AFFINE GRASSMANNIANS AND SPLITTING MODELS FOR TRIALITY GROUPS

By

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A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

Mathematics — Doctor of Philosophy

2021

ABSTRACT

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This thesis concerns the study of affine Grassmannians and of local models for ramified triality groups. The triality groups we consider are groups of type ${}^3D_4$, so they are forms of the orthogonal or the spin groups in 8 variables. They can be given as automorphisms of certain twisted composition algebras obtained from the octonion algebra. Using these composition algebras, we give descriptions of the affine Grassmannians and of the global affine Grassmannians for these triality groups as functors classifying suitable lattices in a fixed space. We combine these descriptions with the Pappas-Zhu construction, to obtain a corresponding description of local models for triality groups; the singularities of these models are supposed to model the singularities of certain orthogonal Shimura varieties.

Moreover, we give a definition of a corresponding splitting model in terms of linear algebra data; this splitting model is expected to provide a partial resolution of the local model. By explicit calculations, we find equations that describe affine charts of the splitting model. Using these calculations, we show that the splitting model is isomorphic to the blow-up of a quadratic hypersurface along a specific smooth closed subscheme of its special fiber. It follows that the splitting model is regular and has special fiber which is the union of two smooth irreducible components that intersect transversely.

ACKNOWLEDGMENTS

I would like to thank my family, without whom I would never have gotten to this point in my life. I could not have accomplished without my parents understanding, sacrifices, and love. I would also like to express my very great appreciation to my advisor, George Pappas, for his enthusiastic encouragement during my research, for his valuable suggestions and insightful criticism of my work, and his patient guidance through this whole trip.

I also want to thank the rest of my committee members, Prof. Rajesh Kulkarni, Aaron Levin, and Michael Shapiro, for their encouragement and suggestions in these six years. I want to thank my friends, Zheng Xiao, for all great time we had together, Ioannis Zachos, for all discussion we had. Thank all members of the student arithmetic geometry seminar, the student algebra seminar. Thank my friends in China, Zishen Zheng, for his support and all wonderful trips we had in several summer vacations. It seems like yesterday when I came to MSU. I am sure this six-year trip will be an unforgettable memory in my life. Thanks also to everyone I forgot to include. I am sure there are a lot of you.

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KEY TO SYMBOLS

p	an odd prime
F_0	a field with characteristic $\neq 2, 3$
$\mathcal{O}_0$	the ring of integers of F_0
F	a cubic Galois extension of F_0
$\mathcal{O}$	the ring of integers of F
π_0	a uniformizer of F_0
π	a uniformizer of F
$\mathbb{Q}_p^{unr}$	the maximal unramified extension of $\mathbb{Q}_p$
κ	the residue field of F
$(V, *)$	the normal twisted composition algebra obtained from split para-Cayley algebra
k	a field with characteristic $\neq 2, 3$
$k[[t]]$(resp. $k[[u]]$)	the power series with indeterminate t (resp. u)
$k((t))$(resp. $k((u))$)	the Laurent power series with indeterminate t (resp. u)
$\mathbb{L}$	the $\mathcal{O}_0[v]$-lattice given by $\mathbb{L} = \oplus_{i=1}^{8} \mathcal{O}_0[v] \cdot e_i$ in $\tilde{V} := \oplus_{i=1}^{8} \mathcal{O}_0[v^{\pm 1}] \cdot e_i$
Λ	the $\mathcal{O}$-lattice given by $\Lambda = \oplus_{i=1}^{8} \mathcal{O} \cdot e_i$ in V
G	the triality group for the orthogonal group
$\mathscr{G}$	a parahoric subgroup given by the lattice $\mathbb{L}$
$M(\mu)$	the Pappas-Zhu local model for triality groups
$\mathcal{M}^{\mathrm{split}}$	the splitting model for triality groups
Q	the projective quadratic hypersurface in $\mathbb{P}^7$

Chapter 1

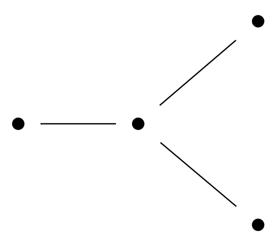
Introduction

Local models are certain projective schemes over the spectrum of a discrete valuation ring which have a homogeneous space for a reductive group as generic fiber. Their singularities are supposed to model the singularities of natural integral models of Shimura varieties with parahoric level structure, i.e., each point on the integral model of the Shimura variety should have an étale neighborhood which is isomorphic to an étale neighborhood of a corresponding point on the local model. So the problem of studying singularities of reductions of Shimura varieties becomes studying singularities of corresponding local models. Therefore, it is interesting to study good properties of local models, such as flatness or Cohen-Macaulayness. Local models for Shimura varieties of PEL type were given by Rapoport and Zink in [27]; sometimes, these are called "naive local models". Naive local models are defined directly in terms of linear algebra data; they are closed subschemes in the product of Grassmannian varieties and they can be calculated explicitly. Unfortunately, naive local models are not always flat (see [21]). Some of these non-flat examples arise due to the fact that the group defining the Shimura variety is non-split over p. In the ramified PEL case, corrected local models for classical groups have been studied case by case. The local structure of local models was considered in several papers by Görtz[7], [8], [9], by Krämer [16], by Smithling [30], by Arzdorf [1] and others. We refer the survey [24] for an overview and more references.

In [26], Pappas and Zhu gave a uniform group-theoretic construction of local models for tamely ramified groups; we call these “PZ-local models”. Roughly speaking, PZ-local models are given as the Zariski closure of a homogeneous space in a global affine Grassmannian over a ring of p-adic integers. Pappas and Zhu showed that PZ-local models have many good properties. However, these local models are not defined directly in terms of linear algebra data. One of our goals in this work is to give a linear algebra description of PZ-local models for certain orthogonal groups. A similar result, for classical orthogonal (split) groups has been recently obtained by Zachos, and Zachos-Pappas in [25], [32]. Here, we consider the harder case of ramified triality groups.

What are triality groups? Let G_0 be an adjoint Chevalley group of type D_4 over a field F_0. Consider the Dynkin diagram:

D_4



The Dynkin diagram of type D_4 has a symmetry not shared by other Dynkin diagrams: it admits automorphisms of order 3. Since the automorphism of the Dynkin diagram of type D_4 is isomorphic to the symmetric group S_3, there is a split exact sequence of algebraic groups:

$$1 \to G_0 \to \mathrm{Aut}(G_0) \xrightarrow{f} S_3 \to 1.$$

Thus, G_0 admits outer automorphisms of order 3, which we call trialitarian automorphisms. The fixed elements of G_0 under such an outer automorphism, define groups of type G_2:

$$G_2 \quad \bullet \Lleftarrow \bullet$$

Consider the Galois cohomology set $H^1(F_0, \mathrm{Aut}(G_0)) := H^1(\Gamma_0, \mathrm{Aut}(G_0))$, where Γ_0 is the absolute Galois group $\mathrm{Gal}(F_{0,sep}/F_0)$. Adjoint algebraic groups of type D_4 over F_0 are classified by $H^1(F, \mathrm{Aut}(G_0))$ (29.B, [14]), and the map induced by f in cohomology $f^1 : H^1(F_0, \mathrm{Aut}(G_0)) \to H^1(F_0, S_3)$ associates to G_0 of type D_4 the isomorphism class of a cubic étale F_0-algebra F, see [14]. The possibilities of F are summarized as follows:

F	type G_0
$F_0 \times F_0 \times F_0$	${}^1D_4$
$F_0 \times \Delta$	${}^2D_4$
Galois field ext.	${}^3D_4$
non-Galois field ext.	${}^6D_4$

The group G_0 is said to be of type ${}^1D_4$ if F is split, of type ${}^2D_4$ if $F \cong F_0 \times \Delta$ for some quadratic separable field extension Δ/F_0, of type ${}^3D_4$ if F is a cyclic field extension over F_0, and of type ${}^6D_4$ if F is a non-cyclic field extension. In our paper, we consider the ${}^3D_4$ case and call the corresponding G_0 the triality group.

These triality groups are often studied by composition algebras. By composition algebras, we mean algebras (not necessarily associative) with a nonsingular quadratic form q such that $q(x \cdot y) = q(x)q(y)$ for all x, y in this algebra. We give a review of different types of composition algebras in §2. Composition algebras can be used to describe exceptional groups. For example, Springer shows the automorphism of an octonion algebra is of type G_2 (§2.3, [31]). Here the octonion algebra is an 8-dimensional composition algebra. We can view this automorphism group as the fixed subgroup of a spin group of an octonion algebra under outer automorphisms (Proposition 35.9, [14]). In §3, we extend this result and show that the subgroup of a spin group of a normal twisted composition algebra, which is fixed under outer automorphisms, is of type ${}^3D_4$. This will be our main tool to study affine Grassmannians and local models for triality groups.

Before considering local models for triality groups in global affine Grassmannians, we will give an explicit description of the triality affine Grassmannian in terms of lattices with extra structure. Corresponding explicit descriptions of affine Grassmannians/flag varieties are known and have been quite useful. Lusztig [18] first showed that affine Grassmannians for simple Lie algebras can be described in terms of certain orders, which are lattices closed under the Lie bracket. Here we aim for an explicit description in terms of lattices in the standard representation which is more in line with such descriptions known for classical groups. For example, Pappas and Rapoport gave such descriptions for affine Grassmannians and affine flag varieties for unitary groups in [23] using lattices (or lattice chains) which are self-dual for a hermitian form. See also work of Görtz [8] and of Smithling [29] for the symplectic and the split orthogonal groups, respectively. It turns out that the case of the ramified triality group, which we consider here, is considerably more complicated. We believe that these objects are of independent interest and we begin by discussing them in detail.

Let k be a field with characteristic $\mathrm{char}(k) \neq 2, 3$. We assume the cubic primitive root ξ is in k. We set $F_0 = k(\!(t)\!)$ (resp. $F = k(\!(u)\!)$) the ring of Laurent power series, with ring of integers $k[\![t]\!]$ (resp. $k[\![u]\!]$). Set $u^3 = t$ so that F/F_0 is a cubic Galois extension with $\mathrm{Gal}(F/F_0) = \langle \rho \rangle \cong A_3$, where ρ acts on u by $\rho(u) = \xi u$. In §2.3, we define the normal twisted composition algebra $(V, *)$ over F. Here $(V, *)$ is an 8-dimensional vector space with an F_0-bilinear product $*$ and a nonsingular quadratic form q satisfying certain properties (see Definition 2.3.1). We also fix a finitely generated projective $k[\![u]\!]$-module $\mathbb{L}$ in V, which we call the standard lattice in V. Denote by $\langle\ ,\ \rangle$ the bilinear form associated to q. We show that the spin group $\mathbf{Spin}(V,*)$ over F has an outer automorphism, and the subgroup of $\mathrm{Res}_{F/F_0}\mathbf{Spin}(V,*)$, which is fixed under the outer automorphism, is the triality group G we are interested in, i.e.,

$$G = \mathrm{Res}_{F/F_0}\mathbf{Spin}(V,*)^{A_3}.$$

We now choose the parahoric group scheme $\mathscr{G}$ over $\mathrm{Spec}(k[\![t]\!])$ given by the lattice $\mathbb{L}$. This is a smooth group scheme with $\mathscr{G} \otimes_{k[\![t]\!]} k(\!(t)\!) = G$. Set $\mathscr{G}_\eta$ the generic fiber of $\mathscr{G}$. We consider the associated loop group $L\mathscr{G}_\eta$ (resp. positive loop group $L^+\mathscr{G}$), which is the ind-scheme representing the functor:

$$R \mapsto L\mathscr{G}_\eta(R) := \mathscr{G}_\eta(R(\!(t)\!)), \ \text{(resp. } R \mapsto L^+\mathscr{G}(R) := \mathscr{G}(R[\![t]\!])),$$

for any k-algebra R. The quotient fpqc sheaf $L\mathscr{G}_\eta/L^+\mathscr{G}$ is by definition the affine Grassmannian for the triality group over $\mathrm{Spec}(k)$. Our first main theorem is:

Theorem 1.0.1. *There is an $L\mathscr{G}_\eta$-equivariant isomorphism*

$$L\mathscr{G}_\eta/L^+\mathscr{G} \simeq \mathscr{F}$$

where the functor $\mathscr{F}$ sends a k-algebra R to the set of finitely generated projective $R[\![u]\!]$-modules L (i.e., $R[\![u]\!]$-lattices) of $V \otimes_k R \cong R(\!(u)\!)^8$, such that

(1) L is self dual under the bilinear form $\langle\ ,\ \rangle$, i.e., $L \simeq \mathrm{Hom}_{R[\![u]\!]}(L, R[\![u]\!])$.

(2) L is closed under multiplication, $L * L \subset L$.

(3) There exists $a \in L$, such that $q(a) = 0$, $\langle a * a, a\rangle = 1$.

(4) For a as in (3), let $e = a + a * a$. Then, we have $\overline{e * x} = -\bar{x} = \overline{x * e}$ for any $\bar{x}$ satisfying $\langle \bar{x}, \bar{e} \rangle = 0$. (Here, $\bar{x}$ is the image of x under the canonical map $L \to L/uL$.)

This theorem is proven in §5.2. In particular, it gives a bijection between k-points in the affine Grassmannian for triality groups and a certain set of $k[\![u]\!]$-lattices in V that satisfy some special conditions.

We are now ready to give definitions of global affine Grassmannians and local models for triality groups. Let $F_0 = \mathbb{Q}_p^{unr}$ be the maximal unramified extension of $\mathbb{Q}_p$ for $p \neq 2, 3$. Let $K/\mathbb{Q}_p$ be a cubic extension and $F = K\mathbb{Q}_p^{unr}$. Then F/F_0 is a cubic Galois extension. Choose a uniformizer π (resp. π_0) in the ring of integers $\mathcal{O}$ (resp. $\mathcal{O}_0$) of F (resp. F_0). In the second part of the paper, we construct a group scheme H over F. This is a generalized group scheme containing $\mathbf{Spin}(V, *)$. Here H sits in the following exact sequence:

$$1 \to \mathbf{Spin}(V, *) \to H \to \mathbb{G}_m^{\times 3}.$$

The Galois group $\mathrm{Gal}(F/F_0) \cong A_3$ also acts on H, and we denote by G the subgroup of $\mathrm{Res}_{F/F_0} H$ fixed under the Galois group. We call G the triality group for the general orthogonal group over $\mathrm{Spec}(F_0)$.

Consider the affine line $\mathbb{A}^1_{\mathcal{O}_0} = \mathrm{Spec}(\mathcal{O}_0[u])$ and its cover $\mathrm{Spec}(\mathcal{O}_0[v]) \to \mathrm{Spec}(\mathcal{O}_0[u])$ given by $u \mapsto v^3$. We can get the cubic field extension F/F_0 from $\mathcal{O}_0[v^{\pm 1}]/\mathcal{O}_0[u^{\pm 1}]$ by base changing via $v \mapsto \pi$. Thus, there is a normal twisted composition algebra $\tilde{V}$ over $\mathcal{O}_0[v^{\pm 1}]$ with a bilinear form $\langle\ ,\ \rangle$, such that the base change $\tilde{V} \otimes_{\mathcal{O}_0[v^{\pm 1}]} F$ is isomorphic to V. We fix an $\mathcal{O}_0[v]$-lattice in $\tilde{V}$, and still call it the standard lattice $\mathbb{L}$ for simplicity. Following [26], we construct a reductive group scheme $\underline{G}$ over $\mathrm{Spec}(\mathcal{O}_0[u^{\pm 1}])$. It is a quasi-split (split after a tamely ramified extension) group scheme such that the base change $\underline{G} \otimes_{\mathcal{O}_0[u^{\pm 1}]} F_0$ given by $\mathcal{O}_0[u^{\pm 1}] \to F_0, u \mapsto \pi_0$, is isomorphic to G. Having given $\underline{G}$, we can now choose the parahoric subgroup $\underline{\mathscr{G}}$ over $\mathrm{Spec}(\mathcal{O}_0[u])$ given by $\mathbb{L}$. The global affine Grssmannian $\mathrm{Gr}_{\underline{\mathscr{G}}}$ is now defined as the quotient fpqc sheaf $L\underline{\mathscr{G}}/L^+\underline{\mathscr{G}}$. Our second main theorem is:

Theorem 1.0.2. *Suppose R is an $\mathcal{O}_0$-algebra. There is an $L\underline{\mathscr{G}}$-equivariant isomorphism between $\mathrm{Gr}_{\underline{\mathscr{G}}}(R)$ and the set of pairs $(L, [\lambda])$, where L is a $R[\![u-\pi_0]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v]$-lattice of $\tilde{V} \otimes_{\mathcal{O}_0[u^{\pm 1}]} R(\!(u-\pi_0)\!)$, and λ is in $(R(\!(u-\pi_0)\!) \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v])^*$, which satisfy:*

(1) *Under the bilinear form $\langle \ , \ \rangle$, we have*

$$\langle \ , \ \rangle : L \otimes L \to \rho(\lambda)\theta(\lambda)(R[\![u-\pi_0]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v])$$

which is perfect, i.e., $L \cong \mathrm{Hom}(L, \rho(\lambda)\theta(\lambda)(R[\![u-\pi_0]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v]))$. Here the tensor $\otimes$ and Hom are for the $R[\![u-\pi_0]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v]$-mod structure.

(2) *We have $L * L \subset \lambda L$.*

(3) *There exists $a \in L$, such that $q(a) = 0, \langle a * a, a\rangle = \lambda\rho(\lambda)\theta(\lambda)$.*

(4) *For a as in (3), let $e = a + \lambda^{-1}(a * a)$. Thus, we have $\bar{\lambda}^{-1} \cdot \overline{e * x} = -\bar{x} = \bar{\lambda}^{-1} \cdot \overline{x * e}$, for any $\bar{x}$ satisfying $\langle \bar{x}, \bar{e}\rangle = 0$, where $\bar{x}$ is the image of x under the canonical map $L \to L/(u-\pi_0, v)L$.*

We refer §6.2 for notations in detail, and the proof is in §6.3. This theorem generalizes our first main theorem.

To define local models for triality groups, we need to fix a coweight μ of G. This coweight gives a morphism $\mu : \mathbb{G}_{m,F} \to G \otimes_{F_0} F$, which gives a F-valued point of $L\underline{G}$. After showing $\underline{\mathscr{G}}_{F_0,\pi_0} := \underline{\mathscr{G}} \otimes_{\mathcal{O}_0[u]} F_0[\![u-\pi_0]\!] \cong G \otimes_{F_0} F_0[\![u-\pi_0]\!]$ in §7.1, we get an F-valued point in

$L\underline{\mathscr{G}}_{F_0,\pi_0}$. Denote by s_μ the corresponding F-valued point in $L\underline{\mathscr{G}}_{F_0,\pi_0}$. We consider the orbit $X_\mu = L^+\underline{\mathscr{G}}_{F_0,\pi_0}[s_\mu]$ in the generic fiber of $\mathrm{Gr}_{\underline{\mathscr{G}}} \otimes_{\mathcal{O}_0[u]} \mathcal{O}$. Then following the definition in [26], the local model for triality groups is the Zariski closure of the orbit X_μ in the ind-scheme $\mathrm{Gr}_{\underline{\mathscr{G}},\mathcal{O}} := \mathrm{Gr}_{\underline{\mathscr{G}}} \otimes_{\mathcal{O}_0[u]} \mathcal{O}$. The description of the global affine Grassmannian as a functor classifying lattices given by the theorem above, now also implies a corresponding description of the local model as classifying such lattices whose distance from the standard lattice is "bounded by μ".

Although the definition of local models for triality groups looks complicated, the generic fiber of this local model has a simple description. Let C be an 8-dimensional vector space equipped with a nondegenerate symmetric quadratic form q. Let Q be the projective quadratic hypersurface defined by q. There are two different orthogonal Grassmannians, which contain maximal isotropic subspaces of C. We denote them by Q^+, Q^-. Then, we have a "triple graph":

$$\begin{tikzcd} & Q \arrow[dl] & \\ Q^+ \arrow[rr] & & Q^- \arrow[ul] \end{tikzcd}$$

(see §20.3, [5]). The outer automorphism of triality groups can be viewed as a counterclockwise action in the above "triple graph". Hence, the fixed subgroup under outer automorphism is isomorphic to the quadratic hypersurface Q, and the generic fiber of local models for triality groups is also isomorphic to the quadratic hypersurface Q. See §7.2 for details.

The last part of this paper is about "splitting models" for triality groups. The original purpose of introducing splitting models by Pappas and Rapoport ([22]) was to modify "naive local models" in the ramified case, so that the modified models are flat and have reasonable singularities. We can view splitting models as "partial resolutions of local models". In [22], Pappas and Rapoport consider the cases where the quasi-split form of G is the general linear group GL_d or the general symplectic group GSp_{2n}. We will give the definition of splitting models $\mathcal{M}^{\mathrm{split}}$ for triality groups in terms of linear algebra data. Our last main theorem is

Theorem 1.0.3. *The scheme $\mathcal{M}^{\mathrm{split}}$ is isomorphic to the blow-up $\tilde{Q}$ of Q along Z.*

Here Q is the quadratic hypersurface in $\mathbb{P}^7_{\mathcal{O}}$, and Z is the closed subscheme that contains all isotropic lines orthogonal to the para-unit e in the special fiber of Q (see §8.2). It easily follows that $\mathcal{M}^{\mathrm{split}}$ is regular and has special fiber which is the union of two smooth irreducible components that intersect transversely. Although we believe that there should be "partial resolution" (birational) morphism: $\mathcal{M}^{\mathrm{split}} \to \mathcal{M}^{\mathrm{loc}}$, we were not able to establish that yet.

The organization of the paper is as follows. In §2, we review the definition and basic propositions of composition algebras, including unital composition algebras, symmetric composition algebras, and twisted composition algebras. We are particular interested in normal twisted composition algebras, since the spin group of normal twisted composition algebras is an ingredient to construct triality groups. We explain relations between isotropic subspaces in the normal twisted composition algebras (we call it "triality triple") in §2.4. In §3.1 – 3.2, we review orthogonal groups and give the principle of triality proposition, which we will use them to explain our construction of triality groups. In §3.3 – 3.4, we give the definition of triality groups, both for the special orthogonal groups and for the general orthogonal groups. In §4 we review loop groups and affine Grassmannians. In §5, we first review Galois cohomology theory, which we will use to prove our first main theorem. We fix the parahoric subgroup for triality groups by picking a lattice, and show our first main theorem in §5.2, to identify points in affine Grassmannian for triality groups with lattices satisfying some special conditions. In §6, we discuss global affine Grassmannians for triality groups. We first recall the general construction by [26] in §6.1. Our special construction with the statement of our second main theorem is in §6.2, whose proof occupies §6.3. This proof is similar but more general than the proof in §5.2. Finally, we give the definition of local models for triality groups in §7. We show the generic fiber of $\mathcal{M}^{\mathrm{loc}}$ is isomorphic to the triality triple in some sense. We explain our motivation for splitting models in §8.1, and define splitting models for triality groups $\mathcal{M}^{\mathrm{split}}$ being the flat closure of some "naive splitting models" $\mathcal{M}$. They are isomorphic in the generic fiber, which are exactly the generic fiber of quadratic hypersurface Q. Our last main theorem is in §8.2. We conclude the paper in §8.3-8.6, which include calculation results that we need for the proof of our last main theorem.

Chapter 2

Composition Algebras

The main topic of this section is composition algebras. We give the definition of unital composition algebras, symmetric composition algebras and normal twisted composition algebras. We will see that they have close connection with each other.

Let F be a field, and suppose $\mathrm{char}(F) \neq 2, 3$. In this and the following sections, by an F-algebra A we mean (unless further specified) a finite dimensional vector space over F equipped with an F-bilinear multiplication $m : A \times A \to A$. Here m is not necessarily associative. Later we will use different notations for the multiplication to distinguish different composition algebras. We do not assume that the algebra A has an identity.

Definition 2.0.1. *An involution on an algebra A over a field F is a map $\sigma : A \to A$ such that*

(1) $\sigma(x + y) = \sigma(x) + \sigma(y)$,

(2) $\sigma(xy) = \sigma(y)\sigma(x)$,

(3) $\sigma^2(x) = x$,

for any $x, y \in A$.

A morphism with involution is a morphism of algebras which commutes with the involution. A quadratic form on A over F is a mapping $q : A \to F$ with the properties:

(1) $q(\lambda x) = \lambda^2 q(x)$ for $\lambda \in F$, $x \in A$.

(2) The mapping $\langle \ , \ \rangle : A \times A \to A$ defined by

$$\langle x, y \rangle = q(x+y) - q(x) - q(y)$$

is bilinear.

We always assume q is nonsingular in this paper, i.e., if $\langle x, y \rangle = 0$ for all $y \in A$, we have $x = 0$. An element x in A is called isotropic if $q(x) = 0$ and anisotropic if $q(x) \neq 0$. The quadratic form q is said to be isotropic if there exist nonzero isotropic elements in A.

Definition 2.0.2. *A composition algebra A over a field F with multiplication $x \cdot y = m(x, y)$ is an algebra with a nonsingular quadratic form q on A satisfying:*

$$q(x \cdot y) = q(x)q(y).$$

This quadratic form q is often referred to as the norm on A, and the associated bilinear form $\langle \ , \ \rangle$ is called the inner product.

A subalgebra of a composition algebra is an F-subspace that is closed under multiplication, and the homomorphisms between composition algebras are the F-linear maps that preserve the multiplication.

2.1 Unital composition algebras

Let A be a composition algebra over F with identity e, and denote by $x \diamond y$ the multiplication $m(x, y)$. We call the triple $(A, \diamond, q)$ a unital composition algebra. It turns out that every element of a unital composition algebra satisfies a quadratic polynomial. This is the minimal polynomial if the element is not a scalar multiple of the identity. (A minimal polynomial of an element $x_0 \in A$ is the unique irreducible monic polynomial $p(x) \in F[x]$ of smallest degree such that $p(x_0) = 0$.)

Proposition 2.1.1. *Every element x of a unital composition algebra $(A, \diamond, q)$ satisfies*

$$x \diamond x - \langle x, e\rangle x + q(x)e = 0.$$

For $x, y \in A$, we have

$$x \diamond y + y \diamond x - \langle x, e\rangle y - \langle y, e\rangle x + \langle x, y\rangle e = 0.$$

Proof. See Proposition 1.2.3, [31]. □

By using this proposition, we can define an involution $r : x \mapsto r(x)$ by

$$r(x) := \langle x, e\rangle e - x,$$

for $x \in A$. We call $r(x)$ the conjugate of x. The following results hold in every unital composition algebra $(A, \diamond, q)$:

Lemma 2.1.2. *We have*

(1) $x \diamond r(x) = r(x) \diamond x = q(x)e,$

(2) $r^2(x) = x,$

(3) $q(r(x)) = q(x),$

(4) $r(x+y) = r(x) + r(y)$,

(5) $\langle r(x), r(y)\rangle = \langle x, y\rangle$,

(6) $r(x \diamond y) = r(y) \diamond r(x)$,

for all $x, y \in A$.

Proof. See Lemma 1.3.1, [31]. □

From the above, we see that $r : x \mapsto r(x)$ is indeed an involution. We list some useful equations that we will use later in the following lemmas (See §1.2, §1.3, [31]):

Lemma 2.1.3. *We have*

(1) $\langle x \diamond z, y \diamond z\rangle = \langle x, y\rangle q(z)$,

(2) $\langle z \diamond x, z \diamond y\rangle = q(z)\langle x, y\rangle\rangle$,

(3) $\langle x \diamond z, y \diamond w\rangle + \langle x \diamond w, y \diamond z\rangle = \langle x, y\rangle\langle z, w\rangle$,

for all $x, y, z \in A$.

Proof. See §1.2, [31]. □

Lemma 2.1.4. *We have*

(1) $x \diamond (r(x) \diamond y) = q(x)y$,

(2) $(x \diamond r(y)) \diamond y = q(y)x$,

(3) $x \diamond (r(y) \diamond z) + y \diamond (r(x) \diamond z) = \langle x, y\rangle z$,

(4) $(x \diamond r(y)) \diamond z + (x \diamond r(z)) \diamond y = \langle y, z\rangle x$,

for all $x, y, z \in A$.

Proof. See Lemma 1.3.3, [31]. $\square$

Unital composition algebras are described by the Cayley-Dickson process. Suppose that (A, r) is a unital composition algebra with an involution r. Let $\lambda \in F^*$. The Cayley-Dickson algebra $CD(A, \lambda)$ associated to (A, r) and λ is the vector space

$$CD(A, \lambda) := A \oplus vA,$$

where v is a new symbol, endowed with the multiplication:

$$(a + vb) \diamond (a' + vb') := (a \diamond a' + \lambda b' \diamond r(b)) + v(r(a) \diamond b' + a' \diamond b),$$

for a, a', b and $b' \in A$. We set $q(a + vb) := q(a) - \lambda q(b)$, and $r(a + vb) := r(a) - vb$. One can check that $(CD(A, \lambda), \diamond, q)$ is an algebra with identity $1 = e + v \cdot 0$. The algebra A is contained in $CD(A, \lambda)$. This process from A to $CD(A, \lambda)$ is called a Cayley-Dickson process. We refer §33.C, [14] or §1.5, [31] for details. By using the Cayley-Dickson process, we now come to the well-known classification of unital composition algebras:

Theorem 2.1.5. *Every unital composition algebra over* F *is obtained by the Cayley-Dickson process. The possible dimensions are* $1, 2, 4$ *and* 8. *Composition algebras of dimension* 1 *or* 2 *are commutative and associative, those of dimension* 4 *are associative but not commutative, and those of dimension* 8 *are neither commutative nor associative.*

See §1.5, [31] and §33.C, [14] for the proof. By repeating the process from $A = F \cdot e$, we get a quadratic étale algebra, a quaternion algebra, and a Caylay algebra corresponding to the unital composition algebra of dimension 2, 4 and 8. The Cayley-Dickson process applied to a Cayley algebra does not yield a composition algebra.

Now we consider the isomorphism classes of unital composition algebras. Let $(A, \diamond, q)$, $(A', \diamond', q')$ be two composition algebras. A similitude is an F-linear map: $g : (A, \diamond, q) \to (A', \diamond', q')$ for which there exists a constant $\alpha \in F^*$ such that $\langle g(x), g(y)\rangle' = \alpha\langle x, y\rangle$ for all $x, y \in A$. We call α the multiplier of a similitude g. A similitude with multiplier $\alpha = 1$ is called an isometry.

Proposition 2.1.6. *Let $(A, \diamond, q)$, $(A', \diamond', q')$ be two composition algebras. The following claims are equivalent:*

(1) $g : A \to A'$ is an isomorphism.

(2) $g : A \to A'$ is a similitude.

(3) $g : A \to A'$ is an isometry.

Proof. See Theorem 33.19, [14]. □

Proposition 2.1.7. *If the quadratic form of a unital composition algebra is isotropic, it is hyperbolic.*

Proof. See Proposition 33.23, [14]. □

It follows from the above propositions that in each possible dimension, there is only one isomorphism class of unital composition algebras with isotropic quadratic form. We are specifically interested in Cayley algebras in this paper. We call the (unique up to isomorphism) Cayley algebra with isotropic norm the split Cayley algebra.

2.2 Symmetric composition algebras

In this section we discuss a special class of composition algebras without identity. Let (S, q) be a composition algebra over F, and denote by $x \star y$ the multiplication $m(x, y)$.

Definition 2.2.1. *A symmetric composition algebra $(S, \star, q)$ is a composition algebra satisfying*

$$\langle x \star y, z \rangle = \langle x, y \star z \rangle,$$

for all $x, y, z \in S$.

Similar to Lemma 2.1.2, Lemma 2.1.3, the following results hold in every symmetric composition algebra $(S, \star, q)$:

Lemma 2.2.2. *We have*

(1) $\langle x \star z, y \star z \rangle = \langle x, y \rangle q(z)$,

(2) $\langle z \star x, z \star y \rangle = q(z)\langle x, y \rangle\rangle$,

(3) $\langle x \star z, y \star w \rangle + \langle x \star w, y \star z \rangle = \langle x, y \rangle \langle z, w \rangle$,

for all $x, y, z \in S$.

Lemma 2.2.3. *We have*

(1) $(x \star y) \star z + (z \star y) \star x = \langle x, z \rangle y$,

(2) $x \star (y \star z) + z \star (y \star x) = \langle x, z \rangle y$,

for all $x, y, z \in S$. In particular, we have $(x \star y) \star x = x \star (y \star x) = q(x)y$.

See Lemma 34.1, [14] for the proof. Starting from a unital composition algebra $(A, \diamond, q)$, we can get a symmetric composition algebra $(A, \star, q)$ by defining $x \star y = r(x) \diamond r(y)$. It satisfies $\langle x \star y, z\rangle = \langle y \star z, x\rangle$ since we have $\langle x \diamond y, r(z)\rangle = \langle y \diamond z, r(x)\rangle$ for any $x, y, z \in A$. We say that $(A, \star, q)$ is a para-quadratic algebra (resp. para-quaternion algebra or para-Cayley algebra) if it is obtained from $(A, \diamond, q)$ a quadratic algebra (resp. quaternion algebra or Cayley algebra). It turns out that the identity element $e \in (A, \diamond, q)$ plays an important role in the corresponding symmetric composition algebra $(A, \star, q)$: By $x \star y = r(x) \diamond r(y)$, it is easy to see that e is an idempotent ($e \star e = e$) and satisfies $e \star x = x \star e = -x$ for any $x \in A$ orthogonal to e ($\langle x, e\rangle = 0$). We call an element which satisfies the above condition a para-unit.

Not every symmetric composition algebra can be obtained in this way. For example, we have Okubo algebras in dimension 8. This is shown in §34, [14]. We call a symmetric composition algebra a para-Hurwitz algebra if it is obtained from a unital composition algebra by defining $x \star y = r(x) \diamond r(y)$.

Proposition 2.2.4. *A symmetric composition algebra is para-Hurwitz if and only if it admits a para-unit.*

Proof. See Proposition 34.8, [14]. $\square$

We are specifically interested in para-Cayley algebras in this paper. By Proposition 34.4, [14], any isomorphism of unital composition algebras is an isomorphism of the corresponding para-Hurwitz algebras. Conversely, when dimension ≥ 4, any isomorphism of para-Hurwitz algebras is an isomorphism of the corresponding unital composition algebras. Hence there is an equivalence of groupoids of unital composition algebras and para-Hurwitz algebras of dimension 4 or 8. Since the split Cayley algebra is unique up to isomorphism, we get

Table 2.1: The split para-Cayley algebra $(C, \star)$ with multiplication $x \star y$

		y							
		e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
	e_1	$\cdot$	$\cdot$	$\cdot$	$-e_1$	$\cdot$	$-e_2$	e_3	$-e_4$
	e_2	$\cdot$	$\cdot$	e_1	$\cdot$	$-e_2$	$\cdot$	$-e_5$	$-e_6$
	e_3	$\cdot$	$-e_1$	$\cdot$	$\cdot$	$-e_3$	$-e_5$	$\cdot$	e_7
	e_4	$\cdot$	$-e_2$	$-e_3$	e_5	$\cdot$	$\cdot$	$\cdot$	$-e_8$
x	e_5	$-e_1$	$\cdot$	$\cdot$	$\cdot$	e_4	$-e_6$	$-e_7$	$\cdot$
	e_6	e_2	$\cdot$	$-e_4$	$-e_6$	$\cdot$	$\cdot$	$-e_8$	$\cdot$
	e_7	$-e_3$	$-e_4$	$\cdot$	$-e_7$	$\cdot$	e_8	$\cdot$	$\cdot$
	e_8	$-e_5$	e_6	$-e_7$	$\cdot$	$-e_8$	$\cdot$	$\cdot$	$\cdot$

the corresponding algebra is also unique up to isomorphism, and denote by $(C, \star)$ the split para-Cayley algebra. The multiplication table of the split Cayley algebra is given by Table 2.1.

2.3 Twisted composiiton algebras

Twisted composition algebras were introduced by Springer in his 1963 lecture notes [31], to get a new description of Albert algebras. We recall the definition from [31] and [15]. Let F_0 be a field with $\mathrm{char}(F_0) \neq 2, 3$, and let F be a separable cubic field extension of F_0. The normal closure of F over F_0 is $F' = F(d)$, where d satisfies a separable quadratic equation over F_0 (see Theorem 4.13, [11]). We can take $d = \sqrt{D}$, the square root of the discriminant D of F over F_0. We set $F_0' = F_0(d)$. So either F is the Galois extension of F_0 with cyclic Galois group of order 3, and then $F' = F, F_0' = F_0$; or F' and F_0' are quadratic extensions of F and F_0, respectively, and F' is the Galois extension of F_0'. We will focus on the case that the separable cubic extension F/F_0 is also normal, and call algebras of this type "normal twisted composition algebras".

Let F/F_0 be a cubic Galois extension. We set $\Gamma = \mathrm{Gal}(F/F_0)$, with ρ the generator of Γ. Set $\theta = \rho^2$, then $\Gamma = \{1, \rho, \theta\}$.

Definition 2.3.1. *A normal twisted composition algebra (of dimension 8) is a 5-tuple $(A, F, q, \rho, *)$, where A is a vector space of dimension 8 over F with a nonsingular quadratic form q, and associated bilinear form $\langle\ ,\ \rangle$. We have an F_0-bilinear product $* : A \times A \to A$ on F with the following properties:*

(1) The product $x * y$ is ρ-linear in x and θ-linear in y, that is:

$$(\lambda x) * y = \rho(\lambda)(x * y),\ \ x * (\lambda y) = \theta(\lambda)(x * y),$$

(2) We have $q(x * y) = \rho(q(x))\theta(q(y))$,

(3) We have $\langle x * y, z\rangle = \rho(\langle y * z, x\rangle) = \theta(\langle z * x, y\rangle)$

for all $x, y, z \in A$, and $\lambda \in F$.

Let $A' = (A', F, q', \rho', *')$ be another normal twisted composition algebra. A similitude $A \to A'$ is defined to be an F-linear isomorphism $g : A \to A'$, for which there exists $\lambda \in F^*$, such that

$$q'(g(x)) = \rho(\lambda)\theta(\lambda)q(x), \quad g(x) *' g(y) = \lambda g(x * y),$$

for all $x, y \in A$. We denote by $A' = A_\lambda$. The scalar λ is called the multiplier of the similitude. Similitudes with multiplier 1 are called isometries.

It turns out that a normal twisted composition algebra can be obtained by scalar extension from a symmetric composition algebra: Given a symmetric composition $(S, \star, q)$ over

F_0. We define a normal twisted composition algebra $\tilde{S} = S \otimes (F, \rho)$ as follows:

$$S \otimes (F, \rho) = (S \otimes_{F_0} F, F, q_F, \rho, *)$$

where q_F is the scalar extension of q to F and $*$ is defined by extending $\star$ linearly to $S \otimes_{F_0} F$ and setting

$$x * y = (id_S \otimes \rho)(x) \star (id_S \otimes \theta)(y), \quad \text{for all } x, y \in S \otimes_{F_0} F$$

(see §2, [15]). A normal twisted composition algebra A over F is said to be reduced if there exist a symmetric composition algebra S over F_0 and $\lambda \in F^*$ such that A is isomorphic to $\tilde{S}_\lambda$.

We denote by $(V, *)$ the normal twisted composition algebra obtained from the para-Cayley algebra. Similar to unital composition algebras and symmetric composition algebras, we list some general properties for normal twisted composition algebras before we move on. Let $(A, F, q, \rho, *)$ be a normal twisted composition algebra.

Lemma 2.3.2. *We have*

(1) $\langle x * z, y * z \rangle = \rho(\langle x, y \rangle)\theta(q(z))$,

(2) $\langle z * x, z * y \rangle = \theta(\langle x, y \rangle)\rho(q(z))$,

(3) $\langle x * z, y * w, \rangle + \langle x * w, y * z \rangle = \rho(\langle x, y \rangle)\theta(\langle z, w \rangle)$,

for all $x, y, z, w \in A$.

Proof. See Lemma 4.1.2, [31]. □

Lemma 2.3.3. *We have*

(1) $x*(y*x)=\rho(q(x))y, \quad (x*y)*x=\theta(q(x))y,$

(2) $x*(y*z)+z*(y*x)=\rho(\langle x,z\rangle)y, \quad (x*y)*z+(z*y)*x=\theta(\langle x,z\rangle)y,$

(3) $(x*x)*(x*x)=T(x)x-q(x)(x*x),$ *where* $T(x):=\langle x*x,x\rangle\in F_0,$

for all $x,y,z\in A$.

Proof. See Lemma 4.1.3, [31]. $\square$

Remark 2.3.4. Let $(A,F,q,\rho,*)$ be a normal twisted composition algebra. Consider the extended algebra $A'=A\otimes_{F_0}F$. We claim this extension algebra A' is also a twisted composition algebra. In fact, we have a nice description of A'. Consider an isomorphism of F-algebras

$$\nu: F\otimes_{F_0}F \xrightarrow{\sim} F\times F\times F \quad \text{given by} \quad r_1\otimes r_2\mapsto(r_1r_2,\rho(r_1)r_2,\theta(r_1)r_2).$$

Note that $\rho\otimes id_F$ is identified with the map defined by $\tilde{\rho}(r_1,r_2,r_3)=(r_2,r_3,r_1)$ to make the diagram commutative:

$$\begin{array}{ccc} F\otimes_{F_0}F & \xrightarrow{\rho\otimes id_F} & F\otimes_{F_0}F \\ \downarrow & & \downarrow \\ F\times F\times F & \xrightarrow{\tilde{\rho}} & F\times F\times F. \end{array}$$

We define the twisted vector spaces ${}^{\rho}A$ and ${}^{\theta}A$

$${}^{\rho}A=\{{}^{\rho}x\mid x\in A\}, \quad {}^{\theta}A=\{{}^{\theta}x\mid x\in A\},$$

with the operations: ${}^{\rho}(rx)=\rho(r){}^{\rho}x$, ${}^{\rho}(x+y)={}^{\rho}x+{}^{\rho}y$, and ${}^{\theta}(rx)=\theta(r){}^{\theta}x$, ${}^{\theta}(x+y)={}^{\theta}x+{}^{\theta}y$, for all $x,y\in A, r\in F$. Then there exists an F-isomorphism $A\otimes_{F_0}F\xrightarrow{\sim}A\times{}^{\rho}A\times{}^{\theta}A$ given by:

$$x \otimes r \mapsto (rx, r(^{\rho}x), r(^{\theta}x))$$

(see Remark 2.3, [15]). To describe the multiplication in $A \otimes_{F_0} F$ and $A \times {}^{\rho}A \times {}^{\theta}A$, we need to consider F-bilinear maps:

$$*_{id} : {}^{\rho}A \times {}^{\theta}A \to A, \quad *_{\rho} : {}^{\theta}A \times A \to {}^{\rho}A, \quad *_{\theta} : A \times {}^{\rho}A \to {}^{\theta}A,$$

given by

$${}^{\rho}x *_{id} {}^{\theta}y = x * y, \quad {}^{\theta}x *_{\rho} y = {}^{\rho}(x * y), \quad x *_{\theta} {}^{\rho}y = {}^{\theta}(x * y),$$

for all $x, y \in A$. Then the product $\diamond : (A \times {}^{\rho}A \times {}^{\theta}A) \times (A \times {}^{\rho}A \times {}^{\theta}A) \to A \times {}^{\rho}A \times {}^{\theta}A$ given by

$$(x, {}^{\rho}x, {}^{\theta}x) \diamond (y, {}^{\rho}y, {}^{\theta}y) = ({}^{\rho}x *_{id} {}^{\theta}y, {}^{\theta}x *_{\rho} y, x *_{\theta} {}^{\rho}y),$$

will make the following diagram commutative:

$$\begin{array}{ccc}
(A \otimes_{F_0} F) \times (A \otimes_{F_0} F) & \xrightarrow{*\otimes id_F} & A \otimes_{F_0} F \\
\downarrow & & \downarrow \\
(A \times {}^{\rho}A \times {}^{\theta}A) \times (A \times {}^{\rho}A \times {}^{\theta}A) & \xrightarrow{\diamond} & A \times {}^{\rho}A \times {}^{\theta}A.
\end{array}$$

Finally, define quadratic forms ${}^{\rho}q : {}^{\rho}A \to F$ and ${}^{\theta}q : {}^{\theta}A \to F$ by

$${}^{\rho}q({}^{\rho}x) = \rho(q(x)), \quad {}^{\theta}q({}^{\theta}x) = \theta(q(x)).$$

We have an isomorphism:

$$(A \otimes_{F_0} F, F \otimes_{F_0} F, q_F, \rho \otimes id_F, * \otimes id_F) \simeq \left(A \times {}^{\rho}A \times {}^{\theta}A, F \times F \times F, q \times^{\rho} q \times^{\theta} q, \tilde{\rho}, \diamond\right).$$

2.4 Triality triple

In this section we will discuss isotropic subspaces in twisted composition algebras. The main results in [31] concerning isotropic subspaces in the split Cayley algebra. Matzri and Vishne translate them to arbitrary composition algebras, specially to symmetric composition algebras (see [20]). We are going to translate them further to normal twisted composition algebras.

Set F/F_0 a cubic Galois extension with Galois group $\Gamma = \mathrm{Gal}(F/F_0) = \{1, \rho, \theta\}$. Let $(C, \star)$ be the split para-Cayley algebra over F_0 with a quadratic form q, and $(V, *)$ be the normal twisted composition algebra obtained from $(C, \star)$, i.e., $V = C \otimes_{F_0} F$, where $x * y = \rho(x) \star \theta(y)$ for all $x, y \in V$. If $\langle\ ,\ \rangle$ is the bilinear form corresponding to the quadratic form q, we have the scalar extension of $\langle\ ,\ \rangle$ to F as the bilinear form of $(V, *)$.

Recall that an element x is said to be isotropic if $q(x) = 0$. A subspace U is said to be isotropic if $q(x) = 0$ for all $x \in U$. A maximal isotropic subspace is an isotropic subspace with the maximal dimension. All maximal isotropic subspaces of V have the same dimension, which is called the Witt index of q. This index is at most equal to $\frac{1}{2} \dim V$. In our case, the maximal isotropic subspaces have dimension 4. We first classify all isotropic subspaces of the split para-Cayley algebra $(C, \star)$.

Proposition 2.4.1. *Every maximal isotropic subspace of $(C, \star)$ is of the form $x \star C$ or $C \star x$, where x is an isotropic element. Furthermore, $x \star C = y \star C$ if and only if $xF_0 = yF_0$.*

Proof. See Theorem 3.1, Proposition 3.2, [20] $\square$

For the intersection of two maximal isotropic subspaces, we will get isotropic subspaces of dimension 0 or 2 when they are same types, and dimension 1 or 3 when there are different types. Denote by $U^{\perp}$ the orthogonal subspace to U, i.e., $U^{\perp} = \{x \in C \mid \langle x, y\rangle = 0 \text{ for all } x \in U\}$.

Proposition 2.4.2. *Let x, y be linearly independent isotropic elements in $(C, \star)$, then*

(1) If $\langle x, y\rangle = 0$, then $x \star C \cap y \star C$ is equal to $x \star (C \star y) = y \star (C \star x)$, which has dimension 2. Otherwise, $x \star C \cap y \star C = 0$.

(2) If $\langle x, y\rangle = 0$, then $C \star x \cap C \star y$ is equal to $(y \star C) \star x = (x \star C) \star y$, which has dimension 2. Otherwise, $C \star x \cap C \star y = 0$.

Proof. See Proposition 3.7, [20]. $\square$

Proposition 2.4.3. *Let x, y be linearly independent isotropic elements in $(C, \star)$, then $x \star C \cap C \star y$ is:*

(1) The 1-dim isotropic subspace $(x \star y)F_0$, if $x \star y \neq 0$.

(2) The 3-dim isotropic subspace $x \star y^{\perp} = x^{\perp} \star y$, if $x \star y = 0$.

Proof. See Proposition 3.8, [20]. $\square$

In fact, all isotropic subspaces of dimension 1,2,3 can be obtained from the intersection of maximal subspaces (Proposition 4.1, [20]). Hence we classify all isotropic subspaces of the split para-Cayley algebra. We are particular interested in the relations between 1-dim isotropic lines and 4-dim maximal isotropic subspaces. That is why we introduce the following multiplication operators L_x, R_x:

$$L_x(y) := x \star y, \quad R_x(z) = z \star x,$$

for all $x, y, z \in (C, \star)$.

