ON THE MTW CONDITIONS OF MONGE-AMPÈRE TYPE EQUATIONS

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A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

Mathematics – Doctor of Philosophy

2021

ABSTRACT

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The MTW condition was introduced in [9] to study the regularity theory of the optimal transportation problem, and the MTW condition was used by many researchers to study other regularity properties of the optimal transportation problem. For example, the MTW condition was used by G. Loeper, A. Figalli, Y-H. Kim, R. McCann and other researchers to show Hölder regularity of the potential function, and by A. Figalli and De Phillippis to show Sobolev regularity of the potential function.

I present two of my results about the MTW condition in this dissertation. The first result concerns about the synthetic expressions of the MTW condition. The cost function of the optimal transportation problem need a high regularity assumption (C^4) to define the MTW condition. There are some expressions of MTW condition, however, which only need much weaker regularity assumption to define, but equivalent to the MTW condition when the cost function has enough regularity. We call these conditions synthetic MTW conditions. Although the synthetic MTW conditions are equivalent to the MTW condition under some assumption, it was not shown that if the synthetic MTW conditions are equivalent under weak regularity assumption which is not enough to define the MTW condition. I present a proof of the equivalence of the synthetic MTW conditions under $C^{2,1}$ assumption on the cost function in chapter 3.

The other result is about the Hölder regularity of solutions to generated Jacobian equations. In generated Jacobian equations, we study more general structure than the optimal transportation problem. Some examples of generated Jacobian equations which is more complicated than the optimal transportation problem can be found in geometric optics problems. The Hölder regularity result was proved by G. Loeper in [8] in the optimal transporta-

tion problem case and this can be generalized to generated Jacobian equations. Since the structure of generated Jacobian equations has more non-linearlity than the structure of the optimal transportation problem, however, there are some difficulties to apply Loeper's idea to generated Jacobian equations. We discuss about the difficulties and suggest a way to go around the problems in chapter 4. Then I generalize Loeper's idea to more general generated Jacobian equations and show that we can have a similar local Hölder regularity result.

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Chapter 1

Introduction

The Monge-Ampère equation is a Partial Differential Equation (PDE) of the form

$$\det\left(D^2\phi(x)\right) = f(x). \tag{MA}$$

It is known that the Monge-Ampère equation is elliptic over the family of convex functions, and it is fully non-linear. Moreover, the Monge-Ampère equation is degenerate elliptic, so that the methods which are developed for uniformly elliptic equations, for example the Evans-Krylov theorem, do not work for the Monge-Ampère equation.

It is well-known that the Monge-Ampère equation is closely related to optimal transportation problems and geometric optics problems. In fact, the potential functions of the solutions to optimal transportation problems satisfy PDEs of the form

$$\det\left(D^2\phi(x) - \mathcal{A}(x, D\phi(x))\right) = \psi(x, D\phi(x)) \tag{c-MA}$$

for some ψ, where $\mathcal{A}(x,p) = -D^2_{xx}c(x, \exp^c_x(p))$ is a matrix valued function defined using the cost function from the optimal transportation problem. Also, the solutions from the geometric optics problems have potentials that satisfy PDEs of the form

$$\det\left(D^2\phi(x) - \mathcal{A}(x, D\phi(x), \phi(x))\right) = \psi(x, D\phi(x), \phi(x)) \tag{GJE}$$

for some ψ where $\mathcal{A}(x,p,u) = D^2_{xx}G\left(x, exp^G_{x,u}(p), Z_x(p,u)\right)$ is a matrix valued function defined using the generating function from the geometric optics problems.

The difficulties that arise from fully non-linearity and degenerate ellipticity along with application to optimal transportation problems and geometric optics problems made research about Monge-Ampère type equations very attractive and active.

To study regularity theory of Monge-Ampère type equations, it is needed to define notions of convexity for each Monge-Ampère type equation, which are called c-convexity and G-convexity. Like the Monge-Ampère equation case, (c-MA) and (GJE) are elliptic over c-convex functions and G-convex functions respectively. Moreover, some additional structural conditions are needed. The MTW condition is one of these structural conditions and the MTW condition is a very important condition for studies about Hölder regularity theory of Monge-Ampère type equations. The MTW condition is a condition about sign of a (2,2)-tensor, which is called the MTW tensor, that contains 4th order derivative of c or G. The MTW condition is first discovered in [9] and used to prove the regularity result in the paper. Meaning of the MTW condition was not clear when it was first discovered, but what the MTW condition means geometrically was found later, for instance in [6] and [8]. What is more, it is proved that the MTW condition is a *necessary* and *sufficient* condition for Hölder regularity of solutions to (c-MA) in [8].

In this thesis, I present my works regarding the MTW condition. In the next chapter, connections of optimal transportation problems and (c-MA) will be introduced with structural conditions on (c-MA). In chapter 3, we discuss synthetic expressions of the MTW condition, and prove that these synthetic expressions are equivalent even when the cost fuction c does not have enough regularity to define the MTW tensor. In the last chapter, the result of Loeper in [8] about the Hölder regularity of solutions to (c-MA) with certain density conditions on the source measure μ will be generalized to a similar result about the Hölder regularity of solutions to (GJE).

Chapter 2

Optimal transportation problem

2.1 Optimal transportation problem

In this section, we introduce the optimal transportation problem and c-convexity. Let X and Y be two compact sets with non-empty interior in $\mathbb{R}^n$, and let $\mu \in \mathcal{P}(X)$ and $\nu \in \mathcal{P}(Y)$ where $\mathcal{P}(X)$ is the set of Borel probability measures on X. We first define the push-forward of a measure.

Definition 2.1.1. Let $T : X \to Y$ be a measurable function. We define the push-forward measure $T_\sharp\mu$ by

$$T_\sharp\mu[A] = \mu\left[T^{-1}(A)\right]$$

for any measurable set $A \subset Y$.

Let $c : X \times Y \to \mathbb{R}$ be a continuous function, which we will call the cost function. The optimal transportation problem which was introduced by G. Monge in 1781 asks to find a function T which minimizes the total transportation cost caused by distributing mass μ to ν.

Problem 1 (Monge problem)**.** *Find a measurable function $T : X \to Y$ which minimizes the following quantity:*

$$\int_X c(x, S(x))d\mu, \tag{2.1}$$

among the family of functions $S(\mu,\nu) = \{S : X \to Y | S_\sharp\mu = \nu\}$.

The Monge problem can be easily applied to real situations such as delivering some products from factories to customers. However, there was not a lot of progress until the 1940s due to high non-linearity of the problem. It was L. Kantorovich who made a break through in the optimal transportation problem in 1942. He introduced a relaxed version of Monge problem, which we call the *Kantorovich problem.*

Problem 2 (Kantorovich problem)**.** *Find a measure π which minimizes the following quantity:*

$$\int_{X\times Y} c(x,y)d\gamma \tag{2.2}$$

among the family of measures

$$\Gamma(\mu,\nu) = \left\{\gamma \in \mathcal{P}(X\times Y) | \mathrm{Proj}_{X\sharp}[\gamma] = \mu \text{ and } \mathrm{Proj}_{Y\sharp}[\gamma] = \nu\right\},$$

where $\mathrm{Proj}_X : X \times Y \to X$ *and* $\mathrm{Proj}_Y : X \times Y \to Y$ *are the projections onto* X *and* Y *respectively.*

Kantorovich used measures γ on $X \times Y$ instead of functions $S : X \to Y$, and let the total cost (2.2) depend on γ linearly. What is more, the Kantorovich problem always admits a solution while the Monge problem does not admit a solution in some cases (for example, $\mu = \delta_0$ and $\nu = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$). We call solutions to the Kantorovich problem and Monge problem, Kantorovich solutions and Monge solutions respectively. Since Kantorovich solutions always exist, we can try to find information about Monge solutions from Kantorovich solutions. In fact, if a Kantorovich solution is of the form $\pi = (Id \times T)_\sharp\mu$, then T will be a Monge solution. Therefore, to deduce the existence of a Monge solution from a Kantorovich solution, we should observe the support of the Kantorovich solution. To achieve this, we introduce another result of Kantorovich called Kantorovich duality.

Problem 3 (Dual problem)**.** *Find a pair of functions (ϕ,ψ) such that $\phi \in L^1(d\mu)$, $\psi \in L^1(d\nu)$, and maximizes the following*

$$-\int_X \phi' d\mu - \int_Y \psi' d\nu, \tag{2.3}$$

among the family of pairs of functions

$$\Phi_c(\mu,\nu) = \left\{ (\phi',\psi') \in L^1(d\mu)\times L^1(d\nu) \;\middle|\; \begin{array}{c} \phi'(x)+\psi'(y) \geq -c(x,y), \\ d\mu\otimes d\nu \;\; a.e. \;\; (x,y)\in X\times Y \end{array} \right\}.$$

Theorem 2.1.2 (Kantorovich duality)**.** *The miminum total Kantorovich cost* (2.2) *equals the maximum of* (2.3)*.*

$$\inf_{\gamma\in\Gamma(\mu,\nu)} \int_{X\times Y} c(x,y)d\gamma = \sup_{(\phi',\psi')\in\Phi_c(\mu,\nu)} \left(-\int_X \phi' d\mu - \int_Y \psi' d\nu \right). \tag{2.4}$$

By Kantorovich duality, we can expect to obtain some information about the Kantorovich solution by solving the dual problem. In the dual problem, we consider pairs of L^1 functions ϕ and ψ. However, we can reduce the family of pairs of functions $\Phi_c(\mu,\nu)$ to a smaller set. Note that

$$\begin{aligned} &\phi(x)+\psi(y) \geq -c(x,y) \\ \Rightarrow &\phi(x) \geq -c(x,y)-\psi(y) \\ \Rightarrow &\phi(x) \geq \sup_{y\in Y}\{-c(x,y)-\psi(y)\} = \psi^c(x). \end{aligned}$$

Therefore, we have

$$-\int_X \phi(x)d\mu - \int_Y \psi(y)d\nu \leq -\int_X \psi^c(x)d\mu - \int_Y \psi(y)d\nu.$$

Hence, we only need to consider pairs of the form (ψ^c,ψ). Similarly, we define $\phi^c(y) = \sup_{x\in X}\{-c(x,y)-\phi(x)\}$ for $\phi : X \to \mathbb{R}$, then a similar argument shows that we only need to consider the pairs (ϕ^{cc},ϕ^c).

Definition 2.1.3. A function $\phi : X \to \mathbb{R}$ is called *c-convex* if

$$\phi(x) = \sup_{y \in Y}\{-c(x,y) - \psi(y)\} \tag{2.5}$$

for some $\psi : Y \to \mathbb{R} \cup \{\infty\}$ such that $\psi \not\equiv \infty$. A function $\psi : Y \to \mathbb{R}$ is called c^**-convex* if

$$\psi(y) = \sup_{x \in X}\{-c(x,y) - \phi(x)\} \tag{2.6}$$

for some $\phi : X \to \mathbb{R} \cup \{\infty\}$ such that $\phi \not\equiv \infty$.

Hence, we only need to consider pairs of c and c^*-convex functions for the dual problem. In fact, existence of a solution to the dual problem of the form (ϕ, ϕ^c) can be proved by considering maximizing sequence of pairs of c and c^*-convex functions. See, for instance, [12] Chapter 1.

As the name and the definition of $c(c^*)$-convex function suggest, there are many properties analogous to properties of convex functions. For example, the analogy of the subdifferential is the c-subdifferential:

Definition 2.1.4. Let $\phi : X \to \mathbb{R}$ be a c-convex function, and let $x_0 \in X$. Then there exists some $y_0 \in Y$ such that

$$\phi(x) \geq -c(x, y_0) + c(x_0, y_0) + \phi(x_0). \tag{2.7}$$

We say y_0 belongs to the *c-subdifferential* of ϕ at x_0, and we denote

$$\partial_c\phi(x_0) = \{y_0 \in Y | \phi(x) \geq -c(x, y_0) + c(x_0, y_0) + \phi(x_0), \forall x \in X\}.$$

If $A \subset X$, we denote $\partial_c\phi(A) = \bigcup_{x \in A} \partial_c\phi(x)$.

Proposition 2.1.5. *Let $\phi : X \to \mathbb{R}$ be a c-convex function. Let $x \in X$, then*

$$y \in \partial_c\phi(x) \Leftrightarrow \phi(x) + \phi^c(y) = -c(x,y). \tag{2.8}$$

Equation (2.8) and the Kantorovich duality provides very important information about the support of the Kantorovich solution. Let π be a Kantorovich solution and (ϕ, ϕ^c) be a pair of c and c^*-convex functions that solves the dual problem. Then the marginal condition on the Kantorovich solution π implies that we have

$$\int_{X\times Y} \phi(x) + \phi^c(y) + c(x,y)d\pi = 0.$$

This shows that the Kantorovich solution π is concentrated in the set $\{\phi(x) + \phi^c(y) = -c(x,y)\}$. Noting (2.8), we obtain

$$\text{spt}(\pi) \subset \{(x,y)|y \in \partial_c\phi(x)\}.$$

Therefore, as we discussed earlier, if $\partial_c\phi$ is single valued, then the Monge solution is given by $T(x) = \partial_c\phi(x)$. Moreover, if c is C^1, then (2.7) shows

$$D\phi(x_0) = -D_x c(x_0, y_0).$$

Hence if $y \mapsto -D_x c(x,y)$ is injective, then we obtain

$$\partial_c\phi(x_0) = y_0 = [-D_x c(x_0, \cdot)]^{-1}(D\phi(x_0)) \tag{2.9}$$

which implies single valuedness of the c-subdifferential $\partial_c\phi$ at each differentiable points of ϕ. If we assume more differentiability on the cost function c, then (2.7) implies semi-convexity of ϕ so that ϕ is differentiable almost everywhere. Then the Monge solution can be defined dx a.e. in X.

2.2 Monge-Ampère type equations

In this section, we explain the connection between the optimal transportation problem and the c-Monge-Ampère equation, and we present the structural conditions for the c-Monge-Ampère equation.

To see the relation between optimal transportation problem and the c-Monge-Ampère equation, let us derive (c-MA) formally from the optimal transportation problem. Let $d\mu = f(x)dx$ and $d\nu = g(y)dy$, and let $T : X \to Y$ be a Monge solution. From the push-forward condition, we obtain

$$\int_X u(T(x))f(x)dx = \int_Y u(y)g(y)dy,$$

for any continuous function $u \in C(Y)$. On the other hand, we use the change of variable formula with $y = T(x)$ to obtain

$$\int_Y u(y)g(y)dy = \int_X u(T(x))g(T(x)) \det(DT(x))dx.$$

Therefore, we obtain

$$\det(DT(x)) = \frac{f(x)}{g(T(x))}. \tag{2.10}$$

Noting that T is given by a c-subdifferential of a c-convex function ϕ and equation (2.9), we obtain the expression

$$DT(x) = \left[-D^2_{xy}c(x, T(x))\right]^{-1} \left(D^2\phi(x) + D^2_{xx}c\,(x, T(x))\right).$$

Denoting $\mathcal{A}(x, p) = -D^2_{xx}c\,(x, [D_x c(x, \cdot)]^{-1}(p))$, we obtain

$$\det\left(D^2\phi(x) - \mathcal{A}(x, D\phi(x))\right) = \frac{f(x)}{g(T(x))} \det\left(-D^2_{xy}c(x, T(x))\right). \tag{2.11}$$

From the above formal derivation of the c-Monge-Ampère equation (2.11), we observe that the cost function c should be at least C^2, and we can deduce that we need the following conditions on the cost function.

$$y \mapsto -D_x c(x, y) \text{ is injective } \forall x \in X, \tag{Twist}$$

$$\det\left(-D^2_{xy} c(x, y)\right) \neq 0. \tag{Non-deg}$$

Note that we can define a condition which is symmetric to the condition (Twist).

$$x \mapsto -D_y c(x, y) \text{ is injective } \forall y \in Y. \tag{Twist*}$$

(Twist) and (Twist*) condition imply inverse functions of $-D_x c(x, \cdot)$ and $-D_y c(\cdot, y)$. We call these inverse functions c-exponential maps.

Definition 2.2.1. Define $Y^*_x \subset \mathbb{R}^n$ by

$$Y^*_x = -D_x c(x, Y).$$

Then $-D_x c(x, \cdot) : Y \to Y^*_x$ is bijective by (Twist). We define the *c-exponential map* $\exp^c_x : Y^*_x \to Y$ by the inverse function of $-D_x c(x, \cdot)$:

$$-D_x c(x, \exp^c_x(p)) = p. \tag{2.12}$$

We call $\exp^c_x$ the c-exponential map focused at x. Similarly, define $X^*_y \subset \mathbb{R}^n$ by

$$X^*_y = -D_y c(X, y).$$

Then $-D_y c(\cdot, y) : X \to X^*_y$ is bijective. We define the *c^*-exponential map* $\exp^{c^*}_y : X^*_y \to X$

by the inverse function of $-D_y c(\cdot, y)$:

$$-D_y c(\exp_y^{c^*}(q), y) = q. \tag{2.13}$$

We call $\exp_y^{c^*}$ the c^*-exponential map focused at y.

Definition 2.2.2. Let $x \in X$ and $y_0, y_1 \in Y$, and let $p_i = -D_x c(x, y_i)$. The *c-segment* $\{y_\theta | \theta \in [0,1]\}$ focused at x that connects y_0 and y_1 is the image of the segment $[p_0, p_1]$ under the c-exponential map $\exp_x^c$:

$$\{y_\theta | \theta \in [0,1]\} = \exp_x^c([p_0, p_1]).$$

We say that y_θ is a c-segment if there is no confusion.

Remark 2.2.3. (Non-deg) implies that the c-exponential maps are differentiable and

$$D_p \exp_x^c(p) = \left[-D_{xy}^2 c(x, \exp_x^c(p))\right]^{-1}.$$

Moreover, compactness of X and Y with (Non-deg) implies that we have a constant λ such that

$$\begin{gathered}
\frac{1}{\lambda} \le |D_{xy}^2 c| \le \lambda, \\
\frac{1}{\lambda}|y_1 - y_0| \le |-D_x c(x, y_1) + D_x c(x, y_0)| \le \lambda |y_1 - y_0|, \\
\frac{1}{\lambda}|x_1 - x_0| \le |-D_y c(x_1, y) + D_y c(x_0, y)| \le \lambda |x_1 - x_0|.
\end{gathered}$$

The conditions (Twist), (Twist*), and (Non-deg) are enough to obtain a Monge solution that is defined almost everywhere, and observe the relation with the Monge-Ampère equation. To study regularity theory, however, we need one more condition called the MTW condition. This condition first appeared in [9] in (A3s) form and is named after the authors of the paper.

