THE HOMOLOGY POLYNOMIAL AND THE BURAU REPRESENTATION FOR PSEUDO-ANOSOV BRAIDS

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ABSTRACT

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The homology polynomial is an invariant for pseudo-Anosov mapping classes [3]. We study the homology polynomial as an invariant for pseudo-Anosov braids and its connection to the Burau representation. Given a pseudo-Anosov braid $\beta \in B_n$, we determine necessary and sufficient conditions under which the homology polynomial of β is equal to the the characteristic polynomial of the image of β under the Burau representation. In particular, we build upon [1] and show that the orientation cover associated to a pseudo-Anosov braid is equivalent to a quotient to the Burau cover when the measured foliations associated to β have odd-ordered singularities at each puncture and any singularity that occurs in the interior of D_n is even-ordered. We next construct an algorithm which allows us to determine the homology polynomial from the Burau representation for an arbitrary pseudo-Anosov braid. As an application, we show how to easily determine the homology polynomial for large family of pseudo-Anosov braids.

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CHAPTER 1

INTRODUCTION

The homology polynomial for a pseudo-Anosov mapping class $[f]$ on a surface S is an integer polynomial invariant $h(x)$ introduced in [3]. If the stable and unstable foliations of $[f]$ are orientable, then $h(x)$ is associated to the induced action of $[f]$ on $H_1(S,\mathbb{R})$. It is the product of two additional polynomial invariants $h(x) = p(x)s(x)$ each with topological meaning. By identifying B_n with the mapping class group on the n-punctured disk, the homology polynomial becomes an invariant for pseudo-Anosov braids.

However determining the homology polynomial for an arbitrary mapping class can be difficult or impossible in practice. Computing $h(x)$ involves an application of the Bestvina-Handel algorithm ([2]). Software limitations make this impractical for many mapping classes.

We will show that for many pseudo-Anosov braids it is trivially easy to determine the homology polynomial using the (reduced) Burau representation. The (reduced) Burau representation is map

$$\Psi : B_n \to GL_{n-1}(\mathbb{Z}[t, 1/t])$$

The image of $\beta \in B_n$ is $\psi_\beta(t)$ and is called the *Burau matrix* of β. It is an $(n-1) \times (n-1)$ matrix with entries from the ring of Laurent polynomials $\mathbb{Z}[t, 1/t]$. Our first result is a connection between the characteristic polynomial of $\psi_\beta(t)$ and $h(x)$. In Section 3.2 we prove the following:

Theorem 1.0.1. *Suppose $\beta \in B_n$ is pseudo-Anosov with stretch factor λ and homology polynomial $h(\beta, x)$. Let $(\mathcal{F}^u, \mu_u)$ and $(\mathcal{F}^s, \mu_s)$ be the singular measured foliations for β. Finally, let $\psi_\beta(t)$ be the Burau matrix for β and let*

$$\chi(\psi_\beta(t)) = |xI - \psi_\beta(t)|$$

Then the following are equivalent

(1) $\chi(\psi_\beta(\eta)) = h(\beta, x)$ *for some root of unity* η.

(2) $\chi(\psi_\beta(-1)) = h(\beta, x)$ *and* -1 *is the only root of unity at which this equality occurs.*

(3) $\mathrm{sr}(\psi_\beta(-1)) = \lambda$ *where* $\mathrm{sr}(\psi_\beta(-1))$ *is the spectral radius of* $\psi_\beta(-1)$.

(4) *The singularites of* $\mathcal{F}^u$ *and* $\mathcal{F}^s$ *are odd-ordered if they occur at a puncture and even-ordered if they occur in the interior of* D_n.

(5) $D_n^{(2)}$ *is the orientation double-cover of* τ *(after attaching a punctured disk to the boundary of* D_n*).*

With the above in mind, if $\mathcal{F}^u$ and $\mathcal{F}^s$ above have an even-ordered singularity at a puncture or odd-ordered singularity in the interior we will say that β *produces* a *bad* singularity. Suppose that $\beta \in B_n$ is pseudo-Anosov and produces at least one bad singularity. In this case we cannot recover $h(x)$ from $\psi_\beta(t)$ and the direct computation may be difficult. However we can still use the Burau representation to compute $h(x)$ which is our next result.

Theorem 1.0.2. *Let* $\beta_0 \in B_n$ *be a pseudo-Anosov braid identified with its pseudo-Anosov representative* $\beta_0 : D_n \to D_n$. *Suppose the measured foliations for* β_0 *have* p *odd-ordered singularities occurring at interior points* $x_1, \ldots, x_p$ *and* q *even-ordered singularities occurring at punctures* $p_1, \ldots, p_q$. *Let* $\beta = \beta_0^k$ *where* $k \geq 1$ *is chosen so that* β *fixes each* p_i *and* x_i *pointwise.*

Identify D_{n+q-r} *with* $(D_n \cup \{p_1, \ldots, p_r\}) - \{x_1, \ldots, x_q\}$. *Since* β *fixes each* x_i *and* p_i *pointwise it induces a map* $\beta' : D_{n+q-r} \to D_{n+q-r}$.

The braid $\beta' \in B_{n+q-r}$ *is pseudo-Anosov with*

$$h(\beta, x) = (1 + x)^\varepsilon \, h(\beta', x) = |xI - \psi_{\beta'}(-1)|$$

where $\varepsilon \geq 0$ *is the number of order-2 singularities occurring at a puncture.*

In other words, if β is pseudo-Anosov we can either recover $h(x)$ from $\psi_\beta(-1)$ or we can construct a new braid β' and recover $h(x)$ from $\psi_{\beta'}(-1)$.

Of course, determining the types of singularities produced by a pseudo-Ansov braid is not necessarily an easier task than computing its homology polynomial directly. The usefulness of Theorem 1.0.2 is demonstrated in Chapter 4. In particular, in Section 4.1 we present a large family of pseudo-Anosov braids and use Theorem 1.0.2 to trivialize the computation of $h(x)$ (regardless of the singularity types they produce).

1.1 Organization of Dissertation

In Chapter 2 we prove an brief overview of the homology polynomial and the reduced Burau representations along with any necessary preliminaries. In Chapter 3 we prove both Theorem 1.0.1 and Theorem 1.0.2. Finally, in Chapter 4 we present examples and applications of Theorem 1.0.1 and Theorem 1.0.2.

CHAPTER 2

PRELIMINARIES

2.1 Pseudo-Anosov mapping classes

We assume a basic familiarity with braid groups [6] and mapping class groups [4]. Unless stated otherwise, we assume all surfaces are closed, connected, and orientable with a disjoint collection of finitely many points and open disks removed.

Definition 2.1.1. Let S be a surface. Let $\mathrm{Hom}^{+}(S, \partial S)$ be the collection of orientatation preserving homeomorphisms on S. The *mapping class group* on S is

$$\mathrm{Mod}(S) = \pi_0(\mathrm{Hom}^{+}(S, \partial S)) = \mathrm{Hom}^{+}(S, \partial S)/\sim$$

where $f \sim g$ if there is an isotopy from f to g which fixes all punctures and boundary components pointwise. If $[f], [g] \in \mathrm{Mod}S$ then $[f][g]$ is defined as composition, which is to say $[f][g] = [f \circ g]$.

Let D_n denote the disk with $n \geq 3$ points removed. It is well known that the braid group on n strands B_n is represented as a mapping class group on D_n, that is

$$B_n \simeq \mathrm{Mod}(D_n).$$

For convenience, we identify $\beta \in B$ with its representative isotopy class in $\mathrm{Mod}(D_n)$. See [6] Section 1.6 for more details.

Definition 2.1.2. Let $\gamma \subset S$ be a simple closed curve. We say γ is *essential* if it is not homotopic to a point, a puncture, or a boundary component of S. We say an isotopy class of curves is essential if it has an essential representative γ.

For example, if $S_{g,n}$ is the genus g surface with n points removed, then $S_{0,n}$ has no essential curves for $n = 0, 1, 2, 3$. On a torus, the meridinal and longitudinal curves are both essential.

2.1.1 The Nielsen-Thurston Classification

The Nielsen-Thurston classification says that all mapping classes can be classified as one of three types. For convenience we provide the statement below. For more information see [4], Section 13.3.

