theta}\left|\square_I^{\sigma,\mathbf{b}}f\right|\sigma\right)\\
\lesssim&\left\|\square_J^{\omega,\mathbf{b}^*}g\right\|_{L^2(\omega)}\sqrt{|J|_\omega}\cdot2^{-(1-\varepsilon(n+1-\alpha))s}\mathrm{P}^{\alpha}\left(I_0,\mathbf{1}_{I_\theta}\mathbf{E}_{I_\theta}^{\sigma}f\cdot\sigma\right)
\end{aligned}$$

Here we are using (5.5.22) in the third line, which applies since $J \subset I_0$, and we have used (5.5.21) in the fourth line and the shorthand notation

$$\mathbf{E}^{\sigma}_{I_\theta} f \equiv \left| E^{\sigma}_{I_\theta} \widehat{\square}^{\sigma,\flat,\mathbf{b}}_{I} f \right| + \mathbf{1}_{\mathfrak{C}_A(A)} \left(I_\theta \right) \ E^{\sigma}_{I_\theta} \left| f \right|$$

where the cube I on the right hand side is determined uniquely by the cube $I_\theta \in \theta\left(I_J\right)$.

In the sum below, we keep the side lengths of the cubes J fixed at 2^{-s} times that of I_0, and of course take $J \subset I_0$. We also keep the underlying assumptions that $J \in \mathcal{C}^{\mathcal{G},shift}_{A}$ and that $J \in \mathcal{G}^{\mathcal{D}}_{(\kappa(I_J,J),\varepsilon)-good}$ in mind without necessarily pointing to them in the notation. Matters will shortly be reduced to estimating the following term:

$$\begin{aligned} A(I, I_0, I_\theta, s) &\equiv \sum_{J\,:\,2^{s+1}\ell(J)=\ell(I):J\subset I_0} \left| \langle T^{\alpha}_{\sigma} \left(\mathbf{1}_{I_\theta} \square^{\sigma,\mathbf{b}}_{I} f \right), \square^{\omega,\mathbf{b}^*}_{J} g \rangle_\omega \right| \\ &\leq 2^{-(1-\varepsilon(n+1-\alpha))s} \left(\mathbf{E}^{\sigma}_{I_\theta} f \right) \mathrm{P}^{\alpha}(I_0, \mathbf{1}_{I_\theta}\sigma) \sum_{\substack{J:J\subset I_0 \\ 2^{s+1}\ell(J)=\ell(I)}} \left\| \square^{\omega,\mathbf{b}^*}_{J} g \right\|_{L^2(\omega)} \sqrt{|J|_\omega} \\ &\leq 2^{-(1-\varepsilon(n+1-\alpha))s} \left(\mathbf{E}^{\sigma}_{I_\theta} f \right) \mathrm{P}^{\alpha}(I_0, \mathbf{1}_{I_\theta}\sigma) \sqrt{|I_0|_\omega} \Lambda(I, I_0, I_\theta, s) \end{aligned}$$

where $\Lambda(I, I_0, I_\theta, s)^2 \equiv \sum_{J\in\mathcal{C}^{\mathcal{G},shift}_{A}:\ 2^{s+1}\ell(J)=\ell(I):\ J\subset I_0} \left\| \square^{\omega,\mathbf{b}^*}_{J} g \right\|^2_{L^2(\omega)}$.

The last line follows upon using the Cauchy-Schwarz inequality and the fact that $J \in \mathcal{C}^{\mathcal{G},shift}_{A}$. We also note that since $2^{s+1}\ell\left(J\right) = \ell\left(I\right)$,

$$\begin{aligned} \sum_{I_0\in\mathfrak{C}_{\mathcal{D}}(I)} \Lambda(I, I_0, I_\theta, s)^2 &\equiv \sum_{J\in\mathcal{C}^{\mathcal{G},shift}_{A}:\ 2^{s+1}\ell(J)=\ell(I):\ J\subset I} \left\| \square^{\omega,\mathbf{b}^*}_{J} g \right\|^2_{L^2(\omega)} \qquad (5.5.23) \\ \sum_{I\in\mathcal{C}_A} \sum_{I_0\in\mathfrak{C}_{\mathcal{D}}(I)} \Lambda(I, I_0, I_\theta, s)^2 &\leq \left\| \mathsf{P}^{\omega,\mathbf{b}^*}_{\mathcal{C}^{\mathcal{G},shift}_{A}} g \right\|^{\bigstar 2}_{L^2(\omega)} \end{aligned}$$

Using (5.4.2) we obtain

$$\left| E^{\sigma}_{I_\theta}\left(\widehat{\square}^{\sigma,\flat,\mathbf{b}}_{I} f\right)\right| \leq \sqrt{E^{\sigma}_{I_\theta}\left|\widehat{\square}^{\sigma,\flat,\mathbf{b}}_{I} f\right|^2} \lesssim \left\|\square^{\sigma,\mathbf{b}}_{I} f\right\|^{\bigstar}_{L^2(\sigma)} \ |I_\theta|_\sigma^{-\frac{1}{2}} \tag{5.5.24}$$

and hence

$$\begin{aligned}
\mathbf{E}^{\sigma}_{I_\theta} f &\equiv \left| E^{\sigma}_{I_\theta(I_J)} \widehat{\square}^{\sigma,\flat,\mathbf{b}}_{I} f\right| + \mathbf{1}_{\mathfrak{C}_{A}(A)}(I_\theta)\ \ E^{\sigma}_{I_\theta}|f| \\
&\lesssim \left(\left\|\square^{\sigma,\mathbf{b}}_{I} f\right\|^{\bigstar}_{L^2(\sigma)} + \mathbf{1}_{\mathfrak{C}_{A}(A)}(I_\theta)\ \ |I_\theta|_\sigma^{\frac{1}{2}} E^{\sigma}_{I_\theta}|f|\right) |I_\theta|_\sigma^{-\frac{1}{2}}
\end{aligned}$$

and thus $A(I, I_0, I_\theta, s)$ is bounded by

$$\begin{aligned}
&2^{-(1-\varepsilon(n+1-\alpha))s}\left(\left\|\square^{\sigma,\mathbf{b}}_{I} f\right\|^{\bigstar}_{L^2(\sigma)} + \mathbf{1}_{\mathfrak{C}_{A}(A)}(I_\theta)\ \ |I_\theta|_\sigma^{\frac{1}{2}} E^{\sigma}_{I_\theta}|f|\right) \cdot \\
&\qquad \cdot \Lambda(I, I_0, I_\theta, s)\, |I_\theta|_\sigma^{-\frac{1}{2}}\, \mathrm{P}^{\alpha}(I_0, \mathbf{1}_{I_\theta}\sigma)\sqrt{|I_0|_\omega} \\
&\lesssim \sqrt{\mathfrak{A}^{\alpha}_{2}}\, 2^{-(1-\varepsilon(n+1-\alpha))s}\left(\left\|\square^{\sigma,\mathbf{b}}_{I} f\right\|^{\bigstar}_{L^2(\sigma)} + \mathbf{1}_{\mathfrak{C}_{A}(A)}(I_\theta)\ \ |I_\theta|_\sigma^{\frac{1}{2}} E^{\sigma}_{I_\theta}|f|\right) \Lambda(I, I_0, I_\theta, s)
\end{aligned}$$

since $\mathrm{P}^{\alpha}(I_0, \mathbf{1}_{I_\theta}\sigma) \lesssim \dfrac{|I_\theta|_\sigma}{|I_\theta|^{1-\frac{\alpha}{n}}}$ shows that

$$|I_\theta|_\sigma^{-\frac{1}{2}}\, \mathrm{P}^{\alpha}(I_0, \mathbf{1}_{I_\theta}\sigma)\ \sqrt{|I_0|_\omega} \lesssim \frac{\sqrt{|I_\theta|_\sigma}\sqrt{|I_0|_\omega}}{|I_\theta|^{1-\frac{\alpha}{n}}} \lesssim \sqrt{\mathfrak{A}^{\alpha}_{2}}$$

where the implied constant depends on α and the dimension. An application of Cauchy-

Schwarz to the sum over I using (5.5.23) then shows that

$$\sum_{I\in\mathcal{C}_A}\sum_{\substack{I_0,I_\theta\in\mathfrak{C}_{\mathcal{D}}(I)\\ I_0\neq I_\theta}} A(I,I_0,I_\theta,s)$$

$$\lesssim\sqrt{\mathfrak{A}_2^\alpha}2^{-(1-\varepsilon(n+1-\alpha))s}\sqrt{\sum_{I\in\mathcal{C}_A}\left\|\square_I^{\sigma,\mathbf{b}}f\right\|_{L^2(\sigma)}^{\bigstar 2}+\sum_{I_\theta\in\mathfrak{C}_A(A)}|I_\theta|_\sigma\left(E_{I_\theta}^\sigma|f|\right)^2}\cdot$$

$$\cdot\sqrt{\sum_{I\in\mathcal{C}_A}\left(\sum_{\substack{I_0,I_\theta\in\mathfrak{C}_{\mathcal{D}}(I)\\ I_0\neq I_\theta}}\Lambda(I,I_0,I_\theta,s)\right)^2}$$

$$\lesssim\sqrt{\mathfrak{A}_2^\alpha}2^{-(1-\varepsilon(n+1-\alpha))s}\sqrt{\left\|\mathsf{P}_{\mathcal{C}_A}^\sigma f\right\|_{L^2(\sigma)}^{\bigstar 2}+\sum_{A'\in\mathfrak{C}_A(A)}|A'|_\sigma\left(E_{A'}^\sigma|f|\right)^2}\cdot$$

$$\cdot\sqrt{\sum_{I\in\mathcal{C}_A}\left(\sum_{\substack{I_0\in\mathfrak{C}_{\mathcal{D}}(I)\\ I_0\neq I_\theta}}\Lambda(I,I_0,I_\theta,s)\right)^2}$$

$$\lesssim\sqrt{\mathfrak{A}_2^\alpha}2^{-(1-\varepsilon(n+1-\alpha))s}\left(\|\mathsf{P}_{\mathcal{C}_A}^\sigma f\|_{L^2(\sigma)}^{\bigstar}+\sqrt{\sum_{A'\in\mathfrak{C}_A(A)}|A'|_\sigma\left(E_{A'}^\sigma|f|\right)^2}\right)\left\|\mathsf{P}_{\mathcal{C}_A^{\mathcal{G},shift}}^{\omega,\mathbf{b}^*}g\right\|_{L^2(\omega)}^{\bigstar}$$

This estimate is summable in $s\geq\mathbf{r}$ since $\varepsilon<\frac{1}{n+1-\alpha}$, and so the proof of

$$\left|\mathsf{B}_{neighbour}^A(f,g)\right|\leq\sum_{I\in\mathcal{C}_A}\sum_{\substack{I_0\text{ and }I_\theta\in\mathfrak{C}_{\mathcal{D}}(I)\\ I_0\neq I_\theta}}\sum_{s=\mathbf{r}}^\infty A(I,I_0,I_\theta,s) \tag{5.5.25}$$

$$\lesssim\sqrt{\mathfrak{A}_2^\alpha}\left(\left\|\mathsf{P}_{\mathcal{C}_A}^\sigma f\right\|_{L^2(\sigma)}^{\bigstar}+\sqrt{\sum_{A'\in\mathfrak{C}_A(A)}|A'|_\sigma\,\alpha_{\mathcal{A}}(A')^2}\right)\left\|\mathsf{P}_{\mathcal{C}_A^{\mathcal{G},shift}}^{\omega,\mathbf{b}^*}g\right\|_{L^2(\omega)}^{\bigstar}$$

is complete since $E^{\sigma}_{A'}|f| \lesssim \alpha_{\mathcal{A}}(A')$.

Now if we sum in $A \in \mathcal{A}$ the inequalities (5.5.19), (5.5.25) and (5.5.18) we get

$$\begin{aligned}
&\sum_{A\in\mathcal{A}} \left|\mathsf{B}^{A}_{\Subset_{\mathbf{r},\varepsilon}}(f,g) + \mathsf{B}^{A}_{stop}(f,g)\right| \\
\lesssim\ & \left(\mathfrak{T}^{\mathbf{b}}_{T^{\alpha}} + \sqrt{\mathfrak{A}^{\alpha}_{2}}\right) \sqrt{\sum_{A\in\mathcal{A}} \left\|\mathsf{P}^{\omega,\mathbf{b}^*}_{\mathcal{C}^{\mathcal{G},shift}_{A}} g\right\|^{\bigstar 2}_{L^2(\omega)}} \cdot \\
&\cdot \sqrt{\sum_{A\in\mathcal{A}} \left\{\alpha_{\mathcal{A}}(A)^2 |A|_{\sigma} + \left\|\mathsf{P}^{\sigma}_{\mathcal{C}_A} f\right\|^{\bigstar 2}_{L^2(\sigma)} + \sum_{A'\in\mathfrak{C}_{\mathcal{A}}(A)} \alpha_{\mathcal{A}}(A')^2 |A'|_{\sigma}\right\}} \\
\lesssim\ & \left(\mathfrak{T}^{\mathbf{b}}_{T^{\alpha}} + \sqrt{\mathfrak{A}^{\alpha}_{2}}\right) \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)}
\end{aligned}$$

The stopping form is the subject of the following section.

5.6 The stopping form

Here we deal with the stopping form. We modify the adaptation of the argument of M. Lacey in to apply in the setting of a Tb theorem for an α-fractional Calderón-Zygmund operator T^{α} in $\mathbb{R}^n$ using the Monotonicity Lemma 5.1.23, the energy condition, and the weak goodness of Hytönen and Martikainen [24]. We directly control the pairs (I,J) in the stopping form according to the $\mathcal{L}$-coronas (constructed from the 'bottom up' with stopping times involving the energies $\left\|\square^{\omega,\mathbf{b}^*}_{J}\right\|^2_{L^2(\omega)}$) to which I and $J^{\maltese}$ are associated. However, due to the fact that the cubes I need no longer be good in any sense, we must introduce an additional top/down 'indented' corona construction on top of the bottom/up construction of M. Lacey, and in connection with this we introduce a Substraddling Lemma. We then control the stopping form by absorbing the case when both I and $J^{\maltese}$ belong to the same $\mathcal{L}$-corona, and by using the Straddling and Substraddling Lemmas, together with the Orthogonality Lemma, to control the case when I and $J^{\maltese}$ lie in different coronas, with a geometric gain coming from the separation of the coronas. This geometric gain is where the new 'indented' corona is required.

Apart from this change, the remaining modifications are more cosmetic, such as

- the use of the weak goodness of Hytönen and Martikainen [24] for pairs (I,J) arising in the stopping form, rather than goodness for all cubes J that was available in [27], [64], [66] and [67]. For the most part definitions such as admissible collections are modified to require $J^{\maltese} \subset I$;

- the pseudoprojections $\square_I^{\sigma,\mathbf{b}}, \square_J^{\omega,\mathbf{b}^*}$ are used in place of the orthogonal Haar projections, and the frame and weak Riesz inequalities compensate for the lack of orthogonality.

Fix grids $\mathcal{D}$ and $\mathcal{G}$. We will prove the bound

$$\left|\mathsf{B}_{stop}^{A}(f,g)\right| \lesssim \mathcal{NTV}_{\alpha} \left\|\mathsf{P}_{\mathcal{C}_A^{\mathcal{D}}}^{\sigma,\mathbf{b}} f\right\|_{L^2(\sigma)}^{\bigstar} \left\|\mathsf{P}_{\mathcal{C}_A^{\mathcal{G},shift}}^{\sigma,\mathbf{b}} g\right\|_{L^2(\omega)}^{\bigstar}, \tag{5.6.1}$$

where we recall that the nonstandard 'norms' are given by,

$$\begin{aligned}
\left\|\mathsf{P}_{\mathcal{C}_A^{\mathcal{D}}}^{\sigma,\mathbf{b}} f\right\|_{L^2(\sigma)}^{\bigstar 2} &\equiv \sum_{I\in\mathcal{C}_A^{\mathcal{D}}} \left\|\square_I^{\sigma,\mathbf{b}} f\right\|_{L^2(\sigma)}^2, \\
\left\|\mathsf{P}_{\mathcal{C}_A^{\mathcal{G},shift}}^{\sigma,\mathbf{b}} g\right\|_{L^2(\omega)}^{\bigstar 2} &\equiv \sum_{J\in\mathcal{C}_A^{\mathcal{G},shift}} \left\|\square_J^{\omega,\mathbf{b}^*} g\right\|_{L^2(\omega)}^2,
\end{aligned}$$

and that the stopping form is given by

$$\begin{aligned}
\mathsf{B}_{stop}^{A}(f,g) &\equiv \sum_{\substack{I\in\mathcal{C}_{A}^{\mathcal{D}} \text{ and } J\in\mathcal{C}_{A}^{\mathcal{G},shift} \\ J^{\maltese}\subsetneqq I \text{ and } \ell(J)\leq 2^{-\rho}\ell(I)}} \left(E_{I_J}^{\sigma}\widehat{\square}_{I}^{\sigma,\flat,\mathbf{b}}f\right)\left\langle T_{\sigma}^{\alpha}\left(b_A\mathbf{1}_{A\setminus I_J}\right),\square_{J}^{\omega,\mathbf{b}^{*}}g\right\rangle_{\omega} \\
&= \sum_{\substack{I:\ \pi I\in\mathcal{C}_{A}^{\mathcal{D}} \text{ and } J\in\mathcal{C}_{A}^{\mathcal{G},shift} \\ J^{\maltese}\subsetneqq I \text{ and } \ell(J)\leq 2^{-(\rho-1)}\ell(I)}} \left(E_{I}^{\sigma}\widehat{\square}_{\pi I}^{\sigma,\flat,\mathbf{b}}f\right)\left\langle T_{\sigma}^{\alpha}\left(b_A\mathbf{1}_{A\setminus I}\right),\square_{J}^{\omega,\mathbf{b}^{*}}g\right\rangle_{\omega}
\end{aligned}$$

where we have made the 'change of dummy variable' $I_J \to I$ for convenience in notation (recall that the child of I that contains J is denoted I_J). Changing $\rho-1$ to ρ we have:

$$\mathsf{B}_{stop}^{A}(f,g) = \sum_{\substack{I:\ \pi I\in\mathcal{C}_{A}^{\mathcal{D}} \text{ and } J\in\mathcal{C}_{A}^{\mathcal{G},shift} \\ J^{\maltese}\subsetneqq I \text{ and } \ell(J)\leq 2^{-\rho}\ell(I)}} \left(E_{I}^{\sigma}\widehat{\square}_{\pi I}^{\sigma,\flat,\mathbf{b}}f\right)\left\langle T_{\sigma}^{\alpha}\left(b_A\mathbf{1}_{A\setminus I}\right),\square_{J}^{\omega,\mathbf{b}^{*}}g\right\rangle_{\omega},$$

For $A\in\mathcal{A}$ recall that we have defined the *shifted* $\mathcal{G}$-corona by

$$\mathcal{C}_{A}^{\mathcal{G},shift} \equiv \left\{J\in\mathcal{G} : J^{\maltese}\in\mathcal{C}_{A}^{\mathcal{D}}\right\},$$

and also defined the *restricted* $\mathcal{D}$-corona by

$$\mathcal{C}_{A}^{\mathcal{D},restrict} \equiv \mathcal{C}_{A}\setminus\{A\} \equiv \mathcal{C}_{A}^{\prime}.$$

Definition 5.6.1. *Suppose that* $A\in\mathcal{A}$ *and that* $\mathcal{P}\subset\mathcal{C}_{A}^{\mathcal{D},restrict}\times\mathcal{C}_{A}^{\mathcal{G},shift}$. *We say that the collection of pairs* $\mathcal{P}$ *is* A -admissible *if*

- (good and (ρ,ε)-deeply embedded) For every $(I,J)\in\mathcal{P}$, and $J^{\maltese}\subset I\subsetneqq A$.

- (tree-connected in the first component) if $I_1\subset I_2$ and both $(I_1,J)\in\mathcal{P}$ and $(I_2,J)\in\mathcal{P}$, then $(I,J) \in \mathcal{P}$ for every I in the geodesic $[I_1, I_2] = \{I \in \mathcal{D} : I_1 \subset I \subset I_2\}$.

From now on we often write $\mathcal{C}_A$ and $\mathcal{C}'_A$ in place of $\mathcal{C}_A^{\mathcal{D}}$ and $\mathcal{C}_A^{\mathcal{D},restrict}$ respectively when there is no confusion. The basic example of an admissible collection of pairs is obtained from the pairs of cubes summed in the stopping form $\mathsf{B}_{stop}^{A}(f,g)$,

$$\mathcal{P}^A \equiv \left\{(I,J) : I \in \mathcal{C}'_A \text{ and } J \in \mathcal{G}_{(\rho,\varepsilon)-good}^{\mathcal{D}} \cap \mathcal{C}_A^{\mathcal{G},shift} \text{ where } J \Subset_{\rho,\varepsilon} I\right\}. \tag{5.6.2}$$

Definition 5.6.2. *Suppose that $A \in \mathcal{A}$ and that $\mathcal{P}$ is an A* -admissible *collection of pairs. Define the associated* stopping *form $\mathsf{B}_{stop}^{A,\mathcal{P}}$ by*

$$\mathsf{B}_{stop}^{A,\mathcal{P}}(f,g) \equiv \sum_{(I,J)\in\mathcal{P}} \left(E_I^{\sigma} \widehat{\square}_{\pi I}^{\sigma,\flat,\mathbf{b}} f\right) \left\langle T_{\sigma}^{\alpha}\left(b_A \mathbf{1}_{A\setminus I}\right), \square_J^{\omega,\mathbf{b}^*} g\right\rangle_{\omega} .$$

Proposition 5.6.3. *Suppose that $A \in \mathcal{A}$ and that $\mathcal{P}$ is an A*-admissible *collection of pairs. Then the stopping form $\mathsf{B}_{stop}^{A,\mathcal{P}}$ satisfies the bound*

$$\left|\mathsf{B}_{stop}^{A,\mathcal{P}}(f,g)\right| \lesssim \left(\mathcal{E}_2^{\alpha} + \sqrt{\mathfrak{A}_2^{\alpha}}\right) \left\|\mathsf{P}_{\mathcal{C}_A}^{\sigma,\mathbf{b}} f\right\|_{L^2(\sigma)}^{\bigstar} \left\|\mathsf{P}_{\mathcal{C}_A^{\mathcal{G},shift}}^{\omega,\mathbf{b}^*} g\right\|_{L^2(\omega)}^{\bigstar} \tag{5.6.3}$$

With the above proposition in hand, we can complete the proof of (5.6.1) by summing over the stopping cubes $A \in \mathcal{A}$ with the choice $\mathcal{P}^A$ of A-admissible pairs for each A:

$$\begin{aligned}
&\sum_{A\in\mathcal{A}} \left| \mathsf{B}_{stop}^{A,\mathcal{P}^A}(f,g) \right| \\
\lesssim\ & \sum_{A\in\mathcal{A}} \left(\mathcal{E}_2^{\alpha} + \sqrt{\mathfrak{A}_2^{\alpha}} \right) \left\| \mathsf{P}_{\mathcal{C}_A}^{\sigma,\mathbf{b}} f \right\|_{L^2(\sigma)}^{\bigstar} \left\| \mathsf{P}_{\mathcal{C}_A^{\mathcal{G},shift}}^{\omega,\mathbf{b}^*} g \right\|_{L^2(\omega)}^{\bigstar} \\
\lesssim\ & \left(\mathcal{E}_2^{\alpha} + \sqrt{\mathfrak{A}_2^{\alpha}} \right) \left(\sum_{A\in\mathcal{A}} \left\| \mathsf{P}_{\mathcal{C}_A}^{\sigma,\mathbf{b}} f \right\|_{L^2(\sigma)}^{\bigstar 2} \right)^{\frac{1}{2}} \left(\sum_{A\in\mathcal{A}} \left\| \mathsf{P}_{\mathcal{C}_A^{\mathcal{G},shift}}^{\omega,\mathbf{b}^*} g \right\|_{L^2(\omega)}^{\bigstar 2} \right)^{\frac{1}{2}} \\
\lesssim\ & \left(\mathcal{E}_2^{\alpha} + \sqrt{\mathfrak{A}_2^{\alpha}} \right) \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)}
\end{aligned}$$

by the lower Riesz inequality $\sum_{A\in\mathcal{A}} \left\| \mathsf{P}_{\mathcal{C}_A}^{\sigma,\mathbf{b}} f \right\|_{L^2(\sigma)}^{\bigstar 2} \lesssim \|f\|_{L^2(\sigma)}^2$, quasi-orthogonality $\sum_{A\in\mathcal{A}} \alpha_{\mathcal{A}}(f)^2 |A|_{\sigma} \lesssim \|f\|_{L^2(\sigma)}^2$ in the stopping cubes $\mathcal{A}$, and by the pairwise disjointedness of the shifted coronas $\mathcal{C}_A^{\mathcal{G},shift}$: $\sum_{A\in\mathcal{A}} \mathbf{1}_{\mathcal{C}_A^{\mathcal{G},shift}} \leq \mathbf{1}_{\mathcal{D}}$.

To prove Proposition 5.6.3, we begin by letting

$$\begin{aligned}
\Pi_1 \mathcal{P} &\equiv \left\{ I \in \mathcal{C}_A^{\mathcal{D},restrict} : (I,J) \in \mathcal{P} \text{ for some } J \in \mathcal{C}_A^{\mathcal{G},shift} \right\}, \\
\Pi_2 \mathcal{P} &\equiv \left\{ J \in \mathcal{C}_A^{\mathcal{G},shift} : (I,J) \in \mathcal{P} \text{ for some } I \in \mathcal{C}_A' \right\},
\end{aligned}$$

consist of the first and second components respectively of the pairs in $\mathcal{P}$, and writing

$$\begin{aligned}
\mathsf{B}_{stop}^{A,\mathcal{P}}(f,g) &= \sum_{J\in\Pi_2\mathcal{P}} \left\langle T_{\sigma}^{\alpha} \varphi_J^{\mathcal{P}}, \square_J^{\omega,\mathbf{b}^*} g \right\rangle_{\omega}; \\
\text{where } \varphi_J^{\mathcal{P}} &\equiv \sum_{I\in\mathcal{C}_A' :\ (I,J)\in\mathcal{P}} b_A E_I^{\sigma} \left(\widehat{\square}_{\pi I}^{\sigma,\flat,\mathbf{b}} f \right) \mathbf{1}_{A\setminus I} \ \text{(since } b_I = b_A \text{ for } I \in \mathcal{C}_A).
\end{aligned}$$

By the tree-connected property of $\mathcal{P}$, and the telescoping property of dual martingale differences, together with the bound $\alpha_{\mathcal{A}}(A)$ on the averages of f in the corona $\mathcal{C}_A$, we have

$$\left|\varphi_J^{\mathcal{P}}\right| \lesssim \alpha_{\mathcal{A}}(A) \, 1_{A \setminus I_{\mathcal{P}}(J)}, \tag{5.6.4}$$

where $I_{\mathcal{P}}(J) \equiv \bigcap \{I : (I,J) \in \mathcal{P}\}$ is the smallest cube I for which $(I,J) \in \mathcal{P}$. It is important to note that J is good with respect to $I_{\mathcal{P}}(J)$ by our infusion of weak goodness above. Another important property of these functions is the sublinearity:

$$\left|\varphi_J^{\mathcal{P}}\right| \leq \left|\varphi_J^{\mathcal{P}_1}\right| + \left|\varphi_J^{\mathcal{P}_2}\right|, \qquad \mathcal{P} = \mathcal{P}_1 \dot{\cup} \mathcal{P}_2 \ . \tag{5.6.5}$$

Now apply the Monotonicity Lemma 5.1.23 to the inner product $\left\langle T_\sigma^\alpha \varphi_J, \square_J^\omega g \right\rangle_\omega$ to obtain

$$\begin{aligned} \left|\left\langle T_\sigma^\alpha \varphi_J, \square_J^{\omega,\mathbf{b}^*} g \right\rangle_\omega\right| &\lesssim \frac{\mathrm{P}^\alpha\left(J, |\varphi_J| \mathbf{1}_{A \setminus I_{\mathcal{P}}(J)} \sigma\right)}{|J|^{\frac{1}{n}}} \left\| \triangle_J^{\omega,\mathbf{b}^*} x \right\|_{L^2(\omega)}^{\spadesuit} \left\| \square_J^{\omega,\mathbf{b}^*} g \right\|_{L^2(\omega)}^{\bigstar} \\ &+ \frac{\mathrm{P}_{1+\delta}^\alpha\left(J, |\varphi_J| \mathbf{1}_{A \setminus I_{\mathcal{P}}(J)} \sigma\right)}{|J|^{\frac{1}{n}}} \left\| x - m_J^\omega \right\|_{L^2(\mathbf{1}_J \omega)} \left\| \square_J^{\omega,\mathbf{b}^*} g \right\|_{L^2}^{\bigstar} \end{aligned}$$

Thus we have

$$\begin{aligned} \left|\mathsf{B}_{stop}^{A,\mathcal{P}}(f,g)\right| &\leq \sum_{J \in \Pi_2 \mathcal{P}} \frac{\mathrm{P}^\alpha\left(J, |\varphi_J| \mathbf{1}_{A \setminus I_{\mathcal{P}}(J)} \sigma\right)}{|J|^{\frac{1}{n}}} \left\| \triangle_J^{\omega,\mathbf{b}^*} x \right\|_{L^2(\omega)}^{\spadesuit} \left\| \square_J^{\omega,\mathbf{b}^*} g \right\|_{L^2(\omega)}^{\bigstar} \\ &+ \sum_{J \in \Pi_2 \mathcal{P}} \frac{\mathrm{P}_{1+\delta}^\alpha\left(J, |\varphi_J| \mathbf{1}_{A \setminus I_{\mathcal{P}}(J)} \sigma\right)}{|J|^{\frac{1}{n}}} \left\| x - m_J^\omega \right\|_{L^2(\mathbf{1}_J \omega)} \left\| \square_J^{\omega,\mathbf{b}^*} g \right\|_{L^2}^{\bigstar} \\ &\equiv |\mathsf{B}|_{stop,1,\triangle^\omega}^{A,\mathcal{P}}(f,g) + |\mathsf{B}|_{stop,1+\delta,\mathsf{P}^\omega}^{A,\mathcal{P}}(f,g), \end{aligned} \tag{5.6.6}$$

where we have dominated the stopping form by two sublinear stopping forms that involve the

Poisson integrals of order 1 and $1+\delta$ respectively, and where the smaller Poisson integral $\mathrm{P}_{1+\delta}^{\alpha}$ is multiplied by the larger quantity $\left\|x-m_J^{\omega}\right\|_{L^2}(\mathbf{1}_J\omega)$. This splitting turns out to be successful in separating the two energy terms from the right hand side of the Energy Lemma, because of the two properties (5.6.4) and (5.6.5) above. It remains to show the two inequalities:

$$\left|\mathsf{B}\right|_{stop,\triangle\omega}^{A,\mathcal{P}}(f,g)\lesssim\left(\mathcal{E}_2^{\alpha}+\sqrt{\mathfrak{A}_2^{\alpha}}\right)\left\|\mathsf{P}_{\pi(\Pi_1\mathcal{P})}^{\sigma,\mathbf{b}}f\right\|_{L^2(\sigma)}^{\bigstar}\left\|\mathsf{P}_{\Pi_2\mathcal{P}}^{\omega,\mathbf{b}^*}g\right\|_{L^2(\omega)}^{\bigstar}, \tag{5.6.7}$$

for $f\in L^2(\sigma)$ satisfying where $E_I^{\sigma}|f|\leq\alpha_{\mathcal{A}}(A)$ for all $I\in\mathcal{C}_A$; and where $\pi(\Pi_1\mathcal{P})\equiv\{\pi_{\mathcal{D}}I:I\in\Pi_1\mathcal{P}\}$; and

$$\left|\mathsf{B}\right|_{stop,1+\delta,\mathsf{P}\omega}^{A,\mathcal{P}}(f,g)\lesssim\left(\mathcal{E}_2^{\alpha}+\sqrt{\mathfrak{A}_2^{\alpha}}\right)\left\|\mathsf{P}_{\mathcal{C}_A^{\mathcal{D}}}^{\sigma,\mathbf{b}}f\right\|_{L^2(\sigma)}\left\|\mathsf{P}_{\mathcal{C}_A^{\mathcal{G},shift}}^{\omega,\mathbf{b}^*}g\right\|_{L^2(\omega)} \tag{5.6.8}$$

where we only need the case $\mathcal{P}=\mathcal{P}^A$ in this latter inequality as there is no recursion involved in treating this second sublinear form. We consider first the easier inequality (5.6.8) that does not require recursion.

5.6.1 The bound for the second sublinear inequality

Now we turn to proving (5.6.8), i.e.

$$\left|\mathsf{B}\right|_{stop,1+\delta,\mathsf{P}\omega}^{A,\mathcal{P}}(f,g)\lesssim\left(\mathcal{E}_2^{\alpha}+\sqrt{\mathfrak{A}_2^{\alpha}}\right)\left\|\mathsf{P}_{\mathcal{C}_A^{\mathcal{D}}}^{\sigma,\mathbf{b}}f\right\|_{L^2(\sigma)}\left\|\mathsf{P}_{\mathcal{C}_A^{\mathcal{G},shift}}^{\omega,\mathbf{b}^*}g\right\|_{L^2(\omega)}$$

where since

$$\left|\varphi_J\right| = \left|\sum_{I\in\mathcal{C}'_A:\ (I,J)\in\mathcal{P}} E_I^\sigma\left(\widehat{\square}_{\pi I}^{\sigma,\flat,\mathbf{b}} f\right)\ b_A \mathbf{1}_{A\setminus I}\right| \leq \sum_{I\in\mathcal{C}'_A:\ (I,J)\in\mathcal{P}} \left|E_I^\sigma\left(\widehat{\square}_{\pi I}^{\sigma,\flat,\mathbf{b}} f\right) b_A\ \mathbf{1}_{A\setminus I}\right|,$$

the sublinear form $|\mathsf{B}|_{stop,1+\delta,\mathsf{P}^\omega}^{A,\mathcal{P}}$ can be dominated and then decomposed by pigeonholing the ratio of side lengths of J and I:

$$\begin{aligned}
&|\mathsf{B}|_{stop,1+\delta}^{A,\mathcal{P}}(f,g) \\
= &\sum_{J\in\Pi_2\mathcal{P}} \frac{\mathrm{P}_{1+\delta}^\alpha\left(J, \left|\varphi_J\right| \mathbf{1}_{A\setminus I_{\mathcal{P}}(J)}\sigma\right)}{|J|^{\frac{1}{n}}} \left\|x-m_J\right\|_{L^2(\mathbf{1}_J\omega)} \left\|\square_J^{\omega,\mathbf{b}^*} g\right\|_{L^2(\omega)}^\bigstar \\
\leq &\sum_{(I,J)\in\mathcal{P}} \frac{\mathrm{P}_{1+\delta}^\alpha\left(J, \left|E_I^\sigma\left(\square_{\pi I}^{\sigma,\flat,\mathbf{b}} f\right)\right| \mathbf{1}_{A\setminus I}\sigma\right)}{|J|^{\frac{1}{n}}} \left\|x-m_J\right\|_{L^2(\mathbf{1}_J\omega)} \left\|\square_J^{\omega,\mathbf{b}^*} g\right\|_{L^2(\omega)}^\bigstar \\
\equiv &\sum_{s=0}^\infty |\mathsf{B}|_{stop,1+\delta}^{A,\mathcal{P};s}(f,g);
\end{aligned}$$

We will now adapt the argument for the stopping term starting on page 42 of [32], where the geometric gain from the assumed 'Energy Hypothesis' there will be replaced by a geometric gain from the smaller Poisson integral $\mathrm{P}_{1+\delta}^\alpha$ used here.

First, we exploit the additional decay in the Poisson integral $\mathrm{P}_{1+\delta}^\alpha$ as follows. Suppose that $(I,J)\in\mathcal{P}$ with $\ell(J)=2^{-s}\ell(I)$. We then compute

$$\begin{aligned}
\frac{\mathrm{P}_{1+\delta}^{\alpha}\left(J,\left|b_A\right|\mathbf{1}_{A\setminus I}\sigma\right)}{\left|J\right|^{\frac{1}{n}}} &\approx \int_{A\setminus I}\frac{\left|J\right|^{\frac{\delta}{n}}}{\left|y-c_J\right|^{n+1+\delta-\alpha}}\left|b_A(y)\right|d\sigma(y) \\
&\leq \int_{A\setminus I}\left(\frac{\left|J\right|^{\frac{1}{n}}}{\operatorname{dist}(c_J,I^c)}\right)^{\delta}\frac{1}{\left|y-c_J\right|^{n+1-\alpha}}\left|b_A(y)\right|d\sigma(y) \\
&\lesssim \left(\frac{\left|J\right|^{\frac{1}{n}}}{\operatorname{dist}(c_J,I^c)}\right)^{\delta}\frac{\mathrm{P}^{\alpha}\left(J,\left|b_A\right|\mathbf{1}_{A\setminus I}\sigma\right)}{\left|J\right|^{\frac{1}{n}}},
\end{aligned}$$

and using the goodness of J in I,

$$d(c_J,I^c)\geq 2\ell(I)^{1-\varepsilon}\ell(J)^{\varepsilon}\geq 2\cdot 2^{s(1-\varepsilon)}\ell(J),$$

to conclude, using accretivity, that

$$\left(\frac{\mathrm{P}_{1+\delta}^{\alpha}\left(J,\left|b_A\right|\mathbf{1}_{A\setminus I}\sigma\right)}{\left|J\right|^{\frac{1}{n}}}\right)\lesssim 2^{-s\delta(1-\varepsilon)}\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{A\setminus I}\sigma\right)}{\left|J\right|^{\frac{1}{n}}}. \tag{5.6.9}$$

We next claim that for $s\geq 0$ an integer,

$$\left|\mathsf{B}\right|_{stop,1+\delta,\mathsf{P}\omega}^{A,\mathcal{P};s}(f,g)\;\lesssim\;2^{-s\delta(1-\varepsilon)}\;\left(\mathcal{E}_2^{\alpha}+\sqrt{\mathfrak{A}_2^{\alpha}}\right)\left\|\mathsf{P}_{\mathcal{C}_A^{\mathcal{D}}}^{\sigma,\mathbf{b}}f\right\|_{L^2(\sigma)}\left\|\mathsf{P}_{\mathcal{C}_A^{\mathcal{G},shift}}^{\omega,\mathbf{b}^*}g\right\|_{L^2(\omega)}$$

from which (5.6.8) follows upon summing in $s\geq 0$. Now using both

$$\left|E_I^{\sigma}\left(\widehat{\square}_{\pi I}^{\sigma,\flat,\mathbf{b}}f\right)\right|\frac{1}{\left|I\right|_{\sigma}}\int_I\left|\square_{\pi I}^{\sigma,\flat,\mathbf{b}}f\right|d\sigma\leq\left\|\square_{\pi I}^{\sigma,\flat,\mathbf{b}}f\right\|_{L^2(\sigma)}\frac{1}{\sqrt{\left|I\right|_{\sigma}}},$$

$$\sum_{I\in\mathcal{D}}\left\|\square_{\pi I}^{\sigma,\flat,\mathbf{b}}f\right\|_{L^2(\sigma)}^2\lesssim\sum_{I\in\mathcal{D}}\left(\left\|\square_{\pi I}^{\sigma,\mathbf{b}}f\right\|_{L^2(\sigma)}^2+\left\|\nabla_{\pi I}^{\sigma}f\right\|_{L^2(\sigma)}^2\right)\approx\left\|f\right\|_{L^2(\sigma)}^2\ ,$$

we apply Cauchy-Schwarz in the I variable above to see that

$$\begin{aligned}
&\left[|\mathsf{B}|^{A,\mathcal{P};s}_{stop,1+\delta,\mathsf{P}^{\omega}}(f,g)\right]^2\\
&\lesssim\left\|\mathsf{P}^{\sigma,\mathbf{b}}_{\mathcal{C}^{\mathcal{D}}_A}f\right\|_{L^2(\sigma)}\left[\sum_{I\in\mathcal{C}'_A}\left(\frac{1}{\sqrt{|I|_\sigma}}\sum_{\substack{J:\ (I,J)\in\mathcal{P}\\ \ell(J)=2^{-s}\ell(I)}}\frac{\mathrm{P}^{\alpha}_{1+\delta}\left(J,\mathbf{1}_{A\setminus I}\sigma\right)}{|J|^{\frac{1}{n}}}\|x-m_J\|_{L^2(\mathbf{1}_J\omega)}\left\|\square^{\omega,\mathbf{b}^*}_J g\right\|^{\bigstar}_{L^2(\omega)}\right)^2\right]^{\frac{1}{2}}
\end{aligned}$$

Using the frame inequality for $\square^{\omega,\mathbf{b}^*}_J$ we can then estimate the sum inside the square brackets by

$$\begin{aligned}
&\sum_{I\in\mathcal{C}'_A}\left\{\sum_{\substack{J:\ (I,J)\in\mathcal{P}\\ \ell(J)=2^{-s}\ell(I)}}\left\|\square^{\omega,\mathbf{b}^*}_J g\right\|^{\bigstar 2}_{L^2(\omega)}\right\}\sum_{\substack{J:\ (I,J)\in\mathcal{P}\\ \ell(J)=2^{-s}\ell(I)}}\frac{1}{|I|_\sigma}\left(\frac{\mathrm{P}^{\alpha}_{1+\delta}\left(J,\mathbf{1}_{A\setminus I}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2\|x-m_J\|^2_{L^2(\mathbf{1}_J\omega)}\\
&\qquad\lesssim\quad\left\|\mathsf{P}^{\omega,\mathbf{b}^*}_{\Pi_2\mathcal{P}}g\right\|^{\bigstar 2}_{L^2(\omega)}A(s)^2,
\end{aligned}$$

where

$$A(s)^2\equiv\sup_{I\in\mathcal{C}'_A}\sum_{\substack{J:\ (I,J)\in\mathcal{P}\\ \ell(J)=2^{-s}\ell(I)}}\frac{1}{|I|_\sigma}\left(\frac{\mathrm{P}^{\alpha}_{1+\delta}\left(J,\mathbf{1}_{A\setminus I}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2\|x-m_J\|^2_{L^2(\mathbf{1}_J\omega)}$$

Finally then we turn to the analysis of the supremum in last display. From the Poisson decay (5.6.9) we have

$$\begin{aligned}
A(s)^2&\lesssim\sup_{I\in\mathcal{C}'_A}\frac{1}{|I|_\sigma}2^{-2s\delta(1-\varepsilon)}\sum_{\substack{J:\ (I,J)\in\mathcal{P}\\ \ell(J)=2^{-s}\ell(I)}}\left(\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_{A\setminus I}\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2\|x-m_J\|^2_{L^2(\mathbf{1}_J\omega)}\\
&\lesssim 2^{-2s\delta(1-\varepsilon)}\left[(\mathcal{E}^{\alpha}_2)^2+\mathfrak{A}^{\alpha}_2\right],
\end{aligned}$$

Indeed, from Definition 5.1.14, as $(I,J)\in\mathcal{P}$, we have that I is *not* a stopping cube in $\mathcal{A}$, and hence that (5.1.28) *fails* to hold, delivering the estimate above since $J\Subset_{\rho,\varepsilon} I$ good must be contained in some $K\in\mathcal{M}_{(\mathbf{r},\varepsilon)-deep}(I)$, and since $\frac{\mathrm{P}^{\alpha}\left(J,|b_I|\mathbf{1}_{A\setminus I}\sigma\right)}{|J|^{\frac{1}{n}}}\approx\frac{\mathrm{P}^{\alpha}\left(K,|b_I|\mathbf{1}_{A\setminus I}\sigma\right)}{|K|^{\frac{1}{n}}}$. The terms $\left\|\mathsf{P}^{\omega}_{J}x\right\|^2_{L^2(\omega)}$ are additive since the $J's$ are pigeonholed by $\ell(J)=2^{-s}\ell(I)$.

5.6.2 The bound for the first sublinear inequality

Now we turn to proving the more difficult inequality (5.6.7). Denote by $\mathfrak{N}^{A,\mathcal{P}}_{stop,\triangle\omega}$ the best constant in

$$|\mathsf{B}|^{A,\mathcal{P}}_{stop,\triangle\omega}(f,g)\leq\mathfrak{N}^{A,\mathcal{P}}_{stop,\triangle\omega}\left\|\mathsf{P}^{\sigma,\mathbf{b}}_{\pi(\Pi_1\mathcal{P})}f\right\|^{\bigstar}_{L^2(\sigma)}\left\|\mathsf{P}^{\omega,\mathbf{b}^*}_{\Pi_2\mathcal{P}}g\right\|^{\bigstar}_{L^2(\omega)},\tag{5.6.10}$$

where $f\in L^2(\sigma)$ satisfies $E^{\sigma}_I|f|\leq\alpha_{\mathcal{A}}(A)$ for all $I\in\mathcal{C}_A$, and $g\in L^2(\omega)$ and $\pi(\Pi_1\mathcal{P})=\{\pi I:I\in\Pi_1P\}$. We refer to $\mathfrak{N}^{A,\mathcal{P}}_{stop,\triangle\omega}$ as the *restricted* norm relative to the collection $\mathcal{P}$. Inequality (5.6.7) follows once we have shown that $\mathfrak{N}^{A,\mathcal{P}}_{stop,\triangle\omega}\lesssim\mathcal{E}^{\alpha}_2+\sqrt{\mathfrak{A}^{\alpha}_2}$.

The following general result on mutually orthogonal admissible collections will prove very useful in establishing (5.6.7). Given a set $\{\mathcal{Q}_m\}_{m=0}^{\infty}$ of admissible collections for A, we say that the collections $\mathcal{Q}_m$ are *mutually orthogonal*, if each collection $\mathcal{Q}_m$ satisfies

$$\mathcal{Q}_m\subset\bigcup_{j=0}^{\infty}\{\mathcal{A}_{m,j}\times\mathcal{B}_{m,j}\}$$

where the sets $\{\mathcal{A}_{m,j}\}_{m,j}$ and $\{\mathcal{B}_{m,j}\}_{m,j}$ are each pairwise disjoint in their respective dyadic grids $\mathcal{D}$ and $\mathcal{G}$:

$$\sum_{m,j=0}^{\infty}\mathbf{1}_{\mathcal{A}_{m,j}}\leq\mathbf{1}_{\mathcal{D}}\text{ and }\sum_{m,j=0}^{\infty}\mathbf{1}_{\mathcal{B}_{m,j}}\leq\mathbf{1}_{\mathcal{G}}.$$

Lemma 5.6.4. *Suppose that* $\{\mathcal{Q}_m\}_{m=0}^{\infty}$ *is a set of admissible collections for* A *that are* mutually orthogonal. *Then* $\mathcal{Q} \equiv \bigcup_{m=0}^{\infty} \mathcal{Q}_m$ *is admissible, and the sublinear stopping form* $|\mathsf{B}|_{stop,\triangle^\omega}^{A,\mathcal{Q}}(f,g)$ *has its restricted norm* $\mathfrak{N}_{stop,\triangle^\omega}^{A,\mathcal{Q}}$ *controlled by the* supremum *of the restricted norms* $\mathfrak{N}_{stop,\triangle^\omega}^{A,\mathcal{Q}_m}$*:*

$$\mathfrak{N}_{stop,\triangle^\omega}^{A,\mathcal{Q}} \leq \sup_{m\geq 0} \mathfrak{N}_{stop,\triangle^\omega}^{A,\mathcal{Q}_m}.$$

Proof. If $J \in \Pi_2\mathcal{Q}_m$, then $\varphi_J^{\mathcal{Q}} = \varphi_J^{\mathcal{Q}_m}$ and $I_{\mathcal{Q}}(J) = I_{\mathcal{Q}_m}(J)$, since the collection $\{\mathcal{Q}_m\}_{m=0}^{\infty}$ is mutually orthogonal. Thus we have

$$\begin{aligned}
|\mathsf{B}|_{stop,\triangle^\omega}^{A,\mathcal{Q}}(f,g) &= \sum_{J\in\Pi_2\mathcal{Q}} \frac{\mathrm{P}^\alpha\left(J,\left|\varphi_J^{\mathcal{Q}}\right|\mathbf{1}_{A\setminus I_{\mathcal{Q}}(J)}\sigma\right)}{|J|^{\frac{1}{n}}} \left\|\triangle_J^{\omega,\mathbf{b}^*}x\right\|_{L^2(\omega)}^{\spadesuit} \left\|\square_J^{\omega,\mathbf{b}^*}g\right\|_{L^2(\omega)}^{\bigstar} \\
&= \sum_{m\geq 0}\sum_{J\in\Pi_2\mathcal{Q}_m} \frac{\mathrm{P}^\alpha\left(J,\left|\varphi_J^{\mathcal{Q}_m}\right|\mathbf{1}_{A\setminus I_{\mathcal{Q}_m}(J)}\sigma\right)}{|J|^{\frac{1}{n}}} \left\|\triangle_J^{\omega,\mathbf{b}^*}x\right\|_{L^2(\omega)}^{\spadesuit} \left\|\square_J^{\omega,\mathbf{b}^*}g\right\|_{L^2(\omega)}^{\bigstar} \\
&= \sum_{m\geq 0} |\mathsf{B}|_{stop,\triangle^\omega}^{A,\mathcal{Q}_m}(f,g),
\end{aligned}$$

and we can continue with the definition of $\widehat{\mathfrak{N}}_{stop,\triangle^\omega}^{A,\mathcal{Q}_m}$ and Cauchy-Schwarz to obtain

$$\begin{aligned}
|\mathsf{B}|_{stop,\triangle^\omega}^{A,\mathcal{Q}}(f,g) &\leq \sum_{m\geq 0} \widehat{\mathfrak{N}}_{stop,\triangle^\omega}^{A,\mathcal{Q}_m} \left\|\mathsf{P}_{\pi(\Pi_1\mathcal{Q}_m)}^{\sigma,\mathbf{b}}f\right\|_{L^2(\sigma)}^{\bigstar} \left\|\mathsf{P}_{\Pi_2\mathcal{Q}_m}^{\omega,\mathbf{b}^*}g\right\|_{L^2(\omega)}^{\bigstar} \\
&\leq \left(\sup_{m\geq 0}\widehat{\mathfrak{N}}_{stop,\triangle^\omega}^{A,\mathcal{Q}_m}\right) \sqrt{\sum_{m\geq 0}\left\|\mathsf{P}_{\pi(\Pi_1\mathcal{Q}_m)}^{\sigma,\mathbf{b}}f\right\|_{L^2(\sigma)}^{\bigstar 2}} \sqrt{\sum_{m\geq 0}\left\|\mathsf{P}_{\Pi_2\mathcal{Q}_m}^{\omega,\mathbf{b}^*}g\right\|_{L^2(\omega)}^{\bigstar 2}} \\
&\leq \left(\sup_{m\geq 0}\widehat{\mathfrak{N}}_{stop,\triangle^\omega}^{A,\mathcal{Q}_m}\right) \sqrt{\left\|\mathsf{P}_{\pi(\Pi_1\mathcal{Q})}^{\sigma,\mathbf{b}}f\right\|_{L^2(\sigma)}^{\bigstar 2}} \sqrt{\left\|\mathsf{P}_{\Pi_2\mathcal{Q}}^{\omega,\mathbf{b}^*}g\right\|_{L^2(\omega)}^{\bigstar 2}}.
\end{aligned}$$

□

Now we turn to proving inequality (5.6.7) for the sublinear form $\left|\mathsf{B}\right|_{stop,\triangle^{\omega}}^{A,\mathcal{P}}(f,g)$, i.e.

$$\begin{aligned}
\left|\mathsf{B}\right|_{stop,\triangle^{\omega}}^{A,\mathcal{P}}(f,g) &\equiv \sum_{J\in\Pi_2\mathcal{P}} \frac{\mathrm{P}^{\alpha}\left(J,\left|\varphi_J\right|\mathbf{1}_{A\setminus I_{\mathcal{P}}(J)}\sigma\right)}{|J|} \left\|\triangle_J^{\omega,\mathbf{b}^*}x\right\|_{L^2(\omega)}^{\spadesuit} \left\|\square_J^{\omega,\mathbf{b}^*}g\right\|_{L^2(\omega)}^{\bigstar} \\
&\lesssim \left(\mathcal{E}_2^{\alpha}+\sqrt{\mathfrak{A}_2^{\alpha}}\right) \left\|\mathsf{P}_{\pi(\Pi_1\mathcal{P})}^{\sigma,\mathbf{b}}f\right\|_{L^2(\sigma)}^{\bigstar} \left\|\mathsf{P}_{\Pi_2\mathcal{P}}^{\omega,\mathbf{b}^*}g\right\|_{L^2(\omega)}^{\bigstar};
\end{aligned}$$

$$\text{where } \varphi_J \equiv \sum_{I\in\mathcal{C}'_A:\ (I,J)\in\mathcal{P}} \left(E_I^{\sigma}\widehat{\square}_{\pi I}^{\sigma,\flat,\mathbf{b}}f\right) b_A\ \mathbf{1}_{A\setminus I} \text{ is supported in } A\setminus I_{\mathcal{P}}(J)$$

and $I_{\mathcal{P}}(J)$ denotes the smallest cube $I\in\mathcal{D}$ for which $(I,J)\in\mathcal{P}$. We recall the stopping energy from (5.1.30),

$$\mathbf{X}_{\alpha}(\mathcal{C}_A)^2 \equiv \sup_{I\in\mathcal{C}_A} \frac{1}{|I|_{\sigma}} \sup_{I\supset\dot{\cup}J_r} \sum_{r=1}^{\infty} \left(\frac{\mathrm{P}^{\alpha}(J_r,\mathbf{1}_A\sigma)}{|J_r|}\right)^2 \left\|x-m_{J_r}\right\|_{L^2\left(\mathbf{1}_{J_r}\omega\right)}^2,$$

where the cubes $J_r\in\mathcal{G}$ are pairwise disjoint in I.