Lemma 2.4.4. *Let $x \neq 0$ be an isotropic element in $(C, \star)$, then*

$$\ker(L_x) = \operatorname{im}(R_x), \quad \ker(R_x) = \operatorname{im}(L_x).$$

Proof. By Lemma 2.2.3, the composition $R_x \circ L_x = L_x \circ R_x$ is a multiplication by $q(x) = 0$. So $\operatorname{im}(R_x) \subset \ker(R_x)$. We claim that $\ker(L_x)$ is an isotropic subspace. For any $y, z \in \ker(L_x)$, we have $x \star a = x \star b = 0$, which implies $q(a)x = a \star (x \star a) = 0$, $q(b)x = b \star (x \star b) = 0$, and also $q(a + b)x = 0$. Since $x \neq 0$, we get $a, b, a + b$ are isotropic elements. Therefore $\langle a, b \rangle = q(a + b) - q(a) - q(b) = 0$.

Since $\operatorname{im}(R_x) = C \star x$ is a 4-dim isotropic subspace, and $\dim(\ker(L_x)) \leq 4$ by $\ker(L_x)$ isotropic, we have $\ker(L_x) = \operatorname{im}(R_x)$. The argument for $\ker(R_x) = \operatorname{im}(L_x)$ is similar. $\square$

Now let us turn to the intersection of an arbitrary number of maximal isotropic subspaces.

Definition 2.4.5. *Let U be any isotropic subspace in $(C, \star)$. We define:*

$$\mathcal{L}(U) = \cap_{x \in U}(C \star x), \quad \mathcal{R}(U) = \cap_{x \in U}(x \star C).$$

It is easy to see that $\mathcal{L}(U) = \{y \mid y \in \operatorname{im}(R_x) \text{ for any } x \in U\} = \{y \mid y \in \ker(L_x) \text{ for any } x \in U\} = \{y \mid U \star y = 0\}$. Similarly, $\mathcal{R}(U) = \{z \mid z \star U = 0\}$. Furthermore, we have

Proposition 2.4.6.

(1) For every isotropic line xF_0, we have $\mathcal{L}(xF_0) = C \star x$, $\mathcal{R}(xF_0) = x \star C$;

(2) For every maximal isotropic subspace, we have

$$\mathcal{L}(C \star x) = 0, \quad \mathcal{R}(C \star x) = xF_0,$$
$$\mathcal{L}(x \star C) = xF_0, \quad \mathcal{R}(x \star C) = 0.$$

Proof. (1) follows directly from the definition. (2) For any $y \in \mathcal{L}(C \star x)$, it is equivalent to $(C \star x) \star y = 0$. Since $\mathcal{L}(U)$ is the intersection of isotropic subspaces, every element in it is also isotropic. Hence

$$C \star x \subset \ker(R_y) = \operatorname{im}(L_y) = y \star C,$$

if $y \neq 0$. But it is contradiction to Proposition 2.4.3 since the dimension of the intersection of different types of maximal subspaces is 1 or 3. Hence $y = 0$. Similarly, for any $z \in \mathcal{R}(C \star x)$, it is equivalent to $C \star x \subset C \star z$, which gives $z \in xF_0$ by Proposiition 2.4.1. The proof for other half part is the same. $\square$

By Proposition 2.4.6, we have the following diagram:

$$\begin{tikzcd}
& xF_0 \arrow[dl, "\mathcal{L}"'] & \\
C \star x \arrow[rr, "\mathcal{R}\circ\mathcal{R}"] & & x \star C. \arrow[ul, "\mathcal{L}"']
\end{tikzcd}$$

We call it the geometric triality graph.

Now let us consider the normal twisted composition algebra $(V, *)$. We can view $(V, \star)$ as a split para-Cayley algebra over F without twisting, and have $x * y = \rho(x) \star \theta(y)$ for all $x, y \in V$. Then $x \star V = \theta(x) * V$, $V \star x = V * \rho(x)$. From Proposition 2.4.1, it is immediately to get:

Proposition 2.4.7. *Every maximal isotropic subspace of* $(V, *)$ *is of the form* $x * V$ *or* $V * x$, *where* x *is an isotropic element. Furthermore,* $x * V = y * V$ *if and only if* $xF = yF$.

Similarly, we can check that $x * V \cap y * V = \rho(x) \star V \cap \rho(y) \star V$, which is equal to $\rho(x) \star (V \star \rho(y)) = x * (V * y)$. Similarly, $x * V \cap V * y$ is equal to $\rho(x) \star V \cap V \star \theta(y)$. We have $\rho(x) \star \theta(y)^{\perp} = x * y^{\perp}$. Hence by Proposition 2.4.2, 2.4.3, we obtain:

Proposition 2.4.8. *Let* x, y *be linearly independent isotropic elements in* $(V, *)$, *then*

(1) If $\langle x, y \rangle = 0$, *then* $x * V \cap y * V$ *is equal to* $x * (V * y) = y * (V * x)$, *which has dimension 2. Otherwise,* $x * V \cap y * V = 0$.

(2) If $\langle x, y \rangle = 0$, *then* $V * x \cap V * y$ *is equal to* $(y * V) * x = (x * V) * y$, *which has dimension 2. Otherwise,* $V * x \cap V * y = 0$.

Proposition 2.4.9. *Let* x, y *be linearly independent isotropic elements in* $(V, *)$, *then* $x * V \cap V * y$ *is:*

(1) The 1-dim isotropic subspace $(x * y)F$, *if* $x \star y \neq 0$.

(2) The 3-dim isotropic subspace $x * y^{\perp} = x^{\perp} * y$, *if* $x * y = 0$.

To give the geometric triality graph for normal twisted composition algebras, we need to define multiplication operators L'_x, R'_x as:

$$L'_x(y) := x * y, \quad R'_x(z) = z * x,$$

for all $x, y, z \in (V, *)$.

Lemma 2.4.10. *Let* $x \neq 0$ *be an isotropic element in* $(V, *)$, *then*

$$\ker(L'_x) = \mathrm{im}(R'_x), \quad \ker(R'_x) = \mathrm{im}(L'_x).$$

Proof. By $\ker(L'_x) = \{z \in V \mid x * z = \rho(x) \star \theta(z) = 0\}$, we have $\theta(x) \star z = 0$, hence $z \in \ker(L_{\theta(x)}) = \operatorname{im}(R_{\theta(x)})$. And $\operatorname{im}(R'_x) = \{w \in V \mid w \in V * x\} = V \star \theta(x) = \operatorname{im}(R_{\theta(x)})$, which gives us $\ker(L'_x) = \operatorname{im}(R'_x)$. The proof for the other half part is the same. $\square$

Example 2.4.11. Take $x = \pi e_1 + e_2 \in V$, where π is the uniformizer of the valuation ring $\mathcal{O}_F$, with $\rho(\pi) = \pi\xi$. We have

$$\ker(L'_x) = \operatorname{im}(R'_x) = F\langle e_1, e_2, \pi\xi^2 e_3 + e_4, \pi\xi^2 e_5 - e_6\rangle,$$
$$\ker(R'_x) = \operatorname{im}(L'_x) = F\langle e_1, e_2, \pi\xi e_3 - e_5, \pi\xi e_4 + e_6\rangle.$$

Similar to Definition 2.4.5, we now consider the intersection of maximal isotropic subspaces in $(V, *)$:

Definition 2.4.12. *Let U be any isotropic subspace in $(V, *)$. We define:*

$$\mathcal{L}'(U) = \cap_{x \in U}(V * x), \quad \mathcal{R}'(U) = \cap_{x \in U}(x * V).$$

Proposition 2.4.13.

(1) For every isotropic line xF, we have $\mathcal{L}'(xF) = V * x$, $\mathcal{R}'(xF) = x * V$;

(2) For every maximal isotropic subspace, we have

$$\mathcal{L}'(V * x) = 0, \quad \mathcal{R}'(V * x) = xF,$$
$$\mathcal{L}'(x * V) = xF, \quad \mathcal{R}'(x * V) = 0.$$

Proof. (1) follows directly from the definition. (2). From $\mathcal{L}'(V * x) = \cap_{y \in V * x}(V * y)$, we can see that $y \in V * x = V \star \theta(x)$. Hence $y \in V * x$ is equivalent to $\theta(y) \in V \star \rho(x)$ by acting θ

on both sides. So

$$\mathcal{L}'(V * x) = \cap_{y \in V*x}(V * y) = \cap_{\theta(y) \in V \star \rho(x)}(V \star \theta(y)) = \mathcal{L}(V \star \rho(x)).$$

By Proposition 2.4.6, we get $\mathcal{L}'(V * x) = 0$. The proof for the rest is similar. $\square$

From the above discussion, the geometric triality graph for normal twisted composition algebras is:

$$\begin{tikzcd} & xF \arrow[dl, "\mathcal{L}'"] & \\ V * x \arrow[rr, "\mathcal{R}' \circ \mathcal{R}'"] & & x * V. \arrow[ul, "\mathcal{L}'"'] \end{tikzcd}$$

We call $(xF, V * x, x * V)$ the triality triple for $(V, *)$.

Chapter 3

Orthogonal groups, similitudes and triality

3.1 Preliminaries

Let (V, q) be a vector space with a nonsingular quadratic form q over a field F, $\mathrm{char}(F) \neq 2$. Denote by $\langle\ ,\ \rangle$ the bilinear form corresponding to q. Similar to composition algebras, an element x is called isotropic if $q(x) = 0$. A subspace W of V is said to be isotropic if $q(x) = 0$ for all $x \in W$. A maximal isotropic subspace is an isotropic subspace with the maximal dimension.

For any $f \in \mathrm{End}_F(V)$, there exists an element $\sigma_q(f) \in \mathrm{End}_F(V)$ such that $\langle x, f(y)\rangle = \langle \sigma_b(f)(x), y\rangle$. We can see this using matrices: If $b \in \mathrm{GL}(V)$ denotes the Gram matrix of $\langle\ ,\ \rangle$ with respect to a fixed basis, then $\langle x, y\rangle = x^t by$. Let $\sigma_q(f) = b^{-1} f^t b$. We have $\langle x, f(y)\rangle = x^t bf(y) = \langle \sigma_b(f)(x), y\rangle$. It is easy to see that $\sigma_q : \mathrm{End}_F(V) \to \mathrm{End}_F(V)$ given by $f \mapsto \sigma_q(f)$ is an involution of $\mathrm{End}_F(V)$.

The orthogonal group $O(V, q)$ is the subgroup of the isomorphism group $\mathrm{Isom}(V, q)$ that preserves the form $\langle\ ,\ \rangle$:

$$O(V, q) := \{g \in \mathrm{Isom}(V, q) \mid \langle g(x), g(y)\rangle = \langle x, y\rangle\}.$$

Since $\det(g) = \pm 1$ for $g \in O(V,q)$, we have the special orthogonal group consists of $g \in O(V,q)$ with $\det g = 1$, denoted by $\mathrm{SO}(V,q)$ or $O^+(V,q)$. Elements in $\mathrm{SO}(V,q)$ are called proper isometries. The universal covering of $\mathrm{SO}(V,q)$ is the spin group $\mathrm{Spin}(V,q)$, which will be used to define the triality group. We give a short review of Clifford algebras before introducing the spin group.

Definition 3.1.1. *The Clifford algebra $C(V,q)$ is the quotient of the tensor algebra $T(V) = \oplus_{n\geq 0} V^{\otimes n}$ by the ideal $I(q)$ generated by all the elements of the form $v \otimes v - q(v)\cdot 1$ for $v \in V$.*

Since $T(V)$ is a graded algebra, we have $T(V) = T_0(V) \oplus T_1(V)$, where $T_0(V) = T(V \otimes V)$ and $T_1(V) = V \otimes T_0(V)$. It induces a $\mathbb{Z}/2\mathbb{Z}$-grading of $C(V,q)$:

$$C(V,q) = C_0(V,q) \oplus C_1(V,q).$$

We call $C_0(V,q)$ the even Clifford algebra and $C_1(V,q)$ the odd Clifford algebra. When $\dim V = n$, we have $\dim C(V,q) = 2^n$, and $\dim C_0(V,q) = 2^{n-1}$ (see Chapter IV, [13]). For every quadratic space (V,q), the identity map on V extends to involution on the tensor algebra $T(V)$ which preserve the ideal $I(q)$: $(v_1 \otimes \cdots \otimes v_r)^t := v_r \otimes \cdots \otimes v_1$ for $v_1, \ldots, v_r \in V$. It is therefore inducing a canonical involution of the Clifford algebra $\tau : C(V,q) \to C(V,q)$ given by $\tau(v_1 \cdots v_d) = v_d \cdots v_1$. Then the spin group is a subgroup of $C_0(V,q)^*$:

$$\mathrm{Spin}(V,q) = \{c \in C_0(V,q)^* \mid cVc^{-1} = V, \tau(c)c = 1\}.$$

For any $c \in \mathrm{Spin}(V,q)$, we have a linear map $\chi_c : x \mapsto cxc^{-1}$. This is an element in $\mathrm{SO}(V,q)$ since $q(\chi_c(x)) = cxc^{-1}cxc^{-1} = q(x)$, and we can show that $\mathrm{Spin}(V,q) \to SO(V,q)$ given by

$c \mapsto \chi_c$ is surjective. We have an exact sequence:

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Spin}(V,q) \to \mathrm{SO}(V,q) \to 1.$$

The orthogonal group scheme $\mathbf{O}(V,q)$ and the special orthogonal group scheme $\mathbf{SO}(V,q)$ over F are defined by:

$$\mathbf{O}(V,q)(R) := \{g \in \mathrm{Isom}(V_R, q) \mid \langle g(x), g(y)\rangle = \langle x, y\rangle\}.$$

$$\mathbf{SO}(V,q)(R) := \{g \in \mathrm{Isom}(V_R, q) \mid \langle g(x), g(y)\rangle = \langle x, y\rangle, \det g = 1\}.$$

for any F-algebra R. Similarly, we have $\mathbf{Spin}(V,q)(R) := \{c \in C_0(V_R,q)^* \mid cV_Rc^{-1} = V_R, \tau(c)c = 1\}$, where $V_R = V \otimes_F R$.

More generally, a similitude of (V,q) is a linear map $g : V \to V$ for which there exist a constant $\mu(g) \in F^*$ such that $\langle g(x), g(y)\rangle = \mu(g)\langle x, y\rangle$ for all $x, y \in V$. We define the general orthogonal group scheme over F as:

$$\mathbf{GO}(V,q)(R) := \{g \in \mathrm{Isom}(V_R, q) \mid \langle g(x), g(y)\rangle = \mu(g)\langle x, y\rangle \text{ for some } \mu(g) \in R^*\}.$$

for $R \in \mathrm{Alg}_F$. The factor $\mu(g)$ is called the multiplier of the similitude g. A similitude with multiplier 1 is called an isometry, i.e., $g \in \mathbf{O}(V,q)$. If $b \in \mathrm{GL}(V)$ denotes the Gram matrix of $\langle\ ,\ \rangle$ with respect to a fixed basis, then $\langle g(x), g(y)\rangle = \mu(g)\langle x, y\rangle$ is equivalent to $g^t bg = \mu(g)b$, hence

$$\mu(g) = b^{-1}g^t bg = \sigma_q(g)g.$$

By taking the determinant on both sides, we obtain $(\det g)^2 = \mu(g)^n$ where $\dim V = n$. It follows that the determinant of an isometry is ± 1 and that, $\det(g) = \pm\mu(g)^{n/2}$ if n is even for $g \in \mathrm{GO}(V,q)$. We say $g \in \mathbf{GO}(V,q)(R)$ is a proper similitude if $\det(g) = \mu(g)^{n/2}$. Thus, the group of proper similitudes is defined as:

$$\mathbf{GO}^+(V,q)(R) := \{g \in \mathrm{Isom}(V_R,q) \mid \langle g(x), g(y)\rangle = \mu(g)\langle x,y\rangle, \det(g) = \mu(g)^{n/2}\},$$

if n is even. For any similitude $f \in \mathbf{GO}(V,q)$, we have an automorphism $C_0(f) : C_0(V_R,q) \to C_0(V_R,q)$ of the even Clifford algebra given by

$$C_0(f)(v_1 \cdots v_{2r}) := \mu(f)^{-r} f(v_1) \cdots f(v_{2r}),$$

(see Proposition (13.1) in [14]). Let $\mathbf{PGO}(V,Q)$ be the quotient group $\mathbf{GO}(V,q)/\mathbb{G}_m$. It is easy to see that the automorphism $C_0(f)$ only depends on the image $[f] \in \mathbf{PGO}(V,q)$, and we shall use the notation $C_0[f]$ for $C_0(f)$.

3.2 The principle of triality

In this section we deal with algebraic triality for the special orthogonal groups and general orthogonal groups. Algebraic triality defines outer automorphisms of $\mathbf{PGO}(V,q)$. In [31, Chapter 3], T.A.Springer defined algebraic triality for the Cayley algebra. In [4], Knus and Tignol defined algebraic triality for the para-Cayley algebra $(C,\star)$. We are going to consider algebraic triality for the normal twisted composition algebra.

Recall that F/F_0 is a cubic Galois extension, and set $\Gamma = \mathrm{Gal}(F/F_0) = \langle\rho\rangle$, $\theta = \rho^2$. Let $(V,*)$ be a normal twisted composition algebra. We define the twisted vector spaces ${}^{\rho}V$, ${}^{\theta}V$

in Remark 2.3.4. For $x \in V$, consider the F-linear maps

$$l_x : {}^{\rho}V \to {}^{\theta}V, \quad r_x : {}^{\theta}V \to {}^{\rho}V,$$

given by

$$l_x({}^{\rho}y) = {}^{\theta}(x * y) \quad \text{and} \quad r_x({}^{\theta}z) = {}^{\rho}(z * x).$$

By §3,[15], the map

$$x \mapsto \begin{pmatrix} 0 & r_x \\ l_x & 0 \end{pmatrix} \in \mathrm{End}_F({}^{\rho}V \oplus {}^{\theta}V)$$

extends to an isomorphism of algebras with involution:

$$\alpha : (C(V,q), \tau) \xrightarrow{\sim} (\mathrm{End}_F({}^{\rho}V \oplus {}^{\theta}V), \sigma_{{}^{\rho}q \perp {}^{\theta}q}),$$

since $\alpha(x)^2 = q(x)id$. In particular, if we restrict this isomorphism to the even Clifford algebra $C_0(V,q)$, we get

$$\alpha : (C_0(V,q), \tau) \xrightarrow{\sim} (\mathrm{End}_F({}^{\rho}V), \sigma_{{}^{\rho}q}) \times (\mathrm{End}_F({}^{\theta}V), \sigma_{{}^{\theta}q}),$$

where $\sigma_{{}^{\rho}q}, \sigma_{{}^{\theta}q}$ are the involutions corresponding to the quadratic forms ${}^{\rho}q, {}^{\theta}q$, respectively.

Proposition 3.2.1. *(The principle of triality) For $g_1, g_2, g_3 \in \mathbf{GO}(V,q)^+(F)$, the following statements are equivalent:*

1) There exist a scalar $\lambda_1 \in F^*$ such that

$$\lambda_1 g_1(x * y) = g_2(x) * g_3(y), \quad \textit{for any } x, y \in V.$$

2) There exist a scalar $\lambda_2 \in F^*$ such that

$$\lambda_2 g_2(x * y) = g_3(x) * g_1(y), \quad \textit{for any } x, y \in V.$$

3) There exist a scalar $\lambda_3 \in F^*$ such that

$$\lambda_3 g_3(x * y) = g_1(x) * g_2(y), \quad \textit{for any } x, y \in V.$$

4) The following diagram commutes:

$$\begin{array}{ccc}
C_0(V,q) & \xrightarrow{\alpha} & \mathrm{End}_F({}^{\rho}V) \times \mathrm{End}_F({}^{\theta}V) \\
\Big\downarrow{\scriptstyle C_0(g_1)} & & \Big\downarrow{\scriptstyle \mathrm{Int}({}^{\rho}g_2)\times\mathrm{Int}({}^{\theta}g_3)} \\
C_0(V,q) & \xrightarrow{\alpha} & \mathrm{End}_F({}^{\rho}V) \times \mathrm{End}_F({}^{\theta}V).
\end{array}$$

When these properties hold, the scalars λ_i and the multipliers $\mu(g_i)$ are related by

$$\mu(g_i) = \rho(\lambda_{i+1})\theta(\lambda_{i+2}).$$

Remark 3.2.2. We may change the scalars λ_i by scaling g_i in the proposition. For instance, we can let $\lambda_1 = 1$, then the multiplier $\mu(g_i)$ satisfies $\mu(g_1) = \rho(\mu(g_2))\theta(\mu(g_3))$. If, as in [14, Propostion (36.17)], we let $\lambda_i = \mu(g_i)^{-1}$, then the multipliers are related by $1 = \mu(g_1)\rho(\mu(g_2))\theta(\mu(g_3))$.

Proof. 1) $\Rightarrow$ 2): By multiplying each side of 1) on the left by $g_3(y)$ and using Lemma 2.3.2, we obtain:

$$\theta(\lambda_1)(g_3(y) * g_1(x * y)) = \rho(q(g_3(y)))g_2(x) = \rho(\mu(g_3)\rho(q(y)))g_2(x).$$

Let $X = y, Y = x * y$. Then, we have $X * Y = \rho(q(y))x$ by Lemma 2.3.3, and we derive from the preceding equation:

$$\theta(\lambda_1)(g_3(X) * g_1(Y)) = \rho(\mu(g_3))g_2(X * Y).$$

Hence, there exist $\lambda_2 \in F^*$ such that $\lambda_2 g_2(x * y) = g_3(x) * g_1(y)$, where $\lambda_2 \cdot \theta(\lambda_1) = \rho(\mu(g_3))$, i.e.,

$$\mu(g_3) = \rho(\lambda_1)\theta(\lambda_2).$$

Similar arguments yield 2) $\Rightarrow$ 3), 3) $\Rightarrow$ 1), with $\mu(g_i) = \rho(\lambda_{i+1})\theta(\lambda_{i+2})$.

Now, assume 1), 2), 3) hold. For any $xy \in C_0(V, q)$, we have $C_0(g_1)(xy) = \mu(g_1)^{-1}g_1(x)g_1(y)$. Since $\alpha(xy) = (r_x l_y, l_x r_y)$, 4) is equivalent to

$$\frac{1}{\mu(g_1)} r_{g_1(x)} l_{g_1(y)} = {}^{\rho}g_2 \cdot r_x l_y \cdot ({}^{\rho}g_2)^{-1}, \quad \frac{1}{\mu(g_1)} l_{g_1(x)} r_{g_1(y)} = {}^{\theta}g_3 \cdot l_x r_y \cdot ({}^{\theta}g_3)^{-1}.$$

For any ${}^{\rho}z \in {}^{\rho}V$, we obtain

$$\begin{aligned}
{}^{\rho}(g_2((y * g_2^{-1}(z)) * x)) &= \frac{1}{\rho(\lambda_2)} {}^{\rho}(g_3(y * g_2^{-1}(z)) * g_1(x)) \\
&= \frac{1}{\rho(\lambda_2)} {}^{\rho}([\frac{1}{\lambda_3}(g_1(y) * z)] * g_1(x)) \\
&= \frac{1}{\rho(\lambda_2)\theta(\lambda_3)} {}^{\rho}([(g_1(y) * z)] * g_1(x)) \\
&= \frac{1}{\mu(g_1)} {}^{\rho}([(g_1(y) * z)] * g_1(x)).
\end{aligned}$$

Hence $\mu(g_1)^{-1} r_{g_1(x)} l_{g_1(y)} = {}^{\rho}g_2 \cdot r_x l_y \cdot ({}^{\rho}g_2)^{-1}$. Similarly, we get $\mu(g_1)^{-1} l_{g_1(x)} r_{g_1(y)} = {}^{\theta}g_3 \cdot l_x r_y \cdot ({}^{\theta}g_3)^{-1}$.

Finally, assume 4) holds. Consider the map

$$\beta : x \mapsto \begin{pmatrix} 0 & r_{g_1(x)} \\ \frac{1}{\mu(g_1)} l_{g_1(x)} & 0 \end{pmatrix}.$$

It is easty to check that $\beta^2(x) = q(x)id$, so we can extend β to an isomorphism of algebras with involution $\beta : (C(V,q),\tau) \to (\mathrm{End}_F({}^{\rho}V \oplus {}^{\theta}V), \sigma_{\rho_q \perp \theta_q})$ by the universal property of Clifford algebras. Then the automorphism $\beta \cdot \alpha^{-1}$ is inner by the Skolem-Noether theorem (see Theorem 1.4, [14]). Hence there exist $\varphi, \psi \in \mathrm{End}_F(V,q)$, such that

$$\beta \cdot \alpha^{-1} = \mathrm{Int} \begin{pmatrix} {}^{\rho}\varphi & 0 \\ 0 & {}^{\theta}\psi \end{pmatrix}.$$

It is equivalent to $\varphi(x * y) = \psi(x) * g_1(y)$, $\rho(\mu(g_1))\psi(x * y) = g_1(x) * \varphi(y)$. Set $g_2 = \lambda_2^{-1}\varphi, g_3 = \psi$. We get

$$\lambda_2 g_2(x * y) = g_3(x) * g_1(y), \ \lambda_3 g_3(x * y) = g_1(x) * g_2(y).$$

Therefore, 4) implies 1), 2), 3). □

From the principle of triality, we can directly get:

Corollary 3.2.3. *For $g_1, g_2, g_3 \in \mathbf{SO}(V,q)(F)$, the following statements are equivalent:*

(1) $g_i(x * y) = g_{i+1}(x) * g_{i+2}(y), i = 1,2,3 \pmod 3$ *for any* $x, y \in V$.

(2) The following diagram commutes:

$$\begin{array}{ccc} C_0(V,q) & \xrightarrow{\alpha} & \mathrm{End}_F({}^{\rho}V) \times \mathrm{End}_F({}^{\theta}V) \\ \Big\downarrow C_0(g_1) & & \Big\downarrow \mathrm{Int}({}^{\rho}g_2)\times\mathrm{Int}({}^{\theta}g_3) \\ C_0(V,q) & \xrightarrow{\alpha} & \mathrm{End}_F({}^{\rho}V) \times \mathrm{End}_F({}^{\theta}V). \end{array}$$

3.3 Special orthogonal groups and triality

We will discuss triality for the special orthogonal group in this section. We continue with the same notations. Let $(V, *)$ be a normal twisted composition algebra over F. Recall that the spin group $\mathbf{Spin}(V, *)$ is defined as

$$\mathbf{Spin}(V,*)(R) = \{c \in C_0(V)^*_R \mid cV_Rc^{-1} = V_R, \tau(c)c = 1\}$$

for any F-algebra R, where $V_R = V \otimes_F R$. It turns out that the isomorphism α in the above section gives a nice description of $\mathrm{Spin}(V, *) := \mathbf{Spin}(V, *)(F)$.

Theorem 3.3.1. *There is an isomorphism*

$$\mathrm{Spin}(V,*) \cong \{(g_1,g_2,g_3) \in \mathrm{SO}(V,q)^{\times 3} \mid g_i(x*y) = g_{i+1}(x) * g_{i+2}(y),\ \textit{for any } x,y \in V\}$$

Proof. Let $c \in C_0(V)^*$. Using the isomorphism with involution α, we obtain ${}^{\rho}g_2 \in \mathrm{End}_F({}^{\rho}V)$ and ${}^{\theta}g_3 \in \mathrm{End}_F({}^{\theta}V)$ such that

$$\alpha(c) = \begin{pmatrix} {}^{\rho}g_2 & 0 \\ 0 & {}^{\theta}g_3 \end{pmatrix} \in \mathrm{End}_F({}^{\rho}V) \times \mathrm{End}_F({}^{\theta}V).$$

We have

$$\alpha(\tau(c)c) = \begin{pmatrix} {}^{\rho}\sigma_q(g_2) & 0 \\ 0 & {}^{\theta}\sigma_q(g_3) \end{pmatrix} \begin{pmatrix} {}^{\rho}g_2 & 0 \\ 0 & {}^{\theta}g_3 \end{pmatrix} = I,$$

which implies $\sigma_q(g_2)g_2 = 1, \sigma_q(g_3)g_3 = 1$, i.e., g_2, g_3 are isometries. Consider $\chi_c(x) = cxc^{-1} \in V$. By applying α on both sides, we have $\alpha(\chi_c(x)) = \alpha(c)\alpha(x)\alpha(c^{-1})$, which gives us:

$$\begin{pmatrix} 0 & r_{\chi_c(x)} \\ l_{\chi_c(x)} & 0 \end{pmatrix} = \begin{pmatrix} 0 & {}^{\rho}g_2 \cdot r_x \cdot {}^{\theta}\sigma_q(g_3) \\ {}^{\theta}g_3 \cdot l_x \cdot {}^{\rho}\sigma_q(g_2) & 0 \end{pmatrix}.$$

It is equivalent to ${}^{\rho}g_2 r_x = r_{\chi_c(x)} {}^{\theta}g_3$, ${}^{\theta}g_3 l_x = l_{\chi_c(x)} {}^{\rho}g_2$, i.e.,

$$g_2(x * y) = g_3(x) * \chi_c(y), \quad g_3(x * y) = \chi_c(x) * g_2(y).$$

Finally, χ_c is an isometry since $q(\chi_c(x)) = cxc^{-1}cxc^{-1} = q(x)$. Thus, let $g_1 = \chi_c$. We get related equations as above. We now send $c \mapsto (g_1, g_2, g_3)$ that gives as above. This giving map is an injective group homomorphism since α is an isomorphism. It is also surjective, since, given any (g_1, g_2, g_3) satisfying $g_i(x * y) = g_{i+1}(x) * g_{i+2}(y)$, there exist $c \in C_0(V)$ such that $\alpha(c) = \begin{pmatrix} {}^{\rho}g_2 & 0 \\ 0 & {}^{\theta}g_3 \end{pmatrix}$. □

From the above theorem, we have an isomorphism between group schemes over F_0:

$$\mathrm{Res}_{F/F_0}(\mathbf{Spin}(V,*))(R) \cong \{(g_1, g_2, g_3) \in \mathrm{Res}_{F/F_0}(\mathbf{SO}(V,q)(R)^{\times 3} \mid g_i(x*y) = g_{i+1}(x)*g_{i+2}(y)\}$$

for any F_0-algebra R. The transformation $\tilde{\rho} : (g_1, g_2, g_3) \mapsto (g_2, g_3, g_1)$ is an outer automorphism of $\mathrm{Res}_{F/F_0}(\mathbf{Spin}(V,*))$ satisfying $\tilde{\rho}^3 = 1$. Here $\tilde{\rho}$ generate a subgroup of $\mathrm{Aut}(\mathrm{Res}_{F/F_0}(\mathbf{Spin}(V,*)))$ which is isomorphic to A_3. Consider the fixed points of the corestriction of $\mathbf{Spin}(V,*)$ from F to F_0 under $A_3 = \langle\tilde{\rho}\rangle$. We obtain the triality group for the special orthogonal group G:

$$\begin{aligned} G(R) &:= \mathrm{Res}_{F/F_0}(\mathbf{Spin}(V,*))^{A_3}(R) \\ &\cong \{g \in \mathbf{SO}(V,q)(R \otimes_{F_0} F) \mid g(x*y) = g(x)*g(y) \text{ for all } x,y \in V \otimes_{F_0} R\}. \end{aligned}$$

for any F_0-algebra R.

3.4 General orthogonal groups and triality

We keep the same notations as in the previous sections. We want to construct a similar triality type group scheme for $\mathbf{GO}^+(V,q)$. It should be a group scheme that contains $\mathbf{Spin}(V,*)$ and should have an action of A_3 acting on it. Define the group:

$$H(R) = \{(c,\lambda) \in C_0(V_R)^* \times R^* \mid cV_R\tau(c) = V_R, \alpha(\tau(c)c) = \begin{pmatrix} \mu & 0 \\ 0 & \nu \end{pmatrix} \in \mathrm{End}({}^{\rho}V_R) \times \mathrm{End}({}^{\theta}V_R)\ \}$$

where R is an F-algebra, $\mu, \nu \in R$. The group scheme H plays the similar role as $\mathbf{Spin}(V,*)$. In fact, we have:

Theorem 3.4.1. *There is an isomorphism*

$$H(R) \cong \{(g_1,g_2,g_3) \in \mathbf{GO}^+(V,q)(R)^{\times 3} \mid \lambda_i g_i(x*y) = g_{i+1}(x)*g_{i+2}(y),\ i = 1,2,3 \mod 3\},$$

for any $x,y \in V_R, \lambda_i \in R^*$, where $\lambda_1 = \lambda^{-1}, \lambda_2 = \theta(\lambda\nu), \lambda_3 = \rho(\lambda\mu) \in R^*$, and the multipliers $\mu(g_2) = \theta(\mu),\ \mu(g_3) = \rho(\nu), \mu(g_1) = \lambda^2\mu\nu$.

Proof. The proof is similar to the spin group case. Let $c \in C_0(V,q)^*_R$. Then, there exist ${}^{\rho}g_2 \in \mathrm{End}_R({}^{\rho}V_R)$ and ${}^{\theta}g_3 \in \mathrm{End}_R({}^{\theta}V_R)$ such that

$$\alpha(c) = \begin{pmatrix} {}^{\rho}g_2 & 0 \\ 0 & {}^{\theta}g_3 \end{pmatrix} \in \mathrm{End}_R({}^{\rho}V_R) \times \mathrm{End}_R({}^{\theta}V_R).$$

We have

$$\alpha(\tau(c)c) = \begin{pmatrix} {}^{\rho}\sigma_q(g_2) & 0 \\ 0 & {}^{\theta}\sigma_q(g_3) \end{pmatrix} \begin{pmatrix} {}^{\rho}g_2 & 0 \\ 0 & {}^{\theta}g_3 \end{pmatrix} = \begin{pmatrix} \mu & 0 \\ 0 & \nu \end{pmatrix},$$

which implies ${}^{\rho}(\sigma_q(g_2)g_2) = \mu, {}^{\theta}(\sigma_q(g_3)g_3) = \nu$, i.e., g_2, g_3 are similitudes with multipliers $\mu(g_2) = \theta(\mu), \mu(g_3) = \rho(\nu)$. Define $f(x) = cx\tau(c) \in V_R$. By applying α on both sides, we obtain $\alpha(f(x)) = \alpha(c)\alpha(x)\alpha(\tau(c))$. Thus,

$$\begin{pmatrix} 0 & r_{f(x)} \\ l_{f(x)} & 0 \end{pmatrix} = \begin{pmatrix} 0 & {}^{\rho}g_2 \cdot r_x \cdot {}^{\theta}\sigma_q(g_3) \\ {}^{\theta}g_3 \cdot l_x \cdot {}^{\rho}\sigma_q(g_2) & 0 \end{pmatrix}.$$

It is equivalent to ${}^{\rho}g_2 r_x = r_{f(x)}{}^{\theta}(\mu(g_3)^{-1}g_3)$, ${}^{\theta}g_3 l_x = l_{f(x)}{}^{\rho}(\mu(g_2)^{-1}g_2)$, i.e.,

$$\theta(\nu)g_2(x * y) = g_3(x) * f(y),$$

$$\rho(\mu)g_3(x * y) = f(x) * g_2(y).$$

Finally, consider $q(f(x)) = cx\tau(c)cx\tau(x)$. Since $\alpha(\tau(c)c) = \begin{pmatrix} \mu & 0 \\ 0 & \nu \end{pmatrix}$, we obtain

$$\alpha(x\tau(c)cx) = \begin{pmatrix} \nu \cdot r_x l_x & 0 \\ 0 & \mu \cdot l_x r_x \end{pmatrix} = q(x) \begin{pmatrix} \mu & 0 \\ 0 & \nu \end{pmatrix}$$

by Lemma 2.3.3. Hence $\alpha(q(f(x))) = \mu\nu q(x) \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$, which implies $q(f(x)) = \mu\nu \cdot q(x)$. Therefore, f is a similitude with multiplier $\mu(f) = \mu\nu$. Then, set $g_1 = \lambda f$. We get $\mu(g_1) = \lambda^2\mu\nu$, and

$$\lambda_i g_i(x * y) = g_{i+1}(x) * g_{i+2}(y),\ i = 1, 2, 3 \mod 3.$$

Consider the map $(c, \lambda) \mapsto (g_1, g_2, g_3)$ giving as above. It is injective: If we have (c_1, λ_1), $(c_2, \lambda_2) \in C_0(V_R, q)$ such that $\alpha(c_1) = \alpha(c_2)$, then $c_1 = c_2$ since α is an isomorphism. We also have $\lambda_1 c_1 x\tau(c_1) = \lambda_2 c_2 x\tau(c_2)$, which implies $\lambda_1 = \lambda_2$. It is also surjective: For any $(g_1, g_2, g_3) \in \mathbf{GO}^+(V, q)(R)^{\times 3}$, we will get $c \in C_0(V_R, q)$ such that $\alpha(c) = \begin{pmatrix} {}^{\rho}g_2 & 0 \\ 0 & {}^{\theta}g_3 \end{pmatrix}$, and λ such that $g_1(x) = \lambda f(x) = \lambda cx\tau(c)$. $\square$

The group H sits in the following exact sequence:

$$1 \to \mathbf{Spin}(C, *) \to H \to \mathbb{G}_m^{\times 3},$$

where the first map is $c \mapsto (c, 1)$, and the last map is $(c, \lambda) \mapsto (\lambda, \mu, \nu)$. There exists an outer automorphism on H by $\tilde{\rho} : (g_1, g_2, g_3) \mapsto (g_2, g_3, g_1)$. Consider the generating group $\Gamma = \langle \tilde{\rho} \rangle$ and the fixed points of $\mathrm{Res}_{F/F_0} H$ under Γ. We can define the triality group for the general orthogonal group G:

$$\begin{aligned} G(R) &:= (\mathrm{Res}_{F/F_0} H)^{\Gamma}(R) \\ &\cong \{g \in \mathbf{GO}_8^+(F \otimes_{F_0} R) \mid \text{there exist } \lambda \in (F \otimes_{F_0} R)^* \text{ such that } \lambda g(x * y) = g(x) * g(y)\}, \end{aligned}$$

for any F_0-algebra R.

Chapter 4

Affine Grassmannians

In this section we review affine Grassmannians for general linear algebraic groups. We will show that the affine Grassmannian is representable by an ind-scheme and is a quotient of loop groups in the case the group is smooth. Our main references in this section are [23], [33].

4.1 Loop groups

Let k be a field. We consider the field $K = k(\!(t)\!)$ of Laurent power series with indeterminate t and coefficients in k. Let $\mathcal{O}_K = k[\![t]\!]$ be the discretely valued ring of power series with coefficients in k. For any k-algebra R, we set $\mathbb{D}_R = \mathrm{Spec}(R[\![t]\!])$, resp. $\mathbb{D}_R^* = \mathbb{D}_R \setminus \{t = 0\} = \mathrm{Spec}(R(\!(t)\!))$, which we picture as an R-family of discs, resp. an R-family of punctured discs.

Let X be a scheme over K. We consider the functor LX from the category of k-algebras to that of sets given by

$$R \mapsto LX(R) := X(R(\!(t)\!)).$$

If $\mathcal{X}$ is a scheme over $\mathcal{O}_K$, we denote by $L^+\mathcal{X}$ the functor from the category of k-algebras to that of sets given by

$$R \mapsto L^+\mathcal{X}(R) := \mathcal{X}(R[\![t]\!]).$$

The functors $LX, L^+\mathcal{X}$ give sheaves of sets for the fpqc topology on k-algebras. In what follows, we will call such functors "k-spaces" for simplicity.

Definition 4.1.1. *A ind-scheme is a functor* $Y : \mathrm{AffSch}^{\mathrm{op}} \to \mathrm{Sets}$ *from the category of affine schemes which admits a presentation* $Y \simeq \mathrm{colim}_{i\in I} Y_i$ *as a filtered colimit of schemes. The ind-scheme is strict if all transition maps* $Y_i \to Y_j, i \leq j$*, are closed immersions.*

If $\mathcal{X} = \mathbf{A}^r_{\mathcal{O}_K}$ is the affine space of dimension r over $\mathcal{O}_K$, then $L^+\mathcal{X}$ is the infinite dimensional affine space $L^+\mathcal{X} = \prod_{i=0}^{\infty} \mathbf{A}^r$, via:

$$L^+\mathcal{X}(R) = \mathrm{Hom}_{k[\![t]\!]}(k[\![t]\!][T_1, ..., T_r], R[\![t]\!]) = R[\![t]\!]^r = \prod_{i=0}^{\infty} R^r = \prod_{i=0}^{\infty} \mathbf{A}^r(R).$$

Let $\mathcal{X}$ be the closed subscheme of $\mathbf{A}^r_{\mathcal{O}_K}$ defined by the vanishing of polynomials $f_1, ..., f_n$ in $k[\![t]\!][T_1, ..., T_r]$. Then $L^+\mathcal{X}(R)$ is the subset of $L^+\mathbf{A}^r(R)$ of $k[\![t]\!]$-algebra homomorphisms $k[\![t]\!][T_1, ..., T_r] \to R[\![t]\!]$ which factor through $k[\![t]\!][T_1, ..., T_r]/(f_1, ..., f_n)$. If X is an affine K-scheme, LX is represented by a strict ind-scheme.

Definition 4.1.2. *Let G be a linear algebraic group over K. The loop group associated to G is the ind-scheme LG over* $\mathrm{Spec}(k)$*.*

We list some properties of loop groups:

(1) $L(X \times_k Y) = LX \times_k LY$;

(2) If k' is a k -field extension, then we have an isomorphism of ind-schemes over k'

$$LG \times_k \mathrm{Spec}(k') \simeq L(G \times_{k(\!(t)\!)} \mathrm{Spec}(k'(\!(t)\!)));$$

(3) Assume that K'/K is a finite extension of K, where $K' = k(\!(u)\!)$. If $G = \mathrm{Res}_{K'/K} H$ for some linear algebraic group H over K', then we have an isomorphism of ind-schemes

over k:

$$LG \simeq LH,$$

by

$$(LG)(R) = G(R(\!(t)\!)) = H(R(\!(t)\!) \otimes_{k(\!(t)\!)} k(\!(u)\!)) = H((R(\!(u)\!)) = LH(R).$$

4.2 Affine Grassmannians

Now let G be a flat affine group scheme of finite type over $k[\![t]\!]$. Let G_η denote the generic fiber of G, which is a group scheme over $k(\!(t)\!)$. We consider the quotient sheaf over $\mathrm{Spec}(k)$:

$$\mathcal{F}_G := LG_\eta / L^+G.$$

This is the fpqc sheaf associated to the presheaf which to a k -algebra R associates the quotient $G(R(\!(t)\!))/G(R[\![t]\!])$. Generally, we define affine Grassmannians as follows:

Definition 4.2.1. *Let G be an affine group scheme over $k[\![t]\!]$. The affine Grassmannian for G is the functor $\mathrm{Gr}_G : \mathrm{Alg}_k \to \mathrm{Sets}$ which associates to a k-algebra R the isomorphism classes of pairs $(\mathcal{E}, \alpha)$ where $\mathcal{E} \to \mathbb{D}_R$ is a left fppf G-torsor and $\alpha \in \mathcal{E}(\mathbb{D}_R^*)$ is a section.*

Here a pair $(\mathcal{E}, \alpha)$ is isomorphic to $(\mathcal{E}', \alpha')$ if there exists a morphism of G -torsors $\pi : \mathcal{E} \to \mathcal{E}'$ such that $\pi \circ \alpha = \alpha'$. The datum of a section $\alpha \in \mathcal{E}(\mathbb{D}_R^*)$ is equivalent to the datum of an isomorphism of G -torsors

$$\mathcal{E}_0|_{\mathbb{D}_R^*} \xrightarrow{\simeq} \mathcal{E}|_{\mathbb{D}_R^*}, \quad g \mapsto g \cdot \alpha,$$

where $\mathcal{E}_0 := G$ is viewed as the trivial G -torsor. The loop group LG acts on the affine

Grassmannian via $g \cdot [(\mathcal{E}, \alpha)] = [(\mathcal{E}, g\alpha)]$.

Proposition 4.2.2. *If $G \to \mathrm{Spec}(k[\![t]\!])$ is a smooth affine group scheme, then the map $LG \to \mathrm{Gr}_G$ given by $g \mapsto [(\mathcal{E}_0, g)]$ induces an isomorphism of fpqc quotients:*

$$\mathcal{F}_G \cong \mathrm{Gr}_G.$$

Proof. See Proposition 1.3.6, [33]. □

Here are a few observations:

(1) If $\rho : G \to H$ is a map of group schemes which are flat of finite presentation over $k[\![t]\!]$, then there is a map of functors:

$$\mathrm{Gr}_G \to \mathrm{Gr}_H, \quad (\mathcal{E}, \alpha) \mapsto (\rho_* \mathcal{E}, \rho_* \alpha),$$

where $\rho_* \mathcal{E} = H \times^G \mathcal{E}$ denotes the push out of torsors, and $\rho_* \alpha = (id, \alpha) : (H \times^G \mathcal{E}_0)|_{\mathbb{D}_R^*} \to (H \times^G \mathcal{E})|_{\mathbb{D}_R^*}$ in this description.

(2) If k' is a k-field extension, then we have:

$$\mathrm{Gr}_G \times_k \mathrm{Spec}(k') \simeq \mathrm{Gr}_{G \times_{k[\![t]\!]} \mathrm{Spec}(k'[\![t]\!])}.$$

When $G = \mathrm{GL}_n$, a G -bundle on $\mathcal{E} \to \mathbb{D}_R$ is canonically given by a rank n vector bundle, i.e., a rank n locally free $R[\![t]\!]$-module L. The trivialization α induces an isomorphism of $R(\!(t)\!)$-modules $L[t^{-1}] \simeq R(\!(t)\!)^n$. By taking the image of $L \subset L[\![t]\!]$ under this isomorphism, we obtain a well defined finite locally free $R[\![t]\!]$ -module $\Lambda = \Lambda_{(\mathcal{E},\alpha)} \subset R(\!(t)\!)^n$ such that $\Lambda[t^{-1}] = R(\!(t)\!)^n$. Note that Λ depends only on the class of $(\mathcal{E}, \alpha)$.

Chapter 5

Affine Grassmannians for triality groups

5.1 Galois cohomology

Let us recall some basic definitions in the Galois cohomology theory, since we will use them later. Our main reference is [28].

A topological group which is the projective limit of finite group, each given the discrete topology, is called a profinite group. Such a group is compact and totally disconnected. Conversely, a compact totally disconnected topological group is profinite. For example, let L/F be a Galois extension of fields. The Galois group $\mathrm{Gal}(L/F)$ of this extension is the projective limit of the Galois groups $\mathrm{Gal}(L_i/F)$ of the finite Galois extensions L_i/F which are contained in L/F. Thus, $\mathrm{Gal}(L/F)$ is a profinite group.

Let Γ be a profinite group. A Γ-group A is a discrete group on which Γ acts continuously, with a group structure invariant under Γ, i.e., ${}^s(xy) = {}^sx{}^sy$ for any $s \in \Gamma$. A homomorphism $A \to A'$ is a group homomorphism which commutes with the action of Γ.

We put $H^0(\Gamma, A) = A^\Gamma$, the set of elements of A fixed under Γ, and we call 1-cocycle of Γ in A a map $\alpha : \Gamma \to A$ given by $s \mapsto \alpha_s$, which is continuous and satisfies $\alpha_{st} = \alpha_s{}^s\alpha_t$ for all $s, t \in \Gamma$. The set of these cocycles will be denoted $Z^1(\Gamma, A)$. Two cocycles α and α' are said to be cohomologous if there exists $b \in A$ such that $\alpha'_s = b^{-1}\alpha_s {}^s b$. Denoted by $\alpha'_s \sim \alpha_s$. It is easy to see that this is an equivalence relation in $Z^1(\Gamma, A)$, and the quotient set is denoted by $H^1(\Gamma, A)$. This is the first cohomology set of Γ in A. We can check that

$$H^1(\Gamma, A) = \varinjlim H^1(\Gamma/U, A^U)$$

for U running over the set of open normal subgroups of Γ (see §5,[28]).

