TEACHER THINKING AND VIRTUAL MANIPULATIVES: HOW DO TEACHERS CONSIDER AFFORDANCES?

By

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ABSTRACT

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A recent review of research on virtual manipulatives (VMs), defined as technology-based interactive representations of mathematical ideas (Moyer-Packenham, & Bolyard, 2016), suggests strengths relative to other instructional tools in supporting student learning of mathematics (Moyer-Packenham & Westenskow, 2013). Researchers have identified affordances of VMs that may help to explain their value and role in learning. Three such affordances are *efficient precision,* or support for creating precise representations quickly and easily, *focused constraints,* or limits placed on how VMs can be manipulated in order to focus attention on mathematical ideas or processes, and *linked representations*, or sets of representations that are dynamically connected such that one changes in response to the other (Sarama & Clements, 2009). These affordances hold particular promise for supporting student learning of fraction concepts, as VMs can help students overcome whole-number biases when operating with fractions (Hansen, Mavrikis, & Geraniou, 2016) and support accurate estimation of magnitudes of fractions and fraction sums (Braithwaite & Siegler, 2021).

Although the evidence of the relationship between VMs and student learning is promising, the impact of all educational resources is shaped by how teachers use them (Cohen, Raudenbush, & Ball, 2003). As such, it is critical to understand how teachers think about VM affordances and how they plan to use them in their instruction. In this dissertation, I explored the thinking of six fourth- and fifth-grade teachers about fractions VMs three contexts: exploration, problem solving, and lesson planning. I employed the professional noticing framework (Jacobs,

Lamb, & Philipp, 2010) to analyze how teachers thought about VM features in each context. I coded the VM features to which they attended, synthesized their interpretations of features reflecting each of the three affordances, and analyzed how they used these features in problem solving and lesson planning.

Teachers attended to features reflecting all three affordances across contexts, but how they interpreted them varied by context and by affordance. Teachers had positive interpretations of features reflecting efficient precision across contexts because they relieved burdens on students for creating equal parts or carefully aligning models. Only in lesson planning, however, did some teachers reflect on how features reflecting efficient precision might support students in reaching more ambitious learning goals. Teachers' interpretations of features reflecting focused constraints were negative in exploration and problem solving because they interfered with their familiar problem-solving strategies. By contrast, teachers noted both positive and negative potential effects of focused constraints on student thinking during lesson planning. Teachers attended to linked representations in exploration but showed minimal interest in these features in the other contexts.

These results provided helpful guidance for teacher professional learning about VMs in teaching mathematics. Teacher thinking about VMs was more flexible and included more consideration of student thinking in the lesson planning context, establishing the importance of connecting professional learning about VMs to teachers' day-to-day practice. Additionally, since teachers often connected VM affordances to their existing practices rather than thinking about how to use them to introduce new strategies or learning goals, they will likely need additional support for conceptualizing why the VM affordances are important and how VMs might be used to their best advantage.

To the STEM Fatales.
Nevertheless, we persist.

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INTRODUCTION

While physical manipulatives have been a mainstay in elementary mathematics classrooms for many years (Carbonneau, Morley, & Selig, 2013), virtual versions of these tools are becoming more popular (Moyer-Packenham & Bolyard, 2016). These virtual manipulatives (VMs) hold promise for supporting student learning because they can mimic their physical counterparts while also possessing features that are difficult or impossible to replicate with physical tools. These features are discussed as *affordances* of VMs. For example, some VMs have *linked representations,* or representations that are connected such that one representation dynamically changes in response to changes in the other (Zbiek, Heid, Blume, & Dick, 2007). These linked representations can provide useful feedback to students about how representations are related and reflect different aspects of the underlying mathematical ideas (Sarama & Clements, 2009). Moreover, many VMs also allow students to offload tedious tasks, such as making sure the parts of a fraction model are equal in size, to the technology. This affordance is called *efficient precision* and can support students in creating and organizing multiple examples and discovering patterns (Moyer-Packenham & Suh, 2012). Some VMs intentionally constrain student actions to support them in focusing on a particular mathematical idea or process—an affordance called *focused constraints* (Hansen, Mavrikis, & Geraniou, 2016).

Research on student learning with VMs suggests these VM affordances may support valuable types of student thinking and learning (Moyer-Packenham & Westenskow, 2013). For example, the discourse of students who used VMs with linked representations showed higher levels of explanation and generalization than the discourse of students using VMs without linked representations (Anderson-Pence & Moyer-Packenham, 2016). The connections between affordances of VMs and rich mathematical student thinking and learning makes them a compelling focus for additional mathematics education research. This is particularly true for fractions content, which presents well-documented challenges for students (Fazio & Siegler, 2011), and for which VM affordances such as linked representations hold promise for supporting students' learning (Rau & Matthews, 2017).

Even so, we know little about how teachers facilitate rich learning experiences with VMs. Teachers are key players in determining how students use any educational resource (Cohen, Raudenbush, & Ball, 2003), and VMs are no exception. The affordances teachers notice as they examine digital tools can influence which tools they choose to use in their classrooms and how they use them (Webel, Krupa, & McManus, 2013). Teacher thinking as they plan and enact lessons with technology tools shapes how they relate the tools to opportunities for student learning in their teaching (Hansen, Mavrikis, & Geraniou, 2016). Yet, there has been little research examining how teachers think about VMs in their planning and instruction (Rich, 2021). This dissertation explored elementary teachers' thinking about VMs, with specific attention on how teachers think about the VM affordances of efficient precision, focused constraints, and linked representations.

Overview of the Study

I explored six elementary teachers' thinking about fractions VMs in multiple contexts related to teaching: free exploration of VMs, problem solving, and planning lessons. The lesson planning context included some attention to anticipating student thinking and responding to it. The teachers' exploration, problem solving, and lesson planning were supported by carefully planned professional development designed to facilitate teachers' access to and appreciation of the promising affordances of a set of VMs. To support teachers in relating VM features to learning goals about challenging mathematical content, I focused the professional learning, problem solving, and lesson planning on the topics of fraction comparison, equivalence, and addition and subtraction. I conducted detailed interviews with the teachers to capture their thinking about the fractions VMs in each context.

My research questions and analysis were guided by the professional noticing framework (Jacobs, Philipp, & Lamb, 2010), which outlines three interrelated skills teachers apply as they make sense of classroom resources and artifacts: *attending* to particular elements, *interpreting* the elements to which they attend, and deciding how to *respond*. I applied qualitative data analysis techniques to answer four research questions that I paraphrase here and present in detail in a later chapter: (1) To what features of VMs do teachers attend in each context? (2) How do teachers interpret features that reflect promising VM affordances? (3) How do teachers respond to the features reflecting the affordances? (4) How do teachers interpret and respond to features with potential for supporting students in connecting symbolic and visual representations? My aim was to better understand whether and how teachers thought about promising features of VMs, with the goal of informing the design of professional learning experiences that support teachers in using VMs to their best advantage.

Organization of the Dissertation

This dissertation is organized in eight chapters. In Chapter 1, I review related literature, discuss the framework guiding the study, and present the four research questions. Chapter 2 provides a detailed description of my methods, which includes attention to my research perspective and positionality, participant recruitment, data collection, and data analysis. In Chapters 3 through 6, I present the results of my study; each chapter presents the results related to one research question. Chapter 3 summarizes my analysis of the VM features to which teachers attended in the three contexts of the study. Chapter 4 addresses how teachers interpreted the VM features I coded as reflecting affordances of VMs identified in prior research. Chapter 5 discusses how teachers used the features reflecting VM affordances (or not) as they solved problems and planned lessons related to fraction comparison, equivalence, and addition and subtraction. Chapter 6 discusses how teachers thought about VM features with potential for supporting students in making connections between visual and symbolic representations.

In Chapter 7, I discuss the results of the study, connecting it to prior research and making recommendations for future research, design of VMs, and design of professional learning experiences for teachers. Lastly, in Chapter 8, I present some concluding remarks and discuss the limitations of the study.

CHAPTER 1: BACKGROUND AND FRAMING

In this chapter, I provide the empirical foundations and theoretical framing for the dissertation. I begin by reviewing three bodies of literature: the first on fractions learning, the second on manipulatives and virtual manipulatives, and the third on teacher thinking about technology. Next, I present my framework for the dissertation based on models of how teachers use instructional resources and make sense of classroom artifacts. To conclude, I draw on the background literature and framework to state the study purpose and research questions.

Literature Review

In this section, I situate my study within relevant existing literature. I first discuss literature on fractions learning to provide rationale for the mathematical context of the study. I chose fractions as the mathematical content focus because it is a perennially challenging topic for students and teachers (Fazio & Siegler, 2011) and has a rich representational nature that may benefit from the multiple representations embedded in VMs (Rau & Matthews, 2017). Next, I review conceptual discussions and empirical research on physical and virtual manipulatives in elementary mathematics education to introduce the three affordances of these tools on which the study is based—efficient precision, focused constraints, and linked representations. Finally, I review literature on teacher thinking about technology resources more generally to provide justification for choosing the theoretical frameworks and research design.

Challenges in Fractions Learning

Fractions are a particularly difficult topic for mathematics teachers to teach and students to learn. Students and teachers both tend to exit their schooling with superficial understanding of fractions and fraction computation that rely heavily on part-whole conceptions of fractions and memorization of procedures without meaning (Fazio & Siegler, 2011). Many struggles and misunderstandings with fractions stem from students making incomplete and incorrect attempts to reconcile their developing knowledge of fractions with their existing knowledge of the properties of whole numbers (Stafylidou & Vosniadou, 2004). Unlike whole numbers, fractions do not have a unique smallest positive number. Fractions also do not have a unique successor. There is no unique “next” fraction that comes after ½ the way that 3 comes after 2—there are always more fractions between any given pair of fractions (indeed, an infinite number of them). Such inconsistencies make many of students’ intuitive strategies for working with fractions invalid.

A task commonly used by researchers to elicit evidence of students’ whole-number reasoning is asking them to compare or order fractions presented symbolically (Clarke & Roche, 2009; Stafylidou & Vosniadou, 2004; Van Hoof, Verschaffel, & Van Dooren, 2015). Common strategies used by students in late elementary and middle grades—and, often, also by older students and adults—include attending only to the numerators and saying the fraction with the larger numerator is the larger fraction (Stafylidou & Vosniadou, 2004) or judging the relative size of fractions based on the size of the “gap” between the numerator and denominator (e.g., saying ⅘ is greater than 4/7 because 4 is closer to 5 than to 7; Clarke & Roche, 2009). These strategies reflect attention to numerators and denominators as independent whole numbers, rather than to the relationship between the numerators and denominators. Students using these strategies do not tend to see fraction as *one* number with a particular value or magnitude (Jordan et al., 2013; Siegler, Fazio, Bailey, & Zhou, 2013). By contrast, students who successfully compare fractions mentally tend to use strategies such as comparing the magnitudes of each fraction to familiar benchmarks such as ½ or 1 whole (Clarke & Roche, 2009). These strategies depend on thinking of the value of the fraction in terms of the relation between the numerator and denominator.

A common mistake students make with fraction addition is to add numerators and denominators (e.g., reasoning that $\frac{1}{2} + \frac{2}{5} = 3/7$; Siegler et al., 2013). Students' tendencies to think about numerators and denominators as whole numbers (without clear relation to each other) contributes to the persistence of this kind of erroneous strategy. When students do not consider how the relationship between the numerator and denominator define a single value for the fraction, they are unable to make reasonable estimates for sums such as $\frac{1}{2} + \frac{2}{5}$. Without fraction estimation skills, they are unable to judge the reasonableness of their answers or whether their answers make sense (Fazio & Siegler, 2011). Given the connections between knowledge of fraction magnitudes and success with tasks such as fraction comparison and addition, some researchers have argued that the most successful fraction interventions will focus on developing students' senses of fraction magnitudes (Siegler et al., 2013). A common recommendation for improving students' facility with fraction magnitudes is to include more work with locating fractions on the number line (Cramer, Aherendt, Monson, Wyberg, & Miller, 2017; Shaughnessy, 2011).

Improving student understanding of and performance on fraction addition tasks, however, may take more than a focus on fraction magnitudes. Even when students make accurate estimates for the magnitude of individual fractions, they are less accurate when making estimates of sums of the same fractions (Braithwaite, Tian, & Siegler, 2018). This finding suggests "improving children's ability to estimate and calculate fraction sums likely requires improving not only knowledge of individual fraction magnitudes, but also understanding of what fraction addition does" (Braithwaite, Tian, & Siegler, 2018, p. 8). Typical fraction addition instruction in classrooms does not address such conceptual issues, but rather is highly rule- and procedure-based (Suh, Moyer, & Heo, 2005). Most elementary teachers in one study could accurately apply the standard fraction addition algorithm (Young & Zientek, 2011), but nearly half of the teachers in a different study provided either no explanation or an incomplete explanation for *why* the algorithm requires students to find equivalent fractions with common denominators (Copur-Gencturk, 2021). Many teachers' explanations included some mention of common denominators or same-size pieces, but only about half were able to connect the common denominator to the idea of constructing like units of the same whole.

The well-documented and persistent challenges in fractions teaching and learning, the importance of developing students' senses of fraction magnitudes, and the lack of teachers' and students' conceptual understanding of fraction addition procedures all suggest that fractions are a content domain that may benefit from use of more visual representations such as those available in virtual manipulatives. Incorporation of visual representations into mathematics instruction may reduce dependence on whole-number reasoning that is often elicited when students work with symbolic (numeric) fractions. Evidence from studies of human cognition suggest that elementary students and adults process visual representations of fraction magnitude more easily than symbolic representations (Rau & Matthews, 2017). Additionally, a recent study suggested allowing students to create visual representations of fractions sums by concatenating the addends along a number line improved students' abilities to make accurate estimates of the sums (Braithwaite & Siegler, 2021). More generally, using visual representations of fractions is a research-based strategy for helping students connect procedural knowledge of fraction computation methods to conceptual understanding (Fazio & Siegler, 2011).

In this study, I will focus on uses of virtual manipulatives (VMs) specifically to support understanding of fraction equivalence, comparison, and addition and subtraction, rather than more basic fraction representation or magnitude concepts, for two reasons. First, tasks involving comparison, equivalence, and addition and subtraction require relating multiple fractions and so can invite purposeful consideration of fraction magnitudes. Such tasks may elicit thinking and misconceptions in a way that merely representing single fractions may not (as suggested by researchers' consistent use of fraction comparison tasks to elicit evidence of whole-number biases). Second, comparing fractions, generating equivalent fractions, and adding and subtracting fractions are specific curricular topics that are likely to be familiar to teachers and are mentioned directly in the *Common Core State Standards for Mathematics* (CCSS-M; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). Because standards related to these topics appear primarily in Grades 4 and 5, I focused on fourth- and fifth-grade teachers' thinking about VMs.

Manipulatives and Virtual Manipulatives

In this section, I first summarize the learning theories that underlie the use of manipulatives in elementary mathematics classrooms. Next, I present theoretical and empirical arguments for how virtual versions of manipulatives may provide advantages for students learning in certain contexts and for certain learning goals. This discussion also introduces the three affordances of virtual manipulatives that have been emphasized in the research literature are the focus of this dissertation.

Learning theories underlying manipulative use in elementary mathematics education. Manipulatives are physical objects used in classrooms to help children learn mathematics (Uttal, Scudder, & DeLoache, 1997). Many of the earliest physical manipulatives were designed with the specific purpose of serving as physical and tangible instantiations of abstract mathematical ideas. For example, Diénès designed base-10 blocks (and blocks of other bases) for the purpose of embodying the concept of place value, and physical manipulation of the blocks was intended to support students in constructing an isomorphic mental representation of place value (Sriraman & Lesh, 2007). The use of manipulatives to support learning of mathematics is based in part on the work of Piaget (1964), who argued young children need experiences with physical materials to develop abstract thought because abstract ideas were seen broadly as internalized actions on physical objects.

Building on Piaget's ideas, Bruner (1964) developed a theory about how children represented their interactions with the environment. He proposed three kinds of representations that children construct sequentially. *Enactive* representations are the memories of the physical actions taken. *Iconic* representations use mental imagery to summarize the important elements of the actions. *Symbolic* representations consist of symbols that bear little or no resemblance to the physical actions, but nonetheless stand for the interactions in the mind of the learner. According to Bruner's theory, organizing mathematics instruction around manipulatives allows children to begin by creating an enactive representation of a mathematical idea through working with the materials. Children also build iconic representations through these interactions, and when manipulatives are designed to highlight important mathematical ideas (which Bruner called *structures*), attention to these ideas enriches the mental images (Resnick & Ford, 1981). Mathematics educators, including Diénès (1971), carefully designed manipulatives to reflect specific mathematical structures (Sriraman & Lesh, 2007). The manipulatives were intended to support teachers in creating experiences that supported children in building iconic and symbolic representations of concepts through enactive experiences.

Researchers developed Bruner's (1964) three-phase representational theory into an instructional framework called the Concrete Representational Abstract (CRA) framework (Agrawal & Morin, 2016). Following this framework, instruction on a mathematical topic should begin with hands-on activities where children use concrete manipulatives. This corresponds to Bruner's call for children to have opportunities to create enactive representations. According to CRA, once children have mastered the use of concrete representations, instruction should proceed to incorporating visual representations such as drawings. These correspond to Bruner's iconic representations and are intended to serve as a bridge between manipulatives and abstract symbols. When children have mastered the mathematical concept using visual representations, CRA instruction proceeds to its "main objective" (Agrawal & Morin, 2016, p. 35), which is to support students in understanding and accessing numbers and symbols (*abstract* representations in the CRA framework, corresponding to Bruner's *symbolic* representations) without the need for support from concrete or visual representations. To facilitate this transition, the CRA framework also recommends presenting symbolic notations alongside the concrete and visual representations in early stages of instruction. Use of concrete and visual representations before relying exclusively on symbols is intended to support the development of conceptual understanding alongside procedural fluency (Agrawal & Morin, 2016).

Despite robust learning theories and frameworks underlying their use, research makes clear that incorporating manipulatives into instruction does not always lead to increases in learning (Puchner, Taylor, O'Donnell, & Fick, 2008; Resnick & Omanson, 1986; Uttal, Scudder, & DeLoache, 1997). To learn with manipulatives, students need to actively construct connections between manipulatives and mathematical ideas (Ball, 1992; Nührenbörger & Steinbring, 2008; Puchner et al., 2008; Uttal et al., 2009). They must develop internal representations of their work with manipulatives that reflect important mathematical ideas. When symbolic representations are used alongside manipulatives, students must also connect the manipulatives to the symbolic representations (Uttal, O'Doherty, Newland, Hand, & DeLoache, 2009).

Unfortunately, children do not make these connections spontaneously. When manipulatives are used in mathematics instruction, teachers usually bear the responsibility of supporting children in constructing connections between concrete, visual, and symbolic representations of mathematical ideas. Children's grasp of representational relationships may be related to how those representations are introduced. For example, Uttal, Scudder, and DeLoache (1997) presented evidence that children made stronger connections between manipulatives and mathematics (specifically, base-10 computation and fraction concepts) when the connections were introduced immediately. In successful programs, connecting numbers and mathematical symbols to manipulatives was presented to children as the first step in learning. Children were also not expected to make the connections between manipulatives and symbols without explicit, consistent instruction. For example, Resnick and Omanson (1986) had limited success supporting students in connecting base-10 block manipulations to symbolic algorithms for multi-digit subtraction via mapping instruction, but they found that verbalization of connections may relate to stronger learning. Despite decades of research, it is not yet clear what kinds of instruction and support most effectively help children make the desired connections between mathematical symbols and physical objects.

Virtual manipulatives. Moyer-Packenham and Bolyard (2016) defined a *virtual manipulative* (VM) as, "an interactive, technology-enabled visual representation of a dynamic mathematical object, including all of the programmable features that allow it to be manipulated, that presents opportunities for constructing mathematical knowledge" (p. 13). Many VMs are screen-based versions of traditional physical mathematics manipulatives found in many classrooms, such as counters, base-10 blocks, and fraction circle pieces. Indeed, VMs often are created with the purpose of mimicking their physical counterparts. Even so, a metaanalysis revealed a moderate positive effect on learning for students using VMs over other instructional treatments and a small positive effect on learning for students using VMs over physical manipulatives (Moyer-Packenham & Westenskow, 2013). Moreover, several studies have shown that using virtual manipulatives supported upper elementary students' conceptual and procedural learning about fractions (Moyer-Packenham & Suh, 2012; Reimer & Moyer, 2005; Suh, Moyer, & Heo, 2005), although these studies did not directly compare VMs to other fractions tools and representations.

Researchers have offered several theories about why VMs may offer an instructional advantage over their physical counterparts. One possible explanation is that VMs are a unique form of representation that exhibits characteristics of both concrete and visual representations (Moyer-Packenham, Salkind, & Bolyard, 2008). Because they are screen-based, VMs have the appearance of a visual representation (such as a drawing or diagram), yet they also can be manipulated like a concrete representation. In the CRA framework, the representational (drawing) phase is intended to act as a bridge between concrete manipulatives and abstract symbols. Bouck et al. (2017) found that students with mild intellectual disabilities were able to transition directly from using virtual manipulatives to using abstract symbols to add fractions with unlike denominators, without an intervening representational phase to support this transition. One explanation for the successful transition from VMs to abstract symbols might be that the drawing-like characteristics of the VMs supported the transition in the same way drawings might have. More generally, the hybrid representational nature of VMs may support students in grasping mathematical concepts underlying the representations by allowing them to take advantage of features that otherwise would not co-exist, such as freely moving models in relation to each other (as with physical manipulative) and annotating the representations (as with a drawing).

In addition to their hybrid representational nature, Sarama and Clements (2009) argued that VMs have particular affordances that explain what makes VMs good for learning. Specifically, they claimed VMs have affordances that promote the development of *integrated-concrete knowledge,* or knowledge of manipulatives that is connected to other mathematical ideas. In essence, Sarama and Clements argued that VMs have affordances that may help children connect the manipulatives to underlying concepts—something they often do not do when using physical manipulatives. In the subsections that follow, I discuss three of the affordances of VMs identified by Sarama and Clements (2009). I describe one empirical study of whether or how each affordance supported student learning, with a focus on studies of fractions learning. I define an *affordance* as "whatever it is about the environment that contributes to the kind of interaction that occurs" (Greeno, 1994, p. 338). I chose these three affordances due to their alignment with my chosen definition of affordance and the existence of empirical evidence supporting their relationship to student learning.

Efficient precision. First, Sarama and Clements (2009) discussed the *efficient precision* of VMs, or their ability to create quick, precise representations. For example, some VMs allow students to instantly produce a reflection image of a figure or instantly partition a shape into equal-sized parts, removing the time and effort it takes for students to create similar images with physical tools. Sarama and Clements noted that VMs allowed students to "work with more precision and exactness" (p. 147), suggesting that this exactness may support students in generating more complete explanations of mathematical phenomena. Moyer-Packenham and Westenskow (2013) synthesized the results of 25 studies involving this affordance to argue that the ability to quickly create many precise representations can support learning in several ways: by alleviating difficulties in motor skills, allowing students to solve more problems or generate more solutions in the same amount of time (Suh, 2010), allowing students to be more methodical and purposeful in their creation of examples (Naftaliev & Yerushalamy, 2011), and supporting students in noticing mathematical patterns as they create and organize examples (Moyer-Packenham & Suh, 2012).

At least one study suggested efficient precision can support students in learning concepts of fraction equivalence. Moyer-Packenham and Suh (2012) studied the behavior and learning of fifth graders using several fractions VMs. One task asked students to use a VM (with functionality similar to this VM) to generate two equivalent fractions. Many students, but particularly high-achieving students, used the efficient precision of the VM to generate more examples of equivalent fractions than required, commenting that they did not "have to worry about taking a long time to change to different fractions" (Moyer-Packenham & Suh, 2012, p. 49). As students generated and examined many examples of equivalent fractions, they noticed the proportional relationships between the numerators and denominators and made generalizations such as "only the numerator and denominator changes, but the number does not" (p. 50). These students used the patterns to generate more equivalent fractions without using the VM. Students' use of multiple examples to make and apply a generalization suggests that the efficient precision of the VM may have supported them in connecting visual aspects of a representation to mathematical symbols and underlying ideas quickly and efficiently. In particular, students' abilities to generate equivalent fractions based on proportional relationships (after employing the efficient precision affordance) suggests that this affordance supported students in connecting the symbolic fractions to their magnitudes.

Focused constraints. Second, VMs can guide or constrain students' actions in ways that bring mathematical ideas and processes into conscious awareness that might be ignored or overlooked when children use physical manipulatives (Sarama & Clements, 2009). Moyer-Packenham and Westenskow (2013) described this set of ideas as an affordance called *focused constraint.* They cited 17 studies providing evidence that constraints embedded into particular VMs brought a mathematical idea into focus. These studies suggested focused constraints could shape student learning by facilitating or discouraging particular problem-solving strategies (Hansen, Mavrikis, & Geraniou, 2016; Manches, O'Malley, & Benford, 2010) or prompting use of mathematical vocabulary (Evans & Wilkins, 2011; Kaur, 2015).

For example, the fraction addition tool within a VM called *Fractions Lab* was designed to produce the numeric sum of two fractions only if the fractions had common denominators (Hansen, Mavrikis, & Geraniou, 2016). If students attempted to add fractions with unlike denominators, visual representations of the fractions were combined—the rectangular pieces representing each fraction were adjoined—but the VM did not produce a symbolic answer. This constraint was "designed to encourage children to create like denominators (using the partition tool) to carry out addition of fractions with uncommon denominators" (Hansen, Mavrikis, & Geraniou, 2016, p. 210). This example illustrates how the focused constraint affordance of VMs might be used to address a common error students make when adding fractions: adding the numerators and adding the denominators, often in an attempt to apply whole-number reasoning to fraction arithmetic (Siegler et al., 2013). This constraint in the addition of fractions tool was designed to prompt a snag in such whole-number thinking, potentially bringing the conceptual ideas underlying the fraction addition algorithm into focus.

While Hansen, Mavrikis, and Geraniou (2016) did not provide direct evidence of student thinking or learning, they described a case study of a teacher's choices while interacting with students using the VM. The teacher felt the constraint helped students develop a conceptual understanding of addition and subtraction of fractions because the visual representation of the sum "allowed them to freely understand why that was happening *[why a symbolic sum was not shown]* and what was happening" (p. 218). He felt that without the visual illustration of the impact of partitioning the addends, students would have to rely on poorly understood symbol-based procedures for finding common denominators.

Moyer-Packenham and Suh (2012) described a similar constraint in the fraction addition VM used in their study. They described how low-achieving students began solving addition problems such as ½ + ⅓ by creating equivalent fractions for each addend without creating a common denominator (e.g., ½ = 2/4 and ⅓ = 2/6). Constraints built into the VM prevented students from using these equivalent fractions to add; when they tried, the VM gave them feedback that guided them to try again until they found a common denominator. The authors argued these constraints "seemed to help them learn the procedure for finding the common denominator" (p. 51) through a process of trial and error. Thus, there is limited empirical evidence that the focused constraint affordance may help students overcome their tendencies to apply erroneous fraction addition algorithms and develop conceptual understanding of why the algorithm works. Although these studies did not provide direct evidence of students' abilities to explain the algorithm, that they could successfully apply the algorithm using the focused constraint affordance suggests that the visual and manipulable representations in the VMs may provide a way for students to make sense of the algorithm.

Linked representations. Third, Sarama and Clements (2009) discussed linked representations as an affordance of VMs. VMs with *linked representations* have at least two representations of the same mathematical concept. For example, a fractions VM may show both the symbolic representation of a fraction and an area model for that fraction. The representations are connected such that when a student changes one representation, the other dynamically changes to match it. Some VMs link visual and symbolic representations. Others link multiple visual representations.

An important aspect of linked representations is the feedback they provide to students (Sarama & Clements, 2009; Zbiek, Heid, Blume, & Dick, 2007). For example, when students watch the size of the parts of a fraction area model decrease as they progressively increase the denominator, they receive immediate, visual feedback about the impact of their changes. Whereas children working with physical manipulatives often do not connect the physical manipulations of the objects to symbolic manipulations, the automatic feedback offered through linked representations may make it more difficult for students to "overlook the consequences of their actions" (Sarama & Clements, 2009, p. 148). Moyer-Packenham and Westenskow (2013) extended these arguments to add that the feedback provided by linked representations may support students in reinterpreting representations when their initial ideas about them are incorrect or incomplete.

Anderson-Pence and Moyer-Packenham (2016) analyzed the discourse of pairs of fifth graders using three kinds of VMs: (1) pictorial VMs, which closely resemble physical manipulatives with few additional features, (2) linked VMs, which contain linked representations, and (3) tutorial VMs, which are embedded in directed activities that walk students through solution steps. The mathematical content focus of the VMs was fractions, geometry, or division. The students using linked VMs showed significantly higher levels of justification, generalization, and collaboration in their discourse. In the case of generalization, this result indicates that students using linked VMs more often made comparisons and generalizations, as opposed to observations of one representation or idea, than did students who used the other types of VMs. This, in turn, suggests that these students were more often making attempts to actively construct connections between representations.

Anderson-Pence and Moyer-Packenham (2016) did not disaggregate their results by mathematical content, so their study does not speak directly to the potential impact of linked representations on fractions understanding. Even so, the researchers did report the specific VMs used in the study. The fractions VM with linked representations focused on fraction equivalence. With this VM, students manipulate fraction area models and watch the corresponding points on a fraction number line shift in response. When they have created two equivalent fractions, the corresponding points on the number line coincide. Several researchers have argued that area models tend to support a part-whole understanding of fractions whereas number line representations support a magnitude understanding of fractions (e.g., Cramer et al., 2017). While it is unclear *how* the linked representations in Anderson-Pence and Moyer-Packenham (2016) supported students in making generalizations, it is possible that the feedback provided by the linked representations helped students connect the part-whole understanding of fractions highlighted by the area models to fraction magnitudes highlighted by the number lines.

Reflection on discussion of VM affordances. This discussion presented theoretical and empirical arguments for how three affordances of VMs may support students learning mathematical concepts, with a particular focus on fractions. These results are promising, yet the focus on interactions with affordances gives only minimal attention to a key idea about student learning with manipulatives: Students must not only take actions on the manipulatives, but also make sense of those actions—on their own or in interaction with others (Leung, 2017; Sarama & Clements, 2009). The relationship between any of the VM affordances and student learning will depend on what actions students take *and* whether and how students make sense of their actions. Linked representations provide feedback about how representations relate (Zbiek et al., 2007), for example, but a student must manipulate a VM in particular ways for the linked representations to be evident. Moreover, even if those manipulations take place, if a student attends only to copying down "matching" pairs of representations and not the relationships between them, the feedback may not support construction of connections between representations.

Relatedly, while the affordances facilitate certain kinds of learner interactions with VMs, they also constrain the opportunities to undertake other actions. For example, fractions VMs often employ efficient precision to allow students to quickly produce accurate area models for fractions, which can help students notice patterns (Moyer-Packenham & Suh, 2012). However, the efficient precision of the VMs also means students are not undertaking the physical or cognitive actions needed to create equal parts—kinds of thinking and action that may be important for certain learning goals. While students can use efficient precision features to identify patterns, those same features may offload too much constructive work to the tool, making important actions invisible and potentially preventing students from undertaking important thinking.

The kinds of student thinking and learning that take place through use of VMs are a result of *how* the resources are used (Cohen, Raudenbush, & Ball, 2003). As such, VM affordances may lead to gains or losses of learning opportunities, depending on the ways in which they are used. Teachers are key players in determining how educational resources are used (Cohen, Raudenbush, & Ball, 2003). With respect to teaching fractions using VMs, they are pivotal in shaping how students see and interact with those resources. I consider teachers next.

Teacher Thinking About Technology Tools

Although the research on VMs summarized above did not focus on the teacher's role, several studies highlight the ways that teachers can affect how students interact with VMs, whether students encounter or make use of affordances, and whether and how students make sense of their actions. Teachers' choices of tasks for students to complete with VMs will shape which features the use, and the questions they ask and comments they make during instruction will guide how students interpret their interactions with VMs. For example, the teacher in Hansen, Mavrikis, & Geraniou's (2016) study of *Fractions Lab* chose to sequence the fraction addition problems presented to a student such that the student solved a like-denominator problem first, then a problem where one denominator was a multiple of the other, then a problem where both denominators had to be changed in order to solve the problem. This choice affected how the student interacted with and interpreted the focused constraint because it set up an opportunity for the teacher to prompt the student to think about how partitioning the area models could support her in finding an equivalent addition problem with the same sum.

Given the importance of the role of the teacher in shaping students' experiences with VMs, it is pertinent to consider how teachers think about these tools and the kinds of professional learning experiences that may support teachers in using the VMs effectively.

Studies of how teachers think about VMs are sparse (Rich, 2021). However, as the number and availability of technology resources for teaching and learning has increased, researchers have begun to explore how teachers think about and use multiple kinds of technology in classrooms. In this section, I review research on teaching thinking about technology tools. I begin by summarizing themes in research about how teachers evaluate, select, and use technology tools designed for mathematics, including (but not only) VMs. Next, I highlight three individual and contextual factors that can impact teachers' use of any technology in the classroom. I refer to research across K-12 but, given the nature of this study, focus on research in K-5 when possible.

Research on teachers' thinking about technology tools for mathematics. There are three key themes in research about teachers' thinking about technology tools for teaching mathematics: (1) consistency in the criteria teachers use to select and evaluate mathematics technology tools and resources; (2) inconsistency in whether and how teachers relate those criteria to each other, and (3) the need for supports for teachers in evaluating, selecting, using technology tools effectively.

Consistency in selection and evaluation criteria. Teachers' criteria for evaluating and selecting mathematics technology tools typically involve two main considerations: (1) students' expected experiences with and reactions to the tool (e.g., engagement) and (2) the mathematics involved in use of the tool (Johnston & Suh, 2009; Shapiro, Sawyer, Dick & Wismer, 2019; Smith, Shin, & Kim, 2017b; Webel, Krupa, & McManus, 2015). The ways teachers thought about these criteria was different across studies, but these two categories were consistently reported.

In relation to student experiences, several studies highlighted the importance of student engagement in teacher's evaluations and selection criteria (Johnston & Suh, 2009; Shapiro et al., 2019; Webel, Krupa, & McManus, 2015). Teachers consistently said that they chose tools they thought would engage students. It is not always clear what researchers (or the participating teachers) meant by student engagement—specifically, whether teachers were concerned with surface level features that would capture student interest or rather with how students would engage with the embedded mathematics. Johnston and Suh (2009) said the majority of their elementary preservice teacher participants cited a tool being "fun, engaging, and/or motivating" (p. 3565) as part of their reason for selecting a technology resource, suggesting the teachers in this study were more concerned with surface engagement rather than engagement with the mathematics. However, the elementary teachers surveyed by Shapiro et al. (2019) cited "perceived student engagement" as the second-most important criterion for selecting an online resource, but "fun activity" and "visual appeal" (p. 680) as the fourth- and seventh-most important criteria. Thus, teachers in this study seemed to distinguish engagement with mathematics from more surface-level engagement.

Webel, Krupa, and McManus (2015) found that student needs was one of four criteria named by fifth grade teachers as they evaluated digital resources for teaching fractions. The researchers' description of this criteria included attention to how a resource bore similarities to a video game (suggesting surface-level engagement) and whether students would be able to interpret the mathematical representations (suggesting mathematical engagement). Lastly, Smith, Shin, and Kim (2017b) found that many of the criteria current and prospective secondary mathematics teachers used to evaluate mathematics technology resources related to pedagogical fidelity, or "whether students could use [a resource] with little difficulty, remain focused on the mathematics with few distractions, and not be constrained by technological features." Although a few of the criteria placed in this category had to do with aesthetics, most had to do with scaffolding student thinking toward a mathematical idea or how the tool facilitated students' interactions with mathematics.

With the second criterion, the mathematics involved in the tool or resource, there was also variation across studies in how this criterion was described and accounted for. Webel, Krupa, and McManus (2015) and Shapiro et al. (2019) both found that teachers cited alignment to the CCSSM as important in their selection of mathematics technology resources. The elementary teachers surveyed in Shapiro et al.'s study ranked alignment to standards as the most important criterion they use to select a resource. Fifth grade teachers described connections to the CCSSM as they evaluated resources for teaching fractions, usually with reference to the representations (such as number lines) named in the standards. While it is unclear from these studies how attentive teachers were to the mathematical details in the standards, the focus on alignment to standards does establish that teachers included some consideration of mathematics as they selected technology resources.

Smith, Shin, and Kim (2017b) found that current and prospective secondary mathematics teachers developed evaluation criteria for resources related to mathematical fidelity, or "how well an object in the [resource] represents the underlying mathematical properties of the object with mathematical accuracy" (p. 668). This focus reflects a more specific attention to mathematics than does alignment with standards, which may be a reflection of the teachers in this study evaluating resources designed to support learning of a specific mathematical topic (the triangle inequality) rather than evaluating a broader pool of resources. Johnston and Suh's (2009) study suggests that the nature of the resources has an impact on whether and how teachers consider mathematics as a criterion for selecting or evaluating resources. While only five of the 15 preservice teachers in Johnston and Suh's study said they chose a resource because it supported visual or conceptual learning—showing more attention to mathematics than the more commonly cited criterion of fun, engagement, or motivation—four of those five teachers chose a tool that targeted a specific mathematics topic, such as a virtual manipulative for generating equivalent fractions. These teachers differed from others who chose tools that were general (e.g., interactive whiteboards) or focused on drill and practice (e.g., online games). Thus, teachers' attention to mathematics may be better supported when they are asked to evaluate tools that have specific purposes.

Inconsistency in whether and how teachers relate the criteria to each other. While many studies on teacher thinking about mathematics technology resources found that teachers consider both student engagement and connection to mathematical topics, there is less available evidence about whether and how teachers relate these ideas to each other. Webel, Krupa, and McManus (2015) noted the fifth-grade teachers in their study tended to focus on one criterion at a time when evaluating technology resources: student engagement or the CCSS (or occasional other criteria), but not both at the same time. There were a few exceptions, but generally when teachers talked "about student engagement, they did not tend to consider whether the resources that they thought would engage students were aligned" to the CCSS (Webel, Krupa, & McManus, 2015, p. 59). Relatedly, Smith, Shin, and Kim (2017b) found that the criteria secondary mathematics teachers developed to evaluate applets for teaching the triangle inequality concerned pedagogical fidelity (whether students could use the resource without difficulty) or mathematical fidelity (whether the resource accurately reflected the mathematics). However, none of the criteria were categorized as related to *cognitive* fidelity, which the researchers defined as "how well the [applet] reflects students' cognitive actions (and possible choices while using the tool), emphasizing… mathematical thinking processes [rather] than simply arriving at the final results." One interpretation of this omission is that teachers thought about student experiences and about mathematics, but not specifically about how the students might think about mathematics when using the tool.

Other studies suggest possible explanations for why teachers do not often consider how technology resources may shape student thinking about mathematics (Johnston & Suh, 2009; Moyer-Packenham, Salkind, & Bolyard, 2008; Shapiro et al., 2019; Smith, Shin, & Kim, 2017b; Webel, Krupa, & McManus, 2015). One explanation is that teachers consider their own familiarity with tools and with the possible approaches to mathematical problems they support, rather than broadly considering different ways the tool might support students in approaching mathematical ideas. In a study of how K-8 teachers used VMs in their mathematics instruction, Moyer-Packenham, Salkind, and Bolyard (2008) found teachers tended to use VMs in the core parts of their lesson where students were learning mathematical content, rather than only during extra practice or as enrichment. This suggests teachers related the VMs to the mathematics in the lesson. However, Moyer-Packenham and colleagues (2008) also noted that the most common VMs used by teachers were those that had familiar physical counterparts, such as pattern blocks and base-10 blocks. The researchers claimed this pattern suggested teachers "may have believed the choice of these tools allowed their students to act mathematically in a way that was closely aligned with teachers' previous practices" (p. 212). Choosing VMs (or other technology tools) that easily fit into their existing classroom practices may direct teachers away from thinking about a variety of ways *students* may use the VMs.

Webel, Krupa, and McManus (2015) also found that teachers cited their own needs, along with student needs, as criteria for evaluating and selecting resources. The ways a resource fit into their existing practices versus exemplifying unfamiliar strategies and practices was a consideration for some teachers, but their reactions to unfamiliar strategies and practices varied. Some teachers chose resources specifically because they corresponded to their existing ways of teaching. Other teachers valued resources that showed a different way of approaching a mathematical idea. The example of this latter point of view, however, was in reference to a video students would watch, as opposed to a tool they might use to do mathematics. The authors specifically said this teacher valued the video because it "provided instruction that was different from her typical practice" (p. 59). Thus, the resource itself did not provide much incentive for this teacher to think through an unfamiliar way a student might solve a problem, as she was not providing the instruction herself.

Another possible explanation for why studies have not produced much evidence of teachers considering mathematics and student experiences together as they think about mathematics technology resources is that the methods used in the studies were not well suited for capturing this kind of nuanced thinking. The studies discussed above collected data about the criteria teachers used to evaluate and select resources by asking them to rank a list of predetermined criteria (Shapiro et al., 2019), develop their own sets of criteria to use to evaluate resources (Smith, Shin, & Kim, 2017b), or describe the perceived benefits of the technology resources they chose (Johnston & Suh, 2009). All of these methods emphasize thinking about one criterion at a time, either for the teachers (as they rank or develop lists of criteria) or the researchers (as they develop codes for describing the perceived benefits of a technology resource). Webel, Krupa, and McManus (2015) studied the criteria teachers used to evaluate resources by analyzing open-ended conversations teachers had about the resources. Although they concluded teachers mostly attended to one criterion at a time, this was also the only study that cited some exceptions to this pattern.

In contrast to the studies discussed above, Smith and colleagues (Smith, Shin, & Kim, 2017a; Smith, Shin, Kim, & Zawodniak, 2018) analyzed secondary mathematics teachers' thinking about technology tools in a more detailed way that revealed more nuance about their thinking. They asked trios of current and prospective secondary mathematics teachers first to develop their own criteria for analyzing technology resources (Smith, Shin, & Kim, 2017b), then to analyze four applets for exploring the triangle inequality, and finally to select one to use in their classrooms. They recorded the teachers' conversations and conducted stimulated recall interviews with the participants to learn more about their thinking during the conversations. The researchers analyzed the transcripts of both the initial group conversations and the follow-up interviews, focusing on the features of the applets to which teachers attended, how they interpreted those features, and in what ways they responded. These three processes—attending, interpreting, and responding—make up the professional noticing framework. (More detail on this framework is included in a later section.)

Smith and colleagues (2017a, 2018) found that teachers began by focusing on the appearance of the applets and the basics of how it worked. These comments were often accompanied with simple surface-level evaluations of what they liked or disliked about the applets. As the conversations progressed, however, teachers' interpretations of the applets became more sophisticated, with greater attention to more subtle features, how the applets represented mathematics, and student thinking (as opposed to student experiences more broadly). For example, one group of teachers eventually discussed how some of the applets allowed students to set the side length of a triangle at zero, how students might interpret an example with a side length of zero, and the classroom discussion they might have about this issue. As this group of teachers debated which of the four applets they would use in the classroom, they also discussed the adaptations they might make to each applet. Ultimately they chose the applet they felt would best focus student thinking on the triangle inequality without much additional framing to be done by the teachers themselves. Thus, analysis of teachers' thinking using rich discussion data and the professional noticing framework revealed some of the ways teachers were thinking about students and the mathematics in concert, rather than one at a time.

Supports for teachers in selecting, evaluating, and using technology resources. Several research teams have either stated a need for supports for teachers in evaluating, selecting, and using mathematical technology resources or explored mechanisms for providing those supports. Johnston and Suh (2009) argued that their participants' emphasis on selecting mathematics technology resources because they were fun, engaging, and motivating suggested teachers were not using those tools to their full potential in supporting students' mathematics learning. Relatedly, Shapiro et al. (2019) found that elementary teachers were selecting mathematics technology resources from sites that were not vetted by education experts, such as *Teachers Pay Teachers*, and this pattern established a need to prepare teachers to vet their own resources.

Suh (2016) provided further evidence of the need to support teachers in selecting high-quality mathematics technology resources. She asked preservice mathematics teachers (across K-12) to choose applets available on the Internet and reflect on how they would use the applets in a classroom. Twelve of the applets chosen were drill-and-practice games that offered minimal opportunities for student thinking, whereas 28 of the chosen applets had VMs embedded or otherwise supported conceptual thinking as students practiced procedures. After analyzing teachers' evaluations of the applets, Suh argued the overall themes showed teachers examined the applets critically and with student learning in mind. For example, one teacher explained how she thought an algebra balance applet would give students new ways to think about solving equations. Even so, such reflections were more common for the applets with VMs embedded than for the drill-and-practice games. Suh (2016) noted the teachers who chose the drill-and-practice applets did so because they thought the game-like features would engage students. Thus, preservice teachers were able to connect applet features to student thinking *if* they chose applets with features that supported that thinking. However, they did not consistently choose those applets and made weaker connections to student thinking when they chose drill-and-practice applets.

Suh (2016) suggested one way to support teachers' future selection of technology resources could be to ask them to categorize the resources using the cognitive demand framework (Smith & Stein, 1998), which considers whether an activity or task provides opportunities for high-level mathematical thinking. Other researchers have also suggested or tried using other frameworks as tools for supporting teachers in evaluating, selecting, and using mathematics technology resources. Thomas and Edson (2019) engaged K-8 teachers in developing examples of how various technology tools might be used to enact effective teaching practices in mathematics in ways that replaced, amplified, or transformed similar practices without technology. They found crossing the effective teaching practices with the replace, amplify, transform (RAT) framework supported teachers in seeing how teaching might shift from using static representations to using dynamic, connected, or interactive representations. Shin, Smith, and Kim (2018) described how teachers might consider pedagogical, mathematical, and cognitive fidelity as they evaluate potential tools. Originally proposed by Dick (2008), these three types of fidelity refer to how a tool allows students to do mathematics without difficulty or distraction, how accurately the tool represents the mathematics, and how closely the tool reflects the ways students might think about the mathematics, respectively. Shin and colleagues argued that use of this framework could help teachers select technology tools best suited to particular learning goals rather than thinking about them more generally.

Rather than introducing a framework, Hansen, Mavrikis, and Geraniou (2016) engaged 23 elementary school teachers in professional development (PD) and codesign of virtual manipulative *Fractions Lab.* Teachers interacted with the features of the VM, which included multiple dynamic representations of fractions. Twenty teachers said that using the VM challenged their thinking about fractions as well as their approach to teaching fractions. For example, the multiple representations embedded in the app made the teachers think about these representations in more detail and make connections between visual representations (e.g., splitting a rectangle) and equivalent fractions. The teachers also reported considering how different representations might be used more frequently in their teaching and said the interactivity of the app helped them think about how children might explore fractions on their own rather than being told what to do.

While engaging teachers in PD around a specific VM does not directly support them in evaluating or selecting VMs, this study does suggest that guided or collaborative exploration of a technology resource, and engaging teachers as users of the resource and learners of mathematics, can support them in thinking about how the tool might be used productively in their own teaching. Moreover, participating teachers also discussed how their explorations of the VM may impact the way they taught (e.g., by allowing students to do more explorations on their own) or the representations they might use for teaching fractions (e.g., connecting splitting rectangles to equivalent fractions), suggesting PD about a particular VM might also lead to learning that teachers could apply to their evaluation, selection, and use of other VMs.

Individual and contextual influences on teacher thinking about technology. The above discussion focused mostly on teachers' thinking about technology and students, without reference to other influences on their thinking. However, studies of the ways teachers use technology in their teaching suggest that there are at least three important factors that influence their thinking, both within and beyond the specific context of mathematics: teachers' knowledge, beliefs, and professional contexts. While relationships between these factors and teacher thinking will not be the focus of this study, I acknowledge their importance here and describe ways I attended to them in Chapter 2 (Methods).

First, teachers' knowledge affects much of what they do. Teaching requires knowledge of the content to be taught, as well as other specialized kinds of knowledge specific to teaching. For example, pedagogical content knowledge (PCK) is content knowledge particularly germane to teaching, including knowledge of various representations of the content and of conceptions and misconceptions commonly held by students (Shulman, 1986). Mathematics education researchers expanded on the construct of PCK, identifying areas of specialized content knowledge needed for teaching mathematics (Ball, Thames, & Phelps, 2008). Mishra and Koheler (2006) argued that instructional technology tools introduce new demands on teacher knowledge, and that teachers' thinking about how they use technology in instruction is developed in relation to both content and pedagogy. Thus, using VMs in instruction requires several types of specialized knowledge, and teachers' levels of knowledge in these multiple areas will certainly have an impact on how they think about VMs.

Second, teachers' beliefs about teaching and learning impacts much of what they do in the classroom. For example, teachers' beliefs about how students learn can influence the kinds of activities they provide to students, and their beliefs about the role of the teacher can influence how they evaluate and take up different modes of teaching (Calderhead, 1996). A recent review highlighted the complexity of the relationship between a teacher's pedagogical beliefs and their use of technology (Tondeur, van Braak, Ertmer, & Ottenbreit-Leftwich, 2017). As with other instructional resources, teachers' pedagogical beliefs have an influence on the ways they use technology in their classrooms and can hinder or prohibit use of technology. However, Tondeur et al. (2017) found evidence that the relationship was bidirectional: use of technology in classrooms has sometimes led to changes in teachers' beliefs when, for example, affordances of technology allowed teachers to see new teaching strategies as viable possibilities.

Given the relationship between pedagogical beliefs and the ways teachers use technology, pedagogical beliefs are likely to play a role in how teachers think about and use the affordances of VMs in their teaching. The learning theory underlying the use of manipulatives relies on active construction of mathematical relationships by the learner (Piaget, 1964), and the instructional power of the focal affordances of VMs may be in supporting students in constructing those relationships. As such, teachers' beliefs about the nature of learning—and how well those beliefs align with constructivist learning theory—are likely to influence how teachers use VMs and whether or not those uses have the potential to support the kinds of learning student-focused studies of VMs have suggested are possible.

Lastly, the systems within which teachers work, both local to their schools and wider to district, state, and national education systems, can also influence the choices they make. Cuban (2018) argued that while technology has supported some incremental changes in teaching and learning, enduring structures in the educational system in the United States (like the grade-level system) have played a big role in preventing broader, transformational change. In this vein, if teachers do not feel supported in adopting a teaching style that allows students to actively construct their own knowledge, the introduction of VMs with particular affordances is unlikely to have an impact on how they think about designing learning activities, even if they see the value in a constructivist approach. Indeed, studies unrelated to technology suggest that teachers feel constrained by institutional pressures to align instruction with standards and produce high test scores, and this affects how they use resources (Amador, 2016; McGee, Wang, & Polly, 2013; Priestley, Edwards, Priestley, & Miller, 2012).

Theoretical Frameworks

This dissertation focused on how teachers think about a particular instructional resource, fractions virtual manipulatives. As such, I based the design of this study on existing frameworks for (1) the mental activities teachers undertake as they make sense of resources and classroom artifacts and (2) the mutual influence that teachers and instructional resources have on instruction. I describe these frameworks here and discuss how they impacted the design of the study in the next section.

How Teachers Make Sense of Resources and Artifacts: Professional Noticing

A common framework used to organize analysis of teachers' thinking about instructional resources and artifacts is the professional noticing framework. This framework stems from the idea that an individual's knowledge and experiences, and in particular their profession, are related to what they notice about their surroundings and artifacts they are asked to examine. The frameworks shared by members of a profession to shape what they notice in complex situations is known as their *professional vision* (Goodwin, 1994). Sherin, van Es, and colleagues (e.g. Colestock & Sherin, 2009; van Es & Sherin, 2008) used the construct of professional vision to examine how mathematics teachers made sense of classroom video, focusing on what teachers identified as important, how they interpreted those elements, and how they connected specific events to broader teaching and learning principles.

Building on this work, Jacobs, Lamb, and Phillipp (2010) argued that focusing on how teachers notice particular elements of classroom video would allow research attention to be shifted away from cataloging the variety of things teachers notice and toward unpacking how and to what extent teachers notice those elements. With a focus on students' mathematical thinking, Jacobs, Lamb, and Philipp (2010) identified three specific skills teachers use in making sense of lesson video: *attending* to mathematical elements of children's strategies, *interpreting* children's understanding, and deciding how to *respond* to children. These three activities have been applied by other mathematics education researchers to better understand how teachers attend to other aspects of student video, such as children's participation (Wager, 2014) or students' work with technology (Wilson, Lee, & Hollebrands, 2011).

The professional noticing framework has also been adapted to study how teachers make sense of resources and artifacts other than classroom video, such as curriculum materials (Males, Earnest, Dietiker, & Amador, 2015) and technology tools (Smith, Shin, & Kim, 2017a). Changing the artifact that teachers are noticing (from children's thinking or participation within a video to an instructional resource) requires respecifying what teachers may attend to and what it means to interpret or respond to what they attend. With classroom video, teachers *attend* to what children say or do, *interpret* what that means about children's thinking or understanding, and *respond* with a particular teaching move they would enact if they were in the classroom (Jacobs, Lamb, & Philipp, 2010). With curriculum materials, by contrast, teachers *attend* to written or visual aspects of the materials, *interpret* those aspects by making sense of them, and *respond* with curricular decisions that result in a lesson plan or enactment (Males et al., 2015). With a technology resource such as a VM or interactive applet, teachers *attend* to features of the resource, *interpret* the features in terms of the ways they could impact instruction or student learning (often evaluating the VM or its features in the process), and *respond* with decisions about whether and how to use the resource for instruction (Smith, Shin, & Kim, 2017a).

Mutual Influence of Teachers and Resources on Instruction: Teacher-Curriculum Relationship and Instrumental Orchestrations

While it is clear that the instructional resources available to teachers can impact the nature of instruction in classrooms, it is equally clear that the ways in which teachers take up resources also affects the resulting instruction (e.g., Cohen, Raudenbush, & Ball, 2003). Several researchers have proposed models and frameworks for accounting for the mutual influence of teachers and resources on instruction. Remillard (2005), after completing a review of literature on how teachers use mathematics curriculum materials, proposed that research should examine not the influence of the curriculum materials *or* the teacher on mathematics instruction, but rather focus on the *teacher-curriculum relationship.* She argued that the relationship between teachers and curriculum materials is participatory, and that both are “active players” in instruction (Remillard, 2005, p. 236).

While Remillard’s (2005) model focused on printed curriculum materials, and not on digital tools such as VMs, other similar frameworks have been proposed and used to account for the mutual influence of teachers and technology resources on instruction. For example, Drijvers and colleagues (Drijvers, Doorman, Boon, Reed, & Gravemeijer, 2010; Drijvers, Tacoma, Besamusca, Doorman, & Boon, 2013) used the theory of *instrumental orchestration* to study how teachers used technology resources in mathematics classrooms. Instrumental orchestration is based on the theory of instrumentation, which says that as a student (or any user) learns to use a technology tool for doing mathematics, the student develops schemes of use. The tool itself and the students' schemes together form an *instrument*, which is a construct distinct from either the tool or the user alone. When teachers plan and enact instruction using technology tools, they necessarily influence the ways students view the tools and the schemes they develop. The ways the teacher sets up the classroom and instruction are called *instrumental orchestrations*. The tools characteristics and affordances and the teachers' thinking and choices each shape the orchestrations.

Study Purpose, Study Design, and Research Questions

In this section, I relate the research and frameworks discussed above to specific elements of the study purpose, design, and research questions.

Study Purpose: Exploring Teacher Thinking about VMs and Their Affordances

The purpose of this study was to gain a better understanding of how elementary teachers think about fractions virtual manipulatives and how they might use them to support student learning of fraction concepts. I aimed to better understand whether and how teachers think about the features of virtual manipulatives that reflect affordances other studies have identified as useful for promoting student thinking and learning: efficient precision, focused constraints, and linked representations (Moyer-Packenham & Westenskow, 2013). By providing teachers with a brief introduction to the affordances before the study, I also conducted an initial exploration of how the affordances might be used as a framework to guide teacher thinking about VMs. Placing a focus on these VM affordances allowed me to account for the mutual influence of teachers and instructional resources, as highlighted in the teacher-curriculum relationship and instrumental orchestration frameworks (Drijvers et al., 2010; Remillard, 2005). The study focused on teacher thinking about VMs, which positioned their thinking as a primary influence on any instruction they create with VMs. However, by placing VM affordances at the center of the analysis of teacher thinking, I aimed to understand how features unique to VMs influenced teachers' thinking *and* how teachers' thinking about those features played into their plans for instruction.

Study Design: Situating Teacher Thinking in Multiple Contexts

I designed the study to build on the themes in existing research on teacher thinking about technology discussed above. To help focus teachers' thinking on specific mathematical ideas rather than broad relationships to standards (e.g., Shapiro et al., 2019; Smith, Shin, & Kim, 2017a), I asked them to explore technology tools designed to target specific mathematical topics (fractions VMs) and to plan lessons related to particular mathematical ideas (fraction comparison, equivalence, and addition and subtraction). In an effort to learn more about teachers' thinking about how students will engage with mathematical ideas (as opposed to surface level student engagement), I collected data about teachers' thinking in three contexts pertinent to teaching and learning mathematics: (1) as teachers explore virtual manipulatives they are encountering for the first time, (2) as they use virtual manipulatives to solve problems, and (3) as they select virtual manipulatives to use during a specific lessons and anticipate how students may use them—and how they (the teachers) might respond.

I chose the three contexts because they position teachers to have different orientations toward the VMs that the literature review suggested would elicit different kinds of thinking from teachers. When teachers are exploring virtual manipulatives, they may not be thinking about using the VMs for particular purposes, but rather attempting to gain a general sense of their features and evaluating their overall potential (Smith et al., 2017a). When teachers are using VMs to solve mathematical problems, they are positioned as an end user of the tool (much as their students will often be) and are likely looking for features that allow them to take certain kinds of actions on the VM's representations. This is a shift in perspective that may support teachers in attending to different features (Smith et al., 2017a). Teachers are also positioned as learners with a fractions VM, and this positioning was productive in eliciting teacher thinking about fractions and fraction pedagogy in at least one other study (Hansen et al., 2016).

Lastly, when teachers are selecting a tool and planning a lesson, they are potentially negotiating the relationships between particular learning goals for the lesson, features of the virtual manipulatives, and tasks they will ask students to complete. Guiding teachers to focus on the relationship between VMs and learning goals has been suggested by other researchers to elicit productive teacher thinking about VMs (Shin, Smith, & Kim, 2018). The sequence of the three contexts was guided by a natural temporal sequence (explore, then use, then plan a lesson) and also modeled after a study highlighting three sequential assignments given to teachers in a mathematics methods course, which focused on evaluating a technology resource, sequencing three related technology resources, and incorporating a technology resource into a lesson plan (Suh, 2016).

