THE RELATIONSHIP BETWEEN PRODUCTIVE DISCIPLINARY ENGAGEMENT AND MIDDLE GRADES STUDENTS' CONSTRUCTION OF WRITTEN JUSTIFICATIONS

By

Taren McKenna Going

A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

Mathematics Education—Doctor of Philosophy

2021

ABSTRACT

THE RELATIONSHIP BETWEEN PRODUCTIVE DISCIPLINARY ENGAGEMENT AND MIDDLE GRADES STUDENTS' CONSTRUCTION OF WRITTEN JUSTIFICATIONS

By

Taren McKenna Going

Engaging students in mathematical justification is extremely important from a learning-focused and a disciplinary perspective. Prior research has commonly found that middle grades students struggle to produce viable justifications, arguments, and proofs, yet these studies often consider students' justification in only one modality (oral or written). This presents an incomplete picture of students' abilities to justify and emphasizes mathematical justifications as a finished product rather than students' processes to produce them. This study concerns middle grades students' mathematical justification in both oral and written forms and attends to the relationship between students' collaborative work in small groups and the ways that they engage in the process of constructing written justifications. To this end, I qualitatively investigated students' work to construct written justifications in the context of small group work in a digital collaborative environment designed to support students' productive disciplinary engagement on open mathematics problems.

I examined students' work in two teachers' classrooms and considered students' collaborative work and written justifications at multiple levels. First, I considered students' overall engagement in collaborative work and written justification over the course of mathematical tasks, and second, I considered localized instances where they made changes to their written justifications. To do this, I utilized a novel framework for considering productive disciplinary engagement in terms of student actions and frameworks to describe the

mathematical completeness and validity of students' written justifications (Toulmin, 1958; Kosko & Zimmerman, 2019).

At a general level, I found that as the level of problematizing, authority, accountability, and resources demonstrated during each group's collaborative work increased, students began to construct written justification that went beyond providing claims to attend to how and why the mathematics worked. Additionally, in this study as students encountered uncertainties about how to justify in written form and attended to the mathematical validity of their written justifications, they demonstrated increased levels of productive disciplinary engagement in collaborative work with peers. This engagement with oral and written justification was facilitated by features of the collaborative digital environment that supported students to develop authority for expressing and explaining their own mathematical ideas, gain accountability to question peers' verbally- or digitally-expressed ideas, and develop their own resources and mathematical inscriptions (or representations) for making sense of problems. These findings suggest that mathematical writing is a complex task that goes beyond simply transcribing the results of oral work, but also that engaging in written justification can help students to more deeply engage in collaborative work to make sense of mathematical ideas.

Dedicated to my family: Kameron, Evelyn, and Cecily

ACKNOWLEDGEMENTS

I did not expect completing a dissertation to be easy, but completing a dissertation in the midst of a global pandemic has been one of the most challenging things I've ever done. I could not have gone through this process without the support of many mentors, colleagues, and family members. First, I thank my spouse Kameron for taking on the lion's share of caring for our family during this journey and always reminding me that I'm capable of anything I set my mind to. I'm incredibly grateful to him and to the numerous friends and family members who cared for my children while I worked.

I especially want to thank my dissertation committee: Dr. Kristen Bieda (chair), Dr. Alden J. Edson, Dr. Beth Herbel-Eisenmann, and Dr. Shiv Karunakaran. Their encouragement and thoughtful feedback throughout this process has been instrumental in helping me reach this point. In particular, I want to thank my advisor Dr. Kristen Bieda for the countless hours she spent with me as she helped me think through and write this study. She has shown me what it means to be a part of this academic community and has provided me with the best example of mentorship through her empathy, intelligence, and insight. She has helped me to develop my voice as a teacher, researcher, and scholar over the past five years and I cannot thank her enough for all she's taught me.

This dissertation study would not have been possible without the support of my mentors and colleagues in the Connected Mathematics Project. I've benefited immensely from the conversations about research, mathematics, and teaching that I've had Dr. Elizabeth Phillips, Dr. Alden J. Edson, Dr. Kristen Bieda, Yvonne Slanger-Grant, and Elizabeth Lozen as they guided my work on this dissertation study and the grant projects in which this study is embedded. I thank them for using their years of experience to help me develop this study and for their support as I navigated each step of my graduate school journey. I'd also like to thank my colleagues Merve Kursav and Chuck Fessler for the incredible amounts of work we've done together to build analytic frameworks.

This dissertation benefited greatly from Dr. Raven McCrory's feedback, and I thank her for her time and insightful comments throughout the writing process. I'd also like to thank the Graduate School for awarding me with a Dissertation Completion Fellowship to provide support through the final stages of analysis and writing. Finally, I want to thank the teachers and students who made this study possible. I am continually impressed and excited by the work they do in their classrooms, and their zeal for teaching and learning.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

KEY TO ABBREVIATIONS

CMP	Connected Mathematics Project
MR	Mathematical Reflection
MSA	*Moving Straight Ahead*
PDE	Productive Disciplinary Engagement
PDE-CW	Productive Disciplinary Engagement in Collaborative Work
SS	*Stretching and Shrinking*

CHAPTER 1:

INTRODUCTION

Engaging students in justification, argumentation, and proof can help them learn mathematics more deeply and communicate mathematics more effectively (Staples et al., 2012; Stylianides et al., 2016). For this reason, scholars and policy documents widely call for the increased presence of these practices in mathematics classrooms at all levels (e.g. CCSSI, 2010; Nardi & Knuth, 2017; Stylianides et al., 2016). Despite the importance of these practices, some studies suggest that students struggle to produce viable justifications, arguments, and proofs (Healy & Hoyles, 2000; Lannin, 2005; Sen & Guler, 2015). These studies largely consider only one modality (either written or oral) when considering students' abilities to justify their mathematical thinking. However, the nature of students' arguments in these two modalities can often differ greatly (Campbell et al., 2020). This is potentially problematic, since students' classroom experiences with justification commonly occur orally and formal assessments that involve justification (e.g. statewide assessments or classroom assessments) often occur in written form. At the same time, classrooms are becoming increasingly digital, with students being called to communicate and justify their mathematical ideas in multiple modalities (Engelbrecht et al., 2020; Goos & Bennison, 2008; Thomas & Palmer, 2014). Although existing research has identified differences in the types of justifications that students produce in written and oral modalities, we know less about why these differences might occur and how students themselves approach the process of justifying in multiple modalities. In this study, I examine the relationship between students' justifying activity as they work in small groups in mathematics classrooms and the nature of their written justifications.

In order to better understand these connections, I considered the nature of students' productive disciplinary engagement (PDE; Engle & Conant, 2002) and how it shapes their written justifications. PDE involves the principles of authority (which includes being willing and able to make claims within their small group environment), accountability (which includes a necessity to defend any claims they make and request defense for others' claims), problematizing (which includes making sense of the conceptual mathematics underlying their claims), and resources (which includes finding appropriate evidence to make and support their claims). As I will explain in subsequent chapters, high levels of PDE have a theoretical relationship with the strength of students' written justifications. This work builds on a novel framework that describes PDE in terms of student actions and outcomes that allows me to characterize any potential connections between students' activity to produce oral justifications and their activity to produce written justifications. The following question guided the development of this study and my analysis of results: *What is the relationship between middle grades students' PDE during small group work and their construction of written justifications in a collaborative mathematics learning environment?*

Overview of the Study

The focus of this study is on instances of mathematical justification, which I consider as "the process of supporting your mathematical claims and choices when solving problems or explaining why your claim or answer makes sense" (Bieda & Staples, 2020, p. 103). Much of the research on students' abilities to justify is based on only their written work, particularly large-scale studies (e.g. Healy & Hoyles, 2000; Knuth et al., 2009). Whereas research on how teachers support students to justify or construct arguments is based largely on whole-class oral work (e.g. Conner et al., 2014; Ellis et al., 2019). These are not necessarily the same kinds of activity, nor will they necessarily produce the same kinds of results. For example, in whole-class oral justifications, the teacher or other students can quickly press students to further explain their thinking if the justification is not deemed sufficient. When students are producing written justifications, such immediate feedback might not be possible in the same form. Further, students themselves may not approach the oral justification and written justification in the same way. Research that has attended to students' justifications, arguments, and proofs in both written and oral modalities (Campbell et al., 2020; Stylianides, 2019) finds that students' oral reasoning is more mathematically complete than their written reasoning, regardless of the order in which students participate in both modalities. Because of these differences, existing work that examines students' justification in only one modality may not offer a complete picture of students' abilities to engage in justification and the factors that shape their participation in this important practice. In this study, I examined students' processes to construct written justifications in the context of collaborative group work, where the need to engage in oral justification with peers is likely to emerge. Examining students' in-the-moment writing actions as they construct written justifications gives insight into how students approach justification in each modality, and why differences might arise.

Although oral and written modalities for justification cannot be viewed as equivalent, I operate under the theoretical assumption that oral justification in whole-class or small-group settings has an effect on students' written justifications, and vice versa. This study draws on some of the ideas of emergent perspective (e.g. Cobb & Yackel, 1996; Stephan & Cobb, 2003), in which individual learning from a psychological constructivist perspective is coordinated with social activity in mathematics classrooms from an interactionist perspective. Oral justification in this perspective is considered a collective social process, but one that is fundamentally shaped by the mathematical knowledge and ways of doing mathematics of individual students. Written justification in this perspective is a process that reflects students' individual ways of knowing and doing mathematics, but one that is fundamentally shaped by the norms around justification within the larger classroom community.

The data for this study were collected as part of a larger NSF-funded grant project to design a digital workspace that allows middle school students to represent, communicate, and develop their mathematical ideas with inscriptional tools as they work on open problems in a collaborative setting with a small group of their peers. These open problems are adapted from the Connected Mathematics Project curriculum (CMP; Lappan et al., 2014), and are structured to help students articulate, deepen, and extend their mathematical thinking in multiple stages as they collaborate with their peers (in short, to enhance students' PDE). Because of the focus on digital collaborative environments, students have multiple ways to interact with their peers' mathematical ideas (e.g. orally, through access to peers' written work, through use and modification of peers' inscriptions). Because of the problem structure and content of problems, students have multiple opportunities to find patterns, make predictions, and provide support for claims - all critical aspects of the practice of justification (Stylianides, 2009). Taken together, the context of this grant project allows middle school students multiple opportunities to collaboratively construct oral justifications and to construct their own written justifications within the platform and provides a space where social learning practices and individual learning practices can be tightly connected.

One purpose of this study was to examine the process of creating written justifications in the context of project classrooms and identify social, group-related factors within small group work that allowed students to improve their justifications over time. As such, I relied heavily on screen recordings of students' digital workspaces as they work on justification tasks as a data source. Screen recordings capture students' construction of written justifications as it happens (rather than just the final product) as well as a real-time record of a small group's collaborative work. It also allows me to foreground students' participation in PDE and justification, rather than teachers' facilitation of these practices, which is more common in existing work on PDE (e.g. Cornelius & Herrenkohl, 2004; Venturini & Amade-Escot, 2014; Williams-Candek, 2015) and justification (e.g. Conner et al., 2014; Ellis et al., 2019; James et al., 2016).

I completed two "layers" of analysis on students' work in the digital collaborative environment in order to examine the relationship between the nature of small group collaborative work and the nature of students' written justifications. First, I examined students' work on overall tasks (i.e. the group's uninterrupted work to complete one part of a problem from the digitally-presented CMP curriculum). This includes characterizing the overall nature of the group's PDE throughout the task as well as characterizing the written justifications constructed by two focal students in the group in terms of their mathematical completeness and validity. Second, I examined students' work in smaller, localized periods of time surrounding focal students' writing actions (i.e. moments when students began, added, modified, or deleted elements of their written justifications). This includes characterizing the writing action itself in terms of how it affected the mathematical completeness or validity of the overall written justification, as well as characterizing the specific ways that the group's speech or digital actions in those moments demonstrated PDE. By including analysis at the level of the overall task and at the localized level of specific writing actions, I sought to represent students' processes to construct written justifications as well as the final product of those justifications.

My analysis revealed several ways that the types of written justifications that students constructed and the level of PDE demonstrated by students during small group work were related. As groups demonstrated greater levels of each principle of PDE (i.e. problematizing, authority, accountability and resources) on overall tasks, students were able to write more mathematically complete justifications, particularly by beginning to include evidence and warrants that explained how and why their mathematical claims worked. Although students wrote mathematically valid justifications in many different circumstances of their group's PDE, as students made attempts to address the validity of their written justification, their small group work showed increased levels of PDE. All of this was facilitated by features of the collaborative digital environment that helped students to develop authority for expressing and explaining their own mathematical ideas, gain accountability to question peers' verbally- or digitally-expressed ideas, and develop their own resources and mathematical inscriptions (or representations) for making sense of problems.

In the following chapters, I share my work to answer the research question posed above. Chapter 2 contains a review of relevant literature, drawing from work in several areas of mathematics education, including the practice of justification, students' mathematical communication in multiple modalities, and the nature of students' collaborative small-group work in a digital environment. Chapter 3 details the methodology of this study, including the context of the larger research project in which data were collected, information on participants, and methods used for data sampling and analysis. The results of analysis and explanation of those results are described in Chapters 4, 5, and 6, with each chapter addressing a distinct research sub-question. Finally, Chapter 7 includes discussion of these results in light of related research work in mathematics education and offers directions for further research.

CHAPTER 2:

LITERATURE REVIEW

This study draws on work from several areas in mathematics education, including the practice of justification, students' mathematical communication in both written and oral modalities, and the nature of students' collaborative small-group work in a digital environment.

The Practice of Mathematical Justification and its Relationship to Other Types of Reasoning

Justification is a core mathematical practice, both within the discipline of mathematics and for learning mathematics. From a disciplinary perspective, justification is used to verify mathematical claims, provide insight into results and claims, and test and revise new conjectures, often as a part of formal proof (Hanna, 2000; Stylianides, 2009). From a learning-focused perspective, justification is "a means by which students enhance their understanding of mathematics and their proficiency at doing mathematics" (Staples et al., 2012, p. 447). The emphasis of justification in classrooms can help students to develop and express complex mathematical thinking (Ball & Bass, 2003; Staples & Newton, 2016; Wood et al., 2006) and promote more equitable learning outcomes for diverse students (Boaler & Staples, 2008). Because of the importance of mathematical reasoning in the discipline of mathematics and for student learning, reform documents increasingly call for justification, argumentation, and proof to play central roles at all levels of schooling (e.g. NCTM, 2000; CCSSI, 2010).

Within the field of mathematics education, there is some ambiguity about how to differentiate between the related concepts of justification, argumentation, and proof (Cai & Cirillo, 2014; Staples et al., 2017). Many definitions of each construct are currently utilized in research, which each afford different ways to make meaning of mathematical activity with different emphases (Balacheff, 2008). As such, it is important that I clarify the meaning of justification within the context of this study and its relationship to other reasoning practices. In this study, I adopted Bieda and Staples (2020) process-oriented view of mathematical justification as "the process of supporting your mathematical claims and choices when solving problems or explaining why your claim or answer makes sense" (p. 103).

This view of mathematical justification is similar to many views of mathematical argumentation. For example, Cross (2009) states that mathematical argumentation is "characterized by the sharing, explaining, and justifying of mathematical ideas" (p. 908). Focusing on classroom interactions, Conner and colleagues (2014) define collective argumentation as "any instance where students and teachers make a mathematical claim and provide evidence to support it" (p. 404). Others define argumentation in terms of its structure, so that argumentation is the process of supporting claims with data, warrants, and backings (e.g. Weber et al., 2008). Each of these definitions highlight supporting mathematical claims and communicating this support to others, which are also highlighted by how I consider mathematical justification. Because of these similarities, I utilized analytic tools commonly associated with research on mathematical argumentation, for example, Toulmin's (1958) model of the structure of arguments. Bieda and Staples' (2020) definition also highlights that students might justify their own mathematical choices or explain why their mathematical work makes sense, which are fundamentally connected to students' processes as they engage in justification. As such, it is important to consider not only the structure of the finished justification, but also how this structure emerges in students' written work as they work to make sense of mathematics.

As a practice, mathematical justification also shares many similarities with more formalized mathematical proof (Hanna, 2000; Staples et al., 2017). In mathematics education research, proof is often defined as an argument with specific characteristics (e.g. Harel & Sowder, 1998; Stylianides, 2007; Czocher & Weber, 2020). Some researchers also assert that the processes of argumentation and justification are vital to producing a proof (e.g. Boero et al., 2010). The main difference between these conceptions of proof and conceptions of other written justifications or arguments is some level of rigor agreed upon by a community (Stylianides, 2007; Staples et al., 2017), indicating that proofs are a specialized form of arguments or written justifications. Further, major functions of proof are to give insight into why a claim is true and convince others of this truth (Hanna, 2000; Harel & Sowder, 2007). Justification has the same explanatory and convincing functions (Staples et al., 2012), indicating that proof and justification might differ in specifics of form rather than differing in purpose. This is particularly relevant within the context of elementary and secondary education, where "a mathematical proof is a formal way of expressing particular kinds of reasoning and justification" (NCTM, 2000, p. 56). Although in this study I did not attend to whether or not students' mathematical justifications meet the requirements for proof within their community, prior research on students' proof schemes and abilities to produce proofs inform my understanding of students' mathematical justification. Since many of the purposes of justification and proof are similar, prior research on the nature of student activity as students go about proving a mathematical statement also gives insight into how students go about justifying a mathematical statement in order to make sense of mathematical ideas and convince themselves and others. Because there are less specific disciplinary requirements on the form that justifications must take as compared to acceptable forms for proof, however, prior research on students' abilities to write capital-P "Proofs" is not as strongly applicable to understanding the ways in which they construct written justifications.

Justification in Multiple Modalities

Students' mathematical justification can take place in a variety of modalities, including orally and in writing. Further, justification in both modalities may incorporate inscriptions (which are graphs, tables, diagrams, notation, and other graphical displays that can be physically produced and reproduced on paper or a computer screen, see Roth and McGinn (1988)). Rather than considering writing as the student reporting out their final results, in this study I considered writing as a potentially powerful part of students' reasoning processes. In this view, mathematical writing "helps students generate and connect their thoughts and ideas and consolidate their thinking" (Cross, 2009, p. 907). This is consistent with the view that writing goes beyond "knowledge-telling" and can in fact be "knowledge-transforming" (Bereiter & Scardamalia, 1987) as a part of students' mathematical justification and overall mathematics learning. In the context of this study, students had the opportunity to engage in oral justification within their small groups and written justification within the digital collaborative environment at the same time, so both modalities need to be considered to effectively interpret the ways that students approach the construction of written justification. Students might engage in writing for many purposes in such an environment.

Although existing research does not often differentiate between students' writing in paper-and-pencil format or digital format, some studies have shown that engaging in mathematical writing for explanation and justification can deepen students' mathematics learning. Johanning (2000) found that as students engaged in writing activities where they reviewed and revised their written explanations with feedback from peers, they demonstrated greater conceptual understanding and wrote more thorough mathematical explanations. In her experimental study, Cross (2009) found similarly that students who participated in classroom activities that involved oral argumentation and written argumentation demonstrated larger gains in their mathematical knowledge than those who participated only in oral argumentation. Interestingly, students who participated only in classroom activities involving written argument also demonstrated larger gains in their mathematical knowledge than those who participated only in oral argumentation, although the difference was less pronounced. In the current study I did not make claims regarding students' mathematical knowledge, but these studies point to the potential benefits of engaging in written justification as part of mathematics classes, particularly when supported by collaborative work.

Despite the potential role of writing as a part of student learning through justification, much of current teaching and assessment practice is aligned with a knowledge-telling view of mathematical writing. Historically, teachers have infrequently incorporated writing into mathematics lessons and mainly promoted note-taking and other passive writing activities (Pearce & Davison, 1987; Swinson, 1992). More recent surveys show that mathematical writing opportunities in the classroom continue to be infrequent, with teachers often citing lack of time as the reason that they do not incorporate mathematics writing into their instruction more regularly (Bakewell, 2008; Kosko, 2016). Additionally, in high-stakes or large-scale assessments students are often asked to write to demonstrate their mathematical competence (Powell et al., 2017). This dependence on writing for accurate mathematical assessment assumes that students will be unproblematically be able to communicate their mathematical thinking and ideas through writing without additional instruction, or at least transfer their writing skills from other content areas to accomplish the task, but these assumptions may not be accurate (Powell & Hebert, 2016). The assumption that writing is a straightforward task of transcribing the results of oral mathematical work stems from the view of writing as knowledge-telling, but this is different from the knowledge-transforming view that writing about mathematics is mathematical work that comes with its own challenges and opportunities for learning.

Considering writing from a knowledge-transforming perspective was particularly important in the context of this study, where students had access to the digital collaborative workspace where they constructed their written justifications for the duration of their work on each task. Because students had the opportunity to participate in mathematical justification orally or in writing at any point, students' work in multiple modalities were intertwined rather than occurring sequentially with oral justification proceeding written justification. Students' writing actions often occurred as part of students' justification processes in collaborative work (e.g. prompting further episodes of oral justification, alerting students to uncertainties about their mathematical claims or support for those claims) or preceding or following these processes, so I argue the nature and qualities of a finished written justification are tied to students' mathematical justification within the classroom.

Because students are participating in multiple modalities, the ways they use mathematical language (including word choice, specificity, and use of mathematical terminology) is fundamentally connected to the context in which they are communicating. The Language Spectrum (described by Herbel-Eisenmann and colleagues (2017) and adapted from Gibbons' (2009) mode continuum) describes this idea that the mathematical features of students' discourse are shaped by the context and the mode (i.e., written or spoken) in which it is produced. Students' communication contexts include the language of interaction (typically used when students are speaking to their peers as they work in small groups), the language of recounting experience (typically used when students "report out" orally to others who were not involved in small group work), the language of generalizing (typically used when students begin to formalize their ideas for final presentation that might include writing), and the mathematics register (which could be used by students to write about their mathematical ideas as if in a mathematics textbook). Across this continuum, as students gain more experience with the problem they begin to communicate about mathematics in new contexts (e.g. with small groups, with the whole class) and they often move away from talking about their own experiences to solve problems and move toward talking about mathematical objects and relationships. Additionally as students have less shared experience with their assumed reader/listener, they might become more precise and explicit in their use of mathematical language (Gibbons, 2009). This is important when examining students' justifying activity within their small groups in comparison to their written justifications. Considering the audience-related aspect of the language spectrum, students' oral justifying activity would be likely to occur in communication context 1 (or at most communication context 2 if students were "reporting out" to the teacher during small group work), whereas students' written justification occurs in communication context 3. That is, students' written justifications are likely to be more precise, organized and understandable to mathematically informed readers than their overall justifying activity within their group.

In collaborative learning environments, students often begin by working orally in small groups, then discussing and presenting orally with the broader classroom community, and finally they engage in writing to generalize what they have learned from their previous exploration (Stein et al., 2008). This general order for activities (exploring mathematical ideas, explaining their thinking, connecting to disciplinary ideas) follows the orders for communications contexts along the continuum of the language spectrum, although classroom instruction might not strictly follow this order. As the nature of classroom activity shifts, the context for students' communication would also shift (Herbel-Eisenmann et al., 2017). In this study students had the opportunity to engage in writing as an early part of their work on open problems in the digital environment, so it was particularly important to consider the ways that students' written communication occurred as a process. In these classrooms, the opportunity to write was presented and encouraged throughout students' collaborative work. This provided an opportunity to investigate how students' construction of written justifications was related to the nature of their mathematical experiences and justifying activity with peers during small group work. In the digital context of this study, it is an open question how students' written justifications are shaped by the progression of their shared mathematical experiences with peers as compared to how students' written justifications are shaped by the simple fact that they are in written rather than oral form. As such, I have reviewed prior research on middle grades students' mathematical and argumentative writing, their participation in the practice of mathematical justification, and how justification emerges in digital contexts.

Prior Research on Mathematical and Argumentative Writing

Studies that focus particularly on middle and secondary students' mathematical writing suggest that the development of students' writing might follow a similar progression to that of the language spectrum. Waywood (1992, 1994) found that in mathematical journal writing, students' work generally progressed over multiple years from recounting their problem-solving processes, to generalized explanations that utilized examples, and finally to descriptions of mathematical relationships. Other research on expository mathematical writing has found similar progressions in students' writing from simple recounts to more complex summaries of mathematical ideas (Clarke et al., 1993; Kosko & Zimmerman, 2019; Shield & Galbraith, 1998). Rather than investigating how individual students' writing progressed over time, these studies characterize how writing progresses for students more generally. It is important to note that in these studies, changes in the nature of students' writing took place over the course of years, whereas similar changes in students' communication due to different communication contexts could occur over the course of a few days. This study offers additional insight into factors (beyond the passage of time) that allow students to improve their written justifications by examining the specific writing actions that students take to increase the mathematical completeness and validity of their written justifications.

Although the above studies of mathematical writing give some indication of the kinds of written justifications that students might produce, mathematics education research focuses less on how teachers might support writing in classrooms. Because written argumentation is a critical practice in literacy (Feretti & Graham, 2019; Newell et al., 2011), research in literacy education more commonly focuses on classroom instruction around written arguments and justifications. Historically, research on how to support students in constructing written arguments has fallen into two major categories: cognitively-focused studies where students learn about the structure or criteria for arguments, and socially-focused studies where students participate in argument or analyze the purpose of their arguments (Newell et al., 2011).

Cognitively-focused studies have examined students' use of graphic organizers to represent the structure of arguments (e.g. Yeh, 1988), instruction on criteria for acceptable written arguments (e.g. Wilder & Wolfe, 2009), and instruction on specific strategies to consider parts of an argument (e.g. Nussbaum & Schraw, 2007). An underlying assumption of these studies is that if students are to produce written arguments, then they first need to know about the features and form of written arguments (VanDerHeide, 2017). In their review of cognitively-focused writing interventions, Newell and colleagues (2011) found mixed effects, where teacher instruction about the form or structure of written arguments supported students to write more effective arguments in some situations, but not others. One potential reason for these mixed effects is that students are called on to "transfer" what they have learned about written argumentation in one context to new or unfamiliar contexts, which may not occur through rote application of rules of form or structure (VanDerHeide, 2017; Yeh, 1988).

The importance of transfer between contexts is highlighted in socially-focused studies on written argumentation, which have addressed how students produce arguments in particular classroom contexts (e.g. VanDerHeide, 2017), according to the demands of particular audiences or readers (e.g. Wollman-Bonilla, 2001), or as students engage with particular goals and issues (e.g. Felton & Kuhn, 2001). In these studies, teachers supported students' written argumentation by engaging them in collaboratively planning written arguments and through moves such as questioning and revoicing. In different types of collaborative work or with different types of teacher support, the nature of students' written arguments has been shown to shift significantly (Nystrand & Graff, 2001) as students adopt writing behaviors that mirror their classroom work with oral argumentation. Despite the potentially shaping influence of students' participation in oral argumentation on how they participate in written argumentation, less research in literacy education has attended to the relationship between modalities (Newell et al., 2011). Although this study does not foreground how teachers support students in oral and written justification, it does offer insight into the ways that students participate in constructing written justifications and the context of small group work (including attempts at oral justification) in which they are constructed.

Prior Research on Middle Grades Students' Mathematical Justification

A dominant theory that has shaped investigations into secondary and middle grades students' mathematical justification and proof is Harel and Sowder (1998)'s proof schemes theory. In this work, a student's proof schemes represent their views on how to go about determining the truth or validity of mathematical conjectures and convincing others of its validity. Broadly speaking, proof schemes can be based on external conviction (e.g. appeal to authority figures or textbooks), empirical evidence (e.g. use of examples), or analytical in nature (e.g. use of deductive logic). Although Harel & Sowder's (1998) work was dominantly based on the experience of college students (and one secondary student), it has been taken up to describe the proving abilities and conceptions of secondary students more broadly (e.g. Ellis, 2007; Sen & Guler, 2015; Stylianou, 2013). Many researchers claim that secondary students often hold external or empirical proof schemes (e.g. Flores, 2006; Healy & Hoyles, 2000; Liu & Manouchehri, 2013), and frame this as a problem to be overcome if students are to develop greater understanding of proof and abilities to produce proofs.

When we consider students' justifying and proving activity more broadly (and not only their ability to produce a "Proof"), such deficit framing is potentially problematic. In a written survey of middle grades students, Liu and Manouchehri (2013) found that students typically preferred analytical justifications when selecting what they viewed as the "best" justification, but often produced empirical justifications. Students in their study accurately recognized the limitations of empirical justifications, but often described them as "convincing" rather than "mathematically complete". In oral interviews with middle grades students, Bieda and Lepak (2014) similarly found that students often preferred empirical justifications, and valued examples for their explanatory power. Taken together, these studies indicate that middle grades students might evaluate and produce justifications in nuanced ways that take into account multiple purposes for justification, including explanation and communication. Additionally, research shows that students might adopt different ways of justifying on different types of mathematical tasks (Knuth et al., 2009), when utilizing different inscriptions as part of their justifying activity (Stylianou, 2013), or in different mathematical content areas (Dawkins & Karunakaran, 2016). This research suggests that when considering justification as a learning practice, the nature of students' justifying activity will likely influence the nature of their written justifications.

In this study, I examined students' mathematical justification in oral and written forms as they occurred in a collaborative digital environment. Prior research on students' justification generally examines only one modality (Campbell et al., 2020). Large-scale studies on secondary students' proving and justifying abilities are based primarily on students' written justifications (e.g. Healy & Hoyles, 2000; Knuth et al., 2009), whereas studies that examine how students engage in justification are based primarily on oral justifications (e.g. Conner et al., 2014; Stylianou, 2013) and often focus on how teachers facilitate justification within their classrooms (e.g. Ellis et al., 2017; Staples, 2007). As previously discussed, written and oral modalities offer different affordances and opportunities for students, so there is a need for additional research on students' mathematical justification that considers both modalities.

Prior research that compares secondary students' construction of mathematical arguments in both modalities has not considered the case where oral and written activity occurs concurrently as in the context of this study, but rather how students engage in one modality and then the other. Stylianides (2019) considered first the written arguments of 14-16 year old students, and then considered the nature of their orally-presented arguments. Campbell and colleagues (2020) considered first the collaborative oral activity of 13-14 year old students to develop an argument, and then the nature of collaboratively-constructed written arguments. In both studies, students' oral arguments were more thorough and closer to mathematical proofs than their written arguments, regardless of the order of written and oral work. Soto-Johnson and

Fuller (2012) found similar patterns in their work with college algebra students, where students produced more thorough oral justifications and were able to engage in oral justification to some degree, even if they did not also produce a written proof. Of these studies, two focus on students' individual work (Soto-Johnson & Fuller, 2019; Stylianides, 2019) and one focuses on students' collaborative work (Campbell et al., 2020). Although these studies establish that there are differences in the ways that students justify in written and oral modalities, they do not identify specific challenges that led students to produce less mathematically complete written justifications, nor do they explicitly consider how students' justification in one modality influences justification in other modalities. This study examines the nature of the connection between modalities, particularly between students' collaborative justifying activity and their construction of written justifications.

Justification in Digital Contexts

Recent research on the use of digital technology in mathematics education has concerned the use of a wide variety of different technologies, including internet-based platforms for student learning, programs and applications to teach specific mathematics topics, and long-standing mathematics tools (e.g. dynamic geometry environments, graphing calculators) that can be used to explore a variety of different mathematics topics (Sinclair & Yerushalmy, 2016). Each of these technologies has different affordances for enhancing student learning and experiences with mathematical justification, but they are often used as tools for exploration, conjecturing, and testing ideas (Baccaglini-Frank & Mariotti, 2010; Olivero & Robutti, 2007). Specifically referencing dynamic geometry environments, Hanna (2000) described that technologies that allow students to make choices about how to test their own ideas and explore the limits of their mathematical ideas had great potential for supporting students to engage in the process of mathematical justification. Research on justification within these types of technologies generally concerns oral justification and does not go on to consider how students might then approach written justification after engaging with technology for mathematics learning.

