INVOLUTIONS AND HEEGAARD FLOER HOMOLOGY

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ABSTRACT

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This Ph.D. dissertation studies the relationship of an involution acting on a 3-manifold (or a knot K) with the Heegaard Floer homology. There are three main aspects of this project: *strong cork* detection, studying *homology bordism group of diffeomorphisms* and explicitly computing the action of symmetry on the Knot Floer complex for symmetric knots. In Chapter 2, we study pairs (Y^3, τ) of an integer homology sphere equipped with an involution $\tau : Y \to Y$ modulo equivariant homology cobordisms. We show that equivalence classes of the above relation form an abelian group under the group operation as disjoint union. We refer to this group as the *homology bordism group of involutions*, $\Theta^{\tau}_{\mathbb{Z}}$. This group can be thought of as a generalized version of the bordism group of diffeomorphisms, which was first studied by Browder. We define two Floer-theoratic invariants of $\Theta^{\tau}_{\mathbb{Z}}$, h_{τ} and $h_{\iota\circ\tau}$ using the framework of involutive Heegaard Floer homology, recently developed by Hendricks and Manolescu [19].

Corks play an important role in the study of exotic smooth structures on 4-manifolds. As shown by Matveyev [28] and Curtis-Freedman-Hsiang-Stong [6], any two smooth structures on a simply connected topological 4-manifold are related by the action of *cork twist*. In [25] Lin-Ruberman-Saveliev studied a more generalised version of a cork, called the strong cork. These are corks for which the cork-twist involution does not extend over any homology 4-ball that the cork may bound. They [25] also constructed the first example of such a strong cork by studying the induced action of a cork-twist on monopole Floer homology. In Chapter 3, we show that the invariants h_{τ}, and $h_{\iota\circ\tau}$ developed in Chapter 2 also detect strong corks. We then go on to establish several new families of corks and prove that various known examples corks are actually strong. Our main computational tool is a monotonicity

theorem which constrains the behavior of our invariants under equivariant negative-definite cobordisms, and an explicit method to construct equivariant cobordisms. The contents of Chapter 2, and Chapter 3 are from a joint work of the author with Irving Dai and Matthew Hedden [7]. In Chapter 4 we study symmetric knots. We show that each symmetry of a knot induces a map on the knot Floer complex. We further show that these induced maps behave differently according to how the fixed set of the symmetry intersects knot. We then explicitly compute some of those maps.

To my late mother: (Late) Tanushree Mallick

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CHAPTER 1

BACKGROUND

1.1 Heegaard Floer homology

Given a based 3-manifold (Y, z), Ozsváth and Szabó defined an invariant [34] called the *Heegaard Floer homology.* These invariants come in different flavors, $\widehat{HF}$, HF^- and HF^+. For example, HF^- assigns an $\mathbb{F}_2[U]$-module to (Y, z). The input for this invariant is a Heegard tuple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, J, z, \mathfrak{s})$.

- Here $\Sigma \subset Y$ is an embedded, oriented surface of genus g, which splits Y as two different handlebodies U_0 and U_1.
- $\boldsymbol{\alpha} = \{\alpha_1, \alpha_2, \cdots, \alpha_g\}$ is a g-tuple of simple closed curves on Σ which bound disks in U_1.
- $\boldsymbol{\beta} = \{\beta_1, \beta_2, \cdots, \beta_g\}$ is a similar set of curves which bound disks in U_2. The α and the β curves only intersect transversely.
- z is a basepoint on $\Sigma - \boldsymbol{\alpha} - \boldsymbol{\beta}$.
- J is an *almost-complex structure* on $\mathrm{Sym}^g(\Sigma)$ (here g is the genus of Σ).
- $\mathfrak{s}$ is a spin^c-structure on Y.

We will abbreviate a Heegaard tuple by $\mathcal{H}$. Associated to $\mathcal{H}$, Ozsváth and Szabó define a chain complex $CF^-(\mathcal{H})$ over $\mathbb{F}_2[U]$. The chain complex is generated by the intersection points between the α and the β curves, roughly the differential counts the homolomorphic disks bounded by the α and β curves. Ozsváth and Szabó then show that

Theorem 1.1.1 ([34]). *The isomorphism class of the homology $HF^-(\mathcal{H})$ of the above chain complex is independent of the choices made in the definition, namely $\mathcal{H}$.*

For a long time it was unknown whether their is a canonical isomorphism between two Heegaard Floer homology group corresponding to the same based 3-manifold (Y, z). Later [21] and [19] showed that the homotopy equivalence class of the chain complex $CF^-(\mathcal{H})$ is an invariant of (Y, z). More precisely they showed

Theorem 1.1.2 ([21], [19]). *Let $\mathcal{H}_i$ be any any three Heegaard tuple representing (Y, z), for $i = 1, 2, 3$. Then there is a chain homotopy equivalence,*

$$\Phi(\mathcal{H}_1, \mathcal{H}_2) : CF^-(\mathcal{H}_1) \to CF^-(\mathcal{H}_2)$$

which is unique up to chain homotopy and satisfies the following relations

- $\Phi(\mathcal{H}_1, \mathcal{H}_2) \circ \Phi(\mathcal{H}_2, \mathcal{H}_3) \simeq \Phi(\mathcal{H}_1, \mathcal{H}_3)$.
- $\Phi(\mathcal{H}_1, \mathcal{H}_1) \simeq \mathrm{id}$.

The relations above imply that the chain complexes $CF^-(\mathcal{H})$ associated to (Y, z), form a transitive system in the homotopy category of chain complexes of $\mathbb{F}_2[U]$-modules. We can then refer to $CF^-(Y, z)$ as the inverse limit of the above system.

The authors in [35, Theorem 3.1] also define maps $F_{W,\mathfrak{s}}$ associated to a cobordism W between two 3-manifolds $(Y_1, \mathfrak{s}_1, z_1)$ and $(Y_2, \mathfrak{s}_1, z_2)$. Here $\mathfrak{s}_i$ are certain spin^c-structures on Y_i which extend to $\mathfrak{s}$ on W.

$$F_{W,\mathfrak{s}} : HF^-(Y_1, \mathfrak{s}_1, z_1) \to HF^-(Y_2, \mathfrak{s}_2, z_2).$$

Later Zemke [41, Theorem A] showed that maps $F_{W,\mathfrak{s}}$ are actually well-defined on the chain complex level. More specifically, in [41] the author showed that, given two tuples $(Y_i, \mathfrak{s}_i, z_i)$ as before and a cobordism $(W, \mathfrak{s}, \gamma)$ (here the extra information γ is a path between the basepoints z_1 and z_2) there is a cobordism map on the chain level

$$f_{W,\mathfrak{s},\gamma} : CF^-(Y_1, \mathfrak{s}_1, z_1) \to CF^-(Y_2, \mathfrak{s}_2, z_2).$$

which descends to the map $F_{W,\mathfrak{s}}$ defined above, in homology. The map $f_{W,\mathfrak{s},\gamma}$ is defined using a handle decomposition of W.

Theorem 1.1.3. *[41, Theorem A] The map $f_{W,\mathfrak{s},\gamma} : CF^-(Y_1, \mathfrak{s}_1, z_1) \to CF^-(Y_2, \mathfrak{s}_2, z_2)$ is independent of the choice of the handle-decomposition of W up to chain homotopy.*

1.2 Knot Floer homology

Given a doubly-based knot (K, w, z) inside a 3-manifold Y, Ozsváth and Szabó defined an invariant [33] called the *Knot Floer homology.* As before the input for this invariant is again the Heegaard data, $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, J, z, w, \mathfrak{s})$, with the exception that now it has two basepoints instead of one. The output again is a chain complex which comes in different flavors. For this study, we will need to consider the *infinity version* CFK^∞, which is a $\mathbb{Z} \oplus \mathbb{Z}$-filtered chain complex over $\mathbb{F}_2[U, U^{-1}]$. The associated graded homology of this chain complex is called the Knot Floer homology. Instead of going too much into the theory of knot Floer homology, we recommend the reader to [26] for a great introduction.

1.3 Involutive Heegaard Floer homology

In [27] Manolescu proved the triangulation conjecture using a the construction of a Pin(2)-equivariant version of Seiberg-Witten Floer homology. On the other hand, by the work of Kutluhan-Lee-Taubes [24], there is an isomorphism between the Heegaard Floer homology and the monopole Floer homology. Motivated by this, one can then ask for a Pin(2)-equivariant version of Heegaard Floer homology. The [19] Hendricks and Manolescu provide a partial answer to this question by constructing a $\mathbb{Z}_4$- equivariant version of Heegaard Floer homology, where $\mathbb{Z}_4$ is thought of as a subgroup of the Pin(2)-group. The authors refer to $\mathbb{Z}_4$-equivariant version of Heegaard Floer homology as *Involutive Heegaard Floer homology.* We now focus on the construction of involutive Heegaard Floer homology.

Recall from Theorem 1.1.2 that we know that the chain homotopy class of the Heegaard Floer chain complex CF^- is an invariant of $(Y, \mathfrak{s}, z)$, where $\mathfrak{s}$ is a spin^c-structure on Y. Note that given a spin^c-structure on Y, there is an associated spin^c-structure $\bar{\mathfrak{s}}$ on Y, called the *conjugate* of $\mathfrak{s}$. This yields a *conjugation symmetry* on the Heegaard Floer homology groups,

$$\mathcal{J} : HF^-(Y, \mathfrak{s}, z) \to HF^-(Y, \bar{\mathfrak{s}}, z).$$

In [33, Proposition 3.9.], the authors show that the above isomorphism is induced by switching the orientation of the Heegaard surface and the interchanging roles of α and β curves.

$$(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z) \to (-\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z).$$

$\mathcal{J}^2 = \mathrm{id}$, so the spin^c-conjugation induces an involution on the Heegaard Floer chain complex. Hendricks and Manolescu define this action on the chain complex level

$$\iota : CF^-(Y, \mathfrak{s}, z) \to CF^-(Y, \bar{\mathfrak{s}}, z).$$

so that ι induces the map $\mathcal{J}$ in homology. We briefly discuss the construction on the chain level.

Note that $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ and $(-\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z)$ are both Heegaard diagrams of the same based 3-manifold (Y, z). Informally, this corresponds to turning the handlebody upside down. Now let $\mathcal{H}$ and $\bar{\mathcal{H}}$, respectively, denote the original Heegaard diagram and its conjugate. Note that there is always an abstract isomorphism η

$$\eta : CF^-(\mathcal{H}, \mathfrak{s}) \to CF^-(\bar{\mathcal{H}}, \bar{\mathfrak{s}}).$$

induced by the one-to-one correspondence between the intersection points of the Heegaard diagrams of $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ and $(-\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z)$. Moreover from 1.1.2 we know that there is a chain homotopy equivalence , well defined up to chain homolopy

$$\Phi(\bar{\mathcal{H}}, \mathcal{H}) : CF^-(\bar{\mathcal{H}}, \bar{\mathfrak{s}}) \to CF^-(\mathcal{H}, \mathfrak{s})$$

We then compose Φ with η and define $\iota := \Phi \circ \eta$.

$$\begin{tikzcd}
CF^-(\mathcal{H},\mathfrak{s}) \arrow[r, "\eta"] \arrow[d, "\iota"'] & CF^-(\bar{\mathcal{H}},\bar{\mathfrak{s}}) \arrow[dl, "\Phi"] \\
CF^-(\mathcal{H},\mathfrak{s}) &
\end{tikzcd}$$

This defines an automorphism of the chain complex associated to spin^c-conjugation. Hendricks and Manolescu showed that

Proposition 1.3.1. *[19, Lemma 2.5.] ι is a homotopy involution i.e. $\iota^2 \simeq \mathrm{id}$.*

The main idea of the proof of the above proposition is that since $\eta^2 = \mathrm{id}$ the composition

$$\eta \circ \Phi(\bar{\mathcal{H}},\mathcal{H}) \circ \eta$$

conjugates $\Phi(\bar{\mathcal{H}},\mathcal{H})$ which is homotopic to $\Phi(\mathcal{H},\bar{\mathcal{H}})$. The result then follows by appealing to the Theorem 1.1.2 as

$$\Phi(\mathcal{H},\bar{\mathcal{H}}) \circ \Phi(\bar{\mathcal{H}},\mathcal{H}) \simeq \mathrm{id}.$$

Let us now restrict to the case where the spin^c-structure is self-conjugate, i.e. $\mathfrak{s} = \bar{\mathfrak{s}}$ [1]. Note that in this case ι is an automorphism of $CF^-(\mathcal{H},\mathfrak{s})$. Hendricks and Manolescu then consider the mapping cone of the map

$$\mathrm{id} + \iota : CF^-(\mathcal{H},\mathfrak{s}) \to CF^-(\mathcal{H},\mathfrak{s})$$

and denote it as $CFI^-(\mathcal{H},\mathfrak{s})$. They show the following

Theorem 1.3.2. *[19, Proposition 2.8] The quasi-isomorphism class of $CFI^-(\mathcal{H},\mathfrak{s})$ is an invariant of the pair $(Y,\mathfrak{s})$.*

[1] This restriction is not necessary to carry out the construction of involutive Heegaard Floer homology, but it turns out the invariant is determined by the original HF homology in a straight forward manner in the case $\mathfrak{s} \neq \bar{\mathfrak{s}}$. See [19, Proposition 4.5.]

We briefly discuss the proof here, which relies on the fact that chain homotopic maps induce quasi-isomorphic mapping cone complexes. Hence it suffices to show that given two different Heegaard diagrams $\mathcal{H}_1$ and $\mathcal{H}_2$ of (Y, z), the corresponding ι maps are chain homotopic. The authors use Theorem 1.1.2 to conclude that the following square commutes up to chain homotopy.

$$\begin{tikzcd} CF^-(\mathcal{H}_1, \mathfrak{s}) \arrow[r, "\iota"] \arrow[d, "\Phi(\mathcal{H}_1,\mathcal{H}_2)"'] & CF^-(\mathcal{H}_1, \mathfrak{s}) \arrow[d, "\Phi(\mathcal{H}_1,\mathcal{H}_2)"] \\ CF^-(\mathcal{H}_2, \mathfrak{s}) \arrow[r, "\iota"'] & CF^-(\mathcal{H}_2, \mathfrak{s}) \end{tikzcd}$$

This paired with the fact that commutation of ι with η is tautological, completes the proof.

Homology of $CFI^-(\mathcal{H}, \mathfrak{s})$ is called the *involutive Heegaard Floer homology* $HFI^-(Y, z)$ of the pair (Y, z). Unlike Theorem 1.1.2, at the time of writing it is still not known whether there is higher order naturality for the involutive Floer chain complex CFI^- is an invariant. It is conjectured to have such properties, see [19].

Another interesting aspect of involutive Floer homology is that, the action of ι is somewhat natural with respect to the cobordism map. This in turn shows that there is a map between involutive chain complexes associated to a cobordism. More specifically in [19] the authors show that

Theorem 1.3.3. *[19, Proposition 4.9.] Let $(W, \mathfrak{s}, \gamma)$ be a cobordism between $(Y_1, \mathfrak{s}_1, z_1)$ and $(Y_2, \mathfrak{s}_2, z_2)$, where $\mathfrak{s}$ is a self-conjugate* spin^c*-structure on W which restricts to the self-conjugate* spin^c*-structures $\mathfrak{s}_i$ on Y_i, and γ is a path between the basepoints z_1 and z_2. Then there is an associated cobordism map*

$$F^I_{(W,\mathfrak{s},\gamma)} : HFI^-(Y_1, \mathfrak{s}_1, z_1) \to HFI^-(Y_2, \mathfrak{s}_2, z_2)$$

Again unlike Theorem 1.1.3 in Heegaard Floer homology, it is not known whether the this cobordism map is independent of the choice of the handle decomposition that is used to define it. Let us now discuss the main ideas of the proof.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ represent the Heegaard diagrams for (Y_1, z_1) and (Y_2, z_2) and $f_{W,\mathfrak{s},\gamma}$ be the map induced on the Heegaard Floer chain complexes

$$f_{W,\mathfrak{s},\gamma} : CF^-(\mathcal{H}, \mathfrak{s}_1, z_1) \to CF^-(\mathcal{H}, \mathfrak{s}_2, z_2)$$

First, the authors show that the following diagram commutes up to chain homotopy.

$$\begin{array}{ccc} CF^-(\mathcal{H}_1, \mathfrak{s}_1) & \xrightarrow{f_{W,\mathfrak{s},\gamma}} & CF^-(\mathcal{H}_2, \mathfrak{s}_2) \\ {\scriptstyle \Phi(\mathcal{H}_1,\overline{\mathcal{H}_1})}\downarrow & & \downarrow{\scriptstyle \Phi(\mathcal{H}_2,\overline{\mathcal{H}_2})} \\ CF^-(\overline{\mathcal{H}_1}, \mathfrak{s}_1) & \xrightarrow[f_{W,\mathfrak{s},\gamma}]{} & CF^-(\overline{\mathcal{H}_2}, \mathfrak{s}_2) \end{array}$$

To see this, the authors invoke the diffeomorphism invariance of cobordism maps on Heegaard Floer homology; see Theorem 1.1.3. Here one regards $\Phi(\mathcal{H}_2, \overline{\mathcal{H}_2}) \circ f_{W,\mathfrak{s},\gamma}$ and $f_{W,\mathfrak{s},\gamma} \circ \Phi(\mathcal{H}_1, \overline{\mathcal{H}_1})$ as maps associated to the cobordism W. Hence they must be homotopic to each other. This in turn implies that ι intertwines with the cobordism map $f_{W,\mathfrak{s},\gamma}$ up to chain homotopy. The argument now follows from standard homological algebra, which shows that $f_{W,\mathfrak{s},\gamma}$ induces a map on the mapping cones of $(\mathrm{id} + \iota)$, i.e. between $CFI^-(Y_1, \mathfrak{s}_1, z_1)$ and $CFI^-(Y_2, \mathfrak{s}_2, z_2)$.

CHAPTER 2

HOMOLOGY BORDISM AND HEEGAARD FLOER HOMOLOGY

2.1 Introduction

An area of intense study in low-dimensional topology is the *homology cobordism group* $\Theta^3_{\mathbb{Z}}$ in dimension 3. To define it, one introduces equivalence relation on the set of $\mathbb{Z}HS^3$ by saying, Y_1 is equivalent to Y_2 if there exists a cobordism W such that $\partial W = Y_1 \sqcup -Y_2$ and the inclusions $Y_i \hookrightarrow W$ induce isomorphisms in homology. The equivalence classes under this relation form an abelian group under the connect sum operation which is referred to as the homology cobrodism group. This group has been studied using various tools:

- The first known result about the structure of $\Theta^3_{\mathbb{Z}}$ was the existence of the Rokhlin homomorphism $\mu : \Theta^3_{\mathbb{Z}} \to \mathbb{Z}/2\mathbb{Z}$.

Several mathematicians have since improved our knowledge about the structure of this group. For example using the techniques from gauge theory the following theorems were proved.

- $\Theta^3_{\mathbb{Z}}$ is infinite. (Fintushel-Stern)[12]
- There is a $\mathbb{Z}^\infty$-subgroup of $\Theta^3_{\mathbb{Z}}$. (Furuta, Fintushel-Stern)[15], [13]
- There is a $\mathbb{Z}$-summand in $\Theta^3_{\mathbb{Z}}$. (Frøyshov)[14]

One of the recent success has been the following theorems

Theorem 2.1.1. *[27] If $\mu(Y) = 1$, then Y is not of order 2 in $\Theta^3_{\mathbb{Z}}$.*

Theorem 2.1.2. *[8] $\Theta^3_{\mathbb{Z}}$ contains a direct summand isomorphic to $\mathbb{Z}^\infty$.*

For the former theorem Manolescu uses *Pin(2)-equivariant Seiberg-Witten Floer theory* and the latter by Dai-Hom-Stoffregen-Truong uses its counterpart in Heegaard Floer theory, *involutive Heegaard Floer homology.* Despite these developments several questions about the structure of $\Theta^3_{\mathbb{Z}}$ still remain open, including whether it contains any torsion.

To this end, one can ask for a generalization of the cobordism where the manifolds are equipped with diffeomorphism. Let (M_1, f_1) and (M_2, f_2) be pair of 3-manifold each equipped with a diffeomorphism (here, we do not require M_i to be connected.) We say that (M_1, f_1) is *bordant* to (M_2, f_2) if there exist a pair (W, f) where W is a cobordism between M_1 and M_2 and f is a diffeomorphism on it, extending the boundary diffeomorphisms. The *3-dimensional bordism group* Δ_3, is an abelian group whose underlying set consist of bordism classes, endowed with the operation induced by disjoint union. This group can be defined and understood in all dimensions, by work of Kreck [22] (for $n \geq 4$), Melvin [29] ($n = 3$), and Bonahon [3] (for $n = 2$). Notably Melvin showed

Theorem 2.1.3. *[29]* $\Delta_3 = 0$.

It is natural to ask whether placing homological restrictions on the manifolds and bordisms in question results in a richer group structure. This parallels the situation in which the three-dimensional oriented cobordism group is trivial, but understanding the homology cobordism group $\Theta^3_{\mathbb{Z}}$ is difficult. In this chapter we define and study the *homology bordism group of diffeomorphisms* in 3-dimensions using Heegaard Floer and involutive Heegaard Floer homology.

2.2 Homology bordism group of diffeomorphisms

We now discuss a precise definition of the homology bordism group. First, let us recall the definition of *bordism* which was first popularized by Browder.

Definition 2.2.1. Let M_1 and M_2 be two closed, oriented n-manifolds, each equipped with an orientation-preserving diffeomorphism f_i ($i = 1, 2$). We say that (M_1, f_1) and (M_2, f_2) are *bordant* if there exists a bordism W between them which admits an orientation-preserving diffeomorphism restricting to f_i on M_i. Here, neither the M_i nor W are assumed to be connected.

Note that the bordism is an equivalence relation, where transitivity follows from uniqueness of collar neighborhoods of boundary components.

Definition 2.2.2. ([4, pg. 22] or [23, Definition 1.4]) The 3-*dimensional bordism group of orientation-preserving diffeomorphisms* Δ_3 is the abelian group whose underlying set consists of bordism classes of pairs (M^3, f), endowed with the addition operation induced by disjoint union. The empty 3-manifold serves as the identity, and inverses are given by orientation reversal.

