

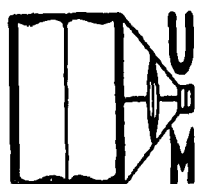
DOCTORAL DISSERTATION SERIES

TITLE SYNTHESIS OF A RESISTANCE-CAPACI-
TANCE FILTER WHOSE POWER INSERTION
RATIO APPROXIMATES A PRESCRIBED
FUNCTION OF FREQUENCY

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SYNTHESIS OF A RESISTANCE - CAPACITANCE FILTER
WHOSE POWER INSERTION RATIO APPROXIMATES A
PRESCRIBED FUNCTION OF FREQUENCY

BY
NOAH HERBERT KRAMER

A THESIS

Submitted to the School of Graduate Studies of Michigan
State College of Agriculture and Applied Science
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for the degree of
DOCTOR OF PHILOSOPHY

Department of Electrical Engineering
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To Dr. J. A. Strelzoff, whose help and
encouragement made this thesis possible.

INTRODUCTION

(1) NETWORK SYNTHESIS.

There are several aspects to the general field of Network Synthesis.

One interesting facet is the Approximation Problem. The desired characteristic is usually given in a graphical form. The problem requires the construction of a mathematical expression to describe a network that possesses the desired characteristic, within a permissible approximation. Elliptic functions, Tschebychef polynomials, and Butterworth functions often appear in this type problem.

Another aspect of synthesis is the Realization Problem. What mathematical or graphical functions can lead to a physically realizable network? What geometric forms can this network assume? Much of the pioneer work in the Realization Problem has been done by O. Brune. (1)

This thesis will deal with both aspects of Network Synthesis.

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1 Brune, O. Synthesis of a Finite Two Terminal Network Whose Driving Point Impedance is a Prescribed function of Frequency. Journal of Math and Phys., vol. 10, pp1191-235

In this thesis, the response of an ideal filter is approximated by a mathematical expression (Approximation problem). This expression is used to realize a symmetrical lattice, composed of resistances and capacitances (Realization problem).

The restriction to these two elements is not just an academic choice. R - C (Resistance-Capacitance) networks can be designed to give many responses, and the additional loss that they introduce, due to dissipation in the resistances, can be compensated by subsequent amplification. In many cases, where only a frequency discriminating circuit is desired, the loss introduced by the resistances is of no importance and can be ignored. The exclusion of inductances might introduce economic and construction advantages. Certainly the problem of magnetic pickup is reduced when there are no inductances.

The terminology in Network Synthesis has become confused, since Mathematicians, Physicists, and Engineers are all contributing to the field. It is advisable to clarify terms at this point.

(2) FOUR TERMINAL NETWORKS

A Four Terminal Network, or Quadripole, has a pair of terminals designated as Input, and the other pair designated as Output.

Associated with the input terminals are the Input Voltage E_1 and the Input Current I_1 .

The output terminals provide Output voltage E_2 and Output Current I_2 .

In the usual case a generator E with an internal Generator Impedance Z_1 is connected to the input. A Load Impedance Z_2 is connected to the output. This thesis restricts these impedances to pure resistances R_1 and R_2 .

Figure 1 illustrates the notation that will be employed. Since the direction of the currents is arbitrary, the notation shown is chosen.

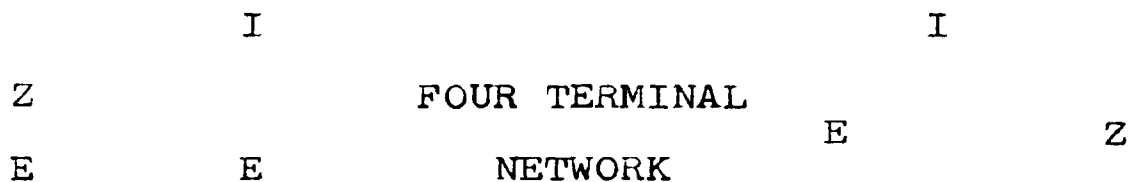


Figure 1 Four Terminal Network with Load.

(3) NETWORK RELATIONS

The relations between the input current and voltage, and the output current and voltage are found in any reference on Network Theory - 2,3,4.

$$\begin{aligned} I_1 &= y_{11} E_1 + y_{12} E_2 & E_1 &= z_{11} I_1 + z_{12} I_2 \\ I_2 &= y_{21} E_1 + y_{22} E_2 & E_2 &= z_{21} I_1 + z_{22} I_2 \end{aligned} \quad \begin{array}{l} \text{Eq.1} \\ \text{Eq.2} \end{array}$$

y_{11}, y_{22} are the short circuit driving point admittances.

y_{12}, y_{21} are the short circuit transfer admittances.

z_{11}, z_{22} are the open circuit driving point impedances.

z_{12}, z_{21} are the open circuit transfer impedances.

In a passive network $y_{12} = y_{21}$, $z_{12} = z_{21}$.

Gewertz (5) has shown how the four admittances or impedances uniquely characterize a network, and has set up realization techniques when these parameters are known.

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- 2 Guillemin, Ernst A. Communication Networks
Vol. II 1935 New York: John Wiley
 - 3 Guillemin, Ernst A. Communication Networks
Vol. II 1935 New York: John Wiley
 - 4 Everitt, W. L. Communication Engineering
Second Edition 1937 New York McGraw-Hill
 - 5 Gewertz Synthesis of a Finite Four Terminal
Network Jour. Math and Phys. Vol. 12
1932 - 1933 pp. 1-257

(4) TRANSFER IMPEDANCE

Consider the generator impedance and the load impedance absorbed in the quadripole, as in figure 2. The ratio of the generator voltage to the load current is E/I . This ratio is defined to be the Transfer Impedance Z_t of the quadripole.

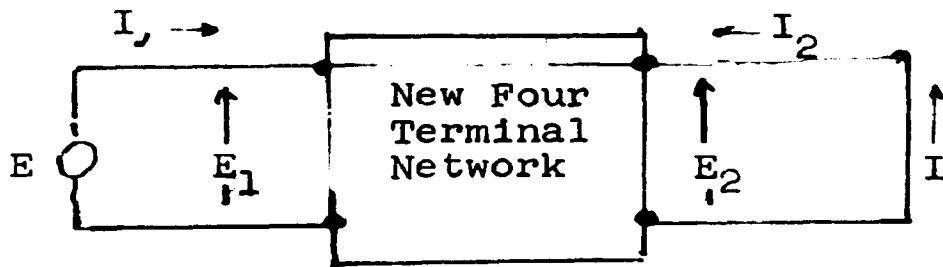


Figure 2 Quadripole for Transfer Impedance

The output voltage of E_2 of this quadripole is zero, for there is a short circuit across the new output terminals. Also $E = E_1$.

When these values are inserted into equation 2, there is obtained $I_2 = E_1 y_{21}$ $Z_t = E/I_2 = 1/y_{21}$
Assuming a passive circuit...

$$Z_t = 1/y_{12} \quad \text{Eq. 3}$$

(5) LOSSES

The most efficient way of coupling the generator to the load is through an ideal transformer. The power delivered to the load in that case will be called P_o . A quadripole inserted between the generator and load will transmit a power P_L to the load.

The ratio P_L/P_o is less than or equal to unity. The equality holds when the quadripole is an ideal transformer.

It is shown (6) that in the ideal case,
 $Z_t = 2\sqrt{Z_1 Z_2}$.

Consider the ratio of powers P_L/P_o .

It is effectively $\frac{|I_L|^2 Z_L}{|I_o|^2 Z_o} = \left| \frac{I_L}{I_o} \right|^2 = \frac{P_L}{P_o}$

Divide the fraction by E^2/E^2 and there results

$$\frac{P_L}{P_o} = \left| \frac{I_L}{I_o} \frac{E^2}{E^2} \right|^2 = \left| \frac{1/Z_t}{1/2\sqrt{Z_1 Z_2}} \right|^2 \cdot \left| \frac{2\sqrt{Z_1 Z_2}}{Z_t} \right|^2 \cdot \left| \frac{2\sqrt{Z_1 Z_2}}{Z_t} \right|^2$$

The ratio P_L/P_o can be defined -

$$\text{Power Insertion Ratio} = \left| 2\sqrt{Z_1 Z_2} Y_t \right|^2 = \frac{P_L}{P_o} \quad \text{Eq. 4}$$

.....

6 Mason, W. P. Electromechanical Filters and Wave Transducers 1st Edition 1942
New York: D. Van Nostrand. p24

In the study of four terminal networks, the Insertion Ratio is often considered. This ratio compares the current that would flow in the load due to ideal transformer coupling, to the current that actually flows when the quadripole is inserted between generator and load impedances. It can be evaluated, as follows:

$$\left| \frac{I_o}{I_1} \right| = \left| \frac{I_o / E}{I_1 / E} \right| = \left| \frac{Z_t}{2 \sqrt{Z_1 Z_2}} \right| \quad \text{then } \left| \frac{I_1}{I_o} \right| = 2 \sqrt{Z_1 Z_2} Y_t \quad \text{Eq. 5}$$

This is the Insertion Loss Ratio^o.

Bode (7) prefers to deal with the inverse ratio, $\frac{I_o}{I_1} = \frac{Z_t}{2 \sqrt{Z_1 Z_2}}$. He defines $\theta = \log \frac{Z_t}{2 \sqrt{Z_1 Z_2}} = A + jB$.

A and B are respectively the transfer loss and the transfer phase. Bode does not use the absolute values in the ratio of currents but treats the currents, and the ratio, as complex numbers. However, the logarithm of a complex number Z 10 is given by $\log |Z| + j0$.