Research Questions

The professional noticing framework (Jacobs et al., 2010) served as the main organizing structure for the research questions. I applied the framework in ways similar to how it has been used to study teacher thinking about curriculum materials and technology tools (Males et al., 2015; Smith et al., 2017a). However, to sharpen the focus of the study, I added greater specificity to how I used the three components, especially responding. I analyzed the VM features to which teachers attended, accounting for all features that were attended at least once. I examined interpretations in terms of how teachers made sense of and evaluated the features, but I focused on interpretations of features that I coded as reflecting efficient precision, focused constraints, or linked representations (the three affordances of VMs described above). When analyzing teachers' responses to these features, I focused specifically on how they *used* the features as they solved fraction comparison, equivalence, and addition problems and how they *planned to use* those features during instruction. The uses within their plans might include ways they expected students to use the features or how they (the teachers) anticipated directing student attention to the features during instruction.

I acknowledge there are different kinds of responses I might have analyzed, such as adaptations teachers wanted to make to the VMs. I included such comments from teachers when they were relevant to provide additional insight to teachers' thinking. However, placing primary focus on responses about how teachers used the VM features to do and teach fraction comparison, equivalence, and addition allowed me to directly examine how the mathematical content focus shaped teacher thinking about VMs.

The first three research questions correspond to the three components of the noticing framework: attending, interpreting, and responding. In accordance with the interactive nature of qualitative research design (Maxwell, 2013), the fourth research question emerged during the analysis and focuses on how teachers interpreted and responded to other VM features to which they attended. I describe the emergence of this question in Chapter 2 (Methods). The research questions for this dissertation are as follows:

1. To what features of fractions virtual manipulatives (VMs) do fourth and fifth grade teachers attend as they explore a VM, use a VM to solve fraction comparison and addition problems, and plan and anticipate student responses to a lesson involving use of a VM on fraction comparison, equivalence, or addition and subtraction?

2. How, if at all, do teachers interpret features of fractions VMs that reflect efficient precision, focused constraint, and linked representations, and how do those interpretations differ across contexts?
3. How, if at all, do teachers relate features reflecting the affordances of efficient precision, focused constraints, and linked representations to the tasks of doing and teaching fraction equivalence, comparison, and addition and subtraction?
4. How do teachers think about the features of VMs that could support students in connecting visual and symbolic representations of fractions?

CHAPTER 2: METHODS

In this chapter, I present the methods I used to answer the research questions, including the recruitment of participants, and data collection strategies, and data analysis procedures.

Research Perspective

I approached this study from an interpretivist perspective, focusing on understanding the views of the teacher participants. I focused my analytic attention on how teachers think about fractions virtual manipulatives (VMs) in multiple contexts to produce a rich, holistic view (Snape & Spencer, 2003) of teachers' thinking around these tools. Guided by the professional noticing framework (Jacobs, Lamb, & Phillip, 2010), my goal was to describe which features of VMs the teacher participants *attended*, their *interpretations* of those features, and their *responses* to them. I included consideration of two units of analysis: the participating teachers in aggregate and individual teachers (or when appropriate, subgroups of teachers with shared characteristics). I included the latter unit of analysis to ensure I had opportunities to detect ways individuals' thinking may interact differently with the affordances of VMs to produce differences in instructional plans, thereby accounting for the mutual influence of resources and teachers on instruction (Drijvers et al., 2010; Remillard, 2005).

While my primary focus was on understanding teachers' perspectives, I determined which features of the VMs reflected each of the affordances through a process of coding the tools that will be described below. As such, the analytic work of identifying which features reflected the affordances was not conducted by the teacher participants. Rather, I constructed the connection between the theoretical categories of affordances and specific features of the VMs examined by the teachers. Thus, my own interpretations played a role in determining the features about which I examined teacher thinking in detail. To counteract the potential bias introduced by limiting my analysis of teachers' interpretations and responses only to features I identified as important (because of their connection to an affordance), I coded teachers' attention to *all* features of the VMs. I also examined teachers' interpretations of features to which they showed high attention (determined by standards that will be detailed below). This allowed me to examine interpretations of features of the VMs the teachers found important or interesting, regardless of whether the features were connected to the affordances.

Statement of Positionality

I am a white, female graduate student who began a PhD program in her mid-30s, after spending nearly ten years as a developer of reform-oriented mathematics curriculum materials for both elementary and secondary school teachers. Over the course of my career, digital alternatives increasingly began to be offered to the paper- and physical-object-based activities outlined in the curriculum projects on which I worked. In many cases, these digital alternatives were advertised by publishers to offer equivalent experiences for teachers and students, but my observations in classrooms led me to believe the experiences of learners and teachers changed significantly when these digital alternatives were used. For example, often each child worked with their own device when using digital tools, whereas they usually shared physical materials. Additionally, children could "break apart" a digital base-10 tens block into 10 ones, but children had to exchange a physical base-10 tens block for 10 physical base-10 cubes. I became interested in exploring in more detail how introducing technology into elementary mathematics curricula affected student and teacher experiences.

I devoted many of my course projects during my graduate school experience to exploring research related to this issue. This project represents a culmination of these pursuits, and therefore its design has been strongly influenced by both my professional and academic experiences to this point. The project is personally meaningful to me, which I hope contributed to the sincerity and credibility I brought to the analyses (Tracy, 2010). Conversely, my interest in and enthusiasm for the potential of technology, and VMs specifically, had the potential to introduce positive biases in my interpretation of the data. I made efforts to become aware of and counteract these potential biases though several means, such as including features teachers found important in my analysis (not only the features I thought were important), utilizing a second coder for some of my analytic steps, and intentionally searching for discrepant data after identifying and elaborating the major findings of the project.

Participant Recruitment

I recruited participants via an informational flyer circulated through four professional networks: (1) members of the *Everyday Mathematics* Virtual Learning Community; (2) graduates of the University of Chicago's Urban Teacher Education Program; (3) Michigan State University's network of mentor teachers for teacher candidates; and (4) followers of the #iteachmath and #MTBoS (Math-Twitter-Blog-o-Sphere) hashtags on Twitter. The flyer focused on finding fourth- and fifth-grade teachers to participate in the first phase of the study, which included three virtual professional learning workshops in October of 2020 and a follow-up interview. The second and third phases were mentioned on the flyer, but presented as optional.

Nineteen teachers completed the interest form linked on the recruitment flyer. I sent a follow up email to each interested teacher that included a page of common questions about the study and the teacher participant consent form. After receiving this information, thirteen teachers agreed to participate in Phase 1 of the study. A summary of location, years of experience, and teaching circumstances for each of these thirteen teachers is shown in Table 1.

Table 1: Summary of participating teachers

Pseudonym	Location within United States	Grade Level	Years of Experience	Teaching Circumstances at Time of Study
Alison	Southwest	5	30+	Fully remote followed by hybrid model. Self-contained classroom.
Brenda	New England	3 - 4	20+	Some students face-to-face, some joining online. Taught math only.
Colleen*	New England	5	3	Some students face-to-face, some online. Self-contained classroom.
Denise**	West Coast	4 - 5	20+	Fully remote, self-contained G4-5 split class. First year teaching after many years in administration.
Erica	New England	5	23	Fully remote. Self-contained classroom.
Tammy*	New England	4	28	Face-to-face, then switched to remote. Self-contained classroom.
Gretchen	Midwest	4	9	Fully remote. First year teaching math in a self-contained classroom.
Sheila*	Midwest	3 - 5	6	Fully remote, teaching math only to G3-5.
Nancy*	New England	4	18	Fully remote, teaching math, social studies, and science to two classes.
Janice*	Midwest	4	2	Hybrid, one day fully remote. Self contained classroom.
Karla*	New England	4	14	Fully remote. Self-contained classroom.
Louise	Midwest	4	5	Fully remote. First year teaching math in a self-contained classroom.
Marcy	New England	5	4	Face-to-face, self- contained classroom.

*Participated in Phases 2 and 3 and data was included in analysis.
**Participated in Phases 2 and 3 but data was excluded from analysis.

After the first phase, I offered all 13 teachers the opportunity to continue into the lesson-planning and student-response phases of the study (described in more detail below). The seven teachers who opted to continue are marked with an asterisk in Table 1. Teachers received a $100 stipend for each phase of the study they completed, making the total stipend received by each teacher either $100 or $300. All teachers provided verbal informed consent at the beginning of each phase and received a written consent form detailing their rights and responsibilities.

Research Activities

Research activities took place in three phases: Professional learning (which includes teacher exploration of VMs and using the VMs to solve problems), lesson planning, and responding to hypothetical student scenarios. The three phases of data collection do not directly correspond to the three contexts for teaching I explored in the analysis. Rather, the data related to the exploration and problem solving contexts was collected in phase 1, and the data related to the lesson planning context was collected across phases 2 and 3. For clarity, from this point on I refer to the three contexts in which teachers explored VMs using title case labels: Exploration, Problem Solving, and Lesson Planning.

I conducted two interviews, the lesson plan interview and the student response interview, with each participant to explore a single context, Lesson Planning, in part because I hope the dual interviews would allow me to support teachers in articulating more of their thinking about their lesson plans. Unlike for the Exploration and Problem Solving contexts, the teachers' thinking related to Lesson Planning was not happening in the moment during the interviews, but was being recalled and restated by the participants during the lesson plan interviews. The student response interviews allowed me to surface some in-the-moment thinking from teachers related to lesson planning.

Phase 1: Professional Learning

Before the professional learning sessions, each teacher participated in a brief interview focused on their backgrounds, teaching contexts, and successes and challenges with teaching fractions. Because this research took place during the COVID-19 pandemic, these interviews also covered whether teachers were currently teaching face-to-face or remotely (or some combination) and what technology was available to their students both in the physical classroom and at home. The protocol for this interview is included in Appendix A. Each introductory interview lasted about 20 minutes. All but one of the participating teachers also submitted fractions content knowledge and mathematics teaching belief surveys (described in more detail under Data Sources below) before the start of the professional learning. The remaining teacher did not complete the surveys even after multiple requests.

The professional learning consisted of three 60-minute synchronous virtual sessions on Zoom with additional activities that teachers completed between sessions. These sessions are described in more detail below.

Session 1. The first session focused on building community among the participants and a base of shared knowledge around fractions. During the first 45 minutes, teachers introduced themselves to each other, described their current teaching situation (and any associated frustrations they felt), and shared successes and challenges they have experienced while teaching fractions. During the last part of the session, we briefly discussed ideas about why manipulatives (of any sort) are useful for teaching mathematics. Then teachers chose to explore either VMs that use a fraction circle model or VMs that used a fraction bar or strip model. The VMs that were the focus of their explorations are shown in Table 2. The VMs in the two sets were chosen due to sharing the same model and some visual similarities but having some key differences in functionality I hoped teachers would notice. For example, the Phoenix College Fraction Circles have pieces that can be stacked on top of each other, while Toy Theater Fraction Circles' pieces cannot be overlapped or stacked. After a few minutes of group exploration, the session ended and teachers were asked to continue exploring the differences between the VMs as homework.

Table 2: VMs explored in the first two professional learning sessions

Virtual Manipulative	**Source**	**Model**
Fraction Circles	Toy Theater	Fraction Circles
Fraction Circles	Phoenix College Math Blog	Fraction Circles
Fraction Models (with "Region" chosen)	National Council of Teachers of Mathematics	Fraction Circles
Fraction Bars	Toy Theater	Fraction Bars / Strips
Fraction Strips	Toy Theater	Fraction Bars / Strips
Fraction Strips	Mathies	Fraction Bars / Strips

Session 2. Between sessions, I synthesized the challenges teachers shared about teaching fractions during the first session. This list is included in Appendix B. I began Session 2 by sharing the synthesized list and explaining we would use the list as a reference as we discussed the VMs. I explained that my goal was for teachers to connect the features of VMs to the fractions challenges and to spark ideas about how the VMs might be used to address the challenges. Next, I put teachers back into the small groups they were in during Session 1 to discuss what they had noticed as they continued to compare the VMs on their own. During the subsequent full-group share-out, we collaboratively generated a table of features that varied across the VMs and reasons why the differences might be relevant to students' learning. Table 3 shows the result of this effort. During the discussion, I made as many explicit connections of the features to the list of fractions challenges (generated by teachers in Session 1; see Appendix B) as I could. For example, when teachers commented that some VMs allowed for showing multiple

wholes while others did not, I encouraged teachers to think about how this feature might support students in creating (and eventually visualizing mentally) models of fractions greater than 1.

Table 3: Features of VMs and why they are important to notice (generated during professional learning sessions)

What to look for (VM features)	**Why do I care?**
What is the whole? How many wholes can I show?	Kids sometimes struggle to visualize fractions greater than 1, so showing multiple wholes can be useful. However, it has to be clear what the whole is.
Can I combine unlike pieces?	Combining unlike pieces is useful for thinking about addition and subtraction with unlike denominators.
Which denominators can I show?	Digital spaces allow for representing more fractions than physical tools (which can support generalization).
Can I change the color? Are the colors consistent within the website?	Control over the color could help you and students coordinate thinking across multiple models.
Are the pieces labeled?	This affects whether a VM is good for thinking about shifting the whole. Automated labels will refer to a specific whole.
Can I overlap the pieces?	Overlapping pieces can be very useful for making comparisons and finding equivalent fractions, but transparency is also important for clarity.
Will the interface be overwhelming for kids?	VMs with many features may be better for demonstration because students will be easily distracted when using it themselves.
What do students manipulate? Is it the symbol, the pictorial model, or both?	Manipulating the symbols in order to change pictorial model can be helpful, but can also allow students to ignore the connection between them.

At the end of Session 2, I introduced two of the VM affordances discussed in Chapter 1: focused constraints and linked representations. As an example of focused constraints, I highlighted teachers' discussion of how some VMs allowed overlapping pieces while others did not. We discussed how this constraint might be good for some learning goals but not for others. For example, pieces that do not overlap are less useful for thinking about equivalence but may be helpful when thinking about addition and subtraction or filling up a whole. As an example of linked representations, I highlighted the way one of the fraction bar VMs showed the pieces of the bar getting smaller as students changed the denominator in the symbolic fraction. We discussed how this link might be useful in helping students overcome the misconception that a larger denominator means a larger fraction. As homework, I asked teachers to compare the fractions ⅔ and ⅗, first using one of the fraction bar VMs and then by folding paper fraction strips. I invited them to consider how the thinking was different according to the tool they used. This task was designed to get them thinking about the last VM affordance, efficient precision.

Session 3. The last session began with teachers discussing their experiences with the homework task in small groups and then as a full group. Teachers commented on how challenging it was to fold paper fraction strips into three or five equal parts, but with the digital strips, the partitioning was done for them. I used this discussion to talk about the third affordance of VMs, efficient precision, and invited teachers to consider when it may or may not be useful for learning. Next, I shared a classroom anecdote (Ball, 1992) about students who thought about solving fraction problems with manipulatives as an entirely separate process from solving fraction problems symbolically. This launched a discussion about how teachers might encourage students to think about how their work with manipulatives is connected to the underlying mathematical ideas and their symbolic representations, either through the tasks they give students to solve (e.g., do not just generate multiple examples of equivalent fractions, instead look for and explain patterns in the examples) or the questions they ask students as they work (e.g., Why does it make sense that the dot on the number line moves to the right when you shade in a new piece in the area model?). I ended the session by sharing a list of fractions VMs (see Table 4) and some strategies for collecting student work with the VMs.

Table 4: Fraction VMs provided to teachers at the end of the professional learning sessions

Source	Name	Model(s)	PC/Mac browser	Mobile browser	Apple app	Google Play app
Toy Theater (easiest and simplest to run on almost any device)	**Fraction strips**	Bar / area	x	x		
	Fraction bars	Bar / area	x	x		
	Fractions circles	Circle	x	x		
	Two-color counters	Set	x	x		
	Decimal strips	Bar / area	x	x		
	Percentage strips	Bar / area	x	x		
Math Learning Center (has sharing features)	**Number line**	Number line	x	x	x	x
	Fractions	Bar / area, Circle	x	x	x	x
Phoenix College (less polished, but more being added all often)	**Fraction Bars**	Bar / area	x	x		
	Fraction Circles	Circle	x	x		
	Cuisinaire Rods	Bar / area	x	x		
NCTM Illuminations (many require Flash but are being updated)	**Equivalent Fractions**	Bar / area, Circle, Number line	x			
	Fraction Models	Bar / area, Circle, Set	x	x		
Mathies (require Flash in browser, but apps included for some)	**Fractions Strips**	Bar / area	x		x	x
	Pouring Containers	Liquid volume	x			
	Relational Rods	Bar / area	x		x	x
	Color Tiles	Set	x		x	x

After the last session, each teacher signed up for an individual follow up noticing interview where they shared their takeaways from the professional learning sessions and explored a fractions VM that was included on the list in Table 4 but had not been discussed during the professional learning sessions. (A description of this VM and my reasons for choosing to use it in the noticing interviews are included in the Data Sources section below.) Each teacher thought out loud as she freely explored the tool. I then asked each teacher specifically to consider whether she saw the three VM affordances discussed in the professional learning. Finally, each teacher continued to think aloud as she solved one comparison task (order the fractions 3/10, 1/4, and 2/6) and one addition task (solve 1/2 + 1/5). All of these interviews occurred within a week of the conclusion of the professional learning sessions and lasted 40-60 minutes. The protocol for these interviews is included in Appendix C. At the end of these interviews, I also asked each teacher if she wished to continue into the next two phases of the study.

Phase 2: Lesson Planning

Teachers who expressed interest in continuing with the study received written directions and a lesson planning template, which are included in Appendices D and E, respectively. The template was a simplified version of the *Thinking Through a Lesson Protocol* (Smith, Bill, & Hughes, 2008). This protocol places the focus of teachers' attention on the tasks students will work on and how they might approach them, as well as what the teachers will do to support students in completing the tasks. I chose this protocol to prompt teachers' thinking about both the tasks they might pose and the questions they might ask students as they worked—the two contexts discussed in the professional learning to support students in making connections between the VMs and the underlying ideas and symbolic representations. The directions asked teachers to choose a VM from the list provided at the end of the professional learning (see Table

4), specify the context in which they expected to teach the lesson (e.g., remotely or in person), and choose a learning goal about fraction equivalence, comparison, or addition and subtraction. Teachers could either adapt a lesson from their curriculum materials to suit the VM they chose or create an entirely new lesson. The seven teachers who participated in Phases 2 and 3 returned these lesson plans to me within six weeks of their noticing interviews.

After I received the written lesson plans from each teacher, I developed a personalized interview protocol based on what was written in the plan. I used a set of common questions to guide the structure of each interview, but also included specific questions to probe teachers' thinking about the details articulated in their written plan. A sample lesson-plan interview protocol is included in Appendix F. My goal was to understand why each teacher chose the VM she did and how she expected to use it in the lesson. These interviews took place within a week of receiving teachers' written plans and lasted 30-90 minutes. After each interview, I made annotations on the associated lesson plan and sent this annotated version to each teacher and invited her to comment. This served as a form of member-checking (Tracy, 2010). At the end of each interview, I also asked teachers if they were interested in continuing to Phase 3. All seven teachers who completed Phase 2 continued to Phase 3.

Phase 3: Responding to Hypothetical Student Scenarios

As a result of the COVID-19 pandemic, I was unable to collect classroom data as teachers taught the lessons they planned. As an alternative, after each lesson planning interview I developed three short vignettes of situations I thought may arise when teachers taught their planned lesson. My goal was to pose situations that would stimulate the kind of teacher thinking that might happen in the moment-to-moment teaching of a lesson. Each vignette was based on one of the following:

- The sample student strategies the teacher described in her lesson plan (and, when necessary for me to understand them, elaborated in her lesson plan interview).
- Common fractions misconceptions and difficulties highlighted by teachers in the professional learning sessions or in the fractions research literature.
- Issues that might be introduced by the features of the VM, when student strategies with the VM could diverge from what they might do with physical manipulatives or drawings.

A sample vignette, with its rationale, is included in Appendix G. I presented these scenarios to teachers in an interview by sharing my screen and demonstrating what a student might do with the VM while narrating what they might say. After posing each situation, I asked teachers the following questions:

- What do you think is going on with the students' thinking?
- What would you do next?
    - *(if needed)* What might you suggest the student try with the VM?
- Do you think this issue might have come up even if students weren't using the VM? Why or why not?
- What's useful about the VM for addressing this issue?

These 20-30 minute interviews took place within a few days of the lesson plan interviews.

Data Sources

I collected six forms of data during the research activities.

Introductory Interviews

My first contact with the 13 Phase 1 participants was a 20-30 minute semi-structured interview (Lichtman, 2006) in which I probed their thinking about issues they encountered when teaching fractions and why they wanted to participate in the study. I gathered some background information about their teaching context and how and how often they use technology in their classrooms. A protocol for these interviews is included in Appendix A. The interviews were conducted via Zoom and recorded. Because the introductory interviews were mostly for the purpose of understanding teachers' backgrounds and did not serve as a main data source for direct analysis, they were not transcribed.

Fractions Content Knowledge Survey and Mathematics Teaching Belief Survey

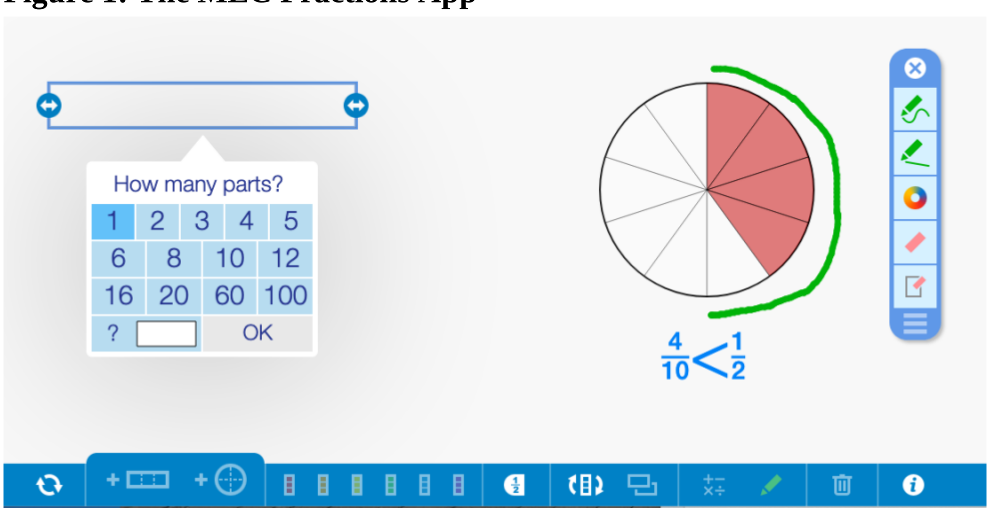
The content knowledge survey measure administered to teachers before the professional learning sessions consisted of items from the rational number section of the *Mathematics Knowledge for Teaching Measures* (Learning Mathematics for Teaching Projects [LMTP], n.d.). This measure focuses on gauging teachers' general common content knowledge about rational numbers as well as content knowledge specifically needed for teaching (Ball, Thames, & Phelps, 2008). I included the items targeting basic conceptual issues related to fractions, fraction comparison and equivalence, and fraction addition and subtraction. Because they were not relevant to the focal topics of the study, I did not include the items focused on decimals, ratios, or fraction multiplication or division. While including these items may have given me more information about teachers' content knowledge, I believed the potential benefits were outweighed by the risk of intimidating or alienating teachers by asking them to complete a lengthy content knowledge survey before the professional learning began.

The mathematics teaching beliefs survey was recently validated and focused on elementary teachers' pedagogical beliefs about teaching mathematics (Schoen & LaVenia, 2019). The three scales included in this survey focus on constructivist versus transmissionist beliefs about teaching, the extent to which teachers believe facts should be learned before solving word problems and understanding procedures, and the extent to which teachers believe they should follow versus deviate from the instructional sequence in a textbook. See Appendix H for sample items from both surveys. While the analysis did not focus on the relationship between content knowledge or beliefs and teacher thinking about VMs, I used the results of these measures as a resource for interpreting differences in thinking among participants.

Noticing Interviews

In the noticing interviews (conducted as a follow up shortly after the conclusion of the professional learning), teachers explored the Math Learning Center (MLC) Fractions App and used it to solve one comparison problem and one addition problem. When first opening this VM, a user sees a mostly blank screen and a blue tool bar of buttons at the bottom. A user presses buttons to add a circular or bar-shaped fraction model to the screen, then uses a pop-up keypad or a free-response textbox to choose a number of parts into which the model will be divided. (See Figure 1, left.) After choosing a number of parts, a user can use the other buttons on the bottom of the screen to add colored fill to parts of the models, toggle numeric fraction labels on and off, rotate or duplicate models, or add annotations to the screen with text and free-write tools. See Figure 1, right, for a model with fill (red) and examples of text (blue) and free-write (green) annotations. Users can freely drag models around the screen and overlap and resize them.

Figure 1: The MLC Fractions App



I chose this VM for the noticing interviews for three reasons. First, it has a wide variety of features, which gave teachers many things to explore and had the potential to elicit a lot of teacher thinking. Second, it has at least one feature that I believed reflected each of the three affordances. For example, the VM automatically divides the models into equal parts (efficient precision), it does not allow a user to use two colors of fill in one model (focused constraints), and the numeric labels automatically update when a user changes the fill within a model (linked representations). Lastly, the MLC Fraction App includes tools for creating both bar and circle models of fractions. This was a benefit for the problem-solving phase of the noticing interview because I could allow teachers to choose a model to use for each problem and also ask them to try using the other model after their first solution attempt. This elicited more thinking from the teachers as they solved problems and sometimes pushed them to use new strategies.

The noticing interviews were conducted online. Teachers shared their screens as they explored the VM and Zoom was used to record the primary speaker video and the shared screen. Verbatim transcripts of the spoken text were created by a paid transcription service, and I reviewed them for accuracy. For each interview, I also created a “cursor transcript,” which described what the teacher was doing with the VM as she spoke. I created these cursor transcripts because I used cursor activity as evidence of attention to VM features, and initial viewing of the data showed that teachers did not always verbally articulate what they were doing. Having a cursor transcript made it easier to code cursor activity during the analysis.

Written Lesson Plans

I collected two versions of the lesson plan created by each teacher: One as the teacher sent it to me, and another with annotations I added after the lesson plan interview. Because the annotations were member-checked by the participants, I used the annotated plans in my analysis.

Lesson Plan Interviews

The lesson plan interviews were conducted online, and Zoom was used to record the primary speaker and shared screen. Teachers shared their screens showing their chosen VM as they described to me why they chose it and what they expected students to do with it during their lesson. This screen sharing facilitated clarity of communication because teachers could demonstrate or gesture with their cursors as needed. As with the noticing interviews, a paid transcription service created verbatim transcripts of the spoken text and I created an accompanying cursor transcript so that teachers' activity with the VM could be used as evidence of attention to VM features.

Student Response Interviews

The student response interviews were also conducted via Zoom, and the audio, shared screen, and primary speaker were recorded. A paid transcription service created verbatim transcripts of the spoken text. Because I was the one screen-sharing during these interviews, and I did not count my own activity with the VMs as evidence of teacher attention, I did not create cursor transcripts for the student response interviews.

Data Used for This Dissertation

Although I had introductory and noticing interviews from 13 teachers, surveys from 12 teachers, and lesson plans and student response interviews from 7 teachers, for the purpose of this dissertation I analyzed only the data from 6 of the 7 teachers who completed all phases of the study. I excluded the seventh teacher (called Denise and marked with a double asterisk in Table 1) from the analysis for two reasons. First, she completed the lesson plan and student response interviews after teaching her lesson, which gave her a different perspective on the VM than the other teachers. Initial review of her interviews revealed it would be challenging to sort out when she was speaking from her experiences with students and when she was anticipating interactions that did not happen in her classroom. Second, her lesson focused on fraction multiplication rather than the requested topics of fraction equivalence, comparison, or addition and subtraction, making it difficult to conduct the same analyses of teacher thinking about mathematical content as the other teachers.

Analysis

In this section, I begin with a high-level summary of my analysis process to orient the reader to how the analysis aligns with the research questions. Next, I describe the software I used for video analysis and how its capabilities supported the analysis. Then I detail the specifics of each step in the analysis.

Analysis Overview

My analysis was guided by the order of my research questions. I include reference to each research question (RQ) in parentheses as I provide this overview to help clarify my analytic goals in each step. The full text of the RQs is at the end of Chapter 1, but the questions related to (1) attention to features, (2) interpretations of features reflecting affordances, (3) responses to features reflecting affordances, and (4) interpretations and responses to other features teacher to which teachers showed high attention.

I began my analysis by coding teachers' attention to various VM features across all three interviews. In this initial phase of my analysis, I attempted to characterize teachers' attention to all VM features, regardless of how often or how many teachers attended to them or whether I thought they reflected any of the affordances (RQ1). I used a teacher's use of a feature or verbalization about the feature as evidence of attention.

After coding for attention, I turned to the more interpretive analysis of how teachers made sense of certain features—that is, what they said about them (how they *interpreted* them; RQs 2 and 4) or how they used them (how they *responded* to them; RQs 3 and 4). I used two criteria to determine which features would be included in these analyses: (1) I included all features I coded as reflecting one of the three affordances (the process for this coding is discussed below; RQs 2-3), and (2) I included features I identified as being of particular interest to teachers based on the analysis of attendance (those criteria are detailed below; RQ 4).

To analyze teachers' interpretations of these features, I gathered the transcripts of all the interview video clips that included teacher talk or activity related to a particular feature. I developed narrative summaries of teachers' interpretations of these features, once with a focus on understanding everything they had to say about them (RQs 2 and 4) and a second time with a focus on how teachers related them to the tasks of solving or teaching fraction equivalence, comparison, or addition and subtraction (RQ 3). Lastly, I looked for patterns in teachers' interpretations of features in the same category (e.g., those I coded as reflecting efficient precision) and for differences according to individual teachers or across contexts (RQs 2-4).

Data Analysis Software

To code the interview data, I used *Transana,* a video coding program that coordinates video with written transcripts and allows coding across multiple transcripts. Codes are applied to data by selecting sections of one or more transcripts and applying a keyword (*Transana*'s language for codes). This creates a *clip,* or a section of the video along with its transcript sections. Additional keywords can be added to clips, and clips can be organized into collections. *Transana*'s features allowed me to do at least three key things that were important to my analysis of the interviews.

First, within a clip I could associate excerpts of a video with portions of its verbatim audio transcript, cursor transcript, or both. This meant that I could indicate, through my transcript selections, what I was using as evidence of attention to or interpretation of a feature. For example, if I created a clip tagged with a particular feature and associated only an excerpt from the cursor transcript, this was an indication (when I revisited the clip later) that the evidence was what the teacher was doing with the VM, not what she was verbalizing. This was very helpful in making sense of my coding when I returned to parts of the data after a long delay.

Second, I utilized *Transana*'s capabilities to create multiple forms of organization for the coded portions of the interview views. I used *Transana*'s keywords (codes) to label clips as including attention to and/or interpretation of particular features. I applied the same code for a feature to all videos, so searches for a particular keyword produced all clips across videos with that code. At the same time, I used *Transana*'s collections to organize (and name) my clips according to which participant and interview they were associated with. Because *Transana* organizes its search results according to the collections in which the clips reside, a search for a keyword allowed me quickly to see which teachers had attended to a particular feature and when. Furthermore, when I played the clips gathered by a keyword search, *Transana* showed me the interview from which the clip came, providing partial context for the clip without having to locate its specific location in an interview.

Lastly, *Transana* has several clip playback features that allowed me to locate and contextualize parts of the data quickly and easily. Clicking on any portion of a transcript or any clip automatically syncs the associated video to the appropriate timestamp, making for easy playback of the video and audio to revisit the nuances that transcription misses. *Transana* can also locate a clip within a video and open the full transcript, which allows for viewing what came before and after the clip. This was very helpful when making sense of a clip required additional context than was available from the clip's transcripts or keywords. For example, when necessary, it was easy for me to locate a clip within an episode and quickly scroll back to see what interview question had solicited the comment from the teacher.

Analysis Steps

This section provides details about each step in the analysis process. Table 5 maps each step onto the research questions it helped to answer and the data sources it used.

Table 5: Map of analysis steps to research questions and data sources

Step	**Research Question(s)**	**Data Sources**		
		Interviews	**Lesson plans**	**Knowledge & Belief Surveys**
Coding for attention to features	1	●		
Analysis of attention to features	1	●		
Coding for interpretations	2, 3	●		
Reconciling attention and interpretation clips	2, 3	●		
Coding VM features as reflecting affordances	2, 3	●		
Synthesis of interpretations of features reflecting affordances	2	●	●	●
Coding interpretations for mathematical topic	3	●		
Describing how teachers connected affordances to mathematical topic	3	●	●	●
Reexamining features of particular interest to teachers	4	●	●	●
Final bias check	1-4	●		

Coding for attention to features. To gain an understanding of the VM features that drew the attention of the participating teachers, I started by coding the noticing interviews where teachers had explored the MLC Fractions App and used it to solve problems. For each interview, I created clips of each time a teacher mentioned (as indicated in a verbatim transcript, e.g., "I like this tab on the side that says it.") or used (as indicated in a cursor transcript, e.g., "clicks fill button and fills in first three parts of a bar in red") a specific feature of the VM. As I identified clips, I listed the VM features that were mentioned in at least one interview. Each clip was coded with a feature name and a participant name. I organized these attending clips into collections according to whether they occurred during teachers' free exploration, when I prompted them to think about each of the three affordances, or when they were solving the problems I posed to them (see questions 4-5, 6, and 7 on the protocol in Appendix C). I called these three phases of the interview *free exploration, prompting,* and *problem solving*, respectively. In later analyses, I examined the *free exploration* and *prompting* clips as part of the broader exploration context, and the *problem solving* clips as the problem-solving context.

As I developed the list of features, I noted that while some were associated with buttons or other visual elements of the VM that directly invited interaction, others were not immediately apparent and only noticeable when teachers interacted with the VM in certain ways. I grouped the features according to whether they were *buttons, secondary buttons, discoverable, inherent,* or an *automated process.* Definitions for each category are shown in Table 6. These groupings were useful analytically because articulating the nature of each category helped me to develop reasonable standards for what counted as evidence of attention to the feature (e.g., pressing a button indicates attention to that button, but watching an automated process does not in and of itself indicate attention to that process). These standards of evidence are also shown in Table 6.

Table 6: Categories of features of MLC Fractions App (VM used in noticing interviews)

Category	Definition	Example(s)	Evidence of Attention
Button	A button that is visible on the VM's interface when its webpage is first loaded.	The MLC Fractions app has buttons on a menu bar that create a circle or bar model.	A participant presses the button or talks about doing so or what happens when she does so.
Secondary button	A button, slider, or other feature inviting interaction that appears after pressing a main button.	After users press the bar or circle model buttons, the MLC Fractions App offers a set of buttons to choose a number of parts for the model.	A participant presses the secondary button or talks about doing so or what happens when she does so.
Discoverable feature	A feature of the VM that becomes apparent only after particular interactions with the VM that are outside of its core functions.*	A user will only discover that the bar models snap together when in close proximity if the user moves the bars close enough together to trigger the snapping.	A participant interacts with the VM in a way that provides visual evidence of the feature, and also either comments on it or engages in actions that suggest experimenting with it. (e.g., A participant may discover that bars snap and then experiment with how close the bars have to be to snap together.)
Inherent feature	A feature that is apparent when the VM first opens or through core functions,* but is not connected to a particular button or interactive feature.	The MLC Fractions App uses small pictures or icons to label its buttons, rather than text.	A participant must verbally mention the feature.
Automated processes	A process that is automatically triggered via particular interactions with the VM.	The MLC Fractions App automatically divides a model into equal parts when a number of parts is chosen.	A participant must verbally mention the feature or use language that indicates acknowledgment that the process occurred.

*Core functions are defined as creating a model, breaking it into parts, shading some of the parts in, and repositioning the models on the screen.

After categorizing the features, I revisited the data and adjusted my first round of coding to apply the standards for evidence consistently. I also removed the attention and feature codes from any clips that showed a teacher attending to a feature after I prompted them to do so. (I occasionally invited teachers to explore certain features they had not used anywhere in the interview, and during the problem-solving phase, I prompted all teachers to solve the problems a second time with the model they had not chosen for themselves. For example, if a teacher solved the comparison problem with bars, I then prompted her to try to solve it using circles).

After coding the noticing interviews, I applied a similar process to each lesson-plan and student-response interview, although a few changes were necessary. First, five out of the six teacher participants chose to plan a lesson using a VM other than the MLC Fractions App, which meant I had to develop different feature lists for each additional tool. As I developed these lists, I also found I needed an additional category in the list of feature types. While the MLC Fractions App starts with a blank screen (the only visible elements are buttons), the other tools have at least one feature on the screen when the tool is first loaded, such as draggable fraction circle pieces or interactive area models. Thus, I added *visible feature* as a category, using the same standards of evidence for attention as the *button* category (see Table 6).

Analysis of attention to features. When coding for attention was complete, I used related, but distinct, processes for identifying themes and patterns in the coded clips for the noticing interviews versus the lesson plan and student response interviews.

Noticing interviews. For the noticing interviews, I compiled a table indicating whether each participant attended to each feature, and how many times she attended to it, during Exploration (which included the prompted phase of the interview) and during Problem Solving. I did not examine the length of time teachers spent attending to each feature for two reasons. First, the evidence for teachers' attention to a feature was sometimes spread across several cursor actions which were interrupted by intervening talk about unrelated experiences or what was going on at the teacher's location (e.g., an apology for the car alarm going off in the background). Clips containing these episodes of intervening talk may have given an inflated sense of attention to some features. Second, especially during Problem Solving, teachers often used many features in rapid succession, making it very difficult to capture clips showing attention to just one feature. As a result, many clips are coded with several features. Attributing the full length of the clip to each feature would have been similarly misleading.

After compiling the number of instances of attention to each feature by each individual teacher, I looked for differences in attention among participants. To do so, I calculated the range across individuals in the number of times they attended to each feature in each context. I plotted each collection of ranges on a line plot to identify any features that showed high individual differences in attention in each context. I defined a noticeably high range as one that was in an upper tail of the line plot's distribution or in a cluster near the high end of the plot. (The line plots and further description of what I counted as a noticeably high range are in Appendix I.) I examined the individual differences in attention for the features with a high range and wrote analytic memos (Maxwell, 2013) about each for consideration in further analysis.

Having looked for and documented individual differences in attention, I shifted my unit of analysis from individual teachers to the group of teachers in aggregate. I compiled a table showing how many of the participants attended to each feature during Exploration and during Problem Solving to identify any collective patterns in attendance to particular features or categories of features in each of these two contexts. I also calculated the total number of instances of attention to each feature (across teachers) in each context. I used this table to look for patterns in teachers' attention to features according to context (Exploration versus Problem Solving) and according to feature category (button, secondary button, discoverable, inherent, and automated process). I used the patterns I identified to answer Research Question 1: To what features of fractions virtual manipulatives (VMs) do fourth and fifth grade teachers attend as they explore a VM, use a VM to solve fraction comparison and addition problems, and plan and anticipate student responses to a lesson involving use of a VM on fraction comparison, equivalence, or addition and subtraction?

I also used this tabulation to identify features that seemed of particular interest to teachers in each context, in order to identify any features—other than those reflecting the three affordances of interest—that I would explore further by examining teachers' interpretations and responses in the next phase of analysis. I used two criteria to identify features of particular interest to teachers. First, I noted all features to which all six teachers had attended to the feature during at least one phase (Exploration or Problem Solving). Second, I noted all features for which the total number of times teachers collectively attended to a feature was at least three greater than the next-lowest count. That is, I ordered the features according to how many times teachers collectively noticed them in each context and looked for a jump of at least three between consecutive numbers in this list. In each context, there was a cluster of features near the high end that was separated from the rest by a jump of at least three. Seven features were in this cluster for Exploration and five were in the cluster for Problem Solving. All jumps in frequency before these clusters were either one or two, and the jumps within the cluster were up to 14, so the three-or-more criterion seemed to identify clusters of features that were meaningfully different in frequency of use.

Lesson plan and student response interviews. I collated the instances of attention differently in the lesson planning and student response interviews in three ways. First, I did not tabulate or report a number of times a teacher attended to a feature in each interview, but instead only noted whether each feature gained a teacher's attention anywhere within each interview. I made this choice because the protocols for the lesson planning and student response interviews were much more structured than the noticing interviews. They included questions that may have directed teachers to mention a feature more or less often than they would have without the researcher's intervention. Thus, the number of times a feature was mentioned felt a less meaningful indication of a teacher's interest in a feature than in the noticing interviews.

Second, because each teacher used a different VM, I could not meaningfully tabulate how many teachers attended to each feature. Instead, I merely produced a list of features mentioned across each teacher's lesson plan and student response interviews, noting whether the teacher attended to it in her lesson planning interview, her student response interview, or both. Because teachers attended to different features in these interviews, it was not possible to meaningfully examine or describe individual differences in attention to specific features. Instead, I relied on examining differences in attention by individual teachers to features falling in different categories (visible, button, discoverable, inherent, and automated process). I used these observations to answer Research Question 1.

Lastly, I used different criteria to identify features of particular interest to teachers. Because I could not rely on the number of times a teacher attended to a feature to gauge her level of interest, I instead made note of any feature that was mentioned by a teacher in *both* her lesson plan and student response interview. This criterion identified five to eight features of interest per teacher, which were manageable numbers for analysis.

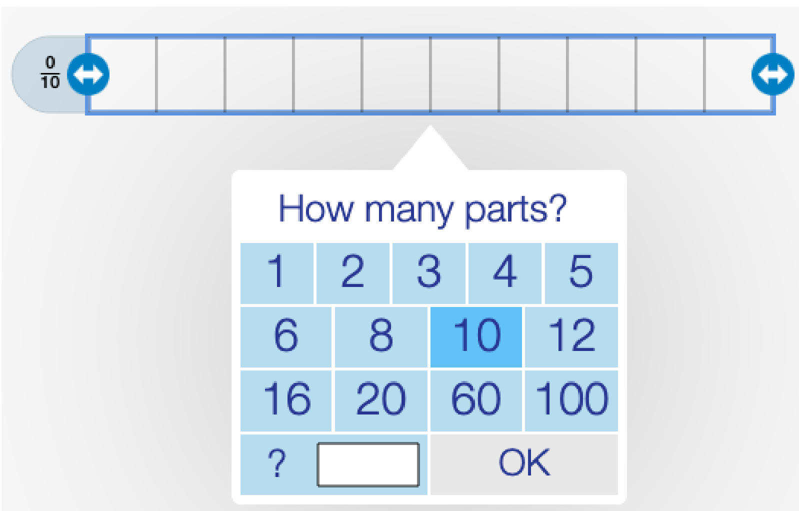
Coding for interpretations. To code for teachers' interpretations of the VMs and their features in each interview, I made an additional coding pass. Without looking at where I had previously identified attending clips, I created a new clip each time a teacher said something about the VM that went beyond a description of what was happening on the screen. These interpretation clips included evaluations, statements about what the VM might be useful for (or not useful for), thoughts about how students may interact with the tool, comparisons to other VMs explored in the professional learning or other tools they have used in their classroom, and occasional other miscellaneous comments. I used the previously established lists of VM features to code each interpretation clip with the features I felt teachers were commenting on. I applied these feature codes without referencing the existing attending clips.

Reconciling attention and interpretation clips. After finishing the interpretation coding, I examined all the attending and interpreting clips in each interview to look for discrepancies between the feature codes applied to the clips. For example, I looked to see if there were any clips coded as interpreting a feature in a section of the interview where there was no attending clip for that feature. In such cases, I either added an attending clip for that feature or removed the code from the interpretation clip, as was warranted by the evidence in the clip. My working assumption was that if interpretation happened, then attending must also have happened, *unless* teachers only interpreted a feature after I prompted them to attend to it. This process of reconciling the two independently coded clips served as a check on my own coding reliability, particularly since several weeks passed between the attention and interpretation coding. After this process, I adjusted the tables summarizing teachers' attention to features as needed.

Coding VM features as reflecting affordances. For the analyses for Research Questions 2 and 3, I needed to determine which VM features I would consider as reflecting each of the three affordances. These were (1) *efficient precision,* or support for generating quick, precise representations, (2) *focused constraints*, or limitations on how the VMs could be manipulated that had potential to focus attention on mathematical ideas, and (3) *linked representations,* or two representations that are linked such that one changes in response to manipulations of the other. I analyzed the VM used in the noticing interviews and the VMs used by the Phase 2 and 3 participants in their lesson plans. Using the definitions and examples of each affordance described in the research literature (see Appendix J), I identified the aspects of each of these VMs that embodied the three affordances. I mapped each of these aspects onto a feature coded in the analysis for attention. For example, in my analysis of the MLC Fraction App, I identified the automated process of creating equal parts in the models in VM as an example of efficient precision because it supports the efficient creation of precise models of fractions with any denominator. Teachers had spoken about this feature in their noticing interviews, and a feature code for it, called *Automatic dividing*, already existed. Thus, I added the automatic dividing feature to my list of features reflecting efficient precision.

Next, I reviewed each interview video and listened for teacher talk that may have indicated that a feature could reflect one of the affordances. For example, if a teacher talked about a way in which the VM *limited* what they were able to do, I considered whether the feature the teacher was discussing met the definition of *focused constraint.* If it did, I added the feature to the list and added a code for focused constraint each time a teacher attended to it. This step of listening to teachers' talk to identify additional features reflecting affordances also revealed instances where the affordances did not map directly onto coded features. For example, if users of the MLC Fractions App click several of the available choices for number of parts before clicking "OK," they can watch the number of parts in the model change (and if the labels are on, also the denominator of the symbolic fraction change—see Figure 2). This is an example of a linked representation but does not correspond to one feature but rather to the interaction of the denominator buttons, the automatic dividing, and the labels. Additionally, teachers occasionally talked about specific aspects of the previously coded features in ways that aligned with the affordances. For example, several teachers mentioned that clicking an area model instantly fills a whole piece of the model—a subfeature of the fill feature already coded that had not yet been captured separately. Because this subfeature supports the efficient creation of precisely shaded fraction models, I considered it an example of a feature reflecting efficient precision.

Figure 2: If a user clicks several numbers on the keypad before clicking "OK," the user sees the number of parts in the area model and the symbolic denominator change.



I added these subfeatures and interactions of features to my lists of features reflecting each affordance and created keywords for them in *Transana*. For simplicity, in the text that follows, I refer to the subfeatures and interactions of features collectively as *subfeatures.* Because the attention coding process had not identified these subfeatures as distinct objects of attention, my next step was to comprehensively code all the relevant data (i.e., the interviews in which the relevant VM was used) for attention to each subfeature. To accomplish this, I reexamined each clip where any of the component features was an object of attention. I added a code for the subfeature any time a participant used the broader feature(s) in a way that would make the affordance apparent or discussed something the VM did that reflected the affordance.

To increase the trustworthiness of my choices for which features reflected each affordance, which would have significant impact on the remainder of the analysis, I presented a second researcher with the full list of features and subfeatures for four of the five VMs used and discussed by a teacher in any interview. (The fifth VM, the NCTM Equivalent Fractions App, had been removed from the NCTM website by this point in the analysis, likely because it used the outdated Adobe Flash Player to run.) I also gave the second researcher the definitions of the three affordances from Appendix J. To develop shared meaning for the affordances, I randomly selected 20% of the features and the second researcher and I discussed whether and why we would code each of those features as reflecting an affordance. The second researcher then coded the remaining 68 features and subfeatures on her own. This researcher was a PhD student in curriculum and instruction with experience as an elementary teacher and research interests in teaching and learning of elementary mathematics.

We initially agreed on the coding of 48 out of the 68 features, or 70.6%. Closer examination of the agreements and discrepancies revealed the second coder had identified 18 out of 19, or 94.7% of the features I had coded as reflecting an affordance. Thus, we had agreed on nearly all of the features on my lists of those reflecting affordances, but there were many features that the second coder had identified as reflecting an affordance when I had not. Without mentioning specific features, I made one clarification to the definition of linked representations and asked the second coder to revisit her coding. This resolved two discrepancies, raising the level of agreement to 73.5%.

As I reviewed the remaining 18 discrepancies, I realized that many of them may have been due to the descriptions of the features and subfeatures not including enough detail to communicate how I had parsed out related features. For example, one comment left by the second coder said that she had coded the denominator buttons in the MLC Fractions App as reflecting efficient precision because pressing one of them automatically divided the model into equal parts. While this is true, as I coded the data, I had identified the automatic dividing process as a feature separate from the buttons themselves as a way to track evidence of attention to that process—pressing the button was evidence of attention to that button, but verbal references to the automatic dividing were necessary to code for attention to the process.

To determine whether such relatedness between features might resolve additional discrepancies in the feature coding, in a meeting the second coder and I discussed each of the features she had coded as reflecting an affordance (whether we had agreed or not). I asked the second coder to describe to me the specific aspects of each feature that had led her to code it as reflecting an affordance. When she described functionality that I had attributed to another feature or subfeature (and she had already coded that feature or subfeature as reflecting an affordance), I removed her code from the feature. Through this the discussion we also discovered a necessary clarification to the definition of focused constraints: a feature must constrain a potential action on the VM *and* have the potential to focus attention on a particular mathematical process or idea. The clarifications in the relationships between features and the definition of focused constraint resolved another 11 discrepancies, raising the level of agreement to 89.7%.

We resolved the remaining seven discrepancies through discussion. This process led me to add one additional subfeature to my list of features reflecting focused constraints, but the list of features reflecting efficient precision and linked representations remained the same. This final list of features reflecting each affordance served as the basis for selecting the interpretation clips I would include in the analysis for Research Questions 2 and 3. Information about these features and the rationale for why they were coded as reflecting an affordance is included in Chapter 4 (Results, Interpretations of Affordances). For simplicity of expression, from this point on I refer to these features as "features reflecting the affordances," "features reflecting efficient precision," and so on. I acknowledge, however, that I—and not the teacher participants—mapped the features onto the affordances. I discuss the implications of my imposition of this analytic lens when relevant in the Results, Discussion, and Limitations.

Synthesis of interpretations of features reflecting affordances. After identifying the collection of VM features reflecting the affordances, I turned my attention to Research Question 2: How, if at all, do teachers interpret features of fractions VMs that reflect efficient precision, focused constraint, and linked representations, and how do those interpretations differ across contexts? To begin examining teachers' interpretations of the features reflecting each affordance, I compiled a spreadsheet listing each relevant feature and the clips containing interpretations of each feature. I noted the participant and the interview from which the clip came and also pasted the verbatim interview transcript text from each clip into the spreadsheet. I grouped the features according to which affordance they were coded as reflecting so that I could look across interpretations of features within each affordance group and analyze how teachers thought about the broader class of affordances as well as individual features.

I synthesized teachers' interpretations of each affordance category first by *context* (Exploration, Problem Solving, and Lesson Planning) and then by *individual teacher*. I began by focusing on the contexts one at a time. Starting with the features reflecting efficient precision interpreted by teachers during Exploration, I read all the interpretations of each feature and wrote a detailed narrative summary of the interpretations. I looked for similar interpretations of the feature across teachers and grouped them, writing a paragraph or two about each grouping. For example, when multiple teachers described the automatic dividing feature as helpful in supporting children who struggle to draw representations of fractions, I wrote a description of that interpretation. I included transcript excerpts and screenshots from the interviews as I felt appropriate to support my descriptions. I noted the individuals whose interpretations I had synthesized in an effort to support my later attention to individuals, but at this stage I did not attempt to track individual teachers' interpretations across features, contexts, or affordances.

Once I had completed narrative summaries of each feature reflecting efficient precision noticed during Exploration, I wrote an overall summary of how teachers had interpreted features reflecting that affordance in Exploration. I continued synthesizing feature by feature for the other two affordances in the Exploration context, and then repeated this process for the Problem Solving context and the Lesson Planning context.

In my second step, I returned to my detailed narrative summaries but read them with attention to how individual teachers interpreted features reflecting the affordances in each context. I constructed a grid for the interpretations of efficient precision, with individual teachers as the rows and the three contexts as the columns. As I read my narrative summaries, I filled in each cell with the features reflecting each affordance that were interpreted by each teacher and brief notes about the nature of their interpretations. I returned to the spreadsheet of interpretations, annotated lesson plans, and/or the original data to clarify details about individuals that had been lost in my first synthesis. A sample row of the efficient precision grid is shown in Table 7. I created similar grids for the focused constraint and linked representations affordances.

Table 7: Sample row from efficient precision grid used to answer Research Question 2

	EXPLORATION		PROBLEM SOLVING		LESSON PLANNING	
Parti-cipant	Features	Interpretations	Features	Interpretations	Features	Interpretations
Karla	Automatic dividing Free choice denomina-tor	Overcome difficulty of creating equal parts Flexibility (ref. to curriculum)	Automatic dividing Bar snapping Full fill	Makes it easy to compare & add Helpful for comparisons & making same-size wholes Easy to fill in	Piece snapping Exact size	Eases burden but may also contribute to mindless strategy - had plan for that. With labels, supported focus on mathematical ideas, avoided problems she had in the past

I looked across grid rows to identify similarities or differences in how the teachers interpreted features reflecting the same affordance in different contexts. I looked down grid columns to identify similarities or differences among individual teachers in how they interpreted features reflecting each affordance in each context. When I identified differences among individuals, I looked to three sources of information to theorize possible explanations of the differences: (1) teachers' responses to the content knowledge and teaching belief surveys; (2) information I learned about each teacher through their introductory interviews and my interactions with them in professional learning and before and after interviews; and (3) analytic memos I had written about individual differences in the patterns of attendance features I identified while answering Research Question 1. Examining these sources of information in relation to individual differences had potential to shed light on how teacher characteristics such as content knowledge, teaching beliefs, years of experience, and prior uses of technology could be related to how teachers interpret VM features. Through examination of the grids and related supplemental data, I identified and elaborated themes to answer Research Question 2.

Coding interpretations for mathematical topic. My next step was to analyze the data with attention to Research Question 3, which focuses on how teachers related the features reflecting affordances to the tasks of solving and teaching three topics: (1) fraction comparison, (2) fraction equivalence, and (3) fraction addition and subtraction. Details about how I defined and differentiated between these three topics is in Chapter 5 (Results, Responses to Affordances).

I returned to the video clips that highlighted teachers' interpretations of the features reflecting VMs in each context and added codes for whether teachers made connections between these features and fraction equivalence, comparison, or addition and subtraction. For example, if a teacher described the way a feature would be useful when comparing fractions, I coded the clip with the fraction comparison code. The full list of criteria I used as evidence for making a connection to one of these mathematical topics in each context are in Table 8.

I added these codes to my previously created spreadsheet of interpretations. This provided some of the context that was lost when extracting the transcript excerpts from the broader interview and allowed me to identify the interpretations that provided insight into how teachers *responded* to the features of the VM for a particular fractions task. Specifically, it helped me locate sections of the interviews where teachers discussed whether and how they might use features of the VMs to help them teach, and to support students in learning, about fraction equivalence, comparison, or addition and subtraction.

Table 8: Criteria for evidence of a connection to fraction comparison, or addition and subtraction in teachers' interpretations of VM features

	Context		
Topic	Exploration	Problem solving	Lesson planning & student response
Fraction comparison	Directly mentions comparison OR Evaluates a feature after using the VM to explore how she might compare fractions	Directly mentions comparison OR Mentions the feature in response to being asked what features were helpful for solving the comparison problem	Directly mentions comparison OR Planned a lesson about fraction comparison and mentions the feature when describing teaching actions or student strategies
Fraction equivalence	Directly mentions equivalence OR Evaluates a feature after using the VM to explore how she might generate equivalent fractions or check for equivalence	Directly mentions equivalence OR Mentions the feature in response to being asked what features were helpful for solving the addition problem, with specific reference to finding equivalent addends with a common denominator	Directly mentions equivalence OR Planned a lesson about fraction equivalence and mentions the feature when describing teaching actions or student strategies
Fraction addition & subtraction	Directly mentions addition or subtraction OR Evaluates a feature after using the VM to explore how she might add or subtract fractions	Directly mentions addition or subtraction OR Mentions the feature in response to being asked what features were helpful for solving the addition problem	Directly mentions addition or subtraction OR Planned a lesson about fraction addition or subtraction and mentions the feature when describing teaching actions or student strategies

Describing how teachers connected affordances to mathematical topics. I reviewed the interpretations that I coded for attention to each of the three mathematical topics and added details to each of my affordance grids about the ways teachers related the features reflecting affordances to the mathematical topics. I again looked for patterns both across rows (for similarities and differences in a teacher's responses across contexts) and down columns (for similarities and differences among teachers in a particular context). I used the patterns I noticed as a frame for answering Research Question 3: How, if at all, do teachers relate features reflecting the affordances of efficient precision, focused constraints, and linked representations to the tasks of doing and teaching fraction equivalence, comparison, and addition and subtraction?

One pattern I noticed was that several teachers' responses to some of the features reflecting efficient precision and focused constraints in the MLC Fractions App in Exploration were not repeated in Problem Solving. To develop a potential explanation for this, I examined teachers' interpretations of features to which they had shown particularly high attention in Problem Solving (that were not coded as reflecting an affordance). I repeated the analysis I had done of the features reflecting affordances for these features (that is, gathering the interpretations into a spreadsheet and synthesizing them with attention to similarities and differences by individual) in the Problem Solving context only. The way teachers used these features to solve the problems partially explained their changes in response to the features reflecting efficient precision and focused constraints—their use of these features helped them make use of precision or overcome the constraints. As such, I integrated my analysis of these features into my answer to Research Question 3. More details about this process, including identification of the particular features, are included in the Chapter 5 (Results, Responses to Affordances).

Reexamining features of particular interest to teachers. At this point, I returned to the list of features I identified as being of particular interest to teachers during my prior analyses of attention—those that were attended by all teachers or a high number of times across teachers in Exploration or Problem Solving, and those that were noticed by a teacher in both her lesson plan and student response interview. Many of these features were coded as reflecting an affordance, and so I had already examined interpretations of and responses to them. Two others had been incorporated into the analysis for Research Question 3. Several more were what I called *core features,* or features for which it is difficult to explore or accomplish anything with the VM without using. (For example, the circle and bar model buttons in the MLC Fraction App.) Because teachers' high attention to these features was expected, I excluded them from further analysis.