The specific features and affordances of the technology students use will also have a shaping influence on the ways that students engage in mathematical activity. As students use digital tools and technologies in mathematics classrooms, "using a tool for doing mathematics not only changes the way to do mathematics but also requires a specific appropriation of the tool" (Laborde, 2003). In the case of justification this means the affordances of the digital collaborative environment will shape the way that students engage in justification, but also that the way students *use* the digital collaborative environment will shape the way that students engage in justification. In this study, the digital collaborative environment offers a way for students to access materials from the curriculum, create and modify inscriptions (e.g. graphs, tables, diagrams), view and copy elements from peers' mathematical work, and access their past mathematical work on related problems. To understand more about why students produce the kinds of written justifications observed in settings where they are using digital tools, it is important to attend to what tools they access and when they access them in the process of producing written justifications.

The features of tools that support students to justify orally and the way that they shape student learning have often been studied in dynamic geometry environments. Olivero and Robutti (2007) studied students' use of measuring, and found that the way that measurements of shapes were approximated in the environment was an important influence on whether and how students justified, especially when the displayed measurement conflicted with some known property of the shape. In these situations, differences between theoretical expectations and experimental measurements sometimes supported students to justify, and sometimes hindered this activity. Baccaglini-Frank and Mariotti (2010) found that as students dragged some features of geometric shapes and maintained others, their activity helped them to formulate conjectures in "if… then…" form and their dragging activity helped them to begin establishing conditional links between parts of their conjecture. That is, students' activity within the platform acted as warrants or backing for their conjecture. Similarly, Patsiomitou (2011) found that when students engaged in "theoretical dragging" where they purposefully chose how to modify shapes rather than at random, this acted as a non-linguistic warrant for students as they justified with peers. Although the collaborative digital environment in this study has different affordances and capabilities than a strictly dynamic geometry environment, it also allows students to build inscriptions, modify them in flexible ways, and test their ideas about mathematical relationships. In oral justification their digital actions might constitute warrants or backing, but they do not necessarily serve the same role in a written justification once students are no longer actively building these inscriptions.

In addition to tools for exploring open problems and producing inscriptions, the digital collaborative environment also contains ways for students to share and view work of peers in their groups, to publish work to the larger classroom community, and to easily revise or delete elements from their written justifications. Engaging in cycles of collaborative feedback and revision can support students in written justification and explanation (Johanning, 2000), so digital features that allow for this have the potential to do the same. Edson, Rogers, and Browning (2016) examined how pre-service elementary teachers engaged in cycles of feedback on their written justifications using VoiceThread, a platform which allowed them to view peers' digitally-recorded justifications and give feedback to peers. As the pre-service teachers engaged in these cycles of feedback and revised their classroom's rubric for what they considered as a complete and valid justification, they began to attend more to warrants and backing as structural components of justification and to place more importance on unpacking and finding connections between mathematical inscriptions. That is, the opportunity to closely examine and revise their past digital work supported them to develop new ideas about what it means to justify and what "counts" as an acceptable justification. In the context of the digital collaborative environment in this study, students also had the opportunity to examine peers' work, incorporate ideas from peers' work, and to revise their own written justifications. In a digital context, then, these activities around feedback and revision are important to consider as they might support students to construct more mathematically complete or valid written justifications.

Collaborative Small Group Work in Mathematics Classrooms

This study used data from students' work with small groups of peers as they collaborated to solve open mathematical problems. This type of learning environment is increasingly called for in curricular materials (e.g. Dieteker et al., 2006; Lappan et al., 2014) and mathematics education policy (e.g. NCTM, 2000, 2014). The use of small groups in mathematics instruction has been shown to have positive outcomes for students' mathematics learning and achievement (Boaler & Staples, 2008; Cohen, 1994; Francisco, 2013; Wood et al., 1993). However, group work is not always productive (Barron, 2003) nor does it always result in equitable learning outcomes for students (Esmonde, 2009; Johnson et al., 2010).

Factors that Support Productive Small Group Work

Three features of small group work that allow for students' mathematical justification have been identified that are pertinent to this study. Although this is discussed further in Chapter 3 regarding the methods for this study, the digital collaborative environment was intentionally designed to allow for these features, as they are critical to students' deep engagement in making sense of mathematical ideas. Further, because these features are present in students' productive small group work, they are related to the way we conceptualized PDE in the PDE-CW framework. Each of these three features should be present in a classroom setting that allows for students to effectively engage in collaborative justification.

First, it is important that students engage in rich, open-ended tasks that invite students to make sense of the mathematics (Mueller & Maher, 2009; Weber et al., 2010). DeJarnette and Gonzalez (2015) found that when students are focused on making sense of multiple mathematical ideas in collaborative small groups, they are able to voice their own ideas and challenge others' thinking in sophisticated ways. They also found that when students are focused only on task completion they are less likely to engage in this type of mathematical justification. As such, rote tasks are unlikely to promote justification in small groups. Further, Francisco and Maher (2005) suggest that rich tasks be revisited over time in different contexts or increasing complexity, to enhance students' opportunities to engage in justification. Each of these features of mathematical tasks is present within the CMP curriculum and was specifically emphasized in the design of the CMP STEM problem format (which will be discussed further in future sections).

Second, it is important that students engage in mathematical discussion where they have the opportunity to make claims and discuss their mathematical thinking (Francisco, 2013; Mueller, 2009; Wood et al., 1993). In such settings, students are able to work together to make sense of mathematical claims and support them (Francisco, 2013; Mueller et al., 2014) - that is, to justify collaboratively. When students justify collaboratively, it implies a joint production of ideas where students work towards a shared meaning or understanding (Staples, 2007), which goes beyond cooperatively completing a task. Although this study foregrounds students, the classroom teacher must allow for students' collaborative justification by establishing and reinforcing the classroom norm that students are responsible for deciding when ideas make sense (Weber et al., 2008; Wood et al., 1993).

Finally, it is important that students are able to create and use representations as they make sense of and communicate mathematical ideas (Mueller, 2009). Selling (2016) found that when students were engaged in small-group work, the creation and modification of graphs, diagrams, and other physical representations of mathematical phenomena allowed them to deepen their participation in justification over time. Students utilized representations to better defend mathematical claims, question and interpret others' claims, and make sense of complex mathematical ideas. Forman and Ansell (2002) similarly found that representations played a central role in students taking argumentative positions and explaining why their ideas made sense in whole-class discussion. In many instances, they found that representations provided a productive entry point for students to begin to understand and communicate about new ideas and methods. Representations such as graphs, equations, or tables are often interpreted differently by different students or teachers, and discussions about such representations offer rich opportunities for justification (Moschkovich, 2008). In this study and the larger grant study in which it is embedded, I follow Roth and McGinn (1998) in referring to graphs, tables, diagrams, notation, and other graphical displays that can be physically produced and reproduced on paper or a computer screen as "inscriptions". This is to differentiate physical representations (that is, inscriptions) from mental representations.

To summarize these factors, rich tasks that encourage mathematical sense-making, ample time for mathematical discussion, and the opportunity to create/interpret inscriptions are all vital to creating an environment for promoting student justification in small group collaborative work.

But there is still much to learn about the specific nature of student-student interactions through which small group work can effectively promote all students' engagement in mathematical justification (Campbell & Hodges, 2020). This study sought to address this need by focusing on the learning practice of justification as it occurs in small group work. Further, the research that has been conducted to arrive at these factors has largely focused on evidence of mathematical reasoning offered either orally or in written form, not across both modalities nor examining the relationship between justification in one modality and what students are able to produce in another modality. Thus by incorporating the aspect of students' written justifications as they occur in the context of small group work, I examine students' justification practices in a fine-grained way across multiple modalities.

Theoretical Framework

In this study I investigated how the nature of middle grades students' PDE during small group work in a collaborative mathematics learning environment relates to their construction of written justifications. An assumption embedded within this research question is that students' collaborative work might actually influence their individual activity to construct written justifications. As such, the proposed study is informed by the emergent perspective (Cobb & Yackel, 1996) as a theory of learning, in which justification can be viewed simultaneously as a social practice for learning and as an individual practice for learning. The primary way in which I considered the social aspect of justification was through the concept of productive disciplinary engagement (PDE), which I used to characterize students' small group work. The primary way in which I considered the individual aspect of justification was through students' construction of written justifications, which I attended to through Toulmin's (1958) argumentative structure. Each of these constructs is described below.

Emergent Perspective: Justification as Simultaneously Social and Individual

In the emergent perspective (e.g. Cobb & Yackel, 1996; Stephan & Cobb, 2003) social processes and norms within the mathematics classroom are coordinated with individual learning processes. Although social processes and norms are shaped by teacher actions, students further negotiate norms during small group work. Small group work is the primary setting in which I considered the social side of the emergent perspective, in which students take part in collective processes to communicate and learn mathematics. Students' written justifications are the primary way in which I considered the individual side of the emergent perspective.

A key assumption of the emergent perspective is that the nature of social processes within the classroom will influence how individual students come to know and do mathematics. In turn, the ways in which individual students know and do mathematics have a shaping influence on the social processes that come to constitute normative ways of doing mathematics within the classroom. Norms and practices from the social perspective are thus correlated with beliefs, conceptions, and activity from an individual perspective and will mutually influence each other.

Classroom social norms describe the normative participation structures for classroom communication (e.g. explain and justify solution methods, make sense of others' justifications, ask questions when uncertain). From the individual perspective, this correlates with the teacher's and students' beliefs about their role in classroom communication. Classroom sociomathematical norms describe the normative participation structures for communication about mathematics (e.g. what counts as an acceptable justification). From the individual perspective, this correlates with the mathematical beliefs and values of teachers and students. Classroom mathematical practices describe common ways of doing mathematics that are utilized by many students in the class (e.g. use of symbolic algebra, any mathematical idea that has been developed by the classroom community until it becomes a taken-as-shared understanding). From the individual perspective, this correlates with particular students' ways of doing mathematics. In the context of this study, I examined classroom norms and practices that were represented within small group interactions (as opposed to whole-class instruction) to consider this social aspect.

From this theoretical view, students' mathematical justifications exist in the intersection of two frames, coordinating social perspective and individual perspective. This is because mathematical justification as a classroom practice can facilitate students' deep learning of content as well as their understanding of how to communicate mathematical ideas to others (Staples & Newton, 2016). I see this as particularly true of written justifications. Written justifications in the context of a class assignment or assessment are often completed with an explicit emphasis on the intended audience (Driscoll, 1999), namely that they must be sensible and convincing to some reader other than the student themself (e.g. the teacher, fellow students). Because of this social intent, the ways in which students construct written justifications reflect the class's sociomathematical norms around when a justification qualifies as "convincing" and "complete", as well as classroom practices around communicating mathematical thought (e.g. to what extent students are expected to unpack or describe mathematical representations, how notation is used within the community). From the psychological perspective, though, students' mathematical writing is also an attempt to accurately display mathematical content knowledge and give insight into what they know and can do. In the digital context of this study, students had a heightened ability to revise and reorder their individual written justifications at will in response to work with their small group. As such, to examine students' written justifications (and more broadly how their justification processes unfold) it is necessary to consider both the social and individual levels of mathematical learning.

In the context of the digital collaborative environment, the nature of the reciprocal relationship between social and individual aspects of students' mathematical justification might be shaped by specific features. In this case, these features allow students to easily view peers' work, copy and adapt inscriptions collaboratively, publish their own work to the class, and communicate digitally in other ways. All of this digital communication takes place in addition to the existing verbal, written, and gestural communication that would regularly take place in a mathematics classroom. Like in the emergent perspective, social norms or sociomathematical norms that emerge around digital communication would have an influence on how students think about their role in the mathematics classroom and how they "talk" about mathematics digitally. Norms around digital communication can have a powerful influence on students' written justifications in this study, because students' written justifications are constructed in a digital environment. Further, these norms might be different than norms around collaborative oral justification.

Given classroom teachers' strong role in establishing and maintaining classroom norms, much of the work that considers how to promote and enable students to justify focuses on teachers' interactions with students, particularly during whole-class discussions (Melhuish et al., 2019; Mueller et al., 2014). But students themselves are also responsible for maintaining productive classroom norms around justification (Wood et al., 1993; Weber et al., 2009), particularly during small group work when the teacher is not consistently present. By focusing primarily on small group work, this study offers additional insight into how students engage in the social aspects of justification, and further, which social aspects of justification support students in constructing and improving their own written justifications.

Productive Disciplinary Engagement

As discussed previously, multiple factors related to mathematical tasks, the nature of group discussion, and use of inscriptions can affect how students work collaboratively in small groups, and thus might also affect how students engage in justification. To describe these multiple factors from the social perspective, I utilized the concept of productive disciplinary engagement (PDE; Engle & Conant, 2002) to characterize students' mathematical activity as they work in small groups. PDE is a way to describe the phenomenon of "students' deep involvement in, and progress on, concepts and/or practices characteristic of the discipline they [are] learning about" (Engle, 2011). This kind of authentic engagement with the discipline of mathematics is called for in mathematics policy and by broad areas of research but can be difficult to foster in classrooms. Although students may be considered "engaged" in many types of classroom learning (marked by their participation in classroom learning structures and completion of tasks), PDE goes beyond this to include students working towards conceptual understanding of important mathematical ideas. Engle and Conant (2002) propose that classroom environments should embody four principles in order to foster PDE: problematizing, authority, accountability, and resources.

The principle of problematizing reflects actions that encourage students to grapple with disciplinary uncertainties that relate to "big ideas" of mathematics (Engle, 2011; Zaslavsky, 2005). There are many ways to promote problematizing, including asking them to solve novel problems or design something new, but each of these centers on students' uncertainties about mathematically important ideas. Students may be uncertain about what to do, how to justify what they are doing, the larger meaning of their conclusions, or competing alternatives among any of the previous categories (Engle & Adiredja, 2008; Zaslavsky, 2005). As students engage with more substantial uncertainties that involve justification, meaning of conclusions, and competing alternatives, this allows for greater levels of problematizing and greater potential for meaningful, conceptual learning (Going et al., 2018). As students attempt to resolve uncertainties, they might use a disciplinary perspective where they adopt practices intrinsic to being a mathematician such as developing algorithms, proving, defining, or other types of problem solving (Lockwood et al., 2019; Rasmussen et al., 2015). As stated above, the opportunity to engage in cognitively-demanding tasks that encourage multiple solution strategies (and thus multiple potential uncertainties) is important for small-group learning to promote justification. But these same opportunities within small-group work can also increase the group's problematizing, and by doing so, facilitate greater levels of PDE.

The principle of authority reflects the idea that when students are engaged in meaningful mathematical problems and uncertainties, they can develop intellectual authority to identify, formulate, and solve these problems for themselves (Engle, 2011). Intellectual authority might include authoring their own ideas, contributing to the co-development of ideas within the group, deciding and justifying whether particular mathematical ideas are reasonable, or pressing one another for justification (Langer-Osuna, 2017). Because of the nature of these activities, the development of higher levels of intellectual authority would allow for the greater presence of justification in a classroom, although issues of privilege and equity can affect how authority develops during small group work. Engle, Langer-Osuna, & McKinney de Royston (2014) found that students were positioned to have authority during small group work for multiple reasons, including the negotiated merits of peers' contributions, degree of intellectual authority, access to the conversational floor, and degree of spatial privilege. In short, students were acknowledged as authorities within their group for many reasons that would not typically be considered mathematical. The way that students developed authority, however, shaped the nature of the mathematical conversation. As such authority is related to both how students contribute to mathematical conversations and how peers receive and respond to these contributions.

The principle of accountability reflects the idea that students' mathematical ideas are made accountable to others and to the norms of the discipline. Engle (2011) noted that if students have a great degree of intellectual authority without an appropriate balance of accountability, they might make unsubstantiated claims and be unlikely to revise and improve their ideas. But when students are held accountable to their own ideas, those of peers, and those of the discipline of mathematics, it can be a constant push to better explain their thinking, examine when ideas do and don't make sense, and to revise the way they make and defend claims. Social norms that have been found to support students in productive discussion in small group work in prior research (e.g. that students should ask questions if they don't understand, that students need to explain their own ideas and make sense of others' ideas) are strongly connected to the principle of accountability. Because of this, the principle of accountability is central to the practice of justification.

Finally, the principle of resources reflects the idea that students must have adequate resources to complete the tasks. In paper-and-pencil classrooms described by Engle and Conant (2002) this has included time, space, and physical tools and materials, but in the context of the digital environment, this also includes tools for creating digital inscriptions within the environment. Other work on PDE in science education has considered more situated resources, such as when the teacher introduces or discourages particular ideas in discussion with students (Venturini & Amade-Escot, 2019) or allows for a meeting with a disciplinary expert (described as an example of resources in Koretsky et al., (2019)). In the digital context of this study, resources might be either inscriptions or tools for interpreting inscriptions (e.g. definitions, rulers, angle measurement tools) Further, because of the collaborative nature of the digital environment and CMP classrooms more generally, these resources may be provided by peers or the teacher.

Classroom environments that embody all four principles to some substantial degree have been shown to foster PDE, while those that do not embody these principles or have a disproportionate emphasis among the principles do not effectively foster PDE (Engle, 2011). The guiding principles of PDE have been used to understand effective classroom learning environments in science education (Koretsky et al., 2019; Venturini & Amade-Escot, 2014) and less often in mathematics education (Williams-Candek, 2015). Because in the context of this study students engaged in justification with their small groups at the same time as they constructed written justifications, the nature of PDE demonstrated by small groups can influence the ways that students construct written justifications and the overall structure of their justifications.

The Structure of Written Justifications

In the context of this study, I sought to examine students' written justifications and the activity to produce them. In the digital collaborative environment, students did not type their mathematical justification all at once in their final form. Rather, they constructed written justifications through a series of writing actions throughout small group work. Each of these writing actions can alter the mathematical completeness of students' written justifications through the addition of new structural elements to their written justification. In this study, I considered the structure of students' written justifications in terms of Toulmin's (1958) parts of an argument and Kosko and Zimmerman's (2019) classification scheme for students' mathematical writing, which are described further in Chapter 3. I utilized these tools to represent both the processes that students used to construct written justifications and the qualities of their finished written justification at the end of small group work. By understanding the ways that students put together the structure of their arguments, we gain insight into their mathematical justification (Conner et al., 2014; Krummheuer, 2007).

Because of the features of the digital environment in which students worked with their small groups and complete their written justifications, students were able to easily insert additional inscriptions or typed words at any point in their work. This ease of revision and editing can encourage students to continue to add new elements to their written justifications through multiple writing actions during small group work as they make sense of mathematical ideas more deeply. I view these as important moments in terms of students' mathematical justification, and sought to identify when students add new structural elements to their written justifications or modify existing components of their written justifications. An important part of my research question includes identifying what features of PDE allow for this increased level of sophistication to emerge in the structure of students' written justifications.

In his model, Toulmin (1958) does not differentiate between written argumentation and verbal argumentation, as an argument can exist in either form. Although the structural elements of an argument are the same, the process of constructing a written argument and constructing a verbal argument could unfold somewhat differently. This is similarly true for justification. In typical settings for oral justification, listeners can easily press the speaker for clarification or additional support (Conner et al., 2014; Ellis et al., 2019), whereas this might not be the case in written justification. Another potential difference between oral and written justification is the level of permanence. If a student types or writes down a warrant, for example, they will be able to examine that warrant at some point in the future in its same form with little difficulty. If, on the other hand, the student verbally communicated that warrant to a peer, it would be unlikely that they would be able to easily and accurately recall the warrant in the same form. With the differences between the process of justification in each modality, the structure of students' written and oral justifications can differ significantly.

Summary of Literature Review

In this chapter I have reviewed literature on justification as a mathematical practice, students' engagement with justification in multiple modalities, and collaborative work in small groups. Prior research on these topics has identified several ways that students can be supported in oral justification through the use of open mathematical tasks, collaboration with peers, and the opportunity to create and investigate inscriptions. We know less, however, about students' processes for justifying in written and digital form and how to support students effectively in constructing written justifications. Prior research on mathematical writing in general indicates that constructing written justifications is not a trivial task that will automatically proceed from high-quality oral justification. Rather, the modality in which students are justifying is an important factor in how they communicate their mathematical ideas.

In addition, I have described the theoretical framework for how prior research on students' mathematical justification and collaborative group work shapes the factors I attended to in this study. The level of PDE demonstrated by a group of students and the specific behaviors they engage in during collaborative work describes the social context of their work, which can influence the work of individual students to construct written justifications. The overall structure and qualities of their written justifications and the specific writing actions they take as they engage in the process of constructing written justifications describe the individual context of their work, which can influence the group's collaborative work on open mathematics problems utilizing tools from the digital environment. By coordinating these two perspectives (social and individual), this study explored the way that students make decisions as they construct written justifications and the factors in collaborative work that might support students to effectively construct written justifications.

CHAPTER 3:

METHODS

This study explored the relationship between students' productive disciplinary engagement during small group work on open problems and their construction of written justifications, all in the context of a novel digital collaborative environment. I examined both of these factors at the level of each task, where I described the relationship between the overall nature of students' PDE during small group work and the mathematical completeness and validity of their written justifications as a finished product. Additionally, I examined both of these factors at a localized level, where I considered the relationship between the writing actions students take during their processes to construct written justifications and the specific ways that they demonstrate PDE during small group work immediately preceding these writing actions. The goal of this work was to better understand the shaping influences on students' construction of written justifications in a process-oriented way, which are not yet well documented for middle grades students. This goal was served by data from a larger research grant concerning the development of a digital collaborative environment to enhance students' engagement with meaningful mathematical problems.

I examined the following research question through qualitative methods inspired by the qualitative foundations of quantitative ethnography (Shaffer, 2017). Data were selected from a larger research project involving multiple researchers, and utilizes frameworks developed in the context of that project. I completed all stages of analysis for this dissertation study. As such, who I am as a researcher and my prior experience with the data set are important to consider as influences to my analytical frame. As a graduate research assistant, I have been connected to the larger grant project since it was initially funded. This means I have been part of the iterative cycles of development of the collaborative digital workspace that lead to its current form, and am sensitive to the affordances and limitations of its features. I have worked with other researchers to develop a framework for considering students' PDE during collaborative work, so my use of this framework is informed by knowledge of all stages of its development. Because of my long-term involvement with the larger research project, I have participated in much of the data collection and organization. As such, I am familiar with all of the teachers involved and most of the physical spaces and students. The larger research project concerns students' PDE during small group, and this study additionally considers how that relates to students' written justification. My views on justification are strongly informed by my past experiences as a middle school teacher, where I often emphasized proof and written justification with my students. In addition to the strong theoretical basis for justification as a learning practice that I have previously discussed, my own positive teaching experiences with justification in middle school inform my interest in this phenomenon.

The research question with which this study is concerned is: *What is the relationship between middle grades students' PDE during small group work and their construction of written justifications in a collaborative mathematics learning environment?* Through the methods described in this chapter, I attended to three related sub-questions:

1. *What is the relationship between middle grades students' PDE during small group work and their construction of written justifications in a collaborative mathematics learning environment in terms of mathematical completeness?*
2. *What is the relationship between middle grades students' PDE during small group work and their construction of written justifications in a collaborative mathematics learning environment in terms of mathematical validity?*

3. *How do middle grades students seek to resolve uncertainties about written justification during small group work in a collaborative mathematics learning environment?*

Study Context

This study was situated within the larger context of a recently concluded project funded by the National Science Foundation. This project utilized design research to develop a digital collaborative environment that would support students' meaningful engagement with mathematical tasks. As part of this, we studied how a problem format that encouraged deepening uncertainties over time might encourage students' PDE. Two sequences of connected problems from the 7th grade CMP curriculum (Lappan et al., 2014) were developed and tested using the digital collaborative environment, within several different teachers' classrooms. This study utilized data that were collected during the 2018-19 and 2019-20 school years, during which the digital collaborative environment (including the CMP STEM problem format) became well-developed and was usable by students and teachers on multiple curricular problems in a row.

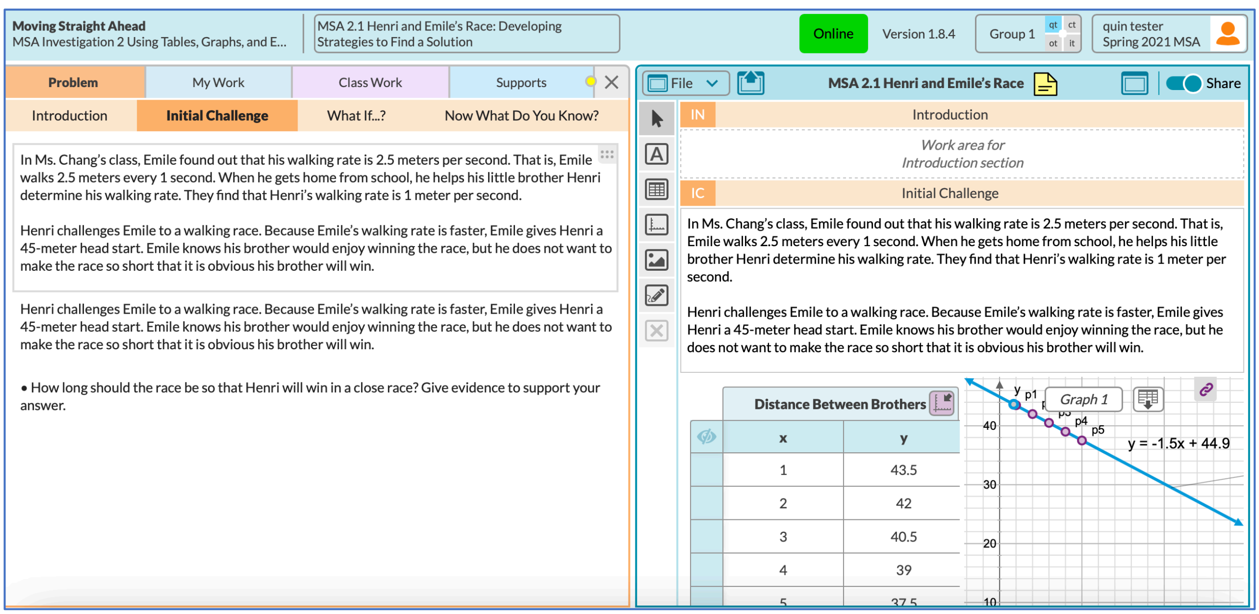
The Digital Collaborative Environment

Design research work to develop the digital collaborative environment occurred as a collaboration between CMP and Concord Consortium over a period of five years. A major goal of this design work was to create an environment that would allow for students' collaborative work to occur digitally as they engaged with meaningful mathematical problems. This goal was attended to through the development of several types of features: the CMP STEM problem format, inscriptional tools, sharing capabilities within small groups, publishing capabilities within the class, learning logs and other records of student work, and problem supports.

The CMP STEM problem format presents open problems from the CMP curriculum using three components: (1) Initial Challenge, which represents students' first opportunity to engage with the problem and its mathematical context, (2) What If…?, which unpacks the embedded mathematics by focusing on new aspects to the problem or other solution strategies, and (3) Now What Do You Know, which reflects on the major mathematical ideas of the problem and connect to prior or future topics. Students can access the problem statements at any point during their work in the digital environment, as shown in Figure 1 below.

Figure 1:

The CMP STEM Problem Format and Inscriptional Tools in the Digital Collaborative Environment[1]



Students can also use a variety of inscriptional tools (e.g. text, tables, graphs, images, drawings) to create their own mathematical work in the digital collaborative environment.

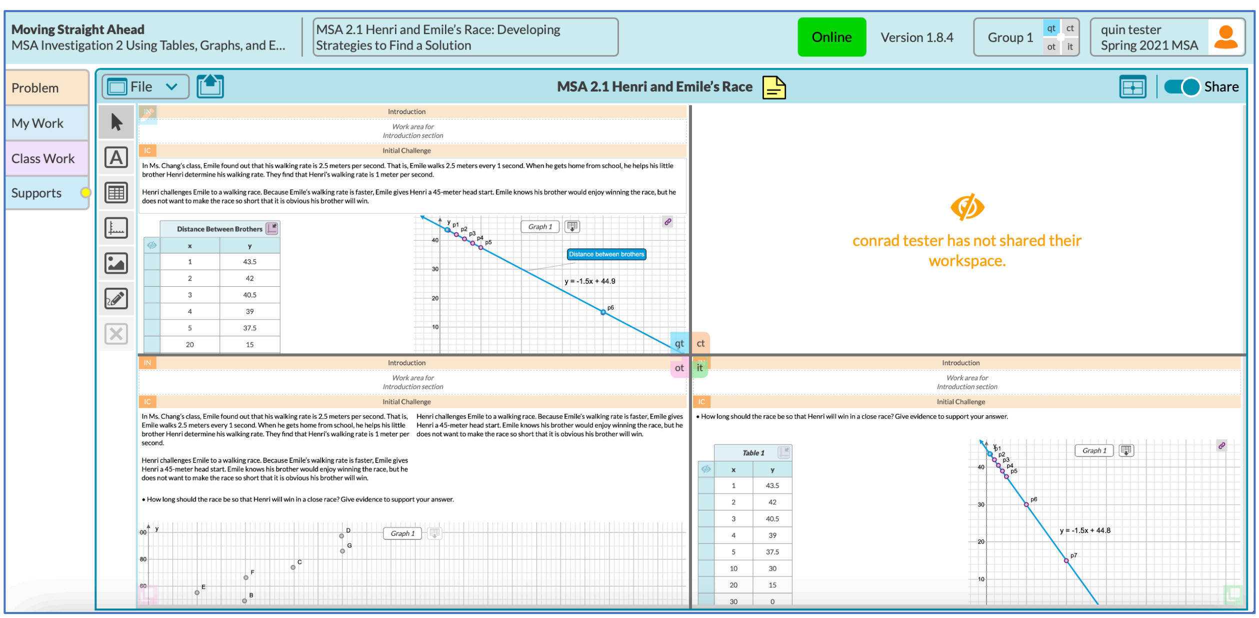
[1] Images obtained from data collected as part of work supported by the National Science Foundation under Grant No. DRL 1660926 (PI: E. Phillips). Any opinions, findings, and conclusions or recommendations expressed in this dissertation are those of the author and do not necessarily reflect the views of the National Science Foundation.

Inscriptions that already exist within the problem statement or peers' work can be moved into students' individual workspaces by clicking and dragging. Students can then continue to modify, mark-up, or build on these inscriptions in their own workspace. A variety of these inscriptional tools that were created in the student's individual workspace are shown in Figure 1. Note that the most recent version of the digital collaborative environment is used to illustrate features in these figures. Although the appearance of the environment changed somewhat between years of testing, the features have remained the same.

Another feature of the digital collaborative environment is sharing capabilities within a group of 4 or fewer students. Students can choose if and when their work is viewable by others in their group. When this feature is activated, all students in the group are able to see their peers' workspaces in real-time by using the collaborative view (often called the "4-up" view). In this view, the student's own word remains visible in the upper-left quadrant, and they can view or utilize work from any peers in their group who have chosen to share (shown in Figure 2).