Let A and B be two Γ-groups, and let $f : A \to B$ be a Γ-homomorphism. If $a \in A$ is fixed by Γ, then so is $f(a) \in B$. Therefore, f restrict to a map:

$$f^0 : H^0(\Gamma, A) \to H^0(\Gamma, B).$$

Moreover, there is an induced map:

$$f^1 : H^1(\Gamma, A) \to H^1(\Gamma, B),$$

which carries the cohomology class of any 1-cocycle α to the cohomology class of the 1-cocycle $f^1(\alpha)$ defined by $f^1(\alpha)_s := f(\alpha_s)$. Set $\alpha'_s = b^{-1}\alpha_s {}^s b$ for some $b \in A$. We have $f(\alpha_s) = f(b)^{-1} f(\alpha_s)^s f(b)$, so it is well defined.

Let B be a Γ-group. We call $A \subset B$ a Γ-subgroup if A is a subgroup of B and ${}^s a \in A$ for all $s \in \Gamma, a \in A$. Let B/A be the Γ-set of left cosets of A in B, i.e., $B/A = \{b \cdot A \mid b \in B\}$. The natural projection of B onto B/A induces a map of pointed sets $B^\Gamma \to (B/A)^\Gamma$. Let $b \cdot A \in (B/A)^\Gamma$. We have $b \cdot A = {}^s b \cdot A$, i.e., $b^{-1} \cdot {}^s b \in A$ for any $s \in \Gamma$. Define a map $\alpha : \Gamma \to A$ given by $\alpha_s := b^{-1} \cdot {}^s b$. This is a 1-cocycle with values in A, whose class $[\alpha] \in H^1(\Gamma, A)$ is independent of choice of b in $b \cdot A$. Hence we have a map of pointed sets:

$$\delta : (B/A)^{\Gamma} \to H^1(\Gamma, A), \quad b \cdot A \mapsto [\alpha], \quad \text{where} \quad \alpha_s = b^{-1} \cdot {}^s b.$$

By definition, the kernel $\ker(g)$ of a map of pointed sets $g : B \to C$ is the subset of all $b \in B$ such that $g(b)$ is the base point of C. A sequence of maps of pointed sets $A \xrightarrow{f} B \xrightarrow{g} C$ is exact if $\mathrm{im}(f) = \ker(g)$. From that, the sequence $A \xrightarrow{f} B \to 1$ is exact if and only if f is surjective. The sequence $1 \to B \xrightarrow{g} C$ is exact if and only if the base point of B is the only element in B mapped to the base point of C. Note that this condition does not imply that g is injective.

Proposition 5.1.1. *If A is normal in B, and set $C = B/A$, then the sequence of pointed sets:*

$$1 \to H^0(\Gamma, A) \to H^0(\Gamma, B) \to H^0(\Gamma, C) \to H^1(\Gamma, A) \to H^1(\Gamma, B) \to H^1(\Gamma, C)$$

is exact.

Proof. See Proposition 38, [28]. □

Now we consider Galois groups. Let G be a finite Galois group. Let $H \subset G$ be a subgroup and A a commutative G-group. The action of G restricts to a continuous action of H. The obvious inclusion $A^G \subset A^H$ is called restriction:

$$\mathrm{Res} : H^0(G, A) \to H^0(H, A).$$

Moreover, the restriction of a 1-cocycle $\alpha \in Z^1(G, A)$ to H is a 1-cocycle of H with values in A. Thus, there is a restriction map:

$$\text{Res} : H^1(G, A) \to H^1(H, A).$$

Conversely, we have the corestriction map defined by the norm:

$$N_{G/H} : a \mapsto \sum_{s \in G/H} {}^s a,$$

form A^H to A^G. We can extend this corestriction in $\text{Cor} = N_{G/H} : H^0(H, A) \to H^0(G, A)$ to a unique map $\text{Cor} : H^1(H, A) \to H^1(G, A)$. Since the cohomological functor $H^i(H, -)$ is effaceable in degree $\geqslant 1$ (see §1.6, [10]).

Proposition 5.1.2. *Let $m = [G : H]$ be the index of H in G. Then the composite* $\text{Cor} \circ \text{Res}$ *is the multiplication by m in $H^1(G, A)$.*

Proof. See Theorem 1.48, [10]. □

Let $H = \{1\}$. We have $|G| \cdot H^1(G, A) = 0$ for G a finite Galois group and A a commutative G-group from Proposition 5.1.2, i.e., $H^1(G, A)$ is $|G|$-torsion. In particular, if A is n -torsion with n prime to $|G|$, we obtain $H^1(G, A) = 0$.

Proposition 5.1.3. *If U is a connected unipotent G-group over F, where char(F) prime to $|G|$. Then we have*

$$H^1(G, U) = 0$$

Proof. We prove this by induction. In a connected unipotent group U, there is a sequence of normal subgroups

$$U = U_1 \supset U_2 \supset \ldots \supset U_n = \{e\},$$

such that all quotients U_i/U_{i+1} are one-dimensional. Every connected one-dimensional unipotent algebraic group is isomorphic to $\mathbb{G}_a$. Since $\mathbb{G}_a$ over F and $\mathrm{char}(F)$ prime to $|G|$, we get $H^1(G, \mathbb{G}_a) = 0$. By Proposition 5.1.1, we have an exact sequence:

$$1 \to H^0(G, U_2) \to H^0(G, U_1) \to H^0(G, U_1/U_2) \to H^1(G, U_2) \to H^1(G, U_1) \to H^1(G, U_1/U_2).$$

Since $H^1(G, U_2) = 0$ by induction and $H^1(G, U_1/U_2) \cong H^1(G, \mathbb{G}_a) = 0$, we have $H^1(G, U_1) = 0$. $\square$

5.2 Statement of the main theorem

In this section we will give a explicitly description of the affine Grassmannian for triality groups. In what follows, let k be a field with $\mathrm{char}(k) \neq 2, 3$. Suppose that the cubic primitive root ξ is in k. We set $F = k(\!(u)\!), F_0 = k(\!(t)\!)$ with $u^3 = t$. Thus $F/F_0 = k(\!(u)\!)/k(\!(t)\!)$ is a cubic Galois field extension. Set $\Gamma = \mathrm{Gal}(F/F_0)$ with generator ρ with $\rho(u) = \xi u$. Then $k[\![t]\!]$ (resp. $k[\![u]\!]$) is the ring of integers of F_0 (resp. F).

Recall that $(V, *)$ is a normal twisted composition algebra obtained from the para-Cayley algebra over F, i.e., there is a basis $\{e_1, ..., e_8\}$ of $(V, *)$ in the Table 2.1, with the multiplication

$$x * y = (id_C \otimes \rho)(x) \star (id_C \otimes \theta)(y) \quad \text{for all } x, y \in C \otimes_{F_0} F,$$

where $(C, \star)$ is the split para-Cayley algebra. The quadratic form of $(V, *)$ is determined by the multiplication by Lemma 2.3.3. Denote by $\langle\ ,\ \rangle$ the bilinear form: $\langle\ ,\ \rangle : V \otimes V \to F$ corresponding to the quadratic form. Let R be an F_0-algebra. Notice that the base change $V \otimes_{F_0} R$ is isomorphic to $R(\!(u)\!)^8$. A finitely generated projective submodule in $V \otimes_{F_0} R$

is called a lattice in $V \otimes_{F_0} R$. We set $\mathbb{L} = \oplus_{i=1}^{8} R[\![u]\!]e_i$, and call it the standard lattice in $V \otimes_{F_0} R$.

In §3.3, we defined the triality group for the special orthogonal group over F_0:

$$\begin{aligned} G(R) &= \mathrm{Res}_{F/F_0}(\mathbf{Spin}(V,*))^{A_3}(R) \\ &\cong \{g \in \mathbf{SO}(V,q)(R \otimes_{F_0} F) \mid g(x * y) = g(x) * g(y) \text{ for all } x, y \in V \otimes_{F_0} R\}, \end{aligned}$$

for any F_0-algebra R. With $F = k(\!(u)\!), F_0 = k(\!(t)\!)$, we can rewrite this triality group. Let $\mathscr{G}$ be the affine group scheme over $k[\![t]\!]$ that represents the functor from $k[\![t]\!]$-algebras to groups that sends R to

$$\mathscr{G}(R) := \{g \in \mathbf{SO}_8(k[\![u]\!] \otimes_{k[\![t]\!]} R) \mid g(x * y) = g(x) * g(y) \text{ for all } x, y \in \mathbb{L}\}.$$

We will prove that this affine group scheme is smooth in the next section. In fact, this affine group $\mathscr{G}$ is the parahoric subgroup of G given by $\mathbb{L}$ by Proposition 1.3.9, [12]. The generic fiber $\mathscr{G}_\eta$ is equal to G. We denote by $L\mathscr{G}_\eta$ (resp. $L^+\mathscr{G}$) the functor from the category of k-algebras to groups given by $L\mathscr{G}_\eta(R) = \mathscr{G}_\eta(R(\!(t)\!))$ (resp. $L^+\mathscr{G}(R) = \mathscr{G}(R[\![t]\!])$). Then the quotient fpqc sheaf $L\mathscr{G}_\eta/L^+\mathscr{G}$ is by definition the affine Grassmannian for the triality group $\mathscr{G}$. Our main theorem in this section is:

Theorem 5.2.1. *There is an $L\mathscr{G}_\eta$-equivariant isomorphism*

$$L\mathscr{G}_\eta/L^+\mathscr{G} \simeq \mathscr{F}$$

where the functor $\mathscr{F}$ sends a k-algebra R to the set of finitely generated projective $R[\![u]\!]$-

modules L (i.e., $R[\![u]\!]$-lattices) of $V \otimes_k R \cong R(\!(u)\!)^8$, such that

(1) L is self dual under the bilinear form $\langle\ ,\ \rangle$, i.e., $L \simeq \mathrm{Hom}_{R[\![u]\!]}(L, R[\![u]\!])$.

(2) L is closed under multiplication, $L * L \subset L$.

(3) There exists $a \in L$, such that $q(a) = 0$, $\langle a * a, a\rangle = 1$.

(4) For a as in (3), let $e = a + a * a$. Then, we have $\overline{e * x} = -\bar{x} = \overline{x * e}$ for any $\bar{x}$ satisfying $\langle \bar{x}, \bar{e}\rangle = 0$. (Here, $\bar{x}$ is the image of x under the canonical map $L \to L/uL$.)

Proof. To prove the theorem it suffices to check the following two statements:

(i) For any R, $g \in L\mathscr{G}_\eta(R)$, $L = g(\mathbb{L})$ satisfies condition (1)-(4).

(ii) For any $L \in \mathscr{F}(R)$ with (R, m) a local henselian ring with the maximal ideal m, there exists $g \in L\mathscr{G}_\eta(R)$ such that $L = g(\mathbb{L})$.

Part (i) is easy to prove, since g preserves the bilinear form $\langle\ ,\ \rangle$ and the product $*$. For any $x, y \in L$, let $x = g(x_0), y = g(y_0)$ where $x_0, y_0 \in \mathbb{L}$. Then $x * y = g(x_0) * g(y_0) = g(x_0 * y_0) \in L$, so (2) satisfied. (1) is obvious via $\langle g(x), g(y)\rangle = \langle x, y\rangle$. For (3), let $a = g(e_4)$. Then $\langle a * a, a\rangle = \langle g(e_4) * g(e_4), g(e_4)\rangle = \langle g(e_5), g(e_4)\rangle = \langle e_4, e_5\rangle = 1$, and $q(a) = q(e_4) = 0$. For $g(e) = g(a) + g(a * a)$, we have $g(e) * g(x) + g(x) = g(x) * g(e) + g(x) = 0$ for any $g(x)$ satisfying $\langle g(x), g(e)\rangle = \langle x, e\rangle = 0$.

To prove part (ii), the key is to find a basis in L such that the multiplication table under the basis is the same as Table 2.1, i.e., we need to find a basis $\{f_i\} \in L$ such that $f_i * f_j = f_k$ for $e_i * e_j = e_k$ in the Table 2.1. Thus we can define $g(e_i) = f_i$, and g is then in $L\mathscr{G}_\eta(R)$.

We claim that a as in assumption (3) is a primitive element in L (an element in L that extends to a basis of L). Consider the quotient map

$$R[\![u]\!] \to R \to R/m = \kappa,$$

where κ is the residue field of R. There is a base change $L \to L \otimes_{R[\![u]\!]} \kappa$, and we still denote by $\bar{x}$ the image of $x \in L$. Consider $\bar{a} \in L \otimes_{R[\![u]\!]} \kappa$. We have $\langle \overline{a * a}, \bar{a} \rangle = 1$, hence $\bar{a} \neq 0$. By Nakayama's lemma, we can extend a to a basis of L. Similarly, we can show that $a * a$ is also a primitive element. Here $a, a * a$ are independent by $\langle a, a * a \rangle = 1$. Let $v_1, ..., v_6$ be any base extension for $a, a * a$. We define a sublattice $L_0 \subset L$:

$$L_0 := \{x \in L \mid \langle x, a \rangle = 0, \langle x, a * a \rangle = 0\}.$$

For any $x \in L$, we can write x as $\sum_{i=1}^{6} r_i v_i + r_7 a + r_8(a * a)$ for some $r_i \in R[\![u]\!]$. Consider $v_i' = v_i - \langle a, v_i \rangle a * a - \langle a * a, v_i \rangle a$. It is easy to see that $\langle v_i', a \rangle = 0, \langle v_i', a * a \rangle = 0$, so $v_i' \in L_0$. And $v_i', a, a * a$ are linear independent. We obtain

$$x = \sum_{i=1}^{6} r_i v_i' + (r_7 + \sum_{i=1}^{6} r_i \langle v_i, a * a \rangle) a + (r_8 + \sum_{i=1}^{6} r_i \langle v_i, a \rangle)(a * a).$$

Therefore, $L = R[\![u]\!]a \oplus R[\![u]\!](a * a) \oplus L_0$, where L_0 is a sublattice of rank 6.

Set $f_1 = a, f_2 = a * a$. Here f_1, f_2 play similar roles as for e_4, e_5 in the Table 2.1. By Lemma 2.3.2 and Lemma 2.3.3, we obtain a hyperbolic subspace $R[\![u]\!]a \oplus R[\![u]\!](a * a)$ with:

$$f_1 * f_1 = f_2, \quad f_2 * f_2 = f_1,$$

$$f_1 * f_2 = f_2 * f_1 = 0,$$

$$q(f_1) = q(f_2) = 0, \quad \langle f_1, f_2 \rangle = 1.$$

Lemma 5.2.2. *We have*

$$L_0 * f_i \subset L_0, \quad f_i * L_0 \subset L_0,$$

for $i = 1, 2$.

Proof. For any $x \in L_0$, we have $\langle x * f_i, f_i \rangle = \rho(\langle f_i * f_i, x \rangle) = 0$, and $\langle x * f_i, f_{i+1} \rangle = \rho(\langle f_i * f_{i+1}, x \rangle) = 0$ by Lemma 2.3.3. Similarly for $f_i * x$. $\square$

Define the ρ-linear transformations $t_i : L_0 \to L_0$, given by $t_i(x) = x * f_i$ for $i = 1, 2$. Here the ρ-linear transformation means $t_i(rx) = \rho(r)t_i(x)$ for $r \in R[\![u]\!], x \in L_0$. Take $L_i = t_i(L_0) = L_0 * f_i$. Trivially, $t_i(L_i) \subset L_i$. Both L_i are isotropic with rank $(L_i) \leq 3$ since f_i is an isotropic element. For any $x \in L_0$, we have

$$(f_2 * x) * f_1 + (f_1 * x) * f_2 = \theta(\langle f_1, f_2 \rangle)x = x,$$

by Lemma 2.3.3. So $L_0 = L_1 + L_2$. Since $\text{rank}(L_i) \leq 3$, we must have a direct sum composition: $L_0 = L_1 \oplus L_2$.

Lemma 5.2.3.

(1) For any $x \in L_0$, $t_i^2(x) = -f_{i+1} * x$ $(i = 1, 2 \mod 2)$.

(2) For any $x \in L_i$, $t_i^3(x) = -x$.

(3) From (2), t_i *is a* $R[\![t]\!]$*-isomorphism when restricted at* L_i*, more precisely, we have* $t_i : L_i \to L_i,\ x \mapsto x * f_i$*. The inverse map* $t_i^{-1} = -t_i^2$ *is a* θ*-linear transformation.*

(4) For $x \in L_1, y \in L_2$*, we have* $\langle t_1(x), t_2(y) \rangle = \rho(\langle x, y \rangle)$.

Proof. (1) For $x \in L_0$, we have $t_1^2(x) = ((x * f_1) * f_1) = -((f_1 * f_1) * x) = -(f_2 * x)$ by Lemma 2.3.3. A similar argument gives $t_2^2(x) = -f_1 * x$.

(2) For any $x \in L_1$, we have $t_1^3(x) = -((f_2 * x) * f_1)$. Consider

$$(f_2 * x) * f_1 + (f_1 * x) * f_2 = \theta(\langle f_1, f_2 \rangle)x = x,$$

by Lemma 2.3.3. Let $x = z * f_1 \in L_1$ for some $z \in L_0$. Then $f_1 * x = f_1 * (z * f_1) = 0$ by $q(f_1) = 0$. Hence $(f_2 * x) * f_1 = x$, and we obtain $t_1^3(x) = -x$. Similar calculations for $y \in L_2$, and gives $t_2^3(y) = -y$.

Part (3) follows from (2) immediately. For (4), we know that $\langle t_1(x), t_2(y) \rangle = \langle x * f_1, y * f_2 \rangle = \rho(\langle f_1 * (y * f_2), x \rangle)$, and

$$f_1 * (y * f_2) = -t_2^2(y * f_2) = -t_2^2 \cdot t_2(y) = -t_2^3(y) = y,$$

by (1) and (2). Hence $\langle t_1(x), t_2(y) \rangle = \rho(\langle x, y \rangle)$. □

Remark 5.2.4. (1) From the proof of above Lemma, we can see that $f_i * L_i = 0$, and $L_i * f_{i+1} = 0$ for $i = 1, 2 \mod 2$.

(2) Since L_1, L_2 are isotropic and $\langle\ ,\ \rangle$ restricted to L_0 is nondegenerate, the L_i are in duality by the isomorphism $L_1 \to L_2^\vee$ given by $x \mapsto \langle x, - \rangle$. Hence $L_1 \simeq \mathrm{Hom}(L_2, R[\![u]\!])$.

Lemma 5.2.5. *We have*

(1) $L_1 * L_2 \subset R[\![u]\!] f_1, \quad L_2 * L_1 \subset R[\![u]\!] f_2,$

(2) $L_i * L_i \subset L_{i+1}$ $(i = 1, 2 \mod 2)$.

Proof. (1) For any $x \in L_1, y \in L_2$, we write x as $x = x_1 * f_1$ with $x_1 \in L_1$, and y as

$y = y_1 * f_2$ with $y_1 \in L_2$. Consider

$$x * y = (x_1 * f_1) * (y_1 * f_2) = -((y_1 * f_2) * f_1) * x_1 + \theta(\langle x_1, y_1 * f_2\rangle) f_1,$$

by Lemma 2.3.3. Notice that $(y_1 * f_2) * f_1 \in L_2 * f_1 = 0$. Thus we have $x * y = \theta(\langle x_1, y_1 * f_2\rangle) f_1$. Further,

$$\begin{aligned}\langle x_1, y_1 * f_2\rangle &= \theta(\langle t_1(x_1), t_2(y_2 * f_2)\rangle) \\ &= \theta(\langle x, t_2(y)\rangle),\end{aligned}$$

by Lemma 5.2.3 (4). Hence $x * y = \rho(\langle x, t_2(y)\rangle) f_1$. Similarly, we have $y * x = \rho(\langle t_1(x), y\rangle) f_2$.

(2) For any $x_1, x_2 \in L_1$, we first claim that $x_1 * x_2 \in L_0$. Consider $\langle x_1 * x_2, f_1\rangle = \theta(\langle f_1 * x_1, x_2\rangle) = 0$ by $f_1 * L_1 = 0$, and $\langle x_1 * x_2, f_2\rangle = \rho(\langle x_2 * f_2, x_1\rangle) = 0$ by $L_1 * f_2 = 0$. Using Lemma 2.3.3, we find that

$$\begin{aligned}t_1(x_1) * t_1(x_2) &= (x_1 * f_1) * (x_2 * f_1) \\ &= -f_1 * (x_2 * (x_1 * f_1)) \\ &= f_1 * (f_1 * (x_1 * x_2))\end{aligned}$$

by $\langle x_1 * f_1, f_1\rangle = 0$ and $\langle f_1, x_2\rangle = 0$. We also have $f_1 * (f_1 * (x_1 * x_2)) = f_1 * (-t_2^2(x_1 * x_2)) = t_2^4(x_1 * x_2) = -t_2(x_1 * x_2)$. Therefore,

$$t_1(x_1) * t_1(x_2) = -t_2(x_1 * x_2).$$

Since $x_1 * x_2 \in L_0$, we obtain that $t_2(x_1 * x_2) \in L_2$. Hence $L_1 * L_1 \subset L_2$. Similarly,

$L_2 * L_2 \subset L_1$. $\square$

We now prove that L has the same multiplication table as the Table 2.1: We want to find a basis $\{x_1, x_2, x_3\}$ for L_1 (resp. $\{y_1, y_2, y_3\}$ for L_2) such that $t_1(x_i) = -id$ (resp. $t_2(y_i) = -id$). Consider the quotient map $R[\![u]\!] \to \kappa = R[\![u]\!]/(m, u)$. We set $\bar{L} = L \otimes_{R[\![u]\!]} \kappa$, $\bar{L}_i = L_i \otimes_{R[\![u]\!]} \kappa$ with multiplication $\bar{x} \star \bar{y} = \overline{x * y}$, and

$$\bar{t}_i : \bar{L}_i \to \bar{L}_i, \quad \text{given by} \quad \bar{t}_i(\bar{x}) = \bar{x} \star \bar{f}_i,$$

for $i = 0, 1, 2$.

Proposition 5.2.6. *Given $(L, *, \langle\ ,\ \rangle)$ satisfying (1)-(4) as above. Then $(\bar{L}, \star)$ is isomorphic to the split para-Cayley algebra.*

Proof. It is easy to see that $q(\bar{x} \star \bar{y}) = q(\bar{x})q(\bar{y})$, and $\langle \bar{x} \star \bar{y}, \bar{z} \rangle = \langle \bar{y} \star \bar{z}, \bar{x} \rangle$, so $\bar{L}$ is a symmetric composition algebra. By Proposition 2.2.4, a symmetric algebra is a para-Cayley algebra if and only if it admits a para-unit, i.e., there exist an element $\bar{e} \in \bar{L}$, such that

$$\bar{e} \star \bar{e} = \bar{e}, \quad \bar{e} \star \bar{x} = \bar{x} \star \bar{e} = -\bar{x},$$

for all $\bar{x} \in \bar{L}$ satisfying $\langle \bar{e}, \bar{x} \rangle = 0$. Set $e = f_1 + f_2$ in our case. We can see that e is an idempotent element by $e \star e = (f_1 + f_2) * (f_1 + f_2) = f_1 + f_2 = e$. By condition (4), we get $\bar{e} \star \bar{x} = \bar{x} \star \bar{e} = -\bar{x}$, for all $\bar{x} \in \bar{L}$ satisfying $\langle \bar{e}, \bar{x} \rangle = 0$. Thus $\bar{e}$ is a para-unit in $\bar{L}$, and $\bar{L}$ is a para-Cayley algebra. It is split since q is an isotropic norm. $\square$

Lemma 5.2.7. *For $\bar{t}_i : \bar{L}_i \to \bar{L}_i$, we have $\bar{t}_i(\bar{x}) = \bar{x} \star \bar{f}_i = -\bar{x}$ for any $\bar{x} \in \bar{L}_i$. Then $\bar{L}_i = \bar{L}_0 \star \bar{f}_i = \{\bar{x} \in \bar{L}_0 \mid \bar{x} \star \bar{f}_i = -\bar{x}\}$, $i = 1, 2$.*

Proof. By Lemma 34.8, [14], we can define $\bar{x} \diamond \bar{y} = (\bar{e} \star \bar{x}) \star (\bar{y} \star \bar{e})$ as a unital composition algebra with the identity element $\bar{e}$. We have $\bar{x} \star \bar{y} = r(\bar{x}) \diamond r(\bar{y})$, where $r(\bar{x}) = \langle \bar{e}, \bar{x} \rangle \bar{e} - \bar{x}$ is the conjugation of $\bar{x}$. By Proposition 2.1.1,

$$\bar{x} \diamond \bar{y} + \bar{y} \diamond \bar{x} - \langle \bar{x}, \bar{e} \rangle \bar{y} - \langle \bar{y}, \bar{e} \rangle \bar{x} + \langle \bar{x}, \bar{y} \rangle \bar{e} = 0.$$

Using $\bar{x} \star \bar{y} = r(\bar{x}) \diamond r(\bar{y})$ and $\langle r(\bar{x}), r(\bar{y}) \rangle = \langle \bar{x}, \bar{y} \rangle$, we obtain

$$\bar{x} \star \bar{y} + \bar{y} \star \bar{x} = \langle \bar{e}, \bar{x} \rangle r(\bar{y}) + \langle \bar{e}, \bar{y} \rangle r(\bar{x}) - \langle \bar{x}, \bar{y} \rangle \bar{e}.$$

Let $\bar{y} = \bar{f}_i$. We get $\bar{x} \star \bar{f}_i + \bar{f}_i \star \bar{x} = r(\bar{x})$. Therefore, if $\bar{x} \in \bar{L}_0 * \bar{f}_i$, we have $\bar{f}_i \star \bar{x} = 0$ by $q(\bar{f}_i) = 0$, and

$$\bar{x} \star \bar{f}_i = \bar{x} \star \bar{f}_i + \bar{f}_i \star \bar{x} = \langle \bar{e}, \bar{x} \rangle \bar{e} - \bar{x} = -\bar{x}.$$

This implies $\bar{L}_0 \star \bar{f}_i \subset \{\bar{x} \in \bar{L}_0 \mid \bar{x} \star \bar{f}_i = -\bar{x}\}$. It is obvious that $\{\bar{x} \in \bar{L}_0 \mid \bar{x} \star \bar{f}_i = -\bar{x}\} \subset \bar{L}_0 \star \bar{f}_i$. Hence we get

$$\bar{L}_i = \bar{L}_0 \star \bar{f}_i = \{\bar{x} \in \bar{L}_0 \mid \bar{x} \star \bar{f}_i = -\bar{x}\},$$

and $\bar{t}_i = -id$. $\square$

So far we know $t_i : L_i \to L_i$ is a ρ-linear isomorphism with $t_i^3 = -id$, and $\bar{t}_i = -id$. We will use non-abelian Galois cohomology to prove that t_i and $-id$ are the same up to ρ-conjugacy. More precisely, if we fix a basis for $L_i \cong R[\![u]\!]^3$ and let $A_i \in \mathrm{GL}_3(R[\![u]\!])$ represent t_i, we can find a new basis for L_i with transition matrix $b \in \mathrm{GL}_3(R[\![u]\!])$, such that

$$-I = b^{-1} A_i \rho(b).$$

Let $\Gamma = \{1, \rho, \theta\}$ be the cyclic group. Set $B = \mathrm{Aut}(L_1) = \mathrm{GL}_3(R[\![u]\!])$. Consider the quotient map $R[\![u]\!] \to \kappa$. Since $(R[\![u]\!], (u)), (R, m)$ are henselian pairs, we obtain the exact sequence:

$$1 \to U \to \mathrm{GL}_3(R[\![u]\!]) \to \mathrm{GL}_3(\kappa) \to 1$$

where U is the kernel of $\mathrm{GL}_3(R[\![u]\!]) \to \mathrm{GL}_3(\kappa)$. Here Γ acts on $\mathrm{GL}_3(R[\![u]\!])$ by $\rho(u) = u\xi$, and Γ acts trivially on $\mathrm{GL}_3(\kappa)$. We obtain the exact sequence of pointed sets:

$$1 \to U^\Gamma \to \mathrm{GL}_3(R[\![u]\!])^\Gamma \to \mathrm{GL}_3(\kappa)^\Gamma \to H^1(\Gamma, U) \to H^1(\Gamma, \mathrm{GL}_3(R[\![u]\!]) \to H^1(\Gamma, \mathrm{GL}_3(\kappa)).$$

Since U is a unipotent group over $k[\![u]\!]$ with $\mathrm{char}(k) \neq 3$, we have $H^1(\Gamma, U) = 1$ by Proposition 5.1.2. Hence the only element mapped to the base point of $H^1(\Gamma, \mathrm{GL}_3(k))$ is the base point of $H^1(\Gamma, \mathrm{GL}_3(R[\![u]\!])$, i.e., for any $[a_s] \in H^1(\Gamma, \mathrm{GL}_3(R[\![u]\!]))$ satisfying $[\bar{a}_s] = 1$, we have $[a_s] = 1$.

Consider $t_1 : L_1 \to L_1$. The subgroup of $\mathrm{GL}_3(R[\![u]\!])$ generated by t_1 is $\{1, t_1^2, -id, -t_1, -t_1^2, id\}$ given by $t_1^3 = -id$. If we fix the basis and use A_1 to represent t_1, we get $t_1^2 = A_1\rho(A_1)$, $t_1^3 = A_1\rho(A_1)\theta(A_1) = -I$. Define a map:

$$a : \Gamma \to \mathrm{GL}_3(R[\![u]\!])$$

given by $\rho \mapsto a_\rho = -A_1$. Using $a_{st} = a_s {}^s a_t$, we get $\theta \mapsto a_\theta = a_\rho \rho(a_\rho) = A_1\rho(A_1)$, and $1 \mapsto a_1 = I$. Hence the image of $\Gamma = \{\rho, \theta, 1\}$ is the subgroup $\{t_1^4 = -t_1, t_1^8 = t_1^2, t_1^{12} = id\} \subset \langle t \rangle$. This is a 1-cocycle. Take the image $[\bar{a}]$ of $[a]$ under the injective map

$$H^1(\Gamma, \mathrm{GL}_3(R[\![u]\!]) \to H^1(\Gamma, \mathrm{GL}_3(k)).$$

We get $[\bar{a}_\rho] = -[\bar{t}] = 1$ by Lemma 5.2.7. Therefore $[a_\rho] = 1$. In matrix language, there exist $b \in \mathrm{GL}_3(R[\![u]\!])$ such that

$$I = b^{-1}(-A_1)\rho(b), \quad t_1 \sim -id.$$

We have a similar conclusion for t_2.

Using the above we see that there exist a basis $\{x_1, x_2, x_3\}$ for L_1, and a dual basis $\{y_1, y_2, y_3\}$ for L_2, such that $t_1(x_i) = -x_i$, $t_2(y_i) = -y_i$, with $\langle x_i, y_j \rangle = \delta_{ij}$. By Lemma 5.2.3, we have

$$x_i * f_1 = -x_i, \quad f_1 * x_i = 0,$$

$$x_i * f_2 = 0, \quad f_2 * x_i = -x_i,$$

$$y_i * f_1 = 0, \quad f_1 * y_i = -y_i,$$

$$y_i * f_2 = -y_i, \quad f_2 * y_i = 0.$$

By Lemma 5.2.5, we have

$$x_i * y_j = -\delta_{ij} f_1, \quad y_i * x_j = -\delta_{ij} f_2.$$

It reminds to calculate the terms in $L_i * L_i$. To approach this goal, we define a wedge product $\wedge : L_i \times L_i \to L_{i+1}$ given by

$$u \wedge v := t_i^{-1}(u) * t_i(v),$$

for any $u, v \in L_i$. Let $u \in L_1$. It is immediate to get

$$
\begin{aligned}
u \wedge u &= t_1^{-1}(u) * t_1(u) \\
&= (f_2 * u) * (u * f_1) \\
&= ((u * f_1)) * u) * f_2 \\
&= f_1 * f_2 = 0
\end{aligned}
$$

by $\langle f_2, u * f_1 \rangle = 0, q(u) = 0$. By linearizing the equation, we find $u \wedge v = -v \wedge u$ for $u, v \in L_1$. A similar argument can be made for $u, v \in L_2$. Now define a trilinear function $\langle\ ,\ ,\ \rangle$ on L_i by $\langle u, v, w \rangle := \langle u, v \wedge w \rangle$. It is an alternating trilinear function since $\langle u, w, v \rangle = \langle u, w \wedge v \rangle = -\langle u, v \wedge w \rangle = -\langle u, v, w \rangle$, and

$$
\begin{aligned}
\langle v, u, w \rangle &= \langle v, u \wedge w \rangle \\
&= \langle v, t_i^{-1}(u) * t_i(w) \rangle \\
&= \rho(\langle t_i(w) * v, t_i^{-1}(u) \rangle) \\
&= \langle t_{i+1}(t_i(w) * v), u \rangle \\
&= \langle t_i^2(w) * t_i(v), u \rangle \\
&= \langle w \wedge v, u \rangle = -\langle u, v \wedge w \rangle.
\end{aligned}
$$

We can now calculate the terms in $L_i * L_i$. Consider $x_1 * x_2$. We have $\langle x_1 * x_2, x_1 \rangle = -\langle x_1 * x_2, t_1(x_1) \rangle = -\langle x_1 * x_2, x_1 * f_1 \rangle = 0$ by $\langle x_2, f_1 \rangle = 0$. Similarly $\langle x_1 * x_2, x_2 \rangle = 0$. Hence we have $x_1 * x_2 = b y_3$ for some $b = \langle x_1 * x_2, x_3 \rangle \in R[\![u]\!]$. Multiplying by y_1 on the right side, we obtain $(x_1 * x_2) * y_1 = (b y_3) * y_1$. Since $(x_1 * x_2) * y_1 + (y_1 * x_2) * x_1 = \theta(\langle x_1, y_1 \rangle) x_2 = x_2$, and $y_1 * x_2 = 0$, we have

$$x_2 = \rho(b)(y_3 * y_1).$$

Therefore $b, \rho(b)^{-1} \in R[\![u]\!]$, which implies $b \in R[\![u]\!]^*$. Let $b = -1$, and get $x_1 * x_2 = -y_3$ (replace by_3 by $-y_3$, and also replace $b^{-1}x_3$ by $-x_3$). We can perform similar calculations for the other $x_i * x_j$ and $y_i * y_j$. By using the alternating trilinear form, we obtain

Table 5.1: Multiplication table $x_i * x_j$

$*$	x_1	x_2	x_3
x_1	0	-y_3	y_2
x_2	y_3	0	-y_1
x_3	$-y_2$	y_1	0

Table 5.2: Multiplication table $y_i * y_j$

$*$	y_1	y_2	y_3
y_1	0	$-x_3$	x_2
y_2	x_3	0	$-x_1$
y_3	$-x_2$	x_1	0

Therefore, we complete the multiplication table of L. By letting $g_(e_4) = f_1, g(e_5) = f_2$, and

$$g(e_1) = x_1, \quad g(e_6) = x_2, \quad g(e_7) = x_3,$$

$$g(e_8) = y_1, \quad g(e_3) = y_2, \quad g(e_2) = y_3.$$

We obtain $g(e_i) * g(e_j) = g(e_i * e_j)$. So, there exist $g \in L\underline{\mathscr{G}}(R)$ such that $L = g(\mathbb{L})$. $\square$

5.3 Smoothness of triality groups

In this section, we will show that the affine group scheme $\mathscr{G}(R)$ is smooth over $k[\![t]\!]$. Recall that $\mathbb{L} = \oplus_{i=1}^{8} R[\![u]\!]e_i$ is the standard lattice in $V \otimes_{F_0} R$.

Theorem 5.3.1. *The functor from* $\mathrm{Alg}_{k[\![t]\!]}$ *to the groups that send* R *to*

$$\mathscr{G}(R) = \{g \in \mathbf{SO}_8(k[\![u]\!] \otimes_{k[\![t]\!]} R) \mid g(x * y) = g(x) * g(y) \textit{ for all } x, y \in V \otimes_{F_0} R\},$$

is smooth.

We want to show that $\mathscr{G}$ is formally smooth, i.e., for any surjective ring hommorphism $S \to R$ with nilpotent kernel I, we can lift an $\alpha \in \mathscr{G}(R)$ to $\tilde{\alpha} \in \mathscr{G}(S)$. The idea is to lift basis from R-modules to the S-modules.

Proposition 5.3.2. *Let M be the S-module satisfying conditions (1)-(4). Assume $I \subset S$ is an ideal with $I^2 = 0$, then the R-module $\bar{M} = M/IM$ also satisfies condition (1)-(4). Assume that $\bar{f}_1 = \bar{a}$, $\bar{f}_2 = \bar{a} * \bar{a}$, $\bar{e} = \bar{f}_1 + \bar{f}_2 \in \bar{M}$ satisfying*

$$q(\bar{f}_1) = 0,\ \langle \bar{f}_1, \bar{f}_2 \rangle = 1,\ \bar{e} * \bar{x} + \bar{x} = 0,\ \bar{x} * \bar{e} + \bar{x} = 0,$$

then there exist $f_1, f_2 \in M$ such that $f_i \mod I$ is $\bar{f}_i$ and

$$q(f_1) = 0,\ \langle f_1, f_2 \rangle = 1,\ e * x + x \in IM, x * e + x \in IM$$

for any x (resp. $\bar{x}$) satisfying $\langle e, x \rangle = 0$ (resp. $\langle \bar{e}, \bar{x} \rangle = 0$).

Proof. Let f_1 be any liftes of $\bar{f}_1$. Then $\langle f_1, f_1 \rangle = m \in I[\![u]\!]$. Consider $f_1' = f_1 + y$, where $y \in IM$. We have $\langle f_1 + y, f_1 + y \rangle = m + 2\langle f_1, y \rangle$ by $\langle y, y \rangle \in I^2 = 0$. By perfectness of the form, any linear form on M with values in $I[\![u]\!]$ is of the type $\langle y, - \rangle$ for some $y \in IM$. Hence we can choose y such that $\langle f_1, y \rangle = -\frac{m}{2}$ to make f_1' isotropic.

Let $f_1'' = f_1' + z$ for some $z \in IM$, and $f_2'' = f_1'' * f_1'' = f_1' * f_1' + f_1' * z + z * f_1'$. Suppose that $\langle f_1', f_1' * f_1' \rangle = 1 + n$, where $n \in I[\![u]\!]$, also notice that $n \in S[\![t]\!]$, since $\langle f_1', f_1' * f_1' \rangle =$

$\rho(\langle f_1', f_1' * f_1' \rangle) = \theta(\langle f_1', f_1' * f_1' \rangle)$. Thus, we get

$$\begin{aligned}\langle f_1'', f_2'' \rangle &= 1 + n + \langle f_1' * z, f_1' \rangle + \langle z * f_1', f_1' \rangle + \langle f_1' * f_1', z \rangle \\ &= 1 + n + \theta(\langle f_1' * f_1', z \rangle) + \rho(\langle f_1' * f_1', z \rangle) + \langle f_1' * f_1', z \rangle.\end{aligned}$$

We can find $z \in IM$ such that $\langle f_1' * f_1', z \rangle = -\frac{n}{3}$. We also need that f_1'' is isotropic, i.e., $\langle f_1'', f_1'' \rangle = 2\langle f_1', z \rangle = 0$. Therefore, it is enough to find z such that $\langle f_1', z \rangle = 0$, $\langle f_1' * f_1', z \rangle = -\frac{n}{3}$. Such z exists by perfectness of the form. Then, set $f_1'' = f_1' + z$, $f_2'' = f_1'' * f_1''$. We obtain $q(f_1'') = 0$, $\langle f_1'', f_2'' \rangle = 1$. The last equation is obvious satisfied since f_1'', f_2'' are liftes of $\bar{f}_1, \bar{f}_2$. $\square$

Proof of Theorem 5.3.1: For any $S \to R$ surjective morphism with nilpotent kernel I (we can just assume $I^2 = 0$), consider the automorphism $\alpha : \mathbb{L} \to \mathbb{L}$ with $R \cong S/I$. Set $M = \mathbb{L} \otimes_R S$. Let $f_1 \in M$ satisfying

$$q(f_1) = 0, \quad \langle f_1, f_2 \rangle = 1, \quad e * x + x \in IM, \quad x * e + x \in IM.$$

By the construction in the proof of Theorem 5.2.1, we have $M \simeq S[\![u]\!]f_1 \oplus S[\![u]\!]f_2 \oplus L_1 \oplus L_2$. Then the projection map $M \to \mathbb{L}$ maps f_1 to $\bar{f}_1$ also satisfying the above condition. We have $\mathbb{L} \cong R[\![u]\!]\bar{f}_1 \oplus R[\![u]\!]\bar{f}_2 \oplus \bar{L}_1 \oplus \bar{L}_2$ where $\bar{f}_2 = \bar{f}_1 * \bar{f}_1$.

$$\begin{tikzcd} M \arrow[r, dashed, "\alpha_2"] \arrow[d] & M \arrow[d] \\ \mathbb{L} \arrow[r, "\alpha"] & \mathbb{L} \end{tikzcd}$$

Notice that we can lift $\alpha(\bar{f}_1)$ to M, denoted by h_1 by Proposition 5.3.2. So there is another construction for M: $M \simeq S[\![u]\!]h_1 \oplus S[\![u]\!]h_2 \oplus L_1' \oplus L_2'$. Then we have an automorphism

$\alpha_2 : M \to M$ satisfying $\alpha(f_1) = h_1$ by Theorem 5.2.1. It is obvious to see that $\bar{\alpha}_2 = \alpha$. Thus, we can lift an $\alpha \in \mathscr{G}(R)$ to $\alpha_2 \in \mathscr{G}(S)$. Hence $\mathscr{G}$ is formally smooth.

Chapter 6

Global affine Grassmannians

In this section, we will discuss global affine Grassmannians for triality groups. Global affine Grassmannians were introduced by Beilinson-Drinfeld in [3].

6.1 General construction

Our main reference in this section is [26]. Suppose that F_0 is either a p-adic field (i.e. a finite extension of $\mathbb{Q}_p$) or the field of Laurent power series $k(\!(t)\!)$ with k finite. In either case the residue field κ has cardinality $q = p^m$ for some m. Let $\mathcal{O}_0$ be the valuation ring of F_0. We fix a separable closure $\bar{F}_0^s$ of F_0 and denote by F_0^{unr} the maximal unramified extension of F_0 in $\bar{F}_0^s$, with the valuation ring $\mathcal{O}_0^{unr}$.

Let G be a connected reductive group over F_0. Denote by H the Chevalley group scheme over $\mathbb{Z}$ which is the split form of G. We will assume that:

Tameness hypothesis: G splits over a finite tamely ramified Galois extension F/F_0, i.e., $G \otimes_{F_0} F \cong H \otimes_{\mathbb{Z}} F$.

Let π_0 be a uniformizer of $\mathcal{O}_0$. Pappas and Zhu show that there exist a reductive group $\underline{G}$ over $\mathrm{Spec}(\mathcal{O}_0[u^{\pm}])$, which extends G in the sense that its base change

$$\underline{G} \otimes_{\mathcal{O}_0[u^{\pm}]} F_0, \quad u \mapsto \pi_0,$$

is isomorphic to G (see §3, [26]). Denote by $\tilde{F}_0$ the maximal unramified extension of F_0 that is contained in F, and by $\tilde{\mathcal{O}}_0$, $\mathcal{O}$ the valuation rings of $\tilde{F}_0, F$ respectively. Set $e = [F : \tilde{F}_0]$ and let γ_0 be a generator of $\mathrm{Gal}(F/\tilde{F}_0)$. Recall that by Steinberg's theorem, the group $G_{F_0^{unr}} := G \otimes_{F_0} F_0^{unr}$ is quasi-split. By possibly enlarging the splitting field F, we can now assume that:

(1) $G_{\tilde{F}_0}$ is quasi-split;

(2) F/F_0 is Galois with group $\Gamma = \mathrm{Gal}(F/F_0) = \langle\sigma\rangle \rtimes \langle\gamma_0\rangle$ which is the semi-direct product of $\langle\sigma\rangle \simeq \mathbb{Z}/(r)$, where σ is a lift of the (arithmetic) Frobenius $\mathrm{Frob}_q \in \mathrm{Gal}(\tilde{F}_0/F_0)$, with the normal inertia subgroup $I := \mathrm{Gal}(F/\tilde{F}_0) = \langle\gamma_0\rangle \simeq \mathbb{Z}/(e)$, with relation $\sigma\gamma_0\sigma^{-1} = \gamma_0^q$;

(3) there is a uniformizer π of F such that $\pi^e = \pi_0$.

Without further mention, we will assume that the extension F/F_0 is as above. Then we also have $\mathcal{O} = \tilde{\mathcal{O}}_0[\pi] = \tilde{\mathcal{O}}_0[x]/(x^e - \pi_0)$ and $\tilde{\mathcal{O}}_0$ contains a primitive e-th root of unity $\xi = \gamma_0(\pi)\pi^{-1}$. Consider the affine line $\mathbf{A}^1_{\mathcal{O}_0} = \mathrm{Spec}(\mathcal{O}_0[u])$ and its cover

$$\mathrm{Spec}(\tilde{\mathcal{O}}_0[v]) \to \mathrm{Spec}(\mathcal{O}_0[u]),$$

given by $u \mapsto v^e$. The Galois group Γ described as above acts on $\tilde{\mathcal{O}}_0[v]$ by

$$\sigma(\sum_i a_i v^i) = \sum_i \sigma(a_i) v^i, \quad \gamma_0(\sum_i a_i v^i) = \sum_i a_i \xi^i v^i.$$

We have $\tilde{\mathcal{O}}_0[v]^\Gamma = \mathcal{O}_0[u]$. The Restriction of this cover over the open subscheme $u \neq 0$ gives us:

$$\mathrm{Spec}(\tilde{\mathcal{O}}_0[v, v^{-1}]) \to \mathrm{Spec}(\mathcal{O}_0[u, u^{-1}]).$$

In what follows, we use $\mathcal{O}_0[u^{\pm 1}]$ (resp. $\tilde{\mathcal{O}}_0[v^{\pm 1}]$) to denote $\mathcal{O}_0[u, u^{-1}]$ (resp. $\tilde{\mathcal{O}}_0[v, v^{-1}]$) for simplicity. The indexed root datum for G gives a group homomorphism $\tau : \mathrm{Gal}(F/F_0) \to \mathrm{Aut}_{\mathcal{O}_0}(H)$. Then we define a group scheme over $\mathcal{O}_0[u^{\pm 1}]$

$$\underline{G}^* = (\mathrm{Res}_{\tilde{\mathcal{O}}_0[v]/\mathcal{O}_0[u]}(H \otimes_{\mathcal{O}_0} \tilde{\mathcal{O}}_0[v^{\pm 1}]))^{\Gamma},$$

where $\gamma \in \Gamma$ acts diagonally via $\tau(\gamma) \otimes \gamma$. Set $\hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z}$. We define the functor from $\mathrm{Alg}_{\mathcal{O}_0[u^{\pm 1}]}$ to groups that sends R to:

$$\underline{G}(R) = \underline{G}^*(\mathcal{O}_0^{unr}[u^{\pm 1}] \otimes_{\mathcal{O}_0[u^{\pm 1}]} R)^{\hat{\mathbb{Z}}}.$$

Then $\underline{G}$ is the reductive group we wanted (see §3.3.4, [26]). By descent, the group scheme $\underline{G}$ is reductive over $\mathcal{O}_0[u^{\pm 1}]$ with base change to $\mathcal{O}_0^{unr}[u^{\pm 1}]$ isomorphic to $\underline{G}^* \otimes_{\mathcal{O}_0[u^{\pm 1}]} \mathcal{O}_0^{unr}[u^{\pm 1}]$.

Since $\underline{G}_{F_0} = \underline{G} \otimes_{\mathcal{O}_0[u^{\pm 1}]} F_0 \cong G$, we fix a point x in the Bruhat- Tits building $\mathcal{B}(G, F_0)$. The parahoric group scheme $\mathcal{P}_x$ of G is a group scheme over $\mathrm{Spec}(\mathcal{O}_0)$ with generic fiber isomorphic to G such that $\mathcal{P}_x(\mathcal{O}_0) \subset \underline{G}(F_0)$ is the connected stabilizer of x.

