ON ORTHOGONAL LOCAL MODELS OF SHIMURA VARIETIES

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ABSTRACT

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We study local models that describe the singularities of Shimura varieties of non-PEL type for orthogonal groups at primes where the level subgroup is given by the stabilizer of a single lattice. In particular, we use the Pappas-Zhu construction and we give explicit equations that describe an open subset around the "worst" point of orthogonal local models given by a single lattice. These equations display the affine chart of the local model as a hypersurface in a determinantal scheme. Using this we prove that the special fiber of the local model is reduced and Cohen-Macaulay.

Moreover, by using the explicit description of this affine chart, we resolve the singularities of our local model. By combining results of Kisin and Pappas, this leads to the construction of regular p-adic integral models for the corresponding orthogonal Shimura varieties.

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KEY TO SYMBOLS

p	an odd prime
F	a finite field extension of $\mathbb{Q}_p$
$\mathcal{O}_F$	the ring of integers of F
π	a uniformizer of $\mathcal{O}_F$
κ_F	the residue field of F
$\bar{F}$	an algebraic closure of F
$\breve{F}$	the completion of the maximal unramified extension of F in $\bar{F}$
$\mathcal{O}$	the ring of integers of $\breve{F}$
k	the residue field of $\breve{F}$
d	the dimension of the F-vector space V
Λ	an $\mathcal{O}_F$-lattice in V
$\Lambda^\vee$	the dual of Λ in V
l	the distance of the lattice Λ to its dual $\Lambda^\vee$
$\mathbb{L}$	the $\mathcal{O}[u]$-lattice given by $\mathbb{L} = \oplus_{i=1}^d \mathcal{O}[u] \cdot \bar{e}_i$
$\mathcal{O}[B_1\|B_2]$	the polynomial ring over $\mathcal{O}$ with variables the entries of the matrix $(B_1\|B_2)$
$LT(f)$	the leading term of the polynomial f
$\wedge^2(B_1\|B_2)$	the 2×2 minors of the matrix $(B_1\|B_2)$
$\mathrm{M}^{\mathrm{loc}}(\Lambda)$	the Pappas-Zhu local model
$U_{d,l}$	an affine chart of $\mathrm{M}^{\mathrm{loc}}(\Lambda)$ around the worst point
$\overline{U}_{d,l}$	the special fiber of $U_{d,l}$

Chapter 1

Introduction

Local models of Shimura varieties are projective flat schemes over the spectrum of a discrete valuation ring. These projective schemes are expected to model the singularities of integral models of Shimura varieties with parahoric level structure. The definition of local model was formalized to some degree by Rapoport and Zink in [20]. However, it was soon realized that the Rapoport-Zink construction is not adequate when the group of the Shimura variety is ramified at p and in many cases of orthogonal groups. Indeed, then the corresponding integral models of Shimura varieties are often not flat ([14]). In [19], Pappas and Zhu gave a general group theoretic definition of local models. These local models appear as subschemes of global ("Beilinson-Drinfeld") affine Grassmannians and are associated to *local model triples.* A LM-triple over a finite extension F of $\mathbb{Q}_p$, for $p \neq 2$, is a triple $(G, \{\mu\}, K)$ consisting of a reductive group G over F, a conjugacy class of cocharacters $\{\mu\}$ of G over an algebraic closure of F, and a parahoric subgroup K of $G(F)$. We denote by $\mathrm{M}^{\mathrm{loc}}_K(G, \{\mu\})$ the corresponding local model.

In the present thesis, we study local models for Shimura varieties for forms of the orthogonal group which are of Hodge but not PEL type. An example of such a Shimura variety is the following: Consider the group $\mathbf{G} = \mathrm{GSpin}(\mathbf{V})$, where $\mathbf{V}$ is a (non-degenerate) orthogonal space of dimension $d \geq 7$ over $\mathbb{Q}$ and the signature of $\mathbf{V}_{\mathbb{R}}$ is $(d-2, 2)$. Let $\mathbf{D}$ be the space of oriented negative definite planes in $\mathbf{V}_{\mathbb{R}}$. Then the pair $(\mathbf{G}, \mathbf{D})$ is a Shimura datum of Hodge type. Further, consider a $\mathbb{Z}_p$-lattice Λ in $V = \mathbf{V} \otimes_{\mathbb{Q}} \mathbb{Q}_p$, for which

$$p\Lambda^\vee \subset \Lambda \subset \Lambda^\vee,$$

where $\Lambda^\vee$ is the dual of Λ for the corresponding symmetric form. We denote by l the distance of the lattice Λ to its dual $\Lambda^\vee$, i.e. $l = \lg_{\mathbb{Z}_p}(\Lambda^\vee/\Lambda)$ and we set $l_* = \min(l, d-l)$. We let K_1 be the connected stabilizer of Λ in $\mathrm{SO}(\mathbf{V})(\mathbb{Q}_p)$ and let K be the corresponding parahoric subgroup of $\mathbf{G}(\mathbb{Q}_p)$. The group $\mathcal{G}$ is the smooth connected "Bruhat-Tits" group scheme over $\mathrm{Spec}\,(\mathbb{Z}_p)$ such that $\mathcal{G} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \mathbf{G} \otimes_{\mathbb{Q}} \mathbb{Q}_p$ and $\mathcal{G}(\mathbb{Z}_p) = K$. Now, for a compact open subgroup $\mathbf{K} \subset \mathbf{G}(\mathbb{A}_f)$ of the form $\mathbf{K} = K^p \cdot K_p$ where $K_p = K$ and K^p is sufficiently small, the corresponding Shimura variety is

$$\mathrm{Sh}_{\mathbf{K}}(\mathbf{G}, \mathbf{D}) = \mathbf{G}(\mathbb{Q}) \backslash (\mathbf{D} \times \mathbf{G}(\mathbb{A}_f)/\mathbf{K}).$$

This complex space has a canonical structure of an algebraic variety over the reflex field $\mathbb{Q}$ (see [13]). The work of Kisin and Pappas [10] gives that orthogonal Shimura varieties as above admit integral models $\mathscr{S}_{\mathbf{K}}(\mathbf{G}, \mathbf{D})$, whose singularities are the "same" as those of the corresponding PZ local models; see Theorem 1.0.1 below where the properties (a) and (b) imply that $\mathscr{S}_{\mathbf{K}}(\mathbf{G}, \mathbf{D})$ and the corresponding local model are locally isomorphic for the étale topology. Note that there is a central extension (see [11])

$$1 \to \mathbb{G}_m \to \mathrm{GSpin}(V) \to \mathrm{SO}(V) \to 1.$$

Hence, by [8, Proposition 2.14], the local model that pertains to the above Shimura variety is $\mathrm{M}^{\mathrm{loc}}(\Lambda) = \mathrm{M}^{\mathrm{loc}}_{K_1}(\mathrm{SO}(V), \{\mu\})$ for the LM triple $(\mathrm{SO}(V), \{\mu\},\ K_1)$ where $V,\ K_1,$

are as above and we take the minuscule coweight $\mu : \mathbb{G}_m \to \mathrm{SO}(V)$ to be given by $\mu(t) = \mathrm{diag}(t^{-1}, 1, \ldots, 1, t)$. In fact, we will consider a more general situation in which $\mathbb{Q}_p$ is replaced by a finite field extension F of $\mathbb{Q}_p$ with integers $\mathcal{O}_F$. As a special case of [10, Theorem 4.2.7] we have the following:

Theorem 1.0.1. *There is a scheme $\mathscr{S}_{\mathbf{K}}(\mathbf{G}, \mathbf{D})$, flat over $\mathrm{Spec}\,(\mathbb{Z}_p)$, with*

$$\mathscr{S}_{\mathbf{K}}(\mathbf{G}, \mathbf{D}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \mathrm{Sh}_{\mathbf{K}}(\mathbf{G}, \mathbf{D}) \otimes_{\mathbb{Q}} \mathbb{Q}_p,$$

and which supports a "local model diagram"

$$\begin{array}{ccc} & \widetilde{\mathscr{S}}_{\mathbf{K}}(\mathbf{G}, \mathbf{D}) & \\ {}^{\pi_{\mathbf{K}}}\swarrow & & \searrow^{q_{\mathbf{K}}} \\ \mathscr{S}_{\mathbf{K}}(\mathbf{G}, \mathbf{D}) & & \mathrm{M}^{\mathrm{loc}}(\Lambda) \end{array} \tag{1.0.0.1}$$

such that:

a) $\pi_{\mathbf{K}}$ is a $\mathcal{G}$-torsor for the parahoric group scheme $\mathcal{G}$ that corresponds to K_p,

b) $q_{\mathbf{K}}$ is smooth and $\mathcal{G}$-equivariant.

Let us add that the integral model $\mathscr{S}_{\mathbf{K}}(\mathbf{G}, \mathbf{D})$ satisfies several additional properties, see [10] and [18, §7]. It is also "canonical" in the sense of [16]. At this point, we want to mention that one application of such orthogonal Shimura varieties lies in arithmetic intersection theory. For example, orthogonal Shimura varieties are used in the proof of the averaged Colmez conjecture (see [3] and [2]).

In the rest of the thesis, we will mainly consider local models and Shimura varieties will only appear again in Chapter 11 where we discuss how our results apply to GSpin Shimura varieties. Moreover, we want to mention that the results in Chapters 10 and 11 are from the joint work [18].

In this thesis, we first give an explicit description of $\mathrm{M}^{\mathrm{loc}}(\Lambda)$. The difficulty in this task arises from the fact that the construction of PZ local models is inexplicit and group theoretical. In particular, in order to define the PZ local model we have to take the reduced Zariski closure of a certain orbit inside a global affine Grassmannian. We refer the reader to Section 2.1 where the construction of the PZ local models is reviewed. In the case of local models of PEL type one can use the standard representation of the group to quickly represent the local model as a closed subscheme of certain linked (classical) Grassmannians (see [19]). This is not possible here since the composition $i \cdot \mu$, where $i : \mathrm{SO}(V) \hookrightarrow \mathrm{GL}(V)$ is the natural embedding, is not a minuscule coweight and we have to work harder. Nevertheless, we give explicit equations for an affine chart of the "worst" point of the local model. These equations display this chart as a quadric hypersurface given by the vanishing of a trace in a determinantal scheme of 2×2 minors. Using this and classical results on determinantal varieties we prove that the special fiber of the affine chart is reduced and Cohen-Macaulay. This implies that the special fiber of the local model is reduced and Cohen-Macaulay. Note here that the "reduced" result follows from Pappas-Zhu paper [19], which in turn uses Zhu's proof of the Pappas-Rapoport coherence conjecture (see [22]). We want also to mention the recent work of Haines and Richarz [7], where the authors prove in a more general setting that the special fiber of the PZ local models is reduced and Cohen-Macaulay.

Here, we give an independent elementary proof of these properties by using the explicit equations which, as we said above, describe an open subset around the "worst" point of our local model. We also calculate the number of the irreducible components of the special fiber of the affine chart. This is equal to the number of irreducible components of the special fiber of the local model. The reason behind these implications lies in the construction of the local model. In particular, as discussed in [19] the geometric special fiber of the PZ local model is a union of affine Schubert varieties. Among those there is a unique closed orbit which consists of a single point, the "worst" point. The one-point stratum lies in the closure of every other stratum. It follows that, if the special fiber of the local model has a certain nice property at the worst point (for example reducedness), then this should hold everywhere (see for example [6]).

Moreover, from the above discussion and the construction of the local models in [19], we deduce that our affine chart is dense and hence it "captures all the singularities" of $\mathrm{M}^{\mathrm{loc}}(\Lambda)$. By using the explicit description of this affine chart, we prove that the blow up of $\mathrm{M}^{\mathrm{loc}}(\Lambda)$ at the worst point resolve the singularities (see Theorem 1.0.3), which in turn leads to the construction of regular integral models for the Shimura varieties $\mathrm{Sh}_{\mathbf{K}}(\mathbf{G}, \mathbf{D})$ over the p-adic integers $\mathbb{Z}_p$. We expect that this construction will find applications to the study of arithmetic intersections of special cycles and Kudla's program.

Below we denote by $\mathcal{O}$ the ring of integers of $\breve{F}$, which is the completion of the maximal unramified extension of F in a fixed algebraic closure, and by k the residue field of $\breve{F}$.

The thesis is organized as follows: In Chapter 2 we review the definition of the PZ local models. In Chapter 3 we show how we derive the explicit equations. We describe an affine chart of the worst point $*$ of our orthogonal local model in the cases where $(d, l) = (\text{even}, \text{even})$, $(d, l) = (\text{odd}, \text{odd})$, $(d, l) = (\text{even}, \text{odd})$ and $(d, l) = (\text{odd}, \text{even})$. Note that when l is even the symmetric form on $V \otimes_F \breve{F}$ splits and when l is odd the symmetric form on $V \otimes_F \breve{F}$ is quasi-split but not split.

The case that $l_* \leq 1$, has been considered by Madapusi Pera in [11] and also in the joint work of He, Pappas and Rapoport [8]. In the last chapter of [8], the authors easily prove that in the case $l_* = 0$ the local model is isomorphic to a smooth quadric. With some more work they prove that in the case $l_* = 1$ the local model is isomorphic to a quadric which is singular in one point. Here, we assume that $l_* > 1$ (and also $d \geq 5$) and extend these results.

Before stating our main theorems we need some more notation. Thus, let $n = \lfloor d/2 \rfloor$, $r = \lfloor l/2 \rfloor$ and X be a $d \times d$ matrix of the form:

$$X = \left[\begin{array}{c|c|c} E_1 & O_1 & E_2 \\ \hline B_1 & A & B_2 \\ \hline E_3 & O_2 & E_4 \end{array}\right],$$

where $E_i \in \mathrm{Mat}_{(n-r)\times(n-r)}$, $O_j \in \mathrm{Mat}_{(n-r)\times l}$, $B_\ell \in \mathrm{Mat}_{l\times(n-r)}$ and $A \in \mathrm{Mat}_{l\times l}$. We write $\mathcal{O}[X]$, $\mathcal{O}[B_1|B_2]$ for the polynomial rings over $\mathcal{O}$ with variables the entries of the matrices X and $(B_1|B_2)$ respectively. We also write $\wedge^2(B_1|B_2)$ for the 2×2 minors of $(B_1|B_2)$ and J_m for the unit antidiagonal matrix of size m,

$$J_m := \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}.$$

In the introduction, we state our results in the case that d and l have the same parity, so $d = 2n$ and $l = 2r$, or $d = 2n + 1$ and $l = 2r + 1$. The results when d and l have different parity are a bit more involved to state; we refer the reader to Theorem 3.2.2 and Chapter 8.