In analogy with $\Theta^3_{\mathbb{Z}}$, one would like to refine the three-dimensional group Δ_3 by requiring M to be a homology sphere and W to be a homology cobordism. However, this presents certain technical difficulties due to the fact that the connected sum of (M_1, f_1) and (M_2, f_2) may not be well-defined in general. Indeed, note that in order to form $(M_1 \# M_2, f_1 \# f_2)$, one must first isotope each f_i to fix a ball $B_i \subseteq M_i$. If f_i and f_i' are isotopic, then (M_i, f_i) and (M_i, f_i') are certainly bordant via the diffeomorphism of the cylinder $M_i \times I$ induced by the isotopy. However, it does *not* follow that the homology cobordism class of $(M_1 \# M_2, f_1 \# f_2)$ is independent of the choice of isotopy. To see this, let f_i and f_i' $(i = 1, 2)$ be two diffeomorphisms of Y_i fixing B_i. Suppose that f_i and f_i' are isotopic, but that the intermediate stages of this isotopy do not fix any ball in M_i. Then it is not clear how to define a diffeomorphism on $(M_1 \# M_2) \times I$ restricting to $f_1 \# f_2$ and $f_1' \# f_2'$ at either end. We thus instead follow Definition 2.2.2 and take disjoint union to be our group operation.

We will consider the following equivalence relation:

Definition 2.2.3. [7] Consider the class of pairs (Y, f), where:

1. Y is a compact (possibly empty) disjoint union of oriented integer homology 3-spheres; and,

2. f is an orientation-preserving diffeomorphism of Y which fixes each component of Y setwise.

We say that two such pairs (Y_1, f_1) and (Y_2, f_2) are *pseudo-homology bordant* if there exists a pair (W, g) with the following properties:

1. W is a compact, oriented cobordism between Y_1 and Y_2 with $H_2(W) = 0$; and,

2. g is an orientation-preserving diffeomorphism of W such that:

    a) g restricts to f_i on each Y_i; and,

    b) g induces the identity map on $H_1(W, \partial W)$.

In this situation, we write $(Y_1, f_1) \sim (Y_2, f_2)$. It is clear that $\sim$ is an equivalence relation.

Note that $H^2(W) = H_2(W, \partial W) = H_2(W) = 0$. In particular, W has only one spin$^{\mathrm{c}}$-structure.

Remark 2.2.4. Readers might think that the most natural extension of homology cobordism in the context of disconnected boundaries is *homology punctured* S^4. But one can check that composition of two such punctured homology spheres need not be a punctured homology sphere. Hence we are forced to consider the definition above, note that in the case for homology punctured spheres, the $H_2(W)$ still vanish.

Let us now define the homology bordism group.

Definition 2.2.5. The *(3-dimensional) homology bordism group of orientation-preserving diffeomorphisms* $\Theta_{\mathbb{Z}}^{\mathrm{diff}}$ is the abelian group whose underlying set consists of pseudo-homology bordism classes of pairs (Y, f) as in Definition 2.2.3, endowed with the addition operation induced by disjoint union. The empty 3-manifold serves as the identity, and inverses are given by orientation reversal. The *(3-dimensional) homology bordism group of orientation-preserving involutions* $\Theta_{\mathbb{Z}}^{\tau}$ is then defined to be the subgroup of $\Theta_{\mathbb{Z}}^{\mathrm{diff}}$ generated by involutions.

2.3 Invariants of the homology bordism group of involutions

We devote this section to defining an two invariants of the homology bordism group of involutions, discussed in Section 2.2. These invariants are derived using the tools of involutive Heegaard Floer homology. We refer the readers to Section 1.3 for a quick introduction to the theory.

Let us now state the main theorem of this section, whose definition and proof occupy rest of the section.

Theorem 2.3.1. *[7, Theorem 1.1] Let Y be an integer homology sphere with involution $\tau : Y \to Y$. Then there are two Floer-theoretic invariants*

$$h_\tau(Y) = [(CF^-(Y)[-2], \tau)] \text{ and } h_{\iota\circ\tau}(Y) = [(CF^-(Y)[-2], \iota \circ \tau)]$$

associated to the pair (Y, τ). If either $h_\tau(Y) \neq 0$ or $h_{\iota\circ\tau}(Y) \neq 0$, then τ does not extend to a diffeomorphism of any homology ball bounded by Y. In fact, the both invariants h_τ and $h_{\iota\circ\tau}$ constitute homomorphisms

$$h_\tau, h_{\iota\circ\tau} : \Theta^\tau_{\mathbb{Z}} \to \mathfrak{I}.$$

Remark 2.3.2. Here $CF^-(Y)[-2]$ represents the Heegaard Floer chain complex of Y, with a grading shift as indicated. This is a matter of convention, since the highest graded generator of $HF^-(S^3)$ lies in grading -2, instead of 0.

In subsection below and the one succeeding it, we will define the invarints h_τ and $h_{\iota\circ\tau}$ and the group $\mathfrak{I}$.

2.3.1 Local equivalence and the group $\mathfrak{I}$.

We now focus on the definition of the invariants h_τ and $h_{\iota\circ\tau}$. From this point onward in this chapter and the ones subsequent to it, we will only consider 3-manifolds that are integer homology spheres. In order to define our invariants, we need the notion of *local maps*.

Definition 2.3.3. [20, Definition 8.1] An ι*-complex* is a pair (C, ι), where

1. C is a (free, finitely generated, $\mathbb{Z}$-graded) chain complex over $\mathbb{F}_2[U]$, with
$$U^{-1}H_*(C) \cong \mathbb{F}_2[U, U^{-1}].$$
Here, U has degree -2.

2. $\iota : C \to C$ is a ($\mathbb{F}_2[U]$-equivariant, grading-preserving) homotopy involution; that is, ι^2 is U-equivariantly chain homotopic to the identity.

There is also a notion of homotopy equivalence between two different ι-complexes.

Definition 2.3.4. Two ι-complexes (C, ι) and (C', ι') are called *homotopy equivalent* if there exist chain homotopy equivalences
$$f : C \to C', \quad g : C' \to C$$
that are homotopy inverses to each other, and such that
$$f \circ \iota \simeq \iota' \circ f, \quad g \circ \iota' \simeq \iota \circ g,$$
where $\simeq$ denotes $\mathbb{F}_2[U]$-equivariant chain homotopy.

Given a Heegaard data $\mathcal{H}$ for (Y, z), in [19] the authors show that the homotopy equivalence class of $(CF^-(\mathcal{H}, \iota))$ is independent of the choice of $\mathcal{H}$. Hence, one can unambiguously refer to the homotopy type of $(CF(Y), \iota)$.

In [20], Hendricks, Manolescu, and Zemke define an equivalence relation on the set of ι-complexes, called *local equivalence*. This notion captures the algebraic relationship imposed on ι-complexes by the presence of a homology cobordism between homology spheres.

Definition 2.3.5. [20, Definition 8.5] Two ι-complexes (C, ι) and (C', ι') are called *locally equivalent* if there exist (U-equivariant, grading-preserving) chain maps
$$f : C \to C', \quad g : C' \to C$$

such that

$$f \circ \iota \simeq \iota' \circ f, \quad g \circ \iota' \simeq \iota \circ g,$$

and f and g induce isomorphisms on homology after localizing with respect to U. We call a map f as above a *local map* from (C, ι) to (C', ι'), and similarly we refer to g as a local map in the other direction.

The authors in [20] then consider set $\mathfrak{I}$ defined as

$$\mathfrak{I} = \{\text{(abstract) } \iota\text{-complexes}\} \ / \text{ local equivalence}$$

consisting of all possible ι-complexes modulo local equivalence. ι-complex of Y thus gives an element of $\mathfrak{I}$, which we denote by $h(Y)$:

$$Y \mapsto h(Y) = [(CF^-(Y)[-2], \iota)].$$

In [20, Section 8], it was shown that $\mathfrak{I}$ admits a group structure, with the group operation being given by tensor product of the complexes over $\mathbb{F}_2[U]$. The identity element, denoted throughout by 0, is the local equivalence class of S^3 or, more algebraically, the complex $\mathbb{F}_2[U]$ with trivial differential and identity involution. With this group structure, Hendricks, Manolescu, and Zemke show that h is an invariant of the homology cobordism group $\Theta^3_{\mathbb{Z}}$, in fact it is a homomorphism

$$h : \Theta^3_{\mathbb{Z}} \to \mathfrak{I}.$$

2.3.2 Defining the invariants h_τ and $h_{\iota\circ\tau}$.

Having defined the ι-complexes and local equivalence class in Subsection 2.3.1, we now move on to defining the invariants mentioned in Theorem 2.3.1. Roughly, we will show that that given an involution τ on a integer homology sphere Y. There is an action of τ on the Heegaard Floer chain complex $CF^-(Y)$, which is a homotopy involution.

$$\tau : CF^-(Y) \to CF^-(Y)$$

This action puts the structure of an ι-complex on the pair $(CF^-(Y), \tau)$. The invariant h_τ will then be the local equivalence class of $[(CF^-(Y), \tau)]$. The definition of $h_{\iota\circ\tau}$ is slightly more involved, as discussed later, nonetheless it represents the local equivalence of the pair $[(CF^-(Y), \iota \circ \tau)]$ where as we will see, $\iota \circ \tau$ acts as an homotopy involution on $CF^-(Y)$.

Firstly, let us define the action of τ of the Heegaard Floer chain complex. Recall that input for (see Section 1.1) Heegaard Floer chain complex is the Heegaard data $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, J)$. The fact that an involution on Y induces (the homotopy class of) a homotopy involution $\tau : CF^-(Y) \to CF^-(Y)$ follows from the work of Juhász, Thurston, and Zemke [21], who showed that the (based) mapping class group acts naturally on Heegaard Floer homology. Let $\mathcal{H}$ be a choice of Heegaard data for Y, and suppose that τ fixes the basepoint z of $\mathcal{H}$. Applying τ to $\mathcal{H}$, we obtain a "pushforward" set of Heegaard data which we denote by $t\mathcal{H}$. Explicitly, we think of Σ as embedded in Y, so that τ maps Σ to another embedded surface $\tau(\Sigma)$ in Y with the obvious pushforward α- and β-curves. We similarly pushforward the family of almost complex structures J on $\mathrm{Sym}^g(\Sigma)$ using the diffeomorphism between Σ and $\tau(\Sigma)$ effected by τ. There is a tautological chain isomorphism

$$t : CF^-(\mathcal{H}) \to CF^-(t\mathcal{H})$$

given by the map sending an intersection point in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ to its corresponding pushforward intersection point. The action of τ is then defined to be the homotopy class of the chain map

$$\tau = \Phi(t\mathcal{H}, \mathcal{H}) \circ t : CF^-(\mathcal{H}) \to CF^-(\mathcal{H}),$$

where $\Phi(t\mathcal{H}, \mathcal{H})$ is the Juhász-Thurston-Zemke homotopy equivalence from $CF^-(t\mathcal{H})$ to $CF^-(\mathcal{H})$. Theorem 1.5 of [21] shows that induced map τ_* on homology is well-defined, and an invariant of the pointed mapping class represented by τ. The proof of their result, however, shows that the homotopy class of τ is also invariant. (See [19, Proposition 2.3].)

In the case that τ does not fix a point on Y, we first consider an isotopy $h_s : Y \to Y$ that moves τz back to z along some arc γ. Composing τ with the result of this isotopy gives an isotoped diffeomorphism $\tau_\gamma = h_1 \circ \tau$, which now fixes the basepoint z. We then define the action of τ to be the mapping class group action of τ_γ:

$$\tau = \Phi(t_\gamma \mathcal{H}, \mathcal{H}) \circ t_\gamma : CF^-(\mathcal{H}) \to CF^-(\mathcal{H}),$$

where t_γ is the tautological pushforward associated to τ_γ. The fact that this is independent of γ follows from work of Zemke [41], who showed that for a homology sphere Y, the π_1-action on $CF^-(Y)$ is trivial up to U-equivariant homotopy. Explicitly, let

$$f_\gamma : CF^-(t\mathcal{H}) \to CF^-(t_\gamma \mathcal{H})$$

be the pushforward map associated to isotopy along γ, so that $t_\gamma = f_\gamma \circ t$. Let γ' be a different arc connecting τz to z. Then $t_{\gamma'}\mathcal{H}$ is obtained from from $t_\gamma \mathcal{H}$ by an isotopy which pushes z around the closed loop $\gamma^{-1} * \gamma'$. The basepoint-moving action of $\gamma^{-1} * \gamma'$ on $CF^-(t_\gamma \mathcal{H})$ is equal to

$$(\gamma^{-1} * \gamma')_* \simeq \Phi(t_{\gamma'}\mathcal{H}, t_\gamma \mathcal{H}) \circ f_{\gamma'} \circ f_\gamma^{-1}.$$

Since Y is a homology sphere, this is U-equivariantly homotopic to the identity by [41, Theorem D]. We thus have

$$\Phi(t_{\gamma'}\mathcal{H}, \mathcal{H}) \circ f_{\gamma'} \simeq \Phi(t_\gamma \mathcal{H}, \mathcal{H}) \circ \Phi(t_{\gamma'}\mathcal{H}, t_\gamma \mathcal{H}) \circ f_{\gamma'} \simeq \Phi(t_\gamma \mathcal{H}, \mathcal{H}) \circ f_\gamma.$$

Composing both sides of this with t shows that $\Phi(t_{\gamma'}\mathcal{H}, \mathcal{H}) \circ t_{\gamma'} \simeq \Phi(t_\gamma \mathcal{H}, \mathcal{H}) \circ t_\gamma$, as desired. For the purposes of Floer theory, we will thus generally think of τ as having been isotoped to fix a basepoint of Y, and in such situations we will blur the distinction between τ and τ_γ.

Lemma 2.3.6. *Let Y be a homology sphere equipped with an involution $\tau : Y \to Y$. Then the map $\tau : CF^-(Y) \to CF^-(Y)$ constructed above is a well-defined homotopy involution.*

Proof. A similar argument as in [19, Section 2] shows that τ is well-defined up to homotopy equivalence (upon changing the choice of Heegaard data $\mathcal{H}$). If $\tau : Y \to Y$ fixes the basepoint of Y, then the action of τ on CF^- is simply defined to be the usual mapping class group action of τ. In this case, the rest of the claim follows from the fact that the action of the (based) mapping class group satisfies $(f \circ g)_* \simeq f_* \circ g_*$. If τ does not have a fixed point, then the action of τ is instead defined to be the mapping class group action of $\tau_\gamma = h_1 \circ \tau$. Now, τ_γ^2 is evidently isotopic to the identity via

$$H_s = (h_s \circ \tau) \circ (h_s \circ \tau) : Y \to Y.$$

However, this isotopy does *not* necessarily fix the basepoint z, so some care is needed. Define a modified isotopy H'_s as follows. For each s, let a_s be the arc traced out by $H_r(z)$ as r ranges from s back to zero. At time s, let H'_s be equal to H_s, followed by the result of an isotopy pushing $H_s(z)$ back to z along a_s. Then H'_s fixes z for all s. Clearly, H'_1 is equal to H_1 composed with an isotopy pushing z around the closed curve a_1. Since the π_1-action on $CF^-(Y)$ is trivial, it follows that the induced actions of H'_1 and τ_γ^2 coincide (up to U-equivariant homotopy). However, the former action is homotopy equivalent to the identity, since H'_1 is isotopic to the identity via a basepoint-preserving isotopy. □

We thus obtain:

Definition 2.3.7. Let Y be a homology sphere with an involution τ. We define the τ-*complex* of (Y, τ) to be the pair $(CF^-(Y), \tau)$, where $\tau : CF^-(Y) \to CF^-(Y)$ is the homotopy involution defined above. We denote the local equivalence class of this complex by

$$h_\tau(Y) = [(CF^-(Y)[-2], \tau)].$$

As we will see in Lemma 2.3.9, ι and τ homotopy commute. Hence their composition is another well-defined homotopy involution. We thus have:

Definition 2.3.8. Let Y be a homology sphere with an involution τ. We define the $\iota \circ \tau$-*complex* of (Y, τ) to be the pair $(CF^-(Y), \iota \circ \tau)$. We denote the local equivalence class of this complex by

$$h_{\iota\circ\tau}(Y) = [(CF^-(Y)[-2], \iota \circ \tau)].$$

Note that ι and τ homotopy commute, so nothing is gained by considering the homotopy involution $\tau \circ \iota$. This is just a re-phrasing of the fact that ι is well-defined up to homotopy (so that conjugating by any diffeomorphism replaces ι with a homotopy equivalent map). We make this explicit in the following lemma

Lemma 2.3.9. *Let Y be a homology sphere with an involution τ. Then $\iota \circ \tau \simeq \tau \circ \iota$.*

Proof. Let $\mathcal{H}$ be a choice of Heegaard data for Y, and let $t\mathcal{H}$ be as above. For notational convenience, let $\eta\mathcal{H}$ denote the conjugate Heegaard data $\overline{\mathcal{H}}$. Note that we also have the Heegaard data $\eta t\mathcal{H}$, which consists of first pushing forward via τ and then interchanging the α- and β-curves (and conjugating the almost complex structure). Similarly, we have the Heegaard data $t\eta\mathcal{H}$ which is formed by first conjugating and then pushing forward. However, it is evident that $\eta t\mathcal{H} = t\eta\mathcal{H}$, and moreover that t and η commute as isomorphisms of the relevant Floer complexes. Now choose any sequence of Heegaard moves from $\eta\mathcal{H}$ to $\mathcal{H}$. Taking the pushforward sequence of Heegaard moves gives the commutative diagram on the left in Figure 2.1. Similarly, choosing any sequence of Heegaard moves from $t\mathcal{H}$ to $\mathcal{H}$ and then applying η gives the commutative diagram on the right.

$$\begin{tikzcd}
CF^-(\eta\mathcal{H}) \arrow[r, "\Phi(\eta\mathcal{H},\mathcal{H})"] \arrow[d, "t"'] & CF^-(\mathcal{H}) \arrow[d, "t"] \\
CF^-(t\eta\mathcal{H}) \arrow[r, "\Phi(t\eta\mathcal{H},t\mathcal{H})"'] & CF^-(t\mathcal{H})
\end{tikzcd}
\qquad
\begin{tikzcd}
CF^-(t\mathcal{H}) \arrow[r, "\Phi(t\mathcal{H},\mathcal{H})"] \arrow[d, "\eta"'] & CF^-(\mathcal{H}) \arrow[d, "\eta"] \\
CF^-(\eta t\mathcal{H}) \arrow[r, "\Phi(\eta t\mathcal{H},\eta\mathcal{H})"'] & CF^-(\eta\mathcal{H})
\end{tikzcd}$$

Figure 2.1: Commutative diagrams for t and η.

We thus have

$$\begin{aligned}
\tau \circ \iota &= \Phi(t\mathcal{H}, \mathcal{H}) \circ t \circ \Phi(\eta\mathcal{H}, \mathcal{H}) \circ \eta \\
&= \Phi(t\mathcal{H}, \mathcal{H}) \circ \Phi(t\eta\mathcal{H}, t\mathcal{H}) \circ t \circ \eta \\
&= \Phi(t\mathcal{H}, \mathcal{H}) \circ \Phi(\eta t\mathcal{H}, t\mathcal{H}) \circ \eta \circ t \\
&\simeq \Phi(\eta\mathcal{H}, \mathcal{H}) \circ \Phi(\eta t\mathcal{H}, \eta\mathcal{H}) \circ \eta \circ t \\
&= \Phi(\eta\mathcal{H}, \mathcal{H}) \circ \eta \circ \Phi(t\mathcal{H}, \mathcal{H}) \circ t \\
&= \iota \circ \tau.
\end{aligned}$$

Here, in the fourth line we have used the fact that the maps $\Phi(t\mathcal{H}, \mathcal{H}) \circ \Phi(\eta t\mathcal{H}, t\mathcal{H})$ and $\Phi(\eta\mathcal{H}, \mathcal{H}) \circ \Phi(\eta t\mathcal{H}, \eta\mathcal{H})$ are chain homotopic, since they are both induced by sequences of Heegaard moves from $\eta t\mathcal{H}$ to $\mathcal{H}$. $\square$

2.4 Equivariant graph cobordism and the τ-local equivalence

In [35] the authors defined maps in Heegaard Floer homology associated to cobordisms. Unfortunately, their formalism only works when the boundaries of the cobordism have only one connected component. In [41] Zemke generalized the cobordism maps to a version called the *graph cobordism*, which allow the boundaries of the cobordism to be disconnected. Roughly speaking these maps depend on the extra information of an embedded graph in the cobordism, which has ends in different connected components of the boundary on either side. Cobordism maps of this form are necessary for our framework, in order to study the group $\Theta^\tau_\mathbb{Z}$ (recall from Section 2.2 that the ends of $\Theta^\tau_\mathbb{Z}$ are allowed to be disconnected). We begin this section with a brief overview of graph cobordisms, and the induced maps on Heegaard Floer homology. In the following subsection we show that an equivariant graph cobordism produces local map.

2.4.1 Introduction to graph cobordism.

In what follows, we allow each manifold Y to have a collection of basepoints $\mathbf{w}$. Usually, one introduces different U-variables to keep track of the different basepoints, but here we will identify all of these into a single U-variable. In the terminology of [41], this is called the *trivial coloring*.

Let W be a cobordism between two (possibly disconnected) 3-manifolds $(Y_1, \mathbf{w}_1)$ and $(Y_2, \mathbf{w}_2)$. A *ribbon graph* in W is an embedded graph Γ whose intersection with each Y_i is precisely $\mathbf{w}_i$. We also require that Γ be given a *formal ribbon structure*, which is a choice of cyclic ordering at every internal vertex of Γ. We refer to the pair (W, Γ) as a *ribbon graph cobordism*. Associated to any such (W, Γ), Zemke constructs a chain map

$$F^A_{W,\Gamma,\mathfrak{s}} : CF^-(Y_1, \mathbf{w}_1, \mathfrak{s}|_{Y_1}) \to CF^-(Y_2, \mathbf{w}_2, \mathfrak{s}|_{Y_2}).$$

This is well-defined up to U-equivariant homotopy and is an invariant of the smooth isotopy class of Γ in W [41, Definition 3.4]. In fact, F^A is invariant under a weaker notion of equivalence called *ribbon equivalence*; see [42, Corollary D]. There is another version of the graph cobordism maps $F^A_{W,\Gamma,\mathfrak{s}}$, the 'B'-version, which is constructed similarly as the 'A' version. Since we will only use the A-version, we choose to omit details about the B-version. Interested readers can look at [41].

