TWO WEIGHTED CARLESON EMBEDDINGS ON MULTI-TREES AND MULTI-DISK

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ABSTRACT

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Given two measures μ, w on a multi-tree $\mathcal{T}^n$ we prove a two weighted multi-parameter dyadic embedding theorem for the Hardy operator, assuming w is a product weight and a certain "Box" condition holds. The main result has been long proven for dimension $n = 1$, however, for higher dimensions the result was not known. There was a general feeling such an embedding was not possible under the Box condition, due to a famous counterexample by Lennart Carleson. In this counterexample, the measure μ was the two-dimensional Lebesgue measure, which is a product measure along with a non-product weight w. Shortly after, A. Chang imposed a (strictly) more general condition than the Box one and showed it is sufficient to get the same embedding in dimension $n = 2$. This was later used by A. Chang and R. Fefferman to characterize the dyadic n-dimensional product BMO, denoted by $BMO^d_{\text{prod}}(\mathbb{R}^n)$.

Recently, the question of embedding the Dirichlet space on the bi-disk $\mathbb{D}^2$ into $L^2(\mathbb{D}^2)$ appeared. This is equivalent to proving a general measure μ is "Carleson" for the Dirichlet space on $\mathbb{D}^2$. It was shown that proving the (discrete) analogue of the embedding on a bi-tree is enough to get the same for the bi-disc. To do this, however, we need to change the restrictions on the measures; we will assume μ to be general and w to be a product weight. Given these restrictions, we managed to prove the surprising result that the Box condition is enough to imply the embedding for dimensions $n = 2, 3$. This is not contradictory to Carleson's counterexample as the weight w was non-product.

Αυτός ο άνθρωπος ακίνητος στο πλάι
μες το δωμάτιο που βλέπει στο γιαλό
είναι ο πατέρας μου που φεύγει αργά
από ένα κόσμο αναίτιο - μαγικό.

– Sokratis Malamas

To my father

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KEY TO SYMBOLS

The following is a list of some of the notation used throughout this paper.

- $\mathbb{R}^n$ the real coordinate space of dimension n
- $\mathbb{D}^n$ the n-disk, i.e. the Cartesian product of n unit disks in the complex plane
- $\mathcal{T}$ a tree of finite depth (for example, a dyadic tree)
- $\mathcal{T}^n$ is the Cartesian product of n trees
- I, $\mathbb{I}$, $\mathbb{I}$ the one, two and three-dimensional Hardy operator respectively
- I^*, $\mathbb{I}^*$, $\mathbb{I}^*$ the adjoints of the operators above
- $A \lesssim B$ if there exists a universal constant C such that $A \leq CB$
- $A \sim B$ when $A \lesssim B$ and $B \lesssim A$
- $D(\mathbb{D}^2)$ the Dirichlet space over the bi-disk
- $\mathrm{Cap}(E)$ the discrete bi-logarithmic capacity of a set $E \subseteq \mathcal{T}^n$ or the $(\frac{1}{2}, .., \frac{1}{2})$-Bessel capacity of a closed set $E \subseteq (\partial\mathbb{D})^n$, according to context
- $\mathbb{V}^\mu$ the potential of measure μ
- $\mathcal{E}[\mu]$ the total energy of measure μ
- $\mathbb{V}^\mu_\delta$ the cut-off of the potential at level δ
- $\mathcal{E}_\delta[\mu]$ the cut-off of the energy at level δ
- $\leq, \geq$ the ordering on the n-tree $\mathcal{T}^n$
- $\mathrm{ch}(\alpha)$ the set of maximal elements strictly smaller than α
- $|\Omega|$ the one-dimensional Lebesgue measure of the set Ω
- $m_2(\Omega)$ the two-dimensional Lebesgue measure of the set Ω
- $|\mu|$ the total mass of a measure μ
- $[w, \mu]_{Box}$ the Box condition constant
- $[w, \mu]_C$ the Carleson condition constant
- $[w, \mu]_{HC}$ the hereditary Carleson (or restricted energy condition) constant
- $[w, \mu]_{CE}$ the Carleson embedding constant

Chapter 1

Introduction

This thesis deals with the boundedness of a specific *dyadic paraproduct*, under a certain condition. Dyadic paraproducts are a special form of *paraproducts* which are arguably one of the most important classes of operators in harmonic analysis: their boundedness properties are at the core of many problems. Initially, they appeared through PDE questions, such as the Leibniz rule for fractional derivatives (see below). A typical example of a paraproduct operator appears when one deals with the classical $T1$ theorem of David and Journé: For a function $b \in BMO(\mathbb{R}^n)$ we define

$$Lf = \int_0^\infty \psi_t * \big((\psi_t * b)(\phi_t * f)\big)\frac{dt}{t}$$

(for formal details see [12]). The operator L is a paraproduct and can be shown to be a bounded Calderón-Zygmund operator on L^2 with $L1 = b$.

However, in the literature paraproducts usually appear as bi-linear operators and one needs to prove the next bound for paraproducts:

$$\|\Pi(f,g)\|_r \lesssim \|f\|_p\|g\|_q, \quad 1 < p, q \leq \infty, \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \quad 0 < r < \infty. \tag{1.1}$$

where Π is a paraproduct operator. For example, consider the **fractional** derivative of order α: $f \in \mathcal{S}$, $\hat{\mathcal{D}}^\alpha f(\xi) = |\xi|^\alpha \hat{f}(\xi)$, where $\mathcal{S}$ is the class of Schwartz functions. Then, the Leibniz rule is:

$$\|\mathcal{D}^{\alpha}(fg)\|_r \lesssim \|\mathcal{D}^{\alpha} f\|_p \|g\|_q + \|f\|_p \|\mathcal{D}^{\alpha} g\|_q \tag{1.2}$$

for r, p, q as above. This rule appears when one works on regularity questions for linear and non-linear PDE (see [17], [18]). In order to prove (1.2) one actually needs paraproducts and bounds as in (1.1). Such estimates are proven in the works [15], [18] and [13].

The need to study more complicated operators, namely *multi-parameter* paraproducts, appears naturally — again primarily from PDE. These are easy to describe, but difficult to work with. For $n = 2$, a *bi-parameter* paraproduct is an operator given by the coordinate-wise tensor product of two one-dimensional paraproduct operators. Proving the boundedness as in (1.1) of those operators is much harder, but still feasible (see [22] or [23]), and it was used by Kenig in [16] to treat the well-posedness of the Kadomtsev–Petviashvili equation describing non-linear wave motions. For this, another type of Leibniz rule was required, which was again reduced to the boundedness of paraproducts; this time of the aforementioned multi-parameter (tensor) type.

It has been noticed that the right model for studying paraproducts are the so-called *dyadic* paraproducts. For a dyadic n-rectangle $R = I_1 \times \cdots \times I_n \subseteq [0,1)^n$ let $h_R(x_1, \ldots, x_n) := h_{I_1}(x_1) \cdots h_{I_n}(x_n)$ where h_{I_i} is the Haar function in $\mathbb{R}$ associated with the dyadic interval I_i. Then, a simple example of a *dyadic* multi-parameter paraproduct, is the operator

$$\Pi_b \, \varphi := \sum_R \langle \varphi \rangle_R \cdot \beta_R \cdot h_R$$

where $\langle \varphi \rangle_R$ is the average of φ over R and $\beta_R := \langle b, h_R \rangle_{L^2}$ are the Haar coefficients of the function b. The boundedness of the operator Π_b has been studied extensively in the last 60 years.

1.1 The one dimensional case

If we consider the case $n = 1$ and a dyadic lattice $\mathcal{D}$ of intervals in $\mathbb{R}$ (or $[0, 1)$ for simplicity), then the operator Π_b is bounded in L^2 if one assumes the following "Carleson Box" condition:

$$\sum_{Q \in \mathcal{D},\, Q \subset P} \beta_Q^2 \leq C\,|P|, \qquad \forall P \in \mathcal{D} \tag{1.3}$$

The boundedness of Π_b is additionally equivalent with $b \in BMO^d(\mathbb{R}^n)$, where d stands for dyadic.

The one dimensional case is well-known and exhausted in every possible way. First, it appeared in the work of L. Carleson (see [8]) in the 60's and was used in complex interpolation and corona results. The underlying measure was the Lebesgue one. Much later, another proof emerged using the Bellman function method (see [25]) and in [2] an equivalent formulation of the problem was proven on a dyadic tree. Finally, in [26] Sawyer considered the weighted situation (with general measure ν) and used it to deal with weighted Calderón–Zygmund operators.

1.2 Higher dimensions

By a simple observation, the arguments above are verbatim the same in the case of $\mathbb{R}^n$, with $n > 1$, if we replace dyadic intervals with dyadic *cubes*. A cube is a Cartesian product of n dyadic intervals of the *same* length. The dyadic lattice produced by these cubes enjoys similar properties to the one-dimensional case; each cube has a single parent and it is covered only once by its 2^n descendants. A question appears here; *What happens if we replace cubes by rectangles?* That is, Cartesian products of dyadic intervals but not necessarily of the same length. In this case, there is much more overlap as each rectangle is covered $2^n - 1$ times by its closest descendants. Another notable difference is this; when $n = 1$ and for I, J dyadic intervals with $I \cap J \neq \varnothing$, we have $I \subseteq J$ or $J \subseteq I$. However, it is obvious this property is not necessarily true if $n > 2$ and I, J are dyadic n-rectangles. This is the setting of the multi-parameter theory.

In the case of higher dimensions one would ask whether the condition (1.3) is enough to imply the boundedness of the operator Π_b for $n = 2$ with the Lebesgue measure. It turns out, this is not true; a counterexample constructed by Carleson in [7] (see also [30]). Hence, the need for a new condition appeared: S.-Y. A. Chang in [10] found the necessary and sufficient condition for (1.1) to be valid:

$$\sum_{R \subseteq \Omega, R \in \mathcal{D}^2} \beta_R^2 \leq C\, m_2(\Omega) \qquad \forall \Omega \text{ dyadic open set}, \tag{1.4}$$

where a dyadic open set, Ω, is any finite union of dyadic rectangles.

Of course, the Chang-Carleson condition (1.4) is a strictly stronger requirement than the Carleson box condition (1.3) as shown by Carleson's counterexample. The same criterion of Chang works for $n > 2$, but again only for Lebesgue measure. Chang's criterion led to the understanding that multi-parameter BMO space, studied by Chang and R. Fefferman, is a much more subtle object than the "usual" BMO. The reader may guess that product BMO^d of Chang-Fefferman (see [11]) consists of the functions b whose Haar coefficients, β_R, satisfy (1.4).

Most of these developments happened in the 80's, and apart from Sawyer's work [27], there were no developments in weighted multi-parameter theory until recently. In particular, the case where Lebesgue measure is replaced by an arbitrary measure was left totally unsettled. The need to find a criterion for the boundedness of Π_b on L^2 for $n \geq 2$ and general measure was initiated by natural questions from several complex variables theory and especially questions about Carleson measures for a certain scale of Hilbert spaces of analytic functions on the bi-disc.

Namely, given $s = (s_1, s_2) \in \mathbb{R}^2$ consider the space $\mathcal{H}_s$ of analytic functions f on the unit bi-disc for which the norm

$$||f||_s^2 := \sum_{n_1, n_2 \geq 0} (n_1 + 1)^{s_1} (n_2 + 1)^{s_2} |\hat{f}(n_1, n_2)|^2$$

is finite. One then can ask for which measures μ the embedding $\mathcal{H}_s \to L^2(\mathbb{D}^2, \mu)$ is bounded. The Hardy space on the bi-disc, $\mathcal{H}_{(0,0)}$, corresponds to the Carleson-Chang-Fefferman case. It turns out that the special form of the second case, with arbitrary measure but without weight, describes the embedding of the Dirichlet space $\mathcal{H}_{(1,1)}$ on the bi-disc (see [4] and [6]). It is worth mentioning that this latter issue is always present when attempting to solve the corona problem in several variables.

The measure μ in [4] was general, as the question there is whether μ is a Carleson measure for the Dirichlet space $D(\mathcal{D}^2)$. For this reason, we change the constraints; instead of the Lebesgue measure and arbitrary weight, we consider an arbitrary measure and a weight w of *product* form. Such weight is the tensor product (coordinate-wise) of n positive functions in $\mathbb{R}$. We examine whether the analogue of the implication (1.3) $\implies$ (1.1) (for $p = q = 2$), still holds true:

$$\sum_{R \in \mathcal{D}^n} w_R \left(\int_R f d\nu \right)^2 \leq c \int_{[0,1)^n} f^2 d\nu, \qquad \forall f \in L^2([0,1)^n, \nu) \tag{1.5}$$

whenever

$$\sum_{P\subseteq R, P\in\mathcal{D}^n} w_P\, \nu(P)^2 \le \nu(R), \quad \forall R\in\mathcal{D}^n \tag{1.6}$$

where $\mathcal{D}^n$, as before, is the Cartesian product of n dyadic intervals which we assume to be subsets of $[0,1)$ (without loss of generality). Note that on the RHS of (1.6) we could have a constant c, which we re-normalize by considering the measure $\tilde{\nu} = \nu/c$, for simplicity.

We now explain why this condition is the correct candidate. For $n = 2$ and $w_P \equiv 1$ on $\mathcal{D}^2$, the authors in [4] were able to prove an equivalent relation to (1.5) on Bi-disc, if they assume a "General Capacitary condition", which we will describe later. However, one can avoid the usage of capacity, if another condition is being assumed (see [3]):

$$\sum_{P\subseteq \Omega, P\in\mathcal{D}^2} w_P\, \nu(P)^2 \le \nu(\Omega), \qquad \forall\Omega \text{ dyadic open set} \tag{1.7}$$

This condition is analogous to the one of S.-Y. A. Chang in [10].

Obviously, (1.7) is stronger than (1.6). Actually, without any restrictions on w, ν it is strictly stronger; L. Carleson in [7] for ν the 2-dimensional Lebesgue measure, constructed a particular sequence w_P such that the condition (1.6) holds, but for any constant C there exists a specific f satisfying opposite inequality of (1.5). The objects in this construction were dyadic 2-rectangles (not of the same side length necessarily). However, Carleson's sequence w_P was equal to $\frac{1}{m_2(P)}$ for certain rectangles and 0 otherwise. In [14] the authors tried to construct an analogous counterexample on bi-trees with $w_P \equiv 1$, however the only managed to construct a weight w with $w_P \in \{0,1\}$.

Nevertheless, as surprising as it seems, such a weight cannot exist. The main result presented in this thesis is this; if $w_P \equiv 1$ then, it is a product weight and for such weights the condition (1.6) implies (1.5). This was proven for $n = 2$ in [5] and for $n = 3$ in [19]. For higher dimensions the question is still open, as we highlight in section 7.

Note that in Carleson's counterexample the weight was *not* of product form. Indeed, the absence of a product weight can trigger several other counterexamples, in similar manner as in [7], see [21].

Next, we give an overview of how we approach the proof of the implication (1.6) $\implies$ (1.5) for $n = 2, 3$. Thanks to an approximation argument we can transfer the implication (1.6) $\implies$ (1.5) to an equivalent implication on finite multi-trees (also called n-trees). Such objects are the Cartesian product of n simple trees of the same finite depth. A finite simple tree $\mathcal{T}$ is a partially ordered set with the following property: For any $\omega \in \partial\mathcal{T}$ the set $\{\alpha : \alpha \geq \omega\}$ is totally ordered, where $\partial\mathcal{T}$ is the set of minimal elements of $\mathcal{T}$. Any element $\alpha \in \mathcal{T}$ is called a *node*. For simplicity we consider dyadic trees, i.e. simple trees with their nodes being dyadic intervals, but the results below can be proven for any partially ordered set with this property. If $\mathcal{T}$ is a finite dyadic tree of depth N, then $\partial\mathcal{T}$ consists of all intervals of side length 2^{-N}.

We continue by defining an n-tree. For $n \in \mathbb{N}$, an n-tree, denoted by $\mathcal{T}^n$ is Cartesian product of n simple trees $\mathcal{T}$ of depth N. All the constants below are independent of N. A measure μ on $\mathcal{T}^n$ is a positive function with domain $\partial\mathcal{T}^n = (\partial\mathcal{T})^n$ (Cartesian products of intervals of side length 2^{-N}). The Hardy operator on $\mathcal{T}^n$ is defined for $f : \mathcal{T}^n \to \mathbb{R}$ as

$$\mathbb{I}f(\beta) = \sum_{\alpha \geq \beta} f(\alpha) \tag{1.8}$$

and its adjoint by

$$\mathbb{I}^* f(\beta) = \sum_{\alpha \leq \beta} f(\alpha) \tag{1.9}$$

Our investigation revolves around the following. Let any $f \in \ell^2(\mathcal{T}^n, \mu)$. We want to check if

$$\sum_{\beta \in \mathcal{T}^n} w(\beta)\big(\mathbb{I}^*(f\mu)(\beta)\big)^2 \lesssim \int_{\mathcal{T}^n} f^2 d\mu \tag{1.10}$$

under the assumption

$$\sum_{\alpha \leq \beta} w(\alpha)\mathbb{I}^*\mu(\alpha)^2 \lesssim \mathbb{I}^*\mu(\beta), \quad \forall \beta \in \mathcal{T}^n \tag{1.11}$$

The smallest constant such the above inequalities hold are denoted by $[w,\mu]_{CE}$ and $[w,\mu]_{Box}$ respectively. The weight w is said to be of product form if for any $\alpha = \alpha_1 \times \cdots \times \alpha_n \in \mathcal{T}^n$ we have $w(\alpha) = w_1(\alpha_1)\cdots w_n(\alpha_n)$ where w_i are positive functions on the simple tree $\mathcal{T}$. The main theorem is this:

Theorem 1.1. *Let $n = 2, 3$ and μ be a measure on $\mathcal{T}^n$. Assume w is a product weight on $\mathcal{T}^n$. Then $[w,\mu]_{CE} \lesssim [w,\mu]_{Box}$*

Before proceeding with our investigation let us make a comment. As we said, the constant in (1.10) is independent of the depth N of the n-tree. Hence, we can easily show the implication (1.11) $\implies$ (1.10) gives the implication (1.6) $\implies$ (1.5). Suppose we have the condition (1.6) and let a positive step function $f \in L^2([0,1)^n, \nu)$. Then, there is N_0 large enough (which depends on f) such that for any fixed $N \geq N_0$, f is constant in each n-dimensional square of side 2^{-N}. These squares form the boundary of an n-tree of depth N. Thus, we can think of f as a function on $\mathcal{T}^n$ with domain $\partial\mathcal{T}^n$. Also we define a measure μ on $\partial\mathcal{T}^n$ by $\mathbb{I}^*\mu(\alpha) := \nu(\alpha)$ for $\alpha \in \mathcal{T}^n$. Additionally, $\mathbb{I}^*(f\mu)(\beta) = \int_\beta f d\nu$ for $\beta \in \mathcal{T}^n$. Hence, as the assumption (1.6) trivially implies (1.11) for the measure μ, and if we assume momentarily that Theorem 1.1 is true, we get (1.10). This becomes, given the definitions above:

$$\sum_{\beta \in \mathcal{T}^n} w(\beta)\big(\int_{\beta} f d\nu\big)^2 \leq C \sum_{\omega \in \partial \mathcal{T}^n} f(\omega)^2 \nu(\omega)$$

and the RHS of this inequality is equal than $\int_{[0,1)^n} f^2 d\nu$. Note on the left-hand we have a positive, increasing sequence (on N) which is also bounded. As the constant C is independent of N and this is true for any $N \geq N_0$, we get (1.5) for any positive step function. By a limiting argument we can extend this for any $f \in L^2([0,1)^n, \nu)$. Therefore the following equivalence holds:

$$\big((1.6) \implies (1.5)\big) \iff \big((1.11) \implies (1.10)\big)$$

Chapter 2

Definitions of different conditions

Apart from the box condition (1.11) other conditions play an important role. First, we have the “Carleson” condition, related to the work of Chang mentioned in the introduction. For any $\mathcal{D} \subseteq \mathcal{T}^n$ with $\mathcal{D}$ a down-set (a set of maximal nodes along with all of their descendants)

$$\sum_{\alpha \in \mathcal{D}} w(\alpha) \mathbb{I}^* \mu(\alpha)^2 \lesssim \mu(\mathcal{D})\,. \tag{2.1}$$

where $\mu(\mathcal{D}) = \sum_{\omega \in \mathcal{D} \cap \partial \mathcal{T}^n} \mu(\omega)$. The smallest constant such the above inequality holds is denoted by $[w, \mu]_C$. The Restricted energy (or Hereditary Carleson) condition holds if

$$\sum_{\alpha \in \mathcal{T}^n} w(\alpha) \big(\mathbb{I}^*(\mu \mathbf{1}_E)(\alpha)\big)^2 \lesssim |\mu \mathbf{1}_E| \tag{2.2}$$

where the latter is the total mass of the restriction of μ on E. The smallest constant such the above inequality holds is denoted by $[w, \mu]_{HC}$. In other words, a measure μ satisfies the Hereditary Carleson condition if for any set $E \subseteq \mathcal{T}^n$ the measure $\mu \mathbf{1}_E$ satisfies the Carleson condition (2.1). The reader might notice the following inequalities are obvious:

$$[w, \mu]_{Box} \leq [w, \mu]_C \leq [w, \mu]_{HC} \leq [w, \mu]_{CE} \tag{2.3}$$

As noted in the introduction, for a product weight w we also prove $[w, \mu]_{CE} \lesssim [w, \mu]_{Box}$ which makes the conditions above equivalent. More precisely we have the equivalence between the constants $[w,\mu]_C$ and $[w,\mu]_{Box}$, which even in the case $w\equiv 1$ is surprising: In the definition (2.1), the set $\mathcal{D}$ could be "wild" in the sense that any covering of $\mathcal{D}$ by dyadic rectangles can have huge overlap, similar to what happens in Carleson's counterexample. Even though this is true, still conditions (1.11) and (2.1) turned out to be equivalent.