Figure 2:

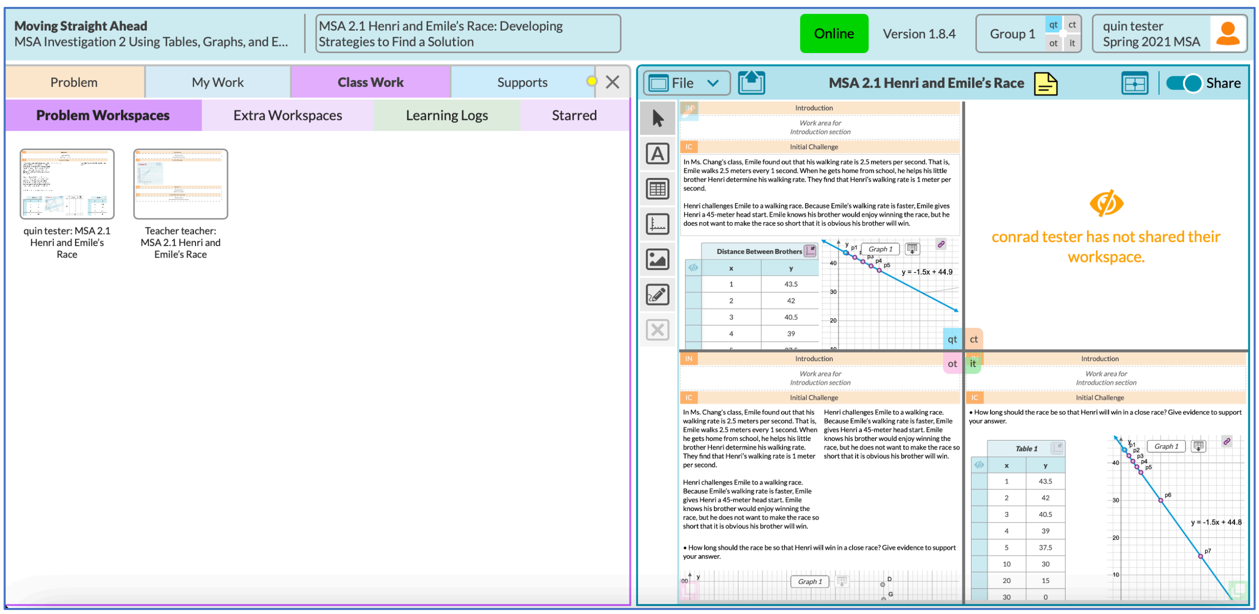
Sharing Capabilities Within Groups in the Digital Collaborative Environment



Students have the option to publish copies of their work to make them visible to the larger classroom community outside of their immediate group. Once published, their work will no longer continue to update in real-time, so students might publish multiple different versions of their work over the course of one problem. Students can access any published work from classmates and integrate inscriptional tools into their own workspaces by clicking and dragging. The class work feature is shown in Figure 3.

Figure 3:

Publishing Capabilities Within the Digital Collaborative Environment



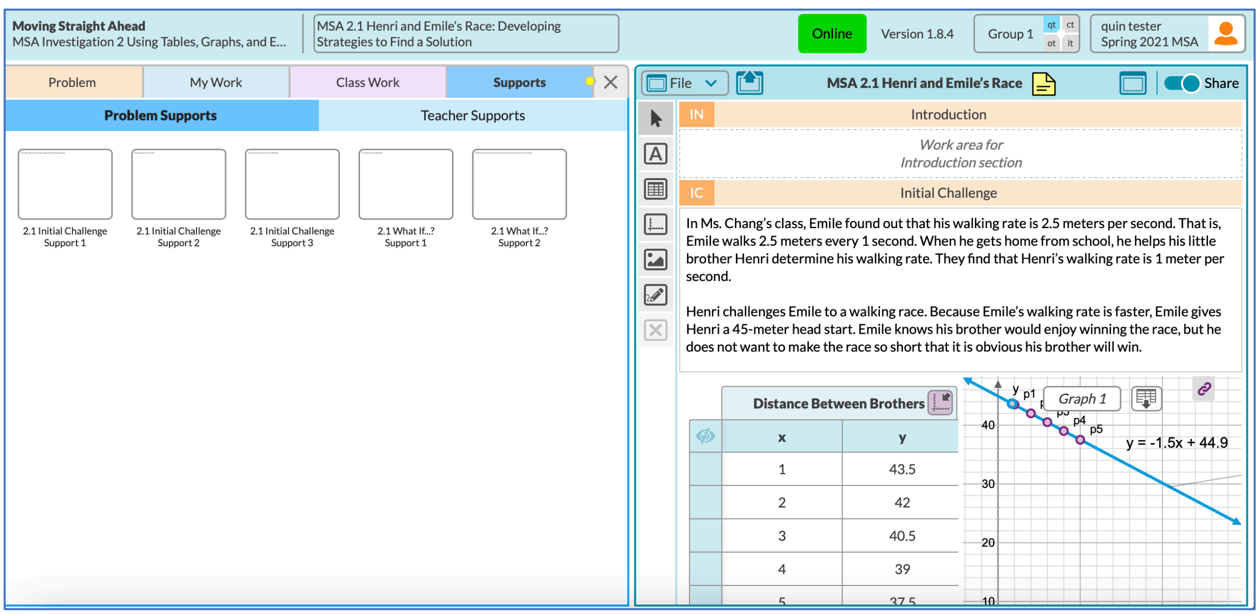
As part of their work in the digital collaborative environment, students can create learning logs which are accessible to them across multiple problems in the curriculum. These are similar to the individual workspaces, where students can use inscriptional tools to create new work or click and drag work from other places. Learning logs, however, are not linked to one specific problem and can be accessed throughout the Investigation. Students can create multiple learning logs, for example to record key terms or summaries or strategies from across the unit.

These can be accessed on the left side of the screen in the digital collaborative environment, similar to what is shown for publishing class work in Figure 3.

Finally, students can access problem supports as they need them during their small group work, often called "just in time" supports. This includes mathematical supports from the curriculum that provide students with additional resources or strategies to consider during problem solving. This can also include supports created by classroom teachers and sent to the whole class, specific groups, or even individual students. These supports can be viewed or incorporated directly into students' workspaces by clicking and dragging.

Figure 4:

Problem Supports in the Digital Collaborative Environment



These features of the digital collaborative environment were intentionally designed to foster PDE based on the four guiding principles. Problematizing is supported through high cognitive-demand tasks that allow students to pursue multiple solution strategies, justify their thinking, and consider alternative ideas as well as problem supports to help students remain mathematically engaged as they encounter uncertainties. Authority is supported through the opportunity for multiple students to contribute mathematical thinking during small group work and to choose their own ways of solving problems. Accountability is supported through the heightened opportunity for students to make sense of multiple peers' work, as well as building on that work through the collaborative view. Resources are supported through the availability of inscriptional tools and a choice of multiple different supports from the problem, peers, and teachers. All of these features are available as a way for groups to develop PDE, but it is most important to consider how students actually use these features.

All data for this study were collected as students and teachers worked on mathematical problems within this digital collaborative environment, so it is an important part of the context for students' small group work. Because the digital collaborative environment has the potential to enhance students' PDE through multiple different features (e.g. promoting problematizing through features of problem structure, promoting accountability to peers through visibility and publishing features), I consider the environment as the context for students' small group work in which there are multiple ways students might develop PDE and eventually come to construct written justifications. As such, it is not the end goal in and of itself that students use any particular feature of the environment. Students' PDE (and by extension their construction of written justifications) could potentially develop through multiple different ways of interacting with the features of the digital collaborative environment.

Participants

In the broader research project, teachers were selected to participate from a pool of volunteers based on a couple factors that would facilitate the design research process of the digital collaborative environment. First, they needed to have taught CMP curriculum for multiple years for greater familiarity with the content, and second, they needed to be teaching multiple class sections of 7th grade mathematics in which to work with the digital collaborative environment. Over the course of the larger project, this included 7 teachers (each with multiple classes) in 5 schools in the Midwestern and the Northwestern United States.

This study concerns students' use of the digital collaborative environment in its well-developed form (that is, during the 2018-19 and 2019-20 school years), during which five teachers participated in testing and data collection. I considered data from units where at least 50% of the problems from that unit were implemented (that is, classrooms where students had a chance to use the environment often and become familiar with it). This enabled me to observe many different instances where students are able to use the digital collaborative environment fairly unproblematically to develop PDE and construct written justifications. The implementation for one teacher's classrooms did not meet this requirement, and data from their classroom were not considered. Two of the remaining teachers implemented only one unit that met this requirement. These two teachers utilized a variety of technological tools (e.g. Google Classroom, Desmos) within their classrooms to a far greater extent than is typical, with each student in their class having a personal laptop available each day. Because I did not consider the role of other types of technology beyond the digital collaborative environment in this study, this heightened degree of technology use was less suitable as a starting point for building theory. As such, I considered data collected in the remaining two teachers' classrooms in this study. I refer to them in this study by the pseudonyms Ms. Price and Ms. Roberts.

Both teachers are highly experienced, with 10 and 20 years of experience teaching middle school mathematics. In particular, they both have considerable experience teaching with the CMP curriculum (at least 10 years) and use it as their primary curriculum. As such, these teachers engage students in small-group collaborative learning on tasks that encourage multiple solution strategies as a routine part of mathematics instruction, and regularly encourage students to share, discuss, and explain their mathematical thinking. Although this study does not foreground the teacher's role in promoting justification, it is important to note that these teachers' use of mathematical tasks and practices to encourage mathematical discussion are in line with existing research on factors that support students' justification during small group work. Further, by attending to classrooms in which students and teachers had the opportunity to use the digital collaborative workspace regularly, I was more likely to observe situations in which students were able to create and use inscriptions to support their justifying activity. Each of these three factors (task features, nature of discussion, and use of inscriptions) has been shown to support small-group justification in mathematics classrooms (Francisco, 2013; Mueller, 2009; Mueller & Maher, 2009; Selling, 2016; Weber et al., 2010). Thus by selecting data from teachers who are experienced with both collaborative small-group instruction and the digital collaborative environment, I aimed for the strongest possibility of observing small-group justifying activity and its relationship to students' construction of written justifications.

Although I considered data from two teachers' classrooms, the focus of this study is not comparative. Instead, data from both teachers' classrooms formed one larger pool of data from which to sample. This is possible because of the many similarities between these teachers. As described above, both teachers have several years of experience teaching open collaborative problems utilizing CMP curriculum. Both teachers' classrooms had similar types of technology (individual student laptops provided by the grant project, document cameras, and digital projectors), but did not regularly implement other technology-focused lessons outside of the project. Both schools are in the same public school district, enroll between 650 and 700 students in 6th through 8th grade, and have approximately 20% of students eligible for free or reduced lunch. At the first teacher's school, students describe their ethnicities as follows: 33% Asian, 12% Black, 5% Hispanic, 40% White, and 10% belonging to two or more groups. At the second teacher's school, students describe their ethnicities as follows: 6% Asian, 12% Black, 7% Hispanic, 63% White, and 12% belonging to two or more groups. Both of these schools have more ethnic diversity among students than surrounding schools and many demographic similarities, with the main difference being the percent of students who describe themselves as Asian and White. Because of these many similarities between both teachers and their school contexts, I considered data from both classrooms in order to examine students' small group work. The main goal of utilizing this larger pool of data was not to compare teachers in any way, but rather to examine as many instances of students' small group as possible to identify features of PDE in small group work that allow students to develop written justifications.

Teachers used the digital collaborative environment with multiple classes of 7th grade mathematics students, who gave personal permission and received permission from their guardians to be video- and audio-recorded for research purposes. Each class had roughly 25-30 students, and students worked on individual computers in collaborative groups of 3-4 students. Because data were collected for small groups of students as opposed to individual students, teachers were asked to form two groups of four students in each class that consisted of only students who had consented to participate. These two groups were the focus of most data collection activities in the class during that unit. Typically, 90-100% of students in each class consented to participate and teachers were able to form groups of students as they normally would (i.e. heterogeneous grouping). In these classes, the groups of students that were recorded reflect the diversity of the broader classroom community in terms of gender, ethnicity, academic background, socioeconomic status, and primary language. One exception to this does occur, where scheduling factors at one school caused a very small class section with only 16 students, less than 50% of whom consented to participate. For this class section the teacher expressed awareness of her limited ability to form collaborative groups as she typically would.

Data Collection and Selection

Design and testing of the digital collaborative environment were completed for two units from the 7th grade CMP curriculum. Each unit consists of a connected sequence of problems that occur in related contexts and are designed to help students deepen their understanding of core mathematical ideas as they progress through the unit. The first unit, *Stretching and Shrinking,* concerns proportionality and students investigate this mathematical idea through considering similar figures and scale factor in graphical contexts. The second unit, *Moving Straight Ahead*, concerns linearity and students explore this mathematical idea through representing known rates of change from realistic contexts (e.g. walking rates, payment plans, design challenges) and investigating unknown patterns of change.

Each unit consists of four Investigations, and each Investigation contains 3-4 problems that occur within a similar realistic context but allow students to consider new aspects of the mathematics at hand. As described above, each problem is formatted in the CMP STEM problem format, which organizes it into three components: 1) Initial Challenge, 2) What If…?, and 3) Now What Do You Know?. This structure for problems gives students access to the main problem situation, then calls on them to consider new situations or variables that might be changed, and then allows for students to consider their mathematical ideas more generally. Finally, at the end of each Investigation there is the option for students to complete a Mathematical Reflection, which calls for students to summarize and synthesize their mathematical ideas up to that point in the unit.

Data Sources

Data were collected in both teachers' classrooms during the 2018-19 and 2019-20 school years in for implementation of the units *Stretching and Shrinking* and *Moving Straight Ahead*. Data sources for the larger researcher project include screen recordings of students' engagement with the digital collaborative environment, video and audio recordings of small-group work and whole-class instruction, audio recordings of teacher instruction throughout class, teacher interviews, and software analytics from the digital collaborative environment. Screen recordings, video and audio recordings, and software analytics were collected on each day that students and teachers used the digital collaborative environment. Interviews were conducted with each teacher after each unit of instruction. Because this study concerns how the nature of students' small group work relates to their construction of written justifications, the main data sources I considered were students' screen recordings, and to a lesser extent, video and audio recordings of small-group work.

The primary data source for this study was screen recordings that show students' use of the digital collaborative environment on each day they use it. Screen recordings were collected using a screen capture app that was installed on students' laptops (Screencastify). These recordings show video of students' onscreen activity coupled with audio from the laptop's microphone and embedded video captured by the laptop's web camera that shows the student's facial expressions and movement. Screen recordings were collected for two focal groups of four students within each class section. Because screen recordings allowed me to hear and see students' activity during small group work as well as all writing actions they took to construct their written justifications, they are the primary data source through which I considered both the nature of students' PDE and their written justifications. By using this common data source, I was able to establish a common timeline between the social perspective of students' small group work and the more individual perspective of their work to construct written justifications.

As needed, I also incorporated information from video- and audio-recordings of small group work in order to clarify my observations from students' screen recordings and eliminate potential confusion or discrepancies. Rather than providing extra information that is independent from screen recordings, video- and audio-recordings of small group work serve as a check on the accuracy of my perception of screen recording data. Given that screen recordings were collected using an internet-based app that occasionally suffered disruptions (detailed further below) and on computers which typically faced only a few members of the group at a time, this check on the accuracy of my perception of small group work was helpful to clarify students' dialogue and ensure correct attribution of words and actions among group members. Audio was captured with a digital audio recorder placed on the table in the middle of each focal group of students, and video was captured with a digital video camera placed 2-3 yards away from the group.

There were several factors that interfered with the collection of complete sets of screen recordings for each focal group on a given day (e.g. a student's absence, an unstable internet connection that caused the screen recording to stop and restart, incorrectly installed security updates that were incompatible with the screen capture app). Although these problems did occur during data collection, it was rare that they occurred on the same day for all members of a given group. The data analysis methods for this study (which will be discussed further below) utilized the screen recordings of two focal students within the group, rather than needing a complete set of four screen recordings. I considered a set of screen recordings viable for analysis if at least two members of the groups have functional screen recordings (i.e. with working video and audio) that show all small group work time that occurred on that day. Screen recordings that were split into multiple files were still considered viable if they were split into only two files and there was a gap of less than 2 minutes of class time between the two files (corroborated through use of video- and audio-recordings of small groups). More than 99% of the data from which I sampled met this viability requirement, so it did not considerably limit my ability to sample and analyze project data. This viability requirement was important, however, to coordinate the words and digital actions of multiple students in the group.

Data Sampling

Each of the curricular problems from the two units that were tested in the larger research project prompts students to engage in practices important for mathematical justification. That is, they all offer students opportunities to make and defend claims, explain why the claims of others are true or false, or to justify problem-solving choices as they make sense of mathematics. Most tasks also specifically prompt students to construct a written justification, by asking them to "justify", "explain why", or "give reasoning". It was not typical for participating teachers to remove any of these prompts, and they often added additional verbal reminders for students to justify during task setup (e.g. "You've got to explain why it works that way", "Convince me. How do you know?"). Because all of the tasks prompt students to engage in justification and/or construct written justification (and are thus suitable for analysis from the individual perspective), I sampled data in order to observe multiple contexts for PDE to emerge in small group work from the social perspective.

The problematizing principle of PDE concerns students engaging with disciplinary uncertainties that might develop over time. Thus, I sampled Investigations rather than isolated days of instruction or problems. This allowed me to examine one group's engagement in a few connected problems in a row. Investigations were selected through a stratified sample to ensure that data from both participating teachers' classrooms were present in the final sample. As described above, data were selected from Investigations in which at least 50% of the problems were taught within the digital collaborative environment, to ensure the teacher's and students' familiarity with the platform. Each teacher taught multiple classes of 7th grade Mathematics throughout the day, and two groups of students from each class section were recorded. From Investigations that were selected through stratified sampling, one class section and one focal group from within it were randomly selected for analysis. Table 1 shows the implemented Investigations for each teacher, as well as the number of class sections for which data was collected and the number of days of instruction in each Investigation. For Investigations where only the partial unit was taught or the teacher additionally opted to engage students in the Mathematical Reflection, that is noted on the table. Investigations that were sampled for analysis are marked in bold, and include three Investigations for each teacher.

Table 1:

Investigations for Data Sampling

Ms. Price	Ms. Roberts
2018-19 *SS* Inv. 1 - 4 class sections, 5 days, includes MR	**2018-19 SS Inv. 1 - 3 class sections, 5 days, includes MR**
2018-19 *SS* Inv. 2 - 4 class sections, 4 days	2018-19 SS Inv. 2 - 3 class sections, 5 days, partial
2018-19 *SS* Inv. 3 - 4 class sections, 4 days	2018-19 MSA Inv. 1 - 3 class sections, 4 days, partial
2018-19 *SS* Inv. 4 - 4 class sections, 5 days includes MR	**2018-19 MSA Inv. 4 - 3 class sections, 3 days**
2018-19 *MSA* Inv. 1 - 4 class sections, 5 days	2019-20 SS Inv. 1 - 4 class sections, 5 days
2018-19 *MSA* Inv. 2 - 4 class sections, 2 days, partial	2019-20 SS Inv. 2 - 4 class sections, 3 days
2018-19 *MSA* Inv. 3 - 4 class sections 5 days	**2019-20 MSA Inv. 2 - 2 class sections, 2 days, partial**
2018-19 *MSA* Inv. 4 - 3 class sections, 4 days	2019-20 MSA Inv. 3 - 2 class sections, 2 days, partial
2019-20 *SS* Inv. 1 - 4 class sections, 3 days	2019-20 MSA Inv. 4 - 2 class sections, 2 days, partial
2019-20 *SS* Inv. 2 - 4 class sections, 4 days	
2019-20 *SS* Inv. 3 - 4 class sections, 3 days	
2019-20 *SS* Inv. 4 - 4 class sections, 3 days	
2019-20 *MSA* Inv. 1 - 4 class sections, 4 days, partial	
2019-20 *MSA* Inv. 2 - 4 class sections, 5 days	
2019-20 *MSA* Inv. 3 - 4 class sections, 2 days, partial	
2019-20 *MSA* Inv. 4 - 4 class sections, 3 days, partial	

Note: *SS* indicates *Stretching and Shrinking, MSA* indicates *Moving Straight Ahead,* and MR indicates "Mathematical Reflection"

As shown on Table 1, six distinct implementations of Investigations were selected. From Ms. Price's classroom this was the 2018-19 implementation of *Stretching and Shrinking* Investigation 4 and the 2019-20 implementations of *Stretching and Shrinking* Investigation 3 and *Moving Straight Ahead* Investigation 2. From Ms. Roberts's classroom this was the 2018-19 implementations of *Stretching and Shrinking* Investigation 1 and *Moving Straight Ahead* Investigation 4 and the 2019-20 implementation of *Moving Straight Ahead* Investigation 2.

Across this sample, there are 23 days of instruction in multiple mathematical contexts. This is a reasonable sample size for analysis by one independent researcher, particularly considering the complexity of the analysis methods for this study. As I describe in the following section, the data analysis for each individual group involved multiple phases, including coordinating multiple group members' screen recordings, attending to talk and actions during small group work, and attending to digital actions and written work within the digital collaborative environment. Analysis occurred for each overall task, but localized analysis also occurred for each writing action taken by two focal students in each group. Although a larger sample size would potentially allow me to make broader claims about students' construction of written justifications, it would also be prohibitive in terms of these analysis processes.

Analytic Framework

In this study, I examined the relationship between the nature of students' collaborative work in small groups and their construction of written justifications. As such, it was important to utilize tools for analysis that could represent the individual actions that students took during small group work and construction of written justification and also the nature of students' work on overall tasks. To consider how groups of students demonstrated evidence of PDE through particular behaviors and throughout an overall task, I utilized a novel framework for describing

PDE in collaborative work. To consider how individual students' processes to construct written justifications and the qualities of their finished written justifications, I utilized Toulmin's (1958) description of the structure of arguments and Kosko and Zimmerman's (2019) classification scheme for students' mathematical writing. Each of these is described below.

PDE in Collaborative Work Framework

To describe the nature of students' collaborative work, I utilized PDE in Collaborative Work (PDE-CW) framework (shown in Table 2) for characterizing the small group work environment in terms of student actions. This framework was developed through iterative coding grounded in data from the grant project in which the current study is embedded, as well as some collaborative group work in a non-digital setting. In this framework, each principle of PDE is described by multiple indicators, each of which can be demonstrated by groups at a high, medium, or low level throughout the overall task.

Table 2:

The PDE in Collaborative Work (PDE-CW) Framework (Bieda et al., revision in preparation)

	High PDE	Medium PDE	Low PDE
Problematizing	1. Students work to understand the task conceptually and procedurally well. 2. Students often adopt a disciplinary perspective. 3. When uncertain or stuck, students push to resolve the issue themselves and/or with help from their peers or teacher. 4. Students' mathematical uncertainties concern what their conclusions mean or competing alternatives.	1. Students work to understand the task procedurally well, with some attempts to understand conceptually. 2. Students sometimes adopt a disciplinary perspective. 3. When uncertain or stuck, students make attempts to resolve the issue but might give up quickly. 4. Students' mathematical uncertainties concern justifying their actions.	1. Students work to complete the procedures, but with few or no attempts to understand conceptually. 2. Students rarely adopt a disciplinary perspective. 3. When uncertain or stuck, students do not attempt to resolve the issue. 4. Students' mathematical uncertainties concern what action to take.

Table 2 (cont'd)

	High PDE	Medium PDE	Low PDE
Authority	1. Most students make mathematical contributions to the group's work, either verbally or nonverbally. 2. Students often talk about the mathematics topic/problem using their own words. 3. Students often seek out, consider, or respond to peers' mathematical ideas. 4. Students react to peers' ideas as worthy of consideration and respond constructively.	1. Some students make mathematical contributions to the group's work, either verbally or nonverbally. 2. Students sometimes talk about the mathematics topic/problem using their own words. 3. Students sometimes seek out, consider, or respond to peers' mathematical ideas. 4. Students react to peers' ideas as worthy of consideration and respond constructively, but not consistently.	1. Few students make mathematical contributions to the group's work, either verbally or nonverbally. 2. Students repeat the mathematics problem statements, rarely using their own words. 3. Students rarely seek out, consider, or respond to peers' mathematical ideas. 4. Students do not react to peers' ideas as worthy of consideration and/or make inferior remarks about peers' contributions.
Accountability	1. Students question each other and explain their mathematical contributions until the idea makes sense to everyone in the group. 2. Students often deliberate about multiple mathematical ideas. 3. Most students participate in each instance of deliberation, with all group members deliberating at some point during the task. 4. Students' ideas are discussed in terms of their utility for problem-solving or their mathematical relevance.	1. Students question each other and explain their mathematical contributions, but exchanges are brief. 2. Students sometimes deliberate about multiple mathematical ideas. 3. Some students participate in each instance of deliberation, with all group members deliberating at some point during the task. 4. Students' ideas are discussed in terms of their utility for problem-solving or their mathematical relevance, but not consistently.	1. Students do not explain their mathematical contributions and/or blindly accept others' contributions. 2. Students rarely deliberate about multiple mathematical ideas. 3. One or more students don't participate in any instance of deliberation during the task. 4. Students' ideas are not discussed in terms of their utility for problem-solving or their mathematical relevance.
Resources	1. Students often use relevant resources from the current task or provided by the teacher. 2. Students often use relevant resources they create or choose. 3. Students use resources in a way that helps them resolve mathematical uncertainties.	1. Students sometimes use relevant resources from the current task or provided by the teacher. 2. Students sometimes use relevant resources they create or choose. 3. Students use resources in a way that doesn't limit their resolution of mathematical uncertainties.	1. Students rarely use relevant resources from the current task or provided by the teacher. 2. Students rarely use relevant resources they create or choose. 3. Students use resources in a way that limits their resolution of mathematical uncertainties.

When PDE is viewed in terms of student actions according to this framework, high quality PDE is characterized to a large extent by students making meaningful attempts to engage in mathematical justification. If they demonstrate problematizing at a high level, then they are necessarily working to make sense of mathematical ideas deeply and comparing multiple ideas or solutions strategies. If they demonstrate a high level of authority within the group, students are necessarily making contributions or claims and working with their peers to consider and defend those claims. If they demonstrate a high level of accountability within the group, students are actively justifying their thinking and pushing their peers to do so in a collaborative way. And if the group demonstrates a high level of resource use, then students are using inscriptions and tools to support themselves in resolving uncertainties (i.e. making sense of mathematical ideas). All of these practices are key components of mathematical justification. From a theoretical perspective, then, high-quality PDE during small group work could support students' engagement in mathematical justification, and eventually support students' construction of written justifications. A major goal of this study is to investigate the nature of this support to identify specific factors that allow students to effectively develop written justifications.

Frameworks for Structure and Completeness of Written Justifications

Much of the research around argumentation has utilized a model based on the work of Toulmin (1958) to analyze the structure of students' arguments and the analytical methods I used to describe the structure of students' written justifications are shaped by this as well. Though his original work was not mathematics specific, it has been heavily utilized in mathematics education (e.g. Boero et al., 2010; Krummheuer, 2007; Weber et al., 2008). Toulmin proposed that the structure of an argument is similar in many fields, consisting of a few main elements. By analyzing the relationships between these elements, it is possible to make sense of arguments beyond formal deductive logic, which would include students' written justifications.

There are three main elements in an argument according to Toulmin's (1958) model: 1) the claim, which is the result or statement being supported, 2), data, which are undoubted statements justifying the claim, and 3) warrants, which are rules of inference that allow data to be connected to claim. Beyond these main elements, there are also backings, which are basic or unquestionable facts that clarify the permissibility of warrants. In terms of the four elements already mentioned, the form for a traditional deductive argument might be "(Data) so (claim) since (warrant) on account of (backing)" (Krummheuer, 2007). Claims, data, warrants, and backing are the four structural elements that were relevant in describing students' writing actions in this study, although Toulmin also detailed other, more specialized elements (e.g. modal qualifiers, which refer to the degree of confidence; rebuttals, which offer specific conditions under which the claim does not hold). Reasoning might proceed deductively, inductively, by analogy, or by other means (Toulmin, Rieke, & Janik, 1984). Further, the way in which students build relationships between elements of an argument can give insight into how they are making sense of the mathematical ideas concerned (Conner et al., 2014; Krummheuer, 2007). That is, the different writing actions that students take give insight into their processes for constructing written justifications.

In addition to considering the writing actions that each student took to construct written justifications, I also categorized their final written justifications (i.e. the written justification in the form it was at the end of the task) according to Kosko & Zimmerman's (2019) classification scheme for children's mathematical writing related to justification. Although the classification scheme was developed based on work with elementary children, the described development of writing sophistication mirrors the small body of existing work on the development of middle and secondary students' writing (as described in Chapter 2). Because of this similarity and its emphasis on written communication rather than meeting disciplinary standards of proof, I find it appropriate to use this classification scheme. Although previously described in general terms, the specific types of written justifications are:

1) Emergent mathematical writing: show some evidence the student is engaging in mathematical thinking, but not interpretable on their own
2) Mathematical statements: include only a claim
3) Mathematical recounts: describe the student's experience to complete the task in non-specific terms and generally include a claim and a warrant
4) Mathematical procedurals: describe the student's experience to complete the task in specific mathematical terms and generally include a claim, warrant, and potentially references to data
5) Mathematical detailing: include a detailed, connected chain of claim and warrant, potentially with explicit references to the data.
6) Mathematical descriptions: include a detailed, connected chain of claim and warrant, potentially with explicit references to the data, and students establish and name mathematical relationships in their writing that they continue to refer to (i.e. nominalization)
7) Mathematical explanations: include a connected chain of claim, data, warrant, and (potentially) backing

An assumption of this classification scheme is that the first category is the least mathematically complete, continuing to the seventh category, which is the most mathematically complete. Categories 1 and 2 include mathematical work or a claim, but do not go beyond this to further justify the work or claims. Categories 3 and 4 go beyond this to address *how* the mathematics works (that is, how the student determined their claim). Categories 5, 6, and 7 go even further to address *why* the mathematics works in terms of underlying structure or mathematical relationships. I used the classification of each written justification as a representation of its mathematical completeness.

I found it analytically useful to incorporate one additional category to this existing classification scheme, which I refer to as "none written" (Category 0 in relation to existing categories). These describe cases where students did mathematical work towards constructing a written justification (for example attempting to construct a graph or diagram of a mathematical relationship, entering calculations or measurements into a table), but had not yet begun to type a written justification. If students also began to type text that indicated they were beginning a written justification (e.g. "A table and a graph could be" or "The equation is"), then their written work was classified as "emergent mathematical writing" as described by Kosko and Zimmerman (2019). Although all students produced some kind of mathematical work during their work on each task in the digital collaborative environment, I differentiated between these categories to separately consider cases where students did mathematical work and also indicated that they were attempting to use that mathematical work to construct a written justification.

Data Analysis

This study concerns how students go about constructing written justifications, and the relationship between their processes to construct written justifications and their work in small groups. To this end, I conducted analysis both at the general level of the overall task and at the more localized level surrounding specific writing actions. At the most general level of analysis, I considered students' PDE on each overall task and the written justifications that students constructed during those tasks. At the more localized level, I identified "writing actions", or distinct additions, modifications, or deletions to a student's written justification after which the student pauses for more than 10 seconds or stops typing for the remainder of the task. For localized analysis, I characterized students' writing actions in terms of how they affected the mathematical completeness and validity of the overall written justification and described the nature of the group's PDE preceding writing actions in terms of specific indicators from the PDE-CW framework (see Table 2).

My analysis focused on students' small group work, which I distinguished from whole-class and individual work through teacher directions to the class (e.g. "Work together to…" or "Discuss with your groups…"). The major unit of analysis within students' work with their small groups is a task, consisting of a group's work to solve one mathematical problem or answer one main question. Most commonly, a task consists of a group's work to solve one component from the CMP STEM problem format (for example, the Initial Challenge), and the task ends when the group moves on to a new task (in this case, "What If…?"), the teacher calls for a shift to whole-class or individual instruction, or class ends. If the group's work on a particular problem component was completed on two different instructional days, then for the purposes of analysis I considered that as two tasks.

Analysis of Overall Tasks

Across the data set for this study, I analyzed 38 tasks in total. The curricular prompts for each of these tasks is shown in Appendix A. Tasks ranged from 3 minutes to 20 minutes in length, with an average length of 10.07 minutes. For each task, I characterized the nature of PDE demonstrated by the group during small group work and examined the written justifications of two focal students in each group in terms of their mathematical completeness and validity. Because I analyzed two focal students per group over 38 tasks, this includes 76 total attempts to construct a written justification.

Characterizing the Nature of PDE During Small Group Work

For each task, I described the overall characteristics of each small group's PDE according to the PDE-CW framework (shown previously in Table 2). Analysis in this phase followed protocols that I helped develop and which have already been tested by myself and other researchers on a variety of tasks from the larger research project data. Full use of the framework and these analysis procedures are described in an in-progress manuscript (Bieda et al., revision in preparation), but I also briefly describe them here. Each indicator of the framework can be described as high-, medium-, or low-level to best describe the overall pattern of student activity throughout the task. The overall pattern of student activity (as represented by the indicator rating) is determined by how much time out of the total small group work on that task includes evidence of a specific indicator. For coding purposes, an indicator occurs "never or rarely" if raters observe it during less than 10% of small group work time for a task, occurs "sometimes" if raters observe it during 10%-40% of small group work time for a task, and occurs "often" if raters observe it during more than 40% of small group work time. In terms of the larger research project, the PDE-CW framework was developed through iterative cycles of analysis grounded in data from the project until it reached its current form. In terms of this dissertation study, I have used this framework to characterize students' PDE on each task without making any further modifications to the form or content of the framework.