Theorem 1.0.2. *Suppose that d and l have the same parity. Then an affine chart of the local model $\mathrm{M}^{\mathrm{loc}}(\Lambda)$ around the worst point $*$ is given by $U_{d,l} = \mathrm{Spec}\,(R)$ where R is the* *quotient ring*

$$R = \mathcal{O}[B_1|B_2]/(\wedge^2(B_1|B_2), Tr(B_2 J_{n-r} B_1^t J_l) + 2\pi).$$

Let us mention here that, for $l_* \neq 0$, none of these models are smooth or semi-stable (as follows from [8, Theorems 5.1, 5.6]).

In Chapters 10 and 11, we resolve the singularities of $\mathrm{M}^{\mathrm{loc}}(\Lambda)$ and $\mathscr{S}_{\mathbf{K}}(\mathbf{G}, \mathbf{D})$ respectively. We consider the blow-up of $\mathrm{M}^{\mathrm{loc}}(\Lambda)$ at the point $*$. This gives a $\mathcal{G}$-birational projective morphism

$$r^{\mathrm{bl}} : \mathrm{M}^{\mathrm{bl}}(\Lambda) \longrightarrow \mathrm{M}^{\mathrm{loc}}(\Lambda).$$

Using the explicit description of $U_{d,l}$ above, we show:

Theorem 1.0.3. *The scheme* $\mathrm{M}^{\mathrm{bl}}(\Lambda)$ *is regular and has special fiber a divisor with normal crossings. In fact,* $\mathrm{M}^{\mathrm{bl}}(\Lambda)$ *is covered by open subschemes which are smooth over* $\mathrm{Spec}\,(\mathbb{Z}_p[u, x, y]/(u^2xy - p))$.

We see that the corresponding blow-up $\mathscr{S}^{\mathrm{reg}}_{\mathbf{K}}(\mathbf{G}, \mathbf{D})$ of the integral model $\mathscr{S}_{\mathbf{K}}(\mathbf{G}, \mathbf{D})$ inherits the same nice properties as $\mathrm{M}^{\mathrm{bl}}(\Lambda)$. In fact, there is a local model diagram for $\mathscr{S}^{\mathrm{reg}}_{\mathbf{K}}(\mathbf{G}, \mathbf{D})$ similar to (1.0.0.1) but with $\mathrm{M}^{\mathrm{loc}}(\Lambda)$ replaced by $\mathrm{M}^{\mathrm{bl}}(\Lambda)$. See Theorem 11.0.1 for the precise statement about the model $\mathscr{S}^{\mathrm{reg}}_{\mathbf{K}}(\mathbf{G}, \mathbf{D})$; this theorem gives regular p-adic integral models for $\mathrm{Sh}_{\mathbf{K}}(\mathbf{G}, \mathbf{D})$. The construction of $\mathscr{S}^{\mathrm{reg}}_{\mathbf{K}}(\mathbf{G}, \mathbf{D})$ from r^{bl} and the local model diagram (11.0.0.1) is an example of a "linear modification" in the sense of [14].

Below we discuss how we derive the equations of Theorem 1.0.2 and then we give the main ingredients of the proof.

We write S_0, S_1 for the antidiagonal matrices of size d,

$$S_0 := \begin{pmatrix} & & 1^{(n-r)} \\ & 0^{(l)} & \\ 1^{(n-r)} & & \end{pmatrix}, \quad S_1 := \begin{pmatrix} & & 0^{(n-r)} \\ & 1^{(l)} & \\ 0^{(n-r)} & & \end{pmatrix}$$

and we define the ideal

$$I^{\text{naive}} = \left(X^2,\, \wedge^2 X,\, X^t S_0 X - 2\pi\,(S_0 + \pi S_1)\,X,\, X^t S_1 X + 2(S_0 + \pi S_1)X\right).$$

Our first step is to show that an affine chart of the PZ local model around the worst point $*$ is given as a closed subscheme of the quotient $\mathrm{M} = \mathcal{O}[X]/I^{\text{naive}}$. We do this in Chapter 3. This $\mathcal{O}$-flat closed subscheme is obtained by adding certain equations to I^{naive}: Set

$$I = I^{\text{naive}} + I^{\text{add}}$$

where

$$I^{\text{add}} = \left(Tr(X),\, Tr(A) + 2\pi,\, B_2 J_{n-r} B_1^t - A J_l\right).$$

We show that I cuts out the $\mathcal{O}$-flat $\mathrm{M}^{\text{loc}}(\Lambda) \cap \mathrm{M}$, which is an open affine subscheme of $\mathrm{M}^{\text{loc}}(\Lambda)$. By an involved but completely elementary manipulation of the relations describing the ideal I we prove that:

Theorem 1.0.4. *Suppose that d and l have the same parity. The quotient $\mathcal{O}[X]/I$ is isomorphic to $\mathcal{O}[B_1|B_2]/\left(\wedge^2(B_1|B_2), Tr(B_2 J_{n-r} B_1^t J_l) + 2\pi\right)$.*

It essentially remains to show that $U_{d,l} = \mathrm{Spec}\,(R)$ is flat over $\mathcal{O}$. By definition, $U_{d,l}$ is a hypersurface in the determinantal scheme $D = \mathrm{Spec}\,(\mathcal{O}[B_1|B_2]\ /(\wedge^2(B_1|B_2))$. Since D is Cohen-Macaulay, see [21, Remark 2.12], we can easily deduce that $U_{d,l}$ and $\overline{U}_{d,l}$ are also Cohen-Macaulay. Flatness of $U_{d,l}$ follows, see Chapter 5. Theorem 1.0.2 quickly follows together with the (essentially equivalent) statement:

Theorem 1.0.5. *Suppose that d and l have the same parity. An affine chart of the local model* $\mathrm{M}^{\mathrm{loc}}(\Lambda)$ *around the worst point is given by* $\mathrm{Spec}\,(\mathcal{O}[X]/I)$*, where I is as above.*

Using Theorem 1.0.2 and the reducedness of the fibers of PZ local models (see [19]) we have that:

Theorem 1.0.6. *The special fiber of $U_{d,l}$ is reduced.*

In Chapter 6 we give an independent proof of this result by using that the special fiber $\overline{U}_{d,l}$ is Cohen Macaulay and generically reduced.

In the course of proving the reducedness of $\overline{U}_{d,l}$, we also determine the number of its irreducible components. We find that when $2 < l_*$, where $l_* = \min(l, d-l)$ and l is the distance of our lattice to its dual, the special fiber $\overline{U}_{d,l}$ has two irreducible components. When $l_* = 2$, $\overline{U}_{d,l}$ has three irreducible components. In fact, we explicitly describe the equations defining the irreducible components of the special fiber. Similar arguments extend to the case that d and l have different parity. We give the corresponding hypersurface in a determinantal scheme and the equations of irreducible components of the special fiber in all cases.

Chapter 2

Preliminaries

Let us fix an odd prime p and consider a finite field extension $F/\mathbb{Q}_p$. Denote with $\mathcal{O}_F$ the ring of integers of F and let π be a uniformizer of $\mathcal{O}_F$. We denote by $\breve{F}$ the completion of the maximal unramified extension of F in an algebraic closure $\bar{F}$. We denote by κ_F the residue field of F and by k the algebraic closure of κ_F which is also the residue field of $\breve{F}$. We also set $\mathcal{O} := \mathcal{O}_{\breve{F}}$ for the ring of integers of $\breve{F}$.

2.1 Local models

We now recall the construction of the Pappas-Zhu local models. For a more detailed presentation we refer the reader to [15] and [19].

Let G be a connected reductive group over F. Assume that G splits over a tamely ramified extension of F. Let $\{\mu\}$ be a conjugacy class of a geometric cocharacter $\mu : \mathbb{G}_{m\bar{F}} \to G_{\bar{F}}$ and assume that μ is minuscule. Define K to be the parahoric subgroup of $G(F)$, which is the connected stabilizer of some point x in the (extended) Bruhat-Tits building $\mathbb{B}(G, F)$ of $G(F)$. Define E to be the extension of F which is the field of definition of the conjugacy class $\{\mu\}$ (the reflex field).

In [19], the authors construct an affine group scheme $\mathcal{G}$ which is smooth over Spec $(O_F[t])$ and which, among other properties, satisfies:

1. The base change of $\mathcal{G}$ by $\mathrm{Spec}\,(O_F) \to \mathrm{Spec}\,(O_F[t]) = \mathbb{A}^1_{O_F}$ given by $t \to \pi$ is the Bruhat-Tits group scheme which corresponds to the parahoric subgroup K (see [1]).

2. The group scheme $\mathcal{G}_{|O_F[t,t^{-1}]}$ is reductive.

Next, they consider the global ("Beilinson-Drinfeld") affine Grassmannian

$$\mathrm{Aff}_{\mathcal{G},\mathbb{A}^1_{O_F}} \to \mathbb{A}^1_{O_F}$$

given by $\mathcal{G}$, which is an ind-projective ind-scheme. By base changing $t \to \pi$, they obtain an equivariant isomorphism

$$\mathrm{Aff}_G \xrightarrow{\sim} \mathrm{Aff}_{\mathcal{G},\mathbb{A}^1_{O_F}} \times_{\mathbb{A}^1_{O_F}} \mathrm{Spec}\,(F)$$

where Aff_G is the affine Grassmannian of G; this is the ind-projective ind-scheme over $\mathrm{Spec}\,(F)$ that represents the fpqc sheaf associated to

$$R \to G(R((t)))/G(R[[t]]),$$

where R is an F-algebra (see also [17]).

The cocharacter μ gives an $\bar{F}[t,t^{-1}]$-valued point of G and thus μ gives an $\bar{F}((t))$-valued point $\mu(t)$ of G. This gives a $\bar{F}$-point $[\mu(t)] = \mu(t)G(\bar{F}[[t]])$ of Aff_G. Since μ is minuscule and $\{\mu\}$ is defined over the reflex field E the orbit

$$G(\bar{F}[[t]])[\mu(t)] \subset \mathrm{Aff}_G(\bar{F}),$$

is equal to the set of $\bar{F}$-points of a closed subvariety X_μ of $\mathrm{Aff}_{G,E} = \mathrm{Aff}_G \otimes_F E$.

Definition 2.1.0.1. *Define the local model* $\mathrm{M}_K^{\mathrm{loc}}(G, \{\mu\})$ *to be the flat projective scheme over* $\mathrm{Spec}\,(O_E)$ *given by the reduced Zariski closure of the image of*

$$X_\mu \subset \mathit{Aff}_G \xrightarrow{\sim} \mathit{Aff}_{\mathcal{G},\mathbb{A}^1_{O_F}} \times_{\mathbb{A}^1_{O_F}} \mathrm{Spec}\,(E)$$

in the ind-scheme $\mathit{Aff}_{\mathcal{G},\mathbb{A}^1_{O_F}} \times_{\mathbb{A}^1_{O_F}} \mathrm{Spec}\,(O_E)$.

The PZ local models have the following property (see [8, Prop. 2.14]).

Proposition 2.1.0.2. *If* F'/F *is a finite unramified extension, then*

$$\mathrm{M}_K^{\mathrm{loc}}(G, \{\mu\}) \otimes_{O_E} O_{E'} \xrightarrow{\sim} \mathrm{M}_{K'}^{\mathrm{loc}}(G \otimes_F F', \{\mu \otimes_F F'\}).$$

Note that here the reflex field E' *of* $(G \otimes_F F', \{\mu \otimes_F F'\})$ *is the join of* E *and* F'*. Also,* K' *is the parahoric subgroup of* $G \otimes_F F'$ *with* $K = K' \cap G$. □

The above proposition allows us to base change to an unramified extension F' over F. This will play a crucial role in the proof of our main theorems.

2.2 Quadratic forms

Let V be an F-vector space with dimension $d = 2n$ or $2n+1$ equipped with a non-degenerate symmetric F-bilinear form $\langle\ ,\ \rangle$. It follows from the classification of quadratic forms over local fields [5] that after passing to a sufficiently big unramified extension F' of F, the base change of $(V, \langle\ ,\ \rangle)$ to F' affords a basis as in one of the following cases:

1. **Split form:** there is a basis f_i with the following relations:

   $\langle f_i, f_{d+1-j} \rangle = \delta_{ij}$, $\forall i, j \in \{1, \ldots, d\}$.

2. **Quasi-split form (for $\mathbf{d = 2n}$):** there is a basis f_i with the relations: $\langle f_i, f_{d+1-j} \rangle = \delta_{ij}$, for i, $j \neq n, n+1$, $\langle f_n, f_n \rangle = \pi$, $\langle f_{n+1}, f_{n+1} \rangle = 1$, $\langle f_n, f_{n+1} \rangle = 0$.

3. **Quasi-split form (for $\mathbf{d = 2n+1}$):** there is a basis f_i with the relations: $\langle f_i, f_{d+1-j} \rangle = \delta_{ij}$, for i, $j \neq n+1$, $\langle f_{n+1}, f_{n+1} \rangle = \pi$.

2.3 Normal forms of quadric lattices

Let V be an F-vector space with dimension $d = 2n$ or $2n+1$ equipped with a non-degenerate symmetric F-bilinear form $\langle\ ,\ \rangle$. We assume that $d \geq 5$. For all the cases below we take the minuscule coweight $\mu : \mathbb{G}_m \to SO(V)$ to be given by $\mu(t) = \mathrm{diag}(t^{-1}, 1, \ldots, 1, t)$, defined over F.

A lattice $\Lambda \subset V$ is called a vertex lattice if $\Lambda \subset \Lambda^\vee \subset \pi^{-1}\Lambda$. By $\Lambda^\vee$ we denote the dual of Λ in V:

$$\Lambda^\vee := \{x \in V | \langle \Lambda, x \rangle \subset O_F\}.$$

Let Λ in V be a vertex lattice. So, $\Lambda \subset_l \Lambda^\vee \subset_{l'} \pi^{-1}\Lambda$ with $l + l' = d$. Here l (respectively l') is the length $l = \lg(\Lambda^\vee/\Lambda)$ (respectively $l' = \lg(\pi^{-1}\Lambda/\Lambda^\vee)$). We assume that $l > 1$ and $l' > 1$.

For the following we refer the reader to Rapoport-Zink's book [20], Appendix on Normal forms of lattice chains. More precisely, by [20, Appendix, Proposition A.21], after an étale base change (i.e an unramified base change) we can find an O_F-basis $\{e_i\}$ of Λ with the following property:

For $d = 2n$:

1. **Split form:** $\Lambda = \oplus_{i=1}^{d} O_F \cdot e_i$ with

$$\langle e_i, e_{d+1-j} \rangle = \delta_{ij}, \ \text{for}\, i \notin [n-r+1, n+r],$$

$$\langle e_i, e_{d+1-j} \rangle = \pi\delta_{ij}, \ \text{for}\, i \in [n-r+1, n+r].$$

We have $\Lambda \subset_l \Lambda^\vee$ where $l = 2r$.