What now follows is an adaptation to our sublinear form $\left|\mathsf{B}\right|_{stop,\triangle^{\omega}}^{A,\mathcal{P}}$ of the arguments of M. Lacey in [27], together with an additional 'indented' corona construction. We have the following Poisson inequality for cubes $B\subset A\subset I$:

$$\begin{aligned}
\frac{\mathrm{P}^{\alpha}\left(A,\mathbf{1}_{I\setminus A}\sigma\right)}{|A|^{\frac{1}{n}}} &\approx \int_{I\setminus A}\frac{1}{(|y-c_A|)^{n+1-\alpha}}d\sigma(y) \\
&\lesssim \int_{I\setminus A}\frac{1}{(|y-c_B|)^{n+1-\alpha}}d\sigma(y) \approx \frac{\mathrm{P}^{\alpha}\left(B,\mathbf{1}_{I\setminus A}\sigma\right)}{|B|^{\frac{1}{n}}}
\end{aligned} \tag{5.6.11}$$

where the implied constants depend on n,α.

Fix $A\in\mathcal{A}$. Following [27] we will use a 'decoupled' modification of the stopping energy

$\mathbf{X}_\alpha(\mathcal{C}_A)$ to define a 'size functional' of an A-admissible collection $\mathcal{P}$. So suppose that $\mathcal{P}$ is an A-admissible collection of pairs of cubes, and recall that $\Pi_1\mathcal{P}$ and $\Pi_2\mathcal{P}$ denote the cubes in the first and second components of the pairs in $\mathcal{P}$ respectively.

Definition 5.6.5. *For an A-admissible collection of pairs of cubes $\mathcal{P}$, and a cube $K \in \Pi_1\mathcal{P}$, define the projection of $\mathcal{P}$ 'relative to K' by*

$$\Pi_2^K\mathcal{P} \equiv \left\{ J \in \Pi_2\mathcal{P} : \ J^{\maltese} \subset K \right\},$$

where we have suppressed dependence on A.

Definition 5.6.6. *We will use as the 'size testing collection' of cubes for $\mathcal{P}$ the collection*

$$\Pi_1^{below}\mathcal{P} \equiv \left\{ K \in \mathcal{D} : K \subset I \text{ for some } I \in \Pi_1\mathcal{P} \right\},$$

which consists of all cubes contained in a cube from $\Pi_1\mathcal{P}$.

Continuing to follow Lacey [27], we define two 'size functionals' of $\mathcal{P}$ as follows. Recall that for a pseudoprojection $\mathsf{Q}_{\mathcal{H}}^{\omega}$ on x we have

$$\begin{aligned}
\left\| \mathsf{Q}_{\mathcal{H}}^{\omega,\mathbf{b}^*} x \right\|_{L^2(\omega)}^{\spadesuit 2} &= \sum_{J\in\mathcal{H}} \left\| \triangle_J^{\omega,\mathbf{b}^*} x \right\|_{L^2(\omega)}^{\spadesuit 2} \\
&= \sum_{J\in\mathcal{H}} \left(\left\| \triangle_J^{\omega,\mathbf{b}^*} x \right\|_{L^2(\omega)}^{2} + \inf_{z\in\mathbb{R}^n} \sum_{J'\in\mathfrak{C}_{brok}(J)} \left| J' \right|_\omega \left(E_{J'}^{\omega} \left| x - z \right| \right)^2 \right)
\end{aligned}$$

Definition 5.6.7. *If $\mathcal{P}$ is A-admissible, define an* initial *size condition $\mathcal{S}_{initsize}^{\alpha,A}(\mathcal{P})$ by*

$$\mathcal{S}_{initsize}^{\alpha,A}(\mathcal{P})^2 \equiv \sup_{K\in\Pi_1^{below}\mathcal{P}} \frac{1}{|K|_\sigma} \left(\frac{\mathrm{P}^\alpha\left(K, \mathbf{1}_{A\setminus K}\sigma\right)}{|K|^{\frac{1}{n}}} \right)^2 \left\| \mathsf{Q}_{\Pi_2^K\mathcal{P}}^{\omega,\mathbf{b}^*} x \right\|_{L^2(\omega)}^{\spadesuit 2}. \tag{5.6.12}$$

The following key fact is essential.

Key Fact #1:

$$\text{If } K \subset A \text{ and } K \notin \mathcal{C}_A, \text{ then } \Pi_2^K \mathcal{P} = \emptyset \ . \tag{5.6.13}$$

To see this, suppose that $K \subset A$ and $K \notin \mathcal{C}_A$. Then $K \subset A'$ for some $A' \in \mathfrak{C}_{\mathcal{A}}(A)$, and so if there is $J' \in \Pi_2^K \mathcal{P}$, then $(J')^{\maltese} \subset K \subset A'$, which implies that $J' \notin \mathcal{C}_A^{\mathcal{G},shift}$, which contradicts $\Pi_2^K \mathcal{P} \subset \mathcal{C}_A^{\mathcal{G},shift}$. We now observe from (5.6.13) that we may also write the initial size functional as

$$\mathcal{S}_{initsize}^{\alpha,A}(\mathcal{P})^2 \equiv \sup_{K \in \Pi_1^{below}\mathcal{P} \cap \mathcal{C}'_A} \frac{1}{|K|_\sigma} \left(\frac{\mathrm{P}^\alpha \left(K, \mathbf{1}_{A \setminus K} \sigma \right)}{|K|^{\frac{1}{n}}} \right)^2 \left\| \mathsf{Q}_{\Pi_2^K \mathcal{P}}^{\omega,\mathbf{b}^*} x \right\|_{L^2(\omega)}^{\spadesuit 2} . \tag{5.6.14}$$

However, we will also need to control certain pairs $(I, J) \in \mathcal{P}$ using testing cubes K which are strictly smaller than $J^{\maltese}$, namely those $K \in \mathcal{C}_A$ such that $K \subset J^{\maltese} \subset \pi_{\mathcal{D}}^{(2)} K$. For this, we need a second key fact regarding the cubes $J^{\maltese}$, that will also play a crucial role in controlling pairs in the indented corona below, and which is that J is always contained in one of the *inner* 2^n grandchildren of $J^{\maltese}$. For $M \in \mathcal{D}$, denote by $M_{\searrow}$ and $M_{\nearrow}$ any of the inner and outer respectively grandchildren of M and by M_J and $M^\flat$ the child and grandchild respectively of M that contains J, provided they exist.

Key Fact #2:

$$3J \subset J^\flat \text{ and } J^\flat \text{ is an inner grandchild of } J^{\maltese} \tag{5.6.15}$$

To see this, suppose that the child $J_J^{\maltese}$ of $J^{\maltese}$ contains J ($J_J^{\maltese}$ exists because J is good in

$J^{\maltese}$). Then observe that J is by definition $\varepsilon - bad$ in $J^{\maltese}_J$, i.e.

$$\operatorname{dist}\left(J, \operatorname{body} J^{\maltese}_J\right) \leq 2\left|J\right|^{\frac{\varepsilon}{n}}\left|J^{\maltese}_J\right|^{\frac{1-\varepsilon}{n}}$$

and so cannot lie in any of the $4^n - 2^n$ outermost grandchildren $J^{\maltese}_{\nearrow}$. Indeed, if $J \subset J^{\maltese}_{\nearrow}$, then

$$\begin{aligned}
\operatorname{dist}\left(J, \operatorname{body} J^{\maltese}\right) &= \operatorname{dist}\left(J, \operatorname{body} J^{\maltese}_J\right) \leq 2\left|J\right|^{\frac{\varepsilon}{n}}\left|J^{\maltese}_J\right|^{\frac{1-\varepsilon}{n}} \\
&= 2^{\varepsilon}\left|J\right|^{\frac{\varepsilon}{n}}\left|J^{\maltese}\right|^{\frac{1-\varepsilon}{n}} < 2\left|J\right|^{\frac{\varepsilon}{n}}\left|J^{\maltese}\right|^{\frac{1-\varepsilon}{n}}
\end{aligned}$$

contradicting the fact that J is $\varepsilon - good$ in $J^{\maltese}$. Thus we must have $J \subset J^{\flat}$, and of course we get that $J^{\flat}$ is an inner grandchild of $J^{\maltese}$, (where the body of $J^{\maltese}$ does not intersect the interior of $J^{\flat}$, thus permitting J to be $\varepsilon - good$ in $J^{\maltese}$). Finally, the fact that J is $\varepsilon - good$ in $J^{\maltese}$ implies that $3J \subset J^{\flat}$.

This second key fact is what underlies the construction of the indented corona below, and motivates the next definition of augmented projection, in which we allow cubes K satisfying $J \subset K \subsetneqq J^{\maltese} \subset \pi^{(2)}_{\mathcal{D}} K$, as well as $K \in C_A$, to be tested over in the augmented size condition below.

Definition 5.6.8. *Suppose $\mathcal{P}$ is an A-admissible collection.*

(1). For $K \in \Pi_1\mathcal{P}$, define the augmented *projection of $\mathcal{P}$ relative to K by*

$$\Pi_2^{K,aug}\mathcal{P} \equiv \left\{J \in \Pi_2\mathcal{P} : J \subset K \text{ and } J^{\maltese} \subset \pi^{(2)}_{\mathcal{D}} K\right\}.$$

(2). Define the corresponding augmented *size functional* $\mathcal{S}_{augsize}^{\alpha,A}(\mathcal{P})$ *by*

$$\mathcal{S}_{augsize}^{\alpha,A}(\mathcal{P})^2 \equiv \sup_{K\in\Pi_1^{below}\mathcal{P}\cap\mathcal{C}'_A} \frac{1}{|K|_\sigma}\left(\frac{\mathrm{P}^\alpha\left(K,\mathbf{1}_{A\setminus K}\sigma\right)}{|K|}\right)^2 \left\|\mathsf{Q}^{\omega,\mathbf{b}^*}_{\Pi_2^{K,aug}\mathcal{P}}x\right\|^{\spadesuit 2}_{L^2(\omega)}$$

We note that the augmented projection $\Pi_2^{K,aug}\mathcal{P}$ includes cubes J for which $J\subset K\subsetneqq J^{\maltese}\subset\pi_{\mathcal{D}}^{(2)}K$, and hence J need not be $\varepsilon-good$ inside K. Then by the second key fact (5.6.15), and using that the boundaries of $J^{\maltese}_{\searrow}$ lie in the *body* of $J^{\maltese}$, we have two consequences,

$$K\in\left\{J^{\maltese}_{J},J^\flat\right\} \text{ and } 3J\subset J^\flat\subset 3J^\flat\subset J^{\maltese} \text{ for } J\in\Pi_2^{K,aug}\mathcal{P},$$

which will play an important role below.

The augmented size functional $\mathcal{S}_{augsize}^{\alpha,A}(\mathcal{P})$ is a 'decoupled' form of the stopping energy $\mathbf{X}_\alpha(\mathcal{C}_A)$ restricted to $\mathcal{P}$, in which the cubes J appearing in $\mathbf{X}_\alpha(\mathcal{C}_A)$ no longer appear in the Poisson integral in $\mathcal{S}_{augsize}^{\alpha,A}(\mathcal{P})$, and it plays a crucial role in Lacey's argument in [27]. We note two essential properties of this definition of size functional:

1. **Monotonicity** of size: $\mathcal{S}_{augsize}^{\alpha,A}(\mathcal{P})\leq\mathcal{S}_{augsize}^{\alpha,A}(\mathcal{Q})$ if $\mathcal{P}\subset\mathcal{Q}$,
2. **Control** by energy and Muckenhoupt conditions: $\mathcal{S}_{augsize}^{\alpha,A}(\mathcal{P})\lesssim\mathcal{E}_2^\alpha+\sqrt{\mathfrak{A}_2^\alpha}$.

The monotonicity property follows from $\Pi_1^{below}\mathcal{P}\subset\Pi_1^{below}\mathcal{Q}$ and $\Pi_2^K\mathcal{P}\subset\Pi_2^K\mathcal{Q}$. The control property is contained in the next lemma, which uses the stopping energy control for the form $\mathsf{B}_{stop}^A(f,g)$ associated with A.

Lemma 5.6.9. *If* $\mathcal{P}^A$ *is as in (5.6.2) and* $\mathcal{P}\subset\mathcal{P}^A$*, then*

$$\mathcal{S}_{augsize}^{\alpha,A}(\mathcal{P})\leq\mathbf{X}_\alpha(\mathcal{C}_A)\lesssim\mathcal{E}_2^\alpha+\sqrt{\mathfrak{A}_2^\alpha}.$$

Proof. We have

$$\begin{aligned}\mathcal{S}_{augsize}^{\alpha,A}(\mathcal{P})^2 &= \sup_{K\in\Pi_1^{below}\mathcal{P}\cap\mathcal{C}'_A} \frac{1}{|K|_\sigma}\left(\frac{\mathrm{P}^\alpha\left(K,\mathbf{1}_{A\setminus K}\sigma\right)}{|K|^{\frac{1}{n}}}\right)^2 \left\|\mathsf{Q}^{\omega,\mathbf{b}^*}_{\Pi_2^K\mathcal{P}\cup\Pi_2^{K,aug}\mathcal{P}}x\right\|^{\spadesuit 2}_{L^2(\omega)}\\ &\lesssim \sup_{K\in\mathcal{C}'_A}\frac{1}{|K|_\sigma}\left(\frac{\mathrm{P}^\alpha\left(K,\mathbf{1}_A\sigma\right)}{|K|^{\frac{1}{n}}}\right)^2 \|x-m_K\|^2_{L^2(\mathbf{1}_K\omega)} \leq \mathbf{X}_\alpha(\mathcal{C}_A)^2,\end{aligned}$$

which is the first inequality in the statement of the lemma. The second inequality follows from (5.1.31). □

There is an important special circumstance, introduced by M. Lacey in [27], in which we can bound our forms by the size functional, namely when the pairs all straddle a subpartition of A, and we present this in the next subsection. In order to handle the fact that the cubes in $\Pi_1^{below}\mathcal{P}\cap\mathcal{C}'_A$ need no longer enjoy any goodness, we will need to formulate a Substraddling Lemma to deal with this situation as well. See **Remark on lack of usual goodness** after (5.6.41), where it is explained how this applies to the proof of (5.6.40). Then in the following subsection, we use the bottom/up stopping time construction of M. Lacey, together with an additional 'indented' top/down corona construction, to reduce control of the sublinear stopping form $|\mathsf{B}|^{A,\mathcal{P}}_{stop,\triangle^\omega}(f,g)$ in inequality (5.6.7) to the three special cases addressed by the Orthogonality Lemma, the Straddling Lemma and the Substraddling Lemma.

5.6.3 ♭Straddling, Substraddling, Corona-Straddling Lemmas

We begin with the Corona-straddling Lemma in which the straddling collection is the set of $\mathcal{A}$-children of A, and applies to the 'corona straddling' subcollection of the initial admissible collection $\mathcal{P}^A$ - see (5.6.2). Define the 'corona straddling' collection $\mathcal{P}^A_{cor}$ by

$$\mathcal{P}^A_{cor} \equiv \bigcup_{A' \in \mathfrak{C}_{\mathcal{A}}(A)} \left\{ (I,J) \in \mathcal{P}^A : J \subset A' \subsetneqq J^{\maltese} \subset \pi^{(2)}_{\mathcal{D}} A' \right\}. \tag{5.6.16}$$

Note that $\mathcal{P}^A_{cor}$ is an A-admissible collection that consists of just those pairs (I,J) for which $J^{\maltese}$ is either the $\mathcal{D}$-parent or the $\mathcal{D}$ -grandparent of a stopping cube $A' \in \mathfrak{C}_{\mathcal{A}}(A)$. The bound for the norm of the corresponding form is controlled by the energy condition.

Lemma 5.6.10. *We have the sublinear form bound*

$$\mathfrak{N}^{A,\mathcal{P}^A_{cor}}_{stop,\triangle^\omega} \leq C\mathcal{E}^\alpha_2 .$$

Proof. The key point here is our assumption that $J \subset A' \subsetneqq J^{\maltese} \subset \pi^{(2)}_{\mathcal{D}} A'$ for $(I,J) \in \mathcal{P}^A_{cor}$, which implies that in fact $3J \subset A'$ since $J \cap \text{body}\left(\pi^{(2)}_{\mathcal{D}} A'\right) = \emptyset$ because J is $\varepsilon - good$ in $\pi^{(2)}_{\mathcal{D}} A'$. We start with

$$|\mathsf{B}|^{A,\mathcal{P}^A_{cor}}_{stop,\triangle^\omega}(f,g) = \sum_{J \in \Pi_2 \mathcal{P}^A_{cor}} \frac{\mathrm{P}^\alpha\left(J, \left|\varphi^{\mathcal{P}^A_{cor}}_J\right| \mathbf{1}_{A \setminus I_{\mathcal{P}^A_{cor}}(J)} \sigma\right)}{|J|} \left\|\triangle^{\omega,\mathbf{b}^*}_J x\right\|^{\spadesuit}_{L^2(\omega)} \left\|\square^{\omega,\mathbf{b}^*}_J g\right\|^{\bigstar}_{L^2(\omega)}$$

$$= \sum_{A' \in \mathfrak{C}_{\mathcal{A}}(A)} \sum_{\substack{J \in \Pi_2 \mathcal{P}^A_{cor} \\ 3J \subset A'}} \frac{\mathrm{P}^\alpha\left(J, \left|\varphi^{\mathcal{P}^A_{cor}}_J\right| \mathbf{1}_{A \setminus I_{\mathcal{P}^A_{cor}}(J)} \sigma\right)}{|J|} \left\|\triangle^{\omega,\mathbf{b}^*}_J x\right\|^{\spadesuit}_{L^2(\omega)} \left\|\square^{\omega,\mathbf{b}^*}_J g\right\|^{\bigstar}_{L^2(\omega)}$$

where

$$\varphi^{\mathcal{P}^A_{cor}}_J \equiv \sum_{I \in \Pi_1 \mathcal{P}^A_{cor}:\ (I,J) \in \mathcal{P}^A_{cor}} b_A E^\sigma_I \left(\widehat{\square}^{\sigma,\flat,\mathbf{b}}_{\pi I} f\right) \mathbf{1}_{A \setminus I} .$$

If $J \in \Pi_2 \mathcal{P}_{cor}^A$ and $J \subset A' \in \mathfrak{C}_{\mathcal{A}}(A)$, then either $A' = J^\flat$ or $A' = J_J^{\maltese}$ and we have

$$\frac{\mathrm{P}^\alpha\left(J, \mathbf{1}_{A \setminus I_{\mathcal{P}_{cor}^A}(J)}\sigma\right)}{|J|^{\frac{1}{n}}} \approx \begin{cases} \frac{\mathrm{P}^\alpha\left(A', \mathbf{1}_{A \setminus I_{\mathcal{P}_{cor}^A}}\sigma\right)}{|A'|^{\frac{1}{n}}} \leq \frac{\mathrm{P}^\alpha\left(A', \mathbf{1}_A \sigma\right)}{|A'|^{\frac{1}{n}}} & \text{if} \quad A' = J^\flat \\ \frac{\mathrm{P}^\alpha\left(A'_J, \mathbf{1}_{A \setminus I_{\mathcal{P}_{cor}^A}}\sigma\right)}{\left|A'_J\right|^{\frac{1}{n}}} \lesssim \frac{\mathrm{P}^\alpha\left(A', \mathbf{1}_A \sigma\right)}{|A'|^{\frac{1}{n}}} & \text{if} \quad A' = J_J^{\maltese} \end{cases}$$

Since $\left|\varphi_J^{\mathcal{P}_{cor}^A}\right| \lesssim \alpha_{\mathcal{A}}(A)\, \mathbf{1}_A$ by (5.6.4), we can then bound $|\mathsf{B}|_{stop,\triangle^\omega}^{A,\mathcal{P}_{cor}^A}(f,g)$ by

$$\begin{aligned} & \alpha_{\mathcal{A}}(A) \sum_{A' \in \mathfrak{C}_{\mathcal{A}}(A)} \left(\frac{\mathrm{P}^\alpha\left(A', \mathbf{1}_A \sigma\right)}{|A'|^{\frac{1}{n}}}\right) \left\|\mathsf{Q}_{\Pi_2 \mathcal{P}_{cor}^A; A'}^{\omega, \mathbf{b}^*} x\right\|_{L^2(\omega)}^{\spadesuit} \left\|\mathsf{P}_{\Pi_2 \mathcal{P}_{cor}^A; A'}^{\omega, \mathbf{b}^*} g\right\|_{L^2(\omega)}^{\bigstar} \\ \leq \; & \alpha_{\mathcal{A}}(A) \left(\sum_{A' \in \mathfrak{C}_{\mathcal{A}}(A)} \left(\frac{\mathrm{P}^\alpha\left(A', \mathbf{1}_A \sigma\right)}{|A'|^{\frac{1}{n}}}\right)^2 \left\|x - m_{A'}^\sigma\right\|_{L^2\left(\mathbf{1}_{A'}\sigma\right)}^2\right)^{\frac{1}{2}} \cdot \\ & \qquad \cdot \left(\sum_{A' \in \mathfrak{C}_{\mathcal{A}}(A)} \left\|\mathsf{P}_{\Pi_2 \mathcal{P}_{cor}^A; A'}^{\omega, \mathbf{b}^*} g\right\|_{L^2(\omega)}^{\bigstar 2}\right)^{\frac{1}{2}} \\ \leq \; & \mathcal{E}_2^\alpha \alpha_{\mathcal{A}}(A) \sqrt{|A|_\sigma} \left\|\mathsf{P}_{\Pi_2 \mathcal{P}_{cor}^A}^{\omega, \mathbf{b}^*} g\right\|_{L^2(\omega)}^{\bigstar} \\ \leq \; & \mathcal{E}_2^\alpha \alpha_{\mathcal{A}}(A) \sqrt{|A|_\sigma} \left\|\mathsf{P}_{\mathcal{C}_A^{shift}}^{\omega, \mathbf{b}^*} g\right\|_{L^2(\omega)}^{\bigstar} \end{aligned}$$

where in the last line we have used the strong energy constant $\mathcal{E}_2^\alpha$ in (5.1.8). □

Definition 5.6.11. *We say that an admissible collection of pairs $\mathcal{P}$ is* reduced *if it contains no pairs from $\mathcal{P}_{cor}^A$, i.e.*

$$\mathcal{P} \cap \mathcal{P}_{cor}^A = \emptyset.$$

Recall that in terms of $J^{\flat}$ we rewrite

$$\begin{aligned}\Pi_2^{K,aug}\mathcal{P} &= \left\{J \in \Pi_2\mathcal{P} : J \subset K \text{ and } J^{\maltese} \subset \pi_{\mathcal{D}}^{(2)}K\right\} \\ &= \left\{J \in \Pi_2\mathcal{P} : J \subset K \text{ and } J^{\flat} \subset K\right\}\end{aligned}$$

Definition 5.6.12. *Given a* reduced admissible *collection of pairs $\mathcal{Q}$ for A, and a subpartition $\mathcal{S} \subset \Pi_1^{below}\mathcal{Q} \cap \mathcal{C}'_A$ of pairwise disjoint cubes in A, we say that $\mathcal{Q}$ $\flat$* ***straddles*** *$\mathcal{S}$ if for every pair $(I,J) \in \mathcal{Q}$ there is $S \in \mathcal{S} \cap [J,I]$ with $J^{\flat} \subset S$. To avoid trivialities, we further assume that for every $S \in \mathcal{S}$, there is at least one pair $(I,J) \in \mathcal{Q}$ with $J^{\flat} \subset S \subset I$. Here $[J,I]$ denotes the geodesic in the dyadic tree $\mathcal{D}$ that connects $J^{\mathcal{D}}$ to I, where $J^{\mathcal{D}}$ is the minimal cube in $\mathcal{D}$ that contains J.*

Definition 5.6.13. *For any dyadic cube $S \in \mathcal{D}$, define the Whitney collection $\mathcal{W}(S)$ to consist of the maximal subcubes K of S whose triples $3K$ are contained in S. Then set $\mathcal{W}^*(S) \equiv \mathcal{W}(S) \cup \{S\}$.*

The following geometric proposition will prove useful in proving the $\flat$ Straddling Lemma 5.6.15 below. For $S \in \mathcal{S}$, let $\mathcal{Q}^S \equiv \left\{(I,J) \in \mathcal{Q} : J^{\flat} \subset S \subset I\right\}$.

Proposition 5.6.14. *Suppose $\mathcal{Q}$ is reduced admissible and $\flat$ straddles a subpartition $\mathcal{S}$ of A. Fix $S \in \mathcal{S}$. Define*

$$\varphi_J^{\mathcal{Q}^S}[h] \equiv \sum_{I \in \Pi_1\mathcal{Q}^S :\ (I,J) \in \mathcal{Q}^S} b_A E_I^{\sigma}\left(\widehat{\square}_{\pi I}^{\sigma,\flat,\mathbf{b}} h\right) \mathbf{1}_{A\setminus I} ,$$

assume that $h \in L^2(\sigma)$ is supported in the cube A, and that there is a cube $H \in \mathcal{C}_A$ with

$H \supset S$ such that

$$E_I^\sigma |h| \leq C E_H^\sigma |h|, \qquad \text{for all } I \in \Pi_1^{below} \mathcal{Q} \cap \mathcal{C}'_A \text{ with } I \supset S.$$

Then

$$\begin{aligned}
&\sum_{J \in \Pi_2 \mathcal{Q}:\ J^\flat \subset S} \frac{\mathrm{P}^\alpha\left(J, \left|\varphi_J^{\mathcal{Q}}[h]\right| \mathbf{1}_{A \setminus I_{\mathcal{Q}}(J)} \sigma\right)}{|J|} \left\| \triangle_J^{\omega,\mathbf{b}^*} x \right\|_{L^2(\omega)}^\spadesuit \left\| \square_J^{\omega,\mathbf{b}^*} g \right\|_{L^2(\omega)}^\bigstar \\
\lesssim\ & \alpha_{\mathcal{H}}(H) \frac{\mathrm{P}^\alpha\left(S, \mathbf{1}_{A \setminus S} \sigma\right)}{|S|} \left\| \mathsf{Q}_{\Pi_2^{S,aug} \mathcal{Q}}^{\omega,\mathbf{b}^*} x \right\|_{L^2(\omega)}^\spadesuit \left\| \mathsf{P}_{\Pi_2^{S,aug} \mathcal{Q}}^{\omega,\mathbf{b}^*} g \right\|_{L^2(\omega)}^\bigstar \\
&+ \alpha_{\mathcal{H}}(H) \sum_{K \in \mathcal{W}(S)} \frac{\mathrm{P}^\alpha\left(K, \mathbf{1}_{A \setminus K} \sigma\right)}{|K|} \left\| \mathsf{Q}_{\Pi_2^{K,aug} \mathcal{Q}}^{\omega,\mathbf{b}^*} x \right\|_{L^2(\omega)}^\spadesuit \left\| \mathsf{P}_{\Pi_2^{K,aug} \mathcal{Q}}^{\omega,\mathbf{b}^*} g \right\|_{L^2(\omega)}^\bigstar .
\end{aligned}$$

The sum over Whitney cubes $K \in \mathcal{W}(S)$ is only required to bound the sum of those terms on the left for which $J^\flat \subset S''$ for some $S'' \in \mathfrak{C}_{\mathcal{D}}^{(2)}(S)$.

Proof. Suppose first that $J^\flat = S \in \mathcal{C}'_A$. Then $3S = 3J^\flat \subset J^{\maltese} \subset I_{\mathcal{Q}}(J)$ and using (5.6.4) with $\alpha_{\mathcal{H}}(H)$ in place of $\alpha_{\mathcal{A}}(A)$, we have

$$\begin{aligned}
\frac{\mathrm{P}^\alpha\left(J, \left|\varphi_J^{\mathcal{Q}}\right| \mathbf{1}_{A \setminus I_{\mathcal{Q}}(J)} \sigma\right)}{|J|^{\frac{1}{n}}} &\lesssim \alpha_{\mathcal{H}}(H) \frac{\mathrm{P}^\alpha\left(J, \mathbf{1}_{A \setminus J^{\maltese}} \sigma\right)}{|J|^{\frac{1}{n}}} \\
&\lesssim \alpha_{\mathcal{H}}(H) \frac{\mathrm{P}^\alpha\left(S, \mathbf{1}_{A \setminus J^{\maltese}} \sigma\right)}{|S|^{\frac{1}{n}}} \leq \alpha_{\mathcal{H}}(H) \frac{\mathrm{P}^\alpha\left(S, \mathbf{1}_{A \setminus S} \sigma\right)}{|S|^{\frac{1}{n}}}.
\end{aligned}$$

Suppose next that $J^\flat = S' \in \mathfrak{C}_{\mathcal{D}}(S)$. Then $3S' = 3J^\flat \subset J^{\maltese} \subset I_{\mathcal{Q}}(J)$ and (5.6.4) give

$$\begin{aligned}
\frac{\mathrm{P}^\alpha\left(J, \left|\varphi_J^{\mathcal{Q}}\right| \mathbf{1}_{A \setminus I_{\mathcal{Q}}(J)}\sigma\right)}{|J|^{\frac{1}{n}}} &\lesssim \alpha_{\mathcal{H}}(H) \frac{\mathrm{P}^\alpha\left(J, \mathbf{1}_{A \setminus J^{\maltese}}\sigma\right)}{|J|^{\frac{1}{n}}} \\
&\lesssim \alpha_{\mathcal{H}}(H) \frac{\mathrm{P}^\alpha\left(S', \mathbf{1}_{A \setminus J^{\maltese}}\sigma\right)}{|S'|^{\frac{1}{n}}} \\
&\leq \alpha_{\mathcal{H}}(H) \frac{\mathrm{P}^\alpha\left(S', \mathbf{1}_{A \setminus S}\sigma\right)}{|S'|^{\frac{1}{n}}} \approx \alpha_{\mathcal{H}}(H) \frac{\mathrm{P}^\alpha\left(S, \mathbf{1}_{A \setminus S}\sigma\right)}{|S|^{\frac{1}{n}}}.
\end{aligned}$$

Thus in these two cases, by Cauchy-Schwarz, the left hand side of our conclusion is bounded by a multiple of

$$\alpha_{\mathcal{H}}(H) \frac{\mathrm{P}^\alpha\left(S, \mathbf{1}_{A \setminus S}\sigma\right)}{|S|^{\frac{1}{n}}} \left(\sum_{\substack{J \in \Pi_2 \mathcal{Q} \\ J^\flat \subset S}} \left\| \triangle_J^{\omega, \mathbf{b}^*} x \right\|_{L^2(\omega)}^{\spadesuit 2} \right)^{\frac{1}{2}} \left(\sum_{\substack{J \in \Pi_2 \mathcal{Q} \\ J^\flat \subset S}} \left\| \square_J^{\omega, \mathbf{b}^*} g \right\|_{L^2(\omega)}^{\bigstar 2} \right)^{\frac{1}{2}}$$

$$= \alpha_{\mathcal{H}}(H) \frac{\mathrm{P}^\alpha\left(S, \mathbf{1}_{A \setminus S}\sigma\right)}{|S|^{\frac{1}{n}}} \left\| \mathsf{Q}^{\omega, \mathbf{b}^*}_{\Pi_2^{S,aug} \mathcal{Q}} x \right\|_{L^2(\omega)}^{\spadesuit} \left\| \mathsf{P}^{\omega, \mathbf{b}^*}_{\Pi_2^{S,aug} \mathcal{Q}} g \right\|_{L^2(\omega)}^{\bigstar}$$

Finally, suppose that $J^\flat \subset S''$ for some $S'' \in \mathfrak{C}^{(2)}_{\mathcal{D}}(S)$. Then $J^{\maltese} \subset S$, and Key Fact #2 in (5.6.15) shows that $3J^\flat \subset J^{\maltese}$, so that $3J^\flat \subset J^{\maltese} \subset S \subset I_{\mathcal{Q}}(J)$. Thus we have $J^\flat \subset K = K[J]$ for some $K \in \mathcal{W}(S)$ and so by (5.6.4) again,

$$\begin{aligned}
\frac{\mathrm{P}^\alpha\left(J, \left|\varphi_J^{\mathcal{Q}}\right| \mathbf{1}_{A \setminus I_{\mathcal{Q}}(J)}\sigma\right)}{|J|^{\frac{1}{n}}} &\lesssim \alpha_{\mathcal{H}}(H) \frac{\mathrm{P}^\alpha\left(J, \mathbf{1}_{A \setminus S}\sigma\right)}{|J|^{\frac{1}{n}}} \\
&\lesssim \alpha_{\mathcal{H}}(H) \frac{\mathrm{P}^\alpha\left(K, \mathbf{1}_{A \setminus S}\sigma\right)}{|K|^{\frac{1}{n}}} \leq \alpha_{\mathcal{H}}(H) \frac{\mathrm{P}^\alpha\left(K, \mathbf{1}_{A \setminus K}\sigma\right)}{|K|^{\frac{1}{n}}}.
\end{aligned}$$

Now we apply Cauchy-Schwarz again, but noting that $J^\flat \subset K$ this time, to obtain that the left hand side of our conclusion is bounded by a multiple of

$$\alpha_{\mathcal{H}}(H)\sum_{K\in\mathcal{W}(S)}\frac{\mathrm{P}^{\alpha}\left(K,\mathbf{1}_{A\setminus K}\sigma\right)}{|K|^{\frac{1}{n}}}\left(\sum_{\substack{J\in\Pi_2\mathcal{Q}\\ J^{\flat}\subset K}}\left\|\triangle_J^{\omega,\mathbf{b}^*}x\right\|_{L^2(\omega)}^{\spadesuit 2}\right)^{\frac{1}{2}}\left(\sum_{\substack{J\in\Pi_2\mathcal{Q}\\ J^{\flat}\subset K}}\left\|\square_J^{\omega,\mathbf{b}^*}g\right\|_{L^2(\omega)}^{\bigstar 2}\right)^{\frac{1}{2}}$$

$$=\alpha_{\mathcal{H}}(H)\sum_{K\in\mathcal{W}(S)}\frac{\mathrm{P}^{\alpha}\left(K,\mathbf{1}_{A\setminus K}\sigma\right)}{|K|^{\frac{1}{n}}}\left\|\mathsf{Q}^{\omega,\mathbf{b}^*}_{\Pi_2^{K,aug}\mathcal{Q}}x\right\|_{L^2(\omega)}^{\spadesuit}\left\|\mathsf{P}^{\omega,\mathbf{b}^*}_{\Pi_2^{K,aug}\mathcal{Q}}g\right\|_{L^2(\omega)}^{\bigstar}.$$

This completes the proof of Proposition 5.6.14. $\square$

Recall the family of operators $\left\{\square_I^{\sigma,\pi,\mathbf{b}}\right\}_{I\in\mathcal{C}_A^{\mathcal{A}}}$, where for $I\in\mathcal{C}_A^{\mathcal{A}}$, the dual martingale difference $\square_I^{\sigma,\pi,\mathbf{b}}$ is defined in (5.1.41), and satisfies

$$\square_I^{\sigma,\pi,\mathbf{b}}f=\left[\sum_{I'\in\mathfrak{C}(I)}\mathbb{F}_{I'}^{\sigma,\pi,\mathbf{b}}f\right]-\mathbb{F}_I^{\sigma,\mathbf{b}}f=\sum_{I'\in\mathfrak{C}(I)}\mathbb{F}_{I'}^{\sigma,b_A}f-\mathbb{F}_I^{\sigma,b_A}f\ .$$

Since $\square_I^{\sigma,\pi,\mathbf{b}}$ is the transpose of $\triangle_I^{\sigma,\pi,\mathbf{b}}$ for $I\in\mathcal{C}_A^{\mathcal{A}}$, the first line of Lemma 5.1.22 (where the superscript π is suppressed for convenience) shows that $\left\{\square_I^{\sigma,\pi,\mathbf{b}}\right\}_{I\in\mathcal{C}_A^{\mathcal{A}}}$ is a family of projections, and the second line of Lemma 5.1.22 shows it is an orthogonal family, i.e.

$$\square_I^{\sigma,\pi,\mathbf{b}}\square_K^{\sigma,\pi,\mathbf{b}}=\begin{cases}\square_I^{\sigma,\pi,\mathbf{b}} & \text{if} \quad I=K\\ 0 & \text{if} \quad I\neq K\end{cases},\qquad I,K\in\mathcal{C}_A^{\mathcal{A}}.$$

The orthogonal projections

$$\mathsf{P}^{\sigma,\pi,\mathbf{b}}_{\pi(\Pi_1\mathcal{Q})}\equiv\sum_{I\in\pi(\Pi_1\mathcal{Q})}\square_I^{\sigma,\pi,\mathbf{b}}=\sum_{I\in\Pi_1\mathcal{Q}}\square_{\pi I}^{\sigma,\pi,\mathbf{b}},$$

$$\text{where } \pi(\Pi_1\mathcal{Q})\equiv\left\{\pi_{\mathcal{D}}I:I\in\Pi_1\mathcal{Q}\right\}\text{ and }\Pi_1\mathcal{Q}\subset\mathcal{C}_A^{\mathcal{A},\prime},$$

thus satisfy the equalities

$$\square_{\pi I}^{\sigma,\pi,\mathbf{b}}f=\square_{\pi I}^{\sigma,\pi,\mathbf{b}}\mathsf{P}_{\pi(\Pi_1\mathcal{Q})}^{\sigma,\pi,\mathbf{b}}f \text{ and } \widehat{\square}_{\pi I}^{\sigma,\pi,\mathbf{b}}f=\widehat{\square}_{\pi I}^{\sigma,\pi,\mathbf{b}}\mathsf{P}_{\pi(\Pi_1\mathcal{Q})}^{\sigma,\pi,\mathbf{b}}f \tag{5.6.17}$$

for $I\in\Pi_1\mathcal{Q}\subset\mathcal{C}_A^{\mathcal{A}restrict}$, which will permit us to apply certain projection tricks used for Haar projections in the proof of $T1$ theorems.

However, in our sublinear stopping form $|\mathsf{B}|_{stop,\triangle\omega}^{A,\mathcal{Q}}$, the dual martingale projections in use in the function

$$\varphi_J^{\mathcal{Q}^S}\equiv\sum_{I\in\Pi_1\mathcal{Q}^S:\ (I,J)\in\mathcal{Q}^S}b_AE_I^\sigma\left(\widehat{\square}_{\pi I}^{\sigma,\flat,\mathbf{b}}f\right)\ \mathbf{1}_{A\setminus I}\ , \tag{5.6.18}$$

given in Proposition 5.6.14 above, are the modified pseudoprojections $\left\{\widehat{\square}_{\pi I}^{\sigma,\flat,\mathbf{b}}\right\}_{I\in\Pi_1\mathcal{Q}}$, where $\square_{\pi I}^{\sigma,\flat,\mathbf{b}}$ differs from the orthogonal projection $\square_{\pi I}^{\sigma,\pi,\mathbf{b}}$ for $I\in\Pi_1\mathcal{Q}$ by

$$\begin{aligned}
&\square_{\pi I}^{\sigma,\flat,\mathbf{b}}f-\square_{\pi I}^{\sigma,\pi,\mathbf{b}}f\\
=&\left\{\left(\sum_{I'\in\mathfrak{C}_{nat}(\pi I)}\mathbb{F}_{I'}^{\sigma,b_A}f\right)-\mathbb{F}_{\pi I}^{\sigma,b_A}f\right\}-\left\{\left(\sum_{I'\in\mathfrak{C}(\pi I)}\mathbb{F}_{I'}^{\sigma,b_A}f\right)-\mathbb{F}_{\pi I}^{\sigma,b_A}f\right\}\\
=&-\sum_{I'\in\mathfrak{C}_{brok}(\pi I)}\mathbb{F}_{I'}^{\sigma,b_A}f.
\end{aligned}$$

But the "box support" $Supp_{box}$ of this last expression $\sum_{I'\in\mathfrak{C}_{brok}(\pi I)}\mathbb{F}_{I'}^{\sigma,b_A}f$ consists of the broken children of πI, $\mathfrak{C}_{brok}(\pi I)$, and is contained in the set

$$\bigcup_{I\in\mathcal{C}'_A}\ \bigcup_{I'\in\mathfrak{C}_{\mathcal{A}}(A)\cap\mathfrak{C}_{\mathcal{D}}(\pi I)}\{I'\}$$

i.e.

$$\begin{aligned} Supp_{box}\left(\sum_{I'\in\mathfrak{C}_{brok}(\pi I)} \mathbb{F}_{I'}^{\sigma,b_A} f\right) &\subset \left\{I' \in \mathfrak{C}_{\mathcal{A}}(A) : I' \in \mathfrak{C}_{brok}(\pi I) \text{ for some } I \in \mathcal{C}'_A\right\} \\ &= \bigcup_{I\in\mathcal{C}'_A} \bigcup_{I'\in\mathfrak{C}_{\mathcal{A}}(A)\cap\mathfrak{C}_{\mathcal{D}}(\pi I)} \{I'\}. \end{aligned}$$

But $I \in \Pi_1 \mathcal{Q}^S \subset \mathcal{C}'_A$ is a *natural* child of πI, and so

$$I \cap Supp_{box}\left(\sum_{I'\in\mathfrak{C}_{brok}(\pi I)} \mathbb{F}_{I'}^{\sigma,b_A} f\right) = \emptyset$$

It now follows that we have

$$E_I^\sigma\left(\widehat{\Box}_{\pi I}^{\sigma,\flat,\mathbf{b}} f\right) = E_I^\sigma\left(\widehat{\Box}_{\pi I}^{\sigma,\pi,\mathbf{b}} f\right), \qquad \text{for } I \in \mathcal{C}'_A \tag{5.6.19}$$

Returning to (5.6.18), we have from (5.6.17) and (5.6.19) the identity

$$\begin{aligned} \varphi_J^{\mathcal{Q}^S} &\equiv \sum_{I\in\Pi_1\mathcal{Q}^S:\ (I,J)\in\mathcal{Q}^S} b_A E_I^\sigma\left(\widehat{\Box}_{\pi I}^{\sigma,\pi,\mathbf{b}} f\right) \mathbf{1}_{A\setminus I} \qquad (5.6.20) \\ &= \sum_{I\in\Pi_1\mathcal{Q}^S:\ (I,J)\in\mathcal{Q}^S} b_A E_I^\sigma\left(\widehat{\Box}_{\pi I}^{\sigma,\pi,\mathbf{b}}\left(\mathsf{P}_{\pi(\Pi_1\mathcal{Q})}^{\sigma,\pi,\mathbf{b}} f\right)\right) \mathbf{1}_{A\setminus I} \end{aligned}$$

which will play a critical role in proving the following $\flat$Straddling and Substraddling lemmas. The $\flat$Straddling Lemma is an adaptation of Lemmas 3.19 and 3.16 in [27].

Lemma 5.6.15. *Let $\mathcal{Q}$ be a reduced admissible collection of pairs for A, and suppose that $\mathcal{S} \subset \Pi_1^{below}\mathcal{Q} \cap \mathcal{C}'_A$ is a subpartition of A such that $\mathcal{Q}$ $\flat$straddles $\mathcal{S}$. Then we have the* *restricted sublinear norm bound*

$$\widehat{\mathfrak{N}}^{A,\mathcal{Q}}_{stop,\triangle\omega} \leq C_{\mathbf{r}} \sup_{S\in\mathcal{S}} \mathcal{S}^{\alpha,A;S}_{locsize}(\mathcal{Q}) \leq C_{\mathbf{r}} \mathcal{S}^{\alpha,A}_{augsize}(\mathcal{Q}), \tag{5.6.21}$$

where $\mathcal{S}^{\alpha,A;S}_{locsize}$ is an S-localized size condition with an S-hole given by

$$\mathcal{S}^{\alpha,A;S}_{locsize}(\mathcal{Q})^2 \equiv \sup_{K\in\mathcal{W}^*(S)\cap\mathcal{C}'_A} \frac{1}{|K|_\sigma}\left(\frac{\mathrm{P}^\alpha\left(K,\mathbf{1}_{A\setminus S}\sigma\right)}{|K|^{\frac{1}{n}}}\right)^2 \sum_{J\in\Pi_2^{K,aug}\mathcal{Q}} \left\|\triangle^{\omega,\mathbf{b}^*}_J x\right\|^{\spadesuit 2}_{L^2(\omega)} \tag{5.6.22}$$

Proof. We begin by using that the reduced collection $\mathcal{Q}$ $\flat$straddles $\mathcal{S}$ to write

$$\begin{aligned}
|\mathsf{B}|^{A,\mathcal{Q}}_{stop,\triangle\omega}(f,g) &= \sum_{J\in\Pi_2\mathcal{Q}} \frac{\mathrm{P}^\alpha\left(J,\left|\varphi^{\mathcal{Q}}_J\right|\mathbf{1}_{A\setminus I_{\mathcal{Q}}(J)}\sigma\right)}{|J|^{\frac{1}{n}}} \left\|\triangle^{\omega,\mathbf{b}^*}_J x\right\|^{\spadesuit}_{L^2(\omega)} \left\|\square^{\omega,\mathbf{b}^*}_J g\right\|^{\bigstar}_{L^2(\omega)}\\
&= \sum_{S\in\mathcal{S}} \sum_{J\in\Pi_2^{S,aug}\mathcal{Q}} \frac{\mathrm{P}^\alpha\left(J,\left|\varphi^{\mathcal{Q}^S}_J\right|\mathbf{1}_{A\setminus I_{\mathcal{Q}}(J)}\sigma\right)}{|J|^{\frac{1}{n}}} \left\|\triangle^{\omega,\mathbf{b}^*}_J x\right\|^{\spadesuit}_{L^2(\omega)} \left\|\square^{\omega,\mathbf{b}^*}_J g\right\|^{\bigstar}_{L^2(\omega)}
\end{aligned}$$

$$\text{where } \varphi^{\mathcal{Q}^S}_J \equiv \sum_{I\in\Pi_1\mathcal{Q}^S:\ (I,J)\in\mathcal{Q}^S} b_A E^\sigma_I\left(\widehat{\square}^{\sigma,\flat,\mathbf{b}}_{\pi I} f\right)\ \mathbf{1}_{A\setminus I}\ .$$

At this point we invoke the identity (5.6.20),

$$\varphi^{\mathcal{Q}^S}_J = \sum_{I\in\Pi_1\mathcal{Q}^S:\ (I,J)\in\mathcal{Q}^S} b_A E^\sigma_I\left(\widehat{\square}^{\sigma,\pi,\mathbf{b}}_{\pi I}\left(\mathsf{P}^{\sigma,\pi,\mathbf{b}}_{\pi(\Pi_1\mathcal{Q})} f\right)\right)\ \mathbf{1}_{A\setminus I}\ ,$$

so that

$$|\mathsf{B}|^{A,\mathcal{Q}}_{stop,\triangle\omega}(f,g) = |\mathsf{B}|^{A,\mathcal{Q}}_{stop,\triangle\omega}(h,g), \qquad \text{where } h \equiv \mathsf{P}^{\sigma,\pi,\mathbf{b}}_{\pi(\Pi_1\mathcal{Q})} f\ .$$

We will treat the sublinear form $|\mathsf{B}|^{A,\mathcal{Q}}_{stop,\triangle\omega}(h,g)$ with $h=\mathsf{P}^{\sigma,\pi,\mathbf{b}}_{\pi(\Pi_1\mathcal{Q})}f$ using a small variation on the corresponding argument in Lacey [27]. Namely, we will apply a Calderón-Zygmund stopping time decomposition to the function $h=\mathsf{P}^{\sigma,\pi,\mathbf{b}}_{\pi(\Pi_1\mathcal{Q})}f$ on the cube A with 'obstacle' $\mathcal{S}\cup\mathfrak{C}_A(A)$, to obtain stopping times $\mathcal{H}\subset\mathcal{C}_A$ with the property that for all $H\in\mathcal{H}\setminus\{A\}$ we have

$$
\begin{aligned}
&H\in\mathcal{C}_A \text{ is not strictly contained in any cube from } \mathcal{S},\\
&E^\sigma_H|h|>\Gamma E^\sigma_{\pi_{\mathcal{H}}H}|h|,\\
&E^\sigma_{H'}|h|\leq\Gamma E^\sigma_{\pi_{\mathcal{H}}H}|h| \text{ for all } H\subsetneqq H'\subset\pi_{\mathcal{H}}H \text{ with } H'\in\mathcal{C}_A.
\end{aligned}
$$

More precisely, define generation 0 of $\mathcal{H}$ to consist of the single cube A. Having defined generation n, let generation $n+1$ consist of the union over all cubes M in generation n of the maximal cubes M' in $\mathcal{C}_A$ that are contained in M with $E^\sigma_{M'}|h|>\Gamma E^\sigma_M|h|$, but are *not* strictly contained in any cube S from $\mathcal{S}$ or contained in any cube A' from $\mathfrak{C}_A(A)$ - thus the construction stops at the obstacle $\mathcal{S}\cup\mathfrak{C}_A(A)$. Then $\mathcal{H}$ is the union of all generations $n\geq 0$.