To code using the PDE-CW framework, I primarily used one focal student's screen recording from within the group for analysis. The focal student was chosen based on completeness of screen recordings and audio quality. If all factors of audio and video quality are otherwise equal, the focal student was chosen randomly. Note that typically, one can hear all other group member's voices on each screen recording and often see other students' mathematical work due to the collaborative nature of the tasks and the digital environment. This feature of screen recordings is important because this framework considers PDE across the entire group rather than for one particular student. Other students' screen recordings and audio- and video-recordings of the whole group were then used to supplement and corroborate information from the focal student's screen recording, rather than being considered as independent data sources.

In this analysis, I first identified all episodes within the task during which there was observable evidence of multiple indicators of medium or high levels of each principle of PDE. Identification of these episodes was completed by creating "clips" in the data analysis platform Transana, which can be saved in collections related to each principle. Evidence can include students' words or actions, including actions within the digital collaborative environment. Even when considering these shorter episodes (rather than the overall patterns for the task) multiple principles can be present at a high level simultaneously or principles can be present in isolation. Once all episodes that showed evidence of medium or high levels for each principle were identified, I then reviewed the full collection of episodes to characterize the level of PDE demonstrated by the group on the task overall. Information from these shorter episodes was used to determine the overall level of PDE demonstrated on a given task, but no further analysis was conducted on these episodes unless they were chronologically close to students' writing actions (detailed below in my description of localized analysis).

Characterizing Students' Overall Written Justifications

For each task, I analyzed two students' construction of written justifications within the group. One of these is the focal student from PDE analysis, as well as one additional student chosen based on similar factors of audio and video quality of screen recordings. I described the mathematical completeness of each written justification according to Kosko and Zimmerman's (2019) classification scheme, described above. Their classifications for written justifications proceed in order of increasing mathematical completeness, from students' attempts to make mathematical claims, then to attempts to describe *how* the mathematics, and then to attempts to also describe *why* the mathematics works.

In addition to mathematical completeness, I also considered the mathematical validity of students' written justification. In the context of this study, written justifications were considered mathematically valid if they included correct claims and if any support for those claims (i.e. evidence, warrants, or backing) met two criteria: (1) the support is accurate without any errors or incorrect statements, and (2) the support is mathematically relevant to the claim. Defining mathematical validity in this way, then, considers mathematical accuracy as well as how students support their claims.

Because of the ability to see students' construction of written justification as it occurred in real-time on screen recordings, I classified each student's written justification according to its overall mathematical validity during its construction. This provides information about the overall validity of the written justification as well as a simple representation of students' processes to produce written justification. I consider whether written justifications were always valid, had emergent validity (that is, they became mathematically valid by the end of small group work on the task), were never mathematically valid, or were not analyzable in terms of validity (which occurred mainly for partially-finished statements or inscriptions). Note that in this study, the mathematical validity of students' written justifications either remained the same throughout the task or improved. There were no cases where a student produced a mathematically valid written justification early in the task and ended the task with a non-valid written justification.

Localized Analysis

I also conducted localized analysis of the nature of PDE preceding specific writing actions (i.e. distinct additions, modifications, or deletions to a student's written justification after which the student pauses for more than 10 seconds or stops typing for the remainder of the task) that students took to construct written justifications throughout the task. All of the written justifications considered in this study were constructed through multiple different additions, modifications, and/or deletions rather than being typed all at once, so considering the nature of these specific writing actions gives insight into students' processes. Additionally, considering the nature of PDE in small group work that precedes these writing actions gives insight into the relationship between specific PDE-related behaviors and writing actions to address the completeness or validity of the written justification.

For localized analysis I examined a subset of 24 tasks where the prompt as written and set up by the teacher asked students to "justify", "explain why", or "give reasoning", rather than only prompting students to create an inscription or provide a claim. Although creating inscriptions and investigating claims are important parts of the process of justification, the analysis for this study focused on students' writing actions as they construct written justifications. As such, I only conducted localized analysis for tasks that explicitly prompted students to construct written justifications. In all, I observed 163 instances where students made changes to written justifications. These included changes to specific elements of the written justification (i.e. claims, evidence, warrants, or backing) that impacted the completeness of the written justification and changes that impacted the validity of written justifications.

Identifying and Characterizing Students' Writing Actions

By identifying the moments when students took distinct writing actions as they constructed written justification, localized analysis established a timeline for the construction of each justification. For each focal student's screen recording, I identified all timecodes when students began, added, modified, or deleted elements of their written justification (i.e. claims, data, warrants, or backings). Note that for analysis purposes, an inscription was only considered as data in a student's written justification if the student explicitly referenced it (e.g. "In the picture…", "My graph shows…"), and not simply because the inscription exists within the student's digital workspace. One outcome of this analytical work will be to create grounded descriptions of the process through which middle grades students construct written justifications (rather than just the quality or characteristics of the finished product), which will complement existing research on students' justification in multiple modalities.

I classified all changes to written justifications based on which component of the written justification (i.e. claims, evidence, warrants, or backing, similar to Toulmin's (1958) description of argument) they were related to. This is a way to consider the order in which students construct written justifications and how they attend to mathematical completeness through specific writing actions. Students' writing actions were classified as relating to the mathematical claims if they added to or modified the result/statement being supported. In the context of this study, claims were often the "answer" to the problem statement from the curriculum. Students' writing actions were classified as relating to evidence if they added or modified the data or factual statements justifying the claim. In the context of this study, evidence was usually presented as mathematical calculations or numerical data organized through mathematical inscriptions (e.g. graphs, tables). Students' writing actions were classified as relating to warrants if they added or modified any references to the mathematical relationships or rules of inference that connected evidence to claims. Warrants were usually typed descriptions of mathematical relationships or structure. Students' writing actions were classified as relating to backing if they added or modified any further support of their warrants. This occurred only once when a student further clarified their warrant about linear relationships with a reference to the group's definition of linearity. If the same writing action altered multiple structural elements of students' written justification, then it was classified according to each of the elements involved (e.g. if a student typed a claim and warrant in order without pausing at any point then the writing action was classified as related to both claims and warrants).

In addition to considering the structural components of justification addressed by each writing action, I consider how the specific writing actions that students took to construct written justifications throughout the task affected the validity of their justifications. I first coded whether the text that was added or modified for each writing action was mathematically accurate or not. Although this coding was helpful, it did not allow me to consider cases where students took writing actions to add support that did not relate to their claim (for example, by adding accurate evidence or warrants to inaccurate claims). As such, I also considered how the writing action affected the validity of students' overall written justifications. Writing actions were placed in three categories: (1) actions that maintained the same validity for the written justification, (2) actions that increased the validity of the written justification, or (3) actions that decreased the validity of the written justification. Moments when students were able to increase or decrease the validity of their written justifications were analytically important for considering the factors related to PDE within small group work that might support or inhibit students in constructing mathematically valid justifications.

Characterizing Localized PDE Preceding Students' Writing Actions

For each writing action I identified, I sought to establish time-based connections between students' specific writing actions during construction of written justifications and the localized nature of PDE during small group work that immediately preceded those actions. Although students' activity during any part of small group work could potentially influence their construction of written justifications, attending to all possible influences at the grain-size of students' unique utterances and actions throughout the entire task is not feasible for one researcher. As such, I utilized tools from quantitative ethnography (Shaffer, 2017) to limit the scope of small group work that I consider as related to each writing action. From preliminary data analysis of 9 tasks (which occurred prior to sampling and were not included in the data set for this study), I found that students did not typically reference events or talk from small group work that occurred more than 2 minutes previously. As such, I considered this a useful boundary for analysis. For localized analysis, then, I considered the localized nature of PDE in small group work that occurred in the 2 minutes leading up to a change in a written justification as social factors that potentially influence individuals' written justifications. As needed, I considered an event from small group work that occurred further back chronologically as a local influence on students' written justification if students explicitly referred to that event (e.g. by accessing a specific inscription that was created previously within the digital collaborative environment, by specifically mentioning previous words or actions). Because students have fairly permanent access to their previous inscriptions and written justifications within the environment, it was important to allow for these exceptions in order to describe how features of the digital collaborative environment might shape students' written justifications.

I transcribed all of the group's speech and actions within the digital collaborative environment (e.g. viewing peers' work, making own work visible to peers, creating a new graph or other inscription, modifying an existing inscription) during the two-minute period preceding each writing action. Transcripts of speech and other digital actions were coordinated with both focal students' writing actions in multiple columns of a common spreadsheet, in order to show all events within the group chronologically. For all transcribed speech and digital actions, I considered whether they showed evidence of any of the indicators of high or medium levels of PDE. While the PDE-CW framework describes the prevalence of certain behaviors across the entire task, students' speech and digital actions provided evidence that those behaviors occurred in localized moments. Thus it is possible, for example, for a group to demonstrate a low level of accountability on an overall task and still engage in localized instances of questioning and explaining their mathematical ideas, talking about the embedded mathematics in the own words, seeking out and considering peers' ideas, or other behaviors that show evidence of the high or medium level indicators of accountability.

Analysis of the Relationship Between PDE and Written Justifications

I used the results of the analysis described above to make comparisons between the level of PDE demonstrated by groups and students' written justifications at the level of overall tasks, and in terms of localized PDE and students' writing actions. By comparing the social factors of how groups demonstrate PDE during small group work and the individual factors of how students construct their own written justifications in that context, I sought to describe the relationship between PDE and written justifications. This included identifying behaviors within small group work that might influence the way that students construct written justification. This also included identifying writing actions that might influence the way that students continue to collaborate in small groups.

At the level of overall tasks, I compared the level of PDE demonstrated by the group with overall measures of the mathematical completeness and validity of students' written justifications. Groups demonstrated high, medium, and low levels of PDE across the data for this study. The mathematical completeness of students' written justifications ranged from "none written" to "mathematical descriptions" in terms of Kosko and Zimmerman's (2019) classification scheme. In terms of mathematical validity, students constructed written justification that were always valid, became mathematically valid by the end of small group work on the task, were never mathematically valid, or were not analyzable in terms of validity. As described in the Theoretical Framework (see Chapter 2), I predicted that when groups demonstrated higher levels of PDE, the students within those groups would write more mathematically complete and valid written justifications. I compared empirically based patterns from the data with these predicted patterns (similar to Yin's (2009) description of pattern matching) in order to examine the relationship between collaborative group work and construction of written justifications. In my results I attend to the ways that empirically based patterns matched with predicted patterns, as well as instances where they did not match in order to identify specific factors of students' work with their groups in the digital collaborative environment that affect the relationship between PDE and written justifications.

At the level of localized analysis, I focused on specific types of writing actions and the evidence of each indicator of PDE that preceded those writing actions. For each writing action, I used transcripts of the group's speech and digital actions to complete counts of how many times students demonstrated evidence of high or medium levels of each of the 15 indicators from the PDE-CW framework in the two minutes preceding that writing action. I used these counts to produce a list of indicators that were demonstrated preceding each writing action (that is, were present as localized influences on that writing action). This allowed me to identify information about how specific indicators and writing actions were related across the overall data set by comparing the prevalence of each indicator of PDE as an influence on different types of writing actions and to the full collection of all writing actions. This also allowed me to identify all cases that showed particular relationships between indicators of PDE and specific writing actions for deeper qualitative study to identify patterns in how these relationships occurred. Both of these analysis methods are drawn from quantitative ethnography (Shaffer, 2017), although I do not use quantitative ethnographic methods of comparing across raters in this single-rater study. In terms of mathematical completeness, I attended to writing actions that concerned students' claims, evidence, and warrants/backing. In terms of mathematical validity, I attended to students' writing actions that increased or decreased the validity of their written justifications.

As I gained familiarity with the data set, I identified two additional types of writing actions that indicated students were encountering uncertainties about written justification. One type of uncertainty I observed was students' uncertainty about how to justify in written form, in which they indicated that they saw a need to write further justification by typing a placeholder for some part of their justification, but then paused and did not continue to type. For example, a student might type, "The scale factor is 5 because" and then pause, unsure of how to continue. By typing "because" the student has indicated that they see a need to include either evidence or a warrant, but at this time they do not include any mathematical details of the evidence or warrant. The other type of uncertainty I observed was students' uncertainty about how to revise their written justifications, in which students attempted to take "out-of-order" writing actions that modified or added to previously-written content rather than simply appending new content to the end of the current written justification. Although students may have encountered other types of uncertainties about written justification, they did not necessarily verbalize or write about them. The two categories to which I attend have observable markers of uncertainty in terms of students' written work that I used to identify these important analytical moments. Because uncertainties are central to students' problematizing as part of PDE and central to their attempts to construct complete, valid written justifications, I also completed comparative analysis for these two types of writing actions. I examined the indicators of PDE that were present preceding writing actions as described above, and I also attended to the nature of the group's PDE as students sought to resolve their uncertainties about written justification.

Methods of Achieving Trustworthiness

In the current study I was the only researcher who conducted analysis (although the data from this study were embedded in the context of a larger research project). As such, establishing measures of inter-rater reliability for measures of PDE and completeness of written justifications was not particularly applicable to the analytical stages of this work. Because a single-rater design minimizes many reliability concerns for coding, the most critical part of analysis on which to address trustworthiness was the description of connections between PDE and written justifications. To avoid making inferential leaps, I shared early stages of this analysis with colleagues and mentors who are familiar with research on PDE and/or familiar with mathematical justification (particularly in written form). This exposed important connections that I did not attend to early in my analysis process. Based on their feedback, I expanded my analysis to attend to students' digital actions as part of collaborative work to a greater degree. Additionally, as I share results I present several extended examples of how I analyzed data and include problematic cases that can illustrate the nuanced and often messy connections between students' PDE and their written justifications (similar to methods of achieving trustworthiness in qualitative research described by Hammer & Berland (2014)). I include these transcripts and extended examples to make my analysis process more transparent to other researchers who review the data or findings.

Summary of Methods

This study incorporates data from a larger grant project to build a digital collaborative environment that can support students in mathematical work on open problems in order to address the relationship between students' processes to construct written justifications and the nature of their small group work. To this end, I examine the connections between the ways that groups demonstrate PDE during small group work and students' work to construct written justifications. I analyzed the connections between these two constructs at the general level of engagement on overall tasks, and at the localized level of students' specific writing actions. Because I conducted multiple layers of analysis and utilized multiple analytical frameworks to examine students' work individually and in small groups, I have summarized the key terms for analysis in Appendix B. In the following three chapters, I discuss the relationship between PDE and written justification in terms of three aspects: mathematical completeness, mathematical validity, and uncertainties about written justification.

CHAPTER 4:

RESULTS ON THE RELATIONSHIP BETWEEN PDE AND MATHEMATICAL COMPLETENESS OF WRITTEN JUSTIFICATIONS

In this chapter I first present an overview of Chapters 4, 5, and 6, which detail the results related to the three research sub-questions for this study. I then turn to results on the connections between PDE during small group work and the mathematical completeness of students' written justification, which I attend to at the level of overall tasks and students' specific writing actions.

Overview of Results Chapters

In the next three chapters, I present the results of data analysis pertaining to the research question: *What is the relationship between middle grades students' PDE during small group work and their construction of written justifications in a collaborative mathematics learning environment?* To answer this question, I utilized screen recordings of students' small group work as they worked to complete open mathematical tasks and construct written justifications, all in the context of a digital collaborative environment. This offers insight into the social perspective of their small-group work through the lens of PDE and the individual perspective of their written justifications, as well as interactions between these two perspectives. I conducted analysis at multiple levels, to investigate both how PDE might relate to students' overall written justifications and how localized PDE might relate to students' specific writing actions as they construct written justifications.

I will attend to three related sub-questions in Chapters 4, 5, and 6 respectively:

1. *What is the relationship between middle grades students' PDE during small group work and their construction of written justifications in a collaborative mathematics learning environment in terms of mathematical completeness?*

2. *What is the relationship between middle grades students' PDE during small group work and their construction of written justifications in a collaborative mathematics learning environment in terms of mathematical validity?*
3. *How do middle grades students seek to resolve uncertainties about written justification during small group work in a collaborative mathematics learning environment?*

At the most general level of analysis, I considered students' PDE on each overall task and the written justifications that students constructed during those tasks. Across the data set, I analyzed 38 tasks and found that groups of students demonstrated low levels of PDE in 10 tasks, medium levels of PDE in 14 tasks, and high levels of PDE in 14 tasks. I examined the digital work and written justifications of two students for each group, which yielded 76 opportunities for students to construct a written justification. All tasks and written justifications were analyzed to determine how the overall level of PDE demonstrated by a group might influence the mathematical completeness or validity of students' written justifications for that task.

As described in the Methods chapter, I conducted localized analysis for 24 tasks where the prompt as written and set up by the teacher asked students to "justify", "explain why", or "give reasoning", rather than only prompting students to create an inscription or provide a claim. Although creating inscriptions and determining/investigating claims are important parts of the process of mathematical justification, this analysis focuses on students' writing actions as they construct written justifications. As such, in my localized analysis I focus on tasks in which students were explicitly prompted to produce written justifications. To isolate the different steps that students took as they constructed written justifications, I identified *writing actions*: moments when students began, added, modified, or deleted elements of their written justifications. In order to identify the ways that the nature of PDE influenced students' writing, I considered indicators of the four principles of PDE demonstrated by the group prior to any writing actions. Across the 24 tasks for localized analysis, I observed 163 instances where students took writing actions. These included changes to specific elements of the written justification (i.e. claims, evidence, warrants, or backing) that impacted the completeness of the written justification and changes that impacted the validity of written justifications. As described in Methods, I completed counts of all indicators of the four principles of PDE that emerged in the two minutes preceding each writing action. I then compared the prevalence of each indicator in cases where students' writing actions concerned claims, evidence, or warrants/backing to the prevalence of those same indicators preceding writing actions overall. By looking across categories of writing actions, I sought to determine how localized features of PDE within the group might influence how students engaged in the process of constructing mathematically complete and valid written justifications.

The Relationship Between PDE During Small Group Work and the Mathematical Completeness of Students' Written Justifications

When students participate in mathematical justification, they go beyond providing isolated mathematical statements and begin to make sense of mathematics as they explain how or why the mathematics works (Bieda & Staples, 2020). For students' written justifications, I consider the extent to which they explained how and why the mathematics works in terms of mathematical completeness. As students incorporate mathematical evidence, warrants, or backing to support mathematical claims, their written justifications are considered more complete (although in the context of this study there is not some "ideal" state of completeness to which students are aiming). In this chapter I attend to the first research sub-question: *What is the relationship between middle grades students' PDE during small group work and their* *construction of written justifications in a collaborative mathematics learning environment in terms of mathematical completeness?* To answer this, I focus first on examining overall qualities of students' written justifications in connection to the nature of their collaborative group work throughout the task, and second, on the specific writing actions that students took to address the mathematical completeness of their written justifications within localized PDE.

Mathematical Completeness of Students' Overall Written Justifications

As described previously in the chapter on Methods, I classified student's written justifications according to Kosko and Zimmerman's (2019) classification scheme for mathematical writing. I considered students' work towards producing mathematical claims and whether their justification addresses how or why the mathematics works, all crucial aspects of justification. The classification of all written justifications, as well as those produced in groups demonstrating low, medium, and high levels of PDE is shown on Table 3 below.

Table 3:

Classification of Students' Written Justifications During Small Group Tasks in Which the Group Demonstrated Low, Medium, and High Levels of PDE

	WHAT: Mathematical statements or other mathematical work			**HOW:** Written justifications that explain **how** the mathematics works		**WHY:** Written justifications that explain **why** the mathematics works		
	None written	Emergent	Statement	Recount	Procedural	Detailing	Descriptions	Total
Low PDE	4 (20%)	2 (10%)	9 (45%)	4 (20%)	1 (5%)	0 (0%)	0 (0%)	20 (100%)
Medium PDE	8 (28.6%)	1 (3.6%)	9 (32.1%)	5 (17.9%)	3 (10.7%)	2 (7.1%)	0 (0%)	28 (100%)
High PDE	9 (32.1%)	5 (17.9%)	2 (7.1%)	6 (21.4%)	3 (10.7%)	2 (7.1%)	1 (3.6%)	28 (100%)
All Tasks	21 (27.6%)	8 (10.5%)	20 (26.3%)	15 (19.7%)	7 (9.2%)	4 (5.3%)	1 (1.3%)	76 (100%)

As the level of PDE observed during small-group work increases, students are more likely to go beyond providing mathematical statements and begin to incorporate explanations of how and/or why the mathematics works to their written justifications. Written justifications were classified as mathematical statements or other mathematical work for 75% of tasks where groups demonstrated low levels of PDE, for 64.3% of tasks where groups demonstrated medium levels of PDE, and for 57.1% of tasks where groups demonstrated high levels of PDE. Written justifications addressed *how* for 25% of tasks where groups demonstrated low levels of PDE, for 28.6% of tasks where groups demonstrated medium levels of PDE, and on 32.1% of tasks where groups demonstrated high levels of PDE. Written justifications never went beyond this to address *why* the mathematics works for tasks where groups demonstrated low levels of PDE, but they did so for 7.1% of tasks where groups demonstrated medium levels of PDE and for 10.7% of tasks where groups demonstrated high levels of PDE. In short, students constructed a higher number of written justifications overall as the level of PDE demonstrated by their groups increased. In the context of this study, high levels of PDE seemed to foster students' construction of mathematically complete written justifications to the greatest degree, but even medium levels of PDE within small group work helped students to construct more mathematically complete written justifications that go beyond providing a claim. In contrast, low levels of PDE during small group work did not seem to have the same effect of supporting students to construct mathematically complete justifications.

When we consider the specific indicators of the PDE-CW framework, high quality PDE is characterized to a large extent by meaningful attempts to engage in mathematical justification during small group work. Because the PDE-CW framework includes several indicators that relate to students' mathematical justification with their peers (for example, indicators Au1, Ac1, Ac2, and Ac4), by definition, groups that demonstrate high levels of PDE are engaged in oral justification to a greater extent than groups that demonstrate lower levels of PDE. In this sense, Table 4 shows that as students engaged in oral justification to a greater extent (that is, produced more mathematically complete oral justifications), the mathematical completeness of their written justifications increased. This is similar to the findings of Stylianides (2019) and Campbell and colleagues (2020), who found that as the quality of students' oral justifications increased, so did the quality of their written justifications (although I do not explicitly compare the quality of justifications across modalities as these studies do). This association between the level of PDE demonstrated by groups of students and the mathematical completeness of individual students' written justifications was expected given my theoretical framing that social ways of doing mathematics might shape individual ways of doing mathematics. Because of this association, my findings suggest that tasks which have the potential to foster high levels of PDE also have the potential to foster high levels of mathematical completeness in students' written justifications. Further, instructional moves that support students to develop increased levels of PDE during small group work might also support students to write more mathematically complete written justifications.

Although in this study students were more likely to write justifications that explain how and why the mathematics works on tasks where their groups demonstrated high levels of PDE, this was not guaranteed. In fact, on tasks where groups demonstrate high levels of PDE, students' work towards a written justification in the digital environment was most likely to be classified as "none written" (32.1% of cases where groups demonstrate high PDE, compared to 28.6% and 20% of medium and low PDE cases). These represent cases where students are likely engaging in high quality oral justification (as evidenced by the group's demonstrated PDE), but for some reason students have not yet begun to construct a written justification. Examining these cases more closely might offer insight into recent findings that students' oral justifications are typically more complete than their written justifications (Campbell et al., 2020; Stylianides, 2019).

One factor that might have influenced students to engage in oral justification prior to constructing a written justification is the nature of the curricular prompt. The kinds of mathematical activity (i.e. justification, calculation, producing inscriptions) that are specified in the wording of the curricular prompt could relate to the ways that students approached the task. There were six different tasks in which the group demonstrated high levels of PDE but one or more students in the group had not began to type a written justification. The curricular prompts for these tasks are shown on Table 4 below. For the Initial Challenge of *Moving Straight Ahead* 4.3, the teacher requested all students in the group to begin working on the same computer in the middle of the group's work on the task. Hence, rather than some feature of the group's PDE, the shift midway through the task resulted in one group member not constructing their own independent written justification. The remaining five tasks were distributed across teachers and across two different Investigations, with three tasks taking place during Ms. Price's instantiation of *Stretching and Shrinking* Investigation 3 (which focuses on scale factor for proportional relationships) and two tasks taking place during Ms. Roberts's instantiation of *Moving Straight Ahead* Investigation 2 (which focuses on the structure of linear relationships). Although the two Investigations occur in slightly different mathematical contexts, the tasks shown on Table 4 involve similar types of mathematical activity. First, all tasks specify that students should "explain" or "support" their reasoning or answer with a written justification. Additionally, these tasks often specify that students should work to create and interpret mathematical inscriptions.

This emphasis on mathematical inscriptions is present in the wording of most prompts and was often emphasized by the teacher as they set up the task during instruction.

Table 4:

Curricular Prompts for Tasks on Which Students Demonstrated High Levels of PDE but did not Begin to Construct Written Justifications.

Problem	Curricular Prompt
Stretching and Shrinking 3.2, Initial Challenge Part 1	For each part, draw a rectangle similar to rectangle A that fits the given description. Explain your reasoning. - Rectangle P: The scale factor from rectangle A to rectangle P is 2.5. - Rectangle R: The area of rectangle R is 1/4 the area of rectangle A.
Stretching and Shrinking 3.2, Initial Challenge Part 2	For each part, draw a triangle similar to triangle B that fits the given description. Explain your reasoning. - Triangle X: The area of triangle X is 16 times the area of triangle B. - Triangle Y: The scale factor from triangle B to triangle X is 1/2. - Triangle Z: The ratio of corresponding side lengths of triangle B to triangle Z is 2:3.
Stretching and Shrinking 3.3, Initial Challenge	What is the distance across the river from Stake 1 to Tree 1? Explain your reasoning.
Moving Straight Ahead 2.1, Initial Challenge	How long should the race be so that Henri will win in a close race? Give evidence to support your answer.
Moving Straight Ahead 2.3, Initial Challenge	What information does the equation give about the pledge plan? Does the plan make sense? Explain your reasoning. How might a table or a graph or the equation help answer questions about the pledge plans?
Moving Straight Ahead 4.3, Initial Challenge	Each situation provides criteria for a relationship. For each situation, decide if the relationship could be linear. For those that are linear, is there exactly one or more than one? Explain. - Situation: A graph that passes through exactly three quadrants.

As written, the prompts for the Initial Challenge of *Stretching and Shrinking* 3.2 and *Moving Straight Ahead* 2.3 call for students to consider particular mathematical inscriptions (e.g. drawings, tables, graphs, equations) and the prompt for the Initial Challenge of *Stretching and Shrinking* 3.3 references features of a particular diagram given in the problem that students must interact with. Although the written prompt for the Initial Challenge of *Moving Straight Ahead* 2.1 does not reference particular types of inscriptions that students should create, Ms. Roberts launched the task with students by specifying, "Your evidence should be a table or a graph" and demonstrating how to use those tools within the digital environment. As such, all of the tasks were either written or set up by the teacher to require students to create some kind of mathematical inscription as part of their work. Although students had the opportunity to examine or create mathematical inscriptions on other tasks as well, these prompts show an increased level of specificity, namely that students should create the inscriptions themselves or that students should use particular types of inscriptions.

Although the wording of the curricular tasks specified that students should produce and make sense of inscriptions, it is important to consider whether this was taken up by students in their small group work. I found that in each group's work on the tasks shown in Table 4, students frequently engaged in making sense of mathematical relationships and inscriptions. In each of these cases, the group demonstrated high levels of problematizing and/or high levels of resources. These features of the groups' PDE indicate that students were often engaged in making sense of mathematical ideas, trying to make sense of or build mathematical inscriptions, and/or using inscriptions as a resource for grappling with complex mathematical problems. These behaviors are a potentially important part of the process of justification, as they can give insight into mathematical claims and relationships (Hanna, 2000; Staples et al., 2012). But for these tasks in particular, students verbally indicated that they did not know how to translate their sense-making experience within the digital environment into a written justification (e.g. "I already showed how [the walking rate] goes up on the table, I just don't know how to explain it" and "I'm stuck. Like, I know what I did on the picture, but I'm stuck on the words"). These students demonstrated that they valued mathematical inscriptions as a resource for PDE with their actions during small group work, but at the same time it was not trivial for them to use inscriptions as resources towards constructing written justifications. So although my findings suggest that higher levels of PDE can be influential towards students' construction of more mathematically complete written justifications, students may require additional support in order to interpret and unpack mathematical inscriptions during the process of constructing written justifications.

In this section I have presented results on PDE at a fairly broad level, where the nature of a group's PDE throughout an entire task is considered as an influence on students' written justifications for that task. I found that when groups demonstrated higher levels of PDE, students were more likely to go beyond providing mathematical statements, to also address how and/or why the mathematics works as part of their written justifications. To better understand *how* this effect occurs as students are engaged in the writing process, I also conducted more fine-grained analysis of students' words and actions as they made changes to their written justifications within the digital environment. In what ways does this localized context of PDE within collaborative work influence students' processes as they work to construct mathematically complete written justifications? I explore this in the following section.

Students' Writing Actions to Address the Mathematical Completeness of Their Written Justifications

In addition to the completeness of students' overall written justifications, I examined the specific writing actions that students took to construct written justifications throughout the task. All of the written justifications considered in this study were constructed through multiple different additions, modifications, and/or deletions rather than being typed all at once, so considering the nature of these specific writing actions gives insight into students' processes. In general, students typed the parts of their written justifications in the same order that those parts appeared in the final form. 134 out of 163 total writing actions (82.2%) can be described as "in-order" actions, where students either began written justifications or added to their written justifications without altering any previously written material. I also observed 29 instances (17.8% of 163 total writing actions) where students took "out-of-order" writing actions that modified or added to previously written content and as a result, students' typing order did not match the order of the final form of the written justification. In-order and out-of-order writing actions had the potential to contribute to the mathematical completeness of students' written arguments through the development of claims, evidence, warrants, and/or backing.

As described in Methods, I classified all changes to written justifications based on which component of the written justification (i.e. claims, evidence, warrants, or backing) they were related to. Because backings were infrequent in students' written justifications and are most closely connected to warrants in terms of structure, I have consolidated students' changes to warrants and backings within the same group for data analysis. If the same writing action altered multiple structural elements of students' written justification, then it was classified according to each of the elements involved (e.g. if a student typed a claim and warrant in order without pausing at any point then the writing action was classified as related to both claims and warrants).

To give insight into how social factors within collaborative group work might relate to students' written justifications, I considered the localized nature of PDE demonstrated by the group in the 2 minutes preceding any writing actions. So for each writing action, I conducted analysis of the indicators of the four principles of PDE that I observed in the group's mathematical conversation and actions within the digital environment. I considered the indicators of PDE that were observed prior to writing actions as social factors within small group work that potentially influence individuals' written justification. This allowed me to examine how the emergence of different indicators of PDE during small group work connects to students' writing actions as they construct written justifications.

To explore how the collaborative environment influenced the completeness of students' written justifications, I sought to address the question: What behaviors at the group level influenced students to engage with the different structural components of their written justification? Table 5 and 6 present findings regarding the indicators that were present when students made changes to their written justifications overall, as well as those that were present when students made changes to claims, evidence, and warrants/backing. Table 5 describes the overall occurrence of indicators of problematizing and authority (see PDE-CE Framework, Table 2) during small group work that preceded changes to written justifications, and Table 6 describes the overall occurrence of indicators of accountability and resources during small group work that preceded changes to written justifications.