2. **Quasi-split form:** $\Lambda = \oplus_{i=1}^{d} O_F \cdot e_i$ with

$$\langle e_i, e_{d+1-j} \rangle = \delta_{ij}, \ \text{for}\, i \in [1, d] \setminus [n-r, n+r+1],$$

$$\langle e_i, e_{d+1-j} \rangle = \pi\delta_{ij}, \ \text{for}\, i \in [n-r, n+r+1] \setminus \{n, n+1\},$$

$$\langle e_n, e_n \rangle = \pi, \quad \langle e_{n+1}, e_{n+1} \rangle = 1, \quad \langle e_n, e_{n+1} \rangle = 0.$$

We have $\Lambda \subset_l \Lambda^\vee$ where $l = 2r + 1$.

For $d = 2n + 1$:

3. **Split form:** $\Lambda = \oplus_{i=1}^{d} O_F \cdot e_i$ with

$$\langle e_i, e_{d+1-j} \rangle = \delta_{ij}, \ \text{for}\, i \notin [n+1-r, n+1+r] \setminus \{n+1\},$$

$$\langle e_i, e_{d+1-j} \rangle = \pi\delta_{ij}, \ \text{for}\, i \in [n+1-r, n+1+r] \setminus \{n+1\}.$$

We have $\Lambda \subset_l \Lambda^\vee$ where $l = 2r$.

4. **Quasi-split form:** $\Lambda = \oplus_{i=1}^{d} O_F \cdot e_i$ with

$$\langle e_i, e_{d+1-j} \rangle = \delta_{ij}, \text{ for } i \notin [n+1-r, n+1+r],$$

$$\langle e_i, e_{d+1-j} \rangle = \pi\delta_{ij}, \text{ for } i \in [n+1-r, n+1+r].$$

We have $\Lambda \subset_l \Lambda^\vee$ where $l = 2r+1$.

From the above discussion, it follows that we can reduce our problem to the above cases by passing to a sufficiently big unramified extension of F. Thus, from now on we will be working over $\breve{F}$. Recall that we denote by $\mathcal{O}$ its ring of integers and by k its residue field.

In all cases, we will denote by S the (symmetric) matrix with entries $\langle e_i, e_j \rangle$ where $\{e_i\}$ is the basis above. We can then write

$$S = S_0 + \pi S_1$$

where S_0, S_1 both have entries only 0 or 1. For example, in case (1) we have the anti-diagonal matrices:

$$S_0 := \begin{pmatrix} & & 1^{(n-r)} \\ & 0^{(2r)} & \\ 1^{(n-r)} & & \end{pmatrix}, \quad S_1 := \begin{pmatrix} & & 0^{(n-r)} \\ & 1^{(2r)} & \\ 0^{(n-r)} & & \end{pmatrix}.$$

Chapter 3

An Affine Chart of $\mathrm{M}^{\mathrm{loc}}(\Lambda)$

3.1 Lattices over $\mathcal{O}[u]$ and orthogonal local models

We can now extend our data to $\mathcal{O}[u, u^{-1}]$. We define $\mathbb{V} = \oplus_{i=1}^{d} \mathcal{O}[u, u^{-1}]\bar{e}_i$ and $\langle\ ,\ \rangle : \mathbb{V} \times \mathbb{V} \to \mathcal{O}[u, u^{-1}]$ a symmetric $\mathcal{O}[u, u^{-1}]$-bilinear form such that the value of $\langle \bar{e}_i, \bar{e}_j \rangle$ is the same as the above for V with the difference that π is replaced by u. Similarly, we define $\bar{\mu}(t) : \mathbb{G}_m \to SO(\mathbb{V})$ by using the $\{\bar{e}_i\}$ basis for $\mathbb{V}$.

We also define $\mathbb{L}$ the $\mathcal{O}[u]$-lattice in $\mathbb{V}$ by $\mathbb{L} = \oplus_{i=1}^{d} \mathcal{O}[u] \cdot \bar{e}_i$. From the above we see that the base change of $(\mathbb{V}, \mathbb{L}, \langle\ ,\ \rangle)$ from $\mathcal{O}[u, u^{-1}]$ to F given by $u \mapsto \pi$ is $(V, \Lambda, \langle\ ,\ \rangle)$.

Let us now define the local model $\mathrm{M}^{\mathrm{loc}}(\Lambda) = \mathrm{M}^{\mathrm{loc}}_K(SO(V), \{\mu\})$ where K is the parahoric stabilizer of Λ. We consider the smooth, as in [19], affine group scheme $\underline{\mathcal{G}}$ over $\mathcal{O}[u]$ given by $g \in SO(\mathbb{V})$ that also preserves $\mathbb{L}$ and $\mathbb{L}^\vee$. If we base change by $u \mapsto \pi$ we obtain the Bruhat-Tits group scheme $\mathcal{G}$ of $SO(V)$ which is the stabilizer of the lattice chain $\Lambda \subset \Lambda^\vee \subset \pi^{-1}\Lambda$. The corresponding parahoric group scheme is the neutral component $\mathcal{G}^0$ of $\mathcal{G}$. The construction of [19] produces the group scheme $\underline{\mathcal{G}}^0$ that extends $\mathcal{G}^0$. By construction, there is a group scheme immersion $\underline{\mathcal{G}}^0 \hookrightarrow \underline{\mathcal{G}}$.

In this case, the global ("Beilinson-Drinfeld") affine Grassmannian

$$\mathrm{Aff}_{\underline{\mathcal{G}}, \mathbb{A}^1_{\mathcal{O}}} \to \mathrm{Spec}\,(\mathcal{O}[u])$$

represents the functor that sends the $\mathcal{O}[u]$-algebra R, given by $u \mapsto r$, to the set of projective finitely generated $R[u]$-modules $\mathcal{L}$ of $\mathbb{V} \otimes_{\mathcal{O}} R$ which are locally free such that $(u-r)^N \mathbb{L}_R \subset \mathcal{L} \subset (u-r)^{-N} \mathbb{L}_R$ for some $N >> 0$ and satisfy

$$\mathcal{L} \subset_l \mathcal{L}^\vee \subset_{l'} u^{-1}\mathcal{L}$$

with all graded quotients R-locally free and of the indicated rank. Here, we set $\mathbb{L}_R = \mathbb{L} \otimes_{\mathcal{O}} R$.

Consider the $\mathcal{O}$-valued point $[\mathcal{L}(0)]$ given by $\mathcal{L}(0) = \bar{\mu}(u-\pi)\mathbb{L}$. Then, as in the Section 2.1 the local model is the reduced Zariski closure of the orbit $[\mathcal{L}(0)]$ in $\mathrm{Aff}_{\underline{\mathcal{G}}^0, \mathbb{A}^1_{\mathcal{O}}} \times_{\mathbb{A}^1_{\mathcal{O}}} \mathrm{Spec}\,(\mathcal{O})$; it inherits an action of the group scheme $\mathcal{G}^0 = \underline{\mathcal{G}}^0 \otimes_{\mathcal{O}[u]} \mathcal{O}$. As in [19], there is a natural morphism $\mathrm{Aff}_{\underline{\mathcal{G}}^0, \mathbb{A}^1_{\mathcal{O}}} \to \mathrm{Aff}_{\underline{\mathcal{G}}, \mathbb{A}^1_{\mathcal{O}}}$ induced by $\underline{\mathcal{G}}^0 \hookrightarrow \underline{\mathcal{G}}$ which identifies $\mathrm{M}^{\mathrm{loc}}(\Lambda)$ with a closed subscheme of $\mathrm{Aff}_{\underline{\mathcal{G}}, \mathbb{A}^1_{\mathcal{O}}} \times_{\mathbb{A}^1_{\mathcal{O}}} \mathrm{Spec}\,(\mathcal{O})$.

By the definition of $\mathcal{L}(0)$ we have

$$(u-\pi)\mathbb{L} \subset \mathcal{L}(0) \cap \mathbb{L} \begin{array}{c} \mathbb{L} \\ \subset \quad \supset \\ \qquad \\ \supset \quad \subset \\ \mathcal{L}(0) \end{array} \mathbb{L} + \mathcal{L}(0) \subset (u-\pi)^{-1}\mathbb{L},$$

where the quotients along all slanted inclusions are $\mathcal{O}$-free of rank 1 (for more details see proof of Proposition 3.1.0.1). Let us define M to be the subfunctor of $\mathrm{Aff}_{\underline{\mathcal{G}}, \mathbb{A}^1_{\mathcal{O}}} \times_{\mathbb{A}^1_{\mathcal{O}}} \mathrm{Spec}\,(\mathcal{O})$ that parametrizes all $\mathcal{L}$ such that

$$(u-\pi)\mathbb{L} \subset \mathcal{L} \subset (u-\pi)^{-1}\mathbb{L}.$$

Then M is represented by a closed subscheme of $\mathrm{Aff}_{\underline{\mathcal{G}}, \mathbb{A}^1_{\mathcal{O}}} \times_{\mathbb{A}^1_{\mathcal{O}}} \mathrm{Spec}\,(\mathcal{O})$ which contains $[\mathcal{L}(0)]$. In that way, $\mathrm{M}^{\mathrm{loc}}(\Lambda)$ is a closed subscheme of M and $\mathrm{M}^{\mathrm{loc}}(\Lambda)$ is the reduced Zariski closure of its generic fiber in M. As in [8, Proposition 12.7], the elements of $\mathrm{M}^{\mathrm{loc}}(\Lambda)$ have the following properties:

Proposition 3.1.0.1. *If $\mathcal{L} \in \mathrm{M}^{\mathrm{loc}}(\Lambda)(R)$, for an $\mathcal{O}$-algebra R, then:*

1. *$\mathcal{L}$ is u-stable,*

2. *$\mathcal{L} \subset_l \mathcal{L}^\vee$, and*

3. $$(u-\pi)\mathbb{L}_R \subset \mathcal{L} \cap \mathbb{L}_R \begin{array}{ccc} & \mathbb{L}_R & \\ \subset & & \subset \\ & \mathcal{L} & \end{array} \mathbb{L}_R + \mathcal{L} \subset (u-\pi)^{-1}\mathbb{L}_R,$$

   where the quotients arising from all slanted inclusions are generated as R-modules by one element (we say that they have rank ≤ 1).

Proof. The first two conditions follow directly from the definition of the local model. By the definition of $\mathcal{L}(0)$ we have $\mathcal{L}(0) = \bar{\mu}(u-\pi)\mathbb{L}$ where $\bar{\mu}(u-\pi) = \mathrm{diag}((u-\pi)^{-1}, 1, \ldots, 1, u-\pi)$. We can easily see that (3) is true for $\mathcal{L}(0)$. Since condition (3) is closed and $\mathcal{G}$-equivariant it also holds for $\mathcal{L}$ and the proposition follows. $\square$

Define $\mathcal{F}'$ to be the image of $\mathcal{L}$ by the map

$$(u-\pi)^{-1}\mathbb{L}_R/(u-\pi)\mathbb{L}_R \xrightarrow{u-\pi} \mathbb{L}_R/(u-\pi)^2\mathbb{L}_R.$$

Define the symmetric bilinear form:

$$\langle\ ,\ \rangle' : \mathbb{L}/(u-\pi)^2\mathbb{L} \times \mathbb{L}/(u-\pi)^2\mathbb{L} \to \mathcal{O}[u]/(u-\pi)^2\mathcal{O}[u],$$

by

$$\langle\ ,\ \rangle' = \langle\ ,\ \rangle \mathrm{mod}(u-\pi)^2.$$

Notice, that condition (2) above means that $\langle \mathcal{L}, \mathcal{L}\rangle \in R[u]$ under the R-base change of the bilinear form $\langle\ ,\ \rangle$. Thus, $\mathcal{F}'$ is isotropic for $\langle\ ,\ \rangle'$ on $\mathbb{L}_R/(u-\pi)^2\mathbb{L}_R \times \mathbb{L}_R/(u-\pi)^2\mathbb{L}_R$, i.e. $\langle \mathcal{F}', \mathcal{F}'\rangle' = 0$. We also observe that $\mathrm{rank}(u-\pi) \leq 1$ where $u - \pi : \mathcal{F}' \to \mathcal{F}'$. That follows from condition (3) and the fact that $(u-\pi)^2\mathbb{L}_R = 0$ in $\mathbb{L}_R/(u-\pi)^2\mathbb{L}_R$.

3.2 The affine chart $U_{d,l}$

For the sake of simplicity we fix $d = 2n$ and $l = 2r$. We get similar results for all the other cases.

For any $\mathcal{O}$-algebra R, let us consider the R-submodule:

$$\mathcal{F} = \{(u-\pi)v + Xv \,|\, v \in R^d\} \subset (u-\pi)R^d \oplus R^d \cong \mathbb{L}_R/(u-\pi)^2\mathbb{L}_R$$

with $X \in \mathrm{Mat}_{d\times d}(R)$.

We ask that $\mathcal{F}$ satisfies the following three conditions:

1. **u-stable:** It suffices to be $(u-\pi)$-stable. Let $(u-\pi)v + Xv \in \mathcal{F}$. Then there exists $w \in R^d$ such that $(u-\pi)^2 v + (u-\pi)Xv = (u-\pi)w + Xw$. This gives $Xuv - X\pi v = uw - \pi w + Xw$ and so:

$$w = Xv,$$

$$-\pi Xv = -\pi w + Xw.$$

By substituting the former equation to the latter, we have $X^2 v = 0$. Because this is correct for every v, we have $X^2 = 0$. Observe that X is the matrix giving multiplication by $(u - \pi)$ on $\mathcal{F}$.

2. **Isotropic:** Let $(u-\pi)v + Xv \in \mathcal{F}$. We want

$$\langle (u-\pi)v + Xv, (u-\pi)v + Xv \rangle' = 0$$

and recall that $\langle\ ,\ \rangle' = \langle\ ,\ \rangle \mathrm{mod} (u-\pi)^2$. By simplifying the above equation we have

$$-2(u-\pi)\langle v, Xv\rangle' = \langle Xv, Xv\rangle'.$$

The above relation holds for any v and so we get:

$$-2(u-\pi)(S_0 + uS_1)X = X^t(S_0 + uS_1)X$$

where S_0, S_1 are the matrices with $S = S_0 + \pi S_1 = (\langle e_i, e_j\rangle)_{i,j}$ as in Section 2.3. By simplifying the above relation we have:

$$2\pi S_0 X + 2\pi^2 S_1 X + u(-2\pi S_1 X - 2S_0 X) = X^t S_0 X + u(X^t S_1 X)$$

which amounts to

$$X^t S_0 X = 2\pi(S_0 X + \pi S_1 X) \text{ and } X^t S_1 X = -2(S_0 X + \pi S_1 X).$$

3. **rank($u-\pi|\ \mathcal{F}'$)$\leq$1:** By the above, this translates to $\wedge^2 X = 0$.