The (A3w) form is appeared in [11]. To define the MTW condition, we need to assume more regularity on the cost function c.

$$c \in C^4(X \times Y). \tag{Regular}$$

Next, we define the MTW tensor, which is a (2,2) tensor that contains 4th derivatives of the cost function. Let $\mathcal{A}(x,p) = -D^2_{xx}c(x, \exp^c_x(p))$. Then the MTW tensor is

$$MTW = D^2_{pp}\mathcal{A}(x,p). \tag{2.14}$$

The MTW condition is a sign condition on the MTW tensor in some directions.

$$MTW[\eta,\eta,\xi,\xi] \geq 0,\ \forall \eta \perp \xi. \tag{A3w}$$

If the cost function c satisfies (A3w) with strict inequality, we say that c satisfies (A3s). In this case, from the compactness of X and Y and the tensorial nature, we obtain a constant $\alpha > 0$ such that

$$MTW[\eta,\eta,\xi,\xi] > \alpha|\eta|^2|\xi|^2,\ \forall \eta \perp \xi. \tag{A3s}$$

To study the c-Monge-Ampère equation, we should define a weak solution for the equation like other PDEs. For the c-Monge-Ampère equation, we can define several different weak solutions. The first weak solution is defined using the mass balance condition (push-forward condition) of the optimal transportation problem.

Definition 2.2.4. A function $\phi : X \to \mathbb{R}$ is called a *Brenier solution* of (2.11) if ϕ satisfies

$$\partial_c \phi_\sharp \mu = \nu. \tag{2.15}$$

Another weak solution can be defined using equation (2.10). If we integrate equation

(2.10) on a measurable set $A \subset X$, then we obtain

$$\int_A \det(DT(x))dx = \int_A \frac{f(x)}{g(T(x))}dx.$$

Using the change of variable formula with $y = T(x)$ on the left hand side and noting that $T = \partial_c \phi$, we obtain

$$|\partial_c \phi(A)| = \int_A \frac{f(x)}{g(T(x))}dx.$$

We use this equation to define another weak solution.

Definition 2.2.5. A c-convex function $\phi : X \to \mathbb{R}$ is called an *Alexandrov solution* of (2.11) if ϕ satisfies

$$|\partial_c \phi(A)| = \int_A \frac{f(x)}{g(T(x))}dx.$$

Note that, in contrast to the Brenier solution, an Alexandrov solution can be defined with more general formulas on the right hand side of (2.10) or (2.11). In addition, a Brenier solution does not have to be an Alexandrov solution. To observe this, suppose f and g are bounded away from 0 and ∞ on each support, and let ϕ be a Brenier solution. Then we have $T_\sharp f(x)dx = g(y)dy$, and

$$\int_A f(x)dx = \int_{T(A)} g(y)dy,$$

so that we have $|A| \sim |T(A) \cap Y|$. If ϕ was an Alexandrov solution, however, Definition 2.2.5 shows that we should have $|A| \sim |T(A)|$. An explicit counter example is explained in [12] and [1]. A Brenier solution becomes an Alexandrov solution when $\partial_c \phi(X) \subset Y$. This, in fact, can be deduced if we add some geometric conditions on X and Y.

Definition 2.2.6. Let $x \in X$, $y \in Y$ and $A \subset X$, $B \subset Y$. B is called *c-convex with respect to x* if the set

$$B_x^* = -D_x c(x, B)$$

is convex. We say that B is *c-convex with respect to A* if B is c-convex with respect to x for any $x \in A$. Similarly, A is called *c^*-convex with respect to y* if the set

$$A_y^* = -D_y c(A, y)$$

is convex, and we say that A is *c^*-convex with respect to B* if A is C^*-convex with respect to y for any $y \in B$.

We add the following conditions on X and Y.

$$Y \text{ is } c\text{-convex with respect to } X. \tag{DomConv}$$

$$X \text{ is } c\text{-convex with respect to } Y. \tag{DomConv*}$$

It is proved, for example in [8], that if a cost function c satisfies (Twist), (Twist*), (Non-deg), (A3w) and (DomConv), then a c-subdifferential $\partial_c \phi(x)$ at a point $x \in X$ of a c-convex function ϕ is c-convex with respect to x. Then we obtain that $\partial_c \phi \subset Y$, and the Brenier solution becomes an Alexandrov solution.

Chapter 3

The synthetic MTW conditions

3.1 The synthetic MTW conditions

In [8], Loeper suggested a condition that is equivalent to the MTW condition when the cost function is C^4. The condition is the following

Definition 3.1.1 (Loeper's condition)**.** Let $x_0, x_1 \in X$ and define a function $F(p) = -c(x_1, \exp^c_{x_0}(p)) + c(x_0, \exp^c_{x_0}(p))$. Then the cost function c is said to satisfy *Loeper's condition* if

$$F(tp_1 + (1-t)p_0) \leq \max\{F(p_0), F(p_1)\}$$

for any $p_0, p_1 \in Y^*_{x_0}$ and for any $x_0, x_1 \in X$.

Technically, we only need C^1 cost function with twisted condition to form Loeper's condition. Therefore, Loeper's condition can be viewed as a synthetic expression of the MTW condition. Moreover, Loeper's condition implies that the c-subdifferential of a c-convex function at a point x_0 is c-convex with respect to x_0. As we can see from Definition 3.1.1, Loeper's condition means quasi-convexity of the function F. We introduce notations for level sets and sublevel sets of the function F:

$$L_{p_0} = \{p \in Y^*_{x_0} | F(p) = F(p_0)\}, \ SL_{p_0} = \{p \in Y^*_{x_0} | F(p) \leq F(p_0)\}. \tag{3.1}$$

Then SL_{p_0} is a convex set, and L_{p_0} is a C^1 manifold. It is proved in [8] that Loeper's condition is equivalent to the MTW condition when the cost function is C^4.

In [2], Kitagawa and Guillen suggested another condition that is equivalent to the MTW condition when the cost function is C^4.

Definition 3.1.2 (Quantitative quasi-convexity (QQconv))**.** Let $x_0, x_1 \in X$ and define $F(p) = -c(x_1, \exp^c_{x_0}(p)) + c(x_0, \exp^c_{x_0}(p))$. Then the cost function c is said to satisfy *QQconv* if there exists a constant $M \geq 1$ such that

$$F(tp_1 + (1-t)p_0) - F(p_0) \leq Mt(F(p_1) - F(p_0))_+ \tag{3.2}$$

for any $p_0, p_1 \in Y^*_{x_0}$ and for any $x_0, x_1 \in X$.

Like Loeper's condition, QQconv makes sense when the cost function is only C^1 with twisted condition. Therefore, QQconv is another synthetic expression of the MTW condition. In fact, QQconv implies Loeper's condition.

Lemma 3.1.3. *Suppose the cost function c satisfies QQconv, Then c also satisfies Loeper's condition.*

Proof. If the cost function c satisfies QQconv, then we have (3.2). If $F(p_1) \leq F(p_0)$, then we have

$$F(tp_1 + (1-t)p_0) \leq F(p_0) = \max\{F(p_1), F(p_0)\}. \tag{3.3}$$

If $F(p_1) \geq F(p_0)$, then we switch the role of p_1 and p_0 in (3.3), and we obtain the same inequality. $\square$

Although both Loeper's condition and QQconv are equivalent to MTW condition when the cost function is C^4, it is not clear if the two synthetic MTW conditions are equivalent under weaker regularity assumptions on the cost function c. The main theorem of this chapter shows that the two synthetic MTW conditions are equivalent under weaker assumption.

$$\left|D^2_{xy}c(x_0, y_0) - D^2_{xy}c(x_1, y_1)\right| \leq \Lambda|(x_1, y_1) - (x_0, y_0)| \tag{Lip hessian}$$

(Lip hessian) condition with non-degeneracy implies Lipschitzness of the inverse matrix of the mixed hessian

$$\left|\left[D^2_{xy}c(x_0, y_0)\right]^{-1} - \left[D^2_{xy}c(x_1, y_1)\right]^{-1}\right| \leq \Lambda'|(x_0, y_0) - (x_1, y_1)| \tag{3.4}$$

Enlarging Λ or Λ' if necessary, we can assume $\Lambda = \Lambda'$.

Now we state the main theorem of this chapter.

Theorem 3.1.4 (Main theorem of Chapter 3)**.** *Let $c : X \times Y \to \mathbb{R}$ be a C^2 cost function that satisfies* (Twist), (Twist*), (Non-deg), *and* (Lip hessian). *Suppose c satisfies Loeper's condition, then c also satisfies QQconv.*

We give the proof of the main theorem in the next section.

3.2 Equivalence of the Synthetic MTW conditions

We start with showing that (Lip hessian) condition implies Lipschitzness of the gradient of the function $F(p) = -c(x_1, \exp^c_{x_0}(p)) + c(x_0, \exp^c_{x_0}(p))$.

Lemma 3.2.1. *For any $x_0, x_1 \in X$ and $p_0, p_1 \in Y^*_{x_0}$, we have*

$$|\nabla F(p_0) - \nabla F(p_1)| \leq C|x_0 - x_1||p_0 - p_1| \tag{3.5}$$

for some constant C that depends on λ and Λ

Proof. Note that F is C^1 with

$$\nabla F(p) = [-D^2_{yx}c(x_0, y)]^{-1}(-D_y c(x_1, y) + D_y c(x_0, y)), \tag{3.6}$$

where $y = \exp^c_{x_0}(p)$. Therefore,

$$\nabla F(p_1) - \nabla F(p_0)$$

$$
\begin{aligned}
&= [-D^2_{xy}c(x_0,y_1)^T]^{-1}\left(-D_yc(x_1,y_1)+D_yc(x_0,y_1)\right) \\
&- [-D^2_{xy}c(x_0,y_0)^T]^{-1}\left(-D_yc(x_1,y_0)+D_yc(x_0,y_0)\right)
\end{aligned} \tag{3.7}
$$

where $y_i = \exp^c_{x_0}(p_i)$. Let $L1$ and $L2$ be the second and third line in (3.7) and let

$$
\begin{aligned}
L1' &= L1 - [-D^2_{xy}c(x_0,y_0)^T]^{-1}\left(-D_yc(x_1,y_1)+D_yc(x_0,y_1)\right), \\
L2' &= L2 + [-D^2_{xy}c(x_0,y_0)^T]^{-1}\left(-D_yc(x_1,y_1)+D_yc(x_0,y_1)\right),
\end{aligned}
$$

so that $\nabla F(p_1) - \nabla F(p_0) = L1' + L2'$. (Lip hessian) implies

$$
\begin{aligned}
|L1'| &= \left|[-D^2_{xy}c(x_0,y_1)^T]^{-1} + [D^2_{xy}c(x_0,y_0)^T]^{-1}\right| |D_yc(x_1,y_1) - D_yc(x_0,y_1)| \\
&\le \Lambda|y_1-y_0| \times \lambda|x_1-x_0| \\
&\le \lambda^2\Lambda|x_0-x_1||p_0-p_1|.
\end{aligned} \tag{3.8}
$$

To get an estimate for $L2'$, we use the fundamental theorem of calculus.

$$
\begin{aligned}
&|L2'| \\
&= \left|[-D^2_{xy}c(x_0,y_0)^T]^{-1}\right| \\
&\times |-D_yc(x_1,y_1) + D_yc(x_0,y_1) + D_yc(x_1,y_0) - D_yc(x_0,y_0)| \\
&= [-D^2_{xy}c(x_0,y_0)^T]^{-1} \\
&\times \left|\int_0^1 [-D^2_{xy}c(x_s,y_0)^T]^{-1}[-D^2_{xy}c(x_s,y_1)^T](q_1-q_0)ds - (q_1-q_0)\right| \\
&= \left|[-D^2_{xy}c(x_0,y_0)^T]^{-1}\right| \\
&\times \left|\int_0^1 [-D^2_{xy}c(x_s,y_0)^T]^{-1}\left([-D^2_{xy}c(x_s,y_1)^T] - [-D^2_{xy}c(x_s,y_0)^T]\right) ds(q_1-q_0)\right|
\end{aligned}
$$

where $q_i = -D_yc(x_i,y_0)$ and x_s is the c^*-segment focused at y_0. Then (Non-deg) with

(Lip hessian) implies

$$|L2'| \leq \lambda^3\Lambda|x_0 - x_1||p_0 - p_1|. \tag{3.9}$$

Combining the two estimate (3.8) and (3.9) with $\nabla F(p_1) - \nabla F(p_0) = L1' + L2'$, we obtain the Lipschitzness of ∇F

$$|\nabla F(p_1) - \nabla F(p_0)| \leq C|x_1 - x_0||p_1 - p_0| \tag{3.10}$$

where $C = \lambda^2\Lambda + \lambda^3\Lambda$. □

Note that (3.6) with (Non-deg) condition implies the following

$$|\nabla F(p)| \sim |x_1 - x_0|. \tag{3.11}$$

In particular, we have a constant C_1 such that

$$|\nabla F(p)| \geq C_1|x_1 - x_0|. \tag{3.12}$$

By Lemma 3.1.3, we only need to consider the case c satisfies Loeper's condition, and show that c satisfies QQconv. However, we do not have to consider arbitrary points $p_0, p_1 \in Y^*_{x_0}$. We use the notation

$$p_t = (1-t)p_0 + tp_1.$$

Lemma 3.2.2. *Suppose the cost function c satisfies Loeper's condition. If c satisfies* (3.2) *for any pair of points $p_0, p_1 \in Y^*_{x_0}$ such that $p_1 \in B^+_r(p_0)$ where*

$$B^+_r(p_0) = \{p||p - p_0| < r, \langle p - p_0, \nabla F(p_0)\rangle \geq 0\}, \tag{3.13}$$

then c satisfies QQconv.

Proof. We divide the proof into three steps.

Step 1) *Claim*: We only need to consider the case $F(p_1) > F(p_0)$.

If $F(p_1) \leq F(p_0)$, then we obtain $(F(p_1) - F(p_0))_+ = 0$. However, by Loeper's condition, we have

$$\begin{aligned} F(p_t) - F(v_0) &\leq \max\{F(p_1), F(p_0)\} - F(p_0) \\ &= F(p_0) - F(p_0) = 0 = Mt(F(p_1) - F(p_0))_+. \end{aligned}$$

Hence (3.2) always holds when $F(p_1) \leq F(p_0)$, and we only need to check the case $F(p_1) > F(p_0)$.

Step 2) *Claim*: If there exist $r > 0$ such that (3.2) holds whenever $|p_1 - p_0| < r$, then c satisfies QQconv.

Suppose (3.2) holds whenever $|p_1 - p_0| < r$. We choose $M' > \frac{1}{r}\mathrm{diam}(Y^*_{x_0})$, and suppose we have p_0 and p_1 which does not satisfy (3.2) with M' instead of M. Note that by *step 1*, we can assume $F(p_1) > F(p_0)$. Then by quasi-convexity of F, we have $F(p_1) \geq F(p_t)$. Therefore

$$M't(F(p_1) - F(p_0)) < F(p_t) - F(p_0) \leq F(p_1) - F(p_0). \tag{3.14}$$

This implies $0 < t < \frac{1}{M'}$. We choose $t' \in (t, \frac{1}{M'}]$ such that

$$\frac{1}{t'}(F(p_{t'}) - F(p_0)) = M'(F(p_1) - F(p_0)). \tag{3.15}$$

Note that such t' exists by the intermediate value theorem. Let $q_1 = p_{t'}$ and $q_0 = p_0$. Then we have

$$|q_1 - q_0| = t'|p_1 - p_0| \leq \frac{1}{M'}\mathrm{diam}(Y^*_{x_0}) < r. \tag{3.16}$$

Therefore, by assumption, we obtain

$$F(q_s) - F(q_0) \leq Ms(F(q_1) - F(q_0))$$

$$= Ms(F(p_{t'}) - F(p_0)) = MM'st'(F(p_1) - F(p_0)).$$

where $s = \frac{t}{t'}$ so that $q_s = p_t$. Hence, we have

$$F(p_t) - F(p_0) \le MM't(F(p_1) - F(p_0)). \tag{3.17}$$

Since MM' does not depend on p_0, p_1, it shows that c satisfies QQconv.

Step 3) *Claim*: We only need to consider the case $p_1 \in B_r^+(p_0)$.

Suppose we have (3.2) for $p_1 \in (B_r^+(p_0)) \cap Y_{x_0}^*$. By *Step 2*, we only need to show (3.2) when $p_1 \in B_r(p_0) \setminus B_r^+(p_0) = \{p||p - p_0| < r, \langle p - p_0, \nabla F(p_0)\rangle < 0\}$. Assume $p_1 \in B_r(p_0) \setminus B_r^+(p_0)$. If $F(p_1) \le F(p_0)$, there is nothing to show by *Step 1*, therefore we assume $F(p_1) > F(p_0)$. Since the function $F : Y_{x_0}^* \to \mathbb{R}$ is C^1, $\langle p_1 - p_0, \nabla F(p_0)\rangle < 0$ implies that $F(p_t) < F(p_0)$ when t is small enough. In addition, convexity of the sublevel sets of the function F implies that there exists t' such that $F(p_{t'}) = F(p_0)$, with $\langle p_1 - p_{t'}, \nabla F(p_{t'})\rangle > 0$. Therefore $p_1 \in B_r^+(p_{t'})$, and we obtain

$$\begin{aligned} F(p_s) - F(p_0) &= F(p_s) - F(p_{t'}) \\ &\le M\frac{s - t'}{1 - t'}(F(p_1) - F(p_{t'})) \le Ms(F(p_1) - F(p_0)). \end{aligned}$$

for any $s \in [t', 1]$. If $s < t'$, then $F(p_{t'}) \le F(p_0)$ so that (3.2) holds. $\square$

Before we start the next proof, we introduce a notation.

$$\mathcal{C}_{k,p_0} = \left\{ p \middle| \langle p - p_0, \nabla F(p_0)\rangle \ge \frac{1}{k}|p - p_0||\nabla F(p_0)| \right\}. \tag{3.18}$$

Lemma 3.2.3. *For any $k \in \mathbb{N}$, there exists $r_k > 0$ such that if $p_0 \in Y_{x_0}^*$ and $p_1 \in \mathcal{C}_{k,p_0} \cap B_{r_k}(p_0) \cap Y_{x_0}^*$, then*

$$F(p_t) - F(p_0) \le 5t(F(p_1) - F(p_0)). \tag{3.19}$$

Proof. We choose $r_k = \frac{C_1}{2kC}$ where C_1 is from (3.12) and C is from Lemma 3.2.1. Then, for any $p \in B_{r_k}(p_0) \cap Y^*_{x_0}$,

$$|\nabla F(p) - \nabla F(p_0)| \leq C|x_1 - x_0||p - p_0| \leq \frac{C_1}{2k}|x_1 - x_0| \leq \frac{1}{2k}|\nabla F(p_0)|. \tag{3.20}$$

Let $p_1 \in (\mathcal{C}_{k,p_0} \cap B_{r_k}(p_0) \cap Y^*_{x_0}) \setminus \mathcal{C}_{\frac{k}{2},p_0}$, and let $v = \frac{p_1 - p_0}{|p_1 - p_0|}$. Then (3.20) gives

$$\begin{aligned} \langle \nabla F(p_1), v \rangle &= \langle \nabla F(p_1) - \nabla F(p_0), v \rangle + \langle \nabla F(p_0), v \rangle \\ &\geq -\frac{1}{2k}|\nabla F(p_0)| + \frac{1}{k}|\nabla F(p_0)| = \frac{1}{2k}|\nabla F(p_0)|. \end{aligned} \tag{3.21}$$

Note that $p_0 + sv \in (\mathcal{C}_{k,p_0} \cap B_{r_k}(p_0) \cap Y^*_{x_0}) \setminus \mathcal{C}_{\frac{k}{2},p_0}$ for $0 \leq s \leq |p_1 - p_0|$ by convexity of $Y^*_{x_0}$, so that we can apply (3.21) with $p_0 + sv$ instead of p_1. Therefore, we have

$$\begin{aligned} F(p_1) - F(p_0) &= \int_0^{|p_1 - p_0|} \langle \nabla F(p_0 + sv), v \rangle ds \\ &\geq \frac{1}{2k}|\nabla F(p_0)||p_1 - p_0|. \end{aligned} \tag{3.22}$$

Moreover, we also have

$$\begin{aligned} \langle \nabla F(p_0 + sv), v \rangle &= \langle \nabla F(p_0 + sv) - \nabla F(p_0), v \rangle + \langle \nabla F(p_0), v \rangle \\ &\leq \frac{1}{2k}|\nabla F(p_0)| + \frac{2}{k}|\nabla F(p_0)| = \frac{5}{2k}|\nabla F(p_0)| \end{aligned}$$

for $0 \leq s \leq |p_1 - p_0|$, where we have used that $p_1 \notin \mathcal{C}_{\frac{k}{2},p_0}$ and (3.20). Therefore,

$$\begin{aligned} F(p_0 + t(p_1 - p_0)) - F(p_0) &= \int_0^{t|p_1 - p_0|} \langle \nabla F(p_0 + sv), v \rangle ds \\ &\leq t|p_1 - p_0| \times \frac{5}{2k}|\nabla F(p_0)|. \end{aligned} \tag{3.23}$$

We combine (3.22) and (3.23).

$$F(p_t) - F(p_0) \leq \frac{5t}{2k}|p_1 - p_0||\nabla F(p_0)| \leq 5t(F(p_1) - F(p_0)).$$

Now note that r_k increases as k decreases. Moreover,

$$\mathcal{C}_{k,p_0} \cap B_{r_k}(p_0) = \bigcup_{i=0}^{\infty} \left[\left(\mathcal{C}_{\frac{k}{2^i} p_0} \cap B_{r_k}(p_0) \right) \setminus \mathcal{C}_{\frac{k}{2^{i+1}},p_0} \right].$$

Therefore, for any $p_1 \in \mathcal{C}_{k,p_0} \cap B_{r_k}(p_0)$, we can repeat the proof with k replaced by $\frac{k}{2^i}$ for some i. $\square$

Note that r_k varies as we choose k. We will choose k later in this chapter.