Table 5:

Frequency of Indicators of Problematizing (P) and Authority (Au) Preceding Students' Writing Actions to Address Claims, Evidence, and Warrants/Backing

	P1	P2	P3	P4	Au1	Au2	Au3	Au4
All writing actions (n=163)	41 (25.2%)	54 (33.1%)	61 (37.4%)	52 (31.9%)	65 (39.9%)	61 (37.4%)	110 (67.5%)	66 (40.5%)
Writing actions addressing claims (n=84)	22 (26.2%)	33 (39.3%)	33 (39.3%)	23 (27.4%)	39 (46.4%)	33 (39.3%)	55 (65.5%)	31 (36.9%)
Writing actions addressing evidence (n=67)	12 (17.9%)	17 (25.4%)	23 (34.3%)	26 (38.8%)	26 (38.8%)	18 (26.9%)	44 (65.7%)	27 (40.3%)
Writing actions addressing warrants or backing (n=40)	15 (37.5%)	18 (45%)	15 (37.5%)	15 (37.5%)	14 (35%)	20 (50%)	28 (70%)	20 (50%)

On Tables 5 and 6, Pn, Aun, Acn, and Rn refer to specific indicators of problematizing, accountability, authority, and resources from the PDE-CW framework. For example, P1 refers to the first indicator of problematizing. Because it was possible for changes to affect multiple elements of the justification at the same time (i.e. belong to more than one of the categories in rows 3-5), the percents within the results for each indicator (i.e. a column) might not add up to 100. Percents are calculated based on the total number of writing actions within that row. For example, in column 2 (P1), row 3 (Claims), the percent is 26.2% = 22/84, which represents that 26.2% of students' changes to claims were preceded by evidence of indicator P1.

As I attended to how each indicator presented during students' writing actions to include claims, evidence, and warrants/backing in their written justifications, I made comparisons to the frequency of each indicator as an influence on students' writing actions overall. Since I did not do fine-grained analysis of students' words and actions during moments when they were not actively making changes to their written justifications, these frequencies do not allow me to make comparisons between moments when students decided to make written changes and moments when they did not. Rather, these overall frequencies serve as a way to consider how students generally use the digital environment when they are engaged in the process of constructing written justifications, as compared to how students use the digital environment when they are engaged with specific structural parts or written justifications (i.e. claims, evidence, warrants, backing). I observed evidence of each of the indicators preceding at least 25% of the changes to students' written justifications, but two indicators (Au3 and R2) were particularly common and were observed preceding at least 50% of the changes to students' written justifications.

Evidence of the indicator Au3 (*Students often seek out, consider, or respond to peers' mathematical ideas)* was observed preceding 67.5% of all changes to students' written justifications. This indicates that it was common for students in this study to engage with peers' ideas verbally or digitally during the process of constructing written justifications. As shown on Table 5, evidence of Au3 was observed with similar frequency regardless of which structural element of justification (i.e. claims, evidence, warrants/backing) students were concerned with at the time. This overall frequency suggests that social norms within small groups and features of the digital environment that support students to access and respond to peers' ideas are a potentially powerful influence on whether or not students engage in the process of constructing written justifications, but not necessarily on the specific types of actions students take to make their written justifications more mathematically complete.

Table 6:

Frequency of Indicators of Accountability (Ac) and Resources (R) Preceding Students' Writing Actions to Address Claims, Evidence, and Warrants/Backing

	Ac1	Ac2	Ac3	Ac4	R1	R2	R3
All writing actions (n=163)	58 (35.6%)	76 (46.6%)	47 (28.8%)	47 (28.8%)	56 (34.4%)	82 (50.3%)	46 (28.2%)
Writing actions addressing claims (n=84)	32 (38.1%)	38 (45.2%)	24 (28.6%)	24 (28.6%)	30 (35.7%)	39 (46.4%)	25 (29.8%)
Writing actions addressing evidence (n=67)	14 (20.9%)	29 (43.3%)	17 (25.4%)	14 (20.9%)	22 (32.8%)	26 (38.8%)	18 (26.9%)
Writing actions addressing warrants or backing (n=40)	20 (50%)	26 (65%)	9 (22.5%)	15 (37.5%)	15 (37.5%)	28 (70%)	14 (35%)

In Table 6, evidence of the indicator R2 (*Students often use relevant resources they create or choose*) was observed preceding 50.3% of all changes to students' written justifications. That is, it was common for students in this study to engage with resources that they had personally worked to make sense of during the process of constructing a written justification. This suggests that social norms within small groups and features of the digital environment that support students to create and interpret their own mathematical inscriptions are a potentially powerful influence on whether or not students engage in the process of constructing written justification. However, as shown on Table 6, evidence of indicator R2 did not occur with consistent frequency across students' engagement with different structural elements of their written justifications. So although the opportunity for students to create or choose their own mathematical inscriptions appears to be important for promoting overall engagement with written justification, it is also important to consider how these opportunities might influence students to address the mathematical completeness of their written justifications through attention to claims, evidence, and warrants/backing.

Students' Writing Actions to Address Claims

The same indicators that were prevalent preceding students' writing actions overall were also prevalent preceding students' writing actions to address claims, and writing actions to address claims generally occurred near the beginning of students' work to construct written justifications. In most cases, the first structural element that students attempted to add to their written justifications was a mathematical claim. Only four exceptions to this occurred across all students' construction of written justifications: one case where a student first typed a warrant ("Since we know that both triangles are similar") and three cases where students first typed evidence (e.g. calculations to show that suggested measurements would meet mathematical design constraints, descriptions of patterns they observed in numerical data). For each of these exceptions, students began to add claims to their written justifications immediately afterward (that is, the second change to their written justification involved mathematical claims). In addition to beginning claims early in the process of constructing written justifications, students also made other changes to claims within their written justifications, including adding additional claims, modifying existing claims, or deleting mathematical claims. Across the full data set for this study, students made 84 changes to mathematical claims within their written justifications.

So, did behaviors within the collaborative group work have a direct influence on the mathematical claims in students' justifications? In the data for this study, the prevalence of each indicator of PDE that preceded writing actions to address claims is approximately the same as the prevalence of the same indicators of PDE that preceded writing actions overall. As shown on

Table 5 and Table 6, the proportion of all changes that were preceded by each indicator differed by less than 7% from the proportion of changes to claims that were preceded by the same indicator. So although collaborative work can support students to write mathematical claims, I did not find evidence of any particular group behaviors (as shown by the occurrence of particular indicators of the PDE-CW framework) that specifically functioned to promote or inhibit students to write mathematical claims. This is not surprising, since students constructed written justifications that were classified as "mathematical statements" for tasks where groups demonstrated low, medium, and high levels of PDE (as shown in Table 3).

Across the data set, students made changes to claims within their mathematical justifications in several different ways: by typing claims to make initial conjectures near the beginning of small group work, by typing claims during small group work after deliberating with their group or investigating mathematical inscriptions, or by modifying their claims to increase readability (i.e. modifying spelling or conventions) or mathematical validity (i.e. modifying the mathematical claim itself). Students were able to take writing actions to address claims in a wide variety of contexts for collaborative work, but across the data for this study it did not appear that there was one specific pattern for the group's demonstration of PDE that supported students to make mathematical claims. For example, when students typed claims after investigating inscriptions, their writing action was preceded by evidence of the principles of problematizing and resources. In contrast, when students typed claims to make initial conjectures at the beginning of small group work, their writing action was preceded by little evidence of any of the indicators of PDE. The presence of typed claims within the group's digital environment, however, could potentially offer opportunities for students to engage with uncertainties about the validity of their claims or how to support their claims more completely with evidence and warrants if they view or discuss these written claims. So although I did not find evidence that particular types of PDE support or inhibit students from incorporating claims into written justifications, students' work to incorporate claims into their written justification might support increased levels of PDE within small group work or other subsequent changes to students' written justifications. For most of the students in this study, adding mathematical claims to their written justifications was a necessary first step before they could continue their work to explain how or why the mathematical claim was valid by incorporating evidence, warrants, or backing.

Students' Writing Actions to Address Evidence

Students' writing actions to address evidence within their written justifications generally occurred later in collaborative work than their writing actions to address claims. As described above, there were three instances where students began constructing written justifications by typing evidence and in all other instances students worked to incorporate evidence once they had already typed a specific claim. Across the full data set for this study, I observed 67 writing actions concerning evidence within their written justifications. It is important to note that I only considered evidence that students explicitly included as part of their written justification, either by typing the evidence as text within their written justification or by making a typed reference to some other inscription as evidence (e.g. "You can see it on my table", "The pattern on the graph is..."). In cases where students worked in the digital environment to create/interpret mathematical inscriptions but did not explicitly include them in written justifications, this mathematical work is represented in my description of the nature of each group's PDE.

The localized nature of students' work in their small groups did not appear to have much influence on their use of evidence in their written justifications. Students' changes to evidence were preceded by fewer indicators of PDE than students' changes to written justifications overall. For 14 of the 15 total indicators of PDE (consisting of 4 indicators of problematizing, 4 indicators of authority, 4 indicators of accountability and 3 indicators or resources, see PDE-CW Framework, Table 2), the proportion of writing actions to address evidence that were preceded by the indicator (Row 3, see Table 5 and Table 6) was equal to or less than the proportion of writing actions overall that were preceded by the same indicator (Row 1, see Table 5 and Table 6). As the only exception, I observed evidence of P4 (*Students' mathematical uncertainties concern what their conclusions mean, competing alternatives, or justifying their actions*) in 38.8% of changes to evidence, and in only 31.9% of changes to written justifications overall. In general then, students were slightly more likely to grapple with mathematical uncertainties about how to justify their thinking or the meaning of their conclusions when making changes to evidence, but less likely to demonstrate all other indicators of PDE.

Although most indicators were slightly less prevalent preceding writing actions concerning evidence than they were preceding writing actions overall, three indicators showed more dramatic differences (greater than 10%) between types of changes. Evidence of Au2 (*Students often talk about the mathematics topic/problem in their own words*) was observed preceding 26.9% of writing actions concerning evidence, as compared to 37.4% of all writing actions. Evidence of Ac1 (*Students question each other and explain their mathematical contributions until the idea makes sense to everyone in the group*) was observed preceding 20.9% of writing actions concerning evidence, as compared to 35.6% of all writing actions. Finally, evidence of R2 (*Students often use relevant resources they create or choose*) was observed preceding 38.8% of writing actions concerning evidence, as compared to 50.3% of all writing actions.

The comparably low occurrence of these indicators within small group work indicates that when students took writing actions to include evidence in their written justifications, they were less likely to describe mathematical ideas in their own words, more likely to accept peers' mathematical contributions blindly, and more likely to use peers' inscriptions or inscriptions from the problem statement without doing their own work to interpret them. During small group work that preceded writing actions to include evidence, students often verbally or digitally shared evidence with their peers and cooperated to collect evidence (e.g. graphs, lists of measurements) that the group could use. In general, though, this was where the conversation stopped. Students cooperated to complete the task by sharing evidence (e.g. inscriptions they had created, their observations or data), but they did not appear to collaborate to develop shared meaning (Staples, 2007) for that evidence within the group.

As an example, consider the work of a group of four students in Ms. Price's classroom to complete the Initial Challenge for *Stretching and Shrinking* 4.1, in which students are asked to identify similar pairs of rectangles and similar pairs of triangles and explain why they are similar. For the first several minutes of the task, students mentioned potential similar pairs of shapes to their peers, who responded and explained the reason for their response (e.g. "Yeah, that one works. The ratios for the sides would be the same.", "They kind of do, but they kind of don't. And they don't have the same angles. So that's a no."). As they discussed similar shapes with peers, multiple students in the group typed these as claims in their own digital environment. Before each of the writing actions to address claims, the group demonstrated indicators of all four principles of PDE by consistently collaborating with peers, explaining their mathematical thinking, and working to interpret resources provided within the task. But as members of the group began to take writing actions to address evidence, the localized nature of their PDE shifted to include less deliberation and explanation. Table 7 shows the conversation of two students in the group as they sought to incorporate evidence, as well as one student's writing actions. Note that only one student's writing actions are included because only one of the students who participated in the conversation shown was selected as a focal student for analysis of written justification. In this and future transcripts, I refer to all students by pseudonyms.

Table 7:

Transcript of Student Dialogue and Writing Actions to Address Evidence During Small Group Work on Stretching and Shrinking 4.1

Speaker	Spoken words	Sayuri's Writing Actions
Leah	So the ratio would be...	
Sayuri	20 to 12.	
Leah	Then 20 to 6.	
Sayuri	The ratios are 20 to 12 and… and what? 15 to 9?	Adds evidence: "The ratios are 20/12"
Leah	20 to 6.	
Sayuri	20 to 6? Wait, wait, wait. 20 to 6? No, 20 to 12.	
Leah	We did 20 to 12.	
Sayuri	Okay, hold on. 20 to 12 and… where's the button? And what?	Modifies evidence: "The ratios are 20:12"
Leah	20 to 6.	
Sayuri	And 20 to 6. Okay, now I need a calculator to find the decimals.	Adds evidence: "and 20:6"

Although these students are sharing evidence with each other and taking writing actions to include evidence in their written justifications, they do not collaborate to explain its meaning or describe their thinking about it. Sayuri does make an attempt to question whether they should include particular evidence (see Table 7, Row 7), but this is not meaningfully addressed by peers and they do not continue to question. As group work progressed over the next two minutes, another student in the group viewed Leah and Sayuri's written justification in the collaborative view and typed this evidence into their own written justification. As they did this, they typed Sayuri's evidence exactly as it appeared, without verbally addressing peers or further examining the graph from which the evidence was determined. Although the group demonstrated medium levels of PDE when considering the task as whole, I observed very few indicators of PDE during localized instances where students took writing actions concerning evidence. If this group had more thoroughly explained their thinking or interpreted evidence through their own mathematical words or use of resources, it is possible they would have realized that these two rectangles cannot possibly be mathematically similar (as they do not show the same ratio between side lengths). In this group's case, "copying" evidence verbatim from peers (either from spoken words or the digital environment) without also seeking to question or explain that evidence was not mathematically productive and contributed to lower localized levels of PDE within the group.

It is important to note that copying from peers' work in the digital environment was not inherently unproductive, but rather that copying was unproductive for students when they did not also make attempts to interpret the material they copied. Across the full data set, I observed instances where viewing or copying from peers' digital environments was mathematically productive for students as they constructed their own written justifications. This occurred, for example, in situations where students viewed peers' work to compare two different inscriptions or written justifications or when students copied inscriptions or written justifications from peers and continued to modify them in their own workspace. But when copying occurred during students' writing actions to address evidence, it was less often preceded by attempts to explain mathematical thinking or meaningfully engage with the digital resources being copied.

These results suggest that students might not view evidence as something that they even *need* to explain. Because evidence in this study often took the form of graphs, tables, measurements from diagrams, or typed calculations, there is the potential for students to consider that this information will be visually interpretable by others to infer the same meaning. In other words, students may view that inscriptions "speak for themselves." This is particularly important when considering written justification, because written justifications are more likely to be read and interpreted by individuals outside of the small groups in which those justifications were constructed. Even in cases where I observed groups making attempts to orally interpret inscriptions, however, it was rare for students to add these details to their written justifications.

As an example, consider the work of four students in Ms. Price's classroom to complete the Initial Challenge of *Moving Straight Ahead* 2.1, in which students are asked to compare the walking rates for two brothers to set up a race that the younger and slower brother can win (with a 45-meter head start, that is). At the beginning of the group's work on this problem, they worked to produce graphs and tables that showed the two brothers' distances at various points in time during the race. Table 8 shows the conversation of all students in the group as they began to discuss their tables and graphs with each other and eventually the teacher. Note that during this conversation, the students in the group did not make any changes to their written justifications.

Table 8:

Transcript of Student Dialogue During Small Group Work on Moving Straight Ahead 2.1

Speaker	Spoken words
Kevin	Dang it, I can tell they intersect there. Right there, at 47 seconds.
Jae-hwa	47? I got 72.5.
Kevin	What?
Jae-hwa	(looking back to their computer) Wait, I might have...

Table 8 (cont'd)

Kevin	No, that's how long the race should be.
Jae-hwa	Yeah.
Kevin	But when do they intersect?
Jae-hwa	(looking at their graphs of walking rate, then hovering their cursor over their tables of walking rate) Oh! If you do it at 30 seconds, they're both 75. And if you do 29, this person will win by 2 seconds.
Lily	(softy to themselves) 28.6… (addressing the group) Okay guys, I think I got it.
Kevin	Which one?
Jae-hwa	(overlapping speech) What did you get?
Lily	Wait, what did you guys get? Let me just listen to you guys (laughs).
Jae-hwa	(overlapping speech) I got seventy… (cuts off suddenly, laughing)
Ms. Price	(approaching the group) How long do we think it's going to be? 70? Is that what you said?
Kevin	Yeah, around 70.
Lily	Yeah, 70. That's what I got.
Ms. Price	You've got something similar?
Lily	No… we could *do* 74, technically.
Jae-hwa	Yeah, I got 74! Good job! (reaches out for a high five)
Lily	That's like, when they… (trails off)
Robert	They match.
Lily	That's right when they do it.
Ms. Price	So they would tie at 74?
Jae-hwa	Oh! We're not supposed to… (trails off) So like, 30 seconds.
Lily	(overlapping speech) No, no, no. 75 is when they tie. 74 is like, 29.6
Ms. Price	(overlapping speech) Ohhhh… So they'd be really close.
Robert	So around 74.

During this conversation, all students in the group contributed their mathematical ideas about how to use graphs and tables to give evidence for their claim that the total distance for the race needs to be 70-74 meters for the younger brother to win. These students did not offer the entire graph or table as evidence of this claim to their peers and the teacher, rather they discussed how particular features of the graph gave evidence. In this case, students verbally highlighted the importance of the intersection point between lines and described its meaning in the context of a race. During collaborative activity where they orally justified, then, these students demonstrated that they saw a need to do further mathematical work to articulate *how* graphs and tables provided evidence rather than letting the graph speak for itself. After this conversation, the group decided that 73 meters would be an appropriate distance for the race and began to type written justifications. Table 9 shows their conversation at this point, as well as one student's writing actions. Note that only one student's writing actions are included because the other focal student for this group had not yet begun to type a written justification at this point.

Table 9:

Transcript of Student Dialogue and Writing Actions During Small Group Work on Moving Straight Ahead 2.1 (Part 2)

Speaker	Spoken words	Jae-hwa's writing actions
Lily	73, right? Yes?	
Jae-hwa	Yes. (pause) Can I just put because and just, like, look at the graph?	Adds claim and placeholder for evidence: "The race should be 73 meters long because"
Lily	Yeah, because the graph already shows it.	
Kevin	Our evidence shows it.	
Jae-hwa	Because look at the graph. I mean, it's a valid answer.	Adds evidence: "The race should be 73 meters long because look at the graph."

The way that these students talked about their inscriptions when making changes to their written justifications is quite different from the way that they talked about inscriptions during their previous attempts to orally justify (see Table 8). This difference continued when the other focal student in the group typed their written justification slightly after this exchange ("The race should be 73 meters long. Because look at the tables and graphs."). These students knew how to *talk* about evidence during collaborative work, but they did not demonstrate a similar awareness of how to *write* about evidence. I also observed this pattern in other groups, where students did collaborative work to unpack the meaning or their interpretation of graphs, tables, diagrams, and other inscriptions, but let the inscriptions speak for themselves in their written justifications (e.g. typing "I made a graph" or "The picture shows it").

Additionally, the resources or tools that students have access to as they engage in written justification can shape how they take writing actions to incorporate evidence. Although prior research on dynamic geometry environments has shown that the affordances of the digital environments in which students do mathematical work can shape their oral justifications (e.g. Baccaglini-Frank & Mariotti, 2010; Patsiomitou, 2011), in this study students also used particular features of the digital collaborative environment in their attempts to construct written justifications. For example, in the digital collaborative environment students had the option to comment directly on features of their graphs. Several students used this option to type their mathematical claims directly onto their graphs, which explicitly linked claims to evidence. This occurred during students' work on the Initial Challenge of *Stretching and Shrinking* 3.1, where students created similar shapes called "rep-tiles" by arranging multiple copies of a small shape. On this problem, students used the comment feature to add mathematical claims directly to their evidence (e.g. "The parallelogram rep-tiles" or "This trapezoid is not similar"), although students did not unpack the meaning of evidence further than this. In this case, the comment feature in the digital collaborative environment afforded students the opportunity to extend their written justifications across multiple different inscriptions and show how they interpreted these inscriptions.

Students' Writing Actions to Address Warrants or Backing

Whereas students' writing actions to address evidence often followed lower occurrence of localized PDE, students' writing actions to address warrants or backing often followed greater occurrence of localized PDE. Students' writing actions to address warrants or backing within their written justifications generally occurred later in collaborative work than their writing actions to address claims (with only one exception where a student began their written justification by typing a warrant). The order of students' writing actions to address evidence and their actions to address warrants/backing was more mixed, with both types of writing actions occurring at multiple points during students' work to construct written justifications. Across the full data set for this study, I observed 40 writing actions concerning warrants/backing within their written justifications.

In contrast to the patterns I observed for students' writing actions to address evidence, students' changes to warrants and backing were preceded by more evidence of PDE than students' changes to written justifications overall. For 13 of the 15 total indicators of PDE, the proportion of changes to warrants/backing that were preceded by the indicator (Row 4, see Table 5 and Table 6) was greater than the proportion of changes overall that were preceded by the same indicator (Row 1, see Table 5 and Table 6). As one exception, I observed evidence of Au1 (*Most students make mathematical contributions to the group's work, either verbally or nonverbally*) in 35% of writing actions to address warrants/backing, and in 39.9% of writing actions overall. As the final exception, I observed evidence of Ac3 (*Most students participate in each instance of deliberation, with all group members deliberating at some point during the task*) in 22.5% of writing actions to address warrants/backing, and in 28.8% of writing actions overall. In general then, it was slightly less likely for all students to consistently contribute their ideas and deliberate with peers, but more likely for students to demonstrate all other indicators of PDE.

Although most indicators were slightly more prevalent preceding writing actions concerning warrants/backing than they were preceding writing actions overall, six indicators showed more dramatic differences (greater than 10%) between types of changes. Indicators P1 (*Students work to understand the task conceptually and procedurally well*), P2 (*Students often adopt a disciplinary perspective*), Au2 (*Students often talk about the mathematics topic/problem using their own words*), Ac1 (*Students question each other and explain their mathematical contributions until the idea makes sense to everyone in the group*), Ac2 (*Students often deliberate about multiple mathematical ideas),* and R2 (*Students often use relevant resources they create or choose*) were respectively 12.3%, 11.9%, 12.6%, 14.4%, 18.4% and 19.7% more likely to occur preceding writing actions concerning warrants/backing than they were preceding writing actions overall. Given the widespread variety of these indicators and that most other indicators of PDE were also slightly higher preceding students' writing actions to address warrants/backing, it is unlikely that one particular principle of PDE that supported students to add warrants/backing. Rather, I propose that the interaction between multiple principles might support students to consider warrants and backings within their written justifications.

As an example, consider the work of four students in Ms. Roberts's classroom to complete the Initial Challenge of *Moving Straight Ahead* 4.3, in which they are asked to determine whether a graph that passes through exactly three quadrants could show a linear relationship (and beyond that a unique linear relationship). Note that the curricula problem included multiple situations to consider, and Ms. Roberts assigned each group a specific situation. Table 10 shows the group's conversation as they begin work on this task, as well as the indicators of PDE that are demonstrated through their spoken words and digital actions.

Table 10:

Transcript of Student Dialogue and the Associated Indicators of PDE During Small Group Work on Moving Straight Ahead 4.3

Speaker	Spoken words and actions	Indicators of Problematizing	Indicators of Authority	Indicators of Accountability	Indicators of Resources
Ora	Okay, Part C was a graph that passes through exactly three coordinates				R1: using problem statement as a resource to begin group work
Lucas	Linear! It's linear!				
Lena	Quadrants				
Ora	Three quadrants. Wait, what is three quadrants? Just like, three points?			Ac1: questioning peers to understand the problem statement	
Liam	Each one of those boxes is a quadrant		Au3: responding to peers	Ac4: terms are mathematical relevant	
Lucas	(overlapping speech) The negative, either the full negative, the negative positive… (gestures to these quadrants and trails off)	P3: pushing to resolve uncertainties by using a visual definition			
Ora	Oh!				

Table 10 (cont'd)

Liam	So we could have it start, just start pretty high up in the first one, come down in the other one, and end in the third one.	P2: disciplinary perspective of finding an example to investigate a general idea			
Lucas	Yeah. So it's technically a line. I mean, it can be linear.	P1: conceptual connection between examples and definitions	Au2: using own words to describe a graph	Ac3: deliberation involves all students	
Ora	Wait, how would it go through three if there's only four boxes? It would have to go like that (points to a proportional line), right?			Ac1: continued questioning	R3: using drawn and gestured lines as resources to express uncertainties
Lucas	No, no, no… (gets up from computer and goes to Ora) It can go... (motions across screen non-proportionally) Boom.			Ac2: deliberating about possible lines	
Ora	(overlapping speech) Oh wait, you're right. I see what, I see what you mean.		Au3: considering peer's visually-presented ideas		
Liam	It's like a cross.		Au1: multiple students contributing		
Lucas	Well... (looks back at his own work) that's a bad example.				
Liam	Yeah, I know what you mean though.		Au4: reacting to peers' ideas constructively		
Lena	Okay, so it is… Yes, it could be linear? Obviously, 'cause it can pass through like that. (makes a diagonal slash motion with hand)	P4: uncertainties about how to justify		Ac4: deliberating with mathematical evidence	

Table 10 (cont'd)

Lucas	Obviously it could not be proportional.	P2: disciplinary perspective of using counter-examples			
Ora	(laughs) Yeah.		Au3: responding to peer's idea		
Liam	(laughs) Yeah.				
Lena	Okay, so… (inserts graphing tool) But we need to do something to show that it passes through. (starts placing points to make a line that goes through quadrants 2, 1, and 4).				R2: creating own resource of a graph

In this task, the teacher asked all students to construct a common written justification, which Lena immediately began typing after this conversation: "Yes, the relationship could be linear because we can see linear relationships on the graph to the left (straight lines, constant slope)." This written justification goes beyond making a claim and giving evidence in the form of a graph and also offers a warrant for why the students believed the relationships shown on their graph were linear. The small group work preceding the construction of this written justification (shown in Table 10) included high levels of each of the four principles of PDE, as students engaged in problematizing and resource use to develop support for their claims and accountability and authority for all peers to deliberate to make sense of this support. Much later in the task, the group also went on to determine that there were multiple possible linear relationships that met this criterion and expanded on their written justification. The presence of one principle or indicator on its own within the group's activity could likely not have prompted this type of written justification. Rather, the presence of evidence of one indicator or principle seemed to lead to increased presence of other indicators or principles, which in this case led to increased levels of mathematical completeness for written justifications.

Summary of Results on Mathematical Completeness

In this chapter, I have shared results to the first sub-question: *What is the relationship between middle grades students' PDE during small group work and their construction of written justifications in a collaborative mathematics learning environment in terms of mathematical completeness?* At a general level, on tasks where groups demonstrate higher levels of PDE, students were more likely to write more mathematically complete justifications that go beyond providing mathematical claims and begin to explain how or why the mathematics works. This may be because students' writing actions to address warrants and backing are preceded by more evidence of almost all indicators of PDE, whereas students' writing actions to incorporate or address claims can be preceded by a variety of levels of PDE. Student behaviors that seem to have an especially important role in supporting students' production of written justifications are considering peers' ideas and creating and using inscriptions of their own choice.

The ways that students dealt with evidence within their written justifications in the digital environment were of particular interest, as in the context of this study students demonstrated views that evidence (and mathematical inscriptions more generally) did not need further justification. When students took writing actions to address evidence within their written justification, the localized PDE within the group was comparably lower than for other types of writing actions. Even when students collaborated to unpack evidence or mathematical inscriptions during the broader task, this did not necessarily translate to their written work, where inscriptions were included without additional comments. These findings are important to consider as we work to better support students' engagement in written justification, particularly since prior research has found that classroom norms around unpacking representations can support students to engage in oral justification. What might these norms look like in the written modality, and how might teachers and students in small group settings reinforce them?

CHAPTER 5:
RESULTS ON THE RELATIONSHIP BETWEEN PDE AND MATHEMATICAL VALIDITY OF WRITTEN JUSTIFICATIONS

A major goal of students' participation in mathematical justification as a learning practice is that they develop their mathematical understanding (Staples, Bartlo, & Thanheiser, 2012). As students develop mathematical understanding, they develop their abilities to make correct claims and support those claims with mathematical reasoning that is valid within their classroom community. When interpreting students' written justification, I considered this in terms of mathematical validity. In the context of this study, written justifications were considered mathematically valid if they included correct claims and if any support for those claims (i.e. evidence, warrants, or backing) was accurate and relevant to the mathematical claim. In this chapter I attend to the second research sub-question: *What is the relationship between middle grades students' PDE during small group work and their construction of written justifications in a collaborative mathematics learning environment in terms of mathematical validity?* To answer this question, I focus first on examining overall validity of students' written justifications in connection to the nature of their collaborative group work throughout the task, and second, on the specific writing actions that students took to address the mathematical validity of their written justifications within the group's localized PDE.

Mathematical Validity of Students' Overall Written Justifications

I classified each student's written justification according to its overall mathematical validity during its construction. This provides a simple representation of students' processes to produce written justifications and I consider whether written justifications were always valid, had emergent validity (i.e. became mathematically valid by the end of small group work on the task), were never mathematically valid, or were not analyzable in terms of validity (which occurred mainly for partially-finished statements or inscriptions). Note that in this study, the mathematical validity of students' written justifications either remained the same throughout the task or improved. There were no cases where a student produced a mathematically valid written justification early in the task and ended the task with a non-valid written justification. The classification of all written justifications, as well as those produced in groups demonstrating low, medium, and high levels of PDE is shown on Table 11 below.

Table 11:

Overall Mathematical Validity of Students' Written Justifications During Small Group Tasks in Which Groups Demonstrated Low, Medium, and High Levels of PDE

	Always mathematically valid	Emergent Validity	Never mathematically valid	Not analyzable for validity	Total
Low PDE	11 (55%)	0 (0%)	5 (25%)	4 (20%)	20 (100%)
Medium PDE	7 (25%)	2 (7.1%)	11 (39.3%)	8 (28.6%)	28 (100%)
High PDE	10 (35.7%)	4 (14.3%)	4 (14.3%)	10 (35.7%)	28 (100%)
All Tasks	26 (34.2%)	6 (7.9%)	22 (28.9%)	22 (28.9%)	76 (100%)

The overall patterns in how the nature of PDE relates to the mathematical validity of students' written justifications are less straightforward than the overall patterns in how the nature of PDE relates to the mathematical completeness of students' written justifications. Students were most likely to construct a written justification that was always valid on tasks where the group demonstrates low PDE (in 55% of cases where the group demonstrated low PDE). This high prevalence of mathematical valid written justifications is notable, but to interpret the result accurately it is important to remember that for each task I analyzed the written justifications of two focal students. The data on Row 1 of Table 11 describes 20 written justifications over 10 different tasks on which groups demonstrated low PDE. For three of these tasks, both focal students wrote justifications that were always valid. In these cases, students did not discuss their mathematical ideas much (likely because they felt confident they were correct) and worked quickly to complete the task, but wrote dominantly mathematical statements in terms of Kosko and Zimmerman's (2019) classification scheme. For one of these tasks, both focal students wrote justifications that were never valid, and the non-valid parts of the written justification were typed verbatim from peers' words. The remaining six tasks, however, showed a disconnect between validity, where one focal student in the group wrote a mathematically valid justification and the other focal student in the group did not.

Given that students that were working in a collaborative digital environment on tasks that the teacher had set up to be completed with peers, this is concerning. When students began their work on the task with a valid justification, they were usually able to complete a valid written justification. If students began the task with mathematical errors of any kind, however, the data suggests that low levels of PDE were not sufficient to help them correct those errors. The overall ratings of specific indicators of PDE can give insight into key mathematical behaviors that might allow for students to write mathematically valid justification. For all tasks in which students demonstrated low levels of PDE, the ratings for indicator P4 (*Students' mathematical uncertainties concern what their conclusions mean, competing alternatives, or justifying their actions*) indicator Ac2 (*Students often deliberate about multiple mathematical ideas*) are low. Low ratings for these indicators show that during their work on these tasks, students only encountered uncertainties about what to do next and rarely or never deliberated about mathematical ideas with peers.