Let $\mathcal{U}^{\mathrm{naive}}$ be the corresponding scheme of $\mathcal{F}$ defined by the $d \times d$ matrices X which satisfy the following relations:

$$X^2 = 0, \quad X^t S_0 X - 2\pi(S_0 X + \pi S_1 X) = 0,$$

$$\wedge^2 X = 0, \quad X^t S_1 X + 2(S_0 X + \pi S_1 X) = 0.$$

We denote by I^{naive} the ideal generated by the entries of the above relations.

The conditions (1)-(3) are necessary but not always sufficient for $\mathcal{L}$ to correspond to an R-valued point of $\mathrm{M}^{\mathrm{loc}}(\Lambda)$. Indeed, the generic fiber of $\mathcal{U}^{\mathrm{naive}}$ contains the additional $\breve{F}$-point $\mathcal{L} = \mathbb{L}_{\breve{F}}$ which is not in the orbit $[\mathcal{L}(0)]$ of $\bar{\mu}$ in the affine Grassmannian Aff_G. Also, calculations in low dimensions show that $\mathcal{U}^{\mathrm{naive}}$ has non-reduced special fiber.

Our goal is to calculate the $\mathcal{O}$-flat closed subscheme $\mathcal{U} = \mathrm{M}^{\mathrm{loc}}(\Lambda) \cap \mathcal{U}^{\mathrm{naive}}$ of $\mathcal{U}^{\mathrm{naive}}$ by adding some explicit relations in the ideal I^{naive}. The resulting $\mathcal{U}$ is an open subscheme of $\mathrm{M}^{\mathrm{loc}}(\Lambda)$.

Observe that the point $\mathbb{L}$ is fixed by the action of the group scheme $\mathcal{G}^0$ and so its our worst point. Thus, $\mathcal{U}$ is an open neighborhood around the worst point $\mathbb{L}$. Then these additional relations, together with I^{naive}, give explicit equations that describe an open subset around the worst point of our local model $\mathrm{M}^{\mathrm{loc}}(\Lambda)$.

We introduce some notation that will help us defining those relations. We first rewrite our matrix $X := (x_{ij})_{1 \leq i,j \leq d}$ as follows:

$$X = \left[\begin{array}{c|c|c} E_1 & O_1 & E_2 \\ \hline B_1 & A & B_2 \\ \hline E_3 & O_2 & E_4 \end{array}\right]$$

where $E_i \in \mathrm{Mat}_{(n-r)\times(n-r)}$, $O_j \in \mathrm{Mat}_{(n-r)\times l}$, $B_\ell \in \mathrm{Mat}_{l\times(n-r)}$ and $A \in \mathrm{Mat}_{l\times l}$. We denote by $\mathcal{O}[X]$ the polynomial ring over $\mathcal{O}$, with variables the entries of the matrix X. We also write J_m for the unit antidiagonal matrix of size m,

$$J_m := \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}.$$

We will show that by adding the following relations:

$$Tr(X) = 0, \quad Tr(A) + 2\pi = 0, \quad B_2 J_{n-r} B_1^t - AJ_l = 0,$$

we get the desired $\mathcal{O}$-flat scheme $\mathcal{U}$ in the cases where (d,l) =(even,even) and (d,l) = (odd,odd). By adding similar relations we get the corresponding result in cases where (d,l) =(even,odd) and (d,l) =(odd,even). Next, we state the main theorems of this thesis.

Theorem 3.2.1. *Suppose that d and l have the same parity so $d = 2n$ and $l = 2r$, or $d = 2n+1$ and $l = 2r+1$. Then an affine chart of the local model $\mathrm{M}^{\mathrm{loc}}(\Lambda)$ around the worst point is given by $U_{d,l} = \mathrm{Spec}\,(\mathcal{O}[X]/I)$, which is defined by the quotient of the polynomial ring $\mathcal{O}[X] = \mathcal{O}[(x_{i,j})_{1\le i,j\le d}]$ by the ideal*

$$I = I^{naive} + I^{add}$$

where

$$I^{add} = \Big(Tr(X),\, Tr(A) + 2\pi,\, B_2 J_{n-r} B_1^t - AJ_l\Big).$$

Next, we state the theorems for the cases where d and l have different parity. In each case we consider $d \times d$ matrices X. In order to define the submatrices (E_i, O_j, B_ℓ, A) giving the block decomposition of X we set:

$$r' = \begin{cases} r & \text{if } l = 2r \\ r+1 & \text{if } l = 2r+1. \end{cases}$$

Then write the matrix X as before, with blocks $E_i \in \mathrm{Mat}_{(n-r')\times(n-r')}$, $O_j \in \mathrm{Mat}_{(n-r')\times(l+1)}$, $A \in \mathrm{Mat}_{(l+1)\times(l+1)}$ and $B_\ell \in \mathrm{Mat}_{(l+1)\times(n-r')}$.

We denote by A' the $l \times l$ matrix which is obtained from A by erasing the part that is in the $(n+1)$-row and $(n+1)$-column of X. Similarly we denote by B_1', B_2' the $l \times (n-r')$ matrices which are obtained from B_1, B_2 by erasing the part that is on the $(n+1)$-row of X. Lastly, we denote by Q the $(r'+1)$-column of A and Q' the $(r'+1)$-column of A with the $(n+1)$-entry erased.

Theorem 3.2.2. *Suppose that d and l have opposite parity, so $d = 2n+1$ and $l = 2r$ or $d = 2n$ and $l = 2r+1$. An affine chart of the local model $\mathrm{M}^{\mathrm{loc}}(\Lambda)$ around the worst point $\mathcal{L} = \mathbb{L}$ is given by $U_{d,l} = \mathrm{Spec}\,(\mathcal{O}[X]/I)$, which is defined by the quotient of the polynomial ring $\mathcal{O}[X]$ by the ideal*

$$I = I^{naive} + I^{add}$$

where

$$I^{add} = (Tr(X), Tr(A') + 2\pi, B_2' J_{n-r'} (B_1')^t + \frac{1}{2} Q'(Q')^t - A' J_l).$$

In chapters 4-6 we carry out the proof of Theorem 3.2.1 for the case $(d, l) = (\text{even}, \text{even})$. The proof of the remaining cases of parity for d and l is given in Chapter 8.

Using Theorems 3.2.1, 3.2.2 and the fact that PZ local models have reduced special fiber, see [19], we obtain:

Theorem 3.2.3. *The special fiber of* $U_{d,l}$ *is reduced.*

Note that in the above theorem we do not specify the parity of d and l. In Chapter 6 we give an independent proof of this theorem, for the case $(d, l) = (\text{even}, \text{even})$, by showing that the special fiber is Cohen Macaulay and generically reduced. A similar argument works for the rest of the cases of parity for d and l.

Chapter 4

Reduction of Relations of $U_{d,l}$

In all of Chapter 4, we assume $d = 2n$ and $l = 2r$. Our goal in this chapter is to prove the simplification of equations given by Theorem 4.0.1. (This corresponds to Theorem 1.0.4 of the introduction.)

We are working over the polynomial ring $S := \mathcal{O}[(x_{i,j})_{1\leq i,j\leq d}]$. We also set

$$S'' := \mathcal{O}[(x_{t,s})_{t\in Z, s\in Z^c}]$$

where $Z := \{n-(r-1), \ldots, n, n+1, \ldots, d-n+r\}$ and $Z^c := \{1, 2, 3, \ldots, d\} \setminus Z$.

Recall that

$$I = (X^2,\ \wedge^2 X,\ Tr(X),\ Tr(A) + 2\pi,\ B_2 J_{n-r} B_1^t - A J_{2r},$$

$$X^t S_0 X - 2\pi(S_0 X + \pi S_1 X),\ \ X^t S_1 X + 2(S_0 X + \pi S_1 X)).$$

We set

$$I'' = \left(\wedge^2(B_1 \vdots B_2), Tr(B_2 J_{n-r} B_1^t J_{2r}) + 2\pi\right)$$

where

$$\wedge^2(B_1 \vdots B_2) := (x_{i,j}x_{t,s} - x_{i,s}x_{t,j})_{i,t\in Z,\, j,s\in Z^c}.$$

Theorem 4.0.1. *There is an $\mathcal{O}$-algebra isomorphism $S/I \cong S''/I''$.*

Proof. We define the ideal:

$$I' = \Big(\wedge^2 X,\, Tr(X),\, Tr(A) + 2\pi,\, B_2 J_{n-r} B_1^t - A J_{2r},$$

$$X^t S_1 X + 2(S_0 X + \pi S_1 X) \Big).$$

The proof will be done in two steps:

1. Show $I = I'$.

2. Show $S/I' \simeq S''/I''$.

4.1 $I = I'$.

Our first reduction is to prove that $I' = I$, which will be given in Proposition 4.1.0.8. To do that, we are going to show that the entries of X^2, $X^t S_0 X - 2\pi(S_0 X + \pi S_1 X)$ are in the ideal I'. Proposition 4.1.0.8 will easily follow. The first relation is more straightforward:

Lemma 4.1.1. *The entries of X^2 are in the ideal I'.*

Proof. Let $(z_{i,j})_{1 \le i,j \le d} := X^2$, where $z_{i,j} = \sum_{a=1}^{d} x_{i,a} x_{a,j}$. Now, set

$$t_{i,j} := x_{i,j} Tr(X) \in I'.$$

Notice also that

$$s_a^{i,j} := x_{i,a} x_{a,j} - x_{i,j} x_{a,a} \in I'$$

from the minors relations. Therefore

$$t_{i,j} + \sum_{a=1}^{d} s_a^{i,j} = z_{i,j} \in I'.$$

□

We have to work harder in order to show that the entries of $X^t S_0 X - 2\pi(S_0 X + \pi S_1 X)$ are in the ideal I'. The first step is as follows. By a simple direct calculation the relation $X^t S_1 X + 2S_0 X + 2\pi S_1 X = 0$ implies that:

$$E_1 = -\frac{1}{2} J_{n-r} B_2^t J_{2r} B_1, \tag{4.1.0.1}$$

$$E_2 = -\frac{1}{2} J_{n-r} B_2^t J_{2r} B_2, \tag{4.1.0.2}$$

$$E_3 = -\frac{1}{2} J_{n-r} B_1^t J_{2r} B_1, \tag{4.1.0.3}$$

$$E_4 = -\frac{1}{2} J_{n-r} B_1^t J_{2r} B_2, \tag{4.1.0.4}$$

$$O_1 = -\frac{1}{2} J_{n-r} B_2^t J_{2r} A, \tag{4.1.0.5}$$

$$O_2 = -\frac{1}{2} J_{n-r} B_1^t J_{2r} A. \tag{4.1.0.6}$$

Therefore, all the entries from E_i for $i \in \{1, 2, 3, 4\}$ and O_1, O_2 can be expressed in terms of the entries of B_1, B_2. The second step is the following lemma.

Lemma 4.1.2. *Assume that all the* 2×2 *minors of the matrix* X *are* 0. *Then, the matrix* $B_1 J_{n-r} B_2^t$ *is symmetric.*

Proof. Set $(\theta_{ij})_{1 \le i,j \le 2r} := B_1 J_{n-r} B_2^t$. By direct calculations we find

$$\theta_{ij} = \sum_{t=1}^{n-r} x_{n-r+i,n-r-t+1} x_{n-r+j,n+r+t}.$$

So,

$$\theta_{ji} = \sum_{t=1}^{n-r} x_{n-r+j,n-r-t+1} x_{n-r+i,n+r+t}.$$

From the minor relations we have that $x_{i,j}x_{t,s} = x_{i,s}x_{t,j}$. Using this and the description of the θ_{ij}, θ_{ji} we can easily see that $\theta_{ij} = \theta_{ji}$. □

A useful observation, which will be used in the following lemma, is that the condition $\wedge^2 X = 0$ together with the fact that the blocks B_1, A, and B_2, all share the same rows of X, easily give

$$AB_1 = \mathrm{Tr}(A)B_1, \quad AB_2 = \mathrm{Tr}(A)B_2. \tag{4.1.0.7}$$

We are now ready to show:

Lemma 4.1.3. *The entries of $X^t S_0 X - 2\pi(S_0 X + \pi S_1 X)$ are in the ideal I'.*

Proof. Using the block form of the matrix X and the relation $X^t S_1 X + 2(S_0 X + \pi S_1 X) = 0$ modulo I', it suffices to prove that:

(i) $E_1^t J_{n-r} E_3 + E_3^t J_{n-r} E_1 - 2\pi J_{n-r} E_3 = 0$,

(ii) $E_2^t J_{n-r} E_4 + E_4^t J_{n-r} E_2 - 2\pi J_{n-r} E_2 = 0$,

(iii) $E_1^t J_{n-r} E_4 + E_3^t J_{n-r} E_2 - 2\pi J_{n-r} E_4 = 0$,

(iv) $E_2^t J_{n-r} E_3 + E_4^t J_{n-r} E_1 - 2\pi J_{n-r} E_1 = 0$,

(v) $O_1^t J_{n-r} E_3 + O_2^t J_{n-r} E_1 - 2\pi^2 J_{2r} B_1 = 0$,

(vi) $O_1^t J_{n-r} E_4 + O_2^t J_{n-r} E_2 - 2\pi^2 J_{2r} B_2 = 0$,

(vii) $O_1^t J_{n-r} O_2 + O_2^t J_{n-r} O_1 - 2\pi^2 J_{2r} A = 0$

in the quotient ring S/I'. We prove the first relation (i) and with the same arguments we can prove the relations (ii)-(iv). Below we use the relations (1) and (3) for E_1, E_3 from above, the relations $B_2 J_{n-r} B_1^t = A J_{2r}$, $AB_1 = Tr(A) B_1$ and Lemma 4.1.2.

$$\begin{aligned}
& E_1^t J_{n-r} E_3 + E_3^t J_{n-r} E_1 - 2\pi J_{n-r} E_3 \\
= \; & \frac{1}{4} B_1^t J_{2r} B_2 J_{n-r} B_1^t J_{2r} B_1 + \frac{1}{4} B_1^t J_{2r} B_1 J_{n-r} B_2^t J_{2r} B_1 + \pi B_1^t J_{2r} B_1 \\
= \; & \frac{1}{2} B_1^t J_{2r} B_2 J_{n-r} B_1^t J_{2r} B_1 + \pi B_1^t J_{2r} B_1 \\
= \; & \frac{1}{2} B_1^t J_{2r} A B_1 + \pi B_1^t J_{2r} B_1 = \frac{1}{2} Tr(A) B_1^t J_{2r} B_1 + \pi B_1^t J_{2r} B_1 = 0.
\end{aligned}$$

The last equality holds because $Tr(A) + 2\pi = 0$.