Theorem 2.1.3 (The Nielsen-Thurston Classification). *Let S be a compact, orientable surface with possibly finitely many punctures and let $[f] \in \mathrm{Mod}(S)$. Then there is a representative homeomorphism $f : S \to S$ that is* periodic, reducible, *or* pseudo-Anosov. *Furthermore if f is pseudo-Anosov then it is neither periodic nor reducible.*

We say a mapping class $[f]$ is *periodic* if there is some positive integer k such that $[f^k]$ has a representative isotopic to the identity. We say $[f]$ is *reducible* if there a non-trivial collection of isotopy classes of essential simple closed curves $\{c_1, \dots, c_k\}$ so that $\{f(c_1), \dots, f(c_k)\} = \{c_1, \dots, c_k\}$ up to isotopy. Our main focus will be on pseudo-Anosov mapping classes. A basic overview of what it means to be pseudo-Anosov is given below.

2.1.2 Singular measured foliations and pseudo-Anosov mapping classes

Definition 2.1.4. A *singular foliation* $\mathcal{F}$ on a surface S is a decomposition of S into a disjoint union of subsets of S called *leaves* along with a finite collection of *singular points* $\{x_1, \dots, x_m\} \subset S$ so that

1. For every nonsingular point $p \in S$ there is a smooth chart from a neighborhood of p to $\mathbb{R}^2$ which sends each leaf to a horizontal line segment.

2. For each singluar point $x_i \in S$ there is a smooth chart from a neighborhood of p to R^2 which sends leaves to level sets of a k-pronged saddle with $k \geq 3$.

A smooth arc $\alpha \in S$ is *transverse* to $\mathcal{F}$ if it is transverse to each leaf of $\mathcal{F}$ and is disjoint from the singular points of $\mathcal{F}$. A *singular measured foliation* is a pair $(\mathcal{F}, \mu)$ where μ is a

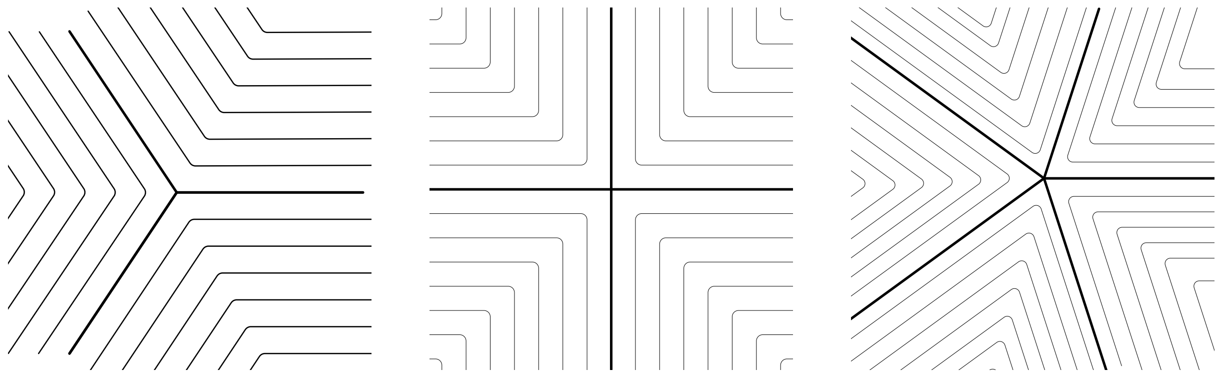

Figure 2.1: Saddles for $n = 3, 4, 5$

measure that assigns a positive value to each smooth arc α transverse to $\mathcal{F}$ with μ invariant under any leaf-preserving isotopy.

In Chapter 3 we will wish to keep track of the types of singularities that occur.

Definition 2.1.5. Let $\mathcal{F}$ be a singular foliation on a surface S with a singular point $x \in S$ with a chart sending the leaves in a neighborhood of x to the the level sets of a k-pronged saddle. Then we say x is an *order* k singularity. See Figure 2.1.

Definition 2.1.6. A homeomorphism $f : S \to S$ is *pseudo-Anosov* if there is a pair of transverse measured foliations $(\mathcal{F}^u, \mu_u)$ $(\mathcal{F}^s, \mu_s)$ on S and a real number $\lambda > 1$ so that

$$f \cdot (\mathcal{F}^u, \mu_u) = (\mathcal{F}^u, \lambda\mu_u) \quad \text{and} \quad f \cdot (\mathcal{F}^s, \mu_s) = \left(\mathcal{F}^s, \lambda^{-1}\mu_s\right)$$

A mapping class is pseudo-Anosov if it has a pseudo-Anosov representative.

The measured foliations $(\mathcal{F}^u, \mu_u)$ and $(\mathcal{F}^s, \mu_s)$ are called the *unstable* and *stable* foliations respectively. The number λ is called the *dilitation* or *stretch factor* of $[f]$.

2.2 Train Tracks

See [8] Chapter 1 for more information on the definitions given in this section.

Definition 2.2.1. Let S be a closed orientable surface of genus g with finitely many punctures and let $\tau \in S$ be an embedded smooth, closed, 1-complex. We will refer to the vertices of τ as switches and denote the set of all switches by $\mathrm{Sw}(\tau)$. Then $\tau - \mathrm{Sw}(\tau)$ consists of a disjoint collection of smooth open arcs. These components will be refered to as the *branches* of τ and we will denote set of all branches by $\mathrm{Br}(\tau)$. Then τ is a *train track* in S if

1. For each switch v there is an open neighborhood U of v and a well defined tangent line $L \in T_v(S)$ so that $\tau \cap U$ is the union of a finite collection of open arcs, each tangent to L at v.

2. The components of $S - \tau$ are either once-punctured k-gons with $k \geq 1$ or unpunctured k-gons with $k \geq 3$.

Example 2.2.2. See Figure 2.2. On the left is a train track in the 4-punctured sphere. On the right is a 1-complex in the 3-punctured sphere which is not a train track ($S_{0,3} - \tau$ contains a punctured disk with no cusps).

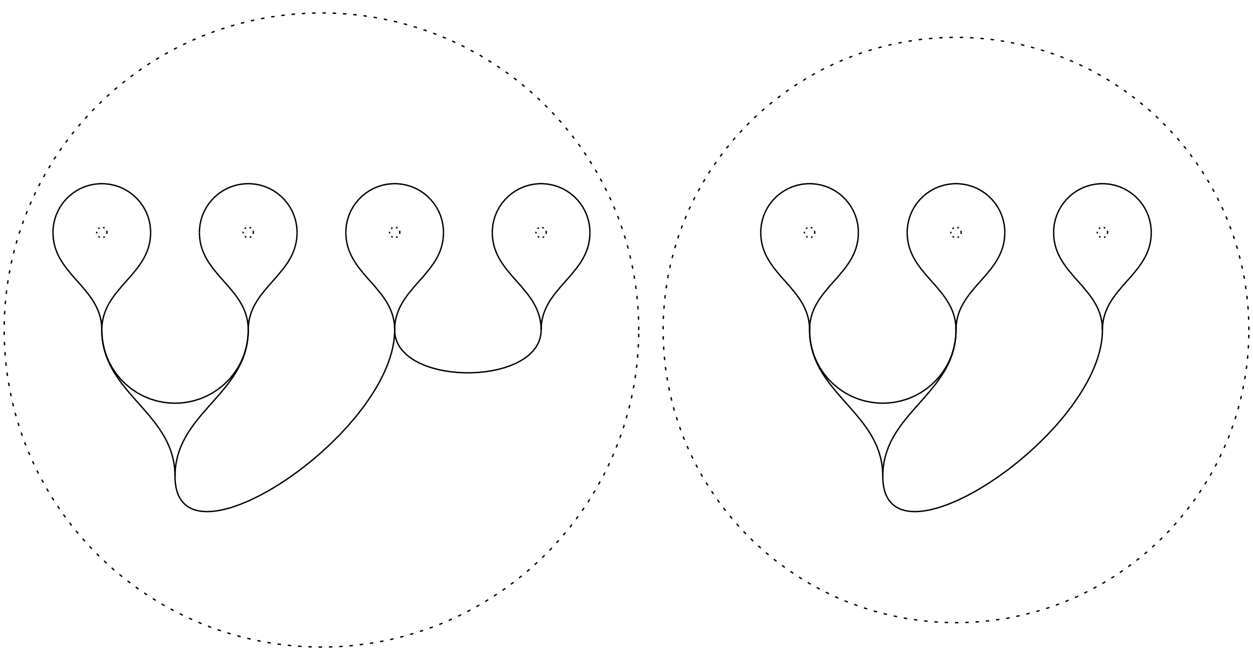

Figure 2.2: A train track in $S_{0,4}$ (left) and a 1-complex that is not a train track in $S_{0,3}$ (right)

Definition 2.2.3. Let C be a family of smooth simple closed curves disjointly embedded in a surface S so that no component belonging to C is homotopic to a point or a puncture. We say that a train track τ *carries* C if there is a smooth map $\phi : S \to \S$, called the *supporting map*, so that

1. $\phi(C) \subseteq \tau$
2. ϕ is homotopic to the identity map
3. The restriction of the differential $d\phi_p$to the tangent line to C at p is nonzero for every $p \in C$.