Now let Y_1 and Y_2 be disjoint unions of homology spheres, and equip each connected component of Y_1 and Y_2 with a single basepoint. Let f be a diffeomorphism of W restricting to τ_i on each Y_i. If τ_i fixes the basepoints of Y_i, then it follows from [41, Theorem A] (together with the well-definedness of graph cobordism maps up to U-equivariant homotopy) that

$$\tau_2 \circ F^A_{W,\Gamma,\mathfrak{s}} \simeq F^A_{W,f(\Gamma),f_*(\mathfrak{s})} \circ \tau_1. \tag{2.1}$$

See [41, Equation 1.2]. If τ_i does not fix the basepoints of Y_i, then (2.1) is not quite correct, since in this case we have defined the action of τ_i on CF^- using an isotoped version of τ_i instead. Clearly, however, we can isotope f so that it restricts to the isotoped versions of

τ_i at either end. Thus, (2.1) holds after replacing $f(\Gamma)$ with a slightly altered graph $f(\Gamma)$ which has the same endpoints as Γ. (Usually, we will be sloppy and continue to write $f(\Gamma)$ despite this difference.)

In order to define the F^A-maps, Zemke first defines graph cobordism maps in the case of a product cobordism $Y \times I$. In this situation, we can use the projection map to view Γ as being embedded in Y (after perturbing slightly, if necessary). In [41], Zemke introduces a set of auxiliary maps on $CF^-(Y)$ which can be used to associate to any such graph an endomorphism $\mathfrak{A}_{\mathcal{G}_\Gamma}$ of $CF^-(Y)$. These auxiliary maps include the *free stabilization maps* $S_w^{\pm}$, as well as the *relative homology maps* A_λ. We will assume some familiarity with these constructions; the reader is referred to [20, Section 3] for a concise and helpful summary.

In order to understand F^A for a general cobordism W, it is helpful to keep in mind the desired composition law. Let $(W, \Gamma) = (W_2, \Gamma_2) \cup (W_1, \Gamma_1)$. If $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are spin^c-structures on W_1 and W_2, then the obvious generalization of the usual composition law of Ozsváth and Szabó yields:

$$F^A_{W_2,\Gamma_2,\mathfrak{s}_2} \circ F^A_{W_1,\Gamma_1,\mathfrak{s}_1} \simeq \sum_{\substack{\mathfrak{s} \in \mathrm{spin}^c(W) \\ \mathfrak{s}|_{W_2} = \mathfrak{s}_2 \\ \mathfrak{s}|_{W_1} = \mathfrak{s}_1}} F^A_{W,\Gamma,\mathfrak{s}}. \tag{2.2}$$

To this end, consider a parameterized Kirby decomposition for W, and split

$$W = W_2 \circ W_1,$$

where W_1 is the subcobordism consisting of all 0- and 1-handles. We denote the outgoing boundary of W_1 by Y. Note that for such a splitting, a spin^c-structure $\mathfrak{s}$ on W is uniquely determined by its restrictions $\mathfrak{s}_i$ to each W_i.

The underlying Morse function on W provides a gradient-like vector field $\vec{v}$ on W. After a small perturbation, we can assume that Γ is disjoint from the descending manifolds of the index-one critical points, the ascending manifolds of the index-three critical points, and both the ascending and descending manifolds of the index-two critical points. Using $\vec{v}$, we flow each point of Γ backwards or forwards so that it hits Y. This gives (possibly after a small perturbation) an embedded graph in Y, which we may think of as a ribbon graph in $Y \times (-\epsilon, \epsilon)$. We connect this to the basepoints of the Y_i via arcs going along the flow lines of $\vec{v}$. Denote these collections of arcs by Γ_1 and Γ_2. The map $F^A_{W,\Gamma,\mathfrak{s}}$ is then equal to the composition

$$F^A_{W,\Gamma,\mathfrak{s}} \simeq F^A_{W_2,\Gamma_2,\mathfrak{s}_2} \circ \mathfrak{A}_{\mathcal{G}_\Gamma} \circ F^A_{W_1,\Gamma_1,\mathfrak{s}_1}. \tag{2.3}$$

Here, $\mathfrak{A}_{\mathcal{G}_\Gamma} : CF^-(Y) \to CF^-(Y)$ is the graph action map associated to the flowed image of Γ in Y, and should be thought of as defining the cobordism map in the case where $W = Y \times I$. When no confusion is possible, we will sometimes suppress notation and write the outer two maps as $F^A_{W_1,\mathfrak{s}_1}$ and $F^A_{W_2,\mathfrak{s}_2}$. See Figure 2.2.

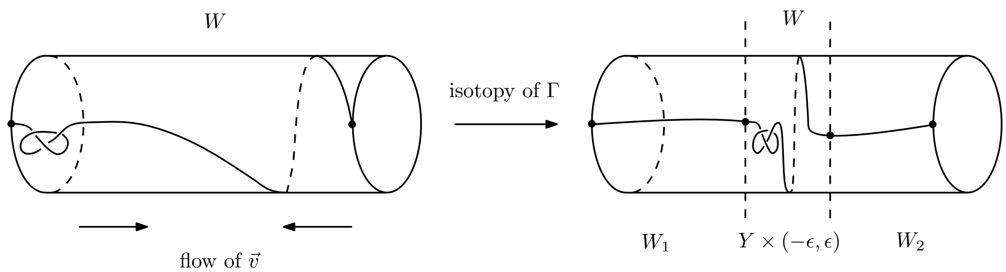


Figure 2.2: Schematic depiction of flowing Γ into Y. In actuality, Y will have some topology and Γ need not be a path.

Roughly speaking, we think of the whole procedure as isotoping Γ so that it is uninteresting outside of Y; the maps associated to (W_1, Γ_1) and (W_2, Γ_2) can then be defined using only a slight modification of the construction of Ozsváth and Szabó. In what follows, we similarly use the technique of flowing Γ so that it is "concentrated" in a convenient slice. In particular, note that if Γ and Γ' are two ribbon graphs in W, then their flowed versions agree outside of Y.

For convenience, we also record the grading shift formula established in [20, Proposition 4.1] which will be useful to us later. Let (W, Γ) be a ribbon graph cobordism from $(Y_1, \mathbf{w}_1)$ to $(Y_2, \mathbf{w}_2)$ and let $\mathfrak{s}$ be a spin^c-structure on W. Define the *reduced Euler characteristic of* Γ to be

$$\widetilde{\chi}(\Gamma) = \chi(\Gamma) - \frac{1}{2}(|\mathbf{w}_1| + |\mathbf{w}_2|).$$

The grading shift associated to $F^A_{W,\Gamma,\mathfrak{s}}$ is then given by

$$\Delta(W,\Gamma,\mathfrak{s}) = \frac{c_1(\mathfrak{s})^2 - 2\chi(W) - 3\sigma(W)}{4} + \widetilde{\chi}(\Gamma).$$

Note that if Γ is a path, then the reduced Euler characteristic of Γ is zero.

2.4.2 Independence for paths

In this subsection, we verify that if Γ is a path, then the map $F^A_{W,\Gamma}$ depends only on the homology class $[\Gamma] \in H_1(W, \partial W)/\text{Tors}$. This is rather well-known to experts, but we record it here for completeness. Note that if Γ is a path, then F^A and F^B are homotopy equivalent and coincide with the usual construction of Ozsváth and Szabó by [41, Theorem B]. In this situation we will thus write F instead of F^A.

Lemma 2.4.1. *Let W be a cobordism between two singly-based (connected) 3-manifolds (Y_1, w_1) and (Y_2, w_2). Let γ and γ' be two paths in W from w_1 to w_2. Suppose that*

$$[\gamma - \gamma'] = 0 \in H_1(W)/\mathit{Tors}.$$

Then

$$F_{W,\gamma,\mathfrak{s}} \simeq F_{W,\gamma',\mathfrak{s}}.$$

Proof. Decompose W as before. Flow γ and γ' into Y and denote the images of w_1 and w_2 in Y by v_1 and v_2. We obtain two arcs in Y that go between v_1 and v_2 which, by an abuse of notation, we continue to denote by γ and γ'. Let $\mathfrak{A}_{\mathcal{G}}$ and $\mathfrak{A}_{\mathcal{G}'}$ be the graph action maps on $CF^-(Y)$ associated to γ and γ'. Note that $c = \gamma * (\gamma')^{-1}$ is a closed loop in Y which is zero when included into $H_1(W)/\text{Tors}$. We now have:

$$\begin{aligned}
F_{W,\gamma,\mathfrak{s}} - F_{W,\gamma',\mathfrak{s}} &\simeq F_{W_2,\mathfrak{s}_2} \circ \mathfrak{A}_{\mathcal{G}} \circ F_{W_1,\mathfrak{s}_1} - F_{W_2,\mathfrak{s}_2} \circ \mathfrak{A}_{\mathcal{G}'} \circ F_{W_1,\mathfrak{s}_1} \\
&= F_{W_2,\mathfrak{s}_2} \circ (\mathfrak{A}_{\mathcal{G}} - \mathfrak{A}_{\mathcal{G}'}) \circ F_{W_1,\mathfrak{s}_1} \\
&= F_{W_2,\mathfrak{s}_2} \circ S^-_{v_1}(A_\gamma - A_{\gamma'})S^+_{v_2} \circ F_{W_1,\mathfrak{s}_1} \\
&= F_{W_2,\mathfrak{s}_2} \circ S^-_{v_1} A_c S^+_{v_2} \circ F_{W_1,\mathfrak{s}_1} \\
&\simeq F_{W_2,\mathfrak{s}_2} \circ A_c S^-_{v_1} S^+_{v_2} \circ F_{W_1,\mathfrak{s}_1}
\end{aligned}$$

Here, in the third line, we have used the definition of $\mathfrak{A}_{\mathcal{G}_\Gamma}$ [41, Equation 7.5], while in the fourth and fifth lines we have used [41, Lemma 5.3] and [41, Lemma 6.13], respectively.

Note that A_c is the usual $H_1(Y)/\text{Tors}$-action on $CF^-(Y)$. We claim that the map $F_{W_2,\mathfrak{s}_2} \circ A_c$ is U-equivariantly nullhomotopic. For this, we use the following result from [17]. Let W be a cobordism from Y to Y', and let $c \subseteq Y$ and $c' \subseteq Y'$ be two closed curves that are homologous in W. Then [17, Theorem 3.6] states that

$$F_{W_2,\mathfrak{s}} \circ A_c \simeq A_{c'} \circ F_{W_2,\mathfrak{s}},$$

where $F_{W_2,\mathfrak{s}}$ is the usual cobordism map of Ozsváth and Szabó As written, [17, Theorem 3.6] deals with the total homology map on $\widehat{HF}$. However, the proof is easily modified to hold on the level of U-equivariant homotopy (for CF^-), and can be refined to take into account individual spin^c-structures. See [17, Remark 3.7]. In our case, note that W_1 consists of adding 1-handles to Y_1. A Mayer-Vietoris argument then shows that the inclusion of $H_1(W_2)$ into $H_1(W)$ is injective. Hence some multiple of $[c]$ is actually nullhomologous in W_2. The claim then follows from the above commutation relation by choosing c' in Y_2 to be empty (or a small unknot). $\square$

2.4.3 Proof of invariance

In this subsection, we prove that $h_{\mathcal{T}}$ and $h_{\iota\circ\mathcal{T}}$, defined in Subsection 2.3.2 are invariants of the group $\Theta^{\tau}_{\mathbb{Z}}$. Moreover we will show that the invariants induce homomorphisms

$$h_{\mathcal{T}}, h_{\iota\circ\mathcal{T}} : \Theta^{\tau}_{\mathbb{Z}} \to \mathfrak{I}.$$

which will complete thr proof of Theorem 2.3.1.

Our strategy for proving Theorem 2.3.1 is showing that any pseudo-homology bordism induces a local equivalence between the τ-complexes (and $\iota\circ\tau$-complexes) of its incoming and outgoing ends. Firstly we will choose a specific graph on the cobordism.

Throughout, let (W, f) be a pseudo-homology bordism between (Y_1, τ_1) and (Y_2, τ_2), where Y_1 and Y_2 are disjoint unions of homology spheres. We equip each connected component of Y_1 and Y_2 with a single basepoint. For simplicity, assume that W itself is connected.

Let W_a be the cobordism formed by an iterated sequence of 1-handle attachments joining together the components of Y_1, as displayed in Figure 2.3. Let W_b be (the reverse of) the analogous cobordism joining together the components of Y_2. Clearly, we can embed W_a and W_b in W to obtain a decomposition

$$W = W_b \circ W_0 \circ W_a,$$

where W_0 is now a cobordism between two homology spheres. Note that the inclusion of W_0 into W induces an isomorphism on H_1.

Definition 2.4.2. We define a ribbon graph Γ in W as follows. On W_a, let Γ be any trivalent 1-skeleton corresponding to the iterated sequence of 1-handle attachments, as displayed in Figure 2.3. For concreteness, we fix an ordering for the connected components of Y_1. (This specifies an order for taking the iterated connected sum, and also a way to choose a cyclic ordering at each internal vertex.) We define Γ on W_b similarly. To define Γ on W_0, first choose a path γ running between the two ends of W_0. Fix an ordered basis $e_1, \cdots, e_n$ of $H_1(W_0)$, and represent each e_k by a simple closed curve c_k that does not intersect γ. We then join each c_k to γ via an arc, which we refer to as a *connecting arc*. Again, for concreteness, fix a cyclic ordering at each internal vertex. We call any Γ constructed in this fashion a *standard graph*. See Figure 2.3.

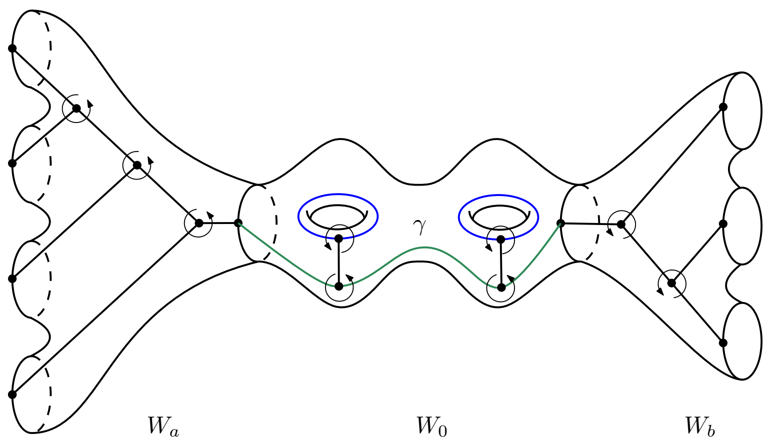


Figure 2.3: Schematic decomposition $W = W_b \circ W_0 \circ W_a$. The path γ is drawn in green, while the curves c_k are drawn in blue. We choose the indicated cyclic ordering at each internal vertex.

Now consider the cobordism map $F^A_{W,\Gamma}$ associated to a standard graph. Our goal will be to show that this is a local map (with respect to both τ and $\iota \circ \tau$). As a first step, it will be helpful for us to have the following alternative formulation of $F^A_{W,\Gamma}$. Let Γ_{red} be the "reduced" ribbon graph formed by replacing the subgraph $\Gamma \cap W_0$ in Definition 2.4.2 with the path γ. Let W_{red} be obtained from W by surgering out the curves c_k. Note that $W_{\text{red}} = W_b \circ W_h \circ W_a$, where W_h is a homology cobordism. The image of Γ_{red} under this surgery defines a ribbon graph in W_{red}, which we also denote by Γ_{red}.

We now prove the Theorem assuming a few Lemmas that we will come back to later.

Proof of Theorem 2.3.1. Let (W, f) be a pseudo-homology bordism from (Y_1, τ_1) to (Y_2, τ_2). We wish to show:

1. $F^A_{W,\Gamma} \circ \iota_1 \simeq \iota_2 \circ F^A_{W,\Gamma}$;

2. $F^A_{W,\Gamma} \circ \tau_1 \simeq \tau_2 \circ F^A_{W,\Gamma}$; and,

3. $F^A_{W,\Gamma}$ maps U-nontorsion elements in homology to U-nontorsion elements in homology (and has zero grading shift).

The first and third claims follow immediately from Lemma 2.4.3 and standard results of Hendricks, Manolescu, and Zemke. Indeed, according to Lemma 2.4.3, we have

$$F^A_{W,\Gamma} \simeq F^A_{W_{\text{red}},\Gamma_{\text{red}}}.$$

The latter cobordism is equal to the composition $W_b \circ W_h \circ W_a$, where the outer two terms are compositions of connected sum cobordisms (or their reverses), and W_h is a homology cobordism equipped with a path γ. By [20, Proposition 5.10] and [19, Proposition 4.9], the maps associated to each of these pieces commutes with ι up to homotopy. Applying the composition law, we thus see that $F^A_{W,\Gamma}$ homotopy commutes with ι also. The third claim is similarly verified by establishing the desired condition for each piece. To prove the second claim, we apply (2.1) and Lemma 2.4.7:

$$F^A_{W,\Gamma} \circ \tau_1 \simeq \tau_2 \circ F^A_{W,f(\Gamma)} \simeq \tau_2 \circ F^A_{W,\Gamma}.$$

This proves that $F^A_{W,\Gamma}$ is a local map with respect to τ. Turning W around shows that $h_{\tau_1}(Y_1) = h_{\tau_2}(Y_2)$, as desired. To show that $F^A_{W,\Gamma}$ preserves $h_{\iota\circ\tau}$, we apply the first and second claims to obtain

$$F^A_{W,\Gamma} \circ (\iota_1 \circ \tau_1) \simeq (\iota_2 \circ \tau_2) \circ F^A_{W,\Gamma}.$$

Hence h_τ and $h_{\iota\circ\tau}$ are well-defined maps from $\Theta^\tau_{\mathbb{Z}}$ to $\mathfrak{I}$. Since CF^- takes disjoint unions to tensor products (for the trivial coloring), this completes the proof. □

We now move on to proving the Lemmas used above.

Lemma 2.4.3. *Let Γ be a standard graph in W. Then*

$$F^A_{W,\Gamma} \simeq F^A_{W_{red},\Gamma_{red}}.$$

Proof. Note that by [41, Proposition 11.1], the cobordism maps F^A are unchanged under puncturing. More precisely, suppose that (W, Γ) is any cobordism from Y_1 to Y_2. Puncture

W at any interior point and equip the new boundary S^3 with a single basepoint. We modify the original ribbon graph Γ by joining this basepoint to Γ via an arc (and choosing any cyclic ordering at the new internal vertex). Let the new incoming boundary be given by $Y_1 \sqcup S^3$. Then it follows from [41, Proposition 11.1] that under the identification of $CF^-(Y_1)$ with $CF^-(Y_1 \sqcup S^3) \simeq CF^-(Y_1) \otimes CF^-(S^3)$, the cobordism map remains unchanged up to U-equivariant homotopy.

In our case, consider the cobordism $W_{S^1 \times B^3}$ from S^3 to $S^1 \times S^2$ formed by puncturing $S^1 \times B^3$ at any interior point. We define a ribbon graph $\Gamma_{S^1 \times B^3}$ on $W_{S^1 \times B^3}$ by taking a closed loop generating $H_1(S^1 \times B^3)$ and joining this to each boundary component via an arc. Now identify a neighborhood of each c_k with $\nu(c_k) \cong S^1 \times B^3$, and puncture W at an interior point of each of these neighborhoods. This punctured version of W may be viewed as the composition of several copies of $(W_{S^1 \times B^3}, \Gamma_{S^1 \times B^3})$, together with the complement of the $\nu(c_k)$ in W. We similarly define $W_{D^2 \times S^2}$ by puncturing $D^2 \times S^2$ at any interior point and equipping this with an arc $\Gamma_{D^2 \times S^2}$ running between the two boundary components. Then W_{red} may be viewed (after puncturing) as several copies of $(W_{D^2 \times S^2}, \Gamma_{D^2 \times S^2})$, together with the same complement as before. By the composition law, to establish the lemma it thus suffices to show that

$$F^A_{W_{S^1 \times B^3}, \Gamma_{S^1 \times B^3}} \simeq F^A_{W_{D^2 \times S^2}, \Gamma_{D^2 \times S^2}}$$

as maps from $CF^-(S^3)$ to $CF^-(S^1 \times S^2)$. This is a standard calculation. $\square$

In light of Lemma 2.4.3, the reader may wonder why we have not simply defined our cobordism maps directly in terms of W_{red} and Γ_{red}, rather than Γ. (Indeed, this corresponds to the usual approach in Floer theory when dealing with cobordisms with $b_1 > 0$; see for example the proof of [32, Theorem 9.1].) The reason is that c_k need not be fixed by f, so the surgered cobordism W_{red} may not inherit an extension of τ_i. Thus, *a priori* there is no reason to think that the surgered cobordism interacts nicely with τ. In actuality, we will show that $F^A_{W,\Gamma}$ homotopy commutes with τ, which implies that $F^A_{W_{\mathrm{red}},\Gamma_{\mathrm{red}}}$ does also.

Alternatively, one can also define $F^A_{W,\Gamma}$ by considering the graph Γ_{red} in W and cutting down via the $H_1(W)/\text{Tors}$-actions of each of the e_k. This is essentially what we do in Lemma 2.4.7, except in a language more amenable to that of [41].

When dealing with the action of f on W, we will thus need to take a slightly different approach. We begin with a more refined decomposition theorem, which is essentially taken from the proof of [32, Theorem 9.1].

Lemma 2.4.4. *Let W be a definite cobordism between two 3-manifolds. Then there exists a decomposition $W = W_2 \circ W_1$ of W for which the following holds:*

1. *W_1 consists of 1- and 2-handles,*

2. *W_2 consists of 2- and 3-handles; and,*

3. *Let Y be the slice given by the outgoing boundary of W_1. Then the map induced by the inclusion of Y into W*

$$i_* : H_1(Y)/\mathit{Tors} \to H_1(W)/\mathit{Tors}$$

   is an isomorphism.