Denote by

$$
\mathcal{C}^{\mathcal{H}}_H\equiv\left\{H'\in\mathcal{C}_A: H'\subset H \text{ but } H'\not\subset H'' \text{ for any } H''\in\mathfrak{C}_{\mathcal{H}}(H)\right\}
$$

the usual $\mathcal{H}$-corona associated with the stopping cube H, but restricted to $\mathcal{C}_A$, and let $\alpha_{\mathcal{H}}(H)=E^\sigma_H|f|$ as is customary for a Calderón-Zygmund corona. Since these coronas $\mathcal{C}^{\mathcal{H}}_H$ are all contained in $\mathcal{C}_A$, we have the stopping energy from the $\mathcal{A}$-corona $\mathcal{C}_A$ at our disposal, which is crucial for the argument. Furthermore, we denote by

$$\mathcal{Q}_H \equiv \left\{ (I,J) \in \mathcal{Q} : J \in \mathcal{C}_H^{\mathcal{H},\flat shift} \right\}, \quad \text{with } \mathcal{C}_H^{\mathcal{H},\flat shift} \equiv \left\{ J \in \Pi_2 \mathcal{Q} : J^\flat \in \mathcal{C}_H^{\mathcal{H}} \right\} \tag{5.6.23}$$

the restriction of the pairs (I,J) in $\mathcal{Q}$ to those for which J lies in the flat shifted $\mathcal{H}$-corona $\mathcal{C}_H^{\mathcal{H},\flat shift}$. Since the $\mathcal{H}$-stopping cubes satisfy a σ-Carleson condition for Γ chosen large enough, we have the quasiorthogonal inequality

$$\sum_{H \in \mathcal{H}} \alpha_{\mathcal{H}} (H)^2 |H|_\sigma \lesssim \|h\|_{L^2(\sigma)}^2 , \tag{5.6.24}$$

which below we will see reduces matters to proving inequality (5.6.21) for the family of reduced admissible collections $\{\mathcal{Q}_H\}_{H \in \mathcal{H}}$ with constants independent of H:

$$\widehat{\mathfrak{N}}_{stop,\triangle^\omega}^{A,\mathcal{Q}_H} \leq C_{\mathbf{r}} \sup_{S \in \mathcal{S}} \mathcal{S}_{locsize}^{\alpha,A;S} (\mathcal{Q}_H) \leq C_{\mathbf{r}} \mathcal{S}_{augsize}^{\alpha,A} (\mathcal{Q}_H) , \qquad H \in \mathcal{H}.$$

Given $S \in \mathcal{S}$, define $H_S \in \mathcal{H}$ to be the minimal cube in $\mathcal{H}$ that contains S, and then define

$$\mathcal{H}_{\mathcal{S}} \equiv \{H_S \in \mathcal{H} : S \in \mathcal{S}\} .$$

Note that a given $H \in \mathcal{H}_{\mathcal{S}}$ may have many cubes $S \in \mathcal{S}$ such that $H = H_S$, and we denote the collection of these cubes by $\mathcal{S}_H \equiv \{S \in \mathcal{S} : H_S = H \}$. We will organize the straddling cubes $\mathcal{S}$ as

$$\mathcal{S} = \bigcup_{H \in \mathcal{H}_{\mathcal{S}}} \bigcup_{S \in \mathcal{S}_H} S$$

where each $S \in \mathcal{S}$ occurs exactly once in the union on the right hand side, i.e. the collections $\{\mathcal{S}_H\}_{H \in \mathcal{H}_{\mathcal{S}}}$ are pairwise disjoint.

We now momentarily fix $H \in \mathcal{H}_{\mathcal{S}}$, and consider the reduced admissible collection $\mathcal{Q}_H$, so that its projection onto the second component $\Pi_2 \mathcal{Q}_H$ of $\mathcal{Q}_H$ is *contained* in the corona $\mathcal{C}_H^{\mathcal{H},\flat shift}$. Then the collection $\mathcal{Q}_H$ $\flat$straddles the set $\mathcal{S}_H = \{S \in \mathcal{S} : H_S = H\}$. Moreover, $\mathcal{Q}_H = \bigcup_{S \in \mathcal{S}:\ S \subset H} \mathcal{Q}_H^S$ and $\Pi_2 \mathcal{Q}_H^S = \Pi_2^{S,aug} \mathcal{Q}_H$.

Recall that a Whitney cube K was required in the right hand side of the conclusion of Proposition 5.6.14 only in the case that $J^\flat \subset S''$ for some $S'' \in \mathfrak{C}_{\mathcal{D}}^{(2)}(S)$, which of course implies $3J^\flat \subset J^{\maltese} \subset S$. In this case we claim that $K \in \mathcal{C}_A$. Indeed, suppose in order to derive a contradiction, that $K \notin \mathcal{C}_A$. Then $J^{\maltese} \not\subset K$, and hence $3J^{\maltese} \not\subset S$. Since $J^{\maltese} \subset S$, it follows that $J^{\maltese}$ shares a common part of the boundary with S (since if not, then $3J^{\maltese} \subset S$, a contradiction). Now Key Fact #2 in (5.6.15) implies that the inner grandchild containing J, $J^\flat$, is contained in K where $K \notin \mathcal{C}_A$. This then implies that the pair (I, J) belongs to the corona straddling subcollection $\mathcal{P}_{cor}^A$, contradicting the assumption that $\mathcal{Q}$ is reduced.

Thus we have $S \in \Pi_1^{below} \mathcal{Q} \cap \mathcal{C}'_A$ and $K \in \mathcal{W}(S) \cap \mathcal{C}'_A$ and we can use Proposition (5.6.14) with $H = H_S$ to bound $|\mathsf{B}|_{stop,\triangle\omega}^{A,\mathcal{Q}}(f,g)$ by first summing over $H \in \mathcal{H}_{\mathcal{S}}$ and then over $S \in \mathcal{S}_H$. Indeed, $\mathcal{Q}_H$ $\flat$straddles $\mathcal{S}_H \equiv \{S \in \mathcal{S} : H_S = H\}$, so that $\left|\varphi_J^{\mathcal{Q}_H}\right| \lesssim \alpha_{\mathcal{H}}(H) \mathbf{1}_{A \setminus I_{\mathcal{Q}_H}}(J)$ by (5.6.4), and so the sum over $S \in \mathcal{S}_H$ of the first term on the right side of the conclusion of Proposition (5.6.14) is bounded by

$$\begin{aligned}
&\alpha_{\mathcal{H}}(H)\sum_{S\in\mathcal{S}_H}\sqrt{|S|_\sigma}\frac{1}{\sqrt{|S|_\sigma}}\left(\frac{\mathrm{P}^\alpha\left(S,\mathbf{1}_{A\setminus S}\sigma\right)}{|S|^{\frac{1}{n}}}\right)\left\|\mathsf{Q}^{\omega,\mathbf{b}^*}_{\Pi_2^{S,aug}\mathcal{Q}_H}x\right\|^{\spadesuit}_{L^2(\omega)}\left\|\mathsf{P}^{\omega,\mathbf{b}^*}_{\Pi_2^{S,aug}\mathcal{Q}_H}g\right\|^{\bigstar}_{L^2(\omega)}\\
\leq\ &\alpha_{\mathcal{H}}(H)\left\{\sup_{S\in\mathcal{S}_H}\frac{1}{\sqrt{|S|_\sigma}}\left(\frac{\mathrm{P}^\alpha\left(S,\mathbf{1}_{A\setminus S}\sigma\right)}{|S|^{\frac{1}{n}}}\right)\left\|\mathsf{Q}^{\omega,\mathbf{b}^*}_{\Pi_2^{S,aug}\mathcal{Q}_H}x\right\|^{\spadesuit}_{L^2(\omega)}\right\}\cdot\\
&\qquad\qquad\cdot\sum_{S\in\mathcal{S}_H}\sqrt{|S|_\sigma}\left\|\mathsf{P}^{\omega,\mathbf{b}^*}_{\Pi_2^{S,aug}\mathcal{Q}_H}g\right\|^{\bigstar}_{L^2(\omega)}\\
\leq\ &\alpha_{\mathcal{H}}(H)\left\{\sup_{S\in\mathcal{S}_H}\mathcal{S}^{\alpha,A;S}_{locsize}(\mathcal{Q}_H)\right\}\sqrt{|H|_\sigma}\left\|\mathsf{P}^{\omega,\mathbf{b}^*}_{\Pi_2\mathcal{Q}_H}g\right\|^{\bigstar}_{L^2(\omega)}
\end{aligned}$$

where $\Pi_2^{K,aug}\mathcal{Q}_H$ is as in Definition 5.6.8, and the corresponding sum over $S\in\mathcal{S}_H$ of the second term is bounded by

$$\begin{aligned}
&\alpha_{\mathcal{H}}(H)\sum_{S\in\mathcal{S}_H}\sum_{K\in\mathcal{W}(S)\cap\mathcal{C}'_A}\frac{\sqrt{|K|_\sigma}}{\sqrt{|K|_\sigma}}\frac{\mathrm{P}^\alpha\left(K,\mathbf{1}_{A\setminus S}\sigma\right)}{|K|^{\frac{1}{n}}}\left\|\mathsf{Q}^{\omega,\mathbf{b}^*}_{\Pi_2^{K,aug}\mathcal{Q}_H^S}x\right\|^{\spadesuit}_{L^2(\omega)}\left\|\mathsf{P}^{\omega,\mathbf{b}^*}_{\Pi_2^{K,aug}\mathcal{Q}_H^S}g\right\|^{\bigstar}_{L^2(\omega)}\\
\lesssim\ &\alpha_{\mathcal{H}}(H)\sup_{S\in\mathcal{S}_{\mathcal{H}}}\mathcal{S}^{\alpha,A;S}_{locsize}(\mathcal{Q}_H)\left(\sum_{S\in\mathcal{S}}\sum_{K\in\mathcal{W}(S)}|K|_\sigma\right)^{\frac{1}{2}}\left\|\mathsf{P}^{\omega,\mathbf{b}^*}_{\Pi_2\mathcal{Q}_H}g\right\|^{\bigstar}_{L^2(\omega)}\\
\leq\ &\left\{\sup_{S\in\mathcal{S}_{\mathcal{H}}}\mathcal{S}^{\alpha,A;S}_{locsize}(\mathcal{Q}_H)\right\}\alpha_{\mathcal{H}}(H)\sqrt{|H|_\sigma}\left\|\mathsf{P}^{\omega,\mathbf{b}^*}_{\Pi_2\mathcal{Q}_{\mathcal{H}}}g\right\|^{\bigstar}_{L^2(\omega)}
\end{aligned}$$

Using the definition of $|\mathsf{B}|^{A,\mathcal{Q}}_{stop,\triangle\omega}(f,g)$, we now sum the previous inequalities over the cubes $H\in\mathcal{H}_{\mathcal{S}}$ to obtain the following string of inequalities (explained in detail after the display)

$$\begin{aligned}
\left|\mathsf{B}\right|_{stop,\triangle\omega}^{A,\mathcal{Q}}(f,g) &\leq \left\{\sup_{S\in\mathcal{S}}\mathcal{S}_{locsize}^{\alpha,A;S}(\mathcal{Q})\right\}\sum_{H\in\mathcal{H}_{\mathcal{S}}}\alpha_{\mathcal{H}}(H)\sqrt{|H|_{\sigma}}\left\|\mathsf{P}_{\Pi_2\mathcal{Q}_H}^{\omega,\mathbf{b}^*}g\right\|_{L^2(\omega)}^{\bigstar} \\
&\leq \left\{\sup_{S\in\mathcal{S}}\mathcal{S}_{locsize}^{\alpha,A;S}(\mathcal{Q})\right\}\sqrt{\sum_{H\in\mathcal{H}_{\mathcal{S}}}\alpha_{\mathcal{H}}(H)^2|H|_{\sigma}}\sqrt{\sum_{H\in\mathcal{H}_{\mathcal{S}}}\left\|\mathsf{P}_{\Pi_2\mathcal{Q}_H}^{\omega,\mathbf{b}^*}g\right\|_{L^2(\omega)}^{\bigstar 2}} \\
&\lesssim \left\{\sup_{S\in\mathcal{S}}\mathcal{S}_{locsize}^{\alpha,A;S}(\mathcal{Q})\right\}\|h\|_{L^2(\sigma)}\sqrt{\sum_{H\in\mathcal{H}_{\mathcal{S}}}\left\|\mathsf{P}_{\Pi_2\mathcal{Q}_H}^{\omega,\mathbf{b}^*}g\right\|_{L^2(\omega)}^{\bigstar 2}} \\
&\leq \left\{\sup_{S\in\mathcal{S}}\mathcal{S}_{locsize}^{\alpha,A;S}(\mathcal{Q})\right\}\left\|\mathsf{P}_{\pi(\Pi_1\mathcal{Q})}^{\sigma,\pi,\mathbf{b}}f\right\|_{L^2(\sigma)}\left\|\mathsf{P}_{\Pi_2\mathcal{Q}}^{\omega,\mathbf{b}^*}g\right\|_{L^2(\omega)}^{\bigstar} \\
&\lesssim \left\{\sup_{S\in\mathcal{S}}\mathcal{S}_{locsize}^{\alpha,A;S}(\mathcal{Q})\right\}\left\|\mathsf{P}_{\pi(\Pi_1\mathcal{Q})}^{\sigma,\mathbf{b}}f\right\|_{L^2(\sigma)}^{\bigstar}\left\|\mathsf{P}_{\Pi_2\mathcal{Q}}^{\omega,\mathbf{b}^*}g\right\|_{L^2(\omega)}^{\bigstar}
\end{aligned}$$

where in the first line we have used $\mathcal{Q}=\bigcup_{H\in\mathcal{H}_{\mathcal{S}}}\mathcal{Q}_H$, which follows from the fact that each $J^{\flat}$ is contained in a unique $S\in\mathcal{S}$; in the third line we have used the quasiorthogonal inequality (5.6.24); in the fourth line we have used that the sets $\Pi_2\mathcal{Q}_H\subset\mathcal{C}_H^{\mathcal{H},\flat shift}$ are pairwise disjoint in H and have union $\Pi_2\mathcal{Q}=\dot{\bigcup}_{H\in\mathcal{H}_{\mathcal{S}}}\Pi_2\mathcal{Q}_H$. In the final line, we have used first the equality (5.1.43), second the fact that the functions $\square_{I,brok}^{\sigma,\pi,\mathbf{b}}f$ have pairwise disjoint supports, third the upper weak Riesz inequality and fourth the estimate (5.1.44) - which relies on the reverse

Hölder property for children in Lemma 5.1.9 - to obtain

$$
\begin{aligned}
\left\|\mathsf{P}_{\pi(\Pi_1\mathcal{Q})}^{\sigma,\pi,\mathbf{b}} f\right\|_{L^2(\sigma)}^2 &= \left\|\sum_{I\in\pi(\Pi_1\mathcal{Q})} \square_I^{\sigma,\mathbf{b}} f - \sum_{I\in\pi(\Pi_1\mathcal{Q})} \square_{I,brok}^{\sigma,\pi,\mathbf{b}} f\right\|_{L^2(\sigma)}^2 \\
&\lesssim \left\|\sum_{I\in\pi(\Pi_1\mathcal{Q})} \square_I^{\sigma,\mathbf{b}} f\right\|_{L^2(\sigma)}^2 + \left\|\sum_{I\in\pi(\Pi_1\mathcal{Q})} \square_{I,brok}^{\sigma,\pi,\mathbf{b}} f\right\|_{L^2(\sigma)}^2 \\
&\lesssim \left\|\mathsf{P}_{\pi(\Pi_1\mathcal{Q})}^{\sigma,\mathbf{b}} f\right\|_{L^2(\sigma)}^2 + \sum_{I\in\pi(\Pi_1\mathcal{Q})} \left\|\square_{I,brok}^{\sigma,\pi,\mathbf{b}} f\right\|_{L^2(\sigma)}^2 \\
&\lesssim \sum_{I\in\pi(\Pi_1\mathcal{Q})} \left\|\square_I^{\sigma,\mathbf{b}} f\right\|_{L^2(\sigma)}^2 + \sum_{I\in\pi(\Pi_1\mathcal{Q})} \left\|\bigtriangledown_I^{\sigma} f\right\|_{L^2(\sigma)}^2 \qquad (5.6.25) \\
&\lesssim \left\|\mathsf{P}_{\pi(\Pi_1\mathcal{Q})}^{\sigma,\mathbf{b}} f\right\|_{L^2(\sigma)}^{\bigstar 2}
\end{aligned}
$$

We now use the fact that the supremum in the definition of $\mathcal{S}_{locsize}^{\alpha,A;S}(\mathcal{Q})$ is taken over $K \in \mathcal{W}^*(S) \cap \mathcal{C}'_A$ to conclude that

$$
\sup_{S\in\mathcal{S}} \mathcal{S}_{locsize}^{\alpha,A;S}(\mathcal{Q}) \leq \mathcal{S}_{augsize}^{\alpha,A}(\mathcal{Q}),
$$

and this completes the proof of Lemma 5.6.15. □

In a similar fashion we can obtain the following Substraddling Lemma.

Definition 5.6.16. *Given a* reduced admissible *collection of pairs $\mathcal{Q}$ for A, and a $\mathcal{D}$-cube L contained in A, we say that $\mathcal{Q}$* ***substraddles*** *L if for every pair $(I,J) \in \mathcal{Q}$ there is $K \in \mathcal{W}(L) \cap \mathcal{C}'_A$ with $J \subset K \subset 3K \subset I \subset L$.*

Lemma 5.6.17. *Let L be a $\mathcal{D}$-cube contained in A, and suppose that $\mathcal{Q}$ is an admissible* *collection of pairs that substraddles L. Then we have the sublinear form bound*

$$\widehat{\mathfrak{N}}_{stop,\triangle\omega}^{A,\mathcal{Q}} \leq C\mathcal{S}_{augsize}^{\alpha,A}(\mathcal{Q}).$$

Proof. We will show that $\mathcal{Q}$ $\flat$straddles the subset $\mathcal{W}_L$ of Whitney cubes for L given by

$$\mathcal{W}^{\mathcal{Q}}(L) \equiv \left\{K \in \mathcal{W}(L) \cap \mathcal{C}'_A : J \subset K \subset 3K \subset I \subset L \text{ for some } (I,J) \in \mathcal{Q}\right\}.$$

It is clear that $\mathcal{W}^{\mathcal{Q}}(L) \subset \Pi_1^{below}\mathcal{Q} \cap \mathcal{C}'_A$ is a subpartition of A. It remains to show that for every pair $(I,J) \in \mathcal{Q}$ there is $K \in \mathcal{W}^{\mathcal{Q}}(L) \cap [J,I]$ such that $J^\flat \subset K$. But our hypothesis implies that there is $K \in \mathcal{W}^{\mathcal{Q}}(L)$ with $J \subset K \subset 3K \subset I \subset L$. We now consider two cases.

Case 1: If $\pi_{\mathcal{D}}^{(3)}K \subset L$, then since K is maximal Whitney cube, it is contained in an *outer* grandchild of $\pi_{\mathcal{D}}^{(3)}K$ and $\pi_{\mathcal{D}}^{(1)}K$ has to share an endpoint with L. Then so does $\pi_{\mathcal{D}}^{(3)}K$. Recall, from Key Fact #2 in (5.6.15), $3J \subset J^\flat$, an *inner* grandchild of $J^{\maltese}$. We thus have $J^{\maltese} \subset \pi_{\mathcal{D}}^{(2)}K$ (If not; $\pi_{\mathcal{D}}^{(2)}K \subset J^{\maltese}$ which implies that $J^\flat$ has the same endpoint as L, a contradiction). This implies that $J^\flat \subset K$.

Case 2: If $\pi_{\mathcal{D}}^{(3)}K \supsetneqq L$, then $K \subset 3K \subset I \subset L$ implies that $I = L = \pi_{\mathcal{D}}^{(2)}K$. Thus we have $J^{\maltese} \subset I = \pi_{\mathcal{D}}^{(2)}K$, which again gives $J^\flat \subset K$.

Now that we know $\mathcal{Q}$ $\flat$straddles the subset $\mathcal{W}^{\mathcal{Q}}(L)$, we can apply Lemma 5.6.15 to obtain the required bound $\widehat{\mathfrak{N}}_{stop,\triangle\omega}^{A,\mathcal{Q}} \leq C\mathcal{S}_{augsize}^{\alpha,A}(\mathcal{Q})$. □

5.6.4 The bottom/up stopping time argument of M. Lacey

Before introducing Lacey's stopping times, we note that the Corona-straddling Lemma 5.6.10 allows us to remove the 'corona straddling' collection $\mathcal{P}_{cor}^{A}$ of pairs of cubes in (5.6.16) from the collection $\mathcal{P}^A$ in (5.6.2) used to define the stopping form $\mathsf{B}^A_{stop}(f,g)$. The collection $\mathcal{P}^A \setminus \mathcal{P}^A_{cor}$ is of course also A-admissible.

We assume for the remainder of the proof that all admissible collections $\mathcal{P}$ are reduced, i.e.

$$\mathcal{P}^A \cap \mathcal{P}^A_{cor} = \emptyset, \text{ as well as } \mathcal{P} \cap \mathcal{P}^A_{cor} = \emptyset \text{ for all } A\text{-admissible } \mathcal{P}. \tag{5.6.26}$$

For a cube $K \in \mathcal{D}$, we define

$$\mathcal{G}[K] \equiv \{J \in \mathcal{G} : J \subset K\}$$

to consist of all cubes J in the other grid $\mathcal{G}$ that are contained in K. For an A-admissible collection $\mathcal{P}$ of pairs, define two atomic measures $\omega_{\mathcal{P}}$ and $\omega_{\flat\mathcal{P}}$ in the upper half space $\mathbb{R}^{n+1}_+$ by

$$\omega_{\mathcal{P}} \equiv \sum_{J \in \Pi_2 \mathcal{P}} \left\| \triangle_J^{\omega,\mathbf{b}^*} x \right\|^{\spadesuit 2}_{L^2(\omega)} \delta_{\left(c_{J^{\maltese}}, \ell\left(J^{\maltese}\right)\right)} \tag{5.6.27}$$

and

$$\omega_{\flat\mathcal{P}} \equiv \sum_{J \in \Pi_2 \mathcal{P}} \left\| \triangle_J^{\omega,\mathbf{b}^*} x \right\|^{\spadesuit 2}_{L^2(\omega)} \delta_{\left(c_{J^{\flat}}, \ell\left(J^{\flat}\right)\right)}, \tag{5.6.28}$$

where $J^{\flat}$ is the inner grandchild of $J^{\maltese}$ that contains J

Note that each cube $J \in \Pi_2 \mathcal{P}$ has its 'energy' $\left\| \triangle_J^{\omega,\mathbf{b}^*} x \right\|^{\spadesuit 2}_{L^2(\omega)}$ in the measure $\omega_{\flat\mathcal{P}}$ assigned to exactly one of the 2^n points $\left(c_{J^{\flat}}, \frac{1}{4}\ell\left(J^{\maltese}\right)\right)$ in the upper half plane $\mathbb{R}^{n+1}_+$ since J is contained in one of $J^{\maltese}_{\searrow}$, namely in $J^{\flat}$, by Key Fact #2 in (5.6.15). Note also that the atomic measure $\omega_{\flat\mathcal{P}}$ differs from the measure μ in (.0.20) in Appendix below - which is used there to control the functional energy condition - in that here we bundle together all the $J's$ having a common $J^{\flat}$. This is in order to rewrite the *augmented* size functional in terms of the measure

$\omega_{\flat\mathcal{P}}$. We can get away with this here, as opposed to in Appendix , due to the 'smaller and decoupled' nature of the augmented size functional to which we will relate $\omega_{\flat\mathcal{P}}$.

Define the tent $\mathbf{T}(L)$ over a cube L to be the convex hull of the cube $L\times\{0\}$ and the point $(c_L, \ell(L)) \in \mathbb{R}^{n+1}_+$. Then for $J \in \Pi_2\mathcal{P}$ we have $J \in \Pi_2^{K,aug}\mathcal{P}$ *iff* $\left\{J \subset K \text{ and } J^{\maltese} \subset \pi_{\mathcal{D}}^{(2)} K\right\}$ *iff* $J^\flat \subset K$ *iff* $\left(c_{J^\flat}, \ell\left(J^\flat\right)\right) \in \mathbf{T}(K)$. We can now rewrite the augmented size functional of $\mathcal{P}$ in Definition 5.6.8 as

$$\mathcal{S}^{\alpha,A}_{augsize}(\mathcal{P})^2 \equiv \sup_{K\in\Pi_1^{below}\mathcal{P}\cap\mathcal{C}'_A} \frac{1}{|K|_\sigma}\left(\frac{\mathrm{P}^\alpha\left(K, \mathbf{1}_{A\setminus K}\sigma\right)}{|K|^{\frac{1}{n}}}\right)^2 \omega_{\flat\mathcal{P}}(\mathbf{T}(K)). \tag{5.6.29}$$

It will be convenient to write

$$\Psi^\alpha(K;\mathcal{P})^2 \equiv \left(\frac{\mathrm{P}^\alpha\left(K, \mathbf{1}_{A\setminus K}\sigma\right)}{|K|^{\frac{1}{n}}}\right)^2 \omega_{\flat\mathcal{P}}(\mathbf{T}(K)),$$

so that we have simply

$$\mathcal{S}^{\alpha,A}_{augsize}(\mathcal{P})^2 = \sup_{K\in\Pi_1^{below}\mathcal{P}\cap\mathcal{C}'_A} \frac{\Psi^\alpha(K;\mathcal{P})^2}{|K|_\sigma}.$$

Remark 5.6.18. *The functional $\omega_{\flat\mathcal{P}}(\mathbf{T}(K))$ is increasing in K, while the functional*

$\frac{\mathrm{P}^{\alpha}\left(K,\mathbf{1}_{A\setminus K}\sigma\right)}{|K|^{\frac{1}{n}}}$ *is 'almost decreasing' in* K*: if* $K_0\subset K$ *then*

$$\begin{aligned}
\frac{\mathrm{P}^{\alpha}\left(K,\mathbf{1}_{A\setminus K}\sigma\right)}{|K|^{\frac{1}{n}}} &= \int_{A\setminus K}\frac{d\sigma\left(y\right)}{\left(|K|^{\frac{1}{n}}+|y-c_K|\right)^{n+1-\alpha}}\\
&\lesssim \int_{A\setminus K}\frac{(\sqrt{n})^{n+1-\alpha}d\sigma\left(y\right)}{\left(|K_0|^{\frac{1}{n}}+\left|y-c_{K_0}\right|\right)^{n+1-\alpha}}\\
&\leq \int_{A\setminus K_0}\frac{C_{\alpha,n}\quad d\sigma\left(y\right)}{\left(|K_0|^{\frac{1}{n}}+\left|y-c_{K_0}\right|\right)^{n+1-\alpha}}=C_{\alpha,n}\frac{\mathrm{P}^{\alpha}\left(K_0,\mathbf{1}_{A\setminus K_0}\sigma\right)}{|K_0|^{\frac{1}{n}}}
\end{aligned}$$

since $|K_0|+\left|y-c_{K_0}\right|\leq|K|+|y-c_K|+\frac{1}{2}\operatorname{diam}\left(K\right)$ *for* $y\in A\setminus K$.

Recall that if $\mathcal{P}$ is an admissible collection for a dyadic cube A, the corresponding sublinear form in (5.6.7) is given by

$$\begin{aligned}
|\mathsf{B}|_{stop,\triangle^{\omega}}^{A,\mathcal{P}}\left(f,g\right) &\equiv \sum_{J\in\Pi_2\mathcal{P}}\frac{\mathrm{P}^{\alpha}\left(J,\left|\varphi_J^{\mathcal{P}}\right|\mathbf{1}_{A\setminus I_{\mathcal{P}}(J)}\sigma\right)}{|J|^{\frac{1}{n}}}\left\|\triangle_J^{\omega,\mathbf{b}^*}x\right\|_{L^2(\omega)}^{\spadesuit}\left\|\square_J^{\omega,\mathbf{b}^*}g\right\|_{L^2(\omega)}^{\bigstar};\\
\text{where }\varphi_J^{\mathcal{P}} &\equiv \sum_{I\in\mathcal{C}_A':\ (I,J)\in\mathcal{P}} b_A E_I^{\sigma}\left(\widehat{\square}_{\pi I}^{\sigma,\flat,\mathbf{b}}f\right)\ \mathbf{1}_{A\setminus I}\ .
\end{aligned}$$

In the notation for $|\mathsf{B}|_{stop,\triangle^{\omega}}^{A,\mathcal{P}}$, we are omitting dependence on the parameter α, and to avoid clutter, we will often do so from now on when the dependence on α is inconsequential.

Recall further that the 'size testing collection' of cubes $\Pi_1^{below}\mathcal{P}$ for the initial size testing functional $\mathcal{S}_{initsize}^{\alpha,A}\left(\mathcal{P}\right)$ is the collection of all subcubes of cubes in $\Pi_1\mathcal{P}$, and moreover, by Key Fact #1 in (5.6.13), that we can restrict the collection to $\Pi_1^{below}\mathcal{P}\cap\mathcal{C}_A'$. This latter set is used for the augmented size functional.

Assumption

We may assume that the corona $\mathcal{C}_A$ is finite, and that each A-admissible collection $\mathcal{P}$ is a finite collection, and hence so are $\Pi_1\mathcal{P}$, $\Pi_1^{below}\mathcal{P}\cap\mathcal{C}'_A$ and $\Pi_2\mathcal{P}$, provided all of the bounds we obtain are independent of the cardinality of these latter collections.

Consider $0<\varepsilon<1$, where $\rho=1+\varepsilon$ will be chosen later in (5.6.37). Begin by defining the collection $\mathcal{L}_0$ to consist of the *minimal* dyadic cubes K in $\Pi_1^{below}\mathcal{P}\cap\mathcal{C}'_A$ such that

$$\frac{\Psi^\alpha\left(K;\mathcal{P}\right)^2}{|K|_\sigma}\geq\varepsilon\mathcal{S}_{augsize}^{\alpha,A}\left(\mathcal{P}\right)^2 .$$

where we recall that

$$\Psi^\alpha\left(K;\mathcal{P}\right)^2\equiv\left(\frac{\mathrm{P}^\alpha\left(K,\mathbf{1}_{A\setminus K}\sigma\right)}{|K|^{\frac{1}{n}}}\right)^2\omega_{\flat\mathcal{P}}\left(\mathbf{T}\left(K\right)\right).$$

Note that such minimal cubes exist when $0<\varepsilon<1$ because $\mathcal{S}_{augsize}^{\alpha,A}\left(\mathcal{P}\right)^2$ is the supremum over $K\in\Pi_1^{below}\mathcal{P}\cap\mathcal{C}'_A$ of $\frac{\Psi^\alpha(K;\mathcal{P})^2}{|K|_\sigma}$. A key property of the minimality requirement is that

$$\frac{\Psi^\alpha\left(K';\mathcal{P}\right)^2}{|K'|_\sigma}<\varepsilon\mathcal{S}_{augsize}^{\alpha,A}\left(\mathcal{P}\right)^2 , \tag{5.6.30}$$

whenever there is $K'\in\Pi_1^{below}\mathcal{P}\cap\mathcal{C}'_A$ with $K'\subsetneqq K$ and $K\in\mathcal{L}_0$.

We now perform a stopping time argument 'from the bottom up' with respect to the atomic measure $\omega_{\mathcal{P}}$ in the upper half space. This construction of a stopping time 'from the bottom up', together with the subsequent applications of the Orthogonality Lemma and the Straddling Lemma, comprise the key innovations in Lacey's argument [27]. However, in our situation the cubes I belonging to $\Pi_1^{below}\mathcal{P}$ are no longer 'good' in any sense, and we must include an additional top/down stopping criterion in the next subsection to accommodate this lack of 'goodness'. The argument in [27] will apply to these special stopping cubes, called 'indented' cubes, and the remaining cubes form towers with a common endpoint, that are controlled using all three straddling lemmas.

We refer to $\mathcal{L}_0$ as the initial or level 0 generation of stopping cubes. Set

$$\rho = 1 + \varepsilon. \tag{5.6.31}$$

As in [64], [66] and [67], we follow Lacey [27] by recursively defining a finite sequence of generations $\{\mathcal{L}_m\}_{m\geq 0}$ by letting $\mathcal{L}_m$ consist of the *minimal* dyadic cubes L in $\Pi_1^{below}\mathcal{P}\cap\mathcal{C}'_A$ that contain a cube from some previous level $\mathcal{L}_\ell$, $\ell < m$, such that

$$\omega_{\flat\mathcal{P}}\left(\mathbf{T}\left(L\right)\right) \geq \rho\omega_{\flat\mathcal{P}}\left(\bigcup_{L'\in\bigcup_{\ell=0}^{m-1}\mathcal{L}_\ell:\ L'\subset L}\mathbf{T}\left(L'\right)\right). \tag{5.6.32}$$

Since $\mathcal{P}$ is finite this recursion stops at some level M. We then let $\mathcal{L}_{M+1}$ consist of all the maximal cubes in $\Pi_1^{below}\mathcal{P}\cap\mathcal{C}'_A$ that are not already in some $\mathcal{L}_m$ with $m \leq M$. Thus $\mathcal{L}_{M+1}$ will contain either none, some, or all of the maximal cubes in $\Pi_1^{below}\mathcal{P}$. We do not of course have (5.6.32) for $A' \in \mathcal{L}_{M+1}$ in this case, but we do have that (5.6.32) fails for subcubes K of $A' \in \mathcal{L}_{M+1}$ that are not contained in any other $L \in \mathcal{L}_m$ with $m \leq M$, and this is sufficient for the arguments below.

We now decompose the collection of pairs (I, J) in $\mathcal{P}$ into collections $\mathcal{P}^{\flat small}$ and $\mathcal{P}^{\flat big}$ according to the location of I and $J^\flat$, but only after introducing below the indented corona $\mathcal{H}$. The collection $\mathcal{P}^{\flat big}$ will then essentially consist of those pairs $(I, J) \in \mathcal{P}$ for which there are $L', L \in \mathcal{H}$ with $L' \subsetneqq L$ and such that $J^\flat \in \mathcal{C}_{L'}^{\mathcal{H}}$ and $I \in \mathcal{C}_L^{\mathcal{H}}$. The collection $\mathcal{P}^{\flat small}$ will consist of the remaining pairs $(I, J) \in \mathcal{P}$ for which there is $L \in \mathcal{H}$ such that $J^\flat, I \in \mathcal{C}_L^{\mathcal{H}}$, along with the pairs $(I, J) \in \mathcal{P}$ such that $I \subset I_0$ for some $I_0 \in \mathcal{L}_0$. This will cover all pairs (I, J) in $\mathcal{P} \subset \mathcal{P}_A$, since for such pairs, $I \in \mathcal{C}'_A$ and $J \in \mathcal{C}_A^{\mathcal{G}shift}$, which in turn implies $I \in \mathcal{C}_L^{\mathcal{H}}$ and $J^\flat \in \mathcal{C}_{L'}^{\mathcal{H}}$ for some $L, L' \in \mathcal{H}$. But a considerable amount of further analysis is required to prove (5.6.7).

First recall that $\mathcal{L} \equiv \bigcup_{m=0}^{M+1} \mathcal{L}_m$ is the tree of stopping $\omega_{\mathcal{P}}$-energy cubes defined above. By the construction above, the maximal elements in $\mathcal{L}$ are the maximal cubes in $\Pi_1^{below}\mathcal{P} \cap \mathcal{C}'_A$. For $L \in \mathcal{L}$, denote by $\mathcal{C}_L^{\mathcal{L}}$ the *corona* associated with L in the tree $\mathcal{L}$,

$$\mathcal{C}_L^{\mathcal{L}} \equiv \left\{ K \in \mathcal{D} : K \subset L \text{ and there is no } L' \in \mathcal{L} \text{ with } K \subset L' \subsetneqq L \right\},$$

and define the $\flat$ *shifted* $\mathcal{L}$-corona by

$$\mathcal{C}_L^{\mathcal{L},\flat shift} \equiv \left\{ J \in \mathcal{G} : J^\flat \in \mathcal{C}_L^{\mathcal{L}} \right\}.$$

Now the parameter m in $\mathcal{L}_m$ refers to the level at which the stopping construction was performed, but for $L \in \mathcal{L}_m$, the corona children L' of L are *not* all necessarily in $\mathcal{L}_{m-1}$, but may be in $\mathcal{L}_{m-t}$ for t large.

At this point we introduce the notion of geometric depth d in the tree $\mathcal{L}$ by defining

$$\begin{aligned}
\mathcal{G}_0 &\equiv \{L \in \mathcal{L} : L \text{ is maximal}\}, \qquad (5.6.33)\\
\mathcal{G}_1 &\equiv \{L \in \mathcal{L} : L \text{ is maximal wrt } L \subsetneqq L_0 \text{ for some } L_0 \in \mathcal{G}_0\},\\
&\vdots\\
\mathcal{G}_{d+1} &\equiv \{L \in \mathcal{L} : L \text{ is maximal wrt } L \subsetneqq L_d \text{ for some } L_d \in \mathcal{G}_d\},\\
&\vdots
\end{aligned}$$

We refer to $\mathcal{G}_d$ as the d^{th} generation of cubes in the tree $\mathcal{L}$, and say that the cubes in $\mathcal{G}_d$ are at depth d in the tree $\mathcal{L}$ (the generations $\mathcal{G}_d$ here are *not* related to the grid $\mathcal{G}$), and we write $d_{geom}(L)$ for the geometric depth of L. Thus the cubes in $\mathcal{G}_d$ are the stopping cubes in $\mathcal{L}$ that are d levels in the *geometric* sense below the top level. While the geometric depth d_{geom} is about to be superceded by the 'indented' depth d_{indent} defined in the next subsection, we will return to the geometric depth in order to iterate Lacey's bottom/up stopping criterion when proving the second line in (5.6.36) in Proposition 5.6.19 below.

5.6.5 The indented corona construction

Now we address the lack of goodness in $\Pi_1^{below}\mathcal{P} \cap \mathcal{C}'_A$. For this we introduce an additional top/down stopping time $\mathcal{H}$ over the collection $\mathcal{L}$. Given the initial generation

$$\mathcal{H}_0 = \{\text{maximal } L \in \mathcal{L}\} = \left\{\text{maximal } I \in \Pi_1^{below}\mathcal{P}\right\},$$

define subsequent generations $\mathcal{H}_k$ as follows. For $k \geq 1$ and each $H \in \mathcal{H}_{k-1}$, let

$$\mathcal{H}_k(H) \equiv \{\text{maximal } L \in \mathcal{L} : 3L \subset H\}$$

consist of the next $\mathcal{H}$-generation of $\mathcal{L}$-cubes below H, and set $\mathcal{H}_k \equiv \bigcup_{H \in \mathcal{H}_{k-1}} \mathcal{H}_k(H)$. Finally set $\mathcal{H} \equiv \bigcup_{k=0}^{\infty} \mathcal{H}_k$. We refer to the stopping cubes $H \in \mathcal{H}$ as *indented* stopping cubes since $3H \subset \pi_{\mathcal{H}} H$ for all $H \in \mathcal{H}$ at indented generation one or more, i.e. each successive such H is 'indented' in its $\mathcal{H}$-parent. This property of indentation is precisely what is required in order to generate geometric decay in indented generations at the end of the proof. We refer to k as the *indented depth* of the stopping cube $H \in \mathcal{H}_k$, written $k = d_{indent}(H)$, which is a refinement of the geometric depth d_{geom} introduced above. We will often revert to writing the dummy variable for cubes in $\mathcal{H}$ as L instead of H. For $L \in \mathcal{H}$ define the $\mathcal{H}$-corona $\mathcal{C}_L^{\mathcal{H}}$ and $\mathcal{H}$-$\flat$shifted corona $\mathcal{C}_L^{\mathcal{H},\flat shift}$ by

$$\begin{aligned}
\mathcal{C}_L^{\mathcal{H}} &\equiv \left\{I \in \mathcal{D} : I \subset L \text{ and } I \not\subset L' \text{ for any } L' \in \mathfrak{C}_{\mathcal{H}}(L)\right\}, \\
\mathcal{C}_L^{\mathcal{H},\flat shift} &\equiv \left\{J \in \mathcal{G} : J^{\flat} \in \mathcal{C}_L^{\mathcal{H}}\right\}.
\end{aligned}$$

We will also need recourse to the coronas $\mathcal{C}_L^{\mathcal{H}}$ restricted to cubes in $\mathcal{L}$, i.e.

$$\mathcal{C}_L^{\mathcal{H}}(\mathcal{L}) \equiv \mathcal{C}_L^{\mathcal{H}} \cap \mathcal{L} = \left\{T \in \mathcal{L} : T \subset L \text{ and } T \not\subset L' \text{ for any } L' \in \mathcal{H} \text{ with } L' \subsetneqq L\right\}.$$

and

$$\mathcal{T}(L) \equiv \mathcal{C}_L^{\mathcal{H},restrict}(\mathcal{L}) = \mathcal{C}_L^{\mathcal{H}}(\mathcal{L}) \backslash \{L\}$$

We emphasize the distinction 'indented generation' as this refers to the indented depth rather than either the level of initial stopping construction of $\mathcal{L}$, or the geometric depth. The point of introducing the tree $\mathcal{H}$ of indented stopping cubes, is that the inclusion $3L \subset \pi_{\mathcal{H}} L$ for all $L \in \mathcal{H}$ with $d_{indent}(L) \geq 1$ turns out to be an adequate substitute for the standard 'goodness' lost in the process of infusing the weak goodness of Hytönen and Martikainen in Subsection 5.2.1 above.

5.6.5.1 Flat shifted coronas

We now define the $\flat$shifted admissible collections of pairs $\mathcal{P}_{L,t}^{\flat \mathcal{H}}$ using the coronas

$$\mathcal{C}_L^{\mathcal{H},\flat shift} \equiv \left\{ J \in \Pi_2 \mathcal{P} : J^\flat \in \mathcal{C}_L^{\mathcal{H}} \right\} \text{ and } \mathcal{C}_L^{\mathcal{L},\flat shift} \equiv \left\{ J \in \Pi_2 \mathcal{P} : J^\flat \in \mathcal{C}_L^{\mathcal{L}} \right\}.$$

In these flat shifted $\mathcal{H}$ and $\mathcal{L}$ coronas, we have effectively shift the cubes J two levels 'up' by requiring $J^\flat \in \mathcal{C}_L^{\mathcal{L}}$, but because $\mathcal{P}$ is admissible, we always have $J^{\maltese} \in \mathcal{C}_A^{\mathcal{A},restrict}$. We define

$$\begin{aligned}
\mathcal{P}_{L,t}^{\flat \mathcal{H}} &\equiv \left\{ (I,J) \in \mathcal{P} : I \in \mathcal{C}_L^{\mathcal{H}}, J \in \mathcal{C}_{L'}^{\mathcal{H},\flat shift} \text{ for some } L' \in \mathcal{H}_{d_{indent}(L)+t} L' \subset L \right\}, \\
\mathcal{P}_{L,0}^{\flat \mathcal{H}} &= \left\{ (I,J) \in \mathcal{P} : I \in \mathcal{C}_L^{\mathcal{H}} \text{ and } J \in \mathcal{C}_L^{\mathcal{H},\flat shift} \right\}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{P}_{L,0}^{\flat \mathcal{H}} &= \mathcal{P}_{L,0}^{\flat \mathcal{H}-small} \dot{\cup} \mathcal{P}_{L,0}^{\flat \mathcal{H}-big}; \\
\mathcal{P}_{L,0}^{\flat \mathcal{H}-small} &\equiv \left\{ (I,J) \in \mathcal{P}_{L,0}^{\flat \mathcal{H}} : \text{there is no } L' \in \mathcal{T}(L) \text{ with } J^\flat \subset L' \subset I \right\} \\
&= \left\{ (I,J) \in \mathcal{P}_{L,0}^{\flat \mathcal{H}} : I \in \mathcal{C}_{L'}^{\mathcal{L}} \setminus \{L'\}, J \in \mathcal{C}_{L'}^{\mathcal{L},\flat shift} \text{ for some } L' \in \mathcal{T}(L) \right\}, \\
\mathcal{P}_{L,0}^{\flat \mathcal{H}-big} &\equiv \left\{ (I,J) \in \mathcal{P}_{L,0}^{\flat \mathcal{H}} : \text{there is } L' \in \mathcal{T}(L) \text{ with } J^\flat \subset L' \subset I \right\},
\end{aligned}$$

with one exception: if $L \in \mathcal{H}_0$ we set $\mathcal{P}_{L,0}^{\flat\mathcal{H}-small} \equiv \mathcal{P}_{L,0}^{\flat\mathcal{H}}$ and $\mathcal{P}_{L,0}^{\flat\mathcal{H}-big} \equiv \emptyset$ since in this case L fails to satisfy (5.6.32) as pointed out above. Finally, for $L \in \mathcal{H}$ we further decompose $\mathcal{P}_{L,0}^{\flat\mathcal{H}-small}$ as

$$\begin{aligned} \mathcal{P}_{L,0}^{\flat\mathcal{H}-small} &= \dot{\bigcup_{L' \in \mathcal{T}(L)}} \mathcal{P}_{L',0}^{\flat\mathcal{L}-small} \\ \text{where } \mathcal{P}_{L',0}^{\flat\mathcal{L}-small} &\equiv \left\{ (I,J) \in \mathcal{P} : I \in \mathcal{C}_{L'}^{\mathcal{L}} \setminus \left\{ L' \right\} \text{ and } J \in \mathcal{C}_{L'}^{\mathcal{L},\flat shift} \right\} \end{aligned}$$

Then we set

$$\begin{aligned} \mathcal{P}^{\flat big} &\equiv \left\{ \bigcup_{L \in \mathcal{H}} \mathcal{P}_{L,0}^{\flat\mathcal{H}-big} \right\} \bigcup \left\{ \bigcup_{t \geq 1} \bigcup_{L \in \mathcal{H}} \mathcal{P}_{L,t}^{\flat\mathcal{H}} \right\}; \\ \mathcal{P}^{\flat small} &\equiv \bigcup_{L \in \mathcal{L}} \mathcal{P}_{L,0}^{\flat\mathcal{L}-small} \end{aligned} \tag{5.6.34}$$

We observed above that every pair $(I,J) \in \mathcal{P}$ is included in either $\mathcal{P}^{small}$ or $\mathcal{P}^{big}$, and it follows that every pair $(I,J) \in \mathcal{P}$ is thus included in either $\mathcal{P}^{\flat small}$ or $\mathcal{P}^{\flat big}$, simply because the pairs (I,J) have been shifted up by two dyadic levels in the cube J. Thus the coronas $\mathcal{P}_{L,0}^{\flat\mathcal{L}-small}$ are now even *smaller* than the regular coronas $\mathcal{P}_{L,0}^{\mathcal{L}-small}$, which permits the estimate (5.6.35) below to hold for the larger augmented size functional. On the other hand, the coronas $\mathcal{P}_{L,0}^{\flat\mathcal{H}-big}$ and $\mathcal{P}_{L,t}^{\flat\mathcal{H}}$ are now bigger than before, requiring the stronger straddling lemmas above in order to obtain the estimates (5.6.36) below. More specifically, we will see that stopping forms with pairs in $\mathcal{P}^{\flat big}$ will be estimated using the $\flat$ Straddling and Substraddling Lemmas (Substraddling applies to part of $\mathcal{P}_{L,0}^{\flat\mathcal{H}-big}$ and $\flat$Straddling applies to the remaining part of $\mathcal{P}_{L,0}^{\flat\mathcal{H}-big}$ and to $\mathcal{P}_{L,t}^{\flat\mathcal{H}}$), and it is here that the removal of the corona-straddling collection $\mathcal{P}_{cor}^{A}$ is essential, while forms with pairs in $\mathcal{P}^{\flat small}$ will be absorbed.

5.6.6 Size estimates

Now we turn to proving the *size estimates* we need for these collections. Recall that the *restricted* norm $\widehat{\mathfrak{N}}^{A,\mathcal{P}}_{stop,\triangle\omega}$ is the best constant in the inequality

$$|\mathsf{B}|^{A,\mathcal{P}}_{stop,\triangle\omega}(f,g) \leq \widehat{\mathfrak{N}}^{A,\mathcal{P}}_{stop,\triangle\omega} \left\| \mathsf{P}^{\sigma,\mathbf{b}}_{\Pi_1\mathcal{P}} f \right\|^{\bigstar}_{L^2(\sigma)} \left\| \mathsf{P}^{\omega,\mathbf{b}^*}_{\Pi_2\mathcal{P}} g \right\|^{\bigstar}_{L^2(\omega)}$$

where $f \in L^2(\sigma)$ satisfies $E^\sigma_I |f| \leq \alpha_{\mathcal{A}}(A)$ for all $I \in \mathcal{C}_A$, and $g \in L^2(\omega)$.