Thus, if one takes the reciprocal of the Insertion Loss Ratio, and then takes the natural logarithm of this new ratio, the Transfer Loss is obtained.

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^o Ibid 3 p. 73

Consider Y_t .

$$Y_t = G_t + jB_t,$$

$$|Y_t| = (G_t^2 + B_t^2)^{1/2}$$

$$\bar{Y}_t = G_t - jB_t,$$

$$\text{Then } Y_t \bar{Y}_t = (G_t + jB_t)(G_t - jB_t) \\ G_t^2 + B_t^2$$

$$\text{So } |Y_t|^2 = Y_t \bar{Y}_t$$

This result will be used later in this thesis
in a method suggested by Bode.

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7 Bode, H. E. Network Analysis and Feedback Amplifier
Design 1945 New York: D. Van Nostrand

Ibid 7 p. 230

(6) R-C QUADRIPOLES

The essential features of R-C dipoles and quadripoles have been established by Cauer (8), and only the points of interest to this thesis are presented here.

The driving-point immittances* of R-C Networks are rational functions of the complex frequency variable $P = j\omega$. Poles and zeros are simple, real, and interlaced.

The immittances can be expanded into partial fractions:

$$Z(p) = a^{(\infty)} + \frac{a^{(0)}}{p} + \sum_{1}^n \frac{a^{(v)}}{p + p_v} \quad \text{Eq. 7}$$

$$Y(p) = b^{(\infty)} \cdot p + b^{(0)} + p \cdot \sum_{1}^n \frac{b^{(v)}}{p + q_v} \quad \text{Eq. 8}$$

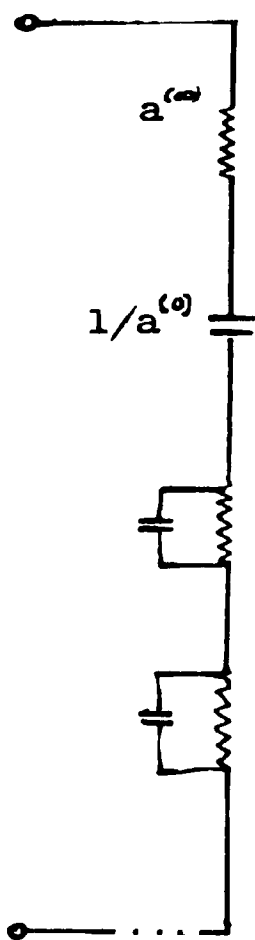
where all the $a^{(v)}$ and $b^{(v)}$ are positive.

These expansions lead to the two canonic forms developed by Cauer for R-C networks, figure 3.

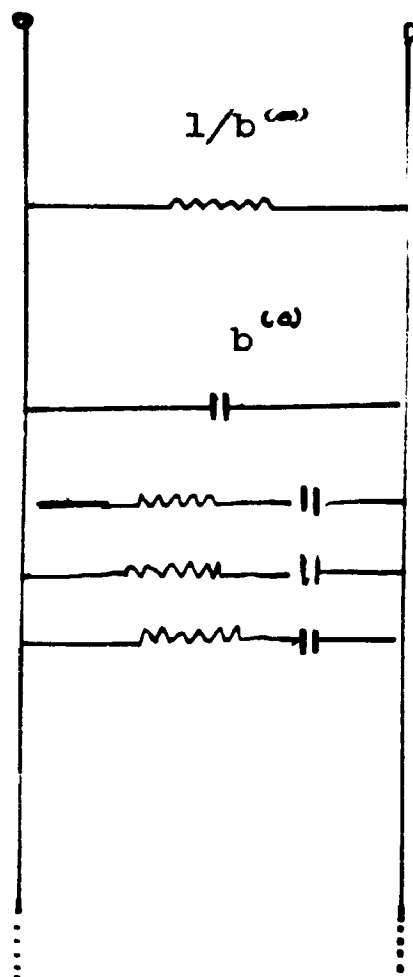
.....

8 Cauer, E. Die Verwirklichung von Wechselstromwiderständen vorgeschriebener Frequenzabhängigkeit. Archiv F. Electrotechnik, vol. 17 p355 1927

* Ibid 7 p.15



(Impedance Form)



(Admittance Form)

Figure 3 Canonic Forms of R-C Networks.

Quadrupole properties are found by inserting the partial fraction expansions, Eq. 7 and Eq. 8, into the general equation 1 and 2.

The terms are:

$$z_{11} = a_{11}^{(\infty)} + \frac{a_{11}^{(0)}}{p} + \sum_{n=1}^{\infty} \frac{a_{11}^{(n)}}{p + p_n}$$

$$z_{12} = a_{12}^{(\infty)} + \frac{a_{12}^{(0)}}{p} + \sum_{n=1}^{\infty} \frac{a_{12}^{(n)}}{p + p_n}$$

$$y_{11} = b_{11}^{(\infty)} \cdot p + b_{11}^{(0)} + p \cdot \sum_{n=1}^{\infty} \frac{b_{11}^{(n)}}{p + q_n}$$

$$y_{12} = b_{12}^{(\infty)} \cdot p + b_{12}^{(0)} + p \cdot \sum_{n=1}^{\infty} \frac{b_{12}^{(n)}}{p + q_n}$$

The very important Residue Theorem^o states:

$$a_{11}^{(n)} \geq 0, \quad a_{22}^{(n)} \geq 0, \quad a_{11}^{(n)} \cdot a_{22}^{(n)} - (a_{12}^{(n)})^2 \geq 0$$

$$b_{11}^{(n)} \geq 0, \quad b_{22}^{(n)} \geq 0, \quad b_{11}^{(n)} \cdot b_{22}^{(n)} - (b_{12}^{(n)})^2 \geq 0$$

for all n .---

as the necessary and sufficient conditions for the physical realizability of a network.

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^o Ibid 3 p. 216

If the R-C quadripole which meets these conditions is inserted between the generator and load, as in figure 2, the transfer impedance is Z_t .

The transfer impedance Z_t is then a rational function of the complex frequency variable, p , with real coefficients. Its zeros are the same as any driving point impedance seen in any mesh of the network, and since the network is R-C, the zeros are real, negative and simple.

Since R_1 is in series with the input to the R-C network, there can be no zero at $p = \infty$.

The poles of Z_t are produced either by the zeros of z_{12} or by the poles of z_{11}, z_{22} which are not in z_{12} .

There are no restrictions on the zeros of z_{12} , for it is not a driving point impedance, so there are no restrictions on the poles of Z_t , other than that they occur in conjugate pairs when complex. Figure 4 shows the possible arrangements. Poles are denoted by crosses, zeros by circles.

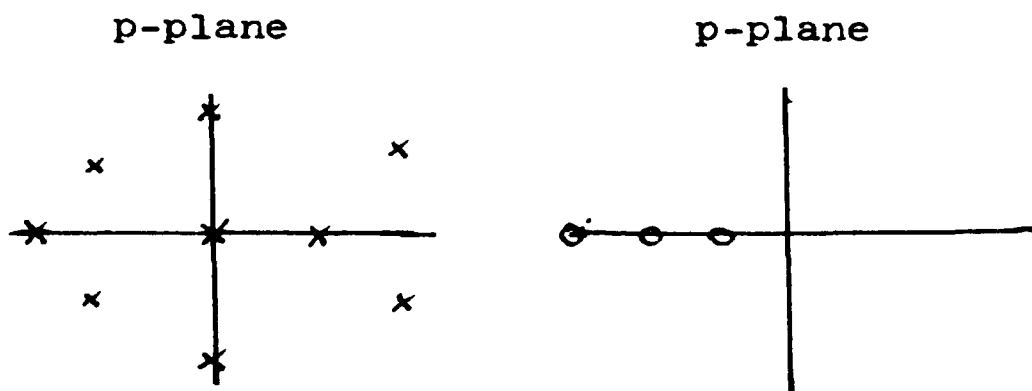


Figure 4 Typical Pole and Zero Patterns

The preceding analysis of R-C networks closely follows the form in a paper by Orchard (9).

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- 9 Orchard, H. J. The Synthesis of R-C Networks
to have Prescribed Transfer Functions
Proc. I.R.E. April 1951 p.428

(7) SYNTHESIS

When the generator and load impedance are absorbed into the quadripole, as in figure 2, the expression for Z_t is $Z_t = 1/y_{12}$.

Knowledge of y_{12} is not sufficient to uniquely determine the quadripole^o, for y_{11} and y_{22} can be any values that satisfy the residue conditions. To complete the synthesis, many devices can be used.

Darlington (10) uses the Z_t and, as additional conditions, he requires that the quadripole consist of pure reactances terminated in, at most, one resistance. This, of course, is not acceptable for R-C networks.

Bode^{oo} requires that the quadripole be a constant resistance lattice. This implies that the arms of the lattice have an inverse relationship to each other. This could introduce inductances and is not acceptable in R-C networks.

Guillemin (11) has a system that results in a ladder formation, but requires a generator of zero internal impedance.

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^o Ibid 5

10 Darlington, S. Synthesis of Reactance Four Poles.
Jour. of Math. Phys. Vol 18 No. 4 Sept. 1939
pp. 257-353.

11 Guillemin, E. A. Synthesis of R-C Networks
Jour. of Math. Phys., Vol 28, p.22 April 1949

^{oo} Ibid 7 p.229

Guillemin's method also restricts his choice of functions to the minimum phase shift function. (All zeros and poles are in the left half of the $p = j\omega$ plane.) This last restriction is due to his choice of a ladder formation, for ^o "Any passive ladder network is a minimum phase shift structure."