I examined the remaining features on the list of those which teachers had shown particular interest. The three remaining features from the MLC Fractions App all had the potential to support a user in viewing or creating symbolic (or numeric) representations alongside the visual representations that were the main features of the app. A few features of other VMs with this same connection to creating or viewing symbolic fractions had been of particular interest to some teachers in the lesson planning and student response interviews. Because helping students to connect their work with manipulatives to the underlying mathematical ideas and the symbolic representations of those same ideas is a perennial challenge in mathematics education (Ball, 1992; Uttal et al., 2009), I developed a research question about whether teachers' attention to these features reflected an interest in addressing this challenge. This is Research Question 4: How do teachers think about the features of VMs that could support students in connecting visual and symbolic representations of fractions? I analyzed teachers' interpretations of these features in the same manner I analyzed the features reflecting each affordance in the analysis for Research Question 2.

The remaining features on the list of those that were particularly interesting to teachers were features of a VM other than the MLC Fractions App and highly noticed by only one teacher. Because of this, I had limited information about teachers' interpretations of each feature, and I excluded them from further analysis.

Final bias check. When analysis was nearly complete, I asked a second coder to examine some interview segments in an effort to identify any major biases in my identification of which features teachers were noticing in the interviews and my descriptions of their interpretations and responses to those features. This coder was a PhD student in mathematics education with research interests in teachers' professional learning. I calculated the amount of time teachers spent talking about their VMs in each interview (excluding, for example, the initial portions of the noticing interviews when teachers told me about their takeaways from the professional learning, before they began exploring the MLC Fractions App). I then calculated 20% of the length and created a video segment of each interview of that length. To select the segments within each interview, I used a random number generator to select the starting point, although I made small adjustments to make sure the clip did not start in the middle of a teacher's thought.

I presented these clips to a second coder along with a spreadsheet for recording her coding. The spreadsheet had a tab for each of the 18 video clips (one of each from 6 noticing, 6 lesson plan, and 6 student response interviews). Each tab showed a list of the features of the VM discussed in each clip. The second coder viewed the videos and noted which features she saw the teacher using or heard the teacher talking about, along with a brief description of what the teacher said or did with that feature in the coder's own words. I reviewed the list of features the second coder identified in relation to the codes I had placed on the interviews and also reviewed her descriptions in relation to my descriptions of the interpretations in my analysis. I identified 39 spreadsheet rows with potentially meaningful differences (out of 488, or 8%). We met to discuss the differences. After this discussion, I adjusted the feature coding on five video clips and wrote analytic memos about the second coder's comments about interpretations of three features. I referred to these memos as I reviewed my interpretive analysis. While I made a few minor changes as a result of this bias check, the major themes and results of the analysis did not change.

CHAPTER 3: RESULTS, ATTENTION TO VM FEATURES

In this chapter, I discuss results related to Research Question 1:

> To what features of fractions virtual manipulatives (VMs) do fourth and fifth grade teachers attend as they explore a VM, use a VM to solve fraction comparison and addition problems, and plan and anticipate student responses to a lesson involving use of a VM on fraction comparison, equivalence, or addition and subtraction?

This question asks which features of the VM teachers attend to in the three sequential contexts created in this study (Exploration, Problem Solving, and Lesson Planning). Although all three contexts are referenced, the analysis focuses primarily on Exploration and Problem Solving because all teachers used the same VM, the MLC Fractions App, in these contexts. Thus, it is possible to aggregate teachers' attention to specific VM features. Teachers all used different VMs in Lesson Planning, so it was more difficult to analyze their attention in aggregate. However, I do incorporate the Lesson Planning data when possible by using categories of features, rather than specific features, to draw parallels or make comparisons across contexts.

I focused the analysis on the features to which teachers *attended.* I did not attempt to account for the features to which teachers *did not attend* for several reasons. First, there was no pre-existing list of VM features for any of the VMs teachers used. The lists I created of features teachers noticed was data-driven. In the case of the MLC Fractions App, which was explored by all six teachers, they collectively attended to many more features than I noticed when I explored the VM on my own. Any attempt I made to identify additional features to which no teachers attended would not be meaningful, as I could not clearly determine that I have accounted for all the features. Granted, not *every* teacher attended to *all* of the features of the MLC Fractions App.

I account for this in the analysis by discussing which features were attended by more teachers than others. Second, when it came to the other VMs teachers used in Lesson Planning, there were a few cases when I noticed features that were not attended by the teacher. These few cases also did not feel meaningful enough to report because in most cases, there was only one teacher using the VM, rather than six. The lesson plan and student response interviews also did not provide an avenue for teachers to talk about everything they noticed, so lack of attention might mean they did not think it was important for their lesson rather than that they did not attend to it at all.

Chapters 4-6 follow up on this chapter's analysis of attention by focusing on the ways teachers interpreted and responded to the features to which they attended. Recall that attending, interpreting, and responding are the three components of the professional noticing framework (Jacobs, Lamb, & Philipp, 2010), which was the analytic framework for the study. Chapters 4-6 are occasionally referenced below.

I organize results related to Research Question 1 into three sections corresponding to three main findings: (1a) teachers attended to a variety of features in each context, but core features got consistently high attention; (1b) changes in context had an impact on the features teachers attended; and (1c) individual differences in attention during Exploration suggest mathematical knowledge and teaching orientation may affect how teachers explore VMs.

Finding 1a: Teachers attended to a variety of features in each context, but core features got consistently high attention across contexts.

The features of the MLC Fraction App attended by teachers in the Exploration and Problem Solving contexts are listed in Table 9. The columns on the left show how many teachers attended to each feature in each context. The columns on the right show the total number of times teachers collectively attended to the feature in each context. The features are arranged by category, with the most immediately visible features (the buttons and secondary buttons) at the top and the less immediately apparent features (those that are discoverable, inherent, or automated processes) at the bottom (on the next page).

Table 9: Attention to MLC Fractions App features in Exploration and Problem Solving

Category	Feature	No. Ts Attending in Exploration	No. Ts Attending in Problem Solving	No. Attending Clips in Exploration	No. Attending Clips in Problem Solving
Button	Bar models	6	6	28	20
	Circle models	6	4	32	14
	Filling of pieces	6	6	46	46
	Pencil	6	3	9	10
	Label button	6	4	24	6
	Text tool	6	1	13	1
	Clear button	3	4	5	12
	Rotation button	5	1	9	2
	Trash can	5	1	10	1
	Copy button	4	0	8	0
	Information button	3	0	3	0
Second-ary button	Denominator buttons	6	6	46	46
	Free choice denominator	5	0	12	0
	Size adjustment - circles	6	3	10	4
	Size adjustment - bars	5	1	10	1
	Circle free rotate	3	5	4	22
	Pencil eraser	4	1	6	1

Table 9 cont'd

Category	Feature	No. Ts Attending in Exploration	No. Ts Attending in Problem Solving		No. Attending Clips in Exploration	No. Attending Clips in Problem Solving
Discoverable	Snapping of bars	1	5		3	13
	Snapping of circles	0	1		0	1
	Removing fill	5	4		18	9
	Free position of fill	4	4		6	6
	Single-color models	6	4		13	7
	Linked symbols	6	2		12	4
	Fixed denominator	3	3		10	3
	Overlap	5	6		13	38
	Pencil layer	1	2		1	4
Inherent	Complete whole	4	4		5	5
	Multiple models	2	2		4	2
	Clean interface	2	1		3	1
	Fluid movement	1	1		2	1
	Button labels	1	0		1	0
Automated process	Automatic dividing	6	5		24	12
	Automatic sizing	1	3		2	3
	Background color	1	0		1	0

Color key: Gray = Core features; Yellow = other features of particular interest to teachers during Exploration; Purple = other features of particular interest to teachers during Problem Solving

A brief look at the number of features listed in Table 9, and the distribution of those features across categories, shows that teachers attended to a wide variety of features in both contexts. Teachers attended to a total of 34 features of the MLC Fractions App across these two contexts, and only half of those features were buttons or secondary buttons that visually invited interaction. The other half either required teachers to interact with the VM in specific ways to discover them or were always apparent but could have easily been taken for granted and not attended by teachers (such as the automated processes). Teachers attended to features related to the ways they could create and manipulate fraction models (e.g., the bar and circle model buttons and the size-adjustment secondary buttons), how those models could interact (e.g., the bar-snapping feature and the overlap feature), and the general look and feel of the VM (e.g., its clean interface and the way the models could be fluidly moved around the screen).

In the Lesson Planning context, each teacher used a different VM and their attention to specific features cannot be aggregated, but it is possible to examine how teachers' attention was distributed across the feature categories. Table 10 shows the number of features in each category each participant attended during Lesson Planning. As in Table 9, the more visible features are at the top and the less visible features are at the bottom. Because the VMs the teachers used did not have the same number of features available to notice, nor the same distribution of features across categories, it is not meaningful to compare the magnitudes of the numbers in Table 10. However, Table 10 does show that each teachers' attention was distributed across easily visible features and features that were less apparent. This pattern was consistent across all three contexts.

Table 10: Number of VM features attended by each teacher in Lesson Planning

	Participants					
Feature Category	**Colleen**	**Janice**	**Karla**	**Nancy**	**Sheila**	**Tammy**
Visible	3	--	2	1	3	5
Button or Secondary Button	6	11	--	6	2	1
Discoverable	1	4	1	1	4	2
Inherent	2	4	4	6	2	9
Automated Process	2	1	2	1	0	2

Even as teachers attended to a wide variety of features, Table 9 also indicates that certain types of features were attended more often. The first three buttons (the bar and circle model buttons and the fill buttons) and the first secondary button (the denominator choices that appear after pressing the bar or circle model buttons; see Figure 2 in Chapter 2) were attended by all six teachers in Exploration, and almost all of them were attended by all six teachers in Problem Solving. (All teachers used the circle models in Problem Solving, but I prompted two of the teachers to use them after they chose to use the bar models for both problems. Because I prompted them, I did not code these instances as reflecting teacher attention.) Because they drew wide attention in both contexts, these four features are marked in gray in Table 9.

The high attention to each of these features makes sense, as they relate to the core function of the VM: creating fraction models. Users cannot accomplish much else with the VM if they do not create either a bar or circle model, use the denominator buttons to divide it into parts, and add fill to the model. Because of their high rates of attention and their connections to the core functions of the VM, I call these four features the *core features* of the MLC Fractions App. The VMs used by the teachers in Lesson Planning also had a set of core features, and all the teachers attended to all the core features in both their lesson plan and student response interviews. Thus, high attention to core features was also consistent across contexts.

Finding 1b: Changes in context had an impact on the features teachers attended.

In contrast to the consistent attention to core features across contexts, there were other features that garnered particularly high attention from teachers in Exploration or Problem Solving, but not both. I describe teachers' attention to these features in this subsection. I also present evidence indicating that changing contexts led teachers to notice some VM features for the first time.

High attention in Exploration (and Lesson Planning). The two features of the MLC Fractions App shaded in dark yellow in Table 9 have quite high overall frequencies of attention in the Exploration context. One is the automated process that divides the fraction models into equal parts when users either press a denominator button or enter a number into the field for free entry of a number of parts. Although this automated process could have been used without being acknowledged or commented on by teachers, all six teachers did comment on it in Exploration. Additionally, the total number of instances of attention in Exploration was 24—a mix of direct comments or use of language that indicated acknowledgement of the process—almost as many as the core features. Attention to this feature continued into Problem Solving, but at a lower rate, suggesting the feature was still noticed by teachers but more often taken for granted.

Teachers were also interested in similar features during Lesson Planning. Five of the six teachers attended to a VM feature related to supporting students in creating equal parts in Lesson Planning, with three attending to the feature in both interviews and two attending to the feature in the lesson plan interview only. The high rates of attention in Exploration and Lesson Planning, and continued, if lower attention in Problem Solving, suggests teachers found features that supported students in creating equal parts particularly valuable or otherwise interesting. Because the MLC Fractions App's automatic dividing feature, and features from other VMs related to it, are on my list of those reflecting efficient precision, further analysis of how teachers thought about these highly attended features is in Chapters 4 and 5.

The other feature highlighted in dark yellow in Table 9 is the label button, which allows users to toggle a tab showing a symbolic fraction on and off on the bar and circle models (see the tab on the left side of the bar in Figure 2 in Chapter 2). All six teachers attended to this feature during Exploration, noticing it a total of 24 times—almost as many times as the core features.

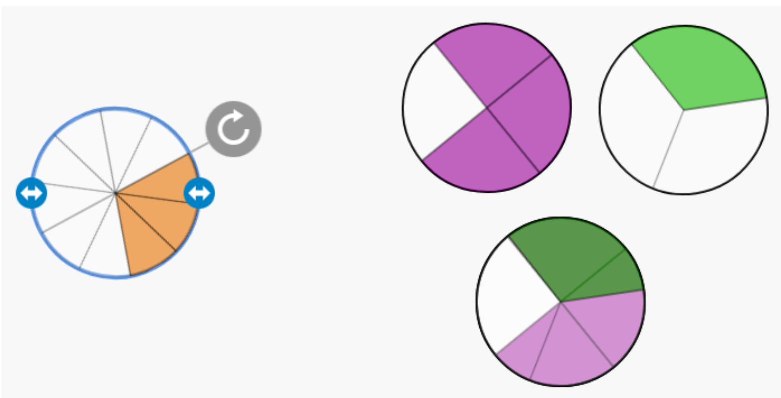
Teachers' high attention to this feature may be particularly relevant, as teachers could use it to support students in making connections between visual and symbolic representations of fractions—a perennial challenge in both fractions teaching and learning (e.g., Rau & Matthews, 2017) and in general use of manipulatives in mathematics education (e.g., Uttal et al., 2009).

Although overall attention was not as high as for the label button, all six teachers also attended to two other features in Exploration that could be used to support students in connecting visual and symbolic representations of fractions (if they were used to create symbolic representations alongside the VM's core visual models): the pencil, which allows users to freely annotate the screen, and the text tool, which allows users to create mathematical expressions using a pop-up keypad. These features are highlighted in lighter yellow in Table 9. Five of the six teacher participants also attended to a feature of the VM they chose for Lesson Planning that related to showing symbolic representations of fractions alongside visual representations. Four teachers attended to such a feature in both their lesson plan and student response interviews, and one attended to such a feature in her lesson plan interview only. Thus, teachers seemed particularly interested in features supporting symbolic labels in Exploration and Lesson Planning, but not in Problem Solving. One interpretation of this is that they thought the labeling features may be helpful instructionally, but they felt they did not need the symbolic representations to support their own mathematical thinking.

The frequent attention to labeling features in the Exploration and Lesson Planning contexts does not necessarily indicate teachers were considering using these features for the purpose of supporting students in making connections between visual and symbolic representations, but this is an issue worthy of further investigation. Chapter 6 provides further analysis of how teachers were thinking about these features.

High attention in Problem Solving. The two MLC Fractions App features highlighted in purple in Table 9 were frequently attended in Problem Solving, suggesting teachers found them particularly useful when they were solving the fraction comparison and addition problems. These features were the secondary button allowing for free rotation of a circle model and the overlap feature. The circle on the left in Figure 3 shows the rotation handle that can be clicked and dragged to freely rotate a circle model, and the circles on the right show two circle models before (top) and after (bottom) they have been overlapped.

Figure 3: The circle free rotation and overlap features of the MLC Fractions App



Only one teacher used the MLC Fractions App in Lesson Planning, and she did not attend to the overlap or circle free rotation features in this new context. This provides a small amount of evidence that these features might not have garnered the same level of interest from teachers in Lesson Planning as they did in Problem Solving—if more had chosen to use it in Lesson Planning. However, the overlap and circle free rotation features do not easily map onto features of the other VMs used by teachers in Lesson Planning, so it is not possible to make an evidence-based claim about the way the other teachers' attention to these or similar features was or was not carried into Lesson Planning. Because teachers' attention to these features was concentrated in the Problem Solving context, they appear to have been particularly salient to teachers when they were focused on solving problems about fraction comparison and addition. For this reason, teachers' thinking about these two features is explored more thoroughly in Chapter 5 (Results, Responses to Affordances).

Attention to features for the first time in Problem Solving. Although Table 9 shows that two features of the MLC Fractions App (circle free rotate and overlap) were particularly highly attended in Problem Solving, this table conceals another interesting pattern that emerged as I examined teachers' attention to VM features in each context. Specifically, there were 12 features of the MLC Fractions App to which at least one teacher attended for the first time during Problem Solving. Table 11 lists these features along with how many teachers attended to them in each context and how many teachers attended to them for the first time in Problem Solving.

Table 11: Features attended to for the first time in Problem Solving

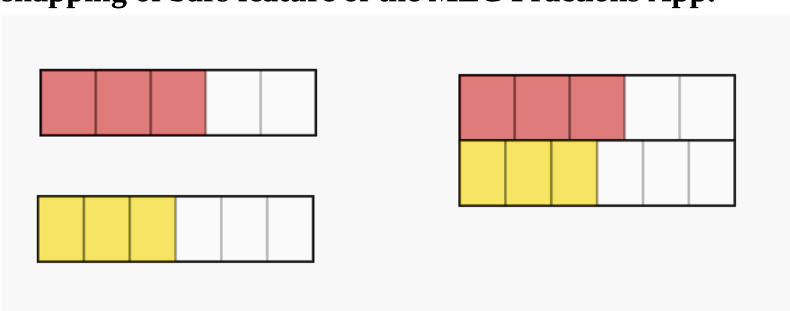
Feature	**Feature Category**	**No. Teachers Attended in Exploration**	**No. Teachers Attended in Problem Solving**	**No. Teachers Attended for First Time in Problem Solving**
Clear button	B	3	4	1
Circle free rotate	SB	3	5	2
Pencil eraser	SB	4	1	1
Overlap	D	5	6	1
Fixed denominator	D	3	3	1
Free position of fill	D	4	4	1
Snapping of bars	D	1	5	4
Snapping of circles	D	0	2	2
Pencil layer	D	1	2	2
Complete whole	I	4	4	2
Fluid movement	I	1	1	1
Automatic sizing	AP	1	3	2

Category code key: B = Button, SB = Secondary Button, D = Discoverable, I = Inherent, AP = Automated process

While a few features in Table 11 are buttons or secondary buttons that are easily visible and invited interaction, half of them are discoverable features that were only noticeable when teachers interacted with the VM in particular ways. This suggests that changing teachers' relationship to the VM—in particular, positioning the VM as a tool for solving mathematical problems and not as an object for exploration—led teachers to try some new things with the VM that they did not spontaneously try in Exploration.

Another interesting pattern in the list of features in Table 11 is that many of them have to do with relating multiple models to each other rather than creating a particular fraction model for the first time. The overlap feature relates to comparing at least two models by overlaying them. The circle rotation, snapping of bars, and snapping of circles features have to do with supporting VM users in aligning models in particular ways. The circle free rotation feature allows a user to align the edges of the shading in two overlapped models. The bar snapping feature draws two bar models together like a magnet, making it easier to precisely align the ends of the models (see Figure 4), and the snapping of circles feature snaps a circle into place, directly aligned with one underneath it. The complete whole feature (a feature reflecting focused constraint) prevents a user from directly moving a piece of one model into another. Finally, the automatic sizing feature refers to the fact that new models added to the screen are the same size as the last model manipulated—a feature that is impossible to notice without comparing the sizes of two models.

Figure 4: Two bar models before (left) and after (right) being snapped together using the snapping of bars feature of the MLC Fractions App.



In general, the transition from the Exploration context to the Problem Solving context seems to have supported teachers in noticing new VM features in two ways: by changing their view of the VM as an object of exploration to a tool for solving a particular problem, and by prompting them to attend to the relationships among multiple models. Indeed, the problems posed to teachers in Problem Solving involved fraction comparison and addition, both of which involve relating fractions.

Because most of the teachers used different VMs in the Lesson Planning context, all features were newly noticed and it is difficult to say whether this additional change in context led them to attend to other features for the first time. One teacher did use the MLC Fractions App for Lesson Planning. She attended to three features for the first time in Lesson Planning: the clear button, which erases the whole screen, the copy button, which duplicates a selected bar or circle model, and the multiple models feature, which refers to the fact that a user can have as many fraction models on the screen on one time as desired. While this is only a small amount of evidence about the potential impact of teachers moving from the Problem Solving to the Lesson Planning context, it does suggest this change in context may also impact what teachers notice. That said, this teacher's lesson was about converting fractions greater than 1 to mixed numbers —a task quite different than those posed during Problem Solving. Thus, the newly attended features in Lesson Planning may have more to do with the nature of the mathematical task than the context itself.

Finding 1c: Individual differences in attention during Exploration suggest mathematical knowledge and teaching orientation may affect how teachers explore VMs.

While examining the teachers' attention to features in aggregate led to Findings 1a and 1b, Finding 1c came about through analysis of teachers' attention to features for which there was wide variation in level of attention across individuals. (See Appendix I for details on how I determined which features to examine.) The MLC Fractions App features with the highest individual variation in attention during Exploration were the bar and circle model buttons, the denominator choice secondary buttons, and the fill buttons—all core features of the VM. These four features are listed along with the number of times each teacher attended to them in Exploration at the top of Table 12.

Table 12: MLC Fractions App Features with high variation in individual teacher attention in Exploration and Problem Solving

	Number of Times Each Teacher Attended						
Feature	**Colleen**	**Janice**	**Karla**	**Nancy**	**Sheila**	**Tammy**	**Range**
EXPLORATION CONTEXT							
Bar models	2	10	4	3	7	2	8
Circle models	2	7	9	5	4	5	7
Denominator choices	3	12	11	9	6	5	9
Filling of pieces	2	12	12	9	5	6	10
PROBLEM SOLVING CONTEXT							
Circle models	0	3	7	3	0	1	7
Filling of pieces	9	5	13	7	6	6	8
Clear button	1	0	7	1	0	3	7
Circle free rotation	0	4	2	9	2	5	9

These core features were highly attended overall, but Table 12 shows that the high variation across teachers was a result of two teachers (Janice and Karla) attending to these core features a high number of times in relation to Colleen. Janice and Karla's counts are highlighted in green in Table 12, and Colleen's are highlighted in red. Review of these three teachers' noticing interviews shows that Janice and Karla engaged in a lot of experimentation with the VM during Exploration and tried to use it to accomplish several tasks, such as generating equivalent fractions or illustrating addition or subtraction of fractions. Colleen, by contrast, worked more systematically through exploring all the features—for example, pressing each button in order to see what it did—but did not try a lot of different tasks.

This individual difference adds nuance to Finding 1b, suggesting that not all teachers necessarily had to move to the Problem Solving context before thinking about how the VM might be used to accomplish specific tasks. Examining the results of the mathematics content knowledge and mathematics teaching beliefs surveys (shown in Table 13) suggests factors that may relate to the individual differences in how teachers explored VMs. Janice and Karla each had higher scores on the fractions content knowledge measure than Colleen, which may suggest that teachers with higher levels of content knowledge are more apt to try to use a VM for multiple purposes. Tammy, who scored comparably to Colleen on the mathematics content knowledge measure, also generally used the core features less than Janice and Karla. This corresponds to a general pattern of teachers with lower content knowledge doing less experimentation during Exploration than teachers with higher content knowledge.

Table 13: Scores on the mathematics knowledge and mathematics teaching belief surveys

Teacher	**Mathematics Content Knowledge Score (out of 10)**	**Teaching Orientation Score (1 = strongly transmissionist, 7 = strongly discovery / constructionist)**	**"Facts First" Belief Score (1 = facts must be learned first, 7 = problem solving can come first)**
Colleen	5	5.3	4.0
Janice	8	5.4	3.3
Karla	7	5.7	4.0
Nancy	8	6.6	5.8
Sheila	7	4.6	4.7
Tammy	5	6.3	3.7

One explanation for this may be that teachers with higher mathematics content knowledge see more mathematical or instructional possibilities for a VM during Exploration—i.e., without prompting to focus on a task or learning goal—than teachers with lower mathematics content knowledge. Another explanation may be that teachers with higher levels of content knowledge are more confident when exploring a VM and thus more apt to try to use it for different purposes.

Sheila, who had the second-lowest number of instances of attention to two of the core features, scored highly on the content knowledge measure. This seems contrary to the pattern of teachers with higher content knowledge doing more experimentation with a VM during Exploration. The results of her mathematics teaching belief survey suggested she was the most strongly transmissionist-oriented, as opposed to discovery-oriented, in her approach to teaching mathematics (see Table 13). This could be a partial explanation for why she did less experimentation during Exploration than the other teachers with higher scores on the content knowledge measure. It could be that her stronger tendency toward telling students how to accomplish specific tasks also led her to try fewer things with the VM during Exploration.

It was more difficult to detect a meaningful pattern among the high individual variation in attention to features during Problem Solving. The four MLC Fraction App features with the highest variation in attention in this context are listed at the bottom of Table 12: the circle model button, the fill buttons, the clear button, and the circle free rotation tool. In each row in the bottom section of Table 12, the highest count of instances of attention is highlighted in green and the lowest count is highlighted in red. Karla had the highest counts of attention for the circle models and fill buttons, two of the core features, as well as the clear button. This is consistent with her problem-solving strategies, which tended to involve deleting models and creating new ones when she did not like how a strategy was playing out. Karla also tended to adjust the position of shading within circle models rather than rotate the models. Nancy's high attention to the circle rotation feature, and Colleen's lack of attention to it, drove the variation in attention to this feature. An explanation of this is that Nancy discovered the rotation feature for the first time during Problem Solving and then experimented with it, and Colleen's strategies for solving the two problems did not require use of the circle rotation feature. All of these differences seem attributable to differences in personality and problem-solving approach for individual teachers and not any other teacher characteristics that my data allows me to identify.

CHAPTER 4: RESULTS, INTERPRETATIONS OF AFFORDANCES

In this chapter, I discuss results related to Research Question 2:

> How, if at all, do teachers interpret features of fractions VMs that reflect efficient precision, focused constraint, and linked representations, and how do those interpretations differ across contexts?

The focus of this research question represents a shift in analytic perspective from Research Question 1 in two important ways. First, while Research Question 1 focused on the first element of the professional noticing framework, *attention,* the emphasis in Research Question 2 is on the second element of the professional noticing framework, *interpretation.* Thus, Research Question 2 moves beyond analyzing *which features* teachers attended to analyze *what they had to say about the features they attended.* To answer Research Question 2, I used teachers' talk as evidence for how they brought meaning to the features (that is, how they interpreted them).

The second shift in analytic focus from Research Question 1 to Research Question 2 is that while the analysis for Research Question 1 took account of all the VM features teachers noticed, the analysis for Research Question 2 focuses on the teachers' interpretations of a subset of features. Specifically, the analysis focuses on only those VM features that I (with the assistance of a second coder) identified as reflecting one of the VM affordances discussed in Chapter 1: efficient precision, focused constraint, and linked representations. While I continued with the overall interpretivist frame and the goal of understanding teachers' perspectives on the features, I imposed my own meaning on the VM features themselves in order to identify them as reflecting the affordances. Thus, my own perspective determined which segments of the data (those where a teacher was interpreting a feature reflecting an affordance) were included in the analysis for this research question.

To begin this chapter, I provide a brief description of the four VMs (other than the MLC Fractions App) that were used by at least one teacher in Lesson Planning, along with a table showing the topic of each teacher's lesson and the VM she used. This information provides useful context for understanding teachers' interpretations of features in the Lesson Planning context. Then, I present the results in three sections that correspond to the three affordances (efficient precision, focused constraint, and linked representations). In each section, I first provide tables of features of the MLC Fractions App and other VMs that I coded as reflecting that affordance with brief explanations how each feature reflects the affordance. Next, I present the finding(s) related to teachers' interpretations of the features reflecting the affordance.

As an aid to the reader, I provide an executive summary of the chapter findings here before delving into the details of each finding below. In accordance with the organization of the chapter, the findings are grouped according to affordance, then ordered by context.

- *Finding 2a:* In Exploration and Problem Solving, teachers generally interpreted features reflecting efficient precision positively in terms of easing the burden of creating visual fraction representations. They felt these features would resolve struggles that they or their students consistently encountered in creating accurate models for fractions. Shifting from Exploration to Problem Solving led some teachers to relate the benefits of these features to specific topics rather than talking about them generally.
- *Finding 2b:* In Lesson Planning, discovery-oriented teachers saw additional value in features reflecting efficient precision for pushing or focusing student thinking. They spoke about how these features could allow them to support students in reaching more

ambitious learning goals or focusing on mathematical ideas rather than struggles with manipulation. This suggests the Lesson Planning context primed teachers' thinking about how they and the VMs could support student thinking, and a discovery orientation may support teachers in taking greater advantage of the efficient precision of VMs.

- *Finding 2c:* In Lesson Planning, the teacher who used the same VM as in Exploration had different interpretations of features reflecting efficient precision. Specifically, one of the efficient precision features she found valuable in Exploration and Problem Solving became problematic for her in Lesson Planning, and another efficient precision feature became more useful in Lesson Planning than it had been in Exploration or Problem Solving. This suggests the specific mathematical tasks and topics she was addressing in each context impacted her interpretation of VM features.
- *Finding 2d:* In Exploration and Problem Solving, teachers largely viewed features reflecting focused constraints negatively because they prohibited them from using their familiar problem solving strategies. This was particularly true for addition strategies. Teachers were frustrated that focused constraints made it difficult for them to show a sum using different colors for the addends, move a model for one addend directly into their model for the other, or partition models to create equivalent fractions for the addends.
- *Finding 2e:* In Lesson Planning, teachers' interpretations of focused constraints highlighted both pros and cons of the constraints for student learning. While they still articulated some negative interpretations of focused constraints, every teacher also articulated a positive interpretation of at least one constraint. The teachers with a discovery orientation and high content knowledge spoke about both pros and cons of a single constraint, while other teachers highlighted pros of one constraint and cons of a

different constraint. This suggests the Lesson Planning context prompts more thinking about different ways students might use VMs and allowed teachers more flexibility in matching VM features to particular tasks. Further, it suggests that the combination of high content knowledge and a discovery orientation may support teachers in more thoroughly considering potential effects of constraints on student thinking.

- *Finding 2f:* Teachers had little to say about linked representations in Exploration or Problem Solving, and only one teacher made direct use of them in Lesson Planning. It is unclear why these features were of less interest to teachers than those reflecting the other two affordances or why one teacher did use them.

Descriptions of VMs Used in Lesson Planning

Table 14 lists the focus of each teacher's lesson along with the VM(s) chose.

Table 14: Lesson foci and VMs used by teachers in Lesson Planning

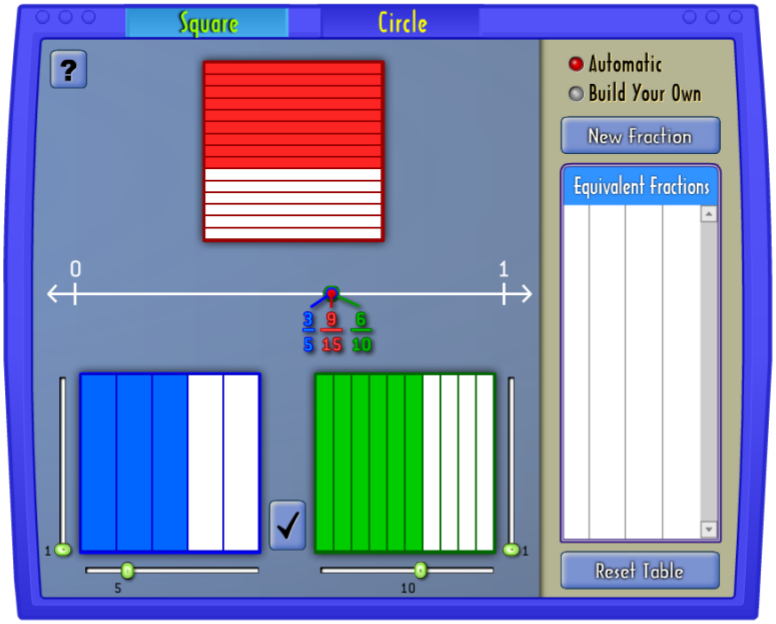
Teacher & Grade Level	Lesson Focus	VM Used
Colleen (G5)	Equivalence: Generating equivalent fractions and realizing fractions with different numerators and denominators can be the same size	NCTM Equivalent Fractions App (no longer available on the web)
Janice (G5)	Equivalence: Converting fractions greater than 1 into equivalent mixed numbers	MLC Fractions App
Karla (G4)	Addition: Partitioning wholes into different size parts & ensuring the parts sum to 1	Toy Theater Fraction Circles
Nancy (G4)	Equivalence: Generating equivalent fractions and developing a sense of which denominators "work together" (e.g., 5ths and 10ths)	Phoenix College Fraction Circles
Sheila (G5)	Equivalence: Generating equivalent fractions and noticing multiplication and division patterns among the numerators and denominators	Toy Theater Fraction Bars
Tammy (G4)	Equivalence: Solving equal sharing problems using different strategies and noting answers that look different (e.g., 3/2 and 1½) are equal	Toy Theater Fraction Circles & Toy Theater Fraction Bars

Janice used the MLC Fractions App—the same VM used by all the teachers during Exploration and Problem Solving. A description of this VM is available in Chapter 2 (Methods). Descriptions of each of the other four VMs are below. Note that the Toy Theater Fraction Circles and Toy Theater Fraction Bars are two different VMs, despite being available on the same website.

NCTM Equivalent Fractions App

Colleen planned a lesson using the NCTM Equivalent Fractions app (see Figure 5). This VM was available at the time of data collection, but has since been removed from the NCTM website, likely because it ran using an extension that is no longer supported by most browsers. When a user opens the VM, three square area models are visible on the screen, one at the top showing a fraction and two at the bottom starting off blank. Between the top and bottom area models is a number line with three dots on it corresponding to the fractions on each area model. The colors of the number line dots and labels correspond to the colors of the area models. As a user partitions and adds shading to one of the bottom models, the corresponding dot on the number line moves. The user's goal is to create two fractions on the bottom models that are equivalent to the fraction shown on the top model. Sliders allow the user to partition the models horizontally and vertically and clicking adds or removes shading from a part of the model. Figure 5 shows what the screen might look like after a user partitions and shades the blue and green area models. When a user creates two fractions equivalent to the top fraction and presses a feedback button (see the check mark button in Figure 5), the three fractions appear in a table to the right of the screen. The VM also has options to switch to circle models (rather than squares) or to switch to a "Build Your Own" mode, where a user chooses the fraction on the top model instead of that fraction being randomly chosen and created by the VM.

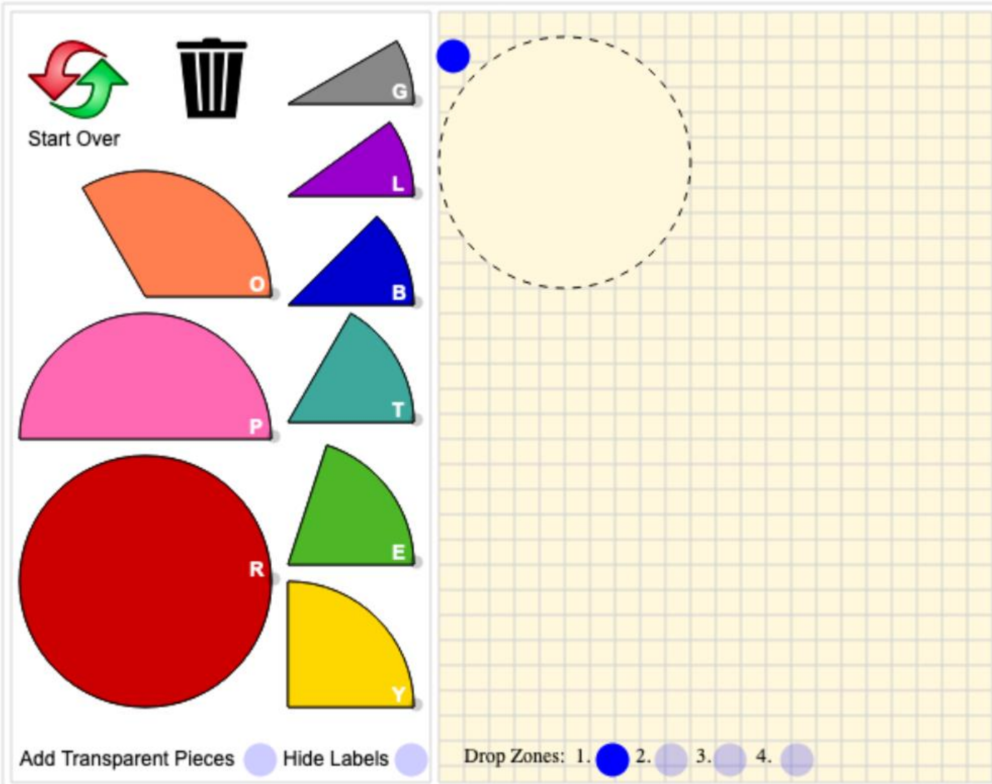
Figure 5: The NCTM Equivalent Fractions App



Phoenix College Fraction Circles

Nancy planned a lesson on generating equivalent fractions using the Fraction Circles manipulative from the Phoenix College Mathematics Blog. When the VM is loaded, a user sees a set of fraction circle pieces on the left (see Figure 6). When a full circle is considered the whole, the pieces represent 1 whole, ½, ⅓, ¼, ⅕, ⅙, ⅛, 1/10, and 1/12, respectively. The pieces are labeled with the first letter of their color (but not with their fraction names), and the labels can be turned off via a button. The pieces can be dragged onto a blank circle on the right side of the screen called a "drop zone." A user can add up to three more drop zones to the canvas via buttons on the bottom of the screen. As pieces are dragged into the drop zones, they "snap" into a position adjacent to other same-color pieces (if there are any already in the drop zone). Small gray dots on the edges of the pieces allow a user to freely rotate them to a new orientation within the circle. The pieces can also be placed on top of each other, and a user can choose to make pieces transparent so the pieces underneath can be seen. The VM also has a trash can a user can use to delete a piece and a start over button that clears all pieces from the drop zones.

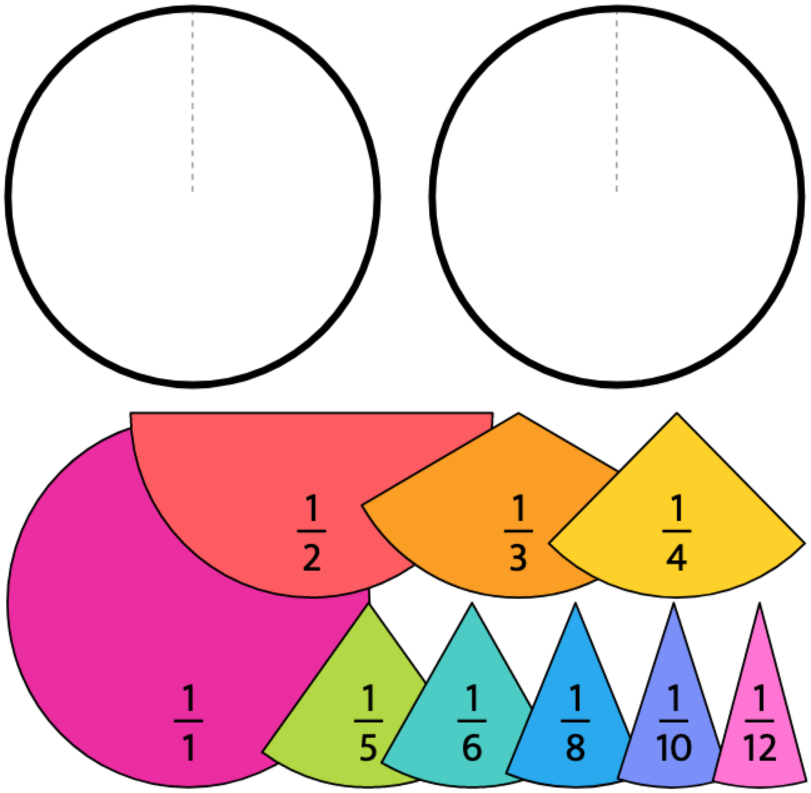
Figure 6: The Phoenix College Fraction Circles



Toy Theater Fraction Circles

Karla used the Toy Theater Fraction Circles to plan her lesson. Tammy also made this VM available to her students (along with one other VM) in her lesson. When a user loads this VM, brightly colored fraction pieces, each labeled with unit fraction, are visible on the screen (see Figure 7). Just above the pieces are two blank circles into which the pieces can be dragged. When pieces are dragged into the circles, they snap into position. The first piece is positioned to start at the top of the circle (aligned with the dotted radius present in each circle) and subsequent pieces snap into place next to existing pieces. The pieces do not overlap and cannot be placed with gaps between them. Rather, the pieces "push" each other out of the way, and pieces that will not fit into the space left in the circle simply disappear. The pieces make a snapping sound when placed into a circle or when they disappear.

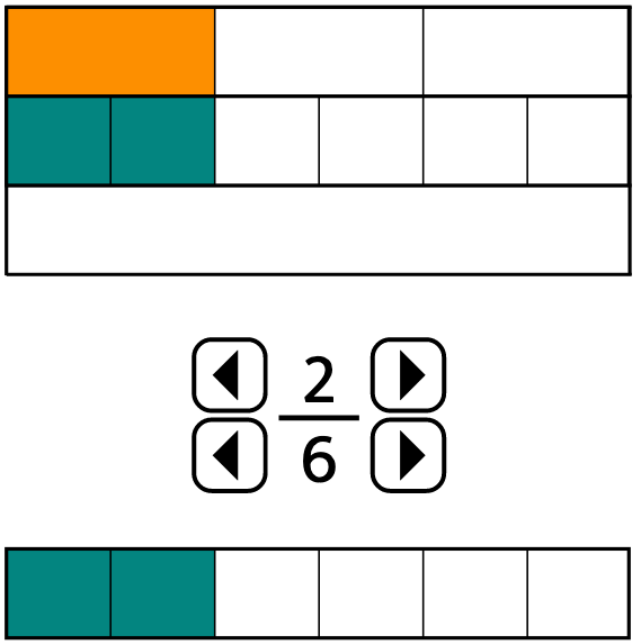
Figure 7: The Toy Theater Fraction Circles



Toy Theater Fraction Bars

Sheila used the Toy Theater Fraction Bars to plan her lesson. Tammy also made this VM available to her students (along with the Toy Theater Fraction Circles) in her lesson. When a user loads this VM, a fraction bar is visible on the bottom of the screen (see Figure 8). Just above this fraction bar, a symbolic fraction is shown with arrow buttons on either side of the numerator and denominator. When a user presses these buttons, the numerator or denominator of the fraction change accordingly, and the number of total parts and shaded parts in the fraction bar at the bottom of the screen also changes to correspond to the symbolic fraction. A user can drag the fraction bar into one of three stacked slots at the top of the screen. Figure 8 shows what the VM looks like after a user drags a ⅔ bar into the top slot and a 2/6 bar into the middle slot. The fraction bars are opaque and cannot be placed on top of each other. If a user places a bar into an already full slot, the bottom bar disappears and is no longer visible when the user moves the top bar off it. It is unclear whether this "disappearing bar" feature was intentionally designed or not, but the effect is that users must compare bars by placing them in adjacent slots, not by overlaying them.

Figure 8: The Toy Theater Fraction Bars



Interpretations of Features Reflecting Efficient Precision

I coded a VM feature as reflecting efficient precision when it supported a VM user in creating a visual representation of a fraction efficiently and precisely. Such features may allow tedious tasks, such as partitioning a whole into equal parts or ensuring parts do not overlap, to be offloaded to the tool. Features reflecting efficient precision may also allow VM users to quickly copy and edit their representations or support the creation of models for fractions with denominators outside those typically supported by physical tools (e.g., 7ths or 23rds). The features of the MLC Fractions App coded as reflecting efficient precision are listed Table 15 along with the rationale for coding them as reflecting this affordance.

Table 15: Features of the MLC Fractions App reflecting efficient precision

Feature	Feature Category	Feature Description	Rationale for Coding as Efficient Precision
Automatic dividing	AP	The tool automatically creates equal parts in the circle or bar model when a number of parts is chosen.	Supports a user in precisely dividing a model into equal parts, as the work of creating the parts is offloaded to the tool.
Free choice denominator	SB	When users create a bar or circle model, a blank field is available to enter any whole number to be used as the number of parts.	Supports a user in precisely dividing a model into *any* number of equal parts, including those not typically provided in physical manipulatives.
Copy button	B	The copy or duplicate button on the menu bar makes a copy of whatever is selected.	Supports a user in making an instant copy of a precise model, preventing a user from having to complete the process again from scratch.
Snapping of bars	D	When bar models are close to one another, they snap onto one another (like magnets).	Supports a user in precisely aligning the ends of bar models, allowing for quick comparisons of fractions close in magnitude.
Automatic sizing	AP	All new models (circle and bar) are sized to match the most recent bar or circle created.	Supports a user in creating several models of the same size, allowing the quick creation of many models with the same whole.
Full fill	SF	When users add or remove fill, one click fills the full piece (or removes the fill from the full piece).	Supports a user in quickly and cleanly adding or removing fill from a part, preventing any conflict size of part and size of the shading.
Snapping of circles	AP	When circle models are placed on top of each other, they snap into place when close to lining up.	Supports a user in precisely aligning two circle models, allowing for easy alignment of shading and partitions via rotation.

Feature category key: B = Button, SB = Secondary Button, D = Discoverable, AP = Automated process, SF = subfeature

The features of the other VMs (used by teachers in Lesson Planning) coded as reflecting efficient precision are listed in Table 16. The MLC Fractions App had more features reflecting efficient precision than the other VMs, likely because it had many more features overall.

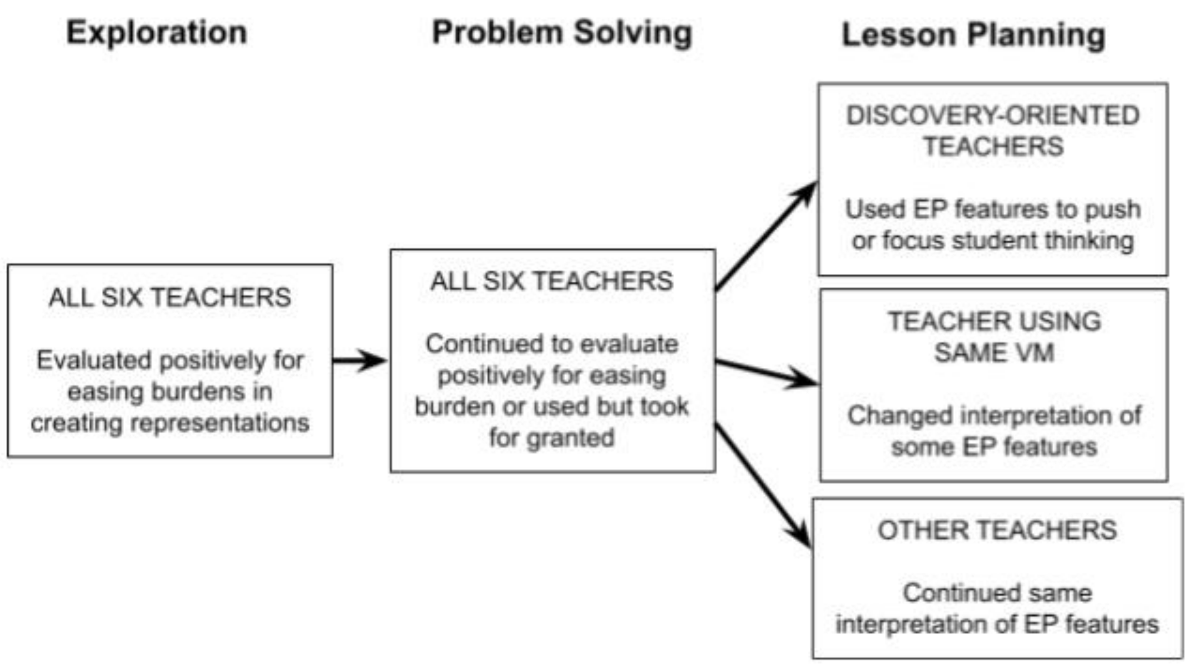
Table 16: Features of other VMs reflecting efficient precision

VM	Feature	Feature Category	Feature Description	Rationale for Coding as Efficient Precision
Toy Theater Fraction Circles	Snapping pieces	AP	The pieces snap into place when dragged into a circle.	Supports a user in aligning pieces.
	Exact size	SF	The circle pieces are exactly the appropriate size for the fraction they are labeled with (when the circle is the whole).	The work of creating appropriately sized parts in relation to the whole is offloaded to the tool.
Toy Theater Fraction Bars	Stacked slots	V	The VM has three adjacent slots for bars to be dragged into that are always visible.	Supports a user in aligning ends of bar models.
	Bar partitions	V	Unless the denominator is set to 1, the bars always have partitions breaking them into equal parts.	Creating equal parts is offloaded to the tool.
Phoenix College Fraction Circles	Snapping pieces	AP	When pieces are dragged into a drop zone, they snap into particular positions adjacent to same size pieces already in the circle.	Supports a user in aligning pieces.
NCTM Equivalent Fractions App	Drag and shade	D	When a user clicks and drags across multiple spaces in an area model, all the selected spaces are filled in.	Allows a user to efficiently shade the desired number of parts.
	Large denominators	SF	The circle models allow a user to create any whole number of pieces 1-24, and the square allows for many (but not all) whole numbers up to 256 (16 by 16).	Supports a user in creating models for fractions with much larger denominators than typical.
	Equal pieces	SF	The partitions within the area models always divide the square or circle into equal-size parts.	Creating equal parts is offloaded to the tool.

Feature category key: V = Visible, D = Discoverable, AP = Automated process, SF = subfeature

A summary of the findings related to teachers' interpretations of features reflecting efficient precision is as follows. Teachers were consistent in their positive views of the features in Table 15 in Exploration and Problem, particularly in terms of how the features eased the burden for students (and themselves) in drawing visual representations of fractions (Finding 2a). Teachers' interpretations of other features reflecting efficient precision in Lesson Planning (shown in Table 16) showed that some teachers carried this general positive interpretation of efficient precision into Lesson Planning. Others—specifically, those who had the strongest orientation toward discovery versus transmission-style teaching—saw potential in features reflecting efficient precision to push student thinking as they discussed the features in Lesson Planning (Finding 2b). Janice, who used the same VM in all three contexts, had different interpretations of the same features reflecting efficient precision in Exploration and Problem Solving versus Lesson Planning (Finding 2c). These findings are summarized in Figure 9 and elaborated in the sections that follow.

Figure 9: Summary of teacher interpretations of features reflecting efficient precision



Finding 2a: In Exploration and Problem Solving, teachers generally interpreted features reflecting efficient precision positively in terms of easing the burden of creating visual fraction representations.

Teachers commented on the first two features listed in Table 15, automatic dividing and free-choice denominator, more than the other features reflecting efficient precision in the first two contexts. Five of the six teachers noted that the automatic dividing feature would help to overcome difficulties they (the teachers) or their students had in creating models with equal parts when drawing by hand, especially when denominators are large. For example, Colleen noted, "The hardest part is just getting them [students] to make equal parts." Similarly, Tammy said, "I can't always draw circles like this on the board, so that's another good thing [about this VM] because I tell them, 'I am not an artist.'" Sheila explicitly pointed out a way in which the automatic dividing feature allowed for representing fractions that free drawing on paper did not: "Like if you're drawing, this would take forever and there's no way you could do hundredths on a circle… I don't know how you would get equal pieces." All the teachers used the automatic dividing feature during Problem Solving, but only three directly mentioned it. These three teachers said the feature was helpful, made creating the fraction representations "easy," or "saves time." This automatic dividing feature seemed to be either taken for granted by teachers or generally appreciated during Problem Solving, but they said little else about it.

Five teachers also rated the free-choice denominator feature positively in Exploration, calling it "cool," "neat," or "nice" without further immediate elaboration. Two teachers later commented that the free-choice denominator feature increased the flexibility of the MLC Fractions App, particularly when compared to the denominators available with most physical fraction manipulatives. Nancy commented,

> I feel like it's a little more versatile than some of the ones that we were looking at [in the professional learning sessions], because it had a pretty wide range of denominators. It also had the option to customize the denominator which we haven't seen one that I recall, being able to customize it completely like that.

Karla mentioned this flexibility as well, further commenting on why such flexibility might offer an advantage to students:

> Well, so I think in terms of things, definitely things that you need is that we're not always working with, you know, even denominators. We're working with odd denominators and they're the toughest ones. So it's nice to have, to be able to see them virtually.

She felt that seeing visual examples with less common denominators (especially odd denominators) may support students' thinking about fractions in a way physical tools could not. Thus teachers felt the automatic dividing feature would make the VM easier to use than drawings, and felt the free-choice denominator feature made it more flexible to use than physical fraction manipulatives. No teachers used or commented on the free-choice denominator feature in Problem Solving, but none of the fractions involved in the two problems they solved had denominators that required use of this feature. All the denominators were available on the denominator choice keypad (see Figure 2 in Chapter 2).

While at least one teacher noticed each of the next four features in Table 15 during Exploration, they had less to say about each one. The comments they did make were consistent with a general positive interpretation of features reflecting efficient precision and feeling the features eased the burden in creating representations of fractions. For example, Nancy called the copy button "helpful," and Sheila said the copy button would allow her to "make a bunch of these [circle models] very quickly." When Janice noticed how the bar models snap together when they are close to each other, she said, “That’s really helpful. I think they [the tool designers] are anticipating us comparing or creating fractions strips that are a little bigger or smaller.” Thus, she interpreted the purpose of this feature to be supporting VM users in creating fraction representations with the same-size whole to facilitate valid comparisons. Janice called the automatic sizing feature “really helpful,” and Sheila said the full fill subfeature was “cool.”

Two other patterns in teachers’ interpretations of features reflecting efficient precision in Exploration and Problem Solving are relevant for framing later discussion. First, four of the seven features listed in Table 15 were attended by at least one teacher for the first time in Problem Solving. Specifically, at least one teacher spoke about the snapping of bars, automatic sizing, and full features for the first time in Problem Solving, and the only teacher (Janice) who mentioned the snapping of circles feature did so in Problem Solving. Teachers’ interpretations of these features were consistent with the general positive evaluations discussed above, but they sometimes said the features were helpful for solving specific kinds of problems. Four additional teachers described the bar snapping feature as helpful during Problem Solving, specifically relating the feature to comparing fractions rather than generally seeing it as useful for creating any fraction representation. Three teachers noted that the automatic sizing feature was helpful for comparisons. For example, when asked what was helpful about the automatic sizing, Colleen said, “We want the wholes to be the same size or else it won’t be accurate comparing them.”

Second, while teachers’ interpretations of features reflecting efficient precision were overwhelmingly positive in Exploration and Problem Solving, there were two examples of a teacher describing a potentially negative effect of the feature. Specifically, a potential drawback of the automatic dividing feature is that because the process of creating equal parts is automated, students may not attend to the equal size of the parts or think about the relevance of the parts being equal or of how the number of equal parts relates to the size of each part. Two teachers showed some evidence of acknowledging this potential drawback as they interpreted this feature. Janice pointed out that the keypad students use to enter the number of parts they want in their model asks, “How many parts?” and commented, “I wish in this text it said, ‘How many equal parts?’ Right, that language. We want to press that hard on students.” This comment could indicate Janice wondered if students would need additional support (outside the tool itself) to recognize that the parts being created were equal in size, although the comment does not directly acknowledge that students using the VM would not be doing the work of creating equal parts. Colleen contrasted the way the MLC Fractions App automatically divided wholes into equal parts to other VMs (and physical manipulatives) that instead allowed students to drag one part at a time into a whole. She said this MLC fractions App would be appropriate for her fifth grade students “once they’re at the point where they understand a fraction, the denominator is the amount of equal parts in that whole,” but may not be appropriate in fourth grade when she thought students needed to build the whole from pieces: “When I was teaching fourth grade, I would want them to have to see, I need to put these 12 parts together to make the whole.”

In summary, in Exploration and Problem Solving teachers generally had positive interpretations of features reflecting efficient precision and felt they would ease the burdens placed on them or their students in creating precise representations of fractions. Sometimes, moving from Exploration to Problem Solving led teachers to relate the helpfulness of a feature reflecting efficient precision to a particular problem or task. While features reflecting efficient precision have the potential to take away thinking opportunities or conceptual demands for students (because some mental or physical work is offloaded to the tool), most teachers did not express concern about this possibility in Exploration or Problem Solving.

Finding 2b: In Lesson Planning, discovery-oriented teachers saw additional value in features reflecting efficient precision for focusing and pushing student thinking.

As illustrated in Table 16, all the VMs used by teachers in Lesson Planning had at least one feature reflecting efficient precision. Many of these features had effects on the representations created with VMs that were comparable to the effects the MLC Fractions App's efficient precision features had. For example, the exact size of the Toy Theater Fraction Circles, the bar partitions of the Toy Theater Fraction Bars, and the equal pieces of the NCTM Equivalent Fractions App all offloaded the work of creating equal or appropriately sized parts to the tool, parallel to the automatic dividing feature in the MLC Fractions App. The snapping pieces in the Toy Theater Fraction Circles and the Phoenix College Fractions Circles and the stacked slots in the Toy Theater Fraction bars supported users in aligning pieces or models, parallel to the bar and circle snapping features of the MLC Fractions App.

Many of teachers' interpretations of the features in Table 16 in Lesson Planning were similar to their interpretations of parallel features of the MLC Fractions App in Exploration and Problem Solving. For example, Colleen attended to the equal pieces feature of the NCTM Equivalent Fractions App as she compared what her lesson would be like using the VM to what it would be like if students were drawing free-hand:

> I like this a lot better because these parts are all even. They're all the same size. And we don't run into where, if we're drawing them ourselves, even when I draw them. And I'm using a ruler and sometimes, I'm like, "Oh! Oh, ignore that. That's not perfect." Because it's hard to draw them all exactly the same, unless you're measuring it out.

This is quite similar to Colleen's and other teachers' comments about the automatic dividing feature in the MLC Fractions App.

As she described why she chose to use the Toy Theater Fraction Bars in her lesson, Sheila mentioned the stacked slots on the VM that support students in aligning the fraction bars for easy comparison: "I really like that you can stack the fraction bars on top of each other. … And then I liked that you could do three at a time." (By "stack on top of each other," Sheila meant place in adjoining slots, not directly overlay—the latter was not possible in this VM.) She went on to explain that she expected students to use the way the partitions on the VM's fraction bars lined up (or did not line up) to judge equivalence. This strategy makes use of the precision afforded by the stacked slots and the bar partitions, but Sheila did not mention this directly, seeming to take it for granted. This is similar to the ways teachers tended to say they liked features reflecting efficient precision during Problem Solving, but not specifically talk about the ways in which the features supported their work with the VM.