Similarly, we say a train track τ' is carried by τ is there is a supporting map $\phi : S \to S$ meeting the conditions given above.

Definition 2.2.4. A mapping class $[f] : S \to S$ is *carried* by a train track $\tau \subset S$ if there is a representative $f \in [f]$ so that $f(\tau)$ is carried by τ.

Note that the supporting map ϕ for a map f carried by τ can always be chosen so that switches are sent to swiches and edges are sent to edge-paths. See [2] for details.

Definition 2.2.5. Suppose β is carried by τ with supporting map ϕ chosen so that edges are sent to edge-paths and vertices are sent to vertices. Then viewing τ as a graph, the map $\beta_* = \phi \circ \beta : |_\tau : \tau \to \tau$ is the *train track map induced by* β.

Example 2.2.6. Let $\beta = \sigma_1 {\sigma_2}^{-1}$. Let $\tau \in D_3$ be the train track depicted in Figure 2.3 (left) with branches labeled as indicated. A representation of $\beta(\tau)$ is shown in Figure 2.3 (below). The induced train track map $\beta_* : \tau \to \tau$ is defined by

$$\begin{array}{llll} e_1 & \mapsto e_3 & e_4 & \mapsto e_5 e_2 \overline{e_4} \\ e_2 & \mapsto e_1 & e_5 & \mapsto e_4 \overline{e_2}\,\overline{e_5}\,\overline{e_3} e_5 \\ e_3 & \mapsto e_2 & & \end{array}$$

Definition 2.2.7. Let $g : G \to G$ be a graph map sending vertices to vertices and edges to edge-paths. Suppose G has edges $e_1, \dots, e_k$. Then the transition matrix for $g : G \to G$ is

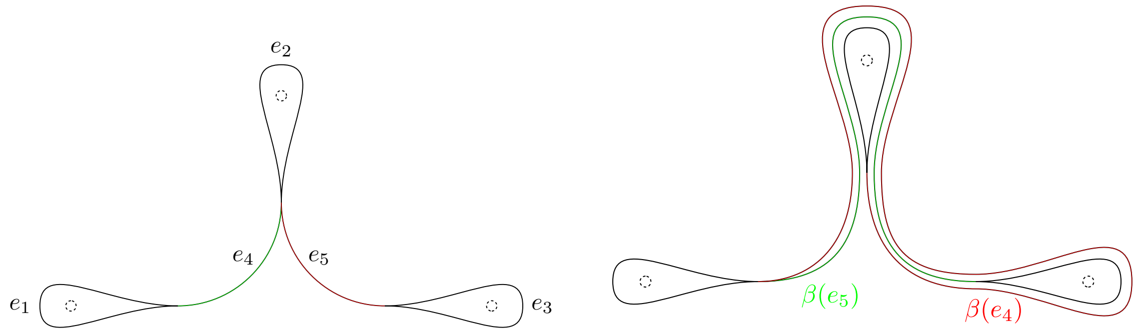

Figure 2.3: $\tau \subset D_3$ (left) and $\beta(\tau)$ (right) where $\beta = \sigma_1\sigma_2^{-1}$

defined as $M = (a_{ij})_{1\le i,j\le k}$ where a_{ij} is the number of times $\beta(e_j)$ passes over e_i (ignoring orientation).

Example 2.2.8. Consider again the braid and train track used in Example 2.2.6 and shown in Figure 2.3. Using the definition above, we have transition matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

2.2.1 Measured train tracks

Let τ be a train track in a surface S and let $v \in \tau$ be a switch. By definition every branch incident to v approaches the switch along a well defined tangent line L_v. This allows us to partition the branches incident to a switch into two *sides*. Let $e_1, \ldots, e_k$ be the branches of τ and let $\boldsymbol{w} = (w_1, ..., w_k)$ be an assignment of real-valued *weights* to the edges of τ with $\boldsymbol{w}(e_i) = w_i$. We say that $\boldsymbol{w}$ satisfies the *switch conditions* if at each switch the sum of the weights of the branches on each side are equal.

Definition 2.2.9. Let τ be a train track with edges $e_1, \dots, e_k$ and let $\boldsymbol{w}$ be an assignment of real-valued weights to the branches of τ. If $\boldsymbol{w}$ satisies the switch conditions, then the pair $(\tau, \boldsymbol{w})$ is called a *measured train track.*

2.3 The Bestvina-Handel Algorithm

In [2] Bestvina and Handel gave an algorithmic proof of the Nielsen-Thurston classification by using train tracks to encode the properties of $(\mathcal{F}^u, \mu_u)$ and $(\mathcal{F}^s, \mu_s)$. Bestvina and Handel's proof showed that given we can always construct a graph $G \subset S$ and a graph map $g : G \to G$ induced by $[f]$ so that g sends vertices to vertices and for any branch b the map $g|_b$ is an immersion. An example of a graph map induced by a braid is given in Example 2.2.6. What follows is a brief overview of how G is constructed.

Definition 2.3.1. A *fibered surface* is a compact surface F decomposed into arcs and polygons modelled after *k-junctions* as shown in Figure 2.4. The components of $F - \{\text{junctions}\}$ are called *strips.*

We will be interested in fibered surfaces which are subsurfaces of S_0 associated to f. We say a fibered surface $F \subset S_0$ *carries* f if

1. $F \hookrightarrow S_0$ is a homotopy equivalence,

2. f sends decomposition elements to decomposition elements

3. The junctions of F are sent to junctions by f

In particular, each arc belonging to a strip in F is sent to an arc or into a junction of F, but junctions must be sent to junctions.

Every fibered surface F is associated to a graph G obtained by crushing each decomposition elements of F to a point. The edges (resp. vertices) of G correspond to the strips (resp. junctions) of F. If F carries f, then there is a map $g : G \to G$ induced by f which sends vertices to vertices and edges to edge-paths defined in the obvious way.

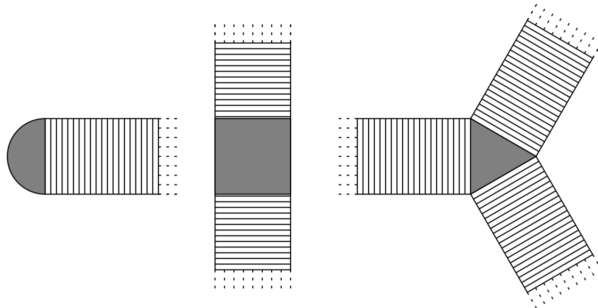

Figure 2.4: k-junctions for $k = 1, 2, 3$

Recall that the *link of a vertex* $v \in G$ is a graph $lk(v, G)$ with vertices corresponding to the edges in G emanating from v. If e_i and e_j emanate from v then the corresponding vertices in $lk(v, G)$ are connected by an edge if e_i and e_j are incident to a common 2-cell. For convenience we will refer to vertex of $\mathrm{Lk}(v, G)$ corresponding to e as $e \in \mathrm{Lk}(v, G)$ when there is no risk of confusion.

With that in mind, suppose v and w are vertices in G with $g(v) = w$ and e is an edge emanating from v. Then $g(v)$ is an edge-path in G with initial edge e' emanating from w. The *derivative* of $g : G \to G$ is the map

$$Dg : \mathrm{Lk}(v, G) \to \mathrm{Lk}(w, G)$$

defined by

$$e \mapsto e'$$

Definition 2.3.2. We say e_i and e_j in $\mathrm{Lk}(v, G)$ belong to the same *gate* if there is some $k > 0$ such that $D(g^k)(e_i) = D(g^k)(e_j)$.

In other words, edges e_i and e_j belong to the same gate if there is some power of g that sends both edges to an edge path with the same initial edge segment in G.

Before proceeding we review some matrix theory. Let $A = (a_{i,j})$ and $B = (b_{i,j})$ be $n \times n$ matrices with non-negative integer entries. We will write $A \geq B$ or $A > B$ to mean that $a_{i,j} \geq b_{i,j}$ for all i, j. By $a^k_{i,j}$ we mean the i, j-th entry of A^k.

We say $A = \{a_{ij}\}$ is *irreducible* if for each $a_{i,j}$ there is a $k > 0$ so that $a^k_{i,j} > 0$. If there is a k for which $A^k > \mathbf{0}$, we say A is primitive.

We can associate to A a directed graph $\mathcal{G}_A$. The graph consists of n vertices $v_1, \dots, v_n$ and an edge oriented from v_j to v_i whenever $a_{i,j}$ is non-zero.