Theorem 6.1.1 (Theorem 4.1, [26]). *There is a unique smooth, affine group scheme $\underline{\mathscr{G}} = \underline{\mathscr{G}}_x$ over $\mathrm{Spec}(\mathcal{O}_0[u])$ (called a Bruhat-Tits group scheme for $\underline{G}$) with connected fibers and with the following properties:*

(1) The generic fiber $\underline{\mathscr{G}} \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[u^{\pm 1}]$ is the group scheme $\underline{G}$.

(2) The base change of $\underline{\mathscr{G}}$ under $\mathrm{Spec}(\mathcal{O}_0) \to \mathrm{Spec}(\mathcal{O}_0[u])$ given by $u \mapsto \pi_0$ is the parahoric

group scheme $\mathcal{P}_x$ for $G = \underline{G} \otimes_{\mathcal{O}_0[u^{\pm 1}]} F_0$.

(3) *The base change of $\underline{\mathscr{G}}$ under $\mathrm{Spec}(\kappa[\![u]\!]) \to \mathrm{Spec}(\mathcal{O}_0[u])$ given by $\mathcal{O}_0[u] \to \kappa[\![u]\!]$ is the parahoric group scheme $\mathcal{P}_{x_{\kappa((u))}}$ for $\underline{G} \otimes_{\mathcal{O}_0[u^{\pm 1}]} \kappa((u))$.*

Suppose that R is an $\mathcal{O}_0$-algebra and denote $r : \mathrm{Spec}(R) \to \mathrm{Spec}(\mathcal{O}_0[u])$ given by $u \mapsto r$. Consider the closed subscheme $\Gamma_r \subset \mathrm{Spec}(R \otimes_{\mathcal{O}_0} \mathcal{O}_0[u])$ given by the graph of r. We have $\Gamma_r = \mathrm{Spec}(R[u]/(u-r))$. The formal completion of $\mathrm{Spec}(R \otimes_{\mathcal{O}_0} \mathcal{O}_0[u])$ along Γ_r is $\hat{\Gamma}_r = \mathrm{Spec}(R[\![u-r]\!])$. There is a natural closed immersion $\Gamma_r \to \hat{\Gamma}_r$ and we denote by $\hat{\Gamma}_r^\circ = \hat{\Gamma}_r - \Gamma_r = \mathrm{Spec}(R((u-r)))$, the complement of the image. When $r = 0$, $\Gamma_r = \mathbb{D}_R, \hat{\Gamma}_r = \mathbb{D}_R^*$ as defined in §4.

Consider the functor that associates to a $\mathcal{O}_0[u]$-algebra R (given by $u \mapsto r$) the group

$$L\underline{\mathscr{G}}(R) = \underline{\mathscr{G}}(\hat{\Gamma}_r^\circ) = \underline{\mathscr{G}}(R((u-r))).$$

Since $\underline{\mathscr{G}} \to \mathrm{Spec}(\mathcal{O}_0[u])$ is smooth and affine, $L\underline{\mathscr{G}}$ is represented by a formally smooth ind-scheme over $\mathrm{Spec}(\mathcal{O}_0[u])$. Next consider the functor that associates to an $\mathcal{O}_0[u]$ -algebra R the group

$$L^+\underline{\mathscr{G}}(R) = \underline{\mathscr{G}}(\hat{\Gamma}_r) = \underline{\mathscr{G}}(R[\![u-r]\!]).$$

We can see that $L^+\underline{\mathscr{G}}$ is represented by a scheme over $\mathrm{Spec}(\mathcal{O}_0[u])$ which is formally smooth. Now we define the global affine Grassmannian associated to the group $\underline{\mathscr{G}}$ over $\mathrm{Spec}(\mathcal{O}_0[u])$ to be the functor from $\mathcal{O}_0[u]$-algebras to groups, which sends R to

$$\mathrm{Gr}_{\underline{\mathscr{G}},\mathcal{O}_0[u]}(R) := \{\text{iso-classes of } (\mathcal{E}, \beta) \mid \mathcal{E} \text{ a } \underline{\mathscr{G}}\text{-torsor on } \hat{\Gamma}_r,\ \beta \text{ a trivialization of } \mathcal{E}\mid_{\hat{\Gamma}_r^\circ}\}.$$

Using this definition, we can see that $L\underline{\mathscr{G}}$ acts on $\mathrm{Gr}_{\underline{\mathscr{G}},\mathcal{O}_0[u]}$ by changing the trivialization β. In fact, consider the fpqc sheaf $L\underline{\mathscr{G}}/L^+\underline{\mathscr{G}}$. We have $\mathrm{Gr}_{\underline{\mathscr{G}},\mathcal{O}_0[u]} \simeq L\underline{\mathscr{G}}/L^+\underline{\mathscr{G}}$.

Proposition 6.1.2 (Proposition 6.5,[26]). *Suppose that $\underline{\mathscr{G}}$ is as in Theorem 6.1.1. The functor $\mathrm{Gr}_{\underline{\mathscr{G}},\mathcal{O}_0[u]}$ on $(\mathrm{Sch}/\mathbf{A}^1_{\mathcal{O}_0})$ is representable by an ind-projective ind-scheme over $\mathbf{A}^1_{\mathbb{Z}_p}$.*

Denote by $\mathrm{Gr}_{\underline{\mathscr{G}},\mathcal{O}_0} \to \mathrm{Spec}(\mathcal{O}_0)$ the base change of $\mathrm{Gr}_{\underline{\mathscr{G}},\mathcal{O}_0[u]} \to \mathrm{Spec}(\mathcal{O}_0[u])$, where the map $\mathrm{Spec}(\mathcal{O}_0) \to \mathrm{Spec}(\mathcal{O}_0[u])$ is given by $u \to \pi_0$. We can use the descent lemma of Beauville-Laszlo [2] and get

Proposition 6.1.3 (Corollary 6.6,[26]).

(1) The generic fiber $\mathrm{Gr}_{\underline{\mathscr{G}},\mathcal{O}_0} \times_{\mathcal{O}_0} F_0$ is equivariantly isomorphic to the affine Grassmannian Gr_G of G over $\mathrm{Spec}(F_0)$.

(2) The special fiber $\mathrm{Gr}_{\underline{\mathscr{G}},\mathcal{O}_0} \times_{\mathcal{O}_0} \kappa$ is equivariantly isomorphic to the affine Grassmannian $\mathrm{Gr}_{\mathcal{P}_\kappa}$ over $\mathrm{Spec}(\kappa)$.

6.2 Global affine Grassmannians for triality groups

In this section, we will construct the global affine Grassmannians for triality groups. Let $K/\mathbb{Q}_p$ be a cubic tamely ramified field extension, $p \neq 2, 3$. For any prime p and $m \in \mathbb{Z}_{\geq 0}$ not divisible by p, there exists a primitive m-th root of unity in $\mathbb{Q}_p$ if and only if m divides $p-1$ by Hensel's lemma. So the 3rd root of unity $\xi \in \mathbb{Q}_p$ if and only if $p \equiv 1 \bmod 3$. Consider the finite unramified extensions of $\mathbb{Q}_p$. These are in one-to-one correspondence with finite extensions of $\mathbb{F}_p$ since $\mathbb{F}_p$ is a perfect field. We know that $\mathbb{F}_p$ has a unique extension of degree n for every n, which is the splitting field of $x^{p^n} - x$. It follows that $\mathbb{Q}_p$ has a unique unramified extension of degree n for each n, obtained as the splitting field of $x^{p^n} - x$, i.e., by adjoining the p^n-1 st roots of unity. Moreover, the maximal unramified extension $\mathbb{Q}_p^{unr}$ of $\mathbb{Q}_p$ corresponds to the separable closure of $\mathbb{F}_p$, and so is obtained by adjoining the p^n-1 st roots of unity for all n. For any integer n with $(n,p)=1$, we have $p^{\varphi(n)}-1\equiv 0 \bmod n$, where $\varphi(n)=n\Pi_{p|n}(1-p^{-1})$ is the Euler's totient function. So we see that $\mathbb{Q}_p^{unr}$ is obtained by adjoining the n-th roots of unity for $(n,p)=1$ for all n. In particular, $\xi\in\mathbb{Q}^{unr}$ since $p\neq 2,3$.

Let $F_0=\mathbb{Q}_p^{unr}$, $F=KF_0$ with the valuation rings $\mathcal{O}_0,\mathcal{O}$ respectively. Let π_0 (resp. π) be a uniformizer of $\mathcal{O}_0$ (resp. $\mathcal{O}$). Then F/F_0 is a cubic Galois extension, with $\pi^3=\pi_0$, and $\mathcal{O}=\mathcal{O}_0[\pi]=\mathcal{O}_0[x]/(x^3-\pi_0)$. The corresponding Galois group $\Gamma=\mathrm{Gal}(F/F_0)=\langle\rho\rangle$, where $\rho(\pi)=\pi\xi$.

Consider the affine line $\mathbf{A}^1_{\mathcal{O}_0}=\mathrm{Spec}(\mathcal{O}_0[u])$ and its cover:

$$\mathrm{Spec}(\mathcal{O}_0[v])\to\mathrm{Spec}(\mathcal{O}_0[u]),$$

given by $u\mapsto v^3$. The Galois group Γ acts on $\mathcal{O}_0[v]$ by $\rho(v)=v\xi$. We have $\mathcal{O}_0[v]^\Gamma=\mathcal{O}_0[u]$. The Restriction of the map over the open subscheme $u\neq 0$ gives us:

$$\mathrm{Spec}(\mathcal{O}_0[v^{\pm 1}])\to\mathrm{Spec}(\mathcal{O}_0[u^{\pm 1}]).$$

Now we construct global affine Grassmannians for triality groups. Recall that $(V,*)$ is the normal twisted composition algebra over F obtained from the para-Cayley algebra. We defined the triality group for general orthogonal groups G in §3.4 to be the group scheme that represents the functor from F_0-algebras to groups, which sends R to

$$
\begin{aligned}
G(R) &:= (\mathrm{Res}_{F/F_0} H)^{\Gamma}(R) \\
&= \{g \in \mathbf{GO}_8^+(F \otimes_{F_0} R) \mid \text{there exist } \lambda \in (F \otimes_{F_0} R)^* \text{ such that } \lambda g(x * y) = g(x) * g(y)\},
\end{aligned}
$$

for all $x, y \in V \otimes_{F_0} R$. Here H represents the functor form F-algebras to groups that sends R to the group

$$
\{(c, \lambda) \in C_0(V_R)^* \times R^* \mid cV_R\tau(c) = V_R, \alpha(\tau(c)c) = \begin{pmatrix} \mu & 0 \\ 0 & \nu \end{pmatrix} \in \mathrm{End}_R({}^{\rho}V_R) \times \mathrm{End}_R({}^{\theta}V_R) \}.
$$

By using the isomorphism $F \otimes_{F_0} F \simeq F \times F \times F$ given by $a \otimes b \mapsto (ab, \rho(a)b, \theta(a)b)$ and Theorem 3.4.1, we can see that

$$
\begin{aligned}
G(F) &\simeq \{(g_1, g_2, g_3) \in \mathrm{GO}_8^+(F)^{\times 3} \mid \text{there exist } \lambda_i \in F^* \text{ such that } \lambda_i g_i(x * y) = g_{i+1}(x) * g_{i+2}(y)\} \\
&\simeq H(F).
\end{aligned}
$$

So G satisfied the Tameness hypothesis.

Consider the ring extension $\mathcal{O}_0[v^{\pm 1}]/\mathcal{O}_0[u^{\pm 1}]$. Since $\mathcal{O}_0[\pi_0^{\pm 1}] = F_0$ and $\mathcal{O}_0[\pi^{\pm 1}] = F$, we can get the Galois extension F/F_0 from $\mathcal{O}_0[v^{\pm 1}]/\mathcal{O}_0[u^{\pm 1}]$ given by $v \mapsto \pi$. Then similarly we constrict the algebra $\tilde{V} := \oplus_{i=1}^{8} \mathcal{O}_0[v^{\pm 1}] e_i$ where the multiplication table of $\{e_i\}_{i=1}^{8}$ is the same as Table 2.1, and observe that the base change isomorphism:

$$
\tilde{V} \otimes_{\mathcal{O}_0[v^{\pm 1}]} F \simeq V,
$$

given by $v \mapsto \pi$. We call $(\tilde{V}, *)$ the normal twisted composition algebra over $\mathcal{O}_0[v^{\pm 1}]$ obtained from the para-Cayley algebra, with the $\mathcal{O}_0[v^{\pm 1}]$-bilinear form $\langle\ ,\ \rangle : \tilde{V} \times \tilde{V} \to \mathcal{O}_0[v^{\pm 1}]$ satisfying $\langle e_i, e_{9-j} \rangle = \delta_{ij}$.

Suppose that $\mathscr{H}$ is the Chevalley split form of H. For any $\mathcal{O}_0$-algebra R, consider R as an $\mathcal{O}_0[u]$-algebra given by $u \mapsto \pi_0$. Define the functor from $\mathcal{O}_0[u^{\pm 1}]$-algebras to groups that sends R to

$$\underline{G}(R) := (\mathrm{Res}_{\mathcal{O}_0[v^{\pm 1}]/\mathcal{O}_0[u^{\pm 1}]}(\mathscr{H} \otimes_{\mathbb{Z}} \mathcal{O}_0[v^{\pm 1}]))^{\Gamma}(R).$$

We have

$$\begin{aligned}\underline{G}(\mathcal{O}_0[u^{\pm 1}]) &= \mathscr{H}(\mathcal{O}_0[v^{\pm 1}])^{\Gamma} \\ &= \{g \in \mathrm{GO}_8(\mathcal{O}_0[v^{\pm 1}]) \mid \text{there exist } \lambda \in \mathcal{O}_0[v^{\pm 1}]^* \text{ such that } \lambda g(x * y) = g(x) * g(y)\},\end{aligned}$$

for all $x, y \in \tilde{V}$. It is easy to see that $\underline{G} \otimes_{\mathcal{O}_0[u^{\pm 1}]} F_0$ is isomorphic to G under the base change $u \mapsto \pi_0$. Since $\mathscr{H}$ is the Chevalley split form of H, by Proposition 1.3.9,[12], we can define a group scheme over $\mathrm{Spec}(\mathcal{O}_0[u])$:

$$\underline{\mathscr{G}}(R) := (\mathrm{Res}_{\mathcal{O}_0[v]/\mathcal{O}_0[u]}(\mathscr{H} \otimes_{\mathbb{Z}} \mathcal{O}_0[v]))^{\Gamma}(R),$$

for any $\mathcal{O}_0[u]$-algebra R. Set the base change $\tilde{V}_{R((u-\pi_0))} := \tilde{V} \otimes_{\mathcal{O}_0[u^{\pm 1}]} R((u-\pi_0))$ and the $R[[u-\pi_0]] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v]$-module

$$\mathbb{L} = \oplus_{i=1}^{8} (R[[u-\pi_0]] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v]) e_i$$

in $\tilde{V}_{R((u-\pi_0))}$. We call $\mathbb{L}$ the standard lattice in $\tilde{V}_{R((u-\pi_0))}$. Then $\underline{\mathscr{G}}$ is the parahoric subgroup of $\underline{G}$ given by the standard lattice $\mathbb{L}$. Consider the functors that associates to an $\mathcal{O}_0[u]$-algebra R the group $L\underline{\mathscr{G}}(R) = \underline{\mathscr{G}}(R(\!(u-\pi_0)\!))$ and $L^+\underline{\mathscr{G}}(R) = \underline{\mathscr{G}}(R[\![u-\pi_0]\!])$. The global affine Grassmannian for $\underline{\mathscr{G}}$ is by definition the quotient fpqc sheaf:

$$\mathrm{Gr}_{\underline{\mathscr{G}}} := L\underline{\mathscr{G}}/L^+\underline{\mathscr{G}}.$$

Remark 6.2.1. Set $t = u - \pi_0$. For any $\mathcal{O}_0[u]$-algebra R, we have

$$\begin{aligned} L\underline{\mathscr{G}}(R) =& \underline{\mathscr{G}}(R(\!(t)\!)) \\ &= \{g \in \mathrm{GO}_8^+(R(\!(t)\!) \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v]) \mid \text{there exist } \lambda \text{ such that } \lambda g(x * y) = g(x) * g(y)\} \end{aligned}$$

for $\lambda \in (R(\!(t)\!) \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v])^*$, $x, y \in \tilde{V}_{R(\!(t)\!)}$, and

$$\begin{aligned} L^+\underline{\mathscr{G}}(R) =& \underline{\mathscr{G}}(R[\![t]\!]) \\ &= \{g \in \mathrm{GO}_8^+(R[\![t]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v]) \mid \text{there exist } \lambda \text{ such that } \lambda g(x * y) = g(x) * g(y)\} \end{aligned}$$

for $\lambda \in (R[\![t]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v])^*$, $x, y \in \mathbb{L}$.

The following theorem gives an explicitly description of the global affine Grassmannian for $\underline{\mathscr{G}}$ in terms of lattices. For any $\lambda \in (R(\!(u-\pi_0)\!) \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v])^*$, we denote by $[\lambda]$ the class of λ mod $(R[\![u-\pi_0]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v])^*$.

Theorem 6.2.2. *Suppose R is an $\mathcal{O}_0$-algebra. There is an $L\underline{\mathscr{G}}$-equivariant isomorphism between $\mathrm{Gr}_{\underline{\mathscr{G}}}(R)$ and the set of pairs $(L, [\lambda])$, where L is a $R[\![u-\pi_0]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v]$-lattice of $\tilde{V} \otimes_{\mathcal{O}_0[u^{\pm 1}]} R(\!(u-\pi_0)\!)$, and λ is in $(R(\!(u-\pi_0)\!) \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v])^*$, which satisfy:*

(1) Under the bilinear form $\langle\ ,\ \rangle$, *we have*

$$\langle\ ,\ \rangle : L \otimes L \to \rho(\lambda)\theta(\lambda)(R[\![u-\pi_0]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v])$$

which is perfect, i.e., $L \cong \mathrm{Hom}(L, \rho(\lambda)\theta(\lambda)(R[\![u-\pi_0]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v]))$. *Here the tensor* $\otimes$ *and* Hom *are for the* $R[\![u-\pi_0]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v]$*-mod structure.*

(2) We have $L * L \subset \lambda L$.

(3) There exists $a \in L$, *such that* $q(a) = 0, \langle a * a, a\rangle = \lambda\rho(\lambda)\theta(\lambda)$.

(4) For a *as in (3), let* $e = a + \lambda^{-1}(a * a)$. *Thus, we have* $\bar{\lambda}^{-1} \cdot \overline{e * x} = -\bar{x} = \bar{\lambda}^{-1} \cdot \overline{x * e}$, *for any* $\bar{x}$ *satisfying* $\langle \bar{x}, \bar{e}\rangle = 0$, *where* $\bar{x}$ *is the image of* x *under the canonical map* $L \to L/(u-\pi_0, v)L$.

6.3 Proof of the main result

This proof is similar to the proof of Theorem 5.2.1. The difference is that we have λ instead of 1 here. It is easy to see that $g(\mathbb{L}) = \mathbb{L}$ for any $g \in L^+\underline{\mathscr{G}}(R) \subset \mathrm{GL}_8(R[\![u-\pi_0]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v])$. So the standard lattice is stable under $L^+\underline{\mathscr{G}}$.

Next, for any $g \in L\underline{\mathscr{G}}(R)$, there exisit $\lambda \in (R(\!(u-\pi_0)\!) \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v])^*$ such that $\lambda g(x*y) = g(x) * g(y)$ for all $x, y \in \tilde{V}_{R(\!(u-\pi_0)\!)}$. Let $L = g(\mathbb{L})$, we will show that L satisfies conditions (1)-(4). We have $L * L = g(\mathbb{L}) * g(\mathbb{L}) \subset \lambda g(\mathbb{L})$, so L satisfies condition (2); the quadratic form of L is determined by the multiplication $*$ since $x * (y * x) = \rho(q(x))y$. Replace x, y by $g(x), g(y)$ and we get $g(x) * (g(y) * g(x)) = \rho(q(g(x)))g(y)$. Meanwhile,

$$g(x) * (g(y) * g(x)) = \theta(\lambda)\lambda \cdot g(x * (y * x)) = \theta(\lambda)\lambda \cdot \rho(q(x)) \cdot g(y).$$

Combining them together, we have

$$\rho(q(g(x))) = \theta(\lambda)\lambda\rho(q(x)),$$

which is equivalent to $\mu(g) = \rho(\lambda)\theta(\lambda)$ by $q(g(x)) = \mu(g)q(x)$. Hence we have $\langle g(x), g(y)\rangle = \rho(\lambda)\theta(\lambda)\langle x, y\rangle$. Thus, we get a perfect bilinear form $\langle\ ,\ \rangle : L\otimes L \to \rho(\lambda)\theta(\lambda)(R[\![u-\pi_0]\!]\otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v])$.

For (3) and (4), we set $a = g(e_4)$. It implies $a * a = g(e_4) * g(e_4) = \lambda g(e_5)$ from the Table 2.1. It is easy to see that $q(a) = 0$, and $\langle a, a * a\rangle = \lambda\langle g(e_4), g(e_5)\rangle = \lambda\rho(\lambda)\theta(\lambda)$. Set

$$e = a + \frac{1}{\lambda}(a * a) = g(e_4 + e_5).$$

By the canonical map: $R[\![u - \pi_0]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v] \to R \otimes_{\mathcal{O}_0} \mathcal{O}_0[v] = R[v] \to R$, the image of $\mathbb{L}$ is the para-Cayley algebra $\overline{\mathbb{L}} = \oplus_{i=1}^{8} R\bar{e}_i$, which satisfied $\overline{(e_4 + e_5) * x} = -\bar{x} = \overline{x * (e_4 + e_5)}$ for any $\bar{x}$ satisfying $\langle \bar{x}, \bar{e}_4 + \bar{e}_5\rangle = 0$. Set $\bar{x} \star \bar{y} := \bar{\lambda}^{-1} \cdot \overline{x * y}$ for any $x, y \in \mathbb{L}$. We get

$$\bar{e} \star \bar{x}_1 = \bar{\lambda}^{-1} \cdot \overline{(g(e_4 + e_5) * g(x))} = g(\overline{(e_4 + e_5) * x}) = -g(\bar{x}) = -\bar{x}_1,$$

for any $x_1 = g(x) \in L$ with $\langle \bar{x}_1, \bar{e}\rangle = 0$. Similarly we have $\bar{x}_1 \star \bar{e} = -\bar{x}_1$. Above all, $L = g(\mathbb{L})$ satisfying (1)-(4).

Conversely, we will show that for any L satisfying condition (1)-(4) with R a local henselian ring, there exist $g \in L\underline{\mathscr{G}}(R)$ such that $L = g(\mathbb{L})$. We want to find a basis in L as we did in §5.2, such that the multiplication table under the basis is the scalaring of the Table 2.1, i.e, there exists a basis $\{f_i\} \in L$ such that $f_i * f_j = \lambda f_k$ for $e_i * e_j = e_k$ in the Table 2.1. Then we can define $g(e_i) = f_i$, where $g \in L\underline{\mathscr{G}}(R)$.

We first claim that $a, \lambda^{-1}(a * a)$ are primitive elements. We omit the proof here since it is similar as we did in §5.2. Since $a, \lambda^{-1}(a * a)$ are linear independent by $\langle a, \lambda^{-1}(a * a)\rangle = \rho(\lambda)\theta(\lambda)$, we have $L = (R[\![u - \pi_0]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v])a + (R[\![u - \pi_0]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v])\lambda^{-1}(a * a) + L_0$, where

$$L_0 := \{x \in L \mid \langle x, a\rangle = 0, \langle x, a * a\rangle = 0\}.$$

Set $f_1 = a, f_2 = \lambda^{-1}(a * a)$. By Lemma 2.3.2 and Lemma 2.3.3, we have

$$\begin{aligned} &f_1 * f_1 = \lambda f_2, \quad f_2 * f_2 = \lambda f_1, \\ &f_1 * f_2 = f_2 * f_1 = 0, \\ &q(f_1) = q(f_2) = 0, \quad \langle f_1, f_2\rangle = \rho(\lambda)\theta(\lambda). \end{aligned}$$

The following lemma is similar to Lemma 5.2.2. We omit the proof here.

Lemma 6.3.1. *We have*

$$\frac{1}{\lambda}(L_0 * f_i) \subset L_0, \quad \frac{1}{\lambda}(f_i * L_0) \subset L_0.$$

Define the ρ-linear transformation $t_i : L_0 \to L_0$ given by $t_i(x) = \lambda^{-1}(x * f_i)$, for $i = 1, 2$. Take $L_i = t_i(L_0) = \lambda^{-1}(L_0 * f_i)$. Both L_i has $\mathrm{rank}(L_i) \leq 3$ since f_i are isotropic. For any $x \in L_0$, we have

$$\frac{1}{\lambda}[(\frac{1}{\lambda}(f_2 * x)) * f_1] + \frac{1}{\lambda}[(\frac{1}{\lambda}(f_1 * x)) * f_2] = \frac{1}{\lambda\rho(\lambda)}\theta(\langle f_1, f_2\rangle)x = x,$$

by Lemma 2.3.3. So $L_0 = L_1 + L_2$, Since $\mathrm{rank}(L_i) \leq 3$, we must have a direct sum composition: $L_0 = L_1 \oplus L_2$.

Lemma 6.3.2.

(1) For any $x \in L_0$, $t_i^2(x) = \lambda^{-1}(f_{i+1} * x)$, $i = 1, 2 \mod 2$.

(2) For any $x \in L_i$, $t_i^3(x) = -x$.

(3) From (2), t_i is a $\mathcal{O}_0[\![u]\!]$-isomorphism when restrict at L_i. The inverse map $t_i^{-1}(x) = -t_i^2(x) = \lambda^{-1}(f_{i+1} * x)$ is θ-linear.

(4) For $x \in L_1, y \in L_2$, we have $\langle t_1(x), t_2(y)\rangle = \rho(\lambda)\lambda^{-1} \cdot \rho(\langle x, y\rangle)$.

Proof. (1) For any $x \in L_0$, we have

$$t_1^2(x) = \frac{1}{\lambda\rho(\lambda)}((x * f_1) * f_1) = -\frac{1}{\lambda\rho(\lambda)}((f_1 * f_1) * x) = -\frac{1}{\lambda}(f_2 * x),$$

by Lemma 2.3.3. A similar argument gives $t_2^2(x) = -\lambda^{-1}(f_1 * x)$.

(2) For any $x \in L_1$, we have $t_1^3(x) = -(\lambda\rho(\lambda))^{-1} \cdot ((f_2 * x) * f_1)$. Consider

$$(f_2 * x) * f_1 + (f_1 * x) * f_2 = \theta(\langle f_1, f_2\rangle)x = \lambda\rho(\lambda)x,$$

by Lemma 2.3.3. Let $x = \lambda^{-1}(z * f_1)$ for some $z \in L_0$. We get $f_1 * x = \theta(\lambda)^{-1}(f_1 * (z * f_1)) = 0$ by $q(f_1) = 0$. Hence $(f_2 * x) * f_1 = \lambda\rho(\lambda)x$, which implies $t_1^3(x) = -x$. Similar calculations for $y \in L_2$ give $t_2^3(y) = -y$.

Part (3) follows from (2) immediately. For (4), we have

$$\langle t_1(x), t_2(y)\rangle = \frac{1}{\lambda^2}\langle x * f_1, y * f_2\rangle = \frac{1}{\lambda^2}\rho(\langle f_1 * (y * f_2), x\rangle).$$

Since

$$f_1 * (y * f_2) = -\lambda t_2^2(y * f_2) = -\lambda t_2^2(\lambda t_2(y)) = -\lambda\theta(\lambda)t_2^3(y) = \lambda\theta(\lambda)y,$$

by (1) and (2), we obtain

$$\langle t_1(x), t_2(y)\rangle = \frac{1}{\lambda^2}\rho(\lambda)\lambda\rho(\langle x, y\rangle) = \frac{\rho(\lambda)}{\lambda}\rho(\langle x, y\rangle).$$

□

Remark 6.3.3. (1) From the proof above Lemma, we can see that $\lambda^{-1}(f_i * L_i) = 0$, and $\lambda^{-1}(L_i * f_{i+1}) = 0$.

(2) Since L_1, L_2 are isotropic and $\langle\ ,\ \rangle$ restricted to L_0 is nondegenerate, the L_i are in duality: $L_1 \cong \mathrm{Hom}(L_2, \rho(\lambda)\theta(\lambda)R[\![u-\pi_0]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v])$.

Lemma 6.3.4. *For the multiplication $L_i * L_j$, we have*

(1) $\lambda^{-1}(L_1 * L_2) \subset (R[\![u-\pi_0]\!]\otimes_{\mathcal{O}_0[u]}\mathcal{O}_0[v])f_1, \quad \lambda^{-1}(L_2 * L_1) \subset (R[\![u-\pi_0]\!]\otimes_{\mathcal{O}_0[u]}\mathcal{O}_0[v])f_2.$

(2) $\lambda^{-1}(L_i * L_i) \subset L_{i+1}$ $(i = 1, 2 \mod 2)$.

Proof. (1) For $x \in L_1, y \in L_2$, we write x as $x = \lambda^{-1}(x_1 * f_1)$ with some $x_1 \in L_1$, and y as $y = \lambda^{-1}(y_1 * f_2)$ with some $y_1 \in L_2$. Consider

$$\frac{1}{\lambda}(x*y) = \frac{1}{\lambda\rho(\lambda)\theta(\lambda)}((x_1*f_1)*(y_1*f_2)) = \frac{1}{\lambda\rho(\lambda)\theta(\lambda)}(-((y_1*f_2)*f_1)*x_1+\theta(\langle x_1, y_1*f_2\rangle)f_1),$$

by Lemma 2.3.3. Notice that $(y_1 * f_2) * f_1 \in L_2 * f_1 = 0$. Thus we have

$$\begin{aligned}\langle x_1, y_1 * f_2\rangle &= \frac{\theta(\lambda)}{\lambda}\theta(\langle t_1(x_1), t_2(y_2 * f_2)\rangle) \\ &= \frac{\theta(\lambda)}{\lambda}\theta(\langle x, t_2(\lambda y)\rangle) \\ &= \theta(\lambda)\theta(\langle x, t_2(y)\rangle).\end{aligned}$$

Hence

$$\frac{1}{\lambda}(x*y) = \frac{1}{\lambda\rho(\lambda)\theta(\lambda)}\rho(\lambda)\rho(\langle x, t_2(y)\rangle)f_1 = \frac{1}{\lambda\theta(\lambda)}\rho(\langle x, t_2(y)\rangle)f_1.$$

Since $\langle x, t_2(y)\rangle \in \rho(\lambda)\theta(\lambda)R[\![u-\pi_0]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v]$, we obtain $(\lambda\theta(\lambda))^{-1}\rho(\langle x, t_2(y)\rangle) \in R[\![u-\pi_0]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v]$. Similarly for $y*x$, we have

$$\frac{1}{\lambda}(y*x) = \frac{1}{\lambda\theta(\lambda)}\rho(\langle t_1(x), y\rangle)f_2.$$

(2) For $x_1, x_2 \in L_1$, we claim that $\lambda^{-1}(x_1 * x_2) \in L_0$. Since $\langle x_1 * x_2, f_1\rangle = \theta(\langle f_1 * x_1, x_2\rangle) = 0$ by $f_1 * L_1 = 0$, and $\langle x_1 * x_2, f_2\rangle = \rho(\langle x_2 * f_2, x_1\rangle) = 0$ by $L_1 * f_2 = 0$. Using Lemma 2.3.3, we find

$$\begin{aligned} t_1(x_1) * t_1(x_2) &= \frac{1}{\rho(\lambda)\theta(\lambda)}((x_1 * f_1) * (x_2 * f_1)) \\ &= \frac{1}{\rho(\lambda)\theta(\lambda)}(-f_1 * (x_2 * (x_1 * f_1))) \\ &= \frac{1}{\rho(\lambda)\theta(\lambda)}(f_1 * (f_1 * (x_1 * x_2))), \end{aligned}$$

by $\langle x_1 * f_1, f_1\rangle = 0$ and $\langle f_1, x_2\rangle = 0$. Since $f_1 * (f_1 * (x_1 * x_2)) = f_1 * (-\lambda t_2^2(x_1 * x_2)) = \lambda\theta(\lambda)t_2^4(x_1 * x_2) = -\lambda\theta(\lambda)t_2(x_1 * x_2)$, we obtain:

$$t_1(x_1) * t_1(x_2) = \frac{-\lambda}{\rho(\lambda)}t_2(x_1 * x_2).$$

Here $x_1 * x_2 \in \lambda L_0$. Let t_2 act on both sides of above equation. We obtain $t_2(x_1 * x_2) \in \rho(\lambda)L_2$. Hence $t_1(x_1) * t_1(x_2) \in \lambda L_2$, which gives us $L_1 * L_1 \subset \lambda L_2$ by t_1 isomorphism. Similarly, $L_2 * L_2 \subset \lambda L_1$. □

So far we discussed the multiplication on L. To make it similar to the Table 2.1, we want to find a basis $\{x_1, x_2, x_3\}$ for L_1 (resp. $\{y_1, y_2, y_3\}$ for L_2) such that $t_1(x_i) = -id$ (resp. $t_2(y_i) = -id$). To do that, we first check $\bar{t_i} = -id$ under the canonical map $L \to L/(u - \pi_0, v)L$.

Recall that we define $\bar{x} \star \bar{y} = \bar{\lambda}^{-1} \cdot \overline{x * y}$. Denote by $\langle\ ,\ \rangle_{\bar{\lambda}} := \bar{\lambda}^{-2}\langle\ ,\ \rangle$ the bilinear form corresponding to $\star$ and $q_{\bar{\lambda}}$ the quadratic form corresponding to $\star$. We have:

Proposition 6.3.5. *Suppose R is a local henselian ring with the maximal ideal m. Given $(L, *, \langle\ ,\ \rangle)$ a $R[\![u - \pi_0]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v]$-lattice satisfying (1)-(4) as above. Set $\bar{L} = L/(u - \pi_0, v, m)L$ with multiplication $\bar{x} \star \bar{y} = \bar{\lambda}^{-1} \cdot \overline{x * y}$ for all $x, y \in L$. We have $(\bar{L}, \star)$ is isomorphic to the split para-Cayley algebra.*

Proof. It is easy to see that $q_{\bar{\lambda}}(\bar{x} \star \bar{y}) = q_{\bar{\lambda}}(\bar{x})q_{\bar{\lambda}}(\bar{y})$, and $\langle \bar{x} \star \bar{y}, \bar{z}\rangle_{\bar{\lambda}} = \langle \bar{y} \star \bar{z}, \bar{x}\rangle_{\bar{\lambda}}$, so $\bar{L}$ is a symmetric composition algebra. We need to find a para-unit, i.e. an element $\bar{e} \in \bar{L}$, such that $\bar{e} \star \bar{e} = \bar{e}$, $\bar{e} \star \bar{x} = \bar{x} \star \bar{e} = -\bar{x}$ for all $\bar{x} \in \bar{L}$ satisfying $\langle \bar{e}, \bar{x}\rangle_{\bar{\lambda}} = 0$. Set $e = f_1 + f_2$. Here e is an idempotent since

$$e \star e = \frac{1}{\lambda}((f_1 + f_2) * (f_1 + f_2)) = \frac{1}{\lambda}(\lambda(f_1 + f_2)) = e.$$

By condition (4), we get $\bar{e} \star \bar{x} = \bar{x} \star \bar{e} = -\bar{x}$. Therefore $\bar{e}$ is a para-unit in $\bar{L}$. Thus $\bar{L}$ is a para-Cayley algebra. It is split since q is an isotropic norm. □

Proposition 6.3.6. *We have $\bar{L}_i = \bar{L}_0 \star \bar{f}_i = \{\bar{x} \in \bar{L}_0 \mid \bar{x} \star \bar{f}_i = -\bar{x}\}$. Hence for*

$$\bar{t}_i : L_i \to L_i,$$

we have $\bar{t}_i = -id$.

Proof. By Lemma 34.8, [14], we can define $\bar{x} \diamond \bar{y} = (\bar{e} \star \bar{x}) \star (\bar{y} \star \bar{e})$ as a unital composition algebra with identity element $\bar{e}$. We have $\bar{x} \star \bar{y} = r(\bar{x}) \diamond r(\bar{y})$, where $r(\bar{x}) = \langle \bar{e}, \bar{x} \rangle_{\bar{\lambda}} \bar{e} - \bar{x}$ is the conjugation of $\bar{x}$. By Proposition 2.1.1,

$$\bar{x} \diamond \bar{y} + \bar{y} \diamond \bar{x} - \langle \bar{x}, \bar{e} \rangle_{\bar{\lambda}} \bar{y} - \langle \bar{y}, \bar{e} \rangle_{\bar{\lambda}} \bar{x} + \langle \bar{x}, \bar{y} \rangle_{\bar{\lambda}} \bar{e} = 0.$$

Using $\bar{x} \star \bar{y} = r(\bar{x}) \diamond r(\bar{y})$ and $\langle r(\bar{x}), r(\bar{y}) \rangle_{\bar{\lambda}} = \langle \bar{x}, \bar{y} \rangle_{\bar{\lambda}}$, we obtain

$$\bar{x} \star \bar{y} + \bar{y} \star \bar{x} = \langle \bar{e}, \bar{x} \rangle_{\bar{\lambda}} r(\bar{y}) + \langle \bar{e}, \bar{y} \rangle_{\bar{\lambda}} r(\bar{x}) - \langle \bar{x}, \bar{y} \rangle_{\bar{\lambda}} \bar{e}.$$

Therefore, if $\bar{x} \in \bar{L}_0 \star \bar{f}_i$, we have $\bar{f}_i \star \bar{x} = 0$ by $q_{\bar{\lambda}}(\bar{f}_i) = 0$, and

$$\bar{x} \star \bar{f}_i + \bar{f}_i \star \bar{x} = \langle \bar{e}, \bar{f}_1 \rangle_{\bar{\lambda}} r(\bar{x}) = r(\bar{x}).$$

Hence $\bar{x} \star \bar{f}_i = \bar{x} \star \bar{f}_i + \bar{f}_i \star \bar{x} = \langle \bar{e}, \bar{x} \rangle_{\lambda} \bar{e} - \bar{x} = -\bar{x}$. Then $\bar{L}_0 \star \bar{f}_i \subset \{\bar{x} \in \bar{L}_0 \mid \bar{x} \star \bar{f}_i = -\bar{x}\}$. It is obvious that $\{\bar{x} \in \bar{L}_0 \mid \bar{x} \star \bar{f}_i = -\bar{x}\} \subset \bar{L}_0 \star \bar{f}_i$. So we get

$$\bar{L}_i = \bar{L}_0 \star \bar{f}_i = \{\bar{x} \in \bar{L}_0 \mid \bar{x} \star \bar{f}_i = -\bar{x}\}$$

and $\bar{t}_i = -id$. $\square$

Now we prove that t_i and $-id$ are the same up to ρ-conjugacy. This part is similar to §5.2. We fix a basis for $L_i \cong (R[\![u - \pi_0]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v])^3$ and let $A_i \in \mathrm{GL}_3(R[\![u - \pi_0]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v])$ representing t_i. We can find a new basis for L_i with transition matrix $b \in \mathrm{GL}_3(R[\![u - \pi_0]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v])$ such that

$$-I = b^{-1} A_i \rho(b).$$

Consider $t_1 : L_1 \to L_1$, the subgroup of $\mathrm{GL}_3(R[\![u-\pi_0]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v])$ generated by t_1 is the cyclic group of order 6 ($t_1^6 = (-id)^2 = id$). If we fix the basis and use A_1 representing t_1, we have $t_1^2 = A_1\rho(A_1), t_1^3 = A_1\rho(A_1)\theta(A_1) = -I$. Consider the map

$$\Gamma \to \mathrm{GL}_3(R[\![u-\pi_0]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v])$$

by $\rho \mapsto a_\rho = -A_1$. Using $a_{st} = a_s {}^s a_t$, we get $\theta \mapsto a_\theta = a_\rho \rho(a_\rho) = A_1\rho(A_1)$, and $1 \mapsto a_1 = I$. The image of $\{\rho, \theta, 1\}$ is $\{t_1^4 = -t, t_1^8 = t^2, t_1^{12} = id\}$, so a_ρ is a 1-cocycle. Denote by $[a_\rho]$ the 1-cocycle in $H^1(\Gamma, \mathrm{GL}_3(R[\![u-\pi_0]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v])$. Using the quotient map

$$R[\![u-\pi_0]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v] \to R \to \kappa,$$

and the fact that R is a local henselian ring. We have $(R[\![u-\pi_0]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v], (u-\pi_0))$, $(R[v], (v)), (R, m)$ are Henselian pairs. Hence we obtain the exact sequence:

$$1 \to U \to \mathrm{GL}_3(R[\![u-\pi_0]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v]) \to \mathrm{GL}_3(\kappa) \to 1$$

where U is the kernel. The group Γ acts on $\mathrm{GL}_3(R[\![u-\pi_0]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v])$ by ρ on $\mathcal{O}_0[v]$, and Γ acts trivially on $\mathrm{GL}_3(\kappa)$. So we obtain the exact sequence of pointed sets:

$$... \to H^1(\Gamma, U) \to H^1(\Gamma, \mathrm{GL}_3(R[\![u-\pi_0]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v]) \to H^1(\Gamma, \mathrm{GL}_3(\kappa)).$$

Since $|\Gamma| = 3$ and $p \neq 3$, we get $H^1(\Gamma, U) = 1$. Hence for any $[a_s] \in H^1(\Gamma, \mathrm{GL}_3(R[\![u-\pi_0]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v])$ satisfying $[\bar{a_s}] = 1$, we get $[a_s] = 1$. Under this observation, we get $[\bar{a_\rho}] = -[\bar{t}] = 1$ by Proposition 6.3.6. Therefore $[a_\rho] = 1$ by the exact sequence. In matrix language, there exist $b \in \mathrm{GL}_3(R[\![u-\pi_0]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v])$ such that

$$I = b^{-1}(-A_t)\rho(b), \quad t_1 \sim -id.$$

We have a similar conclusion for t_2.

Using the above we see that there exist a basis $\{x_1, x_2, x_3\}$ for L_1, and a dual basis $\{y_1, y_2, y_3\}$ for L_2, such that $t_1(x_i) = -id$, $t_2(y_i) = -id$, $\langle x_i, y_j \rangle = \rho(\lambda)\theta(\lambda)\delta_{ij}$. We have

$$x_i * f_1 = -\lambda x_i, \quad f_1 * x_i = 0,$$
$$x_i * f_2 = 0, \quad f_2 * x_i = -\lambda x_i,$$
$$y_i * f_1 = 0, \quad f_1 * y_i = -\lambda y_i,$$
$$y_i * f_2 = -\lambda y_i, \quad f_2 * y_i = 0.$$

By Lemma 6.3.4, we have

$$x_i * y_j = -\lambda \delta_{ij} f_1, \quad y_i * x_j = -\lambda \delta_{ij} f_2.$$

It reminds to calculate the terms in $L_i * L_i$, which we will also define a wedge product $\wedge : L_i \times L_i \to L_{i+1}$ given by

$$u \wedge v := \frac{1}{\lambda}(t_i^{-1}(u) * t_i(v)),$$

for all $u, v \in L_i$. Let $u \in L_1$. It is immediate that

$$\begin{aligned}
u \wedge u &= \frac{1}{\lambda}(t_1^{-1}(u) * t_1(u)) \\
&= \frac{1}{\lambda\rho(\lambda)\theta(\lambda)}((f_2 * u) * (u * f_1)) \\
&= \frac{1}{\lambda\rho(\lambda)\theta(\lambda)}(((u * f_1)) * u) * f_2) \\
&= \frac{q(u)}{\lambda\rho(\lambda)\theta(\lambda)}(f_1 * f_2) = 0
\end{aligned}$$

by $\langle f_2, u * f_1\rangle = 0, q(u) = 0$, and similarly for $u \in L_2$. By linearizing the equation we find $u \wedge v = -v \wedge u$. Now define a trilinear function $\langle\ ,\ ,\ \rangle$ on L_i by $\langle u, v, w\rangle = \langle u, v \wedge w\rangle$. It is an alternating trilinear function (Similar proof as in §5.2).

Consider $\lambda^{-1}(x_1 * x_2)$, we have $\langle \lambda^{-1}(x_1 * x_2), x_1\rangle = -\lambda^{-1}\langle x_1 * x_2, t_1(x_1)\rangle = -\lambda^{-2}\langle x_1 * x_2, x_1 * f_1\rangle = 0$ by $\langle x_2, f_1\rangle = 0$. Similarly $\langle \lambda^{-1}(x_1 * x_2), x_2\rangle = 0$. Hence we have $\lambda^{-1}(x_1 * x_2) = by_3$ for some b:

$$b = \frac{1}{\lambda\rho(\lambda)\theta(\lambda)}\langle x_1 * x_2, x_3\rangle \in R[\![u - \pi_0]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v].$$

Multiplying y_1 on the right side, we obtain $\rho(\lambda)^{-1}((x_1 * x_2) * y_1) = (by_3) * y_1$. Since $(x_1 * x_2) * y_1 + (y_1 * x_2) * x_1 = \theta(\langle x_1, y_1\rangle)x_2 = \lambda\rho(\lambda)x_2$, and $y_1 * x_2 = 0$, we have

$$\lambda x_2 = \rho(b)(y_3 * y_1).$$

Therefore we obtain $b, \rho(b)^{-1} \in R[\![u - \pi_0]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v]$, which implies $b \in (R[\![u - \pi_0]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v])^*$. We can let $b = -1$, and get $x_1 * x_2 = -\lambda y_3$. We can perform similar calculations for the other $x_i * x_j$ and $y_i * y_j$. By using alternating trilinear form, we obtain:

Table 6.1: Multiplication table $x_i * x_j$

$*$	x_1	x_2	x_3
x_1	0	-λy_3	λy_2
x_2	λy_3	0	-λy_1
x_3	$-\lambda y_2$	λy_1	0

Table 6.2: Multiplication table $y_i * y_j$

$*$	y_1	y_2	y_3
y_1	0	$-\lambda x_3$	λx_2
y_2	λx_3	0	$-\lambda x_1$
y_3	$-\lambda x_2$	λx_1	0

Therefore, we complete the multiplication table for L. By letting $g_(e_4) = f_1, g(e_5) = f_2$, and

$$g(e_1) = x_1, \quad g(e_6) = x_2, \quad g(e_7) = x_3,$$

$$g(e_8) = y_1, \quad g(e_3) = y_2, \quad g(e_2) = y_3.$$

We have $g(e_i) * g(e_j) = \lambda g(e_i * e_j)$, so there exist $g \in L\underline{\mathscr{G}}$ such that $L = g(\mathbb{L})$.

Chapter 7

PZ-local models for triality groups

Now we are ready to give the definition of PZ-local models. The generalized local models were introduced by Pappas and Zhu in §7, [26]. We will give an explicit description of PZ local models for triality groups.

7.1 General construction of PZ-local models

We used the same notations as in §6.1. That is, let F_0 be a p-adic field with valuation ring $\mathcal{O}_0$, $p \neq 2, 3$. Let π_0 be a uniformizer of $\mathcal{O}_0$. Set the residue field $\kappa = \mathcal{O}_0/(\pi_0)$. We fix a separable closure $\bar{F}_0^s$ of F_0 and denote by F_0^{unr} the maximal unramified extension of F_0 in $\bar{F}_0^s$, with valuation ring $\mathcal{O}_0^{unr}$. Consider a Galois extension F/F_0. Denote by $\tilde{F}_0$ the maximal unramified extension of F_0 that is contained in F, and by $\tilde{\mathcal{O}}_0$, $\mathcal{O}$ the valuation rings of $\tilde{F}_0, F$ respectively. Set $e = [F : \tilde{F}_0]$. Then there is a uniformizer π of F such that $\pi^e = \pi_0$.

Let G be a connected reductive group over F_0, which splits over a tamely ramified extension. Then $G_{\tilde{F}_0} := G \otimes_{F_0} \tilde{F}_0$ is quasi-split. In [26], it is shown that there exist a reductive group $\underline{G}$ over $\mathrm{Spec}(\mathcal{O}_0[u^{\pm}])$, which extends G in the sense that its base change

$$\underline{G} \otimes_{\mathcal{O}_0[u^{\pm}]} F_0, \quad u \mapsto \pi_0,$$

is isomorphic to G. By fixing a point x in the Bruhat- Tits building $\mathcal{B}(G, F_0)$, Pappas and

Zhu constructed a unique smooth, affine group scheme $\underline{\mathscr{G}} = \underline{\mathscr{G}}_x$ over $\mathrm{Spec}(\mathcal{O}_0[u])$ (called a Bruhat-Tits group scheme for $\underline{G}$) which satisfies the properties in Theorem 6.1.1.