While all of Colleen and Sheila's interpretations of features reflecting efficient precision in Lesson Planning were similar to the general patterns of interpretation of parallel features in Exploration and Problem Solving, three other teachers—Karla, Nancy, and Tammy—offered some interpretations of features reflecting efficient precision that broke with this pattern. To be clear, some of what they had to say paralleled their thinking in the other contexts. For example, as she described why she chose to use the Toy Theater Fraction Circles for her lesson, Karla mentioned the snapping pieces feature: "They [the pieces] snap into place. For a kid, you just grab. I want a 12th and it puts it where it goes [in the circle] rather than some of the others, they sort of have to put it in themselves." She went on to say that because her students use iPads, this feature was especially helpful. This is similar to what Karla and several other teachers said about the bar-snapping feature of the MLC Fractions App—the feature would help students place pieces in helpful alignment with each other. This interpretation is focused on easing the burdens on students in creating fraction representations.

Even so, these three teachers also all mentioned at least one interpretation of a feature reflecting efficient precision that went beyond the general easing of burdens to talk about the implications for what students could think about or learn from the lesson. I describe an example of such an interpretation for each of these three teachers below.

Nancy mentioned the snapping pieces feature of the Phoenix College Fraction Circles as one of the things she liked about this VM: "I liked that it kind of clicks together and doesn't require you to be super precise because in this initial exploration I don't know that kids know to be specifically super precise." Further, she thought the snapping would help make sure students were focused on the mathematical ideas:

> So that way they don't have to focus on all of that manipulation. They can just focus on what they're seeing and what they're actually doing. Because I think if you have to be adjusting it ever so slightly, then the focus is on the tool and not on the actual skill.

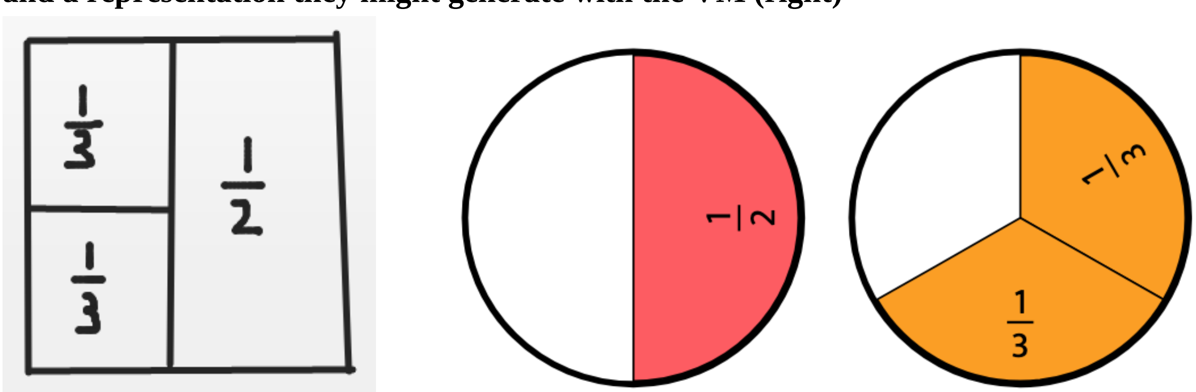
She said this feature made the VM "more forgiving" than physical manipulatives, and the ease with which students could create precise representations might encourage students to generate more examples of equivalent fractions than they would otherwise. Of physical fraction circles, she said:

> They drop them on the floor and they knock them over and they can't get them to fit quite right. And then they're, "Okay. I found one." And then they move on. Because they're happy that they finally got one and then they just move on to the next [problem]. Whereas this one [the VM], I feel like they can kind of keep exploring.

Thus, Nancy started by noting that the piece snapping feature eased manipulation burdens on students but went on to elaborate that as a result of the eased burdens, students may generate more examples of equivalent fractions and "keep exploring." While she did not say so explicitly, the way Nancy framed the continued exploration as a positive thing suggests she thought students' continued exploration and generation of more examples might support them in noticing patterns or otherwise getting more out of the lesson.

Karla similarly explained how she thought a feature of the Toy Theater Fraction Circles reflecting efficient precision may allow her to "go further with the lesson" than she had in the past when students drew their fraction representations. The main task of Karla's lesson was for students to explain why a queen could not give ½ of her kingdom to her oldest daughter and ⅓ of her kingdom to each of her younger daughters. In the past, students tended to draw models of the kingdom by hand, partition their models into three parts, and label those parts as ½, ⅓, and ⅓ without realizing the sum of the three parts was greater than one whole (see Figure 10, left). Karla explained that because the Toy Theater Fraction Circle pieces were exactly the right size in relation to the whole (represented by a circle in the VM), students could not make the same mistake if they used the VM to generate their representations. The two ⅓ pieces would not fit in the whole next to the ½ piece (see Figure 10, right).

Figure 10: A typical student-generated drawing of the queen sharing her kingdom (left) and a representation they might generate with the VM (right)



Because this common mistake would be recognized and avoided, Karla thought students' attention could shift away from their drawings and onto the conceptual issues related to how the fractions in their representations must sum to 1:

> I think I can go further with the lesson with the manipulative because a lot of the focus before was on the drawings. That just didn't make sense. Getting to that piece where you have an equation that equals one was further down the road. And here, I think with seeing the fractions, seeing the relationships, that we'll be able to move the lesson a little further because the struggle will not be the drawing. That's what I thought was great.

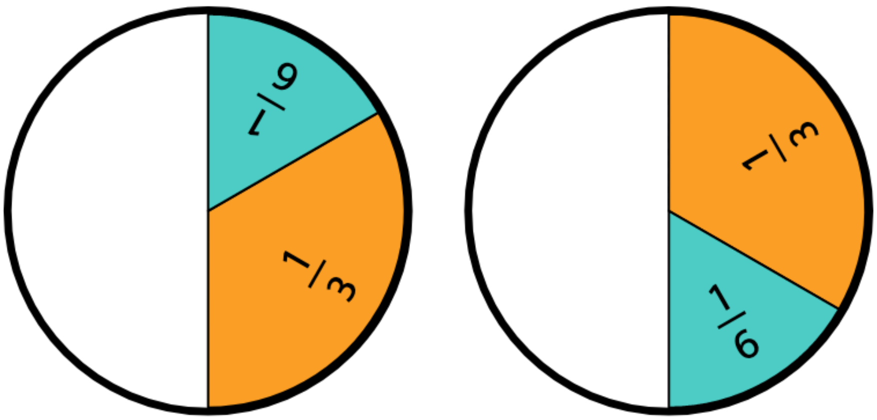
In her student response interview, Karla explicitly attributed this shift in focus to the exact sizing of the pieces:

> I just think that [the VM] provides what the kids can't do for themselves, which is make the fractions the correct sizes. And that's something they just can't do. Adults can't do it either, unless it's quarters and halves. It's not just kids. But it gives them that. So the struggle is not with the drawing, the struggle is ... Or the thinking is all with the fractions, and not with the drawing.

Karla felt the exact size feature of the Toy Theater Fraction Circles allowed students to focus on the way the magnitudes of the fractions related to each other (and to the whole) rather than on understanding how their own self-generated labeling of the pieces in their drawings related to the whole. This, she thought, would allow the students to get to the point of generating number sentences showing sums of 1—a point she had not been able to reach in past implementations of this lesson.

Tammy connected the same VM feature reflecting efficient precision—the exact size of the Toy Theater Fraction Circles—to a way students could explore a new idea in her lesson. One of the problems students would solve in her lesson was, “Two brothers share three pizzas. How much pizza does each brother get?” In Tammy’s student response interview, I posed the following scenario: Suppose a student reasoned each brother would get one whole pizza and was working on sharing the third pizza between the brothers. She starts out by giving each brother ½, but then tries to generate a different solution. She gives each brother one third. She knows she has to make a half for each one, so she makes the representation shown in Figure 11. She asks, “Is this another way we can show the answer? I don’t know if this is right.”

Figure 11: A scenario presented to Tammy about how a student might share a pizza among two brothers



Tammy responded by saying she would accept this answer as a way to share the third pizza among the two brothers, and she would follow up by encouraging students to see if they could find other ways to make ½ with the fraction circle pieces. She explained that she thought the exact sizing of the pieces would support students in finding combinations of fractions that add up to one half using trial-and-error strategies and visual inspection of whether the total was equal to one half: “And with the circle pieces, you know that if you try to put in one-eighth with two one-fifths, that’s not going to work and they have to try something else.” She felt that having those experiences with creating combinations with a particular sum would allow students to some early thinking about fraction addition without “even having that complex process of trying to find a common denominator yet." Even though fraction addition was not the focus of her lesson, Tammy seemed eager for students to explore generating combinations that sum to ½ as part of their work in the lesson, and to think that the exact sizes of the pieces made doing so possible without introducing common denominators.

In sum, Nancy, Karla, and Tammy, all discussed in Lesson Planning a way that a VM feature reflecting efficient precision could contribute to students doing more or different kinds of conceptual explorations of fraction ideas than might have been possible without the VM. These comments went beyond the interpretations of features reflecting efficient precision that they and the other teachers made during Exploration and Problem Solving, which mostly had to do with making it easier to do the same kinds of work students would do without the VM. While one teacher (Tammy) made a brief comment in Exploration on how a feature reflecting efficient precision could help students focus on mathematical ideas rather than details of drawing, this was the only comment that went beyond general notes about easing student burdens in creating representations. No teacher described how that easing of burdens could push students to explore new or different ideas.

This finding raises the question of why the Lesson Planning context spurred a change in interpretation of features reflecting efficient precision for some teachers. One explanation might be that Lesson Planning connected teachers' thinking about VMs to a specific learning goal (unlike Exploration) and also placed them firmly in a teaching mindset (unlike Problem Solving). Their attention to a learning goal may have primed them to think in more detail about ways both they (as teachers) and the VMs could support students in reaching the learning goal.

The fact that only three teachers showed this change in patterns of interpretation in Lesson Planning also raises a question about why these three teachers seemed to see greater potential in features reflecting efficient precision than the others. One potential explanation is suggested by the results of the mathematics teaching belief survey administered at the start of the study. Nancy, Karla, and Tammy had the three highest scores on this survey indicating an orientation toward discovery-based teaching (as contrasted with transmissionist teaching), as shown in the middle column of Table 13 (in Chapter 3). Thus, it may be that these teachers' general orientation to support student's exploration of ideas and discovery of mathematical properties and generalizations may have contributed their seeing ways VM features reflecting efficient precision could be helpful in supporting that work.

Finding 2c: In Lesson Planning, the teacher who used the same VM as in Exploration had different interpretations of features reflecting efficient precision.

One teacher, Janice, neither carried the same patterns of interpretation of features reflecting efficient precision from Exploration and Problem Solving into Lesson Planning (like Colleen and Sheila) nor discussed interpretations in Lesson Planning that related to students getting more out of a lesson (like Nancy, Karla, and Tammy). Instead, her interpretations of specific features changed—from positive to negative and vice versa—from Exploration and Problem Solving to Lesson Planning. Janice used the MLC Fractions App for Lesson Planning, which allowed for direct comparisons of her interpretations of specific features across contexts. These comparisons revealed that the focus of her lesson—which was on converting fractions greater than 1 to mixed numbers, an idea related to equivalence but different from generating equivalent fractions with different denominators—changed the way she thought about some features reflecting efficient precision.

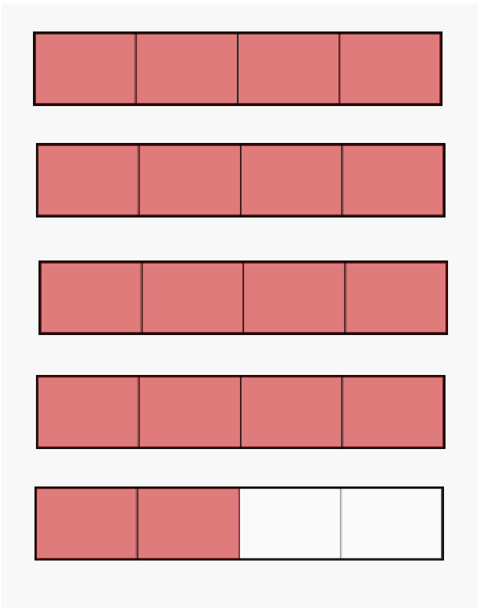
Four of the MLC Fraction App's features reflecting efficient precision came up in Janice's Lesson Planning data. For two of these features, automatic dividing and the free-choice denominator, Janice maintained the same general positive evaluation and connection to relieving student burdens in creating fraction representations as she and the other teachers did in Exploration and Problem Solving. For example, in her lesson plan interview Janice pointed out the automatic dividing feature allowed her students to create more accurate representations than when they drew them freehand:

> Another option in my teaching has been using just a whiteboard slate that they have in front of them and drawing it out. And that has been inconsistent in the way that they've been drawing fractions. Their equal parts are not equal. And this is perfectly equal.

She also pointed out that the free-choice denominator feature made this VM "pretty flexible."

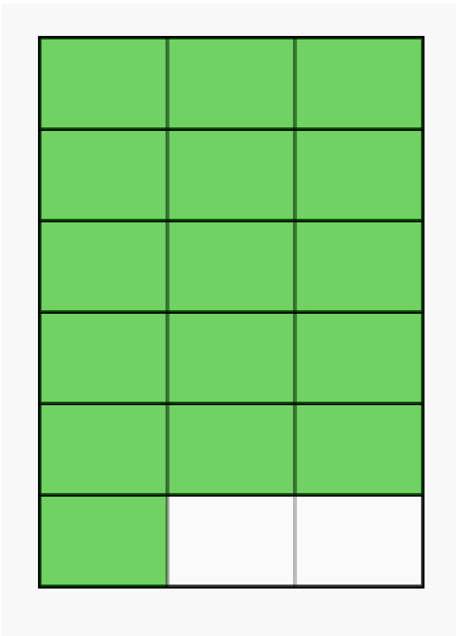
Janice's interpretations of the copy button and bar-snapping features, by contrast, were different in Lesson Planning than they had been in Exploration and Problem Solving. During Problem Solving, Janice pointed out a reason that the copy button was not helpful to her. As she solved the comparison problem, which asked her to put the fractions 3/10, ¼, and 2/6 in order from least to greatest, Janice used the copy button to duplicate her circle model of 2/6. She intended to edit this model so it would show 3/10, but noticed she could not change the number of parts in the circle. She said, "I can't change the two sixths to anything else. I have to start over." She deleted her copied model and created a new circle. By contrast, Janice used the copy button extensively during Lesson Planning as she created representations of fractions greater than 1. For example, she created the representation of 18/4 shown in Figure 12 by creating a fourths bar, filling all four fourths, then pressing the copy button four times to create five bars total. She finished by removing two fourths of shading from the bottom bar. Janice did not specifically comment on the utility of the copy button, but her frequent use of the button suggests she realized the button was useful as long as she did not need to change the number of partitions in the bar or circle.

Figure 12: Janice's representation of 18/4 created using the copy button



Relatedly, during Exploration and Problem Solving, Janice said she found the snapping of bars in the MLC Fractions App helpful, particularly for making comparisons using bar models. But during her student response interview, she did not find the feature helpful for supporting students in converting fractions greater than 1 to mixed numbers. I presented a scenario where a student had created the representation of 16/3 shown in Figure 13 and interpreted this representation as 16/18. Janice described this situation as her "worst nightmare" and explained, "The tool clicks them together. And I think that's why I was thinking worst nightmare, because I've done this before too. The clicking part is so helpful for comparing fractions, but not helpful when we're doing fractions like this." This comment suggests that as Janice was planning her lesson on fractions greater than 1, she had encountered a scenario similar to the one I presented and realized the bar snapping feature is less useful and potentially problematic when the bars are partitioned into the same-size parts, because the snapped bars could easily be interpreted as one contiguous whole.

Figure 13: A representation of 16/3 presented to Janice in a student scenario



Neither of these changes in Janice's interpretations of the VM features are surprising when considered in context. There is a key difference between the representations she created in Problem Solving (representations of fractions with *different* denominators) and those she created in Lesson Planning (representations of multiple wholes divided into the *same* number of parts). The task of representing fractions greater than 1 with multiple bar models versus representing different fractions with multiple bar models changes the potential utility of the copy button and the visual implications of the snapping of bars. Even so, this finding illustrates the potential impact of teachers' interpretation of features depending not only on *whether* they have a particular task and learning goal in mind (as discussed under Finding 2b), but *what* particular task they have in mind. I return to this point in Chapter 7 (Discussion).

Interpretations of Features Reflecting Focused Constraints

I coded a VM feature as reflecting focused constraints when the feature met two criteria: it placed a constraint on how a user could manipulate the VM's representations, and the constraint had the potential to focus a student's attention on a particular mathematical idea or operation related to fractions. Some features met the latter criterion, but not the former, because they did not place a constraint on ways the VMs could be manipulated. For example, the Phoenix College Fraction Circles allow pieces to be stacked, which could focus students' attention on thinking about comparison or equivalence (since stacking pieces on top of each other allows for direct comparison of size). However, this feature did not constrain or prevent a user from taking some other action. By contrast, the complete whole feature of the MLC Fractions App (which refers to the fact that all fraction representations created in the VM show a whole and not just a part separate from its whole) both has the potential to focus a user's attention on the relationship between a part and its whole. The complete whole feature also constrains a user from disembedding a part from a whole or directly combining parts of wholes that were created separately. Thus, the complete whole feature of the MLC Fractions App was coded as reflecting focused constraints, but the stacking feature of the Phoenix College Fraction Circles was not.

Table 17 lists the six features or subfeatures of the MLC Fractions App coded as reflecting focused constraints. Table 18 lists the features of the other fractions VMs (other than the MLC Fractions App) used by teachers in Lesson Planning coded as reflecting focused constraints. Like for efficient precision, the MLC Fractions App had more features reflecting focused constraints than the other VMs, and this was likely due to it having more features overall.

Table 17: Features of the MLC Fractions App reflecting focused constraints

Feature	Feature Category	Feature Description	Rationale for Coding as Focused Constraints
Single-color models	D	When a user tries to add a second color to a model, all the shading changes to match the new color.	Constrains a user from using two colors in a model, which has potential to focus attention on the overall value of the fraction (not how it is composed of unit fractions or other parts).
Complete whole	I	The tool always creates a complete bar or circle as the whole. Users can manipulate and move only complete wholes.	Constrains a user from disembedding a fractional part from its whole, which has the potential to focus attention on the relation between the part and the whole.
Fixed denominator	I	Once a user closes or clicks away from the part choices in a model, the user cannot change how many parts are in the model.	Constrains a user from changing the denominator after creating a fraction model, which has the potential to focus attention on generating a new model and comparing it to the original fraction (instead of changing the original model).
Bar width	SF	The size-change feature for the bars only allows a user to change the length, not the width.	Constrains a user from making a bar model wider or adding horizontal partitions, which has the potential to focus attention on ways of generating equivalent fractions that do not involve partitioning the parts horizontally.
Part choices	SF	The denominator choice keypad shows only a subset of denominators, and excludes denominators such as 7, 9, and 15.	Constrains a user from instantly dividing a whole into a number of parts not included on the denominator keypad (without using the free-choice denominator), which has the potential to focus attention on certain fractions and the relationships between them.
Denominator first	SF	When a user creates a model, they have to choose the denominator before adding any shading.	Constrains a user from constructing a fraction representation by first showing a number of parts (without considering the size of the parts), which has the potential to focus attention on the relation between the number and the size of the parts.

Feature category key: D = Discoverable, I = Inherent, SF = subfeature

Table 18: Features of other VMs reflecting focused constraints

VM	Feature	Feature Category	Feature Description	Rationale for Coding as Focused Constraints
Toy Theater Fraction Circles	Disappearing pieces	D	When a user tries to add a piece to a circle that would result in more than one whole, the piece disappears.	Constrains a user from adding more than a total of 1 to a circle, which has potential to focus attention on fractions less than 1.
	No overlap/ transparency	I	The pieces are not transparent and cannot overlap.	Constrains a user from directly comparing piece sizes, which has potential to focus attention on addition rather than comparison.
	Fixed wholes	SF	The VM always shows two wholes—never one and never more.	Constrains a user from representing fractions greater than 2, which could focus attention fractions less than 2.
Toy Theater Fraction Bars	Numerator & denominator limit	D	The denominator can only be increased through 12, and the denominator and numerator cannot be changed to zero.	Constrains a user from representing a fraction greater than 1, which has potential to focus attention on fractions less than 1.
	Disappearing bar	D	When a bar model is dragged over another in one of the slots, the bottom strip disappears.	Constrains a user from overlapping the bars, which has potential to focus attention on comparing the bars side by side.
	Complete whole	I	A user can only drag a complete strip.	Constrains a user from disembedding a part from its whole, which has potential to focus attention on the relation of part to whole.
	Fixed color	SF	The user can't change the colors of the bars.	Constrains a user from changing the color of bars, keeping a color associated with a size, which has potential to focus attention on comparing piece sizes.

Feature category key: D = Discoverable, I = Inherent, SF = subfeature

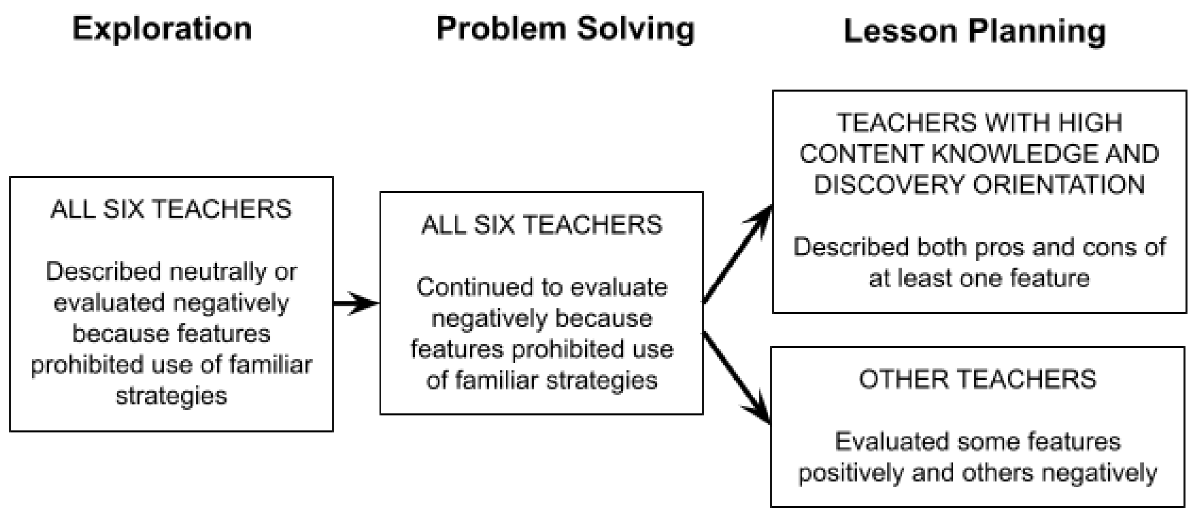
Table 18 cont'd

VM	Feature	Feature Category	Feature Description	Rationale for Coding as Focused Constraints
Phoenix College Fraction Circles	Piece choices	I	There are specific sizes of piece available that do not include, for example, 7ths or 9ths.	Constrains a user from representing fractions with denominators not provided, which has potential to focus attention on certain fractions.
NCTM Equivalent Fractions App	Table limit	SF	A user can only add fractions to the equivalent fractions table if s/he produces three equivalent fractions (not just two).	Constrains a user from showing pairs of equivalent fractions in the table, which has potential to focus attention on patterns across sets of three equivalent fractions.
	Slider limits	SF	The partitioning sliders each have maximum positions (16 on the square sides, 24 on the circle).	Constrains a user from creating representations of fractions with particular denominators, which has potential to focus attention on recognizing patterns within certain fractions.
	No drag	SF	Area models in the VM are set in place. They cannot be dragged onto one another.	Constrains a user from making direct comparisons of the shading in area models, which has potential to focus attention on other ways to make comparisons.

Feature category key: D = Discoverable, I = Inherent, SF = subfeature

A summary of the findings related to teachers' interpretations of features reflecting focused constraints is as follows. Teachers' interpretations of the features in Table 17 in Exploration and Problem Solving were largely negative because focused constraints prohibited them from using their familiar problem-solving strategies (Finding 2d). Some of teachers' interpretations of the features reflecting focused constraints in Table 18 in Lesson Planning continued to be negative, but teachers also noted some potential positive effects of the constraints on student thinking and learning (Finding 2e). Further, teachers with higher content knowledge *and* a tendency toward discovery-oriented teaching described both pros and cons of a single feature reflecting focused constraints in Lesson Planning, whereas the other teachers evaluated some features reflecting focused constraints positively and others negatively. These findings are summarized in Figure 14 and elaborated in the sections that follow.

Figure 14: Summary of teacher interpretations of features reflecting focused constraints



Finding 2d: In Exploration and Problem Solving, teachers largely viewed features reflecting focused constraints negatively because they prohibited them from using familiar problem solving strategies.

Teachers' interpretations of the first three features in Table 17 (single-color models, complete whole, and fixed denominator) were either neutral or negative in Exploration and consistently negative in Problem Solving. However, the particular teachers who gave neutral versus negative interpretations of each feature (or did not mention it) in the two contexts varied by feature. While most of teachers' negative interpretations had to do with the implications of the features for their familiar strategies for solving fraction addition problems, teachers occasionally mentioned the features in relation to other topics, including equal sharing, comparison, and equivalence. Teachers' interpretations of these three features are summarized in Table 19. I describe these interpretations more fully below, starting with the single-color model feature and proceeding to the complete whole feature, then the fixed denominator feature.

Table 19: Teacher interpretations of three features reflecting focused constraints in Exploration and Problem Solving

	Single-color models		**Complete whole**		**Fixed denominator**	
Teacher	**Explor-ation**	**Problem Solving**	**Explora-tion**	**Problem Solving**	**Explora-tion**	**Problem Solving**
Colleen	Neutral	Negative (addition)	Neutral	Not mentioned	Not mentioned	Negative (general)
Janice	Neutral	Not mentioned	Negative (addition)	Negative (addition)	Negative (equiva-lence)	Negative (general)
Karla	Neutral	Not mentioned	Not mentioned	Negative (comparison)	Negative (equal sharing)	Negative (addition)
Nancy	Negative (addition)	Negative (addition)	Not mentioned	Negative (addition)	Not mentioned	Not mentioned
Sheila	Negative (addition)	Negative (addition)	Negative (addition)	Negative (addition)	Neutral	Not mentioned
Tammy	Neutral	Negative (addition)	Neutral	Not mentioned	Not mentioned	Not mentioned

The single-color model feature of the MLC Fractions App refers to a user's inability to use two colors of shading in a model. If a user attempts to add a second color, all the shading changes to that new color. During Exploration, two teachers (Janice and Karla) noticed that it was possible to change the color of the shading but did not seem to realize this meant two colors could not be used together. Two others (Colleen and Tammy) realized they could not place two colors in the same model, but only described the feature without commenting on its implications. For example, Tammy said, "So if I click on one box, it changes all of the boxes to the same color." These interpretations do not suggest positive or negative opinions about the feature.

Nancy and Sheila, by contrast, noted that the single-color model feature would have negative implications for their own or their students' abilities to use the VM to represent addition. After noticing the feature, Nancy said, "Well, that's interesting that you can't use two different colors. Because if you were doing something with fraction addition, I would want to be able to represent two different parts, with two different colors." Similarly, Sheila said, "when I tried to put the different colors together on the bar and it automatically changed, I was like, 'Okay. So I can't add with these, or I can't add on the same bar.'" Both of these teachers seemed to have an existing practice of having students show fraction sums using a different color for each addend. They each interpreted the single-color model feature to mean they could not replicate this practice with the VM. Admittedly, Nancy also noted a potential positive use of this feature, saying it might "cause [students] to realize that whatever you did to one part, it was part of a fraction. So it had to all kind of work together." But overall, she and Sheila seemed to feel this was a limitation of the VM.

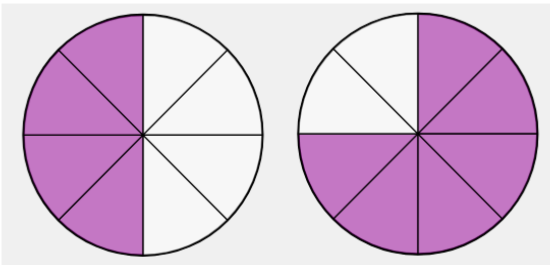
Nancy and Sheila both repeated this negative interpretation of the single-color model feature as they solved $\frac{1}{2} + \frac{1}{5}$ during Problem Solving. Further, two of the teachers who had been neutral about this feature in Exploration echoed this negative interpretation when they tried to add the fractions during Problem Solving. After solving the addition problem, Colleen said, "I do wish you could use different colors on one circle or one of the rectangles" because she liked students "to be able to see this is the one fraction, plus the other fraction, and then look at the total." Similarly, Tammy said, "I can't do the two different colors. I wanted to represent the two tenths [with a different color] and I can't do that." (Janice and Karla did not attend to the single-color model feature in Problem Solving, so their overall interpretation of the feature likely remained neutral.)

The same overall interpretation pattern also appeared with the complete whole feature, which refers to the fact that a user cannot represent a fraction in the MLC Fractions App apart from its whole—all bar and circle models show the whole. However, which teachers fell into each interpretation pattern across contexts varied. Two teachers (Karla and Nancy) did not attend to the complete whole feature in Exploration, while another (Colleen) described it without commenting on its implications. Toward the end of Exploration, Tammy said she thought the MLC Fractions App was "more advanced for the kids that already have knowledge of fractions." When I ask what made her think that, she replied,

> I would say that because there aren't pieces to play with to create fractions. Like in that one where you could actually take the one third and pop it in the circle and overlay it to see what it equals. I would have to know that if I took this chunk and put it over here, this would make a whole and that this would be one fourth leftover, so I think that that's more of an advanced thinking process than what the other ones where they had the pieces here and we could drag them over here.

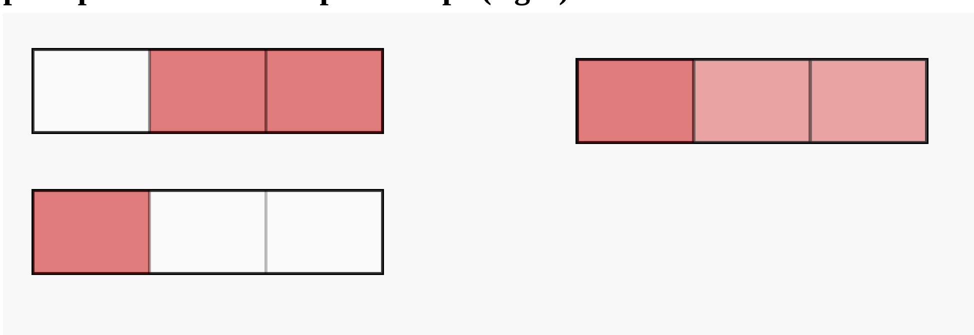
As she spoke the text above, Tammy was gesturing to circle models she had previously created for 4/8 and 6/8 (see Figure 15). She seemed to be pointing out that because she could not move only 2 of the 4 eighths into the 6 eighths model to make a whole (a consequence of the complete whole feature), students would have to imagine this move rather than carry it out. She did not express this as a negative implication, however—her comments seemed a neutral observation.

Figure 15: Tammy's models of 4/8 and 6/8



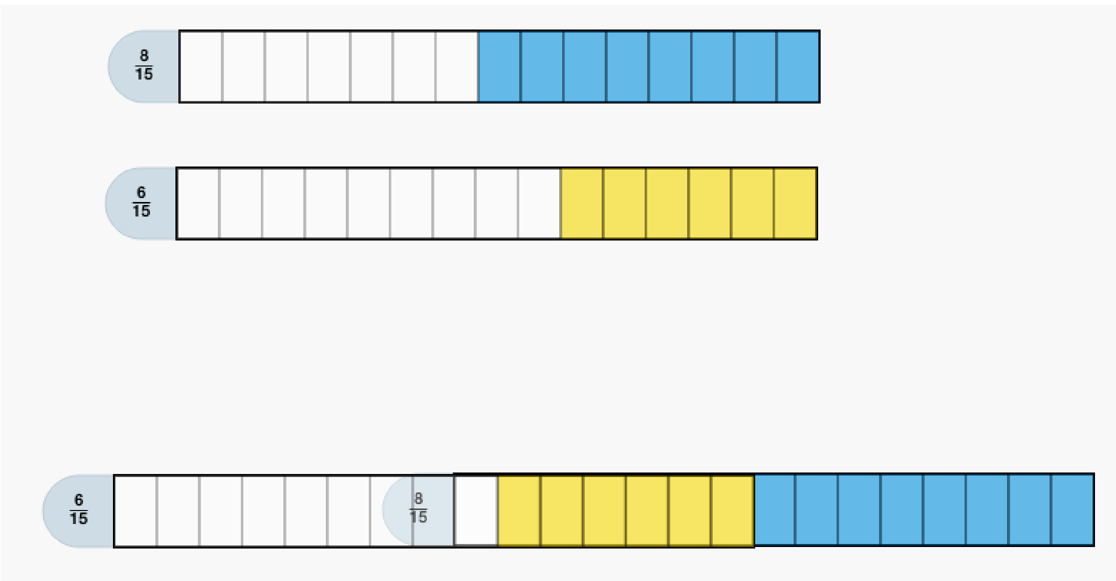
Janice and Sheila interpreted the complete whole feature negatively in Exploration, specifically in relation to how students might think about or represent addition. While attempting to solve the problem ⅔ + ⅓, Janice modeled each addend with a bar model (see Figure 16, left). Looking at her screen, she reflected, “I can see that this can cause students to think, ‘Oh, well this is now three sixths, right?’ … I wish there was a little one third part that I could snap into that blank [in the top fraction strip].” She continued, “it looks like addition and subtraction might be challenging.” To push her thinking, I prompted Janice to overlay the bottom strip onto the top strip to show the sum (see Figure 16, right) and asked for her thoughts. She commented, “that’s an interesting way to do it” but reiterated that she still thought students would question why they could suddenly consider two strips as one.

Figure 16: How Janice modeled the problem ⅔ + ⅓ (left) and how the sum appeared after I prompted her to overlap the strips (right)



As noted in the discussion of the single-color model feature, Sheila said she thought the MLC Fractions App was not well suited for addition and subtraction, mostly due to the inability to show two colors in one bar or circle. The complete whole feature also came into her thinking about this, though, because of the way she positioned the shading within her bar models. While Janice created bars with shading that would not overlap when they were laid atop each other (see Figure 16), Sheila modeled 6/15 and 8/15 with shading starting at the right side in both bars (see Figure 17, top). She considered overlaying the bars—unlike Janice, she did not seem bothered by the existence of unshaded parts that were no longer being considered—but the positioning of the shading meant that she couldn't directly place the strips on top of one another to show the sum. Instead, she tried to drag one strip to the side to align the shading (as shown at the bottom of Figure 17). When she tried this, Sheila commented that "all those extras" extending past the left side of the combined bar "would be confusing" for students. The "extras" are a consequence of the complete whole feature because it prevents a user from moving only the shaded parts of a model.

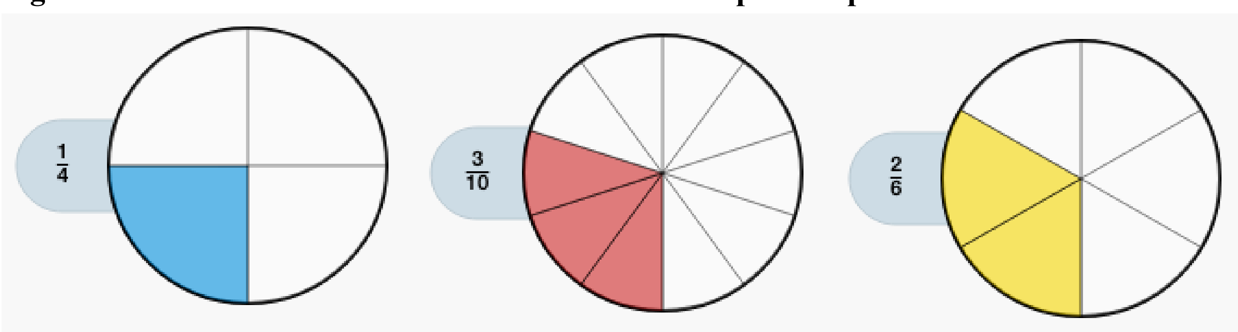
Figure 17: Sheila's models of 6/15 and 8/15 (top) and how she showed the sum (bottom)



Janice and Sheila repeated their negative interpretations of the complete whole feature during Problem Solving. Karla and Nancy, who had not mentioned the complete whole feature in Exploration, articulated negative interpretations for this feature during Problem Solving. Nancy's commentary was similar to Sheila's. She created a representation similar to the bottom of Figure 17 and said, "I guess I could do that. I don't love that, though." Janice, Sheila, and Nancy's negative interpretations of the complete whole feature in relation to addition seemed to relate to what these teachers might do (or have their students do) with physical manipulatives to represent addition: show the two addends separately and then put them together. The complete whole feature meant this existing practice was not easily replicable in the MLC Fractions App.

Karla's negative interpretation of the complete whole feature was different than that of the other teachers because it related to comparison rather than addition. Karla reflected that the inability to manipulate just one fraction piece, rather than having to drag around a full whole, would limit her own or students' abilities to make sense of fraction comparisons. After Karla ordered the fractions ¼, 3/10, and 2/6 by overlapping the circle models shown in Figure 18, I asked her to explain how she could tell the fractions were in the right order. Part of her reasoning was that the red 3/10 model was 2/10 from a half and the yellow 2/6 model was ⅙ from a half. Because 2/10 is equivalent to ⅕, and ⅕ is larger than ⅙, she knew 3/10 was less than 2/6.

Figure 18: Karla's models for the fractions in the comparison problem



Following this explanation, I asked if Karla thought she could use the VM to show why that strategy worked. She said, "You can't pull it apart, right? Yeah, it'd be nice if you could pull it apart and show that one sixth. And then show a fifth and how that was different sizes. But I can't pull that out." By "can't pull that out," she meant she was unable to pull one shaded piece out of any of the wholes. She continued, "I'd like to be able to take that one piece, if I was really comparing fractions, because that's the hard understanding. …. Because they have to think, at the same time, that a tenth is smaller than an eighth when comparing the fractions. That's a lot of thinking." Although Karla had shifted to talking about a different example, I believe in this passage she was making a point parallel to the idea that to use the strategy she described above, students have to reason that $\frac{1}{6}$ is smaller than $\frac{1}{5}$ while still keeping in mind that they are comparing 3/10 and 2/6. She thought having a clearer visual model for the former might help them coordinate their reasoning within the original problem. Karla did not relate her familiar fraction comparison strategies to the VM until I prompted her to do so. However, after the prompting, Karla seemed to interpret the complete whole feature negatively in relation to her existing strategies. Thus, although the mathematical topic was different, Karla's interpretation of the complete whole feature fit into the general pattern of negative interpretations in relation to familiar problem-solving strategies.

Fewer teachers attended to or interpreted the fixed denominator feature of the MLC Fractions App, but again the interpretations were neutral or negative. The fixed denominator feature refers to the fact that a user cannot change the number of parts in a model after it has been created. Some negative interpretations did not relate this feature to a particular type of mathematical problem or strategy. Rather, several teachers generally noted they did not like the inability to change the number of parts. For example, during Problem Solving, Colleen said she "wish[ed] it was easier" to change the number of parts in a model when she "messed it up." She found the fixed denominator feature inconvenient but did not say anything more about why.

However, other interpretations of this feature did fit the pattern of being negative because of how they inhibited familiar problem-solving strategies. For example, during Exploration, Janice said that she was not sure about how she would use the MLC Fractions App to teach a lesson on equivalent fractions:

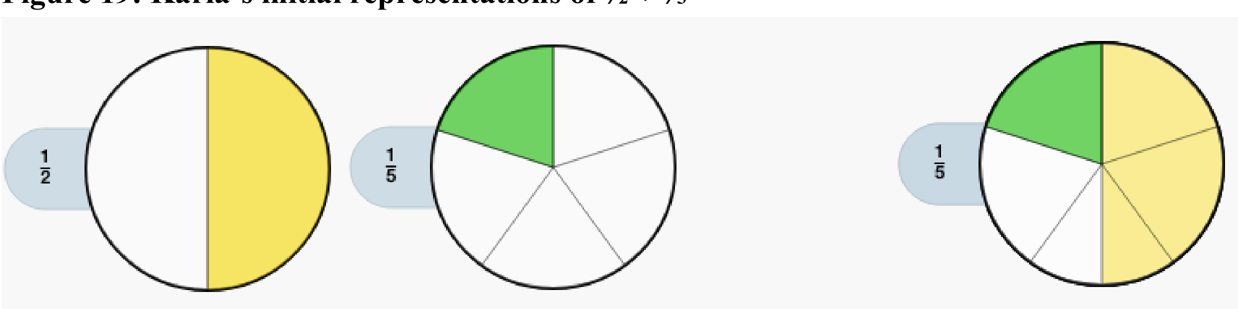
> Let's think, equivalency. Right. When comparing equivalency I would have to already know I would have to think of sixths to use. Man, I can't even wrap my head around how

I would instruct students using this to compare, or to look at equivalent fractions. Or make me an equivalent fraction.

As she said this, Janice was examining a bar model divided in thirds. She was reflecting that the thirds strip could not be changed to be sixths, so students would have to already know to use sixths to create an equivalent fraction. One interpretation of this comment is that Janice saw the fixed denominator feature as in conflict with a strategy she had used in the past for showing equivalent fractions, which was to divide each part of a model into two parts (e.g., divide each third into two sixths). This interpretation is supported by the fact that Janice demonstrated this strategy for generating equivalent fractions elsewhere in the interview.

Karla's interpretation of the fixed denominator feature in Problem Solving provides a final example of how teachers made negative interpretations of features reflecting focused constraints in relation to familiar problem-solving practices. As Karla solved ½ + ⅕, she first represented the two addends with separate circle models (see Figure 19, left). She briefly overlapped the circles (see Figure 19, right) but quickly said, "That doesn't work," and dragged the circles back apart.

Figure 19: Karla's initial representations of ½ + ⅕



After a moment Karla continued:

See, now I can't change this to tenths. Do you know what I mean? Like this is hard to solve with this one, I think. Unless I said okay, I have to get my common denominator first. And that's how I'd have to do it. So I'd have to immediately start with the tenths. And then I'd have to trash these.

Karla continued by deleting her half and fifth circles, creating two tenths circles showing 5/10 and 2/10, and then overlaying them to show 7/10 in one circle. Her interpretation of the fixed denominator feature was that students could not start to solve this addition problem by creating models for the addends (something she'd presumably have them do with other tools). She believed they'd have to find equivalent fractions with a like denominator first, then represent and combine those.

There were far fewer mentions of the remaining three features in Table 17 (bar width, part choices, and denominator first), but the few interpretations of these features reflecting focused constraints mostly followed the same neutral-or-negative pattern. There were two positive interpretations of the denominator-first feature (Janice thought the feature would provide useful feedback to students, and Karla though it might support students in thinking about fraction equivalence). However, these two positive interpretations were rare exceptions to a pattern of frustration and negative evaluations of features reflecting focused constraints in Exploration and Problem Solving.

Finding 2e: In Lesson Planning, teachers' interpretations of features reflecting focused constraints highlighted both pros and cons of the constraints for student learning.

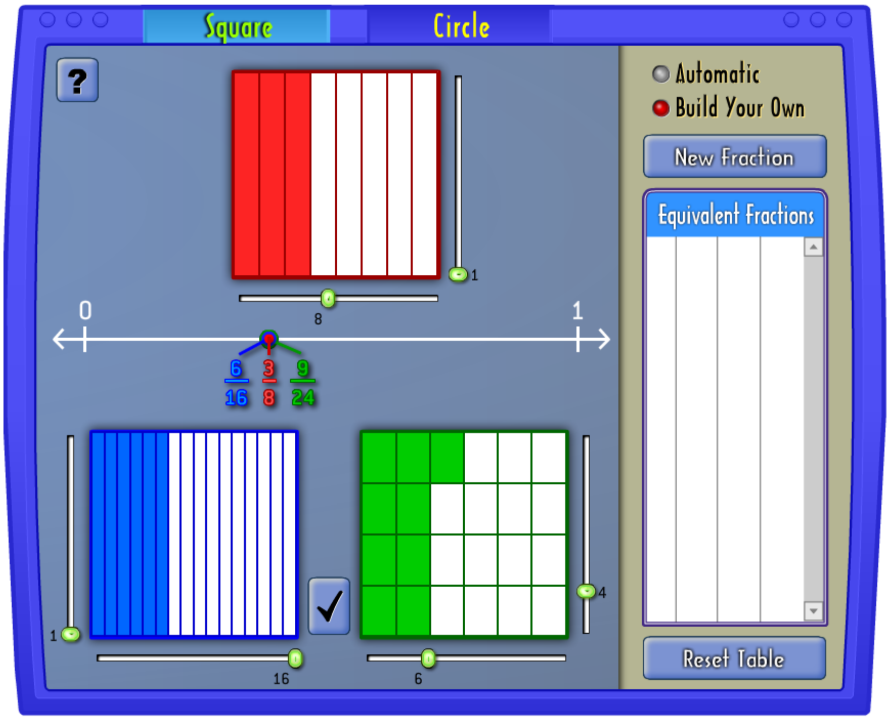
As shown in Table 18, all the VMs teachers used for Lesson Planning had a least one feature coded as reflecting focused constraints. All six teachers mentioned a potential positive effect of at least one feature reflecting focused constraints in Lesson Planning. However, they also discussed negative interpretations of either the same feature or a different feature reflecting focused constraints. This suggests the Lesson Planning context does not fundamentally change teachers' interpretations of features reflecting focused constraints, but does support them in making more balanced evaluations of such features. I describe one positive and one negative interpretation from each teacher, beginning with the three teachers who interpreted one feature positively and a different feature negatively (Colleen, Sheila, and Tammy) and then proceeding to the three teachers who offered a positive and a negative interpretation of a single feature (Janice, Karla, and Nancy).

As she discussed her lesson on equivalent fractions, Colleen described a way one subfeature of the NCTM Equivalent Fractions App reflecting focused constraints, the slider limits, might stimulate student thinking. The sliders that control the number of partitions in the area models have maximum values (or limits)—24 for the slider around the circle models, and 16 for the sliders on each side of the squares. When discussing what she thought students would notice about the VM during her lesson plan interview, Colleen said, "I think the square, you can get a lot larger of a denominator. Because this [circle], you can only go to 24. And this [square], you can go to... 16 [per side, so], 256. So they'll probably notice that." Thus, Colleen seemed to think the differing slider limits for the circle versus square area models in the NCTM Fractions App might spur student thinking about similarities and differences in the model shapes and how they might be used.

Colleen's interpretation of a different subfeature of the NCTM Equivalent Fractions App reflecting focused constraints, called "no drag" in Table 18, was negative in relation to familiar problem-solving strategies, like the majority of interpretations of features reflecting focused constraints in Exploration and Problem Solving. The VM's area models are fixed in place and cannot be dragged closer to each or on top of each other. Colleen mentioned this subfeature in her student response interview as she discussed responding to a student scenario (shown in

Figure 20). In this scenario, a student thought the fractions shown in the blue and green models could not be equivalent because the blue model's shading was in even columns but the green model's shading was not.

Figure 20: A student scenario presented to Colleen



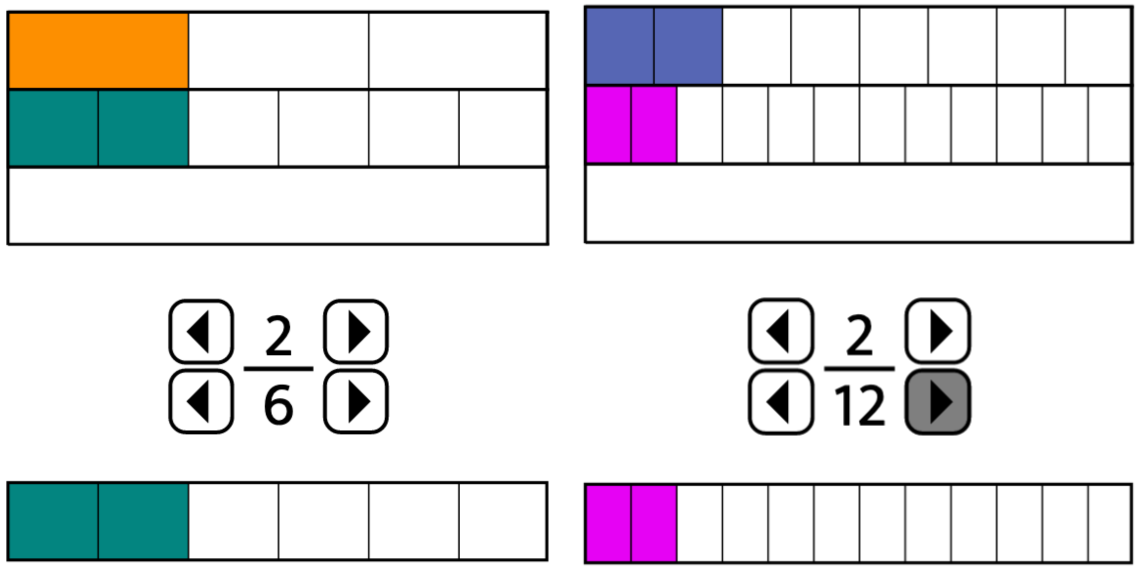
I (as the interviewer) argued that such a student might think it was appropriate to remove the top right piece of green shading and think this showed that 6/16 was equivalent to 8/24. At this point, Colleen said, "I wish that you could drag it or have it right on top of one another or something because... I'm like, 'Oh, *is* that the same?' So I definitely think that kids would think that they're the same if it was set up like this." She thought a potential solution would be for students to overlay the two models but reflected that such an action was not possible in this VM. She also mentioned elsewhere in the interview how in the past she asked students to overlay paper models of fractions to support them in comparing fractions. Colleen's interpretation of the no drag subfeature therefore seemed to fit the pattern of negative interpretations of features that interfere with using familiar problem-solving strategies.

Sheila discussed two features of the Toy Theater Fraction Bars reflecting focused constraints in Lesson Planning, and like Colleen, she evaluated one positively and one negatively. The disappearing bar feature refers to the fact that if a user places a bar into a slot that already contains a bar, the bar underneath disappears—it is not there even if a user removes the bar she placed on top of it. Sheila evaluated the disappearing bar feature negatively, explaining that it caused her to accidentally delete a bar she did not mean to delete. She also commented that the feature meant it was difficult to rearrange the order of the bars in the slots—a reference to being unable to use a familiar problem-solving strategy.

By contrast, she positively evaluated the numerator and denominator limits, which prevent a user from creating a representation of a fraction greater than 1 or of a fraction with a denominator greater than 12. Sheila mentioned this feature as part of her response to one of the student scenarios. In this scenario, a student had been attempting to create equivalent fractions by making the pieces of the bar smaller (i.e., increasing the denominator) until two pieces aligned with one piece of the fraction they had started with. For example, when creating an equivalent fraction for ⅓, the student noticed that two sixths pieces fit into one third piece, as shown at the left a Figure 21. Then the student attempted to use the same strategy to generate an equivalent fraction for 2/8 but claimed to be unable to generate an equivalent fraction because they could not make the pieces small enough on the VM so that two fit in the eighth—12 pieces is the most allowed by the VM. (See Figure 21, right.)

Sheila's response was to direct students' attention to anywhere the partitions in the two bars lined up. However, she mentioned the numerator and denominator limits as contributing to this scenario occurring: "So [the student is] using the doubling strategy but it's limited by what this tool comes up with."

Figure 21: A student's strategy works for finding an equivalent fraction for ⅓, but not for finding an equivalent fraction for 2/8 due to the denominator limit



Rather than seeing this as a problematic issue with the tool, however, Sheila actually appreciated the limitations as a tool for focusing student thinking. After directing a student toward noticing that the right edge of the 2/8 shading aligned with one of the twelfths partitions (see Figure 21, right), Sheila suggested she might have a student "work backward" to see if they could find a fraction with a lower denominator that was equivalent to 2/8. When asked if anything about the VM would support students in using this strategy, Sheila said,

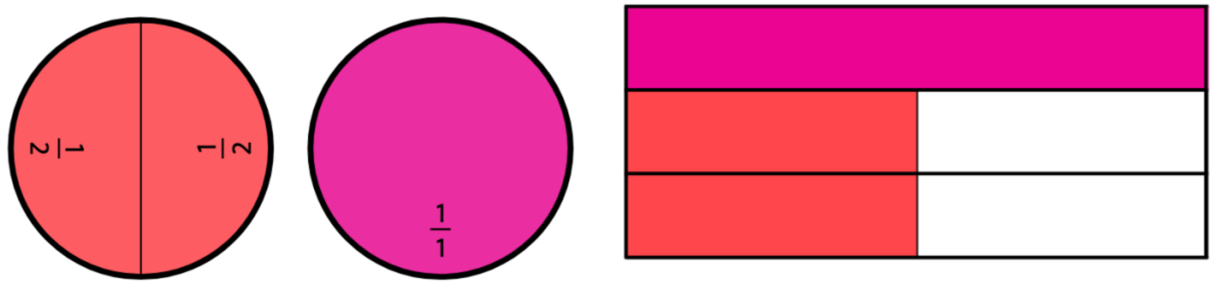
> Definitely. I mean with this model you're restricted by the numbers that are given there. So, you can't work back to zero, or I think it takes a little bit of trying to access what fractions are and what they mean out of it. Which, we need them to do that at some point, but this just takes it into click and figure it out and then analyze or look at that model.

Sheila felt the limits on the numerator and denominator could focus student attention on equivalence, rather than raising issues about what "counts" as a fraction or other basic conceptual ideas about fractions.

As noted in Table 14, Tammy intended to make both the Toy Theater Fraction Bars and the Toy Theater Fraction Circles available to her students during her planned lesson. In one student scenario presented to Tammy, a student used the Toy Theater Fraction Circles to offer $\frac{1}{3}$ and $\frac{1}{6}$ together as an alternative way of showing $\frac{1}{2}$ (see Figure 11 above). Tammy responded by encouraging the student to look for other combinations that would work. She also said she might extend that task, for students who are ready, to finding combinations that sum to one whole. She said, "It might be interesting to see their rationale and their thinking and watching it go, 'Poof! That doesn't work. All right. Let's just play around until I can find something that does work.'" When she said "poof," Tammy was referring to the fact that in Toy Theater Fraction Circles, when a piece larger than the available space in a circle are dragged into that circle, the piece disappears. This is called the disappearing piece feature in Table 18. Tammy's comment suggests she thought the disappearing piece feature may act as useful feedback for students as they explored different ways to make one whole.

On the other hand, Tammy wondered if a different feature reflecting focused constraints would make the Toy Theater Fraction Bars difficult for students to interpret. This feature, called complete whole, refers to the fact that a user can only drag and manipulate a complete fraction strip. There is no way to manipulate just a piece representing a fraction without that fraction being embedded in a whole strip. Tammy mentioned this feature only once, in her lesson plan interview, as she described the ways she would introduce her two chosen VMs to students. She began by showing 1 whole and 2 halves side by side in the fraction circles (see Figure 22, left), then started to show the same thing on the fraction bars. She dragged a bar for 1 whole into the top slot, then a strip for 1 half into the second slot, and finally dragged another 1 half strip into the last slot (see Figure 22, right).

Figure 22: Tammy illustrates equivalence between 1 whole and 2 halves on her two chosen VMs

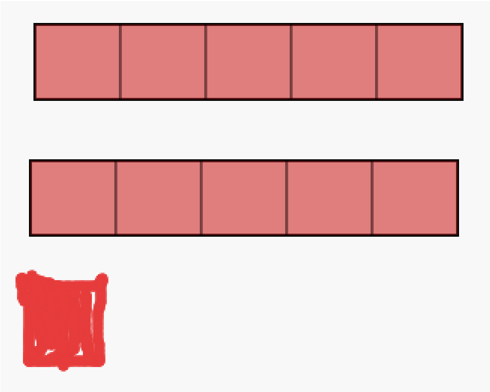


Before dropping the last strip into the third slot, she briefly attempted to place the shaded portion of the strip on the right side of the second slot. She commented, “the only thing that I'm not sure about that is if they’ll be able to see that both of these *(gesturing to the two red halves),* equal these *(gesturing to the two parts of the middle strip)*, but I’m sure that my other kids that are more advanced are going to be able to see, it’s cut down the middle, if I put this piece right here, I should be able to do that.” Tammy seemed to think any difficulties posed by the complete whole feature could be overcome, especially by students she deemed “more advanced,” but her overall interpretation of the feature appeared to be negative.

Rather than interpreting some features reflecting focused constraint positively and others negatively, the remaining three teachers (Janice, Karla, and Nancy) expressed both positive and negative interpretations of the same feature during Lesson Planning. For example, Janice mentioned both positive and negative potential implications of the complete whole feature of the MLC Fractions App for student thinking about fractions greater than 1 (which was the focus of her lesson). On the positive side, Janice identified a way in which the complete whole feature might impact students’ strategies for creating representations of fractions greater than one (see Figure 23 for what Janice is describing):

I think if students were to free draw, then creating the next fifth, they could have drawn a little box, a part of a fifth instead of the whole fraction [bar]. ... And it would've been harder to see the relationship of the part to the whole.

Figure 23: Janice's expectation for how a student might represent 11 fifths without the complete whole feature



In contrast to the mostly negative interpretations of the complete whole feature articulated in Exploration and Problem Solving (some made by Janice herself), here Janice saw a potential benefit of the complete whole feature for supporting student understanding of the relationship between the part and the whole. Even so, Janice also mentioned one way the complete whole feature could impact the way students make sense of representations of fractions greater than 1, such as the one in Figure 12:

I've noticed that some of my students, especially on a problem with that little fraction piece at the bottom, they count that as a whole. They would see four parts making up that one, not [only] the parts shaded in. So I can see them answering .... I think they would say something like five.

Karla expressed both positive and negative potential effects of the disappearing piece feature of the Toy Theater Fraction Circles. Karla attended to this feature in her lesson planning interview when she mimicked how a student might encounter the feature as they tried to place one ½ piece and two ⅓ pieces into a circle:

> If you go and say, okay, Queen Arlene [is] giving ½ to the one daughter, and then there's three daughters so I'm going to give ⅓ to the other one and ⅓ to them … It's not going to work. Then they've got to think about it again.