Lemma 2.3.3.

1. *A is irreducible if and only if for every pair of vertices v_i, v_j in $\mathcal{G}_A$, there is an oriented edge-path connecting v_j to v_i.*

2. *A is primitive if and only if there is an integer n such that for every v_i, v_j in $\mathcal{G}_A$, there is an edge path of length n connecting them.*

A primitive matrix A is *Perron-Frobenius* if it has integer entries.

Theorem 2.3.4 (Perron-Frobenius theorem for primitive matrices)**.** *Let A be a non-negative $n \times n$ matrix. If A is primitive, then there is a eigenvalue $\lambda > 0$ of A such that given any other eigenvalue λ' of A we have $|\lambda'| < \lambda$.*

Note that in the case that A is primitive and has integer entries, then $\lambda > 1$.

By definition the transition matrix M (see Definition 2.2.7) is a square matrix with non-negative integer entries. Therefore if M is irreducible there is a unique positive unit eigenvector with positive eigenvalue λ which is the spectral radius of M and is called the *growth rate* of M. We will also occasionally refer to this value as $\lambda = \lambda(F, f) = \lambda(G, g)$ when it is convenient.

Example 2.3.5. Let $\beta = \sigma_1\sigma_2^{-1} \in B_3$. Let G be the graph depicted in Figure 2.5. A visual representation of the map induced by $\sigma_1\sigma_2^{-1}$ is also shown. However, the actual map sends edges to edge paths along the edges of G. The edges represented as circles are peripheral to punctures and thus do not contribute to the transition matrix for the real edges of the induced train track τ. Comparing $\sigma_1\sigma_2^{-1}(G)$ to G we see that the edgepath $\sigma_1\sigma_2^{-1}(e_1)$ passes once through e_1 and e_2. The edgepath $\sigma_1\sigma_2^{-1}(e_2)$ passes through e_1 once and e_2 twice. Then the transition matrix for the real edges is

$$M = \begin{array}{c c} & \begin{array}{cc} e_1 & e_2 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \end{array} & \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \end{array}$$

As we will see in later sections, the stretch factor for $\sigma_1\sigma_2{}^{-1}$ is the largest real root of the characteristic polynomial of M which is $h(x) = x^2 - 3x + 1$. It will also turn out that $h(x)$ is the homology polynomial for β

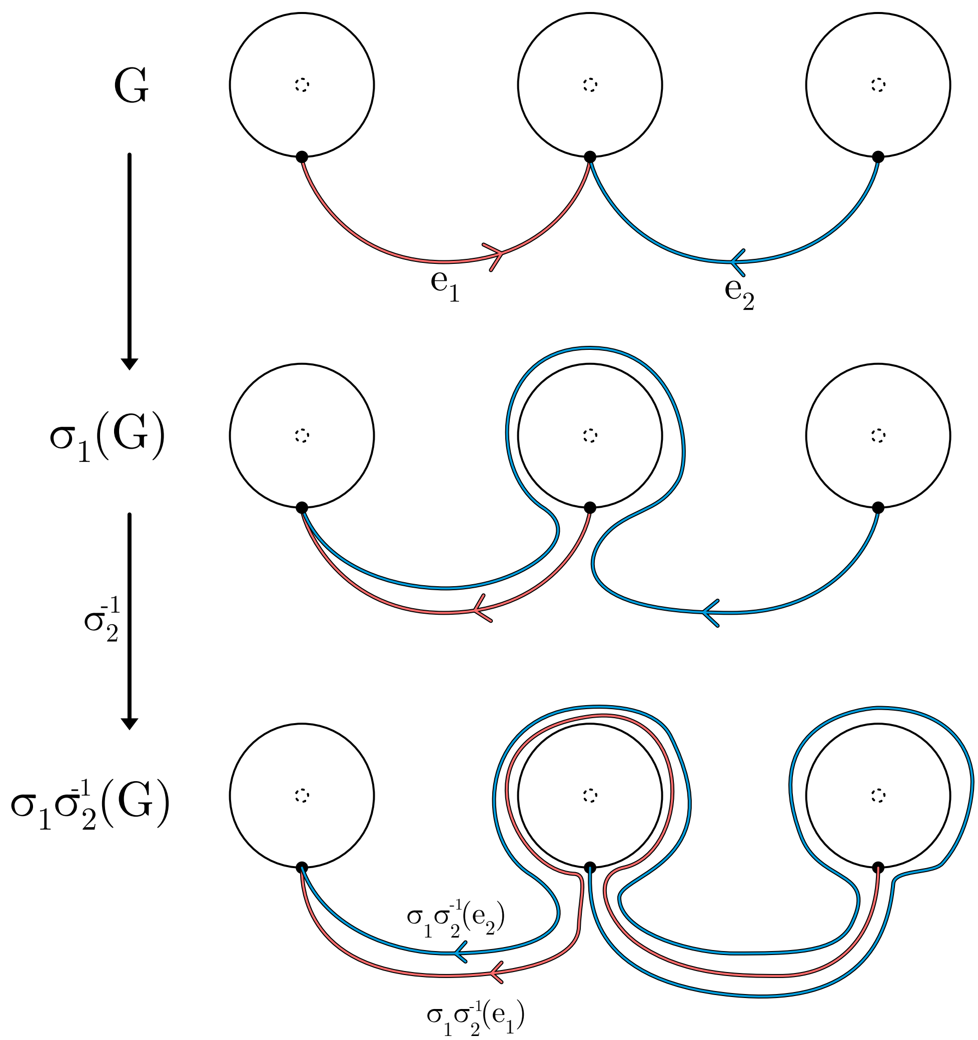

Figure 2.5: A graph map induced by $\sigma_1\sigma_2^{-1}$

The following is a consequence of [2]:

Proposition 2.3.6. *A mapping class $[f]$ is pseudo-Anosov if and only if there is a train track τ invariant under $[f]$ and the transition matrix for the real edges of τ under the train track map is Perron-Frobenius.*

2.3.1 From efficient graph maps to train tracks

The graph map $g : G \to G$ can be used to recover a train track τ that carries β along with a train track map representing β. Suppose v is a vertex of G and that there are k gates at v, $g_1, \ldots, g_k$. Replace v with a small circle C and identify k points $q_1, \ldots, q_k \in C$. We assume the gates are labeled to match the ordering of their associated gates when traveling counterclockwise around C. For each gate g_i, we arrange all edges belonging to the gate to intersect C orthogonally at p_i.

Suppose there is an edge $e_0 \in G$ so that the edgepath $g(e_0)$ passes through v. Then there is some subpath of $g(e_0)$ of the form $e_i e_j$ with $e_i \cap e_j = \{v\}$ Suppose e_i and e_j belong to gates g_i and g_j respectively. Then we add an edge ϵ_{ij} connecting p_i to p_j within the region bounded by C. We assume ϵ_{ij} intersects C orthogonally.

After performing this operation at each vertex of G we have a train track τ. The edges in τ that come from edges in G are called *real edges*. The edges connecting gates are called *infintesimal edges*. At a vertex with k gates, the resulting infintesimal edges form a k-gon (possibly with one edge missing).

Definition 2.3.7. A vertex of G is *odd* (*even* respectively) if the corresponding infintesimal edges form a polygon with an odd (even respectively) number of sides. If v corresponds to a polygon that is missing a side, we say v is *partial*.

Define $\phi : \tau \to \tau$ as follows:

If e is a real edge in τ, then $g(e) = e_i e_j \cdots e_k$ is an edge-path in G. Suppose $e_i e_j$ enters and exits the vertex $v \in G$. By the operation described above, there is an infintesimal edge $\eta \in \tau$

connecting e_i to e_j. Repeating this at each vertex $g(e)$ passes through, we get the edge path $\phi(e) = e_i \eta e_j \cdots$.

If η is an infintesimal edge connecting real edges e_i and e_j. Then there is an edge $e \in G$ for which $g(e)$ contains $e_i e_j$ as a subpath. The map $g^2(e)$ contains the subpath $g(e_i)g(e_j)$ and determines $\phi(\eta)$. See Figure 4.7 and Figure 4.7 for an example of this process for the braid $\sigma_1 \sigma_2 \sigma_3^{-1}$.

After constructing τ, Bestvina and Handel use the train track and map to recover the invariant measured foliations for the pseudo-Anosov mapping class. The following is a consequence:

Proposition 2.3.8. *Let $\beta \in B_n$ be pseudo-Anosov carried by a train track τ. There is a 1-to-1 correspondance between the components of $S_4 - \tau$ and the singularities of $\mathcal{F}^u$ and $\mathcal{F}^s$. In particular, a non-punctured disk with k corners corresponds to an order-k singularity in the interior of S_4 and a punctured disk with k corners corresponds to an order k singularity occurring at a puncture.*

2.4 The homology polynomial

Recall the homology polynomial discussed in the introduction. Let $[f] \in \mathrm{Mod}(S)$. The main result of [3] is the following.