Proof. Give W a handle decomposition consisting of 1-handles, 2-handles, and 3-handles (attached in that order). According to the proof of [32, Theorem 9.1], we can re-index the sequence of 2-handle attachments as follows. Let the 2-handles be denoted by $\{h_i\}_{i=1}^n$, and for each i let S_i be the outgoing boundary obtained after attaching h_i. Let the incoming boundary of the very first 2-handle be denoted by S_0. According to the proof of [32, Theorem 9.1], we may assume that the sequence of Betti numbers $\{b_1(S_i)\}_{i=0}^n$ at first monotonically decreases with i, then is constant, and then finally monotonically increases with i. Ozsváth and Szabó refer to such an ordering of the h_i as a *standard ordering*. This can be achieved whenever W is definite.

We now choose $Y = S_i$ to be any slice in the above sequence for which $b_1(S_i)$ attains its minimum value. This decomposes W into two subcobordisms W_a and W_b that obviously satisfy the first two desired properties. Let the 2-handles h_j for $j > i$ be attached to Y along a link whose components we denote by $\mathbb{K}_j$. We claim that each of these components is rationally nullhomologous in Y. Indeed, the condition $b_1(S_i) \leq b_1(S_{i+1})$ implies that $\mathbb{K}_{i+1}$ is rationally nullhomologous in Y; proceeding by induction, we assume that $\mathbb{K}_{i+1}, \ldots, \mathbb{K}_l$ are rationally nullhomologous in Y. Now, $\mathbb{K}_{l+1}$ is rationally nullhomologous in S_l, which is obtained from Y by integer surgery along $\mathbb{K}_{i+1}, \ldots, \mathbb{K}_l$. The inductive hypothesis then easily implies that $\mathbb{K}_{l+1}$ is rationally nullhomologous in Y also.

It follows immediately that the induced inclusion map $i_* : H_1(Y)/\text{Tors} \to H_1(W_b)/\text{Tors}$ is an isomorphism, since W_b is built from $Y \times I$ via attaching rationally nullhomologous 2-handles and then some 3-handles. Turning the cobordism around, we obtain the same result with W_a in place of W_b. A standard Mayer-Vietoris argument then gives the desired claim. □

Definition 2.4.5. Let Y be any 3-manifold equipped with a collection of incoming basepoints $\mathcal{V}_{\text{in}}$ and outgoing basepoints $\mathcal{V}_{\text{out}}$. We say that a ribbon graph Λ in $Y \times I$ is *star-shaped* if it has a unique internal vertex, which is connected to each basepoint via a single arc. We also fix a formal ribbon structure; this corresponds to a cyclic ordering of $\mathcal{V}_{\text{in}} \cup \mathcal{V}_{\text{out}}$. Note that given any incoming basepoint v_i and outgoing basepoint v_j, there is a unique path in Λ going from v_i to v_j, which we denote by l_{ij}.

The proof of the next technical lemma is similar to that of [41, Lemma 7.13]. The authors would like to thank Ian Zemke for help with the proof and a discussion of the surrounding ideas.

Lemma 2.4.6. *Let Λ and Λ' be two star-shaped graphs in $Y \times I$. Suppose that for any incoming basepoint v_i and outgoing basepoint v_j, we have*

$$[l_{ij} - l'_{ij}] = 0 \in H_1(Y)/\mathit{Tors}.$$

Suppose moreover that Λ and Λ' have the same formal ribbon structure (viewed as cyclic orderings of the set of basepoints). Then for any spin^c*-structure* $\mathfrak{s}$ *on* $Y \times I$*, we have*

$$F^A_{Y\times I,\Lambda,\mathfrak{s}} \simeq F^A_{Y\times I,\Lambda',\mathfrak{s}}.$$

Proof. Without loss of generality, we may isotope Λ and Λ' so that they share the same internal vertex v. For any basepoint v_i, denote the edge of Λ joining v_i to v by e_i.[1] We claim that there is a fixed element $\lambda \in H_1(Y)/\mathrm{Tors}$ such that $[e'_i - e_i] = \lambda$ for all i. Indeed, consider any pair of incoming and outgoing vertices v_i and v_j. Then

$$[e'_i - e_i] - [e'_j - e_j] = [l'_{ij} - l_{ij}] = 0 \in H_1(Y)/\mathrm{Tors}.$$

Set $\lambda = [e'_i - e_i]$. Varying j (and then varying i) gives the claim.

We now turn to the assertion of the lemma. Without loss of generality, let the basepoints of Y be given by $\mathcal{V}_{\mathrm{in}} \cup \mathcal{V}_{\mathrm{out}} = \{v_i\}_{i=1}^n$, and let the cyclic order corresponding to the formal ribbon structure be $v_1, \dots, v_n$. By [41, Equation 7.2],

$$F^A_{Y\times I,\Lambda} = \left(\prod_{x\in\mathcal{V}_{\mathrm{in}}\cup\{v\}} S^-_x\right) \circ A_{e_n} \circ \cdots \circ A_{e_1} \circ \left(\prod_{x\in\mathcal{V}_{\mathrm{out}}\cup\{v\}} S^+_x\right).$$

A similar expression holds for Λ' after replacing each e_i with e'_i. By [41, Lemma 5.3] and the fact that $[e'_i - e_i] = \lambda$, we have $A_{e'_i} \simeq A_{e_i} + A_\lambda$. Hence

$$\begin{aligned} A_{e'_n} \circ \cdots \circ A_{e'_1} &\simeq (A_{e_n} + A_\lambda) \circ \cdots \circ (A_{e_1} + A_\lambda) \\ &\simeq A_{e_n} \circ \cdots \circ A_{e_1} + A_\lambda \circ \left(\sum_i A_{e_n} \circ \cdots \circ \widehat{A}_{e_i} \circ \cdots \circ A_{e_1}\right). \end{aligned}$$

Here, the notation $\widehat{A}_{e_i}$ means that A_{e_i} should be omitted from the composition. In the second line, we have expanded the product and used the fact that $A_\lambda \circ A_\lambda \simeq 0$ whenever λ is a closed curve (see [41, Lemma 5.5]). Substituting this into the expression for $F^A_{Y\times I,\Lambda'}$, it thus clearly suffices to show

$$S^-_v \circ \left(\sum_i A_{e_n} \circ \cdots \circ \widehat{A}_{e_i} \circ \cdots \circ A_{e_1}\right) \circ S^+_v \simeq 0.$$

[1] By [41, Lemma 5.3], note that $A_{-e_i} = -A_{e_i}$. Since this coincides with A_{e_i} mod 2, we will occasionally use e_i to also denote the same edge with reversed orientation.

Throughout, we have used the fact that A_λ commutes with the A_{e_i} and the stabilization maps $S_v^{\pm}$, since λ is a closed curve. (See [41, Lemma 5.4] and [41, Lemma 6.13].)

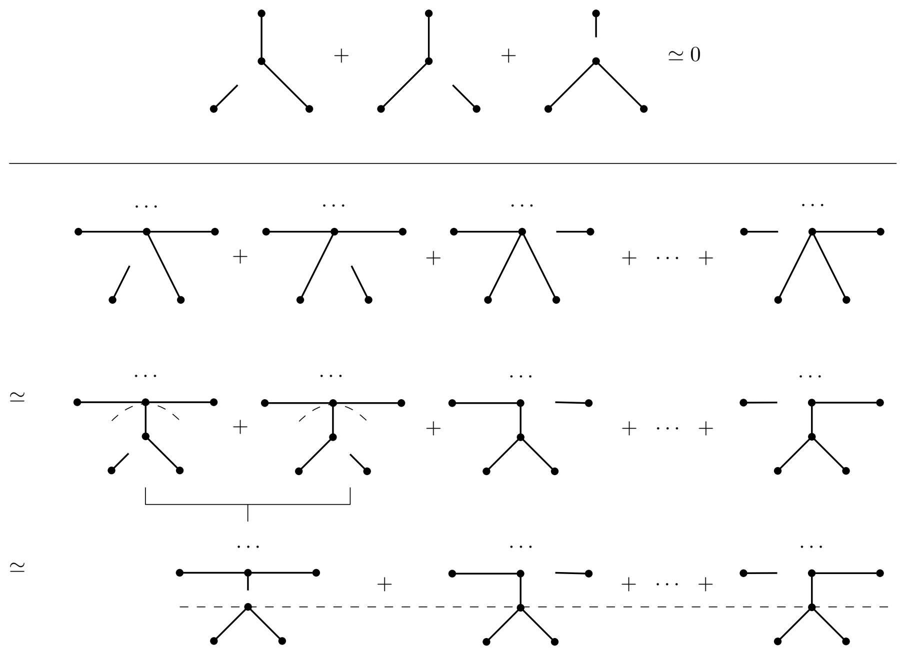


Figure 2.4: Diagrammatic proof of Lemma 2.4.6. The ellipses above each star-shaped graph indicate further edges attached to the interior vertex.

We proceed by induction. For $n = 3$, we claim that

$$S_v^-(A_{e_3}A_{e_2} + A_{e_3}A_{e_1} + A_{e_2}A_{e_1})S_v^+ \simeq S_v^-(A_{e_3} + A_{e_2})(A_{e_2} + A_{e_1})S_v^+.$$

This follows by expanding the right-hand side and noting that $S_v^- A_{e_2}A_{e_2}S_v^+ \simeq US_v^- S_v^+ \simeq 0$ by Lemmas 5.5 and 6.15 of [41]. On the other hand, we have

$$S_v^-(A_{e_3} + A_{e_2})(A_{e_2} + A_{e_1})S_v^+ \simeq S_v^- A_{e_3 * e_2} A_{e_2 * e_1} S_v^+ \simeq A_{e_3 * e_2} A_{e_2 * e_1} S_v^- S_v^+ \simeq 0.$$

Here, to obtain the second homotopy equivalence, we have used [41, Lemma 6.13] and the fact that $e_3 * e_2$ and $e_2 * e_1$ are paths which do not have v as an endpoint. This establishes the base case.

The inductive step is diagrammatically described in Figure 2.4. In the first row of Figure 2.4, we have displayed three graphs corresponding to the three terms in the case $n = 3$. In the second row, we have displayed the sum in question for general n. We modify each of the graphs in the second row by introducing an additional internal vertex and edge, as in the third row of Figure 2.4. Note that this does not change the ribbon equivalence class. We then view the first two terms as composite graphs with the splittings indicated by the dashed arcs, and apply the $n = 3$ case to obtain the fourth row. We similarly view each graph in the fourth row as a composition of two subgraphs, corresponding to the pieces above and below the dashed line. Factoring out the map corresponding to the subgraph below the dashed line, the remaining sum is precisely the inductive hypothesis for $n - 1$. This completes the proof. □

We now come to the central lemma of this section:

Lemma 2.4.7. *Let (W, f) be a pseudo-homology bordism and let Γ be a standard graph in W. Then*

$$F^A_{W,\Gamma} \simeq F^A_{W,f(\Gamma)}.$$

Proof. For convenience, denote $\Gamma' = f(\Gamma)$. Decompose W as in Lemma 2.4.4, and flow Γ and Γ' into the slice Y afforded by Lemma 2.4.4. (Here, we are using the fact that W_a consists of 1- and 2-handles, while W_b consists of 2- and 3-handles.) Without loss of generality, we may thus assume that Γ and Γ' agree outside of $Y \times I$. By abuse of notation, we denote the subgraphs $\Gamma \cap (Y \times I)$ and $\Gamma' \cap (Y \times I)$ by Γ and Γ' also. Applying the composition law, it clearly suffices to prove that $F^A_{Y\times I,\Gamma} \simeq F^A_{Y\times I,\Gamma'}$. Note that we implicitly equip $Y \times I$ with the pullback of the single spin^c-structure on W. See the top-left of Figure 2.5.

Define Γ_{red} to be Γ with the curves c_k and connecting arcs deleted. By [44, Proposition 4.6], we have[2]

$$F^A_{Y\times I,\Gamma} \simeq F^A_{Y\times I,\Gamma_{\text{red}}} \circ \left(\prod_k A_{c_k}\right).$$

Note that Γ' is combinatorially isomorphic to Γ. In particular, Γ' consists of a set of closed loops c'_k, which are joined to an underlying tree via connecting arcs. These loops are in

[2] Compare Figure 2.5 and [44, Figure 4.5]. In our case, contracting each individual connecting arc to a point does not change the ribbon equivalence class.

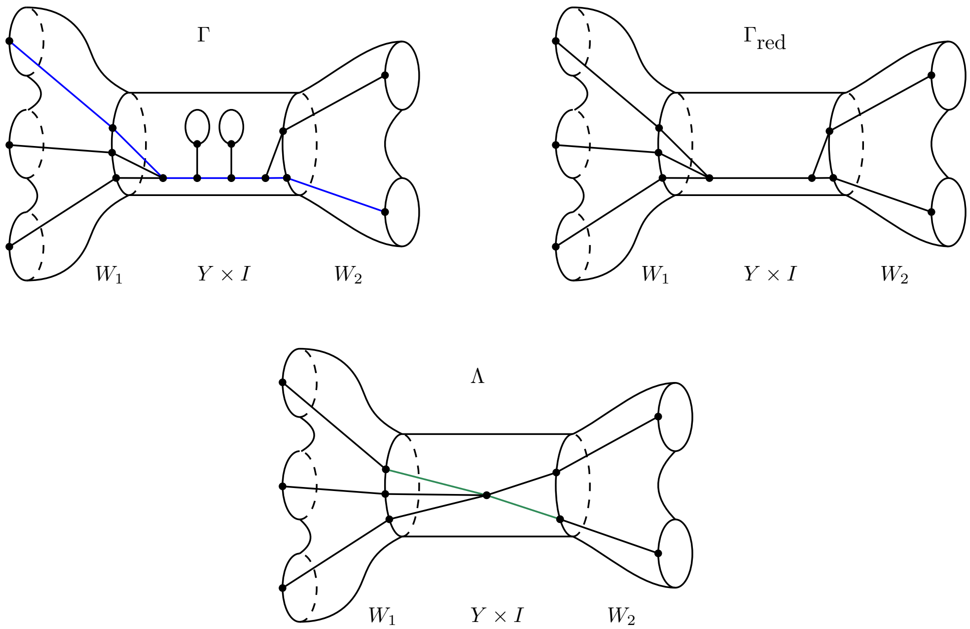


Figure 2.5: Top left: the flowed graph Γ. Top right: the modified graph Γ_{red}. Bottom middle: the graph Λ. The path l_{ij} from the proof of Lemma 2.4.7 is marked in green; the path g_{ij} is marked in blue. In general, Y will have some topology.

correspondence with the analogous loops c_k in Γ. Defining Γ'_{red} similarly, we have

$$F^A_{Y\times I,\Gamma'} \simeq F^A_{Y\times I,\Gamma'_{\text{red}}} \circ \left(\prod_k A_{c'_k}\right).$$

Since f acts as the identity on homology, we have $[c'_k] = [c_k]$ in $H_1(W)$ for each k. By Lemma 2.4.4, this implies that $[c'_k] = [c_k]$ in $H_1(Y)/\text{Tors}$, and thus that $A_{c'_k} \simeq A_{c_k}$ for each k by [41, Proposition 5.8]. Hence to establish the claim, it suffices to prove that $F^A_{Y\times I,\Gamma_{\text{red}}} \simeq F^A_{Y\times I,\Gamma'_{\text{red}}}$. See the top-right of Figure 2.5.

We now contract all of the internal edges in Γ_{red} to obtain a star-shaped graph Λ, as displayed in the second row of Figure 2.5. This does not change the ribbon equivalence class of Γ_{red}. We similarly contract all the edges of Γ'_{red} to obtain a star-shaped graph Λ'. It remains to verify the hypotheses of Lemma 2.4.6. Let v_i be an incoming basepoint in $Y \times I$ and let v_j be an outgoing basepoint. Let g_{ij} be the obvious path in Γ (viewed as a graph in W) going between the corresponding basepoints w_i and w_j of W, as in Figure 2.5. Define g'_{ij} similarly. Note that g_{ij} and g'_{ij} agree outside of $Y \times I$, and $[g_{ij}] = [g'_{ij}] \in H_1(W, \partial W)$

since f acts as the identity on $H_1(W, \partial W)$. Clearly, l_{ij} and $g_{ij} \cap (Y \times I)$ are isotopic in $Y \times I$ (rel boundary), and similarly for l'_{ij} and g'_{ij}. Hence

$$[l_{ij} - l'_{ij}] = [g_{ij} - g'_{ij}] = 0 \in H_1(W).$$

By Lemma 2.4.4, we thus have that $[l_{ij} - l'_{ij}] = 0$ in $H_1(Y)/\text{Tors}$. Applying Lemma 2.4.6 completes the proof. $\square$

CHAPTER 3

CORKS AND HEEGAARD FLOER HOMOLOGY

3.1 Introduction

Smooth structures on 4-manifolds have been of central interest in low-dimensional topology for decades. Much attention, in particular, has been paid to finding pairs of smooth, closed, simply connected 4-manifolds that are homeomorphic but not diffeomorphic. *Corks* are objects of central importance to this study. A cork is a tuple (Y, τ, W) of a 3-manifold Y^3, a contractible 4-manifold W^4 bounded by Y, and an orientation preserving involution τ on Y that does not extend over W as a diffeomorphism. A *cork-twist* is an operation of cutting and re-gluing a cork along its boundary involution, when it is embedded inside a closed 4-manifold. A remarkable theorem by [6, 28] establishes that any two smooth structures of a simply-connected topological 4-manifold are related by a cork-twist. The first example of a cork was found by [1]. Since then numerous examples of corks have been produced using various techniques. The most common way of detecting corks has been to embed them inside a larger closed 4-manifold W and then showing that the cork-twist changes a certain type of smooth 4-manifold invariant, for example the Ozsváth-Szabó 4-manifold invariant [35].

Recently [25] introduced a generalized version of cork, called a *strong cork*, which is a cork where the involution of the boundary does not extend over any homology 4-ball. They also gave an example of a strong cork, by showing that the so-called *Akbulut cork* is strong. Their proof required construction of an appropriate long exact sequence on Monopole Floer homology with $\mathbb{Q}$-coefficients and showing a certain cork-twist changes the 4-manifold invariant.

In this section, we study corks through the lens of homology cobordism. Firstly, we note that the invariants h_τ and $h_{\iota\circ\tau}$ developed in the Chapter 2 are useful in detecting strong corks. We then prove a certain monotonocity theorem, regarded as a computational aid which constrains the behavior of our invariants under equivariant negative-definite cobordisms. Furthermore, we then produce explicit methods of constructing such cobordisms via equivariant surgery. Note that, directly computing the invariants h_{τ} and $h_{\iota\circ\tau}$ by computing the action of τ on the CF^{-} would be quite cumbersome. This is because computing the action would require explicit information of how a Heegaard surface and the α and β curves behave under the action of τ, together with the knowledge of how those interact with the chain complexes. To the best of author's knowledge, no such non-trivial example of directly computing the action of τ on the chain complex CF^{-} (up to chain homotopy), exists the literature. In contrast, via the aforementioned techniques, we compute the invariants for a number of examples which, in turn, yields many new examples of strong corks. Some of them were previously not even known to be (standard) corks.

3.2 Strong corks

Definition 3.2.1. Let Y be an integer homology sphere equipped with an (orientation-preserving) involution $\tau : Y \to Y$, and suppose that Y bounds a compact, contractible manifold W. We say that the triple (Y, W, τ) is a *cork* if τ does not extend over W as a diffeomorphism.

In [25] the authors recently consider a slightly generalized version of corks.

Definition 3.2.2. [25, Section 1.2.2] Let Y be a homology sphere, equipped with an involution $\tau : Y \to Y$. Moreover, assume that Y bounds atleast one contractible 4-manifold W. Then we say that the pair (Y, τ) is a *strong cork* if for *any* homology ball X with $Y = \partial X$, τ does not extend over X as a diffeomorphism.

Note that unlike the definition of a cork, a strong cork is defined using 3-dimensional data. The most common way of detecting a cork has been as follows. Consider an embedding of a cork (Y, W) inside a larger closed 4-manifold X. Then cut out and re-glue via the cork twist. One the calculates the a smooth 4-manifold invariant for the resulting 4-manifold X'

to show that its value is different to that of X. This implies the involution τ on Y cannot possibly extend over W as a diffeomorphism. The process of calculating the 4-manifold invariant for X', is where one requires specific knowledge about the boundary involution τ. For example, when the 4-manifold invariant is the Ozsváth-Szabó 4-manifold invariant one needs to compute the action induced by τ on the Floer homology:

$$\tau : HF^{+}(Y) \to HF^{+}(Y)$$

Moreover since we used the embedding of W inside X, this process only shows that τ on Y does not extend over W. In particular, traditional methods for showing that a standard cork is strong fails. Lin, Ruberman, and Saveliev devised a way to show that Akbulut's first example of a cork is strong [25, Theorem D]. They established a certain long exact sequence for monopole Floer homology with $\mathbb{Q}$-coefficients, and then explicitly understood the induced action of τ in monopole Floer homology, finally they appealed to the Seiberg-Witten 4-manifold invariant to prove that the Akbukut cork is indeed strong.

As we will see in the subsequent sections, our approach does not involve any direct/explicit computation of the induced action of τ on HF^{-} or CF^{-} nor do we refer to the any smooth 4-manifold invariant. In most cases this makes proving that a (Y, τ) is strong much more rather straight forward. Below we state our main obstruction result, which concerns the invariants of the group $\Theta^{\tau}_{\mathbb{Z}}$ defined earlier, see Chapter 2.

Corollary 3.2.3. *Let Y be an integer homology sphere with involution $\tau : Y \to Y$, and suppose Y bounds at least one contractible 4-manifold W. Then, if either $h_{\tau}(Y) \neq 0$ or $h_{\iota\circ\tau}(Y) \neq 0$, the pair (Y, τ) is a strong cork.*

Proof. The proof is immediate from statement of Theorem 2.3.1. □

Note that doing $1/k$ surgery on a slice knot results in a 3-manifold which bounds a contractible 4-manifold [16, Section 6]. Moreover if there is a symmetry on the knot, then it induces a symmetry on the manifold obtained by surgery. A natural question is then, whether one can obtain strong cork from surgery on symmetric knots. We show

Theorem 3.2.4. *[7] For $n > 0$, let $K_{-n,n+1}$ be the family of slice doubly-twist knots displayed in Figure 3.1. For k positive and odd, let $V_{n,k}$ be the $(1/k)$-surgery*

$$V_{n,k} = \begin{cases} S_{1/k}(\overline{K}_{-n,n+1}) & \textit{if } n \textit{ is odd} \\ S_{1/k}(K_{-n,n+1}) & \textit{if } n \textit{ is even.} \end{cases}$$

Equip $V_{n,k}$ with the indicated involutions τ and σ. For n odd, we consider the obvious involutions on the mirrored diagram. Then $(V_{n,k}, \tau)$ and $(V_{n,k}, \sigma)$ are both strong corks.