The main task of Karla's lesson was for students to explain why a queen could not distribute her kingdom to her three daughters such that the oldest got ½ and the two others each got ⅓. Karla thought the disappearing piece feature would act as feedback telling students the proposed distribution of the kingdom is "not going to work"—a potentially positive effect. However, Karla went on to explain that seeing how the second ⅓ piece overlaps with the other pieces when it is placed in a whole—something the disappearing piece feature makes impossible—would have been better feedback for students. This suggests a potential negative effect of the feature on student thinking. Karla was not convinced all students would interpret the disappearing piece feature productively.

The feature Nancy saw with both positive and negative potential effects on student thinking was the specific piece choices available in the Phoenix College Fraction Circles. The available piece sizes constrain students' abilities to test out certain conjectures they may make about fraction relationships. For example, they cannot use the VM to test whether 3/9 is equivalent to ⅓, because there is no piece available to represent 1/9 (if the full circle is considered the whole). Nancy mentioned this constraint on what students could do with the VM as one of the reasons she chose the Phoenix College Fraction Circles for her lesson:

> I do think you're somewhat limited by the fraction circle pieces that are available. And I did consider using the one that allows you to kind of make up your own denominators as far as you want. But then I felt like ... this was an early exposure to equivalent fractions

> and because this was just the explore-and-play [phase of learning about equivalent fractions], I didn't feel like it was important to be able to play endlessly.

Nancy intentionally chose a VM that constrained the possible denominators for students' initial explorations, suggesting she thought the constraint was productive for this lesson.

Later in her lesson planning interview, she also mentioned that the constraint on the piece choices allowed her to challenge students to figure out whether they had found all the fractions equivalent to a given fraction that are possible with the tool. She explained, "That's one of the reasons why I'm okay with this being limited, because [when I ask them how they know they have generated all the possible equivalent fractions] I would want them to say something like, 'Well, I tried all of the pieces or I tried all of the pieces that were factors of whatever.'" She also noted, however, a potential negative effect of using the constraint for that purpose: Students might not realize that the constraint had to do with the tool, not with the concept of equivalent fractions. She reflected that she would need to attend to this potential negative consequence at the end of her lesson: "I would add an additional question obviously saying like, are you ever done finding equivalent fractions? I would want them to realize that this is limited because of the activity, but that equivalent fractions are endless."

In summary, all six teachers offered positive and negative interpretations of features reflecting focused constraints in Lesson Planning. However, some thought about pros of some features and cons of others, and some thought about pros and cons of the same feature. This analysis raises two questions. First, why did teachers begin noticing positive implications for these features in Lesson Planning, when they did not do so in the earlier contexts? Because teachers' interpretations became more consistently negative in Problem Solving, it does not seem that consideration of a specific task is what prompted teachers to consider positive effects. Two additional aspects of the Lesson Planning context could instead explain teachers' more balanced interpretations of features reflecting focused constraints. One aspect is that Lesson Planning, especially since it was guided by prompts the student-focused prompts in the lesson planning template (see Appendix E), asked teachers to consider more than one approach to or way of thinking about a task. This is in contrast to Problem Solving, where only one strategy was required. A particular feature may lead to a positive effect if a student uses one strategy and a negative effect if a different student uses another strategy. This issue may explain Janice's multiple interpretations of the complete whole feature of the MLC Fractions App in Lesson Planning. Janice worried that the complete whole feature might lead some students to misinterpret representations of fractions greater than 1 (considering a partially shaded strip in the same way as a fully shaded strip), but she also said seeing the partially shaded strips might help some students keep the whole in mind. Thus, Janice seemed to be considering multiple ways students could be thinking about the strips.

A second aspect of Lesson Planning that may have contributed to spurring more positive interpretations of features reflecting focused constraints is that in this context, the teachers selected both the task students would be working on and the VM they would use to do so. Thus, they had much greater flexibility in matching aspects of the task to VM features that might be helpful for the task. There is some evidence of teachers actively coordinating features reflecting focused constraints with their tasks. For example, Nancy said she chose the Phoenix College Fraction Circles *because* of the limited piece choices, which is evidence of coordination of the VM features with the tasks she chose. Similarly, Sheila expressed some annoyance and frustration about the disappearing bar feature of the Toy Theater Fraction Bars, but she also discussed many other features she liked about the VM for supporting students in completing her planned tasks for the lesson. Thus, she may have considered the way the disappearing bar feature could interact with her task, but chose the VM anyway, *in spite of* the potential conflict, because she thought so many other features of the VM would support student thinking.

The second question raised by this analysis is why some teachers consider both positive and negative potential effects of a single feature, while the other teachers did not. Interestingly, the three teachers who provided positive and negative interpretations of a single feature—Nancy, Janice, and Karla—are the same teachers who engaged in more experimentation during Exploration than the other teachers. As noted in Chapter 3 (Results, Attention to VM Features), these teachers were those who both scored highly on the mathematics content knowledge measure and also showed a stronger tendency toward discovery-oriented teaching on the mathematics teaching beliefs survey. It seems that this combination of teacher traits may support not only more experimentation as teachers explore VMs, but also more thorough consideration of the implications of particular VM features on student thinking and learning. Such a pattern would make sense, as deeper pedagogical content knowledge would correspondingly strengthen teachers' abilities to imagine multiple scenarios for how a student might interpret a VM, and a discovery orientation may increase their proclivity toward imagining such scenarios.

Interpretations of Features Reflecting Linked Representations

Two fraction representations are linked within a VM when changes in one representation are automatically reflected in the other. These representations could be visual (e.g., an area model or fraction strip), symbolic (i.e., a numerator and denominator), a number line, or verbal. I coded a VM feature as reflecting linked representations when it referred to such a link between two representations. The link had to be dynamic so that a change in one representation was observable when a user made a change in the other. There were fewer features reflecting linked representations in the VMs used in this study than features reflecting the other two affordances. Indeed, two of the VMs used by teachers in Lesson Planning had no features reflecting linked representations. Because there were so few of them across the VMs (6 in total), the features coded as reflecting linked representations are all listed in one table (Table 20).

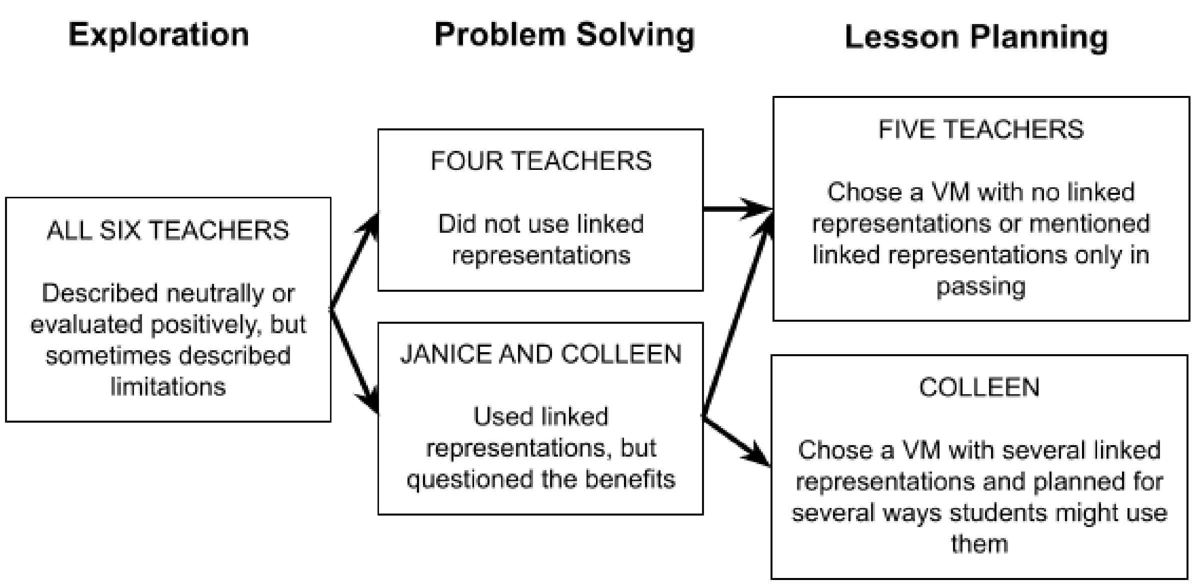
Table 20: Features of VMs reflecting linked representations

VM	Feature	Feature Category	Feature Description	Rationale for Coding as Linked Representations
MLC Fractions App	Linked symbols	D	When labels are turned on, if a user changes the fill in a model, the symbolic label's numerator changes to match it.	User can view the symbolic label change in response to changing the area model.
	Denominator choice partition link	SF	When a user tries several denominator choices before clicking "OK," they can see the position of the model's partitions change.	User can view the partition spacing change in response to changing the number of parts.
Toy Theater Fraction Bars	Symbol bar link	D	As the numerator or denominator change, the partitions and shading in the bar model change.	User can view the fraction bars change in response to changing the symbolic fraction.
NCTM Equivalent Fractions App	Moving dots	AP	When changes are made to the area models, the dots on the number line automatically move.	User can view the dot positions change in response to changing the area models.
	Dynamic labels	AP	When changes are made to the area models, the number line labels dynamically update.	User can view the symbolic labels change in response to changing the area models.
	Equivalent fractions table	V	The VM has a table of equivalent fractions that is populated as students press the feedback button when they have a set of three equivalent fractions.	User can view corresponding symbolic fractions appear when using the feedback button to check the area models.

Feature category key: V = Visible, D = Discoverable, AP = Automated process, SF = subfeature

A summary of teachers' interpretations of features reflecting linked representations is as follows. Although all six teachers noticed a feature reflecting linked representations in Exploration, they did not show much interest in that feature. When they first noticed it, they seemed to like it, but several went on to note a way it was limited. Only Janice and Colleen attended to a feature reflecting linked representations in Problem Solving, which is further evidence of teachers' overall lack of interest in the features. In Lesson Planning, two teachers selected VMs that did not have any features reflecting linked representations, and three chose a VM with at least one linked representation but seldom mentioned these features. Colleen, however, chose a VM with several linked representations and planned for several ways students might use these features. It is not clear what led Colleen to choose a VM with linked representations when the other teachers were not interested in them. This finding (Finding 2f) is summarized in Figure 24 and elaborated below.

Figure 24: Summary of teacher interpretations of features reflecting linked representations



Finding 2f: Teachers had little to say about linked representations in Exploration or Problem Solving, and only one teacher made direct use of them in Lesson Planning.

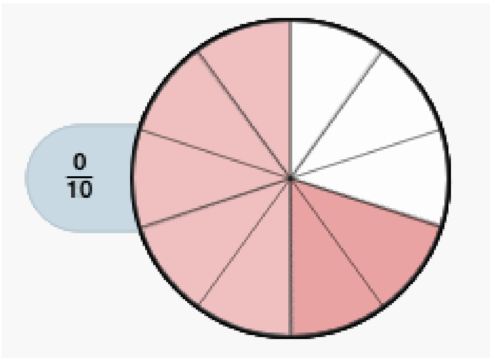
The MLC Fractions App has two features reflecting linked representations. First, when users have the labels turned on, they can observe the numerator in the label changing as they adjust the number of parts that are shaded—a feature called "linked symbols" in Table 20. All six teachers noticed this feature as they explored the VM. Five of them evaluated this link positively, saying it was "helpful," "nice," or "important for kids to see." Sheila specifically pointed out the utility of the label when working with a large denominator (and therefore small pieces): "And it's nice to have the label on the side so I know exactly how many hundredths." The sixth teacher, Tammy, described the way the label changed with the shading but did not say anything further about it.

Alongside these general positive evaluations, four teachers (Janice, Karla, Nancy, and Sheila) saw a limitation in the linked symbols feature. Specifically, they noted that students could not use the linked symbols to observe what happens when the denominator changes—only what happens when the numerator changes—because of the fixed denominator constraint discussed above. Janice pointed out, "I can't change the denominators, right?" and experimented to see if drawing in further divisions with the pencil tool would change the linked symbol's denominator. (It does not.) Nancy and Karla both compared the linked label feature of the MLC Fractions App to a linked label in another VM explored in the professional learning sessions, where a user could change the denominator. Nancy said this difference meant the MLC Fractions App "didn't seem as effective with the labeling." Karla felt similarly, saying about the ability to change the denominator and observe a change in the model: "I liked that part of the other manipulative."

The MLC Fractions App has only one other feature reflecting linked representations, called “denominator choice partition link” in Table 20. As several teachers noted, the link between the bar and circle models and the symbolic labels mostly does not allow for users to observe changes in the number of pieces or denominator of the fraction. However, if users click on several different numbers in the denominator keypad before pressing OK, they can observe how the size of the parts in a strip or circle change according to the number of parts. If the labels are turned on, users can also watch the denominator change as they choose different numbers of parts. Although four teachers showed evidence of attending to this link during Exploration, they did not comment on it. Sheila offered a verbal description, speaking about this feature after being prompted to look for linked representations in the VM: “So I definitely saw that when I’m creating the bar. That automatically changes the denominator.” Overall, though, teachers did not seem to think much about this feature, perhaps because it is only observable while initially creating a model. No one attended to it in Problem Solving.

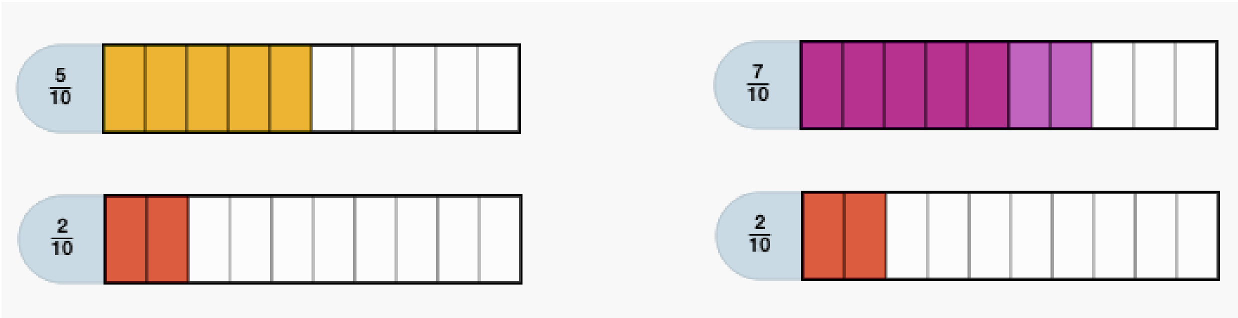
The linked symbols feature, on the other hand, was used by two teachers (Janice and Colleen) during Problem Solving. Janice represented the sum ½ + ⅕ by overlapping models for the addends and placing a blank tenths model on top. Her final model showed 0/10 on the label (see Figure 25). When asked what features of the VM helped her solve the problem, she said, “Not helpful is that these little numbers don’t change. The linked representation does not change once I snap them into place.” She was referring to how the label shows the fraction for only the top circle, and does not update to show the fraction represented by the overlapping models together. Not only did the labels not update when users partitioned a model with the pencil, as she noted during Exploration, it also did not update when users overlapped models. Janice did not see this limitation as entirely negative, however: “But that gives kids the extra ... they need to push themselves to think about how many parts are shaded for the tenths. So perhaps a good not helpfulness."

Figure 25: Janice's model of the sum ½ + ⅕



Colleen also had the linked symbols showing as she solved the addition problem, but in contrast to Janice, she used them in such a way that the labels did show symbolic fractions that matched the visual models. To find an equivalent fraction using tenths for each addend, she overlaid a tenths bar on each of the bars she used to represent the addends. She then reshaded the tenths that covered the ½ and the ⅕, saying, "I might use a different color just to color it in and to tell me how many tenths it is." (See Figure 26, left). The last part of her comment referred to the linked label that counted her tenths for her. When she added two additional tenths to her model of five tenths to represent the sum (see Figure 26, top right), she said, "Then that would equal seven tenths, and it tells you right there." To elicit more of her thinking, I asked, "Is that good or bad?" She answered, "I don't know. I guess it would be good once they understand fractions, but it might be bad because then they don't necessarily ever need to figure it out on their own." Thus, even though Colleen chose to use the linked label as an aid as she created equivalent fractions and found the sum, she was not sure if she felt this feature was a benefit instructionally.

Figure 26: Colleen's model of the addends for ½ + ⅕ (left) and of the sum (top right)



In summary, all six teachers noticed the linked symbols feature in Exploration, mostly giving it a general positive evaluation but sometimes also noting its limitations. Only two used the linked symbols in Problem Solving, and both seemed unsure of its instructional benefit, either because of a limitation or because it may take thinking opportunities away from students.

The same general lack of attention to or enthusiasm for features reflecting linked representations continued into Lesson Planning for five of the six teachers. Two teachers, Karla and Nancy, chose a VM without features reflecting linked representations, suggesting they did not value this class of affordances as they selected a VM and planned their lessons. Three other teachers—Janice, Sheila, and Tammy—did choose a VM with at least one feature reflecting linked representations, but it is unclear whether the presence of linked representations played into their choice. Janice used the MLC Fractions App, but only once during Lesson Planning did she mention the possibility of students turning on the labels. Specifically, when she considered that students may try to create a representation of 5/15 rather than 15/5, she said she might suggest students turn the labels on to see their mistake.

Sheila and Tammy both chose the Toy Theater Fraction Bars, which has one feature reflecting linked representations: As students use the arrow buttons on either side of the numerator and denominator of a symbolic fraction (see Figure 21 above), they can see the fraction bar below change to match. Sheila mentioned this link in her lesson planning interview as one of the reasons she chose to use this VM:

> Some of [the other VMs] had the fractions labeled and some of them didn't. And with this one, you can see, this is 2/5, and now I'm moving 2/5. So even though it doesn't stay labeled, I'm still seeing that fraction together. Whereas the ones where you build and put the pieces in, you're not seeing 2/5. You see 1/5 plus 1/5 plus... You know what I'm saying? So I felt like that was why I liked this one, even though it doesn't stay labeled.

Sheila seemed to appreciate, in particular, that the linked symbolic representation allowed for the "label" to show non-unit fractions, and so students did not have to think of non-unit fractions as being composed of unit fractions. She did note, however, that once students drag the fraction bar away from the bottom of the screen and into a stacked slot, the link between the symbolic fraction and the bar is lost. Overall, Sheila seemed to appreciate the linked representation, but felt its temporary nature limited its utility. She did not say anything else about it.

Tammy did not attend to the linked representations in the Toy Theater Fraction Bars in her lesson plan interview but did discuss it as part of her student response interview. She explained that in the past, she noticed the ways her students were highly dependent on referencing colors of manipulatives as they described their work with fractions, and this concerned her since students would not have colorful manipulatives to work with when they took standardized assessments. She wanted to make sure students could make their own drawings for fractions (eventually) and connect visual representations to symbolic fractions. She thought the linked symbols in the Toy Theater Fraction Bars might help with that issue. However, this comment seemed to be made in passing, and she did not discuss students connecting symbolic representations to visual representations as a goal of her lesson.

In contrast to her peers, Colleen chose a VM, the NCTM Equivalent Fractions App, that had three features reflecting linked representations: moving dots, dynamic labels, and the equivalent fractions table. During Lesson Planning, Colleen seemed to have thoughtfully considered different ways students might use these features and cited them as part of the reason she chose the VM. The moving dots feature refers to the way the position of the dots on the number line changes as the amount of shading in the area models changes. Colleen attended to this feature several times during her lesson plan interview and shared several different ways of thinking about it. She described the moving dots as one of the reasons she chose this VM: "I love that it moves. So it moves the fraction as they go and click through…. I like that, as they click through, the dot moves and shows the fraction." She did not view students' attention to or understanding of this feature as a major goal of her lesson, however. She described "being able to see that as the fraction grows a bit larger, it goes further towards the right of this number line" as a "side goal" of the lesson.

She further pointed out a potential downside of the moving dots. When asked if there was anything she did not like about the VM, Colleen said,

> I go back and forth with what I do and don't like about it. I don't like that it moves. So the kids could just click through and not really think about it until... They would be like, "Oh, that's too far. Let me try a different number." And then, go through and see where... "Oh, okay. Yeah. That matches up." I think that it could take just low-level thinking for some of the kids that would just click through. But then, at the same time, I'm like, "Is that so bad? It gives the kids somewhere to start." And instead of the kids that would typically just sit there and be like, "I don't know how to do it," maybe they would just click through. And then, say, "Oh! Wait a minute. I've just been clicking through. But now, I

> notice a pattern of how I can get to the equivalent fraction." So I think that that could be a positive or a negative.

Colleen seemed to recognize that students may (and likely would) use the moving dots on the number line as a check on whether the fractions they represented with the area models were equivalent; when the dots on the number line coincide, the fractions are equivalent. She reflected that they could depend on this feature and not pay attention to the area models or other ways of judging equivalence. However, she also acknowledged this kind of strategy could provide an entry point to the task of finding equivalent fractions for some students.

The second feature of the NCTM Equivalent Fractions App reflecting linked representations, the dynamic labels, refers to the connection between the area models and the symbolic labels on the number line. The dots themselves, as well as the labels, are color coded to the models, so a user can look at the number line labels to see a symbolic representation for the fraction shown in each area model (without necessarily attending to the position of the dot on the number line). Colleen mentioned the dynamic labels when first noting that she liked the moving dots: "I like that, as they click through, the dot moves and the fraction. It shows them the number [on the label], it shows them the point, moving along the number line." The way she mentioned how the multiple representations (point on the number line, label, area model) work together suggests she valued having multiple linked representations available for student inspection.

This reading of Colleen's comments about the dynamic labels is reinforced by the way she talked about the equivalent fractions table. This feature was also coded as reflecting a linked representation because the fractions entered in the table match those represented by the area models at the time the feedback button is pressed—although the link is lost when the area models are manipulated again. Colleen mentioned this table as a reason she chose this VM. She liked the way it kept a record of students' work with the visual models for reference later:

> I like that you can get it up in the equivalent fractions table. So then, they can see what they did and look back. And I think I would use that, too, as, "Okay. We see the visuals. Now let's look at the table and connect to the rule for how to multiply and get equivalent fractions." So I think it puts everything in one place for them to look at.

Overall, Colleen had both positive and negative interpretations of the linked representations in the NCTM Equivalent Fractions App. She saw potential in the linked representations for supporting students in making connections between area models, number lines, and symbolic fractions. She also acknowledged that the moving dots could be used as a way to generate equivalent fractions that used only "low-level thinking." As discussed further in Chapter 5 (Results, Responses to Affordances), she planned to avoid these outcomes by asking students questions to direct their thinking toward conceptual ideas about fraction equivalence.

It is unclear from my data why Colleen thought so much more about linked representations than the other five teachers. Her scores on the math content knowledge and teaching belief surveys do not distinguish her from all five of the others. Nothing in her introductory interview suggested she had different beliefs or more enthusiasm or experience with technology than the others. The specific goals of her lesson—helping students see patterns in equivalent fractions and understand how fractions with different numerators and denominators can have the same value—were very similar to those for Nancy and Sheila's lessons. The only potential explanation I found in the available data was that Colleen mentioned she chose her VM in part because it was so different from other tools and representations her students had used in the past. The linked representations may have been part of the novelty that drew her to the VM.

CHAPTER 5: RESULTS, RESPONSES TO AFFORDANCES

In this chapter, I discuss results related to Research Question 3:

> How, if at all, do teachers relate features reflecting the affordances of efficient precision, focused constraints, and linked representations to the tasks of doing and teaching fraction equivalence, comparison, and addition and subtraction?

The analysis for this research question, like the analysis for Research Question 2, focused on teacher thinking about a subset of VM features—those that I coded as reflecting the affordances. However, there is a shift in analytic focus from Research Question 2 to Research Question 3. While Research Question 2 focused on the second element of the professional noticing framework, *interpreting*, the emphasis in Research Question 3 is on the third and final element of the professional noticing framework, *responding.* Thus, Research Question 3 shifts from examining *what teachers had to say about features* to examining *how they used them to do or teach mathematics.* To answer Research Question 3, I used teachers' actions with the VMs, along with their verbalizations about what they were doing, to make sense of how they used the features to solve problems (mostly during Problem Solving, but occasionally during Exploration) and in their plans to support students in solving problems (during Lesson Planning).

The analysis also engages in a more detailed way with the three mathematical topics of (1) fraction comparison, (2) fraction equivalence, and (3) fraction addition and subtraction. As a preface, here I provide a brief description of how I see these three topics as distinct from, but related to, each other. *Fraction comparison* refers to the task of deciding whether one fraction is more than, less than, or equal to another fraction. Making fraction comparisons involves making judgements about the magnitudes of fractions in relation to each other. Because comparisons can result in a judgement of two fractions being equal, fraction comparison is related to the second topic, fraction equivalence. However, *fraction equivalence* refers to more than judging two fractions to be equal. In my view, understanding of fraction equivalence also involves realizing that two fractions expressed with different numerals, such as ⅔ and 4/6, can refer to the same value. Rather than making a judgement about how two fraction magnitudes relate, much of the work students do with fraction equivalence involves generating a fraction equivalent to a given fraction and understanding how and why the methods for that generation work.

I view *fraction addition and subtraction* as a single topic, rather than two, because contextualized situations involving, for example, putting together or taking apart are conceptually related and can support additive or subtractive thinking depending on what is known or unknown and how the problem is approached (Carpenter, Fennema, Franke, Levi, & Empson, 2000). Because no teachers planned a lesson on fraction subtraction, and I also did not ask teachers to solve a subtraction problem during Problem Solving, the relationship (or distinction) between addition and subtraction did not have an impact on the analyses for this study. Fraction addition and subtraction is related to fraction equivalence because adding or subtracting fractions with unlike denominators often involves generating equivalent fractions with common denominators to determine an exact numerical sum or difference.

This chapter is organized in three sections that correspond to the three affordances (efficient precision, focused constraint, and linked representations). In each section, I present the finding(s) associated with the analysis of the features reflecting the affordance. As an aid to the reader, I provide an executive summary of the chapter findings here before delving into the details of each finding below. The findings are grouped by affordance and then ordered by context.

- *Finding 3a:* All six teachers used features reflecting efficient precision (sometimes in connection with the overlap and circle free rotation features) to generate and use visual evidence for solving fraction comparison, equivalence, and addition problems. Although they used numeric reasoning for some steps in their strategies, the VM's efficient precision supported them in using visual reasoning about fraction magnitudes for other steps. This lessened their overall reliance on strictly numeric or procedural strategies. Five teachers also planned for students to use similar visual reasoning in Lesson Planning.
- *Finding 3b:* Features reflecting focused constraints challenged the ways all six teachers' thought about fraction addition in Problem Solving. They viewed the constraints as obstacles to work around as they attempted to use the VM to apply familiar addition strategies. The process of working around the constraints led them to confront questions about addition that were not directly related to computation, such as, "What are the implications of representing fraction addition with bar versus circle models?" and "Can I combine two addends that are not made up of same-size parts?" Their consideration of these questions did not seem to spur changes in their existing practices for teaching fraction addition, but they did suggest useful starting points for designing professional learning about fraction addition with VMs.
- *Finding 3c:* Teachers occasionally used features reflecting focused constraints to challenge student thinking in Lesson Planning. Only two teachers specifically mentioned how they might direct student attention to a focused constraint in an effort to shape their thinking, and there is no clear characteristic linking these two teachers. The rareness of

such uses of focused constraints suggests teachers did not often see value in the snags the constraints might cause in their own or their students' thinking.

- *Finding 3d:* Teachers occasionally used features reflecting linked representations to support their own or their students' thinking. Only three teachers explicitly mentioned drawing student attention to linked representations. The lack of teacher attention to this affordance may be due to the fewer available features reflecting linked representations, rather than a trend of not seeing the value in the features.

Responses to Features Reflecting Efficient Precision

The VM features coded as reflecting efficient precision are listed in Tables 15 and 16. Analysis of how teachers used these features in Problem Solving and planned to use them in Lesson Planning led to one finding (Finding 3a) detailed below.

Finding 3a: Teachers used features reflecting efficient precision (sometimes in connection with other features) to generate and use visual evidence for solving fraction comparison, equivalence, and addition problems.

My analysis of how teachers used the MLC Fractions App to solve the problems I presented to them during Problem Solving (order the fractions 3/10, ¼, and 2/6; and add ½ + ⅕) showed they used the features reflecting efficient precision in conjunction with the circle free rotate and overlap features to generate visual evidence to support some steps of their problem-solving strategies. As discussed in Chapter 3 (Results, Attention to VM Features), teachers showed particularly high attention to the overlap and circle free rotate features during Problem Solving. Finding 3a provides an explanation for that high attention: teachers used these features to take advantage of the efficient precision features of the MLC Fractions App as they solved problems.

Teachers used visual reasoning in some problem-solving steps in conjunction with numeric reasoning in other steps. For comparison, teachers used numeric reasoning to generate fraction representations, but then used visual reasoning to compare the fraction magnitudes. For equivalence, teachers relied on numeric reasoning or prior knowledge to identify a common denominator to be used in fraction addition, but they used visual strategies to support them in generating the equivalent fractions with that denominator. For addition, teachers created visual representations of the sum, but then relied on numeric reasoning to generate an exact numeric sum. Five teachers (all but Tammy) teachers also planned to have students use features reflecting efficient precision to generate and use visual evidence in Lesson Planning. I provide examples below of this overall pattern for comparison, equivalence, and addition. I chose these examples to illustrate teachers' use of visual evidence to solve problems, explain the role of features reflecting efficient precision in the generation of that visual evidence, and explain how the overlap and circle free rotation features were used to take advantage of the efficient precision.

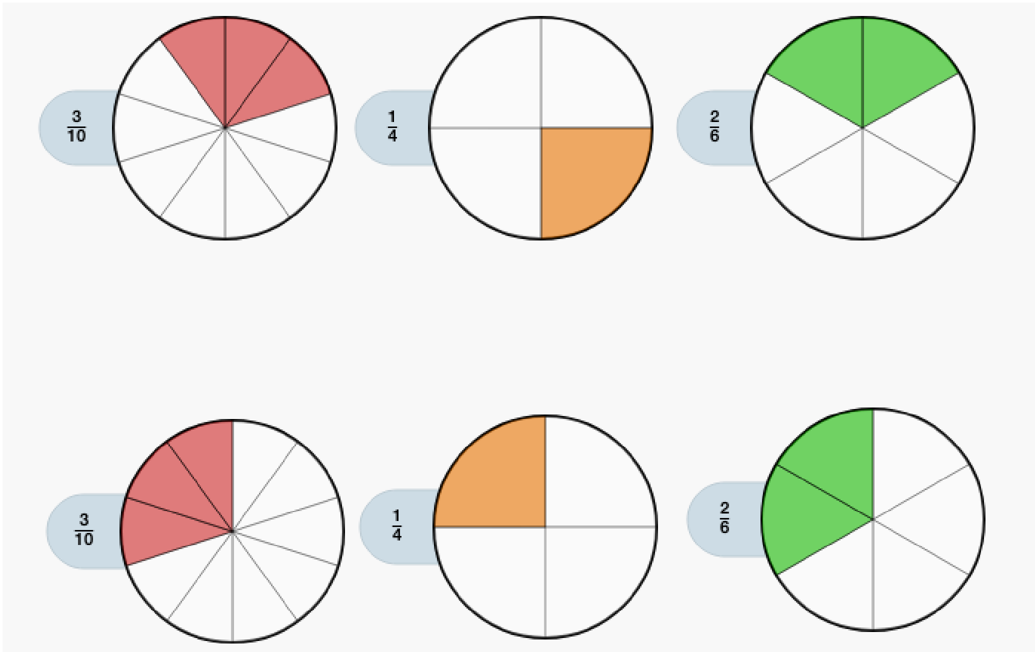
Fraction comparison. The first problem I posed to teachers in Problem Solving was to use the MLC Fractions App to put the following three fractions in order from least to greatest: 3/10, ¼, and 2/6. I chose these three fractions because they were all less than ½ and none of the denominators were multiples of each other. All teachers solved this problem using both the circle models and the bar models available in the VM. (They chose one model first, and then I prompted them to try the other.) The next section explains how teachers used efficient precision to generate evidence for comparison when they used circle models and bar models in their own Problem Solving. A second section highlights how two teachers expected students to use similar kinds of visual evidence as they compared fractions in Lesson Planning.

Visual comparisons in Problem Solving. When using circles to order 3/10, ¼, and 2/6, all six teachers started by creating three separate models—one for each fraction. Doing so required them to make use of the automatic dividing feature and the full fill subfeature, both of which are features reflecting efficient precision. They used numeric reasoning about the numerators and denominators as they chose the number of parts and counted how many parts to fill in. All six teachers proceeded by using some combination of overlapping the models and rotating them so that one edge of the shading on the models aligned. When they overlapped the models, teachers also all used the circle-snapping feature to support them in aligning the circles—although only Janice spoke directly about this feature, and only after solving the addition problem (not the comparison problem). This is part of the explanation of why the overlap and circle free rotation features were so highly attended in Problem Solving; these features supported direct visual comparison of the sizes of the three fractions.

The ways that teachers used the overlap and circle free rotation features showed they made comparisons based on their visual judgements of the amount of shading in each model. For example, Nancy's and Tammy's original models did not have the shading oriented so that it began at a consistent position on the circles (see Figure 27, top). As these two teachers began to compare the fractions, they used the circle free rotation feature to orient the shading consistently. For example, Tammy thought aloud as she reoriented her circles (starting with them oriented as in the top of Figure 27):

> If I am a student who does this, I could bring this so that it looks sort of the same as this one [rotating the red 3/10 model so the shading starts the same position as the shading in the ¼ model] and then I take this and drag it over there [drags ¼ model on top of 3/10 model]. I can see this one [¼] is smaller because I have an overlap.

Figure 27: Models to begin solving the comparison problem that start without consistent orientations (top) versus with consistent orientations (bottom)



The other four teachers (Janice, Karla, Sheila, and Colleen) created models with a consistent orientation at the start of their comparison problem solving (similar to Figure 27, bottom). They did not need to use the rotation feature to align the shading, but they still used an overlapping strategy to make their comparisons, relying on visual evidence of the fraction sizes to place the fractions in order. Two of these teachers, Janice and Karla, acknowledged that since students may not start out with the shading in consistent orientations, the rotation feature was nice to have to support students in aligning the models. For example, Karla noted, “If you didn’t set it up like I did, you can rotate it, which is nice. So if they did need to line them up that’s available.” This suggests these teachers not only used visual comparison strategies themselves, but also expected students to do so if they were using the VM.

As additional evidence that teachers relied on visual evidence to compare fractions, three teachers noted that the overlap strategy was different than other numeric or symbol-based strategies they might have used otherwise. After solving the problem with the VM, Tammy said, “I didn’t even need to find out common denominators. That was cool.” This comment suggests she may not have considered before how fractions could be compared without common denominators. When asked how she knew the fractions were in the right order, Karla said, “Well, visually I know, because I tried.” She went on to talk about some other numeric reasoning strategies that might be used to compare the fractions. Nancy similarly seemed to think of the visual comparisons as a starting point for students: “I love that you can stack them to see. Because I think that’s one of the things until kids have that kind of ability to visualize …. they need more of that scaffold of stacking them or lining them up.” Nancy explained that she mentally converted each of the fractions to decimals to check her work and did not believe her students would do that naturally. She talked about the overlap strategy as a better starting strategy for them to compare fractions.

None of the teachers explicitly pointed out that the automatic dividing feature, the full fill subfeature, or the snapping of circles feature—three features reflecting efficient precision they used to create and align their models—supported their use of their visual reasoning for fraction comparison. However, Colleen and Karla both described the automatic dividing process as something that made creating the fraction representations “easy” after they solved the comparison problem. Tammy named the automatic dividing feature when asked to identify the features that were helpful for solving the comparison problem. These comments suggest the teachers may have recognized the importance of the precision in the size of the parts and the neatness of the fill when using the visual strategies for comparison. Karla also made explicit reference to the closeness in size of two of the fractions in the problem and how it was still possible to see the difference with the VM’s representations:

> It's super easy to put them on top of each other because they're sort of transparent, so that it shows it easily. Especially when it gets close. Like this [2/6 and 3/10], I would say was close for a fourth grader. And that's really easy to see that it's bigger.

This suggests Karla understood, either explicitly or implicitly, that having precise representations of the fractions was important for generating visual evidence for comparisons. Considering that all teachers talked about how valuable it was that the VM supported students in creating equal-size parts because they struggled to do so themselves (see discussion of the automatic dividing feature in Chapter 4), it seems reasonable to assume that if teachers were asked whether the automatic dividing feature was important for supporting students in making visual comparisons, they would agree.

Further, Janice, Nancy, and Colleen directly mentioned another feature reflecting efficient precision, the automatic sizing feature, as important for supporting their comparison strategies using the circle models. Janice and Nancy did not elaborate beyond noting it was helpful, but Colleen specifically connected its value to making comparisons: "We want the wholes to be the same size or else it won't be accurate comparing them." Thus, three of the teachers seemed to acknowledge they used this feature as they made their visual comparisons.

Teachers' strategies for solving the comparison problem with bar models generally did not involve overlapping the models. Rather, they created a bar model for each fraction (again using numeric reasoning and the automatic dividing, automatic sizing, and full fill features) and aligned the models side-by-side to make their comparisons. Five of them (all but Tammy) used the bar-snapping feature to assist with that alignment. All five specifically said they liked this feature or that it was helpful for comparing fractions. Three (Colleen, Karla, and Sheila) said it helped users of the VM to make sure the ends of the bars were aligned or made small differences in the length of the shaded portions easier to see. For example, Colleen explained, “The red and the green [her bars representing 3/10 and 2/6] are kind of close where if they didn’t bring them right exactly together, it might be easy to have it over a little bit where you can barely tell [which is greater].” This comment not only shows that Colleen made a visual judgement to compare, but also implicitly illustrates why the exact size of the pieces and the neatness of the fill was important for supporting that visual strategy.

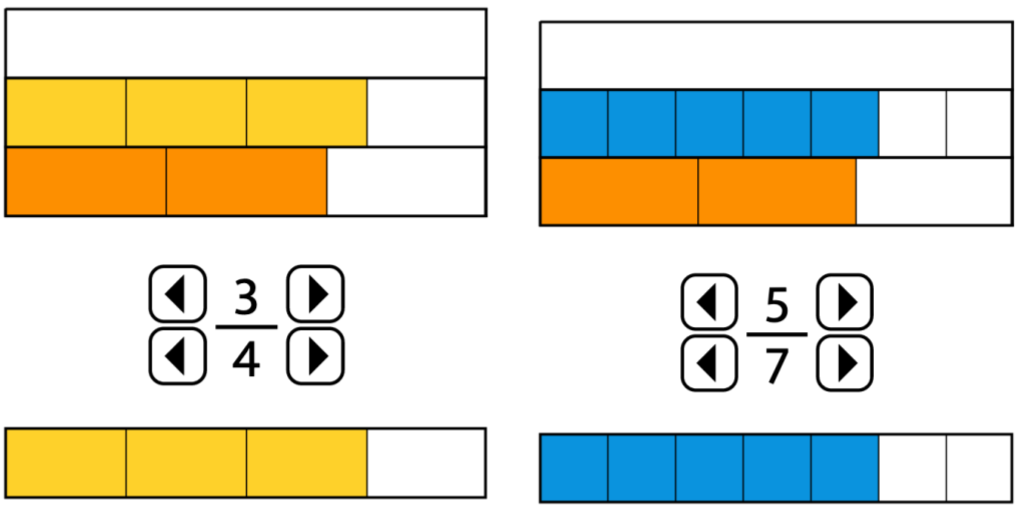
In summary, the visual strategies teachers used to make fraction comparisons in Problem Solving—when using circle models or bar models—were supported by their use of features reflecting efficient precision, including the automatic dividing, automatic sizing, full fill, snapping of circles, and snapping of bars features. Teachers explicitly acknowledged the importance of some of these features for carrying out their strategies and made comments that implied their reliance on others.

Visual comparisons in Lesson Planning. While none of the teachers planned a lesson focused on comparing fractions, two (Sheila and Nancy) planned lessons that involved generating lists of equivalent fractions. The strategies they expected students to use to generate those fractions involved making comparisons. For example, this is how Sheila described how her students might use the Toy Theater Fraction Bars to find examples of fractions equivalent to ⅔ (see Figure 28):

> So I’m thinking a student is going to put 2/3 on here [in the bottom slot], and then I would expect that maybe they are changing the pictures, maybe estimating to say *(adds a ¾ bar to the middle slot)*, “Oh, that’s not quite right. I’m going to try a different one here. *(removes the ¾ bar and adds a 5/7 bar)* That one’s not quite right.” So that’s what I’m thinking that they’re going to be doing.

This suggests Sheila expected students to use visual strategies to compare fractions while they looked for examples that were equivalent. She anticipated students would judge fractions to be "not quite right" as examples of fractions equivalent to ⅔ when they saw that the right end of the shading did not line up. She expected they would use the slacked slots—a feature of this VM reflecting efficient precision because it aids students in precisely aligning the bars—to help them make comparisons.

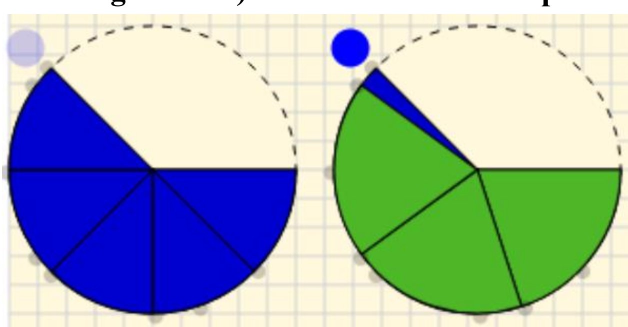
Figure 28: Sheila demonstrates a strategy she expects her students to use to find equivalent fractions for ⅔, trying ¾ (left) and then 4/7 (right)



Sheila also described how she hoped students would be paying attention to the bar partitions—another feature reflecting efficient precision because the partitions automatically divide the bars into equal parts: "So I want them to create on the Toy Theater and to be looking for those similar, the lines where the fractions are lining up and then to use that to say if a fraction is equivalent or not." It is not clear whether Sheila expected students to use visual evidence to generate their guesses as well as check them or if she thought they might use numeric reasoning to generate their guesses (e.g., thinking about the closeness of the numerator and denominator).

Nancy expected her students to generate examples of equivalent fractions by stacking the Phoenix College Fraction Circles to see whether different combinations took up the same amount of space on the circle. She talked several times about how the way the pieces snapped into the circles—a feature reflecting efficient precision—made it easy for students to swap the order that two fraction representations were stacked so that the "extra" part of the larger fraction would be visible. For example, during Nancy's student response interview, I (as a student) placed blue pieces representing ⅝ on top of green pieces representing ⅗ and said it seemed to me that they were equivalent (Figure 29, left). Nancy's response was to have me restack the pieces, with the blue eighths on the bottom and the green fifths on top (Figure 29, right), so that I could visually see that ⅝ was greater.

Figure 29: Nancy asked me to reverse the stacking order of pieces (to go from the left circle to the right circle) to make a visual comparison of ⅗ and ⅝

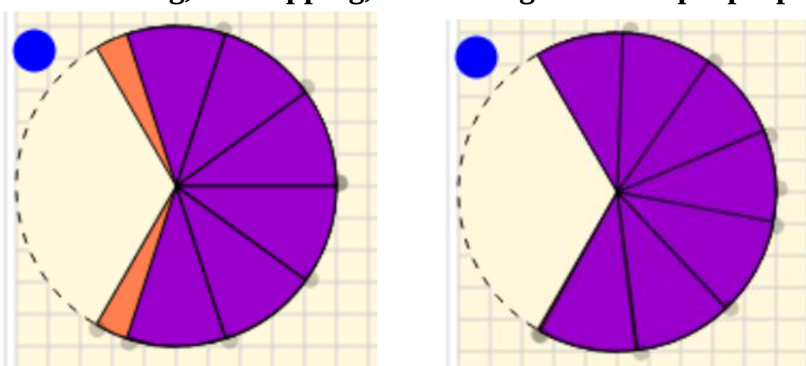


After this, Nancy commented that the way the pieces were easily dragged and snapped into place would make this strategy easier for students than if they used physical manipulatives:

> I think it would be harder to prove because they are clunkier, their fine motor skills are clunkier with the actual physical fractions circle pieces whereas you were able to easily drag them and stack them up and say, "no, that didn't work". The time it would take with physical manipulatives … that would be a lot of time for some kids.

Nancy also suggested that I (as the student) take advantage of the piece snapping feature to generate visual evidence of a comparison in response to another scenario. In this scenario, I (as a student) was attempting to generate an equivalent fraction for ⅔ and began by placing purple tenths pieces atop two orange thirds pieces. The automated snapping of the pieces led to the representation on the left of Figure 30. Seeing the silvers of orange on either side of the purple pieces, I (the student) rotated the purple pieces to shift them to start at the edge of the orange. (The gray dots at the edges of the pieces allow for free rotation of the pieces around the drop zones.) As I shifted the purple pieces, I overlapped them slightly and found there was room for one more purple piece to exactly cover the orange, as shown in on the right of Figure 30. My slight and unintentional overlapping of the purple pieces made room for a seventh piece.

Figure 30: A student scenario presented to Nancy where a student overlapped the edges of purple pieces so 7 tenths fit onto 2 thirds. Left is before rotating the purple pieces; right is after rotating, overlapping, and adding one more purple piece.



Nancy responded by suggesting that I drag the seven purple pieces into a new drop zone and allow "the computer to naturally snap them together," then drag the orange thirds on top of the neatly snapped purple tenths to show that 7 tenths was greater than 2 thirds. Nancy said, "I do really love that [snapping] because in a physical manipulative, the kids have to work hard to line them up sometimes," but with the VM, "they were able to disprove it relatively quickly." (By "it," she meant the false equality between ⅔ and 7/10.) These comments about the snapping feature suggest Nancy valued this feature reflecting efficient precision for supporting visual comparison strategies.

Summary of the role of efficient precision in fraction comparison. The theme of teachers using, or expecting their students to use, visual evidence to compare fractions was present in both the Problem Solving and Lesson Planning data. All teachers used visual judgement to make comparisons during Problem Solving, and two teachers planned for their students to do so during Lesson Planning. Teachers used features reflecting efficient precision—including features that automatically divided models into equal parts or supported alignment of models or pieces— supported the generation and use of visual evidence in both contexts. Three teachers explicitly contrasted their use of visual evidence to compare with other strategies (such as finding common denominators) that rely more heavily on numeric reasoning. The way features reflecting efficient precision supported visual comparison strategies is significant because use of visual evidence suggests teachers were directly comparing the size of the fractions—or the magnitudes of their values—and not fully relying on strategies that involve thinking about the numerators and denominators as unrelated whole numbers. Supporting students (and teachers) to think about fractions as single numbers with magnitudes is an ongoing challenge in fractions teaching and learning (Fazio & Siegler, 2011).

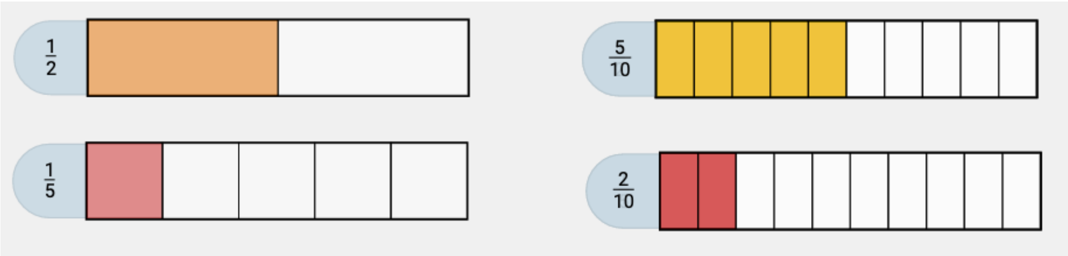
Fraction equivalence. While neither of the problems I presented to teachers during Problem Solving explicitly asked them to generate an equivalent fraction, I chose the addition problem ½ + ⅕ because I thought it would prompt some of the teachers to generate equivalent fractions for ½ and ⅕ as part of their solution process. Four teachers also planned lessons about generating equivalent fractions. In both contexts, most teachers (five in Problem Solving and four in Lesson Planning) described or demonstrated ways of using VMs to generate visual evidence for verifying the equivalence of two fractions or generating equivalent fractions for addends with a common denominator they had already determined. I describe examples in each context below to illustrate this theme and demonstrate the role of features reflecting efficient precision in supporting teachers' visual reasoning about equivalence.

Visual support for equivalence in Problem Solving. One teacher, Colleen, found equivalent fractions for ½ and ⅕ as a distinct step in her addition process both when she used the bar models and when she used the circle models in the MLC Fractions App. After showing ½ and ⅕ on bar models (see Figure 31, left), she said,

> I guess what I would do and I know what my students would do would be to just say, I need my denominator to be tenths… I would maybe then just put the tenths over and I might use another color just to color it in to tell me how many tenths it is.

She overlapped a blank tenths bar on her bar for ½, then filled in the parts that had shading underneath, allowing the linked label to count the tenths for her. She repeated this for her model of ⅕ (Figure 31, right). She used a similar process when I asked her to try the circle models. She did not use the VM to identify the common denominator, but she did use it to determine what the numerator of each equivalent fraction should be (when the denominator was 10). She relied on visual evidence to tell her when she had shaded a number of tenths equivalent to ½ and ⅕.

Figure 31: Colleen showed the addends ½ and ⅕ (left), then overlapped them with tenths bars to help her find equivalent fractions with a common denominator (right).



When prompted to comment on the features of the VM that were helpful for solving the problem, Colleen responded, "It's very helpful for you to be able to overlap the circles or the fraction bars. That way you can actually see why the fractions are equivalent and that amount or the value doesn't change." Her use of the words "actually see" emphasizes that she was using visual evidence to generate the equivalent fractions and verify that they were equivalent. Although Colleen did not acknowledge this specifically, her strategy of overlapping the models to find equivalent fractions relies on the precision in the automatic dividing and filling of the pieces. Colleen did mention that the automatic dividing feature made her problem-solving process "easy" after she solved the addition problem.

Four other teachers (Nancy, Janice, Tammy, and Sheila) used a related strategy to generate equivalent fractions when they solved the addition problem with the circle models. Three (all but Sheila) began by creating a model of each addend and then overlapping them to show the sum, creating a representation similar to the circle on the left of Figure 32. They had varying reactions to this representation, but all of them eventually acknowledged the need to think about tenths. For example, Nancy said, "So that's how I would show it. But now the challenge would be obviously that these are not like denominators. And so what fraction is that? So then you might kind of have a conversation about how tenths could also kind of overlay all of that." She created a tenths circle and placed it on top of her representation of the sum to create the representation on the right of Figure 32. She continued, "And so if you laid over a blank tenth circle, a circle divided into tenths, you could easily see that it would be 7/10."

Figure 32: A representation of the sum ½ + ⅕ created by three teachers, before (left) and after (right) representing equivalent fractions in tenths

Because Nancy referenced “like denominators” but never explicitly acknowledged how her representation showed that ½ is equivalent to 5/10 and ⅕ is equivalent to 2/10, it is not clear whether she was thinking about the equivalent fraction for each addend or rather the equivalency between the full sum and 7/10. Either way, her use of the words “you could easily see” shows she was using visual evidence to think about equivalence (although not to determine that 10 could be used as a common denominator). Further, the visual component of her strategy is dependent on the exact sizing of the pieces (the automatic dividing feature), the neatness of the fill (the full fill feature), and the fact that all her circles are the same size (the automatic sizing feature). It is also facilitated by the way the overlapping circles snap into place (snapping of circles feature). All these features of the VM reflect efficient precision. This is also another way that teachers used the overlap feature to take advantage of the features reflecting efficient precision to support their problem solving.

Janice and Tammy followed a process similar to Nancy’s. Sheila placed each of her addend models onto the same tenths circle, creating a representation like the one on the right of Figure 32 without first showing the sum as on the left. The final teacher, Karla, did not use a visual strategy to generate equivalent fractions with a common denominator. Rather, she deleted her models of ½ and ⅕ and created new models showing 5/10 and 2/10—presumably using numeric reasoning to decide how many parts to create and how many should be shaded. She then overlapped the new models to show the sum. It is particularly interesting that Karla did not see the potential for a visual strategy for equivalence in the context of the addition problem, because she was the only teacher to explicitly articulate a connection between the automatic dividing feature and finding equivalent fractions in Exploration. Specifically, Karla noted that the efficiency in creating visual models of fractions equivalent to ½ might make the MLC Fractions

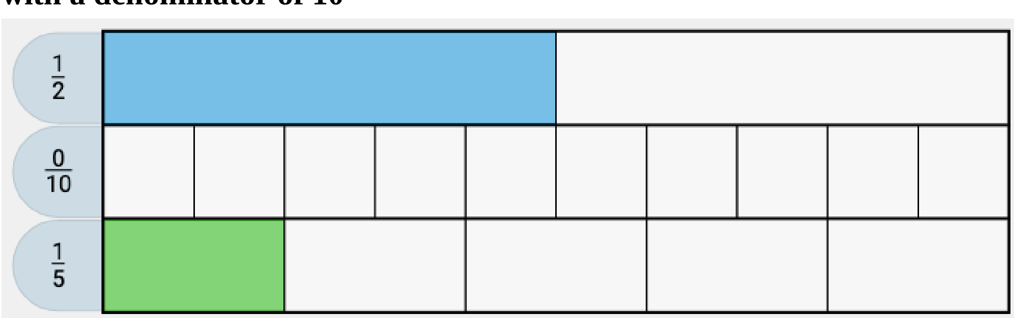
App a good choice for helping students grasp how fractions with different numerators and denominators could be equivalent:

> So if I wanted to do a bunch that were equal to a half, real quick, you know? We find that getting them to realize what's equal to a half takes a long time for some kids. But something like this where you know this is so visual for them to come up and say, "Okay, well let's look at tenths. Now, like, okay, tenths, what's equal to a half?"

She pointed out that the VM allowed a user to pick any denominator and create the parts instantly, allowing students to then focus on how many of a particular size part they needed to fill in to create a fraction equivalent to one half.

As a final example of how teachers used efficient precision features to support the visual steps of their strategies for generating equivalent fractions, Sheila used the bar snapping feature when she solved the addition problem with the bar models. She began by creating bar models of ½ and ⅕. Then she created a tenths bar and snapped it between the two addends (see Figure 33). Looking at this representation and gesturing to the places where the partitions in the tenths bar lined up with the ends of the green and blue shading, she said "I'm going to say that I notice two tenths is the same as one fifth, and five tenths is the same as one half." Thus, she described a visual strategy for generating the numerators of her equivalent fractions and used the bar-snapping feature to support her in aligning the bars.

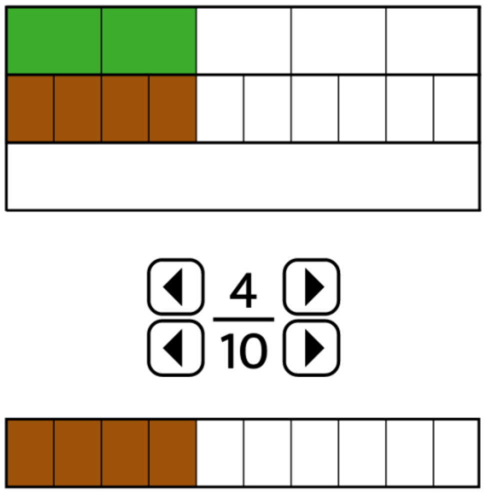
Figure 33: Sheila used the bar snapping feature to find equivalent fractions for ½ and ⅕ with a denominator of 10



Visual support for fraction equivalence in Lesson Planning. Four teachers (Colleen, Sheila, Karla, and Janice) discussed plans for their students to use visual evidence to support their thinking about fraction equivalence in Lesson Planning. They also either explicitly or implicitly related the VM's visual evidence to features reflecting efficient precision. For example, as discussed in Chapter 4 (Finding 2f), Colleen planned to support her students in making sense of equivalent fractions by pushing them to look at both the symbolic fractions generated by the NCTM Equivalent Fractions App and the associated area models. She mentioned the importance of the equal sizes of the parts in the area models (a feature reflecting efficient precision) near the beginning of her lesson plan interview: "I like this a lot better [than hand drawings] because these parts are all even. They're all the same size." She also said she would "talk about looking at the sizes" of the area models to make sure students understood why their lists of equivalent fractions made sense. Her emphasis on fractions sizes suggested she planned to use the area models (which she called "visuals") to support students in thinking about the magnitudes of the fractions and not only the patterns in the numerators and denominators.

As mentioned above, Sheila expected her students to use visual comparison strategies as they looked for examples of equivalent fractions with the Toy Theater Fraction Bars. In addition to describing how students would use the stacked slots and bar partitions (which reflect efficient precision) to make comparisons, Sheila also described how students could reference the bar partitions to visually make sense of why fractions were equivalent. In her student response interview, I presented Sheila with a scenario where a student was generating equivalent fractions by pressing the arrow buttons a number of times equal to the numerator or denominator of the symbolic fraction. For example, if the student started with $\frac{2}{5}$, she pressed the arrow button next to the numerator twice and the one next to the denominator 5 times, generating 4/10 (Figure 34).

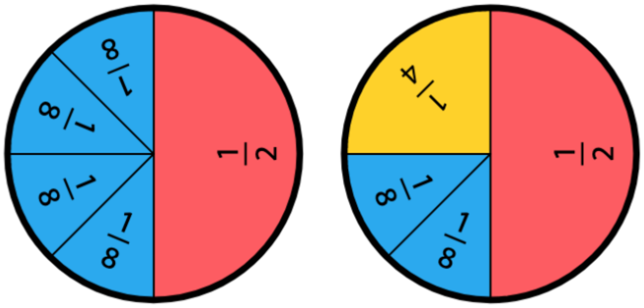
Figure 34: A representation of two equivalent fractions Sheila referenced to help a student understand why a doubling strategy generated an equivalent fraction



Sheila said she would respond by trying to help the student see that they were doubling the numerator and denominator. When I asked how she would help a student see that, Sheila said, “I would probably start with the manipulative because they are, I guess for me I’m looking at it and seeing how the brown makes up two greens and that even the ones that aren’t shaded in there are two tenths that make up one fifth.” She was referencing the equal sizes of the parts created by the bar partitions to explain why multiplying the numerator and denominator by 2 resulted in an equivalent fraction.