The equivalence (2.3) was proven in [5] for $n=2$ and in [19] for $n=3$. Before this, for $n=2$, the inequality $[w,\mu]_{CE}\lesssim [w,\mu]_C$ was proven in [3]. The latter work tried to avoid the usage of *capacity*. For a set $E\subseteq \mathcal{T}^n$, the capacity $\mathrm{Cap}(E)$ is

$$\mathrm{Cap}(E)=\inf_{\varphi}\left\{\|\varphi\|^2_{\ell^2(\mathcal{T}^n)}:\mathbb{I}\varphi(\alpha)\geq 1,\forall\alpha\in E\right\}\tag{2.4}$$

For $n=2$ is was proven in [4] that the embedding (1.5) holds under the assumption

$$\nu(E)\lesssim \mathrm{Cap}(E)\tag{2.5}$$

where E is any subset of $\partial\mathcal{T}^n$. Almost trivially it follows that the embedding implies (2.5). In subsection 4.1 we show the same is true for $n=3$.

If in the definition above we take $E=\{R\}$ with $R=I_1\times\cdots\times I_n$ a dyadic n-rectangle then $\mathrm{Cap}(R)=\frac{1}{(\log_2\frac{1}{|I_1|}+1)\cdots(\log_2\frac{1}{|I_n|}+1)}$. To see this, as noted in Section 2.3 of [4] the minimizer in (2.4) is the unique *equilibrium measure* μ_E of E. This measure satisfies $\mathrm{Cap}(E)=\int_{\mathcal{T}^n}\left(\mathbb{I}^*\mu_E\right)^2$. By Lemma 5.6 of [4], for any other measure μ satisfying $\mathbb{V}^\mu:=\mathbb{I}\mathbb{I}^*\mu\leq 1$ inside E, we have $\mu(E)\leq\mathrm{Cap}(E)$. Let us construct such a measure μ and show the opposite inequality as well, in the case $E=\{R\}$. Suppose $R:=I_1\times\cdots\times I_n$ and k_i are such that $|I_i|=2^{-k_i}$. Then for each $\omega\leq R$, $\omega\in\partial\mathcal{T}^n$ we define $\mu(\omega)=\frac{1}{(k_1+1)\cdots(k_n+1)}\cdot\frac{1}{2^{N-k_1}\cdots 2^{N-k_n}}$ and $\mu(\omega)=0$ if $\omega\not\leq R$. For any $\alpha\geq R$ we set $\varphi(\alpha):=\mathbb{I}^*\mu(\alpha)$ and $\varphi(\alpha)=0$ for $\alpha\not\geq R$. Hence, for any $\alpha\geq R$ we have $\varphi(\alpha)=\frac{1}{(k_1+1)\cdots(k_n+1)}$ and so $\mathbb{I}\varphi(\alpha)=1$, which also means $\mathbb{V}^{\mu}(\alpha) = 1$. Using the result above, we get $\mu(E) \leq \text{Cap}(E)$. Finally, we get $\text{Cap}(E) \leq \int_{\mathcal{T}^n} \varphi^2 = \frac{1}{(k_1+1)\cdots(k_n+1)} = \frac{1}{(\log_2 \frac{1}{|I_1|}+1)\cdots(\log_2 \frac{1}{|I_n|}+1)} = \mu(E)$. Therefore the measure μ is the minimizer of the set $\{R\}$. Our final condition, is called the "box-capacitary condition":

$$\nu(R) \lesssim \frac{1}{(\log_2 \frac{1}{|I_1|} + 1) \cdots (\log_2 \frac{1}{|I_n|} + 1)} \tag{2.6}$$

As we just proved, the RHS is equal to $\text{Cap}(R)$. This condition is the subject of subsection 4.2.

Chapter 3

The proof of Theorem 1.1

Before proving Theorem 1.1 is true we start with the known case of $n = 1$. We give a new proof of the equivalence between (1.5) and (1.11) for the case $n = 1$, using Schur's test. For this idea we are grateful to M. Christ. As we mentioned in the introduction, this result is long-known, proven first in [8], then in [27] and later in [25] and [2] (in the case of a simple tree). The sequence w_I can be arbitrary but as we stated in Theorem 1.1, for $n = 2, 3$ our weight must be of product form. Later, we inspect non-product weights for which the embedding holds (subsection 6.1) and does not hold (section 6).

3.1 The known case of $n = 1$, a new proof

For this section we change our notation a little. Let ν be any measure in $[0, 1)$ and w be any positive weight with domain $\mathcal{D}$, the collection of all dyadic intervals in $[0, 1)$. Also, let $\mathcal{T}$ be a simple tree of finite depth N. For a function $f \in L^2\big([0, 1), \nu\big)$ and a $\beta \in \mathcal{T}$ we define the operator $\mathbb{J}$ as

$$\mathbb{J}f(\beta) = \int_\beta f d\nu$$

As we said before, we are interested to see whether $\mathbb{J}$ is bounded from $L^2\big([0, 1), \nu\big)$ to $\ell^2(\mathcal{T}, w)$, with a norm independent of N, under the (normalized) assumption:

$$\sum_{\alpha \leq \beta} w(\alpha) \cdot \nu(\alpha)^2 \leq \nu(\beta), \quad \forall \beta \in \mathcal{T} \tag{3.1}$$

Note that if we show this for any finite simple tree $\mathcal{T}$ then the implication (1.6) $\implies$ (1.5) is also true by letting $N \to \infty$. To draw a direct connection with the other setting, note that $\|\mathbb{J}\|^2_{L^2([0,1),\nu) \to \ell^2(\mathcal{T},w)} = [w, \mu]_{CE}$ where μ is defined such that $\nu = \mathbb{I}^* \mu$ on $\mathcal{T}$. For a function g on $\mathcal{T}$ and $x \in [0,1)$, its formal adjoint operator is given by

$$\mathbb{J}^*(g)(x) = \sum_{\beta \in \mathcal{T}} (wg)(\beta) \cdot \mathbf{1}_\beta(x)$$

We will show $\mathbb{J} : L^2([0,1),\nu) \to \ell^2(\mathcal{T}, w)$ is bounded by using Schur's test:

Theorem 3.1. ***Schur's Test.*** *Let $1 < p < \infty$ and μ, σ be σ-finite measures on the measurable spaces X, Y resp. Suppose K is an operator of the form $Kf(x) = \int_Y k(x,y) f(y)\, d\mu(y)$, where $k : X \times Y \to \mathbb{R}$ is measurable and non-negative. Then $K : L^p(Y,\mu) \to L^p(X,\sigma)$ is bounded if there are functions u, v with $0 < u < \infty$ for a.e. $x \in X$ and $0 < v < \infty$ for a.e. $y \in Y$ such that*

$$K(v^{p'})(x) \lesssim u^{p'}(x) \quad a.e.\ x \in X$$

and

$$K^*(u^p)(y) \lesssim v^p(y) \quad a.e.\ y \in Y$$

Where K^* is the formal adjoint of K, i.e. the operator $K^*g(y) = \int_X k(x,y) g(x)\, d\sigma(x)$.

We will use this theorem for $p = 2$. We take $K = \mathbb{J}$, $X = \mathcal{T}$, $Y = [0, 1)$ and $\mu = \nu$ and $\sigma = w$ (weighted counting measure on $\mathcal{T}$). Note that $k(\beta, x) = \mathbf{1}_\beta(x)$. For simplicity let $S := \mathbb{J}\mathbb{J}^*$ and note that for $\beta \in \mathcal{T}$:

$$S(g)(\beta) = \sum_{\alpha \in \mathcal{T}} (wg)(\alpha) \cdot \nu(\alpha \wedge \beta)$$

Our goal is to construct an auxiliary function F such that $\mathbb{J}(\mathbb{J}^*(F)) \lesssim F$. Then we will take $u^2 := F$ and $v^2 := \mathbb{J}^*(F)$. To construct such F we use a recursive argument, starting with the function $f_0(\beta) = \nu(\beta)$ for $\beta \in \mathcal{T}$. Then we have:

$$\begin{aligned} S(f_0)(\beta) &= \sum_{\alpha \in \mathcal{T}} (wf_0)(\alpha)\nu(\alpha \wedge \beta) \\ &= \sum_{\alpha \in \mathcal{T}} (w\nu)(\alpha) \cdot \nu(\alpha \wedge \beta) \\ &= \left(\sum_{\alpha > \beta} + \sum_{\alpha \leq \beta} \right) (w\nu)(\alpha) \cdot \nu(\alpha \wedge \beta) \\ &:= f_1(\beta) + \sum_{\alpha \leq \beta} w(\alpha) \cdot \nu(\alpha)^2 \\ &\leq f_1(\beta) + f_0(\beta) \end{aligned}$$

using our assumption. Moreover, for any $i \in \{1, .., m\}$, m to be specified later, we define the functions f_i by the formula

$$f_i(\beta) := \nu(\beta) \cdot \sum_{\alpha > \beta} (wf_{i-1})(\alpha) \tag{3.2}$$

and we will prove

the following recursive formula

$$S(f_i)(\beta) \leq f_{i+1}(\beta) + S(f_{i-1})(\beta) \tag{3.3}$$

To see this we have (as before)

$$S(f_i)(\beta) = f_{i+1}(\beta) + \sum_{\gamma\leq\beta}(wf_i)(\gamma)\cdot\nu(\gamma)$$

and the second term is estimated as follows:

$$\begin{aligned}
&\sum_{\gamma\leq\beta}(wf_i)(\gamma)\cdot\nu(\gamma)\\
&=\sum_{\gamma\leq\beta}(w\nu)(\gamma)\cdot\sum_{\alpha>\gamma}(wf_{i-1})(\alpha)\cdot\nu(\gamma)\\
&=\sum_{\gamma\leq\beta}\sum_{\alpha>\gamma}w(\gamma)\cdot\nu(\gamma)^2\cdot(wf_{i-1})(\alpha)\\
&=\sum_{\gamma\leq\beta}\left(\sum_{\gamma<\alpha\leq\beta}+\sum_{\alpha>\beta}\right)w(\gamma)\cdot\nu(\gamma)^2\cdot(wf_{i-1})(\alpha)
\end{aligned}$$

Now we change the order of summation for both sums and we use our assumption to get:

$$\begin{aligned}
&\sum_{\gamma\leq\beta}\left(\sum_{\gamma<\alpha\leq\beta}+\sum_{\alpha>\beta}\right)w(\gamma)\cdot\nu(\gamma)^2\cdot(wf_{i-1})(\alpha)\\
&=\left(\sum_{\alpha\leq\beta}\sum_{\gamma<\alpha}+\sum_{\alpha>\beta}\sum_{\gamma\leq\beta}\right)w(\gamma)\cdot\nu(\gamma)^2\cdot(wf_{i-1})(\alpha)\\
&\leq\sum_{\alpha\leq\beta}\nu(\alpha)\cdot(wf_{i-1})(\alpha)+\sum_{\alpha>\beta}\nu(\beta)\cdot(wf_{i-1})(\alpha)
\end{aligned}$$

which is equal to

$$
\begin{aligned}
&= \sum_{\alpha \leq \beta} \nu(\alpha \wedge \beta) \cdot (w f_{i-1})(\alpha) + \sum_{\alpha > \beta} \nu(\alpha \wedge \beta) \cdot (w f_{i-1})(\alpha) \\
&= S(f_{i-1})(\beta)
\end{aligned}
$$

which proves (3.3). Now recall that $S(f_0) \leq f_1(\beta) + f_0(\beta)$ and by using (3.3) recursively, we get for every m

$$
S(f_m)(\beta) \leq \sum_{i=0}^{m+1} f_i(\beta) \tag{3.4}
$$

Now we look at the definition of f_i in (3.2). It is defined as a succession of sums, and each such sum is over nodes which are strictly bigger than those of the previous sum. Since the tree $\mathcal{T}$ has finite depth N, there is some $M \leq N$ such that $f_m \equiv 0$ for $m > M$. For minimal such M we define

$$
F(\beta) := \sum_{m=0}^{M} \frac{f_m(\beta)}{2^m} \tag{3.5}
$$

Lets prove that $S(F)(\beta) \lesssim F(\beta)$ using (3.4):

$$
\begin{aligned}
S(F)(\beta) &= \sum_{m=0}^{M} \frac{S f_m(\beta)}{2^m} \\
&\leq \sum_{m=0}^{M} \sum_{i=0}^{m+1} \frac{f_i(\beta)}{2^m} \\
&= \sum_{i=0}^{M} \sum_{m=i-1}^{M} \frac{f_i(\beta)}{2^m} \\
&\leq 4 \sum_{i=0}^{M} \frac{f_i(\beta)}{2^i} = 4F(\beta)
\end{aligned}
$$

Remark 3.2. Here we proved the one-dimensional result for $p = 2$. In addition, the result is true for general $1 < p < \infty$, see for example [25]. The proof there uses the Bellman function method and although the authors take $p = 2$, the same is true for any $1 < p < \infty$ with a slight modification. By generalizing this Bellman function appropriately, [9] proves the same result on dyadic trees.

Remark 3.3. From the proof of Schur's test we can see that $\|\mathbb{J}\|^2_{L^2([0,1),\nu)\to\ell^2(\mathcal{T},w)} \leq 4$. As it is known, this is the sharp constant, see [25]. It is also known that the squared norm of the Maximal function operator is at most 4. For a connection between the Carleson embedding theorem and the Maximal function, see subsection 5.1.

Remark 3.4. Using the inequality $S(F) \lesssim F$ on $\mathcal{T}$ and modifying Schur's test appropriately, we see that $S : \ell^2(T, w) \to \ell^2(T, w)$ is bounded. However, since $\ell^2(T, w)$ is a Hilbert space, the boundedness of S is equivalent with the boundedness of $\mathbb{J} : L^2([0, 1), \nu) \to \ell^2(T, w)$ and moreover, $\|S\|_{\ell^2(\mathcal{T},w)\to\ell^2(\mathcal{T},w)} = \|\mathbb{J}\|^2_{L^2([0,1),\nu)\to\ell^2(\mathcal{T},w)}$.

3.2 The case of higher dimensions - Potential theory on n-tree

For this section we deploy the techniques first presented in [4]. The main tool there is capacity on a bi-tree. A capacity-free theory is developed in [3] and later evolved even more in [5] and [19]. Our presentation is based on the latter.

Recall the operators $\mathbb{I}, \mathbb{I}^*$ from the beginning of the section. This will be the notation for $n = 2$ and for general n too. Specifically, for $n = 3$ we use the symbols $\mathbb{I}$ and $\mathbb{I}^*$. The corresponding operators for dimension $n = 1$ are denoted by I, I^*. Note that for $n = 2$ the operator $\mathbb{I}$ can be written as $\mathbb{I} = I_1 I_2$ and for $n = 3$ we have $\mathbb{I} = I_1 I_2 I_3$. We fix positive a product weight w on $\mathcal{T}^n$ and we define the potential $\mathbb{V}^\mu$ as

$$\mathbb{V}^\mu := \mathbb{I}(w\mathbb{I}^*\mu)$$

(the symbol for the potentials is the same for any dimension) and the energy $\mathcal{E}[\mu]$ as

$$\mathcal{E}[\mu] := \sum_{\alpha\in\mathcal{T}^n} w(\alpha)\mathbb{I}^*\mu(\alpha)^2 = \int_{\mathcal{T}^n} \mathbb{V}^\mu d\mu$$

Moreover, for $\delta > 0$ we define the truncated potential and energy

$$\mathbb{V}^\mu_\delta := \mathbb{I}(\mathbf{1}_{\mathbb{V}^\mu\leq\delta} w\mathbb{I}^*\mu), \quad \mathcal{E}_\delta[\mu] := \int_{\mathcal{T}^n} \mathbb{V}^\mu_\delta d\mu = \sum_{\alpha\in\{\mathbb{V}^\mu\leq\delta\}} w(\alpha)\mathbb{I}^*\mu(\alpha)^2$$

The following is the main result of this section. As we already mentioned the maximum principle fails. However, this quasi- maximum principle holds:

Theorem 3.5. *Let $n = 1, 2, 3$. Let $\mathcal{T}^n$ be an n-tree and μ, ρ positive measures on $\mathcal{T}^n$. Then the following is true for $r = \frac{1}{n}$:*

$$\int_{\mathcal{T}^n} \mathbb{V}^\mu_\delta d\rho \lesssim (\delta|\rho|)^r \Big(\mathcal{E}_\delta[\mu]\mathcal{E}[\rho]\Big)^{\frac{1-r}{2}}$$

By taking $\mu = \rho$ we get

Corollary 3.6. *Let $n = 1, 2, 3$. Let $\mathcal{T}^n$ be an n-tree and μ positive measure on $\mathcal{T}^n$. Then the following is true for $r = \frac{1}{n}$:*

$$\mathcal{E}_\delta[\mu] = \int_{\mathcal{T}^n} \mathbb{V}^\mu_\delta d\mu \lesssim (\delta|\mu|)^{\frac{2r}{1+r}} \mathcal{E}[\mu]^{\frac{1-r}{1+r}}$$

For $n = 1$ the proof of Theorem 3.5 is immediate, as on a simple tree we have $\mathbb{V}_\delta^\mu \leq \delta$ (maximum principle). By using Cauchy-Schwarz inequality we can prove for a positive measure ρ on $\mathcal{T}$ and every $r \in [0, 1]$ that

$$\int_{\mathcal{T}} \mathbb{V}_\delta^\mu d\rho \leq \delta^r |\rho|^r \mathcal{E}_\delta[\mu]^{\frac{1-r}{2}} \mathcal{E}[\rho]^{\frac{1-r}{2}}$$

where $|\rho|$ is the total mass of ρ on the simple tree $\mathcal{T}$. However, the maximum principle fails on the n-tree, for $n \geq 2$ (see section 6.4). Before we give the proof of this theorem, lets see how we can use it in order to get the desired embedding. This part of the proof is purely of combinatorial nature.