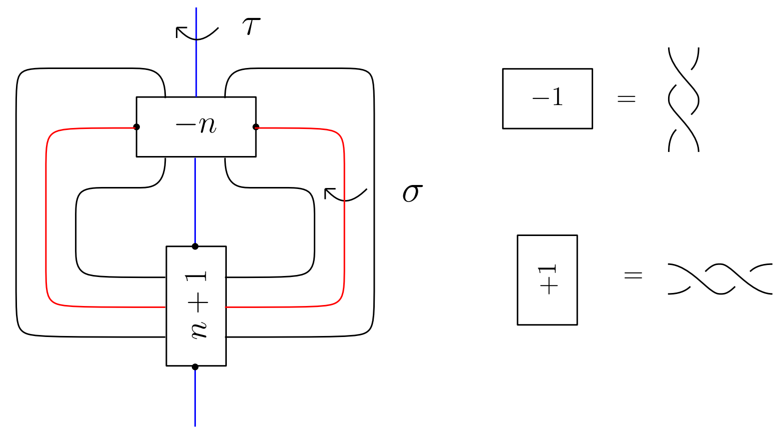


Figure 3.1: Doubly-twist knot $K_{-n,n+1}$. The indicated symmetries τ and σ are given by 180° rotations about the blue and red axes, respectively. In the latter case, it may be helpful to view $K_{-n,n+1}$ as an annular knot; the action of σ is given by rotation about the core of the solid torus. Black dots indicate the intersections of $K_{-n,n+1}$ with the axes of symmetry.

To the best of the authors' knowledge, none of manifolds in Theorem 3.2.4 were previously known to be (the boundaries of) corks, strong or otherwise.

As seen in the previous Theorem, One of our principal ways of finding corks will be to consider surgeries on equivariant slice knots. In this vein, we have

Theorem 3.2.5. *Let K be a knot in S^3 equipped with a strong inversion τ, and let $k \neq -2, 0$. Then we have both $h_\tau(S_{1/(k+2)}(K)) \leq h_\tau(S_{1/k}(K))$ and $h_{\iota\circ\tau}(S_{1/(k+2)}(K)) \leq h_{\iota\circ\tau}(S_{1/k}(K))$.*

We now turn to some examples given by surgeries on links. The family we consider is a generalization of the initial cork from [1]:

Theorem 3.2.6. *For $n > 0$, let M_n be the family of two-component link surgeries displayed on the left in Figure 3.3. Equip M_n with the indicated involution τ. Then (M_n, τ) is a strong cork. In fact, we may modify each M_n by introducing any number of symmetric pairs of negative full twists, as on the right in Figure 3.3, and this conclusion still holds.*

In the theorems mentioned above, we compare the involution on τ on the proposed strong cork with the a certain involutions on certain Brieskorn homology spheres. We also show that it is possible to 'compare' involution on a strong cork to an involution on a manifold that is not a Seifert homology sphere. This example comes from the family of "positron" corks introduced by Akbulut and Matveyev in [2]. Here, we show that the first member of this family is a strong cork.

Theorem 3.2.7. *Let P be the two-component link surgery displayed in the left in Figure 3.2. Equip P with the indicated involution τ. Then (P, τ) is a strong cork. In fact, we may modify P by introducing any number of symmetric pairs of negative full twists, as on the right in Figure 3.2, and this conclusion still holds.*

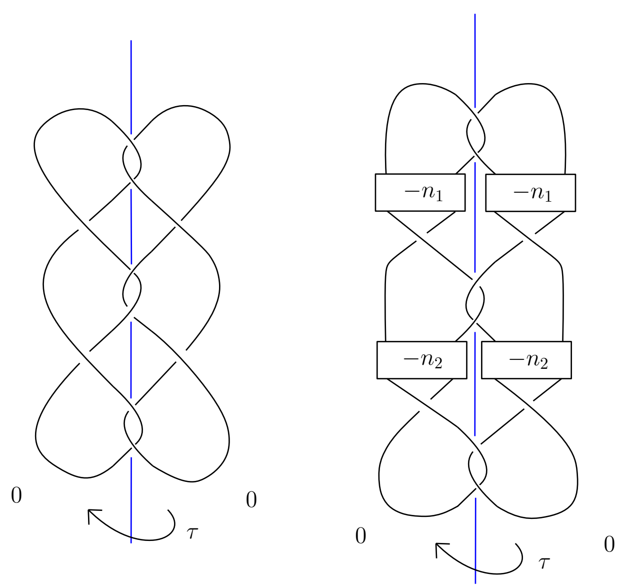


Figure 3.2: Left: the "positron" cork from [2]. Right: adding symmetric pairs of negative full twists to P.

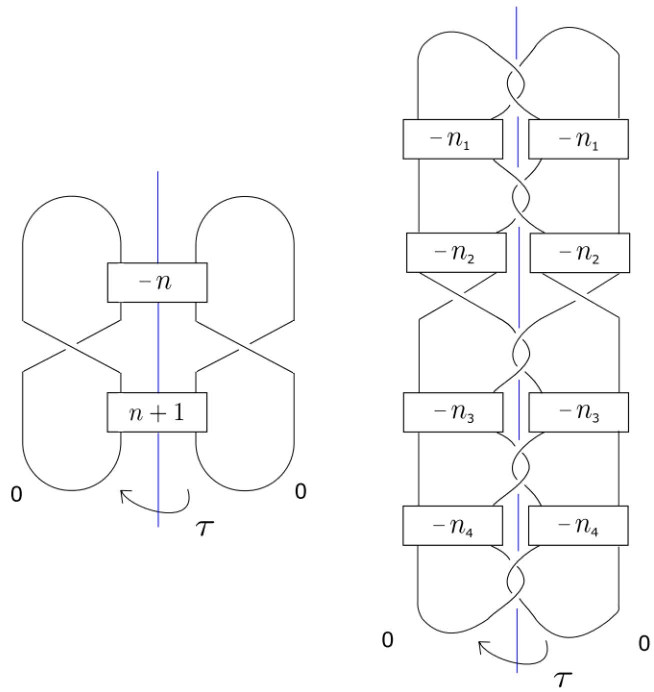


Figure 3.3: Left: the manifold M_n. Right: an example of adding symmetric pairs of negative full twists to M_2.

We hope that the argument in the proof of Theorem 3.2.7 can be adopted for computing the invariants and providing a wide range of strong corks.

3.3 Bordism and equivariant Kirby diagrams

Before moving on to proving the theorems listed in the previous section, we develop a key topological tool which will be useful to us in constraining the behavior of the invariants. Specifically, we focus on constructing explicit bordisms using the Kriby diagram. Here by a bordism we refer to an *equivariant cobordism*, see Chapter 2.

Let K be a knot in a 3-manifold Y, and let τ be an orientation-preserving involution on Y that fixes K setwise. In the case that $Y = S^3$, we will often draw τ as 180° rotation though some axis of symmetry. (By work of Waldhausen, any orientation-preserving involution of S^3 is conjugate to one of this form [39].) Usually, we draw this axis as a line in $\mathbb{R}^3$, but sometimes it will be more convenient to draw the axis of rotation as an unknot, as in Figure 3.1.

In this subsection, we verify that τ induces an involution on any manifold obtained by surgery on K and, similarly, on any cobordism formed from handle attachment along K. This is well-known and implicit in many sources, e.g. [30], but we include the proofs here for completeness.

Definition 3.3.1. An involution τ of (Y, K) is said to be a *strong involution* (or *strong inversion*) if τ fixes two points on K. If instead the action is free on K, we say that τ is a *periodic involution*. Note that a strong involution reverses orientation on K, while a periodic involution preserves orientation. We will sometimes refer to such a K as an *equivariant knot*.

Now let K be an equivariant knot in Y. It is easily checked that there exists an equivariant framing of K, as follows. By averaging an arbitrary Riemannian metric with its pullback under τ, we may assume that τ acts as an isometry on Y, and hence also on the normal bundle to K. If we fix an arbitrary framing of K, we can choose coordinates

$$\nu(K) \cong S^1 \times D^2 = \{(z, w) : |z| = 1, |w| \leq 1\},$$

such that:

1. If τ is strong, then the action of τ on $\nu(K)$ is $\tau(z, w) = (\bar{z}, A_z \bar{w})$.

2. If τ is periodic, then the action of τ on $\nu(K)$ is $\tau(z, w) = (-z, A_z w)$.

In both cases, A_z denotes a continuous family of matrices parametrized by S^1. In the strong case, we have $A_z \in O(2)$ and $\det A_z = -1$, while in the periodic case, we have $A_z \in SO(2)$. If τ is strong, then τ fixes the two discs $\{1\} \times D^2$ and $\{-1\} \times D^2$ setwise, and has two fixed points on the boundary of each. Take any arc γ on $\partial\nu(K)$ running from a fixed point of τ on $\{1\} \times S^1$ to a fixed point of τ on $\{-1\} \times S^1$. (We may also assume that γ projects as a diffeomorphism onto a subarc of K.) Then $\gamma \cup \tau\gamma$ constitutes an equivariant framing of K (and in fact any framing can be realized). If τ is periodic, then we instead take γ to be a similar arc joining an arbitrary point p in $\{1\} \times S^1$ to its image τp in $\{-1\} \times S^1$. Clearly, $\gamma \cup \tau\gamma$ is again an equivariant framing of K.

It follows that we can re-parameterize our neighborhood of K so that the equivariant framing constructed above is given by $S^1 \times \{1\}$. Then:

1. If τ is strong, then the action of τ on $\nu(K)$ is $\tau(z, w) = (\bar{z}, \bar{w})$.

2. If τ is periodic, then the action of τ on $\nu(K)$ is $\tau(z, w) = (-z, w)$.

Lemma 3.3.2. *Let K be an equivariant knot in Y with symmetry τ. Fix any framing K' of K, and let $Y_{p/q}(K)$ be (p/q)-surgery on K with respect to this framing. Then τ extends to an involution on $Y_{p/q}(K)$. This extension is unique up to isotopy.*

Proof. It suffices to prove the claim under the additional assumption that K' is equivariant. Indeed, since the claim of the lemma holds for all surgeries, proving the desired statement for a single framing establishes it for all framings.

On the complement of $\nu(K)$, we define our involution to be equal to τ. Parameterize the boundary of $\nu(K)$ by z and w, as above. The surgered manifold $Y_{p/q}(K)$ is obtained from the complement of K by gluing in the solid torus

$$S^1 \times D^2 = \{(z', w') : |z'| = 1, |w'| \leq 1\}$$

via the boundary diffeomorphism

$$f(z', w') = (z = (z')^s (w')^q, w = (z')^r (w')^p)$$

where r and s are integers such that $ps - qr = 1$. If τ is strong, then we have the obvious extension by complex conjugation

$$\tau(z', w') = (\bar{z}', \bar{w}').$$

If τ is periodic, then we have the extension:

$$\tau(z', w') = \begin{cases} (-z', -w') & \text{if } (p, q, r, s) = (1, 0, 1, 1) \text{ or } (1, 1, 1, 0) \mod 2 \\ (-z', w') & \text{if } (p, q, r, s) = (1, 0, 0, 1) \text{ or } (1, 1, 0, 1) \mod 2 \\ (z', -w') & \text{if } (p, q, r, s) = (0, 1, 1, 0) \text{ or } (0, 1, 1, 1) \mod 2. \end{cases}$$

Note that the diffeomorphism between the gluings corresponding to (p, q, r, s) and $(p, q, r + p, s + q)$ sends (z', w') to $(z', z'w')$. This intertwines τ, so (up to re-parameterization) our extension of τ does not depend on (r, s).

It is easy to check that any two extensions of τ must be isotopic to each other. For example, in the case of a strong involution, τ fixes a meridional curve on the torus boundary setwise. Hence any extension of τ maps the disk D bounded by this curve to some other disk D' with $\partial D = \partial D'$. It is then clear that we can isotope τ (rel boundary) so that it fixes D. Cutting out D, we then use the fact that every diffeomorphism of S^2 extends uniquely over B^3 (up to isotopy). $\square$

Given an equivariant knot, we will thus freely view its symmetry as defining a symmetry on any surgered manifold.

Lemma 3.3.3. *Let K be an equivariant knot in Y with symmetry τ. Fix any framing K' of K. Then τ extends over the 2-handle cobordism given by attaching a 2-handle along K with framing n (relative to K'). The involution on the boundary is the extension of τ to $Y_n(K)$ afforded by Lemma 3.3.2.*

Proof. It suffices to prove the claim under the additional assumption that K' is equivariant. Indeed, since the claim of the lemma holds for all n, proving the desired statement for a single framing establishes it for all framings.

Parameterize the 2-handle by

$$D^2 \times D^2 = \{(z', w') : |z'| \leq 1, |w'| \leq 1\}.$$

The boundary subset $S^1 \times D^2 \subset D^2 \times D^2$ is identified with $\nu(K)$ via the map sending

$$(z', w') \mapsto (z = z', w = (z')^n w').$$

If τ is strong, then the extension is given by

$$\tau(z', w') = (\bar{z}', \bar{w}').$$

If τ is periodic, then the extension is given by

$$\tau(z', w') = \begin{cases} (-z', -w') & \text{if } n \text{ is odd} \\ (-z', w') & \text{if } n \text{ is even,} \end{cases}$$

as desired. $\square$

This shows that if K is an equivariant knot, then equivariant handle attachment along K is well-defined, and that τ moreover extends over the handle attachment cobordism.

We will also consider surgeries on links in which τ exchanges some pairs of link components (with the same framing), in addition to possibly fixing some components. Given the above treatment of the fixed link components, it is clear that such τ extend to involutions on the surgered manifolds and over the handle attachment cobordisms (whenever the surgery coefficients are integral).

3.3.1 Actions on spin^c-structures

We now specialize to the case where Y is a homology sphere. Let K be an equivariant knot in Y with symmetry τ. Let W be the cobordism formed by (-1)-handle attachment along K, relative to the Seifert framing. This is a negative-definite cobordism whose second cohomology $H^2(W)$ is generated by a single element x. Note that the spin^c-structures on W with $c_1(\mathfrak{s}) = \pm x$ have $\Delta(W, \mathfrak{s}) = 0$[1]. We claim that if the involution τ is periodic, then W is spin^c-fixing, while if τ is strong, then W is spin^c-conjugating. To see this, it suffices to understand the action of τ on $H^2(W)$. Under the isomorphism $H^2(W) \cong H_2(W, \partial W)$, the generator x corresponds to the cocore of the attaching 2-handle. In the notation of Lemma 3.3.3, this is given by

$$\{0\} \times D^2 = \{(z', w') : z' = 0, |w'| \leq 1\}.$$

[1] Suppose that W is definite. By a well-known result of Elkies [11], $\Delta(W, \mathfrak{s}) = 0$ if and only if the intersection form of W is diagonalizable (over $\mathbb{Z}$) and $c_1(\mathfrak{s})$ has all coefficients equal to ± 1 in the diagonal basis.

An examination of the extension of τ over W shows that τ reverses orientation on the cocore if τ is strong and preserves orientation if it is periodic. Hence if τ is strong, it acts via multiplication by -1 on $H^2(W)$, and otherwise fixes $H^2(W)$. We thus define:

Definition 3.3.4. Let Y_1 be a homology sphere with involution τ. Let K be an equivariant knot in Y_1. Suppose that Y_2 is obtained from Y_1 by doing (-1)-surgery on K, relative to the Seifert framing. Then the corresponding handle attachment cobordism constitutes an equivariant cobordism from Y_1 to Y_2, where the latter is equipped with the usual extension of τ. This is spin^c-fixing if τ is periodic and spin^c-conjugating if τ is strong. We refer to these as spin^c-*fixing* (-1)-*cobordisms* and spin^c-*conjugating* (-1)-*cobordisms*, respectively.

Similarly, we may consider attaching a pair of handles to Y_1 along a two-component link with algebraic linking number zero whose components are interchanged by τ. In this situation, $H^2(W)$ is generated by two elements x and y, where $\tau x = y$ and $\tau y = x$ (with appropriately chosen orientations). Choosing the spin^c-structure $\mathfrak{s}$ with $c_1(\mathfrak{s}) = x + y$ then yields a spin^c-fixing cobordism, while choosing the spin^c-structure with $c_1(\mathfrak{s}) = x - y$ yields a spin^c-conjugating cobordism.

Definition 3.3.5. Let Y_1 be a homology sphere with involution τ. Let L be a two-component link in Y_1 with algebraic linking number zero whose components are interchanged by τ. Let Y_2 be obtained from Y_1 by doing an additional (-1)-surgery on each component of L, relative to the Seifert framing. Then the corresponding handle attachment cobordism constitutes an equivariant cobordism from Y_1 to Y_2, where the latter is equipped with the usual extension of τ. This is both spin^c-fixing and spin^c-conjugating (with respect to different spin^c-structures). We refer to such a cobordism as an *interchanging* $(-1,-1)$-*cobordism.*

Of course, we have the analogous notion of $(+1)$- and $(+1,+1)$-cobordisms. We obtain a similar set of inequalities (going in the opposite direction) by turning these cobordisms around.

3.3.2 Equivariant blow up/downs

We will occasionally need to compare symmetries in two different surgery descriptions of the same 3-manifold. Although we will not belabor the point, the reader should check that the blow-up and blow-down operations displayed in Figure 3.4 can be performed equivariantly. Note that if u is an equivariant $(1/k)$-framed unknot which is split off from the rest of a surgery diagram, then u can be deleted. Indeed, let u be contained in a ball B^3. Then $(1/k)$-surgery on u is again a ball, equipped with a slightly different extension of the 180-degree-rotation on $S^2 = \partial B^3$. However, every diffeomorphism of ∂B^3 extends uniquely over B^3 up to isotopy rel boundary.

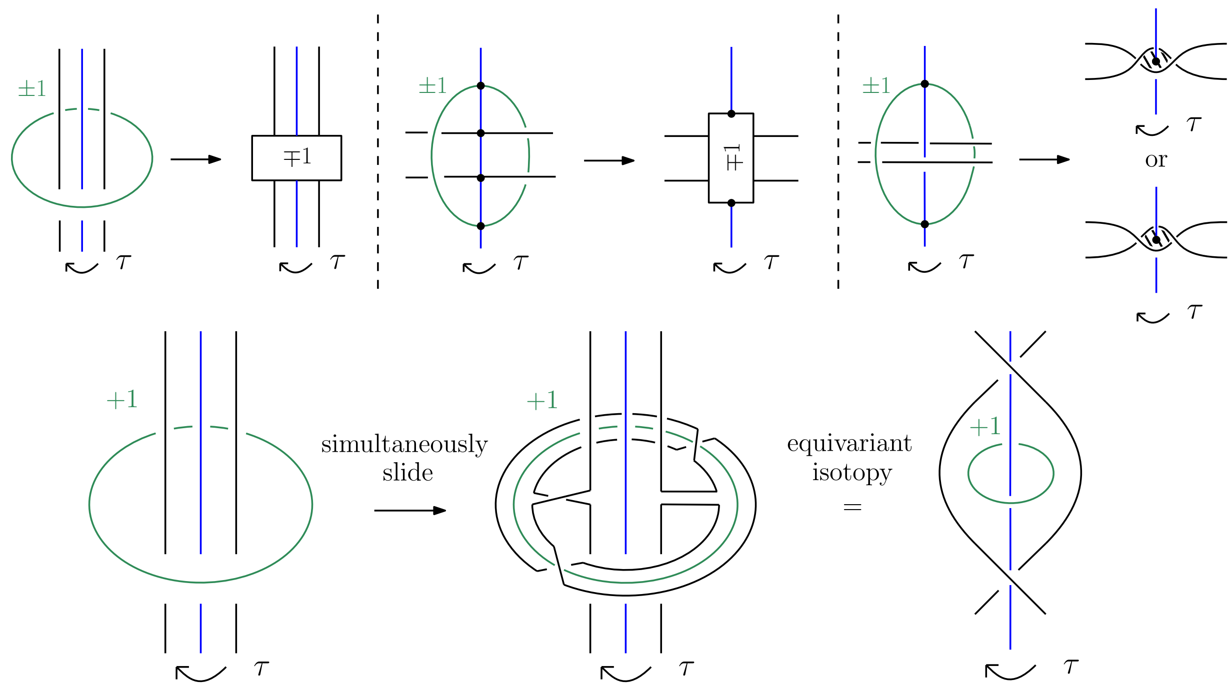


Figure 3.4: Top: various equivariant blow-up/blow-down operations. Bottom: an equivariant (simultaneous) slide followed by an equivariant isotopy.

3.4 Computational aid: A monotonicity theorem

We now prove a result that will constrain the behavior of the invarinats h_τ and $h_{\iota\circ\tau}$ defined in Chapter 2 under negative definite equivariant cobordisms. Before going into the Theorem we discuss a certain property of the group $\mathfrak{I}$, defined in Section 2.3.1.

Definition 3.4.1. Let (C_1, ι_1) and (C_2, ι_2) be two ι-complexes. If there is a local map $f : C_1 \to C_2$, then we write $(C_1, \iota_1) \leq (C_2, \iota_2)$. If, in addition, there does *not* exist any local map from (C_2, ι_2) to (C_1, ι_1), we write the strict inequality $(C_1, \iota_1) < (C_2, \iota_2)$.

Since the composition of two local maps is local, it is clear that the above definition respects local equivalence. Because the tensor product of two local maps is also local, this partial order respects the group structure on $\mathfrak{I}$.

Remark 3.4.2. Note that it is *not* always true that a given ι-complex can be compared to the trivial complex. That is, Definition 3.4.1 does not define a *total* order on $\mathfrak{I}$. See [9, Example 2.7] for further discussion.