Remark 3.2.4. Let $\rho_0 = |\nabla F(p_0)|$. Then (3.20) implies that

$$\forall p \in B_{r_k}(p_0) \cap Y^*_{x_0},\ \nabla F(p) \in B_{\frac{\rho_0}{2k}}(\nabla F(p_0)). \tag{3.24}$$

Let $\nabla F(p) = \nabla F(p_0) + v$, and consider ξ such that $\xi + p_0 \in \mathcal{C}_{k',p_0}$ with $|\xi| = 1$ where $k' \in \mathbb{N}$. Then

$$\langle \xi, \nabla F(p) \rangle = \langle \xi, \nabla F(p_0) \rangle + \langle \xi, v \rangle \geq \frac{1}{k'}|\nabla F(p_0)| - \frac{1}{2k}|\nabla F(p_0)|. \tag{3.25}$$

Therefore, once we fix k and $k' < 2k$, we obtain $\langle \xi, \nabla F(p) \rangle \sim |\nabla F(p_0)|$ for any $p \in B_{r_k}(p_0) \cap Y^*_{x_0}$ and ξ such that $\xi + p_0 \in \mathcal{C}_{k',p_0}$.

Remark 3.2.5. Quasi-convexity of the function F implies that if $p \in B^+_r(p_0) \cap Y^*_{x_0}$, then $F(p) \geq F(p_0)$.

We introduce another notation for a cone:

$$\overline{\mathcal{C}}_{k,p_0}(p_1) = \left\{ p \middle| \langle p - p_1, \nabla F(p_0) \rangle \leq -\frac{1}{k}|p - p_1||\nabla F(p_0)| \right\}. \tag{3.26}$$

Lemma 3.2.6. *Let $4 \leq k' \leq k$, and suppose $B_{r_k}(p_0) \subset Y^*_{x_0}$. Then, for any $p_1 \in B^+_{r_k}(p_0)$, we have*

$$F(p_t) - F(p_0) \leq M_{k'} t(F(p_1) - F(p_0))$$

where $M_{k'}$ is some constant that depends on k'(which will be decided later).

Proof. Note that by Lemma 3.2.3, we only need to check when $p_1 \notin \mathcal{C}_{k,p_0}$. Fix $p_1 \in B^+_{r_k}(p_0) \setminus \mathcal{C}_{k,p_0}$. Consider the cone $\overline{\mathcal{C}}_{k',p_0}(p_1)$ defined in (3.26). We divide the proof into three steps.

Step 1) $\overline{\mathcal{C}}_{k',p_0}(p_1) \cap L_{p_0} \cap B_{r_k}(p_0) \neq \emptyset$.

Note that we have chosen $k' \geq 4$. Then

$$\begin{aligned}\langle (p_0 - \frac{r_k}{2\rho_0}\nabla F(p_0)) - p_1, \nabla F(p_0)\rangle &= -\frac{1}{2} r_k |\nabla F(p_0)| - \langle p_1 - p_0, \nabla F(p_0)\rangle \\ &\leq -\frac{1}{2} r_k |\nabla F(p_0)| \\ &\leq -\frac{1}{4}\left| p_0 - \frac{r_k}{2\rho_0}\nabla F(p_0) - p_1 \right| |\nabla F(p_0)| \end{aligned} \tag{3.27}$$

where $\rho_0 = |\nabla F(p_0)|$. Note that we have used $p_1 \in B^+_{r_k}(p_0)$ in the first inequality and $p_0 - \frac{r_k}{2\rho_0}\nabla F(p_0), p_1 \in B_{r_k}(p_0)$ in the second inequality. (3.27) implies that the point $p_0 - \frac{r_k}{2\rho_0}\nabla F(p_0)$ is in the cone $\overline{\mathcal{C}}_{k',p_0}(p_1)$. In addition, (3.24) shows that $\langle \nabla F(p), \nabla F(p_0)\rangle > 0$ for any $p \in B_{r_k}(p_0)$. Therefore,

$$\begin{aligned}F(p_0) - F\left(p_0 - \frac{r_k}{2\rho_0}\nabla F(p_0)\right) &= \int_{-\frac{r_k}{2\rho_0}}^{0} \langle \nabla F(p_0 + t\nabla F(p_0)), \nabla F(p_0)\rangle \, dt \\ &\geq 0,\end{aligned}$$

so that the point $p_0 - \frac{r_k}{2\rho_0}\nabla F(p_0)$ is in the sublevel set SL_{p_0}. Therefore, by the intermediate value theorem, there is a point q_1 in the segment $[p_1, p_0 - \frac{r_k}{2\rho_0}\nabla F(p_0)]$ such that $F(q_1) = F(p_0)$ i.e. $q_1 \in L_{p_0}$. By convexity of $\overline{\mathcal{C}}_{k',p_0}(p_1) \cap B_{r_k}(p_0)$, q_1 is also in $\overline{\mathcal{C}}_{k',p_0}(p_1) \cap B_{r_k}(p_0)$. This concludes *Step 1*.

*Step 2)*Utilizing convexity of SL_{p_0}.

Let $\xi = \frac{(p_1 - q_1)}{|p_1 - q_1|}$ and consider $p_t - s\xi$. If we set $s = t|q_1 - p_1|$, then we have

$$p_t - s\xi = tq_1 + (1-t)p_0 \in SL_{p_0}, \forall t \in [0,1].$$

Therefore, by intermediate value theorem, for each $t \in [0,1]$, we obtain $s_t \in [0, t|q_1 - p_1|]$ such that $p_t - s_t\xi \in L_{p_0}$. Now, up to an isometry, we can set $\xi = -e_n, p_0 = 0$, and $p_1 = ae_1 + be_n$ for some $a, b \in \mathbb{R}$, $a > 0$. Then we can view the set $\{p_t - s_t\xi | t \in [0,1]\}$ as a graph of a function g on $[0,1]$. Since SL_{p_0} is a convex set, g is a convex function. Note that $s_t = g(at) - bt$ so that s_t is also a convex function of t on $[0,1]$. Convexity of s_t with $s_0 = 0$ implies

$$|p_t - q_t| = s_t \leq ts_1 = t|p_1 - q_1| \tag{3.28}$$

where $q_t = p_t - s_t\xi \in L_{p_0}$.

Step 3) Estimate on the segment $[q_t, p_t]$.

Note that $\xi + p_0 \in \mathcal{C}_{k', p_0}$ and $p_t - s\xi \in B_{r_k}(p_0)$ for $s \in [0, s_t]$. By Remark 3.2.4 and the fundamental theorem of calculus, we obtain

$$\begin{aligned} F(p_t) - F(p_0) &= F(p_t) - F(q_t) \\ &= \int_0^{s_t} \langle \nabla F(q_t + s\xi), \xi \rangle ds \leq s_t \frac{2k+1}{2k} |\nabla F(p_0)| \end{aligned} \tag{3.29}$$

from (3.24), and

$$\begin{aligned} F(p_1) - F(p_0) &= F(p_1) - F(q_1) \\ &= \int_0^{s_1} \langle \nabla F(q_1 + s\xi), \xi \rangle ds \geq s_1 \left(\frac{1}{k'} - \frac{1}{2k} \right) |\nabla F(p_0)| \end{aligned} \tag{3.30}$$

from (3.25). We combine (3.29), (3.30) with (3.28) to obtain

$$F(p_t) - F(p_0) \leq 2s_t |\nabla F(p_0)|$$

$$\leq 2ts_1|\nabla F(p_0)| \leq 4k'F(p_1) - F(p_0).$$

Note that we have used $\dfrac{2k+1}{2k} \leq 2$ and $\dfrac{1}{k'} - \dfrac{1}{2k} \geq \dfrac{1}{2k'}$. Hence, we obtain the lemma with $M_{k'} = 4k'$. $\square$

Lemma 3.2.6 shows that we can obtain (3.2) with a uniform constant M when we only consider the points that stay away from the boundary. When the point p_0 is close to the boundary, however, the proof of Lemma 3.2.6 may not work. The problematic part in the proof of Lemma 3.2.6 is the *Step 1*, because the point $p_0 - \dfrac{r_k}{2\rho_0}|\nabla F(p_0)|$ may not be in $Y^*_{x_0}$. Therefore it is not clear that the point q_1 and the direction ξ exist. To go around this problem, we need to introduce another argument when p_0 is close to the boundary.

The idea in Lemma 3.2.6 is to find a direction ξ so that we can view the level set L_{p_0} as a convex function over the segment $[p_0, p_1]$ with ξ as a vertical direction. When we can not find such a direction ξ, we try to look at the opposite direction, and view the level set L_{p_1} as a function over the segment $[p_0, p_1]$. In this case, the function will be a concave function. We use this idea in the next lemma

Lemma 3.2.7. *Let $p_0 \in Y^*_{x_0}$ and let $p_1 \in B^+_{r_k}(p_0) \setminus \mathcal{C}_{k,p_0}$. Let $k' < k$ and fix ξ such that $\xi + p_0 \in \mathcal{C}_{k',p_0}$ and $|\xi| = 1$. Suppose for any $t \in [0,1], \exists q_t \in L_{p_1} \cap B_{r_k}(p_0)$ such that $q_t = p_t + s_t\xi$ for some $s_t \in \mathbb{R}$. Then*

$$F(p_t) - F(p_0) \leq M_{k,k'}t(F(p_1) - F(p_0))$$

for some constant $M_{k,k'}$.

Proof. Up to an isometry, let $\xi = -e_n$, $p_0 = 0$ and $p_1 = ae_1 + be_n$ for some $a, b \in \mathbb{R}, a > 0$. Then the set $\{q_t | t \in [0,1]\}$ can be viewed as a graph of a C^1 convex function g on $[0,1]$ and $s_t = bt - g(t)$. Therefore, s_t is a concave function of t. In addition, $s_1 = b - g(1)$, and $ae_1 + g(1)e_n = q_1 = p_1 = ae_1 + be_n$ so that $s_1 = 0$. Moreover, $q_t = ate_1 + g(t)e_n$ and

$F(q_t) = F(p_1)$ imply that

$$\frac{d}{dt}F(q_t) = \langle \nabla F(q_t), ae_1 + g'(t)e_n \rangle = 0 \Rightarrow g'(t) = \frac{a\langle \nabla F(q_t), e_1 \rangle}{\langle \nabla F(q_t), \xi \rangle}. \tag{3.31}$$

Note that from our choice of ξ, for any $q \in B_{r_k}(p_0)$,

$$\begin{aligned} \langle \nabla F(q), \xi \rangle &= \langle \nabla F(q) - \nabla F(p_0), \xi \rangle + \langle \nabla F(p_0), \xi \rangle \\ &\geq -\frac{1}{2k}|\nabla F(p_0)| + \frac{1}{k'}|\nabla F(p_0)|, \end{aligned} \tag{3.32}$$

where we have used (3.24) and (3.25). Therefore we combine (3.32) and (3.24) to obtain an upper bound for $|g'|$

$$|g'(t)| \leq \frac{k'(2k+1)r_k}{2k - k'} \leq (2k+1)r_k. \tag{3.33}$$

Next, we use concavity of s_t with $s_1 = 0$ to obtain

$$|q_t - p_t| = s_t \geq (1-t)s_0 = (1-t)|q_0 - p_0|. \tag{3.34}$$

Now we observe that

$$\begin{aligned} &F(p_t) - F(p_0) = (F(p_1) - F(p_0)) - (F(p_1) - F(p_t)) \\ &= (F(q_0) - F(p_0)) - (F(q_t) - F(p_t)) \\ &= \int_0^1 \langle \nabla F(p_0 + (s_0\xi)s), s_0\xi \rangle ds - \int_0^1 \langle \nabla F(p_t + (s_t\xi)s), s_t\xi \rangle ds \\ &= \int_0^1 \langle \nabla F(p_0 + (s_0\xi)s) - \nabla F(p_t + (s_t\xi)s), \xi \rangle s_0 ds \\ &+ \int_0^1 \langle \nabla F(p_t + (s_t\xi)s), \xi \rangle (s_0 - s_t) ds \\ &=: I_1 + I_2. \end{aligned}$$

(3.24) and (3.34) imply that

$$I_2 = \int_0^1 \langle \nabla F(p_t + (s_t\xi)s), \xi \rangle ds \times (s_0 - s_t) \leq 2|\nabla F(p_0)| \times ts_0. \tag{3.35}$$

To estimate I_1, we use Lemma 3.2.1. Recall that $r_k = \frac{C_1}{2kC}$ and $\left|\frac{d}{dt}s_t\right| = |b - g'(t)| \leq 2(k+1)r_k$.

$$\begin{aligned} I_1 &= \int_0^1 \langle \nabla F(p_0 + (s_0\xi)s) - \nabla F(p_t + (s_t\xi)s), \xi \rangle s_0 ds \\ &\leq \int_0^1 C|x_1 - x_0||p_0 - p_t + (s_0 - s_t)s\xi|s_0 ds \\ &\leq \int_0^1 C|x_1 - x_0|t \left(|p_0 - p_1| + \left| \frac{s_0 - s_t}{t} s\xi \right| \right) s_0 ds \\ &\leq \int_0^1 C|x_1 - x_0|t \times 2(k+1)r_k s_0 ds \\ &\leq 2|\nabla F(p_0)| \times ts_0. \end{aligned} \tag{3.36}$$

Finally, we combine (3.36), (3.35), (3.34), and (3.32) to obtain

$$\begin{aligned} F(p_t) - F(p_0) = I_1 + I_2 &\leq 4|\nabla F(p_0)| \times ts_0 \\ &\leq 4t \frac{2kk'}{2k - k'} \int_0^{s_0} \langle \nabla F(p_0 + s\xi), \xi \rangle ds \\ &= \frac{8kk'}{2k - k'} t(F(q_0) - F(p_0)) \\ &= M_{k,k'} t(F(p_1) - F(p_0)). \end{aligned}$$

□

Finally, what is left is to show that one of the cases in Lemma 3.2.3, Lemma 3.2.6, and Lemma 3.2.7 always holds when $|p_1 - p_0| \leq r_k$. To achieve this goal, we should discuss about the boundary of $Y^*_{p_0}$. We first show local Lipschitzness of convex functions.

Lemma 3.2.8. *Let $g : B_l(0) \to \mathbb{R}$ be a bounded convex function. Then for $x, y \in B_{\frac{l}{2}}(0)$, we* *have*

$$|g(x) - g(y)| \leq \frac{4\|g\|_{L^\infty(B_l)}}{l}|x - y|. \tag{3.37}$$

Proof. Let $x \in B_{\frac{l}{2}}(0)$ and p be a subdifferential of g at x. Let $v = \dfrac{p}{|p|}$ be a unit vector, then $x + \dfrac{l}{2}v \in B_l(0)$, and

$$\begin{aligned} & g(x + \frac{l}{2}v) \geq g(x) + \frac{l}{2}\langle p, v\rangle \\ \Rightarrow & g(x + \frac{l}{2}v) - g(x) \geq \frac{l}{2}|p| \\ \Rightarrow & \frac{4\|g\|_{L^\infty}}{l} \geq |p|. \end{aligned}$$

Now, for any $y \in B_{\frac{l}{2}}(0)$, we have

$$\begin{aligned} & g(y) \geq g(x) + \langle p, y - x\rangle \geq g(x) - |p||x - y| \\ \Rightarrow & g(x) - g(y) \leq |p||x - y| \leq \frac{4\|g\|_{L^\infty}}{l}|x - y|. \end{aligned}$$

Note that we can change the role of x and y, and that finishes the proof. $\square$

Lemma 3.2.8 shows that the boundary $\partial Y^*_{x_0}$ is Lipschitz. However, the Lipchitz constant can vary with x_0. In the next lemma, we show that we can chose the Lipschitz constants uniform over $x_0 \in X$, and therefore, at each point on the boundary $\partial Y^*_{x_0}$, we can obtain an interior cone which has uniform opening.