Orchard^{oo} has an ingenious device to synthesize the network. Assume a symmetrical network. Then $y_{11} = y_{22}$. Choose $y_{11} = y_{22}$ to satisfy the equality sign in the residue condition (Eq. 10).

Then $b_{11}^{(n)} = b_{22}^{(n)} = |b_{12}^{(n)}|$. For all n .

From the expansion of $y_{12} = (Z_t)^{-1}$, it is possible to form y_{11} , y_{22} by making all the residues of y_{12} positive. With the specification of y_{11} , y_{22} , y_{12} , y_{21} , it is then possible to synthesize the network in any suitable form.

Guillemin ^o shows that the residue condition is identical to the realization condition of a symmetrical lattice, so if the network is realizable in any form, it is always realizable

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- ^o Ibid 7, p. 243
- ^{oo} Ibid 9
- ^o Ibid 3, p. 381

in the lattice form. This does not rule out other forms, but assures the existence of at least one form, the lattice structure.

The lattice will have arms:

$$Y_A = Y_{11} + Y_{12} , \quad Y_B = Y_{11} - Y_{12} \quad \text{Eq. 11}$$

Orchard then extracts equal resistances from the lattice arms to act as generator and load resistances. The result appears as figure

5.

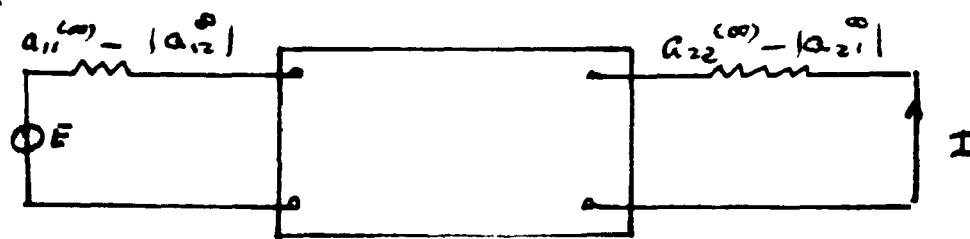


Figure 5 Extraction of Terminating Resistors

The fact that the generator and load resistances are equal is a design limitation of the method.

(8) APPROXIMATION PROCEDURE

This thesis uses a modification of a device of Guillemin^o to approximate a frequency response.

$F(\omega)$ is given graphically over the complete range $-\infty < \omega \leq \infty$.

A change of variable $\omega = \tan \theta/2$ changes the range to $-\pi \leq \theta \leq \pi$, and the $F(\omega)$ is transformed into a periodic function of θ . This $F(\theta)$ is then approximated by Fourier Series. Another transformation, $x = \cos \theta$, changes the series to a polynomial in x , $F(x)$. The last transformation, $x = \frac{1-\omega^2}{1+\omega^2}$, returns the approximation to the ω plane.

The closeness of this approximation is determined by the nearness of the Fourier Series approximation to the prescribed $F(\theta)$. There are many classes of functions that can be approximated by Fourier Series in an arbitrarily close manner, and if the $F(\theta)$ falls into one of these classes, the approximation will be as close as desired.

The mathematical expression that results from the above manipulation is not in a form suitable for R-C networks. The denominator would lead to multiple poles.

.....
^o Ibid 11

In this thesis the approximate $F(\omega)$ will be divided by a function of ω that is very close to unity over the range of ω . The choice of this dividing function will be made to clear the denominator.

There is liberty in choosing this function, as long as the choice does not change the final result to a point that the approximation is not as close as desired.

(19) APPROXIMATION EXAMPLE....LOW PASS FILTER

The graphical representation of the desired filter is given in figure 6.

The filter characteristic has been given in terms of $|Y_r|^2$. $|Y_r|^2$ has been shown to be proportional to the power insertion ratio. (Eq. 4)

For the sake of simplicity the graph as been normalized with respect to frequency and magnitude.

The first change $\omega = \tan \theta/2$ results in the $F(\theta)$ as shown in figure 6b.

There is no difficulty in approximating figure 6b, by Fourier Series. Since $F(\theta)$ is continuous and of bounded variation, a Fourier Theorem (12) states that the series converges uniformly to $F(0)$.

By standard methods the first few terms of the series are....

$$F(\theta) = .500 + .635 \cos \theta - .204 \cos 3\theta + .1215 \cos 5\theta$$

Eq. 11

Since only a few terms will be used, the above approximation might become negative for some θ . this is not allowable for $|Y_r|^2$, so a different form is used.

$$F(\theta) = B_0 + .635 \cos \theta - .204 \cos 3\theta + .1215 \cos 5\theta \quad \text{Eq. 12}$$

B_0 is chosen large enough to keep $F(\theta)$ positive.

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12 Churchill, R. V. Fourier Series and Boundary Value Problems. 1941 New York: McGraw-Hill p.86

At $\theta = \pi/2$, which corresponds to $\omega = 1$, the value of $F(\theta)$ - or $|Y_T(\theta)|^2 = B_\infty$

Make the transformation $x = \cos \theta$. Then $\cos 3\theta = 4x^3 - 3x$, $\cos 5\theta = 16x^5 - 20x^3 + 5x$

The approximation then becomes:

$$\begin{aligned} |Y_T(x)|^2 &= .635x^3 - .816x + .612x + 1.944x^5 - 2.43x^3 + .6075x + B_\infty \\ &= 1.944x^5 - 3.246x^3 + 1.8545x + B_\infty \end{aligned} \quad \text{Eq. 12}$$

For some value of x , and some ω , the $|Y_T(x)|^2$ reaches a minimum. If this minimum is fixed at zero, the $|Y_T(\omega)|^2$ will not be zero at $\omega = \infty$. This is the result of using too few terms in the Fourier Series.

Figure 7 compares the approximation to the $F(\theta)$. The inclusion of additional cosine terms would improve the approximation, but increase the work in the computation.

The minimum is found by differentiating Eq. 12.

$$\frac{d}{dx} |Y_T(x)|^2 = 9.720x^4 - 9.738x^2 + 1.8545 = 0 \quad \text{Eq. 13}$$

This is a quadratic in x^2 , roots $\frac{9.738 \pm 4.972}{19.440}$

The root of interest is $x^2 = .245164$, $x = .495$

At this value of x , B_∞ must be .5823278 to have

$$|Y_T(x)|^2 \underset{x \rightarrow .495}{=} 0$$

CALCULATIONS FOR GRAPHS (Figures 6 & 7)

θ	$\cos \theta$	$.635$ $\cos \theta$	$.204$ $\cos 3\theta$	$.1215$ $\cos 5\theta$	$-.083$ $\cos 7\theta$
0	1.0	.635	-.204	.1215	-.083
10	0.9848	.625	-.177	.073	-.0284
20	0.9397	.596	-.102	-.0215	.0645
30	0.866	.55	0	-.105	.072
40	0.766	.485	.102	-.144	-.0144
50	0.6428	.408	.177	-.0416	-.0816
60	0.500	.317	.204	.0607	-.041
70	0.342	.217	.177	.1195	.0534
80	0.1736	.110	.102	.093	.078
90	0	0	0	0	0
100	-.1736	-.11			.078

From 90 to 180 repeat opposite in sign.

θ	3 term sum	4 term sum	Corresponding w
0	.5525	.4695	0
10	.521	.4926	.087
20	.4725	.5370	.176
30	.445	.517	.268
40	.473	.484	.364
50	.5434	.4618	.466
60	.5827	.5417	.577
70	.5135	.5669	.700
80	.305	.383	.839
90	0	0	1.0
100	-.305	-.383	1.19
110	-.5135	-.5669	1.428
120	-.5827	-.5417	1.732
130	-.5434	-.4618	2.144
140	-.473	-.484	2.747
150	-.445	-.517	3.732
160	-.4725	-.5370	5.6713
170	-.521	-.4926	11.430
180	-.5525	-.4695	∞

The graphs are even functions of ω . Then $F(\omega) = F(-\omega)$. The graphs could be shown to be symmetric in ω about the origin, but only positive ω is of interest.

This leads to:

$$|Y_T(x)|^2 = 1.944x^5 - 3.246x^3 + 1.8545x + .5823278 \quad \text{Eq.14}$$

The next change of variable $x, \frac{1-\omega^2}{1+\omega^2}$ gives $|Y_T(\omega)|^2 =$

$$\frac{1.944(1-\omega^2)^5 - 3.246(1-\omega^2)^3(1+\omega^2)^2 + 1.8545(1-\omega^2)(1+\omega^2)^4 + B(1+\omega^2)^5}{(1+\omega^2)^5} \quad \text{Eq.15}$$

Multiplication and binomial expansion puts the coefficients in a form expressed in the tables below:

	ω^{10}	ω^8	ω^6	ω^4	ω^2	ω^0
1.944	-1	5	-10	10	-5	1
3.246	1	-1	-2	2	1	-1
1.8545	-1	-3	-2	2	3	1

This gives:

ω^{10}	ω^8	ω^6	ω^4	ω^2	ω^0
-1.944	9.720	-19.44	19.44	-9.720	1.944
3.246	-3.246	-6.492	6.492	3.246	-3.246
-1.8545	-5.5635	-3.709	3.709	5.5635	1.8545

The numerator is then formed:

$$(B_\infty - .5525)\omega^{10} + (5B_\infty + .9105)\omega^8 + (10B_\infty - 29.641)\omega^6 + \dots$$

$$\dots (10B_\infty + 29.641)\omega^4 + (5B_\infty + .9105)\omega^2 + (B_\infty + .5525)$$

When the assigned value for B_∞ is inserted ($B_\infty = .5823278$) in the above, there results: $.0298277\omega^{10} + 3.822139\omega^8 - 23.817722\omega^6 + 35.464278\omega^4 + 2.001139\omega^2 + 1.13482$.