Theorem 2.4.1 ([3], Theorem 1.1)**.** *Let $[f]$ be a pseudo-Anosov mapping class in a closed, orientable surface S with possibly finitely many punctures. Let $f : S \to S$ be the pseudo-Anosov representative of $[f]$ with the Bestvina-Handel graph and graph map $g : G \to G$ and transition matrix M. Then*

1. *The characteristic polynomial of M, $|xI - M|$, has a divisor $h(x)$ which is an invariant of $[f]$. The dilatation of $[f]$ is the largest real root of $h(x)$. It is associated to the induced action of f_* on $H_1(X, \mathbb{R})$ where $X = S$ when τ is orientable and X is the orientation cover of τ when is not orientable.*

2. The homology polynomial decomposes as a product $p(x) \cdot s(x)$ of two polynomials, each a topological invariant of $[f]$.

a) $p(x)$ is the puncture polynomial and records the action of g_* on the radical of a skew-symmetric form on $W(G,g)$. It is related to the way f permutes the punctures of S.

b) $s(x)$ is the symplectic polynomial and records the action of g_* on the non-degenerate symplectic space $W(G,g)/Z$ and contains the dilitation of f as its largest real root.

In Chapter 3 we will wish to compare homology polynomials for distinct mapping classes. If $[f],[g] \in \mathrm{Mod}(S)$, we write their homology polynomials as $h([f],x)$ and $h([g],x)$ respectively.

In what follows we shall restrict our attention to braids. As we see, every train track that carries a braid is non orientable. First we recall the Euler-Poincarè-Hopf formula (see [5], Exposè 5, Section 1.6)

Theorem 2.4.2. *Let S be a genus g surface, possibly punctured, with a singular foliation $\mathcal{F}$ and singular points $x_1,\dots,x_k$. For $1 \le i \le k$ let P_i denote the order of x_i. Then*

$$4 - 4g = \sum_{i=1}^{k}(2 - P_i).$$

Lemma 2.4.3. *Let $\beta \in B_n$ be a pseudo-Anosov braid and let τ be a train track that carries β. Then τ is not orientable.*

Proof. It suffices to show that β must produce at least one odd-ordered singularity. By Theorem 2.4.2,

$$4 = \sum_{i=1}^{k}(2 - P_i)$$

If each $P_i > 1$ the above equality fails. Therefore at least one singularity is odd-ordered and τ is not orientable. □

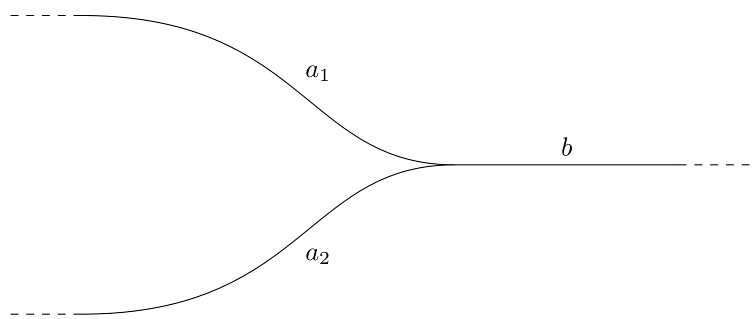

Figure 2.6: The branches a_1 and a_2 form a corner. The branch b does not form a corner with either a_i.

2.4.1 The orientation cover

When τ is not orientable we lift to a special branched double of S determined by τ.

Definition 2.4.4. Let b_1 and b_2 be branches in τ that meet at a switch v. Recall that b_1 and b_2 intersect v along a well defined tangent which allows us to partition the branches meeting v into two *sides.*

If b_1 and b_2 approach v from the same side, we say the angle between them is 0. Otherwise, the angle between them is π. In the latter case we say the branches b_1 and b_2 form a *corner* (see Figure 2.6.

Definition 2.4.5. Let $\tau \subset S$ be a non-orientable train track and fix some basepoint $x \in \tau$. If S is not homotopic to τ, then let S_0 be the surface obtained by puncturing each unpunctured disk component of $S - \tau$. Then S_0 is homotopic to τ and we may identify $\pi_1(S_0, x)$ with $\pi_1(\tau, x)$.

Define $\epsilon : \pi_1(\tau, x) \to \mathbb{Z}/2\mathbb{Z} \simeq \{-1, 1\}$ by

$$\gamma \mapsto (-1)^{\#\text{corners in } \gamma}$$

Then the kernel of ϵ is all loops in τ with an even number of corners. The covering space cooresponding to the kernel of ϵ, after filling in any added punctures, is called the *orientation cover* for τ. It is a two-fold branched cover of S and the fiber of τ is an orientable train track in the cover.

The following is result of Theorem 2.4.1.

Theorem 2.4.6. *Let $\beta \in B_n$ be a pseudo-Anosov mapping class with train track τ. Let $\tilde{D}$ denote the orientation double cover for τ and denote its involution by ι. Then $\iota_* : H_1(\tilde{D}, \mathbb{R}) \to H_1(\tilde{D}, \mathbb{R})$ has two eigenspaces E^+ and E^- corresponding to eigenvalues 1 and -1 respectively. The homology polynomial of β is the characteristic polynomial of $\beta_*|_{E^-}$.*

2.4.2 $W(G, g)$

Given a train track τ constructed from $g : G \to G$, there is a natural surjection $\pi : \tau \to G$ sending real edges to real edges and collapsing all infintesimal polygons to a point.

Let $V(\tau)$ be the $\mathbb{R}$-vector space of real weights on the branches of τ. Define $V(G)$ similarly. Let $W(\tau) \subset V(\tau)$ be the subspace of assignments that satisfy the switch conditions. The surjection $\pi : \tau \to G$ induces a surjection $\pi_* : V(\tau) \to V(G)$.

Definition 2.4.7. We define $W(G, g) = \pi_*(W(\tau))$. It is the subspace of $W(G, g)$ consisting of weight assignments that extend to an assignments of weights on τ that satisfy the switch conditions.

There is a convenient way to determine if an element in $V(G)$ is in $W(G, g)$.

Lemma 2.4.8 ([3], Lemma 2.9)**.** *An element $\eta \in V(G)$ belongs to $W(G, g)$ if and only if for each non-odd vertex the alternating sum of the weights at the incident gates is zero.*

Lemma 2.4.9 ([3], Lemma 2.11)**.** *If τ is orientable, then*

$$\dim W(G, g) = \#(\textit{edges of } G) - \#(\textit{vertices of } G) + 1$$

otherwise

$$\dim W(G, g) = \#(\textit{edges of } G) - \#(\textit{non-odd vertices of } G)$$

2.4.3 The decomposition $h(x) = p(x)s(x)$

Theorem 2.4.10 ([3], Theorem 3.8). *Let $p(x)$ and $s(x)$ be the characteristic polynomials of $g_*|_Z$ and $g_*|_{W(G,g)/Z}$ respectively. The map g_* preserves the direct sum decomposition $W(G,g) \approx Z \oplus (W(G,g)/Z)$ so that $h(x) = p(x)s(x)$. Moreover we have*

1. *The polynomial $p(x)$ is an invariant of the pseudo-Anosov mapping class $[f] \in \mathrm{Mod}(S)$. The restriction $g_*|_Z$ encodes how $[f]$ permutes the puntures whose projections to τ have even numbers of corners. In particular, $g_*|_Z$ is a periodic map, so that all the roots of $p(x)$ are roots of unity and the polynomial $p(x)$ is palindromic or anti-palindromic.*

2. *The polynomial $s(x)$ is an invariant of $[f]$. The skew-symmetric form $\langle \cdot, \cdot \rangle_{W(G,g)}$ naturally induces a symplectic form on $W(G,g)/Z$. The map g_* induces a symplectomorphism of $W(G,g)/Z$. Hence $ks(x)$ is palindromic.*

3. *The homology polynomial $h(x)$ is either palindromic or anti-palindromic.*

2.5 The Burau representation

The Burau representation for braids will play a crucial role in what follows. In particular, there is an equivalent *twisted homological* version of the Burau representation which allows us to represent braids as acting on the first homology group of an infinite cyclic cover

$$D_n^\infty \to D_n$$

with deck group $\langle t \rangle \simeq \mathbb{Z}$. In Chapter 3 we will see that the quotient $D_n^\infty / \langle t^2 \rangle$ is equivalent to the orientation cover for a train track that carries a braid. For more information on this topic see [6] sections 3.1-3.3 and [1].