Lemma 3.2.9. *There exist $\rho > 0$ and $0 < \sigma < 1$ that satisfy the following : For any $x_0 \in X$ and $q \in \partial Y^*_{x_0}$, there exists a unit vector v such that for any $p_0 \in B_\rho(q) \cap Y^*_{x_0}$, we have*

$$\{p \in B_\rho(q) | \langle p - p_0, v\rangle \geq \sigma|p - p_0|\} \subset Y^*_{x_0}. \tag{3.38}$$

Proof. Fix $y \in \mathrm{Int}(Y)$ and $B_l(y) \subset Y$. Then from the bi-Lipschitzness of $D_x c$, we obtain

$$B_{\frac{l}{\lambda}}(p_y) \subset -D_x c(x_0, B_l(y)) \subset Y^*_{x_0} \tag{3.39}$$

where $p_y = -D_x c(x_0, y)$. Denote $\mathcal{H}_q = (p_y - q)^\perp + p_y$, a hyperplane that is perpendicular to $p_y - q$ and containing p_y. Let $B^{n-1}_{\frac{l}{\lambda},q}(p_y) = B_{\frac{l}{\lambda}}(p_y) \cap \mathcal{H}_q$ and $\mathcal{L}^-_p = \{p + (p_y - q)s | s \leq 0\}$. Define

$$\mathcal{D}_{q,l} = \bigcup_{p \in B^{n-1}_{\frac{l}{\lambda},q}(p_y)} \mathcal{L}^-_p, \ \mathcal{Y}_{q,l} = \mathcal{D}_{q,l} \cap \partial Y^*_{x_0}. \tag{3.40}$$

Since $\mathcal{L}^-_p$ is a ray with starting point p in the interior of $Y^*_{x_0}$, $\mathcal{L}^-_p \cap \partial Y^*_{x_0}$ is a singleton by convexity of $Y^*_{x_0}$. Therefore, letting $p_y = 0$ and $(p_y - q) /\!/ e_n$ up to an isometry, $\mathcal{Y}_{q,l}$ can be viewed as a graph of a convex function g on $B^{n-1}_{l,q}(0) \subset \mathbb{R}^{n-1}$. Lemma 3.2.8 shows that the Lipschitz constant of g on $B^{n-1}_{\frac{l}{2\lambda},q}(0)$ is bounded by

$$\frac{2\|g\|_{L^\infty}}{l/(2\lambda)} \leq \frac{4\lambda \mathrm{diam}(Y^*_{x_0})}{l} \leq \frac{4\lambda^2 \mathrm{diam}(Y)}{l} = L.$$

This shows that for any $p_0 \in \mathcal{D}_{q,l/2} \cap Y^*_{x_0}$, the upper Lipschitz cone $\{p | \langle p - p_0, e_n \rangle > \frac{L}{\sqrt{L^2+1}}|p - p_0|\}$ does not intersect with $\mathcal{Y}_{q,\frac{l}{2}}$, the graph of g on $B^{n-1}_{\frac{1}{2\lambda},q}(0)$:

$$\left\{ p | \langle p - p_0, e_n \rangle > \frac{L}{\sqrt{L^2+1}}|p - p_0| \right\} \cap \mathcal{Y}_{q,\frac{l}{2}} = \emptyset.$$

Noting the definition of $\mathcal{Y}_{q,\frac{l}{2}}$, we get the proof with $\rho = \frac{l}{2\lambda}$, $\sigma = \frac{L}{\sqrt{L^2+1}}$, and $v = e_n$. $\square$

Now we show that under appropriate choice of k and k', one of the cases in Lemma 3.2.3, Lemma 3.2.6, and Lemma 3.2.7 must hold.

Lemma 3.2.10. *Let $p_0 \in Y^*_{x_0}$ and take k, k' big enough so that $2 \leq k' < \frac{4}{7}k$, $2r_k < \rho$, and $\frac{1}{k'} < \sqrt{1 - \sigma^2}$ where ρ and σ are from Lemma 3.2.9. Suppose $B_{\frac{r_k}{4}}(p_0) \cap \partial Y^*_{x_0} \neq \emptyset$. If*

$p_1 \in B^+_{\frac{r_k}{4}}(p_0) \setminus \mathcal{C}_{k,p_0}$, then

$$F(p_t) - F(p_0) \leq M_{k,k'} t(F(p_1) - F(p_0))$$

for some constant $M_{k,k'}$.

Proof. We divide the proof into three steps.

Step 1) Let $\mathcal{K}(v) = \{p | \langle p, v\rangle \geq \sigma |p|\}$ and $K(w) = \{p | \langle p, w\rangle \geq \dfrac{1}{k'}|p|\}$ for some unit vectors v and w such that $\langle v, w\rangle \geq 0$. In the first step, we show that $\mathcal{K}(v) \cap K(w) \neq \emptyset$.

WLOG, we can assume that $v = e_n$ and $w = ae_1 + be_n$ for some $a \geq 0$. If $b > \frac{1}{k'}$, then $\langle w, e_n\rangle = b \geq \frac{1}{k'}$ and we obtain that $e_n \in \mathcal{K}(v) \cap K(w)$. Otherwise, denote $w^\perp = -be_1 + ae_n$. Then we can check that $\dfrac{1}{k'}w + (1 - \dfrac{1}{k'^2})^{\frac{1}{2}} w^\perp \in K(w)$. Moreover,

$$\begin{aligned}
\left\langle \frac{1}{k'}w + \left(1 - \frac{1}{k'^2}\right)^{1/2} w^\perp, e_n \right\rangle &= \frac{1}{k'}b + \left(1 - \frac{1}{k'^2}\right)^{1/2} a \\
&= \frac{1}{k'}b + \left(1 - \frac{1}{k'^2}\right)^{1/2} (1 - b^2)^{1/2}.
\end{aligned}$$

The last formula is a concave function of b on $[0, \frac{1}{k'}]$, hence it attains minimum value at the boundary $b = 0$ or $b = \frac{1}{k'}$. Since $k' \geq 2$ and $\frac{1}{k'} < \sqrt{1 - \sigma^2}$, we obtain that $\left(1 - \dfrac{1}{k'^2}\right)^{\frac{1}{2}} > \sigma$, and hence $\dfrac{1}{k'}w + \left(1 - \dfrac{1}{k'^2}\right)^{\frac{1}{2}} w^\perp \in \mathcal{K}(v)$.

Now suppose $q \in B_{\frac{r_k}{4}}(p_0) \cap \partial Y^*_{x_0} \neq \emptyset$ and $p_1 \in B^+_{\frac{r_k}{4}}(p_0) \setminus \mathcal{C}_{k,p_0}$. Then there is a unit vector v which satisfies (3.38). Note $\mathcal{K}(v) + p_0 = \{p | \langle p - p_0, v\rangle > \sigma |p - p_0|\}$. We consider the cases $\langle \nabla F(p_0), v\rangle \geq 0$ and $\langle \nabla F(p_0), v\rangle \leq 0$.

Step 2) Suppose $\langle \nabla F(p_0), v\rangle \geq 0$. Then, by *step 1*, we have $\mathcal{C}_{k',p_0} \cap (\mathcal{K}(v) + p_0) \neq \emptyset$. Let ξ be a unit vector such that $p_0 + \xi \in \mathcal{C}_{k',p_0} \cap (\mathcal{K}(v) + p_0)$. Then by Lemma 3.2.9, $p_t + \frac{1}{2}r_k\xi \in Y^*_{x_0} \cap B_\rho(q)$, $\forall t \in [0, 1]$. Moreover, noting that $|p_t + s\xi - p_0| \leq r_k, \forall s \in [0, \frac{1}{2}r_k]$,

$$F(p_t + \frac{1}{2}r_k\xi) - F(p_t) = \int_0^{\frac{1}{2}r_k} \langle \nabla F(p_t + s\xi), \xi\rangle ds$$

$$
\begin{aligned}
&= \int_0^{\frac{1}{2}r_k} \langle \nabla F(p_0), \xi \rangle ds \\
&+ \int_0^{\frac{1}{2}r_k} \langle \nabla F(p_t + s\xi) - \nabla F(p_0), \xi \rangle ds \\
&\geq \left(\frac{1}{2k'} - \frac{1}{2k} \right) |\nabla F(p_0)| r_k
\end{aligned}
$$

where we have used (3.24). In addition, $p_1 \notin \mathcal{C}_{k,p_0}$ with $|p_1 - p_0| \leq \frac{r_k}{4}$ and $(p_1 - p_t) + p_0 = (1-t)(p_1 - p_0) + p_0 \in \mathcal{C}_{k,p_0}$ implies

$$
\begin{aligned}
F(p_1) - F(p_t) &= \int_0^1 \langle \nabla F(p_t + s(p_1 - p_t)), p_1 - p_t \rangle ds \\
&= \int_0^1 \langle \nabla F(p_0), p_1 - p_t \rangle ds \\
&+ \int_0^1 \langle \nabla F(p_t + s(p_1 - p_t)) - \nabla F(p_0), p_1 - p_t \rangle ds \\
&\leq \left(\frac{1}{k} + \frac{1}{2k} \right) |\nabla F(p_0)| |p_1 - p_t| \\
&\leq \frac{3}{8k} |\nabla F(p_0)| r_k
\end{aligned}
$$

where we have used (3.24). Noting that $k' < \frac{4}{7}k$, we obtain $F(p_t + \frac{1}{2}r_k\xi) > F(p_1)$ and this implies that there exists $w_t \in L_{p_1}$ such that $w_t = p_t + s_t\xi$ for some s_t. Therefore, we can apply Lemma 3.2.7 to obtain the desired inequality.

Step 3) Suppose $\langle \nabla F(p_0), v \rangle \leq 0$. Then by *Step 1*, $\overline{\mathcal{C}}_{k',p_0}(p_1) \cap (\mathcal{K}(v) + p_1) \neq \emptyset$. Let ξ be a unit vector such that $p_1 - \xi \in \overline{\mathcal{C}}_{k',p_0}(p_1) \cap (\mathcal{K}(v) + p_1)$. Then Lemma 3.2.9 and convexity of $Y^*_{x_0} \cap B_\rho(q)$ shows that $p_t - s\xi \in Y^*_{x_0} \cap B_\rho(q)$, $\forall t \in [0,1]$, $\forall s \in [0, \frac{1}{2}r_k]$. In addition, $p_1 - \xi \in \overline{\mathcal{C}}_{k',p_0}(p_1)$ implies $p_0 + \xi \in \mathcal{C}_{k',p_0}$. Therefore, using (3.24) again,

$$
\begin{aligned}
F(p_t) - F(p_t - \frac{1}{2}r_k\xi) &= \int_{-\frac{1}{2}r_k}^0 \langle \nabla F(p_t + s\xi), \xi \rangle ds \\
&= \int_{-\frac{1}{2}r_k}^0 \langle \nabla F(p_0), \xi \rangle ds
\end{aligned}
$$

$$+\int_{-\frac{1}{2}r_k}^{0} \langle \nabla F(p_t + s\xi) - \nabla F(p_0), \xi \rangle ds$$
$$\geq \left(\frac{1}{2k'} - \frac{1}{2k}\right) |\nabla F(p_0)| r_k.$$

Moreover, $p_1 \notin \mathcal{C}_{k,p_0}$ implies that $p_t \notin \mathcal{C}_{k,p_0}$. Therefore

$$\begin{aligned}
F(p_t) - F(p_0) &= \int_0^1 \langle \nabla F(p_0 + s(p_t - p_0)), p_t - p_0 \rangle \\
&= \int_0^1 \langle \nabla F(p_0), p_t - p_0 \rangle ds \\
&+ \int_0^1 \langle \nabla F(p_0 + s(p_t - p_0)) - \nabla F(p_0), p_t - p_0 \rangle ds \\
&\leq \left(\frac{1}{k} + \frac{1}{2k}\right) |\nabla F(p_0)||p_t - p_0| \\
&\leq \frac{3}{8k} |\nabla F(p_0)| r_k.
\end{aligned}$$

Like in *Step 2*, we obtain $F(p_t - \frac{1}{2}r_k\xi) < F(p_0)$ which implies that $\exists w_t \in L_{p_0}$ such that $w_t = p_t - s_t\xi$ for some s_t. Therefore, we can apply *Step 2* and *Step 3* of the proof of Lemma 3.2.6 to obtain the desired inequality. □

Finally, we obtain the proof for the main theorem of this chapter.

Proof of the main theorem of chapter 3. By Lemma 3.2.2, We only need to consider the case $p_1 \in B_r^+(p_0)$ for some $r > 0$. Let $r = \frac{r_k}{4}$. If $p_1 \in \mathcal{C}_{k,p_0}$, then we obtain (3.2) from Lemma 3.2.3. Otherwise, we can apply Lemma 3.2.10 and we obtain (3.2). □

Chapter 4

Local Hölder regularity of generated Jacobian equations

4.1 Generated Jacobian equations

Generated Jacobian equations are Monge-Ampère type equations of the form

$$\det(D^2\phi(x) - \mathcal{A}(x, D\phi(x), \phi(x))) = \psi(x, D\phi(x), \phi(x)) \tag{GJE}$$

where $\mathcal{A}(x,p,u) = D^2_{xx}G(x, T(x,p,u), Z(x,p,u))$ is a matrix valued function. The matrix valued function $\mathcal{A}$ has an extra dependency on u compared to the matrix valued function from the c-Monge-Ampère equation. In fact, the c-Monge-Ampère equation is a special case of generated Jacobian equations. It is easy to see that we can obtain the c-Monge-Ampère equation by setting $G(x,y,z) = -c(x,y) - z$.

Generated Jacobian equations have an application in some geometric optic problem. For example, a generated Jacobian equation was derived in [4] for the near field refractor case and in [5] for the reflector shape design. Like equation (2.10) of c-Monge-Ampère equation, the generated Jacobian equations are derived from the following equations which is called Prescribed Jacobian Equation (PJE):

$$\det\left(D_x(T(x, D\phi(x), \phi(x)))\right) = \psi'(x, D\phi(x), \phi(x)). \tag{PJE}$$

We can derive generated Jacobian equations from (PJE) if there are functions G and Z that satisfy

$$\begin{cases} D_xG(x, T(x,p,u), Z(x,p,u)) = p \\ G(x, T(x,p,u), Z(x,p,u)) = u \end{cases}.$$

Like the optimal transportation problem, the second boundary conditions on (PJE) can be defined using two probability measures μ and ν:

$$T(\cdot, D\phi(\cdot), \phi(\cdot))_{\sharp}\mu = \nu.$$

In case of generated Jacobian equations, the above condition can be written in terms of the G-subdifferential. Therefore, we define the weak solutions of generated Jacobian equations using the G-subdifferentials.

The main theorem in this chapter is local Hölder regularity of solutions to (GJE). This result is proved by Loeper in [8] for the c-Monge-Ampère equation case. we generalize the result in [8] to generated Jacobian equation case. Obtaining the general proof for the local Hölder regularity is not trivial because of the extra non-linearity that comes from the dependency of the matrix valued function $\mathcal{A}$ on the scalar variable u. We discuss this in the next section.

4.2 Structure of generated Jacobian equation

We add some conditions on the generating function G.

$$G \in C^4(X \times Y \times \mathbb{R}), \tag{Regular}$$

$$D_z G < 0. \tag{G-mono}$$

The (Regular) condition is imposed on the set $X \times Y \times \mathbb{R}$ for simplicity. However, there are some examples of the generating functions which are not defined on whole $X \times Y \times \mathbb{R}$, for example, see [7]. Since the argument in this chapter is local, the result of this chapter can be applied to the cases when the generating function is not defined on whole $X \times Y \times \mathbb{R}$. From ($G$-mono), we see that there exists a function $H : X \times Y \times \mathbb{R} \to \mathbb{R}$ such that

$$G(x, y, H(x, y, u)) = u. \tag{4.1}$$

The implicit function theorem implies that $H \in C^4$ and (G-mono) implies

$$D_u H < 0. \tag{H-mono}$$

We also need some conditions on the generating function G which corresponds to (Twist) and (Non-deg) conditions in the optimal transportation problem. However, in contrast to the optimal transportation problem, the structural conditions do not necessarily hold on the whole domain $X \times Y \times \mathbb{R}$, but the structural conditions hold on a subset $\mathfrak{g}$ of $X \times Y \times \mathbb{R}$. Therefore, we assume that there exists a set $\mathfrak{g} \subset X \times Y \times \mathbb{R}$ such that

$$\mathfrak{g} \text{ is relatively open with respect to } X \times Y \times \mathbb{R} \tag{DomOpen}$$

and we assume the following:

$$(y, z) \mapsto (D_x G(x, y, z), G(x, y, z)) \text{ is injective on } \mathfrak{g}_x. \tag{G-twist}$$

$$x \mapsto -\frac{D_y G}{D_z G}(x, y, z) \text{ is injective on } \mathfrak{g}_{y,z}. \tag{G^*-twist}$$

$$\det\left(D^2_{xy} G - D^2_{xz} G \otimes \frac{D_y G}{D_z G}\right) \neq 0 \text{ on } \mathfrak{g}. \tag{G-nondeg}$$

where $\mathfrak{g}_x = \{(y, z) | (x, y, z) \in \mathfrak{g}\}$ and $\mathfrak{g}_{y,z} = \{x | (x, y, z) \in \mathfrak{g}\}$. We denote

$$E = D^2_{xy} G - D^2_{xz} G \otimes \frac{D_y G}{D_z G}.$$

The conditions (G-twist), (G^*-twist), and (G-nondeg) can be written in terms of H instead of G. In fact, we can see that (G-twist) and (G^*-twist) are symmetric conditions like in optimal transportation case by writing the conditions in terms of H. Let $\mathfrak{g}_{x,y} = \{z | (x, y, z) \in \mathfrak{g}\} \subset \mathbb{R}$ and define $\mathfrak{h}$ by

$$\mathfrak{h}_{x,y} = G(x, y, \mathfrak{g}_{x,y}), \tag{4.2}$$

$$\mathfrak{h} = \{(x, y, u) | u \in \mathfrak{h}_{x,y}\}. \tag{4.3}$$

We denote $\mathfrak{h}_{x,u} = \{y | (x, y, u) \in \mathfrak{h}\}$ and $\mathfrak{h}_y = \{(x, u) | (x, y, u) \in \mathfrak{h}\}$. Note that (DomOpen) implies

$$\mathfrak{h} \text{ is relatively open with respect to } X \times Y \times \mathbb{R}. \tag{DomOpen*}$$

(G-twist), (G^*-twist), and (G-nondeg) become (H-twist), (H^*-twist), and (H-nondeg) respectively when we rewrite the conditions in terms of H.

$$y \mapsto -\frac{D_x H}{D_u H}(x, y, u) \text{ is injective on } \mathfrak{h}_{x,u}, \tag{H^*-twist}$$

$$(x, u) \mapsto (D_y H(x, y, u), H(x, y, u)) \text{ is injective on } \mathfrak{h}_y, \tag{H-twist}$$

$$\det\left(D^2_{yx} H - D^2_{yu} H \otimes \frac{D_x H}{D_u H}\right) \neq 0 \text{ on } \mathfrak{h}. \tag{H-nondeg}$$

The conditions (G-twist) and (G^*-twist) allow us to define the inverse maps of the functions in (G-twist) and (G^*-twist).

Definition 4.2.1. We define the maps $exp^G_{x,u}$ and Z_x by

$$\begin{cases} D_x G(x, exp^G_{x,u}(p), Z_x(p, u)) & = p \\ G(x, exp^G_{x,u}(p), Z_x(p, u)) & = u \end{cases}.$$

We call $exp^G_{x,u}$ the *G-exponential map with focus* (x, u). We define another map $exp^{G^*}_{y,z}$ by

$$-\frac{D_y G}{D_v G}(exp^{G^*}_{y,z}(q), y, z) = q.$$

We call $exp^{G^*}_{y,z}$ the *G^*-exponential map with focus* (y, z).