Theorem 3.4.3. *Let (Y_1, τ_1) and (Y_2, τ_2) be homology spheres equipped with involutions τ_1 and τ_2.*

1. *Suppose there is a* spin^c*-fixing (-1)-cobordism from (Y_1, τ_1) to (Y_2, τ_2). Then we have $h_{\tau_1}(Y_1) \leq h_{\tau_2}(Y_2)$.*
2. *Suppose there is a* spin^c*-conjugating (-1)-cobordism from (Y_1, τ_1) to (Y_2, τ_2). Then we have $h_{\iota\circ\tau_1}(Y_1) \leq h_{\iota\circ\tau_2}(Y_2)$.*
3. *Suppose there is an interchanging $(-1, -1)$-cobordism from (Y_1, τ_1) to (Y_2, τ_2). Then we have $h_{\tau_1}(Y_1) \leq h_{\tau_2}(Y_2)$ and $h_{\iota\circ\tau_1}(Y_1) \leq h_{\iota\circ\tau_2}(Y_2)$.*

We will prove a more generalized statement below, from which the Theorem 3.4.3 will follow. Let Y_1 and Y_2 are two homology spheres as before and Let W be a cobordism from Y_1 to Y_2 and let $\mathfrak{s}$ be a spin^c-structure on W. Recall that the associated Heegaard Floer grading shift is given by

$$\Delta(W, \mathfrak{s}) = \frac{c_1(\mathfrak{s})^2 - 2\chi(W) - 3\sigma(W)}{4}.$$

In what follows, we will be concerned with negative-definite cobordisms admitting $\mathfrak{s}$ for which $\Delta(W, \mathfrak{s}) = 0$.

Lemma 3.4.4. *Let Y_1 and Y_2 be two homology spheres equipped with involutions τ_1 and τ_2, respectively. Let (W, f) be a negative-definite cobordism from (Y_1, τ_1) to (Y_2, τ_2) with $b_1(W) = 0$, and let $\mathfrak{s}$ be a* spin^c*-structure on W with $\Delta(W, \mathfrak{s}) = 0$. Then:*

1. *If $f_*\mathfrak{s} = \mathfrak{s}$, then $h_{\tau_1}(Y_1) \leq h_{\tau_2}(Y_2)$.*
2. *If $f_*\mathfrak{s} = \bar{\mathfrak{s}}$, then $h_{\iota\circ\tau_1}(Y_1) \leq h_{\iota\circ\tau_2}(Y_2)$.*

Proof. The proposition is a straightforward consequence of the functorial properties of Heegaard Floer homology under cobordisms. By the proof of [32, Theorem 9.1], the cobordism map

$$F_{W,\mathfrak{s}} : CF^-(Y_1) \to CF^-(Y_2)$$

sends U-nontorsion elements to U-nontorsion elements in homology. By [19, Proposition 4.9], we have

$$F_{W,\bar{\mathfrak{s}}} \circ \iota_1 \simeq \iota_2 \circ F_{W,\mathfrak{s}}.$$

The analogous commutation relation for τ is given by

$$F_{W,f_*\mathfrak{s}} \circ \tau_1 \simeq \tau_2 \circ F_{W,\mathfrak{s}}.$$

Note that implicitly, $F_{W,\mathfrak{s}}$ depends on a choice of path γ from Y_1 to Y_2. The two cobordism maps above should thus be taken with respect to different paths, with the map on the left being taken with respect to $f(\gamma)$. However since $b_1(W) = 0$, it follows from Lemma 2.4.1 that $F_{W,\mathfrak{s}}$ is independent of the choice of path (up to U-equivariant homotopy).

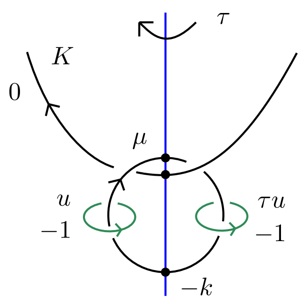
If $f_*\mathfrak{s} = \mathfrak{s}$, then the commutation relation for τ immediately exhibits $F_{W,\mathfrak{s}}$ as the desired local map for the first claim. If $f_*\mathfrak{s} = \bar{\mathfrak{s}}$, we instead observe that

$$F_{W,\mathfrak{s}} \circ (\iota_1 \circ \tau_1) \simeq \iota_2 \circ F_{W,\bar{\mathfrak{s}}} \circ \tau_1 \simeq (\iota_2 \circ \tau_2) \circ F_{W,f_*\bar{\mathfrak{s}}}.$$

Noting that f_* commutes with conjugation, we thus see that $F_{W,\mathfrak{s}}$ effects the desired local map for the second claim. $\square$

We now turn to the proof of Theorem 3.2.5.

Proof of Theorem 3.2.5. Let K be a knot in S^3 with a strong involution τ. Then $(1/k)$-surgery on K is equivariantly diffeomorphic to the two-component link surgery consisting of 0-surgery on K, together with $(-k)$-surgery on a meridian μ of K. Choosing μ to be an equivariant unknot near one of the fixed points on K makes this diffeomorphism τ-equivariant. Let u and τu be an additional pair of (-1)-framed unknots which each link μ, as in Figure 3.5. Blowing down, the resulting manifold is equivariantly diffeomorphic to surgery on K with coefficient $1/(k-2)$. We claim that handle attachment along u and τu constitutes an interchanging $(-1,-1)$-cobordism from $S_{1/k}(K)$ to $S_{1/(k-2)}(K)$. To see this, we equivariantly slide u and τu over K, which algebraically unlinks them from the rest of the diagram (see Figure 3.5). The claim then follows from Theorem 3.4.3. $\square$



$$\begin{array}{lll} (K,\mu)=1 & u' = u - K & (K,u') = (K,\tau u') = 0 \\ (u,\mu) = 1 & \tau u' = \tau u + K & (u',\mu) = (\tau u',\mu) = 0 \\ (\tau u,\mu) = -1 & \longrightarrow & (u',\tau u') = 0 \\ & & (u',u') = (\tau u',\tau u') = -1 \end{array}$$

Figure 3.5: Left: the equivariant cobordism used in the proof of Theorem 3.2.5. Right: handleslides establishing that this is an interchanging $(-1,-1)$-cobordism. Since τ reverses orientation on K, the indicated handleslides are τ-equivariant.

3.5 Constraining the h_τ and $h_{\iota\circ\tau}$ invariants

In this section we prove the Theorems stated in the Section 3.2. Our strategy for showing a pair (Y,τ) is a strong cork will be, by showing that either h_τ or $h_{\iota\circ\tau}$ is non-zero for (Y,τ). In order to bound the invariants from below or above, we first construct an equivariant cobordism from (Y,τ) to a 'simpler' integer homology sphere with an involution, then use the monotonicity theorem from the previous section to obtain inequalities on h_τ or $h_{\iota\circ\tau}$. Finally, we show that inequalities are sharp using the spin^c-conjugation action.

Before diving into the proof, we note the following.

Remark 3.5.1. To rule out the existence of a local map (see Section 2.3.1) between two ι-complexes (C_1, ι_1) to (C_2, ι_2), it suffices to prove that there is no $\mathbb{F}_2[U]$-module map F from $H_*(C_1)$ to $H_*(C_2)$ such that:

1. F maps U-nontorsion elements to U-nontorsion elements; and,
2. F intertwines the actions of $(\iota_1)_*$ and $(\iota_2)_*$.

In light of the above remark, we now give a brief introduction to a certain presentation of the Heegaard Floer homology groups, HF^-. This will be useful to later, while trying to compute the action of τ in homology, for certain Brieskorn homology spheres.

3.5.1 Graded roots

Let G be a weighted graph, and let $Y(G)$ be the boundary of the corresponding plumbing of S^2. In [31] Nemethi computed the Heegaard Floer homology of $Y(G)$, where G has at most one *bad vertex* (Instead of going into the definition of bad vertices we refer readers to [10]), as this particular concept will not be so useful to us for the rest of the discussing). The computation is done by demonstrating an isomorphism between HF^- and a combinatorial object in the shape of an infinite tree. Instead of going deep into the theory, we demonstrate by example such a graded root.

For $i > 0$, consider the chain complex spanned by the generators $v, \iota v$, and α, with

$$\partial\alpha = U^i(v + \iota v).$$

Here, v and ιv lie in Maslov grading zero, while α has grading $-2i + 1$. The action of ι interchanges v and ιv and fixes α. We denote this ι-complex (or sometimes its local equivalence class) by X_i. The homology of X_i is displayed in Figure 3.6; note that the induced action of ι is given by the obvious involution reflection through the vertical axis.

It can be verified that the only self-local equivalences of X_i are isomorphisms. In particular, this shows that the local equivalence classes of the X_i are nonzero and mutually

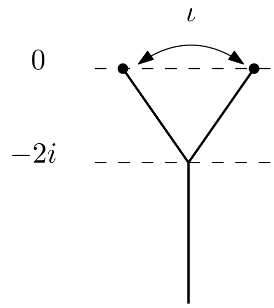


Figure 3.6: Homology of X_i, expressed as a graded root with involution. Vertices of the graph correspond to $\mathbb{F}$-basis elements supported in grading given by the height (shown on the left). Edges between vertices indicate the action of U, and we suppress all vertices forced by this relation. Thus, for instance, the two upper legs of the graded root contain i vertices (excluding the symmetric vertex lying in grading $-2i$). See for example [10, Definition 2.11].

distinct. We can refine their distinction by considering the partial order on $\mathfrak{I}$. It is easily checked that

$$\cdots < X_3 < X_2 < X_1 < 0,$$

where 0 denotes the trivial ι-complex. Indeed, there is evidently a local map showing that $X_1 \leq 0$, by mapping both v and ιv to x and α to zero. However, the only ι-equivariant map in the other direction sends x to $v + \iota v$, which is U-torsion in homology (See Figure 3.7.) Thus, the inequality is strict. The proof that $X_{i+1} < X_i$ is similar.

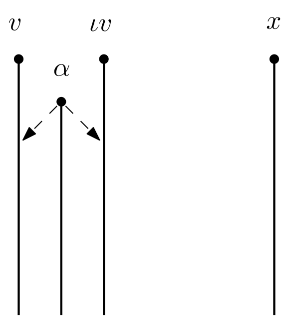


Figure 3.7: Left: the complex X_1. Right: the trivial complex 0.

The classes X_i actually play quite an important role in the study of $\Theta^3_{\mathbb{Z}}$ and $\mathfrak{I}$. In [10, Theorem 1.7], it is shown that the X_i are linearly independent in $\mathfrak{I}$, and in fact they span a $\mathbb{Z}^\infty$-summand of $\mathfrak{I}$ by [9, Theorem 1.1]. In this section, we will use the fact that (-1)-surgery on the right-handed $(2, 2n+1)$-torus knots realize the X_i:

$$h(S_{-1}(T_{2,2n+1})) = X_{\lfloor (n+1)/2 \rfloor}$$

See the proof of [18, Theorem 1.4]. Note that $S_{-1}(T_{2,2n+1})$ can be identified with the Brieskorn sphere $\Sigma(2, 2n+1, 4n+3)$.

3.5.2 Computations

We are now in place to constrain the h_τ and $h_{\iota\circ\tau}$ invariants. Let us firstly consider an example to demonstrate our strategy.

Lemma 3.5.2. *Let $Y_1 = \Sigma(2,3,7)$ be given by $(+1)$-surgery on the figure-eight knot, and let τ and σ be as in Figure 3.8. Then*

1. $h_\tau(Y_1) = h(Y_1) < 0$ *and* $h_{\iota\circ\tau}(Y_1) = 0$
2. $h_\sigma(Y_1) = 0$ *and* $h_{\iota\circ\sigma}(Y_1) = h(Y_1) < 0$.

Proof. Doing $(+1)$-surgery on the unknot indicated on the left in Figure 3.8 (and blowing down) gives a spin^c-conjugating $(+1)$-cobordism from (Y_1, τ) to S^3. Hence $h_{\iota\circ\tau}(Y_1) \geq 0$. Similarly, doing $(+1)$-surgery on the unknot indicated on the right gives a spin^c-fixing $(+1)$-cobordism from (Y_1, σ) to S^3. Hence $h_\sigma(Y_1) \geq 0$. Now, id and ι are the only two possible homotopy involutions on the standard complex of $CF^-(Y_1)$, and the involutive complex corresponding to ι is strictly less than zero (see Figure 3.6, where X_1 is $HF^-(Y_1)$). Hence $\iota \circ \tau = \text{id}$, which shows $\tau = \iota$. Similarly, we have that $0 \leq h_\sigma(Y_1)$, which implies $\sigma = \text{id}$ and $\iota \circ \sigma = \iota$. □

We now turn to our first example of a cork. Let Y_2 be given by $(+1)$-surgery on the stevedore knot 6_1, displayed on the left in Figure 3.9. Note that Y_2 bounds a contractible manifold, see for example [16, §6, Corollary 3.1.1].

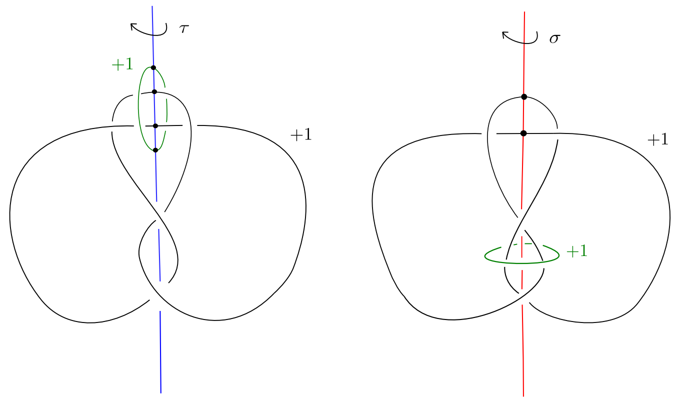


Figure 3.8: Two involutions on the figure-eight knot, with equivariant cobordisms of Lemma 3.5.2.

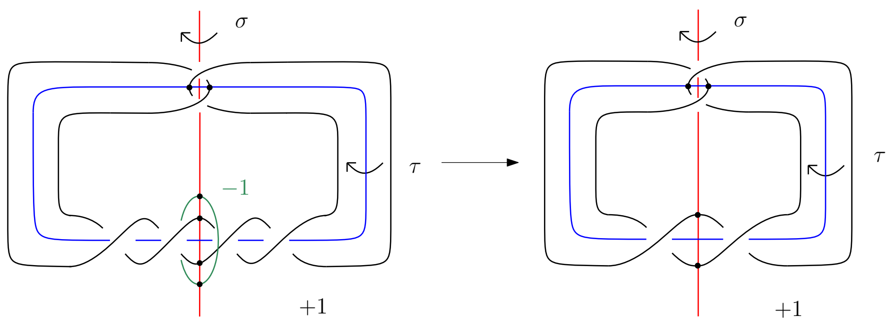


Figure 3.9: Cobordism from $S_{+1}(6_1)$ to $\Sigma(2,3,7)$.

Lemma 3.5.3. *Let $Y_2 = S_{+1}(6_1)$ be given by $(+1)$-surgery on the stevedore knot 6_1, and let τ and σ be as shown on the left in Figure 3.9. Then $h_\tau(Y_2) < 0$ and $h_{\iota\circ\sigma}(Y_2) < 0$. In particular, neither τ nor σ extends over any homology ball that Y bounds.*

Proof. The claim is immediate from Figure 3.9. Doing (-1)-surgery on the indicated unknot gives a spin^c-fixing cobordism from (Y_2, τ) to (Y_1, τ) and a spin^c-reversing cobordism from (Y_2, σ) to (Y_1, σ). It is easily checked that the involutions τ and σ on the right in Figure 3.9 are the same as those defined in Lemma 3.5.2. $\square$

Lemma 3.5.3 already shows that $Y_2 = S_{+1}(6_1)$ is a (strong) cork (with either of the involutions τ and σ). To the best of the authors' knowledge, even the fact that Y_2 bounds a cork was not previously known. Again, we stress here that the entire argument is almost completely formal: the only actual computation we have used so far is the (involutive) Floer homology of $Y_1 = \Sigma(2,3,7)$. In particular, we have not needed to determine the Floer homology of Y_2 (involutive or otherwise).

We now turn to the proof of Theorem 3.2.4. We start with the following Lemma.

Lemma 3.5.4. *Let K_n be the family of twist knots displayed in Figure 3.11, equipped with the indicated involutions τ and σ. Let $A_n = S_{+1}(K_n) = \Sigma(2,3,6n+1)$. For n positive and odd, we have*

1. $h_\tau(A_n) = h_{\iota\circ\sigma}(A_n) = h(A_n) < 0$
2. $h_{\iota\circ\tau}(A_n) = h_\sigma(A_n) = 0.$

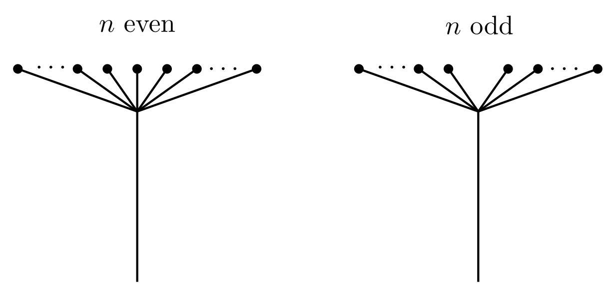


Figure 3.10: Local equivalence class $h(A_n)$.

Proof. The Heegaard Floer homology $HF^-(A_n)$ is displayed in Figure 3.10. This can be computed either by using the usual Heegaard Floer surgery formula, or by using the graded roots algorithm of [5]. The action of ι on $HF^-(A_n)$ is given by reflection across the obvious vertical axis. Using the monotone root algorithm of [10, Section 6], $h(A_n)$ is locally trivial for n even and locally equivalent to $h(\Sigma(2,3,7))$ for n odd. In the latter case, this means that $h(A_n) < 0$. In Figure 3.11, we have displayed a cobordism from A_n to S^3 consisting of n unknots with framing $+1$. Note that this is spin^c-conjugating for τ (since n is odd) and spin^c-fixing for σ. Hence $h_{\iota\circ\tau}(A_n) \geq 0$. This implies $\tau \simeq \iota$, since either $\tau \simeq \iota$ or $\tau \simeq \mathrm{id}$. Similarly, we have $h_\sigma(A_n) \geq 0$, which implies $\sigma \simeq \mathrm{id}$. $\square$

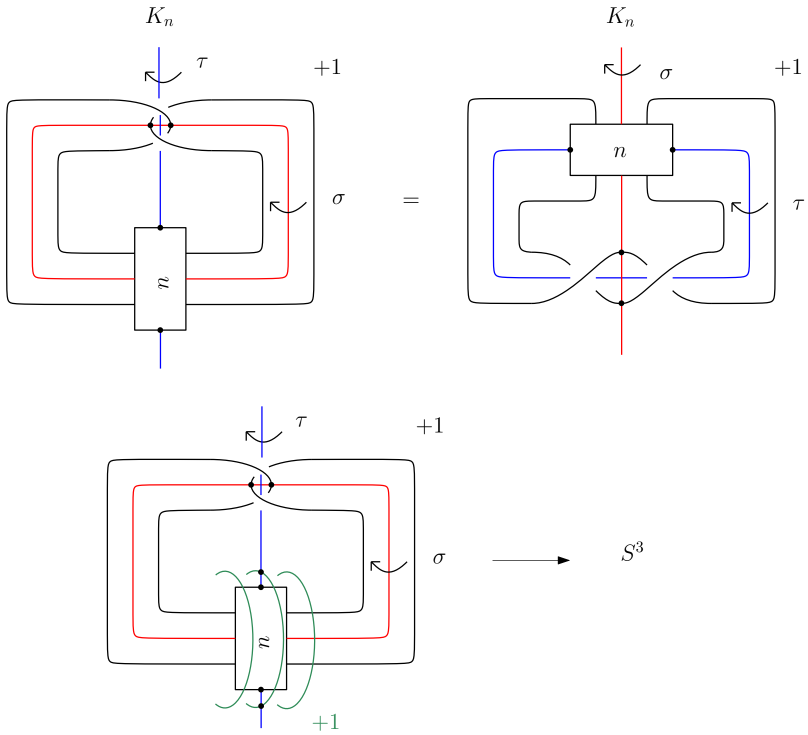


Figure 3.11: Top: two equivalent diagrams for $A_n = S_{+1}(K_n)$. Bottom: cobordism from A_n to S^3.

We are now in place to establish Theorems 3.2.4. Recall that $V_{n,k}$ is defined to be $(1/k)$-surgery on the doubly twist knot:

$$V_{n,k} = \begin{cases} S_{1/k}(\overline{K}_{-n,n+1}) & \text{if } n \text{ is odd} \\ S_{1/k}(K_{-n,n+1}) & \text{if } n \text{ is even.} \end{cases}$$

Each $V_{n,k}$ is equipped with the involutions τ and σ displayed in Figure 3.1 (or rather, the mirrored involutions in the case that n is odd).

Proof of Theorem 3.2.4. We claim that for $k = 1$, we have

1. If n is odd, $h_{\iota\circ\tau}(V_{n,1}) \leq h_{\iota\circ\sigma}(A_n)$ and $h_\sigma(V_{n,1}) \leq h_\tau(A_n)$.

2. If n is even, $h_\tau(V_{n,1}) \leq h_\tau(A_{n+1})$ and $h_{\iota\circ\sigma}(V_{n,1}) \leq h_{\iota\circ\sigma}(A_{n+1})$.