Remove a factor of .0298277 and there is formed:

$$|Y_T(\omega)|^2 = \frac{.0298277(\omega^{10} + 128.14\omega^8 - 798.5\omega^6 + 11.89.71\omega^4 + .\dots67.089908\omega^2 + 38.046}{(1)} \quad \text{Eq. 16}$$

To get in a form for R-C synthesis, divide the above fraction by $C(\omega^2)$.

$$C(\omega^2) = \frac{(\omega^2 + .81)(\omega^2 + .902)(\omega^2 + 1)(\omega^2 + 1.02)(\omega^2 + 1.21)}{(\omega^2 + 1)^5}$$

There is a little latitude in the choice of $C(\omega^2)$, and the above form is chosen for the ease in subsequent calculations. A table of values for $C(\omega^2)$ shows that the choice of this expression is close to unity for all ω .

ω^2	0	1	1.5	2	31
$C(\omega^2)$	.9742	.99952	1.0016	1.0024	1.00288

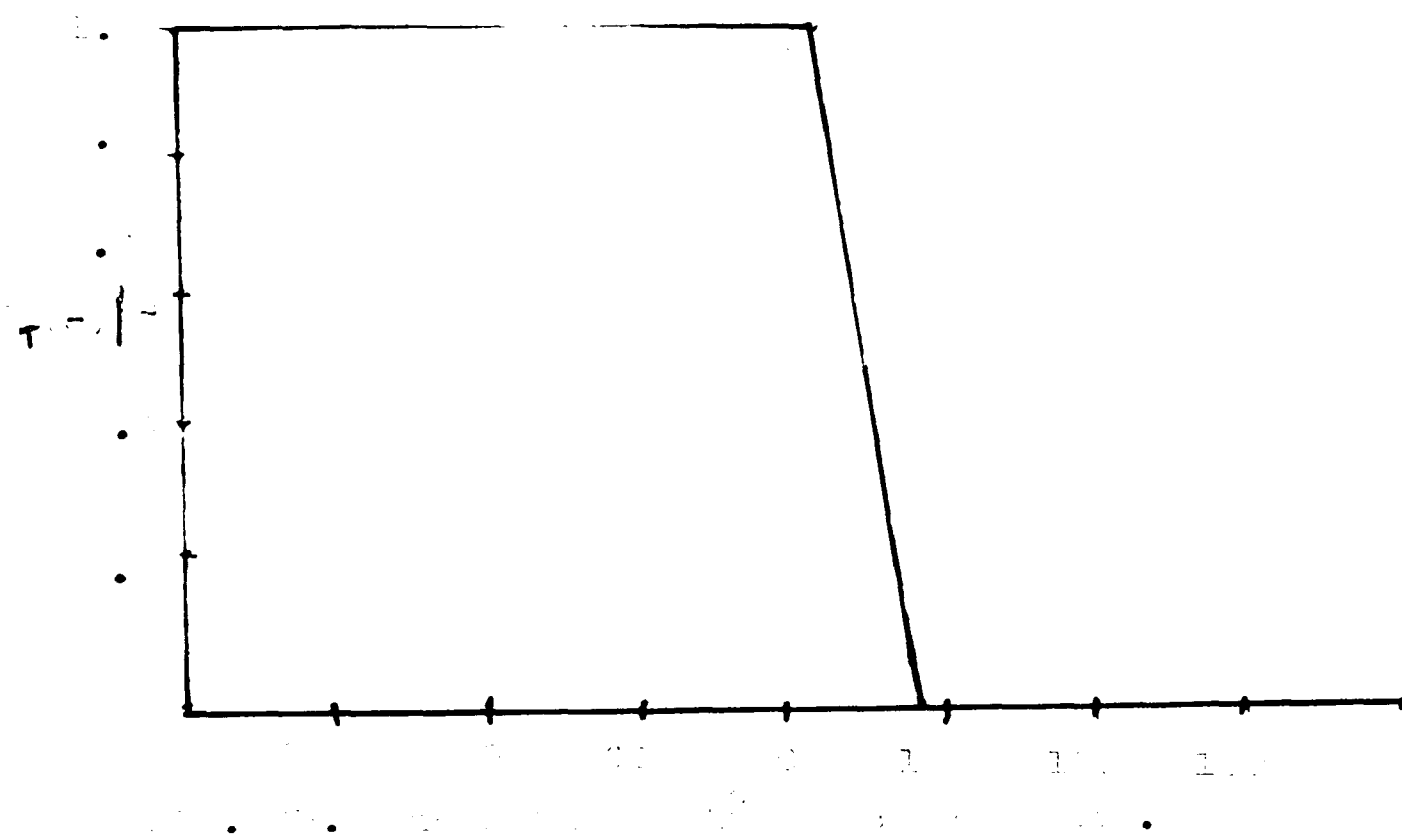
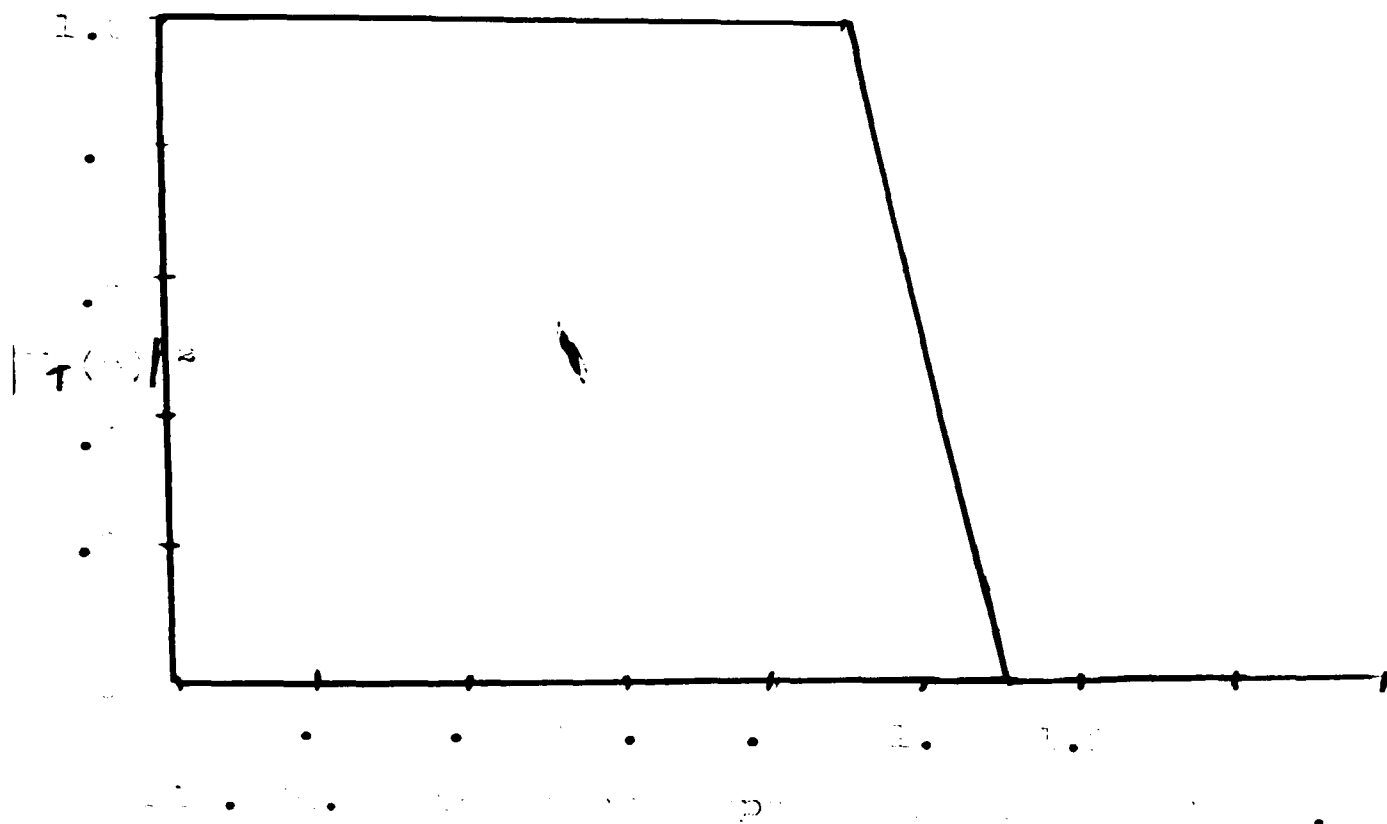
ω^2	5	∞
$C(\omega^2)$	1.00261	1.0

The divisions of Eq. 16 by $C(\omega^2)$ leaves as a final result:

$$|Y_T(\omega)| = \frac{(.0298277)(\omega^{10} + 128.14\omega^8 - 798.506\omega^6 + 1189.71\omega^4 + .\dots67.089686\omega^2 + 38.04595)}{(.81)(.902)(1.02)(1.21)} \quad \text{Eq. 17}$$

This is represented by figure 8 and essentially completes the approximation problem, except for the question of the physical realizability of the trapezoidal filter.

