ON A FAMILY OF INTEGRAL OPERATORS ON THE BALL

By

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ABSTRACT

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In this dissertation, we transform the equation in the upper half space first studied by Caffarelli and Silvestre to an equation in the Euclidean unit ball $\mathbb{B}^n$. We identify the Poisson kernel for the equation in the unit ball. Using the Poisson kernel, we define the extension operator. We prove an extension inequality in the limit case and identify the extremal functions using the method of moving spheres. In addition we offer an interpretation of the limit case inequality as a conformally invariant generalization of Carleman's inequality.

To my parents.

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CHAPTER 1

INTRODUCTION

1.1 Sharp Integral Inequalities for Harmonic Functions

The Laplace equation with Dirichlet boundary in the unit ball is a well known equation. We can solve it with the help of the Poisson kernel. We use $\mathbb{B}^n$ to denote the unit ball in $\mathbb{R}^n$ and $\mathbb{S}^{n-1}$ to denote the unit sphere in $\mathbb{R}^n$. For $\widetilde{f}:\mathbb{S}^{n-1}\to\mathbb{R}$ regular enough, the equation

$$\begin{cases}\Delta\widetilde{u}=0, \text{ in } \mathbb{B}^n\\ \widetilde{u}=\widetilde{f}, \text{ on } \mathbb{S}^{n-1}\end{cases}$$

has the unique solution

$$\widetilde{u}(x)=\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}}\frac{1-|x|^2}{|x-\xi|^n}\widetilde{f}(\xi)d\xi.$$

We can consider the extension operation as an operator that maps functions on the unit sphere to functions in the unit ball. For any $\widetilde{f}$ define

$$\widetilde{P}\widetilde{f}(x)=\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}}\frac{1-|x|^2}{|x-\xi|^n}\widetilde{f}(\xi)d\xi,$$

then it is easy to see that

$$\left\|\widetilde{P}\widetilde{f}\right\|_{L^\infty(\mathbb{B}^n)}\le\left\|\widetilde{f}\right\|_{L^\infty(\mathbb{S}^{n-1})}.$$

This is the well known maximum principle.

In general we want to ask whether we can prove that the extension operator $\widetilde{P}$ is a bounded operator from $L^p(\mathbb{S}^{n-1})$ to $L^q(\mathbb{B}^n)$ for some p and q. Hang, Wang and Yan considered this problem in both the upper half space and the unit ball.

In the upper half space, Hang, Wang and Yan [17] prove that the extension operator $P:L^p\left(\mathbb{R}^{n-1}\right)\to L^{\frac{np}{n-1}}\left(\mathbb{R}^n_+\right)$ is bounded for $1<p<\infty$. Here the operator P is defined as follows, for any $f:\mathbb{R}^{n-1}\to\mathbb{R}$

$$Pf(y)=\frac{2}{n\omega_n}\int_{\mathbb{R}^{n-1}}\frac{y_n}{\left(|y'-w|^2+y_n^2\right)^{n/2}}f(w)dw.$$

Here $y = (y', y_n)$ denote points in the upper half space such that $y' \in \mathbb{R}^{n-1}$ and $y_n > 0$, $w \in \mathbb{R}^{n-1}$, dw denotes the volume form on $\mathbb{R}^{n-1}$ with the Euclidean metric. When $p = \frac{2(n-1)}{n-2}$, there is conformal symmetry in the system, they even identify the sharp constant and classify all the extremal functions. In particular they proved the following

Theorem 1.1. *[17, Theorem 1.1] Assume $n \geq 3$; then for any $f \in L^{\frac{2(n-1)}{n-2}}(\mathbb{R}^{n-1})$*

$$\|Pf\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n_+)} \leq n^{-\frac{n-2}{2(n-1)}} \omega_n^{-\frac{n-2}{2n(n-1)}} \|f\|_{L^{\frac{2(n-1)}{n-2}}(\mathbb{R}^{n-1})}. \tag{1.1.1}$$

Moreover, equality holds if and only if

$$f(\xi) = \frac{c}{(\lambda^2 + |\xi - \xi_0|^2)^{(n-2)/2}}$$

for some constant c, positive constant λ, and $\xi_0 \in \mathbb{R}^{n-1}$. Here ω_n is the Euclidean volume of the unit ball in $\mathbb{R}^n$.

Hang, Wang and Yan [17] proved the existence of the maximizer using two different approaches. In the first approach, they proved it with the concentration compactness principle. In the second approach they proved the existence of the maximizer with the method of symmetrization as in [21].

Hang, Wang and Yan [17] also considered the variational problem

$$c_n = \sup\{\|Pf\|_{\frac{2n}{n-2}(\mathbb{R}^n_+)} : f \in L^{\frac{2(n-2)}{n-2}}(\mathbb{R}^{n-1}), \|f\|_{L^{\frac{2(n-1)}{n-2}}(\mathbb{R}^{n-1})} = 1\},$$

derived the Euler-Lagrange equation for nonnegative critical function f

$$f(\xi)^{\frac{n}{n-2}} = \int_{\mathbb{R}^n_+} P(x,\xi)(Pf)(x)^{\frac{n+2}{n-2}} dx, \tag{1.1.2}$$

and proved the classification result for positive critical functions as follows

Theorem 1.2. *[17, Theorem 1.2] Assume $n \geq 3$, $f \in L^{\frac{2(n-1)}{n-2}}(\mathbb{R}^{n-1})$ is nonnegative, not identically zero and satisfies (1.1.2), then for some $\lambda > 0$ and $\xi_0 \in \mathbb{R}^{n-1}$,*

$$f(\xi) = c(n)\left(\frac{\lambda}{\lambda^2 + |\xi - \xi_0|^2}\right)^{\frac{n-2}{2}},$$

where $c(n)$ is determined by the scaling.

In the unit ball, Hang, Wang and Yan also considered the same extension problem. In [18] they proved the following:

Theorem 1.3. *[18, Theorem 3.1] Assume $n \geq 3$, then for every $\widetilde{f} \in L^{\frac{2(n-1)}{n-2}}(\mathbb{S}^{n-1})$,*

$$\|\widetilde{P}\widetilde{f}\|_{L^{\frac{2n}{n-2}}(\mathbb{B}^n)} \leq n^{-\frac{n-2}{2(n-1)}} \omega_n^{-\frac{n-2}{2n(n-1)}} \|\widetilde{f}\|_{L^{\frac{2(n-1)}{n-2}}(\mathbb{S}^{n-1})}. \tag{1.1.3}$$

Here for any $x \in \mathbb{B}^n$

$$\widetilde{P}\widetilde{f}(x) = \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} \frac{1-|x|^2}{|x-\xi|^n} \widetilde{f}(\xi) d\xi \tag{1.1.4}$$

is the harmonic extension of $\widetilde{f}$, ω_n is the volume of the unit ball in $\mathbb{R}^n$ with the Euclidean metric and $|\mathbb{S}^{n-1}|$ is the volume of the unit sphere in $\mathbb{R}^n$ with the standard metric. Equality holds if and only if

$$\widetilde{f}(\xi) = c(1+\xi\cdot\zeta)^{-\frac{n-2}{2}},$$

for some $c \in \mathbb{R}$ and $\zeta \in \mathbb{B}^n$. Note that here $\xi \cdot \zeta$ denotes the inner product between ξ and ζ with respect to the Euclidean metric.

Inequality (1.1.3) can be considered as a direct consequence of inequality (1.1.1), but in [18] Hang, Wang and Yan proved inequality using the Kazdan-Warner type argument.

1.2 An Extension Problem Related to the Fractional Laplacian

In [3] Caffarelli and Silvestre considered an interesting generalization of Laplace equation in the upper half space.

For a given parameter α such that $2-n \leq \alpha < 1$, in $\mathbb{R}^n_+ = \{y = (y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : y_n > 0\}$, consider the following equation

$$\begin{cases} \operatorname{div}(y_n^\alpha \nabla u) = 0, \ \text{for } y \in \mathbb{R}^n_+, \\ u(y',0) = f(y'), \ \text{for } y' \in \mathbb{R}^{n-1} \end{cases} \tag{1.2.1}$$

For the case $-1 < \alpha < 1$, Caffarelli and Silvestre [3] developed the Poisson formula for this equation

$$(P_\alpha f)\left(y', y_n\right) = c_{n,\alpha} \int_{\mathbb{R}^{n-1}} \frac{y_n^{1-\alpha}}{\left(|y'-w|^2 + y_n^2\right)^{\frac{n-\alpha}{2}}} f(w)\, dw, \tag{1.2.2}$$

where

$$c_{n,\alpha}^{-1} = \left|\mathbb{S}^{n-2}\right| \int_0^\infty \frac{r^{n-2} dr}{\left(1+r^2\right)^{\frac{n-\alpha}{2}}} = \frac{\Gamma(\frac{1-\alpha}{2})\Gamma(\frac{n-1}{2})}{2\Gamma(\frac{n-\alpha}{2})} \left|\mathbb{S}^{n-2}\right|. \tag{1.2.3}$$

Caffarelli and Silvestre [3] also found an interesting relation between the Poisson formula (1.2.2) and the fractional Laplacian $(-\Delta)^{(1-\alpha)/2}$ on $\mathbb{R}^{n-1}$.

For a function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ regular enoght, the fractional Laplacian of f can be defined using the formula

$$(-\Delta)^{(1-\alpha)/2} f(w) = C(n,\alpha) \int_{\mathbb{R}^{n-1}} \frac{f(w) - f(v)}{|w-v|^{n-\alpha}} dv,$$

where $-1 < \alpha < 1$, and $C(n,\alpha)$ is some normalization constant. Note that in this article we used the notation $C(n,\alpha)$ to denote any constant that only depends on n and α, and the constant changes through out the article.

In [3], Caffarelli and Silvestre proved that for the extension $u = P_\alpha f$ given in (1.2.2) we have

$$\begin{aligned} C(n,\alpha)(-\Delta)^{(1-\alpha)/2} f(y') &= \lim_{y_n \to 0+} -y_n^\alpha u_{y^n}(y', y_n) \\ &= \frac{1}{1-\alpha} \lim_{y_n \to 0+} \frac{u(y', y_n) - u(y', 0)}{y_n^{1-\alpha}}. \end{aligned}$$

In Proposition B.4 and Remark 13 in the appendix, we offer an alternative explanation of why the function

$$\Gamma(y) = C(n,\alpha) \frac{1}{|y|^{n-2+\alpha}}$$

is a fundamental solution to the equation

$$div(y_n^\alpha \nabla u) = 0$$

as observed by Caffarelli and Silvestre in [3]. Our explanation does not involve fractional dimension and suggests that the hyperbolic space plays a special role in the family of equations (1.2.1).

1.3 Sharp Integral Inequalities with α

Chen [8] considered the extension problem related to the equation (1.2.1). He defined the extension operator $\widetilde{P}_\alpha$ in the unit ball. For any $\widetilde{f} \in L^{\frac{2(n-1)}{n-2+\alpha}}(\mathbb{S}^{n-1})$ he defined the extension in the unit ball to be $\widetilde{P}_\alpha \widetilde{f}$ such that

$$\left(\widetilde{P}_\alpha \widetilde{f}\right) \circ \phi(y', y_n) = \left| \left(y', y_n + \frac{1}{2}\right) \right|^{n+\alpha-2} P_\alpha \left(\frac{\widetilde{f} \circ \phi(w, \frac{1}{2})}{|(w, \frac{1}{2})|^{n+\alpha-2}} \right), \tag{1.3.1}$$

where $(y', y_n) \in \mathbb{R}^n_+$ and $w \in \mathbb{R}^{n-1}$. Here $\phi : \mathbb{R}^n_+ \to \mathbb{B}^n$ is the projection map

$$\phi(y', y_n) = \frac{\left(y', y_n + \frac{1}{2}\right)}{\left|\left(y', y_n + \frac{1}{2}\right)\right|^2} - (0, 1), \tag{1.3.2}$$

where 0 is assumed to be the origin in $\mathbb{R}^{n-1}$. The extension $P_\alpha \left(\frac{\widetilde{f} \circ \phi(w, \frac{1}{2})}{|(w, \frac{1}{2})|^{n+\alpha-2}} \right)$ is as in (1.2.2).

Note that Chen [8] did not give an explicit formula for the operator $\widetilde{P}_\alpha$. We will use a slightly different projection map to define a similar extension operator $\widetilde{P}_\alpha$ and give the explicit formula in (2.3.1). Chen [8] used the definition (1.3.1) to prove the following:

Theorem 1.4. *[8, Theorem 1] For any $\tilde{f} \in L^{\frac{2(n-1)}{n-2+\alpha}}(\mathbb{S}^{n-1})$, $n \geq 2$, $2 - n < \alpha < 1$, we have the sharp inequality*

$$\|\widetilde{P}_\alpha \tilde{f}\|_{L^{\frac{2n}{n-2+\alpha}}(\mathbb{B}^n)} \leq S_{n,\alpha} \|\tilde{f}\|_{L^{\frac{2(n-1)}{n-2+\alpha}}(\mathbb{S}^{n-1})},$$

where the sharp constant $S_{n,\alpha}$ depends only on n and α. The optimizers are unique up to a conformal transform and include the constant function $f = 1$.

The proof of this theorem uses similar ideas as in [17].

Chen [8] also considered the limit case $\alpha \to 2 - n$, and proved the following.

Theorem 1.5. *[8, Theorem 2] For any F such that $e^F \in L^{n-1}(\mathbb{S}^{n-1})$, $n > 2$, we have*

$$\|e^{\widetilde{I}_n + \widetilde{P}_\alpha F}\|_{L^n(\mathbb{B}_n)} \leq S_n \|e^F\|_{L^{n-1}(\mathbb{S}^{n-1})}.$$

where S_n is a sharp constant that only depends on n, and $\widetilde{I}_n$ is a radial function. Up to a conformal transform any constant is an optimizer.

In [8] Chen gave a way to calculate $\widetilde{I}_n$ but did not give the general formula for it. He also did not prove the uniqueness of optimizers in the limit case.

General formula for $\widetilde{I}_n$ in the even dimension can be found in [25, formula (3.35)]. We will also prove an induction relation for $\widetilde{I}_n$ in all dimensions in Lemma 3.2.

1.4 Fractional Laplacian

In this section I will introduce the fractional Laplacian in conformally compact manifolds following [7] and define the adapted metic following [1] and [5]. The adapted metric in the limit case is as in the work of Fefferman and Graham [11]. I will follow [1] and refer to it as the Fefferman-Graham metric. I will also introduce the Sobolev trace inequality of higher orders proved by Ache and Chang [1] and Yang [25]. The higher order Sobolev trace inequality suggests that there is a natural relation between the hyperbolic harmonic extension of functions and the Fefferman-Graham metric in the unit ball.

The fractional Laplacian in conformally compact manifolds is closely related to the work of Caffarelli and Silvestre as we will see in Lemma 1.6. The Fefferman-Graham metric in the unit ball shows up naturally in the proof of Theorem 1.13 which is one of the main theorem of this dissertation.

1.4.1 Conformally Compact Manifold

The notion of conformally compact manifold generalizes the relation between the Euclidean unit ball and the ball model of hyperbolic space. Given any compact $n-$dimensional manifold $\overline{M}^n$ with interior M and boundary $\partial M = \Sigma^{n-1}$. A function $\rho : \overline{M} \to \mathbb{R}$ is called a defining function of Σ in M if

$$\rho > 0 \text{ in } M,\ \rho = 0 \text{ on } \Sigma,\ d\rho \neq 0 \text{ on } \Sigma.$$

Suppose g^+ is a metric on M, we say that g^+ is a conformally compact metric on M with conformal infinity $(\Sigma, [h])$ if there exists a defining function ρ such that the manifold $(\overline{M}, \overline{g})$ is compact with $\overline{g} = \rho^2 g^+$, and $\overline{g}|_\Sigma \in [h]$. Here $\overline{g}|_\Sigma$ denotes the restriction of $\overline{g}$ to the submanifold Σ, and $[h]$ denotes the conformal class containing the metric h.

As an example we can look at the hyperbolic space. Take $M = \mathbb{B}^n$ with boundary $\Sigma = \mathbb{S}^{n-1}$. We can take $g^+ = \frac{4}{(1-|x|^2)^2} dx^2$, where dx^2 is the standard metric in the Euclidean space and $|x|$ denotes the standard Euclidean norm. We can choose the $\rho = \frac{1-|x|^2}{2}$ as the defining function, then we have $\overline{g} = dx^2$ and h is the standard metric on the unit sphere $\mathbb{S}^{n-1}$.

1.4.2 Connection with Scattering Theory

In a given conformally compact manifold, Case and Chang [5] defined the adapted metric using scattering theory.

It is well know that (Mazzeo and Melrose [22], Graham and Zworski [15]) given $f \in C^\infty(\Sigma)$ and $s \in \mathbb{C}$ such that $Re(s) > \frac{n-1}{2}$, $s - \frac{n-1}{2} \notin \mathbb{N} \cup \{0\}$, and $s(n-1-s)$ is not in the pure point spectrum of Δ_{g^+}, the eigenvalue problem

$$-\Delta_{g^+} u - s(n-1-s)u = 0 \text{ in } M \tag{1.4.1}$$

has a unique solution of the form

$$u = F\rho^{n-1-s} + H\rho^s \text{ for } F,\ H \in C^\infty(M) \text{ and } F|_\Sigma = f.$$

Consider the case $f = 1$, and denote the corresponding solution to (1.4.1) as v_s. When $v_s > 0$ in M (this is the case when M is the unit ball and g^+ is the hyperbolic metric), we can use $y_s = (v_s)^{\frac{1}{n-1-s}}$ as a defining function, and the metric $g_s = y_s^2 g^+$ is called the adapted metric. In the limiting case when n is even and $s \to n-1$ we can define

$$\tau = -\frac{d}{ds}\Big|_{s=n-1} v_s$$

then the metric $g^* = e^{2\tau} g^+$ is as in the work of Fefferman and Graham [11], we will refer to it as the Fefferman-Graham metric.

In [7] Chang and González found a relation between the scattering theory and the extension problem related to the fractional Laplacian. They proved the following

Lemma 1.6. *[7, Lemma 4.1] Let (M^n, g^+) be any conformally compact Einstein manifold with boundary Σ^{n-1}. For any $\gamma \in (0, \frac{n-1}{2})$ such that γ is not an integer, denote $s = \frac{n-1}{2} + \gamma$, $\alpha = 1 - 2\gamma$. For any defining function ρ of Σ, the equation*

$$-\Delta_{g^+} u - s(n-1-s)u = 0 \text{ in } (M, g^+) \tag{1.4.2}$$

is equivalent to

$$-div(\rho^{\alpha} \nabla U) + E(\rho)U = 0 \text{ in } (M, \overline{g}), \tag{1.4.3}$$

where

$$\overline{g} = \rho^2 g^+, \; U = \rho^{s-n+1} u$$

and the derivatives in (1.4.3) are taken with respect to the metric $\overline{g}$. The lower order term is given by

$$E(\rho) = -\Delta_{\overline{g}}(\rho^{\frac{\alpha}{2}})\rho^{\frac{\alpha}{2}} + \left(\gamma^2 - \frac{1}{4}\right)\rho^{-2+\alpha} + \frac{n-2}{4(n-1)} R_{\overline{g}} \rho^{a},$$

here $R_{\overline{g}}$ is the scalar curvature of $\overline{g}$. Or equivalently writing everything in g^+, we have

$$E(\rho) = -\Delta_{g^+}(\rho^{\frac{n-2+\alpha}{2}})\rho^{\frac{-n-2+\alpha}{2}} - \left(\frac{(n-1)^2}{4} - \gamma^2\right)\rho^{-2+\alpha}.$$

Chang and González also proved a similar results for hyperbolic upper half space in theorem 3.1 of their paper [7].

As we will see in Remark 2, equation (2.2.1) obtained by me is a special case of equation (1.4.3).

1.4.3 Sobolev Trace Inequality in 4-dimensional Unit Ball

Ache and Chang found an explicit formula for the Fefferman-Graham metric in 4-dimensional hyperbolic unit ball (proposition 2.2 in [1]). Their calculation make use of the connection (Lemma 1.6) discovered by Chang and González [7]. Ache and Chang also proved a very interesting Sobolev trace inequality of order 4:

Theorem 1.7. *[1, Theorem B] Given $f \in C^\infty(\mathbb{S}^3)$, suppose u is a smooth extension of f to the Euclidean unit ball $(\mathbb{B}^4, dx^2)$. If u satisfies the Neumann boundary condition*

$$\frac{\partial}{\partial \nu} u \Big|_{\mathbb{S}^3} = 0, \tag{1.4.4}$$

here ν is the outer unit normal vector with respect to the Euclidean metric, then we have the inequality

$$\log \left(\frac{1}{2\pi^2} \int_{\mathbb{S}^3} e^{3(f-\overline{f})} d\xi \right) \leq \frac{3}{16\pi^2} \int_{\mathbb{B}^4} (\Delta u)^2 dx + \frac{3}{8\pi^2} \int_{\mathbb{S}^3} |\widetilde{\nabla} f|^2 d\xi. \tag{1.4.5}$$

Here, $d\xi$ is the volume form on the standard sphere, $\overline{f} = \int_{\mathbb{S}^3} f d\xi$ is the average of f, Δ is the Laplacian in the Euclidean unit ball and $\widetilde{\nabla}$ is the gradient on the standard sphere. Moreover, equality holds if and only if u is a biharmonic extension of a function of the form $f_\zeta(\xi) = -\log|1 - \langle \zeta, \xi \rangle| + c$, where $c \in \mathbb{R}$ is a constant, $\zeta \in \mathbb{B}^4$ is a fixed point, $\xi \in \mathbb{S}^3$, and u satisfies the Neumann boundary condition (1.4.4).

Their proof suggests that there is a natural relation between the hyperbolic harmonic extension of functions and the Fefferman-Graham metric in the unit ball. They start with a smooth function $f \in C^\infty(\mathbb{S}^3)$ and consider the biharmonic extension to the unit ball satisfying the boundary condition (1.4.4). Then they do integration by part using the biharmonic extension functions and the Fefferman-Graham metric. Eventually they establish the inequality (1.4.5) using Beckner's generalization of Moser-Trudinger inequality [2].

Based on the calculation in the appendix B, we know that: in $\mathbb{B}^4$ biharmonic extension of a given function satisfying Neumann boundary condition (1.4.4) is exactly the hyperbolic harmonic extension of the same function.

1.4.4 Sobolev Trace Inequality in Even-dimensional Unit Ball

Yang [25] generalized Ache and Chang's result. He found an explicit formula for the Fefferman-Graham metric in all even-dimensional unit ball and used the integration by parts argument in even-dimensional unit ball to prove the following:

Theorem 1.8. *[25, Theorem 1.7] Let $n \geq 4$ be an even integer. Given $f \in C^{\infty}(\mathbb{S}^{n-1})$, suppose u is a smooth extension of f to the unit ball $\mathbb{B}^n$ which also satisfies the Neumann boundary condition*

$$\Delta^k u|_{\mathbb{S}^{n-1}} = (-1)^k \frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n-1}{2}-k\right)}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n}{2}-k\right)} \frac{\mathscr{P}_{n-1}}{\mathscr{P}_{n-1-2k}} f, \text{ for } 0 \leq k \leq [\frac{n-2}{4}], \tag{1.4.6}$$

and

$$\frac{\partial}{\partial \nu}\Delta^k u\Big|_{\mathbb{S}^{n-1}} = (-1)^{k+1} \frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n-1}{2}-k\right)}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n}{2}-k\right)} \frac{\mathscr{P}_{n-1}}{\mathscr{P}_{n-1-2k}} f, \text{ for } 0 \leq k \leq [\frac{n-4}{4}]. \tag{1.4.7}$$

Here Δ denotes the Laplacian in the Euclidean unit ball, ν is the outer unit normal vector on the boundary $\mathbb{S}^{n-1}$ with respect to the Euclidean metric in $\mathbb{B}^n$. For any $\alpha \in \mathbb{R}$, $[\alpha]$ denotes the largest integer that is less than or equal to α. $\mathscr{P}_{2\gamma}$ are operators on $\mathbb{S}^{n-1}$ defined by Beckner [2], such that for any $0 < \gamma < \frac{n-1}{2}$ we have

$$\mathscr{P}_{2\gamma} = \frac{\Gamma(B + 1/2 + \gamma)}{\Gamma(B + 1/2 - \gamma)}, \ B = \sqrt{-\widetilde{\Delta} + \left(\frac{n-2}{2}\right)^2},$$

here $\widetilde{\Delta}$ is the Laplace-Beltrami operator on the standard sphere $\mathbb{S}^{n-1}$.

Then we have the inequality

$$\begin{aligned} &\log\left(\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} e^{(n-1)(f-\overline{f})} d\xi\right) \\ &\leq \frac{n-1}{2^n \pi^{n/2}\Gamma(n/2)}\left(\int_{\mathbb{B}^n} |\nabla^{n/2} u|^2 dx + \int_{\mathbb{S}^{n-1}} f \mathscr{T}_{\frac{n-2}{2}} f d\xi\right). \end{aligned} \tag{1.4.8}$$

Here

$$\nabla^m = \begin{cases} \Delta^{\frac{m}{2}}, \text{ when } m \text{ is even,} \\ \nabla\Delta^{\frac{m-1}{2}}, \text{ when } m \text{ is odd.} \end{cases}$$

$\mathcal{T}_m$ is an operator of order $2m$ on $\mathbb{S}^{n-1}$ defined as follows: when m is an odd integer

$$\begin{aligned}\mathcal{T}_m &= \frac{n-2}{2}\frac{\Gamma(m+1)\Gamma(1/2)}{\Gamma(m+1/2)}\frac{\mathscr{P}_{2m+1}}{\mathscr{P}_1}\\ &+ \sum_{k=1}^{(m-1)/2}(m-2k)\frac{\Gamma(m+1)^2\Gamma(k+1/2)\Gamma(m-k+1/2)}{\Gamma(m+1/2)^2\Gamma(k+1)\Gamma(m-k+1)}\frac{\mathscr{P}_{2m+1}^2}{\mathscr{P}_{2m+1-2k}\mathscr{P}_{2k+1}},\end{aligned}$$

and when m is an even integer

$$\begin{aligned}\mathcal{T}_m &= \frac{n-2}{2}\frac{\Gamma(m+1)\Gamma(1/2)}{\Gamma(m+1/2)}\frac{\mathscr{P}_{2m+1}}{\mathscr{P}_1}\\ &+ \frac{n-2-m}{2}\left(\frac{\Gamma(m+1)\Gamma\left(\frac{m+1}{2}\right)}{\Gamma(m+1/2)\Gamma(m/2+1)}\right)^2\frac{\mathscr{P}_{2m+1}^2}{\mathscr{P}_{m+1}^2}\\ &+ \sum_{k=1}^{m/2-1}(m-2k)\frac{\Gamma(m+1)^2\Gamma(k+1/2)\Gamma(m-k+1/2)}{\Gamma(m+1/2)^2\Gamma(k+1)\Gamma(m-k+1)}\frac{\mathscr{P}_{2m+1}^2}{\mathscr{P}_{2m+1-2k}\mathscr{P}_{2k+1}}.\end{aligned}$$

Moreover, equality holds if and only if

$$u(x) = \pi^{-\frac{n-1}{2}}\frac{\Gamma(n-1)}{2^{n-1}\Gamma(\frac{n-1}{2})}\int_{\mathbb{S}^{n-1}}\frac{(1-|x|^2)^{n-1}}{|x-\xi|^{2n-2}}(-\ln|1-\langle x_0,\xi\rangle|+c)d\xi, \qquad (1.4.9)$$

where $c\in\mathbb{R}$ and $x_0\in\mathbb{B}^n$.

Note that if we use $\left|\mathbb{S}^{n-1}\right|$ to denote the volume of the unit sphere with the standard metric, then we have

$$\pi^{-\frac{n-1}{2}}\frac{\Gamma(n-1)}{2^{n-1}\Gamma(\frac{n-1}{2})} = \left|\mathbb{S}^{n-1}\right|^{-1}.$$

We obtain the same explicit formula for Fefferman-Graham metric in all even dimensional unit ball using a different method in Theorem 3.1. Moreover, we obtain the same boundary conditions (1.4.6) and (1.4.7) using a different method in Proposition B.2 in the appendix

Yang's proof further suggests that there is a natural relation between the hyperbolic harmonic extension of functions and the Fefferman-Graham metric in the unit ball. In particular, (1.4.9) is the hyperbolic harmonic extension of the function $-\ln|1-\langle x_0,\xi\rangle|+c$.

1.5 Introduction to the Method of Moving Spheres

The method of moving spheres is a powerful tool to prove uniqueness of solutions to equations that have conformal symmetry. The method relies on maximum principle and the conformal symmetry of the equation. The method of moving spheres can be considered as a powerful generalization of the method of moving planes, but this dissertation will not talk more about the method of moving planes. For more information about the method of moving planes we refer the readers to the articles [13] [9] [16].

The method of moving spheres is closely related to conformal geometry. In [10] Escobar proved an important result in conformal geometry:

Theorem 1.9. *[10, Theorem 3.1] In $\overline{\mathbb{R}}^n_+ = \{(x', x_n) : x' \in \mathbb{R}^{n-1}, x_n \geq 0\}$ with the Euclidean metric. Let u be a positive solution to the problem*

$$\begin{cases} \Delta u = 0 \;\; on \; \mathbb{R}^n_+, \\ \frac{\partial u}{\partial x_n} + (n-2)u^{n/(n-2)} = 0 \;\; on \; \mathbb{R}^{n-1}, \\ u(x) = O(|x|^{2-n}) \;\; near \; \infty. \end{cases} \tag{1.5.1}$$

Then

$$u(x', x_n) = \left(\frac{\epsilon}{|x' - x_0|^2 + (\epsilon + x_n)^2} \right)^{(n-2)/2},$$

where $\epsilon > 0$ and $x_0 \in \mathbb{R}^{n-1}$.

This uniqueness result is very important in conformal geometry, because it entails the fact that the only metric in $\overline{\mathbb{R}}^n_+$ that is conformal to the Euclidean metric and has zero scalar curvature and constant mean curvature is

$$g_\epsilon = \left(\frac{\epsilon}{|x' - x_0|^2 + (\epsilon + x_n)^2} \right)^2 dx^2,$$

where dx^2 denotes the Euclidean metric.

In [20] Li and Zhu used the method of moving spheres to prove the uniqueness of solutions to the equation (1.5.1). They proved the following:

Theorem 1.10. *[20] For any integer $n \geq 3$, let $u \in C^2(\mathbb{R}^n_+) \cap C^1(\overline{R}^n_+)$ be any nonnegative solution of*

$$\begin{cases} -\Delta u = 0 \;\; in \; \mathbb{R}^n_+, \\ \frac{\partial u}{\partial x_n} = cu^{n/(n-2)} \;\; on \; \mathbb{R}^{n-1}, \end{cases}$$

where $c < 0$. *Then either* $u = 0$ *or*

$$u(x', x_n) = \left(\frac{-(n-2)t_0 c^{-1}}{|x' - x_0|^2 + (x_n + t_0)^2} \right)^{(n-2)/2}$$

for some $t_0 > 0$ *and* $x_0 \in \mathbb{R}^{n-1}$.

Note that in Theorem 1.10 Li and Zhu dropped the asymptotic assumption $u(x) = O(|x|^{2-n})$. But more importantly, the method of moving sphere is very general; it can be applied to a wide range of equations with conformal symmetry. It can be applied to differential equations as well as integral equations. For example in [19] Li proved the following:

Theorem 1.11. *[19, Theorem 1.1] For any integer* $n \geq 3$, $0 < \alpha < n$ *let* $u \in L^\infty_{loc}(\mathbb{R}^n)$ *be a positive solution to the equation*

$$u(x) = \int_{\mathbb{R}^n} \frac{u(y)^{\frac{n+\alpha}{n-\alpha}}}{|x-y|^{n-\alpha}} dy, \text{ for all } x \in \mathbb{R}^n,$$

then

$$u(x) = \left(\frac{a}{d + |x - x_0|^2} \right)^{(n-\alpha)/2},$$

for some $a, d > 0$ *and some* $x_0 \in \mathbb{R}^n$.

In this dissertation I will also use the method of moving spheres to prove Theorem 1.14. My proof combines both approaches for differential equations as in [20] and approaches for integral equations as in [19].

1.6 Main Result

In this article, we revisit the extension problem studied in [8] by a different approach. We firstly derive an explicit formula for $\widetilde{P}_\alpha$ in (2.3.1) and then carry out the analysis on $\mathbb{B}^n$.

In Theorem 2.12, we prove that the following inequality has constant function as optimizers.

Theorem 1.12. *Assume* $n \geq 3$ *and* $\alpha \in (2-n, 1)$. *For every* $f \in L^{\frac{2(n-1)}{n-2+\alpha}}(\mathbb{S}^{n-1})$, *we have*

$$\left\| \widetilde{P}_\alpha f \right\|_{L^{\frac{2n}{n-2+\alpha}}(\mathbb{B}^n)} \leq S_{n,\alpha} \left\| f \right\|_{L^{\frac{2(n-1)}{n-2+\alpha}}(\mathbb{S}^{n-1})}.$$

Where $S_{n,\alpha}$ is a constant that only depends on n and α. Up to conformal transformation any constant is an optimizer..

Our proof of the existence of optimizer relies on subcritical analysis as in [18], while our proof of uniqueness is the same as [8].