Using the local parameter $t = u - \pi_0$, we define $\underline{\mathscr{G}}_{F_0,\pi_0} := \underline{\mathscr{G}} \otimes_{\mathcal{O}_0[u]} F_0[\![t]\!]$ where $\mathcal{O}_0[u] \to F_0[\![t]\!]$ given by $u \mapsto t + \pi_0$. Notice that there is an isomorphism:

$$\tilde{\mathcal{O}}_0[v^{\pm 1}] \otimes_{\mathcal{O}[u^{\pm 1}]} F_0[\![t]\!] \xrightarrow{\sim} F[\![z]\!] \xrightarrow{\sim} F[\![t]\!],$$

(see (6.10), [26]) given by $v \mapsto \pi(1+z)$, and z maps to the power series $(1 + \frac{u-\pi_0}{\pi_0})^{1/e} - 1$, where the e-th root is expressed using the standard binomial formula. This isomorphism also matches the action of Γ on the left side (coming from the cover $\mathcal{O}_0[u] \to \tilde{\mathcal{O}}_0[v]$ by base change), with the action on $F[\![z]\!]$ given by the Galois action on the coefficients F. Using this isomorphism and $\underline{G} \otimes_{\mathcal{O}_0[u^{\pm}]} F_0 \simeq G$, we obtain

$$\underline{\mathscr{G}}_{F_0,\pi_0} \simeq G \otimes_{F_0} F_0[\![t]\!].$$

Let $L\underline{\mathscr{G}}_{F_0,\pi_0}$ be the loop group over $\mathrm{Spec}(F_0)$ representing the functor from F_0- algebras to groups that sends R to

$$L\underline{\mathscr{G}}_{F_0,\pi_0}(R) = \underline{\mathscr{G}}_{F_0,\pi_0}(R(\!(t)\!)),$$

and $L^+\underline{\mathscr{G}}_{F_0,\pi_0}$ over $\mathrm{Spec}(F_0)$ representing the functor from F_0- algebras to groups that sends R to

$$L^+\underline{\mathscr{G}}_{F_0,\pi_0}(R) = \underline{\mathscr{G}}_{F_0,\pi_0}(R[\![t]\!]).$$

Consider $\mathrm{Gr}_{\underline{\mathscr{G}}_{F_0,\pi_0}}(R) := L\underline{\mathscr{G}}_{F_0,\pi_0}/L^+\underline{\mathscr{G}}_{F_0,\pi_0}$ as a fpqc sheaf over $\mathrm{Spec}(F_0)$. By Proposition

6.4, [26], we have an isomorphism:

$$\mathrm{Gr}_{\underline{\mathscr{G}},\mathcal{O}_0[u]} \times_{\mathcal{O}_0[u]} F_0 \xrightarrow{\sim} \mathrm{Gr}_{\underline{\mathscr{G}}_{F_0,\pi_0}},$$

given by $u \mapsto \pi_0$. Combining this isomorphism with Proposition 6.1.3, we obtain $\mathrm{Gr}_{\underline{\mathscr{G}}_{F_0,\pi_0}} \simeq \mathrm{Gr}_G$.

Now we define the PZ-local models. Suppose that $\{\mu\}$ is a geometric conjugacy class of one parameter subgroups of G. Let E be the reflex field of $(G, \{\mu\})$. Since G is quasi-split over the maximal unramified extension $\tilde{F}_0$, we can find a representative of μ over $E' := E\tilde{F}_0$ such that $\mu : \mathbb{G}_{m,E'} \to G_{E'} = G \otimes_{F_0} E'$. Notice that μ gives an $E[z^{\pm 1}]$-valued point of $G_{E'}$, therefore an $E'(\!(z)\!)$-valued point of $G_{E'}$. Hence we have an E'-valued point of the loop group LG. By $\underline{\mathscr{G}}_{F_0,\pi_0} \simeq G \otimes_{F_0} F_0[\![t]\!]$, we have an isomorphism:

$$G(F_0(\!(z)\!)) \xrightarrow{\sim} \underline{\mathscr{G}}_{F_0,\pi_0}(F_0(\!(z)\!)) = \underline{\mathscr{G}}_{F_0,\pi_0}(F_0(\!(t)\!)).$$

We denote by s_μ the corresponding E'-valued point in $L\underline{\mathscr{G}}_{F_0,\pi_0}$, and $[s_\mu]$ the corresponding point in the affine Grassmannian $\mathrm{Gr}_{\underline{\mathscr{G}}_{F_0,\pi_0}} \times_{F_0} E'$. Consider the $L^+\underline{\mathscr{G}}_{F_0,\pi_0}$-orbit: $(L^+\underline{\mathscr{G}}_{F_0,\pi_0})_{E'} \cdot [s_\mu]$. This orbit is contained in $\mathrm{Gr}_{\underline{\mathscr{G}}_{F_0,\pi_0}} \times_{F_0} E'$, which by Theorem 6.1.1, can be identified with the generic fiber of $\mathrm{Gr}_{\underline{\mathscr{G}},\mathcal{O}_0} \times_{\mathcal{O}_0} \mathcal{O}_{E'}$. Since the conjugacy class of μ is defined over E, the same is true for the orbit $(L^+\underline{\mathscr{G}}_{F_0,\pi_0})_{E'} \cdot [s_\mu]$: There is an E-subvariety X_μ of $\mathrm{Gr}_{\underline{\mathscr{G}}_{F_0,\pi_0}} \times_{F_0} E$ such that

$$X_\mu \times_E E' = (L^+\underline{\mathscr{G}}_{F_0,\pi_0})_{E'} \cdot [s_\mu].$$

Definition 7.1.1. *The PZ-local model $M_{\underline{\mathscr{G}},\mu}$ is the reduced scheme over* $\mathrm{Spec}(\mathcal{O}_E)$ *which* *is the Zariski closure of the orbit* X_μ *in the ind-scheme* $\mathrm{Gr}_{\underline{\mathscr{G}},\mathcal{O}_E} = \mathrm{Gr}_{\underline{\mathscr{G}},\mathcal{O}[u]} \times_{\mathcal{O}[u]} \mathcal{O}_E$ *over* $\mathrm{Spec}(\mathcal{O}_E)$.

7.2 PZ-local models for triality groups

We continued with the same notations in §6.2. That is, let $K/\mathbb{Q}_p$ be a cubic tamely ramified field extension, $p \neq 2, 3$. Let $F_0 = \mathbb{Q}_p^{unr}$, $F = KF_0$ with the valuation rings $\mathcal{O}_0, \mathcal{O}$ respectively. Let π_0 (resp. π) be a uniformizer of $\mathcal{O}_0$ (resp. $\mathcal{O}$) with $\pi^3 = \pi_0$. Then F/F_0 is a cubic Galois extension, and $\mathcal{O} = \mathcal{O}_0[\pi]$. The corresponding Galois group $\Gamma = \mathrm{Gal}(F/F_0) = \langle \rho \rangle$, where $\rho(\pi) = \pi\xi$.

Recall that we define the parahoric group given by the standard lattice $\mathbb{L}$:

$$\underline{\mathscr{G}} := (\mathrm{Res}_{\mathcal{O}_0[v]/\mathcal{O}_0[u]}(\mathscr{H} \otimes_{\mathbb{Z}} \mathcal{O}_0[v]))^{\Gamma},$$

as a smooth affine group scheme over $\mathbf{A}^1_{\mathcal{O}_0} = \mathrm{Spec}(\mathcal{O}_0[u])$ (Here $\mathscr{H}$ is the Chevalley form of H). We described the global affine Grassmannian as a fpqc sheaf:

$$\mathrm{Gr}_{\underline{\mathscr{G}}} := L\underline{\mathscr{G}}/L^{+}\underline{\mathscr{G}},$$

and there is a natural identification between the points in $\mathrm{Gr}_{\underline{\mathscr{G}}}(R)$ and the set of $R[\![u - \pi_0]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v]$-lattices satisfying conditions (1)-(4) in §6.2. To describe PZ-local models for triality groups, we need to fix some coweights of $\mathbf{GO}_8$. Set $t = u - \pi_0$.

Suppose that $\{\mu_i\}_{i=1,2,3}$ are coweights of $\mathbf{GO}_8$. We fix $\mu_i : \mathbb{G}_{m,F} \to \mathbf{GO}_8(V, q)$ given by

$$\mu_1(t) = \mathrm{diag}(t^{-1}, 1, 1, 1, 1, 1, 1, t),$$

$$\mu_2(t) = \mathrm{diag}(t^{-1}, t^{-1}, t^{-1}, 1, t^{-1}, 1, 1, 1),$$

$$\mu_2(t) = \mathrm{diag}(t^{-1}, t^{-1}, t^{-1}, t^{-1}, 1, 1, 1, 1).$$

For any $\mathcal{O}$-algebra R, denote by $\mathbb{L}_i = \oplus_{k=1}^{8} R[\![t]\!] e_k$ the standard lattices in the vector space $V_{R(\!(t)\!)} := \oplus_{k=1}^{8} R(\!(t)\!) e_k$ for $i = 1, 2, 3$, where $\{e_k\}_{k=1,...,8}$ satisfies the multiplication in Table 2.1. Similarly, denote $\mathbb{L}_{i,F} = \oplus_{k=1}^{8} F[\![t]\!] e_k$ the base change $\mathbb{L}_i \otimes_R F$. There is a $R(\!(t)\!)$-bilinear form $\langle\ ,\ \rangle : V_{R(\!(t)\!)} \times V_{R(\!(t)\!)} \to R(\!(t)\!)$ given by $\langle e_i, e_{9-j} \rangle = \delta_{ij}$. Let $\mathscr{L}_i(0) = \mu_i(t)\mathbb{L}_i$, and $\mathscr{L}_{i,F}(0) = \mu_i(t)\mathbb{L}_{i,F}$, i.e.,

$$\mathscr{L}_1(0) = R(\!(t)\!)\langle t^{-1}e_1, e_2, ..., e_7, te_8 \rangle,$$

$$\mathscr{L}_2(0) = R(\!(t)\!)\langle t^{-1}e_i, e_j \rangle_{i=1,2,3,5,\ j=4,6,7,8},$$

$$\mathscr{L}_2(0) = R(\!(t)\!)\langle t^{-1}e_i, e_j \rangle_{i=1,2,3,4,\ j=5,6,7,8}.$$

In what follows, we will use the isomorphism $F \otimes_{F_0} F \cong F \times F \times F$. Consider the embeddings $\varphi_i : F \to F \otimes_{F_0} F$ for $i = 1, 2$, which are given by $\varphi_1(f) = f \otimes 1$, $\varphi_2(f) = 1 \otimes f$. We use F_1, F_2 to denote the two isomorphic copies of F, obtained as the image of the embeddings $\varphi_i : F \to F \otimes_{F_0} F$, i.e., $F_1 = \varphi_1(F), F_2 = \varphi_2(F)$. In this identification, we set the uniformizers $\pi = \pi \otimes 1 \in \mathcal{O}_{F_1}, \pi_0^{1/3} = 1 \otimes \pi \in \mathcal{O}_{F_2}$. Notice that there are isomorphisms

$$F_0[\![t]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v] \cong F_1[\![z]\!] \cong F_1[\![t]\!],$$

where the first map is given by $v \mapsto \pi(1+z)$, and the second map is given by

$$z \mapsto (1 + \frac{t}{p})^{1/3} - 1 = \frac{t}{3p} - \frac{t^2}{9p^2} + \frac{5t^3}{81p^3} + ... \in F_2[\![t]\!]. \tag{7.2.1}$$

Applying the tensor product $F_2 \otimes_{F_0} -$ on both sides. We obtain:

$$\mathcal{O}_0[v] \otimes_{\mathcal{O}_0[u]} F_2[\![t]\!] \tilde{\to} F_1 \otimes_{F_0} F_2[\![t]\!] \tilde{\to} F_2[\![t]\!] \times F_2[\![t]\!] \times F_2[\![t]\!],$$

the first part given by $v \mapsto \pi(1+z)$, and the second part given by $\pi \mapsto (\pi_0^{1/3}, \pi_0^{1/3}\xi, \pi_0^{1/3}\xi^2)$. Recall that $\mathbb{L} = \oplus_{k=1}^{8}(R[\![u-\pi_0]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v])e_k$ is the standard lattice in $\tilde{V}_{R(\!(u-\pi_0)\!)}$. Set

$$\mathscr{L}(0) = \mu_1(v-\pi_0^{1/3})\mu_2(v-\pi_0^{1/3}\xi)\mu_3(v-\pi_0^{1/3}\xi^2)\mathbb{L},$$

i.e, $\mathscr{L}(0)$ is a free $R[\![t]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v]$-module with the basis:

$$\left\{\frac{1}{u-\pi_0}e_1, \frac{1}{v^2+\pi_0^{1/3}v+\pi_0^{2/3}}e_i, \frac{1}{v-\pi_0^{1/3}\xi^2}e_4, \frac{1}{v-\pi_0^{1/3}\xi}e_5, e_j, (v-\pi_0^{1/3})e_8\right\}_{i=2,3,\ j=6,7}.$$

When $R = F$, we denote by $\mathscr{L}(0)_F$ the $F[\![t]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v]$-lattice with the same basis as above. By $\mathcal{O}_0[v] \otimes_{\mathcal{O}_0[u]} F[\![t]\!] \simeq F[\![t]\!]^{\times 3}$, we can check that $v-\pi_0^{1/3}$ maps to $(\pi_0^{1/3}z, \pi_0^{1/3}(\xi - 1 + \xi z), \pi_0^{1/3}(\xi^2 - 1 + \xi^2 z))$. It is easy to see that $\xi - 1 + \xi z$ and $\xi^2 - 1 + \xi^2 z$ are units in $F[\![t]\!]$. Hence $(\xi - 1 + \xi z)F[\![t]\!] = (\xi^2 - 1 + \xi^2 z)F[\![t]\!] = F[\![t]\!]$, and $zF_1[\![t]\!] = tF_1[\![t]\!]$ by Equation (7.2.1). Therefore,

$$(F[\![t]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v])(v-\pi_0^{1/3}) \cong (t,1,1)F[\![t]\!]^{\times 3}.$$

Similarly, we have $(F[\![t]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v])(v-\pi_0^{1/3})^{-1} \cong (t^{-1},1,1)F[\![t]\!]^{\times 3}$, and

$$(F[\![t]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v])(v-\pi_0^{1/3}\xi)^{\pm 1} \cong (1,t^{\pm 1},1)F[\![t]\!]^{\times 3},$$
$$(F[\![t]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v])(v-\pi_0^{1/3}\xi^2)^{\pm 1}e_i \cong (1,1,t^{\pm 1})F[\![t]\!]^{\times 3}.$$

Combining the above results, we obtain:

$$\mathscr{L}(0)_F \cong (\mathscr{L}_{1,F}(0), \mathscr{L}_{2,F}(0), \mathscr{L}_{3,F}(0)),$$

by $\mathcal{O}_0[v] \otimes_{\mathcal{O}_0[u]} F[\![t]\!] \simeq F[\![t]\!]^{\times 3}$.

Proposition 7.2.2. *For any $\mathcal{O}$-algebra R, we have:*

(1) $(u-\pi_0)\mathbb{L} \subset \mathscr{L}(0) \subset (u-\pi_0)^{-1}\mathbb{L}$, and $\mathscr{L}(0)$ is a $R[\![u-\pi_0]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v]$-lattice satisfying conditions (1)-(4) in §6.2, i.e., $\mathscr{L}(0) \in \mathrm{Gr}_{\underline{\mathscr{G}}}(R)$.

(2) $\mathscr{L}_i(0)$ satisfy the following diagrams:

(i)

$$\begin{tikzcd}
 & \mathbb{L}_1 & \\
(u-\pi_0)\mathbb{L}_1 \subset \mathscr{L}_1(0) \cap \mathbb{L}_1 \arrow[ur, phantom, sloped, "\subset"] \arrow[dr, phantom, sloped, "\supset"] & & \mathscr{L}_1(0) + \mathbb{L}_1 \subset (u-\pi_0)^{-1}\mathbb{L}_1, \arrow[ul, phantom, sloped, "\supset"] \arrow[dl, phantom, sloped, "\subset"] \\
 & \mathscr{L}_1(0) &
\end{tikzcd}$$

where the quotients arising from all slanted inclusions are generated as $\mathcal{O}$-modules by one element (we say that they have rank 1), and the quotients from $(u-\pi_0)\mathbb{L}_1 \subset \mathscr{L}_1(0) \cap \mathbb{L}_1$, $\mathscr{L}_1(0) + \mathbb{L}_1 \subset (u-\pi_0)^{-1}\mathbb{L}_1$ have rank 7.

(ii)

$$(u-\pi_0)\mathbb{L}_2 \subset \mathscr{L}_2(0)^\vee \subset \mathbb{L}_2 \subset \mathscr{L}_2(0) \subset (u-\pi_0)^{-1}\mathbb{L}_2,$$
$$(u-\pi_0)\mathbb{L}_3 \subset \mathscr{L}_3(0)^\vee \subset \mathbb{L}_3 \subset \mathscr{L}_3(0) \subset (u-\pi_0)^{-1}\mathbb{L}_3,$$

where $L_i(0)^\vee$ is the dual lattice of $L_i(0)$ under the bilinear form: $\langle\ ,\ \rangle : V_{R(\!(t)\!)} \times V_{R(\!(t)\!)} \to R(\!(t)\!)$. The quotients arising from all inclusions have rank 4.

(3) The triple $((\mathscr{L}_{1,F}(0)+\mathbb{L}_{1,F})/\mathbb{L}_{1,F}, \mathscr{L}_{2,F}(0)/\mathbb{L}_{2,F}, \mathscr{L}_{3,F}(0)/\mathbb{L}_{3,F})$ *is isomorphic to the triality triple* $(Fe_1, V{*}Fe_1, Fe_1{*}V)$*. Recall that* $(V,*)$ *is the normal twisted composition algebra obtained from the split para-Cayley algebra.*

Proof. (1) $(u-\pi_0)\mathbb{L} \subset \mathscr{L}(0) \subset (u-\pi_0)^{-1}\mathbb{L}$ is directly from defintion. We can check that $\mathscr{L}(0) * \mathscr{L}(0) \subset \lambda\mathscr{L}(0)$ with $\lambda = (v-\pi_0^{1/3})^{-1}$. In fact, let

$$f_1 = a = \frac{\xi^2}{v-\pi_0^{1/3}\xi^2}e_4, \quad f_2 = \lambda^{-1}(a*a) = \frac{\xi}{v-\pi_0^{1/3}\xi}e_5.$$

We have $q(a)=0, \langle a, a*a\rangle = (u-\pi_0)^{-1}$, and there exist a basis of $\mathscr{L}(0)$ such that:

$$\lambda^{-1}(\tfrac{1}{u-\pi_0}e_1 * f_1) = -\tfrac{1}{u-\pi_0}e_1, \quad \lambda^{-1}(\tfrac{1}{v^2+\pi_0^{1/3}v+\pi_0^{2/3}}e_2 * f_2) = -\tfrac{1}{v^2+\pi_0^{1/3}v+\pi_0^{2/3}}e_2,$$

$$\lambda^{-1}(e_6 * f_1) = -e_6, \qquad \lambda^{-1}(\tfrac{1}{v^2+\pi_0^{1/3}v+\pi_0^{2/3}}e_3 * f_2) = -\tfrac{1}{v^2+\pi_0^{1/3}v+\pi_0^{2/3}}e_3,$$

$$\lambda^{-1}(e_7 * f_1) = -e_7, \qquad \lambda^{-1}((v-\pi_0^{1/3})e_8 * f_2) = -(v-\pi_0^{1/3})e_8.$$

Then $(\mathscr{L}(0), *)$ satisfies conditions (1)-(4) in §6.2, i.e., $\mathscr{L}(0) \in \mathrm{Gr}_{\underline{\mathscr{G}}}(R)$.

(2) We have

$$\mathscr{L}_2(0)^\vee = R[\![t]\!]\langle e_i, te_j\rangle_{i=1,2,3,5,\ j=4,6,7,8},$$
$$\mathscr{L}_3(0)^\vee = R[\![t]\!]\langle e_i, te_j\rangle_{i=1,2,3,4,\ j=5,6,7,8}.$$

One can check the diagrams directly from that.

(3) It is easy to see that

$$(\mathscr{L}_{1,F}(0)+\mathbb{L}_{1,F})/\mathbb{L}_{1,F}=(t^{-1}F[\![t]\!]/F[\![t]\!])e_1\cong Fe_1,$$

$$\mathscr{L}_{2,F}(0)/\mathbb{L}_{2,F}=(t^{-1}F[\![t]\!]/F[\![t]\!])\langle e_1,e_2,e_3,e_5\rangle\cong Fe_1+Fe_2+Fe_3+Fe_5,$$

$$\mathscr{L}_{3,F}(0)/\mathbb{L}_{3,F}=(t^{-1}F[\![t]\!]/F[\![t]\!])\langle e_1,e_2,e_3,e_4\rangle\cong Fe_1+Fe_2+Fe_3+Fe_4.$$

So $((\mathscr{L}_{1,F}(0)+\mathbb{L}_{1,F})/\mathbb{L}_{1,F},\mathscr{L}_{2,F}(0)/\mathbb{L}_{2,F},\mathscr{L}_{3,F}(0)/\mathbb{L}_{3,F})\cong(Fe_1,V*Fe_1,Fe_1*V)$. $\square$

Set $s_\mu=\mu_1(v-\pi_0^{1/3})\mu_2(v-\pi_0^{1/3}\xi)\mu_3(v-\pi_0^{1/3}\xi^2)$. Let $X_\mu=(L^+\underline{\mathscr{G}})\mathscr{L}(0)$ be the orbit of $\mathscr{L}(0)=s_\mu\mathbb{L}$. We now define the PZ-local model for triality groups:

Definition 7.2.3. *The PZ-local model $M(\mu)$ is the Zariski closure of X_μ in the induced scheme $\mathrm{Gr}_{\mathscr{G},\mathcal{O}}=\mathrm{Gr}_{\underline{\mathscr{G}}}\times_{\mathcal{O}_0[u]}\mathcal{O}$ over $\mathrm{Spec}(\mathcal{O})$.*

For any $\mathscr{L}\in M(\mu)(R)$, we have a similar proposition as Proposition 7.2.2:

Proposition 7.2.4. *(1) $(u-\pi_0)\mathbb{L}\subset\mathscr{L}\subset(u-\pi_0)^{-1}\mathbb{L}$.*

(2) Let $\mathscr{L}_F$ be any point of the generic fiber of $M(\mu)$. Then $\mathscr{L}_F$ is a $F[\![t]\!]\otimes_{\mathcal{O}_0[u]}\mathcal{O}_0[v]$-lattice. If $\mathscr{L}_F=(\mathscr{L}_{1,F},\mathscr{L}_{2,F},\mathscr{L}_{3,F})$ given by $F[\![t]\!]\otimes_{\mathcal{O}_0[u]}\mathcal{O}_0[v]\simeq F[\![t]\!]^{\times 3}$, then $\mathscr{L}_{i,F}$ satisfy the following diagrams:

(i)

$$\begin{tikzcd}
 & \mathbb{L}_{1,F} \arrow[dr, phantom, sloped, "\subset"] & \\
(u-\pi_0)\mathbb{L}_{1,F} \subset \mathscr{L}_{1,F} \cap \mathbb{L}_{1,F} \arrow[ur, phantom, sloped, "\subset"] \arrow[dr, phantom, sloped, "\subset"] & & \mathscr{L}_{1,F} + \mathbb{L}_{1,F} \subset (u-\pi_0)^{-1}\mathbb{L}_{1,F}, \\
 & \mathscr{L}_{1,F} \arrow[ur, phantom, sloped, "\subset"] &
\end{tikzcd}$$

(ii)

$$(u-\pi_0)\mathbb{L}_{2,F} \subset \mathscr{L}_{2,F}^{\vee} \subset \mathbb{L}_{2,F} \subset \mathscr{L}_{2,F} \subset (u-\pi_0)^{-1}\mathbb{L}_{2,F},$$

(iii)

$$(u-\pi_0)\mathbb{L}_{3,F} \subset \mathscr{L}_{3,F}^{\vee} \subset \mathbb{L}_{3,F} \subset \mathscr{L}_{3,F} \subset (u-\pi_0)^{-1}\mathbb{L}_{3,F},$$

such that $((\mathscr{L}_{1,F}+\mathbb{L}_{1,F})/\mathbb{L}_{1,F}, \mathscr{L}_{2,F}/\mathbb{L}_{2,F}, \mathscr{L}_{3,F}/\mathbb{L}_{3,F})$ *is isomorphic to the triality triple* $(l, V * l, l * V)$ *for some isotropic line* $l \in V$.

Proof. (1) is obvious. For (2), consider

$$L^+\underline{\mathscr{G}}(F) = \{g = (g_1, g_2, g_3) \in \mathrm{GO}_8(F[\![t]\!]) \mid \lambda_i g_i(x * y) = g_{i+1}(x) * g_{i+2}(y) \bmod 3\}.$$

Suppose that $\mathscr{L}_F = (\mathscr{L}_{1,F}, \mathscr{L}_{2,F}, \mathscr{L}_{3,F}) = g(\mathscr{L}_{1,F}(0), \mathscr{L}_{2,F}(0), \mathscr{L}_{3,F}(0))$ for some $g = (g_1, g_2, g_3) \in L^+\underline{\mathscr{G}}(F)$, then

$$\begin{aligned}
&g_1(\mathscr{L}_{1,F}(0)) + \mathbb{L}_{1,F}/\mathbb{L}_{1,F} = \bar{g}_1(\mathscr{L}_{1,F}(0)) + \mathbb{L}_{1,F}/\mathbb{L}_{1,F}),\\
&g_2(\mathscr{L}_{2,F}(0))/\mathbb{L}_{2,F} = \bar{g}_2(\mathscr{L}_{2,F}(0)/\mathbb{L}_{2,F}),\\
&g_2(\mathscr{L}_{3,F}(0))/\mathbb{L}_{3,F} = \bar{g}_3(\mathscr{L}_{3,F}(0)/\mathbb{L}_{3,F}),
\end{aligned}$$

where $\bar{g}_i = g_i \bmod t$. Denote by l the isotropic line $\bar{g}_1(Fe_1) = \bar{g}_1(\mathscr{L}_{1,F}(0)) + \mathbb{L}_{1,F}/\mathbb{L}_{1,F})$. We get $\bar{g}_2(\mathscr{L}_{2,F}(0)/\mathbb{L}_{2,F}) = \bar{g}_2(V * Fe_1) = V * \bar{g}_1(Fe_1) = V * l$, and $\bar{g}_3(\mathscr{L}_{3,F}(0)/\mathbb{L}_{3,F}) = \bar{g}_3(Fe_1 * V) = \bar{g}_1(Fe_1) * V = l * V$, hence

$$((\mathscr{L}_{1,F} + \mathbb{L}_{1,F})/\mathbb{L}_{1,F}, \mathscr{L}_{2,F}/\mathbb{L}_{2,F}, \mathscr{L}_{3,F}/\mathbb{L}_{3,F}) \cong (l, V * l, l * V).$$

□

Chapter 8

Splitting models for triality groups

The original purpose of introducing splitting models is to modify local models in the ramified case, so that the modified models are flat and have reasonable singularities. Pappas and Zhu discuss the cases where the quasi-split form of G is the general linear group GL_d or the general symplectic group GSp_n. In the following sections, we will consider the splitting model for triality groups. We will see that it is isomorphic to the blow-up of some hypersurface scheme.

8.1 Definition of splitting models for triality groups

Suppose R is an $\mathcal{O}$-algebra. Recall that we set $s_\mu = \mu_1(v-\pi_0^{1/3})\mu_2(v-\pi_0^{1/3}\xi)\mu_3(v-\pi_0^{1/3}\xi^2)$ and $\mathscr{L}(0) = s_\mu\mathbb{L}$ in §7.2. The PZ-local models for triality groups $M(\mu)$ is the Zariski closure of the orbit $X_\mu = (L^+\underline{\mathscr{G}})\mathscr{L}(0)$. To define splitting models, we consider "partial resolutions of $M(\mu)$". More precisely, set

$$\begin{aligned}
\mathscr{L}^{(3)}(0) &= \mathscr{L}(0) + \mathbb{L},\\
\mathscr{L}^{(2)}(0) &= \mu_1(v-\pi_0^{1/3})\mu_2(v-\pi_0^{1/3}\xi)\mathbb{L} + \mathbb{L},\\
\mathscr{L}^{(1)}(0) &= \mu_1(v-\pi_0^{1/3})\mathbb{L} + \mathbb{L}.
\end{aligned}$$

Then we obtain:

$$\mathbb{L} \subset \mathscr{L}^{(1)}(0) \subset \mathscr{L}^{(2)}(0) \subset \mathscr{L}^{(3)}(0) \subset (u-\pi_0)^{-1}\mathbb{L}.$$

We have the following propositions for $\{\mathscr{L}^{(i)}(0)\}_{i=1,2,3}$:

(1) $(v-\pi_0^{1/3}\xi^2)\mathscr{L}^{(3)}(0) \subset \mathscr{L}^{(2)}(0)$, $\quad (v-\pi_0^{1/3}\xi)\mathscr{L}^{(2)}(0) \subset \mathscr{L}^{(1)}(0)$, $\quad (v-\pi_0^{1/3})\mathscr{L}^{(1)}(0) \subset \mathbb{L}$.

(2) $\mathbb{L} * \mathscr{L}^{(1)}(0) \subset \mathscr{L}^{(2)}(0)$, $\quad \mathscr{L}^{(1)}(0) * \mathbb{L} \subset \mathscr{L}^{(3)}(0)$.

(3) $\mathscr{L}(0)$ is self dual with respect to the form $\langle\ ,\ \rangle : \mathscr{L}(0) \otimes \mathscr{L}(0) \to \rho(\lambda)\theta(\lambda)(R[\![t]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v])$, where $\lambda = (v-\pi_0^{1/3})^{-1}$.

These propositions are directly from the definitions of $\mathscr{L}^{(i)}(0)$ we set. Generally, for any $\mathscr{L}$ in the orbit X_μ, we have $\mathscr{L} = g(\mathscr{L}(0))$ for some g in

$$L^+\underline{\mathscr{G}}(R) = \{g \in \mathrm{GO}_8^+(R[\![t]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v]) \mid \text{there exist } \lambda \text{ such that } \lambda g(x*y) = g(x)*g(y)\}$$

for $\lambda \in (R[\![t]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v])^*$, $x, y \in \mathbb{L}$. Set $\mathscr{L}^{(i)} = g(\mathscr{L}^{(i)}(0))$ for $i = 1,2,3$. We observe that $\{\mathscr{L}^{(i)}\}_{i=1,2,3}$ have the same propositions as above:

(1) $\mathbb{L} \subset \mathscr{L}^{(1)} \subset \mathscr{L}^{(2)} \subset \mathscr{L}^{(3)} \subset (u-\pi_0)^{-1}\mathbb{L}$.

(2) $(v-\pi_0^{1/3}\xi^2)\mathscr{L}^{(3)} \subset \mathscr{L}^{(2)}$, $\quad (v-\pi_0^{1/3}\xi)\mathscr{L}^{(2)} \subset \mathscr{L}^{(1)}$, $\quad (v-\pi_0^{1/3})\mathscr{L}^{(1)} \subset \mathbb{L}$.

(3) $\mathbb{L} * \mathscr{L}^{(1)} \subset \mathscr{L}^{(2)}$, $\quad \mathscr{L}^{(1)} * \mathbb{L} \subset \mathscr{L}^{(3)}$.

(1) and (2) are directly from $\mathscr{L}^{(i)} = g(\mathscr{L}^{(i)}(0))$. For (3), consider $\mathbb{L} * \mathscr{L}^{(1)} = g(\mathbb{L}) * g(\mathscr{L}^{(1)}(0)) = \lambda^{-1} g(\mathbb{L} * (\mathscr{L}^{(1)}(0))$, and $g(\mathbb{L} * (\mathscr{L}^{(1)}(0)) \subset g(\mathscr{L}^{(2)}(0)) = \mathscr{L}^{(2)}$. We have

$\mathbb{L} * \mathscr{L}^{(1)} \subset \lambda^{-1}\mathscr{L}^{(2)}$. Since λ is a unit in $R[\![t]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v]$, we obtain $\mathbb{L} * \mathscr{L}^{(1)} \subset \mathscr{L}^{(2)}$. Similarly, $\mathscr{L}^{(1)} * \mathbb{L} \subset \mathscr{L}^{(3)}$.

Furthermore, since $\mathbb{L} \subset \mathscr{L}^{(i)} \subset (u-\pi_0)^{-1}\mathbb{L}$ for $i = 1, 2, 3$, let $\mathcal{F}^i$ be the image of $\mathscr{L}^{(i)}$ under the map:

$$t : (u-\pi_0)^{-1}\mathbb{L} \to \mathbb{L}/(u-\pi_0)\mathbb{L},$$

(recall $t = u - \pi_0$). Set $\Lambda = \mathbb{L}/t\mathbb{L}$. Notice that $\Lambda = \oplus_{i=1}^{8}(R \otimes_{\mathcal{O}_0} \mathcal{O})e_i$, with a bilinear form $\langle\ ,\ \rangle : \Lambda \otimes \Lambda \to R \otimes_{\mathcal{O}_0} \mathcal{O}$ since $\mathcal{O}_0[v]$ is isomorphic to $\mathcal{O}$ by $v^3 = u = \pi_0$. Therefore, we have $R \otimes_{\mathcal{O}_0} \mathcal{O}$-modules $\mathcal{F}^i$ satisfying

$$0 \subset \mathcal{F}^1 \subset \mathcal{F}^2 \subset \mathcal{F}^3 \subset \Lambda.$$

By $R \otimes_{\mathcal{O}_0} \mathcal{O} = R \oplus R\pi \oplus R\pi^2$, we can view Λ as a 24-rank R free module with basis $\{\pi^2 e_i, \pi e_i, e_i\}_{i=1,\ldots,8}$. In particular, the image of $\mathscr{L}^{(i)}(0)$ are:

$$\mathcal{F}^1(0) = R(\pi^2 e_1 + \pi_0^{1/3}\pi e_1 + \pi_0^{2/3} e_1),$$
$$\mathcal{F}^2(0) = R(\pi e_1 - \pi_0^{1/3}\xi^2 e_1) \oplus R(\pi^2 e_1 - \pi_0^{1/3}\xi^2 \pi e_1) \oplus_{k=2,3,5} R(\pi^2 e_k + \pi_0^{1/3}\xi\pi e_k + \pi_0^{2/3}\xi^2 e_k),$$
$$\begin{aligned}\mathcal{F}^3(0) = {} & Re_1 \oplus R\pi e_1 \oplus R\pi^2 e_1 \oplus_{k=2,3} R(\pi e_k - \pi_0^{1/3} e_k) \oplus_{k=2,3} R(\pi^2 e_k - \pi_0^{1/3}\pi e_k)\\ & \oplus R(\pi^2 e_4 + \pi_0^{1/3}\xi^2\pi e_4 + \pi_0^{2/3}\xi e_4) \oplus R(\pi^2 e_5 + \pi_0^{1/3}\xi\pi e_5 + \pi_0^{2/3}\xi^2 e_5),\end{aligned}$$

with $\mathrm{rank}(\mathcal{F}^1) = 1$, $\mathrm{rank}(\mathcal{F}^2) = 5$, $\mathrm{rank}(\mathcal{F}^3) = 9$, when we view $\mathcal{F}^i$ as R-modules.

Collections $\{\mathcal{F}^i\}_{i=1,2,3}$ satisfy some similar propositions as $\{\mathscr{L}^{(i)}\}$. By viewing $\mathcal{F}^i$ as $t\mathscr{L}^{(i)}/t\mathbb{L}$, we have $\Lambda * \mathcal{F}^1 = \mathbb{L} * t\mathscr{L}^{(1)} \mod t\mathbb{L} \subset t\mathscr{L}^{(2)} \mod t\mathbb{L} \subset \mathcal{F}^2$. Similarly $\mathcal{F}^1 * \Lambda \subset \mathcal{F}^3$. Hence:

(1) $0 \subset \mathcal{F}^1 \subset \mathcal{F}^2 \subset \mathcal{F}^3 \subset \Lambda$.

(2) $(\pi - \pi_0^{1/3}\xi^2)\mathcal{F}^3 \subset \mathcal{F}^2, \quad (\pi - \pi_0^{1/3}\xi)\mathcal{F}^2 \subset \mathcal{F}^1, \quad (\pi - \pi_0^{1/3})\mathcal{F}^1 = 0.$

(3) $\Lambda * \mathcal{F}^1 \subset \mathcal{F}^2, \quad \mathcal{F}^1 * \Lambda \subset \mathcal{F}^3$.

Finally, we claim that $\mathcal{F}^i$ are isotropic under the bilinear form: $\langle\ ,\ \rangle : \Lambda \otimes \Lambda \to R \otimes_{\mathcal{O}_0} \mathcal{O}$. Consider the bilinear form $\langle\ ,\ \rangle : \mathscr{L}(0) \otimes \mathscr{L}(0) \to \rho(\lambda)\theta(\lambda)(R[\![t]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v])$, where $\lambda = (v - \pi_0^{1/3})^{-1}$. Since $\mathscr{L}^{(3)}(0) = \mathscr{L}(0) + \mathbb{L}$, we get

$$\langle x, y\rangle \in (u - \pi_0)^{-1}(R[\![t]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v]),$$

for all $x, y \in \mathscr{L}^{(3)}(0)$. It keeps the same for $x, y \in \mathscr{L}^{(3)}$. Therefore, $\langle tx, ty\rangle \in t(R[\![t]\!] \otimes_{\mathcal{O}_0[u]} \mathcal{O}_0[v])$ for all $x, y \in \mathscr{L}^{(3)}$. Set $x' = tx \mod t\mathbb{L}, y' = ty \mod t\mathbb{L}$. We obtain

$$\langle x', y'\rangle = 0, \text{ for all } x', y' \in \mathcal{F}^3.$$

Definition 8.1.1. *Suppose that $\mathcal{M}$ is the functor from $\mathcal{O}$-algebras to sets that sends R to $\mathcal{M}(R)$ of collections $\{\mathcal{F}^i\}_{i=1,2,3}$, where $\mathcal{F}^i$ are $R \otimes_{\mathcal{O}_0} \mathcal{O}$-submodules of Λ, which fit into:*

$$0 \subset \mathcal{F}^1 \subset \mathcal{F}^2 \subset \mathcal{F}^3 \subset \Lambda$$

such that:

(1) $\mathcal{F}^i$ are locally direct summand of Λ, with rank($\mathcal{F}^1$) $= 1$, rank($\mathcal{F}^2$) $= 5$, rank($\mathcal{F}^3$) $= 9$ when we view $\mathcal{F}^i$ as R-modules.

(2) $(\pi - \pi_0^{1/3}\xi^2)\mathcal{F}^3 \subset \mathcal{F}^2, \quad (\pi - \pi_0^{1/3}\xi)\mathcal{F}^2 \subset \mathcal{F}^1, \quad (\pi - \pi_0^{1/3})\mathcal{F}^1 = 0.$

(3) $\Lambda * \mathcal{F}^1 \subset \mathcal{F}^2, \quad \mathcal{F}^1 * \Lambda \subset \mathcal{F}^3$.

(4) $\langle x, y\rangle = 0$, *for all* $x, y \in \mathcal{F}^3$.

We call $\mathcal{M}$ the "naive splitting model for triality groups". Unfortunately, the scheme $\mathcal{M}$ is not flat, so we need to consider its flat closure, and call it the splitting model for triality groups:

Definition 8.1.2. *The splitting model for triality groups* $\mathcal{M}^{\mathrm{split}}$ *is the flat closure of* $\mathcal{M}$.

8.2 Blow-up of a quadratic hypersurface

In this section, we will prove the splitting model $\mathcal{M}^{\mathrm{split}}$ is isomorphic to the blow-up of a quadratic hypersurface. Before we move on to our main result, let us consider the generic fiber of $\mathcal{M}^{\mathrm{split}}$. Recall that the isomorphism $F \otimes_{F_0} F \simeq F \times F \times F$ is given by $r_1 \otimes r_2 \mapsto (r_1 r_2, \rho(r_1) r_2, \theta(r_1) r_2)$, and we use F_1, F_2 to denote the two isomorphic copies of F, obtained by the image of the embeddings $\varphi_i : F \to F \otimes_{F_0} F$, where $\varphi_1(f) = f \otimes 1$, $\varphi_2(f) = 1 \otimes f$. Set the uniformizers $\pi = \pi \otimes 1 \in \mathcal{O}_{F_1}, \pi_0^{1/3} = 1 \otimes \pi \in \mathcal{O}_{F_2}$. By this isomorphism, we get:

$$
\begin{aligned}
&\frac{1}{3\pi_0^{2/3}}(\pi^2 + \pi_0^{1/3}\pi + \pi_0^{2/3}) \mapsto (1, 0, 0),\\
&\frac{\xi}{3\pi_0^{2/3}}(\pi^2 + \pi_0^{1/3}\xi\pi + \pi_0^{2/3}\xi^2) \mapsto (0, 1, 0),\\
&\frac{\xi^2}{3\pi_0^{2/3}}(\pi^2 + \pi_0^{1/3}\xi^2\pi + \pi_0^{2/3}\xi) \mapsto (0, 0, 1).
\end{aligned}
$$

In §8.1, we set $\Lambda = \mathbb{L}/t\mathbb{L} = \oplus_{i=1}^{8}(R \otimes_{\mathcal{O}_0} \mathcal{O})e_i$, with a bilinear form $\langle\, ,\, \rangle : \Lambda \otimes \Lambda \to R \otimes_{\mathcal{O}_0} \mathcal{O}$ for any $\mathcal{O}$-algebra R. Let $\Lambda_F = \oplus_{i=1}^{8}(F \otimes_{\mathcal{O}_0} \mathcal{O})e_i$ be a 24 dimension F-vector space. Consider the linear isomorphism $\pi : \Lambda_F \to \Lambda_F$, where π is represented by the matrix:

$$\pi = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ \pi_0 I & 0 & 0 \end{pmatrix}$$

with respect to the order of basis $\{\pi^2 e_1, ..., \pi^2 e_8, \pi e_1, ..., \pi e_8, ..., e_1, ..., e_8\}$. Here I is the 8×8 identity matrix. We call this order the standard order of basis.

By the characteristic equation $\pi^3 - \pi_0 = (\pi - \pi_0^{1/3})(\pi - \pi_0^{1/3}\xi)(\pi - \pi_0^{1/3}\xi^2) = 0$, it is easy to see that π has 3 eigenvalues $\pi_0^{1/3}, \pi_0^{1/3}\xi$, and $\pi_0^{1/3}\xi^2$. Since

$$\begin{aligned}
&\pi(\pi^2 + \pi_0^{1/3}\pi + \pi_0^{2/3}) = \pi_0^{1/3}(\pi^2 + \pi_0^{1/3}\pi + \pi_0^{2/3}),\\
&\pi(\pi^2 + \pi_0^{1/3}\xi\pi + \pi_0^{2/3}\xi^2) = \pi_0^{1/3}\xi(\pi^2 + \pi_0^{1/3}\xi\pi + \pi_0^{2/3}\xi^2),\\
&\pi(\pi^2 + \pi_0^{1/3}\xi^2\pi + \pi_0^{2/3}\xi) = \pi_0^{1/3}\xi^2(\pi^2 + \pi_0^{1/3}\xi^2\pi + \pi_0^{2/3}\xi),
\end{aligned} \tag{8.2.1}$$

the eigenvectors $f_i^{\xi^k}$ corresponding to eigenvalues $\pi_0^{1/3}\xi^k$ for $k = 0, 1, 2$, $i = 1, .., 8$ are:

$$\begin{aligned}
f_i &= \pi^2 e_i + \pi_0^{1/3}\pi e_i + \pi_0^{2/3} e_i,\\
f_i^{\xi} &= \pi^2 e_i + \pi_0^{1/3}\xi\pi e_i + \pi_0^{2/3}\xi^2 e_i,\\
f_i^{\xi^2} &= \pi^2 e_i + \pi_0^{1/3}\xi^2\pi e_i + \pi_0^{2/3}\xi e_i.
\end{aligned}$$

These three eigenspaces have a close relation with the generic fiber of the splitting model $\mathcal{M}^{\text{split}}$. In fact, we will show that the generic fiber of $\mathcal{M}^{\text{split}}$ is similar to a triality triple in some sense. For example, consider $\{\mathcal{F}_F^i(0)\}_{i=1,2,3}$. We have:

$$\mathcal{F}_F^1(0) = F(\pi^2 e_1 + \pi_0^{1/3}\pi e_1 + \pi_0^{2/3} e_1),$$

$$\mathcal{F}_F^2(0) = F(\pi e_1 - \pi_0^{1/3}\xi^2 e_1) \oplus F(\pi^2 e_1 - \pi_0^{1/3}\xi^2 \pi e_1) \oplus_{k=2,3,5} F(\pi^2 e_k + \pi_0^{1/3}\xi\pi e_k + \pi_0^{2/3}\xi^2 e_k),$$

$$\begin{aligned}\mathcal{F}_F^3(0) = {} & Fe_1 \oplus F\pi e_1 \oplus F\pi^2 e_1 \oplus_{k=2,3} F(\pi e_k - \pi_0^{1/3} e_k) \oplus_{k=2,3} F(\pi^2 e_k - \pi_0^{1/3}\pi e_k) \\ & \oplus F(\pi^2 e_4 + \pi_0^{1/3}\xi^2\pi e_4 + \pi_0^{2/3}\xi e_4) \oplus F(\pi^2 e_5 + \pi_0^{1/3}\xi\pi e_5 + \pi_0^{2/3}\xi^2 e_5),\end{aligned}$$

By using the notations we just set, it is easy to see $\mathcal{F}_F^1(0) = Ff_1$. Observe that

$$F(\pi e_1 - \pi_0^{1/3}\xi^2 e_1) \oplus F(\pi^2 e_1 - \pi_0^{1/3}\xi^2\pi e_1) = Ff_1 \oplus Ff_1^{\xi},$$

so we get $\mathcal{F}_F^2(0) = Ff_1 \oplus_{k=1,2,3,5} Ff_k^{\xi}$. Similarly,

$$\mathcal{F}_F^3(0) = Ff_1 \bigoplus_{i=1,2,3,5} Ff_i \bigoplus_{j=1,2,3,4} Ff_j^{\xi^2}.$$

Recall that a triality triple is $(l, V * l, l * V)$ for some isotropic line l in $(V, *)$. Let $l = f_1$. Then we have $\Lambda_F * f_1 = \oplus_{i=1,2,3,5} Ff_i^{\xi}$ and $f_1 * \Lambda_F = \oplus_{j=1,2,3,4} Ff_j^{\xi^2}$. So $\mathcal{F}_F^2(0) = \mathcal{F}_F^1(0) + \Lambda_F * \mathcal{F}_F^1(0)$, and $\mathcal{F}_F^3(0) = \mathcal{F}_F^1(0) + \Lambda_F * \mathcal{F}_F^1(0) + \mathcal{F}_F^1(0) * \Lambda_F$. Generally, for any point $\{\mathcal{F}_F^i\}$ in the generic fiber $\mathcal{M}_\eta = \mathcal{M} \otimes_{\mathcal{O}} F$, we have:

Theorem 8.2.2. *The generic fiber $\mathcal{M}_\eta$ has dimension 6. For any $\{\mathcal{F}_F^i\} \in \mathcal{M}_\eta(F)$, we have*

$$\mathcal{F}_F^1 \cong l, \quad \mathcal{F}_F^2 \cong l + \Lambda_F * l, \quad \mathcal{F}_F^3 \cong l + \Lambda_F * l + l * \Lambda_F,$$

in Λ_F, where l is an isotropic line in the $\ker(\pi - \pi_0^{1/3}|_{\Lambda_F})$.