APPENDIX III. FIG. 1.1



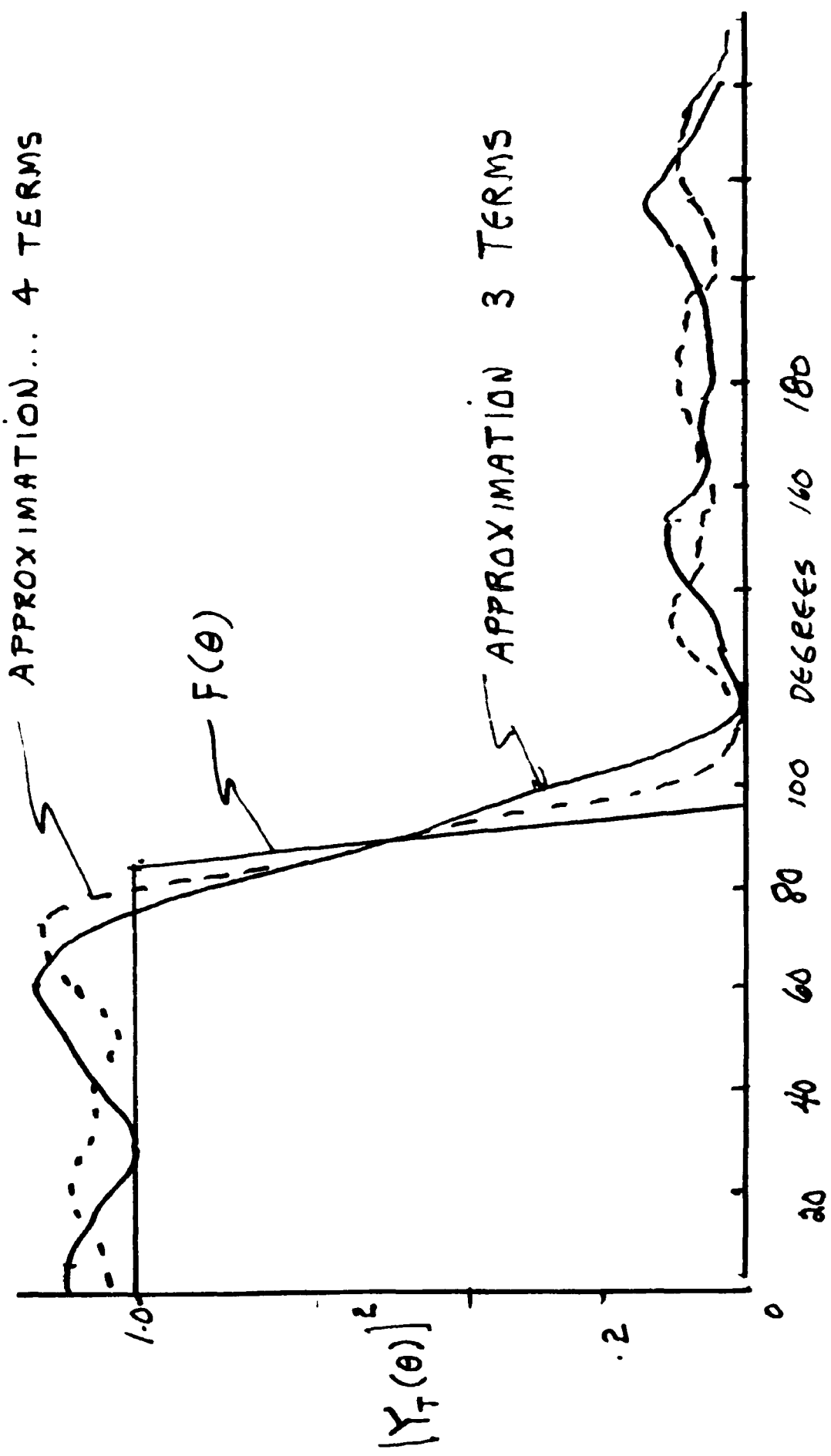


FIG. 7a. COMPARISON OF APPROXIMATIONS TO $F(\theta)$

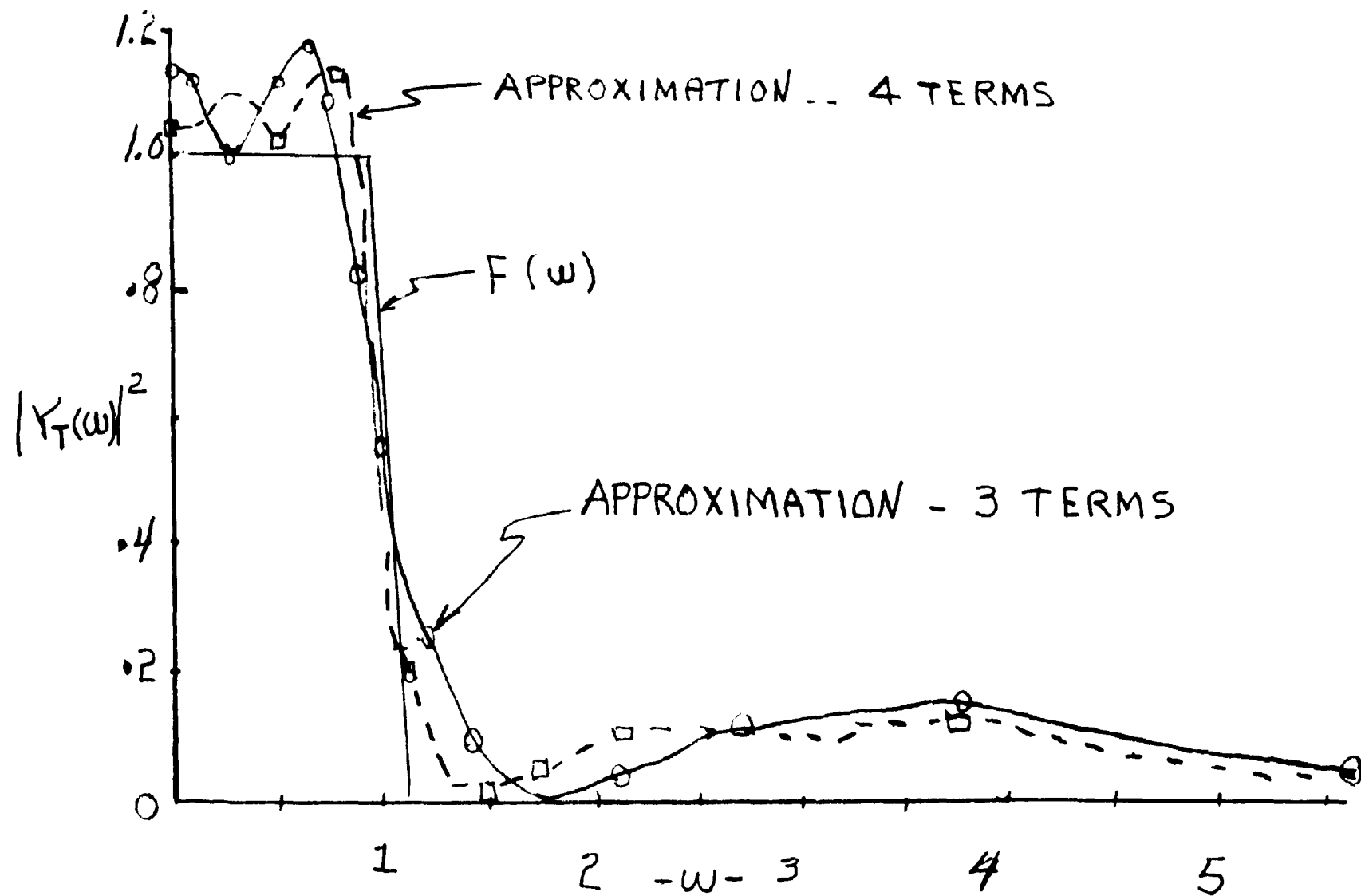


FIG 1b COMPARISON OF APPROXIMATIONS
TO $F(\omega)$.

(10) SYNTHESIS PROCEDURE

The next step is to synthesize a network that is characterized by equation 17.

The numerator of the expression is a polynomial of fifth order in ω^2 . The roots of the polynomial must be found. One root is already known for the $|Y_T(\omega)|^2$ was chosen to have a minimum of zero. This minimum occurred at $x = .495$ which is $\omega^2 = 3.045878$.

An examination of the polynomial shows that there are two changes of sign. One of Descartes rules is applied (13). This rule states that there will be either two or no positive roots for the polynomial. There is assuredly one root, and the expression does not cross the axis in more than one place on the graph. Thus there is a double root at $\omega^2 = 3.045878$.

To find the rest of the roots, the method of Lin (14) is employed. The result is:

$$|Y_T(\omega)|^2 = \frac{.0298277(\omega^2 - 3.0459)(\omega^2 + 134.158)(\omega^4 + .07375\omega^2 + .03056783)}{(\omega^2 + .81)(\omega^2 + .902)(\omega^2 + 1)(\omega^2 + 1.02)(\omega^2 + 1.21)}$$

Eq. 18

.....
13 Pipes, Louis A. Applied Mathematics for Engineers and Physicists. p.98, 1946 New York: McGraw

14 Lin, S. N. Method of Successive Approximations of Evaluating the Real and Complex Roots of Cubic and Higher Order Equations. Jour. Math and Phys. Vol. 20. No. 3 Aug. 1941

The synthesis problem requires the $Y_T(p)$ so since $p = j\omega$, and $\omega^2 = -p^2$, this substitution is then made. This leads to: $|Y_T(p)|^2 =$

$$\frac{.0298277(p^2 + 3.0459)^2 (-p^2 + 134.158)(p^4 - .07375p^2 + .03056783)}{(p^2 - .81)(p^2 - .902)(p^2 - 1)(p^2 - 1.02)(p^2 - 1.21)} \quad \text{Eq. 19.}$$

Equation 6 is applied after the above expression is broken down into its simpler factors.

$$|Y_T(p)|^2 = \left[\frac{(.17271)(p^2 + 3.0459)(p - 11.582)(p^2 + .650710p + .174844)}{(p + .9)(p + .95)(p + 1)(p + 1.05)(p + 1.1)} \right]$$

$$\cdot \left[\frac{(.17271)(p^2 + 3.0459)(p + 11.582)(p^2 - .650710p - .174844)}{(p - .9)(p - .95)(p - 1)(p - 1.05)(p - 1.1)} \right]$$

Eq. 20

For R-C synthesis only the poles in the left half plane can be used. This restricts the denominator to just the positive factors.