3.2.1 Box condition implies Hereditary Carleson for $n = 2, 3$

We start with two definitions

$$\mathbb{V}_\alpha^\mu(\omega) := \sum_{\beta:\omega \leq \beta \leq \alpha} w(\beta)\mathbb{I}^*\mu(\beta), \tag{3.6}$$

$$\mathbb{V}_{\epsilon', good}^\mu(\omega) := \sum_{\alpha \geq \omega: \mathbb{V}_\alpha(\omega) > \epsilon'} w(\alpha)\mathbb{I}^*\mu(\alpha). \tag{3.7}$$

The next results work for any $n \geq 2$, as long as Theorem 3.5 is true. However, we are only able to prove it in the case $n = 2, 3$. Hence, one needs to prove 3.5 for any $n \geq 4$ and this would be sufficient for settling the embedding (more in section 7).

Lemma 3.7. *Let $n \geq 2$ and $\mu : \mathcal{T}^n \to [0, \infty)$. Let $w : \mathcal{T}^n \to [0, \infty)$ be a product weight and assume Theorem 3.5 holds for this n. Suppose that $\mathcal{E}[\mu] \leq |\mu|$ and*

$$\mathbb{V}^\mu \geq 1/3 \quad \textit{on } \operatorname{supp} \mu. \tag{3.8}$$

Then, if ϵ' is small enough, we have

$$\int \mathbb{V}^{\mu}_{\epsilon',good} d\mu \gtrsim |\mu|.$$

Proof. It suffices to show that, for some ϵ' and ϵ_{n-1}, we have

$$\mu\{\omega \in \mathcal{T}^n \mid \mathbb{V}^{\mu}_{\epsilon',good}(\omega) \geq \epsilon_{n-1}\} \geq |\mu|/2.$$

Let $\epsilon > 0$ be chosen later and define

$$\epsilon_1 := \epsilon, \quad \epsilon_2 := \epsilon\epsilon_1^{1/\kappa}, \quad \epsilon_3 := \epsilon\epsilon_2^{1/\kappa}, \ldots$$

where $\kappa = \frac{2r}{1+r}$, $r = 1/n$. By corollary 3.6, we have

$$\int \mathbb{V}^{\mu}_{\epsilon_j} d\mu \lesssim \epsilon_j^{\kappa} |\mu|^{\kappa} \mathcal{E}[\mu]^{1-\kappa} \lesssim \epsilon_j^{\kappa} \int d\mu$$

By Chebyshov's inequality, it follows that

$$\mathbb{V}^{\mu}_{\epsilon_j}(\omega) \leq (\epsilon_j/\epsilon)^{\kappa}/10 \tag{3.9}$$

for a μ- proportion $\geq (1 - C\epsilon^{\kappa})$ of ω's. So we only consider ω's for which (3.9) holds for all $j = 1, \ldots, n-1$. Similarly, we may restrict to those ω's for which $\mathbb{V}^{\mu}(\omega) \lesssim 1$.

Let

$$\epsilon' := \epsilon \cdot \epsilon_1 \cdots \epsilon_{n-1}.$$

For a fixed ω, let

$$\mathcal{U} := \{\alpha \geq \omega \mid \mathbb{V}^{\mu}_{\alpha}(\omega) > \epsilon'\} \tag{3.10}$$

and

$$\mathcal{W}_j := \{\alpha \geq \omega \mid \mathbb{V}^{\mu}(\alpha) \leq \epsilon_j\}, \quad 1 \leq j \leq n-1. \tag{3.11}$$

Note these sets are decreasing in j. For $p \in T^n$, write

$$\uparrow p := \{\alpha \in \mathcal{T}^n \mid \alpha \geq p\}.$$

For $p \in \uparrow \omega$, let

$$\downarrow p := \{\alpha \in \mathcal{T}^n \mid \omega \leq \alpha \leq p\}.$$

If $\mathcal{U} \not\subseteq \mathcal{W}_{n-1}$, then this means that there exists $p \notin \mathcal{W}_{n-1}$ with $\uparrow p \subseteq \mathcal{U}$. Hence,

$$\mathbb{V}^{\mu}_{\epsilon', good}(\omega) \geq \sum_{p' \in \uparrow p} (w\mathbb{I}^* \mu)(p') = \mathbb{V}^{\mu}(p) \geq \epsilon_{n-1}.$$

Assume now that $\mathcal{U} \subseteq \mathcal{W}_{n-1}$. In this case, we will cover $\uparrow \omega \setminus \mathcal{W}_1$ by boundedly many sets of the form $\downarrow q$ with $q \in \uparrow \omega \setminus \mathcal{U}$. This will lead to a contradiction with (3.8), since, by (3.9) and (3.10), the integral of

$$f := w\mathbb{I}^* \mu$$

is small on $\mathcal{W}_1$ and on each such set $\downarrow q$.

For a set of coordinates $J \subseteq \{1, \ldots, n\}$ and a point $p \in \mathcal{T}^n$, let

$$\uparrow_J p := \{q \in \mathcal{T}^n \mid q_j \geq p_j \text{ for } j \in J,\ q_j = p_j \text{ for } j \notin J\}.$$

Given $J \subseteq \{1, \ldots, n\}$ with $J \neq \emptyset$ and $p \in \mathcal{T}^n$, we define a set $\mathcal{Q}_J(p) \subset \mathcal{T}^n$ as follows. If $|J| = 1$, then $\mathcal{Q}_J(p)$ consists of the (unique) maximal element of $\uparrow_J p \setminus \mathcal{U}$, if the latter set is nonempty, and is empty otherwise. If $|J| \geq 2$, then $\mathcal{Q}_J(p)$ is a maximal set of maximal elements of $\uparrow_J p \setminus \mathcal{W}_{n-|J|+1}$ such that the sets $\uparrow_J q \setminus \mathcal{W}_{n-|J|+2}$ are pairwise disjoint for $q \in \mathcal{Q}_J(p)$.

Then, recursively, let $\mathcal{R}_\emptyset(p) := \{p\}$,

$$\mathcal{R}_J(p) := \cup_{J' \subset J} \cup_{p' \in \mathcal{Q}_J(p)} \mathcal{R}_{J'}(p'),$$

where the first union runs ovel all subsets of J with cardinality $|J'| = |J| - 1$.

We claim that, for every $p \in \uparrow \omega$ and every $J \subseteq \{1, \ldots, n\}$ with $J \neq \emptyset$, we have

$$\bigcup_{p' \in \mathcal{R}_J(p)} \downarrow p' \supseteq \uparrow_J p \setminus \mathcal{W}_{n-|J|+1}, \tag{3.12}$$

where we set $\mathcal{W}_n := \mathcal{U}$ to simplify notation. Relation (3.12) is our main combinatorial statement and we prove it by induction on $|J|$. For $|J| = 1$, the claim (3.12) obviously holds. Let now J with $|J| \geq 2$ be given, and suppose that (3.12) is known for all proper subsets of J. Let

$$\mathcal{D} := \bigcup_{p' \in \mathcal{R}_J(p)} \downarrow p', \quad \mathcal{P} := \uparrow_J p \setminus \mathcal{W}_{n-|J|+1}.$$

By the inductive hypothesis,

$$\mathcal{D} \supseteq \uparrow_{J'} p' \setminus \mathcal{W}_{n-|J|+2} \tag{3.13}$$

for every $p' \in Q_J(p)$ and every $J' \subsetneq J$. Suppose that

$$\mathcal{D} \not\supseteq \mathcal{P}. \tag{3.14}$$

Choose a maximal $q \in \mathcal{P} \setminus \mathcal{D}$. Since $\mathcal{D}$ is a down-set, q is also a maximal element of $\mathcal{P}$. We claim that

$$(\uparrow_J q \cap \uparrow_J p') \setminus \mathcal{W}_{n-|J|+2} = \emptyset \text{ for all } p' \in \mathcal{Q}_J(p). \tag{3.15}$$

Indeed, suppose for a contradiction that there exists $q' \in (\uparrow_J q \cap \uparrow_J p') \setminus \mathcal{W}_{n-|J|+2}$, and let q' be minimal with this property. Since $\mathcal{W}_{n-|J|+2}$ is an up-set, q' is also a minimal element of $\uparrow_J q \cap \uparrow_J p'$. Since $q, p' \in \uparrow_J p$ then q' is in fact the coordinatewise maximum of q, p'. Since q and p' are distinct maximal elements of $\mathcal{P}$, in fact q' coincides with p' in at least one coordinate, so $q' \in \uparrow_{J'} p'$ for some $J' \subsetneq J$. Now, (3.13) implies that $q' \in \mathcal{D}$, and, since $\mathcal{D}$ is a down-set and $q' \geq q$, also $q \in \mathcal{D}$, a contradiction.

Therefore, (3.15) holds. But this contradicts the maximality of $\mathcal{Q}_J(p)$. Thus the assumption (3.14) is false, and we obtain (3.12).

Let $p \geq \omega$. For $2 \leq |J| \leq n$, we have

$$\begin{aligned}
1 &\gtrsim \mathbb{V}^{\mu}(\omega) \\
&\geq \mathbb{V}^{\mu}(p) \\
&\geq \sum_{q \in \mathcal{Q}_J(p)} \int_{\uparrow_J q \setminus \mathcal{W}_{n-|J|+2}} f \\
&\geq \sum_{q \in \mathcal{Q}_J(p)} (\mathbb{I}f(q) - \mathbb{I}(f \mathbf{1}_{\mathcal{W}_{n-|J|+2}})(\omega))
\end{aligned}$$

by definition, as $q \notin \mathcal{W}_{n-|J|+1}$ and by (3.9),

$$\geq \sum_{q \in \mathcal{Q}_J(p)} (\epsilon_{n-|J|+1} - (\epsilon_{n-|J|+2}/\epsilon)^{\kappa}/10)$$

$$\gtrsim |\mathcal{Q}_J(p)| \epsilon_{n-|J|+1}.$$

We multiply those inequalities to get:

$$\epsilon_1 \cdots \epsilon_{n-1} |\mathcal{R}_{\{1,\ldots,n\}}(\omega)| \lesssim 1.$$

Hence, by (3.12),

$$\begin{aligned}
\mathbb{V}^{\mu}(\omega) - \mathbb{V}^{\mu}_{\epsilon_1}(\omega) &= \int_{\uparrow\omega \setminus \mathcal{W}_1} f \\
&\leq \sum_{p' \in \mathcal{R}_{\{1,\ldots,n\}}(\omega)} \int_{\downarrow p'} f \\
&= \sum_{p' \in \mathcal{R}_{\{1,\ldots,n\}}(\omega)} \mathbb{V}^{\mu}_{p'}(\omega) \\
&\leq \epsilon' |\mathcal{R}_{\{1,\ldots,n\}}(\omega)| \\
&\lesssim \frac{\epsilon'}{\epsilon_1 \cdots \epsilon_{n-1}} = \epsilon.
\end{aligned}$$

Therefore, by (3.9),

$$1/3 \leq \mathbb{V}^{\mu}(\omega) = (\mathbb{V}^{\mu}(\omega) - \mathbb{V}^{\mu}_{\epsilon_1}(\omega)) + \mathbb{V}^{\mu}_{\epsilon_1}(\omega) \leq C\epsilon + 1/10.$$

This inequality is false if ϵ is sufficiently small, contradicting the assumption $\mathcal{U} \subseteq \mathcal{W}_{n-1}$. □

The following lemma [3, Lemma 3.1] is in place.

Lemma 3.8 (Balancing lemma)**.** *Let $\mu : \mathcal{T}^n \to [0, \infty)$ with*

$$\mathcal{E}[\mu] = \int \mathbb{V}^{\mu} d\mu \geq A|\mu|.$$

Then there exists a down-set $\tilde{E} \subset T^n$ such that for the measure $\tilde{\mu} := \mu \mathbf{1}_{\tilde{E}}$ we have

$$\mathbb{V}^{\tilde{\mu}} \geq \frac{A}{3} \quad \text{on } \tilde{E},$$

and

$$\mathcal{E}[\tilde{\mu}] \geq \frac{1}{3}\mathcal{E}[\mu].$$

With this at hand we complete the proof the implication "Box to Hereditary Carleson".

Theorem 3.9. *Let $n \geq 2$ and μ be a measure on $\mathcal{T}^n$. Let $w : \mathcal{T}^n \to [0, \infty)$ be a product weight and assume Theorem 3.5 holds for this n. Then,*

$$[w, \mu]_{HC} \lesssim [w, \mu]_{Box}.$$

Proof. By scaling, we may assume $[w, \mu]_{Box} = 1$ without loss of generality. Let $A := [w, \mu]_{HC}$. We will show A is bounded by an absolute constant. We start with $E \subset \mathcal{T}^n$ be a subset such that $\mu_E = \mu \mathbf{1}_E \neq 0$ and $\mathcal{E}[\mu_E] = A|\mu_E|$ (such a subset exists because we assume that $\mathcal{T}^n$ is finite). By Lemma 3.8, there exists a further subset $\tilde{E} \subset \mathcal{T}^n$ such that $\tilde{\mu_E} := \mu_E \mathbf{1}_{\tilde{E}}$ satisfies

$$\mathbb{V}^{\tilde{\mu}} \geq \frac{A}{3} \text{ on } \tilde{E}$$

and $\tilde{\mu}_E \neq 0$. Thus, replacing μ_E by $\tilde{\mu_E}$, we may assume $\mathbb{V}^{\mu_E} \geq A/3$ on $\operatorname{supp} \mu_E$.

By Lemma 3.7 applied with $\frac{\mu_E}{A}$ in place of μ, for sufficiently small $\epsilon, \theta > 0$, we have

$$\int \mathbb{V}^{\mu_E}_{\epsilon A, good} d\mu_E \geq 2\theta \mathcal{E}[\mu_E]. \tag{3.16}$$

We claim that, with these values of ϵ and θ, we have

$$\mathcal{E}[\mu_E] \leq \frac{\theta}{1-\theta} \sum_{\alpha:\theta\epsilon A \mathbb{I}^*\mu_E(\alpha) \leq \mathcal{E}_\alpha[\mu_E]} w(\alpha)(\mathbb{I}^*\mu_E(\alpha))^2. \tag{3.17}$$

Indeed, suppose that α is such that

$$\theta\epsilon A \mathbb{I}^*\mu_E(\alpha) > \mathcal{E}_\alpha[\mu_E] = \sum_{\omega \leq \alpha} \mu_E(\omega)\mathbb{V}^{\mu_E}_\alpha(\omega), \quad \mathbb{V}^{\mu_E}_\alpha(\omega) = \sum_{\beta:\omega\leq\beta\leq\alpha} w(\beta)(\mathbb{I}^*\mu_E)(\beta),$$

where the latter definition is from (3.6). Then we have

$$\begin{aligned}\sum_{\omega\leq\alpha:\mathbb{V}^{\mu_E}_\alpha(\omega)\leq\epsilon A} \mu_E(\omega) &= \mathbb{I}^*\mu_E(\alpha) - \sum_{\omega\leq\alpha:\mathbb{V}^{\mu_E}_\alpha(\omega)>\epsilon A} \mu_E(\omega) \\ &\geq \mathbb{I}^*\mu_E(\alpha) - \frac{1}{\epsilon A}\sum_{\omega\leq\alpha} \mathbb{V}^{\mu_E}_\alpha(\omega)\mu_E(\omega) \\ &\geq (1-\theta)\mathbb{I}^*\mu_E(\alpha).\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{\alpha:\theta\epsilon A\mathbb{I}^*\mu_E(\alpha)>\mathcal{E}_\alpha[\mu_E]} w(\alpha)(\mathbb{I}^*\mu_E(\alpha))^2 &\leq \sum_\alpha w(\alpha)\mathbb{I}^*\mu_E(\alpha)\frac{1}{1-\theta}\sum_{\omega\leq\alpha:\mathbb{V}^{\mu_E}_\alpha(\omega)\leq\epsilon A}\mu_E(\omega) \\ &= \frac{1}{1-\theta}\sum_\omega \mu_E(\omega)\sum_{\alpha\geq\omega:\mathbb{V}^{\mu_E}_\alpha(\omega)\leq\epsilon A} w(\alpha)\mathbb{I}^*\mu_E(\alpha)\end{aligned}$$

which is equal to

$$
\begin{aligned}
&= \frac{1}{1-\theta} \sum_{\omega} \mu_E(\omega)(\mathbb{V}^{\mu_E} - \mathbb{V}^{\mu_E}_{good,\epsilon A})(\omega) \\
&\leq \frac{1-2\theta}{1-\theta} \mathcal{E}[\mu_E].
\end{aligned}
$$

This implies the claim (3.17).

By corollary 3.6 again, and since $\mathbb{V}^{\mu_E} \geq A/4$ on $\operatorname{supp} \mu_E$, we also have

$$
\mathcal{E}_{c'A}[\mu_E] \lesssim (c'A)^{\kappa} |\mu_E|^{\kappa} \mathcal{E}[\mu_E]^{1-\kappa} \lesssim (c')^{\kappa} \mathcal{E}[\mu_E]. \tag{3.18}
$$

Taking c' sufficiently small and combining (3.18) with (3.17), we obtain

$$
\mathcal{E}[\mu_E] \lesssim \sum_{\alpha \in \mathcal{R}} w(\alpha)(\mathbb{I}^* \mu_E(\alpha))^2, \quad \mathcal{R} := \{\alpha \in \mathcal{T}^n \mid \theta\epsilon A \mathbb{I}^* \mu_E(\alpha) \leq \mathcal{E}_\alpha[\mu_E], \mathbb{V}^{\mu_E}(\alpha) \geq c'A\}.
$$

For each $\alpha \in \mathcal{R}$, we have

$$
\theta\epsilon A \mathbb{I}^* \mu_E(\alpha) \leq \mathcal{E}_\alpha[\mu_E] \leq \mathcal{E}_\alpha[\nu] \leq [w, \nu]_{Box} \mathbb{I}^* \nu(\alpha) = \mathbb{I}^* \sigma(\alpha),
$$

where $\sigma := \nu \mathbf{1}_F$, $F := \{\beta \in T^n \mid \exists \alpha \in \mathcal{R}, \alpha \geq \beta\}$. It follows that

$$
A^2 \mathcal{E}[\mu_E] \lesssim \mathcal{E}[\sigma]. \tag{3.19}
$$

On the other hand, using the definition of A, the fact that $\mathbb{V}^{\mu_E} \gtrsim A$ on $\operatorname{supp} \sigma$, and the

Cauchy–Schwarz inequality, we obtain

$$\mathcal{E}[\sigma] \le A|\sigma| \lesssim \int \mathbb{V}^{\mu_E} d\sigma \le \mathcal{E}[\mu_E]^{1/2}\mathcal{E}[\sigma]^{1/2}. \tag{3.20}$$

From (3.20), we obtain $\mathcal{E}[\sigma] \lesssim \mathcal{E}[\mu_E]$, and inserting this into (3.19) gives $A \lesssim 1$. □

3.2.2 Hereditary Carleson implies embedding

Theorem 3.10. *Let $n = 1, 2, 3$ and $w : \mathcal{T}^n \to [0, \infty)$ be a positive product weight. Let μ, ρ be positive measures on $\mathcal{T}^n$ with*

$$[w, \mu]_{HC} \le 1, \quad [w, \rho]_{HC} \le 1. \tag{3.21}$$

Then, for some $0 < \kappa \le \frac{1}{4}$, we have

$$\int_{\mathcal{T}^n} \mathbb{V}^{\mu} d\rho \lesssim |\mu|^{1/2-\kappa}|\rho|^{1/2+\kappa}. \tag{3.22}$$

Proof. Let $\delta > 0$ be chosen later and consider the set $E := \{\mathbb{V}^{\mu} > \delta\} \subset \mathcal{T}^n$. As E is a down-set we have $\mathbf{1}_E \mathbb{I}^*\mu \le \mathbb{I}^*(\mu \mathbf{1}_E)$. Thus, by the Hereditary Carleson condition (2.2), we have

$$\int_{\mathcal{T}^n} (\mathbb{V}^{\mu} - \mathbb{V}^{\mu}_{\delta}) d\rho = \sum_{\mathcal{T}^n} w \mathbf{1}_E \mathbb{I}^*\mu \mathbb{I}^*\rho \le \mathcal{E}[\mu \mathbf{1}_E]^{1/2}\mathcal{E}[\rho]^{1/2} \le |\mu \mathbf{1}_E|^{1/2}\mathcal{E}[\rho]^{1/2}$$

Again, using the Hereditary Carleson condition

$$\delta|\mu \mathbf{1}_E| \le \int_E \mathbb{V}^{\mu} d\mu = \sum_{\mathcal{T}^n} w \mathbb{I}^*(\mu \mathbf{1}_E)\mathbb{I}^*\mu \le \mathcal{E}[\mu]^{1/2}\mathcal{E}[\mu \mathbf{1}_E]^{1/2} \le \mathcal{E}[\mu]^{1/2}|\mu \mathbf{1}_E|^{1/2} \tag{3.23}$$

Therefore,

$$|\mu \mathbf{1}_E|^{1/2} \le \delta^{-1}\mathcal{E}[\mu]^{1/2},$$

which implies

$$\int_{\mathcal{T}^n} (\mathbb{V}^\mu - \mathbb{V}^\mu_\delta) d\rho \le \delta^{-1}\mathcal{E}[\rho]^{1/2}\mathcal{E}[\mu]^{1/2}.$$

Next, by Theorem 3.5, Corollary 3.6 and (3.21), we obtain

$$\int_{\mathcal{T}^n} \mathbb{V}^\mu_\delta d\rho \lesssim \delta^r |\mu|^{(1-r)/2}|\rho|^{(1+r)/2}.$$

and thus

$$\int_{\mathcal{T}^n} \mathbb{V}^\mu d\rho \le C\delta^r |\mu|^{(1-r)/2}|\rho|^{(1+r)/2} + \delta^{-1}|\rho|^{1/2}|\mu|^{1/2}.$$

We choose δ which makes the two terms equal, to obtain

$$\int \mathbb{V}^\mu d\rho \lesssim |\mu|^{\frac{1/2}{1+r}}|\rho|^{\frac{1/2+r}{1+r}}.$$

and we take $\kappa := \frac{r}{2(r+1)} \le \frac{1}{4}$. □

Theorem 3.11. *Let $n = 1, 2, 3$. Let $\mu, w : \mathcal{T}^n \to [0, \infty)$. Assume that w is a positive product weight and that the Hereditary Carleson condition* (2.2) *holds. Then*

$$\sum_{\mathcal{T}^n} w(\mathbb{I}^*(f\mu))^2 \lesssim \int_{\mathcal{T}^n} f^2 d\mu.$$

for any $f \in \ell^2(\mathcal{T}^n, \mu)$.