Remark 4.2.2. The G-exponential map can be also defined in the following way

$$-\frac{D_x H}{D_u H}(x, exp^G_{x,u}(p), u) = p.$$

Note that by the implicit function theorem, the functions $exp^G_{x,u}$, Z_x and $exp^{G^*}_{y,v}$ are C^3 on the domain of each function. In fact, computing the derivative of G-exponential map $exp^G_{x,u}$ shows that

$$D_p exp^G_{x,u}(p) = E^{-1}\left(x, exp^G_{x,u}(p), Z_x(p,u)\right)$$

where E is the matrix defined above.

Now we impose one more condition on the generating function G which corresponds to (A3s) condition of optimal transportation problem. We first define the Tru tensor of the generating function G. The Tru tensor generalize the MTW tensor of the optimal transportation problem. The Tru Tensor of the generating function G was introduced by Trudinger in [10]. The Tru tensor is a (2,2)-tensor of the form

$$\mathrm{Tru}(x,p,u) = D^2_{pp}\mathcal{A}(x,p,u)$$

where $\mathcal{A}(x,p,u) = D^2_{xx}G(x, exp^G_{x,u}(p), Z_x(p,u))$ is a matrix valued function. We impose a sign condition on this Tru tensor, which we call (G3s).

$$MTW[\xi,\xi,\eta,\eta] > 0 \text{ for any } \xi \perp \eta. \tag{G3s}$$

In addition to the conditions that we have imposed on the generating function G, We also need to impose some conditions about convexity of the domains X and Y.

Definition 4.2.3. We define the sets $\mathfrak{g}^*_{y,z}$ and $\mathfrak{h}^*_{x,u}$ by

$$\mathfrak{g}^*_{y,z} = -\frac{D_y G}{D_z G}(\mathfrak{g}_{y,z}, y, z), \tag{4.4}$$

$$\mathfrak{h}^*_{x,u} = -\frac{D_x H}{D_u H}(x, \mathfrak{h}_{x,u}, u). \tag{4.5}$$

X is said to be G-convex if $\mathfrak{g}^*_{y,z}$ is convex for any $(y,z) \in Y \times \mathbb{R}$ and Y is said to be G^*-convex if $\mathfrak{h}^*_{x,u}$ is convex for any $(x,u) \in X \times \mathbb{R}$.

We assume that the sets X and Y satisfy G-convex and G^*-convex respectively.

$$X \text{ is } G\text{-convex}, \tag{hDomConv}$$

$$Y \text{ is } G^*\text{-convex}. \tag{vDomConv}$$

Definition 4.2.4. For $x \in X$ and $u \in \mathbb{R}$, let $y_0, y_1 \in \mathfrak{h}_{x,u}$. Let $p_i = -\frac{D_x H}{D_u H}(x, y_i, u)$. The *$G$-segment* that connects y_0 and y_1 with focus (x,u) is the image of $[p_0, p_1]$ under the map $exp^G_{x,u}$:

$$\{exp^G_{x,u}((1-\theta)p_0 + \theta p_0) | \theta \in [0,1]\}.$$

For $y \in Y$ and $z \in \mathbb{R}$, let $x_0, x_1 \in g_{y,z}$ and let $q_i = -\frac{D_y G}{D_z G}(x_i, y, z)$. The *$G^*$-segment* that connects x_0 and x_1 with focus (y,z) is the image of $[q_0, q_1]$ under the map $exp^{G^*}_{y,z}$:

$$\{exp^{G^*}_{y,z}((1-\theta)q_0 + \theta q_0) | \theta \in [0,1]\}.$$

Definition 4.2.5. A function $\phi : X \to \mathbb{R}$ is called *G-convex* if, for any $x_0 \in X$, there exist $y_0 \in Y$ and $w_0 \in \mathbb{R}$ such that

$$\phi(x_0) = G(x_0, y_0, z_0),$$

$$\phi(x) \geq G(x, y_0, z_0).$$

Definition 4.2.6 (G-subdifferential)**.** Let $\phi : X \to \mathbb{R}$ be a G-convex function. The G-

subdifferential of ϕ at $x_0 \in X$ is defined by

$$\partial_G \phi(x_0) = \left\{ y_0 \in Y \;\middle|\; \begin{array}{c} \phi(x) \geq G(x, y_0, H(x_0, y_0, \phi(x_0))), \\ (x_0, y_0, \phi(x_0)) \in \mathfrak{h} \end{array} \right\}.$$

Proposition 4.2.7. *If a G-affine function $G(\cdot, y_0, z_0)$ supports a G-convex function ϕ at x_0 locally,*

$$\phi(x_0) = G(x_0, y_0, z_0),$$
$$\phi(x) \geq G(x, y_0, z_0) \text{ on some neighborhood of } x_0,$$

and if $(x_0, y_0, z_0) \in \mathfrak{g}$, then $y_0 \in \partial_G \phi(x_0)$.

This is proved in [3], but under extra conditions which are called (unif) and (nice).

$$\text{There exist } a, b \in \mathbb{R} \text{ such that } [a, b] \subset \mathfrak{h}_{x,y}, \tag{unif}$$
$$\text{The solution } \phi \text{ is bounded by } a \text{ and } b,: a < \phi < b. \tag{nice}$$

In [3], these condition are used to check that the G-exponential maps they used in the proof are well-defined. In addition, compactness of the set $X \times Y \times [a, b]$ ensures that the norms of derivatives of the generating function are bounded. However, we can weaken the conditions (unif) and (nice) by replacing the constants a and b with some continuous functions $a(x, y)$ and $b(x, y)$.

$$\text{There exist continuous functions } a, b : X \times Y \to \mathbb{R}$$
$$\text{such that } [a(x, y), b(x, y)] \subset \mathfrak{h}_{x,y}, \tag{unifw}$$

$$\text{The solution } \phi \text{ satisfies } a(x, y) < \phi(x) < b(x, y)$$

$$\text{for any } y \in \partial_G \phi(x). \tag{nicew}$$

With these conditions (unifw) and (nicew), the G-exponential maps used in [3] are still well-defined, and we get compact sets

$$\Phi = \{(x, y, u) | u \in [a(x, y), b(x, y)]\} \Subset \mathfrak{h},$$
$$\Psi = \{(x, y, z) | z \in H(x, y, [a(x, y), b(x, y)])\} \Subset \mathfrak{g}.$$

On these compact sets Φ and Ψ, we can bound the norms of derivatives of the generating function G and the function H. Hence Proposition 4.2.7 is still true under the conditions (unifw) and (nicew). In addition, since $X \times Y \times [\min a, \max b]$ is compact, we have a constant $\beta > 0$ such that

$$D_z G < -\beta \tag{4.6}$$

on $X \times Y \times [\min a, \max b]$.

Remark 4.2.8. Let $S \subset \mathfrak{h}$ be a compact set. Then (G-nondeg) implies that we have a constant C_e that depends on S such that

$$\frac{1}{C_e} \leq \|E\| \leq C_e$$

on S where $\|E\|$ is the operator norm of E. This implies that the G-exponential map $exp^G_{x,u}$ is C_e-Lipschitz :

$$\frac{1}{C_e}|p_1 - p_0| \leq |exp^G_{x,u}(p_1) - exp^G_{x,u}(p_0)| \leq C_e|p_1 - p_0| \tag{4.7}$$

when $(x, exp^G_{x,u}(p_\theta), u) \in S$ for any $\theta \in [0, 1]$ where $p_\theta = (1-\theta)p_0 + \theta p_1$. Also, compactness of S with (G3s) implies that we have a constant $\alpha > 0$ that depends on S such that

$$\mathrm{Tru}[\xi, \xi, \eta, \eta] > \alpha|\xi|^2|\eta|^2, \ \forall \xi \perp \eta \tag{4.8}$$

on S.

Proposition 4.2.9. *The subdifferential of ϕ at a point x is a closed subset of Y, and is compactly contained in $\mathfrak{h}_{x,\phi(x)}$:*

$$\partial_G \phi(x) \Subset \mathfrak{h}_{x,\phi(x)}.$$

Proof. We first show that the G-subdifferential $\partial_G \phi(x)$ is closed. Suppose $y \in \overline{\partial_G \phi(x)}$, then $\exists y_i \in \partial_G \phi(x)$ such that $\lim_{i\to\infty} y_i = y$. (unifw) and (nicew) implies

$$\begin{aligned}
\phi(x) &\in \bigcap_{i=1}^{\infty} (a(x, y_i), b(x, y_i)) \\
&\subset [\sup a(x, y_i), \inf b(x, y_i)] \\
&\subset \left[\lim_{i\to\infty} a(x, y_i), \lim_{i\to\infty} b(x, y_i)\right] \\
&= [a(x, y), b(x, y)] \subset \mathfrak{h}_{x,y}.
\end{aligned}$$

Therefore, $(x, y, \phi(x)) \in \mathfrak{h}$. In addition, from Definition 4.2.6,

$$G(x', y_i, H(x, y_i, \phi(x))) \leq \phi(x'), \forall x' \in X.$$

Taking $i \to \infty$, we obtain

$$G(x', y, H(x, y, \phi(x))) \leq \phi(x'), \forall x' \in X.$$

Hence, $y \in \partial_G \phi(x)$ and the G-subdifferential at x is closed, and therefore compact. Noting that the set $\mathfrak{h}_{x,\phi(x)}$ is open, we obtain the desired result. □

Proposition 4.2.10. *Let ϕ be a G-convex function with* (nicew)*. Let $x \in X$, then for $\epsilon > 0$, there exists $\delta > 0$ such that if $|x' - x| \leq \delta$, then*

$$\partial_G \phi(x') \subset \mathcal{N}_\epsilon(\partial_G \phi(x)).$$

Proof. Suppose the proposition is not true. Then there exist sequences $x_i \in X$ and $y_i \in \partial_G\phi(x_i)$ such that $x_i \to x$ as $i \to \infty$, but $y_i \notin \mathcal{N}_\epsilon(\partial_G\phi(x))$ for any i. Since Y is compact, we can assume that $y_i \to y$ for some $y \in Y$. Then $y \notin \mathcal{N}_\epsilon(\partial_G\phi(x))$ and (nicew) implies $(x, y, \phi(x)) \in \mathfrak{h}_{x,\phi(x)}$. Since $y_i \in \partial_G\phi(x_i)$, we have

$$\phi(x') \geq G(x', y_i, H(x_i, y_i, \phi(x_i))), \forall x' \in X.$$

Taking $i \to \infty$, we obtain

$$\phi(x') \geq G(x', y, H(x,y, \phi(x))), \forall x' \in X.$$

Hence $y \in \partial_G\phi(x)$, which contradicts to $y \notin \mathcal{N}_\epsilon(\partial_G\phi(x))$. $\square$

Now we define the weak solutions to generated Jacobian equation. Like in the optimal transportation problem, we define two weak solutions.

Definition 4.2.11. Let $\phi : X \to \mathbb{R}$ be a G-convex function. Then

1. ϕ is called a *weak Alexandrov solution* to (GJE) if

$$\mu(A) = \nu(\partial_G\phi(A)), \forall A \subset X.$$

2. ϕ is called a *weak Brenier solution* to (GJE) if

$$\nu(B) = \mu(\partial_G\phi^{-1}(B)), \forall B \subset Y.$$

Now we state the main theorem of this chapter.

Theorem 4.2.12 (Main theorem of Chapter 4)**.** *Suppose X and Y are compact domains in $\mathbb{R}^n$ and let let μ and ν be probability measures on X and Y respectively. Let $G : X \times Y \times \mathbb{R} \to \mathbb{R}$ be the generating function satisfying* (Regular)*,* (G-mono)*,* (G-twist)*,* (G^*-twist)*,*

(G-nondeg), (G3s), *and* (unifw). *Assume also that* X *and* Y *satisfy* (hDomConv) *and* (vDomConv) *and the target measure* ν *is bounded away from* 0 *and* ∞ *with respect to the Lebesgue measure on* Y. *Let* ϕ *be a weak Alexandrov solution to equation* (GJE) *that satisfies* (nicew). *Then we have the following:*

1. *If there exist* $p \in (n, \infty]$ *and* C_μ *such that* $\mu(B_r(x)) \le C_\mu r^{n(1-\frac{1}{p})}$ *for all* $r \ge 0$, $x \in X$, *then* $\phi \in C^{1,\sigma}_{loc}(X)$.
2. *If there exist* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $\lim_{r\to 0} f(r) = 0$ *and* $\mu(B_r(x)) \le f(r) r^{n(1-\frac{1}{n})}$ *for all* $r \ge 0$, $x \in X$, *then* $\phi \in C^1_{loc}(X)$.

Here, $\sigma = \frac{\rho}{4n-2+\rho}$ *where* $\rho = 1 - \frac{n}{p}$.

4.3 Quantitative Loeper's condition

Before we start this section, we decide some notations. x_m is a point in X, u is a real number, and $y_0, y_1 \in \mathfrak{h}_{x_m,u}$. We denote the G-segment that connects y_0 and y_1 with focus (x_m, u) by y_θ and we denote $z_\theta = H(x_m, y_\theta, u)$. Let $p_\theta = D_x G(x_m, y_\theta, z_\theta)$. Then note that we have $p_\theta = (1-\theta)p_0 + \theta p_1$.

In this section, we will assume that we have a compact set $S \Subset \mathfrak{h}$ such that $(x_m, y_\theta, u) \in S$. Then by Remark 4.2.8, we obtain constants C_e and α which depend on S in the remark.

Lemma 4.3.1. *For some constant* C_1 *that depend on the* C^3 *norm of* G, C^1 *norm of* H, *and* C_e, *we have*

$$\left|\left(D^2_{xx}G(x_m, y_\theta, z_\theta) - D^2_{xx}G(x_m, y_{\theta'}, z_{\theta'})\right)[\xi, \xi]\right| \le C_1 |\theta - \theta'||p_1 - p_0||\xi|^2 \tag{4.9}$$

Proof.

$$\begin{aligned}
&\|D^2_{xx}G(x_m, y_\theta, H(x_m, y_\theta, u)) - D^2_{xx}G(x_m, y_{\theta'}, H(x_m, y_{\theta'}, u))\| \\
&\le \|D^3_{xxy}G\||y_\theta - y_{\theta'}| + \|D^3_{xxz}G\|\|D_y H\||y_\theta - y_{\theta'}| \\
&\le (\|D^3_{xxy}G\| + \|D^3_{xxz}G\|\|D_y H\|) C_e |\theta - \theta'||p_1 - p_0|.
\end{aligned}$$

We set $C_1 = (\|D^3_{xxy}G\| + \|D^3_{xxz}G\|\|D_yH\|)C_e$. $\square$

Lemma 4.3.2. *Let* $\xi_p = \mathrm{Proj}_{p_1-p_0}(\xi)$*, where* Proj_p *is the orthogonal projection on to* p*. Then for some constants* Δ_1 *and* Δ_2 *which depend on* α*, and the* C^4 *norm of* G*, we have*

$$\begin{aligned}
&D^2_{xx}G(x_m, y_\theta, z_\theta)[\xi,\xi] \\
&\leq \left((1-\theta)D^2_{xx}G(x_m, y_0, z_0) + \theta D^2_{xx}G(x_m, y_1, z_1)\right)[\xi,\xi] \\
&\quad + \theta(1-\theta)|p_1 - p_0|^2(-\Delta_1|\xi|^2 + \Delta_2|\xi_p|^2).
\end{aligned}$$

Proof. Define $\mathcal{F}_\xi : [0,1] \to \mathbb{R}$ by

$$\mathcal{F}_\xi(\theta) = D^2_{xx}G(x_m, y_\theta, z_\theta)[\xi,\xi].$$

Let $\xi' = \xi - \xi_p$ so that $\xi' \perp \xi_p$. Then (4.8) implies

$$\mathcal{F}''_{\xi'}(\theta) \geq \alpha|p_1 - p_0|^2|\xi'|^2,$$

which shows that $\mathcal{F}_{\xi'}$ is uniformly convex. Therefore, we obtain

$$\mathcal{F}_{\xi'}(\theta) \leq \theta\mathcal{F}_{\xi'}(1) + (1-\theta)\mathcal{F}_{\xi'}(0) - \frac{1}{2}\alpha|p_1 - p_0|^2|\xi'|^2\theta(1-\theta). \tag{4.10}$$

Let $\mathcal{G}_\xi = \mathcal{F}_\xi - \mathcal{F}_{\xi'}$. Then

$$\begin{aligned}
\mathcal{G}''_\xi(\theta) &= \mathcal{F}''_\xi(\theta) - \mathcal{F}''_{\xi'}(\theta) \\
&= D^2_{pp}\mathcal{A}[\xi,\xi,p_1-p_0,p_1-p_0] - D^2_{pp}\mathcal{A}[\xi',\xi',p_1-p_0,p_1-p_0] \\
&= 2D^2_{pp}\mathcal{A}[\xi',\xi_p,p_1-p_0,p_1-p_0] + D^2_{pp}\mathcal{A}[\xi_p,\xi_p,p_1-p_0,p_1-p_0]
\end{aligned}$$

where $D^2_{pp}\mathcal{A}$ is evaluated at $(x_m, y_\theta, z_\theta)$. We bound $|\xi'|$ by $|\xi|$ to obtain

$$\mathcal{G}_\xi(\theta) \leq \theta\mathcal{G}_\xi(1) + (1-\theta)\mathcal{G}_\xi(0) + \frac{3}{2}|D^2_{pp}\mathcal{A}||p_1 - p_0|^2|\xi||\xi_p|\theta(1-\theta). \tag{4.11}$$

We combine (4.10) and (4.11) to obtain

$$\begin{aligned}
&D^2_{xx}G(x_m, y_\theta, z_\theta)[\xi, \xi] \\
&= \mathcal{G}_\xi + \mathcal{F}_{\xi'} \\
&\leq \theta\mathcal{G}_\xi(1) + (1-\theta)\mathcal{G}(0) \\
&+ \frac{3}{2}|D^2_{pp}\mathcal{A}||p_1 - p_0|^2|\xi||\xi_p|\theta(1-\theta) \\
&+ \theta\mathcal{F}_{\xi'}(1) + (1-\theta)\mathcal{F}_{\xi'}(0) - \frac{1}{2}\alpha|p_1 - p_0|^2|\xi'|^2\theta(1-\theta) \\
&= \theta D^2_{xx}G(x_m, y_1, z_1)[\xi, \xi] + (1-\theta)D^2_{xx}G(x_m, y_0, z_0)[\xi, \xi] \\
&+ \theta(1-\theta)|p_1 - p_0|^2\left(-\frac{\alpha}{2}|\xi'|^2 + \frac{3}{2}|D^2_{pp}\mathcal{A}||\xi||\xi_p|\right) \\
&\leq \theta D^2_{xx}G(x_m, y_1, z_1)[\xi, \xi] + (1-\theta)D^2_{xx}G(x_m, y_0, z_0)[\xi, \xi] \\
&+ \theta(1-\theta)|p_1 - p_0|^2\left(-\frac{\alpha}{2}|\xi|^2 + \left(\frac{3}{2}|D^2_{pp}\mathcal{A}| + \alpha\right)|\xi||\xi_p|\right).
\end{aligned}$$

We use the weighted Young's inequality in the last line of above equation.

$$\left(\frac{3}{2}|D^2_{pp}\mathcal{A}| + \alpha\right)|\xi||\xi_p| \leq \frac{\alpha}{4}|\xi|^2 + \alpha^{-1}\left(\frac{3}{2}|D^2_{pp}\mathcal{A}| + \alpha\right)^2|\xi_p|^2.$$

Then we obtain

$$\begin{aligned}
&D^2_{xx}G(x_m, y_\theta, z_\theta)[\xi, \xi] \\
&\leq \theta D^2_{xx}G(x_m, y_1, z_1)[\xi, \xi] + (1-\theta)D^2_{xx}G(x_m, y_0, z_0)[\xi, \xi] \\
&+ \theta(1-\theta)|p_1 - p_0|^2\left(-\frac{\alpha}{4}|\xi|^2 + \alpha^{-1}\left(\frac{3}{2}|D^2_{pp}\mathcal{A}| + \alpha\right)^2|\xi_p|^2\right).
\end{aligned}$$

Therefore, we obtain the inequality with constants $\Delta_1 = \frac{\alpha}{4}$ and $\Delta_2 = \alpha^{-1}\left(\frac{3}{2}|D^2_{pp}\mathcal{A}| + \alpha\right)^2$.