Definition 2.5.1. Let $\mathbb{Z}[t,1/t]$ denote the ring of Laurent polynomials and let $n \geq 3$. Define $V_1, \dots, V_{n-1} \in \mathrm{GL}_{n-1}(\mathbb{Z}[t,1/t])$ as

$$V_1 = \begin{pmatrix} -t & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & I_{n-3} \end{pmatrix} \quad \text{and} \quad V_{n-1} = \begin{pmatrix} I_{n-3} & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & -t \end{pmatrix}$$

and for $1 < i < n-1$ define

$$V_i = \begin{pmatrix} I_{i-2} & 0 & 0 & 0 & 0 \\ 0 & 1 & t & 0 & 0 \\ 0 & 0 & -t & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & I_{n-2} \end{pmatrix}$$

Then the group homomorphism $\psi_n : B_n \to \mathrm{GL}_{n-1}(\mathbb{Z}[t,1/t])$ denined on the generators as

$$\sigma_i \mapsto V_i$$

for $1 \leq i \leq n-1$ is the *(reduced) Burau representation.*

The above representation is a reduction of the original Burau representation. See [6] for an explaination of the distinction.

Definition 2.5.2. Let $p_1, \dots, p_n \in D$ and let $D_n = D - \{p_1, \dots, p_n\}$. Let $\gamma \in \pi_1(D_n, d)$ where $d \in \partial D$. For each p_i we have $H_1(D - \{p\}; \mathbb{Z}) \simeq \mathbb{Z}$ generated by a loop around p_i oriented counterclockwise. Then γ represents k times the generator of $H_1(D - \{p\}; \mathbb{Z})$. We will call k the *winding number of* γ *around* p_i and we write $w_{p_i}(\gamma) = k$. Then the *total winding number* of γ is the sum

$$w(\gamma) = \sum_{i=1}^{n} w_{p_i}(\gamma).$$

Identify $\mathbb{Z}$ with the multiplicative group generated by t. Consider the map $\varepsilon : \pi_1(D_n, d) \to \mathbb{Z}$ defined by $\varepsilon(\gamma) = t^{w(\gamma)}$. The kernel of this map determines the infinite cyclic covering $D_n^{(\mathrm{inf})} \to D_n$ with deck transformation group identified with $\{t^k\}_{k \in \mathbb{Z}}$. In this way we can view $H_1\left(D_n^{(\mathrm{inf})}; \mathbb{Z}\right)$ as a free module of rank $n-1$ over $\mathbb{Z}[t,1/t]$.

Any braid $\beta \in B_n \simeq \mathrm{Mod}(D_n)$ has a lift $\widetilde{\beta} : D_n^{(\mathrm{inf})} \to D_n^{(\mathrm{inf})}$ which fixes the fiber of d pointwise. This induces a $\mathbb{Z}[t, 1/t]$-module automorphism of $H_1\left(D_n^{(\mathrm{inf})}; \mathbb{Z}\right)$.

Definition 2.5.3. The *twisted homological Burau representation* of B_n is the map

$$\Psi_n : B_n \to \mathrm{Aut}\left(H_1\left(D_n^{(\infty)}; \mathbb{Z}\right)\right)$$

defined by

$$\beta \mapsto \widetilde{\beta}_*$$

Theorem 2.5.4. *The twisted homological Burau representation Ψ_n is equivalent to the (reduced) Burau reprentation ψ_n.*

See [6] section 3.2.5 for a proof.

CHAPTER 3

THE HOMOLOGY POLYNOMIAL AND ITS CONNECTION TO THE BURAU REPRESENTATION

In this chapter we begin by determining exactly when between $h(\beta,x) = |xI - \psi_\beta(-1)|$. In Section 3.1 we summarize a result of Band and Boyland [1] which describes the relationship between the Burau representation and the stretch factor of a pseudo-Anosov braid. In Section 3.2 we expand upon their work to establish a relationship between the Burau representation and the homology polynomial which leads to our first new result. We see that $h(\beta,x) = |xI - \psi_\psi(-1)|$ if and only if all singularities belonging to $\mathcal{F}^u, \mu_u$ and$\mathcal{F}^s, \mu_s$ are odd-ordered if they occur at a puncture and even-ordered if they occur in the interior of D_n. In Section 3.3 we prove Theorem 1.0.2. Finally, in Section 3.3.4 we present an algorithm for constructing β' for an arbitrary pseudo-Anosov β.

Before proceeding we give a simple example demonstrating the construction of β'.

Consider the pseudo-Anosov braid $\beta = \sigma_1\sigma_2\sigma_3^{-1}\sigma_4^{-1}\sigma_3^{-1}$. It has homology polynomial $h(\beta,x) = (1+x)(1-2x-2x^3+x^4)$. However, $|xI - \psi_\beta(-1)| = 1-2x+4x^2-2x^3+x^4$ which has no real roots. Then this cannot be the homology polynomial since $h(\beta,x)$ must have at least one real root (the stretch factor). A train track that carries β is shown in Figure 3.1 (top). The point S represents an order-3 singularity occuring in the interior of D_5. An order-2 singularity occurs at the fourth puncture of D_5 (labeled P). After removing S and filling in P, we get the braid $\beta' = \sigma_1\sigma_2\sigma_1\sigma_3\sigma_4{}^{-1}$ as indicated in Figure 3.1 (right).

Since β' produces no bad singularities we know $h(\beta',x) = \chi(\beta,-1) = 1-2x-2x^3+x^4$. This means $(x+1)h(\beta',x) = h(\beta,x)$ as predicted by Theorem 1.0.2.

3.1 The Burau estimate quotients of the Burau cover

In this section we review a relationship between the stretch factor of a pseudo-Anosov braid and its Burau matrix given by Band and Boyland in [1]. This will give us the foundation we

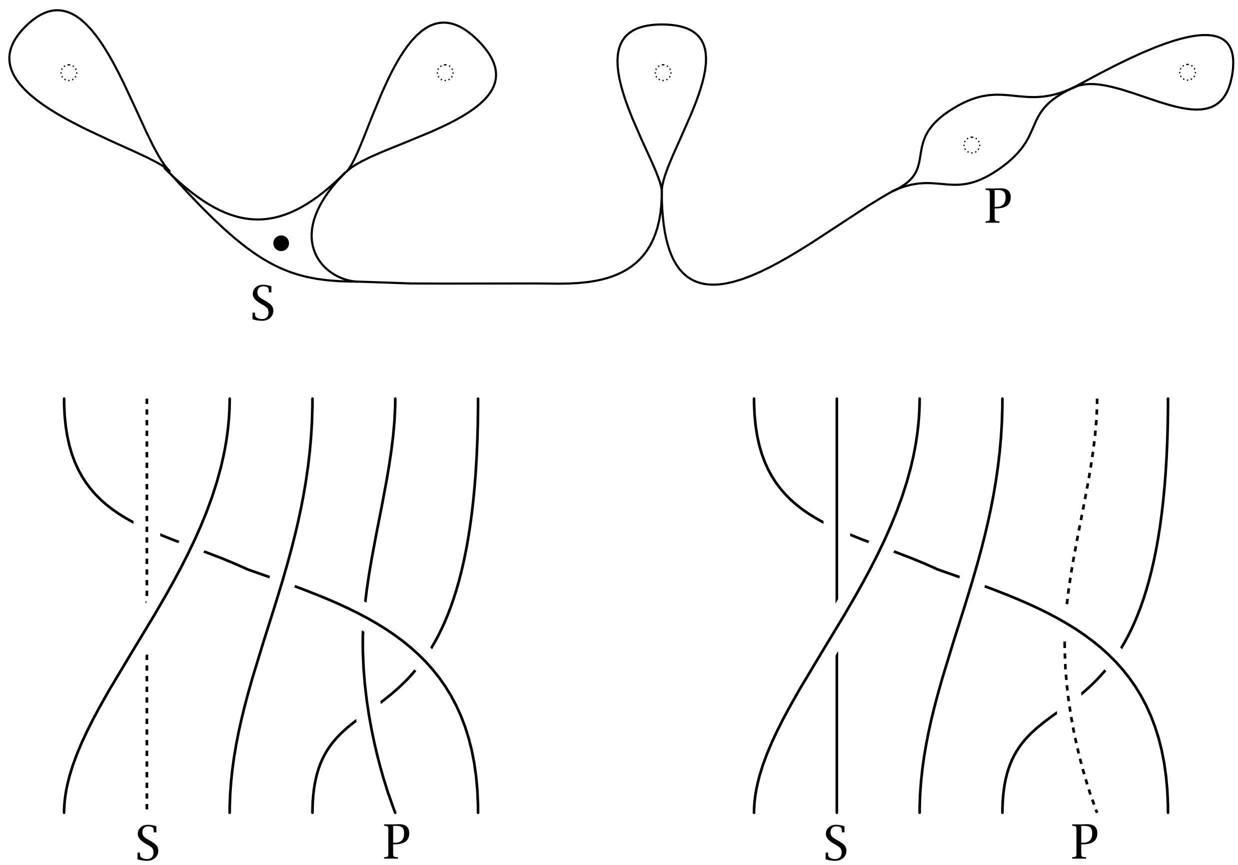

Figure 3.1: β (left), β' (right), τ (above)

need to recover the homology polynomial from the Burau representation.