The argument below is similar to the proof of [1, Theorem 7.1.1].

Proof. Without loss of generality $[w,\mu]_{HC} = 1$. Let $f : \mathcal{T}^n \to [0,\infty)$ and consider

$$f\mu = \int_0^\infty \mu_t dt, \quad \mu_t := \mu \mathbf{1}_{\{f>t\}}.$$

Then $[w,\mu_t]_{HC} \le [w,\mu]_{HC} = 1$ for every $0 < t < \infty$. Using symmetry, we obtain

$$\begin{aligned}
\sum_{\mathcal{T}^n} w(\mathbb{I}^*(f\mu))^2 &= 2\int_0^\infty \int_0^t \int_{\mathcal{T}^n} w(\mathbb{I}^*\mu_s)(\mathbb{I}^*\mu_t) ds dt \\
\text{by Theorem (3.10)} \quad &\lesssim 2\int_0^\infty \int_0^t |\mu_s|^{1/2-\kappa}|\mu_t|^{1/2+\kappa} ds dt \\
s \mapsto rt \text{ and Fubini} \quad &= 2\int_0^1 \int_0^\infty t|\mu_{rt}|^{1/2-\kappa}|\mu_t|^{1/2+\kappa} dt dr \\
&= 2\int_0^1 r^{-(1-2\kappa)} \int_0^\infty (r^2 t|\mu_{rt}|)^{1/2-\kappa}(t|\mu_t|)^{1/2+\kappa} dt dr \\
\text{by Hölder} \quad &\le 2\int_0^1 r^{-(1-2\kappa)} \Big(\int_0^\infty r^2 t|\mu_{rt}| dt\Big)^{1/2-\kappa} \Big(\int_0^\infty t|\mu_t| dt\Big)^{1/2+\kappa} dr \\
&= \Big(\int_0^1 r^{-(1-2\kappa)} dr\Big)\Big(2\int_0^\infty t|\mu_t| dt\Big) \\
&\lesssim 2\int_0^\infty t|\mu_t| dt \\
&= \int_{\mathcal{T}^n} f^2 d\mu.
\end{aligned}$$

□

3.2.3 Proof of Theorem 3.5.

Some more definitions are in order.

Definition 3.12. Given a simple tree $\mathcal{T}$, the set of *children* of a vertex $\beta \in \mathcal{T}$ consists of the maximal elements of $\mathcal{T}$ that are strictly smaller than β:

$$\mathrm{ch}(\beta) := \max\{\beta' \in \mathcal{T} \mid \beta' < \beta\}$$

A function $g : \mathcal{T} \to \mathbb{R}$ is called *superadditive* if for every $\beta \in \mathcal{T}$ we have

$$g(\beta) \geq \sum_{\beta' \in \mathrm{ch}(\beta)} g(\beta')$$

The difference operator is defined by

$$\Delta g(\beta) := g(\beta) - \sum_{\beta' \in \mathrm{ch}(\beta)} g(\beta')$$

Thus, a super-additive function g satisfies $\Delta g \geq 0$ on $\mathcal{T}$.

Lets see our first lemma

Lemma 3.13. *Let $\mathcal{T}$ be a simple tree and $f, g : \mathcal{T} \to [0, \infty)$ be any functions. Then*

$$(If)(Ig) \leq I(f \cdot Ig) + I(If \cdot g)$$

Proof.

$$\begin{aligned} If(\alpha)Ig(\alpha) &= \sum_{\alpha' \geq \alpha} \sum_{\alpha'' \geq \alpha} f(\alpha')g(\alpha'') = \sum_{\alpha' \geq \alpha} \left(\sum_{\alpha'' \geq \alpha'} + \sum_{\alpha \leq \alpha'' < \alpha'} \right) f(\alpha')g(\alpha'') \\ &= \sum_{\alpha' \geq \alpha} \sum_{\alpha'' \geq \alpha'} f(\alpha')g(\alpha'') + \sum_{\alpha' \geq \alpha} \sum_{\alpha \leq \alpha'' < \alpha'} f(\alpha')g(\alpha'') \\ &= \sum_{\alpha' \geq \alpha} f(\alpha')Ig(\alpha') + \sum_{\alpha'' \geq \alpha} (If - f)(\alpha'')g(\alpha'') \\ &= I(f \cdot Ig)(\alpha) + I(If \cdot g)(\alpha) - I(fg)(\alpha) \end{aligned}$$

□

We have the next lemma.

Lemma 3.14 (Partial summation)**.** *Let $\mathcal{T}$ be a simple tree. For any functions*

$f, g : \mathcal{T} \to \mathbb{R}$, *we have*

$$\sum_{\alpha \in \mathcal{T}} f(\alpha) g(\alpha) = \sum_{\alpha \in \mathcal{T}} \Delta f(\alpha) I g(\alpha)$$

Proof. By induction on the size of the tree, one can show

$$f(\alpha) = \sum_{\alpha' \leq \alpha} \Delta f(\alpha')$$

which means I^* is the inverse of Δ (as $\Delta I^* = id$, trivially). It follows that

$$\sum_{\alpha \in \mathcal{T}} f(\alpha) g(\alpha) = \sum_{\alpha \in \mathcal{T}} \sum_{\alpha' \leq \alpha} \Delta f(\alpha') g(\alpha) = \sum_{\alpha' \in \mathcal{T}} \Delta f(\alpha') \sum_{\alpha \geq \alpha'} g(\alpha) = \sum_{\alpha' \in \mathcal{T}} \Delta f(\alpha') I g(\alpha') \quad \square$$

Corollary 3.15. *Let $\mathcal{T}$ be a simple tree and $f, g : \mathcal{T} \to [0, \infty)$ with f superadditive. Then*

$$I^*(fg) \leq I^*(\Delta f \cdot I g)$$

Proof.

$$I^*(fg)(\beta) = \sum_{\alpha \leq \beta} f(\alpha) g(\alpha) = \sum_{\alpha \in \mathcal{T}} f(\alpha) \big(g \mathbf{1}_{\{\gamma : \gamma \leq \beta\}}\big)(\alpha) = \sum_{\alpha \in \mathcal{T}} \Delta f(\alpha) \cdot I(g \mathbf{1}_{\{\gamma : \gamma \leq \beta\}})(\alpha)$$

Where we used the previous lemma on the third equality.

Note f is superadditive on $\mathcal{T}$ and so $\Delta f \geq 0$. Finally, for each $\alpha \in \mathcal{T}$, we have

$I(g \mathbf{1}_{\{\gamma : \gamma \leq \beta\}})(\alpha) \leq I g(\alpha) \cdot \mathbf{1}_{\{\gamma : \gamma \leq \beta\}}(\alpha)$ and so the desired inequality follows.

$\square$

Next we have a two-dimensional analogue of Lemma (3.13).

Lemma 3.16. *Let $\mathcal{T}^2$ be a bi-tree and $f, g : \mathcal{T}^2 \to [0, \infty)$ Then*

$$(\mathbb{I}f)(\mathbb{I}g) \leq \mathbb{I}(\mathbb{I}f \cdot g + I_1 f \cdot I_2 g + I_2 f \cdot I_1 g + f \cdot \mathbb{I}g)$$

Proof. Recall that $\mathbb{I} = I_1 I_2$. We apply Lemma (3.13) first to I_1 and then to I_2 (note I_1, I_2 commute) to get the desired result. □

The following estimate is key for the proof of Theorem 3.5 for $n = 2$.

Lemma 3.17. *Let $\mathcal{T}^2$ be a bi-tree and $f : \mathcal{T}^2 \to [0, \infty)$ a function that is superadditive in each parameter separately. Let w be a positive product weight. Suppose that*
$\operatorname{supp} f \subseteq \{\mathbb{I}(wf) \leq \delta\}$. *Then*

$$\sum_{\mathcal{T}^2} w(I_1(w_1 f))^2 (I_2(w_2 f))^2 \leq 4\delta^2 \sum_{\mathcal{T}^2} w f^2$$

Proof. By Lemma 3.13 and commutativity of operations in different coordinates,

$$\begin{aligned}
\sum_{\mathcal{T}^2} w(I_1(w_1 f))^2 (I_2(w_2 f))^2 &\leq 4 \sum_{\mathcal{T}^2} w I_1(w_1 f \cdot I_1(w_1 f)) \cdot I_2(w_2 f \cdot I_2(w_2 f)) \\
&= 4 \sum_{\mathcal{T}^2} I_1(w_1 f \cdot I_1(wf)) \cdot I_2(w_2 f \cdot I_2(wf)) \\
&= 4 \sum_{\mathcal{T}^2} I_2^*(w_1 f \cdot I_1(wf)) \cdot I_1^*(w_2 f \cdot I_2(wf)) \\
&= 4 \sum_{\mathcal{T}^2} w I_2^*(f \cdot I_1(wf)) \cdot I_1^*(f \cdot I_2(wf))
\end{aligned}$$

By Corollary 3.15, we have

$$I_1^*(f \cdot I_2(wf)) \leq I_1^*(\Delta_1 f \cdot I_1 I_2(wf)).$$

Since f is superadditive, $\Delta_1 f \geq 0$. Hence,

$$I_1^*(f \cdot I_2(wf)) \leq I_1^*(\Delta_1 f \cdot \mathbb{I}(wf)) \leq I_1^*(\Delta_1 f \cdot \delta) = \delta f. \tag{3.24}$$

where we used $\Delta_1 f$ is supported on $\{\mathbb{I}(wf) \leq \delta\}$ (If $\Delta_1 f(\alpha) > 0$ then $f(\alpha) \neq 0$ for otherwise $\Delta_1 f(\alpha) \leq 0$). Arguing similarly for the other term and inserting these in the last estimate above, we obtain the claim.

□

Finally we prove theorem 3.5 for $n = 2$.

Lemma 3.18. *Let μ, ρ be positive measures on $\mathcal{T}^2$ and $\delta > 0$. Let w be a positive product weight. Then*

$$\int_{\mathcal{T}^2} \mathbb{V}_\delta^\mu d\rho \lesssim \delta^{\frac{1}{2}} |\rho|^{\frac{1}{2}} \mathcal{E}_\delta[\mu]^{\frac{1}{4}} \mathcal{E}[\rho]^{\frac{1}{4}} \tag{3.25}$$

Proof. Let $f := \mathbf{1}_{\mathbb{V}\mu \leq \delta} \mathbb{I}^* \mu$. Note that f is superadditive in each parameter. Also $\mathbb{I}(wf) = \mathbb{V}_\delta^\mu \leq \mathbb{V}^\mu \leq \delta$ on *supp*f. Finally, $\mathcal{E}_\delta[\mu] = \sum_{\mathcal{T}^2} wf^2$. Then

$$\begin{aligned}
\int_{\mathcal{T}^2} \mathbb{V}_\delta^\mu d\rho &= \int_{\mathcal{T}^2} \mathbb{I}(wf) d\rho \\
&\leq |\rho|^{1/2} \Big(\int_{\mathcal{T}^2} (\mathbb{I}(wf))^2 d\rho \Big)^{1/2} \\
\text{by Lemma 3.16} \quad &\leq |\rho|^{1/2} \Big(2 \int_{\mathcal{T}^2} \mathbb{I}(I_1(wf) \cdot I_2(wf) + (wf) \cdot \mathbb{I}(wf)) d\rho \Big)^{1/2} \\
&\lesssim |\rho|^{1/2} \Big(\int_{\mathcal{T}^2} w(I_1(w_1 f) \cdot I_2(w_2 f) + f \cdot \mathbb{I}(wf)) \mathbb{I}^* \rho \Big)^{1/2} \\
&\lesssim |\rho|^{1/2} \mathcal{E}[\rho]^{1/4} \Big(\sum_{\mathcal{T}^2} w \big(I_1(w_1 f) \cdot I_2(w_2 f) + f \cdot \mathbb{I}(wf) \big)^2 \Big)^{1/4}
\end{aligned}$$

using the inequality $(a+b)^2 \leq 2(a^2+b^2)$ and Lemma 3.17

$$\lesssim |\rho|^{1/2}\mathcal{E}[\rho]^{1/4}\Big(10\delta^2 \sum_{\mathcal{T}^2} wf^2\Big)^{1/4}$$

$$\lesssim |\rho|^{1/2}\mathcal{E}[\rho]^{1/4}\delta^{1/2}\mathcal{E}_\delta[\mu]^{1/4} \qquad \square$$

Next we develop further our techniques for dimension $n=3$. The next lemma is an analogue of Lemma 3.16.

Lemma 3.19. *Let $\mathcal{T}^3$ be a tri-tree and $f, g : \mathcal{T}^3 \to [0,\infty)$. Then*

$$(\mathbb{I}f)(\mathbb{I}g) \leq \mathbb{I}\Big(\sum_{A\subseteq\{1,2,3\}} I_A f \cdot I_{A^c} g\Big),$$

where $I_A = \prod_{i\in A} I_i$.

Proof. Note that $\mathbb{I} = I_1 I_2 I_3$. We use Lemma (3.13) three times along with the commutativity of I_1, I_2, I_3. $\square$

Corollary 3.20. *Let $0 < \delta \leq \lambda/4$. Let $f : \mathcal{T}^3 \to [0,\infty)$ with $\operatorname{supp} f \subseteq \{\mathbb{I}f \leq \delta\}$. Then*

$$(\mathbb{I}f)\mathbf{1}_{\lambda\leq\mathbb{I}f\leq 2\lambda} \leq 4\lambda^{-1}\mathbb{I}\Big(\sum_{i\in\{1,2,3\}} I_i f \cdot I_{(i)} f \cdot \mathbf{1}_{\mathbb{I}f\leq 2\lambda}\Big),$$

where $I_{(i)} := \prod_{j\neq i} I_j$.

Proof. Substituting $f = g$, Lemma 3.19 implies that

$$(\mathbb{I}f)^2 \leq \mathbb{I}\Big(2\sum_{i\in\{1,2,3\}} I_i f \cdot I_{(i)} f + 2f \cdot \mathbb{I}f\Big)$$

Using the support condition, this implies

$$\begin{aligned}(\mathbb{I}f)\mathbf{1}_{\lambda\le\mathbb{I}f\le 2\lambda} &\le \lambda^{-1}(\mathbb{I}f)^2\mathbf{1}_{\lambda\le\mathbb{I}f\le 2\lambda}\\ &\le \lambda^{-1}\mathbb{I}\Big(2\sum_{i\in\{1,2,3\}} I_i f\cdot I_{(i)}f + 2\delta f\Big)\mathbf{1}_{\lambda\le\mathbb{I}f\le 2\lambda}\\ &\le \Big[\lambda^{-1}\mathbb{I}\Big(2\sum_{i\in\{1,2,3\}} I_i f\cdot I_{(i)}f\Big) + 2\delta\lambda^{-1}\mathbb{I}f\Big]\mathbf{1}_{\lambda\le\mathbb{I}f\le 2\lambda}\end{aligned}$$

Since $2\delta\lambda^{-1}\le 1/2$, this implies

$$\begin{aligned}(\mathbb{I}f)\mathbf{1}_{\lambda\le\mathbb{I}f\le 2\lambda} &\le 4\lambda^{-1}\mathbb{I}\Big(\sum_{i\in\{1,2,3\}} I_i f\cdot I_{(i)}f\Big)\mathbf{1}_{\lambda\le\mathbb{I}f\le 2\lambda}\\ &\le 4\lambda^{-1}\mathbb{I}\Big(\sum_{i\in\{1,2,3\}} I_i f\cdot I_{(i)}f\cdot\mathbf{1}_{\mathbb{I}f\le 2\lambda}\Big)\end{aligned}$$

□

Now we need one more lemma. Denote $w_{(i)} := \prod_{j\ne i} w_j$ and so $w = w_i\cdot w_{(i)}$.

Lemma 3.21. *Let* $f:\mathcal{T}^3\to[0,\infty)$ *be superadditive on each parameter separately. Let* w *be a positive product weight. Suppose that* $\operatorname{supp} f\subseteq\{\mathbb{I}(wf)\le\delta\}$*. Then for every* $i\in\{1,2,3\}$

$$\sum_{\mathcal{T}^3} w\big(I_i(w_i f)\cdot I_{(i)}(w_{(i)}f)\big)^2\mathbf{1}_{\mathbb{I}(wf)\le\lambda}\le 2\delta\lambda\sum_{\mathcal{T}^3} wf^2$$

Proof. By Lemma 3.13 for I_i we have

$$\begin{aligned}&\int w\big(I_i(wf)\cdot I_{(i)}(w_{(i)}f)\big)^2\mathbf{1}_{\mathbb{I}(wf)\le\lambda}\\ &\le 2\int wI_i(w_i f\cdot I_i(w_i f))\cdot(I_{(i)}(w_{(i)}f))^2\mathbf{1}_{\mathbb{I}(wf)\le\lambda}\\ &= 2\int I_i(w_i f\cdot I_i(wf))\cdot(I_{(i)}(w_{(i)}f))\cdot(I_{(i)}(wf))\mathbf{1}_{\mathbb{I}(wf)\le\lambda}\\ &= 2\int w_i f\cdot I_i(wf)\cdot I_i^*\Big((I_{(i)}(w_{(i)}f))\cdot(I_{(i)}(wf))\mathbf{1}_{\mathbb{I}(wf)\le\lambda}\Big)\end{aligned}\tag{3.26}$$

By Corollary 3.15, we have

$$I_i^*\Big((I_{(i)}(w_{(i)}f)) \cdot (I_{(i)}(wf)) \cdot \mathbf{1}_{\mathbb{I}(wf)\le\lambda}\Big) \le I_i^*\Big(\Delta_i(\mathbf{1}_{\mathbb{I}(wf)\le\lambda} \cdot I_{(i)}(w_{(i)}f)) \cdot I_i(I_{(i)}(wf)\Big)$$

As f is superadditive in the i-th coordinate, the function $\tilde{f} := \mathbf{1}_{\mathbb{I}(wf)\le\lambda} \cdot I_{(i)}(w_{(i)}f)$ is also superadditive in the i-th coordinate. Hence $\Delta_i\tilde{f} \ge 0$. Moreover, if $\Delta_i\tilde{f}(\alpha) > 0$ then $\tilde{f}(\alpha) \neq 0$ which means $\alpha \in \{\mathbb{I}(wf) \le \lambda\}$ and so *supp* $\Delta_i\tilde{f} \subseteq \{\mathbb{I}(wf) \le \lambda\}$. Hence,

$$I_i^*\Big(\Delta_i\tilde{f} \cdot I_i\big(I_{(i)}(wf)\big)\Big) \le I_i^*\big(\Delta_i\tilde{f} \cdot \lambda\big) = \lambda\mathbf{1}_{\mathbb{I}(wf)\le\lambda} \cdot I_{(i)}(w_{(i)}f)$$

Using this bound, we obtain

$$\begin{aligned}(3.26) &\le 2\lambda \int w_i f \cdot I_i(wf) \cdot I_{(i)}(w_{(i)}f) \\ &= 2\lambda \int f \cdot I_i(wf) \cdot I_{(i)}(wf) \\ &= 2\lambda \int wf \cdot I_i^*(f \cdot I_{(i)}(wf))\end{aligned}$$

As in (6.10), we see that

$$I_i^*(f \cdot I_{(i)}(wf)) \le \delta f.$$

This implies the conclusion of the lemma. □

The next result is a key tool for dimension 3.