In the limit case $\alpha \to 2-n$. We prove

Theorem 1.13. *For dimension $n \geq 2$, and any function $\widetilde{F} \in L^\infty(\mathbb{S}^{n-1})$ we have*

$$\left\| e^{\widetilde{I}_n + \widetilde{P}_{2-n}\widetilde{F}} \right\|_{L^n(\mathbb{B}^n)} \leq S_n \left\| e^{\widetilde{F}} \right\|_{L^{n-1}(\mathbb{S}^{n-1})}. \tag{1.6.1}$$

Where $\widetilde{I}_n(x) = 2\frac{d\widetilde{P}_\alpha 1}{d\alpha}\big|_{\alpha=2-n}$. When n is even we have

$$\widetilde{I}_n(x) = \sum_{k=1}^{n/2-1} \frac{1}{2k} \cdot \frac{\Gamma\left(\frac{n-2}{2}\right)\Gamma\left(n-k-1\right)}{\Gamma(n-2)\Gamma\left(\frac{n}{2}-k\right)}(1-|x|^2)^k.$$

The sharp constant $S_n = \frac{\left\| e^{\widetilde{I}_n} \right\|_{L^n(\mathbb{B}^n)}}{\left|\mathbb{S}^{n-1}\right|^{\frac{1}{n-1}}}$

Our proof of the limit case inequality is very similar to [8]. When n is even, in addition to proving the inequality, we also found an explicit formula for the function $\widetilde{I}_n$ through induction. The induction formula is in proved in Lemma 3.2. When n is odd, we can calculate $\widetilde{I}_n$ by change of variable. We don't have an explicit formula for $\widetilde{I}_n$ when n is odd, but the induction relation in Lemma 3.2 is true for both when n is even and when n is odd. In particular, we used the induction relation to prove that the function $\widetilde{I}_n + \ln \frac{1-2x_n+|x|^2}{2}$ is hyperbolic harmonic in Lemma 3.6.

We also considered the variational problem

$$S_n = \sup\left\{ \left\| e^{\widetilde{I}_n + \widetilde{P}_{2-n}\widetilde{f}} \right\| : \widetilde{f} \in L^\infty(\mathbb{S}^{n-1}), \left\| e^{\widetilde{f}} \right\|_{L^{n-1}(\mathbb{S}^{n-1})} = 1 \right\},$$

and derived the Euler Lagrange equation

$$e^{(n-1)\widetilde{f}(\xi)} = \int_{\mathbb{B}^n} e^{n\widetilde{I}_n + n\widetilde{P}_{2-n}\widetilde{f}} \widetilde{p}_{2-n}(x,\xi)dx.$$

We prove the following uniqueness result

Theorem 1.14. *For any integer $n \geq 2$, if $\widetilde{f} \in L^{\infty}(\mathbb{S}^{n-1})$ satisfies the equation*

$$e^{(n-1)\widetilde{f}(\xi)} = \int_{\mathbb{B}^n} e^{n\widetilde{I}_n + n\widetilde{P}_{2-n}\widetilde{f}} \widetilde{p}_{2-n}(x,\xi)dx,$$

then for all $\xi \in \mathbb{S}^{n-1}$

$$\widetilde{f}(\xi) = \ln \frac{1-|\zeta|^2}{|\xi-\zeta|^2} + C_n,$$

where $\zeta \in \mathbb{B}^n$ and $C_n = -\frac{1}{n-1}\ln\left|\mathbb{S}^{n-1}\right|$ is a constant. Here $\left|\mathbb{S}^{n-1}\right|$ denotes the volume of the standard sphere.

Note that the choice of C_n is to make sure that $\left\|e^{\widetilde{f}}\right\|_{L^{n-1}(\mathbb{S}^{n-1})} = 1$. We prove this theorem using the moving sphere method in Section 3.5.

1.7 Conformally Invariant Generalization of Carleman's Inequality

In this section we offer an interpretation of the limit case inequality as a conformally invariant generalization of Carleman's inequality.

In [4] Carleman proved the following:

Theorem 1.15. *([4]) For any $u \in C^{\infty}(\overline{\mathbb{B}}^2)$ such that u is harmonic in $\mathbb{B}^2$ with respec to the Euclidean metric then we have*

$$\int_{\mathbb{B}^2} e^{2u}dx \leq \frac{1}{4\pi}\left(\int_{\mathbb{S}^1} e^u d\theta\right). \tag{1.7.1}$$

Where equality holds $u(x) = c$ or $u(x) = -2\ln|x-x_0| + c$ where $c \in \mathbb{R}$ is any constant and $x_0 \in \mathbb{R}^2 \backslash \overline{\mathbb{B}}^2$.

Note that the inequality (1.7.1) is conformally invariant and that it also holds for subharmonic functions. We will refer to (1.7.1) as Carleman's inequality through out the dissertation.

Hang, Wang and Yan [18] proved a higher dimensional generalization of Carleman's inequality for harmonic functions. As an application of inequality (1.1.3), they proved the following result:

Corollary 1.16. *[18, Corollary 3.1] Assume $n \geq 3$, then for $\widetilde{f} \in L^{\infty}(\mathbb{S}^{n-1})$,*

$$\left\| e^{\widetilde{P}\widetilde{f}} \right\|_{L^{\frac{n}{n-1}}(\mathbb{B}^n)} \leq n^{-1}\omega_n^{-\frac{1}{n}} \left\| e^{\widetilde{f}} \right\|_{L^1(\mathbb{S}^{n-1})}.$$

Here $\widetilde{P}\widetilde{f}$ is the harmonic extension of $\widetilde{f}$ as defined in (1.1.4), ω_n is the volume of the unit ball in $\mathbb{R}^n$ with the Euclidean metric. Moreover, equality holds if and only if $\widetilde{f}$ is constant.

Note that the inequality in the corollary also works for subharmonic functions but it is not invariant under conformal transformation.

In [8] Chen proposed another way to generalize the Carleman's inequality in dimension 4, he proved the following:

Corollary 1.17. *[8, Corollary 1] For any $u : \mathbb{B}^4 \to \mathbb{R}$ satisfying $\Delta^2 u \leq 0$ and $-\frac{\partial u}{\partial \nu} \leq 1$,*

$$\left(\int_{\mathbb{B}^4} e^{4u} dx \right)^{\frac{1}{4}} \leq S \left(\int_{\mathbb{S}^3} e^{3u} d\xi \right)^{\frac{1}{3}}.$$

Note that here Δ is the Laplacian in $\mathbb{B}^4$ with the Euclidean metric, ν is the outer unit normal vector with respect to the Euclidean metric. Here S is the sharp constant, and is assumed by the solution to the equation

$$\begin{cases} \Delta^2 u = 0, \ \ in \ \mathbb{B}^4 \\ u = 0, \ \ on \ \mathbb{S}^3 \\ -\frac{\partial u}{\partial \nu} = 1, \ \ on \ \mathbb{S}^3. \end{cases} \tag{1.7.2}$$

Note that Chen did not prove uniqueness of extremal function for this generalization, and as he pointed out at the end of [8] that this generalization works well because the Green's function of equation (1.7.2) is positive. As a result will be difficult for us to find similar generalizations in higher dimensions.

As an application of inequality (1.6.1), we propose another way to generalize the Carleman's inequality:

Corollary 1.18. *Assume $n \geq 3$, then for any $\widetilde{f} \in L^{\infty}(\mathbb{S}^{n-1})$*

$$\left\| e^{\widetilde{I}_n + \widetilde{P}_{2-n}\widetilde{f}} \right\|_{L^n(\mathbb{B}^n)} \leq S_n \left\| e^{\widetilde{f}} \right\|_{L^{n-1}(\mathbb{S}^{n-1})}.$$

The sharp constant $S_n = \dfrac{\left\| e^{\widetilde{I}_n} \right\|_{L^n(\mathbb{B}^n)}}{\left|\mathbb{S}^{n-1}\right|^{\frac{1}{n-1}}}$. *Moreover, equality holds if and only if* $\widetilde{f}(\xi) = -\ln|1-\zeta\cdot\xi| + C$. *Where* $C \in \mathbb{R}$ *is a constant and* $\zeta \in \mathbb{B}^n$.

Note that this inequality is invariant under conformal transformation and that it also holds for hyperbolic subharmonic functions.

The proof of this corollary is simply a combination of Theorem 1.13, Theorem 1.14 and regularity results Proposition C.6 from the appendix. When n is even, we can think of $\|e^{\widetilde{I}_n+\widetilde{P}_2-n\widetilde{f}}\|_{L^n(\mathbb{B}^n)}$ as the L^n norm of $e^{\widetilde{P}_2-n\widetilde{f}}$ measured using the Fefferman-Graham metric [1][11]

$$g = e^{2\widetilde{I}_n}dx^2$$

where dx^2 is the standard Euclidean metric on the unit ball.

CHAPTER 2

FROM THE UPPER HALF SPACE TO THE UNIT BALL

2.1 Chapter Outline and Notation

In this chapter we transform the equation (1.2.1) from the upper half space to the unit ball. We also identify the Poisson kernel of the corresponding equation in the unit ball and study how the Poisson kernel transforms under conformal changes. At the end of this chapter we formulate a conjecture inspired by the Martin boundary theory.

Through out this dissertation, we let

$$\mathbb{R}^n_+ = \{(y', y_n) \in \mathbb{R}^n \text{ such that } y' \in \mathbb{R}^{n-1},\ y_n > 0\},$$

and

$$\mathbb{B}^n = \{x \in \mathbb{R}^n \text{ such that } |x| < 1\},$$

here $|x|$ denotes the norm of x with respect to the Euclidean metric. We also use the notation

$$\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n \text{ such that } |x| = 1\}$$

to denote the unit sphere in $\mathbb{S}^{n-1}$.

Through out this dissertation we use notations like (y', y_n) and (x', x_n) to denote points in $\mathbb{R}^n$ where $y',\ x' \in \mathbb{R}^{n-1}$ and $y_n,\ x_n \in \mathbb{R}$.

Definition 2.1. *For $r \in (0, 1]$, define $\mathbb{S}^{n-1}_r = \{x \in \mathbb{B}^n : |x| = r\}$ and $\mathbb{B}^n_r = \{x \in \mathbb{B}^n : |x| < r\}$.*

Let $\Psi : \mathbb{R}^n_+ \to \mathbb{B}^n$ be the projection map defined by

$$\begin{aligned}(x', x_n) = \Psi\left(y', y_n\right) &= \left(\frac{2y'}{1 + 2y_n + |y|^2}, \frac{-1 + |y|^2}{1 + 2y_n + |y|^2}\right)\\ &= \left(\frac{2y'}{(1 + y_n)^2 + |y'|^2}, 1 - \frac{2(1 + y_n)}{(1 + y_n)^2 + |y'|^2}\right)\end{aligned} \tag{2.1.1}$$

with the inverse $\Psi^{-1}:\mathbb{B}^n\to\mathbb{R}^n_+$

$$\left(y',y_n\right)=\Psi^{-1}\left(x',x_n\right)=\left(\frac{2x'}{(1-x_n)^2+|x'|^2},\frac{1-|x|^2}{(1-x_n)^2+|x'|^2}\right).$$

It is useful to record

$$\frac{1-2x_n+|x|^2}{2}=\frac{2}{1+2y_n+|y|^2}$$

Which means that if we define $[\Psi(y)]_n$ to be the $n-$th component of $\Psi(y)$ (note that by definition $[\Psi(y)]_n=\frac{-1+|y|^2}{1+2y_n+|y|^2}$), then we have

$$\frac{1-2[\Psi(y)]_n+|\Psi(y)|^2}{2}=\frac{2}{1+2y_n+|y|^2}. \tag{2.1.2}$$

The restriction of Ψ on $y_n=0$ is the stereographic projection $\mathbb{R}^{n-1}\to\mathbb{S}^{n-1}$. From the calculation in proposition 2.2, in particular (2.2.2), we see that

$$\Psi^*dx^2=\frac{4dy^2}{\left(1+2y_n+|y|^2\right)^2}.$$

It means that $\Psi:\mathbb{R}^n_+\to\mathbb{B}^n$ is a conformal transformation. Here the conformal factor is very important for our calculation, through out this article we will use $|\Psi'(y)|$ to denote the conformal factor, in particular for any $y\in\mathbb{R}^n_+$ we have

$$|\Psi'(y)|=\frac{2}{1+2y_n+|y|^2}, \tag{2.1.3}$$

and for any $w\in\mathbb{R}^{n-1}=\partial\mathbb{R}^n_+$ we have

$$|\Psi'(w)|=\frac{2}{1+|w|^2}. \tag{2.1.4}$$

We will discuss more about Ψ and other conformal transformation in the appendix.

For a function $\widetilde{f}$ on $\mathbb{B}^n$ or $\mathbb{S}^{n-1}$ we define

$$f\left(y',y_n\right)=\widetilde{f}\circ\Psi\left(y',y_n\right)\left(\frac{2}{1+2y_n+|y|^2}\right)^{\frac{n-2+\alpha}{2}}. \tag{2.1.5}$$

It is easy to check that this map is an isometry from $L^{\frac{2(n-1)}{n-2+\alpha}}\left(\mathbb{S}^{n-1}\right)$ to $L^{\frac{2(n-1)}{n-2+\alpha}}\left(\mathbb{R}^{n-1}\right)$ and from $L^{\frac{2n}{n-2+\alpha}}\left(\mathbb{B}^n\right)$ to $L^{\frac{2n}{n-2+\alpha}}\left(\mathbb{R}^n_+\right)$. The inverse map is

$$\widetilde{f}\left(x',x_n\right)=f\circ\Psi^{-1}\left(x',x_n\right)\left(\frac{2}{1-2x_n+|x|^2}\right)^{\frac{n-2+\alpha}{2}}. \tag{2.1.6}$$

2.2 The Equation in the Unit Ball

Now we are ready to transform the equation (1.2.1) from the upper half space to the unit ball. For any $2-n \le \alpha < 1$ define the operator $\mathcal{L}$ in $\mathbb{B}^n$ such that for any $x = (x', x_n) \in \mathbb{B}^n$ and for any $\widetilde{u} \in C^2(\mathbb{B}^n)$

$$\begin{aligned}\mathcal{L}\widetilde{u} &= \left(\frac{1-2x_n+|x|^2}{2}\right)^{(n+2-\alpha)/2} \\ &\quad\cdot \left[div\left[\left(\frac{1-|x|^2}{2}\right)^{\alpha}\nabla\widetilde{u}\right] + \frac{\alpha\,(2-n-\alpha)}{2}\left(\frac{1-|x|^2}{2}\right)^{\alpha-1}\widetilde{u}\right].\end{aligned}$$

Note that in $\mathbb{B}^n$ we have $\mathcal{L}\widetilde{u} = 0$ if and only if

$$div\left[\left(\frac{1-|x|^2}{2}\right)^{\alpha}\nabla\widetilde{u}\right] + \frac{\alpha\,(2-n-\alpha)}{2}\left(\frac{1-|x|^2}{2}\right)^{\alpha-1}\widetilde{u} = 0. \tag{2.2.1}$$

Proposition 2.2. *(How the operator tranforms) For any $2-n \le \alpha < 1$ and any $u \in C^2(\mathbb{R}^n_+)$ define $\widetilde{u}$ using (2.1.6) then we have*

$$div(y_n^{\alpha}\nabla u) = 0,\ \ in\ \mathbb{R}^n_+$$

if and only

$$\mathcal{L}\widetilde{u} = 0,\ \ in\ \mathbb{B}^n$$

Proof. In the following, for any $y = (y', y_n) \in \mathbb{R}^n$, let $\rho = \left(1 + 2y_n + |y|^2\right)/2$. Then we have $\nabla\rho = \left(y', 1+y_n\right)$. Let $a,\ b,\ c,\ d = 1,\ 2,\ ...,\ n$ be indices. Suppose $x = \Psi(y)$, then by direct calculation we have, when $c \neq n$,

$$\frac{\partial x_c}{\partial y_a} = \begin{cases} -\frac{4y_a y_c}{(1+2y_n+|y|^2)^2}, & a \neq c \text{ and } a \neq n, \\ \frac{2}{1+2y_n+|y|^2} - \frac{4y_c^2}{(1+2y_n+|y|^2)^2}, & c = a \neq n, \\ -\frac{4y_c(y_n+1)}{(1+2y_n+|y|^2)^2}, & a = n, \end{cases}$$

and when $c = n$

$$\frac{\partial x_n}{\partial y_a} = \begin{cases} \frac{2y_a}{1+2y_n+|y|^2} + \frac{2y_a(1-|y|^2)}{(1+2y_n+|y|^2)^2}, & a \neq n, \\ \frac{2y_n}{1+2y_n+|y|^2} + \frac{2(1-|y|^2)(1+y_n)}{(1+2y_n+|y|^2)^2}, & a = n. \end{cases}$$

From it we have

$$\sum_{a=1}^{n} \frac{\partial x_c}{\partial y_a} \frac{\partial x_d}{\partial y_a} = \begin{cases} \rho^{-2}, & c = d, \\ 0, & c \neq d. \end{cases} \tag{2.2.2}$$

$$\begin{aligned} \sum_{a=1}^{n} \frac{\partial x_c}{\partial y_a} \frac{\partial \rho}{\partial y_a} &= \begin{cases} -\rho^{-1} y_c, & c \neq n, \\ \rho^{-1} (1 + y_n) & c = n. \end{cases}, \\ \sum_{a=1}^{n} \frac{\partial x_c}{\partial y_a^2} &= \begin{cases} -(n-2)\rho^{-2} y_c, & c \neq n, \\ (n-2)\rho^{-2}(1 + y_n) & c = n. \end{cases} \end{aligned}$$

We calculate

$$\frac{\partial u}{\partial y_a} = \rho^{-(n+\alpha)/2} \left[\rho \frac{\partial \widetilde{u}}{\partial x_c} \frac{\partial x_c}{\partial y_a} + \left(1 - \frac{n+\alpha}{2} \right) \widetilde{u} \frac{\partial \rho}{\partial y_a} \right],$$

$$
\begin{aligned}
div(y_n^\alpha \nabla u) &= \frac{\partial}{\partial y_a}\left(y_n^\alpha \frac{\partial u}{\partial y_a}\right) \\
&= y_n^\alpha \rho^{-(n+\alpha)/2}\Bigg[\rho \frac{\partial^2 \widetilde{u}}{\partial x_c \partial x_d}\frac{\partial x_c}{\partial y_a}\frac{\partial x_d}{\partial y_a} + \rho \frac{\partial \widetilde{u}}{\partial x_c}\frac{\partial^2 x_c}{\partial y_a^2} \\
&\quad + \left(2 - \frac{n+\alpha}{2}\right)\frac{\partial \rho}{\partial y_a}\frac{\partial \widetilde{u}}{\partial x_c}\frac{\partial x_c}{\partial y_a} + n\left(1 - \frac{n+\alpha}{2}\right)\widetilde{u}\Bigg] \\
&\quad - \frac{n+\alpha}{2} y_n^\alpha \rho^{-(n+\alpha)/2-1}\frac{\partial \rho}{\partial y_a}\left[\rho \frac{\partial \widetilde{u}}{\partial x_c}\frac{\partial x_c}{\partial y_a} + \left(1 - \frac{n+\alpha}{2}\right)\widetilde{u}\frac{\partial \rho}{\partial y_a}\right] \\
&\quad + \alpha y_n^{\alpha-1}\rho^{-(n+\alpha)/2}\left[\rho \frac{\partial \widetilde{u}}{\partial x_c}\frac{\partial x_c}{\partial y_n} + \left(1 - \frac{n+\alpha}{2}\right)\widetilde{u}\frac{\partial \rho}{\partial y_n}\right] \\
&= y_n^\alpha \rho^{-(n+\alpha)/2}\Bigg[\rho^{-1}\Delta \widetilde{u} + \rho \frac{\partial \widetilde{u}}{\partial x_c}\frac{\partial^2 x_c}{\partial y_a^2} \\
&\quad + (2-n-\alpha)\rho^{-1}\left(\frac{\partial \widetilde{u}}{\partial x_n}(1+y_n) - \frac{\partial \widetilde{u}}{\partial x_j}y_j\right) + n\left(1 - \frac{n+\alpha}{2}\right)\widetilde{u}\Bigg] \\
&\quad - (n+\alpha)\left(1 - \frac{n+\alpha}{2}\right) y_n^\alpha \rho^{-(n+\alpha)/2}\widetilde{u} \\
&\quad + \alpha y_n^{\alpha-1}\rho^{-(n+\alpha)/2}\left[\rho \frac{\partial \widetilde{u}}{\partial x_c}\frac{\partial x_c}{\partial y_n} + \left(1 - \frac{n+\alpha}{2}\right)\widetilde{u}\frac{\partial \rho}{\partial y_n}\right] \\
&= y_n^\alpha \rho^{-(n+\alpha)/2} \\
&\quad \cdot \left[\rho^{-1}\Delta \widetilde{u} - \alpha \rho^{-1}\left(\frac{\partial \widetilde{u}}{\partial x_n}(1+y_n) - \frac{\partial \widetilde{u}}{\partial x_j}y_j\right) - \alpha\left(1 - \frac{n+\alpha}{2}\right)\widetilde{u}\right] \\
&\quad + \alpha y_n^{\alpha-1}\rho^{-(n+\alpha)/2}\Bigg[\rho^{-1}(1+y_n)\left(\frac{\partial \widetilde{u}}{\partial x_n}(1+y_n) - \frac{\partial \widetilde{u}}{\partial x_j}y_j\right) \\
&\quad - \frac{\partial \widetilde{u}}{\partial x_n} + \left(1 - \frac{n+\alpha}{2}\right)\widetilde{u}(1+y_n)\Bigg] \\
&= y_n^\alpha \rho^{-(n+\alpha)/2}\rho^{-1}\Delta \widetilde{u} + \alpha y_n^{\alpha-1}\rho^{-(n+\alpha)/2} \\
&\quad \cdot \left[\rho^{-1}\left(\frac{\partial \widetilde{u}}{\partial x_n}(1+y_n) - \frac{\partial \widetilde{u}}{\partial x_j}y_j\right) - \frac{\partial \widetilde{u}}{\partial x_n} + \left(1 - \frac{n+\alpha}{2}\right)\widetilde{u}\right] \\
&= y_n^{\alpha-1}\rho^{-(n+\alpha)/2}\left[\frac{1-|x|^2}{2}\Delta \widetilde{u} - \alpha x_a \frac{\partial \widetilde{u}}{\partial x_a} + \frac{\alpha(2-n-\alpha)}{2}\widetilde{u}\right] \\
&= \left(\frac{1-|x|^2}{2}\right)^{\alpha-1}\left(\frac{1-2x_n+|x|^2}{2}\right)^{(n+2-\alpha)/2} \\
&\quad \cdot \left[\frac{1-|x|^2}{2}\Delta \widetilde{u} - \alpha x_a \frac{\partial \widetilde{u}}{\partial x_a} + \frac{\alpha(2-n-\alpha)}{2}\widetilde{u}\right]
\end{aligned}
$$

□

Remark 1. *For any integer $n \geq 2$ and any $\alpha \in (2-n, 0)$, we can apply theorem 1.1 in [24]* *to show that solution to the equation*

$$\begin{cases} \mathscr{L}u = 0, \ \ in\ \mathbb{B}^n \\ u = f, \ \ in\ \mathbb{S}^{n-1} \end{cases}$$

is unique in $C^2(\mathbb{B}^n) \cap C^0(\overline{\mathbb{B}^n})$.

Remark 2. *Note that the equation (2.2.1) is a special case of equation (4.2) in [7] if we take* $\overline{g}$ *as the Euclidean metric in the unit ball,* g^+ *as the hyperbolic metric in the unit ball, and* $\rho = \frac{1-|x|^2}{2}$ *as the defining function.*

2.3 Poisson Kernel in the Unit Ball

Caffarelli and Silvestre [3] found a Poisson kernel that solves the Dirichlet problem (1.2.1). For any $-1 < \alpha < 1$, $y \in \mathbb{R}^n_+$ and any $\xi \in \mathbb{R}^{n-1}$

$$p_\alpha(y, w) = c_{n,\alpha} \frac{y_n^{1-\alpha}}{|y - w|^{\frac{n-\alpha}{2}}}.$$

For any $f : \mathbb{R}^{n-1} \to \mathbb{R}$ regular enough, we can define

$$P_\alpha f = \int_{\mathbb{R}^{n-1}} p_\alpha(y, \xi) f(w) dw,$$

such that $P_\alpha f$ solves the Dirichlet problem (1.2.1).

We want to find the corresponding Poisson kernel in the unit ball. Define

$$\widetilde{p}_\alpha(x, \xi) = 2^{\alpha-1} c_{n,\alpha} \frac{(1 - |x|^2)^{1-\alpha}}{|x - \xi|^{n-\alpha}},$$

and for any function $\widetilde{\phi} : \mathbb{S}^{n-1} \to \mathbb{R}$, define

$$\widetilde{P}_\alpha \widetilde{\phi}(x) = \int_{\mathbb{S}^{n-1}} \widetilde{p}_\alpha(x, \xi) \widetilde{\phi}(\xi) d\xi. \tag{2.3.1}$$

then we have the following proposition:

Proposition 2.3. *For any integer $n \geq 2$, any $\alpha \in (2-n, 1)$, any $f \in L^{\frac{2(n-1)}{n-2+\alpha}}(\mathbb{R}^{n-1})$, define $\widetilde{f}$ as in (2.1.6), then we have*

$$\widetilde{P_\alpha f} = \widetilde{P}_\alpha \widetilde{f}.$$

Here $\widetilde{P_\alpha f}$ is the transformation of $P_\alpha f$ as defined in (2.1.6), and $\widetilde{P}_\alpha \widetilde{f}$ is the extension of $\widetilde{f}$ in the unit ball as defined in (2.3.1).

Proof. The proof is by direct calculation. Note that for any $w \in \mathbb{R}^{n-1}$ using the fact that (2.1.5) and (2.1.6) are inverse to each other, we have

$$f(w) = \widetilde{f} \circ \Psi(w) \left(\frac{2}{1+|w|^2} \right)^{\frac{n-2+\alpha}{2}}.$$

As a result, we have

$$\begin{aligned}
& (P_\alpha f) \circ \Psi^{-1}(x', x_n) \\
= \; & c_{n,\alpha} \int_{\mathbb{R}^{n-1}} \frac{\left(\frac{1-|x|^2}{1-2x_n+|x|^2} \right)^{1-\alpha}}{\left(\left| \frac{2x'}{(1-x_n)^2+|x'|^2} - w \right|^2 + \left(\frac{1-|x|^2}{1-2x_n+|x|^2} \right)^2 \right)^{\frac{n-\alpha}{2}}} \\
& \cdot \widetilde{f} \circ \Psi(w) \left(\frac{2}{1+|w|^2} \right)^{\frac{n-2+\alpha}{2}} dw \\
= \; & c_{n,\alpha} \int_{\mathbb{S}^{n-1}} \frac{\left(\frac{1-|x|^2}{1-2x_n+|x|^2} \right)^{1-\alpha}}{\left(\left| \frac{2x'}{(1-x_n)^2+|x'|^2} - \frac{\xi'}{1-\xi_n} \right|^2 + \left(\frac{1-|x|^2}{1-2x_n+|x|^2} \right)^2 \right)^{\frac{n-\alpha}{2}}} \\
& \cdot \widetilde{f}(\xi)(1-\xi_n)^{\frac{\alpha-n}{2}} d\xi \\
= \; & 2^{(\alpha-n)/2} c_{n,\alpha} \left(1 - 2x_n + |x|^2 \right)^{\frac{n-2+\alpha}{2}} \int_{\mathbb{S}^{n-1}} \frac{\left(1-|x|^2\right)^{1-\alpha}}{|x-\xi|^{n-\alpha}} \widetilde{f}(\xi) \, d\xi.
\end{aligned}$$

Divide both sides by $\left(\frac{1-2x_n+|x|^2}{2} \right)^{\frac{n-2+\alpha}{2}}$ then we are done. ∎

Remark 3. *For any integer* $n \geq 2$ *and* $\alpha = 2 - n$ *we can prove similar result. For any* $f \in L^{\infty}(\mathbb{R}^{n-1})$, *define*

$$\widetilde{f} = f \circ \Psi^{-1},$$

then the same calculation as in the previous proposition show that

$$(P_{2-n} f) \circ \Psi^{-1} = \widetilde{P}_{2-n} \left(f \circ \Psi^{-1} \right) = \widetilde{P}_{2-n} \widetilde{f}.$$

Remark 4. *We note that*

$$\int_{\mathbb{S}^{n-1}} \widetilde{p}_{\alpha} (x, \xi) \, d\xi$$

is not constant in x *except when* $\alpha = 0$ *or* $2 - n$.

Proposition 2.4. *For any* $f \in C\left(\mathbb{S}^{n-1}\right)$

$$u(x) := \begin{cases} \int_{\mathbb{S}^{n-1}} \widetilde{p}_{\alpha} (x, \xi) \, f(\xi) \, d\xi, x \in \mathbb{B}^n \\ f(x), x \in \mathbb{S}^{n-1} \end{cases}$$

defines a continuous function on $\overline{\mathbb{B}^n}$ *which is smooth in* $\mathbb{B}^n$ *and satisfies* $\mathcal{L} u = 0$.

Proof. The integral $\int_{\mathbb{S}^{n-1}} \widetilde{p}_{\alpha}(x, \xi) d\xi = \frac{1}{r^{n-1}} \int_{\mathbb{S}_r^{n-1}} \widetilde{p}_{\alpha}(x, \xi) dx$ is a function that only depends on $|x|$. Define $h(|x|) = \int_{\mathbb{S}^{n-1}} \widetilde{p}_{\alpha}(x, \xi) d\xi$, then by Lemma 2.7, and remark 6 we know that for $r \in [0, 1]$

$$\left(\frac{2}{1+r} \right)^{n-2+\alpha} \frac{\Gamma\left(\frac{n-\alpha}{2}\right) \Gamma\left(\frac{n-1}{2}\right)}{\Gamma(n-1) \Gamma\left(\frac{1-\alpha}{2}\right)} \leq h(r) \leq \left(\frac{2}{1+r} \right)^{n-2+\alpha} \leq 2^{n-2+\alpha}.$$

By dominated convergence theorem as in Remark 7, we know that for $r \in [0, 1]$, $h(r)$ is continuous and that

$$\lim_{r \to 1} h(r) = 1.$$

By the continuity of f on $\mathbb{S}^{n-1}$, we can choose $\delta > 0$ small, such that when $|\xi_1 - \xi_2| \leq \delta$, we have $|f(\xi_1) - f(\xi_2)| \leq \epsilon$. By the continuity of $h(r)$ on the interval $[0, 1]$ and the fact that it is strictly positive on $[0,1]$ we can choose $\delta > 0$ smaller if needed, such that when $|x - x_0| \leq \delta$ we have $\left|\frac{1}{h(|x|)} - \frac{1}{h(|x_0|)}\right| < \epsilon$. Define

$$M := \|f\|_{L^\infty(\mathbb{S}^{n-1})} \frac{\Gamma(n-1)\Gamma\left(\frac{1-\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}.$$

Note that we have

$$0 < \frac{\Gamma\left(\frac{n-\alpha}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}{\Gamma(n-1)\Gamma\left(\frac{1-\alpha}{2}\right)} \leq h(r),$$

for all $r \in [0,1]$.

Suppose $x_0 \in \mathbb{S}^{n-1}$, and $x \in \mathbb{B}^n$ such that $|x - x_0| < \delta/2$ consider

$$\begin{aligned}
|u(x) - u(x_0)| &= \left|\int_{\mathbb{S}^{n-1}} \widetilde{p}_\alpha(x,\xi) f(\xi)\, d\xi - \int_{\mathbb{S}^{n-1}} \widetilde{p}_\alpha(x,\xi) \frac{u(x_0)}{h(|x|)} d\xi\right| \\
&\leq \int_{|\xi - x_0| \leq \delta} \widetilde{p}_\alpha(x,\xi) \left(|f(\xi) - f(x_0)| + \left|f(x_0) - \frac{f(x_0)}{h(|x|)}\right|\right) d\xi \\
&\quad + \int_{|\xi - x_0| > \delta} \widetilde{p}_\alpha(x,\xi) \left|f(\xi) - \frac{f(x_0)}{h(|x|)}\right| d\xi \\
&\leq C(n,\alpha)\epsilon + \frac{C(n,\alpha) M (1 - |x|^2)^{1-\alpha}}{\delta^{n-\alpha}}
\end{aligned}$$

As a result $u(x)$ is continuous at x_0.

The part that $\mathcal{L}u = 0$ follows from dominated convergence theorem and direct calculation. □

Based on the Martin theory for harmonic functions, we make the following conjecture:

Conjecture 1. *(The representation theorem) Let $\widetilde{u} : \mathbb{B}^n \to \mathbb{R}$ be a positive solution of*

$$\mathcal{L}\widetilde{u} = 0.$$

Then there exists a Borel measure ν on $\mathbb{S}^{n-1}$ s.t.

$$\widetilde{f}(x) = \int_{\mathbb{S}^{n-1}} \widetilde{p}_\alpha(x,\xi)\, d\nu(\xi).$$

2.4 Poisson Kernel under Conformal Transformation

We also want to know how the Poisson kernel transforms under conform transformation. We prove the following:

Proposition 2.5. *For any integer $n \geq 2$, any $\alpha \in [2-n, 1)$, any $y \in \mathbb{R}^n_+$ and any $w \in \mathbb{R}^{n-1}$ we have*

$$\widetilde{p}_\alpha(\Psi(y), \Psi(w)) = p_\alpha(y, w)|\Psi'(y)|^{(2-n-\alpha)/2}|\Psi'(w)|^{(\alpha-n)/2}. \tag{2.4.1}$$

Here Ψ is the conformal transformation defined in (2.1.1), $|\Psi'(y)|$ and $|\Psi'(w)|$ are the conformal factors in (2.1.3) and (2.1.4) respectively.