Next, we prove the relation (v). The relations (vi), (vii) can be proved using similar arguments. We use the relations (1), (3), (5) and (6) from above to express E_1, E_3, O_1, O_2 in terms of B_1 and B_2. We use Lemma 4.1.2 and the relations $B_2 J_{n-r} B_1^t = A J_{2r}$ and $AB_1 = Tr(A) B_1$.

$$\begin{aligned}
& O_1^t J_{n-r} E_3 + O_2^t J_{n-r} E_1 - 2\pi^2 J_{2r} B_1 \\
= \; & \frac{1}{4} A^t J_{2r} B_2 J_{n-r} B_1^t J_{2r} B_1 + \frac{1}{4} A^t J_{2r} B_1 J_{n-r} B_2^t J_{2r} B_1 - 2\pi^2 J_{2r} B_1 \\
= \; & \frac{1}{2} A^t J_{2r} B_2 J_{n-r} B_1^t J_{2r} B_1 - 2\pi^2 J_{2r} B_1 = \frac{1}{2} A^t J_{2r} A B_1 - 2\pi^2 J_{2r} B_1 \\
= \; & -\pi J_{2r} A B_1 - 2\pi^2 J_{2r} B_1 = -\pi (Tr(A) J_{2r} B_1 + 2\pi J_{2r} B_1) = 0.
\end{aligned}$$

□

Proposition 4.1.0.8. *We have $I' = I$.*

Proof. From Lemma 4.1.1 and Lemma 4.1.3 we get the desired result. □

4.2 $S/I' \simeq S''/I''$.

The goal of this section is to prove that S/I' is isomorphic to S''/I''. Recall

$$I' = \Big(\wedge^2 X,\, Tr(X),\, Tr(A) + 2\pi,\, B_2 J_{n-r} B_1^t - A J_{2r},$$

$$X^t S_1 X + 2(S_0 X + \pi S_1 X)\Big).$$

We first simplify and reduce the number of generators of I'. The desired isomorphism will then follow.

Lemma 4.2.1. *The trace $Tr(X)$ belongs to the ideal*

$$\Big(\wedge^2 X,\;\; Tr(A) + 2\pi,\;\; B_2 J_{n-r} B_1^t - A J_{2r},\;\; X^t S_1 X + 2(S_0 X + \pi S_1 X)\Big)\,.$$

Proof. We first write:

$$Tr(X) = Tr(E_1) + Tr(E_4) + Tr(A).$$

By the relations (1), (4) from Section 4.1 we get that the entries of $E_1 + \frac{1}{2} J_{n-r} B_2^t J_{2r} B_1$ and $E_4 + \frac{1}{2} J_{n-r} B_1^t J_{2r} B_2$, belong to the ideal

$$\Big(\wedge^2 X, \quad Tr(A) + 2\pi, \quad B_2 J_{n-r} B_1^t - A J_{2r}, \quad X^t S_1 X + 2(S_0 X + \pi S_1 X)\Big)\,.$$

Also, the element

$$Tr(J_{n-r} B_1^t J_{2r} B_2) - Tr(A)$$

belongs to the above ideal. Thus,

$$\begin{aligned}
Tr(X) &= Tr(E_1) + Tr(E_4) + Tr(A) \\
&= Tr(E_1 + \frac{1}{2}J_{n-r}B_2^tJ_{2r}B_1) + Tr(E_4 + \frac{1}{2}J_{n-r}B_1^tJ_{2r}B_2) + Tr(A) \\
&\quad -\frac{1}{2}Tr(J_{n-r}B_1^tJ_{2r}B_2) - \frac{1}{2}Tr(J_{n-r}B_2^tJ_{2r}B_1) \\
&= Tr(E_1 + \frac{1}{2}J_{n-r}B_2^tJ_{2r}B_1) + Tr(E_4 + \frac{1}{2}J_{n-r}B_1^tJ_{2r}B_2),
\end{aligned}$$

belongs to the above ideal, as desired. □

From the above lemma we obtain

$$I' = \left(\wedge^2 X,\ Tr(A) + 2\pi,\ \ B_2J_{n-r}B_1^t - AJ_{2r},\ X^tS_1X + 2(S_0X + \pi S_1X)\right).$$

Next, we show:

Lemma 4.2.2. *We have* $I' = \left(\wedge^2 X,\ Tr(A) + 2\pi,\ \ B_2J_{n-r}B_1^t - AJ_{2r}\right) + \mathcal{I}'$, *where* $\mathcal{I}'$ *is the ideal generated by the relations* (1)-(6) *from Section 4.1.*

Proof. Using the block form of the matrix X and the relation $X^tS_1X+2(S_0X+\pi S_1X) = 0$, it suffices to prove that:

(a) $A^tJ_{2r}B_1 + 2\pi J_{2r}B_1 = 0$,

(b) $A^tJ_{2r}B_2 + 2\pi J_{2r}B_2 = 0$,

(c) $A^tJ_{2r}A + 2\pi J_{2r}A = 0$,

in the quotient ring of S by $\left(\wedge^2 X,\ Tr(A) + 2\pi,\ \ B_2J_{n-r}B_1^t - AJ_{2r}\right) + \mathcal{I}'$.

We first discuss (a). Recall that $A = B_2 J_{n-r} B_1^t J_{2r}$, $AB_1 = Tr(A)B_1$ and $Tr(A)+2\pi = 0$. Thus,

$$\begin{aligned} A^t J_{2r} B_1 + 2\pi J_{2r} B_1 &= J_{2r} B_1 J_{n-r} B_2^t J_{2r} B_1 + 2\pi J_{2r} B_1 \\ &= J_{2r} A B_1 + 2\pi J_{2r} B_1 \\ &= J_{2r} Tr(A) B_1 + 2\pi J_{2r} B_1 = 0. \end{aligned}$$

Using similar arguments we can prove that the relations (b) and (c) hold. □

The final step is to look more carefully at the minors that come from $\wedge^2 X$.

Lemma 4.2.3. $\wedge^2 X \in \mathcal{I}' + \left(\wedge^2(B_1 \,\vdots\, B_2),\ Tr(A) + 2\pi,\ B_2 J_{n-r} B_1^t - A J_{2r}\right)$.

Proof. In the proof, we use phrases like: "minors *only* from E_c", "minors *only* from A and B_ℓ", or "minors from A and E_c". Let us explain what we mean by these terms. Consider the minor

$$m_{t,s}^{i,j} = \begin{pmatrix} x_{i,j} & x_{i,s} \\ x_{t,j} & x_{t,s} \end{pmatrix} = x_{i,j} x_{t,s} - x_{i,s} x_{t,j}.$$

When we say that "the minor comes *only* from E_c" we mean that *all* of the entries $\{x_{i,j}, x_{t,s}, x_{i,s}, x_{t,j}\}$ are entries of E_c for $c \in \{1,2,3,4\}$. Similarly, when we say "the minor comes *only* from A and B_ℓ" we mean that *all* of $\{x_{i,j}, x_{t,s}, x_{i,s}, x_{t,j}\}$ are entries either of A or of B_ℓ and *at least one* of the $\{x_{i,j}, x_{t,s}, x_{i,s}, x_{t,j}\}$ is an entry of A and *at least one* of the $\{x_{i,j}, x_{t,s}, x_{i,s}, x_{t,j}\}$ is an entry of B_ℓ. On the other hand, when we say that "the minor comes from A and E_c" we mean that *at least one* of the $\{x_{i,j}, x_{t,s}, x_{i,s}, x_{t,j}\}$ is an entry of A and *at least one* of the $\{x_{i,j}, x_{t,s}, x_{i,s}, x_{t,j}\}$ is an entry of E_c for $c \in \{1,2,3,4\}$. We have the following cases of minors:

1. *only* from E_c
2. *only* from A
3. *only* from O_m
4. from E_c and A
5. *only* from A and B_ℓ
6. *only* from E_c and O_m
7. *only* from A and O_m
8. *only* from E_c and B_ℓ

In each case, we will show that the corresponding minors belong to

$$\mathcal{I}' + \left(\wedge^2(B_1 \mathrel{\vdots} B_2),\ Tr(A) + 2\pi,\ B_2 J_{n-r} B_1^t - A J_{2r}\right).$$

We will start by considering case (1), i.e. minors *only* from E_c. It suffices to prove $x_{i,j}x_{t,s} = x_{i,s}x_{t,j}$ in the quotient ring

$$\frac{S}{\mathcal{I}' + \left(\wedge^2(B_1 \mathrel{\vdots} B_2),\ Tr(A) + 2\pi,\ B_2 J_{n-r} B_1^t - A J_{2r}\right)}.$$

By using minors from B_ℓ for $\ell \in \{1, 2\}$ and for all $i, j \in Z^c$, we have the following equation in the above quotient ring:

$$\Big(\sum_{a=n-(r-1)}^{n} x_{d+1-a,d+1-i}x_{a,j}\Big)\Big(\sum_{a=n-(r-1)}^{n} x_{d+1-a,d+1-t}x_{a,s}\Big) =$$

$$\Big(\sum_{a=n-(r-1)}^{n} x_{d+1-a,d+1-i}x_{a,s}\Big)\Big(\sum_{a=n-(r-1)}^{n} x_{d+1-a,d+1-t}x_{a,j}\Big).$$

By using the relations (1)-(4) from Section 4.1 we can express the entries $x_{i,j}$ of E_c as:

$$x_{i,j} = -\sum_{a=n-(r-1)}^{n} x_{d+1-a,d+1-i}x_{a,j}$$

with $i, j \in Z^c$. Using this and the above equality we obtain:

$$x_{i,j}x_{t,s} = x_{i,s}x_{t,j}.$$

The rest of cases (2)-(8) can be handled by similar arguments. More precisely, by using the relations (1)-(6) from Section 4.1 we can express all the entries from E_i for $i \in \{1, 2, 3, 4\}$ and O_1, O_2 in terms of the entries of B_1, B_2. Also, by using $A = B_2 J_{n-r} B_1^t J_{2r}$ we can express all the entries of A in terms of the entries of B_1, B_2. After that, by using the 2×2-minors from the matrix $(B_1|B_2)$ we get the desired result in all the remaining cases. $\square$

End of proof of Theorem 4.0.1: From the above lemma we obtain that

$$I' = \mathcal{I}' + \left(\wedge^2(B_1 \,\vdots\, B_2),\ Tr(A) + 2\pi,\ B_2 J_{n-r} B_1^t - A J_{2r}\right).$$

Observe that an equivalent way of writing I' is:

$$I' = \mathcal{I}' + \left(\wedge^2(B_1 \,\vdots\, B_2),\ Tr(B_2 J_{n-r} B_1^t J_{2r}) + 2\pi,\ B_2 J_{n-r} B_1^t - A J_{2r}\right).$$

Using this and the fact that $I' = I$ the proof of Theorem 4.0.1 follows. $\square$

Chapter 5

Flatness of $U_{d,l}$

We continue to assume $d = 2n$, $l = 2r$.

Recall that $U_{d,l} = \mathrm{Spec}\,(S/I)$. The goal of this chapter is to prove that $U_{d,l}$ is flat over $\mathcal{O}$ as given by Theorem 5.0.4. The simplification that we obtained from Theorem 4.0.1 quickly gives the following, which in turn plays a crucial role for the proof of Theorem 5.0.4.

Theorem 5.0.1. *$U_{d,l}$ is Cohen-Macaulay.*

Proof. Denote by $\mathcal{O}[B_1|B_2]$ the polynomial ring over $\mathcal{O}$ with variables the entries of the matrix $(B_1|B_2)$. From Theorem 4.0.1 we obtain the isomorphism

$$\frac{S}{I} \simeq \frac{\mathcal{O}[B_1|B_2]}{\left(\wedge^2(B_1 \vert B_2), Tr(B_2 J_{n-r} B_1^t J_{2r}) + 2\pi\right)}.$$

Set $\mathcal{R} := \mathcal{O}[B_1|B_2]/(\wedge^2\,(B_1 \vert B_2))$. By [21, Remark 2.12], the ring $\mathcal{R}$ is Cohen-Macaulay and an integral domain. We consider the point P of the determinantal variety which is defined by the relations:

$$\begin{pmatrix} x_{n-r+1,1} & x_{n-r+1,d} \\ x_{n+r,1} & x_{n+r,d} \end{pmatrix} = \begin{pmatrix} 1-\pi & 1-\pi \\ 1 & 1 \end{pmatrix}$$

and we set all the other variables equal to zero. We can easily observe that $Tr(B_2 J_{n-r} B_1^t J_{2r})+$

2π is not zero over the point P. Therefore, we have

$$ht((Tr(B_2 J_{n-r} B_1^t J_{2r}) + 2\pi)) = 1.$$

We apply the fact that if A is Cohen-Macaulay and the ideal $I = (a_1, \ldots, a_r)$ of A has height r then A/I is Cohen-Macaulay ([12, example 3 (16.F)]) to $A = \mathcal{R}$ and $a_1 = Tr(B_2 J_{n-r} B_1^t J_{2r}) + 2\pi$. We obtain that

$$\frac{\mathcal{O}[B_1|B_2]}{\left(\wedge^2(B_1|B_2), Tr(B_2 J_{n-r} B_1^t J_{2r}) + 2\pi\right)}$$

is Cohen-Macaulay. This implies the result. □

Remark 5.0.2. *From the above proof and the standard formula for the dimension of the determinantal varieties (see [21, Proposition 1.1]) we obtain that $dim(S/I) = dim(\mathcal{R}) - 1 = d - 1$. Hence, the dimension of $U_{d,l}$ is $d - 1$.*

Remark 5.0.3. *Mimicking the proof of Theorem 5.0.1 and by considering Remark 5.0.2 we obtain that the special fiber $\overline{U}_{d,l}$ of $U_{d,l}$ is Cohen Macaulay and of dimension $d - 2$.*

Theorem 5.0.4. *$U_{d,l}$ is flat over $\mathcal{O}$.*

Proof. From Remark 5.0.2 we have that $U_{d,l}$ has dimension $d - 1$. From Remark 5.0.3 we have that the dimension of $\overline{U}_{d,l}$ is $d - 2$. Using the fact that $U_{d,l}$ is Cohen Macaulay, see Theorem 5.0.1, we obtain that $ht((\pi)) = 1$. Hence, we have that (π) is a regular sequence i.e π is not a zero divisor (see [9]). From this, flatness of $U_{d,l}$ follows. □

Chapter 6

Reducedness of $\overline{U}_{d,l}$

In all of Chapter 6, we assume $d = 2n$ and $l = 2r$.

We will prove that the special fiber $\overline{U}_{d,l}$ of $U_{d,l} = \mathrm{Spec}\,(S/I)$ is reduced; see Chapter 4 for undefined terms.

Proof of Theorem 3.2.3. From Remark 5.0.3 we know that $\overline{U}_{d,l}$ is Cohen-Macaulay. By using Serre's criterion for reducedness, it suffices to prove that the localizations at minimal primes are reduced. From Lemma 7.1.1 and Lemma 7.2.1 we obtain the minimal primes of $\overline{U}_{d,l}$. Below, we focus on the localization of $\overline{U}_{d,l}$ over I_1 for $1 < r < n-1$ (see Section 7.2 for the notation). In the other cases the proof is similar.