Lemma 3.22 (Small energy majorization on tri-tree)**.** *Let $\mathcal{T}^3$ be a tri-tree and $f : \mathcal{T}^3 \to [0,\infty)$ a function that is superadditive in each parameter separately. Let w be a positive product weight. Suppose that* $\operatorname{supp} f \subseteq \{\mathbb{I}(wf) \le \delta\}$. *Let $\lambda \ge 4\delta$. Then there exists*

$\varphi : \mathcal{T}^3 \to [0, \infty)$ *such that*

$$a)\ \mathbb{I}(w\varphi) \geq \mathbb{I}(wf),\ \textit{where}\ \mathbb{I}(wf) \in [\lambda, 2\lambda],$$

$$b) \sum_{\mathcal{T}^3} w\varphi^2 \leq C\frac{\delta}{\lambda}\sum_{\mathcal{T}^3} wf^2,$$

where C is an absolute constant.

Proof. Put

$$\varphi := 4\lambda^{-1} \sum_{i \in \{1,2,3\}} I_i w_i f \cdot I_{(i)} w_{(i)} f \cdot \mathbf{1}_{\mathbb{I}(wf) \leq 2\lambda}$$

For part $a)$ we use Corollary 3.20 with f replaced by wf. Note $\mathrm{supp}(wf) \subseteq supp(f)$ and thus

$$\mathbb{I}(wf)\mathbf{1}_{\lambda \leq \mathbb{I}(wf) \leq 2\lambda} \leq 4\lambda^{-1}\mathbb{I}\Big(\sum_{i \in \{1,2,3\}} I_i(wf) \cdot I_{(i)}(wf) \cdot \mathbf{1}_{\mathbb{I}(wf) \leq 2\lambda}\Big)$$

$$4\lambda^{-1}\mathbb{I}\Big(w \sum_{i \in \{1,2,3\}} I_i(w_i f) \cdot I_{(i)}(w_{(i)} f) \cdot \mathbf{1}_{\mathbb{I}(wf) \leq 2\lambda}\Big) = \mathbb{I}(w\varphi)$$

For part $b)$ we apply Lemma 3.21. □

Lets prove Theorem 3.5 for $n = 3$

Lemma 3.23. *Let μ, ρ be positive measures on $\mathcal{T}^3$ and $\delta > 0$. Let w be a positive product weight. Then*

$$\int_{\mathcal{T}^3} \mathbb{V}^{\mu}_{\delta} d\rho \lesssim \Big(\delta \mathcal{E}_{\delta}[\mu]\mathcal{E}[\rho]|\rho|\Big)^{\frac{1}{3}} \tag{3.27}$$

Proof. Without loss of generality, $\mathcal{E}_\delta[\mu] \neq 0$ and $\rho \not\equiv 0$. Let $\lambda > 0$ be chosen later.

Let $f := \mathbb{I}^*\mu \cdot \mathbf{1}_{\mathbb{V}^{\mu} \leq \delta}(\alpha)$. This function is superadditive. Also, $\mathbb{I}(wf) = \mathbb{V}^{\mu}_{\delta} \leq \mathbb{V}^{\mu} \leq \delta$ on $\mathrm{supp}\, f$, and $\mathcal{E}_\delta[\mu] = \int wf^2$.

For $m = 0, 1, \dots$ we use Lemma 3.22 with data $(w, f, \delta, 2^m\lambda)$ to get

$$\mathbb{I}(wf) \cdot \mathbf{1}_{2^m\lambda < \mathbb{I}(wf) \leq 2^{m+1}\lambda} \leq \mathbb{I}(w\phi_m),$$

and

$$\sum_{\mathcal{T}^3} w\phi_m^2 \lesssim \frac{\delta}{2^m\lambda} \sum_{\mathcal{T}^3} wf^2.$$

Hence,

$$\begin{aligned}
\int_{\mathcal{T}^3} \mathbb{V}_\delta^\mu d\rho &= \int_{\{\mathbb{V}_\delta^\mu \leq \lambda\}} \mathbb{V}_\delta^\mu d\rho + \sum_{m=0}^{\infty} \int_{\{2^m\lambda < \mathbb{V}_\delta^\mu \leq 2^{m+1}\lambda\}} \mathbb{V}_\delta^\mu d\rho \\
&\leq \lambda|\rho| + \sum_{m=0}^{\infty} \int_{\mathcal{T}^3} \mathbb{I}(w\phi_m) d\rho \\
&= \lambda|\rho| + \sum_{m=0}^{\infty} \int_{\mathcal{T}^3} w\phi_m \mathbb{I}^* d\rho \\
&\leq \lambda|\rho| + \sum_{m=0}^{\infty} \Big(\sum_{\mathcal{T}^3} w\phi_m^2\Big)^{1/2} \mathcal{E}[\rho]^{1/2} \\
&\leq \lambda|\rho| + \sum_{m=0}^{\infty} C(\delta/(2^m\lambda))^{1/2} \mathcal{E}_\delta[\mu]^{1/2} \mathcal{E}[\rho]^{1/2}. \\
&\leq \lambda|\rho| + C(\delta/\lambda)^{1/2} \mathcal{E}_\delta[\mu]^{1/2} \mathcal{E}[\rho]^{1/2}
\end{aligned}$$

Recall we need $\lambda \geq 4\delta$. Let $\lambda := (\delta\mathcal{E}_\delta[\mu]\mathcal{E}[\rho])^{1/3}|\rho|^{-2/3}$. If $\lambda \geq 4\delta$ then we obtain (3.27) by substituting it in the last line above. Otherwise, if $\lambda < 4\delta$, then we automatically get (3.27) without using any lemmata. □

3.3 Improvement in the case of $n = 2$

We state here without a proof an improvement of the energy estimate of Lemma 3.18.

Theorem 3.24. *Let μ be a measure on $\mathcal{T}^2$ and $w \equiv 1$. Then, for every $\tau \in (0,1)$:*

$$\mathcal{E}_\delta[\mu] \lesssim_\tau (\delta|\mu|)^{1-\tau}\mathcal{E}[\mu]^\tau$$

with the implicit constant depending on τ.

Chapter 4

Capacitary conditions and Embedding

4.1 General Capacitary condition implies embedding

As we said this result is known for $n = 2$, first proven in [4]. We prove it here for $n = 3$ following closely the proof of $n = 2$. What would be useful here is Lemma 3.22 which first appeared in [19]. Hence, as we have established the equivalence between the Box condition and the embedding and since we show the General capacitary condition (2.5) is also equivalent to the embedding, then the Box condition implies the general capacitary condition. As we discuss in subsection 4.2 the Box condition is enough to prove the much weaker Box capacitary condition, but here, indirectly we show something much stronger. Here we take $w \equiv 1$.

To begin, suppose we have a measure μ on $\mathcal{T}^3$ and for any $E \subseteq \partial\mathcal{T}^3$ the following condition holds

$$\mu(E) \lesssim \text{Cap}(E)$$

which is condition (2.5). This condition implies for any $f \in \ell^2(\mathcal{T}^3)$

$$\int_{\mathcal{T}^3} (\mathbb{I}f)^2 d\mu \sim \int_0^\infty \lambda\, \mu\{\mathbb{I}f \geq \lambda\} \lesssim \sum_{k\in\mathbb{Z}} 2^{2k} \mu\{\mathbb{I}f \geq 2^k\} \lesssim \sum_{k\in\mathbb{Z}} 2^{2k}\ \text{Cap}\{\mathbb{I}f \geq 2^k\} \tag{4.1}$$

Recall the measure μ is non-zero only on the boundary of $\mathcal{T}^3$ and thus we are interested to estimate the capacity of the sets $E_k := \{\mathbb{I}f \geq 2^k\} \cap \partial\mathcal{T}^3$. More specifically we wish to show

$$\sum_{k\in\mathbb{Z}} 2^{2k} \ \mathrm{Cap}(E_k) \lesssim \|f\|^2_{\ell^2(\mathcal{T}^3)} \tag{4.2}$$

and this would finish the inequality in (4.1). By duality we conclude (1.10) holds. As it is mentioned in [1], for a set E, the minimizer in (2.4) is a function of the form $\mathbb{I}^*\mu_E$ where μ_E is measure on $\mathcal{T}^3$ which is called the *equilibrium measure* of E. The measure μ_E satisfies $\mathbb{V}^{\mu_E} \equiv 1$ on E and also, since E is compact we also have $supp(\mu_E) \subseteq E$. This implies $\mathrm{Cap}(E) = \int (\mathbb{I}^*\mu_E)^2 = \int \mathbb{V}^{\mu_E} d\mu_E = \mu_E(E)$. For every $k \in \mathbb{Z}$ we get such a measure μ_k and by arguments found in the Section 3.1 of [4] we deduce (4.2) as long as the following is proven:

$$\sum_{k\in\mathbb{Z}} \sum_{j\leq k} 2^{j+k} \int_{\mathcal{T}^3} \mathbb{V}^{\mu_k} d\mu_j \lesssim \sum_{k\in\mathbb{Z}} 2^{2k} \int_{\mathcal{T}^3} \mathbb{V}^{\mu_k} d\mu_k \tag{4.3}$$

In turn (4.3) is proven exactly in the same manner as in this paper, if the next Lemma is true;

Lemma 4.1 (Lemma 3.2 of [4]). *Let $F, E \subseteq \partial\mathcal{T}^3$ be two sets such that* $\mathrm{Cap}(F) \leq \mathrm{Cap}(E)$ *and let μ_F, μ_E be their equilibrium measures. Then, we have*

$$\int_{\mathcal{T}^3} \mathbb{V}^{\mu_E} d\mu_F \lesssim |\mu_E|^{1/3} \cdot |\mu_F|^{2/3}$$

Proof. We have seen above that $\mathrm{Cap}(F) = |\mu_F|$. If this number is 0 we have nothing to prove.

Otherwise, we set $\lambda = \dfrac{\int_{\mathcal{T}^3} \mathbb{V}^{\mu_E} d\mu_F}{|\mu_F|}$ and for $k \in \mathbb{N} \cup \{0\}$ we define the sets

$$F_{-1} = \{\omega \in F \ : \ \mathbb{V}^{\mu_E} \leq 1\}$$

$$F_k = \{\omega \in F \ : \ 2^k < \mathbb{V}^{\mu_E} \leq 2^{k+1}\}$$

As $\mathcal{T}^3$ is finite, then there exists some $m \in \mathbb{N}$ such that for any $k \geq m$, $F_k = \varnothing$. Now, by using Proposition 4.2 we see that

$$\mathrm{Cap}(F_k) \lesssim \frac{|\mu_E|}{2^{3k}} \tag{4.4}$$

Next, let j such that $2^j < \lambda \leq 2^{j+1}$ and we write

$$\begin{aligned} \lambda \cdot |\mu_F| = \int_{\mathcal{T}^3} \mathbb{V}^{\mu_E} d\mu_F = \sum_{k \geq -1} \int_{F_k} \mathbb{V}^{\mu_E} d\mu_F \leq \\ \leq \sum_{k \geq -1} 2^{k+1} \cdot \mu_F(F_k) \leq \sum_{k=-1}^{j-2} 2^{k+1} \cdot \mu_F(F_k) + \sum_{k \geq j-1} 2^{k+1} \cdot \mathrm{Cap}(F_k) \end{aligned} \tag{4.5}$$

and the last inequality follows by Lemma 5.6 of [4] : As $\mathbb{V}^{\mu_F} = 1$ on F_k then $\mu_F(F_k) \leq \mathrm{Cap}(F_k)$. Using (4.4) and that the sets F_k are disjoint, we get

$$\begin{aligned} (4.5) \leq 2^{j-1} \cdot \mu_F(F) + C \cdot |\mu_E| \cdot \sum_{k \geq j-1} 2^{k+1} \frac{1}{2^{3k}} \leq \\ \leq \frac{\lambda}{2} \cdot |\mu_F| + C \cdot \frac{|\mu_E|}{\lambda^2} \end{aligned}$$

which implies

$$|\mu_F| \lesssim \frac{|\mu_E|}{\lambda^3}$$

and this finishes the proof given the definition of λ.

□

Proposition 4.2. *Let μ be a measure on $\mathcal{T}^3$ with $\mathbb{V}^\mu \le 1$ on $\operatorname{supp}\mu$. Then, for any $\lambda \ge 1$ we have*

$$\operatorname{Cap}(\{\mathbb{V}^\mu > \lambda\}) \lesssim \frac{\mathcal{E}[\mu]}{\lambda^3}$$

Proof of Proposition 4.2. Consider $f = \mathbb{I}^*\mu$, $\delta = 1$. If $f(\alpha) \neq 0$ then there is $\beta \le \alpha$ such that $\beta \in \operatorname{supp}\mu$. But then by assumption $\mathbb{I}f(\beta) = \mathbb{I}\mathbb{I}^*\mu(\beta) = \mathbb{V}^\mu(\beta) \le 1$. By monotonicity of $\mathbb{I}$ we have that $\mathbb{I}f(\alpha) \le 1$. Hence

$$\operatorname{supp} f \subset \{\mathbb{I}f \le \delta = 1\},$$

and we are in the assumptions of small energy majorization Lemma on tri-tree 3.22. We apply it with data $(f, \delta = 1, \lambda := 2^m\lambda)$ to get functions φ_m, $m = 0, 1, \dots$ such that

$$\mathbb{I}\varphi_m \ge \mathbb{I}f = \mathbb{V}^\mu, \quad \text{where } \mathbb{V}^\mu \in [2^m\lambda, 2^{m+1}\lambda],$$

which means that

$$2^{-m}\lambda^{-1}\mathbb{I}\varphi_m \ge 1, \quad \text{where } \mathbb{V}^\mu \in [2^m\lambda, 2^{m+1}\lambda],$$

Now define $\varphi := \sum_m 2^{-m}\lambda^{-1}\varphi_m$ and hence,

$$\mathbb{I}\varphi \ge 1, \quad \text{where } \mathbb{V}^\mu \in [\lambda, \infty),$$

Now, by Minkowski's integral inequality we get:

$$\int_{\mathcal{T}^3} \varphi^2 \le \left(\lambda^{-1} \sum_{m=0}^{\infty} 2^{-m} \Big(\int_{\mathcal{T}^3} \varphi_m^2\Big)^{1/2}\right)^2 \lesssim$$
$$\lesssim \left(\lambda^{-1} \sum_{m=0}^{\infty} \lambda^{-1/2} 2^{-\frac{3m}{2}} \Big(\int_{\mathcal{T}^3} f^2\Big)^{1/2}\right)^2 \lesssim \lambda^{-3} \int_{\mathcal{T}^3} f^2$$

which finishes the proof, since $f = \mathbb{I}^* \mu$ and so $\int_{\mathcal{T}^3} f^2 = \mathcal{E}[\mu]$. □

4.2 Box condition implies Box Capacitary Condition

Let ν be a finite positive measure in $[0,1)^n$ with $|\nu| \le 1$. In this section we show that the box condition (1.6) on n-tree implies box-capacitary condition (2.6) on n-tree of depth N, for all $n \ge 2$. In other words, if

$$\sum_{P \subseteq R} w_P \, \nu(P)^2 \le \nu(R), \quad \forall R \in \mathcal{T}^n$$

then

$$\nu(R) \lesssim \mathrm{Cap}(R), \quad \forall R \in \mathcal{T}^n$$

Recall the nodes of $\mathcal{T}^n$ have the form $\left[\frac{m}{2^{N-k}}, \frac{m+1}{2^{N-k}}\right)$, $0 \le k \le N$, $0 \le m \le 2^{N-k} - 1$ and we call these dyadic n-rectangles (Cartesian product of dyadic intervals). By construction these are inside the unit n-cube $[0,1)^n$. Recall that

$$\mathrm{Cap}(R) \approx \frac{1}{\log_2 \frac{1}{|I_1|} \cdot \log_2 \frac{1}{|I_2|} \cdot .. \cdot \log_2 \frac{1}{|I_n|}}$$

where $R = I_1 \times .. \times I_n$ and if $|I_i| = 1$ we just replace this term by 1 in the denominator. We will prove our result for a hooked dyadic n-rectangle, i.e. a dyadic n-rectangle which has a vertex at $(0, 0, .., 0) \in \mathbb{Z}^n$. This does not change at all the arguments although it simplifies the calculations.

Let R be a hooked dyadic n-rectangle of the form
$[0, 2^{-N+m_1}] \times [0, 2^{-N+m_2}] \times .. \times [0, 2^{-N+m_n}]$. Then there is an $s \in \mathbb{N}$ such that $2^{-ns} < \nu(R) \leq 2^{-ns+n}$. If we define $a := 2^{-s}$ then we have $a^n < \nu(R) \leq 2^n a^n$.

We connect the points $(m_1, m_2, .., m_n)$ and $(N, N, .., N) \in \mathbb{Z}^n$ by a line L. In parametric form this line is $L : (x_1, x_2, .., x_n) = (m_1, m_2, .., m_n) + t(N - m_1, N - m_2, .., N - m_n)$ where $0 \leq t \leq 1$. We are going to consider points on that line which satisfy some conditions. The points are defined recursively by

$P_k = (m_1 + m_{1,1} + .. + m_{1,k}, m_2 + m_{2,1} + .. + m_{2,k}, .., m_n + m_{n,1} + .. + m_{n,k})$ where $m_{j.k} \in \mathbb{Z}_+$ with $j \in \{1, .., n\}$, $k \in \{1, .., i\}$ and $i \in \mathbb{N}$ is fixed (to be specified). Also, $P_0 = (m_1, .., m_n)$.