Proof. For any $x \in \mathbb{B}^n$ and any $\xi \in \mathbb{S}^{n-1}$, by definition we have

$$\widetilde{p}_\alpha(x, \xi) = 2^{\alpha-1} c_{n,\alpha} \frac{(1-|x|^2)^{1-\alpha}}{|x-\xi|^{n-\alpha}}.$$

From this we have for any $y \in \mathbb{R}^n_+$ and any $w \in \mathbb{R}^{n-1}$

$$\widetilde{p}_\alpha(\Psi(y), \Psi(w)) = 2^{\alpha-1} c_{n,\alpha} \frac{(1-|\Psi(y)|^2)^{1-\alpha}}{|\Psi(y)-\Psi(w)|^{n-\alpha}}.$$

Through direct calculation we have

$$1 - |\Psi(y)|^2 = \frac{4y_n}{1+2y_n+|y|^2} = 2y_n|\Psi'(y)|$$

and

$$\begin{aligned}
|\Psi(y)-\Psi(w)|^2 &= \left|\frac{2}{1+2y_n+|y|^2}(y', -y_n-1) - \frac{2}{1+|w|^2}(w, -1)\right|^2 \\
&= \frac{4}{1+2y_n+|y|^2} + \frac{4}{1+|w|^2} \\
&\quad - \frac{8}{(1+|w|^2)(1+2y_n+|y|^2)}\left(\langle w, y'\rangle + 1 + y_n\right) \\
&= \frac{4}{(1+|w|^2)(1+2y_n+|y|^2)} \cdot \\
&\quad \left(1+|w|^2+1+2y_n+|y|^2-2\langle w, y'\rangle - 2 - 2y_n\right) \\
&= \frac{4}{(1+|w|^2)(1+2y_n+|y|^2)}|y-w|^2 \\
&= |y-w|^2|\Psi'(w)||\Psi'(y)|
\end{aligned}$$

Note that here we use the notation $(y', -y_n - 1)$ and $(w, -1)$ to denote points in $\mathbb{R}^n$ and use the notation $\langle w, y' \rangle$ to denote the Euclidean inner product in $\mathbb{R}^{n-1}$. Combine these calculations together, then we can get (2.4.1). $\square$

We also want to consider how the Poisson kernel transform under the isometry group of $(\mathbb{B}^n, g_h)$. Here $g_h = \frac{4}{(1-|x|^2)} dx^2$ denotes the hyperbolic metric in the unit ball. We use the notation $SO(n,1)$ to denote the isometry group of the unit ball with the hyperbolic metric. For any $\Phi \in SO(n,1)$ any $x \in \mathbb{B}^n$ and any $\xi \in \mathbb{S}^{n-1}$ we use $|\Phi'(x)|$ and $|\Psi'(\xi)|$ to denote the conformal factors in $\mathbb{B}^n$ and $\mathbb{S}^{n-1}$ respectively. We prove the following:

Proposition 2.6. *For any integer $n \geq 2$ any $\alpha \in [2-n, 1)$, any $x \in \mathbb{B}^n$, any $\xi \in \mathbb{S}^{n-1}$ and any $\Phi \in SO(n,1)$ we have*

$$\widetilde{p}_\alpha \left(\Phi\left(x\right), \Phi\left(\xi\right)\right) = \widetilde{p}_\alpha \left(x, \xi\right) \left|\Phi'\left(x\right)\right|^{(2-n-\alpha)/2} \left|\Phi'\left(\xi\right)\right|^{(\alpha-n)/2}. \tag{2.4.2}$$

Proof. For any $\Phi \in SO(n,1)$, since it is an isometry of $\mathbb{B}^n$ with the hyperoblic metric $g_h = \frac{4}{\left(1-|x|^2\right)^2} dx^2$, it is a conformal transformation with respect to the Euclidean metric. We have for any $x \in \mathbb{B}^n$

$$\Phi^* \left(\frac{4}{(1-|x|^2)^2} dx^2 \right) = \frac{4}{(1-|\Phi(x)|^2)^2} \Phi^*(dx^2) = \frac{4}{(1-|x|^2)^2} dx^2,$$

and

$$\Phi^* dx^2 = |\Phi'(x)|^2 dx^2.$$

Here $|\Phi'(x)|$ is the conformal factor, it is a notation similar to $|\Psi'(y)|$. From this we conclude that for any $x \in \mathbb{B}^n$

$$\left|\Phi'\left(x\right)\right| = \frac{1-\left|\Phi\left(x\right)\right|^2}{1-|x|^2}. \tag{2.4.3}$$

For any $x,\ z \in \mathbb{B}^n$, define $d(x,z)$ as the distance between the two points measured by the hyperbolic metric. Then we have

$$\cosh d\left(x, z\right) = 1 + 2 \frac{|x-z|^2}{\left(1-|x|^2\right)\left(1-|z|^2\right)}.$$

Since Φ is an isometry with respect to the hyperbolic metric, we have Thus

$$\frac{|x-z|^2}{\left(1-|x|^2\right)\left(1-|z|^2\right)}=\frac{|\Phi(x)-\Phi(z)|^2}{\left(1-|\Phi(x)|^2\right)\left(1-|\Phi(z)|^2\right)}$$

Letting $z\to\xi\in\mathbb{S}^{n-1}$ yields

$$|\Phi(x)-\Phi(\xi)|^2=\left|\Phi'(\xi)\right|\left|\Phi'(x)\right||x-\xi|^2. \tag{2.4.4}$$

Plug (2.4.3) and (2.4.4) into $\widetilde{p}_\alpha(\Phi(x),\Phi(\xi))$ we have

$$\begin{aligned}
\widetilde{p}_\alpha(\Phi(x),\Phi(\xi)) &= 2^{\alpha-1}c_{n,\alpha}\frac{(1-|\Phi(x)|^2)^{1-\alpha}}{|\Phi(x)-\Phi(\xi)|^{n-\alpha}}\\
&= 2^{\alpha-1}c_{n,\alpha}\frac{(1-|x|^2)^{1-\alpha}|\Phi'(x)|^{1-\alpha}}{|x-\xi|^{n-\alpha}|\Phi'(x)|^{(n-\alpha)/2}|\Phi'(\xi)|^{(n-\alpha)/2}}\\
&= \widetilde{p}_\alpha(x,\xi)|\Phi'(x)|^{(2-n-\alpha)/2}|\Phi'(\xi)|^{(\alpha-n)/2}.
\end{aligned}$$

□

For any $\Phi\in SO(n,1)$ and any $\widetilde{f}\in L^{\frac{2(n-1)}{n-2+\alpha}}(\mathbb{S}^{n-1})$ we define

$$\widetilde{f}_\Phi(\xi)=\widetilde{f}\circ\Phi(\xi)\left|\Phi'(\xi)\right|^{(n-2+\alpha)/2}. \tag{2.4.5}$$

Then it is easy to see that $\widetilde{f}_\Phi\in L^{\frac{2(n-1)}{n-2+\alpha}}(\mathbb{S}^{n-1})$ and that

$$\|\widetilde{f}_\Phi\|_{L^{\frac{2(n-1)}{n-2+\alpha}}(\mathbb{S}^{n-1})}=\|\widetilde{f}\|_{L^{\frac{2(n-1)}{n-2+\alpha}}(\mathbb{S}^{n-1})}.$$

Moreover, the extension of $\widetilde{f}_\Phi$ as defined in (2.3.1) transforms in the following way:

$$\widetilde{P}_\alpha\left(\widetilde{f}_\Phi\right)(x)=\left(\widetilde{P}_\alpha\widetilde{f}\right)\circ\Phi(x)\left|\Phi'(x)\right|^{(n-2+\alpha)/2}. \tag{2.4.6}$$

2.5 Compactness

The goal of this section is to prove that the extension operator $\widetilde{P}_\alpha: L^p(\mathbb{S}^{n-1})\to L^q(\mathbb{B}^n)$ is compact for certain choices of p and q. This is done in Corollary 2.10. Before we can prove Corollary 2.10, we need to prove an estimate in Proposition 2.9. The proof of Proposition 2.9 depends on the following important technical lemma:

Lemma 2.7. *For any $n \geq 2$, $\alpha \in [2-n, 1)$ and $r \in (0,1)$*

$$\int_{\mathbb{S}_r^{n-1}} \widetilde{p}_\alpha(x,\xi)dx \leq 1,$$

The notation $\mathbb{S}_r^{n-1}$ is as in definition 2.1.

Proof.

$$\begin{aligned} &\int_{\mathbb{S}_r^{n-1}} \widetilde{p}_\alpha(x,\xi)dx \\ =\;& 2^{\alpha-1}c_{n,\alpha}|\mathbb{S}^{n-2}|r^{n-1}(1-r^2)^{1-\alpha}\int_0^\pi \frac{\sin^{n-2}(\phi)}{(r^2+1-2r\cos(\phi))^{(n-\alpha)/2}}d\phi. \end{aligned}$$

Using u-substitution, take $u = \tan(\phi/2)$, we get

$$\begin{aligned} &\int_0^\pi \frac{\sin^{n-2}(\phi)}{(r^2+1-2r\cos(\phi))^{(n-\alpha)/2}}d\phi \\ =\;& \int_0^\infty \left(\frac{2u}{1+u^2}\right)^{n-2} \frac{1}{\left(r^2+1-2r\left(\frac{1-u^2}{1+u^2}\right)\right)^{(n-\alpha)/2}} \frac{2du}{1+u^2} \\ =\;& \frac{2^{n-1}}{(1-r)^{n-\alpha}}\int_0^\infty \frac{u^{n-2}}{(1+u^2)^{(n-2+\alpha)/2}} \frac{du}{\left(\left(\frac{1+r}{1-r}\right)^2 u^2+1\right)^{(n-\alpha)/2}} \end{aligned} \tag{2.5.1}$$

Using u-substitution again, take $v = \frac{1+r}{1-r}u$, we have

$$\begin{aligned} &\int_0^\pi \frac{\sin^{n-2}(\phi)}{(r^2+1-2r\cos(\phi))^{(n-\alpha)/2}}d\phi \\ =\;& \frac{2^{n-1}(1-r)^{\alpha-1}}{(1+r)^{n-1}}\int_0^\infty \frac{v^{n-2}}{\left(1+\left(\frac{1-r}{1+r}\right)^2 v^2\right)^{(n-2+\alpha)/2}} \frac{dv}{(v^2+1)^{(n-\alpha)/2}} \\ \leq\;& \frac{2^{n-1}(1-r)^{\alpha-1}}{(1+r)^{n-1}}\int_0^\infty \frac{v^{n-2}dv}{(v^2+1)^{(n-\alpha)/2}} \\ =\;& \frac{2^{n-2}(1-r)^{\alpha-1}\Gamma(\frac{1-\alpha}{2})\Gamma(\frac{n-1}{2})}{(1+r)^{n-1}\Gamma(\frac{n-\alpha}{2})} \end{aligned}$$

Overall, we have

$$\begin{aligned} \int_{\mathbb{S}_r^{n-1}} \widetilde{p}_\alpha(x,\xi)dx \;\leq\;& \frac{2^{n-3+\alpha}r^{n-1}c_{n,\alpha}|\mathbb{S}^{n-2}|\Gamma(\frac{1-\alpha}{2})\Gamma(\frac{n-1}{2})}{(1+r)^{n-2+\alpha}\Gamma(\frac{n-\alpha}{2})} \\ =\;& \frac{2^{n-2+\alpha}r^{n-1}}{(1+r)^{n-2+\alpha}} \\ \leq\;& 1. \end{aligned}$$

Here we used (1.2.3) and the fact that the function $\frac{r^{n-1}}{(1+r)^{n-2+\alpha}}$ is an increasing function of r for $2-n \leq \alpha < 1$ and $0 < r < 1$. $\square$

Remark 5. *From (2.5.1) we see that when $\alpha = 2-n$*

$$\begin{aligned}
&\int_0^\pi \frac{\sin^{n-2}(\phi)}{(r^2+1-2r\cos(\phi))^{n-1}} d\phi \\
&= \int_0^\infty \left(\frac{2u}{1+u^2}\right)^{n-2} \frac{1}{\left(r^2+1-2r\left(\frac{1-u^2}{1+u^2}\right)\right)^{n-1}} \frac{2du}{1+u^2} \\
&= \frac{2^{n-1}}{(1-r)^{2n-2}} \int_0^\infty \frac{u^{n-2}du}{\left(\left(\frac{1+r}{1-r}\right)^2 u^2+1\right)^{n-1}}.
\end{aligned}$$

Now if we take $v = \frac{1+r}{1-r}u$ then we have

$$\begin{aligned}
\int_0^\pi \frac{\sin^{n-2}(\phi)}{(r^2+1-2r\cos(\phi))^{n-1}} d\phi &= \frac{2^{n-1}}{(1-r^2)^{n-1}} \int_0^\infty \frac{v^{n-2}dv}{(1+v^2)^{n-1}} \\
&= \frac{2^{n-2}\left(\Gamma\left(\frac{n-1}{2}\right)\right)^2}{(1-r^2)^{n-1}\Gamma(n-1)},
\end{aligned}$$

where in the last step we used (1.2.3). As a result, for any $x \in \mathbb{B}^n$

$$\int_{\mathbb{S}^{n-1}} \widetilde{p}_{2-n}(x,\xi) = 1.$$

Remark 6. *From the calculation in Lemma 2.7 we can also get a lower bound for the integration. Note that*

$$\begin{aligned}
&\int_0^\pi \frac{\sin^{n-2}(\phi)}{(r^2+1-2r\cos(\phi))^{(n-\alpha)/2}} d\phi \\
= \quad & \frac{2^{n-1}(1-r)^{\alpha-1}}{(1+r)^{n-1}} \int_0^\infty \frac{v^{n-2}}{\left(1+\left(\frac{1-r}{1+r}\right)^2 v^2\right)^{(n-2+\alpha)/2}} \frac{dv}{(v^2+1)^{(n-\alpha)/2}} \\
\geq \quad & \frac{2^{n-1}(1-r)^{\alpha-1}}{(1+r)^{n-1}} \int_0^\infty \frac{v^{n-2}dv}{(v^2+1)^{n-1}} \\
= \quad & \frac{2^{n-2}(1-r)^{\alpha-1}(\Gamma(\frac{n-1}{2}))^2}{(1+r)^{n-1}\Gamma(n-1)}.
\end{aligned}$$

As a result, we have

$$\int_{\mathbb{S}_r^{n-1}} \widetilde{p}_\alpha(x,\xi)dx \geq \frac{2^{n-2+\alpha} r^{n-1}\Gamma(\frac{n-\alpha}{2})\Gamma(\frac{n-1}{2})}{(1+r)^{n-2+\alpha}\Gamma(n-1)\Gamma(\frac{1-\alpha}{2})}$$

Remark 7. *In the calculation of Lemma 2.7, we have*

$$\frac{v^{n-2}}{\left(1+\left(\frac{1-r}{1+r}\right)^2 v^2\right)^{(n-2+\alpha)/2}(v^2+1)^{(n-\alpha)/2}} \leq \frac{v^{n-2}}{(v^2+1)^{(n-\alpha)/2}},$$

for all $r \in [0,1]$ and all $v \geq 0$. As a result, by dominated convergence theorem, for any $r_0 \in [0,1]$ we have

$$\begin{aligned}
&\lim_{r\to r_0}\int_0^\infty \frac{v^{n-2}dv}{\left(1+\left(\frac{1-r}{1+r}\right)^2 v^2\right)^{(n-2+\alpha)/2}(v^2+1)^{(n-\alpha)/2}} \\
=\ &\int_0^\infty \frac{v^{n-2}dv}{\left(1+\left(\frac{1-r_0}{1+r_0}\right)^2 v^2\right)^{(n-2+\alpha)/2}(v^2+1)^{(n-\alpha)/2}},
\end{aligned}$$

In particular, we have

$$\begin{aligned}
&\lim_{r\to 1}\int_0^\infty \frac{v^{n-2}dv}{\left(1+\left(\frac{1-r}{1+r}\right)^2 v^2\right)^{(n-2+\alpha)/2}(v^2+1)^{(n-\alpha)/2}} \\
=\ &\int_0^\infty \frac{v^{n-2}dv}{(v^2+1)^{(n-\alpha)/2}}.
\end{aligned}$$

Before we can prove the estimate in Proposition 2.9 we need to define the weak norm:

Definition 2.8. *Define the weak norm $L^p_W(\mathbb{B}^n)$, such that*

$$|u|_{L^p_W(\mathbb{B}^n)} = \sup_{t>0} t\,||u|>t|^{\frac{1}{p}}.$$

Here $||u|>t|$ is the measure of the set $\{|u|>t\}$.

We are now ready to prove the following estimates, the proof uses the same method as in [18].

Proposition 2.9. *For any $n \geq 2$ and any $2 - n \leq \alpha < 1$ the extension operator $\widetilde{P}_\alpha$ satisfies*

$$\left|\widetilde{P}_\alpha f\right|_{L_W^{\frac{n}{n-1}}(\mathbb{B}^n)} \leq C(n,\alpha)\|f\|_{L^1(\mathbb{S}^{n-1})},$$

and

$$\left\|\widetilde{P}_\alpha f\right\|_{L^{\frac{np}{n-1}}(\mathbb{B}^n)} \leq C(n,\alpha,p)\|f\|_{L^p(\mathbb{S}^{n-1})}$$

Proof. Note that we only need to prove the weak estimate. The strong estimate follows from Marcinkiewicz interpolation theorem and the fact that for any $x \in \mathbb{B}^n$

$$|\widetilde{P}_\alpha f(x)| \leq |f|_{L^\infty(\mathbb{S}^{n-1})} \int_{\mathbb{S}^{n-1}} \widetilde{p}_\alpha(x,\xi)d\xi \leq C(n,\alpha)|f|_{L^\infty(\mathbb{S}^{n-1})},$$

here the last step follows from Lemma 2.7. The constant $C(n,\alpha)$ here only depends on n and α. Note that it is different from the notation $c_{n,\alpha}$, and that $C(n,\alpha)$ changes through out the dissertation. To prove the weak type estimate. Assume that $f \geq 0$ and $|f|_{L^1(\mathbb{S}^{n-1})} = 1$. Note that

$$\begin{aligned}
\widetilde{p}_\alpha(x,\xi) &= 2^{\alpha-1}c_{n,\alpha}\frac{(1-|x|^2)^{1-\alpha}}{|x-\xi|^{n-\alpha}} \\
&\leq C(n,\alpha)\frac{(1-|x|^2)^{1-\alpha}}{(1-|x|)^{n-\alpha}} \\
&\leq \frac{C(n,\alpha)}{(1-|x|)^{n-1}}.
\end{aligned} \tag{2.5.2}$$

As a result, we have

$$0 \leq \widetilde{P}_\alpha f \leq \frac{C(n,\alpha)}{(1-|x|)^{n-1}}. \tag{2.5.3}$$

From (2.5.3) we conclude that

$$|\widetilde{P}_\alpha f > \lambda| = |\{x \in \mathbb{B}^n : 1-|x| < C(n,\alpha)\lambda^{-\frac{1}{n-1}}, \widetilde{P}_\alpha f > \lambda\}|$$

If $1 \leq C(n,\alpha)\lambda^{-\frac{1}{n-1}}$, then we have

$$\begin{aligned}
|\widetilde{P}_\alpha f > \lambda| &\leq \frac{1}{\lambda}\int_{\mathbb{B}^n} \widetilde{P}_\alpha f dx d\xi \\
&\leq \frac{1}{\lambda}\int_{\mathbb{S}^{n-1}} f(\xi)\int_{\mathbb{B}^n} \widetilde{p}_\alpha(x,\xi) dx d\xi \\
&\leq \frac{C(n,\alpha)}{\lambda} \\
&\leq C(n,\alpha)\lambda^{-\frac{n}{n-1}}\lambda^{\frac{1}{n-1}} \\
&\leq C(n,\alpha)\lambda^{-\frac{n}{n-1}}.
\end{aligned}$$

Note that here we used Lemma 2.7 and the fact that $\lambda^{\frac{1}{n-1}} \leq C(n,\alpha)$. Note also that the constant $C(n,\alpha)$ changes along the argument.

If $C(n,\alpha)\lambda^{-\frac{1}{n-1}} < 1$ then we can define $r_0 = 1 - C(n,\alpha)\lambda^{-\frac{1}{n-1}}$, then we have

$$\begin{aligned}
|\widetilde{P}_\alpha f > \lambda| &\leq \frac{1}{\lambda}\int_{\mathbb{B}^n \setminus \mathbb{B}^n_{r_0}} \widetilde{P}_\alpha f dx d\xi \\
&\leq \frac{1}{\lambda}\int_{\mathbb{S}^{n-1}} f(\xi)\int_{\mathbb{B}^n \setminus \mathbb{B}^n_{r_0}} \widetilde{p}_\alpha(x,\xi) dx d\xi \\
&\leq \frac{C(n,\alpha)(1-r_0)}{\lambda} \\
&\leq C(n,\alpha)\lambda^{-\frac{n}{n-1}}.
\end{aligned}$$

This finishes the proof. □

With the help of Proposition 2.9 we can prove the following:

Corollary 2.10. *For any $n \geq 2$, $2-n \leq \alpha < 1$, $1 \leq p < \infty$, and $1 \leq q < \frac{np}{n-1}$ the operator $\widetilde{P}_\alpha : L^p(\mathbb{S}^{n-1}) \to L^q(\mathbb{B}^n)$ is compact.*

Proof. First assume $1 < p < \infty$. Suppose we have a sequence of function $f_i \in L^p(\mathbb{S}^{n-1})$ such that $|f_i|_{L^p(\mathbb{S}^{n-1})} \leq 1$. Then from (2.5.3) we have for all i and all $x \in \mathbb{B}^n$

$$|\widetilde{P}_\alpha f_i(x)| \leq \frac{C(n,\alpha)}{(1-|x|)^{n-1}}.$$

By Schauder estimate, there exists $u \in C^2(\mathbb{B}^n)$ such that $\widetilde{P}_\alpha f_i \to u$ in $C^2_{loc}(\mathbb{B}^n)$. As a result we have: for $r \in (0,1)$

$$
\begin{aligned}
|\widetilde{P}_\alpha f_i - \widetilde{P}_\alpha f_j|_{L^q(\mathbb{B}^n)} &\leq |\widetilde{P}_\alpha f_i - \widetilde{P}_\alpha f_j|_{L^q(\mathbb{B}^n_r)} + |\widetilde{P}_\alpha f_i - \widetilde{P}_\alpha f_j|_{L^q(\mathbb{B}^n \setminus \mathbb{B}^n_r)} \\
&\leq |\widetilde{P}_\alpha f_i - \widetilde{P}_\alpha f_j|_{L^q(\mathbb{B}^n_r)} \\
&\quad + |\widetilde{P}_\alpha f_i - \widetilde{P}_\alpha f_j|_{L^{\frac{np}{n-1}}(\mathbb{B}^n \setminus \mathbb{B}^n_r)} |\mathbb{B}^n \setminus \mathbb{B}^n_r|^{\frac{1}{q} - \frac{n-1}{np}} \\
&\leq |\widetilde{P}_\alpha f_i - \widetilde{P}_\alpha f_j|_{L^q(\mathbb{B}^n_r)} + C(n,\alpha,p) |\mathbb{B}^n \setminus \mathbb{B}^n_r|^{\frac{1}{q} - \frac{n-1}{np}},
\end{aligned}
$$

where we used Holder inequality and Proposition 2.9. Hence

$$
\limsup_{i,j \to \infty} |\widetilde{P}_\alpha f_i - \widetilde{P}_\alpha f_j|_{L^q(\mathbb{B}^n)} \leq C(n,\alpha,p) |\mathbb{B}^n \setminus \mathbb{B}^n_r|^{\frac{1}{q} - \frac{n-1}{np}}.
$$

Letting $r \to 1$, we see that $\widetilde{P}_\alpha f_i$ is a Cauchy sequence in $L^q(\mathbb{B}^n)$, hence $\widetilde{P}_\alpha : L^p(\mathbb{S}^{n-1}) \to L^q(\mathbb{B}^n)$ is compact. □

We can see that $\widetilde{P}_\alpha : L^{\frac{2(n-1)}{n-2+\alpha}}(\mathbb{S}^{n-1}) \to L^{\frac{2n}{n-2+\alpha}}(\mathbb{B}^n)$ is not compact in the following example, which is similar to the example given in [6, Chapter 1].

Remark 8. *We consider a sequence of conformal transformation $\Phi_a : \overline{\mathbb{B}}^n \to \overline{\mathbb{B}}^n$ defined by*

$$
\Phi_a(x) = \frac{a|x-a|^2 + (1-|a|^2)(a-x)}{|a|^2|a^* - x|^2}.
$$

Here $a \in \mathbb{B}^n$ such that $a = (0, ..., 0, 1-\epsilon)$ for some $\epsilon \in (0,1)$, and $a^* = \frac{a}{|a|^2}$. From (2.4.3) we see that for any $x \in \mathbb{B}^n$

$$
|\Phi_a'(x)| = \frac{1 - |\Phi_a(x)|^2}{1 - |x|^2} = \frac{1 - |a|^2}{|a|^2 |a^* - x|^2},
$$

take limit $x \to \xi$ for some $\xi \in \mathbb{S}^{n-1}$ we get

$$
|\Phi_a'(\xi)| = \frac{1 - |a|^2}{|a|^2 |a^* - \xi|^2} = \frac{\epsilon(2+\epsilon)}{(1-\epsilon)^2 |a^* - \xi|^2}.
$$

If $\xi \neq (0, ..., 0, 1)$, then it is easy to see that $\lim_{\epsilon \to 0} |\Phi_a'(\xi)| = 0$. If $\xi = (0, ..., 0, 1) = \frac{a}{|a|}$, then we have

$$
\left| \Phi_a' \left(\frac{a}{|a|} \right) \right| = \frac{\epsilon(2+\epsilon)}{\epsilon^2},
$$

hence $\lim_{\epsilon\to 0}\left|\Phi_a'\left(\frac{a}{|a|}\right)\right| = \infty$.

Now consider the function $\widetilde{f}:\mathbb{S}^{n-1}\to\mathbb{R}$ *such that* $\widetilde{f}=1$. *Define* $\widetilde{f}_{\Phi_a}$ *as in (2.4.5), then it is easy to see that*

$$\|\widetilde{f}_{\Phi_a}\|_{L^{\frac{2(n-1)}{n-2+\alpha}}(\mathbb{S}^{n-1})} = \|\widetilde{f}\|_{L^{\frac{2(n-1)}{n-2+\alpha}}(\mathbb{S}^{n-1})},$$

and that $\widetilde{f}_{\Phi_a}$ *weakly converges to the zero function in* $L^{\frac{2(n-1)}{n-2+\alpha}}(\mathbb{S}^{n-1})$. *For any given* $x\in\mathbb{B}^n$, *think of* $\widetilde{p}_\alpha(x,\xi)$ *as a function of* ξ, *using the* L^∞ *bound (2.5.2) we can show that*

$$\lim_{\epsilon\to 0}\widetilde{P}_\alpha\widetilde{f}_{\Phi_a}(x)=0.$$

Now we can show that $\widetilde{P}_\alpha\widetilde{f}_{\Phi_a}$ *weakly converges to the zero function in* $L^{\frac{2n}{n-2+\alpha}}(\mathbb{B}^n)$. *For any function in the dual space* $h\in L^{\frac{2n}{n+2-\alpha}}(\mathbb{B}^n)$ *and any* $r\in(0,1)$, *we have*

$$\begin{aligned}&\int_{\mathbb{B}^n}\widetilde{P}_\alpha\widetilde{f}_{\Phi_a}(x)h(x)dx\\ &=\int_{\mathbb{B}^n\setminus\mathbb{B}^n_r}\widetilde{P}_\alpha\widetilde{f}_{\Phi_a}(x)h(x)dx+\int_{\mathbb{B}^n_r}\widetilde{P}_\alpha\widetilde{f}_{\Phi_a}(x)h(x)dx\\ &\le\left\|\widetilde{P}_\alpha\widetilde{f}_{\Phi_a}\right\|_{L^{\frac{2n}{n-2+\alpha}}(\mathbb{B}^n)}\|h\|_{L^{\frac{2n}{n+2-\alpha}}(\mathbb{B}^n\setminus\mathbb{B}^n_r)}+\int_{\mathbb{B}^n_r}\widetilde{P}_\alpha\widetilde{f}_{\Phi_a}(x)h(x)dx,\end{aligned}$$

where the second step follows from Hölder's inequality. Note that from (2.4.6) we can see that

$$\left\|\widetilde{P}_\alpha\widetilde{f}_{\Phi_a}\right\|_{L^{\frac{2n}{n-2+\alpha}}(\mathbb{B}^n)}=\left\|\widetilde{P}_\alpha\widetilde{f}\right\|_{L^{\frac{2n}{n-2+\alpha}}(\mathbb{B}^n)}.$$

By dominated convergence theorem we have

$$\lim_{r\to 1}\|h\|_{L^{\frac{2n}{n+2-\alpha}}(\mathbb{B}^n\setminus\mathbb{B}^n_r)}=0.$$

Combine the L^∞ *bound (2.5.3) with dominated convergence theorem we see that for any* $r\in(0,1)$

$$\lim_{\epsilon\to 0}\int_{\mathbb{B}^n_r}\widetilde{P}_\alpha\widetilde{f}_{\Phi_a}(x)h(x)dx=0.$$

Now for any $\delta>0$ *small, we can choose* $r\in(0,1)$ *such that*

$$\|h\|_{L^{\frac{2n}{n+2-\alpha}}(\mathbb{B}^n\setminus\mathbb{B}^n_r)}<\delta.$$

For this given r, we can choose $\epsilon > 0$ small such that

$$\int_{\mathbb{B}^n_r} \widetilde{P}_\alpha \widetilde{f}_{\Phi_a}(x)h(x)dx < \delta.$$

Combine these results, we see that

$$\lim_{\epsilon \to 0} \int_{\mathbb{B}^n} \widetilde{P}_\alpha \widetilde{f}_{\Phi_a}(x)h(x)dx = 0$$

for any $h \in L^{\frac{2n}{n+2-\alpha}}(\mathbb{B}^n)$.

But since

$$\left\| \widetilde{P}_\alpha \widetilde{f}_{\Phi_a} \right\|_{L^{\frac{2n}{n-2+\alpha}}(\mathbb{B}^n)} = \left\| \widetilde{P}_\alpha \widetilde{f} \right\|_{L^{\frac{2n}{n-2+\alpha}}(\mathbb{B}^n)} \neq 0,$$

we conclude that $\widetilde{P}_\alpha : L^{\frac{2(n-1)}{n-2+\alpha}}(\mathbb{S}^{n-1}) \to L^{\frac{2n}{n-2+\alpha}}(\mathbb{B}^n)$ is not compact.

2.6 Extremal Function

Using Corollary 2.10, we can identify the extremal function in the same way as in [18]. In order to do so we need the help of the Kazdan-Warner type condition in the following lemma

Lemma 2.11. *Suppose $\alpha \in (2-n, 1)$, and $K,\ f \in C^1(\mathbb{S}^{n-1})$ such that for any $\xi \in \mathbb{S}^{n-1}$*

$$K(\xi)f(\xi)^{\frac{n-\alpha}{n-2+\alpha}} = \int_{\mathbb{B}^n} \widetilde{p}_\alpha(x,\xi) \left(\widetilde{P}_\alpha f(x) \right)^{\frac{n+2-\alpha}{n-2+\alpha}} dx.$$

Let X be a conformal vector field in $\overline{\mathbb{B}}^n$, then we have

$$\int_{\mathbb{S}^{n-1}} XK \cdot f^{\frac{2(n-1)}{n-2+\alpha}} d\xi = 0.$$

Proof. Consider the functional

$$I(K,f) = \frac{|\widetilde{P}_\alpha f|_{L^{\frac{2n}{n-2+\alpha}}(\mathbb{B}^n)}}{\left(\int_{\mathbb{S}^{n-1}} K \cdot f^{\frac{2(n-1)}{n-2+\alpha}} d\xi \right)^{\frac{n-2+\alpha}{2(n-1)}}}$$

with Euler-Lagrange equation

$$K(\xi)f(\xi)^{\frac{n-\alpha}{n-2+\alpha}} = \int_{\mathbb{B}^n} \widetilde{p}_\alpha(x,\xi)\left(\widetilde{P}_\alpha f(x)\right)^{\frac{n+2-\alpha}{n-2+\alpha}} dx.$$

Consider Φ_t as the 1-parameter family of conformal group generated by X. Define

$$f_{\Phi_t} = f \circ \Phi_t(\xi)|\Phi_t'(\xi)|^{\frac{n-2+\alpha}{2}}.$$

Since f is a critical function for the functional $I(K,f)$, we have

$$\frac{d}{dt}\bigg|_{t=0} I(K, f_{\Phi_t}) = 0.$$

Where according to the calculation of conformal invariance in the beginning, we have

$$\begin{aligned}
I(K, f_{\Phi_t}) &= \frac{|\widetilde{P}_\alpha f_{\Phi_t}|_{L^{\frac{2n}{n-2+\alpha}}(\mathbb{B}^n)}}{\left(\int_{\mathbb{S}^{n-1}} K \cdot f_{\Phi_t}^{\frac{2(n-1)}{n-2+\alpha}} d\xi\right)^{\frac{n-2+\alpha}{2(n-1)}}} \\
&= \frac{|\widetilde{P}_\alpha f|_{L^{\frac{2n}{n-2+\alpha}}(\mathbb{B}^n)}}{\left(\int_{\mathbb{S}^{n-1}} K \circ \Phi_{-t} \circ \Phi_t(\xi) \cdot (f \circ \Phi_t(\xi))^{\frac{2(n-1)}{n-2+\alpha}} |\Phi_t'(\xi)|^{n-1} d\xi\right)^{\frac{n-2+\alpha}{2(n-1)}}} \\
&= \frac{|\widetilde{P}_\alpha f|_{L^{\frac{2n}{n-2+\alpha}}(\mathbb{B}^n)}}{\left(\int_{\mathbb{S}^{n-1}} K \circ \Phi_{-t}(\xi) \cdot f^{\frac{2(n-1)}{n-2+\alpha}} d\xi\right)^{\frac{n-2+\alpha}{2(n-1)}}} \\
&= I(K \circ \Phi_{-t}, f).
\end{aligned}$$

As a result, we have

$$\frac{d}{dt}\bigg|_{t=0} I(K, f_{\Phi_t}) = \frac{d}{dt}\bigg|_{t=0} I(K \circ \Phi_{-t}, f) = 0.$$

From this we can conclude that

$$\int_{\mathbb{S}^{n-1}} XK \cdot f^{\frac{2(n-1)}{n-2+\alpha}} d\xi = 0.$$

□

Now we can find the extremal function and the sharp constant using subcritical approximation as in [18].