Proof. By $(\pi - \pi_0^{1/3})\mathcal{F}_F^1 = 0$, we have $\mathcal{F}_F^1 = F \cdot l$ for some line l in the $\ker(\pi - \pi_0^{1/3}|_{\Lambda_F})$. Since $\langle \mathcal{F}^3, \mathcal{F}^3 \rangle = 0$, we have l is isotropic. Then $\Lambda_F * \mathcal{F}_F^1$ is a 4-dim isotropic subspaces in the eigenspace corresponding to the eigenvalue $\pi_0^{1/3}\xi$. The generators of $\Lambda_F * \mathcal{F}_F^1$ and $\mathcal{F}_F^1$ are linear independent. So we get $\mathcal{F}_F^2 = l + \Lambda_F * l$ since $\mathrm{rank}(\mathcal{F}_F^2) = 5$. Similarly, $\mathcal{F}_F^1 * \Lambda_F$ is a 4-dim isotropic subspaces in the eigenspace corresponding to the eigenvalue $\pi_0^{1/3}\xi^2$. We have $\mathcal{F}_F^3 = l + \Lambda_F * l + l * \Lambda_F$. $\square$

Now we consider the general case. We have π being a root of the Eisenstein polynomial: $P(T) = T^3 - \pi_0$, and

$$\mathcal{O}[T]/(P(T)) \xrightarrow{\sim} \mathcal{O} \otimes_{\mathcal{O}_0} \mathcal{O},$$

given by $T \mapsto \pi$. For $i = 0, 1, 2$, we set

$$P^i(T) = \prod_{j=i}^{2}(T - \pi_0^{1/3}\xi^j), \quad P_i(T) = \prod_{j=0}^{i-1}(T - \pi_0^{1/3}\xi^j), \quad (P_0(T) := 1)$$

so that $P^0(T) = P(T)$, and $P^i(T)P_i(T) = P(T)$. There are exact sequences

$$\mathcal{O}[T]/(P(T)) \overset{P_i(T)}{\to} \mathcal{O}[T]/(P(T)) \overset{P^i(T)}{\to} \mathcal{O}[T]/(P(T)),$$
$$\mathcal{O}[T]/(P(T)) \overset{P^i(T)}{\to} \mathcal{O}[T]/(P(T)) \overset{P_i(T)}{\to} \mathcal{O}[T]/(P(T)).$$

Thus, by $P_1(T) = T - \pi_0^{1/3}$ $P^1(T) = T^2 + \pi_0^{1/3}T + \pi_0^{2/3}$, we get $\ker(P_1(T) \mid \mathcal{O}[T]/(P(T))) = \mathrm{im}(P^1(T) \mid \mathcal{O}[T]/(P(T)))$. In other word, there is an isomorphism:

$$\ker(\pi - \pi_0^{1/3}|_{\Lambda}) \simeq (\pi^2 + \pi_0^{1/3}\pi + \pi_0^{2/3})\Lambda.$$

Thus, $\ker(\pi - \pi_0^{1/3})$ is a free $\mathcal{O}$-module with basis $\{f_i\}$ for $i = 1, 2, ..., 8$. Consider the conditions that $\mathcal{F}^1$ satisfying

$$\operatorname{rank}(\mathcal{F}^1) = 1, \quad (\pi - \pi_0^{1/3})\mathcal{F}^1 = 0, \quad \langle \mathcal{F}^1, \mathcal{F}^1 \rangle = 0.$$

For any $\mathcal{O}$-algebra R, we can set $\mathcal{F}^1 = \sum_{i=1}^8 x_i f_i \in \ker(\pi - \pi_0^{1/3}|_{\Lambda_R})$ for all $x_i \in R$. By $\langle \mathcal{F}^1, \mathcal{F}^1 \rangle = 0$, we get a quadratic equation $Q_0 = x_1x_8 + x_2x_7 + x_3x_6 + x_4x_5 = 0$. Define a group scheme Q over $\operatorname{Spec}(\mathcal{O})$:

$$Q = \{L \in \oplus_{i=1}^8 \mathcal{O} f_i \mid L \text{ is locally direct summand of } \Lambda \text{ with } \operatorname{rank}(L) = 1, \langle L, L \rangle = 0\}.$$

Then Q is a quadratic hypersurface in $\mathbb{P}^7_{\mathcal{O}}$, with homogeneous coordinate ring:

$$S(Q) = \mathcal{O}[x_1, x_2, ..., x_8]/(x_1x_8 + x_2x_7 + x_3x_6 + x_4x_5).$$

Let $U_i = \{f_i + \sum_{j \neq i} x_j f_j\}$ be affine charts in $\mathbb{P}^7_{\mathcal{O}}$ for $i = 1, 2, ..., 8$. We have $\mathbb{P}^7_{\mathcal{O}} = \cup_{i=1}^8 U_i$. In what follows, we consider affine charts $Q \cap U_i$ in Q and still denote by U_i if there is no confusion.

We have a morphism: $\mathcal{M} \to Q$ given by $\{\mathcal{F}^i\}_{i=1,2,3} \mapsto \mathcal{F}^1$. This is an isomorphism over the generic fiber since $\mathcal{F}^2_F$, $\mathcal{F}^3_F$ are determined by $\mathcal{F}^1_F$ by Theorem 8.2.2. It follows that $\mathcal{M} \to Q$ factor through the flat closure $\mathcal{M}^{\text{split}}$:

$$\begin{tikzcd}
\mathcal{M}^{\text{split}} \arrow[rr, hook] \arrow[dr, "\Pi"'] & & \mathcal{M} \arrow[dl] \\
& Q &
\end{tikzcd}$$

Here $\Pi : \mathcal{M}^{\text{split}} \to Q$ is a projective morphism since $\mathcal{M}^{\text{split}} \to \mathcal{M}$ is a closed immersion and $\mathcal{M} \to Q$ is projective. We see that $\Pi : \mathcal{M}^{\text{split}} \to Q$ is also isomorphism in the generic fiber, since $\mathcal{M}^{\text{split}} \otimes F \simeq \mathcal{M} \otimes F$. Similarly, we have morphisms $\mathcal{M} \to \mathrm{Gr}(5, 24)$ and $\mathcal{M} \to \mathrm{Gr}(9, 24)$ given by $\{\mathcal{F}^i\}_{i=1,2,3} \mapsto \mathcal{F}^2$ and $\{\mathcal{F}^i\}_{i=1,2,3} \mapsto \mathcal{F}^3$ by $\mathrm{rank}(\mathcal{F}^2) = 5$, $\mathrm{rank}(\mathcal{F}^3) = 9$. We say that $\mathcal{F}^2$ (resp. $\mathcal{F}^3$) are in some affine chart of the Grassmannian $\mathrm{Gr}(5, 24)$ (resp. $\mathrm{Gr}(9, 24)$) if the image of $\{\mathcal{F}^i\}_{i=1,2,3}$ in $\mathcal{M} \to \mathrm{Gr}(5, 24)$ (resp. $\mathcal{M} \to \mathrm{Gr}(9, 24)$) is in that affine chart.

Consider the closed subscheme Z in the special fiber Q_s of Q, where Z contains all isotropic lines orthogonal to the para-unit e. In our case, $e = e_4 + e_5$ by the Table 2.1, and $l = \sum_{i=1}^{8} x_i \pi^2 e_i \in Q_s$ for all $x_i \in \kappa$. By

$$\langle \pi^2 e_i, e_{9-j} \rangle = \pi^2 \delta_{ij},$$

it is easy to see that $\langle l, e \rangle = 0$ if and only if $x_4 + x_5 = 0$. Then $Z = V(x_4 + x_5, \pi_0^{1/3}) \subset Q$.

Let $\widetilde{Q}$ be the blow-up of the quadratic hypersurface Q along Z. Let $I = (x_4 + x_5, \pi_0^{1/3})$ be the homogeneous ideal in $S(Q)$ and $\mathcal{I}$ be the quasi-coherent sheaf associating to I. Then $\mathcal{B} := \oplus_{n \geq 0} \mathcal{I}^n$ (where $\mathcal{I}^0 = S(Q)$) is a homogeneous $\mathscr{O}_Q$-algebra, and $\widetilde{Q} = \mathrm{Proj}\mathcal{B}$. Our main result is:

Theorem 8.2.3. *The scheme $\mathcal{M}^{\text{split}}$ is isomorphic to the blow-up $\widetilde{Q}$ of Q along Z.*

From Theorem 8.2.3, it is easy to get a corollary by considering the blow-up $\widetilde{Q}$:

Corollary 8.2.4. *The scheme $\mathcal{M}^{\text{split}}$ is regular and has a special fiber, which is the union of two smooth irreducible components.*

To prove Theorem 8.2.3, we first show that there is a morphism $\widetilde{\Pi} : \mathcal{M}^{\text{split}} \to \widetilde{Q}$:

$$\begin{tikzcd} \mathcal{M}^{\text{split}} \arrow[rr, dashed] \arrow[dr, "\Pi"] & & \widetilde{Q} \arrow[dl] \\ & Q & \end{tikzcd}$$

For $\Pi : \mathcal{M}^{\text{split}} \to Q$, recall that $\Pi^{-1}(\mathcal{I})$ is an $\Pi^{-1}(\mathscr{O}_Q)$-module, and $\Pi^*(\mathcal{I}) := \Pi^{-1}(\mathcal{I}) \otimes_{\Pi^{-1}(\mathscr{O}_Q)} \mathscr{O}_{\mathcal{M}^{\text{split}}}$ is an $\mathscr{O}_{\mathcal{M}^{\text{split}}}$-algebra, which we call the inverse image of $\mathcal{I}$ under Π. By the universal property of blow-up, there is a unique morphism from $\mathcal{M}^{\text{split}}$ to $\widetilde{Q}$ if $\Pi^*(\mathcal{I})$ is an invertible sheaf of ideals on $\mathscr{O}_{\mathcal{M}^{\text{split}}}$, i.e., $\Pi^*(\mathcal{I})$ is locally principal ideal sheaf on $\mathscr{O}_{\mathcal{M}^{\text{split}}}$. To prove this, we will check $\Pi^*(\mathcal{I})|_U$ is principal for U running through affine charts in $\mathcal{M}^{\text{split}}$. Fortunately, we just need to consider some special affine charts. The other affine charts have similar results. Let R be a local ring over $\mathcal{O}$ with maximal ideal m.

Proposition 8.2.5. *For any $\{\mathcal{F}^i\}_{i=1,2,3} \in \mathcal{M}^{\text{split}}(R)$, if $\mathcal{F}^1 \in U_1(R)$, then $\mathcal{F}^2$ is either in the affine chart with leading terms$\{\pi^2 e_k, \pi e_1\}_{k=1,2,3,5}$ or in the affine chart with leading terms $\{\pi^2 e_i\}_{i=1,\dots,5}$.*

(1) If $\mathcal{F}^2$ is in the affine chart with leading terms $\{\pi^2 e_k, \pi e_1\}_{k=1,2,3,5}$, then $\mathcal{F}^3$ is in the affine chart with leading terms

$$\{\pi^2 e_i, \pi e_j, e_1\}_{i=1,\dots,5,\ j=1,2,3}.$$

Under this affine chart in $\mathbb{P}^7 \times \mathrm{Gr}(5,24) \times \mathrm{Gr}(9,24)$, the corresponding open subscheme in $\mathcal{M}^{\text{split}}$ is isomorphic to

$$\mathrm{Spec}(\mathcal{O}[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, b_1]/(x_1 - 1, Q_0, (x_4 + x_5) - \pi_0^{1/3}(1-\xi)b_1)).$$

(2) If $\mathcal{F}^2$ is in the affine chart with leading terms $\{\pi^2 e_i\}_{i=1,\dots,5}$, we have $\mathcal{F}^3$ is in the affine chart with leading terms

$$\{\pi^2 e_i, \pi e_1\}_{i=1,\dots,8}.$$

Under this affine chart in $\mathbb{P}^7 \times \mathrm{Gr}(5,24) \times \mathrm{Gr}(9,24)$, the corresponding open subscheme in $\mathcal{M}^{split}$ is isomorphic to

$$\mathrm{Spec}(\mathcal{O}[x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,k_4]/(x_1-1,Q_0,k_4(x_4+x_5)-\pi_0^{1/3}(1-\xi))).$$

Here $Q_0 = x_1x_8 + x_2x_7 + x_3x_6 + x_4x_5$.

Proposition 8.2.6. *If $\mathcal{F}^1 = f_4 + \sum_{i\neq 4} x_i f_i$, where $x_i \in m$, then $\mathcal{F}^2$ is in the affine chart with leading terms $\{\pi^2 e_i\}_{i=1,4,5,6,7}$, $\mathcal{F}^3$ is in the affine chart with leading terms $\{\pi e_5, \pi^2 e_i\}_{i=1,\dots,8}$. Under this affine chart in $\mathbb{P}^7 \times \mathrm{Gr}(5,24) \times \mathrm{Gr}(9,24)$, the corresponding open subscheme in $\mathcal{M}^{split}$ is isomorphic to*

$$\mathrm{Spec}(\mathcal{O}[x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,k_2]/(Q_0,x_4-1,k_2(x_4+x_5)-\pi_0^{1/3}(1-\xi))),$$

where $Q_0 = x_1x_8 + x_2x_7 + x_3x_6 + x_4x_5$.

The proof of Proposition 8.2.5 and 8.2.6 will show in the following sections §8.3, §8.4, §8.5. For other affine charts in $\mathcal{M}^{\mathrm{split}}$, suppose that $\mathcal{F}^1 = \sum x_i f_i$ with $x_i \in R$. If $x_i \in R^*$ for $i \neq 4,5$, then we can transform it to $x_1 \in R^*$, and use the result in Proposition 8.2.5. If all $x_i \in m$ for $i \neq 4,5$, we get $x_4x_5 = -(x_1x_8 + x_2x_7 + x_3x_6) \in m$ by the quadratic equation. Hence at least one of x_4, x_5 is in m. Without loss of generality, we assume $x_4 \in R^*$, $x_5 \in m$, and that comes to Proposition 8.2.6 (see §8.6 for details). Above all, we just need to consider affine charts in Proposition 8.2.5 and 8.2.6.

If $U \subset \mathcal{M}^{\text{split}}$ is the affine chart described in Proposition 8.2.5 (1), then we have

$$x_4 + x_5 = \pi_0^{1/3}(1-\xi)b_1.$$

So $\Pi^*(\mathcal{I})|_U$ is the ideal sheaf corresponding to $I\mathcal{O}_{\mathcal{M}^{\text{split}}}(U) = \pi_0^{1/3}\mathcal{O}_{\mathcal{M}^{\text{split}}}(U)$, which is principal. If $U \subset \mathcal{M}^{\text{split}}$ is the affine chart described in Proposition 8.2.5 (2), then we have

$$\pi_0^{1/3} = (1-\xi)^{-1}k_4(x_4 + x_5),$$

So $\Pi^*(\mathcal{I})|_U$ is the ideal sheaf corresponding to $I\mathcal{O}_{\mathcal{M}^{\text{split}}}(U) = (x_4 + x_5)\mathcal{O}_{\mathcal{M}^{\text{split}}}(U)$. Similarly, if U is the affine chart in Proposition 8.2.6, then $\Pi^*(\mathcal{I})|_U$ is the ideal sheaf corresponding to $(x_4 + x_5)\mathcal{O}_{\mathcal{M}^{\text{split}}}(U)$. Therefore, $\Pi^*(\mathcal{I})$ is locally principal ideal sheaf on $\mathscr{O}_{\mathcal{M}^{\text{split}}}$, and we have a morphism:

$$\widetilde{\Pi} : \mathcal{M}^{\text{split}} \to \widetilde{Q}.$$

Proof of Theorem 8.2.3: Since $\widetilde{\Pi}$ is of finite type with finite fibers, hence it is quasi-finite. By Zariski Main Theorem (Corollary 4.7, [17]), we have $\widetilde{\Pi}$ is a finite morphism since $\widetilde{\Pi}$ is projective. Meanwhile, we claim that $\widetilde{\Pi}$ is flat. Since flatness is a local property, we consider $\widetilde{\Pi}|_U$ for some affine chart U. By Proposition 8.2.5 and 8.2.6, we can see that $\mathscr{O}_{\mathcal{M}^{\text{split}}}(U)$ is a regular local ring, then it is Cohen-Macaulay. Notice that $\widetilde{Q}$ is a regular scheme. Both $\widetilde{Q}$ and $\mathcal{M}^{\text{split}}$ have dimension 7. By Miracle Flatness Theorem (Theorem 23.1, [19]), we get $\widetilde{\Pi}$ is a flat morphism. Since $\widetilde{\Pi}$ is finite and flat, we have that $\widetilde{\Pi}_*\mathscr{O}_{\mathcal{M}^{\text{split}}}$ is locally free over $\mathscr{O}_Q$. In the generic fiber, both $\mathcal{M}^{\text{split}}$ and $\widetilde{Q}$ is isomorphic to Q. Thus $\widetilde{\Pi}_\eta$ is isomorphism.

Therefore, $\widetilde{\Pi}_* \mathscr{O}_{\mathcal{M}^{\text{split}}}$ is locally free of rank 1 over $\mathscr{O}_Q$. We obtain $\widetilde{\Pi}_* \mathscr{O}_{\mathcal{M}^{\text{split}}} \cong \mathscr{O}_Q$, which implies that $\widetilde{\Pi}$ is isomorphism.

8.3 Affine chart U_1, part I.

In what follows, we denote by A_i the i-th column of a matrix A, and $A^{[j]}$ the matrix consisting of the last j rows of A. Likewise, we write $A_i^{[j]}$ for the i-th column with last j rows vector of A. Let R be a local ring with maximal ideal m.

Consider the affine chart $U_1(R) = \{f_1 + \sum_{j \neq 1} x_j f_j\}$ for $x_i \in R$. For any point $\{\mathcal{F}^i\}_{i=1,2,3} \in \mathcal{M}^{\text{split}}(R)$, suppose that $\mathcal{F}^1 \in U_1(R)$, i.e., let $\mathcal{F}^1 = R(f_1 + \sum_{i \neq 1} x_i f_i)$. Consider $\langle \mathcal{F}^1, \mathcal{F}^1 \rangle = 0$. Observe that

$$\begin{aligned} \langle f_i, f_{9-j} \rangle &= (\pi^2 + \pi_0^{1/3}\pi + \pi_0^{2/3})^2 \delta_{ij} \\ &= 3\pi_0^{2/3}(\pi^2 + \pi_0^{1/3}\pi + \pi_0^{2/3})\delta_{ij}. \end{aligned}$$

Then we have $3\pi_0^{1/3}(x_8 + x_2x_7 + x_3x_6 + x_4x_5) = 0$. Since $\mathcal{M}^{\text{split}}$ is the flat closure of $\mathcal{M}$, variables x_i in $\mathcal{M}^{\text{split}}$ are satisfied in equation $x_8 + x_2x_7 + x_3x_6 + x_4x_5 = 0$.

Lemma 8.3.1. *For any $\{\mathcal{F}^i\}_{i=1,2,3} \in \mathcal{M}^{\text{split}}(R)$, if $\mathcal{F}^1 \in U_1(R)$, then $\mathcal{F}^2$ is either in the affine chart with leading terms $\{\pi^2 e_i, \pi e_1\}_{i=1,2,3,5}$ or in the affine chart with leading terms $\{\pi^2 e_i\}_{i=1,...,5}$.*

Proof. Since $\text{rank}(\mathcal{F}^2) = 5$, we want to find 5 generators of $\mathcal{F}^2$, where the leading terms of generators are chosen from $\pi^2 e_i, \pi e_i$, or e_i $(i = 1, ..., 8)$. Consider the point $\mathcal{F}_s^i = \mathcal{F}^i \otimes \kappa$ in the special fiber of the splitting model $\mathcal{M}_s^{\text{split}} = \mathcal{M}^{\text{split}} \otimes \kappa$. We get $\mathcal{F}_s^1 = \kappa(\pi^2 e_1 + \sum_{j \neq 1} x_j \pi^2 e_j)$.

The generators for $\Lambda_s * \mathcal{F}_s^1$ are:

$$\pi^2 e_1 - x_5\pi^2 e_4 + x_6\pi^2 e_6 + x_7\pi^2 e_7,$$
$$\pi^2 e_2 - x_3\pi^2 e_4 - x_4\pi^2 e_6 - x_7\pi^2 e_8,$$
$$\pi^2 e_3 + x_2\pi^2 e_4 + x_4\pi^2 e_7 - x_6\pi^2 e_8,$$
$$\pi^2 e_5 - x_2\pi^2 e_6 + x_3\pi^2 e_7 + x_5\pi^2 e_8.$$

By $\Lambda_s * \mathcal{F}_s^1 \subset \mathcal{F}_s^2$, it is easy to see that $\mathcal{F}_s^2$ contains elements with leading terms $\pi^2 e_k$ for $k = 1, 2, 3, 5$. Hence we only need to consider the last generator for $\mathcal{F}^2$. We start with considering the last generator with leading term e_i for some $i \in \{1, 2, ..., 8\}$. Then by $\pi\mathcal{F}_s^2 \subset \mathcal{F}_s^1$, we obtain an element with leading term πe_i in $\mathcal{F}_s^1 = \kappa(\pi^2 e_1 + \sum_{j\neq 1} x_j\pi^2 e_j)$, which is impossible. Next, suppose that the last generator has the leading term πe_i for some $i \in \{1, 2, ..., 8\}$. Take πe_2 for instance. We can write the last generator as:

$$\pi e_2 + \sum_{i=4,6,7,8} y_i \pi^2 e_i + \sum_{j\neq 2} y'_j \pi e_j + \sum_{k=1,...,8} y''_k e_k.$$

By $\pi\mathcal{F}_s^2 \subset \mathcal{F}_s^1$, we have

$$\pi^2 e_2 + \sum_{j\neq 2} y'_j \pi^2 e_j + \sum_{k=1,...,8} y''_k \pi e_k \in \mathcal{F}_s^1,$$

which implies $y''_k = 0$ for $k = 1, ..., 8$, and $y'_j = y'_1 x_j$ for $j \neq 1, 2$, $y'_1 x_2 = 1$. Thus, y'_1 is a unit in R. By multiplying x_2 to the last generator, we can rewrite it as an element with leading term πe_1. Similarly, if the last generator has the leading term πe_i for some $i \neq 1$, we can rewrite it as an element with leading term πe_1. Namely, $\mathcal{F}_s^2$ is in the affine chart with leading terms $\{\pi^2 e_i, \pi e_1\}_{i=1,2,3,5}$, so is $\mathcal{F}^2$.

Less obvious is the last generator with leading term $\pi^2 e_k$ for some $k \in \{4, 6, 7, 8\}$ to be checked. Take $\pi^2 e_6$ for instance. Before examining the conditions in Definition 8.1.1, we take a look for an equation: Giving $a\pi^2 e_i + b\pi e_i + ce_i$ for some $a, b, c \in R$ and $i \in \{1, ..., 8\}$. Suppose:

$$(\pi - \pi_0^{1/3}\xi)(a\pi^2 e_i + b\pi e_i + ce_i) = m(\pi^2 e_i + \pi_0^{1/3}\pi e_i + \pi_0^{2/3})$$

for some $m \in R$. Then we have:

$$\begin{aligned} m &= -\pi_0^{1/3}\xi a + b, \\ c &= \pi_0^{2/3}\xi^2 a - \pi_0^{1/3}\xi^2 m. \end{aligned}$$

So we can rewrite $a\pi^2 e_i + b\pi e_i + ce_i$ as:

$$\begin{aligned} a\pi^2 e_i + b\pi e_i + ce_i &= a(\pi^2 e_i + \pi_0^{1/3}\xi\pi e_i + \pi_0^{2/3}\xi^2 e_i) + m(\pi e_i - \pi_0^{1/3}\xi^2 e_i) \\ &= af_i^{\xi} + m(\pi e_i - \pi_0^{1/3}\xi^2 e_i). \end{aligned}$$

This calculation result is easy to remember: here $a\pi^2 e_i + b\pi e_i + ce_i$ is separated to 2 parts, the first part is an eigenvector corresponding to eigenvalue $\pi_0^{1/3}\xi$, the second part satisfying $(\pi - \pi_0^{1/3}\xi)(\pi - \pi_0^{1/3}\xi^2)e_i = f_i$. From above discussion, we can rewirite the generators of $\mathcal{F}^2$

(in the affine chart with leading terms $\{\pi^2 e_k\}_{k=1,2,3,5,6}$) as the following form:

$$C_1 = f_1^{\xi} + a_{11} f_4^{\xi} + a_{21} f_7^{\xi} + a_{31} f_8^{\xi} + k_1[(\pi e_1 - \pi_0^{1/3}\xi^2 e_1) + \sum_{i\neq 1} x_i(\pi e_i - \pi_0^{1/3}\xi^2 e_i)],$$
$$C_2 = f_2^{\xi} + a_{12} f_4^{\xi} + a_{22} f_7^{\xi} + a_{32} f_8^{\xi} + k_2[(\pi e_1 - \pi_0^{1/3}\xi^2 e_1) + \sum_{i\neq 1} x_i(\pi e_i - \pi_0^{1/3}\xi^2 e_i)],$$
$$C_3 = f_3^{\xi} + a_{13} f_4^{\xi} + a_{23} f_7^{\xi} + a_{33} f_8^{\xi} + k_3[(\pi e_1 - \pi_0^{1/3}\xi^2 e_1) + \sum_{i\neq 1} x_i(\pi e_i - \pi_0^{1/3}\xi^2 e_i)],$$
$$C_4 = f_5^{\xi} + a_{14} f_4^{\xi} + a_{24} f_7^{\xi} + a_{34} f_8^{\xi} + k_4[(\pi e_1 - \pi_0^{1/3}\xi^2 e_1) + \sum_{i\neq 1} x_i(\pi e_i - \pi_0^{1/3}\xi^2 e_i)],$$
$$C_5 = f_6^{\xi} + a_{15} f_4^{\xi} + a_{25} f_7^{\xi} + a_{35} f_8^{\xi} + k_5[(\pi e_1 - \pi_0^{1/3}\xi^2 e_1) + \sum_{i\neq 1} x_i(\pi e_i - \pi_0^{1/3}\xi^2 e_i)].$$

So that $(\pi - \pi_0^{1/3}\xi)C_i \subset \mathcal{F}^1$ (we have $(\pi - \pi_0^{1/3}\xi)C_i = k_i(f_1 + \sum_{i\neq 1} x_i f_i)$) for some variables $a_{ij}, b_i, k_i \in R$. Next, consider $\Lambda * \mathcal{F}^1 \subset \mathcal{F}^2$. Since $\Lambda * \mathcal{F}^1$ is the maximal isotropic subspace with generators:

$$f_1^{\xi} - x_5 f_4^{\xi} + x_6 f_6^{\xi} + x_7 f_7^{\xi},$$
$$f_2^{\xi} - x_3 f_4^{\xi} - x_4 f_6^{\xi} - x_7 f_8^{\xi},$$
$$f_3^{\xi} + x_2 f_4^{\xi} + x_4 f_7^{\xi} - x_6 f_8^{\xi},$$
$$f_5^{\xi} - x_2 f_6^{\xi} + x_3 f_7^{\xi} + x_5 f_8^{\xi}.$$

Condition $\Lambda * \mathcal{F}^1 \subset \mathcal{F}^2$ is equivalent to:

$$f_1^{\xi} - x_5 f_4^{\xi} + x_6 f_6^{\xi} + x_7 f_7^{\xi} = C_1 + x_6 C_5,$$
$$f_2^{\xi} - x_3 f_4^{\xi} - x_4 f_6^{\xi} - x_7 f_8^{\xi} = C_2 - x_4 C_5,$$
$$f_3^{\xi} + x_2 f_4^{\xi} + x_4 f_7^{\xi} - x_6 f_8^{\xi} = C_3,$$
$$f_5^{\xi} - x_2 f_6^{\xi} + x_3 f_7^{\xi} + x_5 f_8^{\xi} = C_4 - x_2 C_5.$$

Comparison of coefficients of $\pi^2 e_4, \pi^2 e_7, \pi^2 e_8$ and πe_1 in these equations yields:

$$\begin{aligned}
&a_{11}+x_6a_{15}=-x_5, && a_{12}-x_4a_{15}=-x_3, && a_{13}=x_2, && a_{14}-x_2a_{15}=0,\\
&a_{21}+x_6a_{25}=x_7, && a_{22}-x_4a_{25}=0, && a_{23}=x_4, && a_{14}-x_2a_{15}=x_3,\\
&a_{31}+x_6a_{35}=0, && a_{32}-x_4a_{35}=-x_7, && a_{33}=-x_6, && a_{14}-x_2a_{15}=x_5.
\end{aligned} \tag{8.3.2}$$

and

$$k_1+x_6k_5=0, \quad k_2-x_4k_5=0, \quad k_3=0, \quad k_4-x_2k_5=0. \tag{8.3.3}$$

So variables in $\mathcal{F}^2$ are determined by a_{15}, a_{25}, a_{35} and k_5. Finally, we consider condition $\mathcal{F}^1 \subset \mathcal{F}^2$, which is equivalent to:

$$f_1+\sum_{i\neq 1} x_i f_i = C_1+x_2C_2+x_3C_3+x_5C_4+x_6C_5.$$

Comparison of the coefficients of $\pi^2 e_4, \pi^2 e_7, \pi^2 e_8$ and πe_1 in this equation yields

$$\begin{aligned}
&a_{11}+x_2a_{12}+x_3a_{13}+x_5a_{14}+x_6a_{15}=x_4,\\
&a_{21}+x_2a_{22}+x_3a_{23}+x_5a_{24}+x_6a_{25}=x_7,\\
&a_{31}+x_2a_{32}+x_3a_{33}+x_5a_{34}+x_6a_{35}=x_8,\\
&k_1+x_2k_2+x_3k_3+x_5k_4+x_6k_5=\pi_0^{1/3}(1-\xi).
\end{aligned} \tag{8.3.4}$$

From (8.3.3), variables k_1, k_2, k_4 are determined by x_i and k_5. Replace them back into the last equation of (8.3.4), we get:

$$x_2k_5(x_4+x_5)=\pi_0^{1/3}(1-\xi)$$

Similarly, from (8.3.2), variables $a_{11}, a_{12}, a_{13}, a_{14}$ are determined by x_i and a_{15}. Replacing them back into the first equation of (8.3.4) yields:

$$(x_4 + x_5)(x_2 a_{15} - 1) = 0.$$

Thus, by multiplying $x_2 k_5$ on both sides to the above equation, we get $\pi_0^{1/3}(1-\xi)(x_2 a_{15} - 1) = 0$. Since $\mathcal{M}^{\text{split}}$ is the flat closure of $\mathcal{M}$, we have $x_2 a_{15} - 1 = 0$, i.e., a_{15} is a unit in R.

By multiplying x_2 to the generator C_5, we can rewrite C_5 as:

$$C_5 = f_4^{\xi} + x_2 f_6^{\xi} + x_2 a_{25} f_7^{\xi} + x_2 a_{35} f_8^{\xi} + x_2 k_5[(\pi e_1 - \pi_0^{1/3}\xi^2 e_1) + \textstyle\sum_{i\neq 1} x_i(\pi e_i - \pi_0^{1/3}\xi^2 e_i)].$$

In other word, $\mathcal{F}^2$ is in the affine chart with leading terms $\{\pi^2 e_i\}_{i=1,2,3,4,5}$. Similarly, if we choose the last generator with leading terms $\pi^2 e_7$ or $\pi^2 e_8$, we can find the coefficient of $\pi^2 e_4$ in the last generator is also a unit. Thus, we only need to consider the situation where the affine chart with leading terms $\{\pi^2 e_i\}_{i=1,2,3,4,5}$. □

We consider that $\mathcal{F}^2$ is in the affine chart with leading terms $\{\pi^2 e_i, \pi e_1\}_{i=1,2,3,5}$ in this section, and discuss $\mathcal{F}^2$ in the affine chart with leading terms $\{\pi^2 e_i\}_{i=1,\ldots,5}$ in the next section. With respect to the standard order of basis, generators of $\mathcal{F}^2$ can be described as the column span of a 24×5 matrix C having entries in R and being of the following form:

$$C = \begin{pmatrix} A & B \\ A' & B' \\ A'' & B'' \end{pmatrix}$$

where $A_{8\times4}, A'_{8\times4}, A''_{8\times4}$ are $M(R)$-matrices, and $B_{8\times1}, B'_{8\times1}, B_{8\times1}$ are R-vectors. More precisely, we have:

$$A = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ a_{11} & a_{12} & a_{13} & a_{14} \\ & & & 1 \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}, B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ b_1 \\ 0 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}, A' = \begin{pmatrix} 0 & \cdots & 0 \\ a'_{11} & \cdots & a'_{14} \\ \cdots & \cdots & \cdots \\ a_{71'} & \cdots & a'_{74} \end{pmatrix}, B' = \begin{pmatrix} 1 \\ b'_1 \\ \vdots \\ b'_7 \end{pmatrix},$$

For example, the first column C_1 is represented the first generator of $\mathcal{F}^2$:

$$C_1 = \pi^2 e_1 + a_{11}\pi^2 e_4 + a_{21}\pi^2 e_6 + a_{31}\pi^2 e_7 + a_{41}\pi^2 e_8 + \sum_{j=1}^{7} a'_{j1}\pi e_{j+1} + \sum_{k=1}^{8} a''_{k1} e_k.$$

We need to check:

$$(\pi - \pi_0^{1/3}\xi)\mathcal{F}^2 \subset \mathcal{F}^1, \quad \Lambda * \mathcal{F}^1 \subset \mathcal{F}^2, \quad \mathcal{F}^1 \subset \mathcal{F}^2, \quad \langle \mathcal{F}^2, \mathcal{F}^2 \rangle = 0.$$

(1). $(\pi - \pi_0^{1/3}\xi)\mathcal{F}^2 \subset \mathcal{F}^1$. We claim that matrices A', A'', B', B'' are determined by A, B and x_i. Calculation for this condition is similar to what we did in the proof of Lemma 8.3.1. For instance, consider C_1. We have

$$C_1 = (\pi^2 e_1 - \pi_0^{2/3}\xi e_1) + a_{11} f_4^{\xi} + a_{21} f_6^{\xi} + a_{31} f_7^{\xi} + a_{41} f_8^{\xi} - \pi_0^{1/3}\xi \sum_{i=2}^{8} x_i(\pi e_i - \pi_0^{1/3}\xi^2 e_i),$$

such that $(\pi - \pi_0^{1/3}\xi)C_1 = -\pi_0^{1/3}\xi(f_1 + \sum_{i\neq 1} x_i f_i)$. Generally, the generators of $\mathcal{F}^2$ (the columns C_i of C) are:

$$
\begin{aligned}
C_1 =& (\pi^2 e_1 - \pi_0^{2/3}\xi e_1) + a_{11} f_4^\xi + a_{21} f_6^\xi + a_{31} f_7^\xi + a_{41} f_8^\xi - \pi_0^{1/3}\xi \sum_{i=2}^{8} x_i(\pi e_i - \pi_0^{1/3}\xi^2 e_i),\\
C_2 =& f_2^\xi + a_{12} f_4^\xi + a_{22} f_6^\xi + a_{32} f_7^\xi + a_{42} f_8^\xi,\\
C_3 =& f_3^\xi + a_{13} f_4^\xi + a_{23} f_6^\xi + a_{33} f_7^\xi + a_{43} f_8^\xi,\\
C_4 =& f_5^\xi + a_{14} f_4^\xi + a_{24} f_6^\xi + a_{34} f_7^\xi + a_{44} f_8^\xi,\\
C_5 =& (\pi e_1 - \pi_0^{1/3}\xi^2 e_1) + b_1 f_4^\xi + b_2 f_6^\xi + b_3 f_7^\xi + b_4 f_8^\xi + \sum_{i=2}^{8} x_i(\pi e_i - \pi_0^{1/3}\xi^2 e_i).
\end{aligned}
$$

Let X be a vector $(1\ x_2\ x_3\ \cdots\ x_8)^T$. Denote by $(X\ 0\ 0\ 0)$ the matrix where the first column is X and the rest columns are 0. It is easy to see that:

$$
\begin{aligned}
A' &= \pi_0^{1/3}\xi A - \pi_0^{1/3}\xi(X\ 0\ 0\ 0),\\
A'' &= \pi_0^{2/3}\xi^2 A + \pi_0^{2/3}(X\ 0\ 0\ 0),\\
B' &= \pi_0^{1/3}\xi B + X,\\
B'' &= \pi_0^{2/3}\xi^2 B - \pi_0^{1/3}\xi^2 X.
\end{aligned}
$$

Hence A', A'', B', B'' are determined by A, B and x_i.

(2). $\Lambda * \mathcal{F}^1 \subset \mathcal{F}^2$. We show that A is determined by B and x_i. This condition is equivalent to:

$$
\begin{aligned}
f_1^\xi - x_5 f_4^\xi + x_6 f_6^\xi + x_7 f_7^\xi &= C_1 - \pi_0^{1/3}\xi C_5,\\
f_2^\xi - x_3 f_4^\xi - x_4 f_6^\xi - x_7 f_8^\xi &= C_2,\\
f_3^\xi + x_2 f_4^\xi + x_4 f_7^\xi - x_6 f_8^\xi &= C_3,\\
f_5^\xi - x_2 f_6^\xi + x_3 f_7^\xi + x_5 f_8^\xi &= C_4,
\end{aligned}
$$

Comparison of the coefficients of $\pi^2 e_4, \pi^2 e_6, \pi^2 e_7$ yields several identities involving a_{ij} and b_i variables:

$$\begin{aligned} &a_{12} = -x_3, && a_{22} = -x_4, && a_{32} = 0, && a_{42} = -x_7, \\ &a_{13} = x_2, && a_{23} = 0, && a_{33} = x_4, && a_{43} = -x_6, \\ &a_{14} = 0, && a_{24} = -x_2, && a_{34} = x_3, && a_{44} = x_5, \end{aligned} \tag{8.3.5}$$

and

$$\begin{aligned} &a_{11} + \pi_0^{1/3}\xi b_1 = -x_5, && a_{21} + \pi_0^{1/3}\xi b_2 = x_6, \\ &a_{31} + \pi_0^{1/3}\xi b_3 = x_7, && a_{41} + \pi_0^{1/3}\xi b_4 = 0. \end{aligned} \tag{8.3.6}$$

(3). $\mathcal{F}^1 \subset \mathcal{F}^2$. We show that B is determined by b_1 and x_i. For $f_1 + \sum_{i=2}^{8} x_i f_i \in \mathcal{F}^2$, we have the following equation by comparing the coefficients of $\pi^2 e_1$, $\pi^2 e_2$, $\pi^2 e_3$, $\pi^2 e_5$, πe_1:

$$f_1 + \sum_{i=2}^{8} x_i f_i = C_1 + x_2 C_2 + x_3 C_3 + x_5 C_4 + \pi_0^{1/3} C_5.$$

By (8.3.5), the right side of this equation is equivalent to:

$$\begin{aligned} &f_1 + x_2 f_2 + x_3 f_3 + x_5 f_5 + (a_{11} + \pi_0^{1/3} b_1) f_4^\xi + [(a_{21} + \pi_0^{1/3} b_2) - x_2(x_4 + x_5)] f_6^\xi \\ &+ [(a_{31} + \pi_0^{1/3} b_3) + x_3(x_4 + x_5)] f_7^\xi + [(a_{41} + \pi_0^{1/3} b_4) + (x_5^2 - x_2 x_7 - x_3 x_6)] f_8^\xi \\ &+ (\pi_0^{1/3} - \pi_0^{1/3}\xi) \textstyle\sum_{i=4,6,7,8} x_i(\pi e_i - \pi_0^{1/3}\xi^2 e_i). \end{aligned}$$

Compare to the left side, we get:

$$\begin{aligned} x_4 &= a_{11} + \pi_0^{1/3} b_1, \\ x_6 &= (a_{21} + \pi_0^{1/3} b_2) - x_2(x_4 + x_5), \\ x_7 &= (a_{31} + \pi_0^{1/3} b_3) + x_3(x_4 + x_5), \\ x_8 &= (a_{41} + \pi_0^{1/3} b_4) + (x_5^2 - x_2 x_7 - x_3 x_6). \end{aligned} \tag{8.3.7}$$

Replace $a_{11}, a_{21}, a_{31}, a_{41}$ by b_i and x_i from (8.3.6). Equations (8.3.7) yields:

$$x_4 + x_5 = \pi_0^{1/3}(1-\xi)b_1,$$
$$\pi_0^{1/3}(1-\xi)(b_2 - x_2 b_1) = 0,$$
$$\pi_0^{1/3}(1-\xi)(b_3 + x_3 b_1) = 0,$$
$$\pi_0^{1/3}(1-\xi)(b_4 + x_5 b_1) = 0.$$

Since $\mathcal{M}^{\text{split}}$ is the flat closure of $\mathcal{M}$, we see that b_2, b_3, b_4 are determined by x_i, b_1:

$$x_4 + x_5 = \pi_0^{1/3}(1-\xi)b_1, \quad b_2 - x_2 b_1 = 0, \quad b_3 + x_3 b_1 = 0, \quad b_4 + x_5 b_1 = 0.$$

Thus, the generators $C_1, C_2, ..., C_5$ of $\mathcal{F}^2$ are of the forms:

$$\begin{aligned}
C_1 =& (\pi^2 e_1 - \pi_0^{2/3}\xi e_1) - (x_5 + \pi_0^{1/3}\xi b_1) f_4^\xi + (x_6 - \pi_0^{1/3}\xi b_2) f_6^\xi + (x_7 - \pi_0^{1/3}\xi b_3) f_7^\xi - \pi_0^{1/3}\xi b_4 f_8^\xi \\
& - \pi_0^{1/3}\xi \textstyle\sum_{i=2}^8 x_i(\pi e_i - \pi_0^{1/3}\xi^2 e_i), \\
C_2 =& f_2^\xi - x_3 f_4^\xi - x_4 f_6^\xi - x_7 f_8^\xi, \\
C_3 =& f_3^\xi + x_2 f_4^\xi + x_4 f_7^\xi - x_6 f_8^\xi, \\
C_4 =& f_5^\xi - x_2 f_6^\xi + x_3 f_7^\xi + x_5 f_8^\xi, \\
C_5 =& (\pi e_1 - \pi_0^{1/3}\xi^2 e_1) + b_1 f_4^\xi + b_2 f_6^\xi + b_3 f_7^\xi + b_4 f_8^\xi + \textstyle\sum_{i=2}^8 x_i(\pi e_i - \pi_0^{1/3}\xi^2 e_i),
\end{aligned}$$

where variables b_1, b_2, b_3, b_4 satisfy:

$$x_4 + x_5 = \pi_0^{1/3}(1-\xi)b_1, \quad b_2 = x_2 b_1, \quad b_3 = -x_3 b_1, \quad b_4 = -x_5 b_1. \tag{8.3.8}$$

It is easy to check that $C_1, ..., C_5$ already satisfy $\langle C_i, C_j \rangle = 0$ for $i, j \in \{1, 2, 3, 4, 5\}$.