Proposition 5.6.19. *Suppose ρ in (5.6.31) is greater than* 1, *and $\mathcal{P}$ is a* reduced admissible *collection of pairs for a dyadic cube A. Let $\mathcal{P} = \mathcal{P}^{\flat big}\dot{\cup}\mathcal{P}^{\flat small}$ be the decomposition satisfying above, i.e.*

$$\mathcal{P} = \left\{ \bigcup_{L\in\mathcal{H}} \mathcal{P}^{\flat\mathcal{H}-big}_{L,0} \right\} \bigcup \left\{ \bigcup_{t\geq 1} \bigcup_{L\in\mathcal{H}} \mathcal{P}^{\flat\mathcal{H}}_{L,t} \right\} \bigcup \left(\bigcup_{L\in\mathcal{L}} \mathcal{P}^{\flat\mathcal{L}-small}_{L,0} \right)$$

Then all of these collections $\mathcal{P}^{\flat\mathcal{L}-small}_{L,0}$, $\mathcal{P}^{\flat\mathcal{H}-big}_{L,0}$ and $\mathcal{P}^{\flat\mathcal{H}}_{L,t}$ are reduced admissible, and we have the estimate

$$\mathcal{S}^{\alpha,A}_{augsize}\left(\mathcal{P}^{\flat\mathcal{L}-small}_{L,0}\right)^2 \leq (\rho-1)\,\mathcal{S}^{\alpha,A}_{augsize}(\mathcal{P})^2\,, \qquad L \in \mathcal{L} \tag{5.6.35}$$

and the localized norm bounds,

$$\begin{aligned} \widehat{\mathfrak{N}}^{A,\ \bigcup_{L\in\mathcal{H}} \mathcal{P}^{\flat\mathcal{H}-big}_{L,0}}_{stop,\triangle\omega} &\leq C\mathcal{S}^{\alpha,A}_{augsize}(\mathcal{P})\,, \\ \widehat{\mathfrak{N}}^{A,\ \bigcup_{L\in\mathcal{H}} \mathcal{P}^{\flat\mathcal{H}}_{L,t}}_{stop,\triangle\omega} &\leq C\rho^{-\frac{t}{2}}\mathcal{S}^{\alpha,A}_{augsize}(\mathcal{P})\,, \qquad t \geq 1. \end{aligned} \tag{5.6.36}$$

Using this proposition on size estimates, we can finish the proof of (5.6.7), and hence the proof of (5.6.1).

Corollary 5.6.20. *The sublinear stopping form inequality (5.6.7) holds.*

Proof. Recall that $\widehat{\mathfrak{N}}^{A,\mathcal{P}}_{stop,\triangle^\omega}$ is the best constant in the inequality (5.6.10). Since $\left\{\mathcal{P}^{\flat\mathcal{L}-small}_{L,0}\right\}_{L\in\mathcal{L}}$ is a mutually orthogonal family of A-admissible pairs, the Orthogonality Lemma 5.6.4 implies that

$$\widehat{\mathfrak{N}}^{A,\bigcup_{L\in\mathcal{L}}\mathcal{P}^{\flat\mathcal{L}-small}_{L,0}}_{stop,\triangle^\omega} \leq \sup_{L\in\mathcal{L}} \widehat{\mathfrak{N}}^{A,\mathcal{P}^{\flat\mathcal{L}-small}_{L,0}}_{stop,\triangle^\omega}$$

Using this, together with the decomposition of $\mathcal{P}$ and (5.6.36) above, we obtain

$$\begin{aligned}
\widehat{\mathfrak{N}}^{A,\mathcal{P}}_{stop,\triangle^\omega} &\leq \sup_{L\in\mathcal{H}} \widehat{\mathfrak{N}}^{A,\bigcup_{L\in\mathcal{H}}\mathcal{P}^{\flat\mathcal{H}-big}_{L,0}}_{stop,\triangle^\omega} + \sum_{t=1}^{M+1} \sup_{L\in\mathcal{H}} \widehat{\mathfrak{N}}^{A,\bigcup_{L\in\mathcal{H}}\mathcal{P}^{\flat\mathcal{H}}_{L,t}}_{stop,\triangle^\omega} + \widehat{\mathfrak{N}}^{A,\bigcup_{L\in\mathcal{L}}\mathcal{P}^{\flat\mathcal{L}-small}_{L,0}}_{stop,\triangle^\omega} \\
&\lesssim \mathcal{S}^{\alpha,A}_{augsize}(\mathcal{P}) + \left(\sum_{t=1}^{M+1} \rho^{-\frac{t}{2}}\right) \mathcal{S}^{\alpha,A}_{augsize}(\mathcal{P}) + \sup_{L\in\mathcal{L}} \widehat{\mathfrak{N}}^{A,\mathcal{P}^{\flat\mathcal{L}-small}_{L,0}}_{stop,\triangle^\omega}
\end{aligned}$$

Since the admissible collection $\mathcal{P}^A$ in (5.6.2) that arises in the stopping form is finite, we can define $\mathfrak{L}$ to be the best constant in the inequality

$$\widehat{\mathfrak{N}}^{A,\mathcal{P}}_{stop,\triangle^\omega} \leq \mathfrak{L}\mathcal{S}^{\alpha,A}_{augsize}(\mathcal{P}) \text{ for all } A\text{-admissible collections } \mathcal{P}.$$

Now choose $\mathcal{P}$ so that

$$\frac{\widehat{\mathfrak{N}}^{A,\mathcal{P}}_{stop,\triangle^\omega}}{\mathcal{S}^{\alpha,A}_{augsize}(\mathcal{P})} > \frac{1}{2}\mathfrak{L} = \frac{1}{2} \sup_{\mathcal{Q} \text{ is } A\text{-admissible}} \frac{\widehat{\mathfrak{N}}^{A,\mathcal{Q}}_{stop,\triangle^\omega}}{\mathcal{S}^{\alpha,A}_{augsize}(\mathcal{Q})} .$$

Then using $\sum_{t=1}^{M+1} \rho^{-\frac{t}{2}} \leq \frac{1}{\sqrt{\rho}-1}$ we have

$$\begin{aligned}
\mathfrak{L} &< 2\frac{\widehat{\mathfrak{N}}_{stop,\triangle\omega}^{A,\mathcal{P}}}{\mathcal{S}_{augsize}^{\alpha,A}(\mathcal{P})} \leq \frac{C\frac{1}{\sqrt{\rho}-1}\mathcal{S}_{augsize}^{\alpha,A}(\mathcal{P}) + C\sup_{L\in\mathcal{L}} \widehat{\mathfrak{N}}_{stop,\triangle\omega}^{A,\mathcal{P}_{L,0}^{\flat\mathcal{L}-small}}}{\mathcal{S}_{augsize}^{\alpha,A}(\mathcal{P})} \\
&\leq C\frac{1}{\sqrt{\rho}-1} + C\sup_{L\in\mathcal{L}} \mathfrak{L}\frac{\mathcal{S}_{augsize}^{\alpha,A}\left(\mathcal{P}_{L,0}^{\flat\mathcal{L}-small}\right)}{\mathcal{S}_{augsize}^{\alpha,A}(\mathcal{P})} \leq C\frac{1}{\sqrt{\rho}-1} + C\mathfrak{L}\sqrt{\rho-1}
\end{aligned}$$

where we have used (5.6.35) in the last line. If we choose $\rho > 1$ so that

$$C\sqrt{\rho-1} < \frac{1}{2}, \tag{5.6.37}$$

then we obtain $\mathfrak{L} \leq 2C\frac{1}{\sqrt{\rho}-1}$. Together with Lemma 5.6.9, this yields

$$\widehat{\mathfrak{N}}_{stop,\triangle\omega}^{A,\mathcal{P}} \leq \mathfrak{L}\mathcal{S}_{augsize}^{\alpha,A}(\mathcal{P}) \leq 2C\frac{1}{\sqrt{\rho}-1}\left(\mathcal{E}_2^{\alpha} + \sqrt{\mathfrak{A}_2^{\alpha}}\right)$$

as desired, and completes the proof of inequality (5.6.7). □

Thus, in view of Conclusion 5.6.4, it remains only to prove Proposition 5.6.19 using the Orthogonality and Straddling and Substraddling Lemmas above, and we now turn to this task.

Proof of Proposition 5.6.19. We split the proof into three parts.

Proof of (5.6.35): To prove the inequality (5.6.35), suppose first that $L \notin \mathcal{L}_{M+1}$. In the case that $L \in \mathcal{L}_0$ is an initial generation cube, then from (5.6.30) and the fact that every

$I \in \mathcal{P}_{L,0}^{\flat\mathcal{L}-small}$ satisfies $I \subsetneqq L$, we obtain that

$$\begin{aligned}
\mathcal{S}_{augsize}^{\alpha,A}\left(\mathcal{P}_{L,0}^{\flat\mathcal{L}-small}\right)^{2} &= \sup_{K'\in\Pi_{1}^{below}\mathcal{P}_{L,0}^{\flat\mathcal{L}-small}\cap\mathcal{C}_{A}'} \frac{\Psi^{\alpha}\left(K';\mathcal{P}_{L,0}^{\flat\mathcal{L}-small}\right)^{2}}{\left|K'\right|_{\sigma}} \\
&\leq \sup_{K'\in\Pi_{1}^{below}\mathcal{P}\cap\mathcal{C}_{A}':\ K'\subsetneqq L} \frac{\Psi^{\alpha}\left(K';\mathcal{P}_{L,0}^{\flat\mathcal{L}-small}\right)^{2}}{\left|K'\right|_{\sigma}} \\
&\leq \varepsilon\mathcal{S}_{augsize}^{\alpha,A}\left(\mathcal{P}\right)^{2}
\end{aligned}$$

Now suppose that $L \notin \mathcal{L}_0$ in addition to $L \notin \mathcal{L}_{M+1}$. Pick a pair $(I,J) \in \mathcal{P}_{L,0}^{\flat\mathcal{L}-small}$. Then I is in the restricted corona $\mathcal{C}_{L}^{\mathcal{L},'}$ and J is in the $\flat$*shifted* corona $\mathcal{C}_{L}^{\mathcal{L},\flat shift}$. Since $\mathcal{P}_{L,0}^{\flat\mathcal{L}-small}$ is a finite collection, the definition of $\mathcal{S}_{augsize}^{\alpha,A}\left(\mathcal{P}_{L,0}^{\flat\mathcal{L}-small}\right)$ shows that there is an cube $K \in \Pi_{1}^{below}\mathcal{P}_{L,0}^{\flat\mathcal{L}-small} \cap \mathcal{C}_{A}'$ so that

$$\mathcal{S}_{augsize}^{\alpha,A}\left(\mathcal{P}_{L,0}^{\flat\mathcal{L}-small}\right)^{2} = \frac{1}{|K|_{\sigma}}\left(\frac{\mathrm{P}^{\alpha}\left(K,\mathbf{1}_{A\setminus K}\sigma\right)}{|K|^{\frac{1}{n}}}\right)^{2}\omega_{\flat\mathcal{P}}\left(\mathbf{T}\left(K\right)\right).$$

Note that $K \subsetneqq L$ by definition of $\mathcal{P}_{L,0}^{\flat\mathcal{L}-small}$. Now let t be such that $L \in \mathcal{L}_t$, and define

$$t' = t'\left(K\right) \equiv \max\left\{s : \text{there is } L' \in \mathcal{L}_s \text{ with } L' \subset K\right\},$$

and note that $0 \leq t' < t$. First, suppose that $t' = 0$ so that K does not contain any $L' \in \mathcal{L}$. Then it follows from the construction at level $\ell = 0$ that

$$\frac{1}{|K|_{\sigma}}\left(\frac{\mathrm{P}^{\alpha}\left(K,\mathbf{1}_{A\setminus K}\sigma\right)}{|K|}\right)^{2}\omega_{\flat\mathcal{P}}\left(\mathbf{T}\left(K\right)\right) < \varepsilon\mathcal{S}_{augsize}^{\alpha,A}\left(\mathcal{P}\right)^{2},$$

and hence from $\rho = 1 + \varepsilon$ we obtain

$$\mathcal{S}_{augsize}^{\alpha,A}\left(\mathcal{P}_{L,0}^{\flat\mathcal{L}-small}\right)^2 < \varepsilon\mathcal{S}_{augsize}^{\alpha,A}\left(\mathcal{P}\right)^2 = \left(\rho-1\right)\mathcal{S}_{augsize}^{\alpha,A}\left(\mathcal{P}\right)^2 .$$

Now suppose that $t' \geq 1$. Then K fails the stopping condition (5.6.32) with $m = t' + 1$, since otherwise it would contain a cube $L'' \in \mathcal{L}_{t'+1}$ contradicting our definition of t', and so

$$\omega_{\flat\mathcal{P}}\left(\mathbf{T}\left(K\right)\right) < \rho\omega_{\flat\mathcal{P}}\left(\mathbf{V}\left(K\right)\right) \text{ where } \mathbf{V}\left(K\right) \equiv \bigcup_{L'\in\bigcup_{\ell=0}^{t'}\mathcal{L}_\ell:\ L'\subset K} \mathbf{T}\left(L'\right).$$

Now we use the crucial fact that the positive measure $\omega_{\flat\mathcal{P}}$ is *additive* and finite to obtain from this that

$$\omega_{\flat\mathcal{P}}\left(\mathbf{T}\left(K\right)\setminus\mathbf{V}\left(K\right)\right) = \omega_{\flat\mathcal{P}}\left(\mathbf{T}\left(K\right)\right) - \omega_{\flat\mathcal{P}}\left(\mathbf{V}\left(K\right)\right) \leq \left(\rho-1\right)\omega_{\flat\mathcal{P}}\left(\mathbf{V}\left(K\right)\right). \qquad (5.6.38)$$

Now recall that

$$\mathcal{S}_{augsize}^{\alpha,A}\left(\mathcal{Q}\right)^2 \equiv \sup_{K\in\Pi_1^{below}\mathcal{Q}\cap C_A'} \frac{1}{\left|K\right|_\sigma}\left(\frac{\mathrm{P}^\alpha\left(K,\mathbf{1}_{A\setminus K}\sigma\right)}{\left|K\right|^{\frac{1}{n}}}\right)^2 \left\|\mathsf{Q}_{\Pi_2^{K,aug}\mathcal{Q}}^{\omega,\mathbf{b}^*}x\right\|_{L^2(\omega)}^{\spadesuit 2}.$$

We claim it follows that for each $J \in \Pi_2^{K,aug}\mathcal{P}_{L,0}^{\flat\mathcal{L}-small}$ the support $\left(c_{J^\flat},\ell\left(J^\flat\right)\right)$ of the atom $\delta_{\left(c_{J^\flat},\ell\left(J^\flat\right)\right)}$ is contained in the set $\mathbf{T}\left(K\right)$, but not in the set

$$\mathbf{V}\left(K\right) \equiv \bigcup\left\{\mathbf{T}\left(L'\right) : L' \in \bigcup_{\ell=0}^{t'}\mathcal{L}_\ell :\ L' \subset K\right\}.$$

Indeed, suppose in order to derive a contradiction, that $\left(c_{J^\flat},\ell\left(J^\flat\right)\right)\in\mathbf{T}\left(L'\right)$ for some $L'\in\mathcal{L}_\ell$ with $0\leq\ell\leq t'$. Recall that $L\in\mathcal{L}_t$ with $t'<t$ so that $L'\subsetneqq L$. Thus $\left(c_{J^\flat},\ell\left(J^\flat\right)\right)\in\mathbf{T}\left(L'\right)$ implies $J^\flat\subset L'$, which contradicts the fact that

$$J\in\Pi_2^K\mathcal{P}_{L,0}^{\flat\mathcal{L}-small}\subset\Pi_2\mathcal{P}_{L,0}^{\flat\mathcal{L}-small}=\left\{(I,J)\in\mathcal{P}:I\in\mathcal{C}_L^{\mathcal{L}}\setminus\{L\}\text{ and }J\in\mathcal{C}_L^{\mathcal{L},\flat shift}\right\}$$

implies $J^\flat\in\mathcal{C}_L^{\mathcal{L}}$ - because $L'\notin\mathcal{C}_L^{\mathcal{L}}$.

Thus from the definition of $\omega_{\flat\mathcal{P}}$ in (5.6.28), the 'energy' $\left\|\mathsf{Q}^{\omega,\mathbf{b}^*}_{\Pi_2^{K,aug}\mathcal{P}_{L,0}^{\flat\mathcal{L}-small}}x\right\|^{\spadesuit 2}_{L^2(\omega)}$ is at most the $\omega_{\flat\mathcal{P}}$-measure of $\mathbf{T}(K)\setminus\mathbf{V}(K)$. Using now

$$\omega_{\flat\mathcal{P}_{L,0}^{\flat\mathcal{L}-small}}\left(\mathbf{T}(K)\right)=\omega_{\flat\mathcal{P}_{L,0}^{\flat\mathcal{L}-small}}\left(\mathbf{T}(K)\setminus\mathbf{V}(K)\right)\leq\omega_{\flat\mathcal{P}}\left(\mathbf{T}(K)\setminus\mathbf{V}(K)\right)$$

and (5.6.38), we then have

$$\begin{aligned}
&\mathcal{S}_{augsize}^{\alpha,A}\left(\mathcal{P}_{L,0}^{\flat\mathcal{L}-small}\right)^2\leq\\
&\sup_{K\in\Pi_1^{below}\mathcal{P}_{L,0}^{\flat\mathcal{L}-small}\cap\mathcal{C}'_A}\frac{1}{|K|_\sigma}\left(\frac{\mathrm{P}^\alpha\left(K,\mathbf{1}_{A\setminus K}\sigma\right)}{|K|^{\frac{1}{n}}}\right)^2\omega_{\flat\mathcal{P}}\left(\mathbf{T}(K)\setminus\mathbf{V}(K)\right)\\
&\leq(\rho-1)\sup_{K\in\Pi_1^{below}\mathcal{P}_{L,0}^{\flat\mathcal{L}-small}\cap\mathcal{C}'_A}\frac{1}{|K|_\sigma}\left(\frac{\mathrm{P}^\alpha\left(K,\mathbf{1}_{A\setminus K}\sigma\right)}{|K|^{\frac{1}{n}}}\right)^2\omega_{\flat\mathcal{P}}\left(\mathbf{V}(K)\right)
\end{aligned}$$

and we can continue with

$$\begin{aligned}
\mathcal{S}_{augsize}^{\alpha,A}\left(\mathcal{P}_{L,0}^{\flat\mathcal{L}-small}\right)^2&\leq(\rho-1)\sup_{K\in\Pi_1^{below}\mathcal{P}\cap\mathcal{C}'_A}\frac{1}{|K|_\sigma}\left(\frac{\mathrm{P}^\alpha\left(K,\mathbf{1}_{A\setminus K}\sigma\right)}{|K|^{\frac{1}{n}}}\right)^2\omega_{\flat\mathcal{P}}\left(\mathbf{T}(K)\right)\\
&\leq(\rho-1)\,\mathcal{S}_{augsize}^{\alpha,A}(\mathcal{P})^2.
\end{aligned}$$

In the remaining case where $L \in \mathcal{L}_{M+1}$ we can include L as a testing cube K and the same reasoning applies. This completes the proof of (5.6.35).

To prove the other inequality (5.6.36) in Proposition 5.6.19, we will use the $\flat$ Straddling and Substraddling Lemmas to bound the norm of certain 'straddled' stopping forms by the augmented size functional $\mathcal{S}_{augsize}^{\alpha,A}$, and the Orthogonality Lemma to bound sums of 'mutually orthogonal' stopping forms. Recall that

$$\begin{aligned}
\mathcal{P}^{\flat big} &= \left\{ \bigcup_{L\in\mathcal{H}} \mathcal{P}_{L,0}^{\flat\mathcal{H}-big} \right\} \bigcup \left\{ \bigcup_{t\geq 1} \bigcup_{L\in\mathcal{H}} \mathcal{P}_{L,t}^{\flat\mathcal{H}} \right\} \equiv \mathcal{Q}_0^{\flat\mathcal{H}-big} \bigcup \mathcal{Q}_1^{\flat\mathcal{H}-big}; \\
\mathcal{Q}_0^{\flat\mathcal{H}-big} &\equiv \bigcup_{L\in\mathcal{L}} \mathcal{P}_{L,0}^{\flat\mathcal{H}-big}, \qquad \mathcal{Q}_1^{\flat\mathcal{H}-big} \equiv \bigcup_{t\geq 1} \mathcal{P}_t^{\flat\mathcal{H}-big}, \qquad \mathcal{P}_t^{\flat\mathcal{H}-big} \equiv \bigcup_{L\in\mathcal{H}} \mathcal{P}_{L,t}^{\flat\mathcal{H}}
\end{aligned}$$

Proof of the second line in (5.6.36): We first turn to the collection

$$\begin{aligned}
\mathcal{Q}_1^{\flat\mathcal{H}-big} &= \bigcup_{t\geq 1} \bigcup_{L\in\mathcal{H}} \mathcal{P}_{L,t}^{\flat\mathcal{H}} = \bigcup_{t\geq 1} \mathcal{P}_t^{\flat\mathcal{H}-big}; \\
\mathcal{P}_t^{\flat\mathcal{H}-big} &\equiv \bigcup_{L\in\mathcal{L}} \mathcal{P}_{L,t}^{\flat\mathcal{H}}, \qquad t \geq 1,
\end{aligned}$$

where

$$\mathcal{P}_{L,t}^{\flat\mathcal{H}} = \left\{ (I,J) \in \mathcal{P} : I \in \mathcal{C}_L^{\mathcal{H}}, J \in \mathcal{C}_{L'}^{\mathcal{H},\flat shift} \text{ for some } L' \in \mathcal{H}_{d_{indent}(L)+t}, L' \subset L \right\}.$$

We now claim that the second line in (5.6.36) holds, i.e.

$$\widehat{\mathfrak{N}}_{stop,\triangle^\omega}^{A,\mathcal{P}_t^{\flat\mathcal{H}-big}} \leq C\rho^{-\frac{t}{2}} \mathcal{S}_{augsize}^{\alpha,A}(\mathcal{P}), \qquad t \geq 1, \tag{5.6.39}$$

which recovers the key geometric gain obtained by Lacey in [27], except that here we are only gaining this decay relative to the indented subtree $\mathcal{H}$ of the tree $\mathcal{L}$.

The case $t=1$ can be handled with relative ease since decay is not relevant here. Indeed, $\mathcal{P}_{L,1}^{\flat\mathcal{H}}$ straddles the collection $\mathfrak{C}_{\mathcal{H}}(L)$ of $\mathcal{H}$ -children of L, and so the localized $\flat$Straddling Lemma 5.6.15 applies to give

$$\widehat{\mathfrak{N}}_{stop,\triangle^{\omega}}^{A,\mathcal{P}_{L,1}^{\flat\mathcal{H}}} \leq C\mathcal{S}_{augsize}^{\alpha,A}\left(\mathcal{P}_{L,1}^{\flat\mathcal{H}}\right) \leq C\mathcal{S}_{augsize}^{\alpha,A}(\mathcal{P}),$$

and then the Orthogonality Lemma 5.6.4 applies to give

$$\widehat{\mathfrak{N}}_{stop,\triangle^{\omega}}^{A,\mathcal{P}_{1}^{\flat\mathcal{H}-big}} \leq \sup_{L\in\mathcal{H}} \mathfrak{N}_{stop,\triangle^{\omega}}^{A,\mathcal{P}_{L,1}^{\flat\mathcal{H}}} \leq C\mathcal{S}_{augsize}^{\alpha,A}(\mathcal{P}),$$

since $\left\{\mathcal{P}_{L,1}^{\flat\mathcal{H}}\right\}_{L\in\mathcal{L}}$ is mutually orthogonal as $\mathcal{P}_{L,1}^{\flat\mathcal{H}} \subset \mathcal{C}_{L}^{\mathcal{H}} \times \mathcal{C}_{L'}^{\mathcal{H},\flat shift}$ with $L \in \mathcal{H}_k$ and $L' \in \mathcal{H}_{k+1}$ for indented depth $k = k(L)$. The case $t=2$ is equally easy.

Now we consider the case $t \geq 2$, where it is essential to obtain geometric decay in t. We remind the reader that all of our admissible collections $\mathcal{P}_{L,t}^{\flat\mathcal{H}}$ are *reduced* by Conclusion 5.6.4. We again apply Lemma 5.6.15 to $\mathcal{P}_{L,t}^{\flat\mathcal{H}}$ with $\mathcal{S} = \mathfrak{C}_{\mathcal{H}}(L)$, so that for any $(I,J) \in \mathcal{P}_{L,t}^{\flat\mathcal{H}}$, there is $H' \in \mathfrak{C}_{\mathcal{H}}(L)$ with $J^{\flat} \subset H' \subsetneqq I \in \mathcal{C}_{L}^{\mathcal{H}}$. But this time we must use the stronger localized bounds $\mathcal{S}_{locsize}^{\alpha,A;S}$ with an S-hole, that give

$$\begin{aligned}
\widehat{\mathfrak{N}}_{stop,\triangle^{\omega}}^{A,\mathcal{P}_{L,t}^{\flat\mathcal{H}}} &\leq C \sup_{H'\in\mathfrak{C}_{\mathcal{H}}(L)} \mathcal{S}_{locsize}^{\alpha,A;H'}\left(\mathcal{P}_{L,t}^{\flat\mathcal{H}}\right), \qquad t \geq 0; \\
\mathcal{S}_{locsize}^{\alpha,A;H'}\left(\mathcal{P}_{L,t}^{\flat\mathcal{H}}\right)^2 &= \sup_{K\in\mathcal{W}^*(H')\cap\mathcal{C}_A'} \frac{1}{|K|_{\sigma}} \left(\frac{\mathrm{P}^{\alpha}\left(K,\mathbf{1}_{A\setminus H'}\sigma\right)}{|K|^{\frac{1}{n}}}\right)^2 \sum_{J\in\Pi_2^{K,aug}\mathcal{P}_{L,t}^{\flat\mathcal{H}}} \left\|\triangle_J^{\omega,\mathbf{b}^*} x\right\|_{L^2(\omega)}^{\spadesuit 2}
\end{aligned}$$

It remains to show that

$$\sum_{J\in\Pi_2^{K,aug}\mathcal{P}_{L,t}^{\flat\mathcal{H}}}\left\Vert\triangle_J^{\omega,\mathbf{b}^*}x\right\Vert_{L^2(\omega)}^{\spadesuit 2}\leq\rho^{-(t-2)}\omega_{\flat\mathcal{P}}\left(\mathbf{T}\left(K\right)\right),\tag{5.6.40}$$
$$\text{for } t\geq 2,\ K\in\mathcal{W}^*\left(H'\right)\cap\mathcal{C}'_A,\ H'\in\mathfrak{C}_{\mathcal{H}}\left(L\right)$$

so that we then have

$$\begin{aligned}&\frac{1}{\left|K\right|_\sigma}\left(\frac{\mathrm{P}^\alpha\left(K,\mathbf{1}_{A\setminus H'}\sigma\right)}{\left|K\right|^{\frac{1}{n}}}\right)^2\sum_{J\in\Pi_2^{K,aug}\mathcal{P}_{L,t}^{\flat\mathcal{H}}}\left\Vert\triangle_J^{\omega,\mathbf{b}^*}x\right\Vert_{L^2(\omega)}^{\spadesuit 2}\\ \leq\ &\rho^{-(t-2)}\frac{1}{\left|K\right|_\sigma}\left(\frac{\mathrm{P}^\alpha\left(K,\mathbf{1}_{A\setminus K}\sigma\right)}{\left|K\right|^{\frac{1}{n}}}\right)^2\omega_{\flat\mathcal{P}}\left(\mathbf{T}\left(K\right)\right)\leq\rho^{-(t-2)}\mathcal{S}_{augsize}^{\alpha,A}\left(\mathcal{P}\right)^2\end{aligned}$$

by (5.6.29), and hence conclude the required bound for $\mathfrak{N}_{stop,\triangle^\omega}^{A,\mathcal{P}_{L,t}^{\flat\mathcal{H}}}$, namely that

$$\begin{aligned}&\widehat{\mathfrak{N}}_{stop,\triangle^\omega}^{A,\mathcal{P}_{L,t}^{\flat\mathcal{H}}}\qquad(5.6.41)\\ \leq\ &C\sup_{H'\in\mathfrak{C}_{\mathcal{H}}(L)}\ \sup_{K\in\mathcal{W}^*(H')\cap\mathcal{C}'_A}\sqrt{\frac{1}{\left|K\right|_\sigma}\left(\frac{\mathrm{P}^\alpha\left(K,\mathbf{1}_{A\setminus H'}\sigma\right)}{\left|K\right|^{\frac{1}{n}}}\right)^2\sum_{J\in\Pi_2^{K,aug}\mathcal{P}_{L,t}^{\flat\mathcal{H}}}\left\Vert\triangle_J^{\omega,\mathbf{b}^*}x\right\Vert_{L^2(\omega)}^{\spadesuit 2}}\\ \leq\ &C\sqrt{\rho^{-(t-2)}}\mathcal{S}_{augsize}^{\alpha,A}\left(\mathcal{P}\right)=C'\rho^{-\frac{t}{2}}\mathcal{S}_{augsize}^{\alpha,A}\left(\mathcal{P}\right).\end{aligned}$$

Remark on lack of usual goodness: To prove (5.6.40), it is essential that the cubes $H^{k+2}\in\mathcal{H}_{k+2}$ at the next indented level down from $H^{k+1}\in\mathfrak{C}_{\mathcal{H}}\left(L\right)$ are each contained in one of the Whitney cubes $K\in\mathcal{W}\left(H^{k+1}\right)\cap\mathcal{C}'_A$ for some $H^{k+1}\in\mathfrak{C}_{\mathcal{H}}\left(L\right)$. And this is the reason we introduced the indented corona - namely so that $3H^{k+2}\subset H^{k+1}$ for some

$H^{k+1} \in \mathfrak{C}_{\mathcal{H}}(L)$, and hence $H^{k+2} \subset K$ for some $K \in \mathcal{W}\left(H^{k+1}\right)$. In the argument of Lacey in [27], the corresponding cubes were good in the usual sense, and so the above triple property was automatic.

So we begin by fixing $K \in \mathcal{W}^{*}\left(H^{k+1}\right) \cap \mathcal{C}'_{A}$ with $H^{k+1} \in \mathfrak{C}_{\mathcal{H}}(L)$, and noting from the above that each $J \in \Pi_2^{K,aug} \mathcal{P}_{L,t}^{\flat\mathcal{H}}$ satisfies

$$J^{\flat} \subset H^{k+t} \subset H^{k+t-1} \subset \ldots \subset H^{k+2} \subset K$$

for $H^{k+j} \in \mathcal{H}_{k+j}$ uniquely determined by $J^{\flat}$. Thus for $t \geq 2$ we have

$$\begin{aligned} \sum_{J \in \Pi_2^{K,aug} \mathcal{P}_{L,t}^{\flat\mathcal{H}}} \left\| \triangle_J^{\omega,\mathbf{b}^*} x \right\|_{L^2(\omega)}^{\spadesuit 2} &= \sum_{\substack{H^{k+t} \in \mathcal{H}_{k+t} \\ H^{k+t} \subset K}} \sum_{\substack{J \in \Pi_2^{K,aug} \mathcal{P}_{L,t}^{\flat\mathcal{H}} \\ J^{\flat} \subset H^{k+t}}} \left\| \triangle_J^{\omega,\mathbf{b}^*} x \right\|_{L^2(\omega)}^{\spadesuit 2} \\ &\leq \sum_{\substack{H^{k+t} \in \mathcal{H}_{k+t} \\ H^{k+t} \subset K}} \omega_{\flat\mathcal{P}}\left(\mathbf{T}\left(H^{k+t}\right)\right) \end{aligned}$$

In the case $t = 2$ we are done since the final sum above is at most $\omega_{\flat\mathcal{P}}(\mathbf{T}(K))$.

Now suppose $t \geq 3$. In order to obtain geometric gain in t, we will apply the stopping criterion (5.6.32) in the following form,

$$\sum_{L' \in \mathfrak{C}_{\mathcal{L}}(L_0)} \omega_{\flat\mathcal{P}}\left(\mathbf{T}\left(L'\right)\right) = \omega_{\flat\mathcal{P}}\left(\bigcup_{L' \in \mathfrak{C}_{\mathcal{L}}(L_0)} \mathbf{T}\left(L'\right)\right) \leq \frac{1}{\rho} \omega_{\flat\mathcal{P}}\left(\mathbf{T}\left(L_0\right)\right), \quad \text{for all } L_0 \in \mathcal{L} \tag{5.6.42}$$

where we have used the fact that the *maximal* cubes L' in the collection

$$\bigcup_{\ell=0}^{m-1} \left\{L' \in \mathcal{L}_{\ell} : \ L' \subset L_0\right\}$$

for $L_0 \in \mathcal{L}_m$ (that appears in (5.6.32)) are precisely the $\mathcal{L}$-children of L_0 in the tree $\mathcal{L}$ (the cubes L' above are strictly contained in L_0 since $\rho > 1$ in (5.6.32)), so that

$$\bigcup_{L' \in \Gamma} L' = \bigcup_{L' \in \mathfrak{C}_{\mathcal{L}}(L_0)} L' \text{ where } \Gamma \equiv \bigcup_{\ell=0}^{m-1} \left\{ L' \in \mathcal{L}_\ell : \ L' \subset L_0 \right\}.$$

In order to apply (5.6.42), we collect the pairwise disjoint cubes $H^{k+t} \in \mathcal{H}_{k+t}$ such that $H^{k+t} \subset H^{k+2} \subset K$, into groups according to which cube $L^{k'+t-2} \in \mathcal{G}_{k'+t-2}$ they are contained in, where $k' = d_{geom}\left(H^{k+2}\right)$ is the geometric depth of H^{k+2} in the tree $\mathcal{L}$ introduced in (5.6.33). It follows that each cube $H^{k+t} \in \mathcal{H}_{k+t}$ is contained in a unique cube $L^{d_{geom}\left(H^{k+2}\right)+t-2} \in \mathcal{G}_{d_{geom}\left(H^{k+2}\right)+t-2}$. Thus we obtain from the previous inequality that

$$\begin{aligned}
\sum_{J \in \Pi_2^{K,aug} \mathcal{P}_{L,t}^{\flat \mathcal{H}}} \left\| \triangle_J^{\omega,\mathbf{b}^*} x \right\|_{L^2(\omega)}^{\spadesuit 2} &\leq \sum_{\substack{H^{k+t} \in \mathcal{H}_{k+t} \\ H^{k+t} \subset K}} \omega_{\flat \mathcal{P}} \left(\mathbf{T}\left(H^{k+t} \right) \right) \\
&\leq \sum_{\substack{H^{k+2} \in \mathcal{H}_{k+2} \\ H^{k+2} \subset K}} \sum_{\substack{L^{k'+t-2} \in \mathcal{G}_{k'+t-2} \\ L^{k'+t-2} \subset H^{k+2} \\ \text{where } k' = d_{geom}\left(H^{k+2}\right)}} \omega_{\flat \mathcal{P}} \left(\mathbf{T}\left(L^{k'+t-2} \right) \right)
\end{aligned}$$

and this last expression is equal to

$$\sum_{\substack{H^{k+2}\in\mathcal{H}_{k+2}\\ H^{k+2}\subset K}} \sum_{\substack{L^{k'+t-3}\in\mathcal{G}_{k'+t-3}\\ k'+t-3\subset H^{k+2}\\ \text{where } k'=d_{geom}\left(H^{k+2}\right)}} \left\{ \sum_{\substack{L^{k'+t-2}\in\mathcal{G}_{k'+t-2}\\ L^{k'+t-2}\subset L^{k'+t-3}\\ \text{where } k'=d_{geom}\left(H^{k+2}\right)}} \omega_{\flat\mathcal{P}}\left(\mathbf{T}\left(L^{k'+t-2}\right)\right)\right\}$$

$$\leq \sum_{\substack{H^{k+2}\in\mathcal{H}_{k+2}\\ H^{k+2}\subset K}} \sum_{\substack{L^{k'+t-3}\in\mathcal{G}_{k'+t-3}\\ L^{k'+t-3}\subset H^{k+2}\\ \text{where } k'=d_{geom}\left(H^{k+2}\right)}} \left\{\frac{1}{\rho}\omega_{\flat\mathcal{P}}\left(\mathbf{T}\left(L^{k'+t-3}\right)\right)\right\}$$

where in the last line we have used (5.6.42) with $L_0 = L^{k'+t-3}$ on the sum in braces. We then continue (if necessary) with

$$\begin{aligned}
\sum_{J\in\Pi_2^{K,aug}\mathcal{P}_{L,t}^{\flat\mathcal{H}}} \left\|\triangle_J^{\omega,\mathbf{b}^*}x\right\|_{L^2(\omega)}^{\spadesuit 2} &\leq \frac{1}{\rho} \sum_{\substack{H^{k+2}\in\mathcal{H}_{k+2}\\ H^{k+2}\subset K}} \sum_{\substack{L^{k'+t-3}\in\mathcal{G}_{k'+t-3}\\ L^{k'+t-3}\subset H^{k+2}\\ \text{where } k'=d_{geom}\left(H^{k+2}\right)}} \omega_{\flat\mathcal{P}}\left(\mathbf{T}\left(L^{k'+t-3}\right)\right)\\
&\leq \frac{1}{\rho^2} \sum_{\substack{H^{k+2}\in\mathcal{H}_{k+2}\\ H^{k+2}\subset K}} \sum_{\substack{L^{k'+t-4}\in\mathcal{G}_{k'+t-4}\\ L^{k'+t-4}\subset H^{k+2}\\ \text{where } k'=d_{geom}\left(H^{k+2}\right)}} \omega_{\flat\mathcal{P}}\left(\mathbf{T}\left(L^{k'+t-4}\right)\right)\\
&\vdots\\
&\leq \frac{1}{\rho^{t-2}} \sum_{\substack{H^{k+2}\in\mathcal{H}_{k+2}\\ H^{k+2}\subset K}} \sum_{\substack{L^{k'}\in\mathcal{G}_{k'}:\ L^{k'}\subset H^{k+2}\\ \text{where } k'=d_{geom}\left(H^{k+2}\right)}} \omega_{\flat\mathcal{P}}\left(\mathbf{T}\left(L^{k'}\right)\right)
\end{aligned}$$

Since $L^{k'} \subset H^{k+2}$ implies $L^{k'} = H^{k+2}$, we now obtain

$$\begin{aligned}\sum_{J\in\Pi_2^{K,aug}\mathcal{P}_{L,t}^{\flat\mathcal{H}}} \left\|\triangle_J^{\omega,\mathbf{b}^*} x\right\|_{L^2(\omega)}^{\spadesuit 2} &\leq \frac{1}{\rho^{t-2}} \sum_{H^{k+2}\in\mathcal{H}_{k+2}:\ H^{k+2}\subset K} \omega_{\flat\mathcal{P}}\left(\mathbf{T}\left(H^{k+2}\right)\right) \\ &\leq \frac{1}{\rho^{t-2}}\omega_{\flat\mathcal{P}}\left(\mathbf{T}\left(K\right)\right)\end{aligned}$$

which completes the proof of (5.6.40), and hence that of (5.6.41). Finally, an application of the Orthogonality Lemma 5.6.4 proves (5.6.39).

Proof of the first line in (5.6.36): At last we turn to proving the first line in (5.6.36). Recalling that $\mathcal{T}(L) = \mathcal{C}_L^{\mathcal{H}}(\mathcal{L})\setminus\{L\}$, we consider the collection

$$\begin{aligned}&\mathcal{Q}_0^{\flat\mathcal{H}-big} = \bigcup_{L\in\mathcal{H}} \mathcal{P}_{L,0}^{\flat\mathcal{H}-big} \\ \text{where}\qquad &\mathcal{P}_{L,0}^{\flat\mathcal{H}-big} = \left\{(I,J)\in\mathcal{P}_{L,0}^{\flat\mathcal{H}} : \text{there is } L'\in\mathcal{T}(L),\, J^\flat\subset L'\subset I\right\}, L\in\mathcal{H} \\ \text{and}\qquad &\mathcal{P}_{L,0}^{\flat\mathcal{H}} = \left\{(I,J)\in\mathcal{P} : I\in\mathcal{C}_L^{\mathcal{H}} J\in\mathcal{C}_L^{\mathcal{H},\flat shift} \text{ for some } L\in\mathcal{H}\right\}, L\in\mathcal{H}\end{aligned}$$

and begin by claiming that

$$\widehat{\mathfrak{N}}_{stop,\triangle^\omega}^{A,\mathcal{P}_{L,0}^{\flat\mathcal{H}-big}} \leq C\mathcal{S}_{augsize}^{\alpha,A}\left(\mathcal{P}_{L,0}^{\flat\mathcal{H}-big}\right) \leq C\mathcal{S}_{augsize}^{\alpha,A}\left(\mathcal{P}\right), \qquad L\in\mathcal{H}. \tag{5.6.43}$$

To see this, we fix $L\in\mathcal{H}$ and order the cubes of $\mathcal{T}(L) = \left\{L^{k,i}\right\}_{k,i}$, where $1\leq i\leq n_k$ where $L^0 = L$ and $L^{1,i}$ are the maximal cubes in L^0 and then $L^{k+1,i}$ are the maximal cubes inside a cube $L^{k,j}$ of some previous generation. Then $\mathcal{P}_{L,0}^{\flat\mathcal{H}-big}$ can be decomposed as follows, remembering that $J^\flat \subset I \subset L$ for $(I,J) \in \mathcal{P}_{L,0}^{\flat\mathcal{H}-big} \subset \mathcal{P}_{L,0}^{\flat\mathcal{H}}$:

$$\begin{aligned}
\mathcal{P}_{L,0}^{\flat\mathcal{H}-big} &= \dot{\bigcup_{k,i}} \left\{ \mathcal{R}^{\flat\mathcal{L}}_{L^{k,i}_{out,out}} \ \dot{\cup}\ \mathcal{R}^{\flat\mathcal{L}}_{L^{k,i}_{out,in}} \ \dot{\cup}\ \mathcal{R}^{\flat\mathcal{L}}_{L^{k,i}_{in}} \right\} \\
&= \left(\dot{\bigcup_{k,i}} \mathcal{R}^{\flat\mathcal{L}}_{L^{k,i}_{out,out}} \right) \dot{\cup} \left(\dot{\bigcup_{k,i}} \mathcal{R}^{\flat\mathcal{L}}_{L^{k,i}_{out,in}} \right) \dot{\cup} \left(\dot{\bigcup_{k,i}} \mathcal{R}^{\flat\mathcal{L}}_{L^{k,i}_{in}} \right) \ ; \\
\mathcal{R}^{\flat\mathcal{L}}_{L^{k,i}_{out,in}} &\equiv \left\{ (I,J) \in \mathcal{P}_{L,0}^{\flat\mathcal{H}-big} : I \in \mathcal{C}^{\mathcal{L}}_{L^{k-1,i}} \text{ and } J^\flat \subset L^{k,i}_{out,in} \right\}, \\
\mathcal{R}^{\flat\mathcal{L}}_{L^{k,i}_{out,out}} &\equiv \left\{ (I,J) \in \mathcal{P}_{L,0}^{\flat\mathcal{H}-big} : I \in \mathcal{C}^{\mathcal{L}}_{L^{k-1,i}} \text{ and } J^\flat \subset L^{k,i}_{out,out} \right\}, \\
\mathcal{R}^{\flat\mathcal{L}}_{L^{k,i}_{in}} &\equiv \left\{ (I,J) \in \mathcal{P}_{L,0}^{\flat\mathcal{H}-big} : I \in \mathcal{C}^{\mathcal{L}}_{L^{k-1,i}} \text{ and } J^\flat \in \mathcal{C}^{\mathcal{L}}_{L^{k-1,i}} \text{ and } J^\flat \cap L^{k,i} = \emptyset \right\} \\
&= \left\{ (I,J) \in \mathcal{P}_{L,0}^{\flat\mathcal{H}-big} : I = L^{k-1,i} \text{ and } J^\flat \in \mathcal{C}^{\mathcal{L}}_{L^{k-1,i}} \text{ and } J^\flat \cap L^{k-1,i}_{out} = \emptyset, \right\},
\end{aligned}$$

where by $L^{k,i}_{in}$ we denote the union of the children of $L^{k,i}$ that do not touch the boundary of L, by $L^{k,i}_{out,in}$ the union of the grandchildren of $L^{k,i}$ that do not touch the boundary of L while their father does, and by $L^{k,i}_{out,out}$ the grandchildren of $L^{k,i}$ that touch the boundary of L and where in the last line we have used the fact that if $I, J^\flat \in \mathcal{C}^{\mathcal{L}}_{L^{k-1,i}}$ and there is $L' \in \mathcal{T}(L)$ with $J^\flat \subset L' \subset I$, then we must have $I = L^{k-1,i}$. All of the pairs $(I,J) \in \mathcal{P}_{L,0}^{\flat\mathcal{H}-big}$ are included in either $\mathcal{R}^{\flat\mathcal{L}}_{L^{k,i}_{out,in}}$, $\mathcal{R}^{\flat\mathcal{L}}_{L^{k,i}_{out,out}}$ or $\mathcal{R}^{\flat\mathcal{L}}_{L^{k,i}_{in}}$ for some k, since if $J^\flat \supset L^{k,i}$, then $J^\flat$ shares boundary with L, which contradicts the fact that $3J^\flat \subset J^{\maltese} \subset I \subset L$.

We can easily deal with the 'in' collection $\mathcal{Q}^{in} \equiv \dot{\bigcup}_{k=1}^{\infty} \mathcal{R}^{\flat\mathcal{L}}_{L^{k,i}_{in}}$ by applying a *trivial* case of the $\flat$Straddling Lemma to $\mathcal{R}^{\flat\mathcal{L}}_{L^{k,i}_{in}}$ with a single straddling cube, followed by an application of the Orthogonality Lemma to $\mathcal{Q}^{in}$. More precisely, every pair $(I,J) \in \mathcal{R}^{\flat\mathcal{L}}_{L^{k,i}_{in}}$ satisfies $J^\flat \subset L^{k-1,i} = I$, so that the reduced admissible collection $\mathcal{R}^{\flat\mathcal{L}}_{L^{k,i}_{in}}$ $\flat$straddles the trivial choice $\mathcal{S}=\left\{L^{k-1,i}\right\}$, the singleton consisting of just the cube $L^{k-1,i}$. Then the inequality

$$\widehat{\mathfrak{N}}_{stop,\triangle^{\omega}}^{A,\mathcal{R}^{\flat\mathcal{L}}_{L^{k,i}_{in}}}\leq C\mathcal{S}_{augsize}^{\alpha,A}\left(\mathcal{R}^{\flat\mathcal{L}}_{L^{k,i}_{in}}\right),$$

follows from $\flat$Straddling Lemma 5.6.15. The collection $\left\{\mathcal{R}^{\flat\mathcal{L}}_{L^{k,i}_{in}}\right\}_{k,i}$ is mutually orthogonal since

$$\begin{aligned}\mathcal{R}^{\flat\mathcal{L}}_{L^{k,i}_{in}} &\subset \mathcal{C}^{\mathcal{L}}_{L^{k-1,i}}\times\mathcal{C}^{\mathcal{L},\flat shift}_{L^{k-1,i}}\\ \sum_{k=1}^{\infty}\sum_{i=1}^{n_k}\mathbf{1}_{\mathcal{C}^{\mathcal{L}}_{L^{k-1,i}}}\leq\mathbf{1} &\text{ and } \sum_{k=1}^{\infty}\sum_{i=1}^{n_k}\mathbf{1}_{\mathcal{C}^{\mathcal{L},\flat shift}_{L^{k-1,i}}}\leq\mathbf{1}.\end{aligned}$$

Since $\dot{\bigcup}_{k,i}\mathcal{R}^{\flat\mathcal{L}}_{L^{k,i}_{in}}$ is reduced and admissible (each $J\in\Pi_2\left(\dot{\bigcup}_{k,i}\mathcal{R}^{\flat\mathcal{L}}_{L^{k,i}_{in}}\right)$ is paired with a single I, namely the top of the $\mathcal{L}$-corona to which $J^{\flat}$ belongs), the Orthogonality Lemma 5.6.4 applies to obtain the estimate

$$\widehat{\mathfrak{N}}_{stop,\triangle^{\omega}}^{A,\bigcup_{k,i}\mathcal{R}^{\flat\mathcal{L}}_{L^{k,i}_{in}}}\leq\sup_{\substack{1\leq k\\ 1\leq i\leq n_k}}\widehat{\mathfrak{N}}_{stop,\triangle^{\omega}}^{A,\mathcal{R}^{\flat\mathcal{L}}_{L^{k,i}_{in}}}\leq C\sup_{\substack{1\leq k\\ 1\leq i\leq n_k}}\mathcal{S}_{augsize}^{\alpha,A}\left(\mathcal{R}^{\flat\mathcal{L}}_{L^{k,i}_{in}}\right)\leq C\mathcal{S}_{augsize}^{\alpha,A}\left(\mathcal{P}^{\flat\mathcal{H}-big}_{L,0}\right)\tag{5.6.44}$$

Now we turn to estimating the norm of the 'out-in' collection $\mathcal{Q}^{out,in}\equiv\bigcup_{k,i}\mathcal{R}^{\flat\mathcal{L}}_{L^{k,i}_{out,in}}$. First we note that $L^{k,i}_{out,in}\in\mathcal{C}^{\mathcal{A},restrict}_{A}$ if $(I,J)\in\mathcal{R}^{\flat\mathcal{L}}_{L^{k,i}_{out,in}}$ since $\mathcal{R}^{\flat\mathcal{L}}_{L^{k,i}_{out,in}}$ is reduced, i.e. doesn't contain any pairs (I,J) with $J^{\flat}\subset A'$ for some $A'\in\mathfrak{C}_{\mathcal{A}}(A)$. Next we note that $\mathcal{Q}^{out,in}$ is admissible since if $J\in\Pi_2\mathcal{Q}^{out,in}$, then $J\in\Pi_2\mathcal{R}^{\flat\mathcal{L}}_{L^{k,i}_{out,in}}$ for a unique index (k,i), and of course $\mathcal{R}^{\flat\mathcal{L}}_{L^{k,i}_{out,in}}$ is admissible, so that the cubes I that are paired with J are tree-connected. Thus we can apply the Straddling Lemma 5.6.15 to the reduced admissible collection $\mathcal{Q}^{out,in}$

with the 'straddling' set $\mathcal{S} \equiv \left(\bigcup_{k,i} \bigcup_{L' \in L^{k,i}} L' \right) \cap \mathcal{C}_A^{\mathcal{A},restrict}$ to obtain the estimate

$$\widehat{\mathfrak{N}}_{stop,\triangle^\omega}^{A,\bigcup_{k=1}^{\infty} \mathcal{R}^{\flat\mathcal{L}}_{L^{k,i}_{out,in}}} = \widehat{\mathfrak{N}}_{stop,\triangle^\omega}^{A,\mathcal{Q}^{out,in}} \leq C\mathcal{S}_{augsize}^{\alpha,A}\left(\mathcal{Q}^{out,in}\right) \leq C\mathcal{S}_{augsize}^{\alpha,A}\left(\mathcal{P}_{L,0}^{\flat\mathcal{H}-big}\right) \quad (5.6.45)$$

As for the remaining 'out-out' form $|\mathsf{B}|_{stop,\triangle^\omega}^{A,\bigcup_{k,i}\mathcal{R}^{\flat\mathcal{L}}_{L^{k,i}_{out,out}}}(f,g)$, if the cube pair $(I,J) \in \mathcal{R}^{\flat\mathcal{L}}_{L^{k,i}_{out,out}}$, then either $J^\flat \subset L' \in L^{k,i}_{out,out} \subsetneqq J^{\maltese}$ or $J^{\maltese} \subset L' \in L^{k,i}_{out,out}$. But $J^\flat \subset L' \subsetneqq J^{\maltese}$ implies that either $J^\flat = L' \subsetneqq J^{\maltese} \subset I \subset L$, which is impossible since $J^\flat$ cannot share an endpoint with L, or that $J^\flat = L'' \in L'_{in}$ and $J^{\maltese} = L^{k,i}$. So we conclude that if $(I,J) \in \mathcal{R}^{\flat\mathcal{L}}_{L^{k,i}_{out,out}}$, then

$$\text{either } J^{\maltese} \subset L^{k,i}_{out,out} \text{ or } \{J^{\maltese} = L^{k,i} \text{ and } J \subset L^{k,i}_{out,out}\}. \quad (5.6.46)$$

In either case in (5.6.46), there is a unique cube $K[J] \in \mathcal{W}(L)$ that contains J. It follows that there are now two remaining cases:

Case 1: $K[J] \in \mathcal{C}'_A$,

Case 2: $K[J] \subset A' \subsetneqq I$ for some $A' \in \mathfrak{C}_{\mathcal{A}}(A)$.