Theorem 2.12. *Assume $n \geq 3$ and $\alpha \in (2-n, 1)$. For every $f \in L^{\frac{2(n-1)}{n-2+\alpha}}(\mathbb{S}^{n-1})$, we have*

$$\left\| \widetilde{P}_\alpha f \right\|_{L^{\frac{2n}{n-2+\alpha}}(\mathbb{B}^n)} \leq S_{n,\alpha} \|f\|_{L^{\frac{2(n-1)}{n-2+\alpha}}(\mathbb{S}^{n-1})}.$$

Where $S_{n,\alpha}$ is a constant that only depends on n and α. Up to conformal transformation any constant is an optimizer.

Proof. For $p > \frac{2(n-1)}{n-2+\alpha}$, by corollary 2.10, the operator

$$\widetilde{P}_\alpha : L^p(\mathbb{S}^{n-1}) \to L^{\frac{2n}{n-2+\alpha}}(\mathbb{B}^n)$$

is compact. Consider the variational problem

$$S_{n,\alpha} = \sup \left\{ \left\| \widetilde{P}_\alpha f \right\|_{L^{\frac{2n}{n-2+\alpha}}(\mathbb{B}^n)} : f \in L^p(\mathbb{S}^{n-1}) \text{ such that } \|f\|_{L^{\frac{2(n-1)}{n-2+\alpha}}(\mathbb{S}^{n-1})} = 1 \right\}.$$

We show that the supremum is achieved as follows:

Consider a maximizing sequence $f_i \in L^p(\mathbb{S}^{n-1})$, with $f_i \geq 0$,

$$\|f_i\|_{L^p(\mathbb{S}^{n-1})} = 1$$

and

$$\lim_{i \to \infty} \|f_i\|_{L^p(\mathbb{S}^{n-1})} = S_{n,\alpha}.$$

By uniform boundedness of L^p norm, we know that there exists a subsequence f_i weakly converges to some function $f_p \in L^p(\mathbb{S}^{n-1})$. By compactness of $\widetilde{P}_\alpha$ we also know that there exists a subsequence f_i such that $\widetilde{P}_\alpha f_i$ converges to v in $L^{\frac{n}{n-2+\alpha}}(\mathbb{B}^n)$ norm. By the weak L^p convergence of f_i to f_p, we also have $\widetilde{P}_\alpha f_i$ converges to $\widetilde{P}_\alpha f_p$ pointwise. As a result we have $\widetilde{P}_\alpha f_p = v$ and the supremum $S_{n,\alpha}$ is achieved at f_p.

Replacing f_p by f_p^* if necessary, we may assume that f_p is radial symmetric and decreasing. Meaning that for $\xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n$ such that $|\xi| = 1$, the function $f(\xi)$ only depends on ξ_n and that $\frac{\partial f}{\partial \xi_n}(\xi) \leq 0$.

After rescaling, we may assume f_p satisfy the Euler-Lagrange equation

$$\int_{\mathbb{B}^n} \widetilde{p}_\alpha(x,\xi)(\widetilde{P}_\alpha f_p)(x)^{\frac{n+2-\alpha}{n-2+\alpha}} dx = f_p(\xi)^{p-1} = f_p(\xi)^{\frac{n-\alpha}{n-2+\alpha}} f_p(\xi)^{p-\frac{2(n-1)}{n-2+\alpha}}$$

Apply Proposition C.5 from the appendix, we know that $f_p \in C^1(\mathbb{S}^{n-1})$. By Lemma 2.11, we have

$$\int_{\mathbb{S}^{n-1}} \langle \nabla f_p(\xi)^{p-\frac{2(n-1)}{n-2+\alpha}}, \nabla \xi_n \rangle f_p(\xi)^{\frac{2(n-1)}{n-2}} d\xi = 0.$$

Consider the function $g_p(r) = f_p(0, ..., 0, \sin r, \cos r)$ for $r \in [0, \pi]$. The equality becomes

$$\int_0^\pi g_p'(r) g_p(r)^{p-1} \sin^{n-2}(r) dr = 0.$$

Note that $g_p' = -\partial_n f \sin(r) \geq 0$. Hence we know that f_p is actually a constant. $\square$

CHAPTER 3

LIMIT CASE

3.1 Chapter Outline

In this chapter we want to take limit $\alpha \to 2-n$ and study the limit case inequality. We take the limit in section 3.2. In the process of taking the limit, a very special function $\widetilde{I}_n$ shows up. The property of the function $\widetilde{I}_n$ is crucial in the study of the limit case inequality. We prove several important properties of the function $\widetilde{I}_n$ in section 3.3. We prove uniqueness of the limit case inequality using the moving sphere method in section 3.5, but before that we need to transform the limit case inequality to the upper half space in section 3.4.

Through out this chapter, we still use notations $\widetilde{f}$ and f to denote functions on $\mathbb{S}^{n-1}$ and $\mathbb{R}^{n-1}$ respectively, but the relation between them is different from the relation discussed in Chapter 2. We will specify their relation in (3.4.2) below.

3.2 Proof of Theorem 1.13: Limit Case Inequality

We consider the limit case $\alpha \to 2-n$ in the same way as [8], our statement and proof are slightly different.

For any $\widetilde{F} \in L^{\infty}(\mathbb{S}^{n-1})$, define $\widetilde{f} = 1 + \frac{n-2+\alpha}{2}\widetilde{F}$. We have $\widetilde{f} \in L^{\frac{2(n-1)}{n-2+\alpha}}(\mathbb{S}^{n-1})$ for all $\alpha \in (2-n, 1)$. We prove the following theorem for $\widetilde{F}$.

Theorem 3.1. *For dimension $n \geq 2$, and any function $\widetilde{F} \in L^{\infty}(\mathbb{S}^{n-1})$ we have*

$$\|e^{\widetilde{I}_n + \widetilde{P}_{2-n}\widetilde{F}}\|_{L^n(\mathbb{B}^n)} \leq S_n \|e^{\widetilde{F}}\|_{L^{n-1}(\mathbb{S}^{n-1})}. \tag{3.2.1}$$

Where $\widetilde{I}_n(x) = 2\frac{d\widetilde{P}_\alpha 1}{d\alpha}\big|_{\alpha=2-n}$. When n is even we have

$$\widetilde{I}_n(x) = \sum_{k=1}^{n/2-1} \frac{1}{2k} \cdot \frac{\Gamma\left(\frac{n-2}{2}\right)\Gamma\left(n-k-1\right)}{\Gamma(n-2)\Gamma\left(\frac{n}{2}-k\right)}(1-|x|^2)^k.$$

The sharp constant $S_n = \dfrac{\|e^{\widetilde{I}_n}\|_{L^n(\mathbb{B}^n)}}{|\mathbb{S}^{n-1}|^{\frac{1}{n-1}}}$

Proof. For any $\widetilde{F} \in L^\infty(\mathbb{S}^{n-1})$, define $\widetilde{f} = 1 + \frac{n-2+\alpha}{2}\widetilde{F}$. Define $\epsilon = n-2+\alpha$, from theorem 2.12, we have

$$\|\widetilde{P}_\alpha(1+\epsilon\widetilde{F})\|_{L^{\frac{2n}{n-2+\alpha}}(\mathbb{B}^n)} \leq S_{n,\alpha}\|1+\epsilon\widetilde{F}\|_{L^{\frac{2(n-1)}{n-2+\alpha}}(\mathbb{S}^{n-1})},$$

which is equivalent to

$$\left(\int_{\mathbb{B}^n}(\widetilde{P}_\alpha 1)^{\frac{n}{\epsilon}}\left(1+\frac{\epsilon\widetilde{P}_\alpha\widetilde{F}}{\widetilde{P}_\alpha 1}\right)^{\frac{n}{\epsilon}}\right)^{\frac{1}{n}} \leq (S_{n,\alpha})^{\frac{1}{\epsilon}}\left(\int_{\mathbb{S}^{n-1}}(1+\epsilon\widetilde{F})^{\frac{n-1}{\epsilon}}\right)^{\frac{1}{n-1}}.$$

As in [8], we need to find a lower bound for $\widetilde{P}_\alpha 1$ and an upper bound for $(\widetilde{P}_\alpha 1)^{\frac{n}{\epsilon}}$.

We handle the lower bound for $\widetilde{P}_\alpha 1$ firstly. From remark 6, we know that

$$\widetilde{P}_\alpha 1 \geq \frac{\Gamma\left(\frac{n-\alpha}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}{\Gamma(n-1)\Gamma\left(\frac{1-\alpha}{2}\right)}.$$

Since $\frac{\Gamma\left(\frac{n-\alpha}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}{\Gamma(n-1)\Gamma\left(\frac{1-\alpha}{2}\right)}$ is a continuous function of α for all $\alpha \in [2-n, 1)$, and

$$\frac{\Gamma\left(\frac{n-\alpha}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}{\Gamma(n-1)\Gamma\left(\frac{1-\alpha}{2}\right)} > 0$$

for $\alpha < 1$. As a result for some $0 < \alpha_0 < 1$, there exists $m > 0$ such that

$$\frac{\Gamma\left(\frac{n-\alpha}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}{\Gamma(n-1)\Gamma\left(\frac{1-\alpha}{2}\right)} \geq m > 0$$

for all α such that $2-n \leq \alpha < \alpha_0 < 1$. Here m will be the lower bound for the function $\widetilde{P}_\alpha 1(x)$. Note that it does not depend on α or x.

Now we consider the upper bound for $(\widetilde{P}_\alpha 1)^{\frac{n}{\epsilon}}$. From Lemma 2.7, we have

$$\widetilde{P}_\alpha 1 = \frac{\int_{\mathbb{S}_r^{n-1}}\widetilde{p}_\alpha(x,\xi)dx}{r^{n-1}} \leq \left(\frac{2}{1+r}\right)^{n-2+\alpha},$$

for all $\alpha \in [2-n, 1)$. As a result we have an upper bound

$$(\widetilde{P}_\alpha 1)^{\frac{n}{\epsilon}} \leq \left(\frac{2}{1+r}\right)^n \leq 2^n,$$

for all $\alpha \in [2-n, 1)$. If we define $\widetilde{I}_n = 2\frac{d\widetilde{P}_\alpha 1}{d\alpha}|_{\alpha=2-n}$, then with the help of the lower bound for $\widetilde{P}_\alpha 1(x)$ and the upper bound for $(\widetilde{P}_\alpha 1)^{\frac{n}{\epsilon}}$, we can apply dominated convergence theorem to get

$$\lim_{\epsilon \to 0} \int_{\mathbb{B}^n} (\widetilde{P}_\alpha 1)^{\frac{n}{\epsilon}} \left(1 + \frac{\epsilon \widetilde{P}_\alpha \widetilde{F}}{\widetilde{P}_\alpha 1}\right)^{\frac{n}{\epsilon}} = \int_{\mathbb{B}^n} e^{n\widetilde{I}_n + n\widetilde{P}_{2-n}\widetilde{F}}.$$

For the right hand side of the inequality we can get

$$\lim_{\epsilon \to 0} \int_{\mathbb{S}^{n-1}} (1 + \epsilon \widetilde{F})^{\frac{n-1}{\epsilon}} = \int_{\mathbb{S}^{n-1}} e^{(n-1)\widetilde{F}}.$$

In order to find the limit $\lim_{\epsilon \to 0} (S_{n,\alpha})^{\frac{1}{n}}$, first note that since constant is an optimizer in theorem 2.12. As a result, if we take $\widetilde{F} = 0$, then we can have

$$\lim_{\epsilon \to 0} (S_{n,\alpha})^{\frac{1}{\epsilon}} = \lim_{\epsilon \to 0} \frac{\left(\int_{\mathbb{B}^n} (\widetilde{P}_\alpha 1)^{\frac{n}{\epsilon}}\right)^{\frac{1}{n}}}{|\mathbb{S}^{n-1}|^{\frac{1}{n-1}}} = \frac{\|e^{\widetilde{I}_n}\|_{L^n(\mathbb{B}^n)}}{|\mathbb{S}^{n-1}|^{\frac{1}{n-1}}}.$$

When n is an even integer, using mathematical induction and (3.3.10) it is easy to prove that

$$\widetilde{I}_n(x) = \sum_{k=1}^{n/2-1} \frac{1}{2k} \cdot \frac{\Gamma\left(\frac{n-2}{2}\right)\Gamma(n-k-1)}{\Gamma(n-2)\Gamma\left(\frac{n}{2}-k\right)} (1-|x|^2)^k.$$

This finishes the proof. □

3.3 The Function $\widetilde{I}_n$

The function $\widetilde{I}_n$ naturally appears in the process of taking limit; its properties are very important for subsequent analysis. Yang [25] found an explicit formula for the function $\widetilde{I}_n$ when n is an even integer. In the previous section we use a different method to find $\widetilde{I}_n$ when n is even. Our method relies on the induction relation (3.3.10) to be proved in this section. When n is odd we do not have an explicit formula for $\widetilde{I}_n$, but the same induction relation still apply.

In Subsection 3.3.1 we consider how $\widetilde{I}_n$ transforms under conformal transformation. In Subsection 3.3.3 we prove a very useful induction relation, which helps us to determine the explicit formula for $\widetilde{I}_n$ in even dimensional unit ball. The induction relation is the basis for the proofs in Subsection 3.3.5 and Subsection 3.3.6

3.3.1 Conformal Transformation of $\widetilde{I}_n$

In this subsection we want to take a closer look at how the function $\widetilde{I}_n$ changes under conformal transformation $\Phi : \mathbb{B}^n \to \mathbb{B}^n$, as well as how the it changes under the projection map $\Psi : \mathbb{R}^n_+ \to \mathbb{B}^n$. These are given in (3.3.2) and (3.3.3) respectively.

To begin with, we have

$$\widetilde{P}_\alpha 1 = 2^{\alpha-1} c_{n,\alpha} \int_{\mathbb{S}^{n-1}} \frac{(1-|x|^2)^{1-\alpha}}{|x-\xi|^{n-\alpha}} d\xi$$

Taking derivative with respect to α at $\alpha = 2-n$, we get

$$\begin{aligned}
\left.\frac{d\widetilde{P}_\alpha 1}{d\alpha}\right|_{\alpha=2-n} = & \left(\left.\frac{d}{d\alpha}\right|_{\alpha=2-n} (2^{\alpha-1}c_{n,\alpha})\right) \int_{\mathbb{S}^{n-1}} \frac{(1-|x|^2)^{n-1}}{|x-\xi|^{2n-2}} d\xi \\
& -2^{1-n} c_{n,2-n} \int_{\mathbb{S}^{n-1}} \frac{(1-|x|^2)^{n-1} \ln\big(1-|x|^2\big)}{|x-\xi|^{2n-2}} d\xi \\
& +2^{1-n} c_{n,2-n} \int_{\mathbb{S}^{n-1}} \frac{(1-|x|^2)^{n-1} \ln|x-\xi|}{|x-\xi|^{2n-2}} d\xi
\end{aligned}$$

Note that

$$\begin{aligned}
& \left.\frac{d}{d\alpha}\right|_{\alpha=2-n} (2^{\alpha-1}c_{n,\alpha}) \int_{\mathbb{S}^{n-1}} \frac{(1-|x|^2)^{n-1}}{|x-\xi|^{2n-2}} d\xi \\
= & \frac{\frac{d}{d\alpha}|_{\alpha=2-n}(2^{\alpha-1}c_{n,\alpha})}{2^{1-n}c_{n,2-n}} \\
= & \ln(2) - \frac{\psi^0(n-1)}{2} + \frac{\psi^0(\frac{n-1}{2})}{2},
\end{aligned}$$

where $\psi^0(x) = \frac{d}{dx}\ln(\Gamma(x))$ is the polygamma function. Hence the result simplifies to

$$\begin{aligned}
\left.\frac{d\widetilde{P}_\alpha 1}{d\alpha}\right|_{\alpha=2-n} = & \, 2^{1-n} c_{n,2-n} \int_{\mathbb{S}^{n-1}} \frac{(1-|x|^2)^{n-1} \ln|x-\xi|}{|x-\xi|^{2n-2}} d\xi \\
& - \ln\Big(1-|x|^2\Big) + \ln(2) - \frac{\psi^0(n-1)}{2} + \frac{\psi^0(\frac{n-1}{2})}{2}
\end{aligned} \tag{3.3.1}$$

Under conformal transformation Φ, we can see that

$$\begin{aligned}
& \int_{\mathbb{S}^{n-1}} \widetilde{p}_\alpha(\Phi(x),\xi)\ln|\Phi(x)-\xi|d\xi \\
= & \int_{\mathbb{S}^{n-1}} (\widetilde{p}_\alpha(x,\xi)\ln|x-\xi|)|\Phi'(x)|^{\frac{2-n-\alpha}{2}}|\Phi'(\xi)|^{\frac{n-2+\alpha}{2}}d\xi \\
& + \int_{\mathbb{S}^{n-1}} (\widetilde{p}_\alpha(x,\xi)\ln|\Phi'(\xi)|^{\frac{1}{2}})|\Phi'(x)|^{\frac{2-n-\alpha}{2}}|\Phi'(\xi)|^{\frac{n-2+\alpha}{2}}d\xi \\
& + \int_{\mathbb{S}^{n-1}} (\widetilde{p}_\alpha(x,\xi)\ln|\Phi'(x)|^{\frac{1}{2}})|\Phi'(x)|^{\frac{2-n-\alpha}{2}}|\Phi'(\xi)|^{\frac{n-2+\alpha}{2}}d\xi.
\end{aligned}$$

When we take limit $\alpha \to 2-n$, we get

$$\begin{aligned}
\int_{\mathbb{S}^{n-1}} \widetilde{p}_{2-n}(\Phi(x),\xi)\ln|\Phi(x)-\xi|d\xi &= \int_{\mathbb{S}^{n-1}} \widetilde{p}_{2-n}(x,\xi)\ln|x-\xi|d\xi \\
&\quad + \frac{1}{2}\widetilde{P}_{2-n}(\ln|\Phi'(\xi)|) + \frac{1}{2}\ln|\Phi'(x)|.
\end{aligned}$$

From which we can get the conformal change for $\frac{d\widetilde{P}_\alpha 1}{d\alpha}\big|_{\alpha=2-n}$

$$\left.\frac{d\widetilde{P}_\alpha 1}{d\alpha}\right|_{\alpha=2-n} \circ \Phi(x) + \frac{1}{2}\ln|\Phi'(x)| = \left.\frac{d\widetilde{P}_\alpha 1}{d\alpha}\right|_{\alpha=2-n}(x) + \frac{1}{2}\widetilde{P}_{2-n}(\ln|\Phi'(\xi)|).$$

Recall that for any $x \in \mathbb{B}^n$, we define

$$\widetilde{I}_n(x) = 2\left.\frac{d\widetilde{P}_\alpha 1}{d\epsilon}\right|_{\alpha=2-n}(x),$$

then we have

$$\widetilde{I}_n \circ \Phi(x) + \ln|\Phi'(x)| = \widetilde{I}_n(x) + \widetilde{P}_{2-n}(\ln|\Phi'(\xi)|). \tag{3.3.2}$$

From this we see that $\widetilde{I}_n$ is a radial function, since when Φ is a rotation we have $\ln|\Phi'(x)| = 0$ for any $x \in \mathbb{B}^n$ and $\ln|\Phi'(\xi)| = 0$ for any $\xi \in \mathbb{S}^{n-1}$. As a result, we sometimes think of $\widetilde{I}_n(x)$ as $\widetilde{I}_n(r)$ for $r = |x|$.

We also want to consider how $\widetilde{I}_n$ changes under the transformation $\Psi : \mathbb{R}^n_+ \to \mathbb{B}^n$. For any $y \in \mathbb{R}^n_+$ define

$$\begin{aligned}
I_n(y) &= 2\left.\frac{dP_\alpha 1}{d\alpha}\right|_{\alpha=2-n} \\
&= -2\int_{\mathbb{R}^{n-1}} p_{2-n}(y,u)\ln\frac{y_n}{|y-u|}du - \psi^0(n-1) + \psi^0\left(\frac{n-1}{2}\right).
\end{aligned}$$

Note that since

$$\begin{aligned} P_\alpha 1 &= c_{n,\alpha} \int_{\mathbb{R}^{n-1}} \frac{y_n^{1-\alpha}}{(|y'-u|^2+y_n^2)^{\frac{n-\alpha}{2}}} du \\ &= c_{n,\alpha} \int_{\mathbb{R}^{n-1}} \frac{1}{(|\frac{y'}{y_n}-\frac{u}{y_n}|^2+1)^{\frac{n-\alpha}{2}}} \frac{du}{y_n^{n-1}}, \end{aligned}$$

through change of variable, we get

$$P_\alpha 1 = c_{n,\alpha} \int_{\mathbb{R}^{n-1}} \frac{1}{(|u|^2+1)^{\frac{n-\alpha}{2}}} du = c_{n,\alpha}|\mathbb{S}^{n-2}| \int_0^\infty \frac{r^{n-2}dr}{(r^2+1)^{\frac{n-\alpha}{2}}} = 1.$$

Note that in the last step we used (1.2.3). From this it is easy to see that

$$I_n(y) = 2\frac{dP_\alpha 1}{d\alpha}\bigg|_{\alpha=2-n} = 0$$

for all $y \in \mathbb{R}^n_+$. Using (2.4.1) we can show that

$$\widetilde{I}_n \circ \Psi(y) + \ln|\Psi'(y)| = I_n(y) + P_{2-n}\ln|\Psi'(w)| = P_{2-n}\ln|\Psi'(w)|. \tag{3.3.3}$$

Note that here $w \in \mathbb{R}^{n-1}$, while $|\Psi'(y)|$ and $|\Psi'(w)|$ are as in (2.1.3) and (2.1.4) respectively.

3.3.2 Simplify the Function $\widetilde{I}_n$

In this subsection we want to further simplify the function $\widetilde{I}_n$. We write the integration in the polar coordinate in the Euclidean ball $\mathbb{B}^n$, then (3.3.1) becomes

$$\begin{aligned} \frac{d\widetilde{P}_\alpha 1}{d\alpha}\bigg|_{\alpha=2-n} &= \frac{2^{1-n}\Gamma(n-1)}{\Gamma\left(\frac{n-1}{2}\right)^2}(1-r^2)^{n-1}\int_0^\pi \frac{\sin^{n-2}\phi\ln\big(1-2r\cos\phi+r^2\big)d\phi}{(1-2r\cos\phi+r^2)^{n-1}} \\ &\quad -\ln\Big(1-r^2\Big)+\ln(2)-\frac{\psi^0(n-1)}{2}+\frac{\psi^0(\frac{n-1}{2})}{2}. \end{aligned}$$

Note that here we also used the explicit formula for the constant $c_{n,2-n}$ from (1.2.3).

Polygamma functions have two special properties that are useful to us. The first property is

$$\psi^0(n) = -\gamma + \sum_{k=1}^{n-1}\frac{1}{k}, \text{ where } n \in \mathbb{N}^+. \tag{3.3.4}$$

Here γ is the Euler Mascheroni constant. The second property is

$$\psi^0(2z) = \frac{1}{2}\left(\psi^0(z) + \psi^0(z+\frac{1}{2})\right) + \ln(2), \text{ where } z \in \mathbb{C}^*. \tag{3.3.5}$$

For any $n \in \mathbb{N}^+$ such that $n \geq 2$, plug $z = \frac{n-1}{2}$ into (3.3.5) we get

$$\psi^0\left(\frac{n-1}{2}\right) - \psi^0(n-1) = \psi^0(n-1) - \psi^0\left(\frac{n}{2}\right) - \ln(4). \tag{3.3.6}$$

Combine (3.3.6) and (3.3.4) we get that when $n \in \mathbb{N}^+$ is an even integer then

$$\psi^0\left(\frac{n-1}{2}\right) = -\ln(4) - \gamma + \sum_{k=n/2}^{n-2}\frac{1}{k} + \sum_{k=1}^{n-2}\frac{1}{k} \tag{3.3.7}$$

Now combine (3.3.4), (3.3.6) and (3.3.7) we see that for any $n \geq 2$ such that n is an even integer:

$$\psi^0\left(\frac{n-1}{2}\right) - \psi^0(n-1) = -\log(4) + \sum_{k=n/2}^{n-2}\frac{1}{k}.$$

Using (3.3.4) we can see that for any $n \geq 2$ such that n is an odd integer:

$$\psi^0\left(\frac{n-1}{2}\right) - \psi^0(n-1) = -\sum_{k=(n-1)/2}^{n-2}\frac{1}{k}.$$

As a result, when $n \geq 2$ is an even integer, equation (3.3.1) simplifies to

$$\begin{aligned}\left.\frac{d\widetilde{P}_\alpha 1}{d\alpha}\right|_{\alpha=2-n} &= \frac{2^{1-n}\Gamma(n-1)}{\Gamma\left(\frac{n-1}{2}\right)^2}\int_0^\pi \frac{(1-r^2)^{n-1}\sin^{n-2}\phi\ln(1-2r\cos\phi+r^2)}{(1-2r\cos\phi+r^2)^{n-1}}d\phi \\ &\quad - \ln\Big(1-r^2\Big) + \frac{1}{2}\sum_{k=n/2}^{n-2}\frac{1}{k}.\end{aligned} \tag{3.3.8}$$

When $n \geq 2$ is an odd integer, equation (3.3.1) simplifies to

$$\begin{aligned}\left.\frac{d\widetilde{P}_\alpha 1}{d\alpha}\right|_{\alpha=2-n} &= \frac{2^{1-n}\Gamma(n-1)}{\Gamma\left(\frac{n-1}{2}\right)^2}\int_0^\pi \frac{(1-r^2)^{n-1}\sin^{n-2}\phi\ln(1-2r\cos\phi+r^2)}{(1-2r\cos\phi+r^2)^{n-1}}d\phi \\ &\quad - \ln\Big(1-r^2\Big) + \ln 2 - \frac{1}{2}\sum_{k=(n-1)/2}^{n-2}\frac{1}{k}.\end{aligned} \tag{3.3.9}$$

3.3.3 Induction Relation

If we take derivative of $\widetilde{I}_n$ with respect to r, then we have the following induction relation.

Lemma 3.2. *For $n \in \mathbb{N}^+$ such that $n > 3$, $\widetilde{I}_n$ satisfies the induction relation*

$$\widetilde{I}_n = \frac{1-r^4}{4r(n-3)}\frac{d}{dr}(\widetilde{I}_{n-2}) + \widetilde{I}_{n-2} + \frac{1-r^2}{2(n-3)}. \tag{3.3.10}$$

Proof. The main calculation here is to use integration by parts to evaluate the integral

$$\int_0^{\pi} \frac{\sin^{n-2}\phi \ln(1-2r\cos\phi+r^2)}{(1-2r\cos\phi+r^2)^{n-1}} d\phi.$$

Take

$$v = \sin^{n-3}\phi$$

then we have

$$dv = (n-3)\sin^{n-4}\phi\cos\phi d\phi.$$

Take

$$dw = \frac{\sin\phi \ln(1-2r\cos\phi+r^2)}{(1-2r\cos\phi+r^2)^{n-1}} d\phi,$$

then we have

$$w = -\frac{1}{2r}\left(\frac{\ln(1-2r\cos\phi+r^2)}{(n-2)(1-2r\cos\phi+r^2)^{n-2}} + \frac{1}{(n-2)^2(1-2r\cos\phi+r^2)^{n-2}}\right).$$

As a result, we have

$$\begin{aligned}
&\int_0^{\pi} \frac{\sin^{n-2}\phi \ln(1-2r\cos\phi+r^2)d\phi}{(1-2r\cos\phi+r^2)^{n-1}} \\
&= \frac{(n-3)(1+r^2)}{(n-2)4r^2}\int \frac{\sin^{n-4}\phi \ln(1-2r\cos\phi+r^2)d\phi}{(1-2r\cos\phi+r^2)^{n-2}} \\
&- \frac{(n-3)}{(n-2)4r^2}\int \frac{\sin^{n-4}\phi \ln(1-2r\cos\phi+r^2)d\phi}{(1-2r\cos\phi+r^2)^{n-3}} \\
&+ \frac{(n-3)(1+r^2)}{(n-2)^2 4r^2}\int \frac{\sin^{n-4}\phi d\phi}{(1-2r\cos\phi+r^2)^{n-2}} \\
&- \frac{n-3}{(n-2)^2 4r^2}\int \frac{\sin^{n-4}\phi d\phi}{(1-2r\cos\phi+r^2)^{n-3}}.
\end{aligned}$$

Note that here we used

$$\int vdw = vw - \int wdv = -\int wdv,$$

and that $\sin 0 = \sin\pi = 0$. We evaluate each one of the integrals separately in the next subsection. Using the results from next subsection, namely by (3.3.14), (3.3.15) and (3.3.16), we have

$$\begin{aligned}
&\int_0^\pi \frac{\sin^{n-2}\phi\ln(1-2r\cos\phi+r^2)d\phi}{(1-2r\cos\phi+r^2)^{n-1}} \\
&= \frac{(1+r^2)}{(n-2)4r^2}\frac{r}{(1-r^2)}\frac{d}{dr}\left(\int\frac{\sin^{n-4}\phi\ln(1-2r\cos\phi+r^2)d\phi}{(1-2r\cos\phi+r^2)^{n-3}}\right) \\
&+ \frac{(n-3)(1+r^2)}{(n-2)4r^2}\frac{1}{1-r^2}\int\frac{\sin^{n-4}\phi\ln(1-2r\cos\phi+r^2)d\phi}{(1-2r\cos\phi+r^2)^{n-3}} \\
&+ \frac{(n-3)(1+r^2)}{(n-2)4r^2}\frac{2^{n-4}\Gamma(\frac{n-3}{2})^2}{\Gamma(n-3)}\frac{2r^2}{(n-3)(1-r^2)^{n-1}} \\
&- \frac{(n-3)}{(n-2)4r^2}\int\frac{\sin^{n-4}\phi\ln(1-2r\cos\phi+r^2)d\phi}{(1-2r\cos\phi+r^2)^{n-3}} \\
&+ \frac{(n-3)(1+r^2)}{(n-2)^2 4r^2}\frac{2^{n-4}\Gamma(\frac{n-3}{2})^2}{\Gamma(n-3)}\frac{1+r^2}{(1-r^2)^{n-1}} \\
&- \frac{n-3}{(n-2)^2 4r^2}\frac{2^{n-4}\Gamma(\frac{n-3}{2})^2}{\Gamma(n-3)}\frac{1}{(1-r^2)^{n-3}}.
\end{aligned}$$

After reordering, we have

$$\begin{aligned}
&\int_0^\pi \frac{\sin^{n-2}\phi\ln(1-2r\cos\phi+r^2)d\phi}{(1-2r\cos\phi+r^2)^{n-1}} \\
&= \frac{(1+r^2)}{(n-2)4r(1-r^2)}\frac{d}{dr}\left(\int\frac{\sin^{n-4}\phi\ln(1-2r\cos\phi+r^2)d\phi}{(1-2r\cos\phi+r^2)^{n-3}}\right) \\
&+ \frac{(n-3)}{2(n-2)(1-r^2)}\int\frac{\sin^{n-4}\phi\ln(1-2r\cos\phi+r^2)d\phi}{(1-2r\cos\phi+r^2)^{n-3}} \\
&+ \frac{1}{n-2}\frac{2^{n-4}\Gamma(\frac{n-3}{2})^2}{\Gamma(n-3)}\left(\frac{n-3}{n-2}+\frac{1+r^2}{2}\right)\frac{1}{(1-r^2)^{n-1}}
\end{aligned}$$

If we define

$$a_n = \frac{2^{1-n}\Gamma(n-1)}{\Gamma\left(\frac{n-1}{2}\right)^2}(1-r^2)^{n-1}\int\frac{\sin^{n-2}\phi\ln(1-2r\cos\phi+r^2)d\phi}{(1-2r\cos\phi+r^2)^{n-1}},$$

then we have the following induction relation

$$a_n = \frac{1-r^4}{4r(n-3)}\frac{d}{dr}(a_{n-2}) + a_{n-2} + \frac{1}{2(n-2)} + \frac{1+r^2}{4(n-3)} \tag{3.3.11}$$

When n is even, from (3.3.8) we see that

$$\widetilde{I}_n = 2\frac{d\widetilde{P}_\alpha 1}{d\epsilon}|_{\epsilon=0} = 2a_n - 2\ln\left(1-r^2\right) + \sum_{k=\frac{n}{2}}^{n-2}\frac{1}{k}. \tag{3.3.12}$$

Taking derivative with respect to r we see that

$$\frac{d}{dr}(\widetilde{I}_{n-2}) = 2\frac{d}{dr}(a_{n-2}) + \frac{4r}{1-r^2}. \tag{3.3.13}$$

Combine (3.3.11), (3.3.12) and (3.3.13) we get that

$$\begin{aligned}
\widetilde{I}_n &= \frac{1-r^4}{4r(n-3)}\frac{d}{dr}(2a_{n-2}) + 2a_{n-2} + \frac{1}{n-2} + \frac{1+r^2}{2(n-3)} \\
&\quad - 2\ln\left(1-r^2\right) + \sum_{k=n/2}^{n-2}\frac{1}{k} \\
&= \frac{1-r^4}{4r(n-3)}\left(\frac{d}{dr}(\widetilde{I}_{n-2}) - \frac{4r}{1-r^2}\right) \\
&\quad + 2a_{n-2} - 2\ln\left(1-r^2\right) + \sum_{k=(n-2)/2}^{n-4}\frac{1}{k} \\
&\quad + \frac{1}{n-3} + \frac{1+r^2}{2(n-3)} \\
&= \frac{1-r^4}{4r(n-3)}\frac{d}{dr}(\widetilde{I}_{n-2}) + \widetilde{I}_{n-2} + \frac{1-r^2}{2(n-3)}.
\end{aligned}$$

This is the end of the calculation for the case when n is even.