Lemma 8.3.9. *For any $\{\mathcal{F}^i\}_{i=1,2,3} \in \mathcal{M}^{\text{split}}(R)$, if $\mathcal{F}^1 \in U_1(R)$, and $\mathcal{F}^2$ is in the affine chart with leading terms $\{\pi^2 e_1, \pi^2 e_2, \pi^2 e_3, \pi^2 e_5, \pi e_1\}$. Then $\mathcal{F}^3$ is in the affine chart with leading terms*

$$\{\pi^2 e_i, \pi e_j, e_1\} \quad \textit{for } i = 1, ..., 5, j = 1, 2, 3.$$

Proof. By $\mathcal{F}^2 \subset \mathcal{F}^3$, we have elements with leading terms $\{\pi^2 e_1, \pi^2 e_2, \pi^2 e_3, \pi^2 e_5, \pi e_1\}$. Since $f_4^{\xi^2} + x_2 f_6^{\xi^2} - x_3 f_7^{\xi^2} + x_4 f_8^{\xi^2} \in \mathcal{F}^1 * \Lambda \subset \mathcal{F}^3$, we have an element with leafing term $\pi^2 e_4$ in $\mathcal{F}^3$. By $\operatorname{rank}(\mathcal{F}^3) = 9$, it reminds to find the other three generators of $\mathcal{F}^3$. Set:

$$\begin{aligned}
\alpha_1 =& f_2^{\xi} - x_3 f_4^{\xi} - x_4 f_6^{\xi} - x_7 f_8^{\xi}, \\
\alpha_2 =& f_5^{\xi} - x_2 f_6^{\xi} + x_3 f_7^{\xi} + x_5 f_8^{\xi}, \\
\alpha_3 =& f_2^{\xi^2} + x_3 f_5^{\xi^2} + x_5 f_6^{\xi^2} - x_7 f_8^{\xi^2}, \\
\alpha_4 =& f_4^{\xi^2} + x_2 f_6^{\xi^2} - x_3 f_7^{\xi^2} + x_4 f_8^{\xi^2}.
\end{aligned}$$

Here $\alpha_1, \alpha_2 \in \Lambda * \mathcal{F}^1$ and $\alpha_3, \alpha_4 \in \mathcal{F}^1 * \Lambda$, so they are elements in $\mathcal{F}^3$. Consider $\alpha_1 - \alpha_3 + x_3\alpha_2 + x_3\alpha_4$:

$$\begin{aligned}
\alpha_1 - \alpha_3 + x_3\alpha_2 + x_3\alpha_4 =& (f_2^{\xi} - f_2^{\xi^2}) - x_3(f_4^{\xi} - f_4^{\xi^2}) + x_3(f_5^{\xi} - f_5^{\xi^2}) - x_2 x_3(f_6^{\xi} - f_6^{\xi^2}) \\
&+ x_3^2(f_7^{\xi} - f_7^{\xi^2}) - x_7(f_8^{\xi} - f_8^{\xi^2}) - (x_4 f_6^{\xi} + x_5 f_6^{\xi^2}) + x_3(x_5 f_8^{\xi} + x_4 f_8^{\xi^2}).
\end{aligned}$$

Since $f_i^{\xi} - f_i^{\xi^2}$, $x_4 f_i^{\xi} + x_5 f_i^{\xi^2}$, $x_5 f_i^{\xi} + x_4 f_i^{\xi^2}$ for $i \in \{1, ..., 8\}$ are divisible by $\pi_0^{1/3}$ by:

$$\begin{aligned}
f_i^{\xi} - f_i^{\xi^2} &= \pi_0^{1/3}(\xi - \xi^2)(\pi e_i - \pi_0^{1/3} e_i), \\
x_4 f_i^{\xi} + x_5 f_i^{\xi^2} &= (x_4 + x_5)\pi^2 e_i + \pi_0^{1/3}(x_4\xi + x_5\xi^2)\pi e_i + \pi_0^{2/3}(x_4\xi^2 + x_5\xi) e_i, \\
x_5 f_i^{\xi} + x_4 f_i^{\xi^2} &= (x_4 + x_5)\pi^2 e_i + \pi_0^{1/3}(x_5\xi + x_4\xi^2)\pi e_i + \pi_0^{2/3}(x_5\xi^2 + x_4\xi) e_i,
\end{aligned}$$

(here we use the equation $x_4 + x_5 = \pi_0^{1/3}(1-\xi)b_1$), we obtain that $\alpha_1 - \alpha_3 + x_3\alpha_2 + x_3\alpha_4$ is divisible by $\pi_0^{1/3}$. After dividing $\pi_0^{1/3}$, we get an element with leading term πe_2, withr nonzero terms $\pi^2 e_6, \pi^2 e_8$, πe_i for $i = 4, 5, 6, 7, 8$. Similarly, by comparison

$$
\begin{aligned}
&f_3^{\xi} + x_2 f_4^{\xi} + x_4 f_7^{\xi} - x_6 f_8^{\xi},\\
&f_3^{\xi^2} - x_2 f_5^{\xi^2} - x_5 f_7^{\xi^2} - x_6 f_8^{\xi^2},\\
&f_5^{\xi} - x_2 f_6^{\xi} + x_3 f_7^{\xi} + x_5 f_8^{\xi},\\
&f_4^{\xi^2} + x_2 f_6^{\xi^2} - x_3 f_7^{\xi^2} + x_4 f_8^{\xi^2}.
\end{aligned}
$$

We have an element with leading term πe_3 in $\mathcal{F}^3$. The last generator is less obvious to find. Consider $(e_4 + e_5) * (f_1 + \sum_{i \neq 1} x_i f_i)$ and $(f_1 + \sum_{i \neq 1} x_i f_i) * (e_4 + e_5)$. Set:

$$
\begin{aligned}
\beta_1 =& f_1 + \sum_{i \neq 1,4,5} x_i f_i + x_4 f_4 + x_5 f_5,\\
\beta_2 =& -f_1^{\xi} - \sum_{i \neq 1,4,5} x_i f_i^{\xi} + x_5 f_4^{\xi} + x_4 f_5^{\xi},\\
\beta_3 =& -f_1^{\xi^2} - \sum_{i \neq 1,4,5} x_i f_i^{\xi^2} + x_5 f_4^{\xi^2} + x_4 f_5^{\xi^2},
\end{aligned}
$$

By using $x_4+x_5=\pi_0^{1/3}(1-\xi)b_1$, we have:

$$\begin{aligned}\frac{\beta_1+\beta_2}{\pi_0^{1/3}(1-\xi)} &= (\pi e_1-\pi_0^{1/3}\xi^2 e_1)+b_1(\pi^2 e_4+\pi^2 e_5)+(x_4+\pi_0^{1/3}\xi b_1)\pi e_4-(x_4-\pi_0^{1/3}b_1)\pi e_5\\ &+\sum_{i\neq 1,4,5} x_i(\pi e_i-\pi_0^{1/3}\xi^2 e_i)-\pi_0^{1/3}(\xi^2 x_4-\pi_0^{1/3}\xi^2 b_1)e_4+\pi_0^{1/3}(\xi^2 x_4+\pi_0^{1/3}b_1)e_5,\\ \frac{\beta_1+\beta_3}{\pi_0^{1/3}(1-\xi^2)} &= (\pi e_1-\pi_0^{1/3}\xi e_1)-b_1\xi(\pi^2 e_4+\pi^2 e_5)+(x_4-\pi_0^{1/3}b_1)\pi e_4-(x_4+\pi_0^{1/3}\xi b_1)\pi e_5\\ &+\sum_{i\neq 1,4,5} x_i(\pi e_i-\pi_0^{1/3}\xi e_i)-\pi_0^{1/3}(\xi x_4+\pi_0^{1/3}\xi^2 b_1)e_4+\pi_0^{1/3}(\xi x_4-\pi_0^{1/3}\xi b_1)e_5.\end{aligned}$$

Consider

$$\frac{\beta_1+\beta_2}{\pi_0^{1/3}(1-\xi)}-\frac{\beta_1+\beta_3}{\pi_0^{1/3}(1-\xi^2)}+b_1\xi^2(\alpha_2+\alpha_4).$$

We can check that every part in the above equation is divisible by $\pi_0^{1/3}$, and after dividing $\pi_0^{1/3}$, we get an element with leading terms e_1 in $\mathcal{F}^3$. Above all, we have an affine chart with leading terms $\{\pi^2 e_i, \pi e_j, e_1\}$ for $i=1,2,...,5, j=1,2,3$. $\square$

Now we consider conditions of variables in this affine chart for $\mathcal{F}^3$. With respect to the standard order of basis, the form $\mathcal{F}^3$ is represented by the matrix D:

$$D=\begin{pmatrix} I & 0 & 0\\ U & V & W\\ 0 & I & 0\\ U' & V' & W'\\ 0 & 0 & 1\\ U'' & V'' & W'' \end{pmatrix}$$

where $U_{3\times5}, U'_{5\times5}, U''_{7\times5}, V_{3\times3}, V'_{5\times3}, V''_{7\times3}$ are $M(R)$-matrices, and $W_{3\times1}, W'_{5\times1}, W''_{7\times1}$ are R-vectors. More precisely, if we set U_i (resp. U'_i, U''_i, V_i, V'_i, V''_i) the i-th column for U (resp. U', U'', V, V', V''), then the generators of $\mathcal{F}^3$, which is the columns D_i of the matrix D, are of the following forms:

$$D_i = \pi^2 e_i + \sum_{j=1}^{3} u_{ji}\pi^2 e_{j+5} + \sum_{k=1}^{5} u'_{ki}\pi e_{k+3} + \sum_{l=1}^{7} u''_{li} e_{l+1}, \quad \text{for } 1 \le i \le 5,$$
$$D_{i+5} = \pi e_i + \sum_{j=1}^{3} v_{ji}\pi^2 e_{j+5} + \sum_{k=1}^{5} v'_{ki}\pi e_{k+3} + \sum_{l=1}^{7} v''_{li} e_{l+1}, \quad \text{for } 1 \le i \le 3,$$
$$D_9 = e_1 + \sum_{j=1}^{3} w_j\pi^2 e_{j+5} + \sum_{k=1}^{5} w'_k\pi e_{k+3} + \sum_{l=1}^{7} w''_l e_{l+1}.$$

Conditions that we need to check are:

$$\mathcal{F}^1 * \Lambda \subset \mathcal{F}^3, \quad \mathcal{F}^2 \subset \mathcal{F}^3, \quad (\pi - \pi_0^{1/3}\xi^2)\mathcal{F}^3 \subset \mathcal{F}^2, \quad \langle \mathcal{F}^3, \mathcal{F}^3 \rangle = 0.$$

(1). $\mathcal{F}^1 * \Lambda \subset \mathcal{F}^3$. By Table 2.1, the generators of $\mathcal{F}^1 * \Lambda$ are:

$$\begin{aligned}
&f_1^{\xi^2} - x_4 f_5^{\xi^2} + x_6 f_6^{\xi^2} + x_7 f_7^{\xi^2},\\
&f_2^{\xi^2} + x_3 f_5^{\xi^2} + x_5 f_6^{\xi^2} - x_7 f_8^{\xi^2},\\
&f_3^{\xi^2} - x_2 f_5^{\xi^2} - x_5 f_7^{\xi^2} - x_6 f_8^{\xi^2},\\
&f_4^{\xi^2} + x_2 f_6^{\xi^2} - x_3 f_7^{\xi^2} + x_4 f_8^{\xi^2}.
\end{aligned}$$

Then the condition $\mathcal{F}^1 * \Lambda \subset \mathcal{F}^3$ is equivalent to:

$$\begin{aligned}
&f_1^{\xi^2} - x_4 f_5^{\xi^2} + x_6 f_6^{\xi^2} + x_7 f_7^{\xi^2} = D_1 + \pi_0^{1/3}\xi^2 D_6 + \pi_0^{2/3}\xi D_9 - x_4 D_5,\\
&f_2^{\xi^2} + x_3 f_5^{\xi^2} + x_5 f_6^{\xi^2} - x_7 f_8^{\xi^2} = D_2 + \pi_0^{1/3}\xi^2 D_7 + x_3 D_5,\\
&f_3^{\xi^2} - x_2 f_5^{\xi^2} - x_5 f_7^{\xi^2} - x_6 f_8^{\xi^2} = D_3 + \pi_0^{1/3}\xi^2 D_8 - x_2 D_5,\\
&f_4^{\xi^2} + x_2 f_6^{\xi^2} - x_3 f_7^{\xi^2} + x_4 f_8^{\xi^2} = D_4.
\end{aligned} \tag{8.3.10}$$

Before moving on to the calculation, let us take a look at the matrix that represents $\mathcal{F}^1 * \Lambda$.

Set:

$$K = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \\ -x_4 & x_3 & -x_2 & 0 \\ x_6 & x_5 & 0 & x_2 \\ x_7 & 0 & -x_5 & -x_3 \\ 0 & -x_7 & -x_6 & x_4 \end{pmatrix}$$

Recall that we denote by $K^{[j]}$ the matrix consisting of the last j rows of K. With respect to the standard order of basis, $\mathcal{F}^1 * \Lambda$ is represented by the matrix:

$$\mathcal{F}^1 * \Lambda = \begin{pmatrix} K \\ \pi_0^{1/3}\xi^2 K \\ \pi_0^{1/3}\xi K \end{pmatrix}$$

Thus, by comparison of the coefficients of $\pi^2 e_6, \pi^2 e_7, \pi^2 e_8$ in (8.3.10), we obtain:

$$\begin{aligned}
&U_1 + \pi_0^{1/3}\xi^2 V_1 + \pi_0^{2/3}\xi W - x_4 U_5 = K_1^{[3]}, \quad && U_2 + \pi_0^{1/3}\xi^2 V_2 + x_3 U_5 = K_2^{[3]}, \\
&U_3 + \pi_0^{1/3}\xi^2 V_3 - x_2 U_5 = K_3^{[3]}, && U_4 = K_4^{[3]}.
\end{aligned} \tag{8.3.11}$$

By comparison of the coefficients of πe_j for $j = 4, 5, 6, 7, 8$ in (8.3.10), we obtain:

$$\begin{aligned}
&U_1' + \pi_0^{1/3}\xi^2 V_1' + \pi_0^{2/3}\xi W' - x_4 U_5' = \pi_0^{1/3}\xi^2 K_1^{[5]}, \\
&U_2' + \pi_0^{1/3}\xi^2 V_2' + x_3 U_5' = \pi_0^{1/3}\xi^2 K_2^{[5]}, \\
&U_3' + \pi_0^{1/3}\xi^2 V_3' - x_2 U_5' = \pi_0^{1/3}\xi^2 K_3^{[5]}, \\
&U_4' = \pi_0^{1/3}\xi^2 K_4^{[5]}.
\end{aligned} \tag{8.3.12}$$

By comparison of the coefficients of e_k for $k = 2, ..., 8$ in (8.3.10), we obtain:

$$\begin{aligned}
&U_1'' + \pi_0^{1/3}\xi^2 V_1'' + \pi_0^{2/3}\xi W'' - x_4 U_5'' = \pi_0^{2/3}\xi K_1^{[7]}, \\
&U_2'' + \pi_0^{1/3}\xi^2 V_2'' + x_3 U_5'' = \pi_0^{2/3}\xi K_2^{[7]}, \\
&U_3'' + \pi_0^{1/3}\xi^2 V_3'' - x_2 U_5'' = \pi_0^{2/3}\xi K_3^{[7]}, \\
&U_4'' = \pi_0^{2/3}\xi K_4^{[7]}.
\end{aligned} \tag{8.3.13}$$

(2). $\mathcal{F}^2 \subset \mathcal{F}^3$. We show that U, V_1 (resp. U', V_1', U'', V_1'') are determined by V_2, V_3, W (resp. V_2', V_3', W', V_2'', V_3'', W'') and x_i. Since $\mathcal{F}^2 = R\langle C_1, ..., C_5\rangle$, condition $\mathcal{F}^2 \subset \mathcal{F}^3$ comes to:

$$
\begin{aligned}
C_1 &= D_1 - \pi_0^{2/3}\xi D_9 - (x_5 + \pi_0^{1/3}\xi b_1)D_4 - \pi_0^{1/3}\xi x_2 D_7 - \pi_0^{1/3}\xi x_3 D_8,\\
C_2 &= D_2 + \pi_0^{1/3}\xi D_7 - x_3 D_4,\\
C_3 &= D_3 + \pi_0^{1/3}\xi D_8 + x_2 D_4,\\
C_4 &= D_5,\\
C_5 &= D_6 - \pi_0^{1/3}\xi^2 D_9 + b_1 D_4 + x_2 D_7 + x_3 D_8.
\end{aligned}
\tag{8.3.14}
$$

Analogously to $\mathcal{F}^1 * \Lambda \subset \mathcal{F}^3$, we compare the coefficients of (8.3.14). This has to be done carefully since the blocks of the matrix C are of different sizes. Recall that we set $X = (1\ x_2\ \cdots\ x_8)^T$. By comparison of the coefficients of $\pi^2 e_6, \pi^2 e_7, \pi^2 e_8$ in (8.3.14), we obtain:

$$
\begin{aligned}
& U_1 - \pi_0^{2/3}\xi W - (x_5 + \pi_0^{1/3}\xi b_1)U_4 - \pi_0^{1/3}\xi x_2 V_2 - \pi_0^{1/3}\xi x_3 V_3 = A_1^{[3]},\\
& U_2 + \pi_0^{1/3}\xi V_2 - x_3 U_4 = A_2^{[3]},\\
& U_3 + \pi_0^{1/3}\xi V_3 + x_2 U_4 = A_3^{[3]},\\
& U_5 = A_4^{[3]},\\
& V_1 - \pi_0^{1/3}\xi^2 W + b_1 U_4 + x_2 V_2 + x_3 V_3 = B^{[3]},
\end{aligned}
\tag{8.3.15}
$$

By comparison of the coefficients of πe_j for $j = 4, 5, 6, 7, 8$ in (8.3.14), we obtain:

$$\begin{aligned}
&U_1' - \pi_0^{2/3}\xi W' - (x_5 + \pi_0^{1/3}\xi b_1)U_4' - \pi_0^{1/3}\xi x_2 V_2' - \pi_0^{1/3}\xi x_3 V_3' = \pi_0^{1/3}\xi A_1^{[5]} - \pi_0^{1/3}\xi X^{[5]},\\
&U_2' + \pi_0^{1/3}\xi V_2' - x_3 U_4' = \pi_0^{1/3}\xi A_2^{[5]},\\
&U_3' + \pi_0^{1/3}\xi V_3' + x_2 U_4' = \pi_0^{1/3}\xi A_3^{[5]},\\
&U_5' = \pi_0^{1/3}\xi A_4^{[5]},\\
&V_1' - \pi_0^{1/3}\xi^2 W' + b_1 U_4' + x_2 V_2' + x_3 V_3' = \pi_0^{1/3}\xi B^{[5]} + X^{[5]}.
\end{aligned} \tag{8.3.16}$$

By comparison of the coefficients of e_k for $k = 2, ..., 8$ in (8.3.14), we obtain:

$$\begin{aligned}
&U_1'' - \pi_0^{2/3}\xi W'' - (x_5 + \pi_0^{1/3}\xi b_1)U_4'' - \pi_0^{1/3}\xi x_2 V_2'' - \pi_0^{1/3}\xi x_3 V_3'' = \pi_0^{2/3}\xi^2 A_1^{[7]} + \pi_0^{2/3} X^{[7]},\\
&U_2'' + \pi_0^{1/3}\xi V_2'' - x_3 U_4'' = \pi_0^{2/3}\xi^2 A_2^{[7]},\\
&U_3'' + \pi_0^{1/3}\xi V_3'' + x_2 U_4'' = \pi_0^{2/3}\xi^2 A_3^{[7]},\\
&U_5'' = \pi_0^{2/3}\xi^2 A_4^{[7]},\\
&V_1'' - \pi_0^{1/3}\xi^2 W'' + y_4 U_4'' + x_2 V_2'' + x_3 V_3'' = \pi_0^{2/3}\xi^2 B^{[7]} - \pi_0^{2/3}\xi^2 X^{[7]},
\end{aligned} \tag{8.3.17}$$

Consider Equations (8.3.11) and (8.3.15). Equations in (8.3.11) show that U_1, U_2, U_3, U_4 are determined by the matrix $V = (V_1\ V_2\ V_3)$, the vector W, and the last two equations in (8.3.15) show that U_5, V_1 are determined by V_2, V_3, W. Put the expression $U =$

$(U_1\ U_2\ U_3\ U_4\ U_5)$ and V_1 back into the first 3 equations in (8.3.15). We get:

$$\pi_0^{1/3}\xi(1-\xi)V_2 = \begin{pmatrix} -(x_4+x_5) \\ 0 \\ x_3(x_4+x_5) \end{pmatrix}, \quad \pi_0^{1/3}\xi(1-\xi)V_3 = \begin{pmatrix} 0 \\ x_4+x_5 \\ -x_2(x_4+x_5) \end{pmatrix},$$

$$\pi_0^{2/3}W = \pi_0^{2/3}\xi \begin{pmatrix} 0 \\ 0 \\ b_1^2 \end{pmatrix}.$$

It is easy to see that V_2, V_3, W are determined by variables x_i and b_1: Since $x_4+x_5 = \pi_0^{1/3}(1-\xi)b_1$, we obtain

$$V_2 = \xi^2 \begin{pmatrix} -b_1 \\ 0 \\ x_3 b_1 \end{pmatrix}, \quad V_3 = \xi^2 \begin{pmatrix} 0 \\ b_1 \\ -x_2 b_1 \end{pmatrix}, \quad W = \xi \begin{pmatrix} 0 \\ 0 \\ b_1^2 \end{pmatrix}.$$

We can perform similar calculations for (8.3.12) and (8.3.16), and get:

$$V_2' = \begin{pmatrix} -x_3 \\ x_3 \\ -x_2x_3 - (x_4 + \pi_0^{1/3}\xi b_1) \\ x_3^2 \\ -x_7 + x_3(x_5 + \pi_0^{1/3}\xi b_1) \end{pmatrix}, \quad V_2'' = \pi_0^{1/3} \begin{pmatrix} -1 \\ 0 \\ x_3 \\ -x_3 \\ x_2x_3 - (x_5 + \pi_0^{1/3}\xi b_1) \\ -x_3^2 \\ x_3(x_4 + \pi_0^{1/3}\xi b_1) + x_7 \end{pmatrix},$$

$$V_3' = \begin{pmatrix} x_2 \\ -x_2 \\ x_2^2 \\ -x_2x_3 + (x_4 + \pi_0^{1/3}\xi b_1) \\ -x_6 - x_2(x_5 + \pi_0^{1/3}\xi b_1) \end{pmatrix}, \quad V_3'' = \pi_0^{1/3} \begin{pmatrix} 0 \\ -1 \\ -x_2 \\ x_2 \\ -x_2^2 \\ x_2x_3 + (x_5 + \pi_0^{1/3}\xi b_1) \\ -x_2(x_4 + \pi_0^{1/3}\xi b_1) + x_6 \end{pmatrix},$$

$$W' = -\xi^2 \begin{pmatrix} -\xi b_1 \\ \xi^2 b_1 \\ b_1 x_2 \\ -b_1 x_3 \\ -b_1 x_5 - \xi \pi_0^{1/3} b_1^2 \end{pmatrix}, \quad W'' = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{pmatrix} + \pi_0^{1/3}\xi^2 \begin{pmatrix} 0 \\ 0 \\ -\xi b_1 \\ \xi^2 b_1 \\ b_1 x_2 \\ -b_1 x_3 \\ -b_1 x_5 + \pi_0^{1/3} b_1^2 \end{pmatrix}.$$

Above all, all matrices U, U', U'', V, V', V'' and vectors W, W', W'' are determined by x_i, b_1. We can check that $(\pi - \pi_0^{1/3}\xi^2)\mathcal{F}^3 \subset \mathcal{F}^2, \langle \mathcal{F}^3, \mathcal{F}^3 \rangle = 0$ are already satisfied. Therefore, the equations of variables in this affine chart are:

Proposition 8.3.18. *Consider the affine chart in* $\mathbb{P}^7 \times \mathrm{Gr}(5, 24) \times \mathrm{Gr}(9, 24)$ *with the leading terms:*

$$\{\pi^2 e_1\} \times \{\pi^2 e_k, \pi e_1\}_{k=1,2,3,5} \times \{\pi^2 e_i, \pi e_j, e_1\}_{i=1,\ldots,5, j=1,2,3}$$

Under this affine chart, the corresponding open subscheme in $\mathcal{M}^{split}$ is isomorphic to

$$\mathcal{O}[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, b_1]/(x_1 - 1, Q_0, (x_4 + x_5) - \pi_0^{1/3}(1 - \xi)b_1),$$

where $Q_0 = x_1x_8 + x_2x_7 + x_3x_6 + x_4x_5$. Hence it is smooth.

8.4 Affine chart U_1, part II

We continue discuss the affine chart U_1. By Lemma 8.3.1, for any $\{\mathcal{F}^i\}_{i=1,2,3} \in \mathcal{M}^{\mathrm{split}}(R)$, if $\mathcal{F}^1 \in U_1(R)$, then $\mathcal{F}^2$ is either in the affine chart with leading terms $\{\pi^2 e_k, \pi e_1\}_{k=1,2,3,5}$, or in the affine chart with leading terms $\{\pi^2 e_i\}_{i=1,\ldots,5}$. In §8.3, we considered $\mathcal{F}^2$ in the first case. Now we consider the second case, where $\mathcal{F}^2$ is in affine chart with leading terms $\{\pi^2 e_i\}_{i=1,\ldots,5}$.

With respect to the standard order of basis, the generators of $\mathcal{F}^2$ are represented by the columns of a matrix C, where

$$C = \begin{pmatrix} A \\ A' \\ A'' \end{pmatrix}, \text{ with } A = \begin{pmatrix} I_{5\times 5} \\ a_{11} \cdots a_{15} \\ a_{21} \cdots a_{25} \\ a_{31} \cdots a_{35} \end{pmatrix}.$$

Here A, A', A'' are 8×5 matrices in $M(R)$. We can perform similar calculations as in §8.3.

Consider the generators of $\mathcal{F}^2$ are of the following forms by $(\pi - \pi_0^{1/3}\xi)\mathcal{F}^2 \subset \mathcal{F}^1$:

$$C_1 = f_1^\xi + a_{11}f_6^\xi + a_{21}f_7^\xi + a_{31}f_8^\xi + k_1[(\pi e_1 - \pi_0^{1/3}\xi^2 e_1) + \sum_{i\neq 1} x_i(\pi e_i - \pi_0^{1/3}\xi^2 e_i)],$$
$$C_2 = f_2^\xi + a_{12}f_6^\xi + a_{22}f_7^\xi + a_{32}f_8^\xi + k_2[(\pi e_1 - \pi_0^{1/3}\xi^2 e_1) + \sum_{i\neq 1} x_i(\pi e_i - \pi_0^{1/3}\xi^2 e_i)],$$
$$C_3 = f_3^\xi + a_{13}f_6^\xi + a_{23}f_7^\xi + a_{33}f_8^\xi + k_3[(\pi e_1 - \pi_0^{1/3}\xi^2 e_1) + \sum_{i\neq 1} x_i(\pi e_i - \pi_0^{1/3}\xi^2 e_i)],$$
$$C_4 = f_4^\xi + a_{14}f_6^\xi + a_{24}f_7^\xi + a_{34}f_8^\xi + k_4[(\pi e_1 - \pi_0^{1/3}\xi^2 e_1) + \sum_{i\neq 1} x_i(\pi e_i - \pi_0^{1/3}\xi^2 e_i)],$$
$$C_5 = f_5^\xi + a_{15}f_6^\xi + a_{25}f_7^\xi + a_{35}f_8^\xi + k_5[(\pi e_1 - \pi_0^{1/3}\xi^2 e_1) + \sum_{i\neq 1} x_i(\pi e_i - \pi_0^{1/3}\xi^2 e_i)],$$

with variables $k_j \in R$ for $j = 1, ..., 5$. The description of columns C_i implies that A', A'' are determined by the variables in matrix A and x_i, k_j. More precisely, recall that we set $X = (1\ x_2\ \cdots\ x_8)^T$. Denote by $(k_1X\ k_2X\ k_3X\ k_4X\ k_5X)$ the 8×5 matrix where the i-th column is k_iX. Then we have:

$$A' = \pi_0^{1/3}\xi A + (k_1X\ k_2X\ k_3X\ k_4X\ k_5X),$$
$$A'' = \pi_0^{2/3}\xi^2 A - \pi_0^{1/3}\xi^2(k_1X\ k_2X\ k_3X\ k_4X\ k_5X).$$

We still need to check:

$$\Lambda * \mathcal{F}^1 \subset \mathcal{F}^2, \quad \mathcal{F}^1 \subset \mathcal{F}^2, \quad \langle \mathcal{F}^2, \mathcal{F}^2 \rangle = 0.$$

(1). $\Lambda * \mathcal{F}^1 \subset \mathcal{F}^2$. We show that a_{ij} are determined by a_{14}, a_{24}, a_{34}, x_i, and k_1, k_2, k_3, k_5

variables are determined by k_2, x_i. This condition is equivalent to:

$$\begin{aligned}
&f_1^\xi - x_5 f_4^\xi + x_6 f_6^\xi + x_7 f_7^\xi = C_1 - x_5 C_4,\\
&f_2^\xi - x_3 f_4^\xi - x_4 f_6^\xi - x_7 f_8^\xi = C_2 - x_3 C_4,\\
&f_3^\xi + x_2 f_4^\xi + x_4 f_7^\xi - x_6 f_8^\xi = C_3 + x_2 C_4,\\
&f_5^\xi - x_2 f_6^\xi + x_3 f_7^\xi + x_5 f_8^\xi = C_5,
\end{aligned}$$

by comparison of the coefficients of $\pi^2 e_k$ for $k = 1, ..., 5$. Then we get:

$$\begin{array}{llll}
a_{11} - x_5 a_{14} = x_6, & a_{12} - x_3 a_{14} = -x_4, & a_{13} + x_2 a_{14} = 0, & a_{15} = -x_2,\\
a_{21} - x_5 a_{24} = x_7, & a_{22} - x_3 a_{24} = 0, & a_{23} + x_2 a_{24} = x_4, & a_{15} = x_3,\\
a_{31} - x_5 a_{34} = 0, & a_{32} - x_3 a_{34} = -x_7, & a_{33} + x_2 a_{34} = -x_6, & a_{15} = x_5.
\end{array} \tag{8.4.1}$$

and

$$k_1 - x_5 k_4 = 0, \quad k_2 - x_3 k_4 = 0, \quad k_3 + x_2 k_4 = 0, \quad k_5 = 0. \tag{8.4.2}$$

when we compare the coefficients of $\pi^2 e_k$ for $k = 6, 7, 8$, and the coefficient of πe_1.

(2). $\mathcal{F}^1 \subset \mathcal{F}^2$. We show that a_{14}, a_{24}, a_{34} are determined by x_i. For this purpose, we consider the equation:

$$f_1 + \sum_{i \neq 1} x_i f_i = C_1 + x_2 C_2 + x_3 C_3 + x_4 C_4 + x_5 C_5,$$

obtained from the comparison of $\pi^2 e_k$ for $k = 1, 2, 3, 4, 5$. Since $f_i = f_i^\xi + \pi_0^{1/3}(1-\xi)(\pi e_i - \pi_0^{1/3}\xi^2 e_i)$, we get

$$a_{i1} + x_2 a_{i2} + x_3 a_{i3} + x_4 a_{i4} + x_5 a_{i5} = x_{i+5}, \tag{8.4.3}$$

$$k_1 + x_2 k_2 + x_3 k_3 + x_4 k_4 + x_5 k_5 = \pi_0^{1/3}(1-\xi), \tag{8.4.4}$$

for $i = 1, 2, 3$. Combining (8.4.1)-(8.4.3) yields the following equations:

$$\begin{aligned}
&k_4(x_4 + x_5) = \pi_0^{1/3}(1-\xi), \\
&(x_4 + x_5)(a_{14} - x_2) = 0, \\
&(x_4 + x_5)(a_{24} + x_3) = 0, \\
&(x_4 + x_5)(a_{34} + x_5) = 0.
\end{aligned} \tag{8.4.5}$$

By multiplying k_4 on both sides of the last 3 equations in (8.4.5) and using $k_4(x_4+x_5) = \pi_0^{1/3}(1-\xi)$, we get $a_{14} = x_2$, $a_{24} = -x_3, a_{34} = -x_5$. Thus, all variables are determined by x_i and k_4. We can check $\langle C_i, C_j \rangle = 0$ for all $i, j \in \{1, 2, 3, 4, 5\}$, hence $\langle \mathcal{F}^2, \mathcal{F}^2 \rangle = 0$. We rewrite the generators of $\mathcal{F}^2$ (the columns of the matrix C) as the following forms:

$$\begin{aligned}
C_1 &= f_1^\xi + (x_6 + x_2 x_5) f_6^\xi + (x_7 - x_3 x_5) f_7^\xi - x_5^2 f_8^\xi + k_1[(\pi e_1 - \pi_0^{1/3}\xi^2 e_1) \\
&\quad + \sum_{i \neq 1} x_i(\pi e_i - \pi_0^{1/3}\xi^2 e_i)] \\
C_2 &= f_2^\xi + (-x_4 + x_2 x_3) f_6^\xi - x_3^2 f_7^\xi - (x_7 + x_3 x_5) f_8^\xi + k_2[(\pi e_1 - \pi_0^{1/3}\xi^2 e_1) \\
&\quad + \sum_{i \neq 1} x_i(\pi e_i - \pi_0^{1/3}\xi^2 e_i)] \\
C_3 &= f_3^\xi - x_2^2 f_6^\xi + (x_4 + x_2 x_3) f_7^\xi + (-x_6 + x_2 x_5) f_8^\xi + k_3[(\pi e_1 - \pi_0^{1/3}\xi^2 e_1) \\
&\quad + \sum_{i \neq 1} x_i(\pi e_i - \pi_0^{1/3}\xi^2 e_i)] \\
C_4 &= f_4^\xi + x_2 f_6^\xi - x_3 f_7^\xi - x_5 f_8^\xi + k_4[(\pi e_1 - \pi_0^{1/3}\xi^2 e_1) + \sum_{i \neq 1} x_i(\pi e_i - \pi_0^{1/3}\xi^2 e_i)] \\
C_5 &= f_5^\xi - x_2 f_6^\xi + x_3 f_7^\xi + x_5 f_8^\xi,
\end{aligned}$$

with

$$k_4(x_4+x_5) = \pi_0^{1/3}(1-\xi), \quad k_1 = x_5k_4, \quad k_2 = x_3k_4, \quad k_3 = -x_2k_4. \tag{8.4.6}$$

Lemma 8.4.7. *For any $\{\mathcal{F}^i\}_{i=1,2,3} \in \mathcal{M}^{\text{split}}(R)$, if $\mathcal{F}^1 \in U_1(R)$, and $\mathcal{F}^2$ is in the affine chart with leading terms $\{\pi^2 e_i\}_{i=1,..,5}$. Then $\mathcal{F}^3$ is in the affine chart with leading terms*

$$\{\pi^2 e_i, \pi e_1\}_{i=1,\dots,8}.$$

Proof. This proof is similar to the proof of Lemma 8.3.9. The difference is that we do not assume x_4+x_5 is divisible by $\pi_0^{1/3}$ anymore. From the discussion of $\mathcal{F}^2$ as above, we have the equation $k_4(x_4+x_5) = \pi_0^{1/3}(1-\xi)$. By $\mathcal{F}^2 \subset \mathcal{F}^3$, we know that $\mathcal{F}^3$ has elements with leading terms $\pi^2 e_i$ for $i = 1,2,3,4,5$. Analogous to Lemma 8.3.9, we consider $\alpha_1 - \alpha_3 + x_3\alpha_2 + x_3\alpha_4$ where

$$\begin{aligned}
\alpha_1 =& f_2^{\xi} - x_3 f_4^{\xi} - x_4 f_6^{\xi} - x_7 f_8^{\xi}, \\
\alpha_2 =& f_5^{\xi} - x_2 f_6^{\xi} + x_3 f_7^{\xi} + x_5 f_8^{\xi}, \\
\alpha_3 =& f_2^{\xi^2} + x_3 f_5^{\xi^2} + x_5 f_6^{\xi^2} - x_7 f_8^{\xi^2}, \\
\alpha_4 =& f_4^{\xi^2} + x_2 f_6^{\xi^2} - x_3 f_7^{\xi^2} + x_4 f_8^{\xi^2},
\end{aligned}$$

are elements in $\mathcal{F}^3$. Since

$$\begin{aligned}
\alpha_1 - \alpha_3 + x_3\alpha_2 + x_3\alpha_4 =& (f_2^{\xi} - f_2^{\xi^2}) - x_3(f_4^{\xi} - f_4^{\xi^2}) + x_3(f_5^{\xi} - f_5^{\xi^2}) - x_2x_3(f_6^{\xi} - f_6^{\xi^2}) \\
&+ x_3^2(f_7^{\xi} - f_7^{\xi^2}) - x_7(f_8^{\xi} - f_8^{\xi^2}) - (x_4 f_6^{\xi} + x_5 f_6^{\xi^2}) + x_3(x_5 f_8^{\xi} + x_4 f_8^{\xi^2}),
\end{aligned}$$

and $f_i^{\xi} - f_i^{\xi^2}$ is divisible by $\pi_0^{1/3}$. We have $k_4(\alpha_1 - \alpha_3 + x_3\alpha_2 + x_3\alpha_4)$ is divisible by $\pi_0^{1/3}$. More precisely,

$$-\frac{k_4(\alpha_1 - \alpha_3 + x_3\alpha_2 + x_3\alpha_4)}{\pi_0^{1/3}(1-\xi)} = \pi^2 e_6 - x_3\pi^2 e_8 + \sum\nolimits_{j=2,4,5,6,7,8}(y_j\pi e_j + y_j' e_j)$$

for some $y_j, y_j' \in R$. Thus, we have an element with leading term $\pi^2 e_6$ in $\mathcal{F}^3$. Similarly, consider the linear combination of

$$\begin{aligned}
&f_3^{\xi} + x_2 f_4^{\xi} + x_4 f_7^{\xi} - x_6 f_8^{\xi},\\
&f_3^{\xi^2} - x_2 f_5^{\xi^2} - x_5 f_7^{\xi^2} - x_6 f_8^{\xi^2},\\
&f_5^{\xi} - x_2 f_6^{\xi} + x_3 f_7^{\xi} + x_5 f_8^{\xi},\\
&f_4^{\xi^2} + x_2 f_6^{\xi^2} - x_3 f_7^{\xi^2} + x_4 f_8^{\xi^2}.
\end{aligned}$$

We can get an element in $\mathcal{F}^3$: $\pi^2 e_7 - x_2\pi^2 e_8 + \sum_{j=3,4,5,6,7,8}(z_j\pi e_j + z_j' e_j)$ for some $z_j, z_j' \in R$. So we have an element with leading term $\pi^2 e_7$. Next, we claim that there are elements with leading terms πe_1 and $\pi^2 e_8$ in $\mathcal{F}^3$. Recall that we set:

$$\begin{aligned}
\beta_1 =& f_1 + \sum_{i\neq 1,4,5} x_i f_i + x_4 f_4 + x_5 f_5,\\
\beta_2 =& -f_1^{\xi} - \sum_{i\neq 1,4,5} x_i f_i^{\xi} + x_5 f_4^{\xi} + x_4 f_5^{\xi},\\
\beta_3 =& -f_1^{\xi^2} - \sum_{i\neq 1,4,5} x_i f_i^{\xi^2} + x_5 f_4^{\xi^2} + x_4 f_5^{\xi^2}.
\end{aligned}$$

They are elements in $\mathcal{F}^3$ by $(e_4+e_5)*(f_1+\sum_{i\neq 1} x_i f_i) \in \mathcal{F}^3$ and $(f_1+\sum_{i\neq 1} x_i f_i)*(e_4+e_5) \in$

$\mathcal{F}^3$. Consider $-\beta_2+\beta_3$. We obtain:

$$\pi_0^{1/3}\xi(1-\xi)[(\pi e_1-\pi_0^{1/3}e_1)+\sum\nolimits_{i\neq 1,4,5}x_i(\pi e_i-\pi_0^{1/3}e_i)-x_5(\pi e_4-\pi_0^{1/3}e_4)-x_4(\pi e_5-\pi_0^{1/3}e_5)].$$

It is divisible by $\pi_0^{1/3}$. Then we have an element with leading term πe_1 in $\mathcal{F}^3$. Finally, by using $k_4(x_4+x_5)=\pi_0^{1/3}(1-\xi)$, we have:

$$\begin{aligned}
\frac{k_4(\beta_1+\beta_2)}{\pi_0^{1/3}(1-\xi)} &= (\pi^2e_4+\pi^2e_5)+k_4[(\pi e_1-\pi_0^{1/3}\xi^2e_1)-(k_4x_5-\pi_0^{1/3})\pi e_4-(k_4x_4-\pi_0^{1/3})\pi e_5 \\
&\quad +\sum_{i\neq 1,4,5}x_i(\pi e_i-\pi_0^{1/3}\xi^2e_i)]+\pi_0^{1/3}(\xi^2k_4x_5+\pi_0^{1/3})e_4+\pi_0^{1/3}(\xi^2k_4x_4+\pi_0^{1/3})e_5, \\
\frac{k_4(\beta_1+\beta_3)}{\pi_0^{1/3}(1-\xi^2)} &= -\xi(\pi^2e_4+\pi^2e_5)+k_4[(\pi e_1-\pi_0^{1/3}\xi e_1)-(k_4x_5+\pi_0^{1/3}\xi)\pi e_4-(k_4x_4+\pi_0^{1/3}\xi)\pi e_5 \\
&\quad +\sum_{i\neq 1,4,5}x_i(\pi e_i-\pi_0^{1/3}\xi e_i)]+\pi_0^{1/3}(\xi k_4x_5-\pi_0^{1/3}\xi)e_4+\pi_0^{1/3}(\xi k_4x_4-\pi_0^{1/3}\xi)e_5.
\end{aligned}$$

Since

$$\frac{k_4(\beta_1+\beta_2)}{\pi_0^{1/3}(1-\xi)}-\frac{k_4(\beta_1+\beta_3)}{\pi_0^{1/3}(1-\xi^2)}+\xi^2(\alpha_2+\alpha_4)=\xi^2(x_4+x_5)\pi^2e_8+\pi_0^{1/3}(\sum_{j=4,5,6,7,8}r_j\pi e_j+\sum_{k=1,\dots,8}r'_ke_k).$$

for some $r_j,r'_k\in R$. Therefore, we can multiply k_4 on both sides and get an element with leading term π^2e_8 in $\mathcal{F}^3$. Above all, we see that $\mathcal{F}^3$ is in the affine chart with leading terms $\{\pi^2e_i,\pi e_1\}_{i=1,\dots,8}$. □

With respect to the standard order of basis, the generators of $\mathcal{F}^3$ are described as the columns span of the 24×9 matrix D. Here D is of the following form:

$$\mathcal{F}^3 = D = \begin{pmatrix} I & 0 \\ P & Q \\ P' & Q' \end{pmatrix},$$

where $P_{8\times 8}, P'_{8\times 8}$ are matrices, and $Q_{8\times 1}$, $Q'_{8\times 1}$ are vectors. More precisely, we have

$$P = \begin{pmatrix} 0 \quad ... \quad 0 \\ p_{ij} \ {}_{2\le i\le 8,\ 1\le j\le 8} \end{pmatrix}, \quad Q = (1\ q_2\ q_3\ q_4\ q_5\ q_6\ q_7\ q_8)^T,$$

and $P' = (p'_{ij})_{1\le i,j\le 8}$, $Q' = (q'_1 \ \cdots \ q'_8)^T$. Recall that we use P_i (resp. P'_i) to denote the i-th column of the matrix P (resp. P'). Thus, the generators of $\mathcal{F}^3$ (resp. the columns of D) are:

$$\begin{aligned} D_i =& \pi^2 e_i + \sum\nolimits_{j\ne 1} p_{ji}\pi e_j + \sum\nolimits_{k=1}^{8} p'_{ki} e_k, \\ D_9 =& \pi e_1 + \sum\nolimits_{j\ne 1} q_j \pi e_j + \sum\nolimits_{k=1}^{8} q'_k e_k, \end{aligned}$$

for $i = 1, 2, ..., 8$. We need to check:

$$\mathcal{F}^1 * \Lambda \subset \mathcal{F}^3, \quad \mathcal{F}^2 \subset \mathcal{F}^3, \quad \langle \mathcal{F}^3, \mathcal{F}^3 \rangle = 0.$$

(1). $\mathcal{F}^1 * \Lambda \subset \mathcal{F}^3$. Recall that we defined the matrix K as:

$$K = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \\ -x_4 & x_3 & -x_2 & 0 \\ x_6 & x_5 & 0 & x_2 \\ x_7 & 0 & -x_5 & -x_3 \\ 0 & -x_7 & -x_6 & x_4 \end{pmatrix}$$

in §8.3. With respect to the standard order of basis, $\mathcal{F}^1 * \Lambda$ is represented by the matrix:

$$\mathcal{F}^1 * \Lambda = \begin{pmatrix} K \\ \pi_0^{1/3}\xi^2 K \\ \pi_0^{1/3}\xi K \end{pmatrix}$$

We perform similar calculations as what we did in §8.3, but have different leading terms this time. Condition $\mathcal{F}^1 * \Lambda \subset \mathcal{F}^3$ is equivalent to:

$$\begin{aligned}
f_1^{\xi^2} - x_4 f_5^{\xi^2} + x_6 f_6^{\xi^2} + x_7 f_7^{\xi^2} &= D_1 - x_4 D_5 + x_6 D_6 + x_7 D_7 + \pi_0^{1/3}\xi^2 D_9, \\
f_2^{\xi^2} + x_3 f_5^{\xi^2} + x_5 f_6^{\xi^2} - x_7 f_8^{\xi^2} &= D_2 + x_3 D_5 + x_5 D_6 - x_7 D_8, \\
f_3^{\xi^2} - x_2 f_5^{\xi^2} - x_5 f_7^{\xi^2} - x_6 f_8^{\xi^2} &= D_3 - x_2 D_5 - x_5 D_7 - x_6 D_8, \\
f_4^{\xi^2} + x_2 f_6^{\xi^2} - x_3 f_7^{\xi^2} + x_4 f_8^{\xi^2} &= D_4 + x_2 D_6 - x_3 D_7 + x_4 D_8.
\end{aligned} \tag{8.4.8}$$

By comparison of the coefficients of πe_i for $i = 2, ..., 8$ in (8.4.8), we obtain:

$$
\begin{aligned}
&P_1 - x_4 P_5 + x_6 P_6 + x_7 P_7 + \pi_0^{1/3} \xi^2 Q = \pi_0^{1/3} \xi^2 K_1, \\
&P_2 + x_3 P_5 + x_5 P_6 - x_7 P_8 = \pi_0^{1/3} \xi^2 K_2, \\
&P_3 - x_2 P_5 - x_5 P_7 - x_6 P_8 = \pi_0^{1/3} \xi^2 K_3, \\
&P_4 + x_2 P_6 - x_3 P_7 + x_4 P_8 = \pi_0^{1/3} \xi^2 K_4.
\end{aligned} \tag{8.4.9}
$$

Similarly, comparing coefficients of e_i for $i = 1, ..., 8$ yields to:

$$
\begin{aligned}
&P'_1 - x_4 P'_5 + x_6 P'_6 + x_7 P'_7 + \pi_0^{1/3} \xi^2 Q' = \pi_0^{2/3} \xi K_1, \\
&P'_2 + x_3 P'_5 + x_5 P'_6 - x_7 P'_8 = \pi_0^{2/3} \xi K_2, \\
&P'_3 - x_2 P'_5 - x_5 P'_7 - x_6 P'_8 = \pi_0^{2/3} \xi K_3, \\
&P'_4 + x_2 P'_6 - x_3 P'_7 + x_4 P'_8 = \pi_0^{2/3} \xi K_4.
\end{aligned} \tag{8.4.10}
$$

(2). $\mathcal{F}^2 \subset \mathcal{F}^3$. We just need to check that the generators of $\mathcal{F}^2$ are elements in $\mathcal{F}^3$. Compare coefficients of πe_1 and $\pi^2 e_i$ for $i = 1, ..., 8$. This condition is equivalent to:

$$
\begin{aligned}
C_1 &= D_1 + (x_6 + x_2 x_5) D_6 + (x_7 - x_3 x_5) D_7 - x_5^2 D_8 + (\pi_0^{1/3} \xi + k_1) D_9, \\
C_2 &= D_2 + (-x_4 + x_2 x_3) D_6 - x_3^2 D_7 - (x_7 + x_3 x_5) D_8 + k_2 D_9, \\
C_3 &= D_3 - x_2^2 D_6 + (x_4 + x_2 x_3) D_7 + (-x_6 + x_2 x_5) D_8 + k_3 D_9, \\
C_4 &= D_4 + x_2 D_6 - x_3 D_7 - x_5 D_8 + k_4 D_9, \\
C_5 &= D_5 - x_2 D_6 + x_3 D_7 + x_5 D_8,
\end{aligned}
$$

where $k_1 = x_5 k_4, k_2 = x_3 k_4, k_3 = -x_2 k_4$. Recall that the matrix C is of the form

$$C = \begin{pmatrix} A \\ \\ A' \\ \\ A'' \end{pmatrix}, \text{ where } A = \begin{pmatrix} I_{5\times 5} \\ \\ a_{11} \cdots a_{15} \\ \\ a_{21} \cdots a_{25} \\ \\ a_{31} \cdots a_{35} \end{pmatrix},$$

and A', A'' are determined by A:

$$A' = \pi_0^{1/3} \xi A + (k_1 X \ k_2 X \ k_3 X \ k_4 X \ 0),$$
$$A'' = \pi_0^{2/3} \xi^2 A - \pi_0^{1/3} \xi^2 (k_1 X \ k_2 X \ k_3 X \ k_4 X \ 0),$$

for $X = (1 \ x_2 \ \cdots \ x_8)^T$. Then by comaprison of coefficients of πe_i and e_i, we get:

$$\begin{aligned}
&P_1 + (x_6 + x_2 x_5) P_6 + (x_7 - x_3 x_5) P_7 - x_5^2 P_8 + (\pi_0^{1/3} \xi + k_1) Q = \pi_0^{1/3} \xi A_1 + k_1 X, \\
&P_2 + (-x_4 + x_2 x_3) P_6 - x_3^2 P_7 - (x_7 + x_3 x_5) P_8 + k_2 Q = \pi_0^{1/3} \xi A_2 + k_2 X, \\
&P_3 - x_2^2 P_6 + (x_4 + x_2 x_3) P_7 + (-x_6 + x_2 x_5) P_8 + k_3 Q = \pi_0^{1/3} \xi A_3 + k_3 X, \\
&P_4 + x_2 P_6 - x_3 P_7 - x_5 P_8 + k_4 Q = \pi_0^{1/3} \xi A_4 + k_4 X, \\
&P_5 - x_2 P_6 + x_3 P_7 + x_5 P_8 = \pi_0^{1/3} \xi A_5
\end{aligned} \tag{8.4.11}$$

and

$$
\begin{aligned}
&P_1' + (x_6 + x_2x_5)P_6' + (x_7 - x_3x_5)P_7' - x_5^2P_8' + (\pi_0^{1/3}\xi + k_1)Q' = \pi_0^{2/3}\xi^2A_1 - \pi_0^{1/3}\xi^2k_1X,\\
&P_2' + (-x_4 + x_2x_3)P_6' - x_3^2P_7' - (x_7 + x_3x_5)P_8' + k_2Q' = \pi_0^{2/3}\xi^2A_2 - \pi_0^{1/3}\xi^2k_2X,\\
&P_3' - x_2^2P_6' + (x_4 + x_2x_3)P_7' + (-x_6 + x_2x_5)P_8' + k_3Q' = \pi_0^{2/3}\xi^2A_3 - \pi_0^{1/3}\xi^2k_3X,\\
&P_4' + x_2P_6' - x_3P_7' - x_5P_8' + k_4Q' = \pi_0^{2/3}\xi^2A_4 - \pi_0^{1/3}\xi^2k_4X,\\
&P_5' - x_2P_6' + x_3P_7' + x_5P_8' = \pi_0^{2/3}\xi^2A_5.
\end{aligned}
\tag{8.4.12}
$$

It is easy to see that P_i (resp. P_i') for $i = 1, ..., 5$ are determined by P_6, P_7, P_8, Q (resp. P_6', P_7', P_8', Q') from Equations (8.4.11), (8.4.12). Represent P_i by linear combinations of $P_6, P_7,\ P_8, Q$ (resp. P_6', P_7', P_8', Q'), and put them back to equation (8.4.9)-(8.4.10). We get:

$$
\begin{aligned}
&-(x_4 + x_5)P_6 + k_2Q - k_2X = \pi_0^{1/3}\xi(A_2 + x_3A_5) - \pi_0^{1/3}\xi^2K_2,\\
&(x_4 + x_5)P_7 + k_3Q - k_3X = \pi_0^{1/3}\xi(A_3 - x_2A_5) - \pi_0^{1/3}\xi^2K_3,\\
&-(x_4 + x_5)P_8 + k_4Q - k_4X = \pi_0^{1/3}\xi A_4 - \pi_0^{1/3}\xi^2K_4,
\end{aligned}
\tag{8.4.13}
$$

and

$$
(x_4+x_5)(x_2P_6-x_3P_7-x_5P_8)+k_1Q+\pi_0^{1/3}\xi(1-\xi)Q-k_1X = \pi_0^{1/3}\xi(A_1-x_4A_5)-\pi_0^{1/3}\xi^2K_1.
\tag{8.4.14}
$$

Compare equation (8.4.13) and (8.4.14). We can eliminate P_6, P_7, P_8 and get a equation with variables Q and x_i:

$$
\pi_0^{1/3}\xi(1-\xi)Q = \pi_0^{1/3}\xi(1-\xi)(1\ x_2\ x_3\ \ -x_5\ \ -x_4\ x_6\ x_7\ x_8)^T
$$

Thus, Q is determined by x_i. We have $q_i = x_i$ for $i \neq 4, 5$ and $q_4 = -x_5$, $q_5 = -x_4$. Since

$$Q - X = -(x_4 + x_5)(0\ 0\ 0\ 1\ 1\ 0\ 0\ 0)^T$$

Put Q back into equation (8.4.13). By multiplying k_4 on both sides, we can see that P_6, P_7, P_8 are determined by k_4, x_i since $k_4(x_4 + x_5) = \pi_0^{1/3}(1 - \xi)$. Similar calculation for P_i', Q'. We obtain that P_6', P_7', P_8' are determined by Q', x_i, k_4, and

$$Q' = -\pi_0^{1/3}(1\ x_2\ x_3\ \ -x_5\ \ -x_4\ x_6\ x_7\ x_8)^T.$$

Therefore, we have:

Proposition 8.4.15. *Consider the affine chart with leading terms*

$$\{\pi^2 e_1\} \times \{\pi^2 e_k\}_{k=1,\dots,5} \times \{\pi^2 e_i, \pi e_1\}_{i=1,2,\dots,8}.$$

in $\mathbb{P}^7 \times \mathrm{Gr}(24, 5) \times \mathrm{Gr}(24, 9)$. *Under this affine chart, the corresponding open subscheme in* $\mathcal{M}^{split}$ *is isomorphic to*

$$\mathcal{O}[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, k_4]/(x_1 - 1, Q_0, k_4(x_4 + x_5) - \pi_0^{1/3}(1 - \xi))$$

where $Q_0 = x_1x_8 + x_2x_7 + x_3x_6 + x_4x_5$.