However, since $J^\flat \subset K[J]$, as $K[J]$ is the maximal cube whose triple is contained in L, and since $\mathcal{R}^{\flat\mathcal{L}}_{L^{k,i}_{out,out}}$ is reduced, the pairs (I,J) in **Case 2** lie in the 'corona straddling' collection P^A_{cor} that was removed from all A-admissible collections in (5.6.26) of Conclusion 5.6.4 above, and thus there are no pairs in **Case 2** here. Thus we conclude that $K[J] \in \mathcal{C}'_A$.

We now claim that $3K[J] \subset I$ for all pairs $(I,J) \in \bigcup_{k,i} \mathcal{R}^{\flat\mathcal{L}}_{L^{k}_{out,out}}$. To see this, suppose that $(I,J) \in \mathcal{R}^{\flat\mathcal{L}}_{L^{k,i}_{out,out}}$ for some $k \geq 1$, $1 \leq i \leq n_k$. Then by (5.6.46) we have both that $K[J] \subset L^{k,i}_{out,out}$ and $L^{k,i} \subsetneqq I$. But then $K[J] \subset L^{k,i}_{out,out}$ implies that $3K[J] \subset L^{k,i} \subset I$ as claimed.

Now the 'out-out' collection $\mathcal{Q}^{out,out} \equiv \bigcup_{k,i} \mathcal{R}^{\flat\mathcal{L}}_{L^{k,i}_{out,out}}$ is admissible, since if $J \in \Pi_2 \mathcal{Q}^{out,out}$ and $I_j \in \Pi_1 \mathcal{Q}^{out,out}$ with $(I_j, J) \in \mathcal{Q}^{out,out}$ for $j = 1, 2$, then $I_j \in \mathcal{C}^{\mathcal{L}}_{L^{k_j - 1,i}}$ for some k_j and i and all of the cubes $I \in [I_1, I_2]$ lie in one of the coronas $\mathcal{C}^{\mathcal{L}}_{L^{k-1,i}}$ for k between k_1 and k_2. And of course for those coronas we have $J \in L^{k,i}_{out,out}$. Thus $(I, J) \in \mathcal{R}^{\flat\mathcal{L}}_{L^{k}_{out,out}} \subset \mathcal{Q}^{out,out}$ and we have proved the required connectedness. From the containment $3K[J] \subset I \subset L$ for all $(I, J) \in \bigcup_{k,i} \mathcal{R}^{\flat\mathcal{L}}_{L^{k,i}_{out,out}}$, we now see that the reduced admissible collection $\mathcal{Q}^{out,out}$ *substraddles* the cube L. Hence the Substraddling Lemma 5.6.17 yields the bound

$$\widehat{\mathfrak{N}}^{A,\bigcup_{k,i} \mathcal{R}^{\flat\mathcal{L}}_{L^{k,i}_{out,out}}}_{stop,\triangle\omega} = \widehat{\mathfrak{N}}^{A,\mathcal{Q}^{out,out}}_{stop,\triangle\omega} \leq C\mathcal{S}^{\alpha,A}_{augsize}\left(\mathcal{Q}^{out,out}\right) \leq C\mathcal{S}^{\alpha,A}_{augsize}\left(\mathcal{P}^{\flat\mathcal{H}-big}_{L,0}\right). \tag{5.6.47}$$

Combining the bounds (5.6.44), (5.6.45) and (5.6.47), we obtain (5.6.43).

Finally, we observe that the collections $\mathcal{P}^{\flat\mathcal{H}-big}_{L,0}$ themselves are *mutually orthogonal*, namely

$$\begin{aligned} \mathcal{P}^{\flat\mathcal{H}-big}_{L,0} \quad &\subset \quad \mathcal{C}^{\mathcal{H}}_{L} \times \mathcal{C}^{\mathcal{H},\flat shift}_{L}, \qquad L \in \mathcal{H}, \\ \sum_{L\in\mathcal{H}} \mathbf{1}_{\mathcal{C}^{\mathcal{H}}_{L}} \quad &\leq \quad \mathbf{1} \text{ and } \sum_{L\in\mathcal{H}} \mathbf{1}_{\mathcal{C}^{\mathcal{H},\flat shift}_{L}} \leq \mathbf{1}. \end{aligned}$$

Thus an application of the Orthogonality Lemma 5.6.4 shows that

$$\widehat{\mathfrak{N}}^{A,\mathcal{Q}^{\flat\mathcal{H}-big}_{0}}_{stop,\triangle\omega} \leq \sup_{L\in\mathcal{L}} \widehat{\mathfrak{N}}^{A,\mathcal{P}^{\flat\mathcal{H}-big}_{L,0}}_{stop,\triangle\omega} \leq C\mathcal{S}^{\alpha,A}_{augsize}(\mathcal{P}).$$

Altogether, the proof of Proposition 5.6.19 is now complete. □

This finishes the proofs of the inequalities (5.6.7) and (5.6.1).

5.7 Finishing the proof

At this point we have controlled, either directly or probabilistically, the norms of all of the forms in our decompositions - namely the disjoint, nearby, far below, paraproduct, neighbour, broken and stopping forms - in terms of the Muckenhoupt, energy and *functional energy* conditions, along with an arbitrarily small multiple of the operator norm. Thus it only remains to control the functional energy condition by the Muckenhoupt and energy conditions, since then, using $\int (T_\sigma^\alpha f)\, gd\omega = \Theta(f,g) + \Theta^*(f,g)$ with the further decompositions above, we will have shown that for any fixed tangent line truncation of the operator T_σ^α we have

$$\begin{aligned}\left|\int (T_\sigma^\alpha f)\, gd\omega\right| = \boldsymbol{E}_\Omega^{\mathcal{D}} \boldsymbol{E}_\Omega^{\mathcal{G}} \left|\int (T_\sigma^\alpha f)\, gd\omega\right| \quad &\leq \boldsymbol{E}_\Omega^{\mathcal{D}} \boldsymbol{E}_\Omega^{\mathcal{G}} \sum_{i=1}^{3} \left(|\Theta_i(f,g)| + |\Theta_i^*(f,g)|\right) \\ &\leq \left(C_\eta \mathcal{NTV}_\alpha + \eta \mathfrak{N}_{T^\alpha}\right) \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)}\end{aligned}$$

for $f \in L^2(\sigma)$ and $g \in L^2(\omega)$, for an arbitarily small positive constant $\eta > 0$, and a correspondingly large finite constant C_η. Note that the testing constants $\mathfrak{T}_{T^\alpha}$ and $\mathfrak{T}_{T^\alpha,*}$ in $\mathcal{NTV}_\alpha$ already include the supremum over all tangent line truncations of T^α, while the operator norm $\mathfrak{N}_{T^\alpha}$ on the left refers to a *fixed* tangent line truncation of T^α. This gives

$$\mathfrak{N}_{T^\alpha} = \sup_{\|f\|_{L^2(\sigma)}=1} \sup_{\|g\|_{L^2(\omega)}=1} \left|\int (T_\sigma^\alpha f)\, gd\omega\right| \leq C_\eta \mathcal{NTV}_\alpha + \eta \mathfrak{N}_{T^\alpha},$$

and since the truncated operators have finite operator norm $\mathfrak{N}_{T^\alpha}$, we can absorb the term $\eta \mathfrak{N}_{T^\alpha}$ into the left hand side for $\eta < 1$ and obtain $\mathfrak{N}_{T^\alpha} \leq C'_\eta \mathcal{NTV}_\alpha$ for each tangent line truncation of T^α. Taking the supremum over all such truncations of T^α finishes the proof of Theorem 5.1.5.

The task of controlling functional energy is taken up in Appendix, after first establishing weak frame and weak Riesz inequalities for martingale and dual martingale differences (except for the lower weak Riesz inequality for the martingale difference $\triangle_Q^{\mu,\mathbf{b}}$).

APPENDIX

Appendix

Control of functional energy

Now we arrive at one of the main propositions used in the proof of our theorem. This result is proved *independently* of the main theorem, and only using the results on dual martingale differences established in the previous appendix. The organization of the proof is almost identical to that of the corresponding result in [64, pages 128-151], together with the modifications in [66, pages 348-360] to accommodate common point masses, but we repeat the organization here with modifications required for the use of two independent grids, and the appearance of weak goodness entering through the cubes $J^{\maltese}$. Recall that the functional energy constant $\mathfrak{F}_{\alpha}=\mathfrak{F}_{\alpha}^{\mathbf{b}^{*}}(\mathcal{D},\mathcal{G})$ in (5.5.5), $0\leq\alpha<n$, namely the best constant in the inequality (see (.0.7) below for the definition of $\mathcal{W}(F)$),

$$\sum_{F\in\mathcal{F}}\sum_{M\in\mathcal{W}(F)}\left(\frac{\mathrm{P}^{\alpha}(M,h\sigma)}{|M|^{\frac{1}{n}}}\right)^{2}\left\|\mathsf{Q}_{\mathcal{C}_{F}^{\mathcal{G},shift};M}^{\omega,\mathbf{b}^{*}}x\right\|_{L^{2}(\omega)}^{\spadesuit 2}\leq\mathfrak{F}_{\alpha}\|h\|_{L^{2}(\sigma)}, \tag{.0.1}$$

depends on the grids $\mathcal{D}$ and $\mathcal{G}$, the goodness parameter $\varepsilon>0$ used in the definition of $J^{\maltese}$ through the shifted corona $\mathcal{C}_{F}^{\mathcal{G},shift}$, and on the family of martingale differences $\left\{\triangle_{J}^{\omega,\mathbf{b}^{*}}\right\}_{J\in\mathcal{G}}$ associated with $x\in L_{loc}^{2}(\omega)$, but not on the family of dual martingale differences $\left\{\square_{I}^{\sigma,\mathbf{b}}\right\}_{I\in\mathcal{D}}$, since the function $h\in L^{2}(\sigma)$ appearing in the definition of functional energy is not decomposed as a sum of pseudoprojections $\square_{I}^{\sigma,\mathbf{b}}h$. Finally, we emphasize that the pseudoprojection

$$\mathsf{Q}^{\omega,\mathbf{b}^*}_{\mathcal{C}_F^{\mathcal{G},shift};M} \equiv \sum_{J\in\mathcal{C}_F^{\mathcal{G},shift}:\ J\subset M} \triangle^{\omega,\mathbf{b}^*}_J \tag{.0.2}$$

here uses the shifted restricted corona in

$$\begin{aligned} \mathcal{C}_F^{\mathcal{G},shift} &= \left\{J\in\mathcal{G} : J^{\maltese}\in\mathcal{C}_F^{\mathcal{D}}\right\}, \\ \mathcal{C}_F^{\mathcal{G},shift};K &\equiv \left\{J\in\mathcal{C}_F^{\mathcal{G},shift} : J\subset K\right\}, \end{aligned} \tag{.0.3}$$

where $J^{\maltese}$ is defined using the *body* of a cube as in Definition 5.2.8, and where we have defined here the 'restriction' $\mathcal{C}_F^{\mathcal{G},shift};K$ to the cube K of the corona $\mathcal{C}_F^{\mathcal{G},shift}$ (c.f. $\Pi_2^K\mathcal{P}$ in Definition 5.6.5, which uses the stronger requirement $J^{\maltese}\subset K$). Moreover, recall from Notation in 5.1.13.2 and the definition of $\bigtriangledown_J^\omega$ in (5.1.39), that for any subset $\mathcal{H}$ of the grid $\mathcal{G}$,

$$\begin{aligned} \left\|\mathsf{Q}^{\omega,\mathbf{b}^*}_{\mathcal{H}} x\right\|^{\spadesuit 2}_{L^2(\omega)} &\equiv \sum_{J\in\mathcal{H}} \left\|\triangle^{\omega,\mathbf{b}^*}_J x\right\|^{\spadesuit 2}_{L^2(\omega)} \\ &= \sum_{J\in\mathcal{H}} \left(\left\|\triangle^{\omega,\mathbf{b}^*}_J x\right\|^2_{L^2(\omega)} + \inf_{z\in\mathbb{R}} \left\|\widehat{\bigtriangledown}^\omega_J (x-z)\right\|^2_{L^2(\omega)}\right), \end{aligned}$$

so that we never need to consider the norm squared $\left\|\mathsf{Q}^{\omega,\mathbf{b}^*}_{\mathcal{C}_F^{\mathcal{G},shift};M} x\right\|^2_{L^2(\omega)}$ of the pseudoprojection $\mathsf{Q}^{\omega,\mathbf{b}^*}_{\mathcal{C}_F^{\mathcal{G},shift};M} x$, something for which we have no lower Riesz inequality. Note moreover that for $J \in \mathcal{G}$ and an arbitrary cube K, we have by the frame inequality in (5.1.51) ,

$$\begin{aligned} \sum_{J\in\mathcal{G}:\ J\subset K} \left\| \triangle_J^{\omega,\mathbf{b}^*} x \right\|_{L^2(\omega)}^2 &\lesssim \left\| x - m_K^\omega \right\|_{L^2(\mathbf{1}_K\omega)}^2 , &(.0.4) \\ \sum_{J\in\mathcal{G}:\ J\subset K} \inf_{z\in\mathbb{R}} \left\| \widehat{\nabla}_J^\omega (x-z) \right\|_{L^2(\omega)}^2 &\leq \sum_{J\in\mathcal{G}:\ J\subset K} \left\| \widehat{\nabla}_J^\omega \left\{ (x-p)\, \mathbf{1}_K (x) \right\} \right\|_{L^2(\omega)}^2 &(.0.5) \\ &\lesssim \left\| (x-p) \right\|_{L^2(\mathbf{1}_K\omega)}^2 , \quad p \in K, \end{aligned}$$

where the second line follows from (5.1.40).

Important note If $J \in \mathcal{C}_F^{\mathcal{G},shift}$, then in particular $J \Subset_{\rho,\varepsilon} F$ with $\rho = \left[\frac{3}{\varepsilon}\right]$ as mentioned above notation 5.5.8 , and so $J \cap M \neq \emptyset$ for a *unique* $M \in \mathcal{W}(F)$.

We will show that, uniformly in pairs of grids $\mathcal{D}$ and $\mathcal{G}$, the functional energy constants $\mathfrak{F}_\alpha(\mathcal{D},\mathcal{G})$ in (5.5.5) are controlled by $\mathcal{A}_2^\alpha$, $A_2^{\alpha,punct}$ and the large energy constant $\mathfrak{E}_2^\alpha$ - actually the proof shows that we have control by the Whitney plugged energy constant as defined in (.0.16) below. More precisely this is our control of functional energy proposition.

Proposition .0.1. *For all grids $\mathcal{D}$ and $\mathcal{G}$, and $\varepsilon > 0$ sufficiently small, we have*

$$\begin{aligned} \mathfrak{F}_\alpha^{\mathbf{b}^*}(\mathcal{D},\mathcal{G}) &\lesssim \mathfrak{E}_2^\alpha + \sqrt{\mathcal{A}_2^\alpha} + \sqrt{\mathcal{A}_2^{\alpha,*}} + \sqrt{A_2^{\alpha,punct}} , \\ \mathfrak{F}_\alpha^{\mathbf{b},*}(\mathcal{G},\mathcal{D}) &\lesssim \mathfrak{E}_2^{\alpha,*} + \sqrt{\mathcal{A}_2^\alpha} + \sqrt{\mathcal{A}_2^{\alpha,*}} + \sqrt{A_2^{\alpha,*,punct}} , \end{aligned}$$

with implied constants independent of the grids $\mathcal{D}$ and $\mathcal{G}$.

In order to prove this proposition, we first turn to recalling these more refined notions of energy constants.

Various energy conditions

In this subsection we recall various refinements of the strong energy conditions appearing in the main theorem above. Variants of this material already appear in earlier papers, but we repeat it here both for convenience and in order to introduce some arguments we will use repeatedly later on. These refinements represent the 'weakest' energy side conditions that suffice for use in our proof, but despite this, we will usually use the large energy constant $\mathfrak{E}_2^{\alpha}$ in estimates to avoid having to pay too much attention to which of the energy conditions we need to use - leaving the determination of the weakest conditions in such situations to the interested reader. We begin with the notion of 'deeply embedded'. Recall that the goodness parameter $\mathbf{r} \in \mathbb{N}$ is determined by $\varepsilon > 0$ in (5.2.16), and that $0 < \varepsilon < \frac{1}{n+1} < \frac{1}{n+1-\alpha}$.

For arbitrary cubes in $J, K \in \mathcal{P}$, we say that J is (ρ,ε)-*deeply embedded* in K, which we write as $J \Subset_{\rho,\varepsilon} K$, when $J \subset K$ and both

$$\begin{aligned} \ell(J) &\leq 2^{-\rho}\ell(K), \\ d(J,\partial K) &\geq 2\ell(J)^{\varepsilon}\ell(K)^{1-\varepsilon}. \end{aligned} \qquad (.0.6)$$

Note that we use the *boundary* of K for the definition of $J \Subset_{\rho,\varepsilon} K$, rather than the *skeleton* or *body* of K, which would result in a more restrictive notion of (ρ,ε)-deeply embedded. We will use this notion for the purpose of grouping $\varepsilon-$*good* cubes into the following collections. Fix grids $\mathcal{D}$ and $\mathcal{G}$. For $K \in \mathcal{D}$, define the collections,

$$\begin{aligned} \mathcal{M}_{(\rho,\varepsilon)-deep,\mathcal{G}}(K) &\equiv \left\{J \in \mathcal{G} : J \text{ is maximal w.r.t } J \Subset_{\rho,\varepsilon} K\right\}, \\ \mathcal{M}_{(\rho,\varepsilon)-deep,\mathcal{D}}(K) &\equiv \left\{M \in \mathcal{D} : M \text{ is maximal w.r.t } M \Subset_{\rho,\varepsilon} K\right\}, \\ \mathcal{W}(K) &\equiv \{M \in \mathcal{D} : M \text{ is maximal w.r.t } 3M \subset K\} \end{aligned} \qquad (.0.7)$$

where the first two consist of *maximal* (ρ,ε)-deeply embedded dyadic $\mathcal{G}$-subcubes J, respectively $\mathcal{D}$-subcubes M, of a $\mathcal{D}$-cube K, and the third consists of the maximal $\mathcal{D}$-subcubes M whose triples are contained in K.

Let $\gamma > 1$. Then the following bounded overlap property holds where $\mathcal{M}_{(\rho,\varepsilon)-deep}(K)$ can be taken to be either $\mathcal{M}_{(\rho,\varepsilon)-deep,\mathcal{G}}(K)$ or $\mathcal{M}_{(\rho,\varepsilon)-deep,\mathcal{D}}(K)$ or $\mathcal{W}(K)$ throughout.

Lemma .0.2. *Let $0 < \varepsilon \leq 1 < \gamma \leq 1 + 4 \cdot 2^{\rho(1-\varepsilon)}$. Then*

$$\sum_{J \in \mathcal{M}_{(\rho,\varepsilon)-deep}(K)} \mathbf{1}_{\gamma J} \leq \beta \mathbf{1}_{\left[\bigcup_{J \in \mathcal{M}_{(\rho,\varepsilon)-deep}(K)} \gamma J \right]} \tag{.0.8}$$

holds for some positive constant β depending only on γ, ρ and ε. In addition $\gamma J \subset K$ for all $J \in \mathcal{M}_{(\rho,\varepsilon)-deep}(K)$, and consequently

$$\sum_{J \in \mathcal{M}_{(\rho,\varepsilon)-deep}(K)} \mathbf{1}_{\gamma J} \leq \beta \mathbf{1}_K \ . \tag{.0.9}$$

A similar result holds for $\mathcal{W}(K)$.

Proof. We suppose $0 < \varepsilon < 1$ and leave the simpler case $\varepsilon = 1$ for the reader. To prove (.0.8), we first note that there are at most $\frac{2^{n(\rho+1)}-1}{2^n-1}$ cubes J contained in K for which $\ell(J) > 2^{-\rho}\ell(K)$. On the other hand, the maximal (ρ,ε)-deeply embedded subcubes J of K also satisfy the comparability condition

$$\begin{aligned} 2\ell(J)^{\varepsilon}\ell(K)^{1-\varepsilon} \leq d(J,\partial K) \leq d(\pi J,\partial K) + \ell(J) &\leq 2(2\ell(J))^{\varepsilon}\ell(K)^{1-\varepsilon} + \ell(J) \\ &\leq 4\ell(J)^{\varepsilon}\ell(K)^{1-\varepsilon} + \ell(J) . \end{aligned}$$

Now with $0 < \varepsilon < 1$ and $\gamma > 1$ fixed, let $y \in K$. Then if $y \in \gamma J$, we have

$$\begin{aligned} 2\ell(J)^{\varepsilon}\ell(K)^{1-\varepsilon} &\leq d(J,\partial K) \leq \gamma\ell(J) + d(\gamma J,\partial K) \\ &\leq \gamma\ell(J) + d(y,\partial K). \end{aligned}$$

Now assume that $\frac{\ell(J)}{\ell(K)} \leq \left(\frac{1}{\gamma}\right)^{\frac{1}{1-\varepsilon}}$. Then we have $\gamma\ell(J) \leq \ell(J)^{\varepsilon}\ell(K)^{1-\varepsilon}$ and so

$$\ell(J)^{\varepsilon}\ell(K)^{1-\varepsilon} \leq d(y,\partial K).$$

But we also have

$$d(y,\partial K) \leq \gamma\ell(J) + d(J,\partial K) \leq \gamma\ell(J) + 4\ell(J)^{\varepsilon}\ell(K)^{1-\varepsilon} + \ell(J) \leq 6\ell(J)^{\varepsilon}\ell(K)^{1-\varepsilon},$$

and so altogether, under the assumption that $\frac{\ell(J)}{\ell(K)} \leq \left(\frac{1}{\gamma}\right)^{\frac{1}{1-\varepsilon}}$, we have

$$\begin{aligned} \frac{1}{6}d(y,\partial K) &\leq \ell(J)^{\varepsilon}\ell(K)^{1-\varepsilon} \leq d(y,\partial K), \\ \text{i.e. } \left(\frac{1}{6}\frac{d(y,\partial K)}{\ell(K)^{1-\varepsilon}}\right)^{\frac{1}{\varepsilon}} &\leq \ell(J) \leq \left(\frac{d(y,\partial K)}{\ell(K)^{1-\varepsilon}}\right)^{\frac{1}{\varepsilon}}, \end{aligned}$$

which shows that the number of J's satisfying $y \in \gamma J$ and $\frac{\ell(J)}{\ell(K)} \leq \left(\frac{1}{\gamma}\right)^{\frac{1}{1-\varepsilon}}$ is at most $C'\frac{1}{\varepsilon}$. On the other hand, the number of J's contained in K satisfying $y \in \gamma J$ and $\frac{\ell(J)}{\ell(K)} > \left(\frac{1}{\gamma}\right)^{\frac{1}{1-\varepsilon}}$ is at most $C'\frac{1}{1-\varepsilon}(1+\log_2\gamma)$. This proves (.0.8) with

$$\beta = \frac{2^{n(\rho+1)}-1}{2^n-1} + C'\frac{1}{\varepsilon} + C'\frac{1}{1-\varepsilon}(1+\log_2\gamma).$$

In order to prove (.0.9) it suffices, by (.0.8), to prove $\gamma J \subset K$ for all $J \in \mathcal{M}_{(\rho,\varepsilon)-deep}(K)$. But $J \in \mathcal{M}_{(\rho,\varepsilon)-deep}(K)$ implies

$$2\ell(J)^{\varepsilon}\,\ell(K)^{1-\varepsilon} \leq d(J,\partial K) = d(c_J,\partial K) - \frac{1}{2}\ell(J).$$

We wish to show $\gamma J \subset K$, which is implied by

$$\gamma\frac{1}{2}\ell(J) \leq d(c_J, K^c) = d(J,\partial K) + \frac{1}{2}\ell(J).$$

But we have

$$d(J,\partial K) + \frac{1}{2}\ell(J) \geq 2\ell(J)^{\varepsilon}\,\ell(K)^{1-\varepsilon} + \frac{1}{2}\ell(J),$$

and so it suffices to show that

$$2\ell(J)^{\varepsilon}\,\ell(K)^{1-\varepsilon} + \frac{1}{2}\ell(J) \geq \gamma\frac{1}{2}\ell(J),$$

which is equivalent to

$$\gamma - 1 \leq 4\ell(J)^{\varepsilon-1}\,\ell(K)^{1-\varepsilon}.$$

But the smallest that $\ell(J)^{\varepsilon-1}\,\ell(K)^{1-\varepsilon}$ can get for $J \in \mathcal{M}_{(\rho,\varepsilon)-deep}(K)$ is $2^{\rho(1-\varepsilon)} \geq 1$, and so $\gamma \leq 1 + 4 \cdot 2^{\rho(1-\varepsilon)}$ implies $\gamma - 1 \leq 4\ell(J)^{\varepsilon-1}\,\ell(K)^{1-\varepsilon}$, which completes the proof.

The reader can easily verify the same argument works for the Whitney collection $\mathcal{W}(K)$.

□

Now we recall the notion of *alternate* dyadic cubes from [64], which we rename *augmented* dyadic cubes here.

Definition .0.3. *Given a dyadic grid $\mathcal{D}$, the* augmented dyadic grid $\mathcal{AD}$ *consists of those cubes I whose dyadic children I' belong to the grid $\mathcal{D}$.*

Of course an augmented grid is not actually a grid because the nesting property fails, but this terminology should cause no confusion. These augmented grids will be needed in order to use the 'prepare to puncture' argument (introduced in [66]) at several places below.

Now we proceed to recall certain of the definitions of various energy conditions from [62] and [64]. While these definitions are not explicitly used in the proof of functional energy, some of the arguments we give to control them will be appealed to later, and so we take the time to develop these definitions in detail.

Whitney energy conditions

The following definition of Whitney energy condition uses the *Whitney* decomposition $\mathcal{M}_{(\rho,1)-deep,\mathcal{D}}(I_r)$ into $\mathcal{D}$-dyadic cubes in which $\varepsilon = 1$, as well as the 'large' pseudoprojections

$$\mathsf{Q}_K^{\omega,\mathbf{b}^*} \equiv \sum_{J\in\mathcal{G}:\ J\subset K} \triangle_J^{\omega,\mathbf{b}^*}. \tag{.0.10}$$

Definition .0.4. *Suppose σ and ω are locally finite positive Borel measures on $\mathbb{R}^n$ and fix $\gamma > 1$. Then the Whitney energy condition constant $\mathcal{E}_2^{\alpha,Whitney}$ is given by*

$$\left(\mathcal{E}_2^{\alpha,Whitney}\right)^2 \equiv \sup_{\mathcal{D},\mathcal{G}} \sup_{I=\dot{\cup} I_r} \frac{1}{|I|_\sigma} \sum_{r=1}^{\infty} \sum_{M\in\mathcal{W}(I_r)} \left(\frac{\mathrm{P}^\alpha\left(M,\mathbf{1}_{I\setminus\gamma M}\sigma\right)}{|M|^{\frac{1}{n}}}\right)^2 \left\|\mathsf{Q}_M^{\omega,\mathbf{b}^*} x\right\|_{L^2(\omega)}^{\spadesuit 2},$$

where $\sup_{\mathcal{D},\mathcal{G}} \sup_{I=\dot{\cup} I_r}$ is taken over

1. *all dyadic grids $\mathcal{D}$ and $\mathcal{G}$,*

2. *all $\mathcal{D}$-dyadic cubes I,*

3. and all partitions $\{I_r\}_{r=1}^{N \text{ or } \infty}$ *of the cube* I *into* $\mathcal{D}$*-dyadic subcubes* I_r*.*

If the parameter $\gamma > 1$ above is chosen sufficiently close to 1, then the collection of cubes $\{\gamma M\}_{M \in \mathcal{W}(I_r)}$ has bounded overlap β by (.0.9), and the Whitney energy constant $\mathcal{E}_2^{\alpha,Whitney}$ is controlled by the strong energy constant $\mathcal{E}_2^{\alpha}$ in (5.1.8),

$$\mathcal{E}_2^{\alpha,Whitney} \lesssim \mathcal{E}_2^{\alpha}. \tag{.0.11}$$

Indeed, to see this, fix a decomposition of a cube

$$I = \dot{\bigcup}_{1 \leq r < \infty} \dot{\bigcup}_{M \in \mathcal{W}(I_r)} M \tag{.0.12}$$

as in Definition .0.4. Then consider the *sub*decomposition

$$I \supset \dot{\bigcup}_{1 \leq r < \infty} \dot{\bigcup}_{M \in \mathcal{W}(I_r)} M$$

of the cube I given by the collection of cubes,

$$\mathcal{I} \equiv \dot{\bigcup}_{1 \leq r < \infty} \mathcal{W}(I_r).$$

We then have

$$(\mathcal{E}_2^{\alpha})^2 \geq \frac{1}{|I|_\sigma} \sum_{r=1}^{\infty} \sum_{M \in \mathcal{W}(I_r)} \left(\frac{\mathrm{P}^\alpha(M, \mathbf{1}_I \sigma)}{|M|^{\frac{1}{n}}} \right)^2 \left\| x - m_M^\omega \right\|_{L^2(\mathbf{1}_M \omega)}^2 .$$

Now $\mathrm{P}^{\alpha}(M, \mathbf{1}_I \sigma) \geq \mathrm{P}^{\alpha}\left(M, \mathbf{1}_{I \setminus \gamma M} \sigma\right)$ and from (.0.4),

$$\left\| x - m_M^{\omega} \right\|_{L^2(\mathbf{1}_M \omega)}^2 \gtrsim \left\| \mathsf{Q}_M^{\omega, \mathbf{b}^*} x \right\|_{L^2(\omega)}^{\spadesuit 2},$$

and combining these two inequalities, we obtain that

$$(\mathcal{E}_2^{\alpha})^2 \geq c \frac{1}{|I|_{\sigma}} \sum_{r=1}^{\infty} \sum_{M \in \mathcal{W}(I_r)} \left(\frac{\mathrm{P}^{\alpha}\left(M, \mathbf{1}_{I \setminus \gamma M} \sigma\right)}{|M|^{\frac{1}{n}}} \right)^2 \left\| \mathsf{Q}_M^{\omega, \mathbf{b}^*} x \right\|_{L^2(\omega)}^{\spadesuit 2} .$$

Thus we conclude that

$$\frac{1}{|I|_{\sigma}} \sum_{r=1}^{\infty} \sum_{M \in \mathcal{W}(I_r)} \left(\frac{\mathrm{P}^{\alpha}\left(M, \mathbf{1}_{I \setminus \gamma M} \sigma\right)}{|M|^{\frac{1}{n}}} \right)^2 \left\| \mathsf{Q}_M^{\omega, \mathbf{b}^*} x \right\|_{L^2(\omega)}^{\spadesuit 2} \leq \frac{C}{c} \beta \left(\mathcal{E}_2^{\alpha}\right)^2,$$

and taking the supremum over all decompositions (.0.12) as in Definition .0.4, we obtain (.0.11).

There is a similar definition for the dual (backward) Whitney energy conditions that simply interchanges σ and ω everywhere. These definitions of the Whitney energy conditions depend on the choice of $\gamma > 1$.

Commentary on proofs We now introduce a number of results concerning partial plugging of the hole for Whitney energy conditions.

Note that we can 'partially' plug the γ-hole in the Poisson integral $\mathrm{P}^{\alpha}\left(J, \mathbf{1}_{I \setminus \gamma J} \sigma\right)$ for $\mathcal{E}_2^{\alpha, Whitney}$ using the offset A_2^{α} condition and the bounded overlap property (.0.9). Indeed, define

$$\left(\mathcal{E}_2^{\alpha,Whitneypartial}\right)^2 \tag{.0.13}$$
$$\equiv \sup_{\mathcal{D},\mathcal{G}} \sup_{I=\dot{\cup}I_r} \frac{1}{|I|_\sigma} \sum_{r=1}^{\infty} \sum_{M\in\mathcal{W}(I_r)} \left(\frac{\mathrm{P}^\alpha\left(M,\mathbf{1}_{I\setminus M}\sigma\right)}{|M|^{\frac{1}{n}}}\right)^2 \left\|\mathsf{Q}_M^{\omega,\mathbf{b}^*}x\right\|_{L^2(\omega)}^{\spadesuit 2} .$$

Recall from (.0.9) that

$$\gamma M \subset I_r \text{ for all } M \in \mathcal{W}(I_r) \text{ provided } \gamma \leq 5.$$

At this point we need the following analogues of the 'energy A_2^α conditions' from [66], which we denote by $A_2^{\alpha,energy}$ and $A_2^{\alpha,*,energy}$, and define by

$$A_2^{\alpha,energy}(\sigma,\omega) \equiv \sup_{Q\in\mathcal{P}} \frac{\left\|\mathsf{Q}_Q^{\omega,\mathbf{b}^*}\frac{x}{\ell(Q)}\right\|_{L^2(\omega)}^{\spadesuit 2}}{|Q|^{1-\frac{\alpha}{n}}} \frac{|Q|_\sigma}{|Q|^{1-\frac{\alpha}{n}}}, \tag{.0.14}$$
$$A_2^{\alpha,*,energy}(\sigma,\omega) \equiv \sup_{Q\in\mathcal{P}} \frac{|Q|_\omega}{|Q|^{1-\frac{\alpha}{n}}} \frac{\left\|\mathsf{Q}_Q^{\sigma,\mathbf{b}}\frac{x}{\ell(Q)}\right\|_{L^2(\sigma)}^{\spadesuit 2}}{|Q|^{1-\frac{\alpha}{n}}}.$$

Then if $\gamma \leq 5$, we have

$$\begin{aligned}
&\left(\mathcal{E}_2^{\alpha,Whitneypartial}\right)^2 \\
\lesssim\ & \sup_{\mathcal{D},\mathcal{G}} \sup_{I=\dot{\cup}I_r} \frac{1}{|I|_\sigma} \sum_{r=1}^{\infty} \sum_{M\in\mathcal{W}(I_r)} \left(\frac{\mathrm{P}^\alpha\left(M, \mathbf{1}_{I\setminus\gamma M}\sigma\right)}{|M|^{\frac{1}{n}}}\right)^2 \left\|\mathsf{Q}_M^{\omega,\mathbf{b}^*} x\right\|_{L^2(\omega)}^{\spadesuit 2} \\
&+ \sup_{\mathcal{D},\mathcal{G}} \sup_{I=\dot{\cup}I_r} \frac{1}{|I|_\sigma} \sum_{r=1}^{\infty} \sum_{M\in\mathcal{W}(I_r)} \left(\frac{\mathrm{P}^\alpha\left(M, \mathbf{1}_{\gamma M\setminus M}\sigma\right)}{|M|^{\frac{1}{n}}}\right)^2 \left\|\mathsf{Q}_M^{\omega,\mathbf{b}^*} x\right\|_{L^2(\omega)}^{\spadesuit 2} \\
\lesssim\ & \left(\mathcal{E}_2^{\alpha,Whitney}\right)^2 + \sup_{\mathcal{D},\mathcal{G}} \sup_{I=\dot{\cup}I_r} \frac{1}{|I|_\sigma} \sum_{r=1}^{\infty} \sum_{M\in\mathcal{W}(I_r)} A_2^{\alpha,energy} |\gamma M|_\sigma \qquad (.0.15) \\
\lesssim\ & \left(\mathcal{E}_2^{\alpha,deep}\right)^2 + \beta A_2^{\alpha,energy} ,
\end{aligned}$$

by (.0.9).

Plugged energy conditions

We continue to recall some results from [66] and [67] that we will use repeatedly here. For example, we will use the punctured Muckenhoupt conditions $A_2^{\alpha,punct}$ and $A_2^{\alpha,*,punct}$ to control the *plugged* energy conditions, where the hole in the argument of the Poisson term $\mathrm{P}^\alpha\left(M, \mathbf{1}_{I\setminus M}\sigma\right)$ in the partially plugged energy condition above, is replaced with the 'plugged' term $\mathrm{P}^\alpha\left(M, \mathbf{1}_I\sigma\right)$, for example

$$\left(\mathcal{E}_2^{\alpha,Whitneyplug}\right)^2 \equiv \sup_{\mathcal{D},\mathcal{G}} \sup_{I=\dot{\cup}I_r} \frac{1}{|I|_\sigma} \sum_{r=1}^{\infty} \sum_{M\in\mathcal{W}(I_r)} \left(\frac{\mathrm{P}^\alpha\left(M, \mathbf{1}_I\sigma\right)}{|M|^{\frac{1}{n}}}\right)^2 \left\|\mathsf{Q}_M^{\omega,\mathbf{b}^*} x\right\|_{L^2(\omega)}^{\spadesuit 2} . \qquad (.0.16)$$

By an argument similar to that in (.0.15), we obtain

$$\mathcal{E}_2^{\alpha,Whitneyplug} \lesssim \mathcal{E}_2^{\alpha,Whitneypartial} + A_2^{\alpha,energy}. \qquad (.0.17)$$

We first show that the punctured Muckenhoupt conditions $A_2^{\alpha,punct}$ and $A_2^{\alpha,*,punct}$ control respectively the 'energy A_2^{α} conditions' in (.0.14). We will make reference to the proof of the next lemma (for the $T1$ theorem this is from [66, Lemma 3.2 on page 328.]) several times in the sequel. We repeat the proof from [66, Lemma 3.2 on page 328.] but with modifications to accommodate the differences that arise here in the setting of a local Tb theorem. Recall that $\mathfrak{P}_{(\sigma,\omega)}$ is defined below (5.1.6) above.

Lemma .0.5. *For any positive locally finite Borel measures σ,ω we have*

$$\begin{aligned} A_2^{\alpha,energy}(\sigma,\omega) &\lesssim A_2^{\alpha,punct}(\sigma,\omega)\,, \\ A_2^{\alpha,*,energy}(\sigma,\omega) &\lesssim A_2^{\alpha,*,punct}(\sigma,\omega)\,. \end{aligned}$$

Proof. Fix a cube $Q\in\mathcal{D}$. Recall the definition of $\omega\left(Q,\mathfrak{P}_{(\sigma,\omega)}\right)$ in (5.1.6). If $\omega\left(Q,\mathfrak{P}_{(\sigma,\omega)}\right)\geq \frac{1}{2}\left|Q\right|_{\omega}$, then we trivially have

$$\begin{aligned} \frac{\left\|\mathsf{Q}_Q^{\omega,\mathbf{b}^*}\frac{x}{\ell(Q)}\right\|_{L^2(\omega)}^{\spadesuit 2}}{\left|Q\right|^{1-\frac{\alpha}{n}}}\frac{\left|Q\right|_{\sigma}}{\left|Q\right|^{1-\frac{\alpha}{n}}} &\lesssim \frac{\left|Q\right|_{\omega}}{\left|Q\right|^{1-\frac{\alpha}{n}}}\frac{\left|Q\right|_{\sigma}}{\left|Q\right|^{1-\frac{\alpha}{n}}} \\ &\leq 2\frac{\omega\left(Q,\mathfrak{P}_{(\sigma,\omega)}\right)}{\left|Q\right|^{1-\frac{\alpha}{n}}}\frac{\left|Q\right|_{\sigma}}{\left|Q\right|^{1-\frac{\alpha}{n}}}\leq 2A_2^{\alpha,punct}(\sigma,\omega)\,. \end{aligned}$$

On the other hand, if $\omega\left(Q,\mathfrak{P}_{(\sigma,\omega)}\right)<\frac{1}{2}\left|Q\right|_{\omega}$ then there is a point $p\in Q\cap\mathfrak{P}_{(\sigma,\omega)}$ such that

$$\omega\left(\{p\}\right)>\frac{1}{2}\left|Q\right|_{\omega}\ ,$$

and consequently, p is the largest ω-point mass in Q. Thus if we define $\widetilde{\omega}=\omega-\omega\left(\{p\}\right)\delta_p$, then we have

$$\omega\left(Q,\mathfrak{P}_{(\sigma,\omega)}\right)=|Q|_{\widetilde{\omega}}\ .$$

Now we observe from the construction of martingale differences that

$$\triangle_J^{\widetilde{\omega},\mathbf{b}^*}=\triangle_J^{\omega,\mathbf{b}^*},\qquad\text{for all } J\in\mathcal{D}\text{ with } p\notin J.$$

So for each $s\geq 0$ there is a unique cube $J_s\in\mathcal{D}$ with $\ell(J_s)=2^{-s}\ell(Q)$ that contains the point p. Now observe that, just as for the Haar projection, the one-dimensional projection $\triangle_{J_s}^{\omega,\mathbf{b}^*}$ is given by $\triangle_{J_s}^{\omega,\mathbf{b}^*}f=\left\langle h_{J_s}^{\omega,\mathbf{b}^*},f\right\rangle_\omega h_{J_s}^{\omega,\mathbf{b}^*}$ for a unique up to $\pm$ unit vector $h_{J_s}^{\omega,\mathbf{b}^*}$. For this cube we then have

$$\begin{aligned}
\left\|\triangle_{J_s}^{\omega,\mathbf{b}^*}x\right\|_{L^2(\omega)}^2 &= \left|\left\langle h_{J_s}^{\omega,\mathbf{b}^*},x\right\rangle_\omega\right|^2=\left|\left\langle h_{J_s}^{\omega,\mathbf{b}^*},x-p\right\rangle_\omega\right|^2\\
&= \left|\int_{J_s}h_{J_s}^{\omega,\mathbf{b}^*}(x)(x-p)\,d\omega(x)\right|^2=\left|\int_{J_s}h_{J_s}^{\omega,\mathbf{b}^*}(x)(x-p)\,d\widetilde{\omega}(x)\right|^2\\
&\leq \left\|h_{J_s}^{\omega,\mathbf{b}^*}\right\|_{L^2(\widetilde{\omega})}^2\left\|\mathbf{1}_{J_s}(x-p)\right\|_{L^2(\widetilde{\omega})}^2\leq\left\|h_{J_s}^{\omega,\mathbf{b}^*}\right\|_{L^2(\omega)}^2\left\|\mathbf{1}_{J_s}(x-p)\right\|_{L^2(\widetilde{\omega})}^2\\
&\leq \ell(J_s)^2|J_s|_{\widetilde{\omega}}\leq 2^{-2s}\ell(Q)^2|Q|_{\widetilde{\omega}}\,,
\end{aligned}$$

as well as

$$\begin{aligned}
\inf_{z\in\mathbb{R}}\left\|\widehat{\nabla}_{J_s}^\omega(x-z)\right\|_{L^2(\omega)}^2 &\lesssim \|(x-p)\|_{L^2\left(\mathbf{1}_{J_s}\omega\right)}^2=\|(x-p)\|_{L^2\left(\mathbf{1}_{J_s}\widetilde{\omega}\right)}^2\leq\ell(J_s)^2|J_s|_{\widetilde{\omega}}\\
&\leq 2^{-2s}\ell(Q)^2|Q|_{\widetilde{\omega}}\ ,
\end{aligned}$$

from (.0.4). Thus we can estimate

$$
\begin{aligned}
&\left\|\mathsf{Q}_{Q}^{\omega,\mathbf{b}^{*}}\frac{x}{\ell\left(Q\right)}\right\|_{L^{2}(\omega)}^{\spadesuit 2} && (.0.18)\\
\leq\ & \frac{1}{\ell\left(Q\right)^{2}}\left(\sum_{J\in\mathcal{D}:\ J\subset Q}\left\|\triangle_{J}^{\omega,\mathbf{b}^{*}}x\right\|_{L^{2}(\omega)}^{2}+\inf_{z\in\mathbb{R}}\left\|\widehat{\nabla}_{J_{s}}^{\omega}\left(x-z\right)\right\|_{L^{2}(\omega)}^{2}\right)\\
=\ & \frac{1}{\ell\left(Q\right)^{2}}\left(\sum_{J\in\mathcal{D}:\ p\notin J\subset Q}\left\|\triangle_{J}^{\widetilde{\omega},\mathbf{b}^{*}}x\right\|_{L^{2}(\widetilde{\omega})}^{2}+\sum_{s=0}^{\infty}\left\|\triangle_{J_{s}}^{\omega,\mathbf{b}^{*}}x\right\|_{L^{2}(\omega)}^{2}\right.\\
&\qquad\qquad\left.+\inf_{z\in\mathbb{R}}\left\|\widehat{\nabla}_{J_{s}}^{\omega}\left(x-z\right)\right\|_{L^{2}(\omega)}^{2}\right)\\
\lesssim\ & \frac{1}{\ell\left(Q\right)^{2}}\left(\left\|\mathsf{Q}_{Q}^{\widetilde{\omega},\mathbf{b}^{*}}x\right\|_{L^{2}(\widetilde{\omega})}^{\spadesuit 2}+\sum_{s=0}^{\infty}2^{-2s}\ell\left(Q\right)^{2}\left|Q\right|_{\widetilde{\omega}}\right)\\
\lesssim\ & \frac{1}{\ell\left(Q\right)^{2}}\left(\ell\left(Q\right)^{2}\left|Q\right|_{\widetilde{\omega}}+\sum_{s=0}^{\infty}2^{-2s}\ell\left(Q\right)^{2}\left|Q\right|_{\widetilde{\omega}}\right)\\
\leq\ & 3\left|Q\right|_{\widetilde{\omega}}=3\omega\left(Q,\mathfrak{P}_{(\sigma,\omega)}\right),
\end{aligned}
$$

and so

$$
\frac{\left\|\mathsf{Q}_{Q}^{\omega,\mathbf{b}^{*}}\frac{x}{\ell(Q)}\right\|_{L^{2}(\omega)}^{\spadesuit 2}}{\left|Q\right|^{1-\frac{\alpha}{n}}}\frac{\left|Q\right|_{\sigma}}{\left|Q\right|^{1-\frac{\alpha}{n}}}\lesssim\frac{3\omega\left(Q,\mathfrak{P}_{(\sigma,\omega)}\right)}{\left|Q\right|^{1-\frac{\alpha}{n}}}\frac{\left|Q\right|_{\sigma}}{\left|Q\right|^{1-\frac{\alpha}{n}}}\leq 3A_{2}^{\alpha,punct}\left(\sigma,\omega\right).
$$

Now take the supremum over $Q\in\mathcal{D}$ to obtain $A_{2}^{\alpha,energy}\left(\sigma,\omega\right)\lesssim A_{2}^{\alpha,punct}\left(\sigma,\omega\right)$. The dual inequality follows upon interchanging the measures σ and ω. $\square$

We isolate a simple but key fact that will be used repeatedly in what follows:

$$
\sum_{Q\in\mathcal{D}:\ Q\subset P}\ell\left(Q\right)^{2}\left|Q\right|_{\mu}\lesssim\ell\left(P\right)^{2}\left|P\right|_{\mu}\ ,\qquad\text{for }P\in\mathcal{D}\text{ and }\mu\text{ a positive measure.}\qquad(.0.19)
$$

Indeed, to see (.0.19), simply pigeonhole the length of Q relative to that of P and sum. The next corollary follows immediately from Lemma .0.5, (.0.15) and (.0.17).

Corollary .0.6. *Provided* $1 < \gamma \leq 5$*,*

$$\mathcal{E}_2^{\alpha,Whitneyplug} \lesssim \mathcal{E}_2^{\alpha,Whitneypartial} + A_2^{\alpha,punct} \lesssim \mathcal{E}_2^{\alpha,Whitney} + A_2^{\alpha,punct} \ ,$$

and similarly for the dual plugged energy condition.