For every $k \in \{1, .., i\}$, the first condition we impose is $P_k \in L$ and the second condition is

$$m_{1,k} \cdot m_{2,k} \cdot .. \cdot m_{n,k} = \frac{2^n}{a_{k-1}^n}$$

where $a_0 := a$ and $a_k = 2a_{k-1}$ with $k \in \{1, .., i\}$. Working recursively from $k = 1$, we see that for every $j, l \in \{1, .., n\}$ we get

$$\frac{m_{j,k}}{N - m_j} = \frac{m_{l,k}}{N - m_l} \qquad \text{for all } k \in \{1, .., i\} \tag{4.6}$$

Using these equalities and $m_{1,k} \cdot m_{2,k} \cdot .. \cdot m_{n,k} = \frac{2^n}{a_{k-1}^n}$ we deduce the next formula for every $j \in \{1, .., n\}$

$$m_{j,k} = \frac{2}{a_{k-1}} \cdot \frac{(N - m_j)^{\frac{n-1}{n}}}{\prod\limits_{\substack{l \neq j \\ l \in \{1,..,n\}}} (N - m_l)^{\frac{1}{n}}} \tag{4.7}$$

Let $Q_{m'_1, m'_2, .., m'_n}$ be the dyadic n-rectangle with fixed vertex at $(0, 0, .., 0) \in \mathbb{Z}^n$ and of size $2^{N-m'_1} \times 2^{N-m'_2} \times .. \times 2^{N-m'_n}$. For simplicity, let $\nu(m'_1, m'_2, .., m'_n) := \nu(Q_{m'_1, m'_2, .., m'_n})$. Then, for $(m'_1, m'_2, .., m'_n) = (m_1 + m_{1,1}, m_2 + m_{2,1}, .., m_n + m_{n,1})$ we have at least $m_{1,1} \cdot m_{2,1} \cdot .. \cdot m_{n,1}$ other such dyadic n-rectangles which contain R. Now we start using (1.6) for the first time (we normalize the constant to be 1)

$$\nu(m_1 + m_{1,1}, m_2 + m_{2,1}, .., m_n + m_{n,1}) \geq m_{1,1} \cdot m_{2,1} \cdot .. \cdot m_{n,1} \cdot \nu(R)^2 \geq$$

$$m_{1,1} \cdot m_{2,1} \cdot .. \cdot m_{n,1} \cdot a^{2n} = 2^n \cdot a_0^n = a_1^n$$

using that $m_{1,1} \cdot m_{2,1} \cdot .. \cdot m_{n,1} = \frac{2^n}{a_0^n}$ and the definition of a_0, a_1. We continue recursively and after k steps we have

$$\nu(m_1 + m_{1,1} + .. + m_{1,k}, m_2 + m_{2,1} + .. + m_{2,k}, .., m_n + m_{n,1} + .. + m_{n,k}) \geq a_k^n$$

We stop the recursion (and choose $i = k$) at the minimal k such that $m_1 + m_{1,1} + .. + m_{1,k} > N$ (by (4.6) this is equivalent to $m_j + m_{j,1} + .. + m_{j,k} > N$ for any $j \in \{1, .., n\}$) or when $k = s$.

If the first case happens, then by (4.7) we have

$$N - m_1 < m_{1,1} + m_{1,2} + .. + m_{1,i} = \frac{(N-m_1)^{\frac{n-1}{n}}}{\prod\limits_{\substack{l \neq 1 \\ l \in \{1,..,n\}}} (N-m_l)^{\frac{1}{n}}} \cdot \Big(\frac{2}{a_0} + \frac{2}{a_1} + .. + \frac{2}{a_{i-1}}\Big)$$

$$\leq \frac{(N-m_1)^{\frac{n-1}{n}}}{\prod\limits_{\substack{l \neq 1 \\ l \in \{1,..,n\}}} (N-m_l)^{\frac{1}{n}}} \cdot \frac{4}{a}$$

as the sum is a geometric series by the definition of a_i. If we rearrange the two ends of the above inequalities we get

$$a^n \lesssim \frac{1}{(N-m_1) \cdot (N-m_2) \cdot .. \cdot (N-m_n)} = C \cdot cap(R)$$

Keeping in mind that $\nu(R) \leq 2^n a^n$ we have the desired inequality. If the second case happens, namely $i = s$ and since $a_s = 1$ we see that

$$\nu(m_1 + m_{1,1} + .. + m_{1,s}, m_2 + m_{2,1} + .. + m_{2,s}, .., m_n + m_{n,1} + .. + m_{n,s}) \geq 1$$

Our measure is normalized such that $|\nu| \leq 1$. Intuitively this means there is no much room left. If $m_1 + m_{1,1} + .. + m_{1,k} > \frac{N}{2}$ then by reasoning as above we get the same desired inequality. If $m_1 + m_{1,1} + .. + m_{1,k} \leq \frac{N}{2}$ then let $M = \left\lfloor \frac{N}{2} \right\rfloor$. We do one more step in the recursion to get

$$\nu(m_1 + m_{1,1} + .. + m_{1,s} + M, m_2 + m_{2,1} + .. + m_{2,s}, .., m_n + m_{n,1} + .. + m_{n,s}) \geq$$

$$M \cdot \nu(m_1 + m_{1,1} + .. + m_{1,s}, m_2 + m_{2,1} + .. + m_{2,s}, .., m_n + m_{n,1} + .. + m_{n,s}) \geq M$$

which is of course a contradiction as N is large. Therefore the desired inequality has been established.

4.3 Box capacitary does not imply general capacitary condition

One could ask: Since Box condition (1.6) is able to imply the embedding (1.5), is it also true that Box capacitary condition (2.6) also implies the same embedding? The answer is no. This was proven by Stegenga in [28] for $n = 1$. We now give a counterexample (due to P. Mozolyako) in the case of a simple tree.

Consider a simple tree $\mathcal{T}$ of depth 2^N for $N \in \mathbb{N}$. Let us describe the construction of a "Cantor" type set with small capacity and of full-measure. For simplicity we assume the root of the tree is of generation 1. We choose the left-most and right-most boundary points of the tree. Then we move to the 2nd generation and for each of these two nodes we consider the sub-graphs with these as a root. We choose the left-most and right-most boundary points of each sub-graph (we count duplicate nodes just once). Then we go to the 4th generation and we consider the sub-graphs with roots the left-most and right-most descendants of the 2nd generation (hence we look at 4 sub-graphs in total). As before, for each of these sub-graphs we choose the left-most and right-most boundary points. We continue doing this for all generations of order 2^k with $0 \leq k \leq N-1$. The total number of the boundary points we choose is 2^N out of a total 2^{2^N-1} boundary points. Their selection is uniform and we now give the exact positions of these boundary points.

Let us look at the boundary of the simple tree from the left to the right and consider the set E which is comprised of the boundary nodes in positions $m \cdot 2^{2^N-N}, n \cdot 2^{2^N-N}+1$ for

$m = 1, .., 2^{N-1}$ and $n = 0, .., 2^{N-1} - 1$. For $n = 0$ we get the left-most and for $m = 2^{N-1}$ we get the right-most boundary point. The measure μ is constructed as follows: to each node ω of E we set $\mu(\omega) = \frac{1}{2^N}$. Therefore, $|\mu| = 1$. We also set $\nu = \mathbb{I}^*\mu$ and so $|\nu| = \nu(E) = 1$. For all $I \in \mathcal{T}$ we show $\nu(I) \lesssim \text{Cap}(I)$, and also $\text{Cap}(E) \leq \frac{2}{N}$. Hence, the box capacitary condition holds with bounded constant while the general capacitary condition can only hold with a constant which depends on N. We now prove these two facts.

First, let any $I \in \mathcal{T}$. Let ℓ be the generation of I from the top, with $1 \leq \ell \leq 2^N$. Then, there is some k, $0 \leq k \leq N$ such that $2^k \leq \ell < 2^{k+1}$. By our construction, the amount of boundary points which are descendants of I and have μ-mass is 2^{N-k}. Therefore, $\nu(I) = 2^{N-k}\dfrac{1}{2^N} = \dfrac{1}{2^k}$ as each such boundary node has mass $\dfrac{1}{2^N}$. On the other hand, $\text{Cap}(I) = \dfrac{1}{\ell} > \dfrac{1}{2^{k+1}}$ and thus $\nu(I) \leq 2 \cdot \text{Cap}(I)$. Hence ν satisfies the Box capacitary condition.

Second, for any $\omega \in \partial\mathcal{T}$ we show $N \leq \mathbb{V}^\mu(\omega) \leq 2N$ and so $1 \leq \mathbb{V}^{\frac{\mu}{N}} \leq 2$. Using this and the Cauchy-Schwarz inequality we get $\text{Cap}(E) \leq \frac{1}{N} \cdot \int \mathbb{V}^{\frac{\mu}{N}} d\mu \leq \dfrac{2|\mu|}{N}$ which gives the second estimate as $|\mu| = 1$. To prove this, fix $\omega \in \partial\mathcal{T}$ and let any $\alpha \geq \omega$. As before, we have $\mathbb{I}^*\mu(\alpha) = \nu(\alpha) = \dfrac{1}{2^k}$ where k is such that, $2^k \leq \ell < 2^{k+1}$ and ℓ is the generation of α. Next, let $\omega =: \alpha_{2^N} \leq .. \leq \alpha_1$ be an enumeration of the ancestors of ω where α_1 is the root of the tree $\mathcal{T}$. Then we have

$$\mathbb{V}^\mu(\omega) = \sum_{\ell=1}^{2^N} \mathbb{I}^*\mu(\alpha_\ell) = \mathbb{I}^*\mu(\omega) + \sum_{k=0}^{N-1} \sum_{2^k \leq \ell < 2^{k+1}} \mathbb{I}^*\mu(\alpha_\ell) = 1 + \sum_{k=0}^{N-1} (2^{k+1} - 2^k)\frac{1}{2^k} = N + 1$$

Finally, for any $n \in \mathbb{N}$ with $n \geq 2$ and an n-tree $\mathcal{T}^n$, consider a measure μ on $\partial\mathcal{T}^n$ which is a product of measures constructed above. Hence, as capacity is of product nature, the above counterexample gives rise to a counterexample for higher dimensions.

Chapter 5

Connections with known results and applications

5.1 Maximal function and embedding

In this section we consider a set $\mathcal{P}$ which is any finite n-tree with $n \in \mathbb{N}$. More generally one could think of $\mathcal{P}$ as any partially ordered set. Given a measure μ on $\mathcal{P}$ we define the corresponding maximal operator by

$$\mathcal{M}_\mu \psi(\omega) := \sup_{\alpha \geq \omega} \langle |\psi| \rangle_\mu(\alpha), \quad \langle \psi \rangle_\mu(\alpha) := \frac{\mathbb{I}^*(\psi\mu)(\alpha)}{\mathbb{I}^*\mu(\alpha)}, \tag{5.1}$$

with the convention $0/0 = 0$. This definition recovers the usual dyadic maximal operator on the tree and the bi-parameter maximal operator on the bi-tree. We have the following connection between these constants:

Proposition 5.1. *Let μ be a measure on a set $\mathcal{P}$, with $\mathcal{P}$ as above. Then*

$$\sup_{w:[w,\mu]_C \leq 1} [w,\mu]_{CE} = \|\mathcal{M}_\mu\|^2_{L^2(\mu)\to L^2(\mu)}. \tag{5.2}$$

Example 5.2. If $\mathcal{P} = \mathcal{T}$ is a usual tree, then the maximal function $\mathcal{M}_\mu$ is essentially the martingale maximal function, and it is well-known that it is bounded on $L^p(\mu)$ with norm at most p'. In particular the right-hand side of (5.2) equals 4. This is the sharp constant in the Carleson embedding theorem on the tree [25].

Example 5.3. If $\mathcal{P} = \mathcal{T}^n = \mathcal{T}_1 \times .. \times \mathcal{T}_n$ is an n-tree and $\mu = \mu_1 \times .. \times \mu_n$ is a product measure, then the n-parameter maximal operator (5.1) can be majorized by the composition of n one-parameter maximal operators

$$\mathcal{M}_\mu \psi \le \mathcal{M}_{1,\mu_1} \circ .. \circ \mathcal{M}_{n,\mu_n} \psi,$$

which are also defined by (5.1) but on the simple trees $\mathcal{T}_i$. Using L^2 bounds for the one-parameter maximal operators we see that the right-hand side of (5.2) is bounded by 4^n. Hence for product measures μ and arbitrary weights w Proposition 5.1 gives the implication (2.1) $\implies$ (1.10). As the n-dimensional Lebesgue measure is product, this is connected to the results of Chang in the case of the bi-disc (see [10]).

Remark 5.4. As we said in the first example, it is known for $n = 1$ the boundedness of Maximal function is equivalent to the Carleson Embedding theorem. Additionally, by Proposition 5.1 and our results we see the rectangular strong Maximal function on $\mathcal{T}^n$ for $n = 2, 3$ is bounded as soon as the Box condition is true and the weight w is of product form: recall that $[w, \mu]_{Box} \le [w, \mu]_C \lesssim [w, \mu]_{Box}$.

Proof of Proposition 5.1. We begin with the inequality $\le$ in (5.2). Let $\psi : \mathcal{P} \to [0, \infty)$ be a non-negative function with $\|\psi\|_{L^2(\mu)} = 1$. Then

$$\begin{aligned}\sum_{\alpha \in \mathcal{P}} w(\alpha) \mathbb{I}^*(\psi\mu)(\alpha)^2 &= \sum_{\alpha \in \mathcal{P}} w(\alpha) \mathbb{I}^*\mu(\alpha)^2 \int_0^{\langle\psi\rangle_\mu(\alpha)} 2s ds \\ &= \int_0^\infty 2s \sum_{\alpha : \langle\psi\rangle_\mu(\alpha) > s} w(\alpha) \mathbb{I}^*\mu(\alpha)^2 ds\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^\infty 2s \sum_{\alpha:\mathcal{M}_\mu\psi(\alpha)>s} w(\alpha)\mathbb{I}^*\mu(\alpha)^2 ds \\
&\leq [w,\mu]_C \int_0^\infty 2s\mu\{\alpha : \mathcal{M}_\mu\psi(\alpha) > s\}ds \\
&= [w,\mu]_C \|\mathcal{M}_\mu(\psi)\|^2_{L^2(\mu)} \\
&\leq [w,\mu]_C \|\mathcal{M}_\mu\|^2_{L^2(\mu)\to L^2(\mu)}.
\end{aligned}
$$

Here we have used that the superlevel sets $\{\alpha : \mathcal{M}_\mu\psi(\alpha) > s\}$ are down-sets. Taking supremum over all weight w such that $[w,\mu]_C \leq 1$ we obtain the inequality $\leq$ in (5.2).

Now we will show the inequality $\geq$ in (5.2). Let $\psi : \mathcal{P} \to [0,\infty)$ be a non-negative function such that $\|\psi\|_{L^2(\mu)} = 1$. The set $\partial\mathcal{P}$ consists of the minimal elements of $\mathcal{P}$. Recall that μ is non-zero only on $\partial\mathcal{P}$. For each $\alpha \in \mathcal{P}$ with $\mathbb{I}^*(\psi\mu)(\alpha) \neq 0$ let

$$
A'(\alpha) := \{\omega \in \partial\mathcal{P}, \omega \leq \alpha \mid \mathcal{M}_\mu\psi(\omega) = \langle\psi\rangle_\mu(\alpha)\},
$$

and let $A'(\alpha) := \emptyset$ otherwise. Enumerate $\mathcal{P} = \{\alpha_1, \alpha_2, \ldots\}$ and set

$$
A(\alpha_j) := A'(\alpha_j) \setminus \bigcup_{j'<j} A'(\alpha_{j'}).
$$

Then, as the sets $A(\alpha)$ are pairwise disjoint:

$$
\begin{aligned}
\sum_{\omega\in\mathcal{P}} (\mathcal{M}_\mu\psi)^2(\omega)\mu(\omega) &= \sum_{\alpha\in\mathcal{P}} \sum_{\omega\in A(\alpha)} \langle\psi\rangle_\mu(\alpha)^2\mu(\omega) \\
&= \sum_{\alpha\in\mathcal{P}} w(\alpha)(\mathbb{I}^*(\psi\mu)(\alpha))^2,
\end{aligned}
$$

where

$$w(\alpha) := (\mathbb{I}^*\mu(\alpha))^{-2} \sum_{\omega \in A(\alpha)} \mu(\omega)$$

with the convention $w(\alpha) = 0$ if $A(\alpha) = \emptyset$. Since the sets $A(\alpha)$ are disjoint and consist of minimal elements then for every down-set $\mathcal{D} \subseteq \mathcal{P}$ we have

$$\sum_{\alpha \in \mathcal{D}} w(\alpha)(\mathbb{I}^*(\mu)(\alpha))^2 = \sum_{\alpha \in \mathcal{D}} \sum_{\omega \in A(\alpha)} \mu(\omega) \leq \sum_{\omega \in \mathcal{D} \cap \partial\mathcal{P}} \mu(\omega) \leq \mu(\mathcal{D}),$$

and thus $[w, \mu]_C \leq 1$. Also, by the calculations above

$$[w, \mu]_{CE} \geq \|\mathcal{M}_\mu \psi\|^2_{L^2(\mu)}.$$

By first taking the supremum over w with $[w, \mu]_C \leq 1$ and then over ψ we obtain the inequality $\geq$ in (5.2). □

Similarly we can also prove:

Proposition 5.5. *Let μ be a measure on $\partial\mathcal{P}$. Then*

$$\sup_{w:[w,\mu]_C \leq 1} [w, \mu]_{HC} = \sup_{E \subseteq \mathcal{P}} \frac{1}{|\mu \mathbf{1}_E|} \|\mathcal{M}_\mu(\mathbf{1}_E)\|^2_{L^2(\mu)}.$$

5.2 A comparison with a result of E. Sawyer

As we mentioned before, two weighted embeddings have enjoyed much attention in the last 50 years. In a continuous setting the paper [26] of E. Sawyer gives a characterization of the 2-dimensional embedding in terms of three “Box Conditions”. Lets translate his results to the case of bi-trees. Suppose the weight w is supported on a set of the form:

$$\operatorname{supp} w \subset \{\alpha \in \mathcal{T}^2 \mid \alpha \geq \omega_0\}. \tag{5.3}$$

where ω is a fixed point in $\partial\mathcal{T}^2$. Then, in [26] we consider the discrete measure on a bi-tree instead of the Lebesgue measure. The result becomes:

Theorem 5.6 (cf. [26, Theorem 1(A)]). *Suppose that the weight w satisfies the support condition* (5.3)*. Then $[w, \mu]_{CE}$ is finite if and only for some $A < \infty$ and every $\beta \in \mathcal{T}^2$ with $\beta \geq \omega_0$ the following conditions hold:*

$$\mathbb{I}^*\mu(\beta)\mathbb{I}w(\beta) \leq A^2, \tag{5.4a}$$

$$\sum_{\alpha \geq \beta \geq \omega_0} \mu(\alpha)\mathbb{I}w(\alpha)^2 \leq A^2\mathbb{I}w(\beta), \tag{5.4b}$$

$$\sum_{\omega_0 \leq \alpha \leq \beta} w(\alpha)\mathbb{I}^*\mu(\alpha)^2 \leq A^2\mathbb{I}^*\mu(\beta). \tag{5.4c}$$

No two of these conditions suffice to ensure $[w, \mu]_{CE} < \infty$.

Remark 5.7. The last condition (5.4c) is just the box condition (1.11). In other words, if we restrict the weight to be supported only on the hooked rectangles, but drop the requirement that it has a product structure, we see that the single box test (1.11) is getting replaced by three single box tests for the pair w, μ.

Remark 5.8. The second condition (5.4b) is not needed in our setting. Recall that μ is non-zero only on the boundary of the bi-tree and hence (5.4b) is implied by the first one.

Remark 5.9. A careful reading of our proof in section 4.2 shows that for a measure μ on the boundary of the bi-tree, a weight w and for every $\alpha \in \mathcal{T}^2$, we have the following condition:

$$w(\alpha) \cdot \mathbb{I}^* \mu(\alpha) \lesssim \mathrm{Cap}(\alpha)$$

as long as the box condition (1.11) holds. Therefore, if the weight w satisfies the support condition (5.3) and moreover is non-increasing, the inequality above easily implies (5.4a). To summarize: if w is non-increasing and satisfies the support condition (5.3), then the Box condition (5.4c) implies the conditions (5.4a) and (5.4b) and hence the embedding (1.10) holds as well. For an application of this, consider the weight w which satisfies $w \equiv 1$ on the set $\{\alpha \in \mathcal{T}^2 \mid \alpha \geq \omega_0\}$ and 0 otherwise. However, this example is the connecting point between our results and the result in [26]: as this particular weight w is a product weight, we already know the single box condition (1.11) is sufficient to imply (1.10).