There are no restrictions on the zeros of $Y_T(p)$ so there is much liberty in choosing the numerator of the expression.

At this stage, Guillemin's method would require the selection of a numerator that would make Y_T a minimum phase shift function.

Lattice synthesis does not required a minimum phase shift function, but for simplicity in this example, and the material following this example, a minimum phase shift structure is chosen.

The graph of $|Y_T(\omega)|^2$ vs ω , in figure 6, has been normalized with respect to magnitude. Multiply the expressions for $|Y_T(\omega)|^2$ by the factor A^2 to represent the unnormalized expression. The significance of this factor becomes apparent when the full equation for $Y_T(p)$ is considered.

$$Y_T(p) = \frac{1}{2} \frac{P_L}{P_o \sqrt{R_1 R_2}}$$

The above is an extension of equation 4.

The minimum phase shift structure is then:

$$Y_T(p) = \frac{.17271 A(p^2 + 3.04588)(p + 11.52)(p^2 + .6507p + 174844)}{(p + 9)(p + .95)(p + 1)(p + 1.05)(p + 1.1)}$$

Eq. 21

A cutoff frequency (represented by ω_o) could be introduced here. This requires the replacement of ω in equation 18 by ω/ω_o .

When that is done the expression for $\frac{Y_T(p)}{.17271 A}$

is found to be :

$$\frac{p^6 + 12.2327p^4\omega_o + 10.757p^3\omega_o^2 + 39.2206p^2\omega_o^3 + 23.487p\omega_o^4 + 6.16803\omega_o^5}{p^5 + 5p^4\omega_o + 9.9875p^3\omega_o^2 + 9.9625p^2\omega_o^3 + 4.96525p\omega_o^4 + 987525\omega_o^5}$$

Eq. 22

One form of a complete problem would specify the generator and the load resistances. The required cutoff frequency would also be given.

For the purpose of demonstration, let:

$$R_1 = R_2 = 100 \text{ ohms} \quad \dots\dots \text{ then}$$

$$2 \sqrt{R_1 R_2} = 2 R_1 = 200 \text{ ohms.}$$

$$\text{Let } \omega_0 = 1,000,000 \text{ rads / sec.}$$

The value of A cannot be arbitrarily stated when the impedances and the cutoff frequency are fixed. This will become evident as the problem is solved.

The first step is to expand equation 22 by partial fractions.

At $p = 0$, the residue of the right side of equation 22 is equal to 6.2459569. This residue is subtracted from the expression and partial fraction expansion is continued. The details of this expansion are not repeated at this point.

The expanded $Y_T(p) = y_{12}$ is:

$$y_{12} = .17271 A \left[6.24596 - \frac{121,418 p}{(p + .94\omega_0)} + \frac{560,039 p}{\omega_0(p + .95\omega_0)} \dots \right. \\ \left. \frac{-895,069 p}{\omega_0(p + \omega_0)} + \frac{657,443 p}{\omega_0(p + 1.05\omega_0)} - \frac{180,427 p}{\omega_0(p + 1.1\omega_0)} \right]$$

Eq. 23

When all the terms in the previous expressions are made positive, the equations for $y_{11} = y_{22}$ are obtained. They are:

$$y_{11} = y_{22} = .17271 A \left[6.24596 \frac{+121,418 p}{\omega_o(p + .9 \omega_o)} + \frac{560,039 p}{\omega_o(p + .95 \omega_o)} + \frac{895,069 p}{\omega_o(p + \omega_o)} + \frac{657,443 p}{\omega_o(p + 1.05 \omega_o)} + \frac{180,427 p}{\omega_o(p + 1.1 \omega_o)} \right]$$

Eq. 24

The lattice arms can now be formed.

$$Y_A = y_{11} + y_{12} \quad Y_B = y_{11} - y_{12}$$

$$Y_A = .17271 A \left[12.4919 + \frac{1,120,078 p}{\omega_o(p + .95 \omega_o)} + \frac{1,314,886 p}{\omega_o(p + 1.05 \omega_o)} \right]$$

Eq. 25

$$Y_B = .17271 A \left[\frac{242,836 p}{\omega_o(p + .9 \omega_o)} + \frac{1,790,136 p}{\omega_o(p + \omega_o)} + \frac{360,854 p}{\omega_o(p + 1.1 \omega_o)} \right]$$

Eq. 26

The reciprocals are taken in order to write the arm impedances. The result is:

$$Z_A = \frac{5.79005 \left[(p^2 + 2\omega_o p + .997\omega_o^2) \right]}{A \left[p^2 \omega_o (12.4919 + 2,515,666) + \dots \right]}$$

$$p \left(\frac{24.9832\omega_o^2}{2,425,224\omega_o} \right) 12.454 \right]$$

$$Z_B = \frac{A \left[\frac{\omega_o (p^3 + 3\omega_o p^2 + 2.99\omega_o^2 p + .99\omega_o^3)}{2,393,828 p^2 + 4,775,850 \omega_o p + 2,364,122 \omega_o^2} \right] p}{5.79005}$$

Eq. 27

These are driving point impedance expressions for the arms of the lattice. At this point these expressions could undergo partial fraction expansion and be synthesized. This would defeat the purpose of the problem for the generator and load resistors would be absorbed in the network.

A standard theorem* of lattice networks states that if an impedance is common to both arms of a lattice, it can be removed from the lattice and placed in series before and after the new lattice.

This theorem is utilized by finding the minimum value of both arms impedance. Obviously since these are R-C arms the minimum impedance occurs when $\omega = \infty$ then

$$\lim_{\omega \rightarrow \infty} Z_A = \frac{5.79005 \omega_0}{A (12.4919 \omega_0^2 + 2,515,666)}$$

$$\lim_{\omega \rightarrow \infty} Z_B = \frac{5.79005 \omega_0}{A (2,393,888)}$$

Eq. 28

The smaller of these two limits is the first. Then a resistance equal in magnitude to the first part of equation 28 is common to both lattice arms. This minimum resistance can be removed from the lattice.

.....
* Ibid 7, p. 269

This minimum resistance, designated as Z' , is obviously the generator and load resistance. If the Z' turns out to be larger than the desired resistances, only the needed amount need be pulled from the lattice arms.

In the assigned problem, Z' should equal 100.

$$Z' = \frac{5.79005 \omega_0}{A(12.4919 \omega_0 + 2,515,666)} = 100 \text{ ohms}$$

For the assigned ω_0 of 1,000,000, the derived value of A is $A = 0.003858078$

When this value of A is know, it is informative to solve for the Power Insertion Ratio.

At $p = 0$, the equation 18 leads to

$$\begin{aligned} |Y_T(p)|_{p=0}^2 &= \frac{A^2 (.0298277)(3.0459)(134.158)(.03056783)}{.81(.902)(1.02)(1.21)} \\ &= A^2 (.41318617) \end{aligned}$$

$$\text{But } |Y_T(p)|_{p=0}^2 = \frac{P_L}{4 P_o R_1 R_2} = \frac{P_L}{P_o (40,000)} = A^2 (.41318617)$$

Thus at zero frequency the Power Insertion Ratio is found to be $\frac{P_L}{P_o} = .24600596$

From the graph the ratio of $\frac{\text{Maximum } |Y_T(\omega)|^2}{|Y_T(0)|^2}$

is about 1.08/1 so the maximum Power Insertion Ratio is about .265.

This is a result that is not evident from a cursory examination of the problem. It is now seen that for a given load and generator resistance, and a specified cutoff frequency, the maximum allowable Power Insertion Ratio is automatically determined. If it had been required to find a filter with the same load and frequency requirements as the previous problem, but with a higher Power Insertion Ratio, some other form of synthesis would be required. Such a filter as the latter is not realizable by the method just utilized.

It is quite possible that there is no R-C structure that will satisfy a requirement of a higher Power Insertion Ratio. This subject requires further investigation.

The filter problem can now be completed. 100 ohms, or Z' , is to be subtracted from Z_A and Z_B . The modified lattice arms Z'_A and Z'_B are now computed.

With numerical values the arms are:

$$Z_A = \frac{.1500.76 (p^2 + 2 \times 10^6 p + .997 \times 10^{12})}{15.007566 p^2 + 24.983 \times 10^6 p + 12.454}$$

$$Z_B = \frac{1500.76 (p^3 + 3 \times 10^6 p^2 + 2.99 \times 10^{12} p + .99 \times 10^{18})}{2.39382 p^3 + 4.775,85 \times 10^6 p^2 + 2.364,122 \times 10^{12} p}$$

Eq. 30

When 100 is subtracted from equations 30, the modified arms are obtained.

$$Z_A' = \frac{1503.22 \times 10^6 p + 1.4962577 \times 10^{15}}{15.007566 p^2 + 24.983 \times 10^6 p + 12.454}$$

$$Z_B' = \frac{1261.38 p^3 - 4027. \times 10^6 p^2 + 4250.86 \times 10^{12} p + 1485.75 \times 10^6}{2.39382 p^3 + 4,775,850 p^2 + 2,364,122 \times 10^6 p}$$

Eq. 31

The usual methods of partial fraction expansion are then used to realize these driving point impedances.