Using Lemma .0.5 we can control the 'plugged' energy $\mathcal{A}_2^{\alpha}$ conditions:

$$\begin{aligned} \mathcal{A}_2^{\alpha,energyplug}(\sigma,\omega) &\equiv \sup_{Q\in\mathcal{P}} \frac{\left\| \mathsf{Q}_Q^{\omega,\mathbf{b}^*} \frac{x}{\ell(Q)} \right\|_{L^2(\omega)}^{\spadesuit 2}}{|Q|^{1-\frac{\alpha}{n}}} \mathcal{P}^{\alpha}(Q,\sigma), \\ \mathcal{A}_2^{\alpha,*,energyplug}(\sigma,\omega) &\equiv \sup_{Q\in\mathcal{P}} \mathcal{P}^{\alpha}(Q,\omega) \frac{\left\| \mathsf{Q}_Q^{\sigma,\mathbf{b}} \frac{x}{\ell(Q)} \right\|_{L^2(\sigma)}^{\spadesuit 2}}{|Q|^{1-\frac{\alpha}{n}}}. \end{aligned}$$

Lemma .0.7. *We have*

$$\begin{aligned} \mathcal{A}_2^{\alpha,energyplug}(\sigma,\omega) &\lesssim \mathcal{A}_2^{\alpha}(\sigma,\omega) + A_2^{\alpha,energy}(\sigma,\omega), \\ \mathcal{A}_2^{\alpha,*,energyplug}(\sigma,\omega) &\lesssim \mathcal{A}_2^{\alpha,*}(\sigma,\omega) + A_2^{\alpha,*,energy}(\sigma,\omega). \end{aligned}$$

Proof. We have

$$\begin{aligned}
\frac{\left\Vert \mathsf{Q}_{Q}^{\omega,\mathbf{b}^{\ast}}\frac{x}{\ell(Q)}\right\Vert_{L^{2}(\omega)}^{\spadesuit 2}}{\left\vert Q\right\vert^{1-\frac{\alpha}{n}}}\mathcal{P}^{\alpha}\left(Q,\sigma\right) &= \frac{\left\Vert \mathsf{Q}_{Q}^{\omega,\mathbf{b}^{\ast}}\frac{x}{\ell(Q)}\right\Vert_{L^{2}(\omega)}^{\spadesuit 2}}{\left\vert Q\right\vert^{1-\frac{\alpha}{n}}}\mathcal{P}^{\alpha}\left(Q,\mathbf{1}_{Q^{c}}\sigma\right) \\
&+ \frac{\left\Vert \mathsf{Q}_{Q}^{\omega,\mathbf{b}^{\ast}}\frac{x}{\ell(Q)}\right\Vert_{L^{2}(\omega)}^{\spadesuit 2}}{\left\vert Q\right\vert^{1-\frac{\alpha}{n}}}\mathcal{P}^{\alpha}\left(Q,\mathbf{1}_{Q}\sigma\right) \\
&\lesssim \frac{\left\vert Q\right\vert_{\omega}}{\left\vert Q\right\vert^{1-\frac{\alpha}{n}}}\mathcal{P}^{\alpha}\left(Q,\mathbf{1}_{Q^{c}}\sigma\right) + \frac{\left\Vert \mathsf{Q}_{Q}^{\omega,\mathbf{b}^{\ast}}\frac{x}{\ell(Q)}\right\Vert_{L^{2}(\omega)}^{\spadesuit 2}}{\left\vert Q\right\vert^{1-\frac{\alpha}{n}}}\frac{\left\vert Q\right\vert_{\sigma}}{\left\vert Q\right\vert^{1-\frac{\alpha}{n}}} \\
&\lesssim \mathcal{A}_{2}^{\alpha}\left(\sigma,\omega\right) + A_{2}^{\alpha,energy}\left(\sigma,\omega\right).
\end{aligned}$$

□

The Poisson formulation

Recall from Definitions 5.2.8 and 5.5.1 that

$$\mathcal{C}_{F}^{\mathcal{G},shift} = \left\{ J \in \mathcal{G} : J^{\maltese} \in \mathcal{C}_{F} \right\},$$

where $F \in \mathcal{F}$ is a stopping cube in the dyadic grid $\mathcal{D}$. For convenience we repeat here the main result of this section, Proposition .0.1.

Proposition .0.8. *For all grids $\mathcal{D}$ and $\mathcal{G}$, and $\varepsilon > 0$ sufficiently small, we have*

$$\begin{aligned}
\mathfrak{F}_{\alpha}^{\mathbf{b}^{\ast}}\left(\mathcal{D},\mathcal{G}\right) &\lesssim \mathfrak{E}_{2}^{\alpha} + \sqrt{\mathcal{A}_{2}^{\alpha}} + \sqrt{\mathcal{A}_{2}^{\alpha,\ast}} + \sqrt{A_{2}^{\alpha,punct}}\ , \\
\mathfrak{F}_{\alpha}^{\mathbf{b},\ast}\left(\mathcal{G},\mathcal{D}\right) &\lesssim \mathfrak{E}_{2}^{\alpha,\ast} + \sqrt{\mathcal{A}_{2}^{\alpha}} + \sqrt{\mathcal{A}_{2}^{\alpha,\ast}} + \sqrt{A_{2}^{\alpha,\ast,punct}}\ ,
\end{aligned}$$

with implied constants independent of the grids $\mathcal{D}$ and $\mathcal{G}$.

To prove Proposition .0.8, we fix grids $\mathcal{D}$ and $\mathcal{G}$ and a subgrid $\mathcal{F}$ of $\mathcal{D}$ as in (5.5.5), and set

$$\mu \equiv \sum_{F\in\mathcal{F}} \sum_{M\in\mathcal{W}(F)} \left\| \mathsf{Q}_{F,M}^{\omega,\mathbf{b}^*} x \right\|_{L^2(\omega)}^{\spadesuit 2} \cdot \delta_{(c_M,\ell(M))} \text{ and } d\overline{\mu}(x,t) \equiv \frac{1}{t^2} d\mu(x,t) \ , \tag{.0.20}$$

where $\mathcal{W}(F)$ consists of the maximal $\mathcal{D}$-subcubes of F whose triples are contained in F, and where $\delta_{(c_M,\ell(M))}$ denotes the Dirac unit mass at the point $(c_M, \ell(M))$ in the upper half-space $\mathbb{R}_+^{n+1}$. Here $M \in \mathcal{D}$ is a dyadic cube with center c_M and side length $\ell(M)$, and for any cube $K \in \mathcal{P}$, the shorthand notation $\mathsf{P}_{F,K}^{\omega,\mathbf{b}^*}$ (resp. $\mathsf{Q}_{F,K}^{\omega,\mathbf{b}^*}$) is used for the localized pseudoprojection $\mathsf{P}_{\mathcal{C}_F^{\mathcal{G},shift};K}^{\omega,\mathbf{b}^*}$ (resp. $\mathsf{Q}_{\mathcal{C}_F^{\mathcal{G},shift};K}^{\omega,\mathbf{b}^*}$) given in (5.5.9):

$$\mathsf{P}_{F,K}^{\omega,\mathbf{b}^*} \equiv \mathsf{P}_{\mathcal{C}_F^{\mathcal{G},shift};K}^{\omega,\mathbf{b}^*} = \sum_{J\subset K:\ J\in\mathcal{C}_F^{\mathcal{G},shift}} \square_J^{\omega,\mathbf{b}^*} \tag{.0.21}$$

$$\left(\text{resp. } \mathsf{Q}_{F,K}^{\omega,\mathbf{b}^*} \equiv \mathsf{Q}_{\mathcal{C}_F^{\mathcal{G},shift};K}^{\omega,\mathbf{b}^*} = \sum_{J\subset K:\ J\in\mathcal{C}_F^{\mathcal{G},shift}} \triangle_J^{\omega,\mathbf{b}^*} \right). \tag{.0.22}$$

We emphasize that all the subcubes J that arise in the projection $\mathsf{Q}_{F,M}^{\omega,\mathbf{b}^*}$ are good inside the cubes F and beyond since $J^{\maltese} \subset F$. Here $J^{\maltese}$ is defined in Definition 5.2.8 using the body of a cube. Thus every $J \in \mathsf{Q}_F^{\omega,\mathbf{b}^*}$ is contained in a unique $M \in \mathcal{W}(F)$, so that $\mathsf{Q}_F^{\omega,\mathbf{b}^*} = \dot{\bigcup}_{M\in\mathcal{W}(F)} \mathsf{Q}_{F,M}^{\omega,\mathbf{b}^*}$. We can replace x by $x - c$ inside the projection for any choice of c we wish; the projection is unchanged. More generally, δ_q denotes a Dirac unit mass at a point q in the upper half-space $\mathbb{R}_+^{n+1}$.

We will prove the two-weight inequality

$$\left\|\mathbb{P}^{\alpha}\left(f\sigma\right)\right\|_{L^{2}\left(\mathbb{R}_{+}^{n+1},\overline{\mu}\right)}\lesssim\left(\mathfrak{E}_{2}^{\alpha}+\sqrt{\mathcal{A}_{2}^{\alpha}}+\sqrt{\mathcal{A}_{2}^{\alpha,*}}+\sqrt{A_{2}^{\alpha,punct}}\right)\left\|f\right\|_{L^{2}\left(\sigma\right)},\qquad(.0.23)$$

for all nonnegative f in $L^{2}\left(\sigma\right)$, noting that $\mathcal{F}$ and f are *not* related here. Above, $\mathbb{P}^{\alpha}(\cdot)$ denotes the α-fractional Poisson extension to the upper half-space $\mathbb{R}_{+}^{n+1}$,

$$\mathbb{P}^{\alpha}\rho\left(x,t\right)\equiv\int_{\mathbb{R}^{n}}\frac{t}{\left(t^{2}+\left|x-y\right|^{2}\right)^{\frac{n+1-\alpha}{2}}}d\rho\left(y\right),$$

so that in particular

$$\left\|\mathbb{P}^{\alpha}(f\sigma)\right\|_{L^{2}\left(\mathbb{R}_{+}^{2},\overline{\mu}\right)}^{2}=\sum_{F\in\mathcal{F}}\sum_{M\in\mathcal{W}(F)}\mathbb{P}^{\alpha}\left(f\sigma\right)\left(c(M),\ell\left(M\right)\right)^{2}\left\|\mathsf{Q}_{F,M}^{\omega,\mathbf{b}^{*}}\frac{x}{\left|M\right|^{\frac{1}{n}}}\right\|_{L^{2}\left(\omega\right)}^{\spadesuit 2},$$

and so (.0.23) proves the first line in Proposition .0.1 upon inspecting (5.5.5). Note also that we can equivalently write $\left\|\mathbb{P}^{\alpha}\left(f\sigma\right)\right\|_{L^{2}\left(\mathbb{R}_{+}^{2},\overline{\mu}\right)}=\left\|\widetilde{\mathbb{P}}^{\alpha}\left(f\sigma\right)\right\|_{L^{2}\left(\mathbb{R}_{+}^{2},\mu\right)}$ where $\widetilde{\mathbb{P}}^{\alpha}\nu\left(x,t\right)\equiv\frac{1}{t}\mathbb{P}^{\alpha}\nu\left(x,t\right)$ is the renormalized Poisson operator. Here we have simply shifted the factor $\frac{1}{t^{2}}$ in $\overline{\mu}$ to $\left|\widetilde{\mathbb{P}}^{\alpha}\left(f\sigma\right)\right|^{2}$ instead, and we will do this shifting often throughout the proof when it is convenient to do so.

One version of the characterization of the two-weight inequality for fractional and Poisson integrals in [55] was stated in terms of a fixed dyadic grid $\mathcal{D}$ of cubes in $\mathbb{R}$ with sides parallel to the coordinate axes. Using this theorem for the two-weight Poisson inequality, but adapted to the α-fractional Poisson integral $\mathbb{P}^{\alpha}$,[1] we see that inequality (.0.23) requires checking these two inequalities for dyadic cubes $I\in\mathcal{D}$ and boxes $\widehat{I}=I\times\left[0,\ell\left(I\right)\right)$ in the upper half-space

[1] The proof for $0\leq\alpha<1$ is essentially identical to that for $\alpha=0$ given in [55].

$\mathbb{R}_+^{n+1}$:

$$\begin{aligned}\int_{\mathbb{R}_+^2} \mathbb{P}^\alpha\left(\mathbf{1}_I\sigma\right)(x,t)^2 d\overline{\mu}(x,t) &\equiv \left\|\mathbb{P}^\alpha\left(\mathbf{1}_I\sigma\right)\right\|_{L^2(\overline{\mu})}^2 \\ &\lesssim \left((\mathfrak{E}_2^\alpha)^2+\mathcal{A}_2^\alpha+\mathcal{A}_2^{\alpha,*}+A_2^{\alpha,punct}\right)\sigma(I)\,, \qquad (.0.24)\end{aligned}$$

$$\int_{\mathbb{R}}[\mathbb{Q}^\alpha(t\mathbf{1}_{\widehat{I}}\overline{\mu})]^2 d\sigma(x) \lesssim \left((\mathfrak{E}_2^\alpha)^2+\mathcal{A}_2^\alpha+A_2^{\alpha,punct}\right)\int_{\widehat{I}} t^2 d\overline{\mu}(x,t), \qquad (.0.25)$$

for all *dyadic* cubes $I \in \mathcal{D}$, and where the dual Poisson operator $\mathbb{Q}^\alpha$ is given by

$$\mathbb{Q}^\alpha(t\mathbf{1}_{\widehat{I}}\overline{\mu})(x) = \int_{\widehat{I}} \frac{t^2}{\left(t^2+|x-y|^2\right)^{\frac{n+1-\alpha}{2}}} d\overline{\mu}(y,t)\ .$$

It is important to note that we can choose for $\mathcal{D}$ any fixed dyadic grid, the compensating point being that the integrations on the left sides of (.0.24) and (.0.25) are taken over the entire spaces $\mathbb{R}_+^2$ and $\mathbb{R}$ respectively[2].

Poisson testing

We now turn to proving the Poisson testing conditions (.0.24) and (.0.25). Similar testing conditions have been considered in [62], [64], [66] and [67], and the proofs there essentially carry over to the situation here, but careful attention must now be paid to the changed definition of functional energy and the weaker notion of goodness. We continue to circumvent the difficulty of permitting common point masses here by using the energy Muckenhoupt constants $A_2^{\alpha,energy}$ and $A_2^{\alpha,*,energy}$, which require control by the punctured Muckenhoupt constants $A_2^{\alpha,punct}$ and $A_2^{\alpha,*,punct}$. The following elementary Poisson inequalities (see e.g.

[2] There is a gap in the proof of the Poisson inequality at the top of page 542 in [55]. However, this gap can be fixed as in [70] or [31].

[75]) will be used extensively.

Lemma .0.9. *Suppose that J, K, I are cubes in $\mathbb{R}^n$, and that μ is a positive measure supported in $\mathbb{R}^n \setminus I$. If $J \subset K \subset \beta K \subset I$ for some $\beta > 1$, then*

$$\frac{\mathrm{P}^\alpha(J,\mu)}{|J|^{\frac{1}{n}}} \approx \frac{\mathrm{P}^\alpha(K,\mu)}{|K|^{\frac{1}{n}}},$$

while if $J \subset \beta K$, then

$$\frac{\mathrm{P}^\alpha(K,\mu)}{|K|^{\frac{1}{n}}} \lesssim \frac{\mathrm{P}^\alpha(J,\mu)}{|J|^{\frac{1}{n}}}.$$

Proof. We have

$$\frac{\mathrm{P}^\alpha(J,\mu)}{|J|^{\frac{1}{n}}} = \frac{1}{|J|^{\frac{1}{n}}} \int \frac{|J|^{\frac{1}{n}}}{\left(|J|^{\frac{1}{n}} + |x - c_J|\right)^{n+1-\alpha}} d\mu(x),$$

where $J \subset K \subset \beta K \subset I$ implies that

$$|J|^{\frac{1}{n}} + |x - c_J| \approx |K|^{\frac{1}{n}} + |x - c_K|, \qquad x \in \mathbb{R}^n \setminus I,$$

and where $J \subset \beta K$ implies that

$$|J|^{\frac{1}{n}} + |x - c_J| \lesssim |J|^{\frac{1}{n}} + |c_K - c_J| + |x - c_K| \lesssim |K|^{\frac{1}{n}} + |x - c_K|, \qquad x \in \mathbb{R}^n.$$

$\square$

Recall that in the case of the $T1$ theorem in [64], where we assumed *traditional* goodness in a single family of grids $\mathcal{D}$, we had a *strong* bounded overlap property associated with the projections $\mathsf{P}^{\omega,\mathbf{b}^*}_{F,J}$ defined there; namely, that for each cube $I_0 \in \mathcal{D}$, there were a bounded number of cubes $F \in \mathcal{F}$ with the property that $F \supsetneqq I_0 \supset J$ for some $J \in \mathcal{M}_{(\rho,\varepsilon)-deep}(F)$ with $\mathsf{P}_{F,J}^{\omega,\mathbf{b}^*} \neq 0$ (see the first part of Lemma 10.4 in [64]). However, we no longer have this strong bounded overlap property when ordinary goodness is replaced with the *weak* goodness of Hytönen and Martikainen. Indeed, there may now be an *unbounded* number of cubes $F \in \mathcal{F}$ with $F \supsetneqq I_0 \supset J$ and $\mathsf{P}_{F,J}^{\omega,\mathbf{b}^*} \neq 0$, simply because there can be $J' \in \mathcal{G}$ with both $J' \subset I_0$ and $(J')^{\maltese}$ *arbitrarily* large.

What will save us in obtaining the following lemma is that the Whitney cubes M in $\mathcal{W}(F)$ that happen to lie in some $I \in \mathcal{D}$ with $I \subset F$ have one of just two different forms: if I shares an endpoint with F then the cubes M near that endpoint are the same as those in $\mathcal{W}(I)$ - note that F has been replaced with I here - while otherwise there are a bounded number of Whitney cubes M in I, and each such M has side length comparable to $\ell(I)$.

The next lemma will be used in bounding both of the local Poisson testing conditions. Recall from Definition .0.3 that $\mathcal{AD}$ consists of all augmented $\mathcal{D}$-dyadic cubes where K is an augmented dyadic cube if it is a union of 2 $\mathcal{D}$-dyadic cubes K' with $\ell(K') = \frac{1}{2}\ell(K)$.

Lemma .0.10. *Let $\mathcal{D}$ and $\mathcal{G}$ and $\mathcal{F} \subset \mathcal{D}$ be grids and let $\left\{\mathsf{Q}_{F,M}^{\omega,\mathbf{b}^*}\right\}_{\substack{F\in\mathcal{F}\\ M\in\mathcal{W}(F)}}$ be as in (.0.22) above. For any augmented cube $I \in \mathcal{AD}$ define*

$$B(I) \equiv \sum_{F\in\mathcal{F}:\ F\supsetneqq I' \text{ for some } I'\in\mathfrak{C}(I)}\ \sum_{M\in\mathcal{W}(F):\ M\subset I} \left(\frac{\mathrm{P}^{\alpha}(M,\mathbf{1}_I\sigma)}{|M|^{\frac{1}{n}}}\right)^2 \left\|\mathsf{Q}_{F,M}^{\omega,\mathbf{b}^*}x\right\|_{L^2(\omega)}^{\spadesuit 2} . \tag{.0.26}$$

Then

$$B(I) \lesssim \left((\mathfrak{E}_2^{\alpha})^2 + A_2^{\alpha,energy}\right)|I|_{\sigma} . \tag{.0.27}$$

Proof. We first prove the bound (.0.27) for $B(I)$ ignoring for the moment the possible case when $M = I$ in the sum defining $B(I)$. So suppose that $I \in \mathcal{AD}$ is an augmented $\mathcal{D}$-dyadic cube. Define

$$\Lambda^*(I) \equiv \left\{ M \subsetneqq I : M \in \mathcal{W}(F) \text{ for some } F \supsetneqq I',\ I' \in \mathfrak{C}(I) \text{ with } \mathsf{Q}^{\omega,\mathbf{b}^*}_{F,M} x \neq 0 \right\},$$

and pigeonhole this collection as $\Lambda^*(I) = \bigcup_{I' \in \mathfrak{C}(I)} \Lambda(I')$, where for each $I' \in \mathfrak{C}(I)$ we define

$$\Lambda(I') \equiv \left\{ M \subset I' : M \in \mathcal{W}(F) \text{ for some } F \supsetneqq I' \text{ with } \mathsf{Q}^{\omega,\mathbf{b}^*}_{F,M} x \neq 0 \right\}.$$

Consider first the case when $3I' \subset F$, so that $d(I', \partial F) \geq \ell(I')$. Then if $M \in \mathcal{W}(F)$ for some $F \supsetneqq I'$ we have $\ell(M) = d(M, \partial F)$, and if in addition $M \subset I'$, then $M = I'$. Consider the sum over all $F \supsetneqq I' = M$:

$$\begin{aligned} B_M(I) &\equiv \sum_{F \in \mathcal{F}:\ F \supsetneqq M \text{ for some } M \in \mathfrak{C}(I) \cap \mathcal{W}(F)} \left(\frac{\mathrm{P}^\alpha(M, \mathbf{1}_I \sigma)}{|M|^{\frac{1}{n}}} \right)^2 \left\| \mathsf{Q}^{\omega,\mathbf{b}^*}_{F,M} x \right\|^{\spadesuit 2}_{L^2(\omega)} \\ &\leq \left(\frac{\mathrm{P}^\alpha(M, \mathbf{1}_I \sigma)}{|M|^{\frac{1}{n}}} \right)^2 \left\| \mathsf{Q}^{\omega,\mathbf{b}^*}_{M} x \right\|^{\spadesuit 2}_{L^2(\omega)} \lesssim \left(\frac{\mathrm{P}^\alpha(I, \mathbf{1}_I \sigma)}{|I|^{\frac{1}{n}}} \right)^2 \left\| \mathsf{Q}^{\omega,\mathbf{b}^*}_{I} x \right\|^{\spadesuit 2}_{L^2(\omega)} \\ &\lesssim A_2^{\alpha,energy} |I|_\sigma , \end{aligned}$$

where we have used the definitions (.0.22) and (.0.10). Thus we have obtained the bound

$$\sum_{F \in \mathcal{F}:\ F \supsetneqq M \text{ for some } M \in \mathfrak{C}(I) \cap \mathcal{W}(F)} \left(\frac{\mathrm{P}^\alpha(M, \mathbf{1}_I \sigma)}{|M|^{\frac{1}{n}}} \right)^2 \left\| \mathsf{Q}^{\omega,\mathbf{b}^*}_{F,M} x \right\|^{\spadesuit 2}_{L^2(\omega)} \lesssim A_2^{\alpha,energy} |I|_\sigma .$$

Now we turn to the case $3I' \not\subset F$, i.e. when $\partial I' \cap \partial F$ consists of exactly one boundary point. In this case, if both $M \subset I'$ and $M \in \mathcal{W}(F)$ for some $F \supsetneqq I'$, then we must have either $M \in \mathcal{W}(I')$ or $M \in \mathfrak{C}(I')$, since both M and I' are then close to the same boundary point in ∂F. Note that it is here that we use the Whitney decompositions to full advantage. So again we can estimate

$$\sum_{\substack{F\in\mathcal{F}:\ F\supsetneqq I' \text{ for some } I'\in\mathfrak{C}(I)\\ 3I'\not\subset F}} \sum_{M\in\mathcal{W}(F):\ M\subset I'} \left(\frac{\mathrm{P}^{\alpha}\left(M,\mathbf{1}_I\sigma\right)}{|M|^{\frac{1}{n}}}\right)^2 \left\|\mathsf{Q}_{F,M}^{\omega,\mathbf{b}^*}x\right\|_{L^2(\omega)}^{\spadesuit 2}$$

$$\leq \sum_{M\in\{\mathcal{W}(I')\cup\mathfrak{C}(I')\}\cap\mathcal{W}(F)} \left(\frac{\mathrm{P}^{\alpha}\left(M,\mathbf{1}_I\sigma\right)}{|M|^{\frac{1}{n}}}\right)^2 \left\|\mathsf{Q}_{M}^{\omega,\mathbf{b}^*}x\right\|_{L^2(\omega)}^{\spadesuit 2} \lesssim \left(\mathfrak{E}_2^{\alpha}\right)^2 |I|_{\sigma} \ .$$

Finally, we consider the case $M=I$. In this case $I\in\mathcal{D}$ and so $F\supsetneqq I'$ implies $F\supset I$ and we can estimate

$$\begin{aligned}\sum_{F\in\mathcal{F}:\ F\supset I} \left(\frac{\mathrm{P}^{\alpha}\left(I,\mathbf{1}_I\sigma\right)}{|I|^{\frac{1}{n}}}\right)^2 \left\|\mathsf{Q}_{F,I}^{\omega,\mathbf{b}^*}x\right\|_{L^2(\omega)}^{\spadesuit 2} &\leq \left(\frac{\mathrm{P}^{\alpha}\left(I,\mathbf{1}_I\sigma\right)}{|I|^{\frac{1}{n}}}\right)^2 \left\|\mathsf{Q}_{I}^{\omega,\mathbf{b}^*}x\right\|_{L^2(\omega)}^{\spadesuit 2}\\ &\lesssim A_2^{\alpha,energy}|I|_{\sigma} \ .\end{aligned}$$

This completes the proof of Lemma .0.10. $\square$

The forward Poisson testing inequality

Fix $I\in\mathcal{D}$. We split the integration on the left side of (.0.24) into a local and global piece:

$$\int_{\mathbb{R}_+^{n+1}} \mathbb{P}^{\alpha}\left(\mathbf{1}_I\sigma\right)^2 d\overline{\mu} = \int_{\widehat{I}} \mathbb{P}^{\alpha}\left(\mathbf{1}_I\sigma\right)^2 d\overline{\mu} + \int_{\mathbb{R}_+^{n+1}\setminus\widehat{I}} \mathbb{P}^{\alpha}\left(\mathbf{1}_I\sigma\right)^2 d\overline{\mu} \equiv \mathbf{Local}\left(I\right) + \mathbf{Global}\left(I\right)$$

where more explicitly,

$$\mathbf{Local}(I) \equiv \int_{\widehat{I}} \left[\mathbb{P}^{\alpha}(\mathbf{1}_I\sigma)(x,t)\right]^2 d\overline{\mu}(x,t); \qquad \overline{\mu} \equiv \frac{1}{t^2}\mu, \tag{.0.28}$$

$$\text{i.e. } \overline{\mu} \equiv \sum_{F\in\mathcal{F}} \sum_{M\in\mathcal{W}(F)} \left\| \mathsf{Q}^{\omega,\mathbf{b}^*}_{F,M} \frac{x}{\ell(M)} \right\|^{\spadesuit 2}_{L^2(\omega)} \cdot \delta_{(c_M,\ell(M))},$$

where we recall $\mathsf{Q}^{\omega,\mathbf{b}^*}_{F,M}$ is defined in (.0.22) above. Here is a brief schematic diagram of the decompositions, used in this subsection:

$$\begin{array}{ccc}
\mathbf{Local}(I) & & \\
\downarrow & & \\
\mathbf{Local}^{plug}(I) & + & \mathbf{Local}^{hole}(I) \\
\downarrow & & \left(\mathfrak{E}^{\alpha}_2\right)^2 \\
\downarrow & & \\
A & + & B \\
\left(\mathfrak{E}^{\alpha}_2\right)^2 + A^{\alpha,energy}_2 & & \left(\mathfrak{E}^{\alpha}_2\right)^2 + A^{\alpha,energy}_2
\end{array}$$

and

$$\begin{array}{ccccccc}
\mathbf{Global}(I) & & & & & & \\
\downarrow & & & & & & \\
A & + & B & + & C & + & D \\
A^{\alpha}_2 & & \left(\mathfrak{E}^{\alpha}_2\right)^2 + A^{\alpha}_2 + A^{\alpha,energy}_2 & & \mathcal{A}^{\alpha,*}_2 & & \mathcal{A}^{\alpha,*}_2 + A^{\alpha,punct}_2
\end{array}$$

As in our earlier papers [59]-[67] that used a single family of random grids, we have the useful equivalence that

$$(c(M), \ell(M)) \in \widehat{I} \textbf{ if and only if } M \subset I, \tag{.0.29}$$

since M and I live in the common grid $\mathcal{D}$. We thus have

$$\begin{aligned}
&\mathbf{Local}\left(I\right)=\int_{\widehat{I}}\mathbb{P}^{\alpha}\left(\mathbf{1}_{I}\sigma\right)\left(x,t\right)^{2}d\overline{\mu}\left(x,t\right) \\
=&\sum_{F\in\mathcal{F}}\sum_{M\in\mathcal{W}(F):\ M\subset I}\mathbb{P}^{\alpha}\left(\mathbf{1}_{I}\sigma\right)\left(c_{M},\ell\left(M\right)\right)^{2}\left\Vert \mathsf{Q}_{F,M}^{\omega,\mathbf{b}^{\ast}}\frac{x}{\left\vert M\right\vert ^{\frac{1}{n}}}\right\Vert _{L^{2}(\omega)}^{\spadesuit 2} \\
\approx&\sum_{F\in\mathcal{F}}\sum_{M\in\mathcal{W}(F):\ M\subset I}\mathrm{P}^{\alpha}\left(M,\mathbf{1}_{I}\sigma\right)^{2}\Vert \mathsf{Q}_{F,M}^{\omega,\mathbf{b}^{\ast}}\frac{x}{\left\vert M\right\vert ^{\frac{1}{n}}}\Vert _{L^{2}(\omega)}^{\spadesuit 2} \\
\approx&\mathbf{Local}^{plug}\left(I\right)+\mathbf{Local}^{hole}\left(I\right),
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{Local}^{plug}\left(I\right) &\equiv \sum_{F\in\mathcal{F}}\sum_{M\in\mathcal{W}(F)):\ M\subset I}\left(\frac{\mathrm{P}^{\alpha}\left(M,\mathbf{1}_{F\cap I}\sigma\right)}{\left\vert M\right\vert ^{\frac{1}{n}}}\right)^{2}\left\Vert \mathsf{Q}_{F,M}^{\omega,\mathbf{b}^{\ast}}x\right\Vert _{L^{2}(\omega)}^{2}, \\
\mathbf{Local}^{hole}\left(I\right) &\equiv \sum_{F\in\mathcal{F}}\sum_{M\in\mathcal{W}(F):\ M\subset I}\left(\frac{\mathrm{P}^{\alpha}\left(M,\mathbf{1}_{I\setminus F}\sigma\right)}{\left\vert M\right\vert ^{\frac{1}{n}}}\right)^{2}\left\Vert \mathsf{Q}_{F,M}^{\omega,\mathbf{b}^{\ast}}x\right\Vert _{L^{2}(\omega)}^{\spadesuit 2}.
\end{aligned}$$

The 'plugged' local sum $\mathbf{Local}^{plug}\left(I\right)$ can be further decomposed into

$$\begin{aligned}
\mathbf{Local}^{plug}\left(I\right) &= \left\{\sum_{\substack{F\in\mathcal{F}\\ F\subset I}}+\sum_{\substack{F\in\mathcal{F}\\ F\supsetneqq I}}\right\}\sum_{\substack{M\in\mathcal{W}(F)\\ M\subset I}}\left(\frac{\mathrm{P}^{\alpha}\left(M,\mathbf{1}_{F\cap I}\sigma\right)}{\left\vert M\right\vert ^{\frac{1}{n}}}\right)^{2}\left\Vert \mathsf{Q}_{F,M}^{\omega,\mathbf{b}^{\ast}}x\right\Vert _{L^{2}(\omega)}^{\spadesuit 2} \\
&= A+B.
\end{aligned}$$

Then an application of the Whitney plugged energy condition gives

$$\begin{aligned} A &= \sum_{F\in\mathcal{F}:\ F\subset I}\ \sum_{M\in\mathcal{W}(F)} \left(\frac{\mathrm{P}^{\alpha}\left(M,\mathbf{1}_{F\cap I}\sigma\right)}{\left|M\right|^{\frac{1}{n}}}\right)^{2} \left\|\mathsf{Q}_{F,M}^{\omega,\mathbf{b}^{\ast}}x\right\|_{L^{2}(\omega)}^{\spadesuit 2} \\ &\leq \sum_{F\in\mathcal{F}:\ F\subset I} \left(\mathfrak{E}_{2}^{\alpha}+\sqrt{A_{2}^{\alpha,energy}}\right)^{2}\left|F\right|_{\sigma} \lesssim \left(\mathfrak{E}_{2}^{\alpha}+\sqrt{A_{2}^{\alpha,energy}}\right)^{2}\left|I\right|_{\sigma}\ , \end{aligned}$$

since $\left\|\mathsf{Q}_{F,M}^{\omega,\mathbf{b}^{\ast}}x\right\|_{L^{2}(\omega)}^{\spadesuit 2} \leq \left\|\mathsf{Q}_{M}^{\omega,\mathbf{b}^{\ast}}x\right\|_{L^{2}(\omega)}^{\spadesuit 2}$. We also used here that the stopping cubes $\mathcal{F}$ satisfy a σ-Carleson measure estimate,

$$\sum_{F\in\mathcal{F}:\ F\subset F_{0}}\left|F\right|_{\sigma}\lesssim\left|F_{0}\right|_{\sigma}.$$

Lemma .0.10 applies to the remaining term B to obtain the bound

$$B\lesssim\left(\left(\mathfrak{E}_{2}^{\alpha}\right)^{2}+A_{2}^{\alpha,energy}\right)\left|I\right|_{\sigma}\ .$$

Next we show the inequality with 'holes', where the support of σ is restricted to the complement of the cube F.

Lemma .0.11. *We have*

$$\mathbf{Local}^{hole}\left(I\right)\lesssim\left(\mathfrak{E}_{2}^{\alpha}\right)^{2}\left|I\right|_{\sigma}\ . \tag{.0.30}$$

Proof. Fix $I\in\mathcal{D}$ and define

$$\mathcal{F}_{I}\equiv\left\{F\in\mathcal{F}:F\subset I\right\}\cup\left\{I\right\},$$

and denote by πF, for this proof only, the parent of F in the tree $\mathcal{F}_{I}$. Also denote by

$d\left(F,F'\right) \equiv d_{\mathcal{F}_I}\left(F,F'\right)$ the distance from F to F' in the tree $\mathcal{F}_I$, and denote by $d\left(F\right) \equiv d_{\mathcal{F}_I}\left(F,I\right)$ the distance of F from the root I. Since $I\setminus F$ appears in the argument of the Poisson integral, those $F\in\mathcal{F}\setminus\mathcal{F}_I$ do not contribute to the sum and so we estimate

$$S \equiv \mathbf{Local}^{hole}\left(I\right) = \sum_{F\in\mathcal{F}_I}\sum_{M\in\mathcal{W}(F):\ M\subset I}\left(\frac{\mathrm{P}^{\alpha}\left(M,\mathbf{1}_{I\setminus F}\sigma\right)}{\left|M\right|^{\frac{1}{n}}}\right)^2\left\|\mathsf{Q}_{F,M}^{\omega,\mathbf{b}^*}x\right\|_{L^2(\omega)}^{\spadesuit 2}$$

by using $\sum_{F'\in\mathcal{F}:\ F\subset F'\subsetneqq I}\frac{1}{d(F')^2}\leq C$ to obtain

$$\begin{aligned}
S &= \sum_{F\in\mathcal{F}_I}\sum_{\substack{M\in\mathcal{W}(F)\\ M\subset I}}\left(\sum_{F'\in\mathcal{F}:\ F\subset F'\subsetneqq I}\frac{d\left(F'\right)}{d\left(F'\right)}\frac{\mathrm{P}^{\alpha}\left(M,\mathbf{1}_{\pi F'\setminus F'}\sigma\right)}{\left|M\right|^{\frac{1}{n}}}\right)^2\left\|\mathsf{Q}_{F,M}^{\omega,\mathbf{b}^*}x\right\|_{L^2(\omega)}^{\spadesuit 2}\\
&\leq \sum_{F\in\mathcal{F}_I}\sum_{\substack{M\in\mathcal{W}(F)\\ M\subset I}}\left(\sum_{\substack{F'\in\mathcal{F}\\ F\subset F'\subsetneqq I}}\frac{1}{d\left(F'\right)^2}\right)\cdot\\
&\qquad\cdot\left(\sum_{\substack{F'\in\mathcal{F}\\ F\subset F'\subsetneqq I}}d\left(F'\right)^2\left(\frac{\mathrm{P}^{\alpha}\left(M,\mathbf{1}_{\pi F'\setminus F'}\sigma\right)}{\left|M\right|^{\frac{1}{n}}}\right)^2\right)\left\|\mathsf{Q}_{F,M}^{\omega,\mathbf{b}^*}x\right\|_{L^2(\omega)}^{\spadesuit 2}\\
&\leq C\sum_{F'\in\mathcal{F}_I}d\left(F'\right)^2\sum_{\substack{F\in\mathcal{F}\\ F\subset F'}}\sum_{M\in\mathcal{W}(F):\ M\subset I}\left(\frac{\mathrm{P}^{\alpha}\left(M,\mathbf{1}_{\pi F'\setminus F'}\sigma\right)}{\left|M\right|^{\frac{1}{n}}}\right)^2\left\|\mathsf{Q}_{F,M}^{\omega,\mathbf{b}^*}x\right\|_{L^2(\omega)}^{\spadesuit 2}\\
&= C\sum_{F'\in\mathcal{F}_I}d\left(F'\right)^2\sum_{K\in\mathcal{W}(F')}\sum_{\substack{F\in\mathcal{F}\\ F\subset F'}}\sum_{\substack{M\in\mathcal{W}(F)\\ M\subset I}}\left(\frac{\mathrm{P}^{\alpha}\left(M,\mathbf{1}_{\pi F'\setminus F'}\sigma\right)}{\left|M\right|^{\frac{1}{n}}}\right)^2\left\|\mathsf{Q}_{F,M\cap K}^{\omega,\mathbf{b}^*}x\right\|_{L^2(\omega)}^{\spadesuit 2}\\
&\lesssim \sum_{F'\in\mathcal{F}_I}d\left(F'\right)^2\sum_{K\in\mathcal{W}(F')}\left(\frac{\mathrm{P}^{\alpha}\left(K,\mathbf{1}_{\pi F'\setminus F'}\sigma\right)}{\left|K\right|^{\frac{1}{n}}}\right)^2\sum_{\substack{F\in\mathcal{F}\\ F\subset F'}}\sum_{\substack{M\in\mathcal{W}(F)\\ M\subset I}}\left\|\mathsf{Q}_{F,M\cap K}^{\omega,\mathbf{b}^*}x\right\|_{L^2(\omega)}^{\spadesuit 2}
\end{aligned}$$

where in the fifth line we have used that each J' appearing in $\mathsf{Q}_{F,M}^{\omega,\mathbf{b}^*}$ occurs in one of the $\mathsf{Q}_{F,M\cap K}^{\omega,\mathbf{b}^*}$ since each M is contained in a unique K. We have also used there the Poisson inequalities in Lemma .0.9.

We now use the lower frame inequality applied to the function $\mathbf{1}_K\left(x-m_K^{\omega}\right)$ to obtain

$$\sum_{F\in\mathcal{F}:\ F\subset F'}\ \sum_{M\in\mathcal{W}(F):\ M\subset I}\left\|\mathsf{Q}_{F,M\cap K}^{\omega,\mathbf{b}^*}x\right\|_{L^2(\omega)}^{\spadesuit 2}\lesssim\left\|\mathbf{1}_K\left(x-m_K^{\omega}\right)\right\|_{L^2(\omega)}^{\spadesuit 2}\ .$$

Since the collection $\mathcal{F}_I$ satisfies a Carleson condition, namely $\sum_{F\in\mathcal{F}_I}\left|F\cap I'\right|_{\sigma}\leq C\left|I'\right|_{\sigma}$ for all cubes I', we have geometric decay in generations:

$$\sum_{F\in\mathcal{F}_I:\ d(F)=k}\left|F\right|_{\sigma}\lesssim 2^{-\delta k}\left|I\right|_{\sigma}\ ,\qquad k\geq 0. \tag{.0.31}$$

Indeed, with $m>2C$ we have for each $F'\in\mathcal{F}_I$,

$$\sum_{F\in\mathcal{F}_I:\ F\subset F'\text{ and }d(F,F')=m}\left|F\cap F'\right|_{\sigma}<\frac{1}{2}\left|F'\right|_{\sigma}\ , \tag{.0.32}$$

since otherwise

$$\sum_{F\in\mathcal{F}_I:\ F\subset F'\text{ and }d(F,F')\leq m}\left|F\cap F'\right|_{\sigma}\geq m\frac{1}{2}\left|F'\right|_{\sigma}\ ,$$

a contradiction. Now iterate (.0.32) to obtain (.0.31).

Thus we can write

$$\begin{aligned} S &\lesssim \sum_{F'\in\mathcal{F}_I} d\left(F'\right)^2 \sum_{K\in\mathcal{W}(F')} \left(\frac{\mathrm{P}^\alpha\left(K,\mathbf{1}_{\pi F'\setminus F'}\sigma\right)}{|K|^{\frac{1}{n}}}\right)^2 \left\|\mathbf{1}_K\left(x-m_K^\omega\right)\right\|_{L^2(\omega)}^{\spadesuit 2} \\ &= \sum_{k=1}^\infty k^2 \sum_{F'\in\mathcal{F}_I:\ d(F')=k} \sum_{K\in\mathcal{W}(F')} \left(\frac{\mathrm{P}^\alpha\left(K,\mathbf{1}_{\pi F'\setminus F'}\sigma\right)}{|K|^{\frac{1}{n}}}\right)^2 \left\|\mathbf{1}_K\left(x-m_K^\omega\right)\right\|_{L^2(\omega)}^{\spadesuit 2} \\ &\equiv \sum_{k=1}^\infty A_k\ , \end{aligned}$$

where A_k is defined at the end of the above display. Hence using the strong energy condition,

$$\begin{aligned} A_k &= k^2 \sum_{F'\in\mathcal{F}_I:\ d(F')=k} \sum_{K\in\mathcal{W}(F')} \left(\frac{\mathrm{P}^\alpha\left(K,\mathbf{1}_{\pi F'\setminus F'}\sigma\right)}{|K|^{\frac{1}{n}}}\right)^2 \left\|\mathbf{1}_K\left(x-m_K^\omega\right)\right\|_{L^2(\omega)}^{\spadesuit 2} \\ &\lesssim k^2\left(\mathfrak{E}_2^\alpha\right)^2 \sum_{F''\in\mathcal{F}_I:\ d(F'')=k-1} \left|F''\right|_\sigma \lesssim \left(\mathfrak{E}_2^\alpha\right)^2 k^2 2^{-\delta k}\left|I\right|_\sigma\ , \end{aligned}$$

where we have applied the strong energy condition for each $F''\in\mathcal{F}_I$ with $d\left(F''\right)=k-1$ to obtain

$$\sum_{F'\in\mathcal{F}_I:\ \pi F'=F''} \sum_{K\in\mathcal{W}(F')} \left(\frac{\mathrm{P}^\alpha\left(K,\mathbf{1}_{F''\setminus F'}\sigma\right)}{|K|^{\frac{1}{n}}}\right)^2 \left\|\mathbf{1}_K\left(x-m_K^\omega\right)\right\|_{L^2(\omega)}^{\spadesuit 2} \leq \left(\mathfrak{E}_2^\alpha\right)^2\left|F''\right|_\sigma\ . \tag{.0.33}$$

Finally then we obtain

$$S\lesssim \sum_{k=1}^\infty \left(\mathfrak{E}_2^\alpha\right)^2 k^2 2^{-\delta k}\left|I\right|_\sigma \lesssim \left(\mathfrak{E}_2^\alpha\right)^2\left|I\right|_\sigma\ ,$$

which is (.0.30). $\square$

Altogether we have now proved the estimate $\mathbf{Local}\left(I\right)\lesssim\left(\left(\mathfrak{E}_2^\alpha\right)^2+A_2^{\alpha,energy}\right)\left|I\right|_\sigma$ when

$I \in \mathcal{D}$, i.e. for every dyadic cube $I \in \mathcal{D}$,

$$\begin{aligned} \mathbf{Local}(I) &\approx \sum_{F\in\mathcal{F}} \sum_{M\in\mathcal{W}(F):\ M\subset I} \left(\frac{\mathrm{P}^{\alpha}\left(M, \mathbf{1}_{I}\sigma\right)}{|M|^{\frac{1}{n}}} \right)^{2} \left\| \mathsf{Q}_{F,M}^{\omega,\mathbf{b}^{*}} x \right\|_{L^{2}(\omega)}^{\spadesuit 2} \\ &\lesssim \left((\mathfrak{E}_{2}^{\alpha})^{2} + A_{2}^{\alpha,energy} \right) |I|_{\sigma}, \quad I \in \mathcal{D}. \end{aligned} \tag{.0.34}$$

The augmented local estimate

For future use in the 'prepare to puncture' arguments below, we prove a strengthening of the local estimate $\mathbf{Local}(I)$ to *augmented* cubes $L \in \mathcal{AD}$.

Lemma .0.12. *With notation as above and $L \in \mathcal{AD}$ an augmented cube, we have*

$$\begin{aligned} \mathbf{Local}(L) &\equiv \sum_{F\in\mathcal{F}} \sum_{M\in\mathcal{W}(F):\ M\subset L} \left(\frac{\mathrm{P}^{\alpha}\left(M, \mathbf{1}_{L}\sigma\right)}{|M|^{\frac{1}{n}}} \right)^{2} \left\| \mathsf{Q}_{F,M}^{\omega,\mathbf{b}^{*}} x \right\|_{L^{2}(\omega)}^{\spadesuit 2} \\ &\lesssim \left((\mathfrak{E}_{2}^{\alpha})^{2} + A_{2}^{\alpha,energy} \right) |L|_{\sigma}, \quad L \in \mathcal{AD}. \end{aligned} \tag{.0.35}$$

Proof. We prove (.0.35) by repeating the above proof of (.0.34) and noting the points requiring change. First we decompose

$$\mathbf{Local}(L) \lesssim \mathbf{Local}^{plug}(L) + \mathbf{Local}^{hole}(L) + \mathbf{Local}^{offset}(L)$$

where $\mathbf{Local}^{plug}(L)$, $\mathbf{Local}^{hole}(L)$ are analogous to $\mathbf{Local}^{plug}(I)$ and $\mathbf{Local}^{hole}(I)$ above, and where $\mathbf{Local}^{offset}(L)$ is an additional term arising because $L \setminus F$ need not be empty when $L\cap F\neq\emptyset$ and F is not contained in L:

$$\begin{aligned}
\mathbf{Local}^{plug}(L) &\equiv \sum_{F\in\mathcal{F}}\ \sum_{M\in\mathcal{W}(F):\ M\subset L}\left(\frac{\mathrm{P}^{\alpha}\left(M,\mathbf{1}_{L\cap F}\sigma\right)}{\left|M\right|^{\frac{1}{n}}}\right)^{2}\left\Vert \mathsf{Q}_{F,M}^{\omega,\mathbf{b}^{\ast}}x\right\Vert_{L^{2}(\omega)}^{\spadesuit 2}\ ,\\
\mathbf{Local}^{hole}(L) &\equiv \sum_{F\in\mathcal{F}:\ F\subset L}\ \sum_{M\in\mathcal{W}(F):\ M\subset L}\left(\frac{\mathrm{P}^{\alpha}\left(M,\mathbf{1}_{L\setminus F}\sigma\right)}{\left|M\right|^{\frac{1}{n}}}\right)^{2}\left\Vert \mathsf{Q}_{F,M}^{\omega,\mathbf{b}^{\ast}}x\right\Vert_{L^{2}(\omega)}^{\spadesuit 2}\\
\mathbf{Local}^{offset}(L) &\equiv \sum_{F\in\mathcal{F}:\ F\not\subset L}\ \sum_{M\in\mathcal{W}(F):\ M\subset L}\left(\frac{\mathrm{P}^{\alpha}\left(M,\mathbf{1}_{L\setminus F}\sigma\right)}{\left|M\right|^{\frac{1}{n}}}\right)^{2}\left\Vert \mathsf{Q}_{F,M}^{\omega,\mathbf{b}^{\ast}}x\right\Vert_{L^{2}(\omega)}^{\spadesuit 2}
\end{aligned}$$

We have

$$\begin{aligned}
\mathbf{Local}^{plug}(L) &= \left\{\sum_{F\in\mathcal{F}:\ F\subset \text{ some } L'\in\mathfrak{C}(L)}+\sum_{F\in\mathcal{F}:\ F\supsetneqq \text{ some } L'\in\mathfrak{C}_{\mathcal{D}}(L)}\right\}\sum_{M\in\mathcal{W}(F):\ M\subset L}\\
&\qquad\times\left(\frac{\mathrm{P}^{\alpha}\left(M,\mathbf{1}_{F\cap L}\sigma\right)}{\left|M\right|^{\frac{1}{n}}}\right)^{2}\left\Vert \mathsf{Q}_{F,M}^{\omega,\mathbf{b}^{\ast}}x\right\Vert_{L^{2}(\omega)}^{\spadesuit 2}\\
&= A+B.
\end{aligned}$$

Term A satisfies

$$A\lesssim\left(\mathfrak{E}_{2}^{\alpha}+\sqrt{A_{2}^{\alpha,energy}}\right)^{2}\left|L\right|_{\sigma}\ ,$$

just as above using $\left\Vert \mathsf{Q}_{F,M}^{\omega}x\right\Vert_{L^{2}(\omega)}^{2}\leq\left\Vert \mathsf{Q}_{M}^{\omega}x\right\Vert_{L^{2}(\omega)}^{2}$, and the fact that the stopping cubes $\mathcal{F}$ satisfy a σ-Carleson measure estimate,

$$\sum_{F\in\mathcal{F}:\ F\subset L}\left|F\right|_{\sigma}\lesssim\left|L\right|_{\sigma}.$$

Term B is handled directly by Lemma .0.10 with the augmented cube $I = L$ to obtain

$$B \lesssim \left((\mathfrak{E}_2^{\alpha})^2 + A_2^{\alpha,energy} \right) |L|_{\sigma} \ .$$

To handle $\mathbf{Local}^{hole}(L)$, we define

$$\mathcal{F}_L \equiv \{F \in \mathcal{F} : F \subset L\} \cup \{L\} \, ,$$

and follow along the proof there with only trivial changes. The analogue of (.0.33) is now

$$\sum_{F' \in \mathcal{F}_L :\ \pi F' = F''} \sum_{K \in \mathcal{W}(F')} \left(\frac{\mathrm{P}^{\alpha}\left(K, \mathbf{1}_{F'' \setminus F'} \sigma \right)}{|K|^{\frac{1}{n}}} \right)^2 \left\| \mathbf{1}_K \left(x - m_K^{\omega} \right) \right\|_{L^2(\omega)}^{\spadesuit 2} \leq (\mathfrak{E}_2^{\alpha})^2 \left| F'' \right|_{\sigma} \ ,$$

the only change being that $\mathcal{F}_L$ now appears in place of $\mathcal{F}_I$, so that the energy condition still applies. We conclude that

$$\mathbf{Local}^{hole}(L) \lesssim (\mathfrak{E}_2^{\alpha})^2 |L|_{\sigma} \ .$$

Finally, the additional term $\mathbf{Local}^{offset}(L)$ is handled directly by Lemma .0.10, and this completes the proof of the estimate (.0.35) in Lemma .0.12. □

The global estimate

Now we turn to proving the following estimate for the global part of the first testing condition (.0.24):

$$\mathbf{Global}(I) = \int_{\mathbb{R}_+^{n+1} \setminus \widehat{I}} \mathbb{P}^{\alpha} (\mathbf{1}_I \sigma)^2 \, d\overline{\mu} \lesssim \left((\mathfrak{E}_2^{\alpha})^2 + \mathcal{A}_2^{\alpha,*} + A_2^{\alpha,punct} \right) |I|_{\sigma} \, .$$

We begin by decomposing the integral above into four pieces. We have from (.0.29):

$$\begin{aligned}
&\int_{\mathbb{R}_+^{n+1}\setminus\widehat{I}} \mathbb{P}^\alpha\left(\mathbf{1}_I\sigma\right)^2 d\overline{\mu} \\
&= \sum_{M:\ (c_M,\ell(M))\in\mathbb{R}_+^{n+1}\setminus\widehat{I}} \mathbb{P}^\alpha\left(\mathbf{1}_I\sigma\right)\left(c_M,\ell\left(M\right)\right)^2 \sum_{\substack{F\in\mathcal{F}:\\ M\in\mathcal{W}(F)}} \left\| \mathsf{Q}_{F,M}^{\omega,\mathbf{b}^*} \frac{x}{\left|M\right|^{\frac{1}{n}}} \right\|_{L^2(\omega)}^{\spadesuit 2} \\
&= \left\{ \sum_{\substack{M\cap 3I=\emptyset\\ \ell(M)\leq\ell(I)}} + \sum_{M\subset 3I\setminus I} + \sum_{\substack{M\cap I=\emptyset\\ \ell(M)>\ell(I)}} + \sum_{M\supsetneqq I} \right\} \mathbb{P}^\alpha\left(\mathbf{1}_I\sigma\right)\left(c_M,\ell\left(M\right)\right)^2 \cdot \\
&\qquad\qquad \cdot \sum_{\substack{F\in\mathcal{F}:\\ M\in\mathcal{W}(F)}} \left\| \mathsf{Q}_{F,M}^{\omega,\mathbf{b}^*} \frac{x}{\left|M\right|^{\frac{1}{n}}} \right\|_{L^2(\omega)}^{\spadesuit 2} \\
&= A+B+C+D.
\end{aligned}$$

We further decompose term A according to the length of M and its distance from I, and then use the pairwise disjointedness of the projections $\mathsf{Q}_{F,M}^{\omega,\mathbf{b}^*}$ in F (see the definition in (.0.22)) to obtain:

$$\begin{aligned}
A &\lesssim \sum_{m=0}^{\infty}\sum_{k=1}^{\infty} \sum_{\substack{M\subset 3^{k+1}I\setminus 3^kI\\ \ell(M)=2^{-m}\ell(I)}} \left(\frac{2^{-m}\left|I\right|}{d\left(M,I\right)^{n+1-\alpha}} \left|I\right|_\sigma \right)^2 \left|M\right|_\omega \\
&\lesssim \sum_{m=0}^{\infty} 2^{-2m} \sum_{k=1}^{\infty} \frac{\left|I\right|^2 \left|I\right|_\sigma \left|3^{k+1}I\setminus 3^kI\right|_\omega}{\left|3^kI\right|^{2(n+1-\alpha)}} \left|I\right|_\sigma \\
&\lesssim \sum_{m=0}^{\infty} 2^{-2m} \sum_{k=1}^{\infty} 3^{-2k} \left\{ \frac{\left|3^{k+1}I\setminus 3^kI\right|_\omega \left|3^kI\right|_\sigma}{\left|3^kI\right|^{2(1-\alpha)}} \right\} \left|I\right|_\sigma \lesssim A_2^\alpha \left|I\right|_\sigma ,
\end{aligned}$$

where the offset Muckenhoupt constant A_2^α applies because $3^{k+1}I$ has only three times the side length of 3^kI.