Recall the *Burau cover* introduced in Section 2.5, the infinite cyclic cover $D_n^{(\infty)}$ associated to the kernel of the map sending each element in $\pi_1(D_n, d)$ to its winding number in $\mathbb{Z}$. Let t denote the generator of the deck group of $D_n^{(\infty)}/D_n$. Then the (reduced) Burau matrix of β is $\psi[\beta, t] \in \mathrm{GL}_{n-1}(\mathbb{Z}[t, 1/t])$.

If β is pseudo-Anosov with dilatation $\lambda > 1$, it is well known that

$$\sup\{\mathrm{sr}(\psi[\beta, \eta] \mid \eta \text{ a root of unity } \} \leq \lambda \tag{3.1}$$

where $\psi[\beta, \eta]$ is the (reduced) Burau matrix for β and with the substitution $t = \eta$. and $\mathrm{sr}(\psi[\beta, \eta]$ is its spectral radius. The left side of Equation (3.1) is called the *Burau estimate* for the stretch factor of a pseudo-Anosov braid at η.

Let $\phi : \pi_1(D_n, d) \to \mathbb{Z}$ denote the map sending elements to their winding numbers, and for any $k \geq 2$ let $\epsilon_k : \mathbb{Z} \to \mathbb{Z}_k$ denote the standard quotient mapping. Clearly $\ker(\epsilon_k \circ \phi) \subset \ker \phi$.

Definition 3.1.1. For each $k \geq 2$ let $p_k : D_n^{(k)} \to D_n$ denote the covering space of associated to the kernel of the map

$$\epsilon_k \circ \phi : \pi_1(D_n, d) \to \mathbb{Z}_k$$

with

$$D_n^{(k)} = D_n^{(\infty)}/t^k$$

Let $q_k : D_n^{(\infty)} \to D_n^{(k)}$ denote the corresponding covering map.

The main result of [1] that we build upon is

Theorem 3.1.2 ([1], Theorem 5.1)**.** *Let β be a pseudo-Anosov braid with stretch factor λ and (reduced) Burau matrix $\psi[\beta, t]$. Then the following are equivalent*

(1) $\mathrm{sr}(\psi[\beta, \eta] = \lambda$ *for some root of unity* η.

(2) $\mathrm{sr}(\psi[\beta, -1]) = \lambda$ *and* -1 *is the only root of unity for which this equality is true.*

(3) *The invariant foliations $\mathcal{F}^u$ and $\mathcal{F}^s$ have odd-ordered singularities at each puncture and all singularities in the interior of D are even-ordered.*

(4) *$D^{(2)}$ is the orientation double-cover for $\mathcal{F}^u$ and $\mathcal{F}^s$.*

We state two additional results of [1] which we will wish to use in later sections.

Lemma 3.1.3 ([1], Lemma 3.2)**.** *Let T be the generator of the deck group for the covering $p^{(k)} : X^{(k)} \to X$ and let $h^{(k)}$ and $h^{(\infty)}$ be the lifts of h to $X^{(k)}$ and $X^{(\infty)}$. The eigenvalues of T_* restricted to $S_{\mathbb{C}}^{(k)}$ are $1, \eta_k, \eta_k^2, \ldots, \eta_k^{k-1}$ where $\eta_k = e^{2\pi i/k}$. Denote by $E_0, \ldots, E_{k-1}$ the corresponding eigenspaces in $S_{\mathbb{C}}^{(k)}$. Then each subspace E_m is $h_*^{(k)}$-invariant, and the action of $h_*^{(k)}$ on E_m is given by the matrix $M(\eta_k^m)$, obtained by substituting η_k^m into the matrix $M(t)$ of $h_*^{(\infty)}$.*

Theorem 3.1.4 ([1], Theorem 3.4). *Let $h : X \to X$ be a homeomorphism of the locally path-connected, semi-locally simply connected topological space X, whose first homology group we assume to be free and of finite rank. Suppose $\rho : H_1(X) \to \mathbb{Z}$ is a homomorphism which satisfies $\rho h_* = \rho$, and let $X^{(\infty)}$ and $X^{(k)} = X^{(\infty)}/T^k$ denote the covering spaces over X corresponding to ρ and $\xi_k \circ \rho$, with covering projection $q^{(k)} : X^{(\infty)} \to X^{(k)}$. Let $h^{(\infty)}$ and $h^{(k)}$ denote lifts of h to these covering spaces. If $M = M(t) \in GL(r, R)$ denotes the matrix of $h_*^{(\infty)} : H_1(X^{(\infty)}) \to H_1(X^{(\infty)})$ as an R-module isomorphism, then the action of $h_*^{(k)}$ on the invariant subspace $S_{\mathbb{C}}^{(k)} = q_*^{(k)}(H_1(X^{(\infty)}, \mathbb{C}))$ is given by the direct sum*

$$h_*^{(k)} = M(1) \oplus M(\eta_k) \oplus \cdots \oplus M(\eta_k^{k-1})$$

where $M(\eta_k^j)$ denotes the complex matrix obtained by substituting $\eta_k^j = e^{2\pi i j/k}$ into M. Furthermore, any eigenvector of $h_*^{(k)}$ not lying in $S^{(k)}$ has eigenvalue which is a root of unity.

3.2 The homology polynomial from the burau representation

We now build upon Theorem 3.1.2 and show that under the same singularity conditions given in Theorem 3.1.2(3) the characteristic polynomial of the Burau matrix of a pseudo-Anosov braid is the homology polynomial.

Fix some $\beta \in B_n$ be a pseudo-Anosov braid with stretch factor λ and homology polynomial $h(x)$. Suppose $g : G \to G$ is an efficient graph map corresponding to β that induces train track $\tau \subset D_n$. Let $\chi(\beta) = \chi(\psi[\beta, -1]) = |xI - \psi[\beta, -1]|$. That is, $\chi(\beta)$ is the characteristic polynomial of $\psi[\beta, -1]$.

Lemma 3.2.1. *The stretch factor of β is the largest real root of $\chi(\beta)$ if and only if $\chi(\beta) = h(x)$.*

Proof. If $\chi(\beta) = h(x)$ then λ is the largest real root of $\chi(\beta)$ since $h(x)$ always contains λ as its largest real root.

Conversely, suppose that λ is the largest real root of $\chi(\beta)$. Then by Theorem 3.1.2 $D_n^{(2)}$ is the orientation cover for τ and by Lemma 3.1.3 $\psi[\beta, -1]$ represents the action of $\beta_*^{(2)}$ on the eigenspace of $H_1(D_n^{(2)};\mathbb{R})$ corresponding to the eigenvalue -1. Then by Theorem 2.4.6 $\chi(\beta)$ is equal to the homology polynomial. □

Using Lemma 3.2.1 and Proposition 2.3.8. we are able to restate Theorem 3.1.2 in terms of the homology polynomial, graph maps, and train tracks.

Proposition 3.2.2. *Suppose $\beta \in B_n$ is pseudo-Anosov with dilatation λ and homology polynomial $h(x)$. Then the following are equivalent:*

(1) *$\chi(\beta)$ is equal to the homology polynomial for β;*

(2) *The spectral radius of $\psi[\beta, e^{2\pi ij/k}] = \lambda$ for some $0 \leq j < k$;*

(3) *The spectral radius of $\psi[\beta, -1] = \lambda$ and -1 is the only root of unity at which this occurs;*

(4) *The vertices of G occuring at the punctures of D_n and in the interior of D_n are odd and even respectively.*

(5) *$D_n^{(2)}$ is the orientation double-cover of τ (after attacking a punctured disk to the boundary of D_n).*

Proof. By Theorem 3.1.2 conditions (2), (3), and (5) are equivalent. By Lemma 3.2.1 and Theorem 3.1.2 (2) is equivalent to (1). Finally, by Proposition 2.3.8 (4) is equivalent to the third statement of Theorem 3.1.2 which implies (4) is equivalent to (3). □

3.3 Proof of Theorem 1.0.2

3.3.1 Overview

The goal of this section is to prove Theorem 1.0.2. Unless stated otherwise all braids are assumed to be pseudo-Anosov.