Chapter 6

Necessary and sufficient conditions

6.1 A product weight is not necessary

As we know, for $n \geq 2$ the n-tree does not have a nice geometric structure. In the case $n = 1$, for any $\omega \in \partial\mathcal{T}$ the set $\{\alpha : \alpha \geq \omega\}$ is totally-ordered. However for $n \geq 2$ this is not true for any $\omega \in \partial\mathcal{T}^2$. Albeit, one could consider a sub-graph of the n-tree which has this property; We start with $[0,1)^n$ and we keep all its descendants which are cubes, namely Cartesian products of n dyadic sub-intervals of $[0,1)$, of generation at most N and of the **same** length. Thus, if we modify the weight w to be non-zero only on such cubes, then we have a structure similar to a simple tree. Note that such a weight is non-product and that the sum on the LHS of (1.5) is over a partially ordered set $\mathcal{P}$ with the following property; for any $\omega \in \partial\mathcal{P}$ the set $\{\alpha : \alpha \geq \omega\}$ is totally-ordered. As we discussed in the introduction, for a set $\mathcal{P}$ with this property, the same proof as in the case $n = 1$ holds. Therefore, for such a weight w the Box condition (1.6) implies (1.5) which implies a product weight is not a necessary condition for any $n \geq 2$ (although it is sufficient for $n = 2, 3$, as we already proved).

In the last section we see another proof of this fact emerging from our try to push the idea of Shur's test to work for higher dimensions. However such proof might not be possible: As we have seen in the case $n = 1$, the weight and its structure played no role. But the weight

w cannot be general for higher dimensions, given Carleson's counterexample. This might be an issue if one tries to use Schur's test for higher dimensions (more in section 7).

6.2 Box condition does not imply Carleson condition for general weight w

Let m_2 to be the planar Lebesgue measure. As mentioned in the introduction, L. Carleson constructed in [7] families $\mathcal{R}_i$ of dyadic sub-rectangles of $Q = [0,1)^2$ having the following two properties:

$$\sum_{\substack{R \subseteq R_0 \\ R \in \mathcal{R}_i}} m_2(R) \leq m_2(R_0), \quad \forall R_0 \subseteq [0,1)^2 \tag{6.1}$$

but

$$\sum_{R \in \mathcal{R}_i} m_2(R) = 1 >> m_2(\bigcup_{R \in \mathcal{R}_i} R) \tag{6.2}$$

with the latter area a small as one wishes. By taking a finite bi-tree $\mathcal{T}_i^2$, large enough to contain the family $\mathcal{R}_i$, and choosing μ s.t. $\mathbb{I}^*\mu = m_2$ and

$$w_i(R) := \begin{cases} \frac{1}{m_2(R)}, & R \in \mathcal{R}_i, \\ 0, & \text{otherwise} \end{cases}$$

we can identify the left-hand sides of (6.1) and (6.2) with the left-hand sides of (1.11) and (2.1), respectively. The measure μ is fixed, and for any $M > 0$ there is $i \in \mathbb{N}$ such that

$$\sup_{w : [w,\mu]_{Box} \leq 1} [w, \mu]_C \geq [w_i, \mu]_C \geq M$$

6.3 Carleson condition does not imply REC

Our aim here is to show that for general w, μ the Carleson condition (2.1) is no longer sufficient for the Restricted Energy Condition (2.2). This example is quite simple and is inspired by the counterexample of R. Fefferman in [24] for the L^p-boundedness of the bi-parameter maximal function for arbitrary measure μ. Our examples are given in $\mathcal{T}^2$ but we can easily extend these to higher dimensions. Our weight w will not be product, otherwise we know that Hereditary Carleson condition and Carleson condition are equivalent. It will be supported on a very small subset of the bi-tree, which differs greatly from the original graph.

For any $N \in \mathbb{N}$ we construct a measure μ and a weight w such that $[w,\mu]_C \leq 4$ but $[w,\mu]_{HC} \geq N$. Therefore, for any $C > 0$ there is an $N \in \mathbb{N}$ such that

$$\sup_{w,\mu:[w,\mu]_C \leq 1} [w,\mu]_{HC} \geq [w,\mu]_{HC} \geq C$$

We start by letting $Q_i = [0, 2^{-i+1}) \times [0, 2^{-N+i})$ for $i = 1, .., N$ and $Q_0 = [0, 2^{-N})^2$. Let the measure μ satisfy $\mu(Q_0) = \mathbb{I}^*\mu(Q_i^{++}) = 1$ where Q_i^{++} is the upper right quadrant of Q_i, and $\mu \equiv 0$ everywhere else. We also define the weight w to be:

$$w(R) := \begin{cases} 1 & \text{if } R = Q_j \text{ for some } j \in \{0, .., N\}, \\ 0 & \text{otherwise.} \end{cases}$$

So we have $N+1$ nodes α where $w(a)$ is equal to 1. For the node Q_0 we have

$$\sum_{\alpha \in \mathcal{T}_N^2} w(\alpha)\Big(\mathbb{I}^*\big(\mu \mathbf{1}_{Q_0}\big)(\alpha)\Big)^2 = \sum_{i=0}^{N} \big(\mathbb{I}^*\mu(Q_0 \cap Q_i)\big)^2 = (N+1)\cdot 1^2 = (N+1)|\mu \mathbf{1}_{Q_0}|$$

which means $[w,\mu]_{HC} \geq N$. Now, for an arbitrary down-set $\mathcal{D} \subseteq \mathcal{T}_N^2$ we have

$$\sum_{\alpha\in\mathcal{D}} w(\alpha)\mathbb{I}^*\mu(\alpha)^2 = \sum_{\alpha\in\mathcal{D},\, w(\alpha)\neq 0} \mathbb{I}^*\mu(\alpha)^2 = \sum_{j:Q_j\in\mathcal{D}} \mathbb{I}^*\mu(Q_j)^2$$

Then since $Q_i^{++} \cap Q_j = \emptyset$ unless $i = 0, j$, we have

$$\sum_{j:Q_j\in\mathcal{D}} \mathbb{I}^*\mu(Q_j)^2 \leq \sum_{j:Q_j\in\mathcal{D}} 2^2 \leq 4\mu(\mathcal{D})$$

which finishes the counterexample.

6.4 The lack of maximal principle matters

In this section we construct a measure μ which gives an example of the following fact about the potentials on a bi-tree: The maximal principle fails. Another example of such a measure μ can be found in [4] (Proposition 5.2). The weight there satisfies $w \equiv 1$, but here it takes the values $0, 1$. For $s \in \mathbb{N}$ and $N = 2^s$ we construct a measure μ such that

$$\mathbb{V}^\mu \lesssim 1 \quad \text{on } \operatorname{supp}\mu, \tag{6.3}$$

but

$$\max \mathbb{V}^\mu \geq \mathbb{V}^\mu(\omega_0) \gtrsim s. \tag{6.4}$$

where $\omega_0 = [0, 2^{-N})^2$.

We define a collection of rectangles

$$Q_j := [0, 2^{-2^j}] \times [0, 2^{-2^{-j+s}}], \quad j = 1, .., s \tag{6.5}$$

and we define

$$\mathcal{R} := \{R : \ Q_j \subset R \text{ for some } j = 1, .., s\}$$
$$w_Q := \mathbf{1}_{\mathcal{R}}(Q) \tag{6.6}$$
$$\mu(\omega) := \frac{1}{N} \sum_{j=1}^{s} \frac{1}{\# Q_j^{++}} \mathbf{1}_{Q_j^{++}}(\omega).$$

where $\# Q$ denotes the total amount of $\omega \in \partial \mathcal{T}_N^2$ with $\omega \le Q$. Observe that on Q_j the measure is basically a uniform distribution of the mass $\frac{1}{N}$ over the upper right quarter Q_j^{++} of the rectangle Q_j (and these quadrants are disjoint).

To prove (6.3) we fix $\omega \in Q_j^{++}$ and we split: (Q_j^u, Q_j^r are the upper and right half of Q_j resp.)

$$\mathbb{V}^\mu(\omega) = \mathbb{V}^\mu_{Q_j^{++}}(\omega) + \mu(Q_j^u) + \mu(Q_j^r) + \mathbb{V}^\mu(Q_j),$$

where the first term sums up $\mathbb{I}^*\mu(\alpha)$ for α with $\omega \le \alpha \le Q_j^{++}$. It is easy to see that $\mathbb{V}^\mu_{Q_j^{++}}(\omega) \lesssim \frac{1}{N}$ (the left-hand side is a double geometric sum). Trivially $\mu(Q_j^u) + \mu(Q_j^p) \le \frac{2}{N}$. The non-trivial part is the estimate

$$\mathbb{V}^\mu(Q_j) \lesssim 1 \,. \tag{6.7}$$

For each dyadic rectangle $R \ge \omega_0$ and each j' we have

$$\text{either } Q_{j'} \subseteq R, \text{ or } Q_{j'}^{++} \cap R = \emptyset. \tag{6.8}$$

Moreover, since the sides of rectangles Q_j are nested, the set $\{j' : Q_{j'} \subseteq R\}$ is an interval that contains j. For an interval of integers $[m, m+k]$ let

$$C^{[m,m+k]} := \left\{R \geq \omega_0 \mid \{j' : Q_{j'} \subseteq R\} = [m, m+k]\right\}$$

Since each rectangle in $C^{[m,m+k]}$ contains $[0, 2^{-2^m}] \times [0, 2^{-2^{-m-k+s}}]$, we have

$$\#C^{[m,m+k]} \leq (2^m + 1)(2^{-m-k+s} + 1) \lesssim 2^{-k+s} \tag{6.9}$$

It follows that

$$\mathbb{V}^\mu(Q_j) = \sum_{[m,m+k]\ni j} (\#C^{[m,m+k]})(k+1)\frac{1}{2^s} \lesssim \sum_{k\geq 0}(k+1)^2 2^{-k+s}\frac{1}{2^s} \lesssim 1. \tag{6.10}$$

This shows (6.7), and hence (6.3) is also proved.

Now we will estimate $\mathbb{V}^\mu(\omega_0)$ from below. To this end we need a more careful *lower* bound on $\#C^{[m,m+k]}$. The set $C^{\{j\}}$ contains all rectangles R that contain Q_j and are contained in $[0, 2^{-2^{j-1}-1}] \times [0, 2^{-2^{-j-s-1}-1}]$, so

$$\#C^{\{j\}} \geq 2^{j-1} \cdot 2^{-j-s-1} \gtrsim 2^s. \tag{6.11}$$

Hence

$$\mathbb{V}^\mu(\omega_0) \geq \sum_{j=1}^{s}(\#C^{\{j\}})\frac{1}{2^s} \gtrsim s. \tag{6.12}$$

6.5 REC does not imply embedding

In this section we use the same dyadic rectangles $\{Q_j\}$ as above. We also keep the same measure μ to which we add an extra piece of measure. We have a few more definitions:

$$Q_{0,j} := Q_j, \quad \mu_0 := \mu \text{ from the previous section}\,.$$

Next we continue with defining a sequence of collections $\mathcal{Q}_k,\ k = 0, \dots, K \approx \log s$ of dyadic rectangles as follows

$$\mathcal{Q}_k := \left\{ Q_{k,j} := \bigcap_{i=j}^{j+2^k-1} Q_{0,i},\ j = 1, \dots, s - 2^k \right\},\ k = 1, \dots, K. \tag{6.13}$$

In other words, $\mathcal{Q}_k$ consists of the intersections of 2^k consecutive elements of the basic collection $\mathcal{Q}_0$. The total amount of rectangles in $\mathcal{Q}_k$ is denoted by $s_k = s - 2^k + 1$.

For $k = 1, \dots, K$ let

$$\mu_k(\omega) := \frac{2^{-2k}}{N} \sum_{j=1}^{s_k} \frac{1}{\# Q_{k,j}^{++}} \mathbf{1}_{Q_{k,j}^{++}}(\omega), \quad \omega \in \partial \mathcal{T}^2,$$

and define

$$\mu := \mu_0 + \sum_{k=1}^{K} \mu_k.$$

By duality the inequality (1.10) is equivalent to the Carleson embedding inequality

$$\int_{\mathcal{T}^2} (\mathbb{I}(fw))^2 d\mu \le [w, \mu]_{CE} \int_{\mathcal{T}^2} f^2 \cdot w. \tag{6.14}$$

We test the inequality (6.14) with the function

$$f(R) := \mathbb{I}^*\mu_0(R).$$

Using (6.3) we obtain

$$\int_{\mathcal{T}^2} f^2 \cdot w = \int_{\mathcal{T}^2} \mathbb{V}^{\mu_0} d\mu_0 \lesssim |\mu_0| = \frac{s}{N}\,. \tag{6.15}$$

On the other hand, by definition (6.13) and replacing s by 2^k in (6.12) we obtain

$$\mathbb{V}^{\mu_0}(Q_{k,j}) \gtrsim 2^k. \tag{6.16}$$

It follows that

$$\int_{\mathcal{T}^2} (\mathbb{I}(fw))^2 d\mu = \int_{\mathcal{T}^2} (\mathbb{V}^{\mu_0})^2 d\mu = \sum_{k=1}^{K} \int (\mathbb{V}^{\mu_0})^2 d\mu_k \gtrsim \sum_{k=1}^{K} 2^{2k} |\mu_k| \sim \frac{s}{N} \log s. \tag{6.17}$$

Substituting (6.15) and (6.17) in (6.14) we obtain $[w,\mu]_{CE} \gtrsim \log s$.

We claim that $[w,\mu]_{HC} \lesssim 1$. This means that for any collection $\mathcal{A}$ of dyadic rectangles, setting $A := \cup_{R\in\mathcal{A}} R$, we have

$$\mathcal{E}[\mu|A] \lesssim |\mu \mathbf{1}_A|. \tag{6.18}$$

To show (6.18) let $\nu_k := \mu_k|A$, $k = 0, \dots, K$. Then

$$\mathcal{E}[\mu|A] = \sum_{n,k} \int \mathbb{V}^{\nu_n} \nu_k \le 2 \sum_{n\ge k} \int \mathbb{V}^{\nu_n} \nu_k \le 2 \sum_{n\ge k} \int \mathbb{V}^{\mu_n} \nu_k.$$

Since $\operatorname{supp}\nu_k \subseteq \operatorname{supp}\mu_k$ it suffices to show

$$\sum_{n\geq k} \mathbb{V}^{\mu_n} \lesssim 1 \quad \text{on} \quad \operatorname{supp}\mu_k. \tag{6.19}$$

The claim (6.19) has the advantage that it does not depend on $\mathcal{A}$ any more.

For every $R \in \mathcal{R}$ we have

$$\mu_n(R) = 2^{-2n}\#\{Q_{n,j} \subseteq R\} \leq 2^{-2n}(\#\{Q_{0,j} \subseteq R\} + 2^n)$$
$$\leq 2^{-n}(\#\{Q_{0,j} \subseteq R\} + 1) \leq 2 \cdot 2^{-n}\mu_0(R).$$

It follows that

$$\mathbb{V}^{\mu_n}(Q_{k,j}) \lesssim 2^{-n}\mathbb{V}^{\mu_0}(Q_{k,j}) \leq 2^{-n} \sum_{i=j}^{j+2^k-1} \mathbb{V}^{\mu_0}(Q_{0,i}) \lesssim 2^{k-n},$$

where the last inequality follows from (6.3). This implies (6.19) and therefore (6.18).

Finally, for any $C > 0$ there is an $s \in \mathbb{N}$ with $logs \geq C$ and so we get that

$$\sup_{w,\mu:[w,\mu]_{HC}\leq 1} [w,\mu]_{CE} \geq C$$

Chapter 7

Open questions

7.1 Failure of Schur's test for $n \geq 2$

Now let $\mathcal{T}^n$ be a finite n-tree. In analogy to subsection 3.1 we define the operator S as follows: for $\beta \in \mathcal{T}^n$ and a function $f \in \ell^2(\mathcal{T}^n, w)$

$$S(f)(\beta) = \sum_{\alpha \in \mathcal{T}^n} (wf)(\alpha) \cdot \nu(\alpha \wedge \beta)$$

We want to show $S : \ell^2(\mathcal{T}^n, w) \to \ell^2(\mathcal{T}^n, w)$ is bounded if the following "box" condition holds

$$\sum_{\alpha \leq \beta} w(\alpha) \cdot \nu(\alpha)^2 \leq \nu(\beta), \quad \forall \beta \in \mathcal{T}^n$$

For dimension $n = 1$ we used Schur's Test successfully in subsection 3.1. Hence, we try to follow the same scheme for higher dimensions, although we hit an obstacle as we are not able to estimate certain terms which appear only on poly-trees. It would become obvious the problem is the same for any dimension bigger than 2, hence we use $n = 2$ for simplicity.

By $\alpha {>_1} \beta$ we mean inclusion in the first coordinate and by $\alpha {>_2} \beta$ in the second.

As in the case $n = 1$ we start with a function f_0 defined as $f_0(\beta) = \nu(\beta)$.

Then we write

$$
\begin{aligned}
S(f_0)(\beta) &= \sum_{\alpha\in\mathcal{T}^n} (wf_0)(\alpha)\cdot\nu(\alpha\wedge\beta) \\
&= \sum_{\alpha\in\mathcal{T}^n} (w\nu)(\alpha)\cdot\nu(\alpha\wedge\beta) \\
&= \Bigg(\sum_{\alpha>\beta} + \sum_{\alpha\le\beta} + \sum_{\substack{\alpha\le_1\beta\\ \alpha>_2\beta}} + \sum_{\substack{\alpha>_1\beta\\ \alpha\le_2\beta}}\Bigg)(w\nu)(\alpha)\cdot\nu(\alpha\wedge\beta) \\
&:= f_1(\beta) + \sum_{\alpha\le\beta} w(\alpha)\cdot\nu(\alpha)^2 + g_1(\beta)
\end{aligned}
$$

where

$$f_1(\beta) = \nu(\beta)\cdot\sum_{\alpha>\beta}(wf_0)(\alpha)$$

and

$$g_1(\beta) = \Bigg(\sum_{\substack{\alpha\le_1\beta\\ \alpha>_2\beta}} + \sum_{\substack{\alpha>_1\beta\\ \alpha\le_2\beta}}\Bigg)(wf_0)(\alpha)\cdot\nu(\alpha\wedge\beta)$$

The subscripts $1, 2$ mean we sum with respect to the 1st and 2nd coordinate respectively.

As before, the middle term is estimated using our assumption and we thus get

$$S(f_0)(\beta) \le f_1(\beta) + f_0(\beta) + g_1(\beta)$$

We define recursively the functions f_i, g_i by the formula

$$f_i(\beta) = \nu(\beta)\cdot\sum_{\alpha>\beta}(wf_{i-1})(\alpha)$$

and

$$g_i(\beta) = \left(\sum_{\substack{\alpha \leq_1 \beta \\ \alpha >_2 \beta}} + \sum_{\substack{\alpha >_1 \beta \\ \alpha \leq_2 \beta}} \right) (wf_{i-1})(\alpha) \cdot \nu(\alpha \wedge \beta)$$

Then, we see

$$S(f_i)(\beta) = f_{i+1}(\beta) + \sum_{\gamma \leq \beta} (wf_i)(\gamma) \cdot \nu(\gamma) + g_{i+1}(\beta)$$

For the term in the middle we have

$$\begin{aligned} \sum_{\gamma \leq \beta} (wf_i)(\gamma) \cdot \nu(\gamma) &= \sum_{\gamma \leq \beta} w(\gamma) \cdot \nu(\gamma) \cdot \nu(\gamma) \cdot \sum_{\alpha > \gamma} (wf_{i-1})(\alpha) \\ &= \sum_{\gamma \leq \beta} \sum_{\alpha > \gamma} w(\gamma) \cdot \nu(\gamma)^2 \cdot (wf_{i-1})(\alpha) \\ &:= A_1(\beta) + A_2(\beta) \end{aligned}$$

Where

$$A_1(\beta) = \left(\sum_{\gamma \leq \beta} \sum_{\gamma < \alpha \leq \beta} + \sum_{\gamma \leq \beta} \sum_{\alpha > \beta} \right) w(\gamma) \cdot \nu(\gamma)^2 \cdot (wf_{i-1})(\alpha)$$

$$A_2(\beta) = \left(\sum_{\substack{\alpha >_1 \beta \\ \gamma <_2 \alpha \leq_2 \beta}} + \sum_{\substack{\alpha >_2 \beta \\ \gamma <_1 \alpha \leq_1 \beta}} \right) w(\gamma) \cdot \nu(\gamma)^2 \cdot (wf_{i-1})(\alpha)$$

Below we show that

$$A_1(\beta) \leq S(f_{i-1})(\beta)$$
$$A_2(\beta) \leq g_i(\beta)$$

which imply that

$$S(f_i)(\beta) \leq f_{i+1}(\beta) + g_{i+1}(\beta) + S(f_{i-1})(\beta) + g_i(\beta)$$

and as

$$S(f_0)(\beta) \leq f_1(\beta) + f_0(\beta) + g_1(\beta)$$

by recursion we get

$$S(f_m)(\beta) \leq \sum_{i=0}^{m+1} h_i(\beta) \tag{7.1}$$

where $h_0 := f_0$ and for $i \geq 1$, $h_i = f_i + g_i$.