Although the focus of Karla’s lesson was not fraction equivalence, a situation involving equivalence came up in her student response interview. The second problem in her lesson was for students to find a way for a queen to keep ½ of her kingdom, distribute the other half to her three daughters, and make sure the oldest daughter got more than the other two daughters. In response to a scenario where a student suggested giving 2/8 to the first daughter and ⅛ to each of the other daughters (see Figure 35, left), Karla pushed the student to think of another way to express the amount given to the first daughter. She said, “Maybe they could show it even on the other [circle]. Is there another way you can show that, on the other manipulative?” She guided me (as the student) to place the ½ for the queen in the second circle, then the two eighths for the younger daughters. She continued, "And then I think most of them would recognize that then that one fourth would fit. *(See Figure 35, right.)* So it's the same. It's just another way of saying, one fourth is equivalent to two eighths. It's just another way of saying it."

Figure 35: A student's solution to problem 2 (left) and Karla's suggestion of what to do next (right)



Karla thought the student would visually recognize that the space still left in the second circle, after placing a ½ piece and two ⅛ pieces, was equal to ¼. Thus, she guided the student to use a visual strategy for equivalence. This strategy is supported by the exact size and easy snapping of the Toy Theater Fraction Circle pieces—two features reflecting efficient precision.

Finally, as discussed in Chapter 4, Janice used the copy button (another feature reflecting efficient precision) to assist her in generating visual representations of fractions greater than 1. She then referenced those visual representations as she explained how to express those fractions as mixed numbers. For example, after students created a representation of 15/5 (three fully shaded strips, each divided into fifths), this is what Janice said she would do next:

> And I'll say, "Okay, now let's express this in mixed numbers, right? How do we express this in a mixed number? Remember a mixed number is a whole number with a fraction. So how many whole numbers, wholes do we see?" And they'll say, "1, 2, 3 wholes." "Do you see any fractions?" Like, "No." "So zero fifths. But we don't have to write zero fifths. So good." Like, "This is just three. That's the mixed number."

As she mimicked the students counting the wholes, she gestured to the models she had generated with the copy button. She hoped that creating visual models for fractions greater than 1 (which students can do quickly with the copy button) would support them in seeing how the fifths (or whatever denominator they were working with) could be grouped into wholes, sometimes with additional fractional parts left over, to generate an equivalent mixed number.

Summary of the role of efficient precision in equivalence. There was no evidence that teachers used the VMs to support them in identifying 10 as a common denominator as they solved $\frac{1}{2} + \frac{1}{5}$ in Problem Solving. However, five teachers did use features reflecting efficient precision (and the overlap feature) to generate visual evidence for what the numerators of the equivalent fractions would be. In Lesson Planning, Karla and Colleen planned to draw students' attention to how visual representations of equivalent fractions are the same size. These strategies (or steps of strategies) rely on the exact sizing of VM parts, which are generated via features reflecting efficient precision. Further, the emphasis on the size of the visual models draws attention to the shared magnitude of equivalent fractions rather than differences in numerators and denominators. Sheila and Janice discussed how they would draw attention to visual evidence generated by the VM features reflecting efficient precision to explain why numeric strategies worked—doubling the numerator and denominator to generate an equivalent fraction for Sheila, or grouping unit fractions into wholes to convert a fraction to a mixed number for Janice.

As a counterpoint, it is worthy of noting that, as discussed in Chapter 4, Janice also described how the bar snapping feature of the MLC Fractions App created her "worst nightmare" because it could hinder students in using visual strategies for making sense of fractions greater than 1. Specifically, snapping the bar models together made it easy for students to interpret strips intended to represent multiple wholes as one contiguous whole. Thus, features reflecting efficient precision do not always support visual strategies for making sense of fraction equivalence.

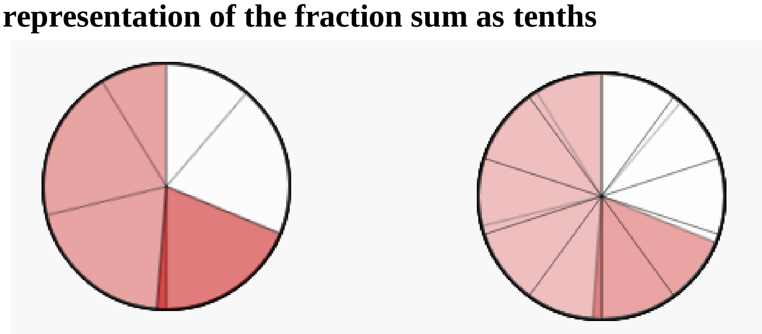
Fraction addition. All six teachers used the MLC Fractions App to generate visual representations of the sum ½ + ⅕ during Problem Solving, and some used the visual representation to count how many tenths were in the sum. Two teachers also created or discussed visual representations of fraction sums during Lesson Planning. They did not fully rely on the visual representations to determine a numeric sum during Problem Solving, but they did describe how students might rely on visual evidence to find sums—and not rely on common denominators—in Lesson Planning. I present examples of these themes in each context below and discuss the role of features reflecting efficient precision, and the overlap and circle free rotation features, in generating the visual sums.

Visual representations of sums in Problem Solving. As mentioned under the discussion of equivalence, four teachers (Janice, Nancy, Sheila, and Tammy) represented the sum ½ + ⅕ by overlapping a circle model for ½, a circle model for ⅕, and a blank tenths circle. The result was a representation similar to that on the right of Figure 32. All four of these teachers also used the rotation feature to help them orient the shading in their ½ and ⅕ models so that the two addends were adjacent and non-overlapping. These uses of the overlap and circle free rotation features to represent addition is another part of the explanation for why these features were so highly attended in Problem Solving—they supported teachers in creating visual representations of the sum ½ + ⅕ when they used circle models.

After creating their representation of the sum, all of these teachers noted that they could see that the sum was 7/10. For example, Janice said, "I'll make a tenths overlay and count, one, two, three, four, five, six, seven. Seven tenths." Teachers did not use the VM or any visual evidence to determine the common denominator, but some (like Janice) did use the visual representation to count how many tenths were in the sum. Like the visual solution steps for comparison and equivalence, this visual step for solving the addition problem was supported by MLC Fractions App features reflecting efficient precision: automatic dividing, automatic sizing, full fill, and snapping of circles.

Colleen and Karla acknowledged the helpfulness of the automatic dividing feature after solving the addition problem. For example, Karla said, "I like the ease of picking the denominators and breaking it apart." Janice pointed out the utility of the snapping of circles feature: "The fact that they snap into place, the circles snap into place when they're a little bit off kilter, was really helpful." She went on to elaborate that the circle snapped into position on the screen but did not snap into a helpful orientation when she rotated the circle: "But once I rotate these guys don't snap into place. So there could be a little overlap and that would make this part [overlaying the tenths circle] really funny." (See Figure 36.) Janice's point further illustrates the importance of the automatic dividing feature for the teachers' visual solutions steps for addition in Problem Solving. If the pieces in the models were not exactly the same size, models created by overlapping circles could end up looking like the ones in Figure 36—even if students had not inadvertently overlapped the shading from the two addends as Janice had.

Figure 36: Janice's illustration of the overlap that might occur because there is no snapping together of shading when circle models are rotated (left) and her resulting "really funny" representation of the fraction sum as tenths



Visual representations of sums in Lesson Planning. Only one teacher, Karla, planned a lesson related to fraction addition in Lesson Planning. Although her task did not center on finding sums of given fractions, it did focus on finding combinations of fractions that summed to 1 (i.e., a valid partitioning of a queen's whole kingdom among herself and her three daughters). During her student response interview, Karla explained how the exact size of the Toy Theater Fraction Circles (a feature reflecting efficient precision) would help students visually judge whether the sum of their fraction pieces was equal to 1 whole. In a student scenario I presented to Karla, a student had created the representation shown in Figure 37 as a way that the queen might keep ½ of her kingdom and distribute the rest among her three daughters. Karla discussed how the exact sizing of the pieces provided visual evidence that the entire kingdom had not been given away in a way that a student's drawing would not:

> It's just the visual of it, where I think when you're trying to explain to a student that it doesn't work and their drawing looks great to them, yeah it does, and to say, "No, it's not exactly, a little bit less, you're close, but it doesn't exactly work," here you can see that and they'd be like, "Oh, okay. It doesn't work. There's some of the kingdom left." But you wouldn't see that in their drawing. It's just not as clear.

Figure 37: A scenario presented to Karla where a student did not give away the whole kingdom

Karla appreciated how the precision of the Toy Theater Fraction Circle pieces would provide visual evidence of the sum's relationship to 1. She recognized there was an alternative, numeric strategy, but she did not think it would be productive: “And they could actually try to add it up to one, but finding a common denominator with all of these numbers would be a little tough. I don’t think they could do that.”

The other five teachers planned lessons focused on fraction equivalence, and so thinking about addition did not come up as often in their Lesson Planning data. However, I did present Tammy with a student scenario that led to discussion of how students might think about fraction addition using the VMs. As discussed in Chapter 4, I asked Tammy how she would respond if a student suggested a combination of a ⅓ piece and a ⅙ piece as an equivalent fraction for ½ (see Figure 11). Tammy said she would accept the ⅓ and ⅙ combination as an answer and encourage students to look for additional combinations. Tammy thought the exact sizing of the pieces, a feature reflecting efficient precision, would support students in finding combinations of fractions that add up to one half: “And with the circle pieces, you know that if you try to put in one-eighth with those two one-fifths, that’s not going to work and they have to try something else.” She felt that having those experiences with creating combinations with a particular sum would allow students to do some early thinking about fraction addition without “even having that complex process of trying to find a common denominator yet.” Ergo, Tammy felt the exact size of the pieces would support students in experimenting with visual strategies for thinking about fraction addition, rather than relying on numeric methods such as finding common denominators. This is a final example of how teachers used features reflecting efficient precision to support students in generating visual evidence to support their strategies for fraction addition.

Summary of teachers’ responses to features reflecting efficient precision. Across the mathematical topics of fraction comparison, equivalence, and addition, teachers’ strategies for solving problems often relied on the efficient precision of the VMs to support visual judgement about fraction magnitudes and relationships. They often used the overlap and circle free rotation features to take advantage of the precision when relating models to each other. In several cases, teachers even pointed out how the visual precision of the representations meant that they could use visual evidence of a fraction's size rather than relying on numeric strategies involving common denominators. A particularly interesting facet of the role of efficient precision is the way teachers mentioned how the features reflecting efficient precision made the VMs more helpful than both using physical manipulatives and creating drawings.

Responses to Features Reflecting Focused Constraints

The VM features coded as reflecting focused constraints are listed in Tables 17 and 18. Analysis of how teachers used these features in Problem Solving led to one finding: Features reflecting focused constraints challenged the ways all six teachers thought about fraction addition (Finding 3b). Analysis of how teachers planned to use these features in Lesson Planning also led to one finding: Teachers occasionally used features reflecting focused constraints to challenge student thinking (Finding 3c). These findings are detailed in the sections that follow.

Finding 3b: Features reflecting focused constraints challenged teachers' thinking about fraction addition in Problem Solving, but not enough to result in significant changes to their existing strategies and practices.

As discussed in Chapter 4 (Results, Interpretations of Affordances), teachers interpreted several of the features of the MLC Fractions App reflecting focused constraints negatively because they interfered with the problem-solving strategies they typically used for fraction addition. The teachers saw the features as limitations of the VM. Teachers' responses to the limitations imposed by focused constraints often consisted of using different features of the VM—the overlap or circle free rotation features—to work around the limitations, producing variations of their familiar strategies. The process of working around the constraints led teachers to confront questions about fraction addition that pushed their thinking about or beyond their familiar problem-solving steps. The questions they confronted included: Why is it important for students to see two addends separately within a sum? What are the implications of representing fraction addition with bar versus circle models? Can I combine two addends that are not made up of same-size parts?

In this section, I provide examples of how three features of the MLC Fractions App reflecting focused constraints—single-color models, complete whole, and fixed denominator—contributed to spurring teachers' thinking about these questions. Teachers' considerations of these questions did not usually lead to significant changes in strategy. Rather, teachers usually found a way to reproduce their familiar strategies, albeit in ways they did not anticipate. Even so, the ways teachers grappled with the questions, which relate to aspects of addition other than the specifics of a computation procedure, are important to consider because they may serve as useful starting points for professional learning experiences.

Why is it important for students to see two addends separately within a sum? The single-color model feature of the MLC Fractions App, which constrains the fill within a bar or circle model to one color, led at least one teacher to consider why showing addends in separate colors within a sum felt important pedagogically. As noted in Chapter 4, four teachers interpreted the single-color model feature as a limitation in how students could represent a fraction sum. When they solved ½ + ⅕ during Problem Solving, three of the teachers (Sheila, Colleen, and Nancy) reluctantly responded to the single-color model feature by settling for representing the sum with one color, at least initially. Their comments reflected varying levels of evidence of grappling with why it would be preferable to show two colors.

As Sheila solved the problem, she said, “I wish I could do two different colors, but I can’t. So I’m going to shade in my five tenths for one half. And then I’m going to add two more for my one fifth.” Her final representation showed 7 tenths that were all one color. She seemed to reason that while it was advantageous to show the sum with two colors, it was acceptable to show it with one. One interpretation of this response is that she could not explain to herself what the consequences of only showing the sum in one color might be for student thinking. Another interpretation is that she had already decided she would not use the MLC Fractions App to teach fraction addition—she mentioned during Exploration that the single-color feature meant she “can’t add on the same bar”—and so she felt she had no reason to consider the consequences for her students’ thinking of representing sums in one color.

Colleen was similarly reluctant to show her sum in one color, but her comments came closer to addressing why the single-color model constraint bothered her. When she tried the problem with circle models, Colleen showed the addends on separate models. She commented that students could “count them up” (referring to the shaded tenths across the two models) to find the sum, but then she decided to create a third model to show the sum in one color. She said: “I guess they could make another circle with tenths and then color in that total. I definitely think it’s useful to see how many pieces in relationship with the whole and actually look at that, but I do wish you could use different colors on one circle.” This comment acknowledged why she wanted students to be able to show the sum in a single circle (so that students could see what the total looked like “in relationship with the whole”—possibly a reference to thinking about the sum’s magnitude) but did not explain why seeing the addends separately within the sum would be helpful. These comments by Sheila and Colleen could serve as a good starting point in professional learning to discuss the pros and cons of using color to show the decomposition of a whole into the two original addends, but it is not clear that these two teachers considered this question on their own.

Nancy's comments more explicitly indicated she was grappling with the question of why she wanted students to be able to see the addends in different colors within the sum. After trying to show the two addends in one strip with different colors and encountering the single-color constraint, Nancy commented, "I don't like that." Upon reflection, she continued:

> I guess it's okay. My initial thought was, I don't like it because I was trying to show the fifth separate from the half. But if I guess if we really want them to see that the half and the fifth, both are equivalent to a denominator of tenths. The fact that the minute that I tried to add the fifth, it changed it all to one color might show more clearly that it's now all one fraction, and it's representing one thing.

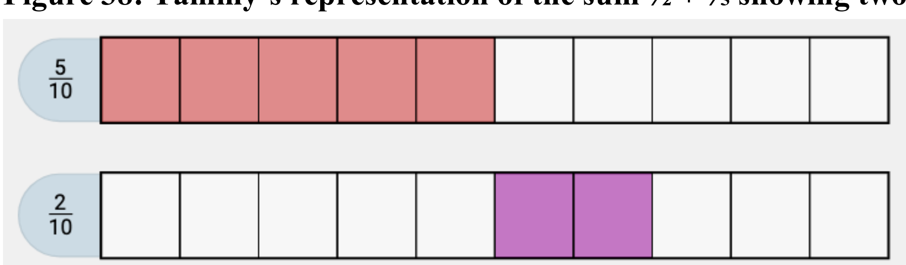
Nancy seemed to be attempting to recognize an advantage of the single-color model feature but remained unconvinced. She went on to say that students' grade levels might relate to whether they would benefit from seeing the two addends separately within the sum:

> I think it depends on how the kids want to tackle the problem. Because if they want to see each addend separately, then they're probably going to want two different colors. But really, really thinking about how each addend can be converted to a common denominator and becomes tenths. Then it doesn't matter if they're two different colors because part of the tenths represent the half, and part of the tenths represent the fifth. And together they're all the sum. And so, yeah, I think it depends on how clearly they're thinking about the problem. So I'm thinking like a fourth grader, they're going to want two different colors. Because they're going to want each thing separately. But maybe as they get more solidified in their fraction thinking then, no, then they'll see it all as one.

Nancy seemed to believe that when fourth graders add fractions, they may not be ready to think about the sum as one quantity without also maintaining a representation of the addends within the sum. In her earlier comment, she acknowledged the single-color model feature could help students see the sum as one number. Here, she stated that kind of push in thinking may be more appropriate for fifth grade and beyond—after students leave her fourth-grade classroom.

Despite the single-color model feature spurring some thinking about the consequences of showing a sum in one versus two colors, teachers' responses often seemed to focus on working *around* the single-color constraint rather than considering how they might adjust their instruction to work *with* it. Three teachers (Colleen, Nancy, and Tammy), after encountering the single-color model feature during Problem Solving, responded by seeking out ways to show the two colors despite the VM's constraint. When using the bar models to represent ½ + ⅕, Colleen had begun by creating a ½ bar and a ⅕ bar using different colors. She then overlaid tenths bars on each addend and re-shaded the same amount in each bar with a different color to create equivalent fractions in tenths (see Figure 26, left). After realizing she couldn't add a second color to either of these bars, Colleen reflected that you could still see a color difference if you added shading to one addend, as the original part of the bar would have the other shading underneath (see Figure 26, right). After realizing she could not use two colors in the same bar model, Tammy represented the sum using two bars, but placed the shading so that the shading from the second addend appeared to continue from the end of the shading from the first addend (see Figure 38).

Figure 38: Tammy's representation of the sum ½ + ⅕ showing two colors



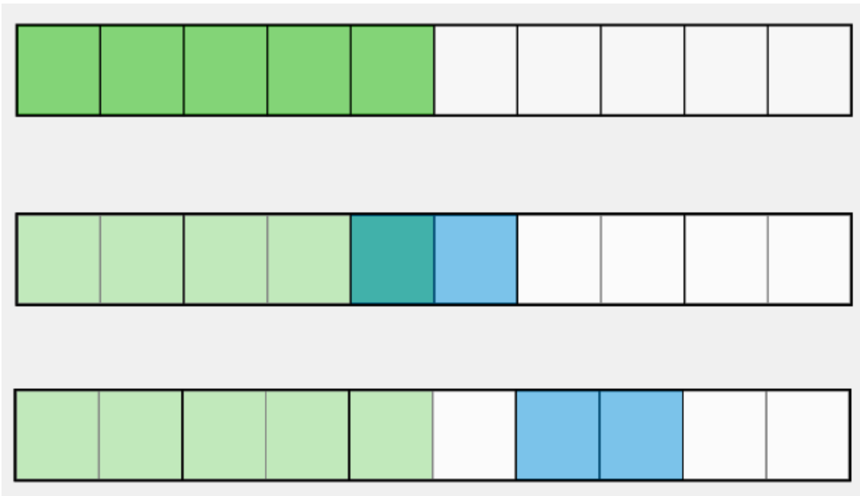
Thus, Colleen and Tammy both found ways to show the addends in separate colors. Colleen used the overlap feature and the semi-transparency of the models to overcome the single-color model constraint, and Tammy used the free positioning of the shading within the model to overcome the single-color model constraint. (Nancy's strategy for overcoming the single-color model constraint is discussed in the next section.) When they could not use their familiar strategy, Colleen and Tammy were challenged to find a different strategy that took advantage of different VM features. Even so, the adjusted strategies reflected a preference for adjusting their use of the VM to reflect their familiar practices rather than adjusting their practices to make use of focused constraints of the VM.

What are the implications of representing addition with bar versus circle models? Two MLC Fractions App features reflecting focused constraints—single-color models and complete whole—spurred three teachers' thinking (Nancy, Sheila, and Janice) about how the choice of bar models versus circle models impacted their representations of fraction addition. Using circle models for addition was part of these teachers' strategies for overcoming limitations of the VM imposed by the focused constraint features. Nancy noted the circle models made it easier to show a sum in two colors (i.e., overcome the single-color constraint). Nancy and Sheila thought the circle models made it easier to show a sum as a contiguous piece of the whole (as they attempted to overcome the single-color constraint). Nancy, Sheila, and Janice all noticed the circle models made it easier to combine the addends while still maintaining a clear view of the whole (a difficulty raised by the complete-whole constraint). In this section, I give examples of teachers explaining or demonstrating each of these consequences of using circle models to represent addition. Teachers said little about *why* these issues were important, but the features did lead them to say or imply that the issues *were relevant,* potentially providing a good starting point for professional learning experiences that build teachers' understanding of fraction addition.

Nancy's strategy for overcoming the single-color model constraint and showing addends in separate colors was to use circle models instead of bar models to represent addition. She created a representation similar to the one in Figure 32 by overlapping circle models. Several teachers used this strategy to represent the sum, but only Nancy mentioned how the circle models made it easier to show the sum with two colors. She said, "On the circles, because I laid them [the addends] separately on two different circles, I can still very clearly see the half and very clearly see the fifth because there are two different colors." She was contrasting her circle representation of the sum, where she overlapped models, with her bar representation, where she tried to use two colors in one model. Thus, Nancy's thinking about the single-color model feature led her to make some comparisons between the bar and circle models for representing addition. She concluded that the circle models were superior because they more easily allowed her to replicate her existing practice of showing the addends in two colors—when used in combination with the overlap feature.

The strategy of using overlapping circle models to show a sum in two colors also led Nancy to grapple with the question of whether sums need to be represented as contiguous pieces of the whole. After realizing that overlapping the circle models allowed her to show the sum in two colors, Nancy attempted to do something similar with the bar models. She reasoned that students might approach solving ½ + ⅕ by first shading in 5 parts of a tenths strip (see Figure 39, top), then overlaying that with a fifths strip and shading one fifth. When attempting to shade the fifth in a position that was adjacent to but not overlapping the 5 tenths underneath, Nancy first produced the representation in the middle of Figure 39, then adjusted it to match the representation at the bottom of Figure 39. She said, “Oh, okay. I guess I’m okay with that. Because it [the bar she overlaid onto the tenths] is in fifths, it didn’t let me. I was looking at the tenths underneath. It wasn’t letting me color what I wanted to color.”

Figure 39: Nancy’s representations of ½ + ⅕ with bar models



Nancy had discovered that because there were no fifths that lined up directly adjacent to the 5 tenths, she was unable to use this technique to show the sum in two colors *and* represent the sum as a contiguous piece of the whole. This challenged her to consider whether the sum needed to be shown as a contiguous piece or not—something she may not have considered otherwise. She reluctantly accepted the non-contiguous representation (“I guess I’m ok with that”), but did not articulate reasons why this representation did not feel ideal to her.

This issue of the inability to show ½ and ⅕ together in a contiguous representation with overlapping bar models also arose for Sheila. She and Nancy both seemed to recognize that it could be misleading to represent the sum with overlapping addends (as in the middle of Figure 39), but both were unsure how they felt about showing the sum with a gap in the shading. Sheila said, “Oh, it doesn’t quite match up,” when she created a representation similar to the one at the bottom of Figure 39. She moved the overlapped bars apart and abandoned the strategy of showing the sum by overlapping the bars. This suggests that she was uncomfortable with showing a sum with a gap, but it is unclear why.

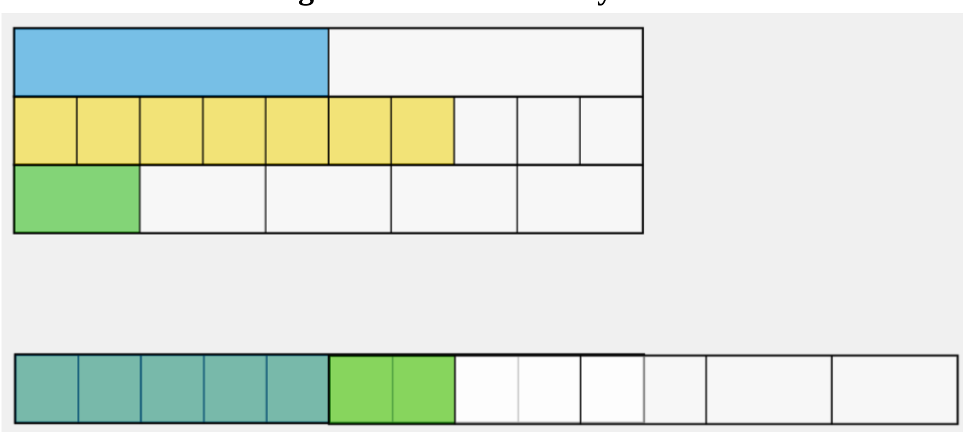
After encountering this issue, both Nancy and Sheila commented that the circle free rotation feature prevented gaps and overlaps in the sum when they used the circle models. For example, Nancy said:

> With the fraction circles, I was able to kind of reorient very, like I was able to finesse it so that even though there is no perfect half in fifths, I was able to line up with the circle, the fifth right next to the edge of the half. So that it was one continuous space. … In the bar, there's not really a nice, neat way to make sure that it's all showing the same size whole and have everything be really neatly and continuously touching.

For Nancy and Sheila, the shape of the models interacted with their strategies for overcoming the single-color model constraint and showing a sum in two colors. They both concluded the circle models were superior for representing addition because they allowed for overlapping to show two colors and also had the rotation feature to support aligning the addends without gaps or overlaps. This is consistent with the pattern of teachers showing a preference for finding ways around the VM's constraints that allow them to use their familiar practices—in this case, showing a sum in two colors *and* as a contiguous piece of a whole. The teachers did not articulate why they thought representing the sum as a contiguous piece of the whole was important, but they each paused as they encountered the issue, suggesting they were considering whether or why the gap mattered.

The differences between bar- and circle-model representations of addition were also part of Nancy's and Sheila's thinking, and also Janice's thinking, in relation to another feature reflecting focused constraints—the complete whole feature. This feature prevents a VM user from moving only the shaded parts of a model; they must move the entire whole. Using the circle models to represent addition allowed Nancy, Sheila, and Janice to address their concerns about the impact of the complete whole feature on their representations of fraction addition. Specifically, these teachers disliked how the complete whole feature sometimes makes it difficult to see the whole when bars representing addends are dragged together to show a sum. When they attempted to show the addition problem with the bar models, they represented each addend with a separate bar and tried dragging one onto the other. When they aligned the edges of the shading from the two bars, they noticed how the unshaded parts of the model for one addend extended past the end of the whole from the other addend, making it difficult to see the original whole. For example, Sheila briefly created the representation at the bottom of Figure 40.

Figure 40: Sheila demonstrates an issue with representing addition with bar models stemming from the complete whole feature. She started with the model on top, then overlaid the blue and green addends on the yellow sum to create the model on the bottom.



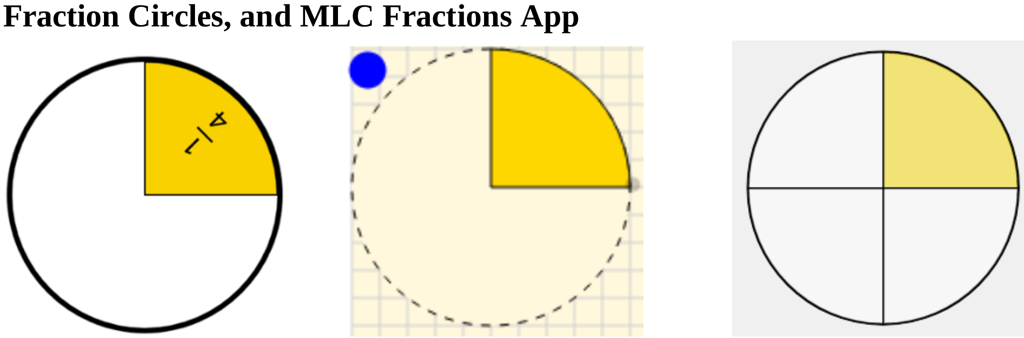
Sheila and Nancy both noted that the circle free rotation feature helped prevent this issue from happening with the circles. For example, Sheila said, “I feel like the circles were more helpful because I could rotate it. Again, I think if I were to use the bars, going back to that overlay, I could do this (overlaps her models to create the representation on the bottom of Figure 40), but then you've got all that extra that’s confusing.” Sheila and Nancy did not articulate a connection between “all that extra” and losing sight of the whole, but Janice did: “The strips, I noticed that when I move it expands the whole. So I couldn’t use these, the strips. Or I couldn't just remove the part, and just put the part in here.” Since the complete whole feature prevented the teachers from moving a shaded piece without moving the associated whole, they reflected that the shape of the circle whole, along with the free rotation feature, was necessary to clearly model addition.

It seems unlikely that these teachers would have noticed these differences in representing addition with the bar versus circle models if it were not for the single-color model and complete whole features of the MLC Fractions App. The differences they noticed—that circles more easily accommodated two colors in a model of a sum, showing the sum without gaps or overlaps, and show the sum while still also clearly showing the whole—mostly had to do with replicating their familiar ways of representing addition and sums. Even so, the contrasts between the models did bring some of their assumptions and conventions into conscious awareness, just as researchers have suggested focused constraints might do (Sarama & Clements, 2009). While confronting the single-color model and complete-whole constraints did not usually lead to teachers to accept deviations from their typical practices, it may have opened opportunities for teachers to consider the reasoning for their practices in professional learning experiences. For example, teachers’ realizations that they wanted to show sums as contiguous pieces of a clearly visible whole could start a conversation about how such representations may make it easier to judge the magnitude of a sum (as compared to non-contiguous representations or representations that do not show the sum in relation to the whole).

Can I combine addends that are not made up of same-size parts? Two features of the MLC Fractions App reflecting focused constraints, the complete whole feature and the fixed denominator feature, spurred teachers’ thinking about when and how converting addends to same-size parts needed to come into their strategies for fraction addition. All teachers were focused on creating a final model of the sum ½ + ⅕ that showed equal-sized parts. However, when confronted with a model of the sum ½ + ⅕ that did not show same-size parts, teachers had varied reactions. Karla and Tammy rejected the representation and revised their strategy so they could create same-size parts *before* creating the model of the sum. They seemed to believe they could not combine addends without first converting them to same-size parts. Nancy and Janice responded by using the overlap feature to show equal parts *within* their representation of the sum. They seemed to be more open to the idea of creating equal parts after representing the sum. I describe this difference and its relationship to focused constraints in this section.

In addition to preventing a user from moving a piece without moving its associated whole, another consequence of the complete whole feature (in combination with the automatic dividing feature and denominator-first constraint) is that all wholes created in the VM are completely partitioned into same-size parts. This is in contrast to some of the other VMs teachers used in this study, such as the Toy Theater Fraction Circles and the Phoenix College Fraction Circles. When users of these VMs place a ¼ piece into a whole circle, for example, they see a colored ¼ and a blank space for the remaining ¾ of the whole. The blank space is not partitioned into three parts. (See Figure 41, left and middle.) In the MLC Fractions App, the corresponding ¾ space is always partitioned. (See Figure 41, right.)

Figure 41: Representations of ¼ on the Toy Theater Fraction Circles, Phoenix College Fraction Circles, and MLC Fractions App



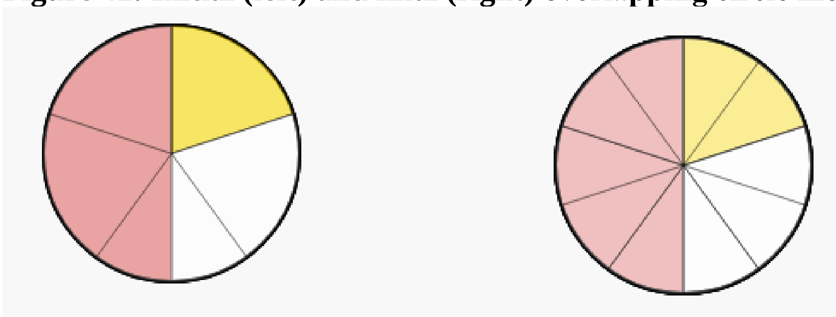
As a result of the wholes in the MLC Fractions App being fully partitioned (and the requirement to move wholes rather than pieces), when teachers represented ½ + ⅕ with circle models, four (all but Colleen and Sheila) produced a representation similar to that on the left side of Figure 42. The teachers had differing responses to this model. Two of them initially said the model was not usable. Karla simply stated, "That doesn't work" and pulled her models of ½ and ⅕ back apart. Karla's answer to the question of whether she could combine addends that were not made up of same-size parts seemed to be "no." She did not spend any time examining the original overlapping representation (Figure 42, left) but rather immediately pulled it apart.

Karla did not directly say why she thought the model "didn't work," but her next comment referenced a common denominator, suggesting her issue with the model was that it did not show same-size parts:

> See now I can't change this to tenths. Do you know what I mean? Like this is hard to solve with this one, I think. Unless I said okay, I have to get my common denominator first. And that's how I'd have to do it. So I'd have to immediately start with the tenths. And then I have to trash these.

When she said she could not "change this to tenths," Karla was referring to the fixed denominator feature—another feature reflecting focused constraints. She knew she could not change the number of parts in her original models and concluded that students would have to find a common denominator before starting to work with the VM so their models could reflect tenths from the start. Karla continued by deleting her original models, creating a 5/10 model and a 2/10 model, and overlapping those to show the sum of 7/10. Her final model looked similar to Figure 42, right.

Figure 42: Initial (left) and final (right) overlapping circle models for the sum ½ + ⅕



As illustrated under Finding 3a, several teachers used overlapping models to think about creating equivalent fractions in the context of solving the addition problem. Karla was the only teacher who specifically mentioned the fixed denominator feature when she was grappling with the addition problem, and she did not seem to think about how overlapping models might be used to show further partitioning in a model. This is one example where a teacher did not find a way around a feature reflecting focused constraints in using her familiar strategies. Instead, she decided some of the work of solving the problem could not be done with the VM.

Tammy also thought her initial representation of combining ½ and ⅕ with circle models was not viable: "I don't know that I could show it with the circles. … I see one, two, three…. They're not even." She was counting the shaded portions of the model and noted one shaded part was smaller than the others (Figure 42, left). She continued by creating a blank tenths model, moving the half model onto it, then moving the fifth model onto that to create a representation similar to the right side of Figure 42. Even though she could have laid the tenths circle on top of her already-overlapping addend models, Tammy chose to move each addend model onto the tenths, one at a time. This may reflect a desire to create addends with equal pieces before combining them, similar to Karla. (Another teacher, Sheila, also layered each addend onto a blank tenths circle to solve this problem, although she did so without first overlapping the half and fifth models.)

Nancy and Janice had different reactions to their initial models of the sum, which looked like the left side of Figure 42. They did not reject the model in the same way Karla and Tammy did—they seemed to recognize it was valid to combine parts that were not yet converted to equal-size parts. However, both teachers did acknowledge they still needed to do more work to find a way to express the sum numerically, even as they accepted they had already created a representation of it. Nancy said, “So that’s how I would show it. But now the challenge would be obviously that they are not like denominators. And so what fraction is that?” Janice hesitated after creating her model, then said, “But now what do I do next?” The next step taken by both Nancy and Janice was to create a tenths circle and place it as a third layer on their already overlapping models. These teachers’ final models looked like the one on the right of Figure 42—visually the same as that created by Karla and Tammy but achieved by different means. They counted the shaded tenths to determine the sum was 7/10.

Nancy and Janice laid their tenths circles on top of their models of the sum and spoke about the sum being 7/10 without explicitly mentioning or representing the addends as 5/10 and 2/10. Thus, they found a strategy to show equal parts *within* the sum, rather than changing the addends to show equal parts before combining them to show the sum. It may be that their thinking was still focused on the replacing ½ with 5/10 and ⅕ with 2/10, and the difference between their thinking and Karla and Tammy’s was just that they were able to temporarily ignore the different-size pieces in the model. This seems likely for Nancy, who questioned whether her students would be able to identify tenths as an appropriate model to overlay to determine how to express the sum. She wondered if her students would “recognize that fifths and tenths are kind of, they go together.” She reflected that she may need to include some instruction in thinking about “compatible” denominators when she taught about equivalent fractions to prepare students for identifying common denominators. Thus, despite reasoning about tenths *after* she visually combined the addends, Nancy was still thinking about common denominators —creating pieces that could be combined—as she produced her solution.

There is stronger evidence that Janice was thinking less about converting each addend to equal parts and more about finding a piece size that could be used to cover the entire sum. She did not seem to reason at all about the addends separately after creating her overlapping model. Her justification for choosing tenths did not reference finding a common denominator for the two addends, but rather had to do with seeing the two smaller parts at the bottom of her original model: "There's these little pieces, and I think those are tenths. So perhaps I'll make a tenths overlay and count." This suggests Janice's thinking as she used the VM may have been more of a departure from the standard algorithm for addition of fractions (which requires converting the addends to equivalent fractions with a common denominator first) than the rest of the teachers. On the other hand, her guess that the smallest pieces were tenths could easily have been based on prior knowledge of how to determine a common denominator for ½ and ⅕.

Two consequences of teachers' differing reactions to the representation of the sum on the left of Figure 42 are relevant to teachers' future professional learning about VMs. First, the most readily apparent difference among teachers' reactions to the original overlapping representation of ½ + ⅕ is that some teachers tolerated the representation with unequal parts (most clearly Janice) and some seemed to reject it (most clearly Karla). This difference may be meaningful, as recent research with students suggests that examining representations of sums that do not partition addends into equal sums supports students in making accurate magnitude estimates for sums (Braithwaite & Siegler, 2021). If some teachers are uncomfortable with doing any reasoning about a representation of a sum that does not show same-size parts, they may need more support in seeing the value of allowing students to create and examine such representations.

Second, despite some differences in responses to the original overlapping model, all teachers did at some point encounter an issue that could be leveraged to help them in thinking about why the fraction addition algorithm requires use of common denominators. For Karla and Tammy, they felt they needed same-size pieces to show a meaningful visual representation of the whole. For Nancy and Jance, they felt they needed same-size pieces to express the sum numerically. Encouraging teachers to reflect on their reactions to models such as those generated with the MLC Fractions App could be a way to develop their understanding of the rationale for the fraction addition algorithm—something recent research suggests many elementary teachers are lacking (Copur-Gencturk, 2021).

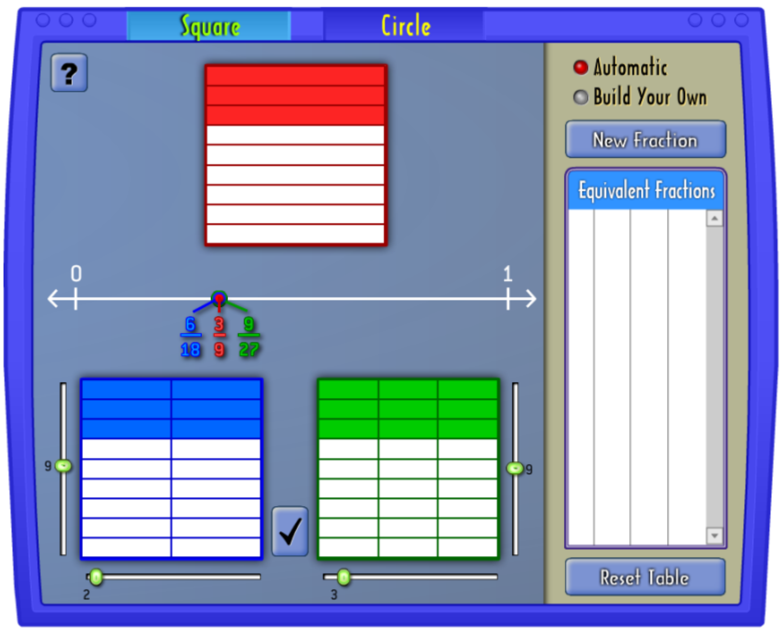
Summary of the effects of features reflecting focused constraints on teachers' thinking about fraction addition. As teachers grappled with features reflecting focused constraints while they worked on the addition problem, their goals were usually to find ways around the constraints to replicate their familiar strategies of showing addends in separate colors, representing sums as contiguous parts of the whole, and constructing representations of addends that show same-size parts. Although their resulting strategies were usually minor variations of their existing strategies, working around the constraints did seem to generate some consideration of *why* they did certain things or how their existing strategies were helpful for student learning. Teachers' work with the VMs on its own was not enough to push teachers' thinking away from their existing practices, but it did introduce opportunities for encouraging teachers to reflect on their thinking if VMs were to be used in professional learning contexts. With appropriate supports, features reflecting focused constraints could be used to deepen teachers' understanding of fraction addition.

Finding 3c: Teachers occasionally used features reflecting focused constraints to challenge student thinking in Lesson Planning.

As discussed in Chapter 4 (Results, Interpretations of Affordances), teachers interpreted some features reflecting focused constraints as potential supports for student thinking during Lesson Planning. For example, Janice thought the complete whole feature of the MLC Fractions App might support students in considering the relationship between the part and the whole. Karla thought the Toy Theater Fraction Circles' disappearing piece feature could provide feedback for students about why the queen could not give ½ of her kingdom to one daughter and ⅓ to each of two others. Sheila thought the numerator and denominator limits on the Toy Theater Fraction Bars could help students focus on ideas about fraction equivalence without having to consider, for example, what it would mean to have a numerator of 0 or a numerator greater than the denominator. In all these cases, teachers did not explicitly indicate what they planned to do to direct student attention to these constraints. Rather, they described a potential effect of the features reflecting focused constraints that they thought students might encounter on their own. Two teachers, Colleen and Nancy, went beyond interpretation and described a specific *response* to a feature reflecting focused constraints—that is, something specific they planned to have students do with the feature—that they believed would support student thinking.

One of the features reflecting focused constraints in the NCTM Equivalent Fractions App—the VM that Colleen used for Lesson Planning—is the slider limits. The sliders that control the partitions in the square and circle models in the VM have maximums (16 for the sliders on the sides of the squares and 24 for the sliders around the circles). Colleen brought up the slider limits in her student response interview. One of the student scenarios presented to her in that interview suggested a student might use the horizontal partitions in the blue and green squares to mimic what was in the red square, then set the blue square to have one vertical partition and the green square to have two vertical partitions (see Figure 43), arguing this strategy would always produce three equivalent fractions.

Figure 43: A student scenario presented to Colleen



After saying she would prompt the student to explain why that strategy worked, Colleen said she might also point out the slider limits to the student: "I think I would also point out though, well, what would you do since you can only go up to 16 on this thing on one, for the width." She did not elaborate on this point, so it is unclear exactly how she thought the slider limit would push students' thinking. She may have been intending to push the student to think about how they could generate more equivalent fractions after reaching that limit. She went on to mention the slider limit for the circle, as well: "I would have them switch to the circle and do the same. ... it only goes up to, I think, 24 pieces. So, that would have to be very strategic." This comment suggests she believed that the circle's slider limitation, which set the maximum denominator for a fraction at 24, would do more to push a student's thinking than the square's slider limitations. She may have believed that the greater restrictions on possible denominators with the circles would serve stronger grist for encouraging students to think strategically rather than relying on the procedural strategy presented in the scenario.

The only feature reflecting focused constraints in the Phoenix College Fraction Circles, the VM Nancy used in Lesson Planning, was the specific piece choices available in the VM. As mentioned in Chapter 4, Nancy pointed out that the constraint on the piece choices allowed her to challenge students to figure out whether they had found all the fractions equivalent to a given fraction that were possible with the tool. She explained, "That's one of the reasons why I'm okay with this being limited, because [when I ask them how they know they have generated all the possible equivalent fractions] I would want them to say something like, 'Well, I tried all of the pieces or I tried all of the pieces that were factors of whatever.'" Nancy's inclusion of a question about how students know they have generated all possible equivalent fractions (that could be shown with the tool) illustrates a specific use of a feature reflecting focused constraint to push student thinking.

It is unclear why it was only Colleen and Nancy who responded to features reflecting focused constraints with plans to use those constraints to push student thinking. These two teachers had quite different scores on the content knowledge and belief surveys (see Table 13) and had different amounts of experience (Nancy had many more years of experience than Colleen). It may be that they had similar responses to features reflecting focused constraints for different reasons. Nancy mentioned her response to the constraint in her lesson plan, without prompting from me, so it may be that her high mathematical content knowledge and years of experience supported her in thinking about how the constraint could be used to push student thinking. Colleen mentioned her response to the constraint in her student response interview, so her response may have been stimulated by the particular scenario presented to her.

One potential explanation for the overall low incidence of responses to features focused constraints to push student thinking is that teachers may view their role as facilitators of student learning and see intentional use of constraints as contrary to that. While the teachers encountered interesting but unfamiliar mathematical situations in Problem Solving in part because of the VM's features reflecting focused constraints (as discussed under Finding 3b), there is little evidence that they considered the thinking spurred by those situations to be useful or productive. If they view their role in terms of facilitating student learning by helping them avoid unfamiliar or confusing situations, it may be that the teachers did not see the value in choosing VMs with focused constraints or in directing student attention to them. It may be that teachers need additional support in connecting features reflecting focused constraints to pedagogical ideas such as productive struggle.

Responses to Features Reflecting Linked Representations

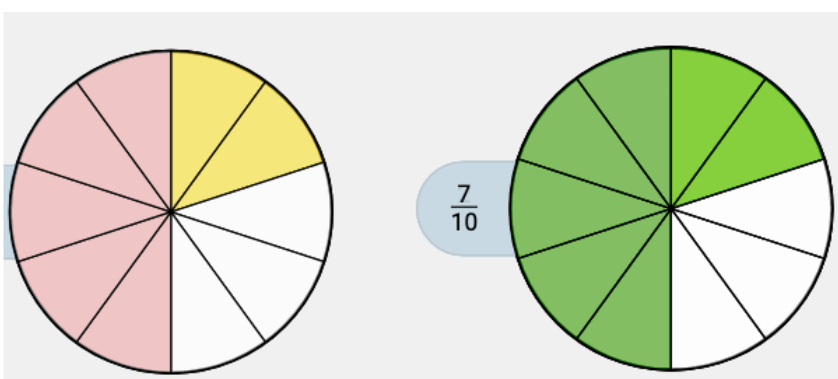
The VM features coded as reflecting linked representations are listed in Table 20. As discussed in Chapter 4 (Results, Interpretations of Affordances), Colleen was the only teacher who showed evidence of thinking much about linked representations. Analysis of how teachers used or planned to use linked representations showed that while they were rare, teachers did occasionally use these features to support their own or their students' thinking (Finding 3d).

Finding 3d: Teachers occasionally used features reflecting linked representations to support their own or their students' thinking.

Two teachers, Colleen and Nancy, used the linked labels of the MLC Fractions App during Problem Solving. As described under Finding 3a, Colleen used the linked labels to support her own thinking about creating equivalent fractions for the addends ½ and ⅕ that used a common denominator of 10. She overlapped tenths bars onto her bar models for ½ and ⅕, then re-shaded over the existing shading and used the linked label to "tell [her] how many tenths it is." (See Figure 31.)

Nancy did not turn on the linked labels on her own during Problem Solving, but she did make a positive comment about them after I suggested she do so. Nancy originally solved the addition problem by creating a model of the sum similar to the one on the left of Figure 44. At the end of her interview, at her request, I shared the way another teacher solved this problem with the VM. (This teacher did not complete phases 2 and 3, so she is not included in the analysis.) I prompted Nancy to turn on the labels and then to re-shade the tenths that were already shaded using a single color. She ended up with the representation at the right of Figure 43. She said, "Oh, I like that. Yeah. I do really like that because that really shows the sum. But if you then, by using that layering technique, you can do this [drags the tenths circle off the two addends] and you could say, see 1/2 + 1/5 = 7/10. Oh, I like that." Thus, Nancy did not use the linked labels to support her own thinking during Problem Solving, but she did express some enthusiasm about how it might be used to support student thinking once I pointed out the possibility.

Figure 44: Nancy's model for the sum ½ + ⅕ before (left) and after (right) I prompted her to turn on the labels and reshade the sum



Two teachers, Janice and Colleen, described ways they planned to use linked representations to support student thinking during Lesson Planning. Janice mentioned the linked symbols in the MLC Fractions App (which is the VM she used for Lesson Planning) only once in passing. She wrote in her lesson plan that one of the mistakes students might make when attempting to represent 15/5 (the first task she planned to give her students) would be to create a model of 5/15 instead. When I asked her about this in her lesson plan interview, Janice said, "And that at that point I'd be like, 'Okay, turn on your fraction label.' They'll turn it on and be like, 'Oh, was that five, 15 fifths? No.'" She felt the linked label might help students correct a common mistake of inverting the numerator and denominator (by providing corrective feedback).

As discussed in Chapter 4 (Finding 2f), Colleen chose a VM for Lesson Planning, the NCTM Equivalent Fractions App, that had several linked representations. Colleen had thoughtful interpretations of these features (the moving dots on the number line, the dynamic number line labels, and the equivalent fractions table) that included attention to ways students might use them productively and unproductively. For example, Colleen described how students might use a trial-and-error strategy to generate equivalent fractions, relying on the moving dots to tell them when they found an equivalent fraction without considering why their answers made sense conceptually. She planned to support students in thinking about how they could use the patterns they saw to be more strategic in their generation of equivalent fractions when they moved on to different problems. When I presented a scenario where a student used a trial-and-error strategy in her student response interview, Colleen responded by asking the student what s/he noticed about the numerators and denominators as s/he used the trial-and-error strategy:

> But if they're just clicking through, I would point out like, "Okay, what did you notice happened, when you were clicking through when you could stop, what do you notice about the numerator, denominator or whatever?" If they could make those connections.

Colleen also said she might refer to the dynamic labels if a student was struggling to move away from the trial-and-error strategy: "If a kid is really stuck, then I would say, 'Okay, the fraction is 10ths [gesturing to the denominator of the red label]. Look at that number 10. What do I know about 10ths?'" These responses could indicate Colleen saw the dynamic labels on the VM as an avenue to connect to students' prior knowledge about multiplication and factors.

Interestingly, during her student response interview, Colleen also suggested there were some situations where she would deemphasize attention to the dynamic labels. One of the student scenarios presented to her, the student had started with 9/15 in the red model (with this fraction generated by the VM), and then used a trial-and-error strategy to find the equivalent fractions ⅗ and 6/10. Even after using the feedback button to check the equivalence, the student expressed confusion on how these could be correct. How could a fraction with a smaller denominator be equivalent if you have to multiply to get equivalent fractions? Colleen said,

> I would emphasize looking at the actual amount that shaded in. So for these, the amount that the color is in the size of that part, rather than the actual digits on the top and the bottom.

She went on to say she might also try encouraging students to think about factors and multiples, a strategy that relies on attention to the symbolic fractions. Although she did not articulate this distinction clearly, Colleen appeared to be grappling with how to get students thinking about the meaning of equivalence as representing the same-size part of a whole and also about how equivalent fractions were related numerically. Her ideas about directing student attention to the area models in some situations and the symbolic fractions in other situations suggests she valued having the two linked representations available for student inspection.

While Colleen planned for several ways students might use linked representations and Janice saw one situation where she thought a linked representation might be useful for supporting student thinking, teachers made little use overall of linked representations in their Problem Solving or Lesson Planning. There are several possible explanations for this pattern. One might be that there were only two examples of linked representations in the MLC Fractions App, which the teachers explored thoroughly in Exploration and used during Problem Solving. Moreover, one of the features of this VM reflecting linked representations (the denominator choice partition link) was only apparent when users first created models. The other (the linked symbols) only linked the numerator of a fraction with the amount of shading—arguably one of the easiest connections for students to make conceptually without the dynamic link between representations. Other relationships that are more challenging conceptually, such as a greater denominator meaning a smaller piece of the whole, were not easily visible in the MLC Fractions App. It may be that the relatively low exposure to linked representations in the noticing interviews did not prime teachers to think about the potential of these features as they planned lessons.

A second explanation for teachers' relative inattention to linked representations in their Problem Solving and Lesson Planning has to do with the ways in which linked representations might be used to offload thinking work to the VM. If the goal of a particular task is to generate an answer, linked representations can often be used to find that answer through means that do not require high-level thinking, as when Colleen predicted her students might use the moving dots on the number line to find equivalent fractions without thinking about them conceptually. Linked representations are most useful for supporting thinking about mathematical relationships, whereas many elementary mathematics lessons focus on answer-getting. It may be that teachers would need more support in generating lessons and tasks that make meaningful use of linked representations than they would in generating lessons and tasks that make meaningful use of efficient precision or focused constraints.

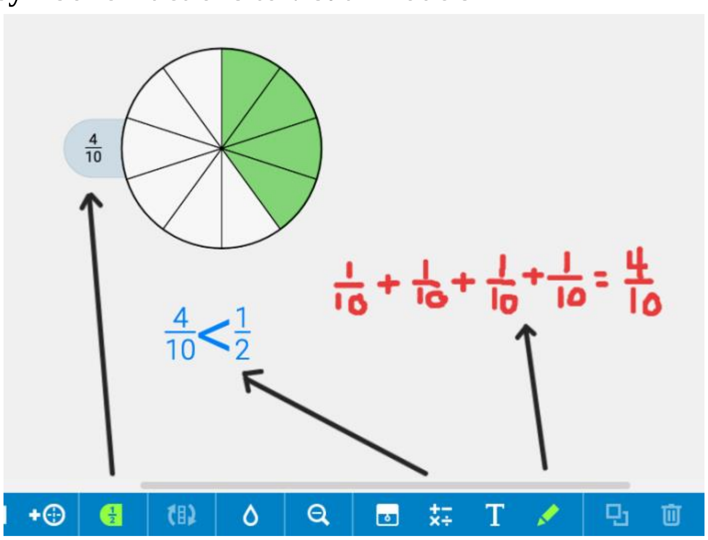
CHAPTER 6:
RESULTS, INTERPRETATIONS AND RESPONSES TO LABELING FEATURES

In this chapter, I discuss results related to Research Question 4:

> How do teachers think about the features of VMs that could support students in connecting visual and symbolic representations of fractions?

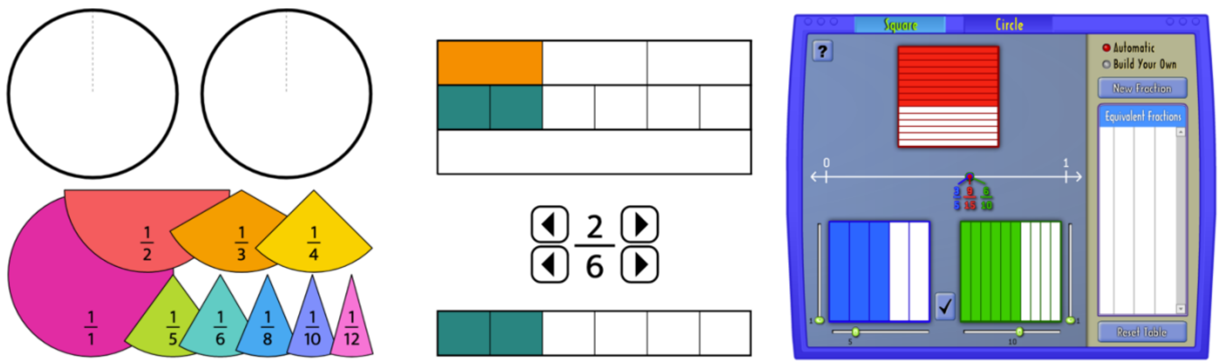
This question stems from one of the results presented in Chapter 3 (Results, Attention to VM Features): During Exploration and Lesson Planning (but less so during Problem Solving), teachers showed particular interest in VM features that could support students in connecting symbolic (numeric) fractions to the visual representations at the core of the VMs. In the MLC Fractions App, used by all teachers in Exploration and Problem Solving, these features were the label button, pencil button, and text tool button. The label button shows and hides the tabs on the left side of the circle and bar models, the text tool button allows a user to type mathematical expressions and equations, and the pencil button allows a user to freely annotate. These features are illustrated, left to right, in Figure 45.

Figure 45: Features of the MLC Fractions App that could support students in connecting symbolic fractions to visual models



All but one of the VMs used by teachers in Lesson Planning also had features that could support students in connecting symbolic fractions to visual representations of fractions, including the labels on the Toy Theater Fraction Circles (see Figure 46, left), the numerator and denominator students could manipulate to change the fraction bars in the Toy Theater Fraction Bars (Figure 46, middle), and the number line labels in the NCTM Equivalent Fractions App (Figure 46, right). For brevity, I use *labeling features* as a term to refer collectively to these features across VMs, but I acknowledge that the features vary both in their nature (e.g., static versus dynamic) and how they might be used (e.g., the pencil in the MLC Fractions App can be used to create any annotation, not only a symbolic fraction).

Figure 46: Features of other VMs that could support students in connecting symbolic fractions to visual models



One of the most challenging aspects of teaching and learning with manipulatives is supporting students in making connections between the manipulatives and the underlying mathematics they are intended to represent (Uttal et al., 2009). One way to facilitate these connections is to consistently ask students to map representations of a mathematical idea onto one another, including visual representations onto symbolic representations (Rau & Matthews, 2017). Given this challenge, I explored whether teachers' high attention to labeling features of VMs was accompanied by interpretations and responses that related to supporting students to create and map symbolic representations onto visual representations or vice versa. Teachers varied in how they thought about labeling features and whether they connected them to the challenge of supporting students in connecting representations, but context did not seem to have an impact on their thinking. I describe below this finding in detail below.

Finding 4: Teachers varied in their attention to and consideration of connecting symbols to visual representations, but interpretations stayed consistent by individual across contexts.

I sorted the six teacher participants into three groups according to the ways they thought about and used the labeling features on the VMs. In each group, teachers' thinking about this issue was most clear in the Lesson Planning context, but evidence of similar thinking was also present during Exploration and sometimes Problem Solving. Two teachers, Colleen and Karla, focused on connecting visual and symbolic representations in their lessons and planned to utilize a feature of their chosen VM to support students in doing so. Two others, Nancy and Janice, also planned to spend time within their lessons supporting students in connecting visual and symbolic fraction representations, but they did not utilize VM features to support this work. The final two, Tammy and Sheila, made occasional comments about the labeling features but did not connect the features specifically to the issue of connecting symbolic and visual representations or did not emphasize these connections in their lesson plans. I describe how each pair of teachers interpreted and responded to the labeling features below. In a final section, I discuss factors that may explain why the teachers were grouped into these three pairs.

Colleen and Karla: Using VM features to support students in connecting visual and symbolic fraction representations

Colleen and Karla shared three similarities in how they discussed labeling features of the VMs they used during Lesson Planning. First, they each mentioned a labeling feature as one of the reasons they chose a particular VM for their lesson. Second, they each made a connection between a labeling feature and a learning goal they had for the lesson. Lastly, they both pointed out how they would encourage students to think about how the visual and symbolic representations in the VMs were connected. Neither teacher assumed the presence of a labeling feature would necessarily lead students to meaningfully connect visual and symbolic fraction representations.