□

The next lemma is the quantitative version of Loeper's condition. We will use (G3s) condition through Lemma 4.3.3 later.

Lemma 4.3.3. *Define* $\overline{\phi}(x) : X \to \mathbb{R}$ *by*

$$\overline{\phi}(x) = \max\{G(x, y_0, z_0), G(x, y_1, z_1)\}$$

Then we have the quantitative Loeper's condition :

$$\overline{\phi}(x) \geq G(x, y_\theta, z_\theta) + \delta_0\theta(1-\theta)|y_1 - y_0|^2|x - x_m|^2 - \gamma|x - x_m|^3 \tag{4.12}$$

for any $\epsilon \in (0, \frac{1}{2})$ *and* $\theta \in [\epsilon, 1-\epsilon]$ *and* $|x - x_m| \leq C\epsilon$ *for some constants* δ_0, γ, C.

Proof. Note that the Taylor expansion theorem yields

$$\begin{aligned} G(x, y_i, z_i) =& u + \langle D_x G(x_m, y_i, z_i), (x - x_m)\rangle \\ &+ \frac{1}{2} D^2_{xx} G(x, y_i, z_i)[x - x_m, x - x_m] + o(|x - x_m|^2). \end{aligned}$$

Therefore,

$$\begin{aligned} \overline{\phi}(x) &\geq \theta G(x, y_0, z_0) + (1-\theta) G(x, y_1, z_1) \\ &= u + \langle \theta p_1 + (1-\theta) p_0, x - x_m\rangle \\ &\quad + \frac{1}{2}\left(\theta D^2_{xx} G(x, y_0, z_0) + (1-\theta) D^2_{xx} G(x, y_1, z_1)\right)[x - x_m, x - x_m] \\ &\quad + o(|x - x_m|^2). \end{aligned}$$

We apply Lemma 4.3.2 to obtain

$$
\begin{aligned}
\overline{\phi}(x) \geq & u + \langle \theta p_1 + (1-\theta)p_0, x - x_m \rangle + \frac{1}{2} D^2_{xx} G(x_m, y_\theta, z_\theta)[x - x_m, x - x_m] \\
& - \frac{1}{2}\theta(1-\theta)|p_1 - p_0|^2 \left(-\Delta_1 |x - x_m|^2 + \Delta_2 |(x - x_m)_p|^2\right) \\
& + o(|x - x_m|^2).
\end{aligned}
\tag{4.13}
$$

Since (4.13) is true for any $\theta \in [0,1]$, we can replace θ with θ' in (4.13). Let us call the inequality that we obtain from (4.13) by replacing θ with θ' (4.13'). We now add and subtract the right hand side of the inequality (4.13) to the right hand side of (4.13'), and rearrange some terms to obtain

$$
\begin{aligned}
\overline{\phi}(x) \geq & u + \langle \theta p_1 + (1-\theta)p_0, x - x_m \rangle + \frac{1}{2} D^2_{xx} G(x_m, y_\theta, z_\theta)[x - x_m, x - x_m] \\
& + \frac{1}{2}\Delta_1 \theta(1-\theta)|p_1 - p_0|^2 |x - x_m|^2 \\
& + (\theta' - \theta)\langle p_1 - p_0, x - x_m \rangle - \frac{1}{2}\theta(1-\theta)\Delta_2 |p_1 - p_0|^2 |(x - x_m)_p|^2 \\
& + \frac{1}{2}\left(D^2_{xx} G(x_m, y_{\theta'}, z_{\theta'}) - D^2_{xx} G(x_m, y_\theta, z_\theta)\right)[x - x_m, x - x_m] \\
& + \frac{1}{2}\Delta_1 \left((\theta'(1-\theta') - \theta(1-\theta)\right) |p_1 - p_0|^2 |x - x_m|^2 \\
& + \frac{1}{2}\Delta_2 \left((\theta(1-\theta) - \theta'(1-\theta')\right) |p_1 - p_0|^2 |(x - x_m)_p|^2 + o(|x - x_m|^2).
\end{aligned}
\tag{4.14}
$$

Let L_i be the i-th line of the right hand side of (4.14). Note that by definition of $(x - x_m)_p$, we have $|p_1 - p_0||(x - x_m)_p| = |\langle p_1 - p_0, x - x_m \rangle|$. Therefore, we can rewrite the third line L_3 as

$$
L_3 = \left(\theta' - \theta - \frac{1}{2}\theta(1-\theta)\Delta_2 \langle p_1 - p_0, x - x_m \rangle\right) \langle p_1 - p_0, x - x_m \rangle.
$$

We choose

$$
\theta' = \theta + \frac{1}{2}\theta(1-\theta)\Delta_2 \langle p_1 - p_0, x - x_m \rangle \tag{4.15}
$$

so that we have $L_3 = 0$. To ensure $\theta' \in [0,1]$, we first assume that θ is away from 0 and 1, i.e. we assume $\theta \in [\epsilon, 1-\epsilon]$ for $\epsilon > 0$. Then we make the second term in (4.15) small by assuming

$$|x - x_m| \leq \frac{4\epsilon}{\Delta_2|p_1 - p_0|} \leq \frac{\epsilon}{\theta(1-\theta)\Delta_2|p_1 - p_0|}.$$

Then we obtain that $\theta' \in [0,1]$ and $L_3 = 0$. We apply Lemma 4.3.1 and (4.15) on the forth line L_4 of (4.14) to obtain

$$\begin{aligned} L_4 &= \frac{1}{2}\left(D^2_{xx}G(x_m, y_{\theta'}, z_{\theta'}) - D^2_{xx}G(x_m, y_\theta, z_\theta)\right)[x - x_m, x - x_m] \\ &\geq -C_1|\theta - \theta'||p_1 - p_0||x - x_m|^2 \\ &\geq -\frac{C_1}{2}\theta(1-\theta)\Delta_2|p_1 - p_0|^2|x - x_m|^3 \\ &\geq -\frac{C_1}{8}\Delta_2|p_1 - p_0|^2|x - x_m|^3. \end{aligned} \tag{4.16}$$

For the fifth line L_5 and sixth line L_6, note that (4.15) implies

$$\begin{aligned} \theta'(1-\theta') - \theta(1-\theta) &= (\theta - \theta')(\theta + \theta' - 1) \\ &= -\frac{1}{2}\theta(1-\theta)\Delta_2\langle p_1 - p_0, x - x_m\rangle(\theta + \theta' - 1), \end{aligned}$$

so that we can obtain

$$\begin{aligned} |L_5| &= \left|\Delta_1\left(\theta'(1-\theta') - \theta(1-\theta)\right)|p_1 - p_0|^2|x - x_m|^2\right| \\ &\leq \frac{1}{2}\theta(1-\theta)(\theta + \theta' - 1)\Delta_1\Delta_2|p_1 - p_0|^3|x - x_m|^3 \\ &\leq \frac{1}{8}\Delta_1\Delta_2|p_1 - p_0|^3|x - x_m|^3, \end{aligned} \tag{4.17}$$

$$\begin{aligned} |L_6| &= \left|\Delta_2\left(\theta(1-\theta) - \theta'(1-\theta')\right)|p_1 - p_0|^2|(x - x_m)_p|^2\right| \\ &\leq \frac{1}{2}\theta(1-\theta)(\theta + \theta' - 1)(\Delta_2)^2|p_1 - p_0|^3|x - x_m|^3 \\ &\leq \frac{1}{8}(\Delta_2)^2|p_1 - p_0|^3|x - x_m|^3. \end{aligned} \tag{4.18}$$

We combine (4.16), (4.17), and (4.18) to bound (4.14) from below

$$\begin{aligned}\overline{\phi}(x) \geq & u + \langle \theta p_1 + (1-\theta)p_0, x - x_m\rangle + \frac{1}{2}D^2_{xx}G(x_m, y_\theta, z_\theta)[x - x_m, x - x_m] \\ & + \Delta_1\theta(1-\theta)|p_1 - p_0|^2|x - x_m|^2 \\ & - C_2(|p_1 - p_0|^2 + |p_1 - p_0|^3)|x - x_m|^3 + o(|x - x_m|^2)\end{aligned} \tag{4.19}$$

where C_2 depends on C_1, Δ_1, and Δ_2. We apply Taylor's theorem on the first line of (4.19) to obtain $G(x, y_\theta, z_\theta)$ with $o(|x - x_m|^2)$ term. Note that the little o term $o(|x - x_m|^2)$ is at least $O(|x - x_m|^3)$ because the generating function G is C^4. Therefore, we can put $|x - x_m|^3$ term in place of $o(|x - x_m|^2)$, and we obtain

$$\begin{aligned}\overline{\phi}(x) \geq & G(x, y_\theta, z_\theta) + \Delta_1\theta(1-\theta)|p_1 - p_0|^2|x - x_m|^2 \\ & - C_2\left(1 + |p_1 - p_0|^2 + |p_1 - p_0|^3\right)|x - x_m|^3\end{aligned}$$

possibly taking larger value for C_2. Finally, we bound $|p_1 - p_0|$ by $C_e \mathrm{diam}(Y)$ from above and $\frac{1}{C_e}|y_1 - y_0|$ from below to obtain

$$\overline{\phi}(x) \geq G(x, y_\theta, z_\theta) + \delta_0\theta(1-\theta)|y_1 - y_0|^2|x - x_m|^2 - \gamma|x - x_m|^3$$

where $\delta_0 = \frac{\Delta_1}{C_e^2}$ and $\gamma = C_2\left(1 + C_e^2 \mathrm{diam}(Y)^2 + C_e^3 \mathrm{diam}(Y)^3\right)$. □

Remark 4.3.4. Lemma 4.3.3 implies G-convexity of G-subdifferentials of a G-convex function ϕ with respect to $(x, \phi(x))$. Suppose $y_0, y_1 \in \partial_G\phi(x_m)$, then $G(x, y_i, z_i)$ supports ϕ at x_m where $z_i = H(x_m, y_i, \phi(x_m))$. Let y_θ be the G-segment connecting y_0 and y_1 with respect to $(x_m, \phi(x_m))$. Fix θ and ϵ such that $\theta \in [\epsilon, 1-\epsilon]$. Then Lemma 4.3.3 shows that we have (4.3.3), which implies that $G(x, y_\theta, z_\theta)$ is a G-affine function that supports ϕ locally. Then Proposition 4.2.7 shows that $y_\theta \in \partial_G\phi(x_m)$.

4.4 G-convex functions

Proposition 4.4.1. *Let ϕ be a G-convex function that satisfies* (nicew). *Then ϕ is semi convex:*

$$\phi(x_t) \leq (1-t)\phi(x_0) + t\phi(x_1) + \frac{1}{2}t(1-t)\|D^2_{xx}G\||x_0 - x_1|^2 \tag{4.20}$$

where $x_t = (1-t)x_0 + tx_1$.

Proof. Since ϕ is G-convex, we have $y \in Y$ and $z \in \mathbb{R}$ such that $(x_t, y, z) \in \Psi$ and

$$\phi(x_t) = G(x_t, y, z),$$

$$\phi(x) \geq G(x, y, z), \forall x \in X.$$

Moreover, we have

$$G(x, y, z) \geq \phi(x_t) + \langle p_t, x - x_t \rangle - \frac{1}{2}\|D^2_{xx}G\||x - x_t|^2$$

where $p_t = D_xG(x_t, y, z)$. Evaluate this at $x = x_0$ and $x = x_1$ and add them with weight $(1-t)$ and t respectively.

$$\begin{aligned}(1-t)\phi(x_0) + t\phi(x_1) &\geq (1-t)G(x_0, y, z) + tG(x_1, y, z) \\ &\geq \phi(x_t) + \langle p_t, (1-t)(x_0 - x_t) + t(x_1 - x_t)\rangle \\ &\quad - \frac{1}{2}\|D^2_{xx}G\| \left((1-t)|x_0 - x_t|^2 + t|x_1 - x_t|^2\right).\end{aligned} \tag{4.21}$$

Note that by the choice of x_t, we have $(1-t)(x - x_t) + t(x_1 - x_t) = 0$ and

$$|x_0 - x_t| = t|x_0 - x_1| \text{ and } |x_1 - x_t| = (1-t)|x_0 - x_1|.$$

Then (4.21) becomes

$$(1-t)\phi(x_0)+t\phi(x_1) \geq \phi(x_t) - \frac{1}{2}t(1-t)\|D^2_{xx}G\||x_0-x_1|^2.$$

Note that this inequality shows that $\phi(x)+\frac{1}{2}\|D^2_{xx}G\||x|^2$ is convex. $\square$

When we use norms of some derivatives of G and H, we need to check that the points we are using are in some compact subset of $X\times Y\times\mathbb{R}$ so that we can have a finite value for the norms. If the point (x,y,u) is on the graph of the G-subdifferential of a (nicew) G-convex function, then the point will be in the set Φ, and therefore, we can use the norms on the set Φ (or Ψ). Later in this section, however, we will need to use some points which might not be in the set Φ (or Ψ). In Lemma 4.4.2, we show that for each point, we can choose a compact subset $S\subset\mathfrak{h}$ which contains the points that we use later. With Lemma 4.4.2, we will be able to obtain finite valued norms.

Lemma 4.4.2. *Let ϕ be a G-convex function with* (nicew) *and let x_c be an interior point of X. Then there exists $\delta(x_c)>0$ and $S\Subset\mathfrak{h}$ such that if $x_0,x_1\in B_{\delta(x_c)}(x_c)$, then*

$$(x_t,y_\theta,G(x_t,y_0,H(x_0,y_0,\phi(x_0)))),(x_t,y_\theta,\phi(x_t))\in S \tag{4.22}$$

for any $x_t=(1-t)x_0+tx_1, t\in[0,1]$ and y_θ, the G-segment connecting y_0 and y_1 with focus $(x_t,\phi(x_t))$ where $y_0\in\partial_G\phi(x_0)$ and $y_1\in\partial_G\phi(x_1)$.

Proof. Note that by (nicew), we have that $(x_c,y_c,\phi(x_c))$ is in the interior of $\mathfrak{h}$ for any $y_c\in\partial_G\phi(x_c)$. Therefore, we have $r_1,r_2,r_3>0$ such that

$$S:=B_{r_1}(x_c)\times(\mathcal{N}_{r_2}(\partial_G\phi(x_c))\cap Y)\times(\phi(x_c)-r_3,\phi(x_c)+r_3)\Subset\mathfrak{h}, \tag{4.23}$$

that is, $\overline{S}$ is compact and $\overline{S}$ is contained in the interior of $\mathfrak{h}$. Therefore, we obtain the constant C_e from Remark 4.2.8. We define

$$\partial_G^*\phi(x_c) = \left(exp^G_{x_c,\phi(x_c)}\right)^{-1}(\partial_G\phi(x_c)).$$

Note that $\partial_G^*\phi(x_c)$ is convex by Remark 4.3.4. From Remark 4.2.8, we obtain

$$\mathcal{N}_{\frac{r_2}{C_e}}\left(\partial_G^*\phi(x_c)\right)\cap \mathfrak{h}^*_{x_c,\phi(x_c)} \subset \left(exp^G_{x_c,\phi(x_c)}\right)^{-1}\left(\mathcal{N}_{r_2}\left(\partial_G\phi(x_c)\right)\cap Y\right).$$

Noting that $D_xG(x,\cdot,H(x,\cdot,u)) = \left(exp^G_{x,u}\right)^{-1}$ and the map

$$(x,y,u)\mapsto D_xG(x,y,H(x,y,u))$$

is uniformly continuous on S, there exist $\delta_x,\delta_u>0$ such that if $|x-x_c|<\delta_x$ and $|u-\phi(x_c)|<\delta_u$, then

$$|D_xG(x,y,H(x,y,u)) - D_xG(x_c,y,H(x_c,y,\phi(x_c)))| < \frac{r_2}{4C_e}$$

for any $y\in\mathcal{N}_{r_2}\left(\partial_G\phi(x_c)\right)\cap Y$. Hence, for any $y\in\mathcal{N}_{r_2}\left(\partial_G\phi(x_c)\right)\cap Y$ such that

$$\left(exp^G_{x_c,\phi(x_c)}\right)^{-1}(y)\in\mathcal{N}_{\frac{r_2}{4C_e}}\left(\partial_G^*\phi(x_c)\right),$$

we have

$$\left(exp^G_{x,u}\right)^{-1}(y)\in\mathcal{N}_{\frac{r_2}{2C_e}}\left(\partial_G^*\phi(x_c)\right) \tag{4.24}$$

if $|x-x_c|<\delta_x$ and $|u-\phi(x_c)|<\delta_u$. Note that $\mathcal{N}_{\frac{r_2}{2C_e}}\left(\partial_G^*\phi(x_c)\right)$ is convex. Remark 4.2.8 shows that

$$\left(exp^G_{x_c,\phi(x_c)}\right)^{-1}\left(\mathcal{N}_{\frac{r_2}{4C_e^2}}\left(\partial_G\phi(x_c)\right)\right)\subset\mathcal{N}_{\frac{r_2}{4C_e}}\left(\partial_G^*\phi(x_c)\right).$$

By Proposition 4.2.10, there exists δ_1 such that if $|x-x_c|<\delta_1$, then

$$\partial_G\phi(x)\subset\mathcal{N}_{\frac{r_2}{4C_e^2}}\left(\partial_G\phi(x_c)\right).$$

Moreover, by continuity of G, H and ϕ, and (nicew) which implies that the range of ϕ is compact, we have δ_2 such that if $|x - x_c| < \delta_2$ and $|x_0 - x_c| < \delta_2$, then for any $y_0 \in \partial_G \phi(x_0)$,

$$|G(x, y_0, H(x_0, y_0, \phi(x_0))) - \phi(x_c)| < \min\{\delta_u, r_3\},$$
$$|\phi(x) - \phi(x_c)| < \min\{\delta_u, r_3\}.$$

We take $\delta(x_c)$ small enough so that $\delta(x_c) \leq \min\{\delta_x, \delta_1, \delta_2, r_1\}$. Suppose $x_0, x_1 \in B_{\delta(x_c)}(x_c)$. Then $|x_t - x_c| < \delta(x_c)$ and therefore

$$|G(x_t, y_0, H(x_0, y_0, \phi(x_0))) - \phi(x_c)| < \min\{\delta_u, r_3\},$$
$$|\phi(x_t) - \phi(x_c)| < \min\{\delta_u, r_3\}.$$

where $y_i \in \partial_G \phi(x_i)$. Hence we obtain that $(x_t, y_i, u) \in S \subset \mathfrak{h}$ for $i = 0, 1$ where u is either $\phi(x_t)$ or $G(x_t, y_0, H(x_0, y_0, \phi(x_0)))$. Moreover, $\partial_G \phi(x_i) \subset \mathcal{N}_{\frac{r_2}{4C_e^2}} (\partial_G \phi(x_c))$ for $i = 0, 1$, (4.24) implies

$$\left(exp^G_{x_t, \phi(x_t)}\right)^{-1} (y_i) \in \mathcal{N}_{\frac{r_2}{2C_e}} \left(\partial^*_G \phi(x_c)\right).$$

Since $\mathcal{N}_{\frac{r_2}{2C_e}} (\partial^*_G \phi(x_c))$ is convex, the segment connecting the two points

$$\left(exp^G_{x_t, \phi(x_t)}\right)^{-1} (y_0) \text{ and } \left(exp^G_{x_t, \phi(x_t)}\right)^{-1} (y_1)$$

lies in $\mathcal{N}_{\frac{r_2}{2C_e}} (\partial^*_G \phi(x_c))$. Therefore, the G-segment y_θ connecting y_0, y_1 with focus $(x_t, \phi(x_t))$ is in $\frac{r_2}{2}$ neighborhood of $\partial_G \phi(x_c)$:

$$y_\theta \in \mathcal{N}_{\frac{r_2}{2}} \left(\partial_G \phi(x_c)\right).$$

Therefore, (4.23) implies (4.22). $\square$

Remark 4.4.3. The constant $\delta(x_c)$ depends on the modulus of continuity of ϕ and $\partial_G \phi$ at x_c.