Remark 7.1. What is missing is an estimate for $S(g_i)$. Given the discussion in the beginning of chapter 6 we believe its impossible to get such an estimate, if w is general. However, as the functions g_i collect all the terms we can not estimate, we might try to "cancel" their contribution by considering a weight w which is non-zero only on sub-squares of $[0,1)^n$. This results to a structure similar to a simple tree: for non-trivially intersecting $\alpha, \beta \in \mathcal{T}^n$ we either have $w(\alpha) = 0$ or $w(\beta) = 0$. This implies that $g_i \equiv 0$ for any $i \in \mathbb{N}$ and so we have $h_i = f_i$. Then, by taking F as in (3.5) and using (7.1) we get S

is bounded from $\ell^2(\mathcal{T}^n, w) \to \ell^2(\mathcal{T}^n, w)$. Of course we saw a much simpler proof of this using the one-dimensional case in subsection 6.1.

7.1.1 Estimates for A_1, A_2

For A_1 we change the order of summation for both double sums and we use our assumption to get:

$$\begin{aligned}
A_1(\beta) &= \\
&= \Big(\sum_{\gamma \le \beta} \sum_{\gamma < \alpha \le \beta} + \sum_{\gamma \le \beta} \sum_{\alpha > \beta} \Big) w(\gamma) \cdot \nu(\gamma)^2 \cdot (wf_{i-1})(\alpha) \\
&= \Big(\sum_{\alpha \le \beta} \sum_{\gamma < \alpha} + \sum_{\alpha > \beta} \sum_{\gamma \le \beta} \Big) w(\gamma) \cdot \nu(\gamma)^2 \cdot (wf_{i-1})(\alpha) \\
&\le \sum_{\alpha \le \beta} \nu(\alpha) \cdot (wf_{i-1})(\alpha) + \sum_{\alpha > \beta} \nu(\beta) \cdot (wf_{i-1})(\alpha) \\
&= \sum_{\alpha \in \mathcal{T}^2} w(\alpha) \cdot \nu(\alpha \wedge \beta) \cdot f_{i-1}(\alpha) \\
&= S(f_{i-1})(\beta)
\end{aligned}$$

Next, recall A_2 is a sum of two double sums:

$$A_2(\beta) = \sum_{\gamma \le \beta} \Big(\sum_{\substack{\alpha >_1 \beta \\ \gamma <_2 \alpha \le_2 \beta}} + \sum_{\substack{\alpha >_2 \beta \\ \gamma <_1 \alpha \le_1 \beta}} \Big) w(\gamma) \cdot \nu(\gamma)^2 \cdot (wf_{i-1})(\alpha)$$

We will take care of the first double sum only as the second follows similarly. By carefully writing the double sum as four sums (each one on the 1-tree) and by changing the order of summation for the second and third sum we have the following:

$$A_2(\beta) = \sum_{\gamma \leq_1 \beta} \sum_{\gamma \leq_2 \beta} \sum_{\gamma <_2 \alpha \leq_2 \beta} \sum_{\alpha >_1 \beta} w(\gamma) \cdot \nu(\gamma)^2 \cdot (wf_{i-1})(\alpha)$$

$$= \sum_{\gamma \leq_1 \beta} \sum_{\alpha \leq_2 \beta} \sum_{\gamma <_2 \alpha} \sum_{\alpha >_1 \beta} w(\gamma) \cdot \nu(\gamma)^2 \cdot (wf_{i-1})(\alpha)$$

Notice the restrictions above would imply $\gamma \leq \alpha \wedge \beta$. By changing the summation again and use our assumption we see

$$\sum_{\alpha >_1 \beta} \sum_{\alpha \leq_2 \beta} \sum_{\gamma \leq_1 \beta} \sum_{\gamma <_2 \alpha} w(\gamma) \cdot \nu(\gamma)^2 \cdot (wf_{i-1})(\alpha)$$

$$\leq \sum_{\alpha >_1 \beta} \sum_{\alpha \leq_2 \beta} \sum_{\gamma \leq \alpha \wedge \beta} w(\gamma) \cdot \nu(\gamma)^2 \cdot (wf_{i-1})(\alpha)$$

$$\leq \sum_{\alpha >_1 \beta} \sum_{\alpha \leq_2 \beta} \nu(\alpha \wedge \beta) \cdot (wf_{i-1})(\alpha)$$

$$= \sum_{\alpha >_1 \beta} \sum_{\alpha \leq_2 \beta} w(\gamma) \cdot \nu(\gamma)^2 \cdot (wf_{i-1})(\alpha)$$

The symmetric of this last sum is obtained by manipulating accordingly the second double sum, which means $A_2(\beta) \leq g_i(\beta)$ as claimed.

7.2 Potential theory and the hurdles for $n \geq 4$

The main problem with pushing the results to n-trees, $n \geq 4$, lies with the analogue of Lemma 3.22.

For exampe for dimension $n = 4$ the analogous candidate φ would be

$$\varphi = \frac{2}{\lambda}\big(I_1 f \cdot \mathbb{I}_{234} f + I_2 f \cdot \mathbb{I}_{134} f + I_3 f \cdot \mathbb{I}_{124} f + I_4 f \cdot \mathbb{I}_{123} f + \\ \mathbb{I}_{12} f \cdot \mathbb{I}_{34} f + \mathbb{I}_{13} f \cdot \mathbb{I}_{24} f + \mathbb{I}_{14} f \cdot \mathbb{I}_{23} f + f \mathbb{I} f\big). \tag{7.2}$$

where $\mathbb{I}_{12} = I_1 I_2$, et cetera and $\mathbb{I} = I_{1234}$. We can show part $a)$ in Lemma 3.21 analogously. But for part $b)$ we need to estimate the energy of each one of these terms. For the first four terms we can do it successfully exactly as in Lemma 3.21. Also, the last term satisfies $\mathbb{I} f \leq \delta$ on $supp(f)$. However, we can not prove the analogue of Lemma 3.21 for the other terms and we have the next question.

Question 7.2. *Let $f : \mathcal{T}^4 \to [0, \infty)$ be superadditive on each parameter separately. Let w be a positive product weight. Suppose that* $\operatorname{supp} f \subseteq \{\mathbb{I}(wf) \leq \delta\}$. *Then*

$$\sum_{\mathcal{T}^4} w\big(I_1 I_2 (w_1 w_2 f) \cdot I_3 I_4 (w_3 w_4 f)\big)^2 \mathbf{1}_{\mathbb{I}(wf) \leq \lambda} \lesssim \delta^{\kappa} \lambda^{r} \sum_{\mathcal{T}^4} w f^2$$

for some appropriate powers κ, r. The problem here is we can not use the same proof as we now have the product of two two-dimensional Hardy operators.

One might ask: can we do it differently? The method above first appeared in [19]. In [5] there is a slightly different majorization which is based on the following lemma:

Lemma 7.3. *Let* $\operatorname{supp} f \subseteq \{Ig \leq \delta\}$. *Let g be a superadditive function. There exists $\varphi : T \to [0, \infty)$ such that*

$$a)\ I\varphi(\omega) \geq If(\omega) \quad \forall \omega \in \partial\mathcal{T} \colon Ig(\omega) \in [\lambda, 2\lambda] \tag{7.3}$$

$$b)\ \int_{\mathcal{T}} \varphi^2 \le C\frac{\delta}{\lambda}\int_{\mathcal{T}} f^2. \tag{7.4}$$

Proof. Put

$$\varphi = \lambda^{-1} If \cdot g \cdot \mathbf{1}_{Ig\le 4\lambda}\,,$$

and see [5]. ☐

Using this lemma, one is able to prove the corresponding of Lemma 3.22 on a bi-tree. However if we try to prove a 2-dimensional analogue of Lemma 7.3 as a way to get Lemma 3.22 then we are going to fail. The analogous of the function φ in Lemma 7.3 is not a good candidate anymore as, according to the methods found in [5], it is impossible to get

$$\int_{\mathcal{T}^2} (\mathbb{I}f\cdot g)^2 \mathbf{1}_{\mathbb{I}g\le\lambda} \lesssim \delta^{\kappa}\lambda^{r}\sum_{\mathcal{T}^2} f^2$$

for some appropriate powers κ, r. Hence, we are, again, in need of different function φ. We have the following question.

Question 7.4. *Suppose* $\operatorname{supp} f \subseteq \{\mathbb{I}g \le \delta\}$ *and* g *be a function which is superadditive for each parameter separately. Then, there exists* $\varphi : \mathcal{T}^2 \to [0,\infty)$ *such that*

$$a)\ \mathbb{I}\varphi(\omega) \ge \mathbb{I}f(\omega) \quad \forall \omega \in \partial\mathcal{T}^2 \colon \mathbb{I}g(\omega) \in [\lambda, 2\lambda] \tag{7.5}$$

$$b)\ \int_{\mathcal{T}^2} \varphi^2 \le C\Big(\frac{\delta}{\lambda}\Big)^{\tau}\int_{\mathcal{T}^2} f^2 \tag{7.6}$$

with some positive τ*.*

The possible positive answer of this question could finish the argument for $n = 3$ (which we already know) but could also open up the road for higher dimensions. Therefore, if we could prove Question 7.4 for $n-1$ (instead of 2) then following Theorem 3.1 of [5] we could get the analogue of Lemma 3.21 on the n-tree.

One of the main issues we encounter is that a function g which is super-additive on each variable satisfies $\Delta_i g \geq 0$ for each coordinate i, but the same is not true if we consider compositions of these operators. For example, our assumptions can not force $\Delta_1\Delta_2 g$ to be non-negative. Hence our methods fail to be generalized in higher dimensions. Therefore, one may wonder if these methods are limited or if the developed potential theory can not answer the question for dimensions $n \geq 4$ and we are again in need of new methods.

Finally, a natural question arises: what if its impossible to prove the embedding for dimension $n \geq 4$?

7.3 Counterexample for $n \geq 4$?

It is natural to ask this question under these circumstances. One might think there is a lot of freedom in four dimensions which might trigger a counterexample. Hence, the following question is also possible:

Question 7.5. *Let $n \geq 4$ and $\mathcal{D}^n$ be the collection of all dyadic n rectangles in $[0,1)^n$. There is a measure μ on $[0,1)^n$ such that for any dyadic n-rectangle R_0:*

$$\sum_{\substack{R\subseteq R_0\\ R\in\mathcal{D}^n}} \mu(R)^2 \leq \mu(R_0)$$

but for any $C>0$ there is some $\psi \in L^2([0,1)^n,\mu)$ with

$$\sum_{R\in\mathcal{D}^n} \Big(\int_R \psi d\mu\Big)^2 > C\int_{[0,1)^n} \psi^2 d\mu$$

Similar questions could be posed by substituting the Box condition with any of the other conditions (Carleson, Hereditary Carleson or even the capacity condition of [4]). So far, we do not know if any of these imply the embedding for $n \geq 4$. However, the methods we used for $n = 2, 3$ reveal that as soon as one proves the analogue of Theorem 3.5 for higher dimensions then we get "Box condition to Embedding" and "General Capacitary to Embedding".

7.4 Embedding for $p \neq 2$

So far our investigation is around the case $p = 2$. As noted in the second chapter the Carleson embedding theorem is true for any $1 < p < \infty$ and a proof can be found in [25]. Although they proved it for $p = 2$ the exact same proof works for general p. In the case of a simple tree and $p = 2$ the proof (which used the same Bellman function) can be found in [2] and for general $1 < p < \infty$ in a recent preprint, see [9].

Hence, as the embedding theorem is true for dimension $n = 2, 3$ a natural question arises: Is it true for any $1 < p < \infty$ other than 2? We do not know the answer. It could be true but our potential theoretic methods lose some important properties. For example, the Box condition becomes, for $\beta \in \mathcal{T}^n$

$$\sum_{\alpha \leq \beta} \mathbb{I}^*\mu(\alpha)^p \lesssim \mathbb{I}^*\mu(\beta)$$

and so one should define the potential to be $\mathbb{V}^\mu = \mathbb{I}\big((\mathbb{I}^*\mu)^{p-1}\big)$. As we can see our potential is not linear anymore and moreover the function $g = (\mathbb{I}^*\mu)^{p-1}$ is superadditive in each variable for $p > 2$, but the energy estimates do not hold with exponent p' (the conjugate of p). One can construct a superadditive f which gives a counterexample of the corresponding of Lemma 3.21 on a simple tree $\mathcal{T}$ and then consider a product of such functions in the case of bi-tree or tri-tree.

On the other hand, if $1 < p < 2$ then the energy estimates hold with power p' and some appropriate powers for δ, λ (by interpolating between 2 and ∞) but g is not superadditive anymore, as the opposite inequality holds (i.e. g is subadditive). Hence $p = 2$ is the only p such the conditional (on p) statements "g superadditive" and "energy estimates hold" are both true. Therefore one might need to find new techniques in order to prove the embedding for general p. These possible new techniques might have the advantage of resolving the question for dimensions $n \geq 4$ as well. Until then, all these interesting questions remain open.

BIBLIOGRAPHY

BIBLIOGRAPHY

[1] Adams, D.R., Hedberg, L.I., 1996: *Function Spaces and Potential Theory*, Grundlehren der mathematischen Wissenschaften; **314**, Springer-Verlag Berlin Heidelberg, 368pp.

[2] Arcozzi, N., Holmes, I., Mozolyako, P., Volberg, A., 2018: *Bellman function sitting on a tree*, arXiv:1809.03397.

[3] Arcozzi, N., Holmes, I., Mozolyako, P., Volberg, A., 2020: *Bi-parameter embedding and measures with restriction energy condition*, Mathematische Annalen **377**, 643-674, doi: 10.1007/s00208-019-01937-x.

[4] Arcozzi, N., Mozolyako, P., Perfekt, K-M., Sarfatti, G., 2018: *Carleson measures for the Dirichlet space on the bidisc*, arXiv:1811.04990.

[5] Arcozzi, N., Mozolyako, P., Psaromiligkos, G., Volberg, A., Zorin-Kranich, P., 2019: *Bi-parameter Carleson embeddings with product weights*, arXiv:1906.11150.

[6] Arcozzi, N., Rochberg, R., Sawyer, E., Wick B.D., 2014: *Potential theory on trees, graphs and Ahlfors-regular metric spaces*, Potential Anal., **41**, no. 2, 317-366, doi: 10.1007/s11118-013-9371-8.

[7] Carleson L., 1974: *Counterexample for measures bounded on H^p for the bi-disc*, Mittag Leffler Report No. **7**.

[8] Carleson L., 1958: *An Interpolation Problem for Bounded Analytic Functions*, American Journal of Mathematics, 80(4), 921-930. doi: 10.2307/2372840.

[9] Cavina, M., 2020: *Bellman function for Hardy's inequality over dyadic trees*, arXiv:2002.07532.

[10] Chang, S-Y. A., 1979: *Carleson measure on the bi-disc,* Ann. of Math. (2), **109**, no. 3, 613-6100 doi: 10.2307/1971229.

[11] Chang, S-Y. A., Fefferman, R., 1980: *A Continuous Version of Duality of H^1 with BMO on the Bidisc* Ann. of Math., (2), **112**, no. 1, 179-201, doi: 10.2307/1971324.

[12] Duoandikoetxea, J., 2000: *Fourier Analysis*, Graduate studies in Mathematics; **29**, Graduate Studies in Mathematics, 210 pp.

[13] Grafakos, L., Torres R., 2002: *Multilinear Calderón–Zygmund theory*, Adv. Math. 165, 124-164, doi: 10.1006/aima.2001.2028.

[14] Holmes, I., Psaromiligkos, G., Volberg, A., 2019: *A comparison of box and Carleson conditions on bi-trees.* Prepint, 2019. arXiv:1903.02478.

[15] Meyer, Y., Coifman, R.R., 1991: *Ondelettes et opérateurs, III: Opérateurs multilinéaires*, Hermann, 270 pp.

[16] Kenig, C.E., 2004: *On the local and global well-posedness for the KP-I equation*, Ann. IHP Anal. Non Lin. **21**, 827-838, doi: 10.1016/j.anihpc.2003.12.002.

[17] Kato, T., Ponce, G., 1988: *Commutator estimates and the Euler and Navier-Stokes equations*, Comm. Pure Appl. Math., **41**, 891-907, doi: 10.1002/cpa.3160410704.

[18] Kenig, C.E., Stein, E. M., 1999: *Multilinear estimates and fractional integration,* Math. Res. Lett. **6**, 10.4310/MRL.1999.V6.N1.A1.

[19] Mozolyako, P., Psaromiligkos, G., Volberg, A., Zorin-Kranich P., 2020: *Carleson embedding on tri-tree and on tri-disc*, arXiv:2001.02373.

[20] Mozolyako, P., Psaromiligkos, G., Volberg, A., Zorin-Kranich P., 2020: *Combinatorial property of all positive measures in dimensions* 2 *and* 3, Comptes Rendus Mathmatique, Tome **358** no. 6, 721-725, doi: 10.5802/crmath.90

[21] Mozolyako, P., Psaromiligkos, G., Volberg A., 2020: *Counterexamples for multi-parameter weighted paraproducts*, Comptes Rendus Mathmatique, Tome **358** no. 5, 529-534, doi: 10.5802/crmath.52

[22] Muscalu, C., Pipher, J, Tao, T. and Thiele, C., 2004: *Bi-parameter paraproducts.* Acta Math. **193**, 269-296, doi: 10.1007/BF02392566.

[23] Muscalu, C., Pipher, J, Tao, T. and Thiele, C., 2006: *Multi-parameter paraproducts.* Rev. Mat. Iberoamericana **22**, no. 3, 963-976, doi: 10.4171/RMI/480.

[24] Fefferman, R., 1981: *Strong Differentiation with Respect to Measures.* American Journal of Mathematics, **103**, no. 1, 33-40, doi: 10.2307/2374188.

[25] Nazarov, F., Treil, S., Volberg A., 1999: *The Bellman functions and two-weight inequalities for Haar multipliers*, J. Amer. Math. Soc, **12**, no. 4, 909-928, doi:10.1090/S0894-0347-99-00310-0.

[26] Sawyer, E.T., 1982: *A characterization of a two weight norm inequality for maximal operators*, Studia Math., **75**, 1-11, doi: 10.4064/sm-75-1-1-11

[27] Sawyer, E.T., 1985: *Weighted inequalities for the two-dimensional Hardy operator*, Studia Math., **82**, no. 1, 1-16, doi: 10.4064/sm-82-1-1-16

[28] Stegenga, D.A., 1980: *Multipliers of the Dirichlet space*, Illinois J. Math. **24**, no. 1, 113-139, doi: 10.1215/ijm/1256047800

[29] Stein, E.M., 1970: *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series; **30**, Princeton University Press, 290 pp.

[30] Tao, T., 1999: *Dyadic product H^1, BMO, and Carleson's counterexample*, unpublished note, available at http://www.math.ucla.edu/~tao/preprints/Expository/product.dvi.