This stands in contrast to the pattern for tasks on which groups demonstrated medium levels of PDE. For these tasks, valid justifications did not appear as often (in only 25% of cases where groups demonstrated medium levels of PDE), but focal students were much more likely to check the validity of their written justifications against their peers' work (either verbally or digitally) and to attempt to reconcile differences by deliberating with peers. The data on Row 3 of Table 11 describes 28 written justifications over 14 different tasks on which groups demonstrated medium PDE. For 9 tasks the validity of both focal students' written justifications were the same throughout the task, for 2 tasks the validity of one focal student's written justification showed emerging validity, and only 3 tasks showed a disconnect between validity by the end of the task. On these tasks, students generally became aware of any potential disconnects (either through discussion or digital activity) and tried to address them, but they were not always successful in doing so.

As an example, consider the work of three students in Ms. Price's classroom to complete the Initial Challenge of *Stretching and Shrinking* 3.1. In this task, students work to create "rep-tiles" (or similar figures of a triangle or quadrilateral made by arranging multiple copies of that same shape). Up to this point, students had created rep-tiles of multiple different triangles, squares, and other quadrilaterals, but had not yet begun to view each other's digital environments. Student P03 had created several larger repeated shapes that were not in fact similar to the original shape, and although the group had previously discussed other similar shapes, this represents their first attempt to discuss inconsistencies between their digital work. Table 12 shows the group's conversation and digital actions.

Table 12:

Transcript of Student Dialogue and Digital Actions During Small Group Work on Stretching and Shrinking 3.1.

Speaker	Speech and Digital Actions
Alex	They all rep-tile, I think.
Seiko	No they're not.
Alex	Yeah.
Seiko	How about the trapezoid?
Alex	(moving the fourth copy of the trapezoid into position) Watch, ready? (rotating trapezoid 180 degrees to place it next to another) It does rep-tile. This way.
Seiko	No, your's is not… you didn't do it right.
Alex	Look, see? (turning computer to face peers) It rep-tiles.
Seiko	Is that… that's how you… Oh! You're allowed to do that?
Alex	Yeah, you can spin it around.
Jiao	Hmmm… Ugh. I can't spell.
Alex	(carries computer to Ms. Price) Wait, for the trapezoid you're allowed to spin it around, right?
Ms. Price	Mm hmm.
Alex	(carrying the computer back to the group) See? It works. They all do. This is so easy.
Ms. Price	(looks at Alex's screen) But that's not a rep-tile.
Alex	Why not?
Ms. Price	Because remember, a rep-tile has to make the *same* shape.
Seiko	Yeah, exactly. That's what I'm saying. It's not...
Ms. Price	Like, this isn't a rep-tile (points to trapezoid). It's not that it makes *a* shape, it has to make the *same* shape.
Seiko	Yeah, exactly. That's what I said. That's why this is not a rep-tile. Because you can't make it.
Alex	Oh, you have to use, like, two of those connected to each other?
Ms. Price	Use as many as you need. But it has to make a triangle that looks like that triangle.
Jiao	(overlapping speech) So a trapezoid is a no-no.
Alex	Ohhhh… (begins to delete incorrect rep-tiles)

In this group's interactions, they made some attempts to address the disconnect between valid claims about rep-tiles in most students' written justifications and Alex's written justifications (for example, on Line 7 of Table 12), but these were not specific enough for the student fully understand why their construction was not a similar figure, and thus not a rep-tile. With some help from the teacher, they were able to resolve this uncertainty and begin to correct the issue in their written claims, but it was not fully resolved before the end of the task a few minutes later.

Because mathematical uncertainty is such an important feature of PDE from a theoretical standpoint, it is possible that the presence of written justifications that are not mathematically valid within a group's digital environment may actually have been a key factor in prompting uncertainties about written justification that allowed groups to develop beyond low levels of PDE. In the case described, as the group grappled with uncertainties about the validity of multiple different inscriptions, they demonstrated increased problematizing which then facilitated increased levels of accountability and resource use as they sought to explain and correct their inscriptions and mathematical claims. In this study I focus on the ways that the nature of students' collaborative work shape students' engagement in written justification, but these results about mathematical validity suggest one way that features of written work might reciprocally influence the nature of collaborative work. The results pertaining to my third sub-question focus specifically on these uncertainties about written justification.

Students were more likely to write mathematically valid justification in groups that demonstrated high levels of PDE than they were in groups that demonstrated medium PDE, with 50% of cases where groups demonstrate high PDE resulting in a written justification that was always mathematically valid or showed emerging validity. Further, as the demonstrated level of

PDE increased it became more likely that students were able to modify a written justification that was not mathematically valid so that it was mathematically valid at the end of small group work. There were no instances of this in tasks where small groups demonstrated low levels of PDE, but this occurred in 7.1% of tasks where small groups demonstrated medium levels of PDE and in 14.3% of tasks where small groups demonstrated medium levels of PDE. This suggests that as PDE increases, students are more likely to notice and attend to any inconsistencies between the multiple written justifications within their small group's digital environment. This increased chance of revision is likely the result of more frequent student behaviors associated with collaborative group work, given that the major difference between medium levels and high levels of each indicator on the PDE-CW framework is the frequency with which specific student behaviors are observed. But to better understand *how* this effect occurs as students are engaged in the writing process and the specific student behaviors that allow for this effect, I also considered the nature of the changes that students made to their written justifications within the digital environment and the associated local context of PDE within their group. In what ways does this localized context of PDE within collaborative work influence students' processes as they attempt to construct mathematically valid written justifications? I attend to this in the following section.

Students' Writing Actions to Address the Mathematical Validity of Their Written Justifications

In addition to the mathematical validity of students' overall written justifications, I considered how the specific writing actions that students took to construct written justifications throughout the task affected the validity of their justifications. Most writing actions (109 out of 163 total writing actions, or 66.9%) maintained the same validity for the written justification.

That is, if the written justification was mathematically valid prior to the writing action, it was still valid after the writing action. If the written justification was not mathematically valid prior to the writing action, it was still not mathematically valid after the writing action. I also observed 35 instances (21.5% of 163 total writing actions) where students took writing actions that decreased the mathematical validity of their written justifications and 18 instances (11.0% of 163 total writing actions) where students took writing actions that increased the mathematical validity of their written justifications.

To give insight into how social factors within collaborative group work might relate to students' written justifications, I considered the localized nature of the group's demonstrated PDE. What social factors within small group work were present in cases where students were able to increase the validity of their written justifications, and thus might support the construction of mathematically valid written justifications? What social factors within small group work might inhibit the construction of mathematically valid written justifications? The indicators that were present as influences on students' changes to their written justifications overall, as well as those present when students took writing actions that decreased or increased the validity of their written justifications are shown on Table 13 and 14 below. Table 13 describes the overall occurrence of indicators of problematizing and authority (see PDE-CW Framework, Table 2) during small group work that preceded changes to written justifications, and Table 14 describes the overall occurrence of indicators of accountability and resources during small group work that preceded changes to written justifications.

Table 13:

Frequency of Indicators of Problematizing (P) and Authority (Au) Preceding Students' Writing Actions that Decreased or Increased the Validity of Their Written Justification

	P1	P2	P3	P4	Au1	Au2	Au3	Au4
All writing actions (n=163)	41 (25.2%)	54 (33.1%)	61 (37.4%)	52 (31.9%)	65 (39.9%)	61 (37.4%)	110 (67.5%)	66 (40.5%)
Writing actions decreased the validity of students' written justifications (n=35)	14 (40%)	11 (31.4%)	20 (57.1%)	11 (31.4%)	3 (8.6%)	15 (45.7%)	21 (60%)	13 (37.1%)
Writing actions that increased the validity of students' written justifications (n=18)	5 (27.8%)	7 (38.9%)	11 (61.1%)	4 (22.2%)	11 (61.1%)	9 (0.5%)	16 (88.9%)	9 (50%)

On Tables 13 and 14, Pn, Aun, Acn, and Rn refer to specific indicators of problematizing, accountability, authority, and resources from the PDE-CW framework. For example, P1 refers to the first indicator of problematizing. Because it was possible for the validity to remain the same when students made changes to their written justifications, not all writing actions are represented in rows 2 or 3. Percents are calculated based on the total number of writing actions within that row. For example, in column 2 (P1), row 2 (Decreased validity), the percent is 40% = 14/35, which represents that 40% of students' changes that decreased the validity of their written justifications were preceded by evidence of indicator P1.

Rather than a general pattern that fit most indicators (as was the case for writing actions to address evidence and writing actions to address warrants/backing), the comparisons for specific indicators according to validity were more mixed. That is, some indicators were comparatively higher for writing actions that increased/decreased validity and some indicators were comparatively lower for writing actions that increased/decreased validity. This is in part because I observed a fairly small number of writing actions that changed the validity of students' written justifications, so it is difficult to be sure of how significant these comparisons are. As such, I focus my analysis on comparisons where the difference between the prevalence of indicators is greater than 10% between the category of writing action I am analyzing and students' overall writing actions.

Table 14:

Frequency of Indicators of Accountability (Ac) and Resources (R) Preceding Students' Writing Actions that Decreased or Increased the Validity of Their Written Justification

	Ac1	Ac2	Ac3	Ac4	R1	R2	R3
All writing actions (n=163)	58 (35.6%)	76 (46.6%)	47 (28.8%)	47 (28.8%)	56 (34.4%)	82 (50.3%)	46 (28.2%)
Writing actions that decreased the validity of students' written justifications (n=35)	15 (42.9%)	19 (54.3%)	13 (37.1%)	13 (37.1%)	12 (34.3%)	12 (34.3%)	12 (34.4%)
Writing actions that increased the validity of students' written justifications (n=18)	8 (44.4%)	14 (77.8%)	9 (50%)	6 (33.3%)	5 (27.8%)	12 (66.7%)	11 (61.1%)

Students' Writing Actions that Decreased the Validity of Their Written Justifications

Two types of writing actions that decreased the validity of students' written justifications were most common across the data set. First, students sometimes decreased the validity of their written justifications by typing claims that were not mathematically valid. Second, students sometimes decreased the validity of their written justifications by typing evidence or warrants that were not mathematically valid (that is, they were inaccurate or did not relate to the student's claim). In this data set, there were no instances where students deleted or modified valid claims or other elements of their written justifications so that they became non-valid. There was one instance, however, where a student deleted one non-valid claim and rapidly replaced it with a different non-valid claim. In general, then, students' writing actions that decreased the validity of their written justifications were new additions rather than modifications or revisions of some kind.

The comparison between localized PDE for writing actions that decreased the validity of students' written justifications and students' overall writing actions was rather mixed. For two indicators of PDE, the proportion of writing actions that decreased the validity of students' written justifications that were preceded by the indicator (Row 2, see Table 13 and Table 14) was greater by more than 10% than the proportion of overall writing actions that were preceded by the same indicator (Row 1, see Table 13 and Table 14). Evidence of P1 (*Students work to understand the task conceptually and procedurally well*) was observed preceding 40% of writing actions that decreased validity, as compared to 25.2% of all writing actions. Additionally, evidence of P3 (*When uncertain or stuck, students push to resolve the issue themselves and/or with help from their peers or teacher*) was observed preceding 57.1% of writing actions that decreased validity, as compared to 37.4% of all writing actions. These comparisons indicate that preceding writing actions that decrease the validity of written justifications, students are more likely to make multiple attempts to understand the conceptual meaning of the mathematics involved. In short, students engaged in a higher level of problematizing to try and understand the mathematics as they engaged in constructing written justification, but this did not always lead them to express their ideas in valid written justifications.

For two indicators of PDE, the proportion of writing actions that decreased the validity of students' written justifications that were preceded by the indicator (Row 2, see Table 13 and Table 14) was more than 10% less than the proportion of overall writing actions that were preceded by the same indicator (Row 1, see Table 13 and Table 14). Evidence of Au1 (*Most students make mathematical contributions to the group's work, either verbally or nonverbally*) was observed preceding only 8.6% of writing actions that decreased validity, as compared to 39.9% of all writing actions. Additionally, evidence of R2 (*Students often use relevant resources they create or choose*) was observed preceding 34.3% of writing actions that decreased validity, as compared to 50.3% of all writing actions. This indicates that preceding writing actions that decreased the validity of written justifications, it was far more likely that a few students' voices dominated the mathematical conversation and less likely that students worked to make sense of resources they created or chose for themselves.

The most striking result from the above analysis is the low occurrence of indicator Au1, which occurred only 3 times preceding writing actions that decreased the validity of students' written justifications. In cases when this indicator was missing, there were two major patterns of engagement during small group work. First, there were groups where one or two students made bids to try and question their group about the mathematical topic at hand but were not successful in getting their peers to engage in longer mathematical conversations or share their digital work. In these groups when peers did not respond, students returned to working individually and did not utilize collaborative features of the digital environment or their group. As the other major pattern of engagement, there were groups where one or two students took up a greater portion of the mathematical conversation and their mathematical ideas were not as frequently doubted or questioned by peers. These students were positioned by themselves and peers as having greater mathematical authority and influence, but it was not necessarily because of the quality or accuracy of their justifications (Engle, Langer-Osuna, & McKinney de Royston, 2014; Langer-Osuna, 2017). In cases where these unduly-influential students suggested mathematical ideas that were not valid, this often led to writing actions that decreased the validity of their written justifications.

As an example, consider the work of a group of four students in Ms. Price's classroom to complete the Initial Challenge of *Moving Straight Ahead* 2.1. In this task, students compared two plans for buying bulk orders of t-shirts for a fictitious class of students: "Mighty Tee" charges a one-time order fee but a lower cost per shirt, and "No Shrink" charges a higher cost per shirt but no other order fees. Students in this group began work by reading the problem statement carefully, and one student (P05) constructed a graph that represented the two plans as intersecting lines. Unfortunately, one of the lines on their graph did not accurately represent the payment plan and they did not appear to realize this at any point during the task. Table 15 shows the group's work as they begin to discuss the problem.

Table 15:

Transcript of Student Dialogue During Small Group Work on Moving Straight Ahead 2.1

Speaker	Spoken words and digital actions
Kevin	So here's what I think. (hovers cursor over the intersection point of their graph) If you're spending more than 33… if you're… I'll just write it down.
Robert	(opens the collaborative view to look at Kevin's graph)
Lily	(leans over to see Kevin's screen) What the heck?
Jae-hwa	Okay, let's say like… let's just say there's like 30 kids in the classroom. Cause I don't know. That's average, about. Or something.
Robert	So which brand? Which brand are we deciding?
Jae-hwa	Uh, do you wanna use 30 people in a classroom?

Table 15 (cont'd)

Kevin	Or no, you should just say if there are less than 30 then you spend said amount. If there are more you… (trails off)
Robert	But what t-shirt brand did we choose?
Jae-hwa	(overlapping speech) You should show the graph.
Ms. Price	(to the whole class) I'm hearing some people in their groups have a conversation about what they're thinking. That's a great way to go about this.
Robert	Which t-shirt brand is it?
Jae-hwa	(typing on calculator, mumbling) That's 79 dollars.
Lily	So wait. So when they have less than 30 dollars…
Kevin	(overlapping speech) People. It's 30 people.
Jae-hwa	30 people (laughs).
Lily	30 people, then.

As they began to type their written justification, all students in the group continued to use 30 students as a tipping point for making a decision between the two companies (e.g. "If there are less than 30 people they should buy from No Shrink.") because of the suggestions of Kevin and Jae-hwa, as well as the fact that this was close to the value for the intersection point of the lines on Kevin's graph. Other students in the group continued to defer to Kevin's graph, even when their own calculations seemed to contradict it. The combined effect of few students in the group contributing their own mathematical ideas (i.e. low evidence of indicator Au1) and few students in the group creating or interpreting their own inscriptions (i.e. low evidence of indicator R2) seems to have contributed to allowing for writing actions that decreased the validity of students' written justifications to occur.

Students' Writing Actions that Increased the Validity of Their Written Justifications

Three types of writing actions that increased the validity of students' written justifications were most common across the data set. First, students sometimes deleted parts of their written argument (i.e. claims, evidence, warrants) that they realized were not mathematically valid without replacing or modifying those parts. Second, students sometimes modified non-valid parts of their argument (i.e. claims, evidence) so that those parts became mathematically valid. Third, students sometimes modified the non-mathematical content of their written justification (e.g. spelling, conventions) so that it became clear that their written justification was mathematically valid. Note that this does not refer to small spelling errors or typos, but more substantial wording and spelling changes that affected the meaning of the written justification. For example, this occurred when a student added "isosceles triangle" to their written justification in place of "the triangle" to make it clear which triangle they were referring to. Although the student's intended meaning had not changed, these writing actions aided with mathematical communication, and made it easier for a reader to correctly interpret the written justification as mathematically valid. In general, then, students' writing actions that increased the validity of their written justifications were new deletions, modifications, or revisions of some kind rather than new additions.

Students' writing actions that increased the validity of their written justifications were preceded by more evidence of PDE than students' writing actions overall. For eight indicators of PDE, the proportion of writing actions that decreased the validity of students' written justifications that were preceded by the indicator (Row 2, see Table 13 and Table 14) was greater by more than 10% than the proportion of overall writing actions that were preceded by the same indicator (Row 1, see Table 13 and Table 14). Specifically, indicators P3 (*When uncertain or stuck, students push to resolve the issue themselves and/or with help their peers or teacher*), Au1

(*Most students make mathematical contributions to the group's work, either verbally or nonverbally*), Au2 (*Students often talk about the mathematics topic/problem using their own words*), Au3 (*Students often seek out, consider, or respond to peers' mathematical ideas*), Ac2 (*Students often deliberate about multiple mathematical ideas*), Ac3 (*Most students participate in each instance of deliberation, with all group members deliberating at some point during the task*), R2 (*Students often use relevant resources they create or choose*), and R3 (*Students use resources in a way that helps them resolve mathematical uncertainties*) were respectively 23.7%, 21.2%, 12.6%, 21.4%, 31.2%, 21.2%, 16.4%, and 32.9% more likely to occur preceding writing actions that increased the validity of students' written justifications than they were preceding writing actions overall. This represents a wide variety of mathematical behaviors, similar to the variety of behaviors that supported students' writing actions to address warrants/backing. So rather than one particular principle or indicator of PDE being supportive of students' construction of mathematically valid written justifications, it is likely the interaction between multiple principles and indicators is supportive of students' writing actions to increase the validity of their written justifications.

Of particular interest are comparisons between indicators Au1 and R2, which describe the extent to which multiple students contribute their mathematical ideas and students use resources that they create or choose themselves. I observed evidence of Au1 preceding 61.1% of writing actions that increased the validity of students' written justifications, but only preceding 8.6% of writing actions that decreased the validity of students' written justifications. Similarly, I observed evidence of R2 preceding 66.7% of writing actions that increased the validity of students' written justifications, but only preceding 34.3% of writing actions that decreased the validity of students' written justifications. In other words, these indicators were particularly common in cases where students took writing actions to improve the validity of their written justification, and particularly uncommon in cases where they did not. So although PDE during collaborative work in general can support students to construct valid mathematical justification, my findings suggest it could be especially important that collaborative group work promote opportunities for all students in the group to share mathematical ideas (rather than only participating in other non-mathematical ways, such as scribing for the group or managing physical materials) and to allow time during collaborative group work for creating and investigating mathematical inscriptions or other resources.

It is not surprising that there was a larger amount localized evidence of PDE preceding writing actions that increased the validity of students' written justifications, given that I only observed cases where students were able to change a non-valid written justification to a mathematically valid written justifications by the end of the task in cases where the group demonstrated medium or high levels of PDE overall (see Table 11). However, the fact that these features of PDE were directly present in moments where students took writing actions to increase the validity of their written justifications indicates a stronger relationship between PDE in small group work and changes in written justifications. In the context of this study, this makes sense because students had the opportunity to engage in collaborative work and individual written work in the digital environment simultaneously. As such, students were able to talk and work with their group immediately preceding any writing actions in order to hear multiple peers' mathematical thinking and to engage with inscriptions and other resources that they chose. As described above, in the context of this study these behaviors were important to supporting students as they sought to increase the validity of their written justification in the context of the digital environment, but how would these behaviors manifest in other contexts for written justification? In situations where students work collaboratively with peers to solve problems, and then work independently to construct their own written justification later, these same localized features of PDE would obviously not be present (or at least not in the same way). This is important to consider, as students more frequently took writing actions that *decreased* the validity of their written justifications when these features of localized PDE were not present. If students are to construct valid written justification in a variety of contexts, we must consider how to best provide opportunities for them to purposefully engage with the thinking of multiple peers and to create and interpret inscriptions with peers. Even if written justifications are constructed independently, this might occur as part of ongoing feedback or revision.

Summary of Results on Mathematical Validity

In this chapter I have shared results to the second research sub-question: *What is the relationship between middle grades students' PDE during small group work and their construction of written justifications in a collaborative mathematics learning environment in terms of mathematical validity?* At a general level, increased levels of PDE during small group work do not necessarily guarantee that students will construct mathematically valid written justifications. It was actually most likely for students to construct mathematically valid written justifications on tasks where their groups demonstrated low levels of PDE (although these were dominantly mathematical statements, rather than going beyond this to explain how or why the mathematics worked). When students constructed mathematical justifications that were not valid, however, only medium and high levels of PDE supported students to attend to the non-valid parts of their written justification and increase the overall validity. In short, mathematical errors occurred during small group work at all levels of PDE, but as the level of PDE increased, I observed that students were better able to notice and attend to these mathematical errors in a productive way. Although increased PDE in general supports students to increase the mathematical validity of their written justifications, some specific behaviors that seem especially important are that all students in the group make mathematical contributions (as evidenced by indicator Au1) and that students are able to use or create resources of their own choice (as evidenced by indicator R2).

The comparably low level of valid written justifications that were constructed in groups that demonstrated medium levels of PDE seems to indicate that when students examine non-valid written justification, it can support increased levels of PDE. On tasks where groups demonstrated low levels of PDE, students often did not notice any discrepancies between the multiple different written justifications within the group's digital workspace. Whereas on tasks where groups demonstrated medium and high levels of PDE, students were more likely to notice discrepancies between the validity of the group's written justification and to become increasingly more successful at resolving them. This points to the importance of written justification as a learning opportunity, similar to Bereiter and Scardamalia's (1987) concept of "writing to learn." My findings suggest that as uncertainties about written justification arise, this increases the level of problematizing within the group, which gives students increased opportunities for meaningful mathematics learning.

CHAPTER 6:

RESULTS ON STUDENTS' ATTEMPTS TO RESOLVE UNCERTAINTIES ABOUT WRITTEN JUSTIFICATION

The results I have shared so far indicate the importance of mathematical uncertainties and deliberation among peers in promoting the mathematical completeness and increased validity of students' written arguments. As such, moments when students encountered uncertainties about writing justifications were analytically important to give insight into how features of the group's PDE during collaborative work supported students to engage in the process of constructing written justifications through interactions with peers and the digital environment. By examining the ways that collaborative work supports students to resolve their uncertainties about written justification, we can gain insight into the social norms that support students to make their written justifications more mathematical complete or valid. Existing research has identified many productive norms for groups of students engaging in oral justifications or for whole-class oral justifications (e.g. Melhuish et al., 2019; Mueller et al., 2014; Wood et al., 1993; Weber et al., 2009), but we know less about productive norms for groups of students engaging in written justification.

Students encountered uncertainties of many kinds across the data for this study (which I have illustrated through transcripts from students' small group work), including uncertainties about whether they were taking valid mathematical actions, how to compare multiple strategies or representations, whether peers had provided valid oral justifications to the group, and the meaning of results to calculations or other processes. These kinds of uncertainties can be a powerful impetus for mathematics learning and engagement (Zaslavsky, 2005), and problematizing (Engle & Adiredja, 2008). In this chapter I attend to the third research sub-question: *How do middle grades students seek to resolve uncertainties about written justification during small group work in a collaborative mathematics learning environment?* To answer this question, I conducted localized analysis of moments where students encountered two types of uncertainties that specifically concerned written justification: placeholder writing actions and out-of-order writing actions. I observed 32 instances (19.6% out of 163 total writing actions) of placeholder writing actions and 29 instances (17.8% out of 163 total writing actions) of out-of-order writing actions. Although students likely encountered other types of uncertainty as they engaged in constructing written justifications, these two types are uncertainties specifically related to their attempts to write.

To give insight into how social factors within collaborative group work might relate to the occurrence of students' uncertainties about written justification and their attempts to resolve these uncertainties, I consider the localized nature of the group's demonstrated PDE. To consider the social context that preceded students' uncertainties about written justification, I attend to how students demonstrated evidence of specific indicators of PDE prior to these writing changes. The indicators that were present as influences on students' writing actions overall, as well as those present when students took placeholder writing actions and out-of-order writing actions are shown on Table 16 and 17 below. Table 16 describes the overall occurrence of indicators of problematizing and authority (see PDE-CW Framework, Table 2) during small group work that preceded changes to written justifications, and Table 17 describes the overall occurrence of indicators of accountability and resources during small group work that preceded changes to written justifications.

Table 16:

Frequency of Indicators of Problematizing (P) and Authority (Au) Preceding Placeholder Writing Actions and Out-of-Order Writing Actions

	P1	P2	P3	P4	Au1	Au2	Au3	Au4
All writing actions (n=163)	41 (25.2%)	54 (33.1%)	61 (37.4%)	52 (31.9%)	65 (39.9%)	61 (37.4%)	110 (67.5%)	66 (40.5%)
Placeholder writing actions (n=32)	10 (31.3%)	16 (50%)	11 (34.4%)	16 (50%)	19 (59.4%)	14 (43.8%)	23 (71.9%)	8 (25%)
Out-of-order writing actions (n=29)	7 (24.1%)	10 (34.5%)	17 (58.6%)	7 (24.1%)	13 (44.8%)	16 (55.2%)	22 (75.9%)	15 (51.7%)

On Tables 16 and 17, Pn, Aun, Acn, and Rn refer to specific indicators of problematizing, accountability, authority, and resources from the PDE-CW framework. For example, P1 refers to the first indicator of problematizing. Because it was possible for writing actions to follow other patterns beyond typing placeholders and making out-of-order additions to written justifications, not all writing actions are represented in rows 2 or 3. Percents are calculated based on the total number of writing actions within that row. For example, in column 2 (P1), row 2 (Placeholder writing actions), the percent is 31.3% = 10/32, which represents that 31.3% of students' writing actions where they added placeholders to their written justifications were preceded by evidence of indicator P1.

For the most part, indicators of PDE were more commonly observed preceding placeholder writing actions and out-of-order writing actions than they were preceding overall writing actions. Although there are some comparisons where indicators of PDE are less commonly observed preceding placeholder writing actions and out-of-order writing actions, these are generally slight differences (smaller than 10% difference between categories). Because I observed a fairly small number of each type of writing action, I focus my analysis on comparisons where the difference between the prevalence of indicators is greater than 10% between the category of writing action I am analyzing and students' overall writing actions. This allowed me to address the most significant patterns in the data and focus on more major differences in the nature of students' collaborative work as I examined their attempts to resolve uncertainties about how to justify in written form and about revising their written justifications.

Table 17:

Frequency of Indicators of Accountability (Ac) and Resources (R) Preceding Placeholder Writing Actions and Out-of-Order Writing Actions

	Ac1	Ac2	Ac3	Ac4	R1	R2	R3
All writing actions (n=163)	58 (35.6%)	76 (46.6%)	47 (28.8%)	47 (28.8%)	56 (34.4%)	82 (50.3%)	46 (28.2%)
Placeholder writing actions (n=32)	12 (37.5%)	13 (40.6%)	9 (28.1%)	7 (21.9%)	14 (43.8%)	21 (65.6%)	13 (40.6%)
Out-of-order writing actions (n=29)	12 (41.4%)	21 (72.4%)	14 (48.3%)	12 (41.4%)	8 (27.6%)	18 (62.1%)	13 (44.8%)

Uncertainties About How to Justify in Written Form

Across the data set, students typed placeholders for claims, evidence, and warrants as a way to begin their work on written justifications or to append new components to their written justifications. When students typed placeholders for claims, these took the form of the beginning of a sentence or equation that did not yet contain any mathematical information (e.g. "x equals" or "I think that"). When students typed placeholders for evidence or warrants, this occurred in one of two ways. First, students often typed a claim and a placeholder for their evidence or warrants as part of the same writing action (e.g. "Yes it is possible because" or "I know x equals 10 because"). Second, in cases where students had already typed claims as part of previous writing actions they often typed a placeholder for their evidence or warrants as a distinct writing action (e.g. "How I found rectangle R is" or "That's because"). After typing placeholders, students paused and then took some other action in the digital environment or with peers in their collaborative groups in order to figure out what to type next.

In general, placeholder writing actions were preceded by more localized evidence of PDE than overall writing actions were. For five indicators of PDE, the proportion of placeholder writing actions that were preceded by the indicator (Row 2, see Table 16 and Table 17) was more than the proportion of overall writing actions that were preceded by the same indicator by more than 10% (Row 1, see Table 16 and Table 17). In particular, indicators P2 (*Students often adopt a disciplinary perspective), P4 (Students' mathematical uncertainties concern what their conclusions mean, competing alternatives, or justifying their actions*), Au1 (*Most students make mathematical contributions to the group's work, either verbally or nonverbally*), R2 (*Students often use relevant resources they create or choose*), and R3 (*Students use resources in a way that helps them resolve mathematical uncertainties*) occurred respectively 16.9%, 18.1%, 19.5%, 15.3%, and 12.4% more frequently preceding placeholder writing actions than they did preceding overall writing actions. The frequent occurrence of indicators of problematizing and resources shows that students commonly made attempts to understand the mathematical problem through a variety of inscriptions or other tools before beginning to construct written justification. In the case of placeholder writing actions, however, they quickly realized that they were not yet ready to complete those written justifications.

Only one indicator of PDE occurred less often by more than 10% preceding placeholder writing actions than preceding overall writing actions. Indicator Au4 (*Students react to peers' ideas as worthy of consideration and respond constructively*) occurred preceding 40.5% of all writing actions, but only preceding 25% of placeholder writing actions. Considering this in conjunction with the increased likelihood of indicator Au1, this indicates that multiple students in the group voiced their mathematical ideas, but that their peers did not appear to consider them. In the localized context preceding placeholder writing actions, students did not dismiss or demean peers' ideas, rather they did not attend carefully to peers' ideas or respond to them. This may mean that when working to construct written justifications, students need to attempt to transcribe their own mathematical thinking before they are ready to engage with peers' mathematical thinking.

Moments when students typed placeholders indicate that students were aware of the expectation to include claims, evidence, or warrants in their written justifications and made attempts to do so. In the localized context of PDE preceding these writing actions, I found that students engaged in trying to understand the mathematical situation through problematizing and resource use, but less commonly engaged in collaboration and deliberation with peers. However, it is important to consider not only the context of small group work that preceded these uncertainties, but also how students responded to such uncertainties. Moments of uncertainty about how to continue justifying in written form (i.e. placeholder writing actions) signal the beginning of students' attempts to resolve uncertainties about written justification. When students got stuck and didn't know what to write next, what actions did they take in the digital environment or with peers in their small groups to try and resolve this? Across the data for this study, in each instance when students encountered uncertainty about how to continue justifying in written form, they always did *something*. They did not sit back and stop work on the problem in response to uncertainty, but they took some kind of action. These actions include examining the problem statement, engaging with a resource of their own choosing or creation, seeking out peers' ideas verbally, or seeking out peers' ideas digitally, which I will describe in more detail below. Students always took one or more of these actions as they sought to resolve their uncertainty about how to continue justifying in written form.

Examining the problem statement allowed students to read supporting details of the mathematical context, clarify vocabulary terms, and identify the specific question to which they were responding with a written justification. Because of the format of the curriculum within the digital environment, students were able to open an embedded window to show the full problem statement as well as copying any text or other inscriptions from the problem statement into their own workspace. Although students did not choose the resources that were available to them in the problem statement (as these were either included from the curriculum or by the teacher), these resources often gave students necessary information to continue constructing their written justifications after placeholder writing actions. This action (that is, responding to uncertainties about written justification with increased levels of indicator R1) occurred when students typed placeholders for claims, evidence, and warrants.