In the case when n is odd, from (3.3.9) we have

$$\widetilde{I}_n = 2\frac{d\widetilde{P}_\alpha 1}{d\epsilon}|_{\epsilon=0} = 2a_n - 2\ln\left(1-r^2\right) + \ln 4 - \sum_{k=(n-1)/2}^{n-2}\frac{1}{k}.$$

Going through similar calculations as in the case when n is even, we can see that for the case n is odd we have exactly the same induction relation. □

Using the induction relation (3.3.10) it is easy to find an explicit formula for $\widetilde{I}_n$ when n is even.

3.3.4 Supplementary Calculation

In this subsection we continue several calculations from previous subsection. We use k to denote any positive integer.

Lemma 3.3. *For any $k \in \mathbb{N}^+$ such that $k \geq 2$*

$$\int_0^\pi \frac{\sin^{k-1}\phi d\phi}{(1-2r\cos\phi+r^2)^k} = \frac{2^k}{|\mathbb{S}^{k-1}|c_{k+1,1-k}} \frac{1}{(1-r^2)^k}. \tag{3.3.14}$$

Proof. This follows directly from Remark 5. □

Lemma 3.4. *For any $k \in \mathbb{N}^+$ such that $k \geq 2$*

$$\int_0^\pi \frac{\sin^{k-1}\phi d\phi}{(1-2r\cos\phi+r^2)^{k+1}} = \frac{2^{k-1}(\Gamma(k/2))^2}{\Gamma(k)} \frac{1+r^2}{(1-r^2)^{k+2}} \tag{3.3.15}$$

Proof. By taking derivative with respect to r, we get

$$\begin{aligned} &\frac{r}{k(1-r^2)}\frac{d}{dr}\left(\int_0^\pi \frac{\sin^{k-1}\phi d\phi}{(1-2r\cos\phi+r^2)^k}\right) \\ &= \int_0^\pi \frac{\sin^{k-1}\phi d\phi}{(1-2r\cos\phi+r^2)^{k+1}} - \frac{1}{1-r^2}\int_0^\pi \frac{\sin^{k-1}\phi d\phi}{(1-2r\cos\phi+r^2)^k}. \end{aligned}$$

Combine this with (3.3.14) then we are done. □

Lemma 3.5. *For any $k \in \mathbb{N}^+$ such that $k \geq 2$*

$$\begin{aligned} &\int_0^\pi \frac{\sin^{k-1}\phi \ln(1-2r\cos\phi+r^2)d\phi}{(1-2r\cos\phi+r^2)^{k+1}} \\ &= \frac{r}{k(1-r^2)}\frac{d}{dr}\left(\int_0^\pi \frac{\sin^{k-1}\phi \ln(1-2r\cos\phi+r^2)d\phi}{(1-2r\cos\phi+r^2)^k}\right) \\ &+ \frac{1}{1-r^2}\int_0^\pi \frac{\sin^{k-1}\phi \ln(1-2r\cos\phi+r^2)d\phi}{(1-2r\cos\phi+r^2)^k} \\ &+ \frac{2^{k-1}(\Gamma(k/2))^2}{\Gamma(k)}\frac{2r^2}{k(1-r^2)^{k+2}}. \end{aligned}$$

Proof. Take derivative of $\int_0^\pi \frac{\sin^{k-1}\phi \ln(1-2r\cos\phi+r^2)d\phi}{(1-2r\cos\phi+r^2)^k}$ with respect to r, we can get

$$\begin{aligned}
&\int_0^\pi \frac{\sin^{k-1}\phi \ln(1-2r\cos\phi+r^2)d\phi}{(1-2r\cos\phi+r^2)^{k+1}}\\
&=\frac{r}{k(1-r^2)}\frac{d}{dr}\left(\int_0^\pi \frac{\sin^{k-1}\phi \ln(1-2r\cos\phi+r^2)d\phi}{(1-2r\cos\phi+r^2)^k}\right)\\
&+\frac{1}{1-r^2}\int_0^\pi \frac{\sin^{k-1}\phi \ln(1-2r\cos\phi+r^2)d\phi}{(1-2r\cos\phi+r^2)^k}\\
&-\frac{1}{k(1-r^2)}\int_0^\pi \frac{\sin^{k-1}\phi d\phi}{(1-2r\cos\phi+r^2)^k}\\
&+\frac{1}{k}\int_0^\pi \frac{\sin^{k-1}\phi d\phi}{(1-2r\cos\phi+r^2)^{k+1}}.
\end{aligned}$$

By (3.3.14) and (3.3.15), we have

$$\begin{aligned}
&\frac{1}{k}\int \frac{\sin^{k-1}\phi d\phi}{(1-2r\cos\phi+r^2)^{k+1}}-\frac{1}{k(1-r^2)}\int \frac{\sin^{k-1}\phi d\phi}{(1-2r\cos\phi+r^2)^k}\\
&=\frac{2^{k-1}(\Gamma(k/2))^2}{\Gamma(k)}\frac{2r^2}{k(1-r^2)^{k+2}}
\end{aligned}$$

Combine these two equations we can get (3.3.16). □

3.3.5 Hyperbolic Harmonic Through Induction

Using the induction relation (3.3.10), we can prove that $\widetilde{I}_n \circ \Psi(y)+\ln|\Psi'(y)|$ is harmonic with respect to the standard hyperbolic metric. We prove it in the unit ball model of hyperbolic space.

Lemma 3.6. *For $n \geq 2$, in $\mathbb{B}^n$ we have*

$$\Delta_{\mathbb{H}}\left(\widetilde{I}_n+\ln\frac{1-2x_n+|x|^2}{2}\right)=0.$$

Here $\Delta_{\mathbb{H}}$ is the Laplacian in hyperbolic space. For any function $u \in C^\infty(\mathbb{B}^n)$ we have

$$\Delta_{\mathbb{H}}u=\left(\frac{1-|x|^2}{2}\right)^2\Delta u+(n-2)\frac{1-|x|^2}{2}\langle x,\nabla u\rangle.$$

Where Δ is the Laplacian in Euclidean space and $\langle x,\nabla u\rangle$ is the inner product in Euclidean space.

Proof. Through direct calculation we have

$$\nabla \ln \frac{1-2x_n+|x|^2}{2} = \frac{2x-2e_n}{1-2x_n+|x|^2},$$

here e_n means the unit vector in the direction x_n.

As a result, we have

$$\Delta \ln \frac{1-2x_n+|x|^2}{2} = \frac{2n-4}{1-2x_n+|x|^2},$$

and

$$\begin{aligned}
\Delta_{\mathbb{H}} \ln \frac{1-2x_n+|x|^2}{2} &= \left(\frac{1-|x|^2}{2}\right)^2 \frac{2n-4}{1-2x_n+|x|^2} \\
&\quad +(n-2)\frac{1-|x|^2}{2}\left\langle x, \frac{2x-2e_n}{1-2x_n+|x|^2}\right\rangle \\
&= \frac{(1-|x|^2)^2(n-2)+(n-2)(1-|x|^2)(2|x|^2-2x_n)}{2(1-2x_n+|x|^2)} \\
&= \frac{(n-2)(1-|x|^2)}{2}
\end{aligned}$$

Next we want to show that

$$\Delta_{\mathbb{H}} \widetilde{I}_n = -\frac{(n-2)(1-|x|^2)}{2}. \tag{3.3.16}$$

Since $\widetilde{I}_n$ is a radial function, we can verify this in polar coordinates in the Euclidean unit ball. Where we have

$$\Delta_{\mathbb{H}} \widetilde{I}_n = \left(\frac{1-r^2}{2}\right)^2 \partial_r^2 \widetilde{I}_n + \left(\frac{1-r^2}{2}\right)^2 \frac{n-1}{r}\partial_r \widetilde{I}_n + \frac{(n-2)r(1-r^2)}{2}\partial_r \widetilde{I}_n$$

For $n=2$ it is easy to see that $\widetilde{I}_2 = 0$, and for $n=3$, we can integrate by part to get $\widetilde{I}_3 = \ln(4) + \frac{(1-r)^2\ln(1-r)-(1+r)^2\ln(1+r)}{2r}$. So it is easy to verify by direct calculation that (3.3.16) is true for $n=2$ and $n=3$.

When $n > 3$, suppose (3.3.16) is true for $\widetilde{I}_{n-2}$, from which we have

$$\begin{aligned}\Delta_{\mathbb{H}}\widetilde{I}_{n-2} &= \left(\frac{1-r^2}{2}\right)^2 \partial_r^2\widetilde{I}_{n-2} + \left(\frac{1-r^2}{2}\right)^2 \frac{n-3}{r}\partial_r\widetilde{I}_{n-2} \\ &\quad + \frac{(n-4)r(1-r^2)}{2}\partial_r\widetilde{I}_{n-2} \\ &= \left(\frac{1-r^2}{2}\right)^2 \partial_r^2\widetilde{I}_{n-2} + \frac{1-r^2}{4r}((n-3)+(n-5)r^2)\partial_r\widetilde{I}_{n-2} \\ &= -\frac{(n-4)(1-r^2)}{2}.\end{aligned}$$

After rearranging we get

$$\partial_r^2\widetilde{I}_{n-2} = -\frac{2(n-4)}{1-r^2} - \frac{2(n-4)r}{1-r^2}\partial_r\widetilde{I}_{n-2} - \frac{n-3}{r}\partial_r\widetilde{I}_{n-2}. \tag{3.3.17}$$

Consider (3.3.10), take derivative with respect to r, we get

$$\partial_r\widetilde{I}_n = \frac{1-r^4}{4r(n-3)}\partial_r^2\widetilde{I}_{n-2} + \left(1 - \frac{1+3r^4}{4r^2(n-3)}\right)\partial_r\widetilde{I}_{n-2} - \frac{r}{n-3},$$

and

$$\begin{aligned}\partial_r^2\widetilde{I}_n &= \frac{1-r^4}{4r(n-3)}\partial_r^3\widetilde{I}_{n-2} + \left(1 - \frac{1+3r^4}{2r^2(n-3)}\right)\partial_r^2\widetilde{I}_{n-2} \\ &\quad + \frac{1-3r^4}{2r^3(n-3)}\partial_r\widetilde{I}_{n-2} - \frac{1}{n-3}.\end{aligned}$$

Using (3.3.17), we have

$$\begin{aligned}\partial_r^3\widetilde{I}_{n-2} &= -\frac{4(n-4)r}{(1-r^2)^2} + \left(\frac{n-3}{r^2} - \frac{2(n-4)(1+r^2)}{(1-r^2)^2}\right)\partial_r\widetilde{I}_{n-2} \\ &\quad - \left(\frac{n-3}{r} + \frac{2(n-4)r}{1-r^2}\right)\partial_r^2\widetilde{I}_{n-2},\end{aligned}$$

and

$$\begin{aligned}&\frac{1-r^4}{4r(n-3)}\partial_r^3\widetilde{I}_{n-2} \\ = &-\frac{(n-4)(1+r^2)}{(n-3)(1-r^2)} + \left(\frac{1-r^4}{4r^3} - \frac{(n-4)(1+r^2)^2}{2r(n-3)(1-r^2)}\right)\partial_r\widetilde{I}_{n-2} \\ &-\left(\frac{1-r^4}{4r^2} + \frac{(n-4)(1+r^2)}{2(n-3)}\right)\partial_r^2\widetilde{I}_{n-2}\end{aligned}$$

As a result, we have

$$\begin{aligned}
\partial_r^2 \widetilde{I}_n &= \left(\frac{-(n+1)r^4+2(n-2)r^2-(n-1)}{4r^2(n-3)}\right)\partial_r^2 \widetilde{I}_{n-2} \\
&\quad + \left(\frac{(n-1)-(3n-9)r^2-(5n-13)r^4-(n-11)r^6}{4r^3(n-3)(1-r^2)}\right)\partial_r \widetilde{I}_{n-2} \\
&\quad - \frac{(n-3)+(n-5)r^2}{(n-3)(1-r^2)}
\end{aligned}$$

Hence

$$\begin{aligned}
&\left(\frac{1-r^2}{2}\right)^2 \partial_r^2 \widetilde{I}_n \\
&= \left(\frac{-(n+1)r^4+2(n-2)r^2-(n-1)}{4r^2(n-3)}\right)\left(\frac{1-r^2}{2}\right)^2 \partial_r^2 \widetilde{I}_{n-2} \\
&\quad + \left(\frac{(n-1)-(3n-9)r^2-(5n-13)r^4-(n-11)r^6}{8r^3(n-3)}\right)\frac{1-r^2}{2}\partial_r \widetilde{I}_{n-2} \\
&\quad - \frac{(n-3)+(n-5)r^2}{2(n-3)}\left(\frac{1-r^2}{2}\right)
\end{aligned}$$

$$\begin{aligned}
&\left(\frac{1-r^2}{2}\right)^2 \frac{n-1}{r}\partial_r \widetilde{I}_n \\
&= \frac{(n-1)(1-r^4)}{4r^2(n-3)}\left(\frac{1-r^2}{2}\right)^2 \partial_r^2 \widetilde{I}_{n-2} \\
&\quad + \left(\frac{(n-1)(1-r^2)^2}{4r} - \frac{(n-1)(1-r^2)^2(1+3r^4)}{16r^3(n-3)}\right)\partial_r \widetilde{I}_{n-2} \\
&\quad - \frac{(n-1)(1-r^2)^2}{4(n-3)}
\end{aligned}$$

$$\begin{aligned}
&\frac{(n-2)r(1-r^2)}{2}\partial_r \widetilde{I}_n \\
&= \frac{(n-2)(1+r^2)}{2(n-3)}\left(\frac{1-r^2}{2}\right)^2 \partial_r^2 \widetilde{I}_{n-2} \\
&\quad + \left(1-\frac{1+3r^4}{4(n-3)r^2}\right)\frac{(n-2)r(1-r^2)}{2}\partial_r \widetilde{I}_{n-2} \\
&\quad - \frac{(n-2)r^2(1-r^2)}{2(n-3)}
\end{aligned}$$

Adding them up, we get

$$\begin{aligned}\Delta_{\mathbb{H}}\widetilde{I}_n &= \frac{(n-2)-r^2}{n-3}\left(\frac{1-r^2}{2}\right)^2\partial_r^2\widetilde{I}_{n-2}\\ &+\frac{-(n-5)r^4+(n^2-8n+13)r^2+(n-2)(n-3)}{2r(n-3)}\left(\frac{1-r^2}{2}\right)\partial_r\widetilde{I}_{n-2}\\ &-\frac{(1-r^2)((n-2)+(n-4)r^2)}{2(n-3)}\end{aligned}$$

Note that we have

$$\begin{aligned}&\frac{(n-2)-r^2}{(n-3)}\cdot\frac{(n-3)+(n-5)r^2}{2r}\\ =\ &\frac{-(n-5)r^4+(n^2-8n+13)r^2+(n-2)(n-3)}{2r(n-3)},\end{aligned}$$

as a result, we have

$$\begin{aligned}&\Delta_{\mathbb{H}}\widetilde{I}_n\\ =\ &\frac{(n-2)-r^2}{n-3}\left(\left(\frac{1-r^2}{2}\right)^2\partial_r^2\widetilde{I}_{n-2}+\frac{(n-3)+(n-5)r^2}{2r}\left(\frac{1-r^2}{2}\partial_r\widetilde{I}_{n-2}\right)\right)\\ &-\frac{(1-r^2)((n-2)+(n-4)r^2)}{2(n-3)}\end{aligned}$$

Use induction assumption in the form

$$\left(\frac{1-r^2}{2}\right)^2\partial_r^2\widetilde{I}_{n-2}+\frac{1-r^2}{4r}((n-3)+(n-5)r^2)\partial_r\widetilde{I}_{n-2}=-\frac{(n-4)(1-r^2)}{2},$$

Then we get

$$\Delta_{\mathbb{H}}\widetilde{I}_n=-\frac{(n-2)(1-r^2)}{2}.$$

Which finishes the proof. □

3.3.6 Boundary Value Through Induction

Using the induction relation (3.3.10), we can find boundary value for $\widetilde{I}_n$. We have the following lemma.

Lemma 3.7. *For $n > 2$ we have*

$$\lim_{r\to 1} \widetilde{I}_n = 0,$$

$$\lim_{r\to 1} \partial_r \widetilde{I}_n = -1,$$

and

$$\lim_{r\to 1} \frac{\widetilde{I}_n - 0}{r-1} = -1.$$

Proof. For $n = 2$, we have $\widetilde{I}_n = 0$. For $n = 3$ we have

$$\widetilde{I}_3 = \ln 4 + \frac{(1-r)^2 \ln(1-r) - (1+r)^2 \ln(1+r)}{2r}.$$

It is easy to see that

$$\lim_{r\to 1} \widetilde{I}_3 = 0,$$

$$\lim_{r\to 1} \partial_r \widetilde{I}_3 = -1,$$

and

$$\lim_{r\to 1} \frac{\widetilde{I}_3 - 0}{r-1} = -1.$$

In general we use induction to prove that

$$\lim_{r\to 1} \widetilde{I}_n = 0$$

and

$$\lim_{r\to 1} \partial_r \widetilde{I}_n = C.$$

Here $C = 0$ for $n = 2$ and $C = -1$ for $n \geq 3$. Suppose the induction assumption holds for $n-2$, then we have

$$\lim_{r\to 1} \widetilde{I}_{n-2} = 0$$

and

$$\lim_{r\to 1} \partial_r \widetilde{I}_{n-2} = C.$$

Now consider $\widetilde{I}_n$. By (3.3.10), we have

$$\lim_{r\to 1} \widetilde{I}_n = \lim_{r\to 1} \frac{1-r^4}{4r(n-3)} \lim_{r\to 1} \partial_r \widetilde{I}_{n-2} + \lim_{r\to 1} \widetilde{I}_{n-2} + \lim_{r\to 1} \frac{1-r^2}{2(n-3)}.$$

Using the induction assumption for $\widetilde{I}_{n-2}$, it is easy to see that

$$\lim_{r\to 1} \widetilde{I}_n = 0.$$

Start with (3.3.10), take derivative with respect to r, we get

$$\partial_r \widetilde{I}_n = \frac{1-r^4}{4r(n-3)}\partial_r^2 \widetilde{I}_{n-2} + \left(1 - \frac{1+3r^4}{4r^2(n-3)}\right)\partial_r \widetilde{I}_{n-2} - \frac{r}{n-3}.$$

Use (3.3.17) to substitute $\partial_r^2 \widetilde{I}_{n-2}$, then we get

$$\begin{aligned}\partial_r \widetilde{I}_n &= \frac{1-r^4}{4r(n-3)}\left(-\frac{2(n-4)}{1-r^2} - \frac{2(n-4)r}{1-r^2}\partial_r \widetilde{I}_{n-2} - \frac{n-3}{r}\partial_r \widetilde{I}_{n-2}\right) \\ &\quad + \left(1 - \frac{1+3r^4}{4r^2(n-3)}\right)\partial_r \widetilde{I}_{n-2} - \frac{r}{n-3}.\end{aligned}$$

Use the induction assumption and take limit we have

$$\lim_{r\to 1} \partial_r \widetilde{I}_n = -1.$$

Now we use induction to show that

$$\lim_{r\to 1} \frac{\widetilde{I}_n(r) - 0}{r-1} = C,$$

again, we have $C = 0$ for $n = 2$ and $C = -1$ for $n \geq 3$. Suppose it is true for $n-2$ then we have the induction assumption

$$\lim_{r\to 1} \frac{\widetilde{I}_{n-2}(r) - 0}{r-1} = C.$$

Use (3.3.10), then we have

$$\frac{\widetilde{I}_n - 0}{r-1} = -\frac{(1+r)(1+r^2)}{4r(n-3)}\partial_r \widetilde{I}_{n-2} + \frac{\widetilde{I}_{n-2} - 0}{r-1} - \frac{1+r}{2(n-3)}.$$

Now take limit and use the induction assumption, then we have

$$\lim_{r\to 1} \frac{\widetilde{I}_n(r) - 0}{r-1} = -\frac{C}{n-3} + C - \frac{1}{n-3}.$$

When $n \geq 5$, we have $\lim_{r\to 1} \partial_r \widetilde{I}_{n-2} = -1$ and $\lim_{r\to 1} \frac{\widetilde{I}_{n-2} - 0}{r-1} = -1$, and as a result we have

$$\lim_{r\to 1} \frac{\widetilde{I}_n(r) - 0}{r-1} = -1.$$

When $n=4$, we have $\lim_{r\to 1}\partial_r \widetilde{I}_{n-2}=0$ and $\lim_{r\to 1}\frac{\widetilde{I}_{n-2}-0}{r-1}=0$, and

$$\lim_{r\to 1}\frac{\widetilde{I}_n(r)-0}{r-1}=-\frac{1}{n-3}=-1.$$

This concludes the proof. □

3.4 Conformal Trasformation of the Inequality

We will prove Theorem 1.14 in the next section. In this section we consider how the inequality (3.2.1) and its corresponding Euler Lagrange equation transforms under conformal transformation $\Psi:\mathbb{R}^n_+\to\mathbb{B}^n$.

Note that in (3.2.1) we have the inequality

$$\|e^{\widetilde{I}_n+\widetilde{P}_{2-n}\widetilde{f}}\|_{L^n(\mathbb{B}^n)}\le S_n\|e^{\widetilde{f}}\|_{L^{n-1}(\mathbb{S}^{n-1})},$$

with Euler-Lagrange euquation

$$e^{(n-1)\widetilde{f}(\xi)}=\int_{\mathbb{B}^n}e^{n\widetilde{I}_n+n\widetilde{P}_{2-n}\widetilde{f}}\widetilde{p}_{2-n}(x,\xi)dx \tag{3.4.1}$$

Consider change of variable

$$\int_{\mathbb{S}^{n-1}}e^{(n-1)\widetilde{f}}d\xi=\int_{\mathbb{R}^{n-1}}e^{(n-1)(\widetilde{f}\circ\Psi+\ln|\Psi'(w)|)}dw.$$

If we define

$$f(w)=\widetilde{f}\circ\Psi(w)+\ln|\Psi'(w)|, \tag{3.4.2}$$

then we have $\left\|e^{\widetilde{f}}\right\|_{L^{n-1}(\mathbb{S}^{n-1})}=\left\|e^{f}\right\|_{L^{n-1}(\mathbb{R}^{n-1})}$. On the other hand, using the same change of variable, we have

$$\begin{aligned}\int_{\mathbb{B}^n}e^{n\widetilde{I}_n+n\widetilde{P}_{2-n}\widetilde{f}}dx &= \int_{\mathbb{R}^n_+}e^{n\widetilde{I}_n\circ\Psi+n(\widetilde{P}_{2-n}\widetilde{f})\circ\Psi+n\ln|\Psi'(y)|}dy\\ &= \int_{\mathbb{R}^n_+}e^{nP_{2-n}(\widetilde{f}\circ\Psi+\ln|\Psi'(w)|)}dy.\end{aligned}$$

Note that in the last step we used (2.4.1) and (3.3.3). Which means we have

$$\|e^{\widetilde{I}_n + \widetilde{P}_{2-n}\widetilde{f}}\|_{L^n(B^n)} = \|e^{P_{2-n}f}\|_{L^n(\mathbb{R}^n_+)}.$$

As a result, for any $\widetilde{f} \in L^\infty(\mathbb{B}^n)$, if we define a corresponding f as in (3.4.2), then we have the corresponding inequality

$$\|e^{P_{2-n}f}\|_{L^n(\mathbb{R}^n_+)} \leq S_n \|e^f\|_{L^{n-1}(\mathbb{R}^{n-1})}. \tag{3.4.3}$$

If we further suppose $\widetilde{f}$ satisfies the Euler-Lagrange equation (3.4.1), then f satisfies the Euler-Lagrange equation

$$e^{(n-1)f(w)} = \int_{\mathbb{R}^n_+} e^{nP_{2-n}f} p_{2-n}(y, w) dy. \tag{3.4.4}$$

3.5 Uniqueness Through the Method of Moving Spheres

In this section, we prove the uniqueness of solutions to equation (3.4.1). We define several notation related to the inversion with respect to a sphere in Subsection 3.5.1.

We prove Theorem 1.14 from Subsection 3.5.2 to Subsection 3.5.5. We will start with a smooth solution $\widetilde{f} \in C^\infty(\mathbb{S}^{n-1})$ to (3.4.1), then define the corresponding f as in (3.4.2). We will prove that the function f is unique up to conformal transformation using the method of moving sphere.

Note that this is different from directly proving uniqueness of smooth solutions to (3.4.4). Since by starting with smooth solutions of (3.4.1) we gain the asymptotic behavior $\ln|\Phi'(u)|$ as in (3.4.2). This asymptotic behavior helps us to start the sphere in subsequent parts.

3.5.1 Notation

For any $w \in \mathbb{R}^{n-1}$ and $\lambda \in \mathbb{R}$, define $v = (w, 0) \in \mathbb{R}^n$. For all $y \in \mathbb{R}^n$ such that $y \neq v$ define the inversion with respect to a sphere centered at v with radius λ as

$$\phi_{\lambda,v}(y) = v + \frac{\lambda^2}{|y - v|^2}(y - v). \tag{3.5.1}$$

Note that $\phi_{\lambda,v}(y)$ maps the upper half space into the itself, and $\phi_{\lambda,v}(y) : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is a conformal transformation. We use $|\phi'_{\lambda,v}(y)|$ to denote the conformal factor such that

$$\phi^*_{\lambda,v} dy^2 = |\phi'_{\lambda,v}(y)|^2 dy^2$$

Note that $|\phi'_{\lambda,v}(y)|^n$ is the Jacobian in $\mathbb{R}^n_+$. Through direct calculation using (3.5.1), we can see that

$$|\phi'_{\lambda,v}(y)|^n = \frac{\lambda^{2n}}{|y-v|^{2n}},$$

and

$$|\phi'_{\lambda,v}(y)| = \frac{\lambda^2}{|y-v|^2} \tag{3.5.2}$$

Note that since $\Psi \circ \phi_{\lambda,v} \circ \Psi^{-1}$ is a conformal transformation that maps $\mathbb{B}^n$ to itself, we have $\Psi \circ \phi_{\lambda,v} \circ \Psi^{-1} \in SO(n,1)$. For any $f : \mathbb{R}^{n-1} \to \mathbb{R}$, we can define $f_{\lambda,v}$ as the conformal transformation of f under $\phi_{\lambda,v}$ such that

$$f_{\lambda,v} = f \circ \phi_{\lambda,v} + \ln |\phi'_{\lambda,v}|. \tag{3.5.3}$$

In addition we define

$$\mathbb{B}_{\lambda,v} = \{y \in \mathbb{R}^n : |y-v| < \lambda\},$$

with $\mathbb{B}^+_{\lambda,v} = \mathbb{B}_{\lambda,v} \cap \mathbb{R}^n_+$ and $\overline{\mathbb{B}}^+_{\lambda,v}$ denotes the closure of $\mathbb{B}^+_{\lambda,v}$ in $\mathbb{R}^n$. We also use $\overline{\mathbb{R}}^n_+$ to denote the closure of $\mathbb{R}^n_+$ in $\mathbb{R}^n$.

3.5.2 Inversion with Respect to Spheres

We now start the proof of Theorem 1.14.

Suppose $\widetilde{f} \in C^\infty(\mathbb{S}^{n-1})$ is a solution to (3.4.1), for $w \in \partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$ define

$$f(w) = \widetilde{f} \circ \Psi(w) + \ln |\Psi'(w)|.$$

Through the discussion in Section 3.4 we know that f is a solution to (3.4.4). Under transformation ϕ_{λ,v_0} as defined in (3.5.1) we can define f_{λ,v_0} as in (3.5.3)

$$\begin{aligned} f_{\lambda,v_0}(w) &= f \circ \phi_{\lambda,v_0}(w) + \ln|\phi'_{\lambda,v_0}(w)| \\ &= \widetilde{f} \circ \Psi \circ \phi_{\lambda,v_0}(w) + \ln|\Psi'(\phi_{\lambda,v_0}(w))| + \ln|\phi'_{\lambda,v_0}(w)| \\ &= \widetilde{f} \circ \Psi \circ \phi_{\lambda,v_0}(w) + \ln\frac{2\lambda^2}{(1+|v_0|^2)|w-v_0|^2 + \lambda^4 + 2\lambda^2\langle v_0, w-v_0\rangle}. \end{aligned}$$

Note that here $|\Psi'(\phi_{\lambda,v_0}(w))|$ is as in (2.1.4) and $|\phi'_{\lambda,v_0}(w)|$ is as in (3.5.2). The notation $\langle v_0, u-v_0\rangle$ denotes the Euclidean inner product in $\mathbb{R}^n$.

In the special case $v_0 = 0$, we have

$$f_{\lambda,0}(v) = \widetilde{f} \circ \Psi \circ \phi_{\lambda,0}(v) + \ln\frac{2\lambda^2}{\lambda^4 + |v|^2}. \tag{3.5.4}$$

Remark 9. *Note that for any $\widetilde{f} \in C^\infty(\mathbb{S}^{n-1})$ and any $\lambda > 0$ the function $f_{\lambda,0}$ is smooth in $\mathbb{R}^{n-1}$. We can see this using the definition of ϕ_{λ,v_0} and Ψ, for any $w \in \mathbb{R}^{n-1}$*

$$\begin{aligned} \Psi \circ \phi_{\lambda,0}(w) &= \Psi\left(\frac{\lambda^2}{|w|^2}w\right) \\ &= \left(\frac{\frac{2\lambda^2}{|w|^2}w}{1+\frac{\lambda^4}{|w|^2}}, \frac{-1+\frac{\lambda^4}{|w|^2}}{1+\frac{\lambda^4}{|w|^2}}\right) \\ &= \left(\frac{2\lambda^2 w}{\lambda^4+|w|^2}, \frac{\lambda^4-|w|^2}{\lambda^4+|w|^2}\right). \end{aligned}$$

If we define $\Psi \circ \phi_{\lambda,0}(0) = (0, ..., 0, 1) \in \mathbb{R}^n$ then it is easy to see that the map

$$\Psi \circ \phi_{\lambda,0} : \mathbb{R}^{n-1} \to \mathbb{S}^{n-1}$$

is smooth. The other parts of the function $f_{\lambda,0}$ is also smooth.

Through similar calculation we can also conclude that for any $\widetilde{f} \in C^\infty(\mathbb{S}^{n-1})$, any $\lambda > 0$ and any $v_0 \in \mathbb{R}^{n-1}$, the function f_{λ,v_0} is smooth in $\mathbb{R}^{n-1}$.

On the other hand, for any $\widetilde{f} \in C^\infty(\mathbb{S}^{n-1})$ define f as in (3.4.2) then we have

$$P_{2-n}(f) = P_{2-n}(\widetilde{f} \circ \Psi) + P_{2-n}(\ln|\Psi'(w)|).$$

Using (3.3.3) and Remark 3, we see that

$$
\begin{aligned}
P_{2-n}(f) &= P_{2-n}(\widetilde{f}\circ\Psi) + \widetilde{I}_n\circ\Psi(y) + \ln|\Psi'(y)| \\
&= (\widetilde{P}_{2-n}\widetilde{f})\circ\Psi + \widetilde{I}_n\circ\Psi(y) + \ln|\Psi'(y)|.
\end{aligned}
\tag{3.5.5}
$$

For f_{λ,v_0}, using (A.0.2) and (A.0.4), we have for any $y\in\mathbb{R}^n_+$

$$
\begin{aligned}
P_{2-n}f_{\lambda,v_0}(y) &= P_{2-n}(f)\circ\phi_{\lambda,v_0}(y) + \ln|\phi'_{\lambda,v_0}(y)| \\
&= P_{2-n}(\widetilde{f}\circ\Psi)\circ\phi_{\lambda,v_0}(y) + \widetilde{I}_n\circ\Psi\circ\phi_{\lambda,v_0}(y) \\
&\quad + \ln|\Psi'(\phi_{\lambda,v_0}(y))| + \ln|\phi'_{\lambda,v_0}(y)| \\
&= P_{2-n}(\widetilde{f}\circ\Psi)\circ\phi_{\lambda,v_0}(y) + \widetilde{I}_n\circ\Psi\circ\phi_{\lambda,v_0}(y) \\
&\quad + \ln\frac{2\lambda^2}{(1+|v_0|^2)|y-v_0|^2+\lambda^4+2\lambda^2\langle v_0, y-v_0\rangle+\lambda^2 y_n}.
\end{aligned}
$$

Using result from Remark 3 we have

$$
\begin{aligned}
P_{2-n}f_{\lambda,v_0}(y) &= \left(\widetilde{P}_{2-n}\widetilde{f}\right)\circ\Psi\circ\phi_{\lambda,v_0}(y) + \widetilde{I}_n\circ\Psi\circ\phi_{\lambda,v_0}(y) \\
&\quad + \ln\frac{2\lambda^2}{(1+|v_0|^2)|y-v_0|^2+\lambda^4+2\lambda^2\langle v_0, y-v_0\rangle+\lambda^2 y_n}.
\end{aligned}
\tag{3.5.6}
$$

Remark 10. *Note that when $\widetilde{f}\in C(\mathbb{S}^{n-1})$, both functions $P_{2-n}f$ and $P_{2-n}f_{\lambda,v_0}$ are continuous in $\overline{\mathbb{R}}^n_+$. We can see this from (3.5.5) and (3.5.6) using similar calculations as in Remark 9.*

Again, note that . When $v_0=0$, we have

$$
P_{2-n}f_{\lambda,0}(y) = \left(\widetilde{P}_{2-n}\widetilde{f}\right)\circ\Psi\circ\phi_{\lambda,0}(y) + \widetilde{I}_n\circ\Psi\circ\phi_{\lambda,0}(y) + \ln\frac{2\lambda^2}{|y|^2+\lambda^4+\lambda^2 y_n} \tag{3.5.7}
$$

We can show the following result, which will be used in subsequent parts.