The relevant equivariant surgeries are displayed in Figure 3.12. (Compare Figure 3.11.) Note that we always attach an odd number of (-1)-framed 2-handles. In the case that n is odd, note that τ acts as a strong involution on a single unknot and interchanges the others in pairs, while σ acts as a periodic involution on each unknot. (The roles of τ and σ are reversed in the case where n is even.) By Lemma 3.5.4, we thus see that all of the above local equivalence classes are strictly less than zero. Applying Theorem 3.2.5 completes the proof. $\square$

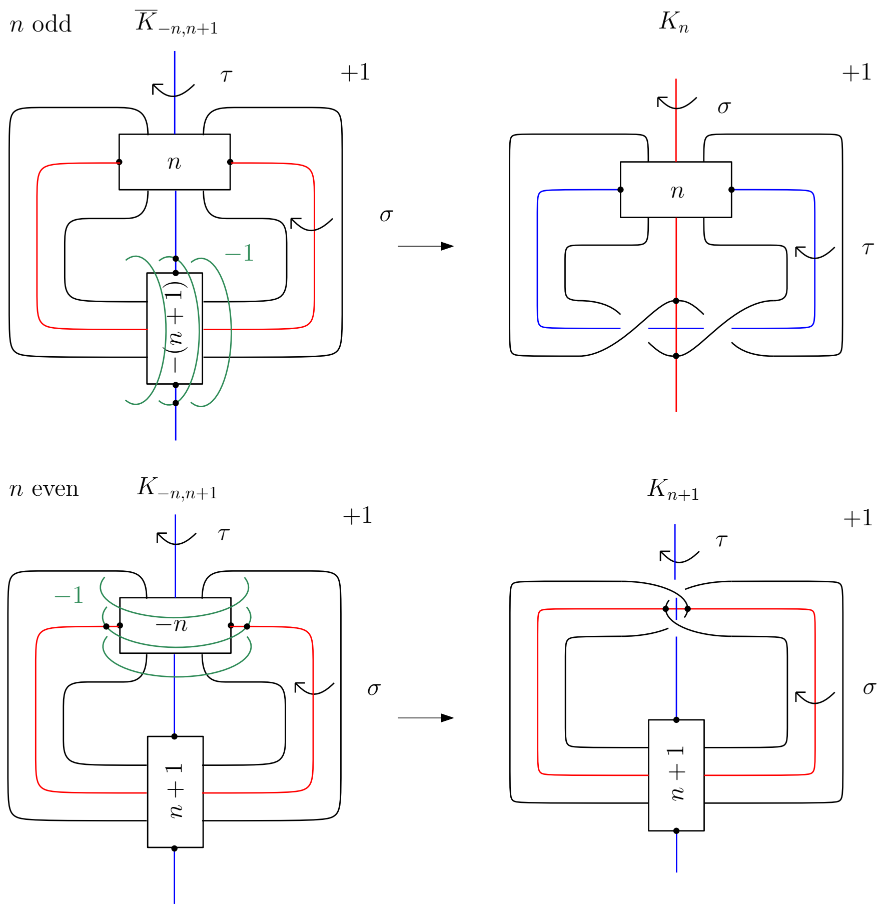
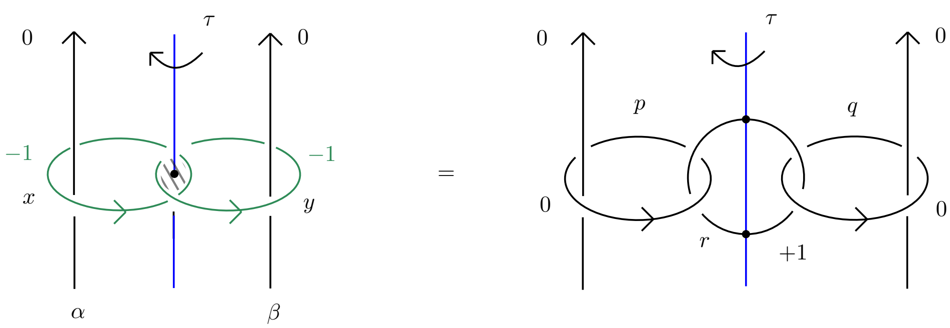


Figure 3.12: Top (n odd): cobordism from $V_{n,1}$ to A_n; there are n green curves. Note that τ on $\overline{K}_{-n,n+1}$ is sent to σ on K_n. Bottom (n even): cobordism from $V_{n,1}$ to A_{n+1}; there are $n-1$ green curves.

We now turn towards strong corks obtained as a surgery on symmetric links. The strategy for constraining the invariants however remain the same.

Proof of Theorem 3.2.6. We begin by describing a handle attachment cobordism on M_n. Let the components of M_n be α and β, oriented such that $(\alpha, \beta) = 1$. Consider a pair of (-1)-framed unknots x and y that link parallel strands of α and β, as displayed on the left in Figure 3.13. We claim that the handle attachment cobordism corresponding to x and y is an interchanging $(-1,-1)$-cobordism from M_n to some manifold Y_n. Indeed, a quick computation shows that sliding x over β and y over α gives the desired claim (see Figure 3.13). On the right in Figure 3.13, we have displayed an alternative diagram for Y_n in which x and y are replaced by two zero-framed unknots p and q, which are themselves linked by a $(+1)$-framed unknot r. As a surgery diagram for Y_n, this is equivariantly diffeomorphic to the previous.



$(\alpha, \beta) = 1$	$x' = x - \beta$	$(\alpha, x') = (\beta, x') = 0$
$(\alpha, x) = 1$	$y' = y - \alpha$	$(\alpha, y') = (\beta, y') = 0$
$(\beta, y) = 1$	$\longrightarrow$	$(x', y') = 0$
$(x, y) = 1$		$(x', x') = (y', y') = -1$

Figure 3.13: Fundamental cobordism in the proof of Theorem 3.2.6.

We attach the configuration of Figure 3.13 to the bottom of the link defining M_n. Clearly, Y_n can be given the alternative equivariant surgery diagram shown in Figure 3.15. We modify this diagram by equivariantly sliding all of the (-1)-framed horizontal unknots over p and q and deleting them. This yields the second diagram in Figure 3.15. Through equivariant isotopy, we transfer the two half-twists of the vertical (-1)-curves onto r, and then slide the horizontal $(+1)$-framed unknots over p and q. We then blow down everything except for r. This yields the final diagram in Figure 3.15.

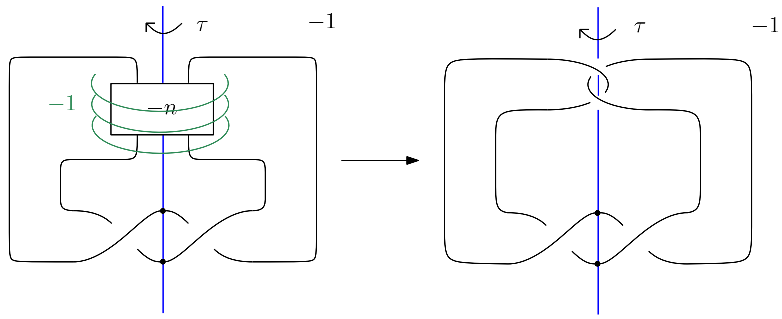


Figure 3.14: Completing the cobordism from M_n to $\Sigma(2,3,7)$.

In Figure 3.14, we display a spin^c-fixing equivariant cobordism from Y_n to $\Sigma(2,3,7)$. This consists of attaching (-1)-framed unknots and blowing down until only one full negative twist remains. The resulting knot is just the right-handed trefoil, equipped with a strong involution. An argument similar to the one given in Lemma 3.5.2 shows that $h_\tau(M_n) < 0$, as desired. Moreover, it is clear that if M' is constructed from M_n by introducing any number of symmetric pairs of negative full twists (as in Figure 3.3), then M' admits a sequence of interchanging $(-1,-1)$-cobordisms to M_n. This completes the proof. □

Note that in all the Theorems above the manifold that we used to constrain the invariants were all Brieskorn homology spheres. For the strong cork in Theorem 3.2.7 however, we use a manifold that is not a Brieskorn homology sphere. Note that the difficulty in this is that unlike $\Sigma(2,3,7)$, we cannot determine the action of ι on it, as needed in the proof of previous Theorems. Hence we adapt to a rather ad-hoc argument, although we are hopeful that this type of argument will lead to more complicated examples of strong corks.

Proof of Theorem 3.2.7. We begin by constructing an interchanging $(-1,-1)$-cobordism from P to another manifold-with-involution. To this end, consider the fundamental cobordism displayed on the left in Figure 3.16. This is formed by attaching two (-1)-handles to parallel strands of P. Figure 3.16 is analogous to Figure 3.13, but differs slightly due to the fact that the two components of P (with the orientations displayed in Figure 3.16) have linking number -1, rather than $+1$. Performing a change-of-basis shows that this is

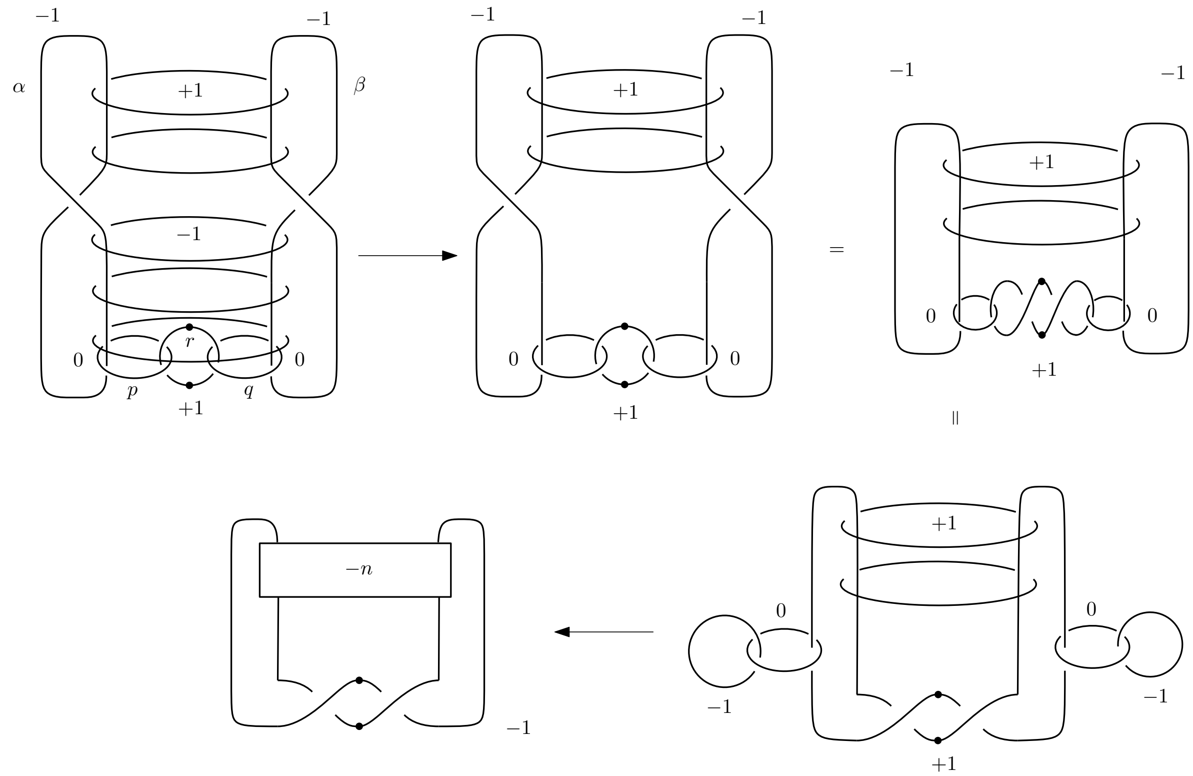


Figure 3.15: Proof of Theorem 3.2.6. In the upper left, there are n horizontal $(+1)$-curves and $n+1$ horizontal (-1)-curves.

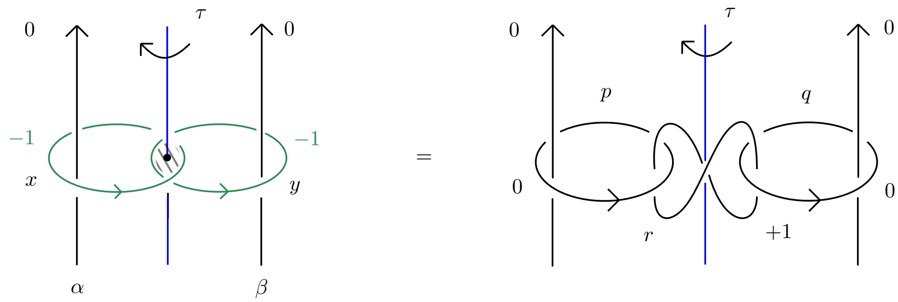
an interchanging $(-1,-1)$-cobordism. On the right in Figure 3.16, we have displayed an alternative surgery diagram for the resulting manifold. The reader should check that this is equivariantly diffeomorphic to the previous.

Using the Kirby calculus manipulations shown in Figure 3.17, one can prove that our new manifold is equivariantly diffeomorphic to $S_{-1}(6_2)$, equipped with the indicated involution τ. Hence by Theorem 3.4.3, we have

$$h_\tau(P) \leq h_\tau(S_{-1}(6_2)) \text{ and } h_{\iota\circ\tau}(P) \leq h_{\iota\circ\tau}(S_{-1}(6_2)).$$

It thus suffices to show that either of the invariants of $S_{-1}(6_2)$ are strictly less than zero. For simplicity, we work on the level of homology by ruling out the existence of an equivariant $\mathbb{F}_2[U]$-module map from the trivial module $\mathbb{F}_2[U]$ (equipped with the identity involution) to $HF^-(S_{-1}(6_2))$ (equipped with either involution τ_* or $\iota_* \circ \tau_*$), as in Remark 3.5.1.

To this end, we first compute the Heegaard Floer homology of $S_{-1}(6_2)$. Since 6_2 is



$$
\begin{array}{lcl}
(\alpha,\beta) = -1 & x' = x + \beta & (\alpha,x') = (\beta,x') = 0 \\
(\alpha,x) = 1 & y' = y + \alpha & (\alpha,y') = (\beta,y') = 0 \\
(\beta,y) = 1 & \longrightarrow & (x',y') = 0 \\
(x,y) = -1 & & (x',x') = (y',y') = -1
\end{array}
$$

Figure 3.16: Fundamental cobordism in the proof of Theorem 3.2.7. Here, α and β are parallel strands in the two components of P. Note the difference in crossings from Figure 3.13.

alternating, its knot Floer complex is determined by its Alexander polynomial. It is then straightforward to calculate $HF^-(S_{-1}(6_2))$ via the usual surgery formula [36], although for technical reasons we display the computation for $HF^+(S_{+1}(\overline{6}_2))$ instead. (See Figure 3.18.) For convenience, denote $K = \overline{6}_2$. Note that since K has genus two, the desired Floer homology is *not* given by the large surgery formula, but rather the homology of the mapping cone $\mathbb{X}^+(1)$ displayed in Figure 3.19. In this case, the desired homology is quasi-isomorphic to the kernel of the (truncated) mapping cone map with domain $H_*(A^+_{-1}) \oplus H_*(A^+_0) \oplus H_*(A^+_{+1})$. The resulting calculation is displayed on the right in Figure 3.18.

We now attempt to obtain partial information regarding the action of ι_* on $HF^+(S_{+1}(K))$. As before, we can compute the action of ι_K on the knot Floer complex of K; this is given by reflection across the obvious diagonal. However, we cannot use the involutive large surgery formula and (at the time of writing) there is not a general involutive surgery formula. We thus resort to the following trick. Observe that there is a map

$$q : \mathbb{X}^+(1) \longrightarrow A^+_0$$

formed by quotienting out $\mathbb{X}^+(1)$ by everything other than A^+_0. In the basis of Figure 3.18, the induced map $q_* : H_*(\mathbb{X}^+(1)) \to H_*(A^+_0)$ sends the two obvious unmarked generators

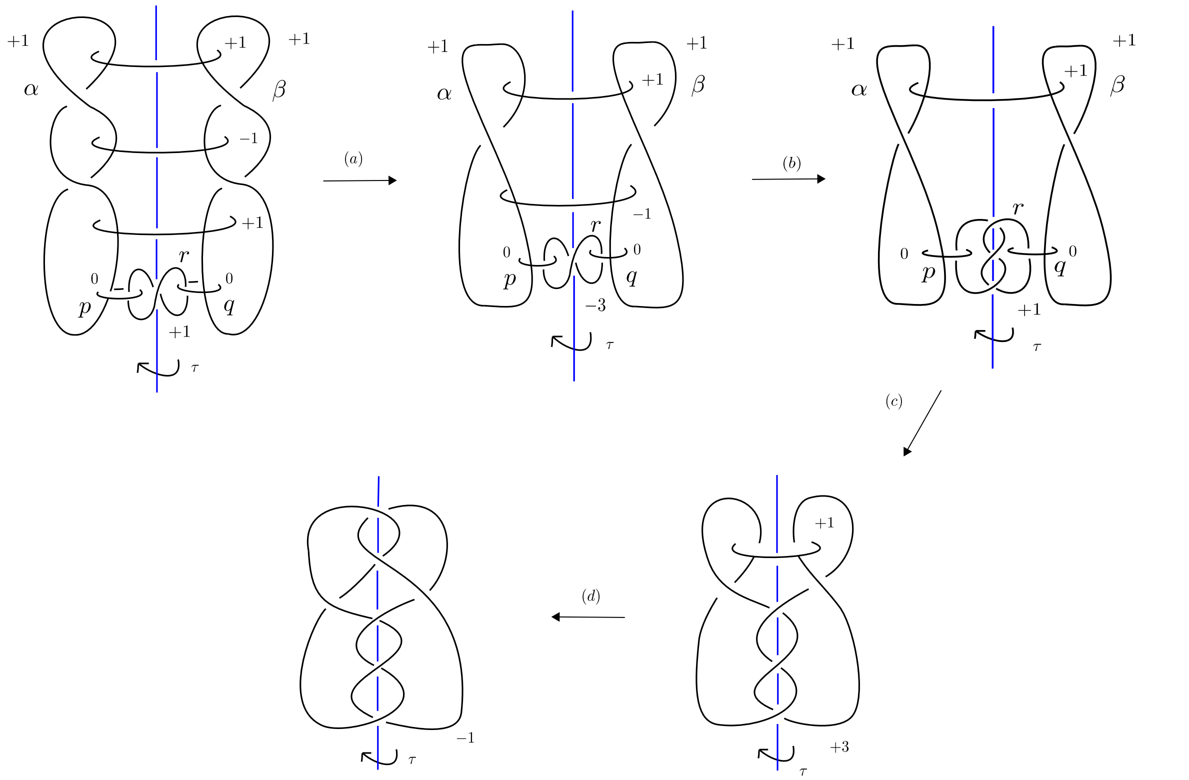


Figure 3.17: Equivariant cobordism used in the proof of Theorem 3.2.7. The first diagram is obtained by attaching the configuration of Figure 3.16 to an alternative surgery diagram for P. In **(a)** we slide the nearest $(+1)$-curve over p and q, blow down, and transfer two of the half-twists in α and β to r. In **(b)** we similarly slide the (-1)-curve over p and q and blow down. In **(c)** we transfer the remaining half-twists in α and β to r, slide the horizontal $(+1)$-curve over p and q, and then blow down the $(+1)$-curves on either side. Finally, in **(d)** we blow down the remaining $(+1)$-curve. This yields (-1)-surgery on a knot which the reader can check is 6_2.

to zero and acts as an isomorphism on the rest of the homology. According to the proof of integer surgery formula in [36], under the identification of $H_*(\mathbb{X}^+(1))$ with $HF^+(S_{+1}(K))$, the quotient map q coincides (on homology) with the triangle-counting map

$$\Gamma_0^+ : CF^+(S_{+1}(K)) \longrightarrow A_0^+$$

defined in [36]. Furthermore, following the proof of [19, Theorem 1.5], one can show that $(\Gamma_0^+)_*$ intertwines the actions of ι_* on $HF^+(S_{+1}(K))$ and $(\iota_K)_*$ on $H_*(A_0^+)$; that is,

$$(\iota_K)_* \circ (\Gamma_0^+)_* = (\Gamma_0^+)_* \circ \iota_*.$$

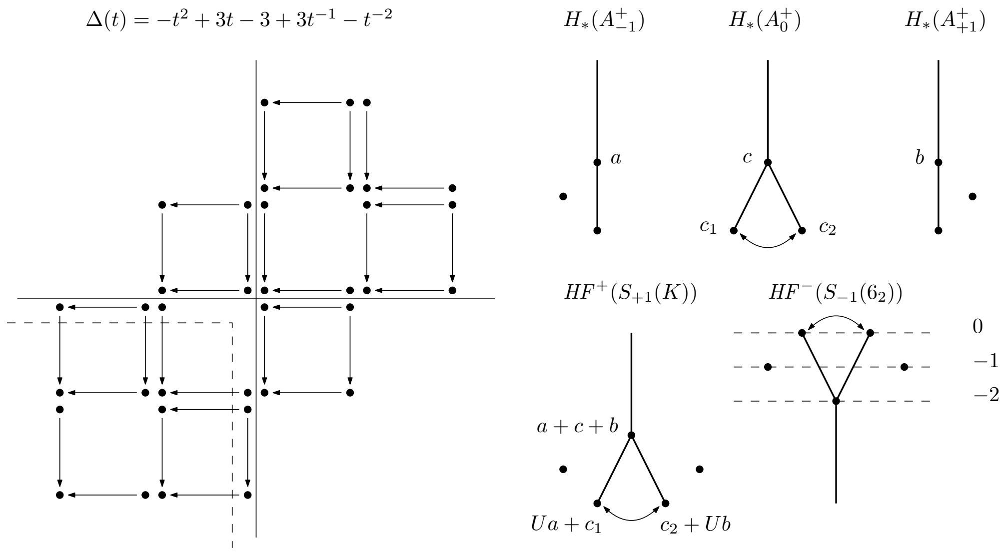


Figure 3.18: Left: the knot Floer complex of K, with the dotted line marking the boundary of the quotient complex A_0^+. Right: various homologies $H_*(A_i^+)$, together with the calculation of $HF^+(S_{+1}(K))$.

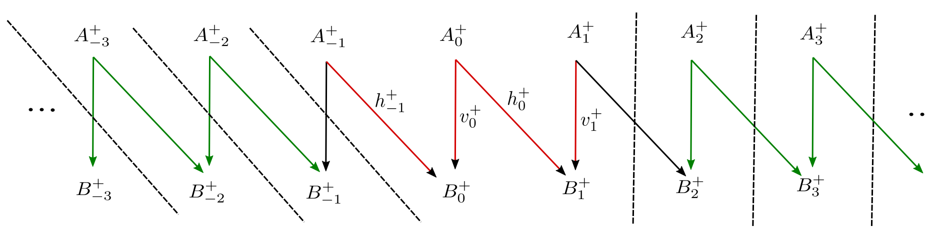


Figure 3.19: The mapping cone $\mathbb{X}^+(1)$. Green arrows are homotopy equivalences. The truncated mapping cone (which carries the homology) consists of the red arrows.

More precisely, Hendricks and Manolescu consider the map $\Gamma^+_{0,p} : CF^+(S_p(K)) \to A_0^+$ when p is large, and show that this intertwines ι and ι_K. However, their proof of this fact does not depend on the surgery coefficient p. Of course, $\Gamma^+_{0,p}$ no longer induces an isomorphism for small surgeries. See [19, Equation 26] and [19, Section 6.6] .