For the first case the impedance is simplified by dividing numerator and denominator by 15.007566.

$$Z_A' = \frac{99.7117 \times 10^6 p + .0997 \times 10^{15}}{p \quad 1,664,747 \quad p \quad .829848}$$

$$\frac{99.7117 \times 10^6 p + .0997 \times 10^{15}}{(p + 1,664,747)(p + .01)}$$

Eq. 32

Equation 32 is expanded by partial fractions.

The result is:

$$Z_A' = \frac{(5.98889) \times 10^{10}}{(p + .01)} \quad \frac{3.98227 \times 10^6}{p + 1.6647 \times 10^6}$$

Eq. 33

This driving point impedance is represented by the figure 8.

$$\begin{aligned}
 R_1 A &= 5.989 \times 10^{12} && \text{ohms} \\
 R_2 A &= 2.3921 && \text{ohms} \\
 C_1 A &= 16.6975 && \text{uufd} \\
 C_2 A &= 0.251113 && \text{ufd}
 \end{aligned}$$

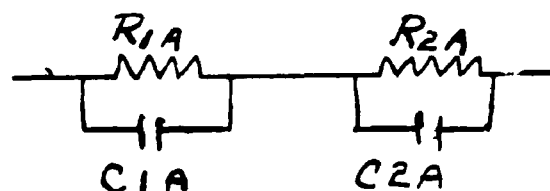


Figure 8 Lattice Arm Z'

A very similar manipulation is done for the other lattice arm. The numerical calculations are a bit more complex, because the denominator of the expression is a cubic polynomial.

Luckily, one of the roots of this denominator is at $p = 0$. This simplifies the cubic solution to the solution of a quadratic.

The expression for Z_B' is found in equation 31. Divide both numerator and denominator by 2.39382 and there is obtained as the driving point impedance of the lattice arm:

$$Z_B' = \frac{526.931p^3 + 1682.2 \times 10^6 p^2 + 1775.77 \times 10^{12} p + 620.66 \times 10^{18}}{p (p^2 + 1,995,074p + 9.875,933 \times 10^{10})}$$

Eq. 34

There is no great difficulty in obtaining the roots of the denominator by the quadratic formula. The roots are:

$$-1,084,063; \quad -911,010.8; \quad 0.$$

The partial fraction expansion, when applied to Z_B' gives the following result:

$$Z_B' = \frac{628.457}{p} + \frac{61,253,150}{p + 1,084,063} + \frac{583,636}{p + 911,010}$$

Eq. 35

The realization of equation 35, which represents the lattice arm Z_B' is shown in figure 9.

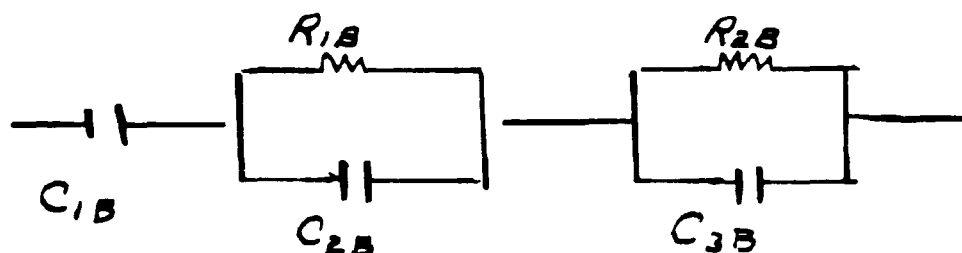


Figure 9 Lattice Arm Z_B'

The numerical values for the components are:

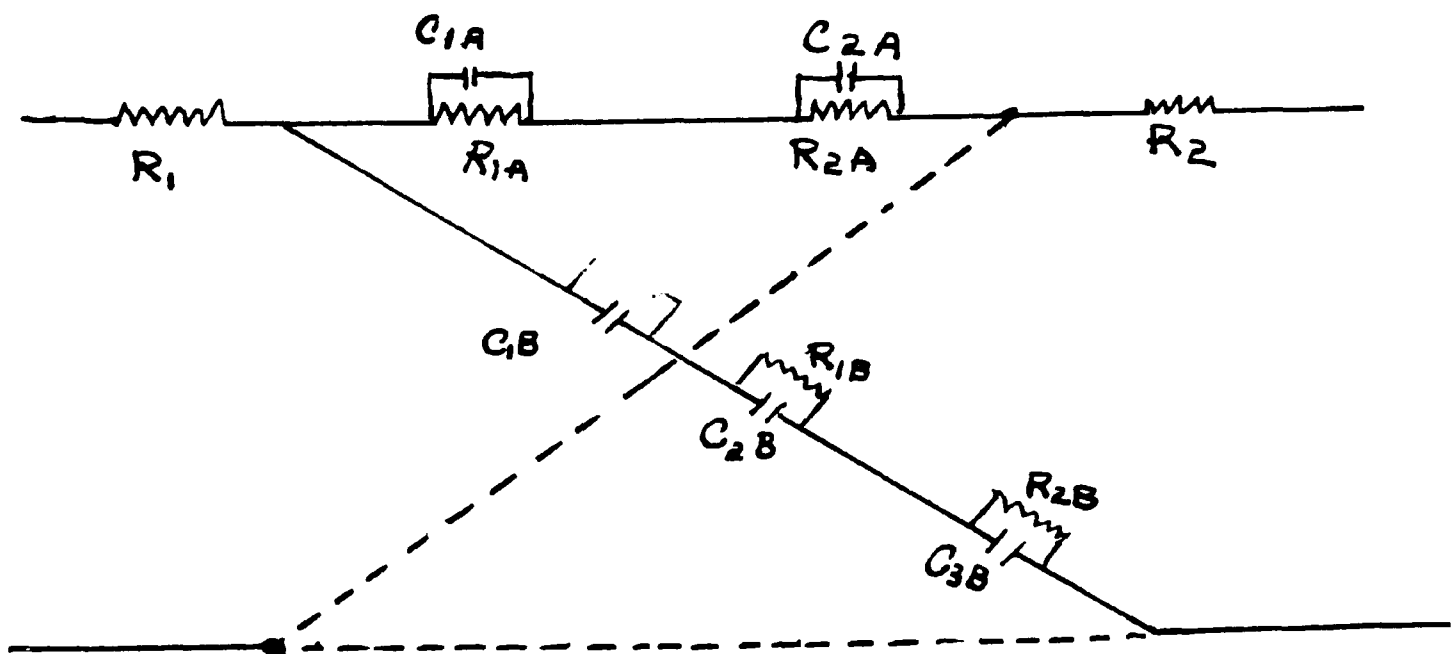
C_{1B}	= 0.00159	farads		
C_{2B}	= 0.0163	farads	R_{1B}	= 56.50 ohms
C_{3B}	= 1.7134	farads	R_{2B}	= 0.6406 ohms

The complete lattice formation is shown in figure 10.

The quadripole realized in this problem fortunately requires the use of components that are available in a laboratory. A less fortunate choice

of cutoff frequency, or of terminal impedances could have led to capacitors in the order of farads, not μ farads; the result might also have required microhm resistors. If that were to be the case, another method of R-C synthesis, not in the form of a lattice, would be in order.

For that reason, a high cutoff frequency was chosen for the demonstration problem.



Lattice Arm A

$$R_{1A} = 5.989 \times 10^{12} \text{ Ohms}$$

$$R_{2A} = 2.3921 \text{ Ohms}$$

$$C_{1A} = 16.698 \text{ } \mu\text{farads}$$

$$C_{2A} = 0.2511 \text{ } \mu\text{farads}$$

Lattice Arm B

$$R_{1B} = 56.50 \text{ Ohms}$$

$$R_{2B} = 0.6406 \text{ Ohms}$$

$$C_{1B} = 0.00159 \text{ } \mu\text{farads}$$

$$C_{2B} = 0.0163 \text{ } \mu\text{farads}$$

$$C_{3B} = 1.7134 \text{ } \mu\text{farads}$$

Terminal Resistors

$$R_1 = 100 \text{ Ohms}$$

$$R_2 = 100 \text{ Ohms}$$

Figure 10

Final Lattice Structure

(11) REALIZABILITY OF A QUADRIPOLE

A glance at figure 7 shows that the synthesized lattice has a frequency response that merely approximates the desired trapezoidal pattern.

This is due to the fact that only a few term of the Fourier Series were used. The inclusion of additional terms would bring the response of the filter more like the desired characteristic, but the inclusion of more terms would require more complex calculation and require the addition of additional circuit components to the lattice.

Since the desired $F(\theta)$ is a continuous function, it was stated before, that the desired response could be approximated arbitrarily close. A Fourier Theorem^o states that if $F(\theta)$ satisfies these conditions: $F(\theta) = F(\theta + 2\pi)$ for all θ , and $F(\theta)$ is sectionally continuous in interval $(-\pi, \pi)$, then the Fourier Series converges to the value $\frac{1}{2}[F(\theta^+) + F(\theta^-)]$ at every point where $F(\theta)$ has a left and right hand derivative.

The trapezoidal pattern fulfills these requirements, so an infinite number of terms could result in a theoretical expression that exactly realizes the desired response.....

Ibid (12) p. 70

(12) EXTENSION TO THEORY OF EQUALIZERS.

Consider the system shown in block diagram form in figure 11. In this system, a signal passes through a Low Pass Filter. Some of this signal is utilized in the filtered form, and the rest of the signal is isolated by a vacuum tube and sent on to an Equalizer. Since the filtered signal passed through a Low Pass Filter, the Equalizer must be a High Pass Filter.