8.5 Affine chart U_4

For other affine charts $U_i = \{f_i + \sum_{j\neq i} x_j f_j\} \subset Q$, suppose that $\mathcal{F}^1 \in U_i(R)$. If $i \neq 4, 5$, we can get similar results as in §8.3, §8.4. Consider $i = 4$. We can assume that all x_i ($i \neq 4, 5$) are in the maximal ideal m (if $x_i \in R^*$ for some $i \neq 4, 5$, we rewrite $\mathcal{F}^1$ and consider $\mathcal{F}^1 \in U_i(R)$). By $Q_0 = 0$, we have $x_5 = -(x_1x_8 + x_2x_7 + x_3x_6) \in m$. So all $x_i \in m$. We have the following lemma:

Lemma 8.5.1. *If $\mathcal{F}^1 = f_4 + \sum_{i\neq 4} x_i f_i$, where $x_i \in m$, then $\mathcal{F}^2$ is in the affine chart with leading terms $\{\pi^2 e_i\}_{i=1,4,5,6,7}$, $\mathcal{F}^3$ is in the affine chart with leading terms $\{\pi e_5, \pi^2 e_i\}_{i=1,\ldots,8}$.*

Proof. In the special fiber, we get $\mathcal{F}^1_s = \kappa(\pi^2 e_4)$. Since $\Lambda_s * \mathcal{F}^1_s = \kappa\langle \pi^2 e_i\rangle_{i=1,5,6,7}$, it is easy to see that $\mathcal{F}^2_s = \kappa\langle \pi^2 e_i\rangle_{i=1,4,5,6,7}$. So $\mathcal{F}^2$ is in the affine chart with leading terms $\{\pi^2 e_i\}_{i=1,4,5,6,7}$. Similarly, we get $\mathcal{F}^s_1 * \Lambda_s = \kappa\langle \pi^2 e_j\rangle_{j=2,3,5,8}$, hence $\pi^2 e_i \in \mathcal{F}^3_s$ for $i = 1, \ldots, 8$. For the last generator in $\mathcal{F}^3_s$. Consider $(e_4 + e_5) * (f_4 + \sum_{i\neq 4} x_i f_i)$ and $(f_4 + \sum_{i\neq 4} x_i f_i) + (e_4 + e_5)$. We obtain:

$$f_5^{\xi} - \textstyle\sum_{i\neq 4,5} x_i f_i^{\xi} + x_5 f_4^{\xi}, \quad f_5^{\xi^2} - \textstyle\sum_{i\neq 4,5} x_i f_i^{\xi^2} + x_5 f_4^{\xi^2},$$

are elements in $\mathcal{F}^3$. Subtracting them gives us:

$$\pi_0^{1/3}(\xi - \xi^2)[(\pi e_5 - \pi_0^{1/3} e_5) - \sum_{i\neq 4,5} x_i(\pi e_i - \pi_0^{1/3} e_i) + x_5(\pi e_4 - \pi_0^{1/3} e_4)].$$

Thus we have

$$(\pi e_5 - \pi_0^{1/3} e_5) - \sum_{i\neq 4,5} x_i(\pi e_i - \pi_0^{1/3} e_i) + x_5(\pi e_4 - \pi_0^{1/3} e_4) \in \mathcal{F}^3.$$

Hence $\pi e_5 \in \mathcal{F}_s^3$. Therefore, we obtain $\mathcal{F}^3 = \kappa\langle \pi^2 e_i, \pi e_5\rangle_{i=1,\dots,8}$, and $\mathcal{F}^3$ is in the affine chart with leading terms $\{\pi^2 e_i, \pi e_5\}_{i=1,2,\dots,8}$. $\square$

For $\mathcal{F}^2$, with respect to the standard order of basis, we again use the columns of matrix C representing the generators of $\mathcal{F}^2$, i.e.,

$$C = \begin{pmatrix} A \\ A' \\ A'' \end{pmatrix}$$

where $A'_{8\times 5} = (a'_{ij})_{1\le i\le 8, 1\le j\le 5}$, $A''_{8\times 5} = (a''_{ij})_{1\le i\le 8, 1\le j\le 5}$, are $M(R)$-matrices, and

$$A = \begin{pmatrix} 1 & & & & \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \end{pmatrix}$$

The conditions for $\mathcal{F}^2$ are:

$$(\pi - \pi_0^{1/3}\xi)\mathcal{F}^2 \subset \mathcal{F}^1, \quad \Lambda * \mathcal{F}^1 \subset \mathcal{F}^2, \quad \mathcal{F}^1 \subset \mathcal{F}^2, \quad \langle \mathcal{F}^2, \mathcal{F}^2\rangle = 0.$$

(1). $(\pi - \pi_0^{1/3}\xi)\mathcal{F}^2 \subset \mathcal{F}^1$. We can perform similar calculations as in the proof of Lemma

8.3.1. We omit calculations here and give the results as follows:

$$
\begin{aligned}
C_1 &= f_1^\xi + a_{11} f_2^\xi + a_{21} f_3^\xi + a_{31} f_8^\xi + k_1[(\pi e_4 - \pi_0^{1/3}\xi^2 e_4) + \textstyle\sum_{i\neq 4} x_i(\pi e_i - \pi_0^{1/3}\xi^2 e_i)],\\
C_2 &= f_4^\xi + a_{12} f_2^\xi + a_{22} f_3^\xi + a_{32} f_8^\xi + k_2[(\pi e_4 - \pi_0^{1/3}\xi^2 e_4) + \textstyle\sum_{i\neq 4} x_i(\pi e_i - \pi_0^{1/3}\xi^2 e_i)],\\
C_3 &= f_5^\xi + a_{13} f_2^\xi + a_{23} f_3^\xi + a_{33} f_8^\xi + k_3[(\pi e_4 - \pi_0^{1/3}\xi^2 e_4) + \textstyle\sum_{i\neq 4} x_i(\pi e_i - \pi_0^{1/3}\xi^2 e_i)],\\
C_4 &= f_6^\xi + a_{14} f_2^\xi + a_{24} f_3^\xi + a_{34} f_8^\xi + k_4[(\pi e_4 - \pi_0^{1/3}\xi^2 e_4) + \textstyle\sum_{i\neq 4} x_i(\pi e_i - \pi_0^{1/3}\xi^2 e_i)],\\
C_5 &= f_7^\xi + a_{15} f_2^\xi + a_{25} f_3^\xi + a_{35} f_8^\xi + k_5[(\pi e_4 - \pi_0^{1/3}\xi^2 e_4) + \textstyle\sum_{i\neq 4} x_i(\pi e_i - \pi_0^{1/3}\xi^2 e_i)],
\end{aligned}
$$

for $k_1, ..., k_5 \in R$. Thus, it is easy to see that A', A'' are determined by A and k_i. Set $Y = (x_1\ x_2\ x_3\ 1\ x_5\ x_6\ x_7\ x_8)^T$. Denote by $(k_1Y\ k_2Y\ k_3Y\ k_4Y\ k_5Y)$ the 8×5 matrix where the i-th column is k_iY. We get:

$$
\begin{aligned}
A' &= \pi_0^{1/3}\xi A + (k_1Y\ k_2Y\ k_3Y\ k_4Y\ k_5Y),\\
A'' &= -\pi_0^{1/3}\xi^2 A - \pi_0^{1/3}\xi^2(k_1Y\ k_2Y\ k_3Y\ k_4Y\ k_5Y).
\end{aligned}
$$

(2). $\Lambda * \mathcal{F}^1 \subset \mathcal{F}^2$. We show that A is determined by its 2nd column, i.e., by variables a_{12}, a_{22}, a_{32} and x_i. This condition is equivalent to:

$$
\begin{aligned}
f_1^\xi + x_6 f_2^\xi - x_7 f_3^\xi + x_8 f_4^\xi &= C_1 + x_8 C_2,\\
f_5^\xi - x_2 f_2^\xi - x_3 f_3^\xi - x_8 f_8^\xi &= C_3,\\
f_6^\xi - x_1 f_2^\xi + x_3 f_4^\xi + x_7 f_8^\xi &= C_4 + x_3 C_2,\\
f_7^\xi + x_1 f_3^\xi + x_2 f_4^\xi - x_6 f_8^\xi &= C_5 + x_2 C_2,
\end{aligned}
$$

by comparison of coefficients of $\pi^2 e_i$. Then we get

$$\begin{array}{lll} a_{11}+x_8a_{12}=x_6, & a_{21}+x_8a_{22}=-x_7, & a_{31}+x_8a_{32}=0, \\ a_{13}=-x_2, & a_{23}=-x_3, & a_{33}=-x_8, \\ a_{14}+x_3a_{12}=-x_1, & a_{24}+x_3a_{22}=0, & a_{34}+x_3a_{32}=x_7, \\ a_{15}+x_2a_{12}=0, & a_{25}+x_2a_{22}=x_1, & a_{35}+x_2a_{32}=-x_6. \end{array} \tag{8.5.2}$$

and

$$k_1+x_8k_2=0, \quad k_3=0, \quad k_4+x_3k_2=0, \quad k_5+x_2k_2=0. \tag{8.5.3}$$

Thus, A is determined by a_{12}, a_{22}, a_{32} and x_i.

(3). $\mathcal{F}^1 \subset \mathcal{F}^2$. We show that a_{12}, a_{22}, a_{32} variables are determined by x_i. This condition is equivalent to:

$$f_4+\sum_{i\neq 4} x_if_i = x_1C_1+C_2+x_5C_3+x_6C_4+x_7C_7.$$

By using $x_1x_8+x_2x_7+x_3x_6+x_5=0$ and (8.5.2), we have

$$(1+x_5)(a_{12}-x_2)=0, \quad (1+x_5)(a_{22}-x_3)=0, \quad (1+x_5)(a_{32}-x_8)=0, \tag{8.5.4}$$

and

$$x_1k_1+x_5k_3+x_6k_4+x_7k_5+k_2=\pi_0^{1/3}(1-\xi). \tag{8.5.5}$$

By (8.5.3), the equation (8.5.5) comes to

$$(1+x_5)k_2=\pi_0^{1/3}(1-\xi). \tag{8.5.6}$$

Hence we obtain $a_{12} = x_2, a_{22} = x_3, a_{32} = x_8$ by multiplying k_2 on both sides of equations in (8.5.4). Thus, the variables in $\mathcal{F}^2$ are only determined by k_2 and x_i. We can check the isotropic condition $\langle \mathcal{F}^2, \mathcal{F}^2 \rangle = 0$ are already satisfied, so the columns of C are:

$$\begin{aligned}
C_1 =& f_1^\xi + (x_6 - x_8 x_2) f_2^\xi - (x_7 + x_8 x_3) f_3^\xi - x_8^2 f_8^\xi - x_8 k_2[(\pi e_4 - \pi_0^{1/3} \xi^2 e_4) \\
&+ \sum_{i \neq 4} x_i (\pi e_i - \pi_0^{1/3} \xi^2 e_i)], \\
C_2 =& f_4^\xi + x_2 f_2^\xi + x_3 f_3^\xi + x_8 f_8^\xi + k_2[(\pi e_4 - \pi_0^{1/3} \xi^2 e_4) + \sum_{i \neq 4} x_i (\pi e_i - \pi_0^{1/3} \xi^2 e_i)], \\
C_3 =& f_5^\xi - x_2 f_2^\xi - x_3 f_3^\xi - x_8 f_8^\xi, \\
C_4 =& f_6^\xi - (x_1 + x_3 x_2) f_2^\xi - x_3^2 f_3^\xi + (x_7 - x_3 x_8) f_8^\xi - x_3 k_2[(\pi e_4 - \pi_0^{1/3} \xi^2 e_4) \\
&+ \sum_{i \neq 4} x_i (\pi e_i - \pi_0^{1/3} \xi^2 e_i)], \\
C_5 =& f_7^\xi - x_2^2 f_2^\xi + (x_1 - x_2 x_3) f_3^\xi - (x_6 + x_2 x_8) f_8^\xi - x_2 k_2[(\pi e_4 - \pi_0^{1/3} \xi^2 e_4) \\
&+ \sum_{i \neq 4} x_i (\pi e_i - \pi_0^{1/3} \xi^2 e_i)],
\end{aligned}$$

with $k_2(1 + x_5) = \pi_0^{1/3}(1 - \xi)$, and $x_1 x_8 + x_2 x_7 + x_3 x_6 + x_5 = 0$.

Now we consider $\mathcal{F}^3$ in the affine chart $\{\pi^2 e_i, \pi e_5\}_{i=1,2,\ldots,8}$. The conditions for $\mathcal{F}^3$ are:

$$\mathcal{F}^1 * \Lambda \subset \mathcal{F}^3, \quad \mathcal{F}^2 \subset \mathcal{F}^3, \quad (\pi - \pi_0^{1/3} \xi^2) \mathcal{F}^3 \subset \mathcal{F}^2, \quad \langle \mathcal{F}^3, \mathcal{F}^3 \rangle = 0.$$

With respect to the standard order of basis, the generators of $\mathcal{F}^3$ are represented by the columns of the matrix D. Here D has the form:

$$D = \begin{pmatrix} I & 0 \\ M & N \\ M' & N' \end{pmatrix}$$

where $M_{8\times 8}, M'_{8\times 8}$ are matrices, and $N_{8\times 1}$, $N'_{8\times 1}$ are vectors. More precisely, we have

$$M = \begin{pmatrix} m_{ij\ 1\le i\le 4,\ 1\le j\le 8} \\ 0 \quad ... \quad 0 \\ m_{ij\ 6\le i\le 8,\ 1\le j\le 8} \end{pmatrix}, \quad N = (n_1\ n_2\ n_3\ n_4\ 1\ n_6\ n_7\ n_8)^T,$$

and $M' = (m'_{ij})_{1\le i,j\le 8}$, $N' = (n'_1\ n'_2\ \cdots\ n'_8)^T$. We use M_i (resp. M'_i) to denote the i-th column for the matrix M (resp. M'). Thus, the generators of $\mathcal{F}^3$ are:

$$\begin{aligned} D_i =& \pi^2 e_i + \sum_{j\neq 5} m_{ji}\pi e_j + \sum_{k=1}^{8} m'_{ki} e_k, \\ D_9 =& \pi e_5 + \sum_{j\neq 5} n_j \pi e_j + \sum_{k=1}^{8} n'_k e_k, \end{aligned}$$

for $i = 1, 2, ..., 8$.

(1). $\mathcal{F}^1 * \Lambda \subset \mathcal{F}^3$. The generators of $\mathcal{F}^1 * \Lambda$ are

$$\begin{aligned} & f_2^{\xi^2} + x_3 f_1^{\xi^2} + x_7 f_4^{\xi^2} - x_8 f_6^{\xi^2}, \\ & f_3^{\xi^2} - x_2 f_1^{\xi^2} + x_6 f_4^{\xi^2} + x_8 f_7^{\xi^2}, \\ & f_5^{\xi^2} - x_1 f_1^{\xi^2} - x_6 f_6^{\xi^2} - x_7 f_7^{\xi^2}, \\ & f_8^{\xi^2} + x_1 f_4^{\xi^2} + x_2 f_6^{\xi^2} - x_3 f_7^{\xi^2}. \end{aligned}$$

So $\mathcal{F}^1 * \Lambda \subset \mathcal{F}^3$ is equivalent to:

$$
\begin{aligned}
f_2^{\xi^2} + x_3 f_1^{\xi^2} + x_7 f_4^{\xi^2} - x_8 f_6^{\xi^2} &= D_2 + x_3 D_1 + x_7 D_4 - x_8 D_6, \\
f_3^{\xi^2} - x_2 f_1^{\xi^2} + x_6 f_4^{\xi^2} + x_8 f_7^{\xi^2} &= D_3 - x_2 D_1 + x_6 D_4 + x_8 D_7, \\
f_5^{\xi^2} - x_1 f_1^{\xi^2} - x_6 f_6^{\xi^2} - x_7 f_7^{\xi^2} &= D_5 - x_1 D_1 - x_6 D_6 - x_7 D_7 + \pi_0^{1/3} \xi^2 D_9, \\
f_8^{\xi^2} + x_1 f_4^{\xi^2} + x_2 f_6^{\xi^2} - x_3 f_7^{\xi^2} &= D_8 + x_1 D_4 + x_2 D_6 - x_3 D_7.
\end{aligned}
\tag{8.5.7}
$$

Set the matrix S:

$$
S = \begin{pmatrix}
x_3 & -x_2 & -x_1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
x_7 & x_6 & 0 & x_1 \\
0 & 0 & 1 & 0 \\
-x_8 & 0 & -x_6 & x_2 \\
0 & x_8 & -x_7 & -x_3 \\
0 & 0 & 0 & 1
\end{pmatrix}.
$$

We can rewrite $\mathcal{F}^1 * \Lambda$ as the matrix:

$$
\mathcal{F}^1 * \Lambda = \begin{pmatrix}
S \\
\pi_0^{1/3} \xi^2 S \\
\pi_0^{2/3} \xi S
\end{pmatrix}.
$$

Recall that S_i is the i-th column in the matrix S. By comparison of coefficients of πe_i in

(8.5.7), we obtain:

$$\begin{aligned}
&M_2 + x_3 M_1 + x_7 M_4 - x_8 M_6 = \pi_0^{1/3} \xi^2 S_1, \\
&M_3 - x_2 M_1 + x_6 M_4 + x_8 M_7 = \pi_0^{1/3} \xi^2 S_2, \\
&M_5 - x_1 M_1 - x_6 M_6 - x_7 M_7 + \pi_0^{1/3} \xi^2 N = \pi_0^{1/3} \xi^2 S_3, \\
&M_8 + x_1 M_4 + x_2 M_6 - x_3 M_7 = \pi_0^{1/3} \xi^2 S_4.
\end{aligned} \tag{8.5.8}$$

By comparison of coefficients of e_i in (8.5.7), we obtain:

$$\begin{aligned}
&M_2' + x_3 M_1' + x_7 M_4' - x_8 M_6' = \pi_0^{2/3} \xi S_1, \\
&M_3' - x_2 M_1' + x_6 M_4' + x_8 M_7' = \pi_0^{2/3} \xi S_2, \\
&M_5' - x_1 M_1' - x_6 M_6' - x_7 M_7' + \pi_0^{1/3} \xi^2 N' = \pi_0^{2/3} \xi S_3, \\
&M_8' + x_1 M_4' + x_2 M_6' - x_3 M_7' = \pi_0^{2/3} \xi S_4.
\end{aligned} \tag{8.5.9}$$

(2). $\mathcal{F}^2 \subset \mathcal{F}^3$. We need to check that $C_i \in \mathcal{F}^3$ for $i = 1, ..., 5$. Compare coefficients in $\pi^2 e_i$. This condition is equivalent to:

$$\begin{aligned}
&C_1 = D_1 + (x_6 - x_2 x_8) D_2 + (-x_7 - x_3 x_8) D_3 - x_8^2 D_8 - x_5 x_8 k_2 D_9, \\
&C_2 = D_4 + x_2 D_2 + x_3 D_3 + x_8 D_8 + x_5 k_2 D_9, \\
&C_3 = D_5 - x_2 D_2 - x_3 D_3 - x_8 D_8 + \pi_0^{1/3} \xi D_9, \\
&C_4 = D_6 + (-x_1 - x_2 x_3) - x_3^2 D_3 + (x_7 - x_3 x_8) D_8 - x_3 x_5 k_2 D_9, \\
&C_5 = D_7 - x_2^2 D_2 + (x_1 - x_2 x_3) D_3 + (-x_6 - x_2 x_8) D_8 - x_2 x_5 k_2 D_9.
\end{aligned}$$

Recall that C is of the form:

$$C = \begin{pmatrix} A \\ A' \\ A'' \end{pmatrix},$$

with

$$A' = \pi_0^{1/3}\xi A + (k_1Y\ k_2Y\ k_3Y\ k_4Y\ k_5Y),$$
$$A'' = -\pi_0^{1/3}\xi^2 A - \pi_0^{1/3}\xi^2(k_1Y\ k_2Y\ k_3Y\ k_4Y\ k_5Y).$$

for $Y = (x_1\ x_2\ x_3\ 1\ x_5\ x_6\ x_7\ x_8)^T$. By comparison of coefficients of πe_i and e_i, we get:

$$\begin{aligned}
&M_1 + (x_6 - x_2x_8)M_2 - (x_7 + x_3x_8)M_3 - x_8^2M_8 - x_5x_8k_2N = \pi_0^{1/3}\xi A_1 - x_8k_2Y.\\
&M_4 + x_2M_2 + x_3M_3 + x_8M_8 + x_5k_2N = \pi_0^{1/3}\xi A_2 + k_2Y,\\
&M_5 - x_2M_2 - x_3M_3 - x_8M_8 + \pi_0^{1/3}\xi N = \pi_0^{1/3}\xi A_3,\\
&M_6 - (x_1 + x_2x_3)M_2 - x_3^2M_3 + (x_7 - x_3x_8)\,M_8 - x_3x_5k_2N = \pi_0^{1/3}\xi A_4 - x_3k_2Y,\\
&M_7 - x_2^2M_2 + (x_1 - x_2x_3)M_3 - (x_6 + x_2x_8)M_8 - x_2x_5k_2N = \pi_0^{1/3}\xi A_5 - x_2k_2Y.
\end{aligned} \tag{8.5.10}$$

and

$$\begin{aligned}
&M_1' + (x_6 - x_2x_8)M_2' - (x_7 + x_3x_8)M_3' - x_8^2M_8' - x_5x_8k_2N' = \pi_0^{2/3}\xi^2 A_1 + \pi_0^{1/3}\xi^2x_8k_2Y,\\
&M_4' + x_2M_2' + x_3M_3' + x_8M_8' + x_5k_2N' = \pi_0^{2/3}\xi^2 A_2 - \pi_0^{1/3}\xi^2k_2Y,\\
&M_5' - x_2M_2' - x_3M_3' - x_8M_8' + \pi_0^{1/3}\xi N' = \pi_0^{2/3}\xi^2 A_3,\\
&M_6' + (-x_1 - x_2x_3)\,M_2' - x_3^2M_3' + (x_7 - x_3x_8)\,M_8' - x_3x_5k_2N' = \pi_0^{2/3}\xi^2 A_4 + \pi_0^{1/3}\xi^2x_3k_2Y,\\
&M_7' - x_2^2M_2' + (x_1 - x_2x_3)\,M_3' + (-x_6 - x_2x_8)\,M_8' - x_2x_5k_2N' = \pi_0^{2/3}\xi^2 A_5 + \pi_0^{1/3}\xi^2x_2k_2Y.
\end{aligned} \tag{8.5.11}$$

We can see that M_1, M_4, M_5, M_6, M_7 (resp. $M_1', M_4', M_5', M_6', M_7'$) are determined by $M_2, M_3,\ M_8, N$ (resp. M_2', M_3', M_8', N') from Equation (8.5.10), (8.5.11). Replace them by the linear combinations of M_2, M_3, M_8, N (resp. M_2', M_3', M_8', N'), and put them back to Equation (8.5.8), (8.5.9). We get:

$$
\begin{aligned}
(1+x_5)M_2 &= \pi_0^{1/3}\xi^2 S_1 - \pi_0^{1/3}\xi(x_3A_1 + x_7A_2 - x_8A_4) + x_5x_7k_2N - x_7k_2Y,\\
(1+x_5)M_3 &= \pi_0^{1/3}\xi^2 S_2 - \pi_0^{1/3}\xi(-x_2A_1 + x_6A_2 + x_8A_5) + x_5x_6k_2N - x_6k_2Y,\\
(1+x_5)M_8 &= \pi_0^{1/3}\xi^2 S_4 - \pi_0^{1/3}\xi(x_1A_2 + x_2A_4 - x_3A_5) + x_5x_1k_2N - x_1k_2Y.
\end{aligned}
\tag{8.5.12}
$$

and

$$
(1+x_5)\sum_{i=2,3,8} x_iM_i = \pi_0^{1/3}\xi^2 S_3 + \pi_0^{1/3}\xi(x_1A_1 + x_6A_4 + x_7A_5 - A_3) - k_2x_5^2N + \pi_0^{1/3}(\xi-\xi^2)N + k_2x_5Y.
\tag{8.5.13}
$$

Sum of 3 equations in (8.5.12) and subtract it from (8.5.13). We obtain:

$$
\pi_0^{1/3}(\xi^2-\xi)\begin{pmatrix} x_1 & x_2 & x_3 & -x_5 & -1 & x_6 & x_7 & x_8 \end{pmatrix}^T + \pi_0^{1/3}(\xi^2-\xi)N = 0.
$$

Hence we have $n_i = -x_i$ for $i = 1,2,3,6,7,8$, and $n_4 = x_5$. By $k_2(1+x_5) = \pi_0^{1/3}(1-\xi)$, we can see that M_2, M_3, M_8 are determined by x_i (Multiply k_2 on both sides of equations in (8.5.12)). Similarly,

$$
\pi_0^{2/3}(\xi-\xi^2)(x_1\ x_2\ x_3\ \ -x_5\ \ -1\ x_6\ x_7\ x_8)^T - \pi_0^{1/3}(\xi-\xi^2)N' = 0.
$$

Hence $n_i' = \pi_0^{1/3}x_i$ for $i = 1,2,3,6,7,8$, and $n_4' = -\pi_0^{1/3}x_5, n_5' = -\pi_0^{1/3}$. We can check that $(\pi - \pi_0^{1/3}\xi^2)\mathcal{F}^3 \subset \mathcal{F}^2$, and $\langle \mathcal{F}^3, \mathcal{F}^3\rangle = 0$ are already satisfied. Therefore, we can see that $\mathcal{F}^i$ are determined by variables x_i and k_2. We have:

Proposition 8.5.14. *Consider the affine chart with leading terms*

$$\{\pi^2 e_4\} \times \{\pi^2 e_i\}_{i=1,4,5,6,7} \times \{\pi e_5, \pi^2 e_i\}_{i=1,\dots,8}.$$

Under this affine chart in $\mathbb{P}^7 \times \mathrm{Gr}(5,24) \times \mathrm{Gr}(9,24)$*, the corresponding open subscheme in* $\mathcal{M}^{split}$ *is isomorphic to*

$$\mathrm{Spec}(\mathcal{O}[x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,k_2]/(Q_0, x_4-1, k_2(x_4+x_5)-\pi_0^{1/3}(1-\xi))),$$

where $Q_0 = x_1x_8 + x_2x_7 + x_3x_6 + x_4x_5$.

8.6 Other affine charts

We calculate the affine charts for U_1 and U_4 above. Calculations for the rest affine charts are similar to the calculations in U_1 and U_4. In this section, we will use the triality group for special orthogonal group G to transfer our calculation results to other affine charts U_i. Thus the ideal sheaf $\Pi^*(\mathcal{I})|_{U_i}$ is principal, and we have a morphism:

$$\widetilde{\Pi} : \mathcal{M}^{\mathrm{split}} \to \widetilde{Q}$$

by the universal property of blow-up. Recall that F/F_0 is the cubic Galois extension with valuation rings $\mathcal{O}, \mathcal{O}_0$, where $\mathcal{O} = \mathcal{O}_0[\pi]$. Consider the triality group for special orthogonal groups $G = \mathrm{Res}_{F/F_0}(\mathbf{Spin}(V,*))^{A_3}$. We have:

$$
\begin{aligned}
G(R) &= \mathrm{Res}_{F/F_0}(\mathbf{Spin}(V,*))^{A_3}(R) \\
&\cong \{g \in \mathbf{SO}(V,q)(R \otimes_{F_0} F) \mid g(x*y) = g(x)*g(y) \text{ for all } x, y \in V \otimes_{F_0} R\},
\end{aligned}
$$

for any F_0-algebra R. Let $\mathscr{G}$ be the parahoric subgroup over $\mathrm{Spec}(\mathcal{O}_0)$ given by $\mathbb{L} = \sum_{i=1}^{8}(\mathcal{O} \otimes_{\mathcal{O}_0} R)e_i$, which represents the functor from $\mathcal{O}_0$-algebras to the groups that sends R to

$$\mathscr{G}(R) = \{g \in \mathbf{SO}_8(\mathcal{O} \otimes_{\mathcal{O}_0} R) \mid g(x*y) = g(x)*g(y) \text{ for all } x, y \in \mathbb{L}\}.$$

For any element $g \in \mathscr{G}(R)$ and $\{\mathcal{F}^i\}_{i=1,2,3} \in \mathcal{M}^{\mathrm{split}}(R)$, we have $\{g(\mathcal{F}^i)\}_{i=1,2,3} \in \mathcal{M}^{\mathrm{split}}(R)$ since $g(x*y) = g(x)*g(y)$ and $\langle g(x), g(y)\rangle = \langle x, y\rangle$. We want to find g such that $g(U_1)$ (resp. $g(U_4)$) is equal to other affine chart $U_i (i \neq 4, 5)$ in $\mathcal{F}^1$. Although there are many elements $g \in \mathscr{F}$ satisfying this requirement, we choose permutation and diagonal groups since they are simple enough.

Recall that a square matrix is called a monomial matrix if there is exactly one non-zero element in each row and column. Any monomial matrix is the product of a diagonal matrix and a permutation matrix. Consider the monomial matrices with the non zero elements are ± 1.

Denote by $\mathcal{S}$ the group of 8×8 monomial matrices with the non zero elements are ± 1. Then any element in $\mathcal{S}$ can be written as $\mathrm{diag}(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)\sigma$, where σ is a permutation in S_8 and $a_i \in \{\pm 1\}$. Set $\mathrm{diag}(a_i)_{i=1}^{8} = \mathrm{diag}(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$. We have a morphism:

$$\mathcal{S} \to \mathrm{End}_{\mathbb{L}}(\mathbb{L}), \quad \mathrm{diag}(a_i)_{i=1}^{8}\sigma \mapsto \mathrm{diag}(a_i)_{i=1}^{8}P_\sigma$$

given by $P_\sigma(e_i) = e_{\sigma(i)}$ for any $\sigma \in S_8$. We want to find the elements in $\mathcal{S}$ that make the image $\mathrm{diag}(a_i)_{i=1}^8 P_\sigma$ sit in $\mathscr{G}(R)$.

Example 8.6.1. Set

$$g = \mathrm{diag}(a, b, c, 1, 1, c, b, a)P_{(12)(36)(45)(78)}$$

for $a, b, c \in \{1, -1\}$, then $g(x * y) = g(x) * g(y)$ if and only if $abc = -1$.

This example is given by Garibaldi in his dissertation [6]. It is an element in $\mathscr{G}$ we wanted. Notice that $g(U_1) = U_2$, and $g(U_4) = U_5$, so we can use our calculation result in Proposition 8.2.5 (1), and transfer the affine chart with leading terms:

$$\{\pi^2 e_1\} \times \{\pi^2 e_k, \pi e_1\}_{k=1,2,3,5} \times \{\pi^2 e_i, \pi e_j, e_1\}_{i=1,\ldots,5, j=1,2,3},$$

to the affine chart with leading terms:

$$\{\pi^2 e_2\} \times \{\pi^2 e_k, \pi e_2\}_{k=1,2,4,6} \times \{\pi^2 e_i, \pi e_j, e_2\}_{i=1,2,4,5,6, j=1,2,6},$$

Similarly, we can transfer affine charts in Proposition 8.2.5 (2) and Proposition 8.2.6. We want to find other elements like Example 8.6.1. In fact, we will prove that $\mathscr{G} \cap \mathcal{S}$ is the dihedral group D_4, and it will transfer U_1 (resp. U_4) to all other affine charts.

Consider $g = \mathrm{diag}(a_i)_{i=1}^8 P_\sigma \in \mathcal{S}$. Suppose that $g(e_4) = a_i e_i$ and $g(e_5) = a_j e_j$ for some $a_i, a_j \in \{1, -1\}$, $i, j \in \{1, 2, ..., 8\}$. We have

$$e_i * e_i = g(e_4) * g(e_4) = g(e_5) = a_j e_j,$$

$$e_j * e_j = g(e_5) * g(e_5) = g(e_4) = a_i e_i.$$

Since $e_k * e_k = 0$ for any $k \neq 4, 5$, and $e_4 * e_4 = e_5$, $e_5 * e_5 = e_4$ by Table 2.1, we get $a_i = a_j = 1$. Either $g(e_4) = e_5, g(e_5) = e_4$ or $g(e_4) = e_4, g(e_5) = e_5$. Set $I = \{1, 6, 7\}$, and $J = \{2, 3, 8\}$. We consider 2 different cases in the following:

(1) $g(e_4) = e_5, g(e_5) = e_4$: Since $e_k * e_5 = -e_k$ for $k \in J$, and $e_k * e_4 = -e_k$ for $k \in I$. We have $e_1 * e_4 = -e_1$, which implies $g(e_1) * e_5 = -g(e_1)$. Then $g(e_1) = a_k e_k$ for some $k \in J$.

(1a): Suppose that $g(e_1) = a_2 e_2$.

By $e_2 * e_5 = -e_2$, we get that $g(e_2) * e_4 = -g(e_2)$, which implies $g(e_2) = a_k e_k$ for $k \in I$. If $g(e_2) = a_1 e_1$, we get $a_1 e_1 * g(e_3) = a_2 e_2$ by $e_2 * e_3 = e_1$. From Table 2.1, it is easy to see that only one e_k satisfies: $e_1 * e_k = a e_2$ for some $a \in \{\pm 1\}$, which is e_6 ($e_1 * e_6 = -e_2$). Hence $g(e_3) = a_6 e_6$, with $a_6 = -a_1 a_2$. Similarly, we get $g(e_6) = a_3 e_3$ by $e_1 * e_6 = -e_2$, where $a_6 = -a_1 a_2$, and $g(e_7) = a_8 e_8$ by $e_1 * e_7 = e_3$, where $a_8 = a_1$, $g(e_8) = a_7 e_7$ by $e_1 * e_8 = -e_4$, where $a_7 = a_2$. Combining them together, we have a monomial matrix in $\mathscr{G}(R)$:

$$g = \mathrm{diag}(a_1, a_2, a_3, 1, 1, a_3, a_2, a_1) P_{(12)(36)(78)(45)}. \tag{8.6.2}$$

with $a_1, a_2, a_3 \in \{\pm 1\}$, $a_1 a_2 a_3 = -1$. It is exactly the monomial matrix in Example 8.6.1.

If $g(e_2) = a_6 e_6$, we have $g(e_6) = a_8 e_8$ by $e_1 * e_6 = -e_2$ with $a_2 a_6 a_8 = 1$. Similarly, we have $g(e_8) = a_7 e_7$ by $e_1 * e_8 = -e_4$ with $a_2 = a_7$, $g(e_7) = a_3 e_3$ by $e_7 * e_2 = -e_4$ with $a_3 = a_6$. Finally, $g(e_3) = a_1 e_1$ by $e_3 * e_2 = -e_1$ with $a_1 a_2 a_6 = 1$. Hence we obtain:

$$g = \mathrm{diag}(a_1, a_2, a_3, 1, 1, a_3, a_2, a_1) P_{(126873)(45)} \tag{8.6.3}$$

with $a_1, a_2, a_3 \in \{\pm 1\}$, $a_1 a_2 a_3 = 1$.

If $g(e_2) = a_7e_7$, by $e_1 * e_6 = -e_2$, we have $a_2e_2 * g(e_6) = a_7e_7$. This equation does not have a solution by Table 2.1.

(1b): Suppose that $g(e_1) = a_3e_3$. Similar calculations as above. Consider $g(e_2) = a_ke_k$ for $k \in I$. Then $g(e_2) = a_1e_1$ or a_7e_7 (for $g(e_2) = a_6e_6$, we don not have a solution by $e_1 * e_6 = -e_2$). If $g(e_2) = a_1e_1$, we obtain:

$$g = \mathrm{diag}(a_1, a_2, a_3, 1, 1, a_3, a_2, a_1)P_{(137862)(45)} \tag{8.6.4}$$

with $a_1, a_2, a_3 \in \{\pm 1\}$, $a_1a_2a_3 = 1$. If $g(e_2) = a_7e_7$, we have:

$$g = \mathrm{diag}(a_1, a_2, a_3, 1, 1, a_3, a_2, a_1)P_{(13)(27)(68)(45)}. \tag{8.6.5}$$

with $a_1, a_2, a_3 \in \{\pm 1\}$, $a_1a_2a_3 = -1$.

(1c): Suppose that $g(e_1) = a_8e_8$. Consider $g(e_2) = a_6e_6$ or $g(e_2) = a_7e_7$. We get:

$$g = \mathrm{diag}(a_1, a_2, a_3, 1, 1, a_3, a_2, a_1)P_{(18)(26)(37)(45)}. \tag{8.6.6}$$

with $a_1, a_2, a_3 \in \{\pm 1\}$, $a_1a_2a_3 = -1$ for $g(e_2) = e_6$, and

$$g = \mathrm{diag}(a_1, a_2, a_3, 1, 1, a_3, a_2, a_1)P_{(8)(27)(36)(45)} \tag{8.6.7}$$

with $a_1, a_2, a_3 \in \{\pm 1\}$, $a_1a_2a_3 = 1$ for $g(e_2) = e_7$.

(2) $g(e_4) = e_4, g(e_5) = e_5$. The second case has similar result to case (1). We omit the

calculation and just list monomial matrices below. We also have 6 matrices:

$$
\begin{aligned}
g &= \mathrm{diag}(a_1, a_2, a_3, 1, 1, a_3, a_2, a_1)P_{(16)(38)}, \\
g &= \mathrm{diag}(a_1, a_2, a_3, 1, 1, a_3, a_2, a_1)P_{(17)(28)}, \\
g &= \mathrm{diag}(a_1, a_2, a_3, 1, 1, a_3, a_2, a_1)P_{(23)(67)},
\end{aligned}
\tag{8.6.8}
$$

with $a_1, a_2, a_3 \in \{\pm 1\}$, $a_1a_2a_3 = -1$, and

$$
\begin{aligned}
g &= \mathrm{diag}(a_1, a_2, a_3, 1, 1, a_3, a_2, a_1)P_{(167)(283)}, \\
g &= \mathrm{diag}(a_1, a_2, a_3, 1, 1, a_3, a_2, a_1)P_{(176)(238)}, \\
g &= \mathrm{diag}(a_1, a_2, a_3, 1, 1, a_3, a_2, a_1)id,
\end{aligned}
\tag{8.6.9}
$$

with $a_1, a_2, a_3 \in \{\pm 1\}$, $a_1a_2a_3 = 1$.

Monomial matrices 8.6.2-8.6.9 are all possible monomial matrices satisfy $g(x * y) = g(x) * g(y)$. They form a subgroup of $\mathscr{G}$. Denote by

$$
\begin{aligned}
g_1 &= \mathrm{diag}(a_1, a_2, a_3, 1, 1, a_3, a_2, a_1)P_{(13)(27)(68)(45)}, \\
g_2 &= \mathrm{diag}(a_1, a_2, a_3, 1, 1, a_3, a_2, a_1)\ P_{(126873)(45)},
\end{aligned}
$$

in (8.6.3) and (8.6.5). They are generators of $\mathscr{G} \cap \mathcal{S}$. It turns out that the subgroup is isomorphic to the dihedral group $D_6 = \{g_1, g_2 \mid g_1^2 = 1, g_2^6 = 1, g_1g_2 = g_2^{-1}g_1\}$. By using

	$a_1a_2a_3 = -1$	$a_1a_2a_3 = 1$
$g(e_4) = e_5$ $g(e_5) = e_4$	g_1 $g_2^4g_1$ $g_2^2g_1$	g_2 g_2^5 g_2^3
$g(e_4) = e_4$ $g(e_5) = e_5$	g_2g_1 $g_2^3g_1$ $g_2^5g_1$	g_2^2 g_2^4 id

dihedral group D_4, it is easy to see that all other affine charts $U_i (i \neq 4, 5)$ in $\mathcal{F}^1$ can be transferred by U_1 and U_4.

Remark 8.6.10. Transferring from one affine chart to another is not unique . For example, affine chart

$$\{\pi^2 e_1\} \times \{\pi^2 e_k, \pi e_1\}_{k=1,2,3,5} \times \{\pi^2 e_i, \pi e_j, e_1\}_{i=1,\ldots,5,j=1,2,3},$$

can be transferred to

$$\{\pi^2 e_2\} \times \{\pi^2 e_k, \pi e_2\}_{k=1,2,4,6} \times \{\pi^2 e_i, \pi e_j, e_2\}_{i=1,2,4,5,6,j=1,2,6},$$

both by (8.6.2) and (8.6.3). But the corresponding open subscheme has the same equations for variables.

We end this section with an example of finding generators of $\mathcal{F}^2$ in the affine chart U_5 by using g in $\mathscr{G} \cap \mathcal{S}$.

Example 8.6.11. Consider $g = \mathrm{diag}(a_1, a_2, a_3, 1, 1, a_3, a_2, a_1)P_{(12)(36)(78)(45)}$. Let $a_1 = a_2 = 1$, $a_3 = -1$. In §8.5, we calculated the affine chart U_4, where the leading terms for $\mathcal{F}^i$ $(i = 1, 2, 3)$ are:

$$\{\pi^2 e_4\} \times \{\pi^2 e_k\}_{k=1,4,5,6,7} \times \{\pi^2 e_i, \pi e_5\}_{i=1,\ldots,8}.$$

Then g acting on this affine chart gives us:

$$\{\pi^2 e_5\} \times \{\pi^2 e_k\}_{k=2,3,4,5,8} \times \{\pi^2 e_i, \pi e_4\}_{i=1,\ldots,8},$$

which is exactly the affine chart U_5 that we want to calculate. More precisely, recall that the generators of $\mathcal{F}^2$ in U_4 have the form:

$$\begin{aligned}
C_1 =& f_1^\xi + (x_6 - x_8 x_2) f_2^\xi - (x_7 + x_8 x_3) f_3^\xi - x_8^2 f_8^\xi - x_8 k_2[(\pi e_4 - \pi_0^{1/3}\xi^2 e_4) + \\
& \sum_{i\neq 4} x_i(\pi e_i - \pi_0^{1/3}\xi^2 e_i)], \\
C_2 =& f_4^\xi + x_2 f_2^\xi + x_3 f_3^\xi + x_8 f_8^\xi + k_2[(\pi e_4 - \pi_0^{1/3}\xi^2 e_4) + \sum_{i\neq 4} x_i(\pi e_i - \pi_0^{1/3}\xi^2 e_i)], \\
C_3 =& f_5^\xi - x_2 f_2^\xi - x_3 f_3^\xi - x_8 f_8^\xi, \\
C_4 =& f_6^\xi - (x_1 + x_3 x_2) f_2^\xi - x_3^2 f_3^\xi + (x_7 - x_3 x_8) f_8^\xi - x_3 k_2[(\pi e_4 - \pi_0^{1/3}\xi^2 e_4) \\
& + \sum_{i\neq 4} x_i(\pi e_i - \pi_0^{1/3}\xi^2 e_i)], \\
C_5 =& f_7^\xi - x_2^2 f_2^\xi + (x_1 - x_2 x_3) f_3^\xi - (x_6 + x_2 x_8) f_8^\xi - x_2 k_2[(\pi e_4 - \pi_0^{1/3}\xi^2 e_4) \\
& + \sum_{i\neq 4} x_i(\pi e_i - \pi_0^{1/3}\xi^2 e_i)],
\end{aligned}$$

with $k_2(1+x_5) = \pi_0^{1/3}(1-\xi)$, and $x_1x_8 + x_2x_7 + x_3x_6 + x_5 = 0$. Consider $g(\mathcal{F}^i)$. We have:

$$g(\mathcal{F}^1) = f_5 + x_2 f_1 + x_1 f_2 - x_6 f_3 + x_5 f_4 - x_3 f_6 + x_8 f_7 + x_7 f_8.$$

Set:

$$\begin{aligned}
y_1 &= x_2, \quad & y_2 &= x_1, \quad & y_3 &= -x_6, \\
y_6 &= -x_3, \quad & y_7 &= x_8, \quad & y_8 &= x_7.
\end{aligned}$$

and $y_4 = x_5$. We can rewrite $g(\mathcal{F}^1) = f_5 + \sum_{i\neq 5} y_i f_i$ with $y_1y_8 + y_2y_7 + y_3y_6 + y_4 = 0$. The generators of $g(\mathcal{F}^2)$ are:

$$
\begin{aligned}
g(C_1) =& f_2^\xi - (y_3 + y_7 y_1) f_1^\xi + (y_8 - y_7 y_6) f_6^\xi - y_7^2 f_7^\xi - y_7 k_2[(\pi e_5 - \pi_0^{1/3}\xi^2 e_5) \\
&+ \sum_{i \neq 5} y_i(\pi e_i - \pi_0^{1/3}\xi^2 e_i)], \\
g(C_2) =& f_5^\xi + y_1 f_1^\xi + y_6 f_6^\xi + y_7 f_7^\xi + k_2[(\pi e_5 - \pi_0^{1/3}\xi^2 e_5) + \sum_{i \neq 5} y_i(\pi e_i - \pi_0^{1/3}\xi^2 e_i)], \\
g(C_3) =& f_4^\xi - y_1 f_1^\xi - y_6 f_6^\xi - y_7 f_7^\xi, \\
-g(C_4) =& f_3^\xi + (y_2 - y_6 y_1) f_1^\xi - y_6^2 f_6^\xi - (y_8 + y_6 y_7) f_7^\xi - y_6 k_2[(\pi e_5 - \pi_0^{1/3}\xi^2 e_5) \\
&+ \sum_{i \neq 5} y_i(\pi e_i - \pi_0^{1/3}\xi^2 e_i)], \\
g(C_5) =& f_8^\xi - y_1^2 f_1^\xi - (y_2 + y_1 y_6) f_6^\xi + (y_3 - y_1 y_7) f_7^\xi - y_1 k_2[(\pi e_5 - \pi_0^{1/3}\xi^2 e_5) \\
&+ \sum_{i \neq 5} y_i(\pi e_i - \pi_0^{1/3}\xi^2 e_i)].
\end{aligned}
$$

with $y_1 y_8 + y_2 y_7 + y_3 y_6 + y_4 = 0,\ k_2(1 + y_4) = \pi_0^{1/3}(1 - \xi)$. We can check that it is exactly $\mathcal{F}^2$ in U_5 by using same calculation we did in §8.5.

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