Lemma 3.8. *Suppose $\widetilde{f}\in C(\mathbb{S}^{n-1})$. Define f, f_{λ,v_0} as in (3.4.2) and (3.5.3) repectively. Define the coresponding extensions $P_{2-n}f$ and $P_{2-n}f_{\lambda,v_0}$ as in (3.5.5), then $P_{2-n}f$ and $P_{2-n}f_{\lambda,v_0}$ are harmonic in $\mathbb{R}^n_+$ with the standard Hyperbolic metric and with boundary values f and f_{λ,v_0} respectively.*

Proof. We consider $P_{2-n}f$ firstly. Using (3.5.5) and (2.4.1) we can see that

$$\begin{aligned} P_{2-n}(f) &= P_{2-n}(\widetilde{f} \circ \Psi) + \widetilde{I}_n \circ \Psi(y) + \ln |\Psi'(y)| \\ &= (\widetilde{P}_{2-n}\widetilde{f}) \circ \Psi + \widetilde{I}_n \circ \Psi(y) + \ln |\Psi'(y)|. \end{aligned}$$

From Proposition 2.4 we can see that $(\widetilde{P}_{2-n}\widetilde{f}) \circ \Psi$ is harmonic in $\mathbb{R}^n_+$ with the Hyperbolic metric and that $\widetilde{P}_{2-n}\widetilde{f} \circ \Psi(u) = \widetilde{f} \circ \Psi(u)$ for all $u \in \partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$.

From Lemma 3.6 we see that in $\mathbb{B}^n$ the fucntion $\widetilde{I}_n(x) + \ln \frac{1-2x_n+|x|^2}{2}$ is hyperbolic harmonic. Since $\Psi : \mathbb{R}^n_+ \to \mathbb{B}^n$ is an isometry between two models of hyperbolic spaces, we have $\widetilde{I}_n \circ \Psi(y) + \ln |\Psi'(y)|$ is also harmonic in $\mathbb{R}^n_+$ with the Hyperbolic metric. Note that here we used the relation (2.1.2).

From Lemma 3.7 we see that $\widetilde{I}_n$ is continuous up to the boundary and that $\widetilde{I}_n \circ \Psi(w) = 0$ for all $w \in \partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$. As a result, we have $\widetilde{I}_n \circ \Psi(w) + \ln |\Psi'(w)| = \ln |\Psi'(w)|$ for all $w \in \partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$. This finishes the proof for $P_{2-n}f$.

Now we consider $P_{2-n}f_{\lambda,v_0}$. By Remark 9, f_{λ,v_0} is continuous on $\partial\mathbb{R}^n_+$. Using the definition of f_{λ,v_0}, (A.0.2) and (A.0.4), we have

$$P_{2-n}f_{\lambda,v_0} = P_{2-n}f \circ \phi_{\lambda,v_0} + \ln |\phi'_{\lambda,v_0}|.$$

Since ϕ_{λ,v_0} is an isometry of $\mathbb{R}^n_+$ with hyperbolic metric, we have $P_{2-n}f \circ \phi_{\lambda,v_0}$ is harmonic in $\mathbb{R}^n_+$ with the hyperbolic metric. On the other hand, using (3.5.2), by direct calculation as in (B.3.1) we can see that $\ln |\phi'_{\lambda,v_0}|$ is harmonic in $\mathbb{R}^n_+$ with the hyperbolic metric.

From (3.5.6) and similar calculations as in Remark 9 we see that $P_{2-n}f_{\lambda,v_0}$ is continuous in $\overline{R}^n_+$. As a result, $P_{2-n}f_{\lambda,v_0}$ is hyperbolic harmonic in $\mathbb{R}^n_+$ with boundary value f_{λ,v_0}.

□

Note that in this proof we only need the asymptotic behavior (3.5.4) and (3.5.7) and Proposition 2.4. We also need the regularity $\widetilde{f} \in C(\mathbb{S}^{n-1})$ in order to use Proposition 2.4, but we do not need to assume $\widetilde{f}$ to be a solution of (3.4.1).

3.5.3 Start the Sphere

Now we start the moving sphere argument. The first step is called start the sphere; and we prove it in the following two propositions. The proof of Proposition 3.9 relies on the asymptotic behavior (3.5.4) while the proof of Proposition 3.10 relies on the asymptotic behavior (3.5.7) and the maximum principle.

Proposition 3.9. *For any $\widetilde{f} \in C^\infty(\mathbb{S}^{n-1})$ any $\lambda > 0$ and any $v_0 \in \partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$, define f and f_{λ,v_0} as in (3.4.2) and (3.5.3) respectively. Then there exists $\lambda_0 > 0$ depending only on v_0, such that for all $0 < \lambda < \lambda_0$*

$$e^{(n-1)f_{\lambda,v_0}}(w) < e^{(n-1)f}(w),$$

for all $|w - v_0| > \lambda$.

Proof. Our proof uses similar ideas as in Lemma 3.1 of [19] and Lemma 2.1 of [20]. Our proof is even simpler since we have the asymptotic behavior (3.5.4).

Consider in polar coordinate of $\mathbb{R}^{n-1}$, $w = (r, \theta) \in \mathbb{R}^{n-1}$ where $r \geq 0$ and $\theta \in \mathbb{S}^{n-2}$. Since $f \in C^\infty(\mathbb{R}^{n-1})$ (actually we only need f to be C^1) and $e^{(n-1)f} > 0$, there exists $r_0 > 0$ such that

$$\frac{d}{dr}\left(r^{n-1}e^{(n-1)f}(r,\theta)\right) > 0$$

for all $0 < r < r_0$ and all $\theta \in \mathbb{S}^{n-2}$.

As a result, when $0 < \lambda < |w| < r_0$ we have

$$\left(\frac{\lambda^2}{|w|}\right)^{n-1} e^{(n-1)f}\left(\frac{\lambda^2 w}{|w|^2}\right) < |w|^{n-1}e^{(n-1)f}(w),$$

and hence using (3.5.3) we have

$$e^{(n-1)f_{\lambda,0}}(w) < e^{(n-1)f}(w)$$

for all $0 < \lambda < |w| < r_0$. (Note that here we used the fact that when $0 < \lambda < |w|$, we have $|\frac{\lambda^2 w}{|w|}|^2 = \frac{\lambda^2}{|w|} < |w|$.)

For $|u| > r_0$, from (3.4.2) and (3.5.4) we have

$$e^{(n-1)f}(w) = \left(\frac{2}{1+|w|^2}\right)^{n-1} e^{(n-1)\widetilde{f}\circ\Psi}(w)$$

and

$$e^{(n-1)f_{\lambda,0}}(w) = \left(\frac{2\lambda^2}{\lambda^4+|w|^2}\right)^{n-1} e^{(n-1)\widetilde{f}\circ\Psi}\left(\frac{\lambda^2 w}{|w|^2}\right).$$

Define $m = \inf_{\xi\in\mathbb{S}^{n-1}} e^{\widetilde{f}}(\xi)$ and $M = \sup_{\xi\in\mathbb{S}^{n-1}} e^{\widetilde{f}}(\xi)$. Note that we have $0 < m \leq M$. Choose λ_0 small enough such that $0 < \lambda_0 < r_0$ and

$$m - \lambda_0^2 M > m/2,$$

and

$$r_0^2 m/2 > \lambda_0^2 M.$$

Then we have for all $|w| > r_0$ and $0 < \lambda < \lambda_0$

$$\begin{aligned}(m-\lambda^2 M)|w|^2 &> (m-\lambda_0^2 M)r_0^2\\ &> mr_0^2/2\\ &> \lambda_0^2 M\\ &> \lambda^2 M\\ &> \lambda^2(M-\lambda^2 m).\end{aligned}$$

As a result, we have for all $0 < \lambda < \lambda_0 < r_0 < |w|$,

$$\frac{2m}{1+|w|^2} > \frac{2\lambda^2 M}{|w|^2+\lambda^4},$$

and hence

$$
\begin{aligned}
e^{(n-1)f}(w) &= \left(\frac{2}{1+|w|^2}\right)^{n-1} e^{(n-1)\widetilde{f}\circ\Psi}(w) \\
&\geq \left(\frac{2m}{1+|w|^2}\right)^{n-1} \\
&> \left(\frac{2\lambda^2 M}{|w|^2+\lambda^4}\right)^{n-1} \\
&\geq \left(\frac{2\lambda^2}{|w|^2+\lambda^4}\right)^{n-1} e^{(n-1)\widetilde{f}\circ\Psi}\left(\frac{\lambda^2 w}{|w|^2}\right) \\
&= e^{(n-1)f_{\lambda,0}}(w).
\end{aligned}
$$

With the chosen λ_0, combining the two cases together, we get for all $0 < \lambda < \lambda_0$ and all $|w| > \lambda$, we have

$$e^{(n-1)f}(w) > e^{(n-1)f_{\lambda,0}}(w).$$

□

Remark 11. *Note that*

$$e^{(n-1)f_{\lambda,v_0}}(w) < e^{(n-1)f}(w)$$

for all $|w - v_0| > \lambda$ *is equivalent to*

$$e^{(n-1)f_{\lambda,v_0}}(w) > e^{(n-1)f}(w)$$

for all $|w - v_0| < \lambda$*. Since for any* w *such that* $|w - v_0| < \lambda$*, we can define* $w_{\lambda,v_0} = \phi_{\lambda,v_0}(w)$*. Then we have*

$$\phi_{\lambda,v_0}(w_{\lambda,v_0}) = \phi_{\lambda,v_0} \circ \phi_{\lambda,v_0}(w) = w, \tag{3.5.8}$$

and

$$|w_{\lambda,v_0} - v_0| = |\phi_{\lambda,v_0}(w) - v_0| = \frac{\lambda^2}{|w - v_0|} > \lambda.$$

As a result, using Theorem 3.9, we have for $|w - v_0| < \lambda$

$$\begin{aligned} e^{(n-1)f_{\lambda,v_0}}(w) &= |\phi'_{\lambda,v_0}(w)|^{n-1} e^{(n-1)f}(w_{\lambda,v_0}) \\ &> |\phi'_{\lambda,v_0}(w)|^{n-1} e^{(n-1)f_{\lambda,v_0}}(w_{\lambda,v_0}) \\ &= |\phi'_{\lambda,v_0}(w)|^{n-1} |\phi'_{\lambda,v_0}(w_{\lambda,v_0})|^{n-1} e^{(n-1)f}(w) \\ &= e^{(n-1)f}(w). \end{aligned}$$

Note that in the last step we used

$$|\phi'_{\lambda,v_0}(w)|^{n-1} |\phi'_{\lambda,v_0}(w_{\lambda,v_0})|^{n-1} = 1, \tag{3.5.9}$$

which follows from (3.5.8) and chain rule.

With the help of the previous remark, we can use maximum principle to prove similar results for $P_{2-n}f$.

Proposition 3.10. *For any* $\widetilde{f} \in C^\infty(\mathbb{S}^{n-1})$*, define* f *and* f_{λ,v_0} *as in (3.4.2) and (3.5.3) respectively.*

For any $v_0 \in \partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$ *there exists* $\lambda_0 > 0$ *such that for all* $0 < \lambda < \lambda_0$ *we have*

$$e^{nP_{2-n}f}(y) < e^{nP_{2-n}f_{\lambda,v_0}}(y)$$

for all $|y - v_0| < \lambda$.

Proof. From Lemma 3.8 we see that $P_{2-n}f$ and $P_{2-n}f_{\lambda,v_0}$ are Hyperbolic harmonic with boundary values f and f_{λ,v_0} respctivly. By the discussion at the beginning of this section, all four functions f, f_{λ,v_0}, $P_{2-n}f$ and $P_{2-n}f_{\lambda,v_0}$ are continuous at the point $v_0 \in \partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$. Choose the same λ_0 as in Theorem 3.9. Using the previous remark we have

$$e^{(n-1)f}(w) < e^{(n-1)f_{\lambda,v_0}}(w),$$

for any $0 < \lambda < \lambda_0$ and for all $w \in \mathbb{R}^{n-1}$ such that $|w - v_0| < \lambda$. Which is equivalent to

$$f(w) < f_{\lambda,v_0}(w)$$

for all $w \in \mathbb{R}^{n-1}$ such that $|w - v_0| < \lambda$. On the other hand, using the definition of f_{λ,v_0}, (A.0.2) and (A.0.4), we can see

$$P_{2-n}f(y) = P_{2-n}f_{\lambda,v_0}(y)$$

for all $y \in \mathbb{R}^n_+$ such that $|y - v_0| = \lambda$. By maximum principle we can see

$$P_{2-n}f(y) < P_{2-n}f_{\lambda,v_0}(y)$$

for all $|y - v_0| < \lambda$. □

Note that in this subsection we only used asymptotic behavior (3.5.4), (3.5.7) and the maximum principle. We did not use the fact that $\widetilde{f}$ is a solution to the integral equation (3.4.1).

3.5.4 The Case $\bar{\lambda}_0 = \infty$

In the previous subsection we showed that $\lambda_0 > 0$ exists. In this subsection we will show that it can not go to infinity.

Define

$$\bar{\lambda}_0 = \sup\{\lambda > 0 \text{ such that } e^{(n-1)f}(w) < e^{(n-1)f_{\lambda,v_0}}(w) \text{ for all } |w - v_0| < \lambda.\} \qquad (3.5.10)$$

In the following lemma we show that $\bar{\lambda}_0$ can not equal to ∞.

Lemma 3.11. *For any $f : \mathbb{R}^{n-1} \to \mathbb{R}$, such that $e^f \in L^{n-1}(\mathbb{R}^{n-1})$, and for all $v_0 \in \partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$, we have $\bar{\lambda}_0 < \infty$.*

Proof. We can prove this by contradiction following [12]. Suppose for some $v_0 \in \partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$, we have $\bar{\lambda}_0 = \infty$, then we can find a sequence $\lambda_i \to \infty$ such that

$$e^{(n-1)f}(w) < e^{(n-1)f_{\lambda_i,v_0}}(w)$$

for all $|w-v_0| < \lambda_i$ and for all i. For a given i, by the previous inequality, (3.5.9) and change of variable we have

$$\begin{aligned} \int_{\mathbb{B}^+_{\lambda_i,v_0}} e^{(n-1)f}(w)dw &< \int_{\mathbb{B}^+_{\lambda_i,v_0}} e^{(n-1)f_{\lambda_i,v_0}}(w)dw \\ &= \int_{\mathbb{R}^n_+ \setminus \mathbb{B}^+_{\lambda_i,v_0}} e^{(n-1)f}(w)dw. \end{aligned}$$

As a result, we have

$$0 < \frac{1}{2}\int_{\mathbb{R}^{n-1}} e^{(n-1)f}(w)dw < \int_{|w-v_0|>\lambda_i} e^{(n-1)f}(w)dw.$$

But the right hand side of the inequality goes to zero as $\lambda_i \to \infty$ by dominated convergence theorem. □

Note that in the this proof we only need $e^f \in L^{n-1}(\mathbb{R}^{n-1})$ in order to use dominated convergence theorem. We don't need $\widetilde{f}$ to be a solution to the integral equation (3.4.1).

3.5.5 The Case $\bar{\lambda}_0 < \infty$

Note that since $\bar{\lambda}_0 > 0$, this is the last case we need to consider. In this subsection we will show that at the critical value $\bar{\lambda}_0$, we have

$$f_{\bar{\lambda}_0,v_0} = f.$$

But firstly we need to show three lemmas. The first lemma is about the kernel function $p_{2-n}(y,u)$.

Lemma 3.12. *For any $v_0 \in \partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$ and any $\lambda > 0$, define ϕ_{λ,v_0} as in (3.5.1). Then for any $y \in \mathbb{B}^+_{\lambda,v_0}$ and any $w \in \partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$ such that $|w-v_0| < \lambda$, we have*

$$p_{2-n}(y,w) - p_{2-n}(\phi_{\lambda,v_0}(y),w) > 0 \tag{3.5.11}$$

Proof. By (A.0.1), we have

$$p_{2-n}(\phi_{\lambda,v_0}(y),w) = p_{2-n}(y,\phi_{\lambda,v_0}(w))|\phi'_{\lambda,v_0}(w)|^{n-1},$$

so we only need to prove

$$\frac{1}{|y-w|^{2n-2}} - \frac{|\phi'_{\lambda,v_0}(w)|^{n-1}}{|y-\phi_{\lambda,v_0}(w)|^{2n-2}} > 0,$$

for any $y \in \mathbb{B}^+_{\lambda,v_0}$ and any $w \in \mathbb{R}^{n-1}$ such that $|w-v_0| < \lambda$. By direct calculation using (3.5.1), we have

$$|\phi'_{\lambda,v_0}(w)| = \frac{\lambda^2}{|w-v_0|^2}.$$

So the proof follows from

$$|y-\phi_{\lambda,v_0}(w)|^2 \frac{|w-v_0|^2}{\lambda^2} - |y-w|^2 = \frac{(\lambda^2-|w-v_0|^2)(\lambda^2-|y-v_0|^2)}{\lambda^2},$$

which is positive when both $|w-v_0| < \lambda$ and $|y-v_0| < \lambda$.

□

If we define

$$K(v_0,\lambda;y,w) = p_{2-n}(y,w) - p_{2-n}(\phi_{\lambda,v_0}(y),w)$$

as in [19] (right before Lemma 3.1 in [19]), then we can show the following result about the derivative of K with respect to w (as in the proof of Lemma 3.2 in [19]).

Lemma 3.13.

$$\langle \overline{\nabla}_w K(v_0,\lambda;y,w), w-v_0\rangle\big|_{|w-v_0|=\lambda} = -\frac{2(n-1)c_{n,2-n}y_n^{n-1}}{|w-y|^{2n}}(|w-v_0|^2-|y-v_0|^2), \quad (3.5.12)$$

for all $y \in \mathbb{R}^n_+$.

Proof. Here $\langle \overline{\nabla}_w K(v_0,\lambda;y,w), w-v_0\rangle$ denote the inner product in $\mathbb{R}^{n-1}$ (or $\mathbb{R}^n$) with respect to the Euclidean metric. And $\overline{\nabla}_w$ denotes the gradient in $\mathbb{R}^{n-1}$ with the Euclidean metric. The subscript w emphasizes the fact that the derivative is taken with respect to w. The proof follows from direct calculation. As in the previous lemma, use A.0.1 we can have

$$\begin{aligned} & K(v_0,\lambda;y,w) \\ = \; & c_{n,2-n}y_n^{n-1}\left(\frac{1}{|w-y|^{2n-2}} - \left(\frac{\lambda}{|w-v_0|}\right)^{2n-2} \frac{1}{\left|y-v_0-\frac{\lambda^2(w-v_0)}{|w-v_0|^2}\right|^{2n-2}}\right). \end{aligned}$$

As a result, we can calculate $\overline{\nabla}_u K$ to be

$$\begin{aligned}
& \frac{\overline{\nabla}_w K}{c_{n,2-n} y_n^{n-1}} \\
= \; & -\frac{2(n-1)(w-y')}{|w-y|^{2n}} - \frac{2(n-1)\lambda^{2n-2}|w-v_0|^{2n-4}}{|(y-v_0)|w-v_0|^2 - \lambda^2(w-v_0)|^{2n-2}}(w-v_0) \\
& + \frac{(n-1)\lambda^{2n-2}|w-v_0|^{2n-2}(4|y-v_0|^2|w-v_0|^2 - 4\lambda^2\langle y-v_0, w-v_0\rangle)}{|(y-v_0)|w-v_0|^2 - \lambda^2(w-v_0)|^{2n}}(w-v_0) \\
& - \frac{2(n-1)\lambda^{2n}|w-v_0|^{2n-2}}{|(y-v_0)|w-v_0|^2 - \lambda^2(w-v_0)|^{2n}}(|w-v_0|^2(y-v_0) - \lambda^2(w-v_0)).
\end{aligned}$$

Here $\langle y-v_0, w-v_0\rangle$ denotes inner product in $\mathbb{R}^n$. Then for the inner product we have

$$\langle \overline{\nabla}_w K, w-v_0\rangle\big|_{|w-v_0|=\lambda} = \frac{-2(n-1)c_{n,2-n}y_n^{n-1}}{|w-y|}(|w-v_0|^2 - |y-v_0|^2).$$

□

In the next lemma, we use change of variable to rewrite equation (3.4.4) into a new form.

Lemma 3.14. *For any $v_0 \in \partial\mathbb{R}^n_+$ and any $\lambda > 0$, define the inversion ϕ_{λ,v_0} as in (3.5.1). For any $\widetilde{f} \in C^\infty(\mathbb{R}^{n-1})$ that is a solution to (3.4.1). Define f and f_{λ,v_0} as in (3.4.2) and (3.5.3) respectively, then we have*

$$\begin{aligned}
& e^{(n-1)f_{\lambda,v_0}} - e^{(n-1)f} \\
= & \int_{\mathbb{B}^+_{\lambda,v_0}} \left(e^{nP_{2-n}f_{\lambda,v_0}} - e^{nP_{2-n}f}\right)(p_{2-n}(y,w) - p_{2-n}(\phi_{\lambda,v_0}(y),w))dy
\end{aligned} \tag{3.5.13}$$

Proof. Since $\widetilde{f}$ is a solution to 3.4.1, through change of variable we can see that f is a solution to (3.4.4). Using equation (3.4.4) we have

$$\begin{aligned}
e^{(n-1)f} &= \int_{\mathbb{B}^+_{\lambda,v_0}} e^{nP_{2-n}f} p_{2-n}(y,w)dy + \int_{\mathbb{R}^n_+\setminus\mathbb{B}^+_{\lambda,v_0}} e^{nP_{2-n}f} p_{2-n}(y,w)dy \\
&= \int_{\mathbb{B}^+_{\lambda,v_0}} \left(e^{nP_{2-n}f} p_{2-n}(y,w) + e^{nP_{2-n}f_{\lambda,v_0}} p_{2-n}(\phi_{\lambda,v_0}(y),w)\right)dy,
\end{aligned}$$

where the second step follows from change of variable and (A.0.4). Similarly

$$\begin{aligned}
e^{(n-1)f_{\lambda,v_0}} &= \int_{\mathbb{B}^+_{\lambda,v_0}} e^{nP_{2-n}f_{\lambda,v_0}} p_{2-n}(y,w)dy + \int_{\mathbb{R}^n_+\setminus\mathbb{B}^+_{\lambda,v_0}} e^{nP_{2-n}f_{\lambda,v_0}} p_{2-n}(y,w)dy \\
&= \int_{\mathbb{B}^+_{\lambda,v_0}} \left(e^{nP_{2-n}f_{\lambda,v_0}} p_{2-n}(y,w) + e^{nP_{2-n}f} p_{2-n}(\phi_{\lambda,v_0}(y),w)\right)dy.
\end{aligned}$$

Subtracting the first inequality from the second one, we get the desired result.

□

We now prove $f = f_{\overline{\lambda}_0, v_0}$ by contradiction.

Proposition 3.15. *Suppose $f \in C^1(\mathbb{R}^{n-1})$ is a solution to (3.4.4). For any $v_0 \in \partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$, with $\overline{\lambda}_0$ defined as in (3.5.10), f and $f_{\overline{\lambda}_0, v_0}$ defined as in (3.4.2) and (3.5.3) respectively, we have*

$$f(w) = f_{\overline{\lambda}_0, v_0}(w),$$

for all $w \in \mathbb{R}^{n-1}$.

Proof. We prove this by contradiction. Our proof is similar to a combination of Lemma 2.4 in [20] and Lemma 3.2 in [19]. Suppose there exists $v \in \partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$ such that

$$f(v) < f_{\overline{\lambda}_0, v_0}(v),$$

then by maximum principle (note that by Lemma 3.8 we can use maximum principle here), we have

$$P_{2-n}f(y) < P_{2-n}f_{\overline{\lambda}_0, v_0}(y),$$

for all $y \in \mathbb{B}^+_{\overline{\lambda}_0, v_0}$. As a result, using (3.4.4) and Lemma 3.12 we see that

$$f(w) < f_{\overline{\lambda}_0, v_0}(w) \tag{3.5.14}$$

for all $w \in \partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$ such that $|w - v_0| < \overline{\lambda}_0$. By (3.5.4), $f_{\overline{\lambda}_0, v_0}$ is continuous on $\partial\mathbb{R}^n_+$. Using compactness of $\overline{\mathbb{B}}^+_{\overline{\lambda}_0/2, v_0}$, there exists $\gamma > 0$ such that

$$P_{2-n}f_{\overline{\lambda}_0, v_0}(y) - P_{2-n}f(y) \geq \gamma > 0$$

for all $y \in \overline{\mathbb{B}}^+_{\overline{\lambda}_0/2, v_0}$.

Consider a sequence $\{\lambda_i\}_{i=1}^{\infty}$ such that for each i we have

$$\lambda_i > \lambda_{i+1} > \overline{\lambda}_0,$$

and that

$$\lambda_i \to \overline{\lambda}_0$$

as $i \to \infty$. We can also require that

$$P_{2-n} f_{\lambda_i, v_0}(y) - P_{2-n} f(y) \geq \frac{\gamma}{2} > 0, \tag{3.5.15}$$

since by (3.5.7) we know that $P_{2-n} f_{\lambda, v_0}(y)$ is continuous function of λ.

For each i by compactness of $\overline{\mathbb{B}}^+_{\lambda_i, v_0}$, there exists $w_i \in \overline{\mathbb{B}}^+_{\lambda_i, v_0}$ such that

$$P_{2-n} f_{\lambda_i, v_0}(w_i) - P_{2-n} f(w_i) = \inf_{\overline{\mathbb{B}}^+_{\overline{\lambda}_0, v_0}} P_{2-n} f_{\lambda_i, v_0} - P_{2-n} f.$$

Since $\overline{\lambda}_0$ is the critical value, we must have

$$P_{2-n} f_{\lambda_i, v_0}(w_i) - P_{2-n} f(w_i) < 0.$$

By maximum principle, we also know that $w_i \in \overline{\mathbb{B}}^+_{\lambda_i, v_0} \cap \partial \mathbb{R}^n_+$, and that

$$f_{\lambda_i, v_0}(w_i) - f(w_i) = P_{2-n} f_{\lambda_i, v_0}(w_i) - P_{2-n} f(w_i) < 0.$$

In addition, by 3.5.15 we have $\frac{\lambda_0}{2} \leq |w_i - v_0| < \lambda_i$. We have strict inequality because $f_{\lambda_i, v_0}(w) = f(w)$ for all $|w - v_0| = \lambda_i$.

Since w_i is an interior minimum for $f_{\lambda_i, v_0} - f$ in $\overline{\mathbb{B}}^+_{\lambda_i, v_0} \cap \partial \mathbb{R}^n_+$, we have

$$\overline{\nabla}(f_{\lambda_i, v_0} - f)(w_i) = 0.$$

Here we use $\overline{\nabla}$ to denote the gradient in $\mathbb{R}^{n-1}$ with the Euclidean metric. It is the same notation as in Lemma 3.13.

By compactness of $\overline{\mathbb{B}}^+_{\lambda_1, v_0}$, we can choose a subsequence (still denote it $\{w_i\}_{i=1}^{\infty}$ and $\{\lambda_i\}_i^{\infty}$) such that $w_i \to w_0$ for some $w_0 \in \overline{\mathbb{B}}^+_{\overline{\lambda}_0, v_0} \cap \partial \mathbb{R}^n_+$. By (3.5.4) we can see that both $f_{\lambda, v_0}(w)$ and $\overline{\nabla} f_{\lambda, v_0}(w)$ are continuous for $\lambda > 0$ and $|w| \neq 0$. As a result, we can take limit $i \to \infty$ to get

$$f_{\overline{\lambda}_0, v_0}(w_0) - f(w_0) \leq 0 \tag{3.5.16}$$

and

$$\overline{\nabla}(f_{\overline{\lambda}_0,v_0} - f)(w_0) = 0. \tag{3.5.17}$$

Because of (3.5.16) and (3.5.14) we have $|w_0 - v_0| = \overline{\lambda}_0$. But by Lemma 3.14 and Lemma 3.13, we can see that

$$\begin{aligned}
& (n-1)e^{(n-1)f}(w_0)\langle\overline{\nabla}(f_{\overline{\lambda}_0,v_0} - f)(w_0), w_0 - v_0\rangle \\
= \ & \langle\overline{\nabla}(e^{(n-1)f_{\overline{\lambda}_0,v_0}} - e^{(n-1)f})(w_0), w_0 - v_0\rangle \\
= \ & \int_{\mathbb{B}^{\pm}_{\overline{\lambda}_0,v_0}} \left(e^{nP_{2-n}f_{\overline{\lambda}_0,v_0}} - e^{nP_{2-n}f}(y)\right)\langle\overline{\nabla}_w K(v_0,\overline{\lambda}_0; y, w_0), w_0 - v_0\rangle dy \\
= \ & -2(n-1)c_{n,2-n}\int_{\mathbb{B}^{\pm}_{\overline{\lambda}_0,v_0}} \left(e^{nP_{2-n}f_{\overline{\lambda}_0,v_0}} - e^{nP_{2-n}f}(y)\right)\frac{y_n^{n-1}}{|w_0 - y|^{2n}}(\overline{\lambda}_0^2 - |y - v_0|^2)dy \\
< \ & 0.
\end{aligned}$$

Which is a contradiction to (3.5.17).

□

3.5.6 Proof of Theorem 1.14

For the proof of Theorem 1.14 we need the following Lemma proved by Li and Zhu [20]

Lemma 3.16. *[20, Lemma 2.5] For any integer $n \geq 3$, suppose $f \in C^1(\mathbb{R}^{n-1})$ satisfying: for any $b \in \mathbb{R}^{n-1}$, there exists $\lambda_b \in \mathbb{R}$ such that*

$$f(w) = \frac{\lambda_b^{n-2}}{|w-b|^{n-2}} f\left(\frac{\lambda_b^2}{|w-b|^2}(w-b) + b\right), \ \textit{for all } w \in \mathbb{R}^{n-1}\backslash\{b\}.$$

Then for some $a \geq 0$, $d > 0$, $w_0 \in \mathbb{R}^{n-1}$,

$$f(w) = \left(\frac{a}{|w-w_0|^2 + d}\right)^{(n-2)/2}, \ \textit{for all } w \in \mathbb{R}^{n-1},$$

or

$$f(w) = -\left(\frac{a}{|w-w_0|^2 + d}\right)^{(n-2)/2}, \ \textit{for all } w \in \mathbb{R}^{n-1}.$$

We restate Theorem 1.14 here for the convenience of the reader:

Theorem 3.17. *For any integer $n \geq 2$, if $\widetilde{f} \in L^{\infty}(\mathbb{S}^{n-1})$ satisfies the equation*

$$e^{(n-1)\widetilde{f}(\xi)} = \int_{\mathbb{B}^n} e^{n\widetilde{I}_n + n\widetilde{P}_{2-n}\widetilde{f}} \widetilde{p}_{2-n}(x,\xi)dx,$$

then for all $\xi \in \mathbb{S}^{n-1}$

$$\widetilde{f}(\xi) = \ln \frac{1-|\zeta|^2}{|\xi - \zeta|^2} + C_n,$$

where $\zeta \in \mathbb{B}^n$ and $C_n = -\frac{1}{n-1} \ln \left|\mathbb{S}^{n-1}\right|$ is a constant. Here $\left|\mathbb{S}^{n-1}\right|$ denotes the volume of the standard sphere.

Proof. By Proposition C.6 in the appendix we know that $\widetilde{f} \in C^1(\mathbb{S}^{n-1})$. For $w \in \mathbb{R}^{n-1}$ define $f(w)$ as in (3.4.2) then we have $f \in C^1(\mathbb{R}^{n-1})$. By discussion at the end of Section 3.4 we know that f satisfies the Euler-Lagrange equation (3.4.4). Then by Proposition 3.15 we know that for any $v_0 \in \partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$, there exists $\overline{\lambda}_0 > 0$ depending on v_0, such that

$$f(w) = f_{\overline{\lambda}_0, v_0}(w),$$

for all $w \in \mathbb{R}^{n-1}$. Note that here

$$\begin{aligned} f_{\overline{\lambda}_0, v_0}(w) &= f \circ \phi_{\overline{\lambda}_0, v_0}(w) + \ln |\phi'_{\overline{\lambda}_0, v_0}(w)| \\ &= f\left(\frac{\overline{\lambda}_0^2}{|w - v_0|^2}(w - v_0) + v_0\right) + \ln \frac{\overline{\lambda}_0^2}{|w - v_0|^2} \end{aligned}$$

is as in (3.5.3).