We first introduce some additional notation:

Denote by $k[B_1|B_2]$ the polynomial ring over k with variables the entries of the matrix $(B_1|B_2)$. We set $Z_1 := Z \setminus \{n-r+1\}$ and $Z_2 := Z^c \setminus \{1\}$. By direct calculations we get

$$Tr(B_2 J_{n-r} B_1^t J_l) = \sum_{n-r+1 \leq i \leq n+r, 1 \leq j \leq n-r} x_{i,j}\, x_{d+1-i,d+1-j}.$$

Set $t_r := \frac{1}{2} Tr(B_2 J_{n-r} B_1^t J_l)$ and $t'_r = t_r - x_{n-r+1,1} x_{n+r,d}$. Lastly, we set $m := x_{n-r+1,d}^{-1} t'_r$. We refer the reader to Section 7.2 for the rest undefined terms.

From Theorem 4.0.1 and Lemma 4.1.2 we obtain that the special fiber $\overline{U}_{d,l}$ is given by the quotient of $k[B_1|B_2]$ by the ideal $(\wedge^2(B_1\dot{|}B_2), t_r)$. Set $J_1 = I_1 + \big(\wedge^2(B_1\dot{|}B_2),\ t_r\big)$ where

$$I_1 = \Big(\big(\sum_{a=n-(r-1)}^{n} x_{d+1-a,d+1-i}x_{a,j}\big)_{i,j\in Z^c},\ \wedge^2(B_1\dot{|}B_2)\Big).$$

By localizing $k[B_1|B_2]/(\wedge^2(B_1\dot{|}B_2), t_r)$ at J_1 we have

$$\left(\frac{k[B_1|B_2]}{\wedge^2(B_1\dot{|}B_2), t_r}\right)_{J_1} \simeq \frac{k[B_1|B_2]_{I_1}}{\big(\wedge^2(B_1\dot{|}B_2), t_r\big)_{I_1}}.$$

In the proof of Lemma 7.2.1 we used the fact that $x_{n-r+1,1} \notin I_1$. Similarly, we have that $x_{n-r+1,d} \notin I_1$.

Claim:

$$\Big(\wedge^2(B_1\dot{|}B_2),\ t_r\Big)_{I_1}$$

$$= \Big((x_{i,j} - x_{n-r+1,1}^{-1}x_{i,1}x_{n-r+1,j})_{i\in Z_1, j\in Z_2},\ (x_{n+r,1}+m)\Big)_{I_1}.$$

Proof of the claim: From the minors we have $x_{i,j}x_{n-r+1,1} - x_{i,1}x_{n-r+1,j} = x_{n-r+1,1}(x_{i,j} - x_{n-r+1,1}^{-1}x_{i,1}x_{n-r+1,j})$. We rewrite t_r as:

$$t_r = x_{n-r+1,1}x_{n+r,d} + t_r'. \tag{6.0.0.1}$$

Combining (6.0.0.1) with the minor $x_{n-r+1,1}x_{n+r,d} = x_{n-r+1,d}x_{n+r,1}$ we obtain

$$x_{n-r+1,1}x_{n+r,d} + t_r' = x_{n-r+1,d}(x_{n+r,1}+m).$$

Now, because $x_{n-r+1,1}, x_{n-r+1,d} \notin I_1$ the claim follows.

Combining all the above we have:

$$\begin{aligned}
&\frac{k[B_1|B_2]_{I_1}}{\left(\wedge^2(B_1|B_2),\ t_r\right)_{I_1}} \\
\simeq\ &\frac{k[B_1|B_2]_{I_1}}{\Big((x_{i,j} - x_{n-r+1,1}^{-1} x_{i,1} x_{n-r+1,j})_{i\in Z_1,\, j\in Z_2},\ (x_{n+r,1} + m)\Big)_{I_1}} \\
\simeq\ &k[(x_{i,1})_{i\in Z\setminus\{n+r\}},\ (x_{n-r+1,j})_{j\in Z^c},\ x_{n-r+1,1}^{-1},\ x_{n-r+1,d}^{-1}]_{J_1}
\end{aligned}$$

where the last one is a reduced ring. Thus, the special fiber of $U_{d,l}$ is reduced. □

Chapter 7

Irreducible Components of $\overline{U}_{d,l}$ Part I

We continue to assume $d = 2n$, $l = 2r$.

Recall that $U_{d,l} = \text{Spec}\,(S/I)$ and $\overline{U}_{d,l}$ is the special fiber of $U_{d,l}$. In this chapter, the main goal is to calculate the irreducible components of $\overline{U}_{d,l}$.

Theorem 7.0.1. *(i) For $r = 1$ and $r = n - 1$, $\overline{U}_{d,l}$ has three irreducible components.*

(ii) For $1 < r < n - 1$, $\overline{U}_{d,l}$ has two irreducible components.

The proof of the theorem will be carried out in Section 7.1 (case (i)) and in Section 7.2 (case (ii)).

7.1 $l = 2$ and $l = d - 2$

In this section we will prove Theorem 7.0.1 in the case $r = 1$. A similar argument works in the case $r = n - 1$. For this section we introduce the following notation. Observe that when $r = 1$, $Z = \{n, n+1\}$ and $Z^c = \{1, 2, 3, \ldots, d\} \setminus Z$. We rename the variables as follows:

$$v_i = x_{n,i}, \text{ for } i \in Z^c \text{ and } w_j = x_{n+1,j}, \text{ for } j \in Z^c.$$

Define the polynomial ring $S_{\text{sim}} = k[v_i, w_j]_{i,j \in Z^c}$. From Theorem 4.0.1 we obtain that the special fiber is isomorphic to $S_{\text{sim}}/I_{\text{sim}}$ where

$$I_{\text{sim}} = \Big(\sum_{i=1}^{n-1} v_i w_{2n+1-i}, \quad (v_i w_j - v_j w_i)_{i,j \in Z^c} \Big).$$

It is not very hard to observe, by using the minors, that

$$I_{\text{sim}} = \Big(\sum_{i=1}^{n-1} v_i w_{2n+1-i},\ (v_i \sum_{j=1}^{n-1} w_j w_{2n+1-j})_{i \in Z^c},\ (w_i \sum_{j=1}^{n-1} v_j v_{2n+1-j})_{i \in Z^c},$$

$$\sum_{i=1}^{n-1} v_i v_{2n+1-i} \sum_{j=1}^{n-1} w_j w_{2n+1-j}, \quad (v_i w_j - v_j w_i)_{i,j \in Z^c} \Big).$$

Next, we set

$$I_1 = \Big((v_i)_{i \in Z^c} \Big), \quad I_2 = \Big((w_i)_{i \in Z^c} \Big)$$

and

$$I_3 = \Big(\sum_{i=1}^{n-1} v_i v_{2n+1-i},\ \sum_{i=1}^{n-1} w_i w_{2n+1-i},\ \sum_{i=1}^{n-1} v_i w_{2n+1-i},\ (v_i w_j - v_j w_i)_{i,j \in Z^c} \Big).$$

Proof of Theorem 7.0.1 (i)*:* From the above, it suffices to calculate the irreducible components of $V(I_{\text{sim}})$.

Observe that the elements

$$(v_i \sum_{j=1}^{n-1} w_j w_{2n+1-j})_{i \in Z^c}, \quad (w_i \sum_{j=1}^{n-1} v_j v_{2n+1-j})_{i \in Z^c},$$

$$\sum_{i=1}^{n-1} v_i v_{2n+1-i} \sum_{j=1}^{n-1} w_j w_{2n+1-j}$$

belong to I_{sim}.

Therefore, we can easily see that

$$V(I_{\text{sim}}) = V(I_1) \cup V(I_2) \cup V(I_3).$$

Observe also that

$$S_{\text{sim}}/I_1 \simeq k[(w_j)_{j \in Z^c}], \qquad S_{\text{sim}}/I_2 \simeq k[(v_j)_{j \in Z^c}].$$

Thus, the closed subschemes $V(I_1), V(I_2)$ are affine spaces of dimension $d-2$ and so they are irreducible and smooth. We have to check that the third one is irreducible and of dimension $d-2$. Notice that I_3 is generated by homogeneous elements. Thus, it suffices to prove that the projectivization

$$V_p(I_3) \subseteq \mathbb{P}_k^{2d-5}$$

of the affine cone $V(I_3)$ is irreducible. Consider

$$V_{v_1} := V_p(I_3) \cap U_{v_1},$$

where $U_{v_1} = D(v_1 \neq 0)$. We can see that it is isomorphic to

$$\operatorname{Spec}\left(k[(v_i)_{i \in Z^c \setminus \{1,d\}}, w_1]\right),$$

and so it is irreducible. By symmetry we have a similar result for every V_{v_i} and V_{w_j} with $i, j \in Z^c$. Moreover, the V_{v_i} and V_{w_j} form a finite open cover of irreducible open subsets of $V_p(I_3)$. Thus $V_p(I_3)$ is irreducible and so $V(I_3) \subseteq \mathbb{A}_k^{2d-4}$ is irreducible. This completes the proof of Theorem 7.0.1 (i). $\square$

We can go one step further and prove that:

Lemma 7.1.1. *The ideals I_1, I_2, I_3 are prime.*

Proof. From the proof of Theorem 7.0.1 (i), I_1, I_2 are clearly prime ideals. It suffices to prove that I_3 is prime. From Theorem 7.0.1 (i) we have that I_3 is a primary ideal and so every zero divisor in $D := k[v_i, w_j]_{i,j\in Z^c}/I_3$ is a nilpotent element. Hence, it suffices to prove that D is reduced. From the proof of Theorem 7.0.1 (i) we have that the scheme $\mathrm{Spec}\,(D)$ is smooth over $\mathrm{Spec}\,(k)$ outside from its closed subscheme of dimension 0 which is defined by the ideal $((v_j)_{j\in Z^c}, (w_i)_{i\in Z^c})$. Therefore, using Serre's criterion for reducedness ([12] 17.I) and the above description it suffices to find a regular element f such that $f \in ((v_j)_{j\in Z^c}, (w_i)_{i\in Z^c})$.

Claim: *We can take $f = w_1$.*

Proof of the claim: Assume that w_1 is not a regular element. Then, because I_3 is primary, we have $w_1^m \in I_3$ for some $m > 0$. We will obtain a contradiction by using Buchberger's algorithm. This is a method of transforming a given set of generators for a polynomial ideal into a Gröbner basis with respect to some monomial order. For more information about this algorithm we refer the reader to [4, Chapter 2].

Set $R = k[v_i, w_j]_{i,j\in Z^c}$ for the polynomial ring. We choose the following order for our variables:

$$v_1 > v_2 > \cdots > v_d > w_1 > \cdots > w_d.$$

Then, the graded lexicographic order induces an order of all monomials in R.

Next, we recall the division algorithm in R. We fix the above monomial ordering. Let $J = (f_1, \ldots, f_s)$ be an ordered s-tuple of polynomials in R. Then every $g \in R$ can be written as

$$g = a_1 f_1 + \ldots + a_s f_s + r,$$

where $a_i, r \in R$ and either $r = 0$ or r is a linear combination, with coefficients in k, of monomials, none of which is divisible by any of $LT(f_1), \dots, LT(f_s)$. By $LT(f_i)$ we denote the leading term of f_i. We will call r a remainder of g on division by J. (See [4, Chapter 2] for more details.)

Recall that the $\mathcal{S}$-polynomial of the pair $f,\ g \in R$ is

$$\mathcal{S}(f,g) = \frac{LCM(LM(f), LM(g))}{LT(f)} f - \frac{LCM(LM(f), LM(g))}{LT(g)} g.$$

Here, by $LM(f)$ we denote the leading monomial of f according to the above ordering.

To find the Gröbner basis for the ideal I_3, we start with the generating set

$$\Big\{ \sum_{i=1}^{n-1} v_i v_{2n+1-i}, \sum_{i=1}^{n-1} w_i w_{2n+1-i}, \sum_{i=1}^{n-1} v_i w_{2n+1-i}, (v_i w_j - v_j w_i)_{i,j \in Z^c} \Big\}.$$

Then, we calculate all the $\mathcal{S}$-polynomials $\mathcal{S}(f,g)$, where f, g are any two generators from the generating set that we have started. If all the $\mathcal{S}$-polynomials are divisible by the generating set then the generating set already forms a Gröbner basis. On the other hand, if a remainder is nonzero we extend our generating set by adding this remainder and we repeat the above process until we have a generating set where all the $\mathcal{S}$-polynomials are divisible by the generating set. In our case, the generators of I_3 are homogeneous polynomials of degree 2. The monomials of those homogeneous polynomials have one of the following forms:

1. $v_i v_j$ with $i \neq j$, or

2. $w_i w_j$ with $i \neq j$, or

3. $w_i v_j$ with $i \neq j$.

Thus, the nonzero remainder of any $\mathcal{S}$-polynomial is a polynomial where each monomial is divisible by at least one monomial of the above three forms. By this observation we can see that, the Gröbner basis cannot have a monomial that looks like w_i^m or v_j^m. Now, let $\{g_1 \dots, g_N\}$ be the Gröbner basis of I_3. From the above we have

$$w_1^m \notin \langle LT(g_1), \dots, LT(g_N) \rangle.$$

Moreover, because $\{g_1 \dots, g_N\}$ is the Gröbner basis of I_3 we have that

$$\langle LT(g_1), \dots, LT(g_N) \rangle = LT(I_3),$$

(see [4, Chapter 5]). By $LT(I_3)$ we denote the ideal generated by all the leading terms of elements of I_3. Therefore,

$$w_1^m \notin LT(I_3).$$

Hence, w_1 is a regular element and so I_3 is a prime ideal. This completes the proof of the claim and by the above the proof of lemma. □

7.2 $2 < l < d - 2$

In this section we will prove Theorem 7.0.1 in the case $1 < r < n - 1$. In this case we have $Z = \{n-(r-1), \dots, n, n+1, \dots, d-n+r\}$ and $Z^c = \{1, 2, 3, \dots, d\} \setminus Z$. For the undefined terms below we refer the reader to Chapters 4 and 6. From Theorem 4.0.1 we obtain that the special fiber $\overline{U}_{d,l}$ is given by the quotient of $k[B_1|B_2]$ by the ideal $I_s = (\wedge^2(B_1 \mathrel{\vdots} B_2), t_r)$.