Students also sought to resolve their uncertainties about how to justify in written form by engaging with a resource that they chose or created. In some cases, students re-examined or added to inscriptions or resources that they had previously created or had been using. This occurred when students typed a placeholder for some part of their written justification and then expressed doubt about the strategy they had used to arrive at their conclusions (e.g. "Wait, did I do that right?" or "No, that's not what I should write down"). In these cases, students often "checked their work" by going through their steps again to make sure they accurately knew what to write. In other cases, after students added a placeholder, they did work to create new inscriptions or resources. This happened when students typed a placeholder for some part of their written justification, and then expressed a lack of information about how to proceed (e.g. "Wait, why *does* that method work?" or "I don't know what to put"). Although this action often involved students engaging with evidence in the form of inscriptions (that is, responding to uncertainties about written justification with increased levels of indicator R2), it occurred in cases where students typed placeholders for claims and warrants.

In other situations, students sought peers' ideas verbally to try to resolve uncertainties about how to justify in written form. Students either asked peers for input generally (e.g. "Guys, I'm stuck on the explanation. What do we say?") or asked for input from a specific peer. (e.g. "Hey [name]. What should we write? Just 'it will work'?") These questions and requests for input generally initiated short conversations with peers (3-5 turns of talk) where students attended to peers' ideas and explained their thinking or longer conversations with peers where they deliberated about multiple options for how to continue typing their written justifications. That is, when students sought peers' ideas verbally it led to increased levels of authority and/or accountability within the group. In the localized context of PDE preceding students' placeholder writing actions, it was comparably less likely for students to demonstrate evidence of indicator Au4 by attending to and considering peers' ideas (as described above). But after encountering uncertainties about how to justify in written form, students purposefully sought out peers' ideas and considered them carefully. In these situations, attempting to construct a written justification seemed to alert students to the need for considering others' mathematical ideas.

Alternatively, students sought peers' ideas digitally when they encountered uncertainties about how to justify in written form. This was done by clicking a button to use the collaborative view within the digital environment, where students were able to view all of their group members' digital workspaces at the same time. Students had the option to make their own workspaces visible or non-visible to peers (and switch this at any time), so when students entered the collaborative view it was possible that they would only see some group members' work. Although there were two instances where students verbally followed up with peers about what they saw in their digital workspaces (e.g. "[Name], how did you get x=10? I thought it was 2."), it was more common for students to silently use the collaborative view to get ideas about how to continue with their written justification. In general, students used it as a low-stakes way to remind themselves of previous collaborative work without interrupting peers. I did observe several instances, however, where students silently entered the collaborative view and copied peers' work verbatim into their own collaborative view, either by copying the text directly into their own workspace or by transcribing it word-for-word. As described in my previous results on students' writing actions to address evidence, I found that this behavior is more likely to lead to the production of written justifications that are not mathematically valid and does not contribute to increased levels of PDE on the task in general. So although the collaborative view was often a useful feature for helping students to resolve their uncertainties about written justification, it was less effective when students did not also make attempts to translate peers' digital work into their own words or deliberate with peers about their digital work.

Uncertainties About Revising Written Justifications

Across the data set, students grappled with uncertainties about revising their written justifications to make out-of-order writing changes in four major ways: First, students modified claims, evidence, or warrants in attempts to make them more mathematically valid. Second, students deleted claims, evidence, or warrants that they no longer believed to be mathematically valid, but without replacing them with other written material as part of the same writing action. Third, students modified the non-mathematical content of the written justifications (that is, grammar, spelling, or other conventions) in attempts to enhance clarity and communication. And as the final type of out-of-order writing action, students added additional evidence or warrants to a written justification that they had already deemed complete. Students' attempts to revise their written justifications through out-of-order writing actions usually improved or maintained mathematical validity for the justification (in 25 out of 29 total out-of-order writing actions), resulting in an overall written justification that was mathematically valid. The remaining four out-of-order writing actions were additions that maintained non-validity for justification, resulting in an overall written justification that was not mathematically valid.

Because students often took out-of-order writing actions when they realized that parts of their written justification were not mathematically valid, this category has significant overlap with the category of writing actions that increased the validity of students' written justifications. But as described above, students also took out-of-order writing actions to improve their written justifications in other ways, such as increasing its clarity or mathematical completeness. Students' out-of-order writing actions signaled the end of attempts to resolve uncertainties about revising written justifications (as opposed to placeholder writing actions, which generally signaled the *start* of attempts to resolve uncertainties about written justification). As such, the localized nature of PDE preceding students' out-of-order writing actions gives insight into how they attempted to resolve their uncertainties.

In general, out-of-order writing actions were preceded by more localized evidence of PDE than overall writing actions were. For eight indicators of PDE, the proportion of out-of-order writing actions that were preceded by the indicator (Row 3, see Table 16 and Table 17) was more than the proportion of overall writing actions that were preceded by the same indicator by more than 10% (Row 1, see Table 16 and Table 17. In particular, indicators P3 (*When uncertain or stuck, students push to resolve the issue themselves and/or with help from their peers or teacher*), Au2 (*Students often talk about the mathematics topic/problem using their own words*), Au4 (*Students react to peers' ideas as worthy of consideration and respond constructively*), Ac2 (*Students often deliberate about multiple mathematical ideas*), Ac3 (*Most students participate in each instance of deliberation, with all group members deliberating at some point during the task*), Ac4 (*Students' ideas are discussed in terms of their utility for problem-solving or their mathematical relevance*), R2 (*Students often use relevant resources they create or choose*), and R3 (*Students use resources in a way that helps them resolve mathematical uncertainties*) occurred respectively 21.2%, 17.7%, 11.2%, 25.8%, 19.4%, 12.5%, 11.8%, and 16.6% more frequently preceding out-of-order writing actions than they did preceding overall writing actions. This represents increased levels of each principle of PDE within small group work preceding out-of-order writing actions.

The most dramatic difference of those described above was in evidence of indicator Ac2, which was observed preceding 72.4% of out-of-order writing actions as compared to 46.6% of overall writing actions. As such, it was extremely likely that students deliberated with peers to compare multiple mathematical ideas as they attempted to resolve their uncertainties about revising written justifications. Because indicators Ac3 and and Ac4 were also comparably more likely preceding out-of-order writing actions, this deliberation often involved most or all of the students in the group and to be focused on the mathematical relevance of students' ideas. In short, my findings suggest that comparing mathematical ideas with multiple peers in verbal or digital form supported students in their attempts to resolve uncertainties about their written justifications.

As an example, consider the work of four students in Ms. Price's class as they worked to complete the Initial Challenge of *Stretching and Shrinking* 4.1, in which students are asked to identify similar pairs of rectangles and similar pairs of triangles and explain why they are similar. For the first several minutes of the task, students mentioned potential similar pairs of rectangles to their peers, who responded and explained the reason for their response. At this point, some students had silently typed claims about similar pairs of triangles, but then they began to talk about these as a group. Table 18 shows the conversation of three students in the group as they sought to incorporate evidence, as well as one student's writing actions. The fourth student attended to the conversation, nodded to confirm peers' claims, and checked their written justification as their peers spoke, but did not speak during this portion of small group work. Note that only one student's writing actions are included because only one of the selected focal students made changes to their written justification during this exchange.

Table 18:

Transcript of Student Dialogue and Writing Actions During Small Group Work on Stretching and Shrinking 4.1

Speaker	Spoken words	Kingsley's writing actions
Leah	Is it C and B or C and D?	Pre-existing written justification: "A and D, C and B, F and G."
Sayuri	For which one? The rectangle? The rectangle or the triangles?	
Leah	For the triangles.	

Table 18 (cont'd)

Sayuri	Yeah, I got C and D because they have the same angles and these ones don't. (pause) Oh, I'm so hungry!	Types warrant: "A and D, C and B, F and G. These shapes go together because they all look alike in some way."
Kingsley	What do you guys have?	
Sayuri	Okay, shapes A and D, C and B...	
Kingsley	(overlapping speech) Same.	
Sayuri	... F and G, and Triangle C and Triangle D.	
Kingsley	Oh, I forgot to say triangles. (opens the problem statement to look at the diagram of all shapes)	
Sayuri	Oh, yes. Much better. We need to say what the ratios are.	Types claim: "A and D, C and B, F and G, Triangles C and D. These shapes go together because they all look alike in some way.
Kingsley	Yeah we do.	

In this exchange, peers compared their mathematical ideas in verbal form. This comparison and deliberation allowed Kingsley to add a correct claim to their written justification, and for the group to establish a plan for supporting these claims by measuring ratios of side lengths for similar shapes. As such the localized level of accountability supported this student to determine that they did not yet have a complete set of mathematical claims and to take action to revise their written justification accordingly. Unfortunately, the group did not continue to demonstrate this same level of accountability as they continued to determine ratios for the next several minutes and students in the group did not make further out-of-order changes to modify the invalid claim that rectangles A and D were mathematically similar (see Table 8 in previous results). Although the group demonstrated a medium level of PDE during the task overall, these two episodes show very different kinds of localized behavior around PDE, and thus led to different types of writing actions for students.

In addition to comparing multiple ideas in verbal form, the collaborative view within the digital environment allowed students to compare ideas in digital form by viewing multiple peers' written justifications and other digital work. As described in the previous section, students often entered the collaborative view when they felt "stuck" about what to write next to continue their written justification. Students also entered the collaborative view when they thought their own written justification might be complete, but were not yet sure (signaled by saying, for example, "I think my answer is done. Should I add anything?" or "Is anything missing from mine?"). This behavior of entering the collaborative view to determine if or how to revise written justifications did not always lead to an out-of-order writing action. Indeed, across the data there were many instances where students entered the collaborative view after finishing their own written justification and decided that their written justification did not need any revision or additions. Obviously these cases did not result in an out-of-order writing action. When they did result in an out-of-order writing action, students did not copy verbatim from peers' workspaces, but found a way to put peers' ideas into their own words as they revised their written justification. This is one potential explanation for the comparatively high levels of indicator Au2 in conjunction with out-of-order writing actions.

Summary of Results on Uncertainties About Written Justification

In this chapter I have shared results to the third research sub-question: *How do middle grades students seek to resolve uncertainties about written justification during small group work in a collaborative mathematics learning environment?* I examined two kinds of uncertainties about written justification: when students began to type some part of a written justification but then experienced uncertainty about how to continue justifying in written form, and when students experienced uncertainty about revising their written justification and took out-of-order writing actions to address this. In resolving both types of uncertainty, student behaviors showed increased levels of multiple principles of PDE. My findings suggest that it was especially important for students to engage with other students' mathematical contributions and to compare and deliberate about multiple mathematical ideas through verbal interactions and/or collaborative features of the digital environment.

In their work to resolve uncertainties about written justification students used multiple different features of the digital collaborative environment (e.g. examining the problem statement, engaging with a resource of their own choosing or creation, seeking out peers' ideas digitally) to successfully continue constructing or revising their written justification. There were three features of the digital environment, however, that I expected students to use as they sought to resolve uncertainties about written justification but that they did not use: "just-in-time" supports, published work, and learning logs. The just-in-time supports were designed specifically to help students when stuck by giving specific ideas about how to interpret the mathematical context or approach the task, but across the data for this study students never accessed them during small group work. Both teachers did show the just-in-time supports to students during the overall data collection for the larger research project, so students were aware of their existence. But for some reason, students did not access them during moments of uncertainty (or at any point during small group work). Across the data for this study, published work (where students are able to share and access the work of class members outside their group) was accessed once during small group work when a student accidentally published their work, but at no other times during small group work. Both teachers used published work as part of whole-class summary discussions, but students themselves did not choose to use it without teacher prompting. Finally, learning logs were created by the students when directed by Ms. Roberts in order to keep a record of vocabulary terms and major mathematical ideas for the unit, but students themselves did not choose to access learning logs unless directed by the teacher during whole-class discussions. Across the data for this study, none of these features were used by students when they encountered uncertainties about written justification. Although these features might have helped students resolve their uncertainties, their use was not normalized during small group work. In this study I considered the digital collaborative environment as a context for students' work in small groups and to construct written justifications, rather than seeking to identify what led students to utilize particular features. The fact that students utilized digital features that allowed them to access resources from their small group and information about the current problem rather than features that allowed access to the larger classroom community or could be used across problems, however, is an important consideration for future research.

It is clear from the prevalence of both types of uncertainties (which accounted for 61 out of 163 total writing actions, or 37.4%) that constructing written justification was not a trivial matter of transcribing the results of their oral collaborative attempts at justification. Rather, for the students in this study constructing written justifications was its own mathematical process that came with its own challenges and uncertainties. As students worked to construct written justifications, they often questioned whether their written work was "reasonable", "understandable", or "enough", and they sought the support of peers and further insight from mathematical inscriptions to address these concerns. With these supports, students were largely successful in taking writing actions that made their written justifications more mathematically valid and complete. I have described students' work to construct written justification in one context: collaborative work in a digital environment with integrated tools for representing their ideas. By better understanding the features and behaviors that supported students to construct written justification in this context, we can gain understanding about what might support students to construct written justification in other contexts as well.

CHAPTER 7:

DISCUSSION

In this chapter, I discuss the significant ways that this study relates to prior research on PDE and students' small group work to construct written justifications. My findings suggest potentially productive social norms and sociomathematical norms that can support students in collaborative work to construct written justifications, which have accompanying implications for teachers and curriculum designers. I then discuss the limitations of the current study and directions for further research suggested by my findings.

Significance of the Study

One of the major purposes of this study was to examine middle grade students' written justifications, both in terms of the writing actions that students took as they engaged in the process and in terms of the qualities of their finished written justifications. Because the process of constructing written justifications occurred in the context of small group work in a collaborative digital environment, the nature of the group's PDE shaped students' individual work to construct written justifications and vice versa. I examined this relationship between PDE and students' written justification through screen recordings of students' collaborative work in a novel digital environment on six different mathematical units from the CMP curriculum in two experienced teachers' classrooms. In this way, I examined the overall nature of students' engagement on tasks and their written justification, as well as the localized nature of their PDE during moments when they took specific writing actions towards constructing written justifications.

As described by Engle & Conant (2002), the concept of PDE offers guiding principles for designing learning environments. The PDE-CW framework expands on their work to consider how students' particular actions and decisions during collaborative work might contribute to overall PDE; in short, to describe what PDE really "looks like" for students. By foregrounding the way that students build and experience PDE, this study complements existing research which has focused on teachers' facilitation of PDE (e.g. Cornelius & Herrenkohl, 2004; Venturini & Amade-Escot, 2014; Williams-Candek, 2015).

My findings suggest that when higher levels of PDE occurred during collaborative group work in the digital environment, students were also able to construct more mathematically complete written justifications. Reciprocally, as students made attempts to construct mathematically valid written justifications, this often prompted increased levels of multiple principles of PDE within collaborative group work. In the context of this study then, integrating written justification and collaborative work on open mathematics problems in a digital environment was supportive for students as they sought to write mathematically valid justifications that included claims, evidence, warrants, and backing. As such, including opportunities for students to engage in mathematical writing as part of collaborative work with peers might be one way to help students deal with the oft-described challenges of written justification (Campbell et al., 2020; Cross, 2009).

As part of my analysis, I focused particularly on students' uncertainties about written justification. These uncertainties show that written justification presents challenges for students beyond simply transcribing the results of their oral collaborative work with peers, and give insight into how students might experience justification differently in different modalities. As students made attempts to resolve their uncertainties about how to continue justifying and revising in written form, they often sought support from the problem context, by creating inscriptions to try and make sense of the problem, and verbally or digitally deliberating with peers. Although these actions occurred in a unique digital context, they contribute to theory about how students engage in the process of constructing written justifications.

Productive Norms for Supporting Written Justification and Their Implications

Existing research on students' participation in collaborative oral justification (either in whole-class settings or in small group settings) has identified several productive social norms and sociomathematical norms that allow students to participate in justification. One of the most commonly-identified classroom norms for supporting students' oral justifications is that students are responsible to make sense of their peers' mathematical ideas (e.g. Weber et al., 2008; Wood et al., 1993). This same social norm seems to support students' written justification, as groups demonstrated increased questioning, deliberation, and other indicators of accountability on tasks where group members constructed more mathematically complete and valid written justifications. In the context of written justification in the digital environment it was also important that students explain and justify their mathematical thinking, even if peers weren't actively making sense of it at that moment. In the digital collaborative environment, multiple students were often authoring written justifications simultaneously, which is different from oral situations where one student can speak at a time and all other students hold them accountable for making sense. So, while in situations involving oral justification it is important that students make sense of others' mathematical thinking, in situations involving written justification it is equally important that students try to be sensible to others. This may take on an even greater level of importance in situations where written justifications are produced outside of collaborative work, without the immediate opportunity for students to question each other about any confusing parts of their written justifications.

Across multiple groups of students, I observed that listening to peers' mathematical ideas or viewing peers' mathematical work in the digital environment was productive for students in increasing the validity or completeness of their written justifications. Copying peers' work verbatim without further attempts to explain or modify it, however, often led to persistent errors in reasoning that students did not notice or correct. This suggests that a productive social norm for supporting students' written justification is that students are responsible to compare and build on peers' ideas. In whole-class oral settings, making connections between multiple students' ideas is an important responsibility of the teacher as they facilitate discussion (Stein et al., 2008), but during collaborative group work and written justification students might need to take more authority for this themselves. In this study, students had mixed success in using features of the digital collaborative environment to do so. But when groups did establish a social norm of comparing and building on peers' ideas (either verbally or digitally), they were able to successfully increase the mathematical completeness and validity of their written justifications.

One more social norm that supported students to construct more mathematically complete and valid written justifications was that students are responsible to review and revise their mathematical thinking. In an oral modality, this behavior might go unrecognized in groups because students do not have to formally erase, delete, or modify their previously spoken words in order to revise their mathematical ideas. But in written modality, this behavior takes on added importance. In groups where students reviewed their written work and considered whether revision was necessary, students constructed more mathematically complete and valid written justifications. One reason for this was that students' revisions almost always increased or maintained the mathematical validity of the written justification, and often added evidence or warrants to support their mathematical claims as students considered whether their written justifications were reasonable or finished. As such, in groups that established and maintained a norm of revising their written work, students were more likely to improve their written justifications over time. It might also be possible that in groups where students engaged in revision of their written work, they were also more likely to write down their initial conjectures and ideas and engage in writing actions overall. From the perspective of writing as knowledge-transforming (Bereiter & Scardamalia, 1987), when students externalize their thinking in this way they are better able to examine it and make sense of it. For either of these potential reasons, when groups established a norm of reviewing and revising their written work, they were better able to construct mathematically complete and valid written justifications.

This study also suggests a potentially productive sociomathematical norm for students' construction of written justification, that students are responsible to explain and interpret evidence rather than letting the evidence speak for itself. It was not common for groups to establish or maintain this norm in the data for this study. Across multiple groups, I observed students offering tables, graphs, and equations without any further written comment or explanation (even in cases where students verbally discussed these inscriptions with their groups). This suggests that it was more common for groups to uphold the sociomathematical norm that evidence counts as an acceptable justification on its own, which can be interpreted the same by any reader.

Although this result is not surprising, it raises questions about how teachers should support students in different modalities for justification. If students verbally discuss and unpack the meaning of inscriptions and other evidence during collaborative oral justification, but do not engage in these behaviors in their written justification, this gives two very different pictures of students' mathematical progress and abilities to justify. How might the teacher reconcile these two different pictures? Further, how might teachers and peers support students to unpack the meaning of inscriptions and other evidence to a greater degree in their written justifications? Written justification (and mathematical writing in general) has become increasingly important as more classrooms adopt asynchronous, hybrid, or virtual models that rely on written text in chat features, responses to peers, and formative or summative assessments. But if students are not supported to communicate the full extent of their reasoning in these digitally-enhanced classrooms, then the written work they produce may not give teachers useful insight into students' ways of knowing and doing mathematics.

Because of this observed disconnect between the way students treat evidence in oral and written modalities, it seems that groups might establish different sociomathematical norms in different modalities. Consider the work of the small group shown in Table 8 and Table 9, as they attempted to find the length of a race that a younger, slower brother could win with a given head start. In oral collaborative work with their small groups, students verbally interpreted specific features of graphs and tables (e.g. intersection points, slope) and used these features to justify the length they had determined for the race. In written justification, however, students offered tables and graphs without further comment. In both modalities, these two very different types of behaviors were enacted by the group and seemed to be viewed as acceptable behavior. But if students are to construct mathematically complete and valid written justifications, the sociomathematical norm that evidence is an acceptable justification that speaks for itself may not be productive for students. In this study, I did not focus on the role of the teacher in establishing and maintaining productive norms for written justification, but these potential social and sociomathematical norms offer insight into how teachers might play an important role in making connections between how students behave in both modalities.

Prior research has already identified that teachers can help to establish and maintain social norms that support students' oral justification in a variety of ways. For example, teachers can help to maintain the norms that students are responsible to make sense of peers' work and that students are responsible to compare and build on peers' ideas through instructional moves that probe specific details of students' thinking, invite students to question peers, and help students to connect their own ideas to those of peers (Mueller et al., 2014; Wood et al., 1993). In these instructional moves, the teacher acts as the orchestrator to shape a mathematical conversation in which they help students to participate. At the same time as teachers provide opportunities for students to engage in productive oral justification, Wood and colleagues (1993) describe that teachers should "talk about talking about mathematics." That is, in order to establish and maintain productive classroom norms teachers need to provide opportunities for students to enact these norms as well as modeling and discussing these norms with students. When we also consider students' written justification, the teacher has an important responsibility to prepare students to participate effectively in future justifications which the students will necessarily take a greater role in shape shaping.

To this end, teachers should talk about *writing* about mathematics with their students as well as modeling effective written justification and providing consistent opportunities for students to construct written justifications. This also aligns with the findings of literacy education research (described in Chapter 2) that students can develop effective argumentative writing moves through explicit discussion of the form/structure of written argumentation and through modeling of effective writing moves in classroom discussion. If, as my findings suggest, it is a productive social norm that students are responsible to make their own mathematical thinking sensible to others, then this implies that teachers should model constructing sensible written justification and talk with students about how they engage in ensuring that their written justification makes sense (for example, through considering potential readers or checking for inaccurate statements). If, as my findings suggest, it is a productive sociomathematical norm for students to explain and interpret evidence rather than letting evidence speak for itself, then this implies that teachers should model different ways of unpacking evidence during collaborative discussion as well as talking with students about how this occurs in written form.

As teachers work to have these discussions about written justification with students, it is important that they provide opportunities for students to construct written justification. Both teachers and curriculum designers can work to incorporate more opportunities for middle grades students to engage in written justification. Further, these opportunities should be structured so that students can develop productive norms around written justification, such as those I describe above. For example, the curricular prompt or the teacher might in provide opportunities for students to review and revise a previously-constructed written justification in light of new mathematical information or a specific imagined reader. As another example, teachers might adapt the practice of "gallery walks" (where students walk around the classroom to view the full variety of peers' strategies on a given problem) to allow opportunities for students to build on peers' ideas in writing by giving students a writing-specific goal such as "Find one strategy or idea on your gallery walk that you can use to improve your written justification." Such a practice, either in the curriculum as written or in the teacher's instantiation of problems, might help students to maintain the productive norm that they are responsible to compare and build on peers' ideas without allowing for the unproductive behavior of verbatim copying.

As digital curriculum and other technology for teaching and learning mathematics digitally continue to be developed, they should also allow for students to develop productive norms around written justification. Although I did not study students' use of specific features, there are many features and tools within the digital collaborative environment that already have the potential to do this. For example, the way that text, graphs, tables, and other inscriptions are treated in the environment allows students to continually revise and revisit any previously-created justifications or inscriptions. Additionally, it allows them to continue modifying and building on any inscriptions that are copied from peers' workspaces. This allows students to maintain the productive social norms described above, although teachers and students would need to collectively establish these norms first. The comment feature, flexibility to determine axes and headings graphs and tables, and the drawing tool give students the ability to be specific about how they are unpacking and interpreting evidence. These features allow students to maintain the productive social norm of explaining and interpreting evidence rather than attempting to let it "speak for itself", provided teachers and students work together to establish this norm first. As mark-up and commenting tools like continue to be developed, developers can work to make them easy to use and accessible, and to ensure that they are usable on a wide variety of inscriptions. Although digital tools cannot establish productive norms within a classroom, the goal should be to make it easier for teachers and students to maintain these productive norms as they construct written justifications.

Limitations of the Study

This study addresses students' construction of written justifications in the context of small group work in a digital collaborative environment. As such, I foreground students' social and individual activity as they work to construct written justifications. By doing so, however, the teacher's role in promoting PDE and justification is less emphasized. Prior research on both PDE and oral justification (as discussed in my literature review) has identified the teacher's powerful role in promoting these practices. Although the teacher set up the tasks that students worked on and interacted with small groups as they worked, I do not explicitly consider the effect that the teacher had on how students enact the principles of problematizing, authority, accountability, and resources. Further, in my analysis of data for this study, I observed some instances where students continued to develop their written justifications during whole-group summary discussions following small group tasks. How did the nature of students' PDE during small group work relate to the way they continued to develop their written justifications in this setting? What role did the teacher play in facilitating this? By foregrounding student activity in small groups, I am largely not able to answer these questions in the scope of this study. By focusing on students' maintenance of classroom norms and engagement in justification, however, this study will complement existing teacher-focused work.

One further limitation of this study is that the overall number of tasks sampled for data analysis was fairly small (a total of 38 tasks). As described in Methods, this was the largest feasible sample size given the complexity of data analysis that incorporated multiple layers of analysis of PDE on overall tasks, overall qualities of students' written justifications and localized analysis of students' writing actions and the PDE surrounding them. Across this sample, most of the tasks analyzed were from the "Initial Challenge" and "What If…?" parts of the CMP STEM problem format (20 and 13 tasks, respectively). Beyond this, two tasks were from the "Introduction" and three tasks were from the "Now What Do You Know?" parts of the CMP STEM problem format. Although each of these parts of the problem offered students the opportunity to justify, students may approach justification differently as their familiarity with the mathematical topic increases throughout different parts of the problem and they synthesize, generalize, and abstract their knowledge (as described in CMP's Arc of Learning by Edson and colleagues (2019)). As students' mathematical knowledge increases and becomes more connected, it is an important open question how the nature of their written justifications might shift. Further, the data were collected from the classrooms of highly experienced middle grades mathematics teachers in the context of a digital collaborative environment that is not widely available in classrooms. Although this context was purposefully chosen because of the heightened opportunity it presents for students to construct written justifications, the findings are not automatically generalizable to students in other classroom contexts. Rather, this study presents a useful starting point for characterizing how students construct written justifications in the context of collaborative small group work.

Because students were working in a particular digital environment in this study, the specific features that were available within the digital environment likely shaped the ways that students attempted to solve problems and construct written justifications. In this collaborative digital environment, students had access to a variety of tools for representing mathematics, individual spaces in which to work, a way to view the workspaces of group members in real-time, and access to material from the curriculum and their past work. This is not the same as, for example, the online digital problem-solving environment described by Schreiber (2013) where groups of students worked in one shared space for producing inscriptions and communicated only through chat features. The nature of mathematical justification that emerged in that environment would likely be very different than what I observed in the context of this study. In other digital environments or paper-and-pencil classroom environments, what types of writing actions might students take as they attempt to construct written justification? Revision of written justifications, verbal or digital comparison of work with other peers, and unpacking evidence emerged as particularly important behaviors for supporting student justification in this setting.

These behaviors might not be as likely in other contexts, however, that do not allow for the same degree of digital collaboration. In these cases, what actions would students take as they attempt to construct mathematically complete and valid written justifications? Findings from this specific context should be built on with future work in classrooms utilizing different technologies, different support for collaborative learning, or with more focus on how teachers prepare students to use the collaborative digital environment.

Directions for Further Research

My findings from students' small group work in the digital collaborative environment show that the mathematical validity and completeness of students' written justifications were connected to the nature of collaboration (as reflected in the level of PDE demonstrated by groups). This relationship was found at a general level on tasks, and increased levels of PDE at a more localized scale supported student students' writing actions to add warrants/backing, increase mathematical validity, and revise their written justifications. Across the data for this study, it has become clear that constructing a written justification poses unique challenges for students (beyond simply transcribing the results of their collaborative work with peers) and students encounter additional mathematical uncertainties as they engage with justification in written and digital modalities. As described above, this study offers insight into several potentially productive social norms and sociomathematical norms for supporting students' construction of written justifications, and future research should address students' written justification processes in other contexts and how these processes develop for students throughout the different stages of their mathematical learning about a topic. Other settings where collaborative work and written justification occur sequentially (rather than at the same time as in this study), and where student collaboration is fully digital or fully non-digital would provide a useful way to consider students' construction of written justifications in multiple disparate contexts.

In this study, the sociomathematical norm that evidence can speak for itself without further interpretation was not necessarily productive for the mathematical completeness and validity of students' written justifications. I found that even when students interpreted evidence verbally during collaborative work, they did not often do so in their writing. When students did question and interpret the meaning of inscriptions and other resources, however, this helped them to increase the mathematical completeness and validity of their written justifications. As such, this phenomenon is of particular interest for future research. In the digital collaborative environment, students used several features to unpack the meaning of mathematical inscriptions during their small group work. These did not often lead to the construction of a written justification, however, and thus they were not often included in localized analysis of students' writing actions. These features include commenting/labelling specific points or shapes on graphs, customizing header and axis labels for tables and graphs, controlling the proximity of different inscriptions by rearranging them within their own workspace. When students utilize features like these that allow them to add their own interpretation of inscriptions or other evidence, how does this affect their writing actions and their overall written justifications? More investigation of these features and the nature of PDE and writing actions surrounding their use would be helpful to understanding how and why students decide to unpack evidence in their collaborative group work and their written justifications.

Further research should also consider how teachers support students to unpack evidence in written justification. Existing research on whole-class oral justification describes that teachers play an important role in supporting students to make contributions to reasoning (Staples, 2007), eliciting and facilitating discussion or particular points of students' reasoning (Ellis et al., 2019), and pressing for further explanation (Conner et al, 2014). Although the classroom teacher was not the focal point of this study, I observed both teachers engage in these practices during whole-class discussions and as they circulated around the classroom during small group discussions, by asking students about particular features of their inscriptions, asking students to further explain their own work or build on peers' work, and inviting multiple students to contribute to classroom oral justifications. These kinds of supports can help students to unpack and interpret evidence during whole-class oral justification, but did not seem to help students to do so in their own written justifications. One potential explanation is that students view this as part of their own sense-making process, but not something that needs to be included in written justifications. Another potential explanation is that during whole-class oral instruction the teacher takes responsibility for shaping how evidence is unpacked (e.g. the particular features of evidence that they attend to, when the evidence has been sufficiently explained) and that students might require additional support to take on responsibility for shaping this process in their own written justifications. Future research should attend to how the teacher can shift authority for making decisions about justification to students, in order to better support them as they construct written justifications more independently.

Conclusion

Given the important role that mathematical justification plays in students' learning of mathematics, it is important that we understand how students engage in this practice in a variety of contexts. The results of this study show that increased levels of problematizing, authority, accountability, and resources that make up students' PDE in collaborative work are related to increased levels of mathematical completeness in students' written justifications. Additionally, students' work to produce mathematically valid written justifications in the context of this digital environment can promote PDE during their collaborative work. This occurs at a generalized level on open mathematical tasks where students can collaboratively create and investigate inscriptions, and during students' specific writing actions as they construct written justifications. This study builds on existing work around students' mathematical justification by focusing on the decisions that students make as they construct written justifications, and purposefully examines the role that modality and technology might play in students' engagement with mathematical justification.