Define

$$\varphi(w) = e^{\frac{n-2}{2} f(w)},$$

then it is easy to check that φ satisfies the assumption of Lemma 3.16. As a result we have

$$\varphi(w) = \left(\frac{a}{|w - w_0|^2 + d}\right)^{(n-2)/2}$$

for some $a > 0$, $d > 0$ and $w_0 \in \mathbb{R}^{n-1}$. From this we know that

$$f(w) = -\ln\Big(|w - w_0|^2 + d\Big) + \ln a.$$

Define

$$\zeta = \Psi(w_0, \sqrt{d}),$$

where $(w_0, \sqrt{d})$ is a point in $\mathbb{R}^n_+$. Then we have

$$\widetilde{f}(\xi) = \ln \frac{1 - |\zeta|^2}{|\xi - \zeta|^2} + C_n, \text{ for all } \xi \in \mathbb{S}^{n-1}.$$

Note that here the constant $C_n = -\frac{1}{n-1} \ln \left|\mathbb{S}^{n-1}\right|$ is determined by the restriction

$$\left\| e^{\widetilde{f}} \right\|_{L^{n-1}(\mathbb{S}^{n-1})} = 1.$$

This finishes the proof. $\square$

APPENDICES

APPENDIX A

CONFORMAL TRANFORMATIONS

Note that we use three different types of conformal transformation in this part

$$\Psi : \mathbb{R}^n_+ \to \mathbb{B}^n,$$

defined in section 2. $\Phi \in SO(n,1)$, and the corresponding $\phi : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ defined by

$$\phi = \Psi^{-1} \circ \Phi \circ \Psi.$$

Using the definition of Ψ, through direct calculation, we have

$$|\Psi'(y)| = \frac{1 - |\Psi(y)|^2}{2y_n}.$$

Lemma A.1. *For any $\Phi \in SO(n,1)$, define*

$$\phi = \Psi^{-1} \circ \Phi \circ \Psi,$$

then for any $y \in \mathbb{R}^n_+$ and any $u \in \mathbb{R}^{n-1}$ we have

$$p_{2-n}(\phi(y), \phi(w)) = p_{2-n}(y,u)|\phi'(w)|^{1-n}, \tag{A.0.1}$$

as a result, for any $f : \mathbb{R}^n_+ \to \mathbb{R}$ we have

$$(P_{2-n}f) \circ \phi = P_{2-n}(f \circ \phi) \tag{A.0.2}$$

Proof. Since we have

$$\Psi \circ \Psi^{-1} = Id,$$

using chain rule, we have

$$\Psi'(\Psi^{-1}(x)) \cdot (\Psi^{-1})'(x) = Id$$

for all $x \in \mathbb{B}^n$. Note that we think of the left hand side of the equation as matrix multiplication. As a result we have

$$|\Psi'(\Psi^{-1}(x))| = |(\Psi^{-1})'(x)|^{-1} \tag{A.0.3}$$

Using both (2.4.1) and (2.4.2) we have

$$\begin{aligned}
p_{2-n}(\phi(y), \phi(w)) &= p_{2-n}(\Psi^{-1} \circ \Phi \circ \Psi(y), \Psi^{-1} \circ \Phi \circ \Psi(w)) \\
&= \widetilde{p}_{2-n}(\Phi \circ \Psi(y), \Phi \circ \Psi(w))|\Psi'(\Psi^{-1} \circ \Phi \circ \Psi(w))|^{n-1} \\
&= \widetilde{p}_{2-n}(\Psi(y), \Psi(w))|\Phi'(\Psi(w))|^{1-n}|\Psi'(\Psi^{-1} \circ \Phi \circ \Psi(w))|^{n-1} \\
&= p_{2-n}(y, u)|\Psi'(w)|^{1-n}|\Phi'(\Psi(w))|^{1-n}|\Psi'(\Psi^{-1} \circ \Phi \circ \Psi(w))|^{n-1} \\
&= p_{2-n}(y, w)|\phi'(w)|^{1-n}.
\end{aligned}$$

Note that in the last step we used chain rule and (A.0.3) by plug in $x = \Phi \circ \Psi(w)$.

Using (A.0.1) we have

$$\begin{aligned}
(P_{2-n}f) \circ \phi(y) &= c_{n,2-n} \int_{\mathbb{R}^{n-1}} p_{2-n}(\phi(y), w) f(w) dw \\
&= c_{n,2-n} \int_{\mathbb{R}^{n-1}} p_{2-n}(\phi(y), \phi(w)) f(\phi(w)) |\phi'(w)|^{n-1} dw \\
&= c_{n,2-n} \int_{\mathbb{R}^{n-1}} p_{2-n}(y, w) f(\phi(w)) dw \\
&= P_{2-n}(f \circ \phi)(y).
\end{aligned}$$

□

Lemma A.2.

$$\ln|\phi'(y)| = P_{2-n} \ln|\phi'(w)| \tag{A.0.4}$$

Proof. Since $\Phi = \Psi \circ \phi \circ \Psi^{-1}$, by (3.3.2), we have

$$\widetilde{I}_n \circ \Phi(x) + \ln|\Phi'(x)| = \widetilde{I}_n(x) + \widetilde{P}_{2-n}(\ln|\Phi'(\xi)|).$$

If we define $y = \Psi^{-1}(x)$, then using (3.3.3) we have

$$\begin{aligned}
\widetilde{I}_n(x) &= \widetilde{I}_n \circ \Psi(y) \\
&= P_{2-n}(\ln|\Psi'(u)|) - \ln|\Psi'(y)|,
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{I}_n \circ \Phi(x) &= \widetilde{I}_n \circ \Phi \circ \Psi(y) \\
&= \widetilde{I}_n \circ \Psi(\phi(y)) \\
&= (P_{2-n} \ln |\Psi'(w)|) \circ \phi(y) - \ln |\Psi'(\phi(y))| \\
&= P_{2-n} \ln |\Psi'(\phi(w))|(y) - \ln |\Psi'(\phi(y))|,
\end{aligned}$$

where the last step follows from (A.0.2).

On the other hand, using chain rule, we have

$$\begin{aligned}
\ln |\Phi'(x)| &= \ln |\Psi'(\phi \circ \Psi^{-1}(x))| + \ln |\phi'(\Psi^{-1}(x))| + \ln |(\Psi^{-1})'(x)| \\
&= \ln |\Psi'(\phi(y))| + \ln |\phi'(y)| + \ln |(\Psi^{-1})'(\Psi(y))|.
\end{aligned}$$

If we define $w = \Psi^{-1}(\xi)$, then using chain rule, we have

$$\ln |\Phi'(\xi)| = \ln |\Psi'(\phi \circ \Psi^{-1}(\xi))| + \ln |\phi'(\Psi^{-1}(\xi))| + \ln |(\Psi^{-1})'(\xi)|,$$

and by (2.4.1) we have

$$(\widetilde{P}_{2-n} \ln |\Phi'(\xi)|) \circ \Psi = P_{2-n} \ln |\Psi'(\phi(w))| + P_{2-n} \ln |\phi'(w)| + P_{2-n} \ln |(\Psi^{-1})'(\Psi(w))|.$$

Putting everything together and using (A.0.3) we have

$$\ln |\phi'(y)| = P_{2-n} \ln |\phi'(w)|.$$

□

APPENDIX B

HARMONIC FUNCTIONS IN HYPERBOLIC SPACE

B.1 Induction Formula

In this section we prove a simple induction formula for hyperbolic harmonic functions. This induction formula will help us relate harmonic functions in hyperbolic space and poly-harmonic functions in Euclidean space in subsequent sections.

For an integer $n \geq 2$, consider the unit ball model of hyperbolic space $(\mathbb{B}^n, g_h)$, where $g_h = \frac{4}{(1-|x|^2)^2}dx^2$. Define $\rho = \frac{1-|x|^2}{2}$, then we have

$$\nabla\rho = -x,$$

and

$$\Delta\rho = -n,$$

Note that here ∇ and Δ are the gradient and Laplacian with respect to the Euclidean metric.

We prove the following induction formula

Lemma B.1. *For $k \in \mathbb{N}_+$ we have*

$$\rho\Delta^k u = -(k-1)(n-2k)\Delta^{k-1}u - (n-2k)\langle x, \nabla\Delta^{k-1}u\rangle. \tag{B.1.1}$$

Note that here the inner product is with respect to the Euclidean metric.

Proof. When $k = 1$ the equation (B.1.1) holds because u is harmonic with respect to the hyperbolic metric.

Now suppose (B.1.1) holds for k. Apply Δ on both sides, we get

$$\Delta\left(\rho\Delta^k u\right) = -n\Delta^k u + \rho\Delta^{k+1}u - 2\langle x, \nabla\Delta^k u\rangle,$$

and

$$\begin{aligned}&\Delta\left(-(k-1)(n-2k)\Delta^{k-1}u - (n-2k)\langle x, \nabla\Delta^{k-1}u\rangle\right)\\ &= -(k-1)(n-2k)\Delta^k u - (n-2k)(2\Delta^k u + \langle x, \nabla\Delta^k u\rangle).\end{aligned}$$

Rearrange terms then we have

$$\rho\Delta^{k+1}u = -(k)(n-2(k+1))\Delta^k u - (n-2(k+1))\langle x, \nabla\Delta^k u\rangle.$$

This finishes the proof. □

B.2 Hyperbolic Harmonic Functions in the Unit Ball

In this section we only consider even dimensional unit ball. From (B.1.1) we see that for any $u \in C^\infty(\mathbb{B}^n)$ if

$$\Delta_{\mathbb{H}} u = 0,$$

then

$$\Delta^{n/2} u = 0.$$

Here Δ denotes the Laplacian with respect to the Euclidean metric, whereas $\Delta_{\mathbb{H}}$ denotes the Laplacian with respect to the hyperbolic metric.

In [25] Yang found the boundary condition satisfied by hyperbolic harmonic extension of smooth functions. Suppose $n \geq 2$ is even and $f \in C^\infty(\mathbb{S}^{n-1})$, if we define

$$u(x) = c_n \int_{\mathbb{S}^{n-1}} \frac{(1-|x|^2)^{n-1}}{|x-\xi|^{2n-2}} f d\xi$$

then u is hyperbolic harmonic, and satisfies the boundary conditions

$$\Delta^k u\Big|_{\mathbb{S}^{n-1}} = (-1)^k \frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n-1}{2}-k\right)}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n}{2}-k\right)} \frac{\mathscr{P}_{n-1}}{\mathscr{P}_{n-1-2k}} f;\ \text{for } 0 \leq k \leq [\frac{n-2}{4}]; \tag{B.2.1}$$

and

$$\frac{\partial}{\partial\nu}\Delta^k u\Big|_{\mathbb{S}^{n-1}} = (-1)^{k+1} \frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n-1}{2}-k\right)}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n}{2}-k\right)} \frac{\mathscr{P}_{n-1}}{\mathscr{P}_{n-1-2k}} f;\ \text{for } 0 \leq k \leq [\frac{n-4}{4}]. \tag{B.2.2}$$

Here ν is the outer normal unit vector. The notation $[\alpha]$ denotes the largest integer that is less than or equal to α. For example $[1/2] = 0$ and $[1] = 1$. $\mathscr{P}_{2\gamma}$ denotes a family of operators on $\mathbb{S}^{n-1}$ such that

$$\mathscr{P}_{2\gamma} = \frac{\Gamma(B+\frac{1}{2}+\gamma)}{\Gamma(B+\frac{1}{2}-\gamma)},\ B = \sqrt{-\widetilde{\Delta} + \frac{(n-2)^2}{4}},$$

here $\widetilde{\Delta}$ is the Laplace-Beltrami opertor on the standard sphere $\mathbb{S}^{n-1}$.

Note that using the property of gamma function, namely $\frac{\Gamma(x+1)}{\Gamma(x)} = x$, we can get

$$\begin{aligned}
\frac{\mathscr{P}_{n-1}}{\mathscr{P}_{n-1-2k}} &= \frac{\Gamma(B+n/2)}{\Gamma(B+n/2-k)} \frac{\Gamma(B-n/2+1+k)}{\Gamma(B-n/2+1)} \\
&= \Pi_{i=1}^{k}(B+n/2-i)\Pi_{i=1}^{k}(B-n/2+i) \\
&= \Pi_{i=1}^{k}(-\widetilde{\Delta}+(i-1)(n-i-1)).
\end{aligned}$$

Plug this into (B.2.1), we have

$$\Delta^k u\Big|_{\mathbb{S}^{n-1}} = (-1)^k \frac{\Pi_{i=1}^{k}(n-2i)}{\Pi_{i=1}^{k}(n-2i-1)} \Pi_{i=1}^{k}[-\widetilde{\Delta}+(i-1)(n-i-1)]u. \tag{B.2.3}$$

B.2.1 Recover the Boundary Conditions

In this subsection we will show that we can recover the boundary conditions (B.2.1) and (B.2.2) using only the induction formula $(B.1.1)$ and results from [23].

Proposition B.2. *For any even integer $n \geq 2$ and any $f \in C^\infty(\mathbb{S}^{n-1})$, if we define*

$$u(x) = c_n \int_{\mathbb{S}^{n-1}} \frac{(1-|x|^2)^{n-1}}{|x-\xi|^{2n-2}} f d\xi$$

then $u \in C^\infty(\overline{B}^n)$ and satisfies the boundary conditions (B.2.1) and (B.2.2). Here $c_n = \frac{2\Gamma(n-1)}{\left|\mathbb{S}^{n-2}\right|\Gamma\left(\frac{n-1}{1}\right)}$ is the normalization constant.

Proof. In proposition 5.1 of [23], in particular (5.14) of the note, Michael Taylor showed that when the unit ball is even dimensional, the hyperbolic extension maps smooth function to functions that are smooth up to the boundary. As a result we have $u \in C^\infty(\overline{\mathbb{B}}^n)$.

Now we showed that u satisfies the boundary conditions (B.2.1) and (B.2.2). It will be easier for us to work in the polar coordinate in the Euclidean unit ball. In this coordinate (B.1.1) becomes

$$\frac{1-r^2}{2}\Delta^k u = -(k-1)(n-2k)\Delta^{k-1}u - (n-2k)r\frac{\partial}{\partial r}\Delta^{k-1}u. \tag{B.2.4}$$

We want to take limit $r \to 1$ to obtain the boundary conditions. Since $u \in C^\infty(\overline{\mathbb{B}}^n)$, for any $k \in \mathbb{N}_+$ both $\Delta^k u$ and $r\frac{\partial}{\partial r}\Delta^k u$ are continuous in $\overline{\mathbb{B}}^n$.

This leads to two simple results, firstly when $k = 1$, by taking limit $r \to 1$ in (B.2.4) we have

$$\frac{\partial}{\partial r}u\bigg|_{\mathbb{S}^{n-1}} = 0,$$

and for any $k \in \mathbb{N}_+$ by taking limit $r \to 1$ in (B.2.4) we have

$$\frac{\partial}{\partial r}\Delta^k u\bigg|_{\mathbb{S}^{n-1}} = -k\Delta^k u\bigg|_{\mathbb{S}^{n-1}}.$$

So now in order to finish the calculation, we only need to calculate $\Delta^k u\Big|_{\mathbb{S}^{n-1}}$ for any $k \in \mathbb{N}_+$.

Consider the case when $k = 1$, from (B.2.4) we have

$$\frac{1-r^2}{2}\Delta u = -(n-2)r\frac{\partial}{\partial r}u,$$

divide both sides by $(1-r^2)/2$, and take limit $r \to 1$ (using the boundary condition that $\frac{\partial}{\partial r}u\Big|_{\mathbb{S}^{n-1}} = 0$), we have

$$\Delta u\bigg|_{\mathbb{S}^{n-1}} = (n-2)\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}u\right)\bigg|_{\mathbb{S}^{n-1}} = (n-2)\frac{\partial^2 u}{\partial r^2}. \tag{B.2.5}$$

In Euclidean polar coordinate we have

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{n-1}{r}\frac{\partial u}{\partial r} + \frac{\widetilde{\Delta} u}{r^2},$$

here $\widetilde{\Delta}$ is the Laplace-Beltrami operator in the standard sphere $\mathbb{S}^{n-1}$. Taking limit $r \to 1$ we have

$$\frac{\partial^2 u}{\partial r^2}\bigg|_{\mathbb{S}^{n-1}} + \widetilde{\Delta} f = \Delta u\bigg|_{\mathbb{S}^{n-1}} = (n-2)\frac{\partial^2 u}{\partial r^2}\bigg|_{\mathbb{S}^{n-1}},$$

where the last step follows from (B.2.5). From which we have

$$\widetilde{\Delta} f = (n-3)\frac{\partial^2 u}{\partial r^2}\bigg|_{\mathbb{S}^{n-1}}$$

and

$$\Delta u\bigg|_{\mathbb{S}^{n-1}} = \frac{n-2}{n-3}\widetilde{\Delta} f.$$

This is exactly (B.2.3) for $k = 1$.

Calculation for $1 < k \le [\frac{n-2}{4}]$ can be done using induction with (B.2.3), we omit the calculation here. This finishes the proof. $\square$

B.2.2 The Other Direction

On the contrary we can prove the following

Proposition B.3. *For any even integer $n \ge 2$ and any $u \in C(\overline{\mathbb{B}}^n) \cap C^\infty(B^n)$ with boundary value $f \in C^\infty(\mathbb{S}^{n-1})$ such that u satisfies the boundary conditions (B.2.1) and (B.2.2), we have*

$$\Delta_{\mathbb{H}} u = 0, \ \ in\ \mathbb{B}^n.$$

Recall that $\Delta_{\mathbb{H}}$ is the Laplacian in the hyperbolic unit ball.

Proof. We consider the hyperbolic harmonic extension of f, define $\widetilde{u} : \mathbb{B}^n \to \mathbb{R}$ by

$$\widetilde{u}(x) = c_n \int_{\mathbb{S}^{n-1}} \frac{(1-|x|^2)^{n-1}}{|x-\xi|^{2n-2}} f d\xi.$$

From the previous subsection we know that $\widetilde{u}$ satisfies the same boundary conditions as u. Since the equation and boundary conditions are all linear, we only need to show that if

$$\begin{cases} \Delta^{n/2} v = 0, \\ \Delta^k v \Big|_{\mathbb{S}^{n-1}} = 0; \ \text{for } 0 \le k \le [\frac{n-2}{4}]; \\ \frac{\partial}{\partial \nu} \Delta^k v \Big|_{\mathbb{S}^{n-1}} = 0; \ \text{for } 0 \le k \le [\frac{n-4}{4}], \end{cases}$$

then $v = 0$.

This can be done by considering the integration

$$\int_{\mathbb{B}^n} v \Delta^{n/2} v = 0.$$

There are two cases, the first case is when $n/2$ is even. By doing integration by parts repeatedly, using the boundary condition $\frac{\partial}{\partial \nu} \Delta^k v \Big|_{\mathbb{S}^{n-1}} = 0$, we get $\int_{\mathbb{B}^n} (\Delta^{n/4} v)^2 = 0$, which means $\Delta^{n/4-1}v$ is harmonic with boundary condition $\Delta^{n/4-1}v\Big|_{\mathbb{S}^{n-1}} = 0$, so $\Delta^{n/4-1}v = 0$. Now use all the boundary conditions $\Delta^k v\Big|_{\mathbb{S}^{n-1}} = 0$ we eventually arrive at the conclusion that $v = 0$.

The second case is when $n/2$ is odd. This is similar, the only difference is in the last step, we only get

$$\int_{\mathbb{B}^n} |\nabla \Delta^{(n-2)/4} v|^2 = 0,$$

which mean $\Delta^{(n-2)/4}v$ is constant, but since $\Delta^{(n-2)/4}v\Big|_{\mathbb{S}^{n-1}} = 0$, we know that $\Delta^{(n-2)/4}v = 0$ in $\mathbb{B}^n$. This finishes the proof. □

B.3 Calculation in the Upper Half Space

We also include some calculation in the upper half space model of hyperbolic space. Consider the space $\left(\mathbb{R}^n_+, \frac{dy^2}{y_n^2}\right)$. Here dy^2 denotes the Euclidean metric; y_n is the last component;

$$\mathbb{R}^n_+ = \{(y', y_n) \in \mathbb{R}^n \text{ such that } y' \in \mathbb{R}^{n-1},\ y_n > 0\}.$$

We show that $\ln |y|^2$ is hyperbolic harmonic in the upper half space. Note that here $|y|^2$ denotes the norm of y with respect to the Euclidean norm.

This is a straight forward calculation. We use $\Delta_{\mathbb{H}}$ to denote the hyperbolic Laplacian in the upper half space.

$$
\begin{aligned}
\Delta_{\mathbb{H}} \ln |y|^2 &= \frac{\sum_{i=1}^n \partial_i \left(y_n^{2-n} \partial_i \ln |y|^2 \right)}{y_n^{-n}} \\
&= y_n^2 \sum_{i=1}^n \partial_i^2 \ln |y|^2 + (2-n) y_n \partial_n \ln |y|^2 \\
&= y_n^2 \sum_{i=1}^n \left(\frac{2}{|y|^2} - \frac{4y_i^2}{|y|^4} \right) + \frac{2(2-n)y_n^2}{|y|^2} \\
&= y_n^2 \left(\frac{2n}{|y|^2} - \frac{4|y|^2}{|y|^4} \right) + \frac{2(2-n)y_n^2}{|y|^2} \\
&= \frac{(2n-4)y_n^2}{|y|^2} + \frac{2(2-n)y_n^2}{|y|^2} \\
&= 0.
\end{aligned}
\tag{B.3.1}
$$

Remark 12. *The calculation in (B.3.1) offers an alternative explanation of the origin of the function $\widetilde{I}_n$ as discussed in Section 3.3.*

It is easy to expect that (B.3.1) is invariant under translation in $\mathbb{R}^{n-1}$. In the sense that for any $v \in \mathbb{R}^{n-1}$ we have

$$\Delta_{\mathbb{H}} \ln |y - v|^2 = 0.$$

It is also easy to expect that (B.3.1) is not invariant under translation in y_n direction, in the sense that for any $\lambda < 0$

$$\Delta_{\mathbb{H}} \ln\Big((y_n - \lambda)^2 + |y'|^2 \Big) \neq 0.$$

Note that here $y = (y', y_n)$ where $y' \in \mathbb{R}^{n-1}$ and $y_n > 0$.

But instead we have when $\lambda = -1$

$$\Delta_{\mathbb{H}} \left(\widetilde{I}_n \circ \Psi(y) - \ln\Big((y_n + 1)^2 + |y'|^2 \Big) \right) = 0.$$

Note that this is the consequence of Lemma 3.6, where we also use the relation (2.1.2).

The calculation (B.3.1) also leads to an interesting observation

Proposition B.4. *For any $y \in \mathbb{R}^n_+$ define $u(y) = \ln |y|$ then*

$$div \left(y_n^{\alpha} \nabla e^{(2-n-\alpha)u} \right) = 0$$

in $\mathbb{R}^n_+$. Where the divergence and ∇ are all defined with respect to the Euclidean metric.

Proof. This is a straight forward calculation. From (B.3.1) we have

$$\Delta u = \frac{(n-2)\partial_n u}{y_n},$$

where Δu denotes the Laplacian with respect to the Euclidean metric. As a result, we have

$$\begin{aligned}
&div\left(y_n^\alpha \nabla e^{(2-n-\alpha)u}\right)\\
&= (2-n-\alpha)y_n^\alpha e^{(2-n-\alpha)u}\left(\Delta u + \alpha\frac{\partial_n u}{y_n} + (2-n-\alpha)|\nabla u|^2\right)\\
&= (2-n-\alpha)y_n^\alpha e^{(2-n-\alpha)u}\left(\frac{(n-2)\partial_n u}{y_n} + \alpha\frac{\partial_n u}{y_n} + (2-n-\alpha)|\nabla u|^2\right).
\end{aligned}$$

We conclude the proof by pointing out the fact that

$$\frac{\partial_n u}{y_n} = |\nabla u|^2.$$

□

Remark 13. *Note that Proposition B.4 offers an alternative explanation of why the function*

$$\Gamma(y) = C(n,\alpha)\frac{1}{|y|^{n-2+\alpha}}$$

is a fundamental solution to the equation

$$div(y_n^\alpha \nabla u) = 0$$

as observed by Caffarelli and Silvestre in [3].

We think of it as the consequence of the fact that $\ln|y|$ is hyperbolic harmonic in the upper half space and of the coincidence that $\frac{\partial_n \ln|y|}{y_n} = |\nabla \ln|y||^2$.

APPENDIX C

REGULARITY

For any integer $n \geq 3$ any $\alpha \in (2-n, 1)$ and any $p > \frac{2(n-1)}{n-2+\alpha}$ we want to prove regularity of the function $\widetilde{f} \in L^p(\mathbb{S}^{n-1})$ such that $\widetilde{f}$ is a solution to the integral equation for any $\xi \in \mathbb{S}^{n-1}$

$$\int_{\mathbb{B}^n} \widetilde{p}_\alpha(x,\xi) \left(\widetilde{P}_\alpha \widetilde{f}\right)(x)^{\frac{n+2-\alpha}{n-2+\alpha}} = \widetilde{f}(\xi)^{p-1}.$$

Here $\widetilde{p}_\alpha(x,\xi)$ is the Poisson kernel .For any $x \in \mathbb{B}^n$ and any $\xi \in \mathbb{S}^{n-1}$ we have

$$\widetilde{p}_\alpha(x,\xi) = 2^{\alpha-1} c_{n,\alpha} \frac{(1-|x|^2)^{1-\alpha}}{|x-\xi|^{n-\alpha}}.$$

The normalizing constant $c_{n,\alpha}$ is as in (1.2.3), we restate it here:

$$c_{n,\alpha}^{-1} = \left|\mathbb{S}^{n-2}\right| \int_0^\infty \frac{r^{n-2} dr}{\left(1+r^2\right)^{\frac{n-\alpha}{2}}} = \frac{\Gamma(\frac{1-\alpha}{2})\Gamma(\frac{n-1}{2})}{2\Gamma(\frac{n-\alpha}{2})} \left|\mathbb{S}^{n-2}\right|.$$

In upper half space the kernel function is

$$p_\alpha(y,w) = c_{n,\alpha} \frac{y_n^{1-\alpha}}{|y-w|^{n-\alpha}}$$

with

$$P_\alpha f(y', y_n) = c_{n,\alpha} \int_{\mathbb{R}^{n-1}} \frac{y_n^{1-\alpha}}{\left(|y'-w|^2 + y_n^2\right)^{\frac{n-\alpha}{2}}} f(w) dw$$

The corresponding integral equation in the upper half space is

$$\left(\widetilde{f} \circ \Psi(w)\right)^{p-1} = \int_{\mathbb{R}^n_+} p_\alpha(y,w) |\Psi'(w)|^{\frac{\alpha-n}{2}} \left((P_\alpha f)(y)\right)^{\frac{n+2-\alpha}{n-2+\alpha}} dy,$$

or equivalently

$$(f(w))^{p-1} \left|\Psi'(w)\right|^{\frac{n-\alpha}{2} - \frac{(p-1)(n-2+\alpha)}{2}} = \int_{\mathbb{R}^n_+} p_\alpha(y,w) \left((P_\alpha f)(y)\right)^{\frac{n+2-\alpha}{n-2+\alpha}} dy, \qquad \text{(C.0.1)}$$

C.1 From L^p to L^∞

Proposition C.1. *For any integer $n \geq 3$, any $\alpha \in (2-n, 1)$ and any $\frac{2(n-1)}{n-2+\alpha} < p < \infty$ suppose $\widetilde{f} \in L^p(\mathbb{S}^{n-1})$ is a solution to the Euler-Lagrange equation*

$$\int_{\mathbb{B}^n} \widetilde{p}_\alpha(x,\xi) \left(\left(\widetilde{P}_\alpha \widetilde{f} \right)(x) \right)^{\frac{n+2-\alpha}{n-2+\alpha}} dx = \widetilde{f}(\xi)^{p-1},$$

then we have $\widetilde{f} \in L^\infty(\mathbb{S}^{n-1})$

Proof. Note that the operator $\widetilde{P}_\alpha : L^p(\mathbb{S}^{n-1}) \to L^{\frac{2n}{n-2+\alpha}}(\mathbb{B}^n)$ is bounded and compact when $\frac{2(n-1)}{n-2+\alpha} < p < \infty$. For $\widetilde{u} \in L^q(\mathbb{B}^n)$ where $1 \leq q < n$, define the operator T_α

$$\widetilde{T}_\alpha \widetilde{u} = \int_{\mathbb{B}^n} \widetilde{p}_\alpha(x,\xi) \widetilde{u}(x) dx.$$

Using a duality argument as in [18] we can prove

$$\|\widetilde{T}_\alpha \widetilde{u}\|_{L^{\frac{(n-1)q}{n-q}}(\mathbb{S}^{n-1})} \leq \|\widetilde{u}\|_{L^q(\mathbb{B}^n)} \tag{C.1.1}$$

Suppose $\widetilde{f} \in L^p(\mathbb{S}^{n-1})$ is a solution to the Euler-Lagrange equation

$$\int_{\mathbb{B}^n} \widetilde{p}_\alpha(x,\xi) \left(\left(\widetilde{P}_\alpha \widetilde{f} \right)(x) \right)^{\frac{n+2-\alpha}{n-2+\alpha}} dx = \widetilde{f}(\xi)^{p-1}.$$

From Proposition 2.9 we see that

$$\widetilde{P}_\alpha \widetilde{f} \in L^{\frac{np}{n-1}}(\mathbb{B}^n).$$

As a result if we define $\gamma = \frac{n-2+\alpha}{n+2-\alpha}$, then we have

$$(\widetilde{P}_\alpha \widetilde{f})^{1/\gamma} \in L^{\frac{np\gamma}{n-1}}(\mathbb{B}^n).$$

Using (C.1.1) we have

$$\widetilde{f}^{p-1} \in L^{\frac{\gamma p(n-1)}{n-1-\gamma p}}(\mathbb{S}^{n-1}),$$

and hence

$$\widetilde{f} \in L^{\frac{\gamma p(p-1)(n-1)}{n-1-\gamma p}}(\mathbb{S}^{n-1})$$

If we keep going we can have $\widetilde{f} \in L^\infty(\mathbb{S}^{n-1})$. □

C.2 Derivative of $\widetilde{p}_\alpha$ with respect to x

Next, we want to prove regularity for f in using the same idea as in the book by Gilbart-Trudinger [14, Chapter 4]. (The way they handle the Newtonian Potential) So firstly, we take derivative of $\widetilde{p}_\alpha$ with respect to x and with respect to ξ. Through direct calculation we get

$$\partial_{y_i} p_\alpha(y,w) = \begin{cases} -(n-\alpha)\frac{y_n^{1-\alpha}(y_i-w_i)}{|y-w|^{n-\alpha+2}}, \ i \neq n \\ -(n-\alpha)\frac{y_n^{2-\alpha}}{|y-w|^{n-\alpha+2}} + (1-\alpha)\frac{y_n^{-\alpha}}{|y-w|^{n-\alpha}}, \ i = n \end{cases}$$

$$\partial_{w_i} p_\alpha(y,w) = (n-\alpha)\frac{y_n^{1-\alpha}(y_i-w_i)}{|y-w|^{n-\alpha+2}}, \ i = 1, \ 2, ..., \ n-1$$

C.3 C^β Regularity for F

Suppose we have the integral equation

$$F(w) = c_{n,\alpha}\int_{\mathbb{R}^n_+} \frac{y_n^{1-\alpha}}{|y-w|^{n-\alpha}} U(y)dy,$$

where $y = (y', y_n) \in \mathbb{R}^n_+$ and $y', w \in \mathbb{R}^{n-1}$. We assume $U(y) \in L^\infty(\mathbb{R}^n_+)$ and $U(y) > 0$ for all $y \in \mathbb{R}^n_+$.

For any $R > 0$ we can write

$$\begin{aligned} F(w) &= c_{n,\alpha}\int_0^R \int_{\mathbb{R}^{n-1}} \frac{y_n^{1-\alpha}}{|y-w|^{n-\alpha}} U(y)dy'dy_n \\ &+ c_{n,\alpha}\int_R^\infty \int_{\mathbb{R}^{n-1}} \frac{y_n^{1-\alpha}}{|y-w|^{n-\alpha}} U(y)dy'dy_n. \end{aligned}$$

Define

$$F_R(w) = c_{n,\alpha}\int_0^R \int_{\mathbb{R}^{n-1}} \frac{y_n^{1-\alpha}}{|y-w|^{n-\alpha}} U(y)dy'dy_n. \tag{C.3.1}$$

It is easy to see that $F - F_R \in C^\infty(\mathbb{R}^{n-1})$, since there is no local singularity, and the singularity at ∞ is still summable after taking derivatives.