Colleen chose the NCTM Equivalent Fractions App (see Figure 46, right) for her lesson on generating equivalent fractions. She mentioned the dynamic labels on the number line as she described her reasons for choosing this VM: “I like that, as they click through, the dot moves and the fraction. It shows them the number [on the label], it shows them the point, moving along the number line.” Colleen was more interested in the dynamic labels than the moving dots for the purposes of the lesson she planned. She described students attending to the position of the dots on the number line as a “side goal” of her lesson. She thought student attention to the labels on the number line was more important than their attention to the position of the dots because students had struggled to connect area models to symbolic fractions in the past:

> I want them to see the picture but also see the numbers, see the fractions. Because I feel like sometimes kids are able to color in a model. But then, I say, or I could, going back to the adding and subtracting that we just did, I could say, “What’s 3/8 plus 2/8?” And they would color it in on a model. And then, some of them wouldn’t be able to tell me the answer.

This comment suggests Colleen saw supporting students in connecting visual and symbolic representations as an ongoing goal in her instruction (if not the main goal of her lesson), and that she saw the dynamic labels as a feature that might support students in this.

In both her lesson plan and student response interviews, Colleen acknowledged that even though the NCTM Equivalent Fractions App showed connected visual and symbolic representations of fractions, students might not attend to both or think meaningfully about either one. She described some trial-and-error strategies students might use to generate equivalent fractions by just clicking around in the area models until they saw the points on the number line coincide. When I presented a scenario in her student response interview of a student using such a trial-and-error strategy, her response suggested she planned to push students to connect the area models to the symbolic fractions on the labels or in the equivalent fractions table on the right side of the screen (see Figure 46, right). She first said she would encourage students to look for numeric patterns in the symbolic fractions:

> But if they're just clicking through, I would point out like, "Okay, what did you notice happened, when you were clicking through when you could stop, what do you notice about the numerator, denominator or whatever?" See if they could make those connections. And then they probably would say, "Oh, well, eight is double four. So, that... And six is double three." So they probably would be able to do it.

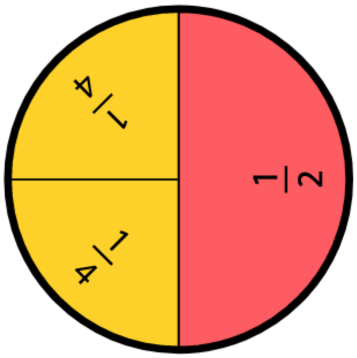
She also acknowledged that directing attention to the symbolic fractions might lead students to ignore the area models and just focus on the multiplication rule, but she had a plan for that possibility as well: "But then again, that kind of leads just straight into the rule rather than them being able to see. But afterwards I would talk about looking at the sizes." In general, throughout her interview, she encouraged student attention to both representations, suggesting she intended to intervene to support students in making connections between representations—or at the least, to prevent students from relying on only one representation.

Karla chose the Toy Theater Fraction Circles VM for her lesson (see Figure 46, left). In her lesson planning interview, she described the symbolic labels on the fraction circle pieces as a key reason she chose the VM, even though it did not offer a different feature she wanted: "I would have liked that this allowed some transparency, but I needed it to have the names of the fraction on it because … we're doing the correspondence between adding them to one." This reasoning provides the first indication that Karla saw the VM's labels as connected to her lesson's learning goals.

Karla discussed this connection in more detail as the lesson plan interview went on and brought it up again in her student response interview. As described in Chapters 4 and 5, Karla's lesson centered on a task that required students to partition one whole into different-size parts to solve a contextualized problem about a queen sharing her kingdom. The first part of the task in Karla's lesson was for students to explain why a queen could not give ½ of her kingdom to one daughter and ⅓ of her kingdom to each of her other daughters. After creating the representation in Figure 47 to illustrate a workable solution (rather than giving the two younger daughters ⅓, they each get ¼), Karla explained,

> If we line that up and say ½ + ¼ + ¼ = 1 whole, [the number sentence] is sort of generated for them. And then I think we have to get into that conversation about, okay, ½ is how many ¼s? And then say, okay, well that's ¼ + ¼ + 2/4 = 4/4 = 1 whole. I think it's easy to see that way, that if they had to use the other [VM, which doesn't have the labels] and figure out what the fraction was in their head and then talk about adding it, I thought that was too many levels of thinking because this is complicated enough that they didn't need that extra piece.

Figure 47: A solution for Karla created to illustrate why the fraction labels were important for her lesson



For Karla, the labels acted as a scaffold to support students through the multiple levels of thinking it took to go from a partitioned whole, to determining what fraction each piece of the whole represented, to writing a number sentence that expressed the sum of the parts being equal to 1. Understanding that the fractions into which a whole is partitioned must sum to one was a main learning goal of her lesson, and Karla felt the labels were key to supporting students' thinking about that idea.

In her student response interview, Karla discussed the labels as part of what would prevent some of the issues that had come up in past implementations of this lesson when students were drawing to solve the problem:

> They'll draw something and divide it whatever way they want. I think when the kids drew it, they would draw the half here and then say a third and a third, and it would be done. Not paying any attention at all to the fraction, just thinking about pieces. "Oh, that's three pieces now", do you know what I mean? But when they have this in front of them, and they put that in, now they see it's a fourth. When they're drawing, they don't see that, and they would just write a third in there.

Karla was describing students making a drawing similar to Figure 47 but writing ⅓ in each of the fourths. She felt that because the labeling was there, students wouldn't be able to make that mistake. She recognized that in past implementations, students had difficulty connecting symbolic fractions to the pieces that represented them, and she felt the VM's labels would help students with that difficulty. (As discussed in Chapter 4, Karla also thought the exact size of the pieces was important for addressing this common mistake.)

Even as Karla appreciated the potential of the VM's labels for supporting student thinking about how the size of the pieces contributed to the whole, she also acknowledged that the labels would not be beneficial even in this lesson unless students attended to them as they worked:

> I hope they're looking at the fractions. I do, I hope that they're just not... You hope that they're paying attention to which fractions actually fit and which don't. And, "Well, the 1/5 doesn't fit in there," or whatever, and thinking about that rather than just quickly popping pieces in and seeing what works, which definitely will happen, I know it will. But I think we'll talk a little bit about thinking as we work and things like that before we begin.

While the labels were important to her, Karla acknowledged their presence wouldn't solve every problem and that students would need to be pushed to attend to them as they worked.

While Colleen and Karla's interest in using labeling features to support students in making connections between visual and symbolic representations was not as clear or elaborated in Exploration or Problem Solving, they each made at least one comment that suggested consistencies with how they interpreted and responded to labeling features in Lesson Planning. While she was exploring the MLC Fractions App, Colleen said the following about the label button that toggles the symbolic labels on and off the circle and bar models (see Figure 45): "I like this tab on the side that says it, but I wish it was always there and not necessarily just where you can click it on and off." This comment is consistent with her choice to use a VM for Lesson Planning that always showed the symbolic fractions, and also aligns with her comment in Lesson Planning about thinking it was important for students to see both visual representations and numbers.

Colleen did not say anything else about the labeling features in Exploration, but as noted in Chapter 3 (Finding 1c), she did not engage in much experimentation with any feature in Exploration. During Problem Solving, Colleen had the linked symbols on as she solved the addition problem ½ + ⅕ and used them as a kind of check or calculator for different parts of her problem-solving process. To find an equivalent fraction using tenths for each addend, she overlaid a tenths bar on each of the bars she used to represent the addends. She then reshaded the tenths that covered the ½ and the ⅕, saying, "I might use a different color just to color it in and to tell me how many tenths it is." The last part of her comment referred to the linked label that counted her tenths for her. When she added two additional tenths to her model of five tenths to represent the sum, she said, "Then that would equal seven tenths, and it tells you right there." To press her thinking, I asked, "Is that good or bad?" She answered, "I don't know. I guess it would be good once they understand fractions, but it might be bad because then they don't necessarily ever need to figure it out on their own." Thus, as in her Lesson Planning interviews, Colleen mentioned attention to both the visual and symbolic fraction representations in Exploration and Problem Solving and, with prompting, acknowledged that merely having the symbolic fractions there did not mean students would connect them to the visual fraction representations.

Like Colleen, as Karla explored the MLC Fractions App, she mentioned the way the label button toggled the symbolic labels on and off. However, rather than wishing the labels stayed on, Karla liked the flexibility of the toggling: "I do like that they can see the fraction, but they don't necessarily have to use that. Only if you want them to." This comment suggests she thought there were some instructional situations where the labels could be a good support and others where it would be beneficial to turn them off. During Exploration, Karla also used circle models to represent the addition problem ½ + ¼, and then used the MLC Fractions App to write a number sentence to represent the problem, saying her students "could write their representation of what they did on there. And then they could send it to me." Thus, as in Lesson Planning, during Exploration Karla showed interest in how VM features could support students in seeing visual and symbolic representations of fractions together. Further, her attention to how the labels could be turned off and her use of the text tool, which does not automatically generate a corresponding fraction, suggested she did not think the automated labeling was appropriate all the time and she may choose to have students create the symbolic representations on their own for certain lessons.

Nancy and Janice: Supporting students in connecting visual and symbolic fraction representations without use of VM features

Like Colleen and Karla, Nancy and Janice also expressed intentions to support students in connecting visual and symbolic representations of fractions during Lesson Planning. In contrast to Colleen and Karla, Nancy and Janice did not select VMs for Lesson Planning based on the availability of features that would support students in making those connections. Their plans for how they would support students in making connections between visual and symbolic representations had little to do with the features of their VMs.

Nancy chose the Phoenix College Fraction Circles for her lesson on generating equivalent fractions. As shown in Figure 6, this VM does not have symbolic fraction labels on the pieces. Rather, the only labels are a letter corresponding to the name of the color of the piece, and those can be hidden. There are also no other features that might support students in creating symbolic representations to accompany the visual representations they create with the circle pieces. While Nancy did not say she chose this VM *because* the pieces did not show fraction labels, she said she liked that the pieces were not labeled because if students had to figure out what each piece was worth for themselves, they would build “a little bit deeper understanding” of why each piece represented the fraction it did:

> If they had it labeled, they wouldn’t necessarily have to understand why it was labeled that. But if they have to stack them up and fit them in the circle themselves and then they go, “Aha, there is five of them. So that means fifths, this is one fifth.” They then have that association that it’s a fifth because I put five of them to make a whole circle. But if it just said fifth, there would be a lot of kids who would just be like, “Oh yeah, that’s a fifth.” They wouldn’t have any clue what they meant.

This excerpt shows that Nancy did think it was important for students to make a connection between the size of the VM’s pieces and symbolic fractions, but unlike Colleen and Karla, she did not think a labeling feature would support them in making those connections or thinking about the piece size or the symbols in relation to the underlying fraction concepts. Rather, she thought labeling features might make students *less* likely to make the connections. To support students in connecting pieces to fraction values, Nancy planned to start her lesson with a discussion of how students could fill a whole to determine what fraction each piece represented.

Nancy also described other ways she would intervene to support students in connecting the Phoenix College Fraction Circles to symbolic fractions in her student response interview. Nancy’s responses to the student scenarios all involved some combination of making suggestions of how students could manipulate the fraction pieces while she connected their work to symbolic annotations she would make with some kind of annotation tool (if she were teaching digitally) or on a whiteboard (if she were face to face with students). For example, after she helped me (as a hypothetical student) use the fraction circle pieces to “prove” that 4/6 was equivalent to ⅔ but ⅗ was not equivalent to either, Nancy said, “So this is where I would probably grab a pen tool. I would want them to see the fraction ⅔ written next to the fraction 4/6 to see if they could make any connections. … I definitely want to start connecting the actual notation to the visual.” Her annotations were number sentences (equations or inequalities, depending on the pair of fractions), and she verbally connected the symbols to the fraction pieces the students were working with and encouraged them to look for patterns.

Janice used the MLC Fractions App for Lesson Planning, the same VM that all teachers used during Exploration and Problem Solving. While this VM has several features that could be used for labeling, Janice rarely mentioned them during Lesson Planning. Her only mention of the text tool was to say some of her students liked it, but she did not (she had used the VM for a few lessons in her classroom before the lesson plan interview). She mentioned the label button once during her lesson plan interview in a way that was more meaningful to the lesson, when she suggested students use the label to help them realize the difference between 15/5 and 5/15. (See Chapter 5, Finding 3d). However, she did not mention the label button again anywhere or her lesson plan or student response interviews.

Janice did mention the third labeling feature of the MLC Fraction App, the pencil tool, during Lesson Planning, and she did use it—or suggest that students use it—to create symbolic fractions to correspond with visual representations. Throughout her lesson plan interview, Janice repeatedly used the pencil tool to write symbolic representations to accompany the visual models she created—especially to write whole or mixed numbers to accompany visual representations of fractions greater than 1 (or suggest that students do so). For example, after students represented

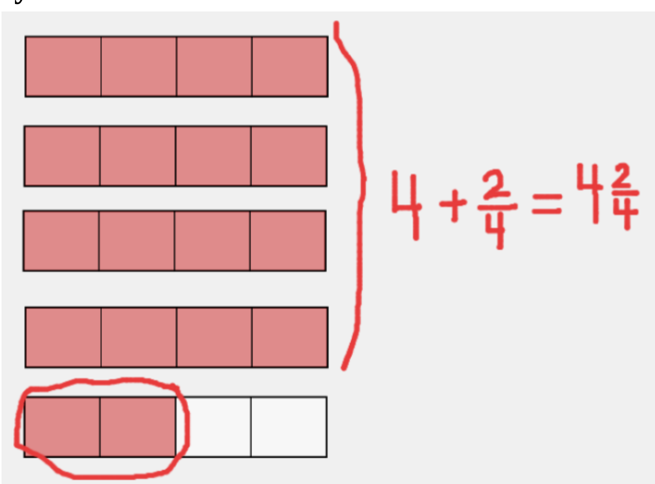
15/5 with bar models, Janice said, “Then I would tell them, ‘Okay, label it.’ We’re not good at labeling right now.” As she said this, she used the pencil tool to write 15/5. Janice’s uses of the pencil in the student response interview were similar to most of her uses of it in the lesson plan interview. In her responses to two of the student scenarios, she suggested the student use the pencil to record symbolic representations of the VM’s visuals.

This use of the pencil tool could be considered use of a VM feature to support students in connecting visual and symbolic representations of fractions. However, two other pieces of information about how Janice thought about the pencil tool suggests she was not thinking about it as a VM-embedded support for connecting visual and symbolic fraction representations in the same way that Colleen and Karla thought about the labeling features of their VMs.

First, Janice mentioned at the beginning of her lesson plan interview that she liked the VM’s pencil tool because it allowed her to treat the VM screen as a whiteboard: “I also really like, in Zoom we use the whiteboard function a lot. And I like how this mimics that Whiteboard function.” This suggests that Janice’s thinking about the pencil tool related more to the ways she could bring her existing annotation practices into VM’s digital space, rather than as a particularly useful feature that was connected to the essence of the VM. She liked that the pencil tool allowed her to treat the VM like a whiteboard, but since she had other tools for doing the same thing, it is not clear that she thought about the annotation ability as specifically connected to an instructional goal of helping students connect symbolic and visual representations of fractions.

Second, in addition to using the MLC Fraction App’s pencil tool to write out symbolic representations of fractions, Janice also used it to make other annotations on the visual representations. For example, when showing me how she would demonstrate using the VM to convert 18/4 to a mixed number, she made a mark grouping the four full fraction strips together before writing 4, then circled the 2/4 fraction strip before adding 2/4 to her annotation (see Figure 48). Janice also mentioned other ways she could envision students using the pencil tool. For example, when demonstrating potential student strategies for representing 15/5 as part of the lesson plan interview, Janice mentioned the possibility of students creating blank fraction bars and using the pencil to partition them. All these uses suggest Janice was thinking quite flexibly about the pencil tool, and not thinking of it specifically as a way to support students in connecting visual and symbolic fraction representations.

Figure 48: Janice used the MLC Fractions App pencil tool to do more than write the symbolic fractions



Consistent with how they thought about labeling features of VMs in Lesson Planning, both Nancy and Janice showed little interest in the label button in the MLC Fraction App during Exploration or Problem Solving. During Exploration, Nancy said the MLC Fractions App “didn’t seem to be as effective with the labeling” as other VMs we explored in the professional learning because students had to take the extra step of turning the labels on. Upon further reflection she did briefly mention she liked the dynamic connection between the label and the fill in the visual models and that she could turn the labels on and off. In her explorations of the MLC Fractions App, Janice was not explicitly negative in her interpretation of the label button but hinted that she was considering the ways that the labels might limit student thinking opportunities. She said the label “tells you what the numerator and denominator ought to look like” and “just gives it to me.” She did not elaborate on the implications of this implied possibility of the labels giving something away to students during exploration, but her comments suggest she may have been thinking about lost student thinking opportunities. This may relate to why she didn’t intend to have students use the linked labels in her planned lesson.

Nancy and Janice were more interested in the text tool and pencil tool of the MLC Fractions App during Exploration than they were in the label button. Both teachers used the text tool once to create a symbolic representation of something they created visually (a comparison number sentence for Nancy, an equivalent fraction for Janice). This is consistent with the interest they both expressed in supporting students in connecting visual and symbolic representations in Lesson Planning. The fact that they used the text tool rather than the label button for this purpose in Exploration is consistent with their interest in students generating the symbolic representations independently, without the assistance of an automatic link or static label within the VM.

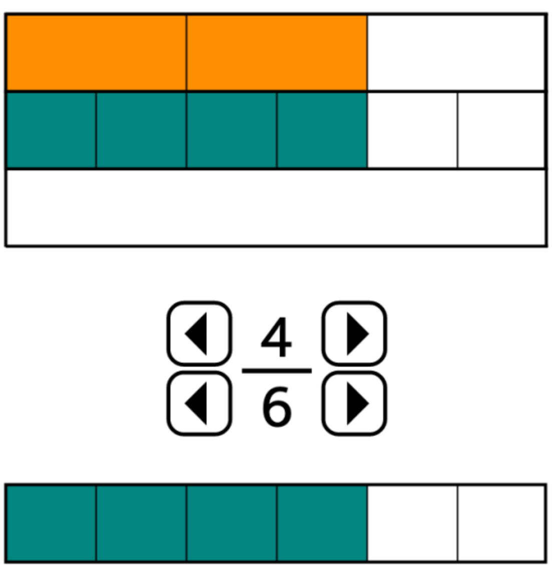
Janice and Nancy also both used the pencil during Exploration, but for purposes other than writing symbolic fractions. Nancy suggested using the pencil to cross out pieces in a circle model to represent subtraction and to draw in additional divisions in a model. Janice also used the pencil to create additional dividing lines in a fraction bar, changing thirds to sixths to show an equivalent fraction. She also used it to color in unshaded parts of a fraction bar as a way to represent addition and briefly experimented with using the pencil to draw a number line. These uses are similar to the flexible ways Janice used the pencil during Lesson Planning, suggesting neither teacher thought about the pencil tool explicitly (or at least exclusively) as a feature that could be used to support students in connecting visual and symbolic representations of fractions.

Sheila and Tammy: Minimal interest in supporting students in connecting visual and symbolic fraction representations.

The final two teachers, Sheila and Tammy, were similar to Colleen and Karla in their thinking about labeling features: Both mentioned a labeling feature as one of the reasons they chose a particular VM for Lesson Planning. In contrast to the other four teachers, however, neither Sheila nor Tammy talked about supporting students in making connections between visual and symbolic representations of fractions as part of their goals for their lessons, and they did not discuss using a VM's labeling features for this purpose.

Sheila chose to use the Toy Theater Fraction Bars for her lesson on generating and seeing patterns within equivalent fractions. With this VM, students use arrow buttons to manipulate the numerator and denominator of a symbolic fraction and the fraction bar beneath the symbolic fraction changes to match (see Figure 49). Students can then drag the fraction bar into one of the slots at the top of the screen and compare it to other fraction bars.

Figure 49: Showing equivalent fractions on the Toy Theater Fraction Bars



Sheila mentioned the symbolic fraction representation and the way it corresponded to the fraction bar in her lesson planning interview as one of the reasons she chose to use this VM:

> Some of [the other VMs] had the fractions labeled and some of them didn't. And this with this one, you can see, this is 2/5, and now I'm moving 2/5. So even though it doesn't stay labeled, I'm still seeing that fraction together. Whereas the ones where you build and put the pieces in, you're not seeing 2/5. You see 1/5 plus 1/5 plus... You know what I'm saying? So I felt like that was why I liked this one, even though it doesn't stay labeled.

Sheila seemed to appreciate, in particular, that the linked symbolic representation allowed for the "label" to show non-unit fractions, and so students did not have to think of non-unit fractions as being composed of unit fractions. She did note, however, that once students drag the fraction bar away from the bottom of the screen and into one of the slots at the top of the screen, the link between the symbolic fraction and the bar is lost, and she said would have preferred that the bars stayed labeled.

This was the extent of Sheila's discussion of the connection between the symbolic fraction and the visual models, however. When she described how she imagined the lesson would proceed, she said students would write down lists of equivalent fractions they found by looking at the "common lines" on the VM. She later clarified that by "common lines," she meant the places where the partitions in the fraction bars aligned. As students recorded symbolic fractions in their notebooks, she said she hoped students would "start noticing or seeing those patterns like we said, 1/2, 2/4 where the numerator is half the denominator or that you can multiply to get from two to four, three to six." When I followed up to ask if she thought students would be seeing those patterns in the VM's visual representations as well as in their lists of symbolic fractions, Sheila replied,

> I could see how the manipulative does show that as well, because there are two blocks here that make up this 1/3 and two here that make up this 1/3. *(See Figure 49.)* I

> definitely think the student could see that pattern as well. I guess I didn't initially think of that because that's not what I'm drawn to. I would be drawn to seeing the numbers, but a student could see that pattern. ... There are always independent thinkers. Students who think very differently about math and especially fractions because it's different than other math. I would say for the majority of them, they would not see that pattern, or the majority of the students that I've had in the past would not have seen it.

This comment clarifies that although she expected students to use the VM to generate their lists of equivalent fractions, she wasn't expecting the subsequent pattern-finding to connect back to the visual representation. Rather, although she acknowledged students might see the pattern in the fraction bars, she did not expect them to do so and did not intend to push for that connection. Her goal was focused on students using models to generate equivalent fractions (with the VM doing the work of matching the symbolic fractions to the models) and then moving on to look for patterns in the symbolic representations.

In short, while Sheila directed students' attention to both the visual and symbolic models, there was little evidence within her lesson plan or student response interviews that she would push students to explain why the multiplication and division patterns made sense in the context of equivalent fractions or to meaningfully connect the patterns they saw in the visual and symbolic representations. Her only comment about connecting numeric patterns to the visual representations created with the VM was in response to a student scenario, and only after I pushed her to think about how she might use the VM in her response.

Tammy planned to allow students to use both the Toy Theater Fraction Circles and the Toy Theater Fraction Bars in her lesson, which focused on students solving equal sharing problems and discussing how different strategies led to different (equivalent) expressions of the same answer (e.g., 3/2, 6/4, and 1½). Tammy said the labels on the Toy Theater Fraction Circles were one of the reasons she chose this VM over other fraction circle VMs. She explained, "With the circles, I think it's important to have [labels] because you can't actually see the chunks like you can in the bars." She felt that because the bars showed the whole fully partitioned, students could more easily determine how much a piece was worth, but in a circle they couldn't easily count the pieces to determine this. In her student response interview, she also mentioned her belief that the Toy Theater Fraction Circle labels might help students overcome the common misconception that a larger denominator should mean a larger piece.

Tammy did not attend to the labeling feature of the Toy Theater Fraction Bars while discussing her lesson plan, but she did discuss it as part of her student response interview. She explained that in the past, she noticed the ways her students were highly dependent on referencing colors of manipulatives as they described their work with fractions, and this concerned her since students would not have colorful manipulatives to work with when they took standardized assessments. She wanted to make sure students could make their own drawings for fractions (eventually) and connect visual representations to symbolic fractions. The link available in the fraction bars VM, in Tammy's opinion, might facilitate students in making that connection: "So I think that a benefit to this manipulative is actually seeing the number and seeing the fractions as opposed to just seeing color."

Even as she occasionally mentioned the issue of students constructing a connection between a visual representation and a symbolic fraction, Tammy did not elaborate on this issue as something she would support or be paying attention to in her lesson. Moreover, she did not describe the VMs in general as central to the student thinking she expected to go on during the lesson. Rather, she planned to offer the two VMs as tools to students but said she would be fine with it if students used other strategies, such as drawing. The benefit of using the VMs, in Tammy's view, was that they would support students in creating more accurate pictures of fractions with denominators that were difficult for them to draw by hand—which, in her view, was anything but halves and fourths. Because the VMs did not play a central role in Tammy's lesson plan, it is not surprising that she did not think much about the way VM features might support students in making connections between visual and symbolic representations of fractions.

Similar to how they thought about labeling features in Lesson Planning, Sheila and Tammy occasionally mentioned or used labeling features of the MLC Fractions App in Exploration, but only in passing. Each teacher used a labeling feature to write a number sentence once during Exploration—Sheila used the pencil tool to write a comparison she showed with fraction bars, and Tammy used the text tool to write an addition equation for a problem she represented with fraction circles. Each of these teachers also observed that the label button showed the symbolic fraction. Sheila specifically pointed out the utility of the label when working with a large denominator (and therefore small pieces): "And it's nice to have the label on the side so I know exactly how many hundredths." But this was the extent of teachers' commentary or use of the label features. They experimented with the labeling features but did not have much to say about them.

Explaining the teacher pairs

The consistency of each individual teachers' thinking about labeling features across contexts suggests that their orientation toward using VM features to support students in connecting visual and symbolic representations of fractions may be connected to personal rather than contextual characteristics. However, it is not clear what those personal characteristics might be for each pair. Janice and Nancy had the two highest scores on the math content knowledge survey (see Table 13). Their high levels of fraction knowledge may be a partial explanation for why they intended to support students in connecting visual and symbolic representations of fractions but not use the VM features to do so. Because they had a deeper understanding of fractions, they may have felt more confident in their abilities to support students in connecting representations without the help of a VM feature in comparison to the other teachers.

A clear individual characteristic linking Karla and Colleen did not emerge to explain why they both intended to use VM features to support students in connecting visual and symbolic representations of fractions. Karla had relatively high fraction knowledge and Colleen had relatively low fraction knowledge according to the math knowledge survey measure (see Table 13). According to the teaching beliefs survey, they had similar orientations toward discovery versus transmissionist teaching and toward believing problem solving could come before fact learning, so a common set of beliefs about teaching may be a partial explanation. Their mid-range scores on discovery versus transmissionist teaching, in particular, may suggest they liked the balance between expecting students to make connections between representations (a constructionist idea) and having a support there on the VM to assist them in doing so (an idea that could bend transmissionist). It may also be that these teachers saw promise in the VM labeling features for addressing a teaching issue they had encountered in past years. Colleen mentioned that she noticed students were often unable to label their drawings with symbolic fractions. In past implementations of her lesson, Karla said students had difficulty coordinating the size of the pieces in their drawings with fraction names.

Like for Colleen and Karla, there is no obvious unifying individual characteristic linking Sheila and Tammy that would explain their relative lack of concern with supporting students in connecting visual and symbolic representations of fractions. They had different levels of fractions knowledge and different belief orientations (see Table 13). Their reasons for not attending to this instructional issue may be different. For example, Tammy positioned the VMs she chose as optional extra tools that students could choose to use (or not) as they solved problems. Thus, in our discussions of the role of the VMs in her lesson, she may not have been considering all aspects of her instruction and the issue of connecting representations may not have come up. Sheila did intend for her chosen VM to serve a primary role in her students' work but seemed most concerned with the connections students might make across examples of equivalent fractions. This may have eclipsed any focus she intended to place on making connections across different representations of the same fraction.

CHAPTER 7: DISCUSSION

From the detailed results discussed in Chapters 3-6, three top-level findings are apparent: (1) teachers' interpretations of features reflecting the three VM affordances were largely consistent across individuals and contexts, but (2) an exception to that finding was that teachers interpreted features reflecting efficient precision and focused constraints in more balanced ways in the Lesson Planning context than in Exploration and Problem Solving, and (3) features reflecting efficient precision and focused constraints disrupted some patterns in teachers' thinking about fractions in productive ways that could be leveraged in professional learning sessions. I summarize each of these three top-level findings and relate them to existing research in the sections that follow. I conclude the chapter with a discussion of implications of this study for future research, design of VMs, and teachers' professional learning.

Teachers' Interpretations of VM Affordances

There was a consistent pattern across individuals and contexts in teachers' interpretations of features reflecting the three focal VM affordances (efficient precision, focused constraints, and linked representations): Teachers had mostly positive interpretations of features reflecting efficient precision (Finding 2a), mostly negative interpretations of features reflecting focused constraints (Finding 2d), and mostly lukewarm interpretations of features reflecting linked representations (Finding 2f). More specifically, teachers valued the ways that features reflecting efficient precision relieved student burdens in creating fraction representations. They discussed how the VMs' automatic partitioning of wholes into equal-size parts and features supporting the precise placement and alignment of models could prevent students from getting bogged down in the details of creating accurate fraction models. By contrast, teachers teachers tended to view features reflecting focused constraints negatively because the constraints prevented them from using their familiar problem-solving strategies and ways of using fraction representations in their teaching, especially for fraction addition. Rarely did teachers see value in a constraint, and they often expressed desires to manipulate the VMs in ways the constraints precluded. Lastly, perhaps because few features reflecting linked representations were available on the VMs they used, teachers had less to say about these features than those reflecting the other two affordances and made less frequent use of them across contexts. I relate this pattern of interpretations to prior research about the impact of each of these affordances in the sections that follow.

Efficient Precision

Previous research suggests the efficient precision afforded by VMs could support student learning in a variety of ways: by alleviating difficulties in motor skills, allowing students to solve more problems or generate more solutions in the same amount of time, leading students to be more methodical and purposeful in their creation of examples, and supporting students in noticing mathematical patterns as they create and organize examples (Moyer-Packenham & Westenskow, 2013). Teachers' initial reactions to fractions VM features reflecting efficient precision tended to focus on the first point. They emphasized how features that supported the creation of equal parts or assisted students in snapping pieces or models into alignment would be helpful, as they often observed students struggling with creating accurate representations when working with drawings or physical manipulatives. Teachers also expressed some appreciation for how quickly and easily they could make fraction representations in the Problem-Solving context. Teachers' general appreciation for the power of VMs to alleviate motor difficulties or procedural burdens suggests these features may serve as a strong starting point for helping teachers notice and appreciate affordances of VMs. On the other hand, teachers did not mention benefits of features reflecting efficient precision beyond augmenting students' motor skills or lessening their difficulties in creating representations in Exploration or Problem Solving. Their lack of attention to other potential benefits of efficient precision suggests they may need greater support appreciating the full potential of this affordance for supporting student learning.

There were a few exceptions to the strong focus on relieving motor difficulties. Specifically, the teachers with the strongest discovery orientations related efficient precision to other benefits during Lesson Planning only. For example, Nancy mentioned how the piece-snapping feature of the Phoenix College Fraction Circles could encourage her fourth-grade students to create and organize more examples of equivalent fractions, similar to how fifth-grade students in one prior study used efficient precision features to generate more examples of equivalent fractions (Moyer-Packenham & Suh, 2012). Two other teachers, Karla and Tammy, discussed during Lesson Planning how features reflecting efficient precision—specifically, the piece snapping and exact size of the Toy Theater Fraction Circles, might support students in purposefully looking for examples of fractions that would sum to ½ or 1 whole. This aligns with Moyer-Packenham and Westenskow's (2013) claim that efficient precision can benefit student learning by supporting them to be purposeful in their generation of examples. Thus, this study produced evidence that a few additional potential effects of efficient precision were noticed and utilized by some teachers—but only during Lesson Planning. I further discuss the importance of the Lesson Planning context in a later section.

Features reflecting efficient precision (particularly the automatic dividing and full fill features of the MLC Fractions App) also supported teachers in this study in generating and using visual evidence to make fraction comparisons and find sums of fractions during Problem Solving. Their use of visual evidence lessened their reliance on strictly numeric strategies. This connection between efficient precision and visual strategies has not been explicitly identified in prior research on VMs. However, Johnston and Suh (2009) did find that preservice teachers who incorporated a VM into a lesson were more likely to say a benefit was that it “allowed students to learn conceptually and visually” (p. 3565) than teachers who instead used a general tool (such as an interactive whiteboard) or a drill and practice game. It could be that researchers and teachers see the connection between VMs and reasoning with visual representations to be obvious, since VMs have been designed for creating visual representations. This may be why research has not focused on connecting specific VM features or affordances to the generation of visual evidence. Alternatively, it may be that the precision of visual representations is particularly important for fractions learning, and that is why this study surfaced a connection between efficient precision and use of visual evidence in problem solving.

Focused Constraints

Researchers have argued that focused constraints can benefit students’ mathematical learning by bringing mathematical features or processes into conscious awareness that could be otherwise overlooked (Sarama & Clements, 2009). While physical manipulatives impose few constraints on how they may be used, VM designers place limits on the ways VMs can be manipulated. These limits can serve as feedback to students and make it more difficult for them to overlook important mathematical features or properties of the VMs. For example, two studies described VMs that did not allow students to generate numeric sums of fractions with unlike denominators (Hansen et al., 2016; Moyer-Packenham & Suh, 2012). This constraint was designed to focus students’ attention on the purpose for finding a common denominator and bring into conscious awareness how common denominators were connected to creating and combining same-size parts of a whole. The feedback of being unable to combine unlike parts (and generate a numeric sum) could disrupt common (but problematic) strategies such as adding numerators and adding denominators, leading students to use different strategies.

In this study, teachers' initial interpretations of VM features reflecting focused constraints did suggest they thought features could disrupt their own and students' use of familiar problem-solving strategies, especially for fraction addition. However, teachers did not see value or purpose in these disruptions. Rather, they evaluated features reflecting focused constraints negatively because they prevented them from using their familiar strategies. The only evidence of teachers seeing value in features reflecting focused constraints for supporting student learning occurred in the Lesson Planning context—a point I discuss further in a later section. Teachers' dominant reaction to focused constraints was to wish that they were not there and to find ways to work around them, suggesting they may need more support for appreciating the value of focused constraints than for appreciating the value of efficient precision.

Despite the lack of evidence that teachers valued the effects of focused constraints on their thinking, the VM features reflecting focused constraints did shape teachers' strategies for fraction addition during Problem Solving. However, the influence of these features on teachers was different from other researchers' (Hansen et al., 2016; Moyer-Packenham & Suh, 2012) claims about their potential effects on students in two important ways. First, the focused constraints of the MLC Fractions App did not disrupt teachers' use of incorrect strategies. Whereas the focused constraints in existing literature on fractions VMs are framed as potential disruptions to incorrect strategies or common misconceptions, the MLC Fractions App's single-color models, fixed denominator feature, and complete whole feature instead disrupted teachers' familiar, but mathematically valid strategies for adding fractions. These strategies, such as coloring in models of sums with a separate color for each addend or mimicking the steps of the standard addition algorithm were not mathematically incorrect but were also not mathematically necessary. They were likely part of the teachers' standard classroom practices and were quite familiar to them. The constraints of the VM, however, problematized familiar practices for teachers. Second, the disruptions the focused constraints did not result in teachers trying entirely new strategies, but rather making some small adjustments to their typical strategies.

In sum, features reflecting focused constraints did challenge teachers' thinking about fraction addition during Problem Solving by problematizing teachers' familiar practices, but they did not redirect teachers toward entirely different strategies. Rather, features reflecting focused constraints seemed to generate disruptions in teachers' thinking that were interpreted as small frustrations to work around. Despite falling short of the disruptive but productive effects researchers have speculated focused constraints might have, this finding is not entirely negative. Any leverage point that could be used to support teachers' grasp of the conceptual underpinnings of fraction computation procedures could be beneficial. I further discuss the potential of these disruptions in teachers' fractions thinking in a later section.

Linked Representations

Researchers have argued that linked representations are a valuable affordance of VMs because they highlight relationships between representations and provide feedback to students about their interpretations of representations (Sarama & Clements, 2009; Zbiek et al., 2007). Teachers in this study did not see much value in linked representations, as they discussed and used features reflecting this affordance much less often than features reflecting the other two affordances. This pattern may reflect the particular VMs made available to teachers. The VM teachers explored and used for problem solving did not prominently feature a linked representation, as the linked labels were quite small in relation to the models and could be toggled on and off. Further, the most apparent link was between the number of shaded parts of a model and the symbolic fraction's numerator, which is a link that is likely more easily understood by students and teachers than the link between the denominator and the size of the part or between number lines and other representations (Cramer et al., 2017). The list of VMs from which teachers chose for Lesson Planning (see Table 4) also included several VMs that did not have linked representations. Thus, the general lack of availability of linked representations may have contributed to teachers lack of noticing them.

Even so, examining the way that Colleen—the teacher who made the most direct use of linked representations in both her problem solving and her lesson planning—interpreted and responded to linked representations suggests other reasons why teachers spent less time thinking about linked representations than the other affordances. Colleen described a couple of ways that students could use linked representations as feedback, but her expectations about how students would interpret the feedback had more to do with judging the correctness of specific answers than evaluating their understanding of the representations themselves. For example, she used the linked labels of the MLC Fractions App during Problem Solving to check her counting of the number of tenths that would be equivalent to ½ and ⅕ but did not reflect on whether or why those counts of tenths made sense. She also noted that students could use the linked number line representation in the NCTM Equivalent Fractions App to use trial-and-error thinking to generate equivalent fractions. She reflected that such a strategy could offer a useful entry point for students but did not offer ideas about how she might push students to think about *why* equivalent fractions share the same point on a number line. Instead, she described attention to the number line as a "side goal" of her lesson.

This suggests that Colleen, and other teachers, could benefit from support in understanding how students might interpret feedback from linked representations in more meaningful ways than as corrective feedback on answers. Zbiek et al. (2007) argued that students could use feedback from linked representations to revise their initial interpretations of representations, which may be incorrect or incomplete. However, if the tasks students are given to complete using VMs do not emphasize understanding of the representations itself as a goal, students may be less inclined to think about the feedback in these terms. Anderson-Pence and Moyer-Packenham (2016) found that students who used VMs with linked representations made more generalizations than students who used VMs without linked representations, but the tasks students were given to solve with VMs were different for each tool and "adapted from tool-specific lesson explorations" (p. 10) provided on each VM's website. Thus, it is quite possible that the increased number of generalizations related not only to the linked representations, but to the ways the tasks shaped students' interactions with and interpretations of the feedback provided by those linked representations. Colleen's discussion of how students might use the linked representations suggests that teachers may need support in developing such tasks.

In sum, while teachers' lack of attention to linked representations may have related to the limited availability of these features or their seeing limited value to such linkages, a third explanation may be that meaningful use of linked representations depends on creating tasks for students that move away from answer-getting and toward analysis of multiple examples and making generalizations. Teachers who are less familiar with creating or orchestrating students' work on such tasks may see less value in linked representations than other VM affordances and benefit from support in creating lessons centered on such tasks.

The Importance of Context—Especially Lesson Planning

The second top-level finding of this study relates to the importance of context, and especially the Lesson Planning context, for shaping teachers' interpretations of features reflecting efficient precision and focused constraints. Shifting from Exploration or Problem Solving into Lesson Planning led teachers toward more balanced interpretations of these features (Findings 2b and 2e). Specifically, some teachers saw benefits in features reflecting efficient precision during Lesson Planning that they had not mentioned in the other contexts. Further, all teachers mentioned both benefits and drawbacks of features reflecting focused constraints in Lesson Planning—a depth of interpretation that was not evident in Exploration or Problem Solving, where they focused only on drawbacks. These additional interpretations of these two affordances occurred only in Lesson Planning and served as the only noticeable deviation from the pattern of interpretations discussed in the previous section. In what follows, I elaborate on the additional interpretations present in Lesson Planning context and discuss some potential explanations for why this context prompted different interpretations from teachers.

Three teachers—specifically, the three teachers with the strongest discovery orientations—saw more value in features reflecting efficient precision during Lesson Planning than they did during Exploration and Problem Solving. Specifically, they discussed not only how those features would help to resolve student difficulties in creating fraction representations, but also how the features might serve to push or focus students' thinking toward a more ambitious learning goal. For example, Kathy discussed how the exact sizing on the Toy Theater Fraction Circles could support students in focusing on how different combinations of unit fractions relate to each other and to the whole (instead of focusing on the creation of appropriately sized and labeled pieces in a drawing. These newly-noticed benefits of features reflecting efficient precision reflect how other researchers (e.g., Moyer-Packenham & Westenskow, 2013) have discussed how efficient precision could be used to transform the kinds of tasks students are asked to do—for example, to *generate* a rule for creating equivalent fractions, rather than merely apply one or discuss its meaning or rationale (Moyer-Packenham & Suh, 2012).

Teachers continued to articulate frustrations with features reflecting focused constraints during Lesson Planning, but their interpretations of these features also tended to include more positive evaluations in this context. Some of teachers' positive interpretations were related to how the features could help students see why their erroneous strategies were not valid, similar to how Hansen et al. (2016) and Moyer-Packenham and Suh (2012) described how a focused constraint could help students see why they could not add numerators and denominators to add fractions. For example, Karla described how the disappearing piece feature of the Toy Theater Fraction Circles could act as feedback to help students see why a queen could not give ½ her kingdom to one daughter and ⅓ of her kingdom to each of her other two daughters.

Thus, for these two affordances, the additional interpretations teachers made in the Lesson Planning context more closely reflected the potential transformative effects of these features identified in previous theorizations (Sarama & Clements, 2009) and empirical studies (Moyer-Packenham & Westenskow, 2013). That is, during Lesson Planning teachers discussed ways that feature reflecting VM affordances could more strongly enhance the activities students might do with fractions using physical manipulatives or drawings (rather than merely replace these materials).

Relating the results of this dissertation to another study of teacher thinking about mathematics technology tools suggests a partial explanation for why the Lesson Planning context elicited additional interpretations of features reflecting VM affordances from teachers: Lesson

Planning prompted teachers to think about using VMs to complete specific tasks. The way teachers' thinking about VM features changed across contexts was similar in some ways to how teachers' thinking about applets progressed across time in Smith et al.'s (2017a) study of teacher noticing of technology resources. Teachers in that study, who discussed the applets in small groups, began by making broad, general comments and blanket evaluations of the applets. As their conversations progressed, teachers' interpretations showed greater attention to more subtle features, how the applets represented mathematics, and student thinking. Similarly, my teachers' interpretations of VM features in Exploration tended toward simple positive evaluations of features reflecting efficient precision and simple negative evaluations of features reflecting focused constraints. As they moved into Problem Solving, they noticed additional discoverable features (Finding 1b), and in Lesson Planning they gave more detailed rationale for their evaluations and showed more nuance in their interpretations (Findings 2b and 2e).

The applets teachers were exploring in the Smith et al. (2017a) study contained directions for completing a specific task, as opposed to being tools that, like VMs, could be used for multiple purposes. Smith and colleagues credited the progression in teachers' thinking about the applets to them first focusing broadly on how the tool worked, before attempting to use it to complete the task. Asking teachers to freely explore a VM seemed to prompt them to think broadly about how the VMs worked. However, because there was no specific task associated with the VM, most of the teachers—Janice and Karla being exceptions—did not try to use it to complete tasks during Exploration. Problem Solving and Lesson Planning did direct teachers to think about how a VM could be used for specific tasks. It seems likely that attention to specific tasks contributed to eliciting teachers' their more detailed considerations of VM features, as teachers' attempts to use Smith et al.'s (2017a) applets to complete the embedded tasks had.

However, this explanation does not account for why additional interpretations of VM features came up only in Lesson Planning, as both the Lesson Planning and Problem Solving contexts would have focused teachers' attention on using VMs to complete specific tasks. An additional, potentially complementary explanation for why the Lesson Planning context elicited additional interpretations is that Lesson Planning prompted greater attention to student thinking than the other contexts. Seeing additional benefits of efficient precision and seeing potential positive effects of focused constraints suggests teachers were thinking about multiple possible student strategies and interpretations in Lesson Planning rather than focusing on their own interpretations or single interpretation they imagined for students in Problem Solving. Because the lesson planning template I provided to teachers (see Appendix E) and my lesson plan interview protocol (see Appendix F) specifically prompted teachers to consider multiple student strategies, it is difficult to discern whether these prompts or the Lesson Planning context more generally is what fostered greater attention to student thinking. Either way, some aspect of the Lesson Planning context supported teachers in thinking more broadly about VM features, which may support them to be more flexible in responding to a variety of student strategies when they use VMs in their classrooms.

While the focus on student thinking the Lesson Planning provided served as a helpful support for teachers, it is also important to note that this context did not entirely negate the need for more targeted supports for teachers in seeing and applying the transformative potential of VMs. Only three teachers, not all six, mentioned benefits of efficient precision in Lesson Planning beyond relieving burdens on students' motor skills. All six teachers mentioned a potential positive effect of a feature reflecting focused constraints in Lesson Planning, but the specific nature of the positive interpretations varied. As mentioned above, some positive interpretations were in line with the productive effects of focused constraints explored by other researchers. Other positive interpretations of focused constraints, however, did not relate to students being able to see why their erroneous strategies did not work, but instead focused on how the features would help students avoid confronting particular mathematical issues. For example, Sheila discussed how the numerator and denominator limits on the Toy Theater Fraction Bars would allow students to avoid thinking about what it would mean for a numerator or denominator to be 0. She felt this was a helpful constraint because she worried student thinking about this issue would distract from the main learning goals of her lesson, which were for students to generate examples of equivalent fractions and begin to notice multiplication and division patterns in sets of equivalent fractions. While there may be value in focusing on one mathematical idea in a lesson, helping students confront misconceptions is also an important practice, especially when it comes to fractions (Fazio & Siegler, 2011). Thus, consciously choosing tools that allow students to avoid confronting difficult conceptual issues may have negative consequences for learning.

In sum, including lesson planning, or other contexts closely connected to teachers' day-to-day work that focus their attention on specific tasks and student thinking, is likely to support teacher learning about VMs in professional learning settings. However, other supports and experiences will likely be required to help teachers use VMs to their best advantage in classrooms. I discuss some ideas for enhancing professional learning in the implications sections at the end of this chapter.

As a final point, this study produced the strongest evidence for the influence of context on teacher thinking about VMs than any other factor. However, it also generated some preliminary evidence that other factors, including content knowledge and teaching beliefs, may also influence teachers' thinking about VMs. Teachers with higher scores on the measure of fractions content knowledge were more reluctant to use labeling features of VMs to support students in making connections between visual and symbolic representations. Teachers with the strongest discovery orientations to teaching saw benefits of features reflecting efficient precision beyond relieving student burdens in creating fraction representations. Teachers with high content knowledge *and* a discovery orientation were more likely to experiment with how a VM could be used to solve various tasks during Exploration. These teachers also described both pros and cons of at least one feature reflecting focused constraints during Lesson Planning.

These differences suggest that teachers with different levels of content knowledge and different teaching beliefs see the potential learning benefits of using fractions VMs (and other VMs) and how particular uses of VMs may exploit those benefits differently. These preliminary results are unsurprising giving the many decades of research establishing the importance of knowledge and beliefs for much of the work teachers do (Calderhead, 1996; Ball, Thames, & Phelps, 2008). Individual teacher differences and professional learning needs according to content knowledge and pedagogical orientation make sense in the context of the TPACK framework (Mishra & Koehler, 2006), which suggests effective use of technology for instruction requires several interconnected types of knowledge. Teachers with lower content knowledge, for example, may need the most support with developing technological content knowledge (TCK) in order to use fractions VMs effectively. Teachers with a weaker orientation toward discovery learning may benefit the most from professional learning targeting technological pedagogical knowledge (TPK). Future research could more carefully examine how content knowledge and teaching beliefs interact with how teachers think about and use VMs.

Disruptions in Teachers' Thinking about Fractions

The final top-level finding of this dissertation is that features reflecting efficient precision and focused constraints prompted minor disruptions in teachers thinking when they used a VM to solve fraction comparison and addition problems. With regards to efficient precision, features reflecting this affordance supported teachers in using visual evidence of fraction magnitudes to make fraction comparisons. To a lesser extent, teachers also used visual evidence to support their reasoning about equivalence and addition. Some teachers contrasted these visually-supported strategies to other numeric strategies they may have used otherwise (Finding 3b), suggesting the availability of reliable visual evidence produced by the VM led them to avoid reliance on those numeric strategies. With regards to focused constraints, features reflecting this affordance contributed to teachers creating a representation of the sum ½ + 1/5 showing unequal parts. This representation felt unfamiliar to teachers—it problematized the addition process for them—and led some to pause and consider how to interpret it (Finding 3b).

These disruptions in teachers' thinking about fractions can be related to two perennial challenges in mathematics education. Specifically, when teachers used visual evidence (generated via use of features reflecting efficient precision) to compare fractions, they may have been thinking about the overall magnitude of each fraction rather than thinking about the numerator and denominator separately. Helping students to develop conceptual understanding of fractions and to think of numerators and denominators in relation to each other (as opposed to as unrelated whole numbers) is one enduring challenge in mathematics education (Siegler et al., 2013; Stafylidou & Vosniadou, 2004). When teachers considered their unfamiliar representation of ½ + 1/5 (generated via features reflecting focused constraints), they may have been considering the rationale behind the steps of the fraction addition algorithm—a form of understanding that is typically underdeveloped for teachers (Copur-Gencturk, 2021) and likely also for their students. This is a second enduring challenge in mathematics education.

In the sections that follow, I discuss in more detail the connections between the VM affordances, the disruptions in teachers' fractions thinking, and the challenging issues of conceptualizing fractions as magnitudes and understanding the fraction addition algorithm. I also offer ideas about how the disruptions in teachers' thinking might be leveraged to support their own understanding of fractions and fraction procedures. Because I did not collect data from the teachers' classrooms when they enacted lessons using VMs, I cannot make claims about how teachers might apply their own conceptual learning in their classrooms to support student learning. However, the analysis of the Lesson Planning data provides initial insights about how teachers might use VMs to create instruction that focuses on helping students to develop conceptual understanding of fraction magnitudes and fraction addition.

Connection between Efficient Precision and Reasoning about Fraction Magnitudes

As they used the MLC Fractions App to order the fractions 3/10, ¼, and 2/6 from least to greatest, all six teachers in this study began by creating a model for each fraction. To create those models, the teachers had to think about the numerator and denominator of each fraction separately when they chose a number of parts (corresponding to the denominator) and decided how many parts to fill in (corresponding to the numerator). In this first part of teachers' problem solving, the VM prompted them to use some basic whole-number reasoning (as they counted parts) and to rely on a part-whole model of fractions. This early stage of using the VMs therefore has the potential to reinforce two problematic issues in the teaching and learning of fractions: applying whole-number reasoning to fractions and relying on part-whole meanings of fractions (Fazio & Siegler, 2011).

After creating the models, however, teachers proceeded by overlapping and aligning the models to make judgements about how the amount of shading in each one compared to the others. They did not count parts, but rather attended to and reasoned about the relative sizes of the fractions in relation to the whole. This use of visual reasoning suggests teachers used the models to reason about the fractions' magnitudes as they placed them in order. Several teachers specifically mentioned numerically-based strategies they might have used otherwise, such as finding common denominators, that may have directed them away from reasoning about fraction magnitudes and focused their thinking about numerators and denominators separately. The ways teachers expected students to use VMs to compare fractions or generate equivalent fractions in their lessons followed a similar pattern. Students could use some whole-number reasoning or part-whole thinking to create fraction models, but teachers indicated they expected them to use visual reasoning about the fraction magnitudes to make comparisons or judge equivalence. This pattern provides preliminary evidence that teachers may have also encouraged students to utilize VMs to consider fraction magnitudes rather than think separately about numerators and denominators.

Several features of the VMs seemed to support teachers' transition from thinking of numerators and denominators separately to thinking of fractions as magnitudes. As discussed in Chapter 5 (Results, Responses to Affordances), several MLC Fractions App features reflecting efficient precision were important for supporting teachers' use of the models to visually compare the fraction magnitudes. The automatic dividing feature and the full fill subfeature, in particular, supported the easy, quick creation of precise models that could be visually compared. Teachers discussed these features in relation to the prospect of using drawings to compare fractions; the VM felt superior to them because of its efficient precision features. The bar and circle snapping features also supported teachers in aligning their models to facilitate visual comparisons. At a more fundamental level, the overlap and circle free rotation features also supported use of visual strategies because they allowed for easy manipulation of the models while keeping the whole intact and the pieces of the models neatly arranged. Several teachers' discussion of how piece-snapping features in the Toy Theater Fraction Circles and the Phoenix College Fraction Circles made those tools superior to physical fraction pieces also illustrated that they valued easy manipulation of the representations created with VMs.

As discussed in Chapter 1 (Background and Framing), VMs exist in a hybrid representational space with characteristics of both physical or concrete representations and drawings (Moyer-Packenham, Salkind, & Bolyard, 2008). In some ways, the VMs mimic the work teachers or students can do with physical objects. Because they are manipulable, they allow learners to create enactive representations of mathematical ideas (Bruner, 1964). Further, the efficient precision affordance of VMs allows teachers and students to be confident that their fraction representations are accurate in size and proportion; they are always the exact appropriate size, just like pre-fabricated physical fraction manipulatives. However, they differ from other manipulatives for fractions, such as physical fraction circle pieces, because they support thinking about the fraction as a single quantity rather than as a count of unit fractions. For example, the Toy Theater Fraction Circles and Phoenix College Fraction Circles assist learners in snapping pieces into a whole such that they are perfectly aligned into a contiguous piece of the whole. The MLC Fractions App and the Toy Theater Fraction Bars allow learners to manipulate fractions representations that stay embedded in their wholes (discussed as the *complete whole* feature in Chapters 4 and 5). These features can support teachers—and potentially students—in thinking about how all the shaded pieces in a fraction model form a quantity with a single magnitude in the way that drawing might (Rau & Matthews, 2017), rather than thinking about individual pieces apart from the whole as they might when using physical manipulatives.

In short, VM models may function like *drawings* (emphasizing magnitudes of fractions) that are *reliably precise* (allowing for valid comparisons) and that a learner can *manipulate* (in order to make direct comparisons via overlap and rotation of models). This combination of features allowed teachers to create enactive representations of fraction magnitudes, which may support the development of iconic and symbolic representations that reflect fraction magnitudes. The efficient precision of VMs, along with their hybrid representational nature, may be part of what disrupted teachers' typical reliance on numerical strategies and supported them in shifting (and planning for students to transition) from thinking about the numerator and denominator separately to thinking about fraction magnitudes.

While this study provides initial evidence of the potential for VMs, and specifically efficient precision, for supporting teacher and student reasoning about fraction magnitudes, the study did not generate evidence that teachers were aware of this connection. Some teachers did mention they liked the precision and easy manipulation of the fraction models for supporting comparison strategies that did not rely on finding common denominators or converting to decimals. However, they did not articulate reasons why the use of visual evidence might be beneficial for students or connect the visual models to thinking of fractions as quantities. Teachers will likely benefit from professional learning opportunities that support them in considering more carefully the benefits of students using visual evidence to compare fractions and connect the use of visual evidence to the development of fraction number sense (Fazio & Siegler, 2011; Rau & Matthews, 2017).

Connection between Focused Constraints and Making Sense of Fraction Addition

In a recent study, only about half of elementary teachers could fully explain the conceptual basis for the standard fraction addition algorithm (Copur-Gencturk, 2021), suggesting that many students are not receiving instruction that emphasizes why the steps of the algorithm make sense and produce fraction sums. As such, it is worthwhile to consider how teachers' thinking about and use of VMs may help them develop their own understanding of the fraction addition algorithm. Because the standard algorithm is familiar to teachers, finding a fraction sum is often not as problematic for them as it is for their students. However, explaining the meaning of steps of the algorithm *is* often a challenge for them. This study generated evidence that some VM features—in particular, features reflecting focused constraints—may create opportunities to push teachers to explain the rationale behind the some (but not all) of the steps of the familiar fraction addition algorithm by problematizing their use of familiar addition procedures and representations.

The *Common Core State Standards for Mathematics* (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) describe the standard algorithm for adding (and subtracting) fractions with unlike denominators as follows: "Add and subtract fractions with unlike denominators … by replacing given fractions with equivalent fractions in such a way as to produce an equivalent sum or difference of fractions with like denominators" (p. 36). To use this algorithm, students must identify a common denominator for the two addends, generate an equivalent fraction for each addend using that denominator, and then combine the new like-denominator versions of the addends. Teachers in this study did not use the MLC Fractions App (the only VM where they thought about the fraction addition algorithm) to support them in identifying a common denominator. In fact, several teachers explicitly noted that the VM would not support their students in identifying 10 as a common denominator for ½ and ⅕.

Most teachers did, by contrast, use the VM to support them in generating equivalent fractions for ½ and ⅕ with a denominator of 10 (or imagine ways their students might use the VM to do so). Five teachers used the overlap feature of the MLC Fractions App to overlay tenths on their models of the addends. Sheila also used the bar-snapping feature to support her reasoning about ½ being equivalent to 5/10 and ⅕ being equivalent to 2/10. Colleen used the linked symbolic labels to help her "count" how many tenths were equivalent to each addend. Thus, most teachers in this study found features reflecting efficient precision or linked representations to be useful in generating the equivalent addends once they chose the common denominator. This is an early indication that teachers could use these features to avoid or correct the mistake of changing the addends to have common denominators but not adjusting the numerators. This error was less common than adding numerators and denominators but was present in the work of some prospective teachers (Young & Zientek, 2011).

The most interesting results of this study related to addition, however, have to do with the third part of the fraction addition algorithm—combining the two addends to form a single sum. Working with the VM—and especially having to deal with or work around the features reflecting focused constraints—seemed to cause the strongest disruptions in teachers' thinking about fraction addition when they grappled with putting the two addends together. Particularly when using the circle models, teachers started by creating representations for ½ and ⅕ and then overlapped them to generate the representation of ½ + ⅕ shown in Figure 42 (left). After doing this, several seemed to realize that this representation did not immediately suggest a numerical value for the sum. Seeing tenths—and the numerical sum 7/10—in this representation was not obvious to them. One teacher noted that the pieces shown in this model of the sum were "not even," so she could not use a simple counting strategy to determine the sum. Other teachers made more indirect references to the lack of same-size pieces in the model, commenting that they could not express the answer without more work. All teachers except one eventually added a tenths circle to their representation to show the sum was equal to 7/10.