For term B we first dispose of the nearby sum B_{nearby} that consists of the sum over those M which satisfy in addition $2^{-\rho}\ell(I) \leq \ell(M) \leq \ell(I)$. But it is a straightforward task to bound B_{nearby} by $CA_2^{\alpha,energy}|I|_\sigma$ as there are at most $2^{\rho+1}$ such cubes M. To bound $B_{away} \equiv B - B_{nearby}$, we further decompose the sum over $F \in \mathcal{F}$ according to whether or not $F \subset 3I \setminus I$:

$$\begin{aligned}
B_{away} &\approx \sum_{M \subset 3I \setminus I \text{ and } \ell(M) < 2^{-\rho}\ell(I)} \left(\frac{\mathrm{P}^\alpha(M, \mathbf{1}_I\sigma)}{|M|^{\frac{1}{n}}}\right)^2 \sum_{\substack{F \in \mathcal{F}:\ F \subset 3I \setminus I \\ M \in \mathcal{W}(F)}} \left\|\mathsf{Q}_{F,M}^{\omega,\mathbf{b}^*} x\right\|_{L^2(\omega)}^{\spadesuit 2} \\
&\quad + \sum_{M \subset 3I \setminus I \text{ and } \ell(M) < 2^{-\rho}\ell(I)} \left(\frac{\mathrm{P}^\alpha(M, \mathbf{1}_I\sigma)}{|M|^{\frac{1}{n}}}\right)^2 \sum_{\substack{F \in \mathcal{F}:\ F \not\subset 3I \setminus I \\ M \in \mathcal{W}(F)}} \left\|\mathsf{Q}_{F,M}^{\omega,\mathbf{b}^*} x\right\|_{L^2(\omega)}^{\spadesuit 2} \\
&\equiv B_{away}^1 + B_{away}^2 .
\end{aligned}$$

To estimate B_{away}^1, let

$$\mathcal{J}^* \equiv \bigcup_{\substack{F \in \mathcal{F} \\ F \subset 3I \setminus I}} \ \bigcup_{\substack{M \in \mathcal{W}(F) \\ M \subset 3I \setminus I \text{ and } \ell(M) < 2^{-\rho}\ell(I)}} \left\{J \in \mathcal{C}_F^{\mathcal{G},shift} : J \subset M\right\} \tag{.0.36}$$

consist of all cubes $J \in \mathcal{G}$ for which the projection $\triangle_J^{\omega,\mathbf{b}^*}$ occurs in one of the projections $\mathsf{Q}_{F,M}^{\omega,\mathbf{b}^*}$ in term B_{away}^1. In order to use $\mathcal{J}^*$ in the estimate for B_{away}^1 we need the following inequality. For any cube $M \in \mathcal{W}(F)$ we have

$$\left(\frac{\mathrm{P}^{\alpha}(M, \mathbf{1}_I \sigma)}{|M|}\right)^2 \left\|\mathsf{Q}^{\omega,\mathbf{b}^*}_{F;M} x\right\|^{\spadesuit 2}_{L^2(\omega)} = \left(\frac{\mathrm{P}^{\alpha}(M, \mathbf{1}_I \sigma)}{|M|}\right)^2 \sum_{J \in \mathcal{C}_F^{\mathcal{G},shift}:\ J \subset M} \left\|\triangle^{\omega,\mathbf{b}^*}_J x\right\|^{\spadesuit 2}_{L^2(\omega)}$$
$$\lesssim \sum_{J \in \mathcal{C}_F^{\mathcal{G},shift}:\ J \subset M} \left(\frac{\mathrm{P}^{\alpha}(J, \mathbf{1}_I \sigma)}{|J|}\right)^2 \left\|\triangle^{\omega,\mathbf{b}^*}_J x\right\|^{\spadesuit 2}_{L^2(\omega)} \quad (.0.37)$$

since

$$\begin{aligned}
\frac{\mathrm{P}^{\alpha}(M, \mathbf{1}_I \sigma)}{|M|^{\frac{1}{n}}} &= \int_I \frac{1}{(\ell(M) + |x - c_M|)^{n+1-\alpha}} d\sigma(x) \\
&\lesssim \int_I \frac{1}{(\ell(J) + |x - c_J|)^{n+1-\alpha}} d\sigma(x) = \frac{\mathrm{P}^{\alpha}(J, \mathbf{1}_I \sigma)}{|J|^{\frac{1}{n}}}
\end{aligned}$$

for $J \subset M$ because

$$\ell(J) + |x - c_J| \lesssim \ell(M) + |x - c_M|, \qquad J \subset M \text{ and } x \in \mathbb{R}^n.$$

We now use (.0.37) to replace the sum over $M \in \mathcal{W}(F)$ in B^1_{away}, with a sum over $J \in \mathcal{J}^*$:

$$\begin{aligned}
B^1_{away} &= \sum_{M \subset 3I \setminus I \text{ and } \ell(M) < 2^{-\rho}\ell(I)} \left(\frac{\mathrm{P}^{\alpha}(M, \mathbf{1}_I \sigma)}{|M|^{\frac{1}{n}}}\right)^2 \sum_{\substack{F \in \mathcal{F}:\ F \subset 3I \setminus I \\ M \in \mathcal{W}(F)}} \left\|\mathsf{Q}^{\omega,\mathbf{b}^*}_{F,M} x\right\|^{\spadesuit 2}_{L^2(\omega)} \\
&\lesssim \sum_{M \subset 3I \setminus I \& \ell(M) < 2^{-\rho}\ell(I)} \sum_{\substack{F \in \mathcal{F}:\ F \subset 3I \setminus I \\ M \in \mathcal{W}(F)}} \sum_{\substack{J \in \mathcal{C}_F^{\mathcal{G},shift} \\ J \subset M}} \left(\frac{\mathrm{P}^{\alpha}(J, \mathbf{1}_I \sigma)}{|J|^{\frac{1}{n}}}\right)^2 \left\|\triangle^{\omega,\mathbf{b}^*}_J x\right\|^{\spadesuit 2}_{L^2(\omega)} \\
&\lesssim \sum_{J \in \mathcal{J}^*} \left(\frac{\mathrm{P}^{\alpha}(J, \mathbf{1}_I \sigma)}{|J|}\right)^2 \left\|\triangle^{\omega,\mathbf{b}^*}_J x\right\|^{\spadesuit 2}_{L^2(\omega)},
\end{aligned}$$

where the final line follows since for each $J \in \mathcal{J}^*$ there is a unique pair (F, M) satisfying the conditions in the second line.

We will now exploit the smallness of $\varepsilon > 0$ in the weak goodness condition by decomposing the sum over $J \in \mathcal{J}^*$ according to the length of J, and then using the fractional Poisson inequality (5.5.22) in Lemma 5.5.10 on the neighbour I' of I containing J. Indeed, for $J \subset I' \subset \mathbb{R}$ and $I \subset \mathbb{R} \setminus I'$, we have

$$\mathrm{P}^{\alpha}\left(J, \mathbf{1}_I \sigma\right)^2 \lesssim \left(\frac{\ell(J)}{\ell(I)}\right)^{2-2(n+1-\alpha)\varepsilon} \mathrm{P}^{\alpha}\left(I, \mathbf{1}_I \sigma\right)^2, \qquad J \in \mathcal{J}^*, \tag{.0.38}$$

where we have used that $\ell\left(I'\right) = \ell(I)$ and $\mathrm{P}^{\alpha}\left(I', \mathbf{1}_I \sigma\right) \approx \mathrm{P}^{\alpha}\left(I, \mathbf{1}_I \sigma\right)$, and that the cubes $J \in \mathcal{J}^*$ are good in I' and beyond, and have side length at most $2^{-\rho}\ell(I)$, all because $J^{\maltese} \subset F \subset 3I \setminus I$ and we have already dealt with the term B_{nearby}. Moreover, we may also assume here that the exponent $2 - 2(n+1-\alpha)\varepsilon$ is positive, i.e. $\varepsilon < \frac{1}{n+1-\alpha}$, which is of course implied by $0 < \varepsilon < \frac{1}{2}$. We then obtain from (.0.38), the inequality $\left\|\triangle_J^{\omega,\mathbf{b}^*} x\right\|_{L^2(\omega)}^{\spadesuit 2} \lesssim |J|^2 |J|_{\omega}$, the pairwise disjointedness of the $M \in \mathcal{W}(F)$, the uniqueness of F with $J \in \mathcal{C}_F^{\mathcal{G},shift}$, and since $F \subset 3I \setminus I$ in the sum over $J \in \mathcal{J}^*$, that

$$\begin{aligned}
B_{away}^1 &\lesssim \sum_{J \in \mathcal{J}^*} \left(\frac{\mathrm{P}^{\alpha}\left(J, \mathbf{1}_I \sigma\right)}{|J|^{\frac{1}{n}}}\right)^2 \left\|\triangle_J^{\omega,\mathbf{b}^*} x\right\|_{L^2(\omega)}^{\spadesuit 2} \\
&\lesssim \sum_{m=\rho}^{\infty} \sum_{\substack{J \in \mathcal{J}^* \\ \ell(J) = 2^{-m}\ell(I)}} \left(2^{-m}\right)^{2-2(n+1-\alpha)\varepsilon} \mathrm{P}^{\alpha}\left(I, \mathbf{1}_I \sigma\right)^2 |J|_{\omega} \\
&\lesssim \sum_{m=\rho}^{\infty} \left(2^{-m}\right)^{2-2(n+1-\alpha)\varepsilon} \left(\frac{|I|_{\sigma}}{|I|^{1-\frac{\alpha}{n}}}\right)^2 \sum_{\substack{J \subset 3I \setminus I \\ \ell(J) = 2^{-m}\ell(I)}} |J|_{\omega} \\
&\lesssim \sum_{m=\rho}^{\infty} \left(2^{-m}\right)^{2-2(n+1-\alpha)\varepsilon} \frac{|I|_{\sigma} |3I \setminus I|_{\omega}}{|3I|^{2\left(1-\frac{\alpha}{n}\right)}} |I|_{\sigma} \lesssim A_2^{\alpha} |I|_{\sigma} ,
\end{aligned}$$

since $2-2(n+1-\alpha)\varepsilon>0$.

To complete the bound for term $B=B_{nearby}+B_{away}^{1}+B_{away}^{2}$, it remains to estimate term B_{away}^{2} in which we sum over $F\not\subset 3I\backslash I$. In this case $F\supsetneqq I'$ for one of the two neighbours I' of I, and so we can apply Lemma .0.10, with I there replaced by the augmented cubes $I'\cup I$, to obtain the estimate

$$B_{away}^{2}\lesssim\left((\mathfrak{E}_{2}^{\alpha})^{2}+A_{2}^{\alpha,energy}\right)|I|_{\sigma}\ .$$

Next we turn to term D. The cubes M occurring here are included in the set of ancestors $A_k\equiv\pi_{\mathcal{D}}^{(k)}I$ of I, $1\leq k<\infty$. Then D is equal to

$$\begin{aligned}
&\sum_{k=1}^{\infty}\mathbb{P}^{\alpha}\left(\mathbf{1}_{I}\sigma\right)\left(c\left(A_k\right),\left|A_k\right|\right)^{2}\sum_{\substack{F\in\mathcal{F}:\\ A_k\in\mathcal{W}(F)}}\left\|\mathsf{Q}_{F,A_k}^{\omega,\mathbf{b}^{*}}\frac{x}{\left|A_k\right|^{\frac{1}{n}}}\right\|_{L^{2}(\omega)}^{\spadesuit 2}\\
=&\sum_{k=1}^{\infty}\mathbb{P}^{\alpha}\left(\mathbf{1}_{I}\sigma\right)\left(c\left(A_k\right),\left|A_k\right|\right)^{2}\sum_{\substack{F\in\mathcal{F}:\\ A_k\in\mathcal{W}(F)}}\sum_{J'\in\mathcal{C}_{F}^{\mathcal{G},shift}:\ J'\subset A_k\backslash I}\left\|\triangle_{J'}^{\omega,\mathbf{b}^{*}}\frac{x}{\left|A_k\right|^{\frac{1}{n}}}\right\|_{L^{2}(\omega)}^{\spadesuit 2}\\
&+\sum_{k=1}^{\infty}\mathbb{P}^{\alpha}\left(\mathbf{1}_{I}\sigma\right)\left(c\left(A_k\right),\left|A_k\right|\right)^{2}\sum_{\substack{F\in\mathcal{F}:\\ A_k\in\mathcal{W}(F)}}\sum_{\substack{J'\in\mathcal{C}_{F}^{\mathcal{G},shift}:\ J'\subset A_k\\ J'\cap I\neq\emptyset\text{ and }\ell\left(J'\right)\leq\ell(I)}}\left\|\triangle_{J'}^{\omega,\mathbf{b}^{*}}\frac{x}{\left|A_k\right|^{\frac{1}{n}}}\right\|_{L^{2}(\omega)}^{\spadesuit 2}\\
&+\sum_{k=1}^{\infty}\mathbb{P}^{\alpha}\left(\mathbf{1}_{I}\sigma\right)\left(c\left(A_k\right),\left|A_k\right|\right)^{2}\sum_{\substack{F\in\mathcal{F}:\\ A_k\in\mathcal{W}(F)}}\sum_{\substack{J'\in\mathcal{C}_{F}^{\mathcal{G},shift}:\ J'\subset A_k\\ J'\cap I\neq\emptyset\text{ and }\ell\left(J'\right)>\ell(I)}}\left\|\triangle_{J'}^{\omega,\mathbf{b}^{*}}\frac{x}{\left|A_k\right|^{\frac{1}{n}}}\right\|_{L^{2}(\omega)}^{\spadesuit 2}\\
\equiv&\ D_{disjoint}+D_{descendent}+D_{ancestor}\ .
\end{aligned}$$

We thus have from the pairwise disjointedness of the projections $\mathsf{Q}_{F,A_k}^{\omega,\mathbf{b}^{*}}$ in F once again,

$D_{disjoint}$ equals

$$
\begin{aligned}
& \sum_{k=1}^{\infty} \mathbb{P}^{\alpha}\left(\mathbf{1}_{I}\sigma\right)\left(c\left(A_{k}\right),\left|A_{k}\right|\right)^{2} \sum_{\substack{F\in\mathcal{F}: \\ A_{k}\in\mathcal{W}(F)}} \sum_{\substack{J'\in\mathcal{C}_{F}^{\mathcal{G},shift}: \\ J'\subset A_{k}\setminus I}} \left\Vert \triangle_{J'}^{\omega,\mathbf{b}^{*}} \frac{x}{\left|A_{k}\right|^{\frac{1}{n}}} \right\Vert_{L^{2}(\omega)}^{\spadesuit 2} \\
\lesssim & \sum_{k=1}^{\infty} \left(\frac{\left|I\right|_{\sigma}\left|A_{k}\right|}{\left|A_{k}\right|^{n+1-\alpha}} \right)^{2} \left|A_{k}\setminus I\right|_{\omega} = \left\{ \frac{\left|I\right|_{\sigma}}{\left|I\right|^{1-\frac{\alpha}{n}}} \sum_{k=1}^{\infty} \frac{\left|I\right|^{1-\frac{\alpha}{n}}}{\left|A_{k}\right|^{2(n-\alpha)}} \left|A_{k}\setminus I\right|_{\omega} \right\} \left|I\right|_{\sigma} \\
\lesssim & \left\{ \frac{\left|I\right|_{\sigma}}{\left|I\right|^{1-\frac{\alpha}{n}}} \mathcal{P}^{\alpha}\left(I,\mathbf{1}_{I^{c}}\omega\right) \right\} \left|I\right|_{\sigma} \lesssim \mathcal{A}_{2}^{\alpha,*}\left|I\right|_{\sigma},
\end{aligned}
$$

since

$$
\begin{aligned}
\sum_{k=1}^{\infty} \frac{\left|I\right|^{1-\frac{\alpha}{n}}}{\left|A_{k}\right|^{2(n-\alpha)}} \left|A_{k}\setminus I\right|_{\omega} &= \int \sum_{k=1}^{\infty} \frac{\left|I\right|^{1-\frac{\alpha}{n}}}{\left|A_{k}\right|^{2(n-\alpha)}} \mathbf{1}_{A_{k}\setminus I}\left(x\right) d\omega\left(x\right) \\
&= \int \sum_{k=1}^{\infty} \frac{1}{2^{2(n-\alpha)k}} \frac{\left|I\right|^{1-\frac{\alpha}{n}}}{\left|I\right|^{2(n-\alpha)}} \mathbf{1}_{A_{k}\setminus I}\left(x\right) d\omega\left(x\right) \\
&\lesssim \int_{I^{c}} \left(\frac{\left|I\right|^{\frac{1}{n}}}{\left[\left|I\right|^{\frac{1}{n}}+d\left(x,I\right)\right]^{2}} \right)^{n-\alpha} d\omega\left(x\right) = \mathcal{P}^{\alpha}\left(I,\mathbf{1}_{I^{c}}\omega\right),
\end{aligned}
$$

upon summing a geometric series with $2\left(n-\alpha\right)>0$.

The next term $D_{descendent}$ satisfies

$$\begin{aligned}
D_{descendent} &\lesssim \sum_{k=1}^{\infty}\left(\frac{|I|_{\sigma}|A_k|}{|A_k|^{n+1-\alpha}}\right)^2 \left\|\mathsf{Q}_{3I}^{\omega,\mathbf{b}^*}\frac{x}{2^k|I|^{\frac{1}{n}}}\right\|_{L^2(\omega)}^{\spadesuit 2} \\
&= \sum_{k=1}^{\infty} 2^{-2k(n+1-\alpha)}\left(\frac{|I|_{\sigma}}{|I|^{n-\alpha}}\right)^2 \left\|\mathsf{Q}_{3I}^{\omega,\mathbf{b}^*}\frac{x}{|I|^{\frac{1}{n}}}\right\|_{L^2(\omega)}^{\spadesuit 2} \\
&\lesssim \left\{\frac{|I|_{\sigma}\left\|\mathsf{Q}_{3I}^{\omega,\mathbf{b}^*}\frac{x}{|I|^{\frac{1}{n}}}\right\|_{L^2(\omega)}^{\spadesuit 2}}{|I|^{2(n-\alpha)}}\right\}|I|_{\sigma} \lesssim A_2^{\alpha,energy}|I|_{\sigma}\ .
\end{aligned}$$

Lastly, for $D_{ancestor}$ we note that there are at most two cubes K_1 and K_2 in $\mathcal{G}$ having side length $\ell(I)$ and such that $K_i \cap I \neq \emptyset$. Then each J' occurring in the sum in $D_{ancestor}$ is of the form $J' = A_i^{\ell} \equiv \pi_{\mathcal{G}}^{(\ell)} K_i$ with $J' \subset A_k$ for some $1 \leq \ell \leq k$ and $i \in \{1,2\}$. Now we write

$$\begin{aligned}
D_{ancestor} &= \sum_{k=1}^{\infty}\mathbb{P}^{\alpha}(\mathbf{1}_I\sigma)(c(A_k),|A_k|)^2 \sum_{\substack{F\in\mathcal{F}:\\ A_k\in\mathcal{W}(F)}} \sum_{\substack{J'\in\mathcal{C}_F^{\mathcal{G},shift}:\ J'\subset A_k\\ J'\cap I\neq\emptyset \text{ and } \ell(J')>\ell(I)}} \left\|\triangle_{J'}^{\omega,\mathbf{b}^*}\frac{x}{|A_k|^{\frac{1}{n}}}\right\|_{L^2(\omega)}^{\spadesuit 2} \\
&\lesssim \sum_{k=1}^{\infty}\left(\frac{|I|_{\sigma}|A_k|}{|A_k|^{n+1-\alpha}}\right)^2 \sum_{i=1}^{2}\sum_{\ell=1}^{k}\left\|\triangle_{A_i^{\ell}}^{\omega,\mathbf{b}^*}\frac{x}{|A_k|^{\frac{1}{n}}}\right\|_{L^2(\omega)}^{\spadesuit 2} \\
&\leq 2\sum_{k=1}^{\infty}\left(\frac{|I|_{\sigma}|A_k|}{|A_k|^{n+1-\alpha}}\right)^2 \left\|\mathsf{Q}_{A_k}^{\omega,\mathbf{b}^*}\frac{x}{|A_k|^{\frac{1}{n}}}\right\|_{L^2(\omega)}^{\spadesuit 2}.
\end{aligned}$$

At this point we need a *'prepare to puncture'* argument, as we will want to derive geometric decay from $\left\|\mathsf{Q}_{J'}^{\omega,\mathbf{b}^*}x\right\|_{L^2(\omega)}^{\spadesuit 2}$ by dominating it by the 'nonenergy' term $|J'|^2|J'|_{\omega}$, as well as using the Muckenhoupt energy constant. For this we define $\widetilde{\omega} = \omega - \omega(\{p\})\delta_p$ where p is an atomic point in I for which

$\omega(\{p\}) = \sup_{q\in\mathfrak{P}_{(\sigma,\omega)}:\ q\in I} \omega(\{q\})$. (If ω has no atomic point in common with σ in I set $\widetilde{\omega}=\omega$.)

Then we have $|I|_{\widetilde{\omega}} = \omega\left(I, \mathfrak{P}_{(\sigma,\omega)}\right)$ and

$$\frac{|I|_{\widetilde{\omega}}}{|I|^{1-\frac{\alpha}{n}}}\frac{|I|_{\sigma}}{|I|^{1-\frac{\alpha}{n}}} = \frac{\omega\left(I, \mathfrak{P}_{(\sigma,\omega)}\right)}{|I|^{1-\frac{\alpha}{n}}}\frac{|I|_{\sigma}}{|I|^{1-\frac{\alpha}{n}}} \leq A_2^{\alpha,punct}.$$

A key observation, already noted in the proof of Lemma .0.5 above, is that

$$\left\|\triangle_K^{\omega,\mathbf{b}^*} x\right\|_{L^2(\omega)}^2 = \begin{cases} \left\|\triangle_K^{\omega,\mathbf{b}^*}(x-p)\right\|_{L^2(\omega)}^2 & \text{if} \quad p\in K \\ \left\|\triangle_K^{\omega,\mathbf{b}^*} x\right\|_{L^2(\widetilde{\omega})}^2 & \text{if} \quad p\notin K \end{cases} \leq \ell(K)^2 |K|_{\widetilde{\omega}}, \tag{.0.39}$$

and so, as in the proof of (.0.18) in Lemma .0.5,

$$\left\|\mathsf{Q}_{A_k}^{\omega,\mathbf{b}^*}\frac{x}{|A_k|^{\frac{1}{n}}}\right\|_{L^2(\omega)}^{\spadesuit 2} \lesssim |A_k|_{\widetilde{\omega}} \ .$$

Then we continue with

$$\begin{aligned} & \sum_{k=1}^{\infty}\left(\frac{|I|_{\sigma}|A_k|}{|A_k|^{n+1-\alpha}}\right)^2 \left\|\mathsf{Q}_{A_k}^{\omega,\mathbf{b}^*}\frac{x}{|A_k|^{\frac{1}{n}}}\right\|_{L^2(\omega)}^{\spadesuit 2} \\ \lesssim\ & \sum_{k=1}^{\infty}\left(\frac{|I|_{\sigma}|A_k|}{|A_k|^{n+1-\alpha}}\right)^2 |A_k|_{\widetilde{\omega}} \\ =\ & \sum_{k=1}^{\infty}\left(\frac{|I|_{\sigma}}{|A_k|^{n-\alpha}}\right)^2 |A_k\setminus I|_{\omega} + \sum_{k=1}^{\infty}\left(\frac{|I|_{\sigma}}{2^{k(n-\alpha)}|I|^{n-\alpha}}\right)^2 |I|_{\widetilde{\omega}} \\ \lesssim\ & \left(\mathcal{A}_2^{\alpha,*} + A_2^{\alpha,punct}\right)|I|_{\sigma}, \end{aligned}$$

where the inequality $\sum_{k=1}^{\infty}\left(\frac{|I|_{\sigma}}{|A_k|^{n-\alpha}}\right)^2 |A_k\setminus I|_{\omega} \lesssim \mathcal{A}_2^{\alpha,*}|I|_{\sigma}$ is already proved above in the display estimating $D_{disjoint}$.

Finally, for term C we will have to group the cubes M into blocks B_i. We first split the sum according to whether or not I intersects the triple of M:

$$\begin{aligned} C &\approx \left\{ \sum_{\substack{M:\ I\cap 3M=\emptyset \\ \ell(M)>\ell(I)}} + \sum_{\substack{M:\ I\subset 3M\setminus M \\ \ell(M)>\ell(I)}} \right\} \left(\frac{|M|^{\frac{1}{n}}}{\left(|M|^{\frac{1}{n}} + d(M,I)\right)^{n+1-\alpha}} |I|_\sigma \right)^2 \\ &\qquad \cdot \sum_{\substack{F\in\mathcal{F}: \\ M\in\mathcal{W}(F)}} \left\| \mathsf{Q}_{F,M}^{\omega\mathbf{b}^*} \frac{x}{|M|^{\frac{1}{n}}} \right\|_{L^2(\omega)}^{\spadesuit 2} \\ &= C_1 + C_2. \end{aligned}$$

We first consider C_1. Let $\mathcal{M}$ consist of the maximal dyadic cubes in the collection $\{Q : 3Q \cap I = \emptyset\}$, and then let $\{B_i\}_{i=1}^{\infty}$ be an enumeration of those $Q \in \mathcal{M}$ whose side length is at least $\ell(I)$. Note in particular that $3B_i \cap I = \emptyset$. Now we further decompose the sum in C_1 by grouping the cubes M into the 'Whitney' cubes B_i, and then using the pairwise disjointedness of the martingale supports of the pseudoprojections $\mathsf{Q}^{\omega,\mathbf{b}^*}_{F,M}$ in F:

$$
\begin{aligned}
C_1 &\leq \sum_{i=1}^{\infty} \sum_{M:\ M\subset B_i} \left(\frac{1}{\left(|M|^{\frac{1}{n}} + d(M,I) \right)^{n+1-\alpha}} |I|_\sigma \right)^2 \sum_{\substack{F\in\mathcal{F}:\\ M\in\mathcal{W}(F)}} \left\| \mathsf{Q}^{\omega,\mathbf{b}^*}_{F,M} x \right\|^{\spadesuit 2}_{L^2(\omega)} \\
&\lesssim \sum_{i=1}^{\infty} \left(\frac{1}{\left(|B_i|^{\frac{1}{n}} + d(B_i,I) \right)^{n+1-\alpha}} |I|_\sigma \right)^2 \sum_{M:\ M\subset B_i} \sum_{\substack{F\in\mathcal{F}:\\ M\in\mathcal{W}(F)}} \left\| \mathsf{Q}^{\omega,\mathbf{b}^*}_{F,M} x \right\|^{\spadesuit 2}_{L^2(\omega)} \\
&\lesssim \sum_{i=1}^{\infty} \left(\frac{1}{\left(|B_i|^{\frac{1}{n}} + d(B_i,I) \right)^{n+1-\alpha}} |I|_\sigma \right)^2 \sum_{M:\ M\subset B_i} |M|^{\frac{2}{n}} |M|_\omega \\
&\lesssim \sum_{i=1}^{\infty} \left(\frac{1}{\left(|B_i|^{\frac{1}{n}} + d(B_i,I) \right)^{n+1-\alpha}} |I|_\sigma \right)^2 |B_i|^{\frac{2}{n}} |B_i|_\omega \\
&\lesssim \left\{ \sum_{i=1}^{\infty} \frac{|B_i|_\omega |I|_\sigma}{|B_i|^{2(n-\alpha)}} \right\} |I|_\sigma \ ,
\end{aligned}
$$

Now since $|B_i| \approx d(x,I)$ for $x \in B_i$,

$$
\begin{aligned}
\sum_{i=1}^{\infty} \frac{|B_i|_\omega |I|_\sigma}{|B_i|^{2(n-\alpha)}} = \frac{|I|_\sigma}{|I|^{1-\alpha}} \sum_{i=1}^{\infty} \frac{|I|^{1-\alpha}}{|B_i|^{2(n-\alpha)}} |B_i|_\omega &\approx \frac{|I|_\sigma}{|I|^{1-\frac{\alpha}{n}}} \sum_{i=1}^{\infty} \int_{B_i} \frac{|I|^{1-\frac{\alpha}{n}}}{d(x,I)^{2(n-\alpha)}} d\omega(x) \\
&\approx \frac{|I|_\sigma}{|I|^{1-\frac{\alpha}{n}}} \sum_{i=1}^{\infty} \int_{B_i} \left(\frac{|I|^{\frac{1}{n}}}{\left[|I|^{\frac{1}{n}} + d(x,I) \right]^2} \right)^{n-\alpha} d\omega(x) \\
&\leq \frac{|I|_\sigma}{|I|^{1-\frac{\alpha}{n}}} \mathcal{P}^\alpha(I, \mathbf{1}_{I^c}\omega) \leq \mathcal{A}_2^{\alpha,*},
\end{aligned}
$$

we obtain

$$C_1 \lesssim \mathcal{A}_2^{\alpha,*} |I|_\sigma \ .$$

Next we turn to estimating term C_2 where the triple of M contains I but M itself does not. Note that there are at most two such cubes M of a given side length. So with this in mind, we sum over the cubes M according to their lengths to obtain

$$\begin{aligned}
C_2 &= \sum_{m=1}^{\infty} \sum_{\substack{M: \\ I\subset 3M\setminus M \\ \ell(M)=2^m\ell(I)}} \left(\frac{|M|^{\frac{1}{n}}}{\left(|M|^{\frac{1}{n}} + \operatorname{dist}(M,I)\right)^{n+1-\alpha}} |I|_\sigma \right)^2 \sum_{\substack{F\in\mathcal{F}: \\ M\in\mathcal{W}(F)}} \left\| \mathsf{Q}_{F,M}^{\omega,\mathbf{b}^*} \frac{x}{|M|^{\frac{1}{n}}} \right\|_{L^2(\omega)}^{\spadesuit 2} \\
&\lesssim \sum_{m=1}^{\infty} \left(\frac{|I|_\sigma}{|2^m I|^{n-\alpha}} \right)^2 |(5\cdot 2^m I)\setminus I|_\omega = \left\{ \frac{|I|_\sigma}{|I|^{1-\frac{\alpha}{n}}} \sum_{m=1}^{\infty} \frac{|I|^{1-\frac{\alpha}{n}} |(5\cdot 2^m I)\setminus I|_\omega}{|2^m I|^{2(n-\alpha)}} \right\} |I|_\sigma \\
&\lesssim \left\{ \frac{|I|_\sigma}{|I|^{1-\frac{\alpha}{n}}} \mathcal{P}^\alpha\left(I, \mathbf{1}_{I^c}\omega\right) \right\} |I|_\sigma \leq \mathcal{A}_2^{\alpha,*} |I|_\sigma ,
\end{aligned}$$

since in analogy with the corresponding estimate above,

$$\sum_{m=1}^{\infty} \frac{|I|^{1-\frac{\alpha}{n}} |(5\cdot 2^m I)\setminus I|_\omega}{|2^m I|^{2(n-\alpha)}} = \int \sum_{m=1}^{\infty} \frac{|I|^{1-\frac{\alpha}{n}}}{|2^m I|^{2(n-\alpha)}} \mathbf{1}_{(5\cdot 2^m I)\setminus I}(x) \ d\omega(x) \lesssim \mathcal{P}^\alpha\left(I, \mathbf{1}_{I^c}\omega\right).$$

The backward Poisson testing inequality

The argument here follows the broad outline of the analogous argument in [64], but using modifications from [66] that involve 'prepare to puncture arguments', using decompositions $\mathcal{W}(F)$ in place of (ρ,ε)-decompositions, and using pseudoprojections $\mathsf{Q}_{F,M}^{\omega,\mathbf{b}^*} x$ (see (.0.22) for the definition). The final change here is that there is no splitting into local and global parts as in [64] - instead, we follow the treatment in [63] in this regard.

Fix $I \in \mathcal{D}$. It suffices to prove

$$\begin{aligned}
\mathbf{Back}\left(\widehat{I}\right) &\equiv \int_{\mathbb{R}^n} \left[\mathbb{Q}^\alpha\left(t\mathbf{1}_{\widehat{I}}\overline{\mu}\right)(y)\right]^2 d\sigma(y) \\
&\lesssim \left\{\mathcal{A}_2^\alpha + \left(\mathfrak{E}_2^\alpha + \sqrt{A_2^{\alpha,energy}}\right)\sqrt{A_2^{\alpha,punct}}\right\}\int_{\widehat{I}} t^2 d\overline{\mu}(x,t). \qquad (.0.40)
\end{aligned}$$

Note that for a 'Poisson integral with holes' and a measure μ built with Haar projections, Hytönen obtained in [22] the simpler bound A_2^α for a term analogous to, but significantly smaller than, (.0.40). Using (.0.29) we see that the integral on the right hand side of (.0.40) is

$$\int_{\widehat{I}} t^2 d\overline{\mu} = \sum_{F\in\mathcal{F}} \sum_{M\in\mathcal{W}(F):\ M\subset I} \|\mathsf{Q}_{F,M}^{\omega,\mathbf{b}^*}x\|_{L^2(\omega)}^{\spadesuit 2}. \qquad (.0.41)$$

where $\mathsf{Q}_{F,M}^{\omega,\mathbf{b}^*}$ was defined in (.0.22).

We now compute using (.0.29) again that

$$\begin{aligned}
\mathbb{Q}^\alpha\left(t\mathbf{1}_{\widehat{I}}\overline{\mu}\right)(y) &= \int_{\widehat{I}} \frac{t^2}{\left(t^2+|x-y|^2\right)^{\frac{n+1-\alpha}{2}}} d\overline{\mu}(x,t) \qquad (.0.42) \\
&\approx \sum_{F\in\mathcal{F}} \sum_{M\in\mathcal{W}(F):\ M\subset I} \frac{\|\mathsf{Q}_{F,M}^{\omega,\mathbf{b}^*}x\|_{L^2(\omega)}^{\spadesuit 2}}{(|M|+|y-c_M|)^{n+1-\alpha}},
\end{aligned}$$

and then expand the square and integrate to obtain that the term $\mathbf{Back}\left(\widehat{I}\right)$ is

$$\sum_{\substack{F\in\mathcal{F}\\ M\in\mathcal{W}(F)\\ M\subset I}} \sum_{\substack{F'\in\mathcal{F}:\\ M'\in\mathcal{W}\left(F'\right)\\ M'\subset I}} \int_{\mathbb{R}} \frac{\left\|\mathsf{Q}_{F,M}^{\omega,\mathbf{b}^*}x\right\|_{L^2(\omega)}^{\spadesuit 2}}{(|M|+|y-c_M|)^{n+1-\alpha}} \frac{\left\|\mathsf{Q}_{F',M'}^{\omega,\mathbf{b}^*}x\right\|_{L^2(\omega)}^{\spadesuit 2}}{\left(|M'|+\left|y-c_{M'}\right|\right)^{n+1-\alpha}} d\sigma(y).$$

By symmetry we may assume that $\ell\left(M'\right) \leq \ell(M)$. We fix a nonnegative integer s, and consider those cubes M and M' with $\ell\left(M'\right)=2^{-s}\ell\left(M\right)$. For fixed s we will control the expression

$$U_s \equiv \sum_{F,F'\in\mathcal{F}} \sum_{\substack{M\in\mathcal{W}(F) \\ M'\in\mathcal{W}\left(F'\right) \\ M,M'\subset I,\ \ell\left(M'\right)=2^{-s}\ell(M)}} \int_{\mathbb{R}^n} \frac{\left\|\mathsf{Q}_{F,M}^{\omega,\mathbf{b}^*}x\right\|_{L^2(\omega)}^{\spadesuit 2}}{\left(\left|M\right|+\left|y-c_M\right|\right)^{n+1-\alpha}} \frac{\left\|\mathsf{Q}_{F',M'}^{\omega,\mathbf{b}^*}x\right\|_{L^2(\omega)}^{\spadesuit 2}}{\left(\left|M'\right|+\left|y-c_{M'}\right|\right)^{n+1-\alpha}} d\sigma\left(y\right) \tag{.0.43}$$

by proving that

$$U_s \lesssim 2^{-\delta s}\left\{\mathcal{A}_2^{\alpha}+\left(\mathfrak{E}_2^{\alpha}+\sqrt{A_2^{\alpha,energy}}\right)\sqrt{A_2^{\alpha,punct}}\right\}\int_{\widehat{I}} t^2 d\overline{\mu}, \qquad \text{where } \delta=\frac{1}{2}. \tag{.0.44}$$

With this accomplished, we can sum in $s\geq 0$ to control the term $\mathbf{Back}\left(\widehat{I}\right)$. We now decompose $U_s=T_s^{proximal}+T_s^{difference}+T_s^{intersection}$ into three pieces.

Our first decomposition is to write

$$U_s=T_s^{proximal}+V_s^{remote}\ , \tag{.0.45}$$

where in the 'proximal' term $T_s^{proximal}$ we restrict the summation over pairs of cubes M,M' to those satisfying $d\left(c_M,c_{M'}\right)<2^{s\delta}\ell\left(M\right)$; while in the 'remote' term V_s^{remote} we restrict the summation over pairs of cubes M,M' to those satisfying the opposite inequality $d\left(c_M,c_{M'}\right)\geq 2^{s\delta}\ell\left(M\right)$. Then we further decompose

$$V_s^{remote}=T_s^{difference}+T_s^{intersection},$$

where in the 'difference' term $T_s^{difference}$ we restrict integration in y to the difference $\mathbb{R} \setminus B\left(M, M'\right)$ of $\mathbb{R}$ and

$$B\left(M, M'\right) \equiv B\left(c_M, \frac{1}{2}d\left(c_M, c_{M'}\right)\right), \tag{.0.46}$$

the ball centered at c_M with radius $\frac{1}{2}d\left(c_M, c_{M'}\right)$; while in the 'intersection' term $T_s^{intersection}$ we restrict integration in y to the intersection $\mathbb{R}^n \cap B\left(M, M'\right)$ of $\mathbb{R}^n$ with the ball $B\left(M, M'\right)$; i.e.

$$\begin{aligned} T_s^{intersection} &\equiv \sum_{F,F' \in \mathcal{F}} \sum_{\substack{M \in \mathcal{W}(F),\ M' \in \mathcal{W}\left(F'\right) \\ M,M' \subset I,\ \ell\left(M'\right) = 2^{-s}\ell(M) \\ d\left(c_M, c_{M'}\right) \geq 2^{s(1+\delta)}\ell\left(M'\right)}} \\ &\int_{B(M,M')} \frac{\left\| \mathsf{Q}_{F,M}^{\omega,\mathbf{b}^*} x \right\|_{L^2(\omega)}^{\spadesuit 2}}{\left(|M| + |y - c_M|\right)^{n+1-\alpha}} \frac{\left\| \mathsf{Q}_{F',M'}^{\omega,\mathbf{b}^*} x \right\|_{L^2(\omega)}^{\spadesuit 2}}{\left(|M'| + \left|y - c_{M'}\right|\right)^{n+1-\alpha}} d\sigma(y). \end{aligned} \tag{.0.47}$$

We will exploit the restriction of integration to $B\left(M, M'\right)$, together with the condition

$$d\left(c_M, c_{M'}\right) \geq 2^{s(1+\delta)}\ell\left(M'\right) = 2^{s\delta}\ell(M),$$

which will then give an estimate for the term $T_s^{intersection}$ using an argument dual to that used for the other terms $T_s^{proximal}$ and $T_s^{difference}$, to which we now turn.

The proximal and difference terms

We have

$$T_s^{proximal} \equiv \sum_{F,F'\in\mathcal{F}} \sum_{\substack{M\in\mathcal{W}(F),\ M'\in\mathcal{W}\left(F'\right) \\ M,M'\subset I,\ \ell\left(M'\right)=2^{-s}\ell(M) \text{ and } d\left(c_M,c_{M'}\right)<2^{s\delta}\ell(M)}} \tag{.0.48}$$
$$\times \int_{\mathbb{R}^n} \frac{\left\|\mathsf{Q}_{F,M}^{\omega,\mathbf{b}^*}x\right\|_{L^2(\omega)}^{\spadesuit 2}}{\left(|M|+|y-c_M|\right)^{n+1-\alpha}} \frac{\left\|\mathsf{Q}_{F',M'}^{\omega,\mathbf{b}^*}x\right\|_{L^2(\omega)}^{\spadesuit 2}}{\left(|M'|+\left|y-c_{M'}\right|\right)^{n+1-\alpha}} d\sigma(y)$$
$$\leq \mathcal{M}_s^{proximal} \sum_{F\in\mathcal{F}} \sum_{\substack{M\in\mathcal{W}(F) \\ M\subset I}} \|\mathsf{Q}_{F,M}^{\omega,\mathbf{b}^*}z\|_{\omega}^{\spadesuit 2} = \mathcal{M}_s^{proximal} \int_{\widehat{I}} t^2 d\overline{\mu},$$

where

$$\mathcal{M}_s^{proximal} \equiv \sup_{F\in\mathcal{F}} \sup_{\substack{M\in\mathcal{W}(F) \\ M\subset I}} \mathcal{A}_s^{proximal}(M);$$
$$\mathcal{A}_s^{proximal}(M) \equiv \sum_{F'\in\mathcal{F}} \sum_{\substack{M'\in\mathcal{W}\left(F'\right) \\ M'\subset I,\ \ell\left(M'\right)=2^{-s}\ell(M) \text{ and} \\ d\left(c_M,c_{M'}\right)<2^{s\delta}\ell(M)}} \int_{\mathbb{R}^n} S_{(M',M)}^{F'}(y)\, d\sigma(y);$$
$$S_{(M',M)}^{F'}(x) \equiv \frac{1}{\left(|M|+|y-c_M|\right)^{n+1-\alpha}} \frac{\left\|\mathsf{Q}_{F',M'}^{\omega}x\right\|_{L^2(\omega)}^{\spadesuit 2}}{\left(|M'|+\left|y-c_{M'}\right|\right)^{n+1-\alpha}},$$

and similarly

$$
\begin{aligned}
T_s^{difference} &\equiv \sum_{F,F'\in\mathcal{F}} \sum_{\substack{M\in\mathcal{W}(F),\ M'\in\mathcal{W}\left(F'\right) \\ M,M'\subset I,\ \ell\left(M'\right)=2^{-s}\ell(M) \text{ and } d\left(c_M,c_{M'}\right)\geq 2^{s\delta}\ell(M)}} \qquad (.0.49)\\
&\times \int_{\mathbb{R}^n\setminus B(M,M')} \frac{\left\|\mathsf{Q}_{F,M}^{\omega,\mathbf{b}^*}x\right\|_{L^2(\omega)}^{\spadesuit 2}}{\left(|M|+|y-c_M|\right)^{n+1-\alpha}} \frac{\left\|\mathsf{Q}_{F',M'}^{\omega,\mathbf{b}^*}x\right\|_{L^2(\omega)}^{\spadesuit 2}}{\left(|M'|+\left|y-c_{M'}\right|\right)^{n+1-\alpha}} d\sigma(y)\\
&\leq \mathcal{M}_s^{difference} \sum_{F\in\mathcal{F}} \sum_{\substack{M\in\mathcal{W}(F)\\ M\subset I}} \|\mathsf{Q}_{F,M}^{\omega,\mathbf{b}^*}z\|_{\omega}^{\spadesuit 2} = \mathcal{M}_s^{difference} \int_{\widehat{I}} t^2 d\overline{\mu};
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{M}_s^{difference} &\equiv \sup_{F\in\mathcal{F}} \sup_{\substack{M\in\mathcal{W}(F)\\ M\subset I}} \mathcal{A}_s^{difference}(M);\\
\mathcal{A}_s^{difference}(M) &\equiv \sum_{F'\in\mathcal{F}} \sum_{\substack{M'\in\mathcal{W}\left(F'\right)\\ M'\subset I,\ \ell\left(M'\right)=2^{-s}\ell(M) \text{ and}\\ d\left(c_M,c_{M'}\right)\geq 2^{s\delta}\ell(M)}} \int_{\mathbb{R}^n\setminus B\left(M,M'\right)} S_{(M',M)}^{F'}(y)\, d\sigma(y).
\end{aligned}
$$

The restriction of integration in $\mathcal{A}_s^{difference}$ to $\mathbb{R}^n\setminus B\left(M,M'\right)$ will be used to establish (.0.51) below.

Notation .0.13. *Since the cubes F,M,F',M' that arise in all of the sums here satisfy*

$$
M\in\mathcal{W}(F),\ \ M'\in\mathcal{W}\left(F'\right) \ \text{ and } \ell\left(M'\right)=2^{-s}\ell(M) \ \text{ and } M,M'\subset I,
$$

we will often employ the notation $\overset{*}{\sum}$ to remind the reader that, as applicable, these four conditions are in force even when they are not explictly mentioned.