Lemma C.2. *For any $2-n \le \alpha < 1$. For any $R>0$ and for $U \in L^\infty(R^n_+)$, and $U>0$, define F_R as in (C.3.1). Then for any β such that $0<\beta<1$, we have $F_R \in C^\beta_{loc}(\mathbb{R}^{n-1})$.*

Proof. For any $v,\ w \in R^{n-1}$, consider

$$F_R(w) - F_R(v) = c_{n,\alpha} \int_0^R \int_{\mathbb{R}^{n-1}} \left(\frac{y_n^{1-\alpha}}{|y-w|^{n-\alpha}} - \frac{y_n^{1-\alpha}}{|y-v|^{n-\alpha}} \right) U(y) dy' dy_n.$$

Define $r = |v-w|$. Suppose we have $0 < r < \frac{R}{2}$. Define

$$A = \{y \in \mathbb{R}^n_+ : \text{ such that } |y-v| < 2r \text{ and } |y-w| < 2r\}.$$

Then we can write

$$\begin{aligned} F_R(w) - F_R(v) &= c_{n,\alpha} \int_0^R \int_A \left(\frac{y_n^{1-\alpha}}{|y-w|^{n-\alpha}} - \frac{y_n^{1-\alpha}}{|y-v|^{n-\alpha}} \right) U(y) dy' dy_n \\ &+ c_{n,\alpha} \int_0^R \int_{\mathbb{R}^{n-1}\setminus A} \left(\frac{y_n^{1-\alpha}}{|y-w|^{n-\alpha}} - \frac{y_n^{1-\alpha}}{|y-v|^{n-\alpha}} \right) U(y) dy' dy_n \\ &:= I + II \end{aligned}$$

For I we have

$$\begin{aligned} & \left| c_{n,\alpha} \int_0^R \int_A \left(\frac{y_n^{1-\alpha}}{|y-w|^{n-\alpha}} - \frac{y_n^{1-\alpha}}{|y-v|^{n-\alpha}} \right) U(y) dy' dy_n \right| \\ \le\ & C(n,\alpha) \int_0^R \int_A \frac{y_n^{1-\alpha}|y-w|^\beta}{|y-w|^{n-\alpha+\beta}} U(y) dy' dy_n \\ & + C(n,\alpha) \int_0^R \int_A \frac{y_n^{1-\alpha}|y-v|^\beta}{|y-v|^{n-\alpha+\beta}} U(y) dy' dy_n \\ \le\ & 2^\beta |w-v|^\beta C(n,\alpha) \|U\|_{L^\infty(\mathbb{R}^n_+)} \Bigg(\int_0^R \int_A \frac{y_n^{1-\alpha}}{|y-w|^{n-\alpha+\beta}} dy' dy_n \\ & + \int_0^R \int_A \frac{y_n^{1-\alpha}}{|y-v|^{n-\alpha+\beta}} dy' dy_n \Bigg), \end{aligned}$$

in the last inequality we used the fact that $|y-w| < 2r = 2|v-w|$, $|y-w| < 2|v-w|$ and

$U>0$ in $\mathbb{R}^n_+$. By change of variable, choose $z'=\frac{y'-w}{y_n}$ with $dz'=\frac{dy'}{y_n^{n-1}}$ then we have

$$\begin{aligned}\int_0^R\int_A \frac{y_n^{1-\alpha}}{|y-w|^{n-\alpha+\beta}}dy'dy_n &\le \int_0^R \frac{dy_n}{y_n^{\beta}}\int_{\mathbb{R}^{n-1}}\frac{dz'}{(|z'|^2+1)^{\frac{n-\alpha+\beta}{2}}}\\ &= |\mathbb{S}^{n-2}|\int_0^R\frac{dy_n}{y_n^\beta}\int_0^\infty \frac{r^{n-2}dr}{(r^2+1)^{\frac{n-(\alpha-\beta)}{2}}}\\ &= |\mathbb{S}^{n-2}|\frac{R^{1-\beta}}{1-\beta}\cdot\frac{\Gamma\left(\frac{1-(\alpha-\beta)}{2}\right)\Gamma(\frac{n-1}{2})}{2\Gamma\left(\frac{n-(\alpha-\beta)}{2}\right)}.\end{aligned}$$

Note that when $2-n\le\alpha<1$ and $0<\beta<1$, we have $\frac{1-(\alpha-\beta)}{2}>0$ and $\frac{n-(\alpha-\beta)}{2}>0$. As a result, we have

$$\int_0^R\int_A\frac{y_n^{1-\alpha}}{|y-w|^{n-\alpha+\beta}}dy'dy_n\le C(n,\alpha,\beta,R).$$

Similarly for v we have

$$\int_0^R\int_A\frac{y_n^{1-\alpha}}{|y-v|^{n-\alpha+\beta}}dy'dy_n\le C(n,\alpha,\beta,R).$$

As a result, we have

$$I\le C(n,\alpha,\beta,R)\|U\|_{L^\infty(\mathbb{R}^n_+)}|w-v|^\beta.$$

Now we consider II. Notice that

$$|D_wp_\alpha(y,w)|\le(n-\alpha)c_{n,\alpha}\frac{y_n^{1-\alpha}}{|y-w|^{n-\alpha+1}}.$$

Using mean value theorem we have: for some w_0 lying on the line segment between v and w

$$|p_\alpha(y,w)-p_\alpha(y,v)|\le|D_wp_\alpha(y,w_0)||w-v|.$$

As a result, we have

$$\begin{aligned}|II| &\le C(n,\alpha)\|U\|_{L^\infty(\mathbb{R}^n_+)}\int_0^R\int_{\mathbb{R}^{n-1}\backslash A}|D_wp_\alpha(y,w_0)||w-v|dy'dy_n\\ &= C(n,\alpha)\|U\|_{L^\infty(\mathbb{R}^n_+)}\int_0^R\int_{\mathbb{R}^{n-1}\backslash A}\frac{y_n^{1-\alpha}}{|y-w_0|^{n-\alpha+1}}|w-v|^{1-\beta}|w-v|^\beta dy'dy_n.\end{aligned}$$

In $\mathbb{R}^n_+\backslash A$, we have

$$|y-w_0|\ge|w-v|=r.$$

We can prove this by contradiction. Suppose we have

$$|y - w_0| < r,$$

then by triangle inequality, we have

$$|y - w| \leq |w - w_0| + |y - w_0| \leq |w - v| + |y - w_0| \leq 2r.$$

Similarly for v, we have

$$|y - v| \leq 2r.$$

As a result, we have $y \in A$, which is a contradiction.

Now we have

$$\begin{aligned} |II| &\leq C(n, \alpha)\|U\|_{L^\infty(\mathbb{R}^n_+)} \int_0^R \int_{\mathbb{R}^{n-1}\setminus A} \frac{y_n^{1-\alpha}}{|y - w_0|^{n-\alpha+1}} |y - w_0|^{1-\beta} |w - v|^\beta dy' dy_n \\ &= C(n, \alpha)\|U\|_{L^\infty(\mathbb{R}^n_+)} |w - v|^\beta \int_0^R \int_{\mathbb{R}^{n-1}\setminus A} \frac{y_n^{1-\alpha}}{|y - w_0|^{n-\alpha+\beta}} dy' dy_n. \end{aligned}$$

By previous calculation we have

$$\begin{aligned} \int_0^R \int_{\mathbb{R}^{n-1}\setminus A} \frac{y_n^{1-\alpha}}{|y - w_0|^{n-\alpha+\beta}} dy' dy_n &\leq \int_0^R \int_{\mathbb{R}^{n-1}} \frac{y_n^{1-\alpha}}{|y - w_0|^{n-\alpha+\beta}} dy' dy_n. \\ &\leq C(n, \alpha, \beta, R) \end{aligned}$$

As a result, we have

$$|II| \leq C(n, \alpha, \beta, R)\|U\|_{L^\infty(\mathbb{R}^n_+)} |w - v|^\beta.$$

All together, for any $0 < \beta < 1$ we have

$$|F_R(w) - F_R(v)| \leq C(n, \alpha, \beta, R)\|U\|_{L^\infty(\mathbb{R}^n_+)} |w - v|^\beta,$$

which means $F_R \in C^\beta_{loc}(\mathbb{R}^{n-1})$. □

C.4 C^β Regularity for U

Lemma C.3. *For some $2-n \le \alpha < 1$, $0<\beta<1$ such that $\alpha+\beta<1$ and for some $f \in C^\beta_{loc}(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$, define*

$$u(y) = P_\alpha f(y) = c_{n,\alpha} \int_{\mathbb{R}^{n-1}} \frac{y_n^{1-\alpha}}{|y-w|^{n-\alpha}} f(w)dw.$$

Then for any $y' \in R^{n-1}$ we have

$$\lim_{y_n \to 0} u(y', y_n) = f(y'),$$

and $u \in C^\beta_{loc}(\overline{\mathbb{R}}^n_+)$.

Proof. Through change of variable we see that

$$|u(y)| = c_{n,\alpha} \left| \int_{\mathbb{R}^{n-1}} \frac{f(y_n w + y')}{(|w|^2+1)^{\frac{n-\alpha}{2}}} dw \right| \le \|f\|_{L^\infty(\mathbb{R}^{n-1})} c_{n,\alpha} \int_{\mathbb{R}^{n-1}} \frac{dw}{(|w|^2+1)^{\frac{n-\alpha}{2}}}.$$

From which we can get

$$\|u\|_{L^\infty(\mathbb{R}^n_+)} \le \|f\|_{L^\infty(\mathbb{R}^{n-1})}.$$

In addition, by dominated convergence theorem, we get

$$\lim_{y_n \to 0} u(y', y_n) = c_{n,\alpha} \int_{\mathbb{R}^{n-1}} \lim_{y_n \to 0} \frac{f(y_n w + y')}{(|w|^2+1)^{\frac{n-\alpha}{2}}} dw = f(y').$$

Since it is easy to see that $U \in C^\infty(\mathbb{R}^n_+)$, we only need to show that U is Hölder continuous up to the boundary.

For any $y \in \mathbb{R}^n_+$ and any $v \in \mathbb{R}^{n-1}$, define $D = \{w \in \mathbb{R}^{n-1}$ such that $|(y_n w + y') - v| < 1\}$. Note that D is a ball in $\mathbb{R}^{n-1}$ centered at $\frac{v-y'}{y_n}$ with radius $\frac{1}{y_n}$. Also, note that for any $w \in D$ we have

$$|y_n w + y'| \le 1 + |v|$$

Choose $R>0$ large enough such that

$$1 + |v| < R$$

then for all $w \in D$ we have

$$|y_n w + y'| \leq 1 + |v| < R.$$

Since $f \in C^{\beta}_{loc}(\mathbb{R}^{n-1})$, we have

$$\sup_{|w_1|<R,\ |w_2|<R} \frac{|f(w_1) - f(w_2)|}{|w_1 - w_2|^{\beta}} = \kappa < \infty.$$

Consider

$$\begin{aligned}
|u(y) - u(v)| &= c_{n,\alpha} \int_{\mathbb{R}^{n-1}} \left| \frac{f(y_n w + y') - f(v)}{(|w|^2+1)^{\frac{n-\alpha}{2}}} \right| dw \\
&\leq c_{n,\alpha} \int_{D} \left| \frac{f(y_n w + y') - f(v)}{(|w|^2+1)^{\frac{n-\alpha}{2}}} \right| dw \\
&\quad + c_{n,\alpha} \int_{\mathbb{R}^{n-1}\setminus D} \left| \frac{f(y_n w + y') - f(v)}{(|w|^2+1)^{\frac{n-\alpha}{2}}} \right| dw \\
&\leq c_{n,\alpha}\kappa \int_{D} \frac{|y_n w + (y'-v)|^{\beta}}{(|w|^2+1)^{\frac{n-\alpha}{2}}} dw \\
&\quad + 2c_{n,\alpha}\|f\|_{L^{\infty}(\mathbb{R}^{n-1})} \int_{\mathbb{R}^{n-1}\setminus D} \frac{|y_n w + (y'-v)|^{\beta}}{(|w|^2+1)^{\frac{n-\alpha}{2}}} dw \\
&\leq C(n,\alpha,\beta,\kappa,\|f\|_{L^{\infty}}) \int_{\mathbb{R}^{n-1}} \frac{|y_n w|^{\beta} + |y'-v|^{\beta}}{(|w|^2+1)^{\frac{n-\alpha}{2}}} dw \\
&\leq C(n,\alpha,\beta,\kappa,\|f\|_{L^{\infty}}) \int_{\mathbb{R}^{n-1}} \left(\frac{|y_n|^{\beta}|w|^{\beta}}{(|w|^2+1)^{\frac{n-\alpha}{2}}} + \frac{|y'-v|^{\beta}}{(|w|^2+1)^{\frac{n-\alpha}{2}}} \right) dw \\
&\leq C(n,\alpha,\beta,\kappa,\|f\|_{L^{\infty}})|y-v|^{\beta} \int_{\mathbb{R}^{n-1}} \frac{|w|^{\beta}}{(|w|^2+1)^{\frac{n-\alpha}{2}}} dw \\
&\quad + C(n,\alpha,\beta,\kappa,\|f\|_{L^{\infty}})|y-v|^{\beta} \int_{\mathbb{R}^{n-1}} \frac{1}{(|w|^2+1)^{\frac{n-\alpha}{2}}} dw.
\end{aligned}$$

Note that in the third inequality, we used subadditivity of concave function. Also, note that

$$\int_{\mathbb{R}^{n-1}} \frac{|w|^{\beta}}{(|w|^2+1)^{\frac{n-\alpha}{2}}} dw = |\mathbb{S}^{n-2}| \frac{\Gamma\left(\frac{1-(\alpha+\beta)}{2}\right)\Gamma\left(\frac{n+\beta-1}{2}\right)}{2\Gamma\left(\frac{n-\alpha}{2}\right)},$$

and

$$\int_{\mathbb{R}^{n-1}} \frac{|w|^{\beta}}{(|w|^2+1)^{\frac{n-\alpha}{2}}} dw = |\mathbb{S}^{n-2}| \frac{\Gamma\left(\frac{1-\alpha}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}{2\Gamma\left(\frac{n-\alpha}{2}\right)}.$$

Both integrals are finite when $2-n \le \alpha < 1$, $0 < \beta < 1$ and $\alpha + \beta < 1$. As a result, we have

$$|u(y) - u(v)| \le C(n, \alpha, \beta, \kappa, \|f\|_{L^\infty})|y - v|^\beta$$

for all $v \in \mathbb{R}^{n-1}$ such that $|v| + 1 < R$ and for all $y \in R^n_+$. Note that since κ depends on R, the function u is only locally Hölder continuous. We have $u \in C^\beta_{loc}(\overline{\mathbb{R}}^n_+)$. □

C.5 C^1 Regularity for F

For any $v \in \mathbb{R}^{n-1} = \partial\mathbb{R}^n$ and for any $R > 0$ define $\mathbb{B}_{R,v} = \{y \in \mathbb{R}^n : |y - v| < R\}$, and $\mathbb{B}^+_{R,v} = \mathbb{B}_{R,v} \cap \mathbb{R}^n_+$. We also define the $n-1$ dimensional ball as $\mathbb{B}^{n-1}_{R,v} = \{w \in \mathbb{R}^{n-1} : |w - v| < R\}$. We also use notation $\mathbb{B}_R$, $\mathbb{B}^+_R$ and $\mathbb{B}^{n-1}_R$ to denote $\mathbb{B}_{R,0}$, $\mathbb{B}^+_{R,0}$ and $\mathbb{B}^{n-1}_{R,0}$ respectively.

We prove $C^1_{loc}(\mathbb{R}^{n-1})$ regularity using the same argument as in [14, Lemma 4.2].

Lemma C.4. *For any $2-n \le \alpha < 1$, $0 < \beta < 1$ and for any $U \in L^\infty(\mathbb{R}^n_+) \cap C^\beta_{loc}(\overline{\mathbb{R}}^n_+)$, $U > 0$ such that*

$$F(w) = c_{n,\alpha} \int_{\mathbb{R}^n_+} \frac{y_n^{1-\alpha}}{|y - w|^{n-\alpha}} U(y)dy$$

is well defined. We have $F \in C^1_{loc}(\mathbb{R}^{n-1})$.

Proof. Note that for any $R > 0$ we can write

$$F(w) = c_{n,\alpha} \int_{\mathbb{B}^+_R} \frac{y_n^{1-\alpha}}{|y - w|^{n-\alpha}} U(y)dy + c_{n,\alpha} \int_{\mathbb{R}^n_+ \setminus \mathbb{B}^+_R} \frac{y_n^{1-\alpha}}{|y - w|^{n-\alpha}} U(y)dy,$$

where

$$c_{n,\alpha} \int_{\mathbb{R}^n_+ \setminus \mathbb{B}_{R,w}} \frac{y_n^{1-\alpha}}{|y - w|^{n-\alpha}} U(y)dy \in C^\infty_{loc}(\mathbb{B}^{n-1}_R),$$

so we only need to consider

$$c_{n,\alpha} \int_{\mathbb{B}^+_R} \frac{y_n^{1-\alpha}}{|y - w|^{n-\alpha}} U(y)dy.$$

Extend $U(y)$ to $\mathbb{R}^n$ by defining

$$\overline{U}(y', y_n) = \begin{cases} U(y', y_n), \text{ for } y_n \ge 0, \\ U(y', -y_n), \text{ for } y_n < 0. \end{cases}$$

for all $y' \in \mathbb{R}^{n-1}$. Then it is easy to see that $\overline{U} \in L^{\infty}(\mathbb{R}^n) \cap C^{\beta}_{loc}(\mathbb{R}^n)$. Extend $p_\alpha(y, w)$ to $\mathbb{R}^n$ in the y variable by defining

$$\overline{p}_\alpha(y, w) = c_{n,\alpha} \frac{|y_n|^{1-\alpha}}{|y - w|^{n-\alpha}}.$$

As a result we have

$$\int_{\mathbb{B}_R^+} p_\alpha(y, w) U(y) dy = \frac{1}{2} \int_{\mathbb{B}_R} \overline{p}_\alpha(y, w) \overline{U}(y) dy,$$

for all $w \in \mathbb{B}_R^{n-1}$. Now we only need to consider $\int_{\mathbb{B}_R} \overline{p}_\alpha(y, w)\overline{U}(y)dy$. We can use the same argument as in Lemma 4.2 of [14] to prove that

$$\begin{aligned} D_i \left(\int_{\mathbb{B}_R} \overline{p}_\alpha(y, w) \overline{U}(y) dy \right) &= \int_{\mathbb{B}_R} D_i \overline{p}_\alpha(y, w) \left(\overline{U}(y) - \overline{U}(w) \right) dy \\ &\quad - U(w) \int_{\partial \mathbb{B}_R} \overline{p}_\alpha(y, w) \nu_i(y) dS_y, \end{aligned}$$

for $i = 1, 2, ..., n-1$. Here derivative is taken with respect to w, $\partial \mathbb{B}_R$ is the boundary of $\mathbb{B}_R$ in $\mathbb{R}^n$, dS_y is the standard measure on $\partial \mathbb{B}_R$. □

C.6 Application to the Non-limit case

Now we are ready to prove regularity results which was used in Theorem 2.12.

Proposition C.5. *For any integer $n \geq 3$, for any $2-n < \alpha < 1$ and any $p > \frac{2(n-1)}{n-2+\alpha}$ suppose we have $\widetilde{f} \in L^{\infty}(\mathbb{S}^{n-1})$, $\widetilde{f} \geq 0$ and that $\widetilde{f}$ is a solution to the Euler-Lagrange equation*

$$\int_{\mathbb{B}^n} \widetilde{p}_\alpha(x, \xi) \left(\left(\widetilde{P}_\alpha \widetilde{f} \right)(x) \right)^{\frac{n+2-\alpha}{n-2+\alpha}} dx = \widetilde{f}(\xi)^{p-1},$$

then $\widetilde{f} \in C^1(\mathbb{S}^{n-1})$.

Proof. If for any $w \in \mathbb{R}^{n-1}$ we define

$$f(w) = \widetilde{f} \circ \Psi(w) \left(\frac{2}{1 + |w|^2} \right)^{\frac{n-2+\alpha}{2}}$$

as in 2.1.5, then by Proposition 2.5 and change of variable we can see that f satisfies the the integral equation

$$(f(w))^{p-1}\,|\Psi'(w)|^{\frac{n-\alpha}{2}-\frac{(p-1)(n-2+\alpha)}{2}} = \int_{\mathbb{R}^n_+} p_\alpha(y,w)\,((P_\alpha f)(y))^{\frac{n+2-\alpha}{n-2+\alpha}}\,dy.$$

As an immediate result, we have $f(w) > 0$ for all $w \in \mathbb{R}^{n-1}$, since otherwise we have $f = 0$. Where $\|P_\alpha f\|_{L^\infty(\mathbb{R}^n_+)} \leq \|f\|_{L^\infty(\mathbb{R}^{n-1})}$, and hence $((P_\alpha f)(y))^{\frac{n+2-\alpha}{n-2+\alpha}} \in L^\infty(\mathbb{R}^n_+)$.

Using Lemma C.2, choose some β such that $0 < \beta < 1$, $\beta < p-1$ and $\frac{\beta}{p-1} + \alpha < 1$, we get that

$$(f(w))^{p-1}\,|\Psi'(w)|^{\frac{n-\alpha}{2}-\frac{(p-1)(n-2+\alpha)}{2}} \in C^\beta_{loc}(\mathbb{R}^{n-1}).$$

Since $|\Psi'(w)| = \frac{2}{1+|w|^2}$ is smooth as a function of w and that it is always positive, we have

$$f^{p-1} \in C^\beta_{loc}(\mathbb{R}^{n-1}),$$

and as a reuslt

$$f \in C^{\frac{\beta}{p-1}}_{loc}(\mathbb{R}^{n-1}).$$

Now apply Lemma C.3, we get

$$P_\alpha f \in C^{\frac{\beta}{p-1}}_{loc}(\mathbb{R}^n_+).$$

Finally apply Lemma C.4 to get

$$(f(w))^{p-1}\,|\Psi'(w)|^{\frac{n-\alpha}{2}-\frac{(p-1)(n-2+\alpha)}{2}} \in C^1_{loc}(\mathbb{R}^n),$$

and hence

$$(f(w))^{p-1} \in C^1_{loc}(\mathbb{R}^n).$$

Lastly, since $f(w) > 0$ for all $w \in \mathbb{R}^{n-1}$, we have

$$f(w) \in C^1_{loc}(\mathbb{R}^{n-1}).$$

Transform f back to the unit ball we see that

$$\widetilde{f} \in C^1(\mathbb{S}^{n-1}).$$

$\square$

C.7 Application to the Limit Case $\alpha = 2 - n$

Proposition C.6. *For any integer $n \geq 3$, suppose we have $\widetilde{f} \in L^\infty(\mathbb{S}^{n-1})$ and that $\widetilde{f}$ is a solution to the Euler-Lagrange equation*

$$e^{(n-1)\widetilde{f}(\xi)} = \int_{\mathbb{B}^n} e^{n\widetilde{I}_n + n\widetilde{P}_{2-n}\widetilde{f}} \widetilde{p}_{2-n}(x, \xi) dx \tag{C.7.1}$$

then $\widetilde{f} \in C^1(\mathbb{S}^{n-1})$.

Proof. Suppose $\widetilde{f} \in L^\infty(\mathbb{S}^{n-1})$ satisfies the integral equation (C.7.1). Define

$$f(w) = \widetilde{f} \circ \Psi(w) + \ln |\Psi'(w)|,$$

then as discussed in Section 3.4 we see that $f(w)$ satisfy the following integral equation:

$$e^{(n-1)f(w)} = \int_{\mathbb{R}^n_+} e^{nP_{2-n}f} p_{2-n}(y, w) dy. \tag{C.7.2}$$

Since $e^{nP_{2-n}f}(y) = |\Psi'(y)|^n e^{n\widetilde{I}_n + n\widetilde{P}_{2-n}\widetilde{f}} \circ \Psi(y) \in L^\infty(\mathbb{R}^n_+)$, we can apply Lemma C.2 to get

$$e^{(n-1)f(w)} \in C^\beta_{loc}(\mathbb{R}^{n-1}).$$

Since $e^{(n-1)f(w)} > 0$ for all $w \in \mathbb{R}^{n-1}$, we have

$$f(w) \in C^\beta_{loc}(\mathbb{R}^{n-1})$$

Now from Lemma C.3 we know that $P_{2-n}f \in C^\beta_{loc}(\overline{\mathbb{R}}^n_+)$. Using Lemma C.4 and equation (C.7.2) we eventually get $f \in C^1_{loc}(\mathbb{R}^{n-1})$. Eventually, transform back to the unit ball we get $\widetilde{f} \in C^1(\mathbb{S}^{n-1})$. $\square$

APPENDIX D

CONFORMALLY COMPACT MANIFOLD

A natural way to generalize observations in the hyperbolic space is to consider the conformally compact manifolds. In this appendix we consider the solvability of Laplace equation with Dirichlet boundary condition. Firstly we define some notations that will be used through out this appendix.

Definition D.1. *By (M, g) is conformally compact to $(\overline{M}, \overline{g})$, we mean: $\overline{M}$ is a compact manifold with nonempty boundary, and M is the interior of $\overline{M}$. We assume $\overline{g}$ to be smooth. With a fixed defining function $r : \overline{M} \to \mathbb{R}$ such that r is smooth and positive in M, where $r = 0$ on $\partial\overline{M}$ and $|dr|_{\overline{g}} = 1$ on $\partial\overline{M}$. Such that $g = r^{-2}\overline{g}$. Note that here we use ∇ and $\Delta = \operatorname{tr}_g \nabla^2$ to denote operators on (M, g) and use $\overline{\nabla}$ and $\overline{\Delta} = \operatorname{tr}_{\overline{g}} \overline{\nabla}^2$ to denote operators on $(\overline{M}, \overline{g})$. Moreover, Sobolev spaces on M and $\overline{M}$ are defined using measure from g and $\overline{g}$ respectively.*

Note that this definition is closely related to Subsection 1.4.1. In conformally compact manifold there exists a special neighborhood called the collar neighborhood. We discuss the collar neighborhood in the following remark.

Remark 14. *Note that by compactness of $\partial\overline{M}$ and the fact that $|dr|_{\overline{g}} = 1$, there exists a collar neighborhood $\Omega = [0, \epsilon) \times \partial\overline{M}$ of $\partial\overline{M}$, such that the metric $g = r^{-2}(dr^2 + h)$ in the collar neighborhood. Here $\epsilon > 0$ is a real number, and h is a metric on $\partial\overline{M}$ that depends on r, we use the h_0 to denote the metric on $\{0\} \times \partial\overline{M}$.*

In order to prove the solvability of Laplace equation in conformally compact manifolds with Dirichlet boundary condition, we firstly prove that we can extend a smooth function on the boundary to be a nice function on the collar neighborhood in the following sense:

Proposition D.2. *Suppose (M, g) is a connected n-dimensional manifold conformally compact to $(\overline{M}, \overline{g})$ as in Definition D.1. For C^∞ function $f : \partial\overline{M} \to \mathbb{R}$, extend f to $f_0 : \Omega \to \mathbb{R}$ by $f_0(t, \theta) = f(\theta)$. Where $(t, \theta) \in [0, \epsilon) \times \partial\overline{M}$. Then there exists functions $a_i : [0, \epsilon) \times \partial\overline{M} \to \mathbb{R}$ for $i = 2, ..., n$, which only depends on $\theta \in \partial\overline{M}$, such that*

$$\Delta\left(f_0 + \sum_{i=2}^{n-2} a_i r^i\right) = O(r^{n-1})$$

Proof. In the collar neighborhood as mentioned in Remark 14, for any C^2 function $u : \Omega \to \mathbb{R}$, we have

$$\Delta u = r^n \partial_r (r^{2-n} \partial_r u) + r^2 \frac{\partial_r \det h}{2 \det h} \partial_r u + r^2 \Delta_h u. \tag{D.0.1}$$

In particular, when $u = r^\alpha$, we have

$$\Delta r^\alpha = \alpha(\alpha - n + 1) r^\alpha + \alpha \frac{\partial_r \det h}{2 \det h} r^{\alpha+1} = \alpha(\alpha - n + 1) r^\alpha + O(r^{\alpha+1}) \tag{D.0.2}$$

Hence we have $\Delta f_0 = r^2 \Delta_h f$. If $n < 4$ then we are done. If $n \geq 4$, then we can take $a_2 = \frac{1}{2(n-3)} \Delta_{h_0} f_0$ such that

$$\Delta(f_0 + a_2 r^2) = \frac{1}{2(n-2)} \left(2r^3 \frac{\partial_r \det h}{\det h} \Delta_{h_0} f_0 + r^4 \Delta_h \Delta_{h_0} f_0\right) = O(r^3).$$

We can keep going until

$$\Delta\left(f_0 + \sum_{i=2}^{n-2} a_i r^i\right) = O(r^{n-1}), \tag{D.0.3}$$

where $a_i : [0, \epsilon) \times \partial\overline{M} \to \mathbb{R}$ does not depend on r, $i = 2, ..., n$. Note that this is the best we can do, since by equation $(D.0.2)$, we have for any $a_{n-1} : [0, \epsilon) \times \partial\overline{M} \to \mathbb{R}$ that does not depend on r,

$$\Delta a_{n-1} r^{n-1} = O(r^n). \tag{D.0.4}$$

So in general we can not cancel the $O(r^{n-1})$ term in the same way. ☐

Now we are ready to prove the solvability of Laplace equation in conformally compact manifolds with Dirichlet boundary condition.

Proposition D.3. *For any C^∞ function $f : \partial\overline{M} \to \mathbb{R}$, there exists a function $\psi : \overline{M} \to \mathbb{R}$ such that $\psi|_{\partial\overline{M}} = f$ and that $\Delta\psi = 0$ in M.*

Proof. Extend f to f_0 as in proposition $D.2$. By proposition $D.2$, there exists functions $a_i : [0,\epsilon) \times \partial\overline{M} \to \mathbb{R}$ for $i = 2, ..., n$, which only depends on $\theta \in \partial\overline{M}$, such that

$$\Delta\left(f_0 + \sum_{i=2}^{n-2} a_i r^i\right) = O(r^{n-1})$$

Choose a cut off function $\chi : \partial\overline{M} \to \mathbb{R}$ such that $0 \le \chi \le 1$ and $\chi = 1$ on $[0, \epsilon/2] \times \partial\overline{M}$ and that χ is supported in $[0,\epsilon) \times \partial\overline{M}$. Define

$$M_\epsilon = \{x \in M |\ r(x) > \epsilon\}.$$

Note that $\overline{M}_\epsilon = \{x \in M \mid r(x) \ge \epsilon\}$. Define

$$U = \chi\left(f_0 + \sum_{i=2}^{n-2} a_i r^i\right),$$

then we can extend U to be a function on $\overline{M}$ such that $U = 0$ on M_ϵ and that

$$\Delta U = O(r^{n-1}).$$

Choose $0 < \epsilon_0 < \epsilon/2$ small enough such that

$$\sup_{\partial\overline{M}_{\epsilon_0}} |U| \le 2 \sup_{\partial\overline{M}} |f|$$

For any $0 < \epsilon' < \epsilon_0$, in $\overline{M}_{\epsilon'}$, by standard theorem on bounded domain, there exists a function $\varphi_{\epsilon'}$ such that

$$\begin{cases} \Delta\varphi_{\epsilon'} = -\Delta U, \text{ in } M_{\epsilon'} \\ \varphi_{\epsilon'} = 0, \text{ on } \partial\overline{M}_{\epsilon'}. \end{cases}$$

Since $\Delta(\varphi_{\epsilon'} + U) = 0$ in $M_{\epsilon'}$, by maximum principle we have

$$\sup_{M_{\epsilon'}} |\varphi_{\epsilon'} + U| \le \sup_{\partial\overline{M}_{\epsilon'}} |U| \le 2 \sup_{\partial\overline{M}} |f|$$

As a result we have

$$|\varphi_{\epsilon'}| \leq 2 \sup_{\partial \overline{M}} |f| + \sup_{M} |U|$$

in $M_{\epsilon'}$ for all $\epsilon' \in (0, \epsilon_0)$. Define $C = 2\sup_{\partial \overline{M}} |f| + \sup_M |U|$. Choose $\alpha \in (0, n-1)$, by equation $(D.0.2)$ we have

$$\Delta r^{\alpha} = \alpha(\alpha - n + 1) r^{\alpha} + O(r^{\alpha+1}).$$

Moreover, since $\Delta U = O(r^{n-1})$ there exists $\epsilon_1 \in (0, \epsilon_0)$ such that

$$\Delta \left(C \frac{r^{\alpha}}{\epsilon_1^{\alpha}} \pm U \right) \leq 0$$

for all $(t, \theta) \in [0, \epsilon_1) \times \partial \overline{M}$. Choose $\epsilon' \in (0, \epsilon_1)$, define $A_{\epsilon', \epsilon_1} = \{x \in M | \ \epsilon' < r(x) < \epsilon_1\}$, then we have

$$\Delta \left(C \frac{r^{\alpha}}{\epsilon_1^{\alpha}} \pm \varphi_{\epsilon'} \right) \leq 0$$

in $A_{\epsilon', \epsilon_1}$. Which mean $C\frac{r^{\alpha}}{\epsilon_1^{\alpha}} \pm \varphi_{\epsilon'}$ is a super harmonic function in $A_{\epsilon', \epsilon_1}$. By maximum principle, we have

$$\inf_{A_{\epsilon', \epsilon_1}} C \frac{r^{\alpha}}{\epsilon_1^{\alpha}} \pm \varphi_{\epsilon'} \geq 0.$$

Note that when $r = \epsilon'$ we have $C\frac{r^{\alpha}}{\epsilon_1^{\alpha}} \pm \varphi_{\epsilon'} = C\frac{\epsilon'^{\alpha}}{\epsilon_1^{\alpha}} > 0$, and when $r = \epsilon_1$ we have $C\frac{r^{\alpha}}{\epsilon_1^{\alpha}} \pm \varphi_{\epsilon'} = C \pm \varphi_{\epsilon'} \geq 0$. As a result we have

$$|\varphi_{\epsilon'}| \leq C \frac{r^{\alpha}}{\epsilon_1^{\alpha}}$$

for all $x \in A_{\epsilon', \epsilon_1}$ and for all $\epsilon' \in (0, \epsilon_1)$. Hence we have

$$|\varphi_{\epsilon'} + U| \leq C \left(\frac{r^{\alpha}}{\epsilon_1^{\alpha}} + 1 \right) \leq 2C$$

for all $x \in A_{\epsilon', \epsilon_1}$ and for all $\epsilon' \in (0, \epsilon_1)$. Since $\varphi_{\epsilon'} + U$ is harmonic in $M_{\epsilon'}$, by maximum principle we have for any $x \in M_{\epsilon'}$

$$|\varphi_{\epsilon'} + U| \leq 2C.$$

By standard elliptic estimate we have

$$\|\varphi_{\epsilon'} + U\|_{C^{2,\alpha}(M_{\epsilon'})} \leq C_1.$$

By Arzela-Ascoli, and a diagonalization argument, there exists a sequence $\epsilon_i \to 0$ such that $\varphi_{\epsilon_i} + U$ converges to some function $\psi : M \to \mathbb{R}$ in C^2 norm. As a result, we have ψ is harmonic in M and that $\psi = f$ on $\partial\overline{M}$. □

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