A system such as this can be handled by the methods of this thesis, for the problem of the interaction between the High Pass and the Low Pass Filter is solved by the use of the Isolation Amplifier.

The system requirement might be that the product of the Power Insertion Ratios of the High and Low Pass Filters be a constant; or the requirement might be that the sum of the ratios be a constant, for all ω .

The characteristic required for the High Pass Filter can then be transformed into a Fourier Polynomial and the synthesis performed.

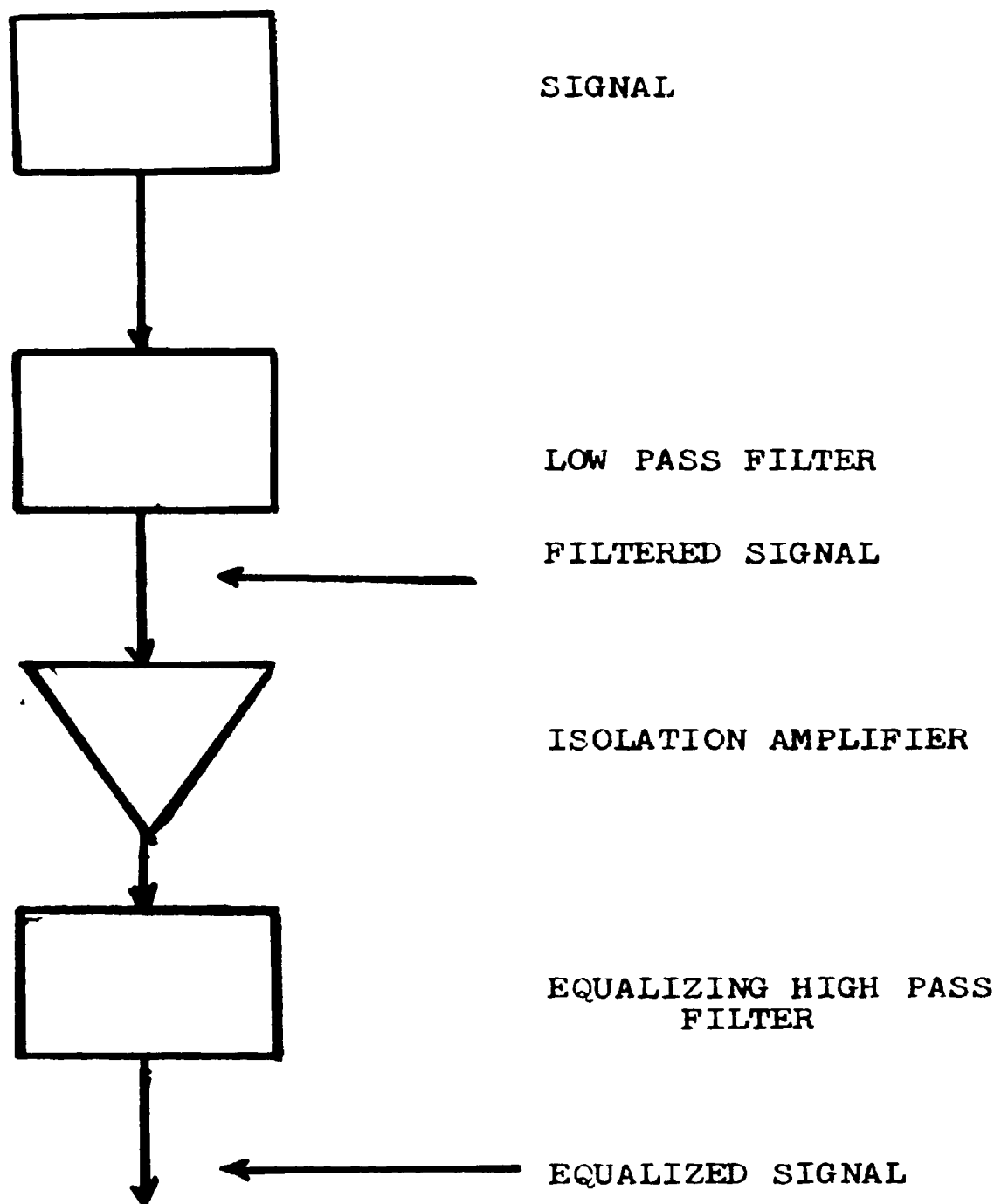


Figure 11 Block System using Equalization

(13) HIGH PASS FILTER SYNTHESIS

It is desired to synthesize a High Pass Filter such that the sum of the Power Insertion Ratio of this filter and the filter previously designed in this thesis be a constant for all.

The Power Insertion Ratio characteristic desired for this filter is shown in figure 12.

The Fourier representation for this filter is readily obtainable from the Fourier representation of the Low Pass Filter, equation 36.

The expression for the High Pass Filter is obtained when the Low Pass characteristic is subtracted from its maximum value.

Thus, if equation 18 were subtracted from unity, the result would be:

$$1 - \frac{.0298277(\omega^2 - 3.0459)^2 (\omega^2 + 134.158)(\omega^4 + .07375\omega^2 + .03056783)}{(\omega^2 + .81)(\omega^2 + .902)(\omega^2 + 1)(\omega^2 + 1.02)(\omega^2 + 1.21)} \quad \text{Eq. 36}$$

This equation would be a starting point for the synthesis procedure. The remainder of the problem would be solved in the same manner as the solution for the Low Pass Filter.

$$|H(f)|^2 = e$$

$$|T(f)|^2$$

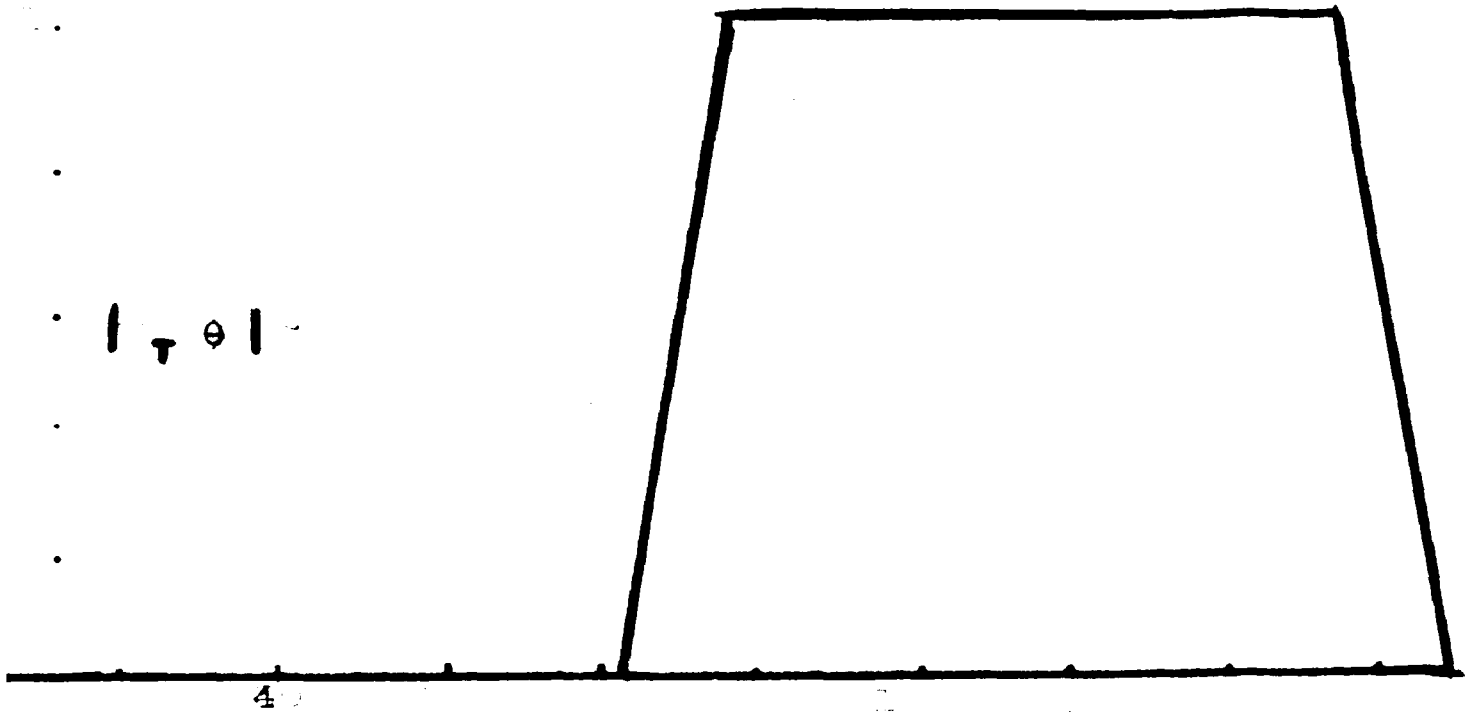


Figure 12 Fourier Representation of High Pass Filter

(14) CONCLUSION

If it is desired to synthesize an R-C quadripole, to have a Power Insertion Ratio, that approximates a given characteristic, there are sufficiency theorems that will permit the synthesis.

If there exists a series of transformations that can put the desired characteristic in a periodic form, and if this periodic form is, at least, sectionally continuous with left and right hand derivatives, it is possible to approximate the desired form by a Fourier Series technique.

As long as the derived expression for the y_{12} is realizable, it is realizable as a lattice. If the expression is also of "minimum phase shift" it is also realizable as a ladder network.

There are certain restrictions on the relations between terminating impedances, cutoff frequency, and maximum Power Insertion Ratio, that should be further investigated.

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