Remark 4.4.4. In the proof of Lemma 4.4.2, we have used that X is a domain in $\mathbb{R}^n$ so that we can use the identification $T^*X = X \times \mathbb{R}^n$. In the general manifold setting, where we do not have this trivialization, we should choose $\delta(x_c)$ small enough so that we have the local trivialization $T^*B_{\delta(x_c)}(x_c) = B_{\delta(x_c)}(x_c) \times \mathbb{R}^n$.

Lemma 4.4.5. *Let ϕ be a G-convex function and let x_c be an interior point. Choose x_0 and x_1 such that $|x_i - x_c| < \delta(x_0)$. Let $G(x, y_0, z_0)$ and $G(x, y_1, z_1)$ be G-affine functions that support ϕ at x_0 and x_1 respectively. Then there exists $x_t \in [x_0, x_1]$ such that*

$$G(x_t, y_0, z_0) = G(x_t, y_1, z_1) =: u.$$

We assume $|y_1 - y_0| \geq |x_0 - x_1|$. Then we have

$$\phi(x_t) - u \leq C_3|x_1 - x_0||y_1 - y_0| \tag{4.25}$$

where C_3 depends on the C^2 norm of G.

Proof. First of all, we show the existence of the point x_t. By the definition of supporting function, we have

$$G(x_0, y_0, z_0) - G(x_0, y_1, z_1) = \phi(x_0) - G(x_0, y_1, z_1) \geq 0,$$
$$G(x_1, y_0, z_0) - G(x_1, y_1, z_1) = G(x_1, y_0, z_0) - \phi(x_1) \leq 0.$$

Therefore x_t exists by the intermediate value theorem. If t was either 1 or 0, then the left hand side of (4.25) is 0. Otherwise, by our choice of x_t and u, we have

$$u = G(x_t, y_0, z_0) = G(x_t, y_0, H(x_0, y_0, \phi(x_0))).$$

Then by Lemma 4.4.2, we know $(x_t, y_i, u) \in S, i = 0, 1$. We use Taylor expansion on

$G(x, y_i, z_i)$ at $x = x_t$ to obtain

$$\phi(x_i) - u \leq \langle D_x G(x_t, y_i, z_i), x_i - x_t \rangle + \frac{1}{2} \|D^2_{xx} G\| |x_i - x_t|^2. \tag{4.26}$$

Also, from Proposition 4.4.1,

$$\begin{aligned} \phi(x_t) - u &\leq (1-t)(\phi(x_0) - u) + t(\phi(x_1) - u) + \frac{1}{2} t(1-t) \|D^2_{xx} G\| |x_1 - x_0|^2 \\ &\leq (1-t)(\phi(x_0) - u) + t(\phi(x_1) - u) + \frac{1}{8} \|D^2_{xx} G\| |x_1 - x_0|^2. \end{aligned} \tag{4.27}$$

If $(1-t)\langle D_x G(x_t, y_0, z_0), x_0 - x_t \rangle + t(\phi(x_1) - u) \leq 0$, then from (4.26) and (4.27), we obtain

$$\begin{aligned} \phi(x_t) - u &\leq (1-t) \left(\langle D_x G(x_t, y_0, z_0), x_0 - x_t \rangle + \frac{1}{2} \|D^2_{xx} G\| |x_0 - x_t|^2 \right) \\ &\quad + t(\phi(x_1) - u) + \frac{1}{8} \|D^2_{xx} G\| |x_1 - x_0|^2 \\ &\leq \frac{1}{2}(1-t) \|D^2_{xx} G\| |x_0 - x_t|^2 + \frac{1}{8} \|D^2_{xx} G\| |x_1 - x_0|^2 \\ &\leq \frac{5}{8} \|D^2_{xx} G\| |y_1 - y_0| |x_1 - x_0|. \end{aligned}$$

Otherwise, since $t(1-t) \leq 1$, we have

$$\begin{aligned} 0 &\leq (1-t)\langle D_x G(x_t, y_0, z_0), x_0 - x_t \rangle + t(\phi(x_1) - u) \\ &\leq \frac{1}{t} \langle D_x G(x_t, y_0, z_0), x_0 - x_t \rangle + \frac{1}{1-t}(\phi(x_1) - u), \end{aligned}$$

so that

$$\begin{aligned} \phi(x_t) - u &\leq (1-t)\langle D_x G(x_t, y_0, z_0), x_0 - x_t \rangle + t(\phi(x_1) - u) \\ &\quad + \left(\frac{1}{8} + \frac{1}{2}(1-t) \right) \|D^2_{xx} G\| |x_1 - x_0|^2 \\ &\leq \frac{1}{t} \langle D_x G(x_t, y_0, z_0), x_0 - x_t \rangle + \frac{1}{1-t}(\phi(x_1) - u) \end{aligned}$$

$$+\left(\frac{1}{8}+\frac{1}{2}(1-t)\right)\|D_{xx}^2G\||x_1-x_0|^2.$$

Here, we use $t=\dfrac{|x_t-x_0|}{|x_1-x_0|}$, $1-t=\dfrac{|x_1-x_t|}{|x_1-x_0|}$, and Taylor expansion to obtain

$$\begin{aligned}\phi(x_t)-u \leq &\langle D_xG(x_t,y_0,z_0),x_0-x_1\rangle+\langle D_xG(x_t,y_1,z_1),x_1-x_0\rangle\\ &+\frac{1}{2}\|D_{xx}^2G\||x_1-x_t||x_1-x_0|+\frac{5}{8}\|D_{xx}^2G\||x_1-x_0|^2.\end{aligned}$$

Now, we use the fundamental theorem of calculus to obtain

$$\begin{aligned}\phi(x_t)-u &\leq \int_0^1 \frac{d}{d\theta}\langle D_xG(x_t,y_\theta,H(x_t,y_\theta,u)),x_1-x_0\rangle d\theta\\ &\quad+\frac{9}{8}\|D_{xx}^2G\||x_1-x_0|^2\\ &=\int_0^1\langle E(x_t,y_\theta,H(x_t,y_\theta,u))\frac{d}{d\theta}y_\theta,x_1-x_0\rangle\\ &\quad+\frac{9}{8}\|D_{xx}^2G\||x_1-x_0|^2\\ &\leq C_e^3|y_1-y_0||x_0-x_t|+\frac{9}{8}\|D_{xx}^2G\||x_1,x_0|^2\\ &\leq\left(C_e^3+\frac{9}{8}\|D_{xx}^2G\|\right)|y_1-y_0||x_1-x_0|\end{aligned}$$

where y_θ is the G-segment connecting y_0 and y_1 with focus (x_t,u). □

Lemma 4.4.6. *Let x_t be as in Lemma 4.4.5. There exist l,r that depend on $|x_1-x_0|$ and $|y_1-y_0|$ and κ such that if $\mathcal{N}_r([x_0,x_1])\subset X$ and*

$$|y_1-y_0|\geq\max\{|x_1-x_0|,\kappa|x_1-x_0|^{1/5}\} \tag{4.28}$$

then , choosing x_1 close to x_0 if necessary, we have

$$\mathcal{N}_l\left(\left\{y_\theta,|\theta\in\left[\frac{1}{4},\frac{3}{4}\right]\right\}\right)\cap Y\subset\partial_G\phi(B_r(x_t))$$

where y_θ is the G-segment connecting y_0 and y_1 with focus (x_t, u) as in the proof of Lemma 4.4.5.

Proof. Let u be as in the proof of Lemma 4.4.5:

$$G(x_t, y_0, z_0) = G(x_t, y_1, z_1) = u.$$

Note that by Lemma 4.4.2, $(x_t, y_\theta, u) \in S \Subset \mathfrak{h}$. By G-convexity of ϕ and Lemma 4.3.3, we have

$$\phi(x) \geq G(x, y_\theta, z_\theta) + \frac{3}{16}\delta_0|y_1 - y_0|^2|x - x_t|^2 - \gamma|x - x_t|^3 \tag{4.29}$$

for $\theta \in \left[\frac{1}{4}, \frac{3}{4}\right]$, $|x - x_t| \leq \frac{1}{4}C$ (notice that we can choose in 4.3.3) where $z_\theta = H(x_t, y_\theta, u)$. Next, we look at the G-affine function $G(x, y, H(x_t, y, \phi(x)))$ on the boundary of the ball $B_r(x_t)$ to compare the value with ϕ. Let $z(y, u) = H(x_t, y, u)$.

$$\begin{aligned} G(x, y, z(y, \phi(x_t))) =& G(x, y, z(y, \phi(x_t))) - G(x, y_\theta, z(y_\theta, \phi(x_t))) \\ &+ G(x, y_\theta, z(y_\theta, \phi(x_t))) - G(x, y_\theta, z(y_\theta, u)) \\ &+ G(x, y_\theta, z(y_\theta, u)). \end{aligned} \tag{4.30}$$

For the first line, noting that $\phi(x_t) = G(x_t, y, z(y, \phi(x_t)))$, we have

$$\begin{aligned} &G(x, y, z(y, \phi(x_t))) - \phi(x_t) \\ &= \int_0^1 \frac{d}{ds}\left(G(x_t + s(x - x_t), y, z(y, \phi(x_t)))\right) ds \\ &= \int_0^1 \langle D_x G(x_t + s(x - x_t), y, z(y, \phi(x_t))), x - x_t\rangle ds \end{aligned} \tag{4.31}$$

and similar equation holds for $G(x, y_\theta, z(y_\theta, \phi(x_t))) - \phi(x_t)$. Therefore, we have

$$G(x, y, z(y, \phi(x_t))) - G(x, y_\theta, z(y_\theta, \phi(x_t)))$$

$$
\begin{aligned}
&= \int_0^1 \left\langle \begin{pmatrix} D_xG(x_t + s(x-x_t), y, z(y, \phi(x_t))) \\ -D_xG(x_t + s(x-x_t), y_\theta, z(y_\theta, \phi(x_t))) \end{pmatrix}, x - x_t \right\rangle ds \\
&= \int_0^1 \int_0^1 \frac{d}{ds'} \left\langle \left(D_xG \begin{pmatrix} x_t + s(x-x_t), y_\theta + s'(y - y_\theta), \\ z(y_\theta + s'(y-y_\theta), \phi(x_t)) \end{pmatrix} \right), x - x_t \right\rangle ds'ds \\
&= \int_0^1 \int_0^1 \left(D_{xy}G + D_{xz}G \otimes D_yH\right)[y - y_\theta, x - x_t] ds'ds \qquad (4.32) \\
&= \int_0^1 \int_0^1 \left(D_{xy}G + D_{xz}G \otimes \frac{D_yG}{-D_zG}\right)[y - y_\theta, x - x_t] ds'ds \\
&\leq C_4|x - x_t||y - y_\theta|,
\end{aligned}
$$

where C_4 depends on the C^2 norm of G and β. Note that the functions in the last integral are evaluated at different points so that we can not simply bound the functions by C_e. For the second line of (4.30), we use (4.25).

$$
\begin{aligned}
&G(x, y_\theta, z(y_\theta, \phi(x_t))) - G(x, y_\theta, z(y_\theta, u)) \\
&= \int_0^1 \frac{d}{ds}\left(G(x, y_\theta, z(y_\theta, u + s(\phi(x_t) - u))\right) ds \\
&= \int_0^1 D_zGD_uH(\phi(x_t) - u)ds \qquad (4.33) \\
&\leq C_5'|\phi(x_t) - u| \\
&\leq C_5|x_1 - x_0||y_1 - y_0|,
\end{aligned}
$$

where C_5 depends on the C^1 norm of G, β, and C_3. We apply (4.32) and (4.33) to (4.30) to obtain

$$
\begin{aligned}
G(x, y, z(y, \phi(x_t))) \leq &G(x, y_\theta, z(y_\theta, u)) + C_4|x - x_t||y - y_\theta| \\
&+ C_5|x_1 - x_0||y_1 - y_0|. \qquad (4.34)
\end{aligned}
$$

We compare (4.29) and (4.34). If we have

$$
\begin{aligned}
&C_4|x-x_t||y-y_\theta| + C_5|x_1-x_0||y_1-y_0| \\
&\le \frac{3}{16}\delta_0|y_1-y_0|^2|x-x_t|^2 - \gamma|x-x_t|^3,
\end{aligned} \tag{4.35}
$$

then we obtain $G(x,y,z(y,\phi(x_t))) \le \phi(x)$. Note that (4.35) is satisfied if we have

$$
C_5|x_1-x_0||y_1-y_0| \le \frac{1}{16}\delta_0|y_1-y_0|^2|x-x_t|^2, \tag{4.36}
$$

$$
C_4|x-x_t||y-y_\theta| \le \frac{1}{16}\delta_0|y_1-y_0|^2|x-x_t|^2, \tag{4.37}
$$

$$
\gamma|x-x_t|^3 \le \frac{1}{16}\delta_0|y_1-y_0|^2|x-x_t|^2. \tag{4.38}
$$

Therefore, we choose

$$
r^2 = \frac{16C_5}{\delta_0}\frac{|x_1-x_0|}{|y_1-y_0|}, \quad l = \frac{\delta_0}{16C_4}r|y_1-y_0|^2, \quad \kappa = \left(\frac{16^3\gamma^2C_5}{\delta_0^3}\right)^{\frac{1}{5}}. \tag{4.39}
$$

Note that the choice of r gives (4.36), then the choice of l gives (4.37). The choice of r and κ gives (4.38). Therefore we obtain

$$
G(x,y,z(y,\phi(x_t))) \le \phi(x)
$$

for $y \in \mathcal{N}_l\left(\{y_\theta | \theta \in [\frac{1}{4},\frac{3}{4}]\}\right)$ and $x \in \partial B_r(x_t)$. Note that κ does not depend on x_0 and x_1. From the condition (4.28), we have

$$
r^2 \le \frac{16C_5}{\kappa\delta_0}|x_1-x_0|^{\frac{4}{5}},\ l \le \frac{\sqrt{\delta_0C_5}}{4C_4}|x_1-x_0|^{\frac{1}{2}}\mathrm{diam}(Y)^{\frac{3}{2}}.
$$

Hence, if we choose x_1 so that $|x_1-x_0| \le \dfrac{4C_4^2r_2^2}{\mathrm{diam}(Y)^3\delta_0C_5}$, we have

$$
l \le \frac{r_2}{2} \tag{4.40}
$$

where r_2 is from the proof of Lemma 4.4.2. Since $G(x_t, y, z(y, \phi(x_t))) = \phi(x_t)$, we get a local maximum of $G(\cdot, y, z(y, \phi(x_t))) - \phi(\cdot)$ at some point $x_y \in B_r(x_t)$ with non-negative value. Then from the proof of Lemma 4.4.2, we obtain

$$\mathcal{N}_l\left(\{y_\theta | \theta \in [0,1]\}\right) \cap Y \subset \mathfrak{h}_{x_y, \phi(x_t)}.$$

If $G(x_y, y, z(y, \phi(x_t))) = \phi(x_y)$, then $G(\cdot, y, z(y, \phi(x_t)))$ is a local support of ϕ at x_y and Proposition 4.2.7 implies that $y \in \partial_G \phi(x_y) \subset \partial_G \phi(B_r(x_t))$. Otherwise, we have the strict inequality $G(x_y, y, z(y, \phi(x_t))) > \phi(x_y)$. In addition, we know $\phi(\cdot) \geq G(\cdot, y, z(y, u))$. We define a function u_y by

$$\begin{aligned} u_y(h) &= \max_{x \in B_r(x_t)} \{G(x, y, z(y, h)) - \phi(x)\} \\ &= \max_{x \in B_r(x_t)} \{G(x, y, H(x_t, y, h)) - \phi(x)\}. \end{aligned}$$

Then $u_y(\phi(x_t)) > 0$ and $u_y(u) \leq 0$. Moreover, since $\|D_v G\|$ and $\|D_u H\|$ are finite on Ψ and Φ, $\{G(x, y, z(y, \cdot)) - \phi(x) | x \in B_r(x_t)\}$ is a family of equicontinuous functions. Therefore, u_y is a continuous function and there exists $h_y \in [u, \phi(x_t)]$ at which we have $u_y(h_y) = 0$. In other words, $G(\cdot, y, z(y, h_y))$ supports ϕ at some point x' in $B_r(x_t)$. From (4.40) and the proof of Lemma 4.4.2, we have $(x', y, h_y) \in \mathfrak{h}$. Hence we obtain $y \in \partial_G \phi(B_r(x_t))$. $\square$

4.5 Proof of the local Holder regularity

The idea of the proof of the main theorem is to use Lemma 4.4.6 to compare the volume of $\mathcal{N}_l\left(\left\{y_\theta, |\theta \in \left[\frac{1}{4}, \frac{3}{4}\right]\right\}\right) \cap Y$ and $B_r(x_t)$. Therefore, we should estimate the volume of the set $\mathcal{N}_l\left(\left\{y_\theta, |\theta \in \left[\frac{1}{4}, \frac{3}{4}\right]\right\}\right) \cap Y$.