Using Hendricks and Manolescu's computation of ι_K for thin knots [19], we can calculate that the action of $(\iota_K)_*$ on $H_*(A_0^+)$ interchanges the two elements of lowest grading. Hence ι_* on $HF^+(S_{+1}(K))$ must also interchange the two elements of lowest grading. Reflecting $HF^+(S_{+1}(K))$ over a horizontal line gives $HF^-(S_{-1}(6_2))$, with the action of ι_* exchanging the two elements of (shifted) grading zero, as displayed in Figure 3.18. Hence one of τ_* or

$(\iota \circ \tau)_*$ on $HF^-(S_{-1}(6_2))$ must also exchange the pair of elements in grading zero. Clearly, there is no map (satisfying the properties of Remark 3.5.1) from the trivial $\mathbb{F}_2[U]$-module, equipped with the identity involution, to $HF^-(S_{-1}(6_2))$, equipped with an involution acting nontrivially on the two elements of highest grading.

This completes the proof that (P, τ) is a strong cork. Moreover, it is clear that if P' is constructed from P by introducing any number of symmetric pairs of negative full twists (as in Figure 3.2), then P' admits a sequence of interchanging $(-1, -1)$-cobordisms to P. $\square$

CHAPTER 4

SYMMETRIC KNOTS AND HEEGAARD FLOER HOMOLOGY

4.1 Introduction

Let K be a knot in S^3. Let τ be an orientation preserving diffeomorphism of order 2 of S^3 which fixes the knot setwise. We refer to such knots as *symmetric knots* of order 2 (or in short symmetric knots), where the restriction of τ to K acts as a symmetry of K. The fixed set of τ can be either the empty set or S^1. When the fixed set of τ is S^1, then the fixed set can either intersect K in two points or be disjoint from K. In the former case we refer to (K, τ) as *strongly invertible knot* and for the later case, we call (K, τ) a *periodic knot*. It is also well-known that surgery on such symmetric knots induce an involution on surgered 3-manifold (see Section 3.3). Montesinos [30] showed that surgery on a strongly invertible knot is always a double branched cover of a knot inside S^3. Moreover, he showed that a 3-manifold is a double branch covering of S^3 if and only if it can be obtained as surgery on a strongly invertible link. In fact, one can identify the covering involution with the induced involution on the surgered manifold. More generally, [37] showed that any 3-manifold with a finite order diffeomorphism can be obtained by doing surgery on a periodic link, where the diffeomorphism on the 3-manifold is conjugate to the induced diffeomorphism on the surgered manifold from the periodic link. This result can be interpreted as an *equivariant* version of the Lickorish–Wallace theorem.

In the context of previously defined invariants and the study of group $\Theta^{\tau}_{\mathbb{Z}}$ where we looked at the induced action of an involution of the Heegaard Floer chain complex of the 3-manifold (see Chapters 2 and 3), one might be interested in studying the induced action of a symmetry on the knot Floer chain complex of K, where K is a symmetric knot. In this chapter we initiate such a study by defining the induced action, and then computing it for several classes of symmetric knots.

4.2 Defining the induced actions on the knot Floer complex

In this section we define the action of a symmetry on the knot Floer chain complex. We will restrict ourselves to knots in integer homology spheres. Let (Y, K, τ, w, z) be a tuple where (Y, K, w, z) represents a knot K with two basepoints z and w embedded in Y , and τ is an orientation preserving involution on (Y, K) which fixes K set-wise. We now consider two separate families of such tuples, defined according to how the invariants act on the knot.

Definition 4.2.1. Given (Y, K, τ, w, z) as above, we say that K is *2-periodic* if τ has no fixed points on K and it preserves the orientation on K [1]. On the other hand, we will say that K is a *strongly invertible* if τ has two fixed points when restricted to K; note that such an involution switches the orientation of K.

Both periodic and strong involutions induce actions on the knot Floer complex. Let us consider the periodic case first. As before we start with a Heegaard data $\mathcal{H}_K$. There is a tautological chain isomorphism

$$t_K : CFK^{\infty}(\mathcal{H}_K) \to CFK^{\infty}(\tau\mathcal{H}_K)$$

Now note that $\tau\mathcal{H}_K$ represents the same knot inside Y although the basepoints (z, w) have moved to $(\tau z, \tau w)$. So we apply a diffeomorphim ρ_1, obtained by isotopy ρ_t taking τz and τw back to z and w along an arc of the knot, following the orientation of the knot. We also require the isotopy to be the identity outside a small neighborhood of the knot. $\rho_1\tau\mathcal{H}_K$ now represents the based knot (Y, K, z, w). So by work of Hendricks-Manolescu [19] and [21] there is a sequence of Heegaard moves relating the $\rho_1\tau\mathcal{H}_K$ and $\mathcal{H}_K$ inducing a chain homotopy equivalence

$$\Phi(\rho_1\tau\mathcal{H}_K, \mathcal{H}_K) : CFK^{\infty}(\rho\tau\mathcal{H}_K) \to CFK^{\infty}(\mathcal{H}_K)$$

[1]Since we are dealing only with involutions in this paper we will abbreviate 2-periodic knots as just periodic.

We now define the τ action to be

$$\tau_K := \Phi(\rho_1 \tau \mathcal{H}_K, \mathcal{H}_K) \circ t_K : CFK^\infty(\mathcal{H}_K) \to CFK^\infty(\mathcal{H}_K)$$

The chain homotopy type of τ_K is independent of the choice of Heegaard data. This is again a consequence of the naturality results shown in [21]. In particular the map descends to a map

$$\tau_K : CFK^\infty(K, w, z) \to CFK^\infty(K, w, z)$$

where $CFK^\infty(K, w, z)$ is the transitive homotopy type of $CFK^\infty(\mathcal{H}_K)$, see for example [45].

Sarkar [38] defined a specific action on the knot Floer complex called the Sarkar map ς obtained by moving the two base points once around the orientation of the knot, which amounts to applying a full Dehn twist along the orientation of the knot. The map ς is a filtered, grading-preserving chain map, which is well-defined up to filtered chain homotopy equivalence. This map was explicitly computed in [43]; in particular we have $\varsigma^2 \simeq \mathrm{id}$. Analogous to the case in 3-manifolds, one can inquire whether τ_K is a homotopy involution. It turns out that it is not a homotopy involution, in general, but $\tau_K^4 \simeq \mathrm{id}$. As a consequence of the following:

Proposition 4.2.2. *Let Y be a $\mathbb{Z}HS^3$ and (Y, K, τ, w, z) be a doubly-based periodic knot in it, then τ_K is a grading preserving, filtered map that is well-defined up to chain homotopy and $\tau_K^2 \simeq \varsigma$.*

Proof. The proof is similar to that of Hendricks-Manolescu [19, Lemma 2.5.] with only cosmetic changes, so we will omit the proof. The main idea is that since the definition τ_K involves the basepoint moving map taking (z, w) to $(\tau z, \tau w)$, τ_K^2 results in moving the pair (z, w) once around the knot K along its orientation, back to (z, w). τ_K is grading preserving and filtered since all the maps involved in its definition are. $\square$

We now define a similar action for strong involutions. Note that in this case τ reverses orientation of the knot K. Since the knot Floer chain complex is an invariant (up to canonical chain homotopy equivalence) of *oriented* knots, we do not *a priori* have an automorphism of K. However it is still possible to engineer an involution on the knot Floer complex induced by τ.

As before we start by taking a Heegaard data $\mathcal{H}_K = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$ for (Y, K, w, z). Recall that the order of the basepoints determine an orientation for the knot, i.e they intersect Σ positively at z and negatively w. Note that $\mathcal{H}_{-K} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, w)$ then represents $(Y, -K, z, w)$ [2], where there is an obvious correspondence between the intersection points of these two diagrams. In order to avoid confusion, for an intersection point $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ for $\mathcal{H}_K$, we will write the corresponding intersection point for $\mathcal{H}_{-K}$ as $\mathbf{x}'$. Now there is a tautological grading preserving *skew-filtered* chain isomorphism,

$$sw : CFK^\infty(\mathcal{H}_K) \to CFK^\infty(\mathcal{H}_{-K})$$

obtained by switching the order of the base-points z and w.

More specifically, recall that $CFK^\infty(\mathcal{H}_K)$ is $\mathbb{Z} \oplus \mathbb{Z}$-filtered chain complex, generated by triples $[\mathbf{x}, i, j]$. So we define $sw[\mathbf{x}, i, j] = [\mathbf{x}', j, i]$. This map is *skew-filtered* in the sense that if we take the filtration on the range to be $\bar{\mathcal{F}}_{z,w}([\mathbf{x}', i, j]) = (j, i)$ then sw is filtration preserving. Notice that in the definition of sw, we are crucially using that Y is an $\mathbb{Z}HS^3$ or atleast that $[K] = 0$, since in general sw takes a torsion spin^c-structure $\mathfrak{s}$ and sends it to $\mathfrak{s} + \mathrm{PD}(K)$, see for example [45, Lemma 3.3.].

Let us now define the action of τ on the knot Floer complex. To simplify the process, we assume that τ switches the basepoints, i.e.

$$(\tau w, \tau z) = (z, w), \text{ as an ordered pair.}$$

We will denote map on the knot Floer complex, as obtained by pushing-forward $\mathcal{H}_{-K}$ by τ, as t_{-K} and the Heegaard data as $\tau\mathcal{H}_{-K}$.

[2]Here $-K$ represents, the knot K with its other orientation.

We start by applying the τ to K to get

$$\tau_K : CFK^\infty(\mathcal{H}_K) \to CFK^\infty(\tau\mathcal{H}_K)$$

So now $\tau\mathcal{H}_K$ represents the based knot $(Y, -K, \tau w, \tau z)$. Then by Theorem 1.1.2 there is a chain homotopy equivalence Φ induced by the sequence of Heegaard moves connecting $\mathcal{H}_{-K}$ and $\tau\mathcal{H}_K$. Finally we apply the sw map to get back to original knot Floer complex. The action τ_K, of τ on the knot Floer complex is then defined to be the composition of the maps above, i.e we τ_K is the following composition

$$CFK^\infty(\mathcal{H}_K) \xrightarrow{t_K} CFK^\infty(\tau\mathcal{H}_K) \xrightarrow{\Phi} CFK^\infty(\mathcal{H}_{-K}) \xrightarrow{sw} CFK^\infty(\mathcal{H}_K)$$

Proposition 4.2.3. *Let Y be a $\mathbb{Z}HS^3$ and (K, τ, w, z) be a doubly-based strongly invertible knot in it. The induced map τ_K an well-defined map up to chain homotopy. Furthermore, it is a grading preserving skew-filtered involution on $CFK^\infty(Y, K)$, i.e $\tau_K^2 \simeq \mathrm{id}$.*

Proof. Firstly, we note that t_K and sw satisfy the following relation, tautologically i.e we have

$$sw \circ t_K \simeq t_{-K} \circ sw$$

The following chain of grading preserving homotopies yields the result

$$\begin{aligned}
\tau_K^2 &= sw \circ \Phi(\tau\mathcal{H}_K, \mathcal{H}_{-K}) \circ t_K \circ sw \circ \Phi(\tau\mathcal{H}_K, \mathcal{H}_{-K}) \circ t_K \\
&\simeq sw \circ \Phi(\tau\mathcal{H}_K, \mathcal{H}_{-K}) \circ t_K \circ sw \circ t_K \circ \Phi(\mathcal{H}_K, \tau\mathcal{H}_{-K}) \\
&\simeq sw \circ \Phi(\tau\mathcal{H}_K, \mathcal{H}_{-K}) \circ sw \circ \Phi(\mathcal{H}_K, \tau\mathcal{H}_{-K}) \\
&\simeq sw \circ \Phi(\tau\mathcal{H}_K, \mathcal{H}_{-K}) \circ \Phi(\mathcal{H}_{-K}, \tau\mathcal{H}_K) \circ sw \\
&\simeq \mathrm{id}
\end{aligned}$$

□

Remark 4.2.4. Readers familiar with the involutive knot action ι_K will recognize that Proposition 4.2.3 implies that τ_K is different from the ι_K in the sense that although both are graded, skew-filtered maps, τ_K does not square to the Sarkar map. In particular, it is less rigid.

4.3 Computations

We now move on to computing the induced action for several symmetric knots. The main strategy is to use the grading and filtration information to pin down the action on CFK^∞ (up to change of basis.) Note also that computing these actions directly, by examining the effect of τ on the Heegaard surface Σ and the α and β curves, is quite cumbersome just as in the case for 3-manifolds. We now provide several examples of the computations.

Remark 4.3.1. In the computations of the actions we often use a particular model of the knot Floer complex that is filtered chain homotopic to $CFK^\infty(\mathcal{H}_K)$. An argument similar to [19, Lemma 6.5.] shows that we can conjugate the action of τ_K on $CFK^\infty(\mathcal{H})$ so that it induces an action on the model complex. We will then unambiguously refer to the conjugated action as τ_K.

4.3.1 Strongly invertible L-space knots and their mirrors

There are several L-space knots that admit a strong involution [40]. Here we show that we can explicitly compute the involution in for those knots and their mirrors.

Recall that L-space knots are the knots for which $S^3_p(K)$ is an L-space, an integer $p > 0$. In particular, sufficiently large surgery on L-space knots are L-spaces. The knot Floer homology for these knots are determined their Alexander polynomial. Specifically the knot Floer homology $CFK^\infty(K)$ of an L-space knot can be regarded as chain homotopic to $C \otimes \mathbb{Z}_2[U, U^{-1}]$. Here C is a chain complex in the shape of a staircase.

In a similar fashion the knot Floer complex of the of a mirror of an L-space knot can be taken to be a copy of a staircase, see Figure 4.1. We refer readers to [19, Section 7] for a description of the relationship between the Alexander polynomial of L-space knots and their

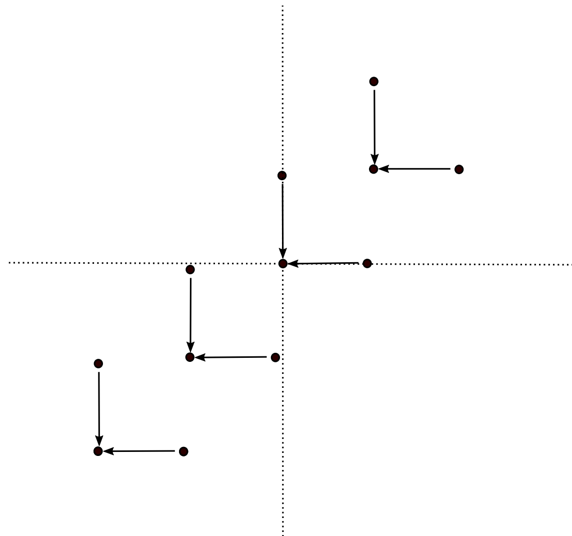


Figure 4.1: Left: CFK^{∞} of the left-handed trefoil

mirrors with the knot Floer complexes. Before moving forward let us recall that we have the definition of the action

$$\iota_K : CFK^{\infty}(K) \to CFK^{\infty}(K)$$

on the knot Floer complex induced by spin^c-conjugation defined by [19]. Starting with a doubly-pointed Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, w)$, there is a tautological map

$$\eta_K : CFK^{\infty}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z) \to CFK^{\infty}(-\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z, w)$$

we then take the basepoint moving map ψ which sends the basepoints (z, w) to (w, z) by performing half-Dehn twist along the orientation of the knot and finally since $\psi(-\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z, w)$ and $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$ represent the same doubly pointed knot (K, w, z), there is an homotopy equivalence $\Phi : CFK^{\infty}(\psi(-\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, z, w)) \to CFK^{\infty}(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$, induced by Heegaard moves.

ι_K is then defined as $\iota_K := \Phi \circ \psi \circ \eta_K$. One can then show this is an well-defined automorphism (independent of the choices made) of CFK^{∞} so that $\iota_K^2 \simeq \varsigma$.

We show the following

Proposition 4.3.2. *Let K be an L-space knot (or the mirror of an L-space knot) that is strongly invertible with the strong-involution τ_K. We have*

$$\iota_K \simeq \tau_K.$$

Proof. Recall that if τ_K is a strong involution on a knot K, the induced map on CFK^∞ is a skew-filtered map that squares to the id. As seen in [19], the fact that ι_K is grading preserving skew-filtered and it squares to identity is enough to uniquely determine it for both L-space knots and the mirrors of L-space knots. The claim then follows from similar computation of ι_K for L-space knots from [19].

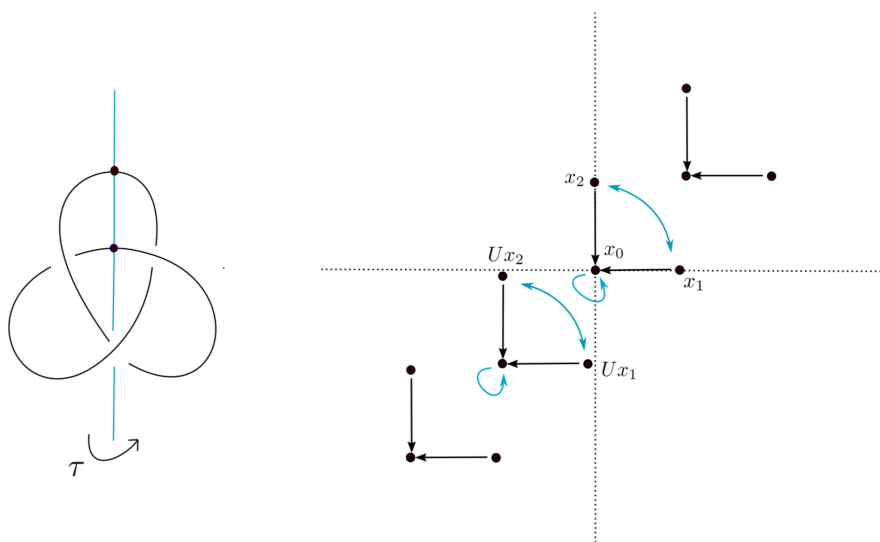


Figure 4.2: Left: left-handed trefoil with the strong inversion on the left, Right: The induced action on the knot Floer complex.

□

4.3.2 Periodic involution on L-space knots and their mirrors

There are several L-space knots which admit a periodic involution. For example $(2, q)$ torus knots are 2-periodic. In this case we have the following

Proposition 4.3.3. *Let K be an L-space knot (or mirror of an L-space knot) with a periodic involution τ_K. we have*

$$\tau_K \simeq \text{id}.$$

Proof. Note that when τ_K is a periodic involution, the induced map on the knot Floer complex is a grading preserving filtered map that squares to the Sarkar map. The conclusion the follows. $\square$

4.3.3 An Example

Example 4.3.4. We now look at the figure-eight knot 4_1 and the periodic involution τ_K on it, as in Figure 4.3.4. We identify this involution on the knot Floer complex of 4_1. To see this first note that, 4_1 is a *Floer homologically thin knot*.

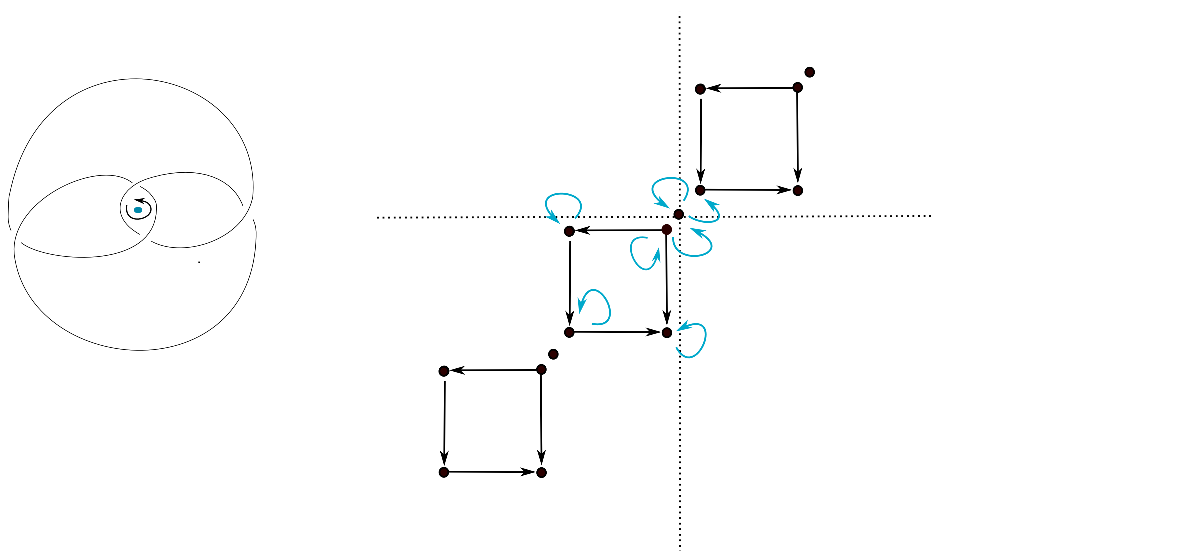


Figure 4.3: Left: Figure-eight knot with a periodic symmetry, Right: Induced action on the Knot Floer complex.

Now τ_K is a filtered grading preserving automorphism on $CFK^{\infty}(K)$. This implies τ_K sends $[\mathbf{x}, i, j]$ to $[\tau_K(\mathbf{x}), i', j']$ where $i' \leq i$ and $j' \leq j$. Coupled with the fact that τ_K

preserves the Maslov grading and K is thin, we get $\tau_K(\mathbf{x})$ lies in the same diagonal line as $\mathbf{x}$ which implies $i' = i$ and $j' = j$.

Furthermore, we know that $\tau_K^2 \simeq \varsigma$ from Proposition 4.2.2. The Sarkar map ς is map known to be identity for staircases. For squares the map takes the form indicated below

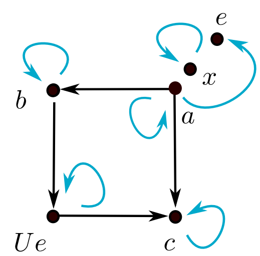


Figure 4.4: Sarkar map, for figure eight knot.

$$\varsigma(a) = a + e,\ \varsigma(b) = b,\ \varsigma(c) = c,\ \varsigma(e) = e.$$

We will use the constraints laid out above to find such candidate τ_K. The calculation similar to that in [19, Section 8] then yields the action as shown in the Figure 4.3.4.

Remark 4.3.5. Note that the action defined in Figure 4.3.4 is different from the involutive action on the knot Floer chain complex of the 4_1 knot, although both actions square to the Sarkar map.

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