Also with direct calculations and by using the minors we can see that

$$I_s = \Big(\wedge^2 (B_1 ¦ B_2), \quad t_r,$$

$$\Big(\sum_{a=n-(r-1)}^{n} x_{d+1-a,d+1-t} x_{a,s} \sum_{b=1}^{n-r} x_{i,b} x_{d+1-j,d+1-b} \Big)_{\substack{i,j \in Z \\ t,s \in Z^c}} \Big).$$

Next, set

$$I_1 = \Big(\Big(\sum_{a=n-(r-1)}^{n} x_{d+1-a,d+1-t} x_{a,s} \Big)_{t,s \in Z^c}, \quad \wedge^2 (B_1 ¦ B_2) \Big)$$

and

$$I_2 = \Big(\Big(\sum_{a=1}^{n-r} x_{i,a} x_{d+1-j,d+1-a} \Big)_{i,j \in Z}, \quad \wedge^2 (B_1 ¦ B_2) \Big).$$

Proof of Theorem 7.0.1 (ii): From the above, it suffices to calculate the irreducible components of $V(I_s)$.

Observe that

$$\Big(\sum_{a=n-(r-1)}^{n} x_{d+1-a,d+1-t} x_{a,s} \sum_{b=1}^{n-r} x_{i,b} x_{d+1-j,d+1-b} \Big)_{\substack{i,j \in Z \\ t,s \in Z^c}} \in I_s.$$

Therefore, we can easily see that

$$V(I_s) = V(I_1) \cup V(I_2).$$

Next, we prove that the closed subschemes $V(I_1)$ and $V(I_2)$ are irreducible of dimension $d-2$. We will start by proving that $V(I_1)$ is an irreducible component. Observe that I_1 is generated by homogeneous elements.

Thus, it suffices to prove that the projectivization

$$V_p(I_1) \subseteq \mathbb{P}_k^{4r(n-r)-1}.$$

of the affine cone $V(I_1)$ is irreducible. We look at

$$V_{x_{n-(r-1),1}} := V_p(I_1) \cap U_{x_{n-(r-1),1}},$$

where $U_{x_{n-(r-1),1}} = D(x_{n-(r-1),1} \neq 0)$. We can see that it is isomorphic to

$$\mathrm{Spec}\,(k[(x_{n-(r-1),j})_{j \in Z^c \setminus \{1\}}, (x_{t,1})_{n-(r-2) \leq t \leq n+r-1}])$$

and so it is irreducible. Because of the symmetry of the relations we will have a similar result for every $V_{x_{t,s}}$ with $t \in Z, s \in Z^c$. Moreover,

$$(V_{x_{t,s}})_{t \in Z,\, s \in Z^c}$$

form a finite open cover of irreducible open subsets of $V_p(I_1)$ and thus we get that $V_p(I_1)$ is irreducible and so $V(I_1)$ is irreducible of dimension $d-2$.

We use a similar argument for $V(I_2)$. I_2 is generated by homogeneous elements and so by using the projectivization

$$V_p(I_2) \subseteq \mathbb{P}_k^{4r(n-r)-1}$$

of the affine cone $V(I_2)$, it suffices to prove that $V_p(I_2)$ is irreducible. We look at

$$V_{x_{n-(r-1),1}} := V_p(I_2) \cap U_{x_{n-(r-1),1}},$$

where $U_{x_{n-(r-1),1}} = D(x_{n-(r-1),1} \neq 0)$. We can see that it is isomorphic to

$$\mathrm{Spec}\,(k[(x_{n-(r-1),j})_{j\in Z^c\setminus\{1,d\}}, (x_{t,1})_{n-(r-2)\leq t\leq n+r}])$$

and so it is irreducible. Therefore, $V(I_2)$ is irreducible of dimension $d-2$. This completes the proof of Theorem 7.0.1 (ii). □

Next, we prove that:

Lemma 7.2.1. *The ideals I_1, I_2 are prime.*

Proof. To see that I_1, I_2 are prime ideals one proceeds exactly as in Lemma 7.1.1. So, it suffices to find a regular element f such that $f \in ((x_{t,s})_{t\in Z, s\in Z^c})$. We claim that $x_{n-r+1,1}$ is a choice for f. Assume for contradiction that $x_{n-r+1,1}$ is not a regular element. Then, because I_i is primary, we should have that $x^m_{n-r+1,1} \in I_i$ for some $m > 0$.

We choose the following order for our variables $(x_{t,s})_{t\in Z, s\in Z^c}$:

$$x_{n-r+1,1} > \ldots > x_{n-r+1,n-r} > x_{n-r+1,n+r+1} > \ldots > x_{n-r+1,d} >$$

$$x_{n-r+2,1} > \ldots > x_{n-r+2,d} > \ldots > x_{n+r,1} > \ldots > x_{n+r,n-r} >$$

$$x_{n+r,n+r+1} > \ldots > x_{n+r,d}.$$

Then, the graded lexicographic order induces an ordering to all the monomials. First, let's consider the ideal I_1. In order to find the Gröbner basis for I_1, we start with the generating set

$$\Big\{\Big(\sum_{a=n-(r-1)}^{n} x_{d+1-a,d+1-i}x_{a,j}\Big)_{i,j\in Z^c}, \quad \wedge^2(B_1 \vert B_2)\Big\}.$$

After that we calculate all the $\mathcal{S}$-polynomials $\mathcal{S}(f,g)$, where f,g are any two generators from the generating set that we have started; so in our case is I_1. The generators of I_1 are homogeneous polynomials of degree 2. The monomials of those homogeneous polynomials have one of the following form:

$$(x_{i,j}x_{t,s})$$

with $i,t \in Z$, $j,s \in Z^c$ and either $i \neq t$ or $j \neq s$. Thus, the nonzero remainder of any $\mathcal{S}$-polynomial is a polynomial where each monomial is divisible by at least one monomial of the above form. By this observation we can see that, the Gröbner basis cannot have a monomial that looks like $x^m_{n-r+1,1}$. Now, by using a Gröbner basis argument as in Lemma 7.1.1 we deduce that $x_{n-r+1,1}$ is a regular element and so I_1 is a prime ideal.

Now, by looking the ideal I_2 we have the generating set:

$$\Big\{\Big(\sum_{a=1}^{n-2} x_{i,a}x_{d+1-j,d+1-a}\Big)_{i,j\in Z}, \quad \wedge^2(B_1|B_2)\Big\}.$$

So, we observe that in this case also the generators of the ideal I_2 are homogeneous polynomials of degree 2. All the monomials have one of the following form:

$$(x_{i,j}x_{t,s})$$

with $i,t \in Z$, $j,s \in Z^c$ and either $i \neq t$ or $j \neq s$. So, by using the same argument we can prove that I_2 is a prime ideal. □

Chapter 8

The Remaining Cases

In this chapter we sketch the proof of Theorems 3.2.1 and 3.2.2 in the remaining cases of parity for d and l. The main point is that in all the cases the affine chart $U_{d,l}$ of the local model is displayed as a hypersurface in a determinantal scheme. The arguments are similar with the proof of the case $(d,l) = (\text{even}, \text{even})$. In fact, in the case that $(d,l) = (\text{odd}, \text{odd})$ the argument is exactly the same. The case of Theorem 3.2.2 (different parity) is somewhat different as we explain below.

8.1 Proof of Theorem 3.2.2

Proof. We use the notation from Section 3.2. We also introduce some new notation. Set

$$Z' := \{n-r', \dots, n, n+2, \dots, d-n+r'\}, \quad (Z')^c := \{1,2,3,\dots,d\} \setminus Z'.$$

Also, define the polynomial ring

$$\mathcal{O}[B_1'|Q'|B_2'] := \mathcal{O}[(x_{t,s})_{t\in Z', s\in (Z')^c}].$$

Lastly, set

$$\wedge^2(B_1'|Q'|B_2') := (x_{i,j}x_{t,s} - x_{i,s}x_{t,j})_{i,t\in Z' \ j,s\in (Z')^c}.$$

We can now sketch the proof. In this case, similar elementary arguments as in the proof of Theorem 4.0.1 give that the quotient $\mathcal{O}[X]/I$ is isomorphic to the quotient of $\mathcal{O}[B_1'|Q'|B_2']$ by the ideal

$$(\wedge^2(B_1'|Q'|B_2'),\ Tr((B_2'J_{n-r'}(B_1')^t + \frac{1}{2}Q'(Q')^t)J_l) + 2\pi).$$

The rest of the argument deducing flatness is the same as before. $\square$

Chapter 9

Irreducible Components of $\overline{U}_{d,l}$ Part II

In this chapter we present the irreducible components of the special fiber $\overline{U}_{d,l}$ of $U_{d,l}$ in the remaining cases. We omit the proofs which are similar to Theorem 7.0.1. For the notation we refer the reader to Chapter 4 and Chapter 8.

9.1 $(d, l) = (\text{odd}, \text{odd})$

9.1.1

When $l < d - 2$, the irreducible components of $\overline{U}_{d,l}$ are the closed subschemes $V(I_1)$ and $V(I_2)$ where:

$$I_1 = \Big((\sum_{a=n-r+1}^{n} x_{d+1-a,d+1-t} x_{a,s} + \frac{1}{2} x_{n+1,d+1-t} x_{n+1,s})_{t,s \in Z^c},\ \wedge^2 (B_1 \vdots B_2) \Big)$$

and

$$I_2 = \Big((\sum_{a=1}^{n-r} x_{i,a} x_{d+1-j,d+1-a})_{i,j \in Z},\ \ \wedge^2 (B_1 \vdots B_2) \Big).$$

9.1.2

When $l = d - 2$, the irreducible components of $\overline{U}_{d,l}$ are the closed subschemes $V(I_1), V(I_2)$

and $V(I_3)$ where:

$$I_1 = \Big((x_{i,1})_{i\in Z}\Big), \quad I_2 = \Big((x_{i,d})_{i\in Z}\Big)$$

and

$$I_3 = \Big(\sum_{a=n-r+1}^{n} x_{a,1}x_{d+1-a,1} + \frac{1}{2}x_{n+1,1}^2, \sum_{a=n-r+1}^{n} x_{a,d}x_{d+1-a,d} + \frac{1}{2}x_{n+1,d}^2,$$

$$\wedge^2(B_1 \mid B_2), \sum_{a=n-r+1}^{n} x_{a,1}x_{d+1-a,d} + \frac{1}{2}x_{n+1,1}x_{n+1,d}\Big).$$

9.2 $(d,l) = (\text{odd}, \text{even})$

9.2.1

When $l > 2$ the irreducible components of $\overline{U}_{d,l}$ are the closed subschemes $V(I_1)$ and $V(I_2)$ where:

$$I_1 = \Big(\Big(\sum_{a=n-(r-1)}^{n} x_{d+1-a,d+1-t}x_{a,s}\Big)_{t,s\in(Z')^c}, \quad \wedge^2((B_1' \mid Q' \mid B_2'),$$

$$\Big(\sum_{a=n-(r-1)}^{n} x_{d+1-a,n+1}x_{a,j}\Big)_{1\le j\le d}\Big)$$

and

$$I_2 = \Big(\wedge^2\,((B_1' \mid Q' \mid B_2'), \ \Big(\sum_{a=1}^{n-r} x_{i,a}x_{d+1-j,d+1-a} + \frac{1}{2}x_{i,n+1}x_{d+1-j,n+1}\Big)_{i,j\in Z'}\Big).$$

9.2.2

When $l = 2$, the irreducible components of $\overline{U}_{d,l}$ are the closed subschemes $V(I_1), V(I_2)$ and $V(I_3)$ where:

$$I_1 = \Big((x_{n,i})_{i\in(Z')^c}\Big), \quad I_2 = \Big((x_{n+2,i})_{i\in(Z')^c}\Big)$$

and

$$I_3 = \Big(\sum_{a=1}^{n-1} x_{n,a}x_{n,d+1-a} + \frac{1}{2}x_{n,n+1}^2, \quad \sum_{a=1}^{n-1} x_{n+2,a}x_{n+2,d+1-a} + \frac{1}{2}x_{n+2,n+1}^2,$$

$$\wedge^2((B_1'\,\vdots\, Q'\,\vdots\, B_2'), \quad \sum_{a=1}^{n-1} x_{n,a}x_{n+2,d+1-a} + \frac{1}{2}x_{n,n+1}x_{n+2,n+1}\Big).$$

9.3 $(d, l) = (\text{even}, \text{odd})$

The irreducible components of $\overline{U}_{d,l}$ are the closed subschemes $V(I_1)$ and $V(I_2)$ where:

$$I_1 = \Big(\Big(\sum_{a=n-r}^{n-1} x_{d+1-a,n+1}x_{a,j} + \frac{1}{2}x_{n,n+1}x_{n,j}\Big)_{1\leq j\leq d}, \quad \wedge^2((B_1'\,\vdots\, Q'\,\vdots\, B_2'),$$

$$\Big(\sum_{a=n-r}^{n-1} x_{d+1-a,d+1-t}x_{a,s} + \frac{1}{2}x_{n,d+1-t}x_{n,s}\Big)_{t,s\in(Z')^c}\Big),$$

$$I_2 = \Big(\Big(\sum_{a=1}^{n-r-1} x_{i,a}x_{n,d+1-a} + \frac{1}{2}x_{i,n+1}x_{n,n+1}\Big)_{i\in Z'}, \quad \wedge^2((B_1'\,\vdots\, Q'\,\vdots\, B_2'),$$

$$\Big(\sum_{a=1}^{n-r-1} x_{i,a}x_{d+1-j,d+1-a} + \frac{1}{2}x_{i,n+1}x_{d+1-j,n+1}\Big)_{i\in Z', j\in Z'\setminus\{n\}}\Big).$$

Chapter 10

The Blow-Up of $\mathrm{M}^{\mathrm{loc}}(\Lambda)$

The statements and the results from this chapter are contained in [18]. The reader is referred to *loc. cit.* §5.3 for more details.

Let

$$r^{\mathrm{bl}} : \mathrm{M}^{\mathrm{bl}}(\Lambda) \to \mathrm{M}^{\mathrm{loc}}(\Lambda)$$

be the blow-up of $\mathrm{M}^{\mathrm{loc}}(\Lambda)$ at the closed point $*$ of its special fiber that corresponds to $\mathcal{L} = \mathbb{L}$. We will show:

Theorem 10.0.1. *The scheme* $\mathrm{M}^{\mathrm{bl}}(\Lambda)$ *is regular and has special fiber a divisor with normal crossings. In fact,* $\mathrm{M}^{\mathrm{bl}}(\Lambda)$ *is covered by open subschemes which are smooth over* $\mathrm{Spec}\,(\mathbb{Z}_p[u, x, y]/(u^2xy - p))$.

Before we start the proof of the above theorem, we first restate Theorems 3.2.1 and 3.2.2 in a different form; see Theorem 10.0.2. Then using this we prove Theorem 10.0.1.