Remark 4.5.1. Lemma 3.2.8 and the proof of Lemma 3.2.9 shows that for a compact convex set $A \subset \mathbb{R}^n$, there exist $r_A > 0$ and L_A such that for any $p \in \partial A$ there exists a unit vector v_p such that $\partial A \cap B_{r_A}(p)$ can be written as a graph of a convex function with Lipschitz constant

L_A in a coordinate system that has v_p as a vertical axis. Moreover, we can choose r_A smaller so that for any $r' \leq r_A$ and $p' \in A \cap B_{r_A}(p)$, $B_{r'}(p') \cap A$ contains a conical sector of a certain size.

$$B_{r'}(p') \cap A \supset \left\{ q \in B_{r'}(p') \middle| \langle q - p', v_p \rangle \geq \frac{L_A}{\sqrt{L_A^2 + 1}} \right\}.$$

This implies that if $r' \leq r_A$ then we have

$$\mathrm{Vol}(B_{r'}(p) \cap A) \geq C_A \mathrm{Vol}(B_{r'}(p)) \tag{4.41}$$

for any $p \in A$.

Lemma 4.5.2. *Let A be a compact convex set and let $\gamma : [0,1] \to A$ be a bi-Lipschitz curve, that is*

$$\underline{L}|s - t| \leq |\gamma(s) - \gamma(t)| \leq \overline{L}|s - t|$$

for some constants $\underline{L}$ and $\overline{L}$. Then there exist K_A and $l_A > 0$ that depend on A, $\overline{L}$, and $\underline{L}$ such that for any $l \leq l_A$, we have

$$\mathrm{Vol}(\mathcal{N}_l(\gamma) \cap A) \geq K_A \underline{L} l^{n-1}. \tag{4.42}$$

Before we start the proof, note that the curve γ does not have to be differentiable. Therefore, we give the next definition.

Definition 4.5.3. Let $\gamma : [0,1] \to \mathbb{R}^n$ be a continuous curve. We define its length by

$$\mathrm{Length}(\gamma) = \sup \left\{ \sum_{i=1}^{n} |\gamma(t_i) - \gamma(t_{i-1})| \middle| a = t_0 \leq t_1 \leq \cdots \leq t_n = b \right\}.$$

It is well known that this definition preserves many properties of arclength of C^1 curves. Note that the length is finite if the curve is Lipschitz.

Proof. We assume $l < \dfrac{r_A}{4}$ where r_A is from Remark 4.5.1. Let $m \in \mathbb{N}$ be the smallest number such that $2\overline{L} \le r_A m$. Then by taking r_A smaller if necessary, we have $m \le \dfrac{4\overline{L}}{r_A}$. Note that we have $\mathrm{Length}(\gamma) \le \overline{L}$. We define

$$\gamma_i(t) = \gamma\left((1-t)\frac{i}{2m} + t\frac{i+1}{2m}\right).$$

Then γ_i is bi-Lipschitz with Lipschitz constants $\dfrac{\underline{L}}{2m}$ and $\dfrac{\overline{L}}{2m}$. In addition, by our choice of m, we have $\mathrm{Length}(\gamma_i) \le \dfrac{r_A}{4}$. Suppose $\mathcal{N}_l(\gamma_i) \cap \partial A \neq \emptyset$. Then we can write ∂A as a graph of a Lipschitz convex function f_A with Lipschitz constant L_A in $B_{r_A}(p)$ for some point $p \in \mathcal{N}_l(\gamma_i) \cap \partial A$. Moreover, for any $p' \in \mathcal{N}_l(\gamma_i)$, there are some $t, s \in [0,1]$ such that

$$\begin{aligned}|p' - p| &\le |p' - \gamma_i(t)| + |\gamma_i(t) - \gamma_i(s)| + |\gamma_i(s) - p| \\ &\le \frac{r_A}{4} + \frac{r_A}{4} + \frac{r_A}{4} = \frac{3}{4} r_A.\end{aligned}$$

Therefore $\mathcal{N}_l(\gamma_i) \cap A$ lies in the epigraph of f_A in $B_{r_A}(p)$. Then Remark 4.5.1 shows that at each t, there exists a conical sector $\mathrm{Sec}_{\gamma_i(t)}$ with vertex $\gamma_i(t)$ in $B_l(\gamma_i(t)) \cap A$ which is a translation of the conical sector Sec_0:

$$\mathrm{Sec}_0 = B_l(0) \cap \left\{ p' \middle| \langle p', v_p \rangle \ge \frac{L_A}{\sqrt{L_A^2 + 1}} \right\}.$$

Note that the inscribed ball in this conical sector has radius $L'_A l$ where L'_A is a constant that depends on L_A so that the inscribed ball is $B_{L'_A l}(q)$ for some $q \in \mathrm{Sec}_0$. Therefore, for each $0 \le i \le 2m-1$, we get q_i such that

$$\mathcal{N}_{L'_A l}(\gamma_i + q_i) \subset \mathcal{N}_l(\gamma) \cap A.$$

Now, if we have $l < \frac{\underline{L}}{4m}$, then for any $p \in \mathcal{N}_l(\gamma_i)$ and $p' \in \mathcal{N}_l(\gamma_j)$ where $|i - j| \ge 2$, we obtain that for some $s, t \in [0,1]$,

$$|p-p'| \geq |\gamma_i(t)-\gamma_j(s)| - (|p-\gamma_i(t)| + |p'-\gamma_j(s)|)$$
$$\geq \frac{\underline{L}}{2m} - 2l > 0.$$

Therefore, $\mathcal{N}_{L'_A l}(\gamma_i) \cap \mathcal{N}_{L'_A l}(\gamma_j) = \emptyset$. Note that each $\mathcal{N}_{L'_A l}(\gamma_i)$ has a volume bounded below by $(L'_A l)^{n-1}|\gamma_i(0)-\gamma_i(1)| \geq \frac{(L'_A)^{n-1}\underline{L}}{2m} l^{n-1}$ so that

$$\begin{aligned}\mathrm{Vol}(\mathcal{N}_l(\gamma) \cap A) &\geq \mathrm{Vol}\left(\bigcup_{i=0}^{2m-1} \mathcal{N}_{L'_A l}(\gamma_i + q_i)\right)\\ &\geq \mathrm{Vol}\left(\bigcup_{i=0}^{m-1} \mathcal{N}_{L'_A l}(\gamma_{2i} + q_{2i})\right)\\ &\geq \frac{(L'_A)^{n-1}\underline{L}}{2m} l^{n-1} \times m = \frac{1}{2}(L'_A)^{n-1}\underline{L} l^{n-1}.\end{aligned}$$

Therefore we get the lemma with $l_A = \min\{\frac{r_A}{4}, \frac{\underline{L}}{4m}\}$ and $K_A = \frac{1}{2}(L'_A)^{n-1}$. □

Lemma 4.5.4. *Let y_θ be as in Lemma 4.4.6 and let $A = \mathfrak{h}^*_{x_c,\phi(x_c)}$. If $l \leq l_A$, then we have*

$$\mathrm{Vol}\left(\mathcal{N}_l\left(\left\{y_\theta \middle| \theta \in \left[\frac{1}{4}, \frac{3}{4}\right]\right\}\right) \cap Y\right) \geq C_V l^{n-1}|y_0 - y_1| \tag{4.43}$$

where C_V depends on x_c, $\mathfrak{h}$, and C_e.

Proof. Note that $\theta \mapsto y_\theta$ is a bi-Lipschitz curve with

$$\frac{1}{C_e^2}|y_1-y_0||\theta-\theta'| \leq |y_\theta - y_{\theta'}| \leq C_e^2|y_1-y_0||\theta-\theta'|.$$

Therefore, the reparametrized curve $\theta \mapsto y_{(1-\theta)\frac{1}{4}+\theta\frac{3}{4}}$ is bi-Lipschitz with Lipschitz constants $\frac{2}{C_e^2}|y_1-y_0|$ and $2C_e^2|y_1-y_0|$. Then the curve

$$\theta \mapsto -\frac{D_x H}{D_u H}\left(x_c, y_{(1-\theta)\frac{1}{4}+\theta\frac{3}{4}}, \phi(x_c)\right)$$

is bi-Lipschitz in $\mathfrak{h}^*_{x_c,\phi(x_c)}$ with Lipschitz constants

$$\underline{L} = \frac{2}{C_e^3}|y_1 - y_0| \text{ and } \overline{L} = 2C_e^3|y_1 - y_0|.$$

Moreover, the function $-\dfrac{D_x H}{D_u H}(x_c, \cdot, \phi(x_c)) = {exp^G_{x_c,\phi(x_c)}}^{-1}(\cdot)$ is bi-Lipschitz with Lipschitz constants $\frac{1}{C_e}$ and C_e, so that we have

$$\begin{aligned}\mathcal{N}_{\frac{l}{C_e}}\left({exp^G_{x_c,\phi(x_c)}}^{-1}\left(\left\{y_\theta \middle| \theta \in \left[\frac{1}{4}, \frac{3}{4}\right]\right\}\right)\right) \cap \mathfrak{h}^*_{x_c,\phi(x_c)} \\ \subset {exp^G_{x_c,\phi(x_c)}}^{-1}\left(\mathcal{N}_l\left(\left\{y_\theta \middle| \theta \in \left[\frac{1}{4}, \frac{3}{4}\right]\right\}\right) \cap Y\right).\end{aligned}$$

Note that by (vDomConv), $\mathfrak{h}^*_{x_c,\phi(x_c)}$ is convex. From Lemma 4.5.2, we obtain a constant K_{x_c} that depends on $\mathfrak{h}^*_{x_c,\phi(x_c)}$ such that

$$\begin{aligned}\mathrm{Vol}\left(\mathcal{N}_{\frac{l}{C_e}}\left({exp^G_{x_c,\phi(x_c)}}^{-1}\left(\left\{y_\theta, \middle| \theta \in \left[\frac{1}{4}, \frac{3}{4}\right]\right\}\right)\right) \cap \mathfrak{h}^*_{x_c,\phi(x_c)}\right) \\ \geq K_{x_c}\frac{2}{C_e^3}|y_1 - y_0|\left(\frac{l}{C_e}\right)^{n-1}.\end{aligned}$$

We use bi-Lipschitzness once more to obtain

$$\mathrm{Vol}\left(\mathcal{N}_l\left(\left\{y_\theta \middle| \theta \in \left[\frac{1}{4}, \frac{3}{4}\right]\right\}\right) \cap Y\right) \geq C_V l^{n-1}|y_1 - y_0|,$$

with $C_V = \dfrac{2K_{x_c}}{C_e^{2n+2}}$. □

Remark 4.5.5. The constant C_V in Lemma 4.5.4 depends on the set $\mathfrak{h}_{x_0,\phi(x_0)}$, in particular, on r_A and Lipschitz constant L_A of the boundary $\partial\mathfrak{h}_{x_c,\phi(x_c)}$. Therefore, if we assume that the constants r_A and L_A are uniform over $\{\mathfrak{h}^*_{x,u}\}_{(x,u)\in X\times[\min a, \max b]}$, the constant C_V does not depend on x_c and $\phi(x_c)$.

Remark 4.5.6. In the proof of the main theorem of this chapter, we use Lemma 4.4.2, Lemma

4.4.6, and Lemma 4.5.4. Therefore, we should choose x_c in X and choose x_0 and x_1 close enough to x_c so that $|x_i - x_c|$ satisfies the assumptions for the lemmas. In particular, we assume $|x_i - x_c| < \delta(x_c)$ to use Lemma 4.4.2, $|x_i - x_c| < \dfrac{2C_4^2 r_2^2}{\mathrm{diam}(Y)^3 \delta_0 C_5}$ to use Lemma 4.4.6, and $|x_i - x_c| < \dfrac{8C_4^2 l_A^2}{C_5 \mathrm{diam}(Y)^3}$ so that l from (4.39) is smaller than l_A in Lemma 4.5.4 with $A = \mathfrak{h}^*_{x_0,\phi(x_0)}$. On the other hand, we also need to assume that $|y_1 - y_0| \geq \max\{|x_1 - x_0|, \kappa|x_1 - x_0|^{1/5}\}$ to use Lemma 4.4.6. Note that if we have points (x_0, y_0) and (x_1, y_1) that do not satisfy this assumption, then we already obtain an inequality for $\frac{1}{5}$-Hölder regularity at these points.

Proof of the main theorem of chapter 4. Let x_c be an interior point of X and choose x_0 and x_1 close to x_c as we have discussed in Remark 4.5.6. Let $y_i \in \partial_G \phi(x_i)$.

Case 1) We deal with the first case of the Theorem. We separate the case $p = \infty$ and $p < \infty$. If $p = \infty$, then we have

$$\mu(B_r(x_t)) \leq C\mathrm{Vol}(B_r(x_t)) \leq C' r^n$$

for some C and C'. Moreover, since ϕ is an Alexandrov solution, Lemma 4.4.6 and Lemma 4.5.4 imply

$$\begin{aligned} \mu(B_r(x_t)) = \nu(\partial_G \phi(B_r(x_t))) &\geq \nu\left(\mathcal{N}_l\left(\left\{y_\theta \middle| \theta \in \left[\frac{1}{4}, \frac{3}{4}\right]\right\}\right) \cap Y\right) \\ &\geq \underline{\nu} C_V l^{n-1} |y_1 - y_0|, \end{aligned} \tag{4.44}$$

where $\underline{\nu} > 0$ is a lower bound of ν with respect to the Lebesgue measure dy, that is $\nu \geq \underline{\nu} dy$. Therefore, we obtain $C' r^n \geq \underline{\nu} C_V l^{n-1} |y_1 - y_0|$. We plug the values of r and l from (4.39) and rearrange to obtain

$$|y_1 - y_0| \leq C|x_1 - x_0|^{\frac{1}{4n-1}}$$

for some constant C. Note that this implies single valuedness and Hölder continuity of $\partial_G \phi$.

Next, we see the case $p < \infty$. In this case, we define

$$F_\mu(V) = \sup\{\mu(B) | B \subset X \text{ a ball of volume } V\}. \tag{4.45}$$

Then we have $F_\mu(\mathrm{Vol}(B_r(x_t))) \geq \mu(B_r(x_t)) = \nu(\partial_G \phi(B_r(x_t)))$. Thus (4.44) implies

$$F_\mu\left(C\frac{|x_1 - x_0|^{\frac{n}{2}}}{|y_1 - y_0|^{\frac{n}{2}}}\right) \geq C'|x_1 - x_0|^{\frac{n-1}{2}}|y_1 - y_0|^{\frac{3n-1}{2}} \tag{4.46}$$

for some constants C and C'. From the condition we have imposed on μ, we have $F(V) \leq C''V^{1-\frac{1}{p}}$ for some C''. This inequality and (4.46) shows that

$$|y_1 - y_0|^{2n-1+\frac{1}{2}\left(1-\frac{n}{p}\right)} \leq C|x_1 - x_0|^{\frac{1}{2}\left(1-\frac{n}{p}\right)}.$$

Therefore, since $p > n$, we get

$$|y_1 - y_0| \leq C|x_1 - x_0|^{\frac{\rho}{4n-2+\rho}}$$

where $\rho = 1 - \frac{n}{p}$. Therefore, we have that for any interior point x_c of X, there exists some constants r_{x_c} and C_{x_c} that depends on x_c, $\phi(x_c)$, continuity of ϕ at x_c such that if $|x_i - x_c| < r_{x_c}$,$i = 0, 1$, we have

$$|y_1 - y_0| \leq C_{x_c}|x_1 - x_0|^{\frac{\rho}{4n-2+\rho}}. \tag{4.47}$$

Note that this inequality shows the single valuedness of G-subdifferential $\partial_G\phi$. Therefore, for any x_c in the interior of X, there exists a ball around x_c on which the function $\partial_G\phi$ is Hölder continuous, hence $\partial_G\phi$ is locally Hölder continuous. To obtain the Hölder regularity of the potential ϕ, we note that $\partial_G\phi(x) = \exp^G_{x,\phi(x)}(D_x\phi(x))$, and Remark 4.2.8.

Case 2) Now we prove the second part of the theorem. Suppose we have $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{r\to 0} f(r) = 0$ and for any $x \in X$ and $r \geq 0$ we have $\mu(B_r(x)) \leq f(r)r^{n\left(1-\frac{1}{n}\right)}$. Note that we can choose f strictly increasing. Then by (4.46), we have

$$F_\mu(V) \leq f\left(\left(\frac{1}{\omega_n}\right)^{\frac{1}{n}} V^{\frac{1}{n}}\right) \times \left(\frac{1}{\omega_n}\right)^{1-\frac{1}{n}} V^{1-\frac{1}{n}} \tag{4.48}$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. Define $\overline{f}$ by

$$\overline{f}(V)^{2n-1} = \left(\frac{1}{\omega_n}\right)^{1-\frac{1}{n}} f\left(\left(\frac{1}{\omega_n}\right)^{\frac{1}{n}} V^{\frac{1}{2}}\right).$$

Then (4.48) becomes

$$F_\mu(V) \leq \left(\overline{f}\left(V^{\frac{2}{n}}\right)\right)^{2n-1} V^{1-\frac{1}{n}}. \tag{4.49}$$

We combine (4.49) with (4.46) to obtain

$$\overline{f}\left(C'\frac{|x_1 - x_0|}{|y_1 - y_0|}\right) \geq C''|y_1 - y_0| \tag{4.50}$$

for some constants $C', C'' > 0$. Note that we can assume that $\dfrac{|x_1 - x_0|}{|y_1 - y_0|} \to 0$ as $|x_1 - x_0| \to 0$ because otherwise, we obtain a Lipschitz estimate. Then (4.50) implies that $\partial_G\phi$ is a single valued map. Let g be the modulus of continuity of G-subdifferential map $\partial_G\phi$ at x_0. Note that if $g(u) \leq \max\{u, \kappa u^{\frac{1}{5}}\}$, then we get $g(u) \to 0$ as $u \to 0$. In the other case, the pairs (x_0, y_0) and (x_1, y_1) satisfies the assumption of Lemma 4.4.6 and we can apply (4.50) to obtain

$$\overline{f}\left(C'\frac{u}{g(u)}\right) \geq C''g(u).$$

Since f is strictly increasing, so is $\overline{f}$, so that $\overline{f}$ is invertible. Therefore, we obtain

$$u \geq \overline{f}^{-1}\left(C''g(u)\right)\frac{g(u)}{C'}.$$

Let ω be the inverse of $z \mapsto \overline{f}^{-1}(C''z)\frac{z}{C'}$. Note that ω is strictly increasing. Therefore, composing ω with the above inequality shows that

$$g(u) \leq \omega(u).$$

Since the function $z \mapsto \overline{f}^{-1}(C''z)\frac{z}{C'}$ is strictly increasing and has limit 0 as $z \to 0$, $\omega(u)$ also has limit 0 as $u \to 0$. Therefore the above inequality implies that $g(u) \to 0$ as $u \to 0$. Hence the modulus of continuity of $\partial_G \phi$ has limit 0 as the variable tends to 0 so that $\partial_G \phi$ is continuous at x_0. $\square$

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