APPENDICES

APPENDIX A:

Curricular Prompts for All Small Group Work

Problem Part	Curricular Prompt	Teacher(s) and Number of Tasks for this Problem Part
SS 1.1, Introduction	Write down two observations from the introduction and define "scale drawing." (Note: this task was introduced by the teacher and was not a curricular prompt)	Ms. Roberts, 1 task
SS 1.1, Initial Challenge	Is Daphne's claim correct? Is it possible to use the magazine and the picture to estimate the teacher's height? Explain.	Ms. Roberts, 2 tasks
SS 1.1, What If…?	Could you have found the height of the teacher if you did not know the magazine was 10 inches tall?	Ms. Roberts, 1 task
SS 1.1, Now What Do You Know?	The Mystery Club advisor says that the picture is similar to the actual scene. What do you suppose the advisor means by similar?	Ms. Roberts, 1 task
SS 1.2, Initial Challenge	How are the original figure and its image alike? Different? How do you know?	Ms. Roberts, 2 tasks
SS 1.2, What If…?	What if you could change the anchor point? How does this affect the image? What if you used three rubber bands? How does this affect the image?	Ms. Roberts, 1 task
SS 1.3, Introduction	What is the definition of corresponding sides and corresponding angles? (Note: this task was introduced by the teacher and was not a curricular prompt)	Ms. Roberts, 1 task
SS 1.3, What If…?	Amelia thinks that the copier creates images that are similar to those made with rubber band stretchers and the photograph. Do you agree? Explain why.	Ms. Roberts, 1 task

SS 1.MR, Mathematical Reflection	Now what do you know about similarity?	Ms. Roberts, 1 task
SS 3.1, Initial Challenge	Which shapes are rep-tiles? Make a sketch to show how the copies fit together. For the shapes that rep-tile, what is the scale factor from the original figure to its image? Explain your reasoning.	Ms. Price, 1 task
SS 3.1, What If…?	Suppose you are given a rectangle or a triangle rep-tile and a scale factor of 5. How many copies of your rep-tile are needed to make the scale copy? Explain your reasoning.	Ms. Price, 1 task
SS 3.2, Initial Challenge Part 1	For each part, draw a rectangle similar to rectangle A that fits the given description. Explain your reasoning. - Rectangle P: The scale factor from rectangle A to rectangle P is 2.5. - Rectangle R: The area of rectangle R is 1/4 the area of rectangle A.	Ms. Price, 2 tasks
SS 3.2, Initial Challenge Part 2	For each part, draw a triangle similar to triangle B that fits the given description. Explain your reasoning. - Triangle X: The area of triangle X is 16 times the area of triangle B. - Triangle Y: The scale factor from triangle B to triangle X is 1/2. - Triangle Z: The ratio of corresponding side lengths of triangle B to triangle Z is 2:3.	Ms. Price, 1 task
SS 3.3, Initial Challenge	What is the distance across the river from Stake 1 to Tree 1? Explain your reasoning.	Ms. Price, 2 tasks
SS 4.1, Initial Challenge	For each set of shapes, which are similar? Why? If two figures are similar, how does the ratio of adjacent side lengths in one figure compare to the corresponding ratio of side lengths of the other figure?	Ms. Price, 1 task
SS 4.2, Initial Challenge	Find the missing side length in each pair. Explain how you know your answers are correct.	Ms. Price, 1 task
SS 4.2, What If…?	Find the value of x. Explain how you found it.	Ms. Price, 1 task

SS 4.3, What If…?	Does this method work? Why? What estimate should Ashton give for the principal's height?	Ms. Price, 1 task
SS 4.MR, Mathematical Reflection	Now what do you know about similarity?	Ms. Price, 1 task
MSA 2.1, Initial Challenge	How long should the race be so that Henri will win in a close race? Give evidence to support your answer.	Ms. Price, 3 tasks; Ms. Roberts, 1 task
MSA 2.1, What If…?	1. How can a table or graph be used to find the length of the race? 2. How does the walking rate of each brother show up in the table or graph? 3. After 20 seconds, how far apart are the brothers? How is this distance represented in the table and on the graph?	Ms. Price, 1 task
MSA 2.2, Initial Challenge	Which company should they choose? Give evidence for your answer.	Ms. Price, 1 task
MSA 2.2, What If…?	1. Explain how a graph or table could be helpful. 2. What if the class only had $120 dollars to spend, how would this affect your answer?	Ms. Price 1 task
MSA 2.3, Initial Challenge	What information does the equation give about the pledge plan? Does the plan make sense? Explain your reasoning. How might a table or a graph or the equation help answer questions about the pledge plans?	Ms. Roberts, 1 task
MSA 2.3, What If…?	1. Which plan has a graph that contains the point (2,4)? What question could you answer by locating that point? 2. Which plan has a graph that you can use to find the solution for 8=5x-3? What would the coordinates of the point that represents the solution tell you about the situation?	Ms. Price 1 task; Ms. Roberts, 2 tasks

MSA 4.1, Initial Challenge	How can you describe the steepness of the stairs? Is the steepness the same between any two consecutive steps? How do the stairs you measured compare to the carpenter's guidelines on the previous page?	Ms. Roberts, 1 task
MSA 4.1, What If…?	A set of stairs is being build for the front of the new Arch Middle School. The ratio of rise to run is 7 to 11. Is this ratio within the carpenter's guidelines? Make a sketch of a set of stairs that meets this ratio. Label the lengths of the rise and run of the step. Create a graph of a line that passes through the origin and whose y values increase by 7 units for every 11 unit change in the x values. Does everyone in your class have the same graph? Explain why.	Ms. Roberts, 1 task
MSA 4.2, What If…?	What is the constant rate of change and y-intercept associated with each representation? Do any of the relationships represent a proportional relationship?	Ms. Roberts, 1 task
MSA 4.3, Initial Challenge	Each letter provides criteria for a relationship. For each situation, decide if the relationship could be linear. For those that are linear, is there exactly one or more than one? Explain. - Situation: A graph that passes through exactly three quadrants.	Ms. Roberts, 1 task

Note: *SS* refers to *Stretching and Shrinking* and *MSA* refers to *Moving Straight Ahead.*

APPENDIX B:

Key Terms for Data Analysis

Item	Categories or terms	Explanation/source/use
Writing Actions	Begin Add Modify Delete	For written justifications, discrete conversation with at least 10 second breaks between actions
Writing Actions	In-order Out-of-order	For written justifications, compare to final order of the justification
Elements of written justification	Claims Evidence Warrants Backing	For written justifications, from Toulmin (1958)
Completeness	None written Emergent Statement Recount Procedural Detailing Description	For written justifications, modified from Kosko and Zimmerman (2019)
Localized analysis	Writing Actions Speech or digital actions	Includes the speech and digital actions or a group during the two minutes preceding students' writing actions.
PDE Indicators	P*n* Problematizing Au*n* Authority Ac*n* Accountability R*n* Resources	For small group work, *n* has values 1-4 for P, Au, and Ac; values 1-3 for R for a total of 15 categories. Each category has Low, Medium, and High levels based on overall prevalence of that behavior
PDE Level	Low Medium High	For small group work, level determined by the prevalence of indicators from the PDE-CW framework during the overall task

Task-level analysis	Written justification Level of PDE	Includes overall qualities of students' finished written justifications and the overall level of PDE during small group work on that task
Uncertainties about written justification	How to justify in written form How to revise	Identified through placeholder writing actions and out-of-order writing actions
Validity	Always Emerging Never Not analyzable	For written justifications, category determined by examining state throughout construction
Validity	Increasing Decreasing	For writing actions, category applied if the validity of the overall justification changes

BIBLIOGRAPHY

BIBLIOGRAPHY

Baccaglini-Frank, A. M., & Mariotti, A. (2010). Generating conjectures in dynamic geometry: The maintaining dragging model. *International Journal of Computers for Mathematical Learning*, *15*(3), 225–253. https://doi.org/10.1007/s10758-010-9169-3

Bakewell, L. L. (2008). Writing Use in the Mathematics Classroom. *Studies in Teaching 2008 Research Digest*, *7*, 7–12. Retrieved from /citations?view_op=view_citation&continue=/scholar?hl=en&start=20&as_sdt=0,5&scilib=1&citilm=1&citation_for_view=-8N_dQUAAAAJ:wyM6WWKXmoIC&hl=en&oi=p

Balacheff, N. (2008). The role of the researcher's epistemology in mathematics education: an essay on the case of proof. *ZDM*, *40*, 501–502. https://doi.org/10.1007/s11858-008-0103-2

Ball, D. L., & Bass, H. (2003). Making mathematics reasonable in school. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A Research Companion to Principles and Standards for School Mathematics* (pp. 27–44). Reston, VA: National Council of Teachers of Mathematics.

Barron, B. (2003). When smart groups fail. *Journal of the Learning Sciences*, *12*(3), 307–359. https://doi.org/10.1207/S15327809JLS1203_1

Bereiter, C., & Scardamalia, M. (1987). *The psychology of written composition*. Hillsdale, NJ: Erlbaum.

Bieda, K. N., & Lepak, J. (2014). Are you convinced ? Middle-grade students' evaluations of mathematical arguments. *School Science and Mathematics*, *114*(4), 166–177.

Bieda, K. N., & Staples, M. (2020). Justification as an equity practice. *The Mathematics Teacher*, *113*(2), 102–108.

Bieda, K. N., Going, T., Kursav, M., Edson, A. J., & Phillips, E. (revision in preparation). Productive disciplinary engagement in technology-enhanced classrooms.

Boaler, J., & Staples, M. (2008). Creating mathematical futures through an equitable teaching approach: The case of railside school. *Teachers College Record*, *110*(3), 608–645. https://doi.org/Article

Boero, P., Douek, N., Morselli, F., Pedemonte, B., Camargo, L., Perry, P., & Samper, C. (2010). Argumentation and proof: A contribution to theoretical perspectives and their classroom implementation. In *Proceedings of PME 34 the 34th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 179–209). Universidade Federal de Minas Gerais: PME. Retrieved from http://www.seminariodidama.unito.it/2011/app/boero34.pdf

Cai, J., & Cirillo, M. (2014). What do we know about reasoning and proving? Opportunities and missing opportunities from curriculum analyses. *International Journal of Educational Research*, *64*. https://doi.org/10.1016/j.ijer.2013.10.007

Campbell, T. G., & Hodges, T. S. (2020). Using positioning theory to examine how students collaborate in groups in mathematics. *International Journal of Educational Research*, *103*, 1–13. https://doi.org/10.1016/j.ijer.2020.101632

Campbell, T. G., King, S., & Zelkowski, J. (2020). Comparing middle grade students' oral and written arguments. *Research in Mathematics Education*, 1–18. https://doi.org/10.1080/14794802.2020.1722960

Clarke, D. J., Waywood, A., & Stephens, M. (1993). Probing the structure of mathematical writing. *Educational Studies in Mathematics*, *25*(3), 235–250. https://doi.org/10.1007/BF01273863

Cobb, P., & Yackel, E. (1996). Constructivist, emergent, and sociocultural perspectives in the context of developmental research. *Educational Psychologist*, *31*(3), 175–190. https://doi.org/10.1207/s15326985ep3103&4_3

Cohen, E. G. (1994). Restructuring the classroom: Conditions for productive small groups. *Review of Educational Research*, *64*(1), 1–35. https://doi.org/10.3102/00346543064001001

Common Core State Standards Initiative. (2010). Common core state standards for mathematics. Retrieved from http://www.corestandards.org/assets/CCSSI_Math Standards.pdf

Conner, A. M., Singletary, L. M., Smith, R. C., Wagner, P. A., & Francisco, R. T. (2014). Teacher support for collective argumentation: A framework for examining how teachers support students' engagement in mathematical activities. *Educational Studies in Mathematics*, *86*(3), 401–429. https://doi.org/10.1007/s10649-014-9532-8

Cornelius, L. L., & Herrenkohl, L. R. (2004). Power in the classroom: How the classroom environment shapes students' relationships with each other and with concepts. *Cognition and Instruction*, *22*(4), 467–498. https://doi.org/10.1207/s1532690xci2204_4

Cross, D. I. (2009). Creating optimal mathematics learning environments: Combining argumentation and writing to enhance achievement. *International Journal of Science and Mathematics Education*, *7*(5), 905–930. https://doi.org/10.1007/s10763-008-9144-9

Czocher, J. A., & Weber, K. (2020). Proof as a cluster category. *Journal for Research in Mathematics Education*, *51*(1), 50–74.

Dawkins, P. C., & Karunakaran, S. S. (2016). Why research on proof-oriented mathematical behavior should attend to the role of particular mathematical content. *Journal of Mathematical Behavior*, *44*, 65–75. https://doi.org/10.1016/j.jmathb.2016.10.003

DeJarnette, A. F., & González, G. (2015). Positioning during group work on a novel task in Algebra II. *Journal for Research in Mathematics Education*, *46*(4), 378–422. https://doi.org/10.5951/jresematheduc.46.4.0378

Dietiker, L., Kysh, J., Sallee, T., & Hoey, B. (2006). *Core Connections Course 2* (3.1 ed.). Sacramento, CA: CPM Educational Program.

Driscoll, M. (1999). *Fostering algebraic thinking: A guide for teachers grade 6-10*. (Heinemann, Ed.). Portsmouth, NH.

Edson, A. J., Phillips, E., Slanger-Grant, Y., & Stewart, J. (2019). The Arc of Learning framework: An ergonomic resource for design and enactment of problem-based curriculum. *International Journal of Educational Research*, *93*, 118–135. https://doi.org/10.1016/j.ijer.2018.09.020

Edson, A. J., Rogers, D. R., & Browning, C. A. (2016). Formative assessment and preservice elementary teachers' mathematical justification. In M. Niess, S. Driskell, & K. Hollebrands (Eds.), *Handbook of Research on Transforming Mathematics Teacher Education in the Digital Age* (pp. 293–323). Hershey, PA: Information Science Reference. https://doi.org/10.4018/978-1-5225-0120-6.ch012

Ellis, A. B. (2007). Connections between generalizing and justifying: Students' reasoning with linear relationships. *Journal for Research in Mathematics Education*, *38*(3), 194–229.

Ellis, A., Özgür, Z., & Reiten, L. (2019). Teacher moves for supporting student reasoning. *Mathematics Education Research Journal*, *31*, 107–132. https://doi.org/10.1007/s13394-018-0246-6

Engelbrecht, J., Llinares, S., & Borba, M. C. (2020). Transformation of the mathematics classroom with the internet. *ZDM - Mathematics Education*, *52*(5), 825–841. https://doi.org/10.1007/s11858-020-01176-4

Engle, R. A. (2011). The productive disciplinary engagement framework: Origins, key concepts, and continuing developments. In D. Y. Dai (Ed.), *Design research on learning and thinking in educational settings: enhancing intellectual growth and functioning* (Vol. 1, pp. 161–200).

Engle, R. A., & Adiredja, A. P. (2008). Increasing the cognitive demand of mathematical tasks. In *Research Pre-session of the National Council of Teachers of Mathematics conference*. Salt Lake City, UT.

Engle, R. A., & Conant, F. R. (2002). Guiding principles for fostering productive disciplinary engagement: Explaining an emergent argument in a community of learners classroom. *Cognition and Instruction*, *20*(4), 399–483. https://doi.org/10.1207/S1532690XCI2004_1

Engle, R. A., Langer-Osuna, J. M., & McKinney de Royston, M. (2014). Toward a model of influence in persuasive discussions: Negotiating quality, authority, privilege, and access within a student-led argument. *Journal of the Learning Sciences*, *23*(2), 245–268. https://doi.org/10.1080/10508406.2014.883979

Esmonde, I. (2009). Mathematics learning in groups: Analyzing equity in two cooperative activity structures. *Journal of the Learning Sciences*, *18*(2), 247–284. https://doi.org/10.1080/10508400902797958

Felton, M., & Kuhn, D. (2001). The development of argumentive discourse skill. *Discourse Processes*, *32*(2–3), 135–153. https://doi.org/10.1080/0163853x.2001.9651595

Ferretti, R. P., & Graham, S. (2019). Argumentative writing: theory, assessment, and instruction. *Reading and Writing*, *32*(6), 1345–1357. https://doi.org/10.1007/s11145-019-09950-x

Flores, A. (2006). How do students know what they learn in middle school mathematics is true? *School Science and Mathematics*, *106*(3), 124–132. https://doi.org/10.1111/j.1949-8594.2006.tb18169.x

Forman, E. A., & Ansell, E. (2002). Orchestrating the multiple voices and inscriptions of a mathematics classroom. *The Journal of the Learning Sciences*, *11*(2/3), 251–274.

Francisco, J. M. (2013). Learning in collaborative settings: Students building on each other's ideas to promote their mathematical understanding. *Educational Studies in Mathematics*, *82*(3), 417–438. https://doi.org/10.1007/s10649-012-9437-3

Francisco, J. M., & Maher, C. A. (2005). Conditions for promoting reasoning in problem solving: Insights from a longitudinal study. *Journal of Mathematical Behavior*, *24*(3–4), 361–372. https://doi.org/10.1016/j.jmathb.2005.09.001

Gibbons, P. (2009). *English learners, academic literacy, and thinking*. Portsmouth, NH: Heinemann.

Going, T., Kursav, M., Slanger-Grant, Y., Bieda, K. N., & Edson, A. J. (2018). Understanding the nature of uncertainty in problem solving situations. In T. E. Hodges, G. J. Roy, & A. M. Tyminski (Eds.), *Proceedings of the 40h annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1335–1338). Greenville, SC: University of South Carolina & Clemson University.

Goos, M., & Bennison, A. (2008). Surveying the technology landscape: Teachers' use of technology in secondary mathematics classrooms. *Mathematics Education Research Journal*, *20*(3), 102–130. https://doi.org/10.1007/BF03217532

Hammer, D., & Berland, L. K. (2014). Confusing claims for data: A critique of common practices for presenting qualitative research on learning. *Journal of the Learning Sciences*, *23*(1), 37–46. https://doi.org/10.1080/10508406.2013.802652

Hanna, G. (2000). Proof, explanation, and exploration: An overview. *Educational Studies in Mathematics Proof in Dynamic Geometry Environments*, *44*(1/2), 5–23. Retrieved from http://www.jstor.org/stable/3483203

Harel, G., & Sowder, L. (2007). Second handbook of research on mathematics teaching and learning: a project of the National Council of Teachers of Mathematics. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 805–842). Reston, VA: National Council of Teachers of Mathematics.

Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. *CBMS Issues in Mathematics Education*, *7*, 234–283.

Healy, L., & Hoyles, C. (2000). A study of proof conceptions in algebra. *Journal for Research in Mathematics Education*, *31*(4), 396–428. https://doi.org/10.2307/749651

Herbel-Eisenmann, B., Cirillo, M., Steele, M. D., Otten, S., & Johnson, K. R. (2017). *Mathematics discourse in secondary classrooms: A practice-based resource for professional learning*. Menlo Park, CA: Math Solutions.

James, C., Philiben, L., & Knievel, M. (2016). Doing the maths: Supporting student justifications. *Mathematics Teaching in the Middle School*, *21*(7), 416. https://doi.org/10.5951/mathteacmiddscho.21.7.0416

Johanning, D. (2000). An analysis of writing and postwriting group collaboration in middle school pre-algebra. *School Science and Mathematics*, *100*(3), 151–160.

Johnson, D. W., Johnson, R. T., & Roseth, C. (2010). Cooperative learning in middle schools: Interrelationship of relationships and achievement. *Middle Grades Research Journal*, *5*(1), 1–18.

Knuth, E. J., Choppin, J. M., & Bieda, K. N. (2009). Middle school students' production of mathematical justifications. In *Teaching and Learning Proof Across the Grades: a K-16 Perspective* (pp. 153–170). Routledge.

Koretsky, M. D., Vauras, M., Jones, C., Iiskala, T., & Volet, S. (2019). Productive disciplinary engagement in high- and low-outcome student groups: Observations from three collaborative science learning contexts. *Research in Science Education*. https://doi.org/10.1007/s11165-019-9838-8

Kosko, K. W. (2016). Writing in mathematics: A survey of K-12 teachers' reported frequency in the classroom. *School Science and Mathematics*, *116*(5), 276–285. https://doi.org/10.1111/ssm.12177

Kosko, K. W., & Zimmerman, B. S. (2019). Emergence of argument in children's mathematical writing. *Journal of Early Childhood Literacy*, *19*(1), 82–106. https://doi.org/10.1177/1468798417712065

Krummheuer, G. (2007). Argumentation and participation in the primary mathematics classroom. Two episodes and related theoretical abductions. *Journal of Mathematical Behavior*, *26*(1), 60–82. https://doi.org/10.1016/j.jmathb.2007.02.001

Laborde, C. (2003). Technology used as a tool for mediating knowledge in the teaching of mathematics: The case of Cabri-geometry. In W. Yang, S. Chu, T. de Alwis, & G. Lee (Eds.), *Proceedings of the 8th Asian Technology Conference in Mathematics* (pp. 23–38). Hsinchu Taiwan ROC: Chung Hua University.

Langer-Osuna, J. M. (2017). Authority, identity, and collaborative mathematics. *Journal for Research in Mathematics Education*, *48*(3), 237–247.

Lannin, J. K. (2005). Generalization and justification: The challenge of introducing algebraic reasoning through patterning activities. *Mathematical Thinking and Learning*, *7*(3), 231–258. https://doi.org/10.1207/s15327833mtl0703_3

Lappan, G., Phillips, E., Fey, J. T., & Friel, S. N. (2014). *Connected Mathematics 3 Grade 7*. Boston, MA: Pearson.

Liu, Y., & Manouchehri, A. (2013). Middle school children's mathematical reasoning and proving schemes. *Investigations in Mathematics Learning*, *6*(1), 18–40. https://doi.org/10.1080/24727466.2013.11790328

Lockwood, E., DeJarnette, A. F., & Thomas, M. (2019). Computing as a mathematical disciplinary practice. *Journal of Mathematical Behavior*, *54*(February 2018), 1–18. https://doi.org/10.1016/j.jmathb.2019.01.004

Melhuish, K., Thanheiser, E., & Fagan, J. (2019). The Student Discourse Observation Tool: Supporting teachers in noticing justifying and generalizing. *Mathematics Teacher Educator*, *7*(2), 57–74. https://doi.org/10.5951/mathteaceduc.7.2.0057

Moschkovich, J. N. (2008). " I went by twos, he went by one": Multiple interpretations of inscriptions as resources for mathematical discussions. *The Journal of the Learning Sciences*, *17*(4), 551–587. https://doi.org/10.1080/10508400802395077

Mueller, M. F. (2009). The co-construction of arguments by middle-school students. *Journal of Mathematical Behavior*, *28*(2–3), 138–149. https://doi.org/10.1016/j.jmathb.2009.06.003

Mueller, M., & Maher, C. (2009). Learning to reason in an informal math after-school program. *Mathematics Education Research Journal*, *21*(3), 7–35.

Mueller, M., Yankelewitz, D., & Maher, C. (2014). Teachers promoting student mathematical reasoning. *Investigations in Mathematics Learning*, *7*(2), 1–20. https://doi.org/10.1080/24727466.2014.11790339

Nardi, E., & Knuth, E. (2017). Changing classroom culture, curricula, and instruction for proof and proving: how amenable to scaling up, practicable for curricular integration, and capable of producing long-lasting effects are current interventions? *Educational Studies in Mathematics*, *96*(2), 267–274. https://doi.org/10.1007/s10649-017-9785-0

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Newell, G. E., Beach, R., Smith, J., & VanDerHeide, J. (2011). Teaching and learning argumentative reading and writing: A review of research. *Reading Research Quarterly*, *46*(3), 273–304. https://doi.org/10.1598/RRQ.46.3.4

Nussbaum, E. M., & Schraw, G. (2007). Promoting argument-counterargument integration in students' writing. *Journal of Experimental Education*, *76*(1), 59–92. https://doi.org/10.3200/JEXE.76.1.59-92

Nystrand, M., & Graff, N. (2001). Report in argument's clothing: An ecological perspective on writing instruction in a classroom. *The Elementary School Journal*, *101*(4), 479–493.

Olivero, F., & Robutti, O. (2007). Measuring in dynamic geometry environments as a tool for conjecturing and proving. *International Journal of Computers for Mathematical Learning*, *12*(2), 135–156. https://doi.org/10.1007/s10758-007-9115-1

Patsiomitou, S. (2011). Theoretical dragging: A non-linguistic warrant leading to 'dynamic' propositions. In *35th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 361–368). Ankara, Turkey: PME.

Pearce, D. L., & Davison, D. M. (1988). Teacher Use of Writing in the Junior High Mathematics Classroom. *School Science and Mathematics*, *88*(1), 6–15. https://doi.org/10.1111/j.1949-8594.1988.tb11772.x

Powell, S. R., Hebert, M. A., Cohen, J. A., Casa, T. M., & Firmender, J. M. (2017). A Synthesis of mathematics writing. *Journal of Writing Research*. https://doi.org/10.17239/j

Powell, S. R., & Hebert, M. A. (2016). Influence of writing ability and computation skill on mathematics writing. *Elementary School Journal*, *117*(2), 310–335. https://doi.org/10.1086/688887

Rasmussen, C., Wawro, M., & Zandieh, M. (2015). Examining individual and collective level mathematical progress. *Educational Studies in Mathematics*, *88*(2), 259–281. https://doi.org/10.1007/s10649-014-9583-x

Roth, W., & McGinn, M. K. (1998). Inscriptions: Toward a theory of representing as social practice. *Review of Educational Research*, *68*(1), 35–59.

Schreiber, C. (2013). Semiotic processes in chat-based problem-solving situations. *Educational Studies in Mathematics*, *82*(1), 51–73.

Selling, S. K. (2016). Learning to represent, representing to learn. *Journal of Mathematical Behavior*, *41*, 191–209. https://doi.org/10.1016/j.jmathb.2015.10.003

Sen, C., & Guler, G. (2015). Examination of secondary school seventh graders' proof skills and proof schemes. *Universal Journal of Educational Research*, *3*(9), 617–631. https://doi.org/10.13189/ujer.2015.030906

Shaffer, D. W. (2017). *Quantitative Ethnography*. Madison, WI: Cathcart Press.

Shield, M., & Galbraith, P. (1998). The analysis of student expository writing in mathematics. *Educational Studies in Mathematics*, *36*(1), 29–52. https://doi.org/10.1023/A:1003109819256

Sinclair, N., & Yerushalmy, M. (2016). Digital technology in mathematics teaching and learning: A decade focused on theorising and teaching. In Á. Gutiérrez, G. C. Leder, & P. Boero (Eds.), *The Second Handbook of Research on the Psychology of Mathematics Education: The Journey Continues* (pp. 235–274). Brill. https://doi.org/10.1007/978-94-6300-561-6_7

Soto-Johnson, H., & Fuller, E. (2012). Assessing proofs via oral interviews. *Investigations in Mathematics Learning*, *4*(3), 1–14. https://doi.org/10.1080/24727466.2012.11790313

Staples, M. (2007). Supporting whole-class collaborative inquiry in a secondary mathematics classroom. *Cognition and Instruction*, *25*(2–3), 161–217. https://doi.org/10.1080/07370000701301125

Staples, M. E., Bartlo, J., & Thanheiser, E. (2012). Justification as a teaching and learning practice: Its (potential) multifacted role in middle grades mathematics classrooms. *Journal of Mathematical Behavior*, *31*(4), 447–462. https://doi.org/10.1016/j.jmathb.2012.07.001

Staples, M., & Newton, J. (2016). Teachers' contextualization of argumentation in the mathematics classroom. *Theory into Practice*, *55*(4), 294–301. https://doi.org/10.1080/00405841.2016.1208070

Staples, M., Newton, J., Conner, A. M., Cirillo, M., Bieda, K., Yopp, D., … Zhuang, Y. (2017). Using artifacts to explore conceptions and consequences of argumentation, justification, and proof. Retrieved from https://www.researchgate.net/profile/Karl-Kosko/publication/317267228_White_Paper_Using_Artifacts_to_Explore_Conceptions_and_Consequences_of_Argumentation_Justification_and_Proof/links/592f0edb0f7e9beee752bcd0/White-Paper-Using-Artifacts-to-Explore-Concep

Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell.

Mathematical Thinking and Learning, *10*(4), 313–340. https://doi.org/10.1080/10986060802229675

Stephan, M., & Cobb, P. (2003). Chapter 3: The methodological approach to classroom-based research. *Journal for Research in Mathematics Education Monograph*, *12*, 36–50.

Stylianides, A. J. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics Education*, *38*(3), 289–321. Retrieved from http://www.jstor.org/stable/30034869

Stylianides, A. J. (2019). Secondary students' proof constructions in mathematics: The role of written versus oral mode of argument representation. *Review of Education*, *7*(1), 156–182. https://doi.org/10.1002/rev3.3157

Stylianides, A. J., Bieda, K. N., & Morselli, F. (2016). Proof and argumentation in mathematical education research. In A. Gutierrez, G. C. Leder, & P. Boero (Eds.), *The Second Handbook of Research on the Psychology of Mathematics Education* (pp. 315–351).

Stylianides, G. J. (2009). Reasoning and proving in school mathematics textbooks. *Mathematical Thinking and Learning*, *11*(4), 258–288.

Stylianou, D. A. (2013). An examination of connections in mathematical processes in students' problem solving: Connections between representing and justifying. *Journal of Education and Learning*, *2*(2), 23–35. https://doi.org/10.5539/jel.v2n2p23

Swinson, K. (1992). An investigation of the extent to which writing activities are used in mathematics classes. *Mathematics Education Research Journal*, *4*(2), 38–49. https://doi.org/10.1007/BF03217238

Thomas, M. O. J., & Palmer, J. M. (2014). Teaching with digital technology: Obstacles and opportunities. In *The Mathematics Teacher in the Digital Age* (pp. 71–90). https://doi.org/10.1007/978-94-007-4638-1_2

Toulmin, S. E. (1958). *The uses of argument* (updated ed). Cambridge: Cambridge University Press.

Toulmin, S. E., Rieke, R., & Janik, A. (1984). *An introduction to reasoning*. New York, NY: Macmillan.

VanDerHeide, J. (2017). Classroom talk as writing instruction for learning to make writing moves in literary arguments. *Reading Research Quarterly*, *53*(3), 323–344. https://doi.org/10.1002/rrq.196

Venturini, P., & Amade-Escot, C. (2014). Analysis of conditions leading to a productive disciplinary engagement during a physics lesson in a disadvantaged area school.

International Journal of Educational Research, *64*, 170–183. https://doi.org/10.1016/j.ijer.2013.07.003

Waywood, A. (1994). Informal writing-to-learn as a dimension of a student profile. *Educational Studies in Mathematics*, *27*(4), 321–340. https://doi.org/10.1007/BF01273376

Waywood, A. (1992). Journal writing and learning mathematics. *For the Learning of Mathematics*, *12*(2), 34–43.

Weber, K., Radu, I., Mueller, M., Powell, A.&Maher, C. (2010). Expanding participation in problem solving in a diverse middle school mathematics classroom: New paradigms for equity research. *Mathematics Education Research Journal*, *22*(1), 91–118.

Weber, K., Maher, C., Powell, A., & Lee, H. S. (2008). Learning opportunities from group discussions: Warrants become the objects of debate. *Educational Studies in Mathematics*, *68*(3), 247–261. https://doi.org/10.1007/s10649-008-9114-8

Wilder, L., & Wolfe, J. (2009). Sharing the tacit rhetorical knowledge of the literary scholar: The effects of making disciplinary conventions explicit in undergraduate writing about literature courses. *Research in the Teaching of English*, *44*(2), 170–209.

Williams-Candek, M. (2015). *Principles of productive disciplinary engagement: Framework for design*.

Wollman-Bonilla, J. E. (2001). Can first-grade writers demonstrate audience awareness? *Reading Research Quarterly*, *36*(2), 184–201. https://doi.org/10.1598/rrq.36.2.4

Wood, T., Cobb, P., Yackel, E., & Dillon, D. (Eds). (1993). Rethinking elementary school mathematics: Insight and issues. *Journal for Research in Mathematics Education Monograph*, *6*.

Wood, T., Williams, G., & McNeal, B. (2006). Children's mathematical thinking in different classroom cultures. *Journal for Research in Mathematics Education*, *37*(3), 222–255.

Yeh, S. S. (1998). Empowering education: Teaching argumentative writing to cultural minority middle- school students. *Research in the Teaching of English*, *33*(1), 49–83.

Yin, F. K. (2009). *Case Study Research: Design and Methods* (4 ed.). Los Angeles: SAGE.

Zaslavsky, O. (2005). Seizing the opportunity to create uncertainty in learning mathematics. *Educational Studies in Mathematics*, *60*(3), 297–321. https://doi.org/10.1007/s10649-005-0606-5