Copur-Gencturk (2021) found that only half of the 303 elementary teachers in her study could produce a complete explanation for why the fraction addition algorithm calls for generating equivalent fractions with like denominators. Many explanations referred to same-size pieces but did not connect the same-size pieces to combining like-size units of the same whole. Comments from teachers in this study about their initial representation of the sum $\frac{1}{2} + \frac{1}{5}$ suggest this representation may have spurred their thinking about *why* common denominators were helpful in finding a numerical sum. Teachers' comments that the parts were "not even" and their questioning of "what fraction is that?" suggests that they may have connected their subsequent steps of incorporating tenths into their solutions to the notion of finding a way to express the sum as a number of like-size unit fractions. The data collected did not allow me to say for certain that teachers made this connection, but it did suggest VMs might support teachers in generating explanations for the "need" for common denominators in fraction addition. Teacher educators could use disruptions in teachers' thinking, such as those generated by the unfamiliar representation of $\frac{1}{2} + \frac{1}{5}$, as opportunities to push teachers to articulate why they wanted the pieces to be the same size and how they could tell what size pieces would exactly cover the sum.

In addition to supporting teachers in developing their understanding of the "need" for common denominators in the fraction addition algorithm, the representation generated by teachers as they put together circle models of the addends may also have some potential for supporting teachers in making magnitude estimates for sums—another gap in many teachers' and students' knowledge about fraction addition (Braithwaite, Tian, & Siegler, 2018). Teachers' overlapping representation of ½ + 1/5 bore some similarities to representations of fraction sums generated by students in a recent study (Braithwaite & Siegler, 2021). In this study, students placed virtual fraction bars representing two addends along a number line to support them in making a magnitude estimate of the sum. For example, to support them in making a magnitude estimate for 3/9 + ½, students placed three 1/9 fraction bars and one ½ fraction bar along a number line and clicked the number line where the fraction bars ended. They continued by making estimates for sums *without* first placing the fraction bars along the number line. The researchers found this intervention supported students in making more accurate magnitude estimates of sums (Braithwaite & Siegler, 2021), and they credited the improvement to students viewing representations of the sums that emphasized the overall magnitude of the sums without partitioning each addend into equal-size pieces.

Despite Braithwaite and Siegler's (2021) result suggesting creating representations emphasizing the magnitude of the sums supported students in making estimates of sums, this study produced no evidence that comparable representations supported teachers' estimation of sums. Teachers noted that their magnitude representations of the sums did not show same-size pieces, but their subsequent steps focused on showing those same-size pieces, usually relying on prior knowledge to determine that tenths could be an appropriate-size piece to use. No teachers talked about using the representation to estimate the sum as, for example, more than ½ or a little less than ¾. Granted, Braithwaite and Siegler prompted students to make estimates by clicking on the number line, whereas the prompt I gave to teachers was to find the sum. Thus, some of the differences in reasoning may have been due to the prompts. However, teachers' focus on finding the appropriate-sized piece that would allow them to express their answer suggests their thinking was focused on that, rather than conceptualizing the overall magnitude of their answer. Teacher educators might consider encouraging teachers to make magnitude estimates for sums after generating their initial VM representations of fraction sums.

In summary, teachers' thinking as they solved ½ + ⅕ suggested that the VM used in Problem Solving (the MLC Fractions App)—and other VMs with similar features reflecting focused constraints—may be useful for supporting teachers to generate meaningful, quantity-based explanations for the steps of the standard fraction addition algorithm, particularly the reasons for finding a common denominator. However, even as teachers used the VM to generate representations of the sum that showed the overall magnitude without showing countable parts, those representations did not seem to spur teacher thinking about the sum's magnitude. This pattern may suggest a limitation of the MLC Fractions App rather than a limitation of all fraction VMs. Regardless of the VMs used, however, it seems likely that teachers will need support from teacher educators to reflect on the representations of fraction addition they create with VMs in a way that facilitates deeper understanding of this operation and its associated procedures.

Implications for Future Research

This study's results suggest several avenues to be pursued in future research. In this section, I outline two potential research foci I believe are productive next steps. The two foci address further understanding how teachers think about features reflecting focused constraints and examining teachers' thinking about VMs during classroom instruction.

Future Research on How Teachers View Features Reflecting Focused Constraints

Teachers in this study generally saw some value in VM features reflecting efficient precision and linked representations for supporting their students' learning across contexts.

Professional learning experiences can likely build on the ideas teachers have about these features to support them in designing instruction that utilizes the potential of these features. However, teachers much less frequently saw value in features reflecting focused constraints. To guide the design professional learning to support teachers in productively utilizing focused constraints, additional research is needed to more fully understand teachers' negative perceptions of these features and how these perceptions might be changed.

A common thread evident in many teachers' negative reactions to features reflecting focused constraints was that they disrupted their use of their familiar problem-solving strategies. The connection between teachers' technology use and their existing classroom practices has been established in prior research (e.g., Moyer-Packenham, Salkind, & Bolyard, 2008; Webel, Krupa, & McManus, 2015). Moyer-Packenham, Salkind, and Bolyard (2008) found the VMs teachers chose to use in their classrooms were often those for which they had a physical alternative (e.g., pattern blocks and base-10 blocks). Webel, Krupa, and McManus (2015) found that teachers varied in whether they valued technology resources that showed solution methods different from what was familiar to them. The unfamiliar methods made some teachers uncomfortable with using the resources, whereas other teachers valued the resources because they could expose students to ideas the teachers felt they could not effectively teach them.

Teachers in this study did not seem to value the way the MLC Fractions App pushed them away from their familiar practices. Future research might use task-based interviews or think-alouds to more thoroughly examine *why* teachers were reluctant to allow focused constraints to shape their strategies and instead found ways to accommodate their existing strategies. One likely explanation is that some teachers lacked the content knowledge or mathematical flexibility to develop other strategies as they used the VMs. If this is the case, professional learning about VMs will benefit from including attention to developing teachers' content knowledge. However, there may be other explanations for teachers' reluctance to adapt their strategies, such as hesitance to deviate from methods presented in their curriculum. These reasons may suggest different professional learning needs.

In addition to understanding more about the basis for teachers' negative reactions to disruptions in their own strategies, future research should also develop strategies for helping teachers to see benefits of disruptions in thinking. Another potential explanation for why teachers reacted negatively to snags in their own thinking—and sometimes valued focused constraints that would allow students to avoid conceptual challenges—is that they saw their role as guiding students smoothly through a problem-solving process. This suggests that professional learning about VMs and focused constraints might also include attention to frameworks that help teachers see value in students making mistakes, confronting misconceptions, or generally struggling with mathematics. Potential frameworks that could be examined in conjunction with teacher learning about focused constraints could include the cognitive demand framework (Smith & Stein, 1998), growth mindset (Dweck, 1999), or productive struggle (Warshauer, 2015). Future studies could incorporate one or more of these frameworks into professional learning sessions similar to those I facilitated at the start of this study and examine whether participating teachers more often articulate value in features reflecting focused constraints.

Future Research on How Teachers Use VMs During Classroom Instruction

This study examined teachers' thinking about VM affordances across three contexts related to teaching and learning: Exploration, Problem Solving, and Lesson Planning. It did not include attention to an important fourth context: classroom instruction. Teachers' thinking changed across contexts and the changes were particularly noticeable when moving from

Exploration or Problem Solving to Lesson Planning. A likely explanation for the differences in teacher thinking during Lesson Planning is the greater and sustained attention to student strategies and thinking. Teachers' thinking would necessarily change again during classroom instruction, as they would be coordinating their attention to student thinking—which may or may not reflect the student strategies and issues they anticipated during planning—with other issues such as time and behavior management. Because it is during classroom instruction that teachers have the most leverage on student learning, it is important to understand how teachers think about VMs in this fourth context.

One way to study how teachers do or do not carry the ways they think about VMs in planning into instruction would be to replicate the professional learning and lesson planning phases of this study and also add a classroom instruction phase. Analyses could include specific attention to exploring whether findings from this study's Lesson Planning phase are also present in teachers' instruction—especially the more promising ways teachers intended to use VM features. For example, some teachers in this study described ways they would use features reflecting efficient precision to push students to achieve more ambitious learning goals. Do other teachers see this potential in features reflecting efficient precision, and do students engage in those learning goals during instruction? If so, in what ways do teachers shape how students interact with the VMs to support those learning goals? Similarly, some teachers in this study described both positive and negative ways a single feature reflecting focused constraints might shape students' thinking. Do those different student interpretations come up in instruction, and if so, how does the teacher respond?

Answers to questions like these about how teachers use VMs during instruction could inform the design of professional learning experiences for teachers about how to use VM features to support student learning. For example, studies could reveal that teachers see potential in features reflecting efficient precision for supporting ambitious learning goals, but have difficulty realizing that potential during instruction because they do not notice specifics about how students are using VMs. Such a finding would parallel the common pattern of teachers planning high cognitive demand tasks but finding it difficult to maintain cognitive demand during instruction (Smith, Grover, & Henningsen, 1996). If this is the case, professional learning should include attention to strategies teachers can use to shape student work with VMs without having a significant impact on their thinking opportunities.

On the other hand, studies of how teachers think about VM features during instruction may reveal that watching their students use VMs supports some teachers in thinking more flexibly about their potential and seeing the value in their features. Such a finding would be consistent with professional development frameworks that emphasize that teachers' beliefs are most easily influenced when they see the impact of instructional changes directly on student learning (e.g., Guskey, 2002). If future studies reveal this as a common pattern in teachers' thinking about VMs, professional learning may benefit from encouraging teachers to experiment with using VMs in their teaching before, during, and after any organized meetings and workshops.

Implications for Design of VMs

While the impact of any educational resource will always reflect an interaction between the resource itself and the teacher or student using it (Drijvers et al., 2010; Remillard, 2005), the results of this study do suggest three implications for designers of virtual manipulatives about how they might better support teachers in using VMs to their full advantage. The first implication relates to how VM designers might address the tension between creating flexible tools for teachers while also including features that prompt teachers to consider tools in relation to specific tasks. As discussed above, connecting VMs to specific tasks was important for eliciting productive thinking about VM affordances from teachers. The second design implication relates to how VM designers might balance teachers' desires to remove constraints (as reviewed in the discussion above) with including features that prompt potentially productive disruptions in teachers' thinking. The last design implication relates to how designers might support teachers in taking advantage of the VM affordance in which they showed the least interest: linked representations.

First, designers of VMs might consider how they can balance the broad utility of a flexible, multi-use tool like a VM with benefits of prompting teachers to think about the features of the tool in relation to specific tasks and learning goals. While asking the teachers in this study to plan a lesson that incorporated use of a VM elicited more teacher thinking about how the VM might be used transformatively (as compared to general exploration of the tool), teachers in another study who explored applets with embedded tasks thought about student strategies during the course of their general explorations (Smith et al., 2017a). Incorporating some kind of prompting to think about tasks within the VMs themselves may support teachers in thinking more carefully about the potential impact of the tool on student thinking and how the VM features might support students in reaching different learning goals. On the other hand, tailoring every VM to a specific task may result in VMs seeming incompatible with teachers' curricula or necessitate daily searches for appropriate technology tools. The teachers in this study often expressed a desire increase the flexibility of the VMs they used by adding features or removing constraints, suggesting they were not enthusiastic about the prospect of undertaking frequent searches for VMs and expected VMs to be multi-purpose rather than tailored to specific lessons.

A middle ground designers might consider is designing VMs with settings or modes that invite certain kinds of interactions without fully defining a task. As one example, the *TouchCounts* app serves as a model for this, as it includes an Enumerating World and an Operating World (Sinclair, Chorney, & Rodney, 2016). These “worlds” have features that promote focus on ordinality and cardinality, respectively, but do not define specific tasks. Fractions VMs could be designed to have “worlds” focused on, for example, magnitude judgements (e.g., a comparison world) and or fraction operations (e.g., an addition and subtraction world). The “worlds” might also be designed with learning trajectories or progressions in mind to support different stages in students’ development of fractions understanding. For example, Confrey et al. (2014) identified partitioning a whole into equal parts and (re)assembling equal parts into a whole as two distinct levels in a learning trajectory for equipartitioning. One teacher in this study, Colleen, noticed that while the MLC Fractions App and its automatic dividing feature emphasized partitioning, other VMs, such as the Toy Theater Fraction Circles, emphasized assembling a whole from equal parts via its draggability and piece-snapping features. If a single fraction VM had “worlds” that supported both partitioning and assembly, the VM’s features might be used grow with the student as their understanding progresses. Moreover, teachers may be able to connect the worlds to different parts of their curriculum, allowing the VM to be aligned with the particular sequences of ideas in the teachers’ support materials.

Second, designers might consider providing some sort of documentation explaining their design decisions. Such documentation could be especially helpful in relation to intentionally designed focused constraints. Teachers in this study did not usually consider the potential benefits of VM features reflecting focused constraints. Instead, the expressed desires to remove the constraints and did not seem to appreciate the potential learning opportunities provided by the disruptions the constraints caused in their thinking. Further, as the researcher identifying the features reflecting focused constraints, I often had to speculate on how these features could focus student thinking on a particular idea and had no means to know whether my speculations aligned with the designer's intentions. When designers intentionally constrain manipulations of a VM, it would be useful to provide information to teachers about why the constraint is present and how the designers intend for the constraint to shape student thinking. Designers could also include information about how features reflecting efficient precision offload potentially important work (such as creating equal parts) from the student to the tool and encourage teachers to consider the implications of this offloading for the cognitive load on students and their opportunities for learning. This documentation would be similar in purpose to the educative information included in educative curriculum materials (Davis & Krajcik, 2005). If teachers engaged with this information, it may make them more likely to see the benefits of features reflecting focused constraints or the transformative potential (or potential detriment) of features reflecting efficient precision and more thoughtfully incorporate them into their lesson plans and instruction.

Lastly, VM designers could more thoughtfully consider the nature and direction of the linked representations they incorporate into their VMs. Linked representations attracted little interest from teachers in this study, and a contributing factor may have been the lack of availability and prominence of the links. The specific nature of the links may also matter. Several teachers mentioned, for example, that they would have found the linked labels in the MLC Fractions App more useful if they allowed students to observe the relationship between the denominator and the size of the parts, rather than only between the numerator and the number of shaded parts. A dynamic link between the denominator and the size of the part would be more useful, as it is this relationship (not the relationship between the numerator and the number of shaded parts) that is in greater conflict with the properties of whole numbers (Stafylidou, & Vosniadou, 2004). Zbiek et al. (2007) also pointed out that students receive more and different kinds of feedback from a set of linked representations when the links are active in both directions—students can manipulate either representation and see change in the other. All the links in the VMs used in this study were one-directional (from visual model to symbolic fraction or vice versa), which may have also placed limits on their utility.

Implications for Teacher Professional Learning

This study suggested lines of research that could continue to inform the design of professional learning opportunities for teachers around VMs. Even so, VMs are widely available now, and the sudden move to remote teaching due to the COVID-19 pandemic likely increased teachers' interest in and use of a variety of digital resources. As such, it is useful to consider direct implications of the current study for the design of teacher professional learning. The results suggest three guidelines for professional learning about VMs.

First, like nearly all professional learning experiences, teacher learning about VMs will likely be most beneficial to teachers if it is connected to their practice and ongoing over time. Several results of this study illustrated the impact of connections to teachers' practice. Most prominently, the Lesson Planning context elicited more detailed thinking from teachers about the benefits of VM affordances for student learning. Thus, professional learning experiences should ask teachers to *use* VMs in their teaching, not merely discuss them. Additionally, the teachers who planned to use VM labeling features to support students in connecting visual and symbolic representations of fractions (see Finding 4) did so, in part, because they had prior struggles in supporting students to make those connections and were looking for new tools and strategies.

Keeping teacher learning connected to their current problems of practice will likely motivate teachers to use VMs for specific purposes and potentially to iterate on their practice. Finally, teachers' interpretations of VM features changed according to the topic and task, suggesting that facilitating teachers' reflection on their use of VMs across time for multiple purposes could promote flexible thinking about the affordances and constraints of using VMs.

Second, professional learning experiences about VMs should invite teachers to compare VMs with other representations they have used in their classrooms. Over the course of their interviews, most teachers mentioned ways VMs might be superior to both physical manipulatives and drawings. Moreover, teachers' problem solving with the VMs suggested they valued the combination of characteristics of manipulatives and drawings as they compared and added fractions. Explicitly pointing out the hybrid representational nature of VMs to teachers may support them in thinking flexibly about how to use them to supplement their use of other representations.

Lastly, professional learning sessions about VMs may benefit from using the three affordances (efficient precision, focused constraints, and linked representations) as a framework for shaping teacher thinking about VMs, but session designers should consider also incorporating other instructional frameworks. Features reflecting each of the affordances have different kinds of uses and benefits (Sarama & Clements, 2009), and differences in teachers' interpretations of the affordances suggest giving specific attention to each may be helpful. Even so, several results of this study also suggest teachers could benefit from further support in thinking about when the affordances are most appropriate to use and how the affordances could be used in a transformational way. For example, teachers discussed the benefits of offloading the work of creating equal parts to a VM, but rarely mentioned potentially detrimental effects of this offloading (e.g., if students do not have to think about equal parts, they may not realize the equality of the parts is an essential feature of a fraction representation; Rau & Matthews, 2017). Moreover, teachers often tried to work around focused constraints to make the VMs better fit into their existing practices, rather than developing new strategies and practices. Crossing the VM affordances with other frameworks that support thinking about whether students have opportunities to engage in high-level thinking (e.g., Stein, Grover, & Henningsen, 1996) or how technology might transform tasks (Thomas & Edson, 2019) could push teachers' thinking about VMs in productive ways.

CHAPTER 8: CONCLUDING REMARKS

Virtual manipulatives hold promise for supporting learning of mathematics and may hold particularly strong promise for supporting learning about the challenging topic of fractions. This study suggested that the hybrid representational nature of VMs may support teachers in thinking about fractions in terms of their magnitudes by combining features of drawings and manipulable tools. It also suggested that the representations teachers created with fractions VMs spurred thinking about the conceptual underpinnings of the standard fraction addition algorithm—particularly the connection between common denominators and combining like-size parts—or generated disruptions in teacher thinking that could serve as starting points for conversations about the rationale for problem-solving steps and familiar practices in professional learning sessions.

Many questions remain about how to best support teachers in realizing the potential benefits of VMs and incorporating VMs into their instruction in ways that allow students opportunities to benefit from the VM features. The teachers in this study identified some benefits of efficient precision, focused constraints, and linked representations with only a minimal introduction to these features through three hours of virtual professional learning. This suggests several productive starting points on which professional learning around VMs might build: Teachers noticed examples of all three affordances, recognized the value in using VMs to support the creation of precise representations, and utilized the VMs' representations to use (and encourage students to use) comparison, equivalence, and addition strategies that relied on visual evidence of fraction magnitudes rather than entirely on numeric or rule-bound reasoning.

However, the study also showed that teachers did not consistently plan to leverage VM affordances to their full potential. Only some teachers (3 of 6) discussed how efficient precision could support students in reaching more ambitious learning goals or acknowledged both pros and cons of individual features reflecting focused constraints. And only one teacher made meaningful use of linked representations in her lesson. Thus, this study both demonstrates productive starting points and continuing needs for teacher professional learning around affordances of VMs. In particular, teachers will need support to learn how to use VMs transformatively rather than use them to replicate their existing practices.

Limitations

This study has several limitations. The detailed analyses of teacher thinking in multiple contexts necessitated limiting the study to six participants. The small sample size allowed me to gather enough data to produce rich descriptions of these teachers' thinking, but it also limited the generalizability of the findings. The sampling methods also likely introduced selection bias, as all teachers volunteered to participate and so likely had at least nominal interest in developing their mathematics teaching skills and exploring the potential of virtual manipulatives. In that sense, the results of this study may represent a best-case scenario of how teachers might think about and use fractions VMs. On the other hand, the online recruitment methods led to a sample of teachers with varying levels of fractions knowledge (according to my content knowledge measure), variations in mathematics teaching beliefs, and teaching positions in both public and private schools in dispersed geographic locations. These features of the sample may make the study findings representative of a greater variety of teachers than a local sample would have.

The data collection and analysis occurred during the height of the COVID-19 pandemic, which placed limitations on the study in at least two important ways. First, the prevalence of online teaching and other school safety precautions during this time prevented me from collecting classroom data as teachers taught their lessons with VMs. Lack of classroom data left me unable to examine how teachers thought about or used the VMs in the context of their classroom practice. The student response interviews were intended to serve as a substitute for data about how teachers might think about VMs as they responded to students in real time. This data was helpful, but not adequate as a substitute for classroom data.

Second, because almost all teachers were forced to teach at least partially online, they had much greater incentive than usual to incorporate technology into their teaching practices. Teachers' beliefs about technology typically influence whether and how they use technology in their classrooms (Drijvers et al., 2010; Tondeur et al., 2017), but the circumstances of the pandemic may have washed out many of the effects of teachers' negative feelings about technology. They had no choice but to expand their use of technology. While these circumstances may have helped in my recruitment (because I was offering professional learning about resources that were very relevant to teachers), it is also unclear whether teachers' general positive views of VMs revealed in this study will continue when more typical classroom teaching resumes.

The analysis methods also introduced a few limitations. To manage the scope of the study, I focused my analysis on interpretations of and responses to features I coded as reflecting three affordances of VMs identified in prior research: efficient precision, focused constraints, and linked representations. The choice to focus only on teacher thinking about these features may have resulted in missing other features that may have influenced teacher thinking in meaningful ways. Additionally, I made the decisions about which features I would treat as reflecting each affordance through a process of coding the VMs. I utilized a second coder to bolster the validity of my choices, but my own influence on the selection of the features still had the potential to introduce biases into the analysis. My own influence was particularly apparent in my choices of which features reflected focused constraints, as identifying such a feature required considering how a limitation could focus student attention on a mathematical idea or process. Identifying those potential foci in student attention was highly interpretative. This identification process might have been improved by speaking to designers of VMs to learn about what kinds of limitations they intentionally imposed on the VM and why.

APPENDICES

APPENDIX A

Introductory Interview Protocol

1. Do you have any questions about the consent form? About the study?
2. Tell me a bit about yourself. What grade level do you teach and where?
    a. In recent years, have you taught all subjects in a self-contained classroom, or is there a different arrangement?
    b. How long have you been teaching elementary school? How long have you been teaching this grade level?
    c. What mathematics curriculum do you use?
    d. What can you tell me about your teaching circumstances for this school year?
3. What made you want to participate in this study?
4. What issues have come up in the past as you taught fractions?
    a. For example, what are some common difficulties students have?
    b. What tools and teaching ideas have you found useful for teaching fractions?
    c. How is fractions taught differently from how you learned fractions?
5. What kinds of technology do you use in your classroom (now and pre-pandemic), and how often?
    a. What technology do you use for mathematics specifically, and how often?
    b. Why do you use those technology tools? For what purpose? What benefits do you see in using them?
    c. What challenges have you encountered using technology in your classroom?
6. In general, what days and times would work for you for professional learning sessions?

APPENDIX B

List of Fractions Challenges Generated by Participating Teachers

- Students in G4-5 are missing foundational understanding of fractions.
    - They don't have visual models for what fractions look like, esp. when > 1
        - And physical manipulatives often don't come with enough pieces!
    - They have imprecise understanding of common terminology (e.g., half is any piece)
- Larger denominator = smaller fraction is counterintuitive
- Fractions feel like *two* numbers, not one.
- Shifting the whole (what is the fraction of?) is challenging
    - Especially for fractions of a set and fractions of a fraction
- Students struggle to understand equivalence and ordering
- All the steps needed means fraction operations get procedural fast, especially for addition and subtraction with unlike denominators.

APPENDIX C

Noticing Interview Protocol

May I record this interview?

This interview will happen in three parts. First, I'll ask you about your takeaways from our professional learning sessions. Then I'll have you share your screen and explore a virtual manipulative so I can learn about what you find interesting or useful about it. Then I'll ask you to solve a couple of problems using the tool.

1. What were your main takeaways from the professional learning sessions?

2. Do you think you'll use VMs in your remote teaching? If so how?

3. Do you think you'll use VMs in your face to face teaching? If so, how?

4. Please open the Fractions manipulative. Explore the virtual manipulative and tell me what you notice.
    a. *Continue with prompts of "anything else?" until the participant has said everything they want to say.*

5. *As needed, follow up with more specific questions to probe the three parts of the noticing framework (attending, interpreting, responding):*
    a. What features caught your attention?
    b. What's interesting about that?
    c. What problems might you ask students to solve with this tool?

6. Did you notice any of the affordances we discussed in the professional learning seminar?
    a. Do you see focused constraints? (Is that important? How would you use it?
    b. Do you see efficient precision? (Is that important? How would you use it?)
    c. Do you see linked representations? (Is that important? How would you use it?)

7. Please use the VM to complete these two tasks, thinking aloud as you do so. ?
    a. Put these three fractions in order from least to greatest: 3/10, 1/4, 2/6
        i. What features were useful for solving this problem?
        ii. Could you solve the problem with the other model (strips vs circles)?
        iii. What additional features did you notice using that model?
    b. Solve: 1/2 + 1/5 = ?
        i. What features were useful for solving this problem?
        ii. Could you solve the problem with the other model (strips vs circles)?
        iii. What additional features did you notice using that model?

Anything else you'd like to share that you haven't said yet?

APPENDIX D

Directions for Phase 2: Lesson Planning

Thank you so much for your interest in continuing with the study. Here are your next steps.

I would deeply appreciate it if you could complete these steps within 2 weeks of receiving these instructions. It is fine if fractions do not come up in your curriculum any time soon -- consider this as a plan for later!

1. Decide the format of the lesson you are planning. For example, will you teach this lesson remotely? Will it include both synchronous time with the students and some asynchronous work time for them? Or alternatively, are you planning a face to face lesson?
   The format is up to you. You can plan for your current circumstances if fractions are coming up soon in your curriculum, or plan for face to face circumstances if you expect this lesson to actually be taught later on when you may see your students face to face. (Or you can choose to plan a face to face lesson to use in later years or just because you want a break from thinking about remote teaching!)

2. Choose the mathematics focus of your lesson. Because I'm particularly interested in how you think about how VMs may help you with the challenging topics of fraction equivalence, comparison, or addition & subtraction, I'd like you to choose something within those areas. If you like, you can choose to base your lesson on something from your math curriculum, just adapting it to make use of a VM. Or you can create something brand new.

3. Choose a VM for students to use during the lesson. You can pick any of the VMs on the list I shared.

4. Record your decisions in this Lesson Plan Template. This link will prompt you to make a copy that you can then share with me (richkat3@msu.edu).

5. Complete the rest of the plan template to think through the **tasks** you will have students do with the VMs, the **strategies** students might use to complete the tasks, how you will **support** students through questioning or other feedback, and what you might want to **discuss** with them after they complete the tasks.

6. *Optional:* If there is an additional format you usually use to plan (e.g., a document that lists the lesson activities in order with estimated time for each), feel free to include that at the end of the lesson plan template document.

7. Notify me with a quick email when you are done. I'll follow up within 48 hours to schedule an interview time.

8. During this interview, I'll ask you to walk me through your plan and ask follow up questions about various choices you made. Once I have an understanding of your thinking that went into the plan, I'll also offer some feedback and support for further developing the plan, if you wish. **It'll be entirely up to you if and how you use your lesson with your students.**

9. After completing this interview, you'll receive another $100. You'll also have the option to continue to Phase 3, which will involve practicing how you might respond to students as they work on the lesson. A few more details:
    a. In a COVID-free world, I would have visited your classroom as you taught your lesson. Given the circumstances, I will instead make some educated guesses about how students might respond to your lesson (e.g., strategies they might use, mistakes they might make).
    b. If you choose to continue, I'll share my ideas about what students might do during your lesson and ask you to practice responding to various situations in another interview. I'll provide more details if you express interest in this after completing Phase 2.

APPENDIX E

Lesson Plan Template

Format In what format will you be teaching this lesson? For example, will students be meeting synchronously with you on a remote platform? Will this lesson happen in a physical classroom?
Goals What are the mathematical goals for this lesson? What do you want students to know and understand about fractions from this lesson? What do students already know that will help them reach the goal?
Tools What virtual manipulative will students use in this lesson? Why did you choose this virtual manipulative?
Tasks What is the main task (or tasks) students will work on in this lesson? How will the virtual manipulative be helpful to students as they work on the task(s)?
Strategies What are the ways students might go about solving the task? Describe as many as you can.
Support How will you support students as they work on the task? What questions will you ask? How will you maintain their focus on the mathematical ideas you want them to explore?
Discussion What ideas do you want to explore or revisit in a discussion after students work on the task?

APPENDIX F

Sample Lesson Plan Interview Protocol

The questions common across participants are shown in italic. The questions in roman font were specific to the participant.

1. *What's the general structure of your remote lessons?*
   a. *How long do you typically spend?*

2. *What is the learning goal for your lesson? What do you hope kids will take away?*
   a. Have they seen any equivalent fractions at all? What representations? Do they grasp the "different numbers, but same amount" idea?
   b. Do students know the rule for equivalent fractions yet? Do you hope they'll derive it themselves?
   c. What did you mean when you said students would see the order of the fractions but it wouldn't be the focus?

3. *What virtual manipulative did you choose?* (For Colleen: Equivalent Fractions) *?*
   a. *Show it to me and tell me what you like about it.*
      1. What do you mean by multiple visuals of the same number?
      2. Why do you like the circle and square options?
      3. Why do you like the equivalent fractions being displayed on the side?
   b. *Is there anything you don't like about it?*
   c. *Why did you choose this particular virtual manipulative for your lesson?*
   d. *Did you choose your lesson to go with the VM, or choose the VM to go with your lesson?*

4. *Tell me more about your task. How will you pose it to students? What will their goal be?*
   a. When you say they'll discuss their thoughts as a group about $\frac{3}{8}$, what do you mean? Thoughts about how to start making something equivalent? Predictions of what an equivalent fraction might look like? Etc.
   b. What do you mean when you say they'll discuss their ideas with a partner? E.g., do you expect the ideas to be about strategies for creating equivalent fractions with the tool? Things they notice about the list on the side? etc.

5. *What are some things you expect students to do with the VM as they work on your task?*
   a. Show me what you mean when you say they might go up one space at a time.
   b. Why might that be helpful? Why might that not be helpful?
   c. Do you have any other ideas of what they might do with the manipulative?
   d. What would you say to a student who skipped over the manipulative?

6. *How do you hope the VM will support students' thinking? What parts of the tool do you hope or intend for them to pay attention to?*

7. Why did you choose to have the kids work in small groups?
   a. Will you group them by ability? Why?

8. What might you expect students to say to each of your prompts?
   a. Can you give an example of what you mean by a more specific guiding question?

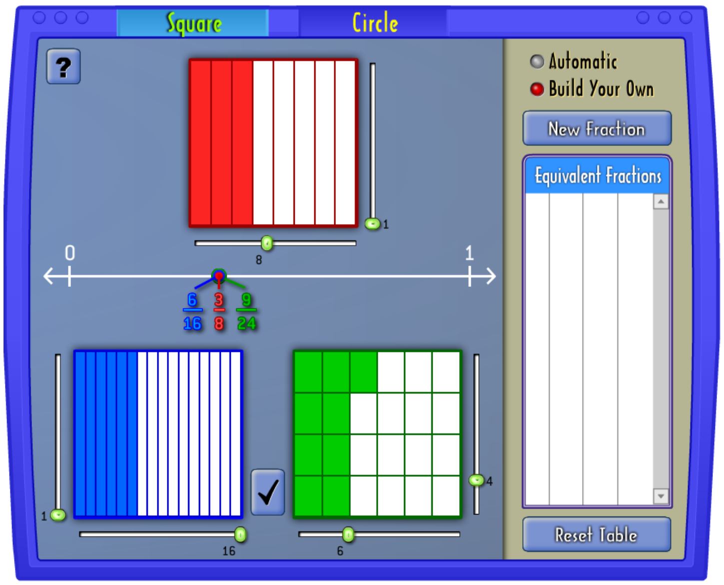
9. *How might you use the manipulative in your concluding discussion? E.g., will you show your screen? Have a student do so?*

10. *How do you think you might have taught this lesson differently if you didn't have this VM?*
    a. Would the goals have changed?
    b. Would the tasks have changed?

APPENDIX G

Sample Vignette for Student Response Interviews

Teacher: Colleen **VM:** NCTM Equivalent Fractions (no longer available online)



Vignette: Imagine that after students use the tool to find equivalent fractions for ⅜, a student makes 6/16 like this. Then they try to make 24ths using just vertical lines, but can't. A partner says they can make 24ths if they divide it both ways, something like this. A student fills in nine 24ths, but then says, "This can't be right. The number line says they are the same but I know equivalent fractions show the same amount. The green can't be the same as the blue and red because it has that piece sticking out. It should be even like the others."

Rationale:

- *Sample strategies from plan:* This is an example that came up in Colleen's lesson plan interview. She didn't seem to be bothered by it or think about possible student responses.

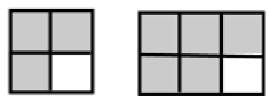
- *Common misconceptions:* The CCSS identifies the idea that same-size pieces do not necessarily have to be the same shape as a standard, albeit for a grade level below 4-5. This suggests students may trip up on this idea.
- *Typical approaches vs VMs:* Many times, when using drawings, lessons focus on children subdividing existing parts, suggesting that the idea that equivalent fractions representations have to look the same might be reinforced. The VM makes this new issue more likely to come up.

APPENDIX H

Sample Items from Fractions Content Knowledge and Mathematics Teaching Belief Surveys

Sample content knowledge item (LMTP,n.d.)

16. Takeem's teacher asks him to make a drawing to compare $\frac{3}{4}$ and $\frac{5}{6}$. He draws the following:



and claims that $\frac{3}{4}$ and $\frac{5}{6}$ are the same amount. What is the <u>most likely</u> explanation for Takeem's answer? (Mark ONE answer.)

a) Takeem is noticing that each figure leaves one square unshaded.

b) Takeem has not yet learned the procedure for finding common denominators.

c) Takeem is adding 2 to both the numerator and denominator of $\frac{3}{4}$, and he sees that that equals $\frac{5}{6}$.

d) All of the above are equally likely.

Sample beliefs items (Schoen & LaVenia, 2019):

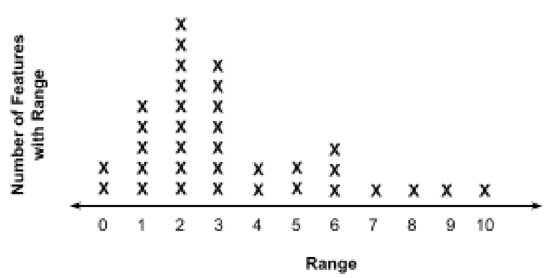
Scale	*Item*
Transmissionist	It is very important for students to discover how to solve math problems in their own ways.
Facts first	Students should master carrying out computational procedures before they are expected to understand why those procedures work.
Fixed instructional plan	If the teacher deviates from the sequence in the textbook, students will not learn the mathematics they are supposed to learn.

APPENDIX I

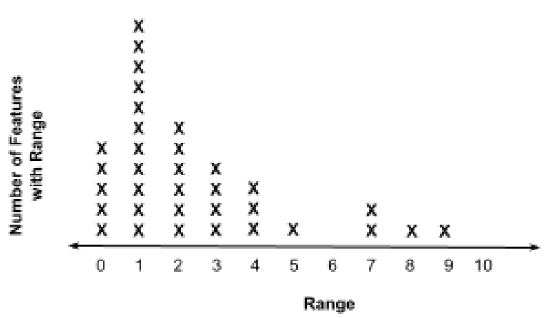
Identification of Individual Differences in Attention

The list of ranges in individual instances of attention to each MLC Fractions App feature are graphed on the line plots below. The first plot shows the ranges in the Exploration phase, and the second plot shows the ranges in the Problem Solving phase. I used a visual analysis to identify noticeably high ranges in each plot. The plot of the Exploration ranges shows four ranges in an upper "tail" of the plot (one range each of 7, 8, 9, and 10). The plot of Problem Solving Ranges shows a cluster of four ranges that are higher than and set apart from the rest (two ranges of 7, one range of 8, and one range of 9). I chose to investigate teachers' attention to the features associated with these ranges to look for evidence of possible relevant differences in attention among individual participants in each context.

Plot of Ranges of Attention to Features in Exploration



Plot of Ranges of Attention to Features in Problem Solving



APPENDIX J

Affordance Definitions Used for Coding of VMs

Efficient precision:

A feature that supports VM users in creating precise representations of fractions quickly and efficiently. These features may allow tedious tasks, such as partitioning a whole into equal parts or ensuring parts do not overlap, to be offloaded to the tool. VMs features reflecting efficient precision may also allow VM users to quickly copy and edit their representations or support the creation of models for fractions with denominators outside those typically supported by tools (e.g., 7ths or 23rds).

- Example: Students can choose a denominator and an area model is partitioned for them.
- Article description: "Virtual manipulatives are faithful to mathematical properties, provide precise mathematical examples, and create multiple copies of dynamic objects efficiently, therefore exhibiting mathematical fidelity" (Moyer-Packenham and Westenskow, 2013, p. 44).

Focused constraints:

These features reflect limitations placed on how a VM user can manipulate a VM's representations. The limitations could have the potential to focus students' attention on a particular idea or to prevent misconceptions or errors.

- Example: Fraction pieces can only be placed next to one another, and cannot overlap with each other or be placed with gaps between pieces.
- Article description: "Constraining and focusing features included: bringing to a specific level of awareness mathematical aspects of the objects which may not have been

observed by the student; and, applets focusing student attention on specific characteristics of mathematical processes or procedures" (Moyer-Packenham & Westenskow, p. 42).

Linked Representations:

In a linked representation, two representations of the same fraction are connected such that when one is changed, the other automatically changes to match it. The representations could be visual (e.g., an area model or fraction strip), symbolic (i.e., a numerator and denominator separated by a vinculum), a number line, or words.

- Example: When a student changes the number of partitions in an area model, the denominator of a linked symbolic fraction changes to match.
- Article description: "This category of affordances included research studies that reported: linking two different dynamic pictorial objects, [and] linking dynamic pictorial objects with symbols" (Moyer-Packenham & Westenskow, 2013, p. 43).

REFERENCES

REFERENCES

Agrawal, J., & Morin, L. L. (2016). Evidence-based practices: Applications of concrete representational abstract framework across math concepts for students with mathematics disabilities. *Learning Disabilities Research and Practice*, *31*(1), 34–44. https://doi.org/10.1111/ldrp.12093

Amador, J. M. (2016). Teachers' considerations of students' thinking during mathematics lesson design. *School Science and Mathematics*, *116*(5), 239–252.

Anderson-Pence, K. L., & Moyer-Packenham, P. S. (2016). The influence of different virtual manipulative types on student-led techno-mathematical discourse. *Journal of Computers in Mathematics and Science Teaching*, *35*(1), 5–31.

Ball, D. L. (1992). Magical hopes: Manipulatives and the reform of math education. *American Educator, Summer,* 14–18, 46–47. Retrieved from https://eric.ed.gov/?id=EJ449539

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes It special? *Journal of Teacher Education*, *59*(5), 389–407. https://doi.org/10.1177/0022487108324554

Bouck, E. C., Park, J., Sprick, J., Shurr, J., Bassette, L., & Whorley, A. (2017). Using the virtual-abstract instructional sequence to teach addition of fractions. *Research in Developmental Disabilities*, *70*(June), 163–174. https://doi.org/10.1016/j.ridd.2017.09.002

Braithwaite, D. W., & Siegler, R. S. (2021). Putting fractions together. *Journal of Educational Psychology*, *113*(3), 556–571. https://doi.org/10.1037/edu0000477

Braithwaite, D. W., Tian, J., & Siegler, R. S. (2018). Do children understand fraction addition? *Developmental Science*, *21*(4), 1–9. https://doi.org/10.1111/desc.12601

Bruner, J. S. (1964). The course of cognitive growth. *American Psychologist*, *19*(1), 1–15. https://doi.org/10.1037/h0044160

Calderhead, J. (1996). Teachers: Beliefs and knowledge. In D. C. Berliner & R. C. Calfee (Eds.), *Handbook of Educational Psychology* (pp. 709–725). New York: Macmillan.

Carbonneau, K. J., Marley, S. C., & Selig, J. P. (2013). A meta-analysis of the efficacy of teaching mathematics with concrete manipulatives. *Journal of Educational Psychology*, *105*(2), 380–400. https://doi.org/10.1037/a0031084

Carpenter, T. P., Fennema, E., Franke, M. L., Levi, L., & Empson, S. B. (2000). *Cognitively Guided Instruction: A research-based teacher professional development program for elementary school mathematics*. Retrieved from http://eric.ed.gov/?id=ED470472

Clarke, D. M., & Roche, A. (2009). Students’ fraction comparison strategies as a window into robust understanding and possible pointers for instruction. *Educational Studies in Mathematics*, *72*(1), 127–138. https://doi.org/10.1007/s10649-009-9198-9

Cohen, D. K., Raudenbush, S. W., & Ball, D. L. (2003). Resources, instruction, and research. *Educational Evaluation and Policy Analysis*, *25*(2), 119–142.

Colestock, A., & Sherin, M. G. (2009). Teachers’ sense-making strategies while watching video of mathematics instruction. *Journal of Technology and Teacher Education*, *17*(1), 7–29.

Confrey, J., Maloney, A. P., & Corley, A. K. (2014). Learning trajectories: A framework for connecting standards with curriculum. *ZDM - International Journal on Mathematics Education*, *46*(5), 719–733. https://doi.org/10.1007/s11858-014-0598-7

Copur-Gencturk, Y. (2021). Teachers’ conceptual understanding of fraction operations: results from a national sample of elementary school teachers. *Educational Studies in Mathematics*. https://doi.org/10.1007/s10649-021-10033-4

Cramer, K., Ahrendt, S., Monson, D., Wyberg, T., & Miller, C. (2017). Making sense of third-grade students’ misunderstandings of the number line. *Investigations in Mathematics Learning*, *9*(1), 19–37. https://doi.org/10.1080/19477503.2016.1245035

Cuban, L. (2018). Change and stability in classrooms, schools, and districts. In *The flight of the butterfly or the path of a bullet? Using technology to transform teaching and learning* (pp. 153–187). Cambridge, MA: Harvard University Press.

Davis, E. A., & Krajcik, J. S. (2005). Designing educative curriculum materials to promote teacher learning. *Educational Researcher*, *34*(3), 3–14. https://doi.org/10.3102/0013189X034003003

Dick, T. P. (2008). Keeping the faith: Fidelity in technological tools for mathematics education. In G. W. Blume & M. K. Heid (Eds.). *Research on technology and the teaching and learning of mathematics: Cases and perspectives, vol. 2* (pp. 333–339). Charlotte, NC: Information Age.

Diénès, Z. P. (1971). An example of the passage from the concrete to the manipulation of formal systems. *Educational Studies in Mathematics*, *3*(3/4), 337–352.

Drijvers, P., Doorman, M., Boon, P., Reed, H., & Gravemeijer, K. (2010). The teacher and the tool: Instrumental orchestrations in the technology-rich mathematics classroom. *Educational Studies in Mathematics*, *75*(2), 213–234. https://doi.org/10.1007/s10649-010-9254-5

Drijvers, P., Tacoma, S., Besamusca, A., Doorman, M., & Boon, P. (2013). Digital resources inviting changes in mid-adopting teachers’ practices and orchestrations. *ZDM - International Journal on Mathematics Education*, *45*(7), 987–1001. https://doi.org/10.1007/s11858-013-0535-1

Dweck, C. S. (1999). Caution—Praise can be dangerous. *American Educator*, *23*(1), 4–9.

Evans, M. A., & Wilkins, J. L. M. (2011). Social interactions and instructional artifacts: Emergent socio-technical affordances and constraints for children's geometric thinking. *Journal of Educational Computing Research*, *44*(2), 141–171. https://doi.org/10.2190/EC.44.2.b

Fazio, L., & Siegler, R. S. (2011). *Teaching fractions*. *Educational Practice Series*. Retrieved from http://www.ibe.unesco.org/fileadmin/user_upload/Publications/Educational_Practices/EdPractices_22.pdf

Goodwin, C. (1994). Professional vision. *American Anthropologist*, *96*(3), 606–633.

Greeno, J. G. (1994). Gibson's affordances. *Psychological Review*, *101*(2), 336–342.

Guskey, T. R. (2002). Professional development and teacher change. *Teachers and Teaching: Theory and Practice*, *8*(3), 381–391. https://doi.org/10.1080/135406002100000512

Hansen, A., Mavrikis, M., & Geraniou, E. (2016). Supporting teachers' technological pedagogical content knowledge of fractions through co-designing a virtual manipulative. *Journal of Mathematics Teacher Education*, *19*(2–3), 205–226. https://doi.org/10.1007/s10857-016-9344-0

Jacobs, V. R., Lamb, L. L. C., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education*, *41*(2), 169–202.

Johnston, C., & Suh, J. (2009). Pre-service elementary teachers planning for math instruction: Use of technology tools. In *Proceedings of the Society for Information Technology & Teacher Education International Conference* (pp. 3561–3566).

Jordan, N. C., Hansen, N., Fuchs, L. S., Siegler, R. S., Gersten, R., & Micklos, D. (2013). Developmental predictors of fraction concepts and procedures. *Journal of Experimental Child Psychology*, *116*(1), 45–58. https://doi.org/10.1016/j.jecp.2013.02.001

Kaur, H. (2015). Two aspects of young children's thinking about different types of dynamic triangles: prototypicality and inclusion. *ZDM Mathematics Education*, *47*(3), 407–420. https://doi.org/10.1007/s11858-014-0658-z

Learning Mathematics for Teaching Projects (LMTP). (n.d.) Mathematics knowledge for teaching measures. Requested from http://www.umich.edu/~lmtweb/

Leung, A. (2017). Exploring techno-pedagogic task design in the mathematics classroom. In *Digital technologies in designing mathematics education tasks* (pp. 3–16). https://doi.org/10.1007/978-3-319-43423-0

Lichtman, M. (2006). Learning about others through interviewing. In *Qualitative Research in Education: A User's Guide* (pp. 115–135). Thousand Oaks, CA: Sage Publications.

Males, L. M., Earnest, D., Dietiker, L., & Amador, J. M. (2015). Examining K-12 prospective teachers' curricular noticing. In *Proceedings of the 37th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 88–95). East Lansing: Michigan State University.

Manches, A., O'Malley, C., & Benford, S. (2010). The role of physical representations in solving number problems: A comparison of young children's use of physical and virtual materials. *Computers and Education*, *54*(3), 622–640. https://doi.org/10.1016/j.compedu.2009.09.023

Maxwell, J. A. (2013). *Qualitative Research Design: An Interactive Approach.* Thousand Oaks, CA: SAGE Publications.

McGee, J. R., Wang, C., & Polly, D. (2013). Guiding teachers in the use of a standards-based mathematics curriculum: Teacher perceptions and subsequent instructional practices after an intensive professional development program. *School Science and Mathematics*, *113*(1), 16–28.

Mishra, P., & Koehler, M. J. (2006). Technological pedagogical content knowledge: A framework for teacher knowledge. *Teachers College Record*, *108*(6), 1017–1054. https://doi.org/10.1111/j.1467-9620.2006.00684.x

Moyer-Packenham, P. S., & Bolyard, J. J. (2016). Revisiting the definition of a virtual manipulative. In P. S. Moyer-Packenham (Ed.), *International Perspectives on Teaching and Learning Mathematics with Virtual Manipulatives* (pp. 3–23). Switzerland: Springer International Publishing.

Moyer-Packenham, P. S., Salkind, G., & Bolyard, J. J. (2008). Virtual manipulatives used by K-8 teachers for mathematics instruction: Considering mathematical, cognitive, and pedagogical fidelity. *Contemporary Issues in Technology and Teacher Education*, *8*(3), 202–218. Retrieved from http://www.editlib.org/p/26057/

Moyer-Packenham, P. S., & Suh, J. M. (2012). Learning mathematics with technology: The influence of virtual manipulatives on different achievement groups. *Journal of Computers in Mathematics and Science Teaching*, *31*(1), 39–59.

Moyer-Packenham, P. S., & Westenskow, A. (2013). Effects of virtual manipulatives on student achievement and mathematics learning. *International Journal of Virtual and Personal Learning Environments*, *4*(3), 35–50. https://doi.org/10.4018/jvple.2013070103

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common Core State Standards for Mathematics.* Washington, DC: Authors. http://www.corestandards.org/Math/

Naftaliev, E., & Yerushalmy, M. (2011). Solving algebra problems with interactive diagrams: Demonstration and construction of examples. *Journal of Mathematical Behavior*, *30*(1), 48–61. https://doi.org/10.1016/j.jmathb.2010.12.002

Nührenbörger, M., & Steinbring, H. (2008). Manipulatives as tools in mathematics teacher education. In D. Tirosh & T. Wood (Eds.). *Tools and processes in mathematics teacher education* (pp. 157-181). Rotterdam, Netherlands: Sense Publishers.

Piaget, J. (1964). Part I: Cognitive development in children: Piaget, development and learning. *Journal of Research in Science Teaching*, *2*(3), 176–186. https://doi.org/10.1002/tea.3660020306

Priestley, M., Edwards, R., Priestley, A., & Miller, K. (2012). Teacher agency in curriculum making: Agents of change and spaces for manoeuvre. *Curriculum Inquiry, 42*(2), 191–214. https://doi.org/10.1111/j.1467-873X.2012.00588.x

Puchner, L., Taylor, A., O'Donnell, B., & Fick, K. (2008). Teacher learning and mathematics manipulatives: A collective case study about teacher use of manipulatives in elementary and middle school mathematics lessons. *School Science and Mathematics*, *108*(7), 313–325. https://doi.org/10.1111/j.1949-8594.2008.tb17844.x

Rau, M. A., & Matthews, P. G. (2017). How to make 'more' better? Principles for effective use of multiple representations to enhance students' learning about fractions. *ZDM - Mathematics Education, 49*(4), 531–544. https://doi.org/10.1007/s11858-017-0846-8

Reimer, K., & Moyer, P. S. (2005). Third-graders learn about fractions using virtual manipulatives: A classroom study. *The Journal of Computers in Mathematics and Science Teaching*, *24*(1), 5–25.

Remillard, J. T. (2005). Examining key concepts in research on teachers' use of mathematics curricula. *Review of Educational Research*, *75*(2), 211–246. Retrieved from http://www.jstor.org/stable/3516049

Resnick, L. B., & Ford, W. W. (1981). Teaching the structures of mathematics. In *The Psychology of Mathematics for Instruction* (pp. 101–127). Hillsdale, NJ: Erlbaum Associates. https://doi.org/10.1016/j.paid.2011.03.037

Resnick, L. B., & Omanson, S. F. (1986). Learning to understand arithmetic. In R. Glaser (Ed.), *Advances in Instructional Psychology* (pp. 41–95). Routledge.

Rich, K. M. (2021). Virtual manipulatives in elementary mathematics: A critical integrative review of research. *Journal of Computers in Mathematics and Science Teaching*, *40*(2), 163–177.

Sarama, J., & Clements, D. H. (2009). "Concrete" computer manipulatives in mathematics education. *Child Development Perspectives, 3*(3), 145–150. https://doi.org/10.1111/j.1750-8606.2009.00095.x

Schoen, R. C., & LaVenia, M. (2019). Teacher beliefs about mathematics teaching and learning: Identifying and clarifying three constructs. *Cogent Education*, *6*(1), 1–29. https://doi.org/10.1080/2331186X.2019.1599488

Shapiro, E. J., Sawyer, A. G., Dick, L. K., & Wismer, T. (2019). Just what online resources are elementary mathematics teachers using? *Contemporary Issues in Technology and Teacher Education*, *19*(4), 670–686.

Shaughnessy, M. M. (2011). Identify fractions and decimals on a number line. *Teaching Children Mathematics*, *17*(7), 428–434.

Shin, D., Smith, R. C., & Kim, S. (2018). Evaluating technology for teaching mathematics. *Mathematics Teaching in the Middle School*, *24*(3), 156–163.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4–14. https://doi.org/10.3102/0013189X015002004

Siegler, R. S., Fazio, L. K., Bailey, D. H., & Zhou, X. (2013). Fractions: The new frontier for theories of numerical development. *Trends in Cognitive Sciences*, *17*(1), 13–19. https://doi.org/10.1016/j.tics.2012.11.004

Sinclair, N., Chorney, S., & Rodney, S. (2016). Rhythm in number: exploring the affective, social and mathematical dimensions of using TouchCounts. *Mathematics Education Research Journal*, *28*(1), 31–51. https://doi.org/10.1007/s13394-015-0154-y

Smith, M. S., Bill, V., & Hughes, E. K. (2008). Thinking through a lesson: Successfully implementing high-level tasks. *Mathematics Teaching in the Middle School*, *14*(3), 132–138.

Smith, R. C., Shin, D., & Kim, S. (2017a). A framework for examining teachers' noticing of mathematical cognitive technologies. *Journal of Computers in Mathematics and Science Teaching*, *36*(1), 41–63.

Smith, R. C., Shin, D., & Kim, S. (2017b). Prospective and current secondary mathematics teachers' criteria for evaluating mathematical cognitive technologies. *International Journal of Mathematical Education in Science and Technology*, *48*(5), 659–681. https://doi.org/10.1080/0020739X.2016.1264635

Smith, R. C., Shin, D., Kim, S., & Zawodniak, M. (2018). Novice secondary mathematics teachers' evaluation of mathematical cognitive technological tools. *Contemporary Issues in Technology and Teacher Education*, *18*(4), 606–630.

Smith, M. S., & Stein, M. K. (1998). Selecting and creating mathematical tasks: From research to practice. *Mathematics Teaching in the Middle School*, *3*(5), 344–350.

Snape, D., & Spencer, L. (2003). The foundations of qualitative research. In J. Ritchie & J. Lewis (Eds.), *Qualitative research practice: A guide for social science students and researchers* (pp. 1–23). London: Sage Publications. https://doi.org/10.4135/9781452230108

Sriraman, B., & Lesh, R. (2007). A conversation With Zoltan P. Diénès. *Mathematical Thinking and Learning*, *9*(1), 59–75. https://doi.org/10.1207/s15327833mtl0901_5

Stafylidou, S., & Vosniadou, S. (2004). The development of students' understanding of the numerical value of fractions. *Learning and Instruction*, *14*(5 SPEC.ISS.), 503–518. https://doi.org/10.1016/j.learninstruc.2004.06.015

Stein, M. K., Grover, B. W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal*, *33*(2), 455–488. https://doi.org/10.3102/00028312033002455

Suh, J. M. (2010). Leveraging cognitive technology tools to expand opportunities for critical thinking on data analysis and probability in elementary classrooms. *Journal of Computers in Mathematics and Science Teaching*, *29*(3), 289–302.

Suh, J. M. (2016). Ambitious teaching: Designing practice-based assignments for integrating virtual manipulatives into mathematics lessons. In S. Ladel & U. Kortenkamp (Eds.), *International Perspectives on Teaching and Learning Mathematics with Virtual Manipulatives* (Vol. 7, pp. 301–321). Cham, Switzerland: Springer. https://doi.org/10.1007/978-3-319-32718-1

Suh, J., Moyer, P. S., & Heo, H.-J. (2005). Examining technology uses in the classroom: Developing fraction sense using virtual manipulative concept tutorials. *Journal of Interactive Online Learning*, *3*(4).

Thomas, A., & Edson, A. J. (2019). A framework for teachers' evaluation of digital instructional materials: Integrating mathematics teaching practices with technology use in K-8 classrooms. *Contemporary Issues in Technology and Teacher Education*, *19*(3), 1–31.

Tondeur, J., van Braak, J., Ertmer, P. A., & Ottenbreit-Leftwich, A. (2017). Understanding the relationship between teachers' pedagogical beliefs and technology use in education: A systematic review of qualitative evidence. *Educational Technology Research and Development*, *65*(3), 555–575. https://doi.org/10.1007/s11423-016-9481-2

Tracy, S. J. (2010). Qualitative quality: Eight "big-tent" criteria for excellent qualitative research. *Qualitative Inquiry*, *16*(10), 837–851. https://doi.org/10.1177/1077800410383121

Uttal, D. H., O'Doherty, K., Newland, R., Hand, L. L., & DeLoache, J. S. (2009). Dual representation and the linking of concrete and symbolic representations. *Child Development Perspectives*, *3*(3), 156–159. https://doi.org/10.1111/j.1750-8606.2009.00097.x

Uttal, D. H., Scudder, K. V., & DeLoache, J. S. (1997). Manipulatives as symbols: A new perspective on the use of concrete objects to teach mathematics. *Journal of Applied Developmental Psychology*, *18*(1), 37–54. https://doi.org/10.1016/S0193-3973(97)90013-7 van Es, E. A., & Sherin, M. G. (2008). Mathematics teachers' "learning to notice" in the context of a video club. *Teaching and Teacher Education*, *24*(2), 244–276. https://doi.org/10.1016/j.tate.2006.11.005

Van Hoof, J., Verschaffel, L., & Van Dooren, W. (2015). Inappropriately applying natural number properties in rational number tasks: Characterizing the development of the natural number bias through primary and secondary education. *Educational Studies in Mathematics*, *90*(1), 39–56. https://doi.org/10.1007/s10649-015-9613-3

Wager, A. A. (2014). Noticing children's participation: Insights Into teacher positionality toward equitable mathematics pedagogy. *Journal for Research in Mathematics Education*, *45*(3), 312–350. https://doi.org/10.5951/jresematheduc.45.3.0312

Warshauer, H. K. (2015). Productive struggle in middle school mathematics classrooms. *Journal of Mathematics Teacher Education*, *18*(4), 375-400.

Webel, C., Krupa, E. E., & McManus, J. (2015). Teachers' evaluations and use of web-based curriculum resources in relation to the common core state standards for mathematics. *Middle Grades Research Journal*, *10*(2), 49–64.

Wilson, P. H., Lee, H. S., & Hollebrands, K. F. (2011). Understanding prospective mathematics teachers' processes for making sense of students' work with technology. *Journal for Research in Mathematics Education*, *42*(1), 39–64.

Young, E., & Zientek, L. R. (2011). Fraction operations: An examination of prospective teachers' errors, confidence, and bias. *Investigations in Mathematics Learning*, *4*(1), 1–23. https://doi.org/10.1080/24727466.2011.11790307

Zbiek, R. M., Heid, M. K., Blume, G. W., & Dick, T. P. (2007). Research on technology in mathematics education: A perspective of constructs. In F. K. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 1169–1207). Charlotte, NC: Information Age Publishing.