In this chapter we use the notation from Section 3.2 and we also introduce some additional notation: we set

$$\mathrm{T}(B_1|B_2) = \mathrm{Tr}(B_2 J_{n-r} B_1^t J_l), \qquad \text{if } d \equiv l \bmod 2,$$

$$\mathrm{T}(B_1'|Q'|B_2') = \mathrm{Tr}((B_2' J_{n-r'} (B_1')^t + \frac{1}{2} Q'(Q')^t) J_l), \qquad \text{if } d \not\equiv l \bmod 2.$$

For simplicity, we define

$$Z = \begin{cases} [B_1|B_2], & \text{if } d \equiv l \bmod 2, \\ [B_1'|Q'|B_2'], & \text{if } d \not\equiv l \bmod 2. \end{cases}$$

Then $Z = (z_{ij}) \in \mathrm{Mat}_{l\times(d-l)}$, in both cases. By an explicit calculation, we find that

$$\mathrm{T}(Z) = \frac{1}{2} \sum_{1\le i\le l, 1\le j\le d-l} z_{i\ d-l+1-j}\, z_{l+1-i\ j}.$$

(The same expression is valid for any pair (d, l).) Finally, denote by $\mathscr{D}^2_{l\times(d-l)} = \{Z \mid \wedge^2 Z = 0\} \subset \mathrm{Mat}_{l\times(d-l)}$ the "determinantal" subscheme of the affine space of matrices Z over $\mathrm{Spec}\,(\mathcal{O})$.

Using the above notation, Theorems 3.2.1 and 3.2.2 are equivalent to the statement:

Theorem 10.0.2. *An affine chart of the local model* $\mathrm{M}^{\mathrm{loc}}(\Lambda)$ *around the worst point* $\mathcal{L} = \mathbb{L}$ *is given by* $U_{d,l}$ *and is isomorphic to the closed subscheme* $\mathscr{D}_{\mathrm{T}}$ *of the determinantal scheme* $\mathscr{D}^2_{l\times(d-l)}$ *which is defined by the quadratic equation*

$$\sum_{1\le i\le l, 1\le j\le d-l} z_{i\ d-l+1-j}\, z_{l+1-i\ j} = -4\pi.$$

Proof. This follows from the proofs of Theorems 3.2.1 and 3.2.2. More precisely, see Theorem 4.0.1 and Chapter 8. □

Now, we are ready to prove Theorem 10.0.1.

Proof. By Theorem 10.0.2, it is enough to show the conclusion of the theorem for the blow-up $\widetilde{\mathscr{D}}_{\mathrm{T}}$ of $\mathscr{D}_{\mathrm{T}}$ at the (maximal) ideal given by (z_{ij}). For simplicity, we write $\mathscr{D}$ for the determinantal scheme $\mathscr{D}^2_{l\times(d-l)}$ over $\mathrm{Spec}\,(\mathcal{O})$. This is the affine cone over the Segre embedding

$$(\mathbb{P}^{l-1}\times\mathbb{P}^{d-l-1})_{\mathcal{O}}\hookrightarrow\mathbb{P}^{l(d-l)-1}_{\mathcal{O}}.$$

Also, we set

$$\mathrm{T}=\frac{1}{2}\sum_{1\le i\le l,1\le j\le d-l} z_{i\ d-l+1-j}\, z_{l+1-i\ j}.$$

Let us consider the blow-up

$$\widetilde{\mathscr{D}}\longrightarrow\mathscr{D}$$

of the determinantal scheme over $\mathrm{Spec}\,(\mathcal{O})$ along the vertex of the cone, i.e. along the subscheme defined by the ideal (z_{ij}). Then, the blow-up $\widetilde{\mathscr{D}}_{\mathrm{T}}$ is isomorphic to the strict transform of the hypersurface $\mathscr{D}_{\mathrm{T}}\subset\mathscr{D}$ given by $\mathrm{T}+2\pi=0$. Let $V_{s,t}$ be the open affine chart of $\widetilde{\mathscr{D}}$ over which the image of z_{st} generates the pull-back of the ideal (z_{ij}). Then

$$V_{s,t}=\mathrm{Spec}\,(\mathcal{O}[(u_{i,j})_{1\le i\le l,1\le j\le d-l}]/((u_{i,j}-u_{s,j}u_{i,t})_{i,j},u_{s,t}-1).$$

The intersection $V_{s,t}\cap\widetilde{\mathscr{D}}_{\mathrm{T}}$ is obtained by substituting $z_{ij}=u_{i,j}z_{st}$ and $u_{i,j}=u_{s,j}u_{i,t}$, for all i, j, in the equation $\mathrm{T}=-2\pi$. This amounts to setting

$$z_{ij}=u_{s,j}u_{i,t}z_{st}$$

and gives

$$4\pi+z_{st}^2(\sum_{1\le i\le l,1\le j\le d-l} u_{s,d-l+1-j}u_{i,t}u_{s,j}u_{l+1-i,t})=0.$$

This is

$$4\pi + z_{st}^2(\sum_{i=1}^{l} u_{i,t}u_{l+1-i,t})(\sum_{j=1}^{d-l} u_{s,j}u_{s,d-l+1-j}) = 0. \tag{10.0.0.1}$$

Note that, since $u_{s,t} = 1$, the two sums in the line above are

$$S_1 = u_{l+1-s,t} + \sum_{i \neq s} u_{i,t}u_{l+1-i,t} \ , \qquad S_2 = u_{s,d-l+1-t} + \sum_{j \neq t} u_{s,j}u_{s,d-l+1-j}.$$

We see that $u \mapsto z_{st}$, $x \mapsto -S_1/2$, $y \mapsto S_2/2$ defines a smooth morphism

$$V_{s,t} \cap \widetilde{\mathscr{D}}_{\mathrm{T}} \longrightarrow \operatorname{Spec}(\mathcal{O}[u,x,y]/(u^2xy - \pi)).$$

□

Chapter 11

Application to Shimura Varieties

In this chapter we present one of the main results from the joint work [18]. For a more detailed presentation we refer the reader to *loc. cit.* §7.

The goal is to construct regular integral models for GSpin Shimura varieties; see Theorem 11.0.1.

We start with an odd prime p and an orthogonal quadratic space V over $\mathbb{Q}$ of dimension $d \geq 5$ and signature $(d-2, 2)$. We take $\mathbf{G} = \mathrm{GSpin}(V)$ and we consider the hermitian symmetric domain

$$\mathbf{D} = \{z \in V_{\mathbb{C}} : \langle z, z\rangle = 0,\ \langle z, \bar{z}\rangle < 0\}/\mathbb{C}^{\times}$$

of dimension $d-2$. The pair $(\mathbf{G}, \mathbf{D})$ defines the spin similitude Shimura datum (for more details see [18, §7.1].).

In addition, we choose a vertex lattice $\Lambda \subset V \otimes_{\mathbb{Q}} \mathbb{Q}_p$ with $\pi\Lambda^{\vee} \subset \Lambda \subset \Lambda^{\vee}$ and $l = \mathrm{length}_{\mathbb{Z}_p}(\Lambda^{\vee}/\Lambda)$, $l_* = \min(l, d-l)$, and assume $l_* \geq 2$. This defines the parahoric subgroup

$$K_p = \{g \in \mathrm{GSpin}(V \otimes_{\mathbb{Q}} \mathbb{Q}_p) \mid g\Lambda g^{-1} = \Lambda,\ \eta(g) \in \mathbb{Z}_p^{\times}\}$$

which we fix below. (Here, $\eta : \mathrm{GSpin}(V \otimes_{\mathbb{Q}} \mathbb{Q}_p) \to \mathbb{Q}_p^{\times}$ is the spinor similitude, and for $v \in V \otimes_{\mathbb{Q}} \mathbb{Q}_p$, gvg^{-1} is defined using the Clifford algebra, see [18, §2.3, §2.5].) The group $\mathcal{G}$ is the smooth connected "Bruhat-Tits" group scheme over $\mathrm{Spec}\,(\mathbb{Z}_p)$ such that

$\mathcal{G} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \mathbf{G} \otimes_{\mathbb{Q}} \mathbb{Q}_p$ and $\mathcal{G}(\mathbb{Z}_p) = K_p$. Choose also a sufficiently small compact open subgroup K^p of the prime-to-p finite adelic points $\mathbf{G}(\mathbb{A}_f^p)$ of $\mathbf{G}$ and set $K = K^p K_p$. The Shimura variety $\mathrm{Sh}_K(\mathbf{G}, \mathbf{D})$ with complex points

$$\mathrm{Sh}_K(\mathbf{G}, \mathbf{D})(\mathbb{C}) = \mathbf{G}(\mathbb{Q})\backslash \mathbf{D} \times \mathbf{G}(\mathbb{A}_f)/K$$

is of Hodge type and has a canonical model over the reflex field $\mathbb{Q}$.

Theorem 11.0.1. *For every K^p as above, there is a scheme $\mathscr{S}_K^{\mathrm{reg}}(\mathbf{G}, \mathbf{D})$, flat over $\mathrm{Spec}\,(\mathbb{Z}_p)$, with*

$$\mathscr{S}_K^{\mathrm{reg}}(\mathbf{G}, \mathbf{D}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \mathrm{Sh}_K(\mathbf{G}, \mathbf{D}) \otimes_{\mathbb{Q}} \mathbb{Q}_p,$$

and which supports a "local model diagram"

$$\begin{array}{ccc} & \widetilde{\mathscr{S}}_K^{\mathrm{reg}}(\mathbf{G}, \mathbf{D}) & \\ {}^{\pi_K^{\mathrm{reg}}}\swarrow & & \searrow^{q_K^{\mathrm{reg}}} \\ \mathscr{S}_K^{\mathrm{reg}}(\mathbf{G}, \mathbf{D}) & & \mathrm{M}^{\mathrm{bl}}(\Lambda) \end{array} \qquad (11.0.0.1)$$

such that:

a) π_K^{reg} is a $\mathcal{G}$-torsor for the parahoric group scheme $\mathcal{G}$ that corresponds to K_p,

b) q_K^{reg} is smooth and $\mathcal{G}$-equivariant.

c) $\mathscr{S}_K^{\mathrm{reg}}(\mathbf{G}, \mathbf{D})$ is regular and has special fiber which is a divisor with normal crossings. In fact, $\mathscr{S}_K^{\mathrm{reg}}(\mathbf{G}, \mathbf{D})$ can be covered, in the étale topology, by schemes which are smooth over $\mathrm{Spec}\,(\mathbb{Z}_p[u, x, y]/(u^2xy - p))$.

In addition, we have:

1) The schemes $\{\mathscr{S}^{\rm reg}_K(\mathbf{G},\mathbf{D})\}_{K^p}$*, for variable* K^p*, support correspondences that extend the standard prime-to-p Hecke correspondences on* $\{\mathrm{Sh}_K(\mathbf{G},\mathbf{D})\}_{K^p}$*. These correspondences extend to the local model diagrams above (acting trivially on* $\mathrm{M}^{\rm bl}(\Lambda)$*).*

2) The projective limit

$$\mathscr{S}^{\rm reg}_{K_p}(\mathbf{G},\mathbf{D}) = \varprojlim_{K^p} \mathscr{S}_{K^pK_p}(\mathbf{G},\mathbf{D})$$

satisfies the "dvr extension property": For every dvr R *of mixed characteristic* $(0,p)$ *we have:*

$$\mathscr{S}^{\rm reg}_{K_p}(\mathbf{G},\mathbf{D})(R) = \mathrm{Sh}_{K^p}(\mathbf{G},\mathbf{D})(R[1/p]).$$

Note that (a) and (b) together amount to the existence of a smooth morphism

$$\bar{q}_K : \mathscr{S}^{\rm reg}_K(\mathbf{G},\mathbf{D}) \to [\mathcal{G}\backslash \mathrm{M}^{\rm bl}(\Lambda)]$$

where the target is the quotient algebraic stack.

Proof. By [10, Theorem 4.2.7], there are schemes $\mathscr{S}_K(\mathbf{G},\mathbf{D})$ which satisfy similar properties, excluding (c), but with $\mathrm{M}^{\rm bl}(\Lambda)$ replaced by the PZ local model $\mathrm{M}^{\rm loc}(\Lambda)$. In particular, we have

$$\begin{array}{ccc} & \widetilde{\mathscr{S}}_K(\mathbf{G},\mathbf{D}) & \\ {}^{\pi_K}\swarrow & & \searrow^{q_K} \\ \mathscr{S}_K(\mathbf{G},\mathbf{D}) & & \mathrm{M}^{\rm loc}(\Lambda) \end{array} \tag{11.0.0.2}$$

with π_K a $\mathcal{G}$-torsor and q_K smooth and $\mathcal{G}$-equivariant. We set

$$\widetilde{\mathscr{S}}^{\rm reg}_K(\mathbf{G},\mathbf{D}) = \widetilde{\mathscr{S}}_K(\mathbf{G},\mathbf{D}) \times_{\mathrm{M}^{\rm loc}(\Lambda)} \mathrm{M}^{\rm bl}(\Lambda)$$

which carries a diagonal $\mathcal{G}$-action. Since $r : \mathrm{M}^{\mathrm{bl}}(\Lambda) \longrightarrow \mathrm{M}^{\mathrm{loc}}(\Lambda)$ is given by a blow-up, is projective, and we can see ([14, §2]) that the quotient

$$\pi_K^{\mathrm{reg}} : \widetilde{\mathscr{S}}_K^{\mathrm{reg}}(\mathbf{G}, \mathbf{D}) \longrightarrow \mathscr{S}_K^{\mathrm{reg}}(\mathbf{G}, \mathbf{D}) := \mathcal{G} \backslash \widetilde{\mathscr{S}}_K^{\mathrm{reg}}(\mathbf{G}, \mathbf{D})$$

is represented by a scheme and gives a $\mathcal{G}$-torsor. (This is an example of a “linear modification”, see [14, §2].) In fact, since blowing-up commutes with étale localization, $\mathscr{S}_K^{\mathrm{reg}}(\mathbf{G}, \mathbf{D})$ is the blow-up of $\mathscr{S}_K(\mathbf{G}, \mathbf{D})$ at the subscheme of closed points that correspond to $* \in \mathrm{M}^{\mathrm{loc}}(\Lambda)$ under the local model diagram (11.0.0.1). This set of points is the discrete Kottwitz-Rapoport stratum of the special fiber of $\mathscr{S}_K(\mathbf{G}, \mathbf{D})$. The projection gives a smooth $\mathcal{G}$-morphism

$$q_K^{\mathrm{reg}} : \widetilde{\mathscr{S}}_K^{\mathrm{reg}}(\mathbf{G}, \mathbf{D}) \longrightarrow \mathrm{M}^{\mathrm{bl}}(\Lambda)$$

which completes the local model diagram. Property (c) follows from Theorem 10.0.1 and properties (a) and (b) which imply that $\mathscr{S}_K^{\mathrm{reg}}(\mathbf{G}, \mathbf{D})$ and $\mathrm{M}^{\mathrm{bl}}(\Lambda)$ are locally isomorphic for the étale topology. The rest of the properties in the statement follow from the corresponding properties for $\mathscr{S}_K(\mathbf{G}, \mathbf{D})$ and the construction. For a more detailed presentation of this proof see the proof of [18, Theorem 7.2.1]. □

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