Now fix M as in $\mathcal{M}_s^{proximal}$ respectively $\mathcal{M}_s^{difference}$, and decompose the sum over M' in $\mathcal{A}_s^{proximal}(M)$ respectively $\mathcal{A}_s^{difference}(M)$ by

$$\begin{aligned}
\mathcal{A}_s^{proximal}(M) &= \sum_{F'\in\mathcal{F}} \sum_{\substack{M'\in\mathcal{W}\left(F'\right) \\ M'\subset I,\ \ell\left(M'\right)=2^{-s}\ell(M) \text{ and} \\ d\left(c_M,c_{M'}\right)<2^{s\delta}\ell(M)}} \int_{\mathbb{R}^n} S_{(M',M)}^{F'}(y)\, d\sigma(y) \\
&= \sum_{F'\in\mathcal{F}} \sum_{\substack{c_{M'}\in 2M \\ d\left(c_M,c_{M'}\right)<2^{s\delta}\ell(M)}}^{*} \int_{\mathbb{R}^n} S_{(M',M)}^{F'}(y)\, d\sigma(y) \\
&\qquad + \sum_{F'\in\mathcal{F}} \sum_{\ell=1}^{\infty} \sum_{\substack{c_{M'}\in 2^{\ell+1}M\setminus 2^{\ell}M \\ d\left(c_M,c_{M'}\right)<2^{s\delta}\ell(M)}}^{*} \int_{\mathbb{R}^n} S_{(M',M)}^{F'}(y)\, d\sigma(y) \\
&\equiv \sum_{\ell=0}^{\infty} \mathcal{A}_s^{proximal,\ell}(M),
\end{aligned}$$

respectively

$$\begin{aligned}
\mathcal{A}_s^{difference}(M) &= \sum_{F'\in\mathcal{F}} \sum_{\substack{M'\in\mathcal{W}\left(F'\right) \\ M'\subset I,\ \ell\left(M'\right)=2^{-s}\ell(M) \text{ and} \\ d\left(c_M,c_{M'}\right)\geq 2^{s\delta}\ell(M)}} \int_{\mathbb{R}^n\setminus B(M,M')} S_{(M',M)}^{F'}(y)\, d\sigma(y) \\
&= \sum_{F'\in\mathcal{F}} \sum_{\substack{c_{M'}\in 2M \\ d\left(c_M,c_{M'}\right)\geq 2^{s\delta}\ell(M)}}^{*} \int_{\mathbb{R}^n\setminus B(M,M')} S_{(M',M)}^{F'}(y)\, d\sigma(y) \\
&\qquad + \sum_{\ell=1}^{\infty}\sum_{F'\in\mathcal{F}} \sum_{\substack{c_{M'}\in 2^{\ell+1}M\setminus 2^{\ell}M \\ d\left(c_M,c_{M'}\right)\geq 2^{s\delta}\ell(M)}}^{*} \int_{\mathbb{R}^n\setminus B(M,M')} S_{(M',M)}^{F'}(y)\, d\sigma(y) \\
&\equiv \sum_{\ell=0}^{\infty}\mathcal{A}_s^{difference,\ell}(M).
\end{aligned}$$

Let $m=2$ so that

$$2^{-m}\leq\frac{1}{3}. \tag{.0.50}$$

Now decompose the integrals over $\mathbb{R}^n$ in $\mathcal{A}_s^{proximal,\ell}(M)$ by

$$\begin{aligned}
\mathcal{A}_s^{proximal,0}(M) &= \sum_{F'\in\mathcal{F}} \sum_{\substack{c_{M'}\in 2M \\ d\left(c_M,c_{M'}\right)< 2^{s\delta}\ell(M)}}^{*} \int_{\mathbb{R}^n\setminus 4M} S_{(M',M)}^{F'}(y)\, d\sigma(y) \\
&\qquad + \sum_{F'\in\mathcal{F}} \sum_{\substack{c_{M'}\in 2M \\ d\left(c_M,c_{M'}\right)< 2^{s\delta}\ell(M)}}^{*} \int_{4M} S_{(M',M)}^{F'}(y)\, d\sigma(y) \\
&\equiv \mathcal{A}_{s,far}^{proximal,0}(M)+\mathcal{A}_{s,near}^{proximal,0}(M),
\end{aligned}$$

and for $\ell \geq 1$

$$\begin{aligned}
\mathcal{A}_s^{proximal,\ell}(M) &= \sum_{F'\in\mathcal{F}} \sum_{\substack{c_{M'}\in 2^{\ell+1}M\setminus 2^{\ell}M \\ d\left(c_M,c_{M'}\right)<2^{s\delta}\ell(M)}}^{*} \int_{\mathbb{R}^n\setminus 2^{\ell+2}M} S_{(M',M)}^{F'}(y)\,d\sigma(y) \\
&+ \sum_{F'\in\mathcal{F}} \sum_{\substack{c_{M'}\in 2^{\ell+1}M\setminus 2^{\ell}M \\ d\left(c_M,c_{M'}\right)<2^{s\delta}\ell(M)}}^{*} \int_{2^{\ell+2}M\setminus 2^{\ell-m}M} S_{(M',M)}^{F'}(y)\,d\sigma(y) \\
&+ \sum_{F'\in\mathcal{F}} \sum_{\substack{c_{M'}\in 2^{\ell+1}M\setminus 2^{\ell}M \\ d\left(c_M,c_{M'}\right)<2^{s\delta}\ell(M)}}^{*} \int_{2^{\ell-m}M} S_{(M',M)}^{F'}(y)\,d\sigma(y) \\
&\equiv \mathcal{A}_{s,far}^{proximal,\ell}(M) + \mathcal{A}_{s,near}^{proximal,\ell}(M) + \mathcal{A}_{s,close}^{proximal,\ell}(M)
\end{aligned}$$

Similarly we decompose the integrals over the difference

$$B^* \equiv \mathbb{R}^n \setminus B\left(M,M'\right)$$

in $\mathcal{A}_s^{difference,\ell}(M)$ by

$$\begin{aligned}
\mathcal{A}_s^{difference,0}(M) &= \sum_{F'\in\mathcal{F}} \sum_{\substack{c_{M'}\in 2M \\ d\left(c_M,c_{M'}\right)\geq 2^{s\delta}\ell(M)}}^{*} \int_{B^*\setminus 4M} S_{(M',M)}^{F'}(y)\,d\sigma(y) \\
&+ \sum_{F'\in\mathcal{F}} \sum_{\substack{c_{M'}\in 2M \\ d\left(c_M,c_{M'}\right)\geq 2^{s\delta}\ell(M)}}^{*} \int_{B^*\cap 4M} S_{(M',M)}^{F'}(y)\,d\sigma(y) \\
&\equiv \mathcal{A}_{s,far}^{difference,0}(M) + \mathcal{A}_{s,near}^{difference,0}(M),
\end{aligned}$$

and

$$\mathcal{A}_{s}^{difference,\ell}(M)=\sum_{F'\in\mathcal{F}}\sum_{\substack{c_{M'}\in 2^{\ell+1}M\setminus 2^{\ell}M\\ d\left(c_M,c_{M'}\right)\geq 2^{s\delta}\ell(M)}}^{*}\int_{B^*\setminus 2^{\ell+2}M}S_{(M',M)}^{F'}(y)\,d\sigma(y)$$

$$+\sum_{F'\in\mathcal{F}}\sum_{\substack{c_{M'}\in 2^{\ell+1}M\setminus 2^{\ell}M\\ d\left(c_M,c_{M'}\right)\geq 2^{s\delta}\ell(M)}}^{*}\int_{B^*\cap\left(2^{\ell+2}M\setminus 2^{\ell-m}M\right)}S_{(M',M)}^{F'}(y)\,d\sigma(y)$$

$$+\sum_{F'\in\mathcal{F}}\sum_{\substack{c_{M'}\in 2^{\ell+1}M\setminus 2^{\ell}M\\ d\left(c_M,c_{M'}\right)\geq 2^{s\delta}\ell(M)}}^{*}\int_{B^*\cap 2^{\ell-m}M}S_{(M',M)}^{F'}(y)\,d\sigma(y)$$

$$\equiv\ \mathcal{A}_{s,far}^{difference,\ell}(M)+\mathcal{A}_{s,near}^{difference,\ell}(M)+\mathcal{A}_{s,close}^{difference,\ell}(M),\qquad \ell\geq 1.$$

We now note the important point that the close terms $\mathcal{A}_{s,close}^{proximal,\ell}(M)$ and $\mathcal{A}_{s,close}^{difference,\ell}(M)$ both *vanish* for $\ell>\delta s$ because of the decomposition (.0.45):

$$\mathcal{A}_{s,close}^{proximal,\ell}(M)=\mathcal{A}_{s,close}^{difference,\ell}(M)=0,\qquad \ell\geq 1+\delta s. \tag{.0.51}$$

Indeed, if $c_{M'}\in 2^{\ell+1}M\setminus 2^{\ell}M$, then we have

$$\frac{1}{2}2^{\ell}\ell(M)\leq d\left(c_M,c_{M'}\right). \tag{.0.52}$$

Now the summands in $\mathcal{A}_{s,close}^{proximal,\ell}(M)$ satisfy $d\left(c_M,c_{M'}\right)<2^{\delta s}\ell(M)$, which by (.0.52) is impossible if $\ell\geq 1+\delta s$ - indeed, if $\ell\geq 1+\delta s$, we get the contradiction

$$2^{\delta s}\ell(M)=\frac{1}{2}2^{1+\delta s}\ell(M)\leq\frac{1}{2}2^{\ell}\ell(M)\leq d\left(c_M,c_{M'}\right)<2^{\delta s}\ell(M).$$

It now follows that $\mathcal{A}_{s,close}^{proximal,\ell}(M)=0$. Thus we are left to consider the term $\mathcal{A}_{s,close}^{difference,\ell}(M)$, where the integration is taken over the set $\mathbb{R}^n\setminus B(M,M')$. But we are also restricted in $\mathcal{A}_{s,close}^{difference,\ell}(M)$ to integrating over the cube $2^{\ell-m}M$, which is contained in $B(M,M')$ by (.0.52). Indeed, the smallest ball centered at c_M that contains $2^{\ell-m}M$ has radius $\frac{1}{2}2^{\ell-m}\ell(M)$, which by (.0.50) and (.0.52) is at most $\frac{1}{4}2^{\ell}\ell(M)\leq\frac{1}{2}d(c_M,c_{M'})$, the radius of $B(M,M')$. Thus the range of integration in the term $\mathcal{A}_{s,close}^{difference,\ell}(M)$ is the empty set, and so $\mathcal{A}_{s,close}^{difference,\ell}(M)=0$ as well as $\mathcal{A}_{s,close}^{proximal,\ell}(M)=0$. This proves (.0.51).

From now on we treat $T_s^{proximal}$ and $T_s^{difference}$ in the same way since the terms $\mathcal{A}_{s,close}^{proximal,\ell}(M)$ and $\mathcal{A}_{s,close}^{difference,\ell}(M)$ both vanish for $\ell\geq 1+\delta s$. Thus we will suppress the superscripts *proximal* and *difference* in the *far*, *near* and *close* decomposition of $\mathcal{A}_{s}^{proximal,\ell}(M)$ and $\mathcal{A}_{s}^{difference,\ell}(M)$, and we will also suppress the conditions $d(c_M,c_{M'})<2^{s\delta}\ell(M)$ and $d(c_M,c_{M'})\geq 2^{s\delta}\ell(M)$ in the proximal and difference terms since they no longer play a role. Using the pairwise disjointedness of the shifted coronas $\mathcal{C}_F^{\mathcal{G},shift}$, we have

$$\sum_{F'\in\mathcal{F}}\left\|\mathsf{Q}_{F',A}^{\omega,\mathbf{b}^*}x\right\|_{L^2(\omega)}^{\spadesuit 2}\lesssim|A|^2|A|_\omega\ ,\qquad\text{for any cube }A.$$

Note that if $c_{M'}\in 2M$, then $M'\subset 3M$. Then with

$$\mathcal{W}_M^s\equiv\bigcup_{F'\in\mathcal{F}}\left\{M'\in\mathcal{W}(F'):M'\subset 3M\text{ and }\ell(M')=2^{-s}\ell(M)\right\},\tag{.0.53}$$

we have

$$\begin{aligned}
\mathcal{A}^0_{s,far}(M) &\leq \sum_{F'\in\mathcal{F}} \sum^{*}_{c_{M'}\in 2M} \int_{\mathbb{R}^n\setminus 4M} S^{F'}_{(M',M)}(y)\,d\sigma(y) \\
&\lesssim \sum_{A\in\mathcal{W}^s_M} \sum_{F'\in\mathcal{F}:\ A\in\mathcal{W}(F')} \int_{\mathbb{R}^n\setminus 4M} \frac{\left\|\mathsf{Q}^{\omega,\mathbf{b}^*}_{F',M'}x\right\|^{\spadesuit 2}_{L^2(\omega)}}{(|M|+|y-c_M|)^{2(n+1-\alpha)}}d\sigma(y) \\
&\lesssim \sum_{A\in\mathcal{W}^s_M} \int_{\mathbb{R}^n\setminus 4M} \frac{|A|^2\,|A|_\omega}{(|M|+|y-c_M|)^{2(n+1-\alpha)}}d\sigma(y) \\
&= \left(\sum_{A\in\mathcal{W}^s_M} |A|^2\,|A|_\omega\right) \int_{\mathbb{R}^n\setminus 4M} \frac{1}{(|M|+|y-c_M|)^{2(n+1-\alpha)}}d\sigma(y)\,.
\end{aligned}$$

Now we use the standard pigeonholing of side length of A to conclude that

$$\begin{aligned}
\sum_{A\in\mathcal{W}^s_M} |A|^2\,|A|_\omega &= \sum_{k=s}^{\infty} \sum_{A\in\mathcal{W}^s_M:\ \ell(A)=2^{-k}\ell(M)} |A|^2\,|A|_\omega \\
&\leq \sum_{k=s}^{\infty} 2^{-2k}\,|M|^2 \sum_{A\in\mathcal{W}^s_M:\ \ell(A)=2^{-k}\ell(M)} |A|_\omega \qquad (.0.54)\\
&\leq \sum_{k=s}^{\infty} 2^{-2k}\,|M|^2\,|3M|_\omega \lesssim 2^{-2s}\,|M|^2\,|3M|_\omega\,,
\end{aligned}$$

so that combining the previous two displays we have

$$\begin{aligned}
\mathcal{A}^0_{s,far}(M) &\lesssim 2^{-2s}\,|M|^2\,|3M|_\omega \int_{\mathbb{R}^n\setminus 4M} \frac{1}{(|M|+|y-c_M|)^{2(n+1-\alpha)}}d\sigma(y) \\
&\leq 2^{-2s}\,|4M|_\omega \int_{\mathbb{R}^n\setminus 4M} \frac{1}{(|M|+|y-c_M|)^{2(1-\alpha)}}d\sigma(y) \\
&\approx 2^{-2s}\frac{|4M|_\omega}{|4M|^{1-\alpha}} \int_{\mathbb{R}^n\setminus 4M} \left(\frac{|M|}{(|M|+|y-c_M|)^2}\right)^{1-\alpha} d\sigma(y) \\
&\lesssim 2^{-2s}\frac{|4M|_\omega}{|4M|^{1-\alpha}}\mathcal{P}^\alpha\left(4M,\mathbf{1}_{\mathbb{R}^n\setminus 4M}\sigma\right) \lesssim 2^{-2s}\mathcal{A}^\alpha_2\ .
\end{aligned}$$

To estimate the near term $\mathcal{A}^0_{s,near}(M)$, we initially keep the energy $\left\|\mathsf{Q}^{\omega,\mathbf{b}^*}_{F',M'}z\right\|^2_{L^2(\omega)}$ and write

$$\begin{aligned}
\mathcal{A}^0_{s,near}(M) &\leq \sum_{F'\in\mathcal{F}}\sum^{*}_{c_{M'}\in 2M}\int_{4M} S^{F'}_{(M',M)}(y)\,d\sigma(y)\\
&\approx \sum_{F'\in\mathcal{F}}\sum^{*}_{c_{M'}\in 2M}\int_{4M}\frac{1}{|M|^{n+1-\alpha}}\frac{\left\|\mathsf{Q}^{\omega,\mathbf{b}^*}_{F',M'}x\right\|^{\spadesuit 2}_{L^2(\omega)}}{\left(|M'|^{\frac{1}{n}}+\left|y-c_{M'}\right|\right)^{n+1-\alpha}}d\sigma(y)\\
&= \sum_{F'\in\mathcal{F}}\frac{1}{|M|^{n+1-\alpha}}\sum^{*}_{c_{M'}\in 2M}\left\|\mathsf{Q}^{\omega,\mathbf{b}^*}_{F',M'}x\right\|^{\spadesuit 2}_{L^2(\omega)}\int_{4M}\frac{d\sigma(y)}{\left(|M'|^{\frac{1}{n}}+\left|y-c_{M'}\right|\right)^{n+1-\alpha}}\\
&= \sum_{F'\in\mathcal{F}}\frac{1}{|M|^{n+1-\alpha}}\sum^{*}_{c_{M'}\in 2M}\left\|\mathsf{Q}^{\omega,\mathbf{b}^*}_{F',M'}x\right\|^{\spadesuit 2}_{L^2(\omega)}\frac{\mathrm{P}^{\alpha}\left(M',\mathbf{1}_{4M}\sigma\right)}{|M'|^{\frac{1}{n}}}.
\end{aligned}$$

In order to estimate the final sum above, we must invoke the 'prepare to puncture' argument above, as we will want to derive geometric decay from $\left\|\mathsf{Q}^{\omega,\mathbf{b}^*}_{M'}x\right\|^{\spadesuit 2}_{L^2(\omega)}$ by dominating it by the 'nonenergy' term $\left|M'\right|^2\left|M'\right|_{\omega}$, as well as using the Muckenhoupt energy constant. Choose an augmented cube $\widetilde{M}\in\mathcal{AD}$ satisfying $\bigcup_{c_{M'}\in 2M} M'\subset 4M\subset\widetilde{M}$ and $\ell\left(\widetilde{M}\right)\leq C\ell(M)$. Define $\widetilde{\omega}=\omega-\omega(\{p\})\,\delta_p$ where p is an atomic point in $\widetilde{M}$ for which

$$\omega(\{p\})=\sup_{q\in\mathfrak{P}_{(\sigma,\omega)}:\ q\in\widetilde{M}}\omega(\{q\}).$$

(If ω has no atomic point in common with σ in $\widetilde{M}$, set $\widetilde{\omega}=\omega$). Then we have $\left|\widetilde{M}\right|_{\widetilde{\omega}}=\omega\left(\widetilde{M},\mathfrak{P}_{(\sigma,\omega)}\right)$ and

$$\frac{\left|\widetilde{M}\right|_{\widetilde{\omega}}}{\left|\widetilde{M}\right|^{1-\frac{\alpha}{n}}}\frac{\left|\widetilde{M}\right|_{\sigma}}{\left|\widetilde{M}\right|^{1-\frac{\alpha}{n}}}=\frac{\omega\left(\widetilde{M},\mathfrak{P}_{(\sigma,\omega)}\right)}{\left|\widetilde{M}\right|^{1-\frac{\alpha}{n}}}\frac{\left|\widetilde{M}\right|_{\sigma}}{\left|\widetilde{M}\right|^{1-\frac{\alpha}{n}}}\leq A_2^{\alpha,punct}.$$

From (.0.39) and (.0.19) we also have

$$\sum_{F'\in\mathcal{F}}\left\|\mathsf{Q}_{F',A}^{\omega,\mathbf{b}^*}x\right\|_{L^2(\omega)}^{\spadesuit 2}\lesssim\ell(A)^2|A|_{\widetilde{\omega}}\ ,\qquad\text{for any cube } A.$$

Now by Cauchy-Schwarz and the augmented local estimate (.0.35) in Lemma .0.12 with $M=\widetilde{M}$ applied to the second line below, and with $\mathcal{W}_M^s$ as in (.0.53), and noting (.0.54), the last sum in (.0.55) is dominated by

$$\begin{aligned}
&\frac{1}{|M|^{n+1-\alpha}}\left(\sum_{F'\in\mathcal{F}}\sum_{c(M')\in 2M}^{*}\left\|\mathsf{Q}_{F',M'}^{\omega,\mathbf{b}^*}x\right\|_{L^2(\omega)}^{\spadesuit 2}\right)^{\frac{1}{2}} \qquad (.0.55)\\
&\qquad\times\left(\sum_{F'\in\mathcal{F}}\sum_{c_{M'}\in 2M}^{*}\left\|\mathsf{Q}_{F',M'}^{\omega,\mathbf{b}^*}x\right\|_{L^2(\omega)}^{\spadesuit 2}\left(\frac{\mathrm{P}^{\alpha}\left(M',\mathbf{1}_{4M}\sigma\right)}{|M'|^{\frac{1}{n}}}\right)^2\right)^{\frac{1}{2}}\\
\lesssim\ &\frac{1}{|M|^{n+1-\alpha}}\left(\sum_{A\in\mathcal{W}_M^s}|A|^2|A|_{\widetilde{\omega}}\right)^{\frac{1}{2}}\sqrt{\left(\mathfrak{E}_2^{\alpha}\right)^2+A_2^{\alpha,energy}}\sqrt{\left|\widetilde{M}\right|_{\sigma}}\\
\lesssim\ &\frac{2^{-s}|M|}{|M|^{n+1-\alpha}}\sqrt{|4M|_{\widetilde{\omega}}}\sqrt{\left(\mathfrak{E}_2^{\alpha}\right)^2+A_2^{\alpha,energy}}\sqrt{\left|\widetilde{M}\right|_{\sigma}}\\
\lesssim\ &2^{-s}\sqrt{\left(\mathfrak{E}_{\alpha}\right)^2+A_2^{\alpha,energy}}\sqrt{\frac{\left|\widetilde{M}\right|_{\widetilde{\omega}}}{\left|\widetilde{M}\right|^{n+1-\alpha}}\frac{\left|\widetilde{M}\right|_{\sigma}}{\left|\widetilde{M}\right|^{n+1-\alpha}}}\\
\lesssim\ &2^{-s}\sqrt{\left(\mathfrak{E}_2^{\alpha}\right)^2+A_2^{\alpha,energy}}\sqrt{A_2^{\alpha,punct}}\ .
\end{aligned}$$

Similarly, for $\ell\geq 1$, we can estimate the far term $\mathcal{A}_{s,far}^{\ell}(M)$ by the argument used for $\mathcal{A}_{s,far}^{0}(M)$ but applied to $2^{\ell}M$ in place of M. For this need the following variant of $\mathcal{W}_M^s$ in

(.0.53) given by

$$\mathcal{W}_M^{s,\ell} \equiv \bigcup_{F' \in \mathcal{F}} \left\{ M' \in \mathcal{W}\left(F'\right) : M' \subset 3\left(2^{\ell} M\right) \text{ and } \ell\left(M'\right) = 2^{-s-\ell} \ell\left(2^{\ell} M\right) \right\}. \tag{.0.56}$$

Then we have

$$\begin{aligned}
\mathcal{A}_{s,far}^{\ell}(M) &\leq \sum_{F' \in \mathcal{F}} \sum_{c_{M'} \in \left(2^{\ell+1} M\right) \setminus \left(2^{\ell} M\right)}^{*} \int_{\mathbb{R}^n \setminus 2^{\ell+2} M} S_{(M',M)}^{F'}(y)\, d\sigma(y) \\
&\lesssim \sum_{A \in \mathcal{W}_M^{s,\ell}} \sum_{F' \in \mathcal{F}:\ A \in \mathcal{W}(F')} \int_{\mathbb{R}^n \setminus 4\left(2^{\ell} M\right)} \frac{\left\| \mathsf{Q}_{F',A}^{\omega,\mathbf{b}^*} x \right\|_{L^2(\omega)}^{\spadesuit 2}}{\left(|M| + |y - c_M|\right)^{2(n+1-\alpha)}} d\sigma(y) \\
&\lesssim \sum_{A \in \mathcal{W}_M^{s,\ell}} \int_{\mathbb{R}^n \setminus 4\left(2^{\ell} M\right)} \frac{|A|^2 |A|_{\omega}}{\left(|M| + |y - c_M|\right)^{2(n+1-\alpha)}} d\sigma(y) \\
&= \left(\sum_{A \in \mathcal{W}_M^{s,\ell}} |A|^2 |A|_{\omega} \right) \int_{\mathbb{R}^n \setminus 4\left(2^{\ell} M\right)} \frac{1}{\left(|M| + |y - c_M|\right)^{2(n+1-\alpha)}} d\sigma(y),
\end{aligned}$$

where, just as for the sum over $A \in \mathcal{W}_M^{s,0}$, we have

$$\begin{aligned}
\sum_{A \in \mathcal{W}_M^{s,\ell}} |A|^2 |A|_{\omega} &= \sum_{k=s}^{\infty} \sum_{A \in \mathcal{W}_M^{s,\ell}:\ \ell(A) = 2^{-k-\ell} \ell\left(2^{\ell} M\right)} |A|^2 |A|_{\omega} \\
&\leq \sum_{k=s}^{\infty} 2^{-2k-2\ell} \left| 2^{\ell} M \right|^2 \sum_{A \in \mathcal{W}_M^{s,\ell}:\ \ell(A) = 2^{-k-\ell} \ell\left(2^{\ell} M\right)} |A|_{\omega} \qquad (.0.57) \\
&\leq \sum_{k=s}^{\infty} 2^{-2k-2\ell} \left| 2^{\ell} M \right|^2 \left| 3\left(2^{\ell} M\right) \right|_{\omega} \lesssim 2^{-2s-2\ell} \left| 2^{\ell} M \right|^2 \left| 3\left(2^{\ell} M\right) \right|_{\omega}.
\end{aligned}$$

Now using $\frac{\left|2^{\ell}M\right|^{2}}{\left(|M|+|y-c_{M}|\right)^{2(n+1-\alpha)}}\leq\frac{1}{\left(\left|2^{\ell}M\right|+\left|y-c_{2^{\ell}M}\right|\right)^{2(1-\alpha)}}$ for $y\notin 2^{\ell+2}M$, we can continue with

$$\begin{aligned}
\mathcal{A}_{s,far}^{\ell}\left(M\right) &\lesssim 2^{-2s}2^{-2\ell}\left|2^{\ell+2}M\right|_{\omega}\int_{\mathbb{R}^{n}\setminus 2^{\ell+2}M}\frac{d\sigma\left(y\right)}{\left(\left|2^{\ell}M\right|+\left|y-c_{2^{\ell}M}\right|\right)^{2(1-\alpha)}}\\
&\approx 2^{-2s}2^{-2\ell}\frac{\left|2^{\ell+2}M\right|_{\omega}}{\left|2^{\ell}M\right|^{1-\alpha}}\int_{\mathbb{R}^{n}\setminus 2^{\ell+2}M}\left(\frac{\left|2^{\ell}M\right|}{\left(\left|2^{\ell}M\right|+\left|y-c_{2^{\ell}M}\right|\right)^{2}}\right)^{1-\alpha}d\sigma\left(y\right)\\
&\lesssim 2^{-2s}2^{-2\ell}\left\{\frac{\left|2^{\ell+2}M\right|_{\omega}}{\left|2^{\ell}M\right|^{1-\alpha}}\mathcal{P}^{\alpha}\left(2^{\ell+2}M,\mathbf{1}_{\mathbb{R}^{n}\setminus 2^{\ell+2}M}\sigma\right)\right\}\lesssim 2^{-2s}2^{-2\ell}\mathcal{A}_{2}^{\alpha}\ .
\end{aligned}$$

To estimate the near term $\mathcal{A}_{s,near}^{\ell}\left(M\right)$ we must again invoke the *'prepare to puncture'* argument. Choose an augmented cube $\widetilde{M}\in\mathcal{AD}$ such that $\ell\left(\widetilde{M}\right)\leq C2^{\ell}\ell\left(M\right)$ and $\bigcup_{c_{M'}\in 2^{\ell+1}M\setminus 2^{\ell}M}M'\subset 2^{\ell+2}M\subset\widetilde{M}$. Define $\widetilde{\omega}=\omega-\omega\left(\{p\}\right)\delta_{p}$ where p is an atomic point in $\widetilde{M}$ for which $\omega\left(\{p\}\right)=\sup_{q\in\mathfrak{P}_{(\sigma,\omega)}:\ q\in\widetilde{M}}\omega\left(\{q\}\right).$

(If ω has no atomic point in common with σ in $\widetilde{M}$ set $\widetilde{\omega}=\omega$.) Then we have $\left|\widetilde{M}\right|_{\widetilde{\omega}}=\omega\left(\widetilde{M},\mathfrak{P}_{(\sigma,\omega)}\right)$, and just as in the argument above following (.0.55), we have from (.0.39) and (.0.19) that both

$$\frac{\left|\widetilde{M}\right|_{\widetilde{\omega}}}{\left|\widetilde{M}\right|^{1-\frac{\alpha}{n}}}\frac{\left|\widetilde{M}\right|_{\sigma}}{\left|\widetilde{M}\right|^{1-\frac{\alpha}{n}}}\leq A_{2}^{\alpha,punct}\text{ and }\sum_{F'\in\mathcal{F}}\left\|\mathsf{Q}_{F',M'}^{\omega,\mathbf{b}^{*}}x\right\|_{L^{2}(\omega)}^{\spadesuit 2}\lesssim\ell\left(M'\right)^{2}\left|M'\right|_{\widetilde{\omega}}\ .$$

Thus using that $m=2$ in the definition of $A^{\ell}_{s,near}(M)$, we see that

$$\mathcal{A}^{\ell}_{s,near}(M) \leq \sum_{F' \in \mathcal{F}} \sum^{*}_{c_{M'} \in 2^{\ell+1}M \setminus 2^{\ell}M} \int_{2^{\ell+2}M \setminus 2^{\ell-m}M} S^{F'}_{(M',M)}(y)\, d\sigma(y)$$

$$\approx \sum_{F' \in \mathcal{F}} \sum^{*}_{c_{M'} \in 2^{\ell+1}M \setminus 2^{\ell}M} \int_{2^{\ell+2}M \setminus 2^{\ell-m}M} \frac{1}{\left|2^{\ell}M\right|^{n+1-\alpha}} \frac{\left\|\mathsf{Q}^{\omega,\mathbf{b}^*}_{F',M'}x\right\|^{\spadesuit 2}_{L^2(\omega)}}{\left(\left|M'\right|^{\frac{1}{n}} + \left|y - c_{M'}\right|\right)^{n+1-\alpha}} d\sigma(y)$$

$$\lesssim \frac{1}{\left|2^{\ell}M\right|^{n+1-\alpha}} \sum_{F' \in \mathcal{F}} \sum^{*}_{c_{M'} \in 2^{\ell+1}M \setminus 2^{\ell}M} \left\|\mathsf{Q}^{\omega,\mathbf{b}^*}_{F',M'}x\right\|^{\spadesuit 2}_{L^2(\omega)} \int_{2^{\ell+2}M} \frac{d\sigma(y)}{\left(\left|M'\right|^{\frac{1}{n}} + \left|y - c_{M'}\right|\right)^{n+1-\alpha}}$$

is dominated by

$$\frac{1}{\left|2^{\ell}M\right|^{n+1-\alpha}} \sum_{F' \in \mathcal{F}} \sum^{*}_{c_{M'} \in 2^{\ell+1}M \setminus 2^{\ell}M} \left\|\mathsf{Q}^{\omega,\mathbf{b}^*}_{F',M'}x\right\|^{\spadesuit 2}_{L^2(\omega)} \frac{\mathrm{P}^{\alpha}\left(M', \mathbf{1}_{2^{\ell+2}M}\sigma\right)}{\left|M'\right|^{\frac{1}{n}}}$$

$$\leq \frac{1}{\left|2^{\ell}M\right|^{n+1-\alpha}} \left(\sum_{F' \in \mathcal{F}} \sum^{*}_{c_{M'} \in 2^{\ell+1}M \setminus 2^{\ell}M} \left\|\mathsf{Q}^{\omega,\mathbf{b}^*}_{F',M'}x\right\|^{\spadesuit 2}_{L^2(\omega)}\right)^{\frac{1}{2}}$$

$$\times \left(\sum_{F' \in \mathcal{F}} \sum^{*}_{c_{M'} \in 2^{\ell+1}M \setminus 2^{\ell}M} \left\|\mathsf{Q}^{\omega,\mathbf{b}^*}_{F',M'}x\right\|^{\spadesuit 2}_{L^2(\omega)} \left(\frac{\mathrm{P}^{\alpha}\left(M', \mathbf{1}_{2^{\ell+2}M}\sigma\right)}{\left|M'\right|^{\frac{1}{n}}}\right)^{2}\right)^{\frac{1}{2}}.$$

This can now be estimated as for the term $\mathcal{A}^{0}_{s,near}(M)$, along with the augmented local estimate (.0.35) in Lemma .0.12 with $M=\widetilde{M}$ applied to the final line above, to get

$$\begin{aligned}
\mathcal{A}^{\ell}_{s,near}(M) &\lesssim 2^{-s}2^{-\ell}\frac{\left|2^{\ell}M\right|}{\left|2^{\ell}M\right|^{n+1-\alpha}}\sqrt{\left|\widetilde{M}\right|_{\widetilde{\omega}}}\sqrt{\left(\mathfrak{E}^{\alpha}_{2}\right)^{2}+A^{\alpha,energy}_{2}}\sqrt{\left|\widetilde{M}\right|_{\sigma}} \\
&\lesssim 2^{-s}2^{-\ell}\sqrt{\left(\mathfrak{E}^{\alpha}_{2}\right)^{2}+A^{\alpha,energy}_{2}}\sqrt{\frac{\left|\widetilde{M}\right|_{\widetilde{\omega}}}{\left|\widetilde{M}\right|^{1-\frac{\alpha}{n}}}\frac{\left|\widetilde{M}\right|_{\sigma}}{\left|\widetilde{M}\right|^{1-\frac{\alpha}{n}}}} \\
&\lesssim 2^{-s}2^{-\ell}\sqrt{\left(\mathfrak{E}^{\alpha}_{2}\right)^{2}+A^{\alpha,energy}_{2}}\sqrt{A^{\alpha,punct}_{2}}\ .
\end{aligned}$$

Each of the estimates for $\mathcal{A}^{\ell}_{s,far}(M)$ and $\mathcal{A}^{\ell}_{s,near}(M)$ is summable in both s and ℓ.

Now we turn to the terms $\mathcal{A}^{\ell}_{s,close}(M)$, and recall from (.0.51) that $\mathcal{A}^{\ell}_{s,close}(M)=0$ if $\ell\geq 1+\delta s$. So we now suppose that $\ell\leq\delta s$. We have, with $m=2$ as in (.0.50),

$$\begin{aligned}
\mathcal{A}^{\ell}_{s,close}(M) &\leq \sum_{F'\in\mathcal{F}}\sum^{*}_{c_{M'}\in 2^{\ell+1}M\setminus 2^{\ell}M}\int_{2^{\ell-m}M}S^{F'}_{(M',M)}(y)\,d\sigma(y) \\
&\approx \sum_{F'\in\mathcal{F}}\sum^{*}_{c_{M'}\in 2^{\ell+1}M\setminus 2^{\ell}M}\int_{2^{\ell-m}M}\frac{1}{\left(|M|^{\frac{1}{n}}+|y-c_M|\right)^{n+1-\alpha}}\frac{\left\|\mathsf{Q}^{\omega,\mathbf{b}^{*}}_{F',M'}x\right\|^{\spadesuit 2}_{L^{2}(\omega)}}{\left|2^{\ell}M\right|^{n+1-\alpha}}d\sigma(y) \\
&= \left(\sum_{F'\in\mathcal{F}}\sum^{*}_{c_{M'}\in 2^{\ell+1}M\setminus 2^{\ell}M}\left\|\mathsf{Q}^{\omega,\mathbf{b}^{*}}_{F',M'}x\right\|^{\spadesuit 2}_{L^{2}(\omega)}\right)\frac{1}{\left|2^{\ell}M\right|^{n+1-\alpha}} \\
&\qquad\times\int_{2^{\ell-m}M}\frac{1}{\left(|M|^{\frac{1}{n}}+|y-c_M|\right)^{n+1-\alpha}}d\sigma(y)\,.
\end{aligned}$$

The argument used to prove (.0.57) gives the analogous inequality with a hole $2^{\ell-1}M$,

$$\sum_{F'\in\mathcal{F}}\sum^{*}_{c_{M'}\in 2^{\ell+1}M\setminus 2^{\ell}M}\left\|\mathsf{Q}^{\omega,\mathbf{b}^{*}}_{F',M'}x\right\|^{\spadesuit 2}_{L^{2}(\omega)}\lesssim 2^{-2s}\left|2^{\ell}M\right|^{\frac{2}{n}}\left|2^{\ell+2}M\setminus 2^{\ell-1}M\right|_{\omega}\ .$$

Thus we get that $\mathcal{A}^{\ell}_{s,close}(M)$ is bounded by

$$\begin{aligned}
&\lesssim\ 2^{-2s}\left|2^{\ell}M\right|^{\frac{2}{n}}\left|2^{\ell+2}M\backslash 2^{\ell-1}M\right|_{\omega}\frac{1}{\left|2^{\ell}M\right|^{\frac{1}{n}(n+1-\alpha)}}\int_{2^{\ell-m}M}\frac{d\sigma(y)}{\left(|M|^{\frac{1}{n}}+|y-c_M|\right)^{n+1-\alpha}}\\
&\lesssim\ 2^{-2s}\left|2^{\ell}M\right|^{\frac{2}{n}}\frac{\left|2^{\ell+2}M\setminus 2^{\ell-1}M\right|_{\omega}}{\left|2^{\ell}M\right|^{\frac{1}{n}(n+1-\alpha)}}\frac{\left|2^{\ell-m}M\right|_{\sigma}}{|M|^{\frac{1}{n}(n+1-\alpha)}}\\
&\lesssim\ 2^{-2s}2^{(n+1-\alpha)\ell}\frac{\left|2^{\ell+2}M\setminus 2^{\ell-1}M\right|_{\omega}}{\left|2^{\ell+2}M\right|^{1-\frac{\alpha}{n}}}\frac{\left|2^{\ell-m}M\right|_{\sigma}}{\left|2^{\ell-m}M\right|^{1-\frac{\alpha}{n}}}\lesssim 2^{-2s}2^{(n+1-\alpha)\ell}A_2^{\alpha},
\end{aligned}$$

provided that $m=2>1$. Note that we can use the offset Muckenhoupt constant A_2^{α} here since $2^{\ell+2}M\setminus 2^{\ell-1}M$ and $2^{\ell-m}M$ are disjoint. If $\ell\le s$, then we have the relatively crude estimate $\mathcal{A}^{\ell}_{s,close}(M)\lesssim 2^{-s}A_2^{\alpha}$ without decay in ℓ. But we are assuming $\ell\le\delta s$ here, and so we obtain a suitable estimate for $\mathcal{A}^{\ell}_{s,close}(M)$ provided we choose $0<\delta\le\frac{1}{n+1-\alpha}$. Indeed, we then have

$$\sum_{l=1}^{\delta s}2^{-2s}2^{(n+1-\alpha)\ell}A_2^{\alpha}=2^{-2s}\left(\sum_{l=1}^{\delta s}2^{(n+1-\alpha)\ell}\right)A_2^{\alpha}\lesssim 2^{-2s}2^{(n+1-\alpha)\delta s}A_2^{\alpha}\le 2^{-s}A_2^{\alpha}\ ,$$

provided $\delta\le\frac{1}{n+1-\alpha}$, and in particular we may take $\delta=\frac{1}{2}$. Altogether, the above estimates prove

$$T_s^{proximal}+T_s^{difference}\lesssim 2^{-s}\left(\mathcal{A}_2^{\alpha}+\sqrt{\left(\mathfrak{E}_2^{\alpha}\right)^2+A_2^{\alpha,energy}}\sqrt{A_2^{\alpha,punct}}\right)\int_{\widehat{I}}t^2d\mu,$$

which is summable in s.

The intersection term

Now we return to the term $T_s^{intersection} \equiv$

$$\sum_{F,F'\in\mathcal{F}} \sum_{\substack{M\in\mathcal{W}(F)\\ M'\in\mathcal{W}\left(F'\right)\\ M,M'\subset I,\ \ell\left(M'\right)=2^{-s}\ell(M)\\ d\left(c_M,c_{M'}\right)\geq 2^{s(1+\delta)}\ell\left(M'\right)}} \int_{B(M,M')} \frac{\left\|\mathsf{Q}_{F,M}^{\omega,\mathbf{b}^*}x\right\|_{L^2(\omega)}^{\spadesuit 2}}{\left(|M|+|y-c_M|\right)^{n+1-\alpha}} \frac{\left\|\mathsf{Q}_{F',M'}^{\omega,\mathbf{b}^*}x\right\|_{L^2(\omega)}^{\spadesuit 2}}{\left(|M'|+\left|y-c_{M'}\right|\right)^{n+1-\alpha}} d\sigma\left(y\right).$$

It will suffice to show that $T_s^{intersection}$ satisfies the estimate,

$$\begin{aligned} T_s^{intersection} &\lesssim 2^{-s\delta}\sqrt{\left(\mathfrak{E}_2^{\alpha}\right)^2+A_2^{\alpha,energy}}\sqrt{A_2^{\alpha,punct}}\sum_{F'\in\mathcal{F}'}\sum_{\substack{M'\in\mathcal{M}_{(\rho,\varepsilon)-deep}\left(F'\right)\\ M'\subset I}}\|\mathsf{Q}_{F',M'}^{\omega,\mathbf{b}^*}x\|_{L^2(\omega)}^{\spadesuit 2} \\ &= 2^{-s\delta}\sqrt{\left(\mathfrak{E}_2^{\alpha}\right)^2+A_2^{\alpha,energy}}\sqrt{A_2^{\alpha,punct}}\int_{\widehat{I}} t^2\overline{\mu}\ . \end{aligned}$$

Recalling $B(M,M') = B\left(c_M, \frac{1}{2}d(c_M, c_{M'})\right)$, we can write (suppressing some notation for clarity) $T_s^{intersection}$ as

$$\begin{aligned}
&= \sum_{F,F'} \sum_{M,M'} \int_{B(M,M')} \frac{\left\| \mathsf{Q}_{F,M}^{\omega,\mathbf{b}^*} x \right\|_{L^2(\omega)}^{\spadesuit 2}}{\left(|M|^{\frac{1}{n}} + |y - c_M| \right)^{n+1-\alpha}} \frac{\left\| \mathsf{Q}_{F',M'}^{\omega,\mathbf{b}^*} x \right\|_{L^2(\omega)}^{\spadesuit 2}}{\left(|M'|^{\frac{1}{n}} + |y - c_{M'}| \right)^{n+1-\alpha}} d\sigma(y) \\
&\approx \sum_{F,F'} \sum_{M,M'} \frac{\left\| \mathsf{Q}_{F,M}^{\omega,\mathbf{b}^*} x \right\|_{L^2(\omega)}^{\spadesuit 2} \left\| \mathsf{Q}_{F',M'}^{\omega,\mathbf{b}^*} x \right\|_{L^2(\omega)}^{\spadesuit 2}}{|c_M - c_{M'}|^{n+1-\alpha}} \int_{B(M,M')} \frac{d\sigma(y)}{\left(|M|^{\frac{1}{n}} + |y - c_M| \right)^{n+1-\alpha}} \\
&\leq \sum_{F'} \sum_{M'} \left\| \mathsf{Q}_{F',M'}^{\omega,\mathbf{b}^*} x \right\|_{L^2(\omega)}^{\spadesuit 2} \sum_{F} \sum_{M} \frac{\left\| \mathsf{Q}_{F,M}^{\omega,\mathbf{b}^*} x \right\|_{L^2(\omega)}^{\spadesuit 2}}{|c_M - c_{M'}|^{n+1-\alpha}} \int_{B(M,M')} \frac{d\sigma(y)}{\left(|M|^{\frac{1}{n}} + |y - c_M| \right)^{n+1-\alpha}} \\
&\equiv \sum_{F'} \sum_{M'} \left\| \mathsf{Q}_{F',M'}^{\omega,\mathbf{b}^*} x \right\|_{L^2(\omega)}^{\spadesuit 2} S_s(M'),
\end{aligned}$$

and since $\int_{B(M,M')} \frac{d\sigma(y)}{\left(|M|^{\frac{1}{n}} + |y - c_M| \right)^{n+1-\alpha}} \approx \frac{\mathrm{P}^\alpha\left(M, \mathbf{1}_{B(M,M')} \sigma \right)}{|M|^{\frac{1}{n}}}$, it remains to show that for each fixed M',

$$\begin{aligned}
S_s(M') &\approx \sum_{F} \sum_{M:\ d\left(c_M, c_{M'}\right) \geq 2^{s(1+\delta)} \ell\left(M'\right)}^{*} \frac{\left\| \mathsf{Q}_{F,M}^{\omega,\mathbf{b}^*} x \right\|_{L^2(\omega)}^{\spadesuit 2}}{|c_M - c_{M'}|^{n+1-\alpha}} \frac{\mathrm{P}^\alpha\left(M, \mathbf{1}_{B(M,M')} \sigma \right)}{|M|^{\frac{1}{n}}} \\
&\lesssim 2^{-\delta s} \sqrt{\left(\mathfrak{E}_2^\alpha \right)^2 + A_2^{\alpha,energy}} \sqrt{A_2^\alpha}\ .
\end{aligned}$$

We write

$$
\begin{aligned}
S_s\left(M'\right) &\approx \sum_{k\geq s(1+\delta)} \frac{1}{\left(2^k\left|M'\right|\right)^{n+1-\alpha}} S_s^k\left(M'\right) \ ; \\
S_s^k\left(M'\right) &\equiv \sum_F \sum_{M:\ d\left(c_M,c_{M'}\right)\approx 2^k\ell(M')}^{*} \left\|\mathsf{Q}_{F,M}^{\omega,\mathbf{b}^*}x\right\|_{L^2(\omega)}^{\spadesuit 2} \frac{\mathrm{P}^\alpha\left(M,\mathbf{1}_{B(M,M')}\sigma\right)}{|M|^{\frac{1}{n}}},
\end{aligned}
$$

where by $d\left(c_M,c_{M'}\right)\approx 2^k\ell\left(M'\right)$ we mean $2^k\ell\left(M'\right)\leq d\left(c_M,c_{M'}\right)\leq 2^{k+1}\ell\left(M'\right)$. Moreover, if $d\left(c_M,c_{M'}\right)\approx 2^k\ell\left(M'\right)$, then from the fact that the radius of $B\left(M,M'\right)$ is $\frac{1}{2}d\left(c_M,c_{M'}\right)$, we obtain

$$
B\left(M,M'\right)\subset C_0 2^k M',
$$

where C_0 is a positive constant ($C_0=6$ works).

For fixed $k\geq s\left(1+\delta\right)$, we invoke yet again the *'prepare to puncture'* argument. Choose an augmented cube $\widetilde{M'}\in\mathcal{AD}$ such that $C_0 2^k M\subset\widetilde{M'}$ and $\ell\left(\widetilde{M'}\right)\leq C2^k\ell\left(M'\right)$. Define $\widetilde{\omega}=\omega-\omega\left(\{p\}\right)\delta_p$ where p is an atomic point in $\widetilde{M'}$ for which

$$
\omega\left(\{p\}\right)=\sup_{q\in\mathfrak{P}_{(\sigma,\omega)}:\ q\in\widetilde{M'}}\omega\left(\{q\}\right).
$$

(If ω has no atomic point in common with σ in $\widetilde{M'}$, set $\widetilde{\omega}=\omega$.) Then we have $\left|\widetilde{M'}\right|_{\widetilde{\omega}}=\omega\left(\widetilde{M'},\mathfrak{P}_{(\sigma,\omega)}\right)$ and so from (.0.39) and (.0.19), for any cube A,

$$
\frac{\left|\widetilde{M'}\right|_{\widetilde{\omega}}}{\left|\widetilde{M'}\right|^{1-\frac{\alpha}{n}}}\frac{\left|\widetilde{M'}\right|_{\sigma}}{\left|\widetilde{M'}\right|^{1-\frac{\alpha}{n}}}\leq A_2^{\alpha,punct} \text{ and } \sum_{F\in\mathcal{F}}\left\|\mathsf{Q}_{F,A}^{\omega,\mathbf{b}^*}x\right\|_{L^2(\omega)}^{\spadesuit 2}\lesssim\ell\left(A\right)^2\left|A\right|_{\widetilde{\omega}}
$$

Now we are ready to apply Cauchy-Schwarz and the augmented local estimate (.0.35) in

Lemma .0.12 with $M=\widetilde{M'}$ to the second line below, and to apply the argument in (.0.57) to the first line below, to get the following estimate for $S_s^k\left(M'\right)$ defined in (.0.58) above:

$$\begin{aligned}
S_s^k\left(M'\right) &\leq \left(\sum_F \sum_{M:\, d\left(c_M,c_{M'}\right)\approx 2^k\ell(M')} \left\|\mathsf{Q}_{F,M}^{\omega,\mathbf{b}^*}x\right\|_{L^2(\omega)}^{\spadesuit 2}\right)^{\frac{1}{2}} \\
&\quad\times\left(\sum_F \sum_{M:\, d\left(c_M,c_{M'}\right)\approx 2^k\ell(M')} \left\|\mathsf{Q}_{F,M}^{\omega,\mathbf{b}^*}x\right\|_{L^2(\omega)}^{\spadesuit 2}\left(\frac{\mathrm{P}^\alpha\left(M,\mathbf{1}_{B(M,M')}\sigma\right)}{|M|^{\frac{1}{n}}}\right)^2\right)^{\frac{1}{2}} \\
&\lesssim \left(2^{2s}\left|\widetilde{M'}\right|^2\left|\widetilde{M'}\right|_{\widetilde{\omega}}\right)^{\frac{1}{2}}\left(\left[\left(\mathfrak{E}_2^\alpha\right)^2+A_2^{\alpha,energy}\right]\left|\widetilde{M'}\right|_\sigma\right)^{\frac{1}{2}} \\
&\lesssim \sqrt{\left(\mathfrak{E}_2^\alpha\right)^2+A_2^{\alpha,energy}}\,2^s\left|\widetilde{M'}\right|\sqrt{\left|\widetilde{M'}\right|_{\widetilde{\omega}}}\sqrt{\left|\widetilde{M'}\right|_\sigma}
\end{aligned}$$

$$\begin{aligned}
&\lesssim \sqrt{\left(\mathfrak{E}_2^\alpha\right)^2+A_2^{\alpha,energy}}\sqrt{A_2^{\alpha,punct}}\,2^s\left|\widetilde{M'}\right|\left|\widetilde{M'}\right|^{1-\alpha} \\
&\approx \sqrt{\left(\mathfrak{E}_2^\alpha\right)^2+A_2^{\alpha,energy}}\sqrt{A_2^{\alpha,punct}}\,2^s 2^{k(1-\alpha)}\left|M'\right|^{n+1-\alpha},
\end{aligned}$$

because $\ell\left(\widetilde{M'}\right)\approx 2^k\ell\left(M'\right)$.

Altogether then we have

$$\begin{aligned}
S_s\left(M'\right) &= \sum_{k\geq(1+\delta)s}\frac{1}{\left(2^k\left|M'\right|\right)^{n+1-\alpha}}S_s^k\left(M'\right) \\
&\lesssim \sqrt{\left(\mathfrak{E}_2^\alpha\right)^2+A_2^{\alpha,energy}}\sqrt{A_2^{\alpha,punct}}\sum_{k\geq(1+\delta)s}\frac{2^s2^{k(1-\alpha)}}{\left(2^k\left|M'\right|\right)^{n+1-\alpha}}\left|M'\right|^{n+1-\alpha} \\
&= \sqrt{\left(\mathfrak{E}_2^\alpha\right)^2+A_2^{\alpha,energy}}\sqrt{A_2^{\alpha,punct}}\sum_{k\geq(1+\delta)s}2^{s-k} \\
&\lesssim 2^{-\delta s}\sqrt{\left(\mathfrak{E}_2^\alpha\right)^2+A_2^{\alpha,energy}}\sqrt{A_2^{\alpha,punct}},
\end{aligned}$$

which is summable in s. This completes the proof of (.0.44), and hence of the estimate for $\mathbf{Back}\left(\widehat{I}\right)$ in (.0.40).

The proof of Proposition .0.1 is now complete.

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