Definition 3.3.1. We say that β *produces* a k-ordered singularity if the invariant foliations associated to β have a k-ordered singularity.

By Proposition 3.2.2 $h(\beta, x) = \chi(\beta)$ if and only if β does not produce certain types of "bad" singularities.

Definition 3.3.2. A singularity produced by β is *bad* if it is odd-ordered and occurs at an interior point of D_n or even-ordered and occurs at a puncture of D_n.

By Theorem 1.0.1 if β does not produce bad singularities $h(\beta, x) = |xI - \psi_{\ beta}(-1)|$. In the case that β does produce bad singularities Theorem 1.0.2 says that we can algorithmically construct some new braid β' which produces no bad singularities so that $h(\beta, x)$ is recoverable from $h(\beta', x)$.

We construct β' from β using two operations. The first involves puncturing D_n at each odd-ordered singularity in the interior of D_n. This can be thought of as "inserting a strand" into β. The second operation involves filling in any punctures of D_n at which β produces an even-ordered singularity. This can be thought of as "forgetting a strand". If β_{i+1} is constructed from β_i using one of these two operations we see that β_{i+1} produces exactly one less bad singularity than β_i.

3.3.2 Odd-ordered singularities in the interior

We now prove assume that a pseudo-Anosov braid $\beta \in B_n$ produces and fixes an odd-ordered singularity at a point s in the interior of D_n. We show that declaring s a new puncture results in a pseudo-Anosov braid $\beta' \in B_{n+1}$ with the same homology polynomial as β.

Lemma 3.3.3. *Let $\beta \in B_n$ be a pseudo-Anosov braid that produces an odd-ordered singularity at a point s in the interior of D_n. Suppose β fixes s and let $D_{n+1} = D_n - \{s\}$. Define $\beta' \in B_{n+1}$ by*

$$\beta' = \beta|_{D_n - \{s\}}.$$

Then

1. *β' is pseudo-Anosov*

2. *β' produces one less bad singularity than β*

3. *We have*

$$p(\beta', x) = p(\beta, x)$$

$$s(\beta', x) = s(\beta, x)$$

$$h(\beta', x) = h(\beta, x)$$

Proof. We first prove that β' is pseudo-Anosov.

Consider β as an element of $\mathrm{Mod}(D_n)$ where $D_n = (D, \{p_1, \ldots, p_n\})$ is the disk with marked points $p_1, \ldots, p_n$. Let $D_{n+1} = (D, \{p_1, \ldots, p_n, S\})$. Let T denote the transition matrix for a train track $\tau \subset D_n$ that carries β. The singular point s is in $D_n - \tau$ so we may embed a copy of τ in D_{n+1}. Since β acts as the identity on some neighborhood of s β' is also carried by the image of τ embedded in D_{n+1}. Then the transition matrix for β' is also represented by T and the submatrix representing the transition matrix corresponding to the real edges of τ is also Perron-Frobenius. Then by Theorem 2.3.4 β is pseudo-Anosov.

The singularities of β' are the same as those of β except we have replaced an odd-ordered singularity in the interior with an odd-ordered singularity at a puncture. Thus β' produces one less bad singularity than β.

It remains to show (c).

If the singularity at s is order k then s cooresponds to a vertex v_s of G with k gates. Define G' as G with the vertex v_s replaced by a k-gon, with k partial vertices $v_1, \ldots, v_k$ cooresponding to the gates of v_s and k edges $e_1, \ldots, e_k$ with e_i connecting v_i to v_{i+1} for $i < k$ and e_k connecting v_k to v_1. See Figure 3.2. Each of there new vertices is partial by the assumption that v_S is odd.

Since the induced train track for a braid is not orientable (Lemma 2.4.3), we know from Lemma 2.4.9 that

$$\dim W(G, g) = \#(\text{edges of } G) - \#(\text{non-odd vertices of } G)$$

By construction, the addition of k real edges and k partial vertices results in

$$\begin{aligned} \#(\text{edges of } G') &= k + \#(\text{edges of } G) \\ \#(\text{non-odd vertices of } G') &= k + \#(\text{non-odd vertices of } G) \end{aligned}$$

which gives

$$\begin{aligned} \dim W(G', g') &= \#(\text{edges of } G') - \#(\text{non-odd vertices of } G') \\ &= \#(\text{edges of } G) + k - (\#(\text{non-odd vertices of } G) + k) \\ &= \dim W(G, g). \end{aligned}$$

Let $\{\eta_1, \ldots, \eta_m\}$ be a basis for $W(G,g)$ and for any edge G let $\eta_i(e)$ denote the weight assigned to e by η_i. Let x_i denote the gate of v_s associated to the vertex $v_i \in G'$ and let w_i be the sum of the weights assigned to the edges of G belonging to x_i.

For each $j = 1, \ldots, m$ we now construct an element $\eta'_j \in W(G', g')$ and show the $\{\eta'_j\}$ form a basis for $W(G', g)$.

First, for every edge e in G' that comes from an edge of G, $\widehat{\eta}_j(e) = \eta_j(e)$. For each $i = 1, \ldots, k$, add $\pm w_i/2$ to $e_1, \ldots, e_k$ as indicated in Figure 3.2. We can do this consistently with $\widehat{\eta}_i \in W(G', g')$ because k is odd. Since $\dim W(G', g') = \dim W(G, g)$, this forms a basis for $W(G', g')$.

By construction, g' fixes the edges $e_1, \ldots, e_k$ and the following diagram commutes.

$$\begin{tikzcd} W(G,g) \arrow[r, "\eta_i \mapsto \widehat{\eta}_i"] \arrow[d, "g_*"] & W(G',g') \arrow[d, "g'_*"] \\ W(G,g) \arrow[r, "\eta_i \mapsto \widehat{\eta}_i"] & W(G',g') \end{tikzcd}$$

It follows that the characteristic polynomial for $g'_*|_{W(G',g')}$ is equal to the characteristic polynomial for $g_*|_{W(G,g)}$ and thus the homology polynomials for β and β' are equal.

We now show that $p(\beta', x) = p(\beta, x)$.

By construction β' produces the same set of singularities as β. The only change is that an odd-ordered singularity that previously occurred in the interior of D_n now occurs at a puncture of D_{n+1}. In particular, β' produced the same collection of even-ordered singularities occurring at punctures and β' permutes them in the same way. By Theorem 2.4.1 this implies $p(\beta', x) = p(\beta, x)$.

From $h(\beta', x) = h(\beta, x)$ and $p(\beta', x) = p(\beta, x)$ it immedietly follows that $s(\beta', x) = s(\beta, x)$.

□

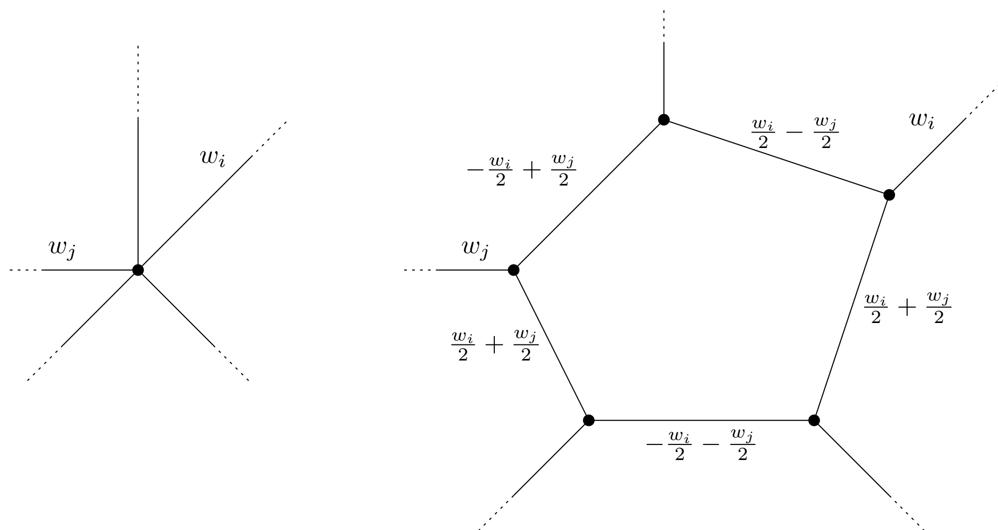

Figure 3.2: Left: η_i, Right: $\widehat{\eta_i}$
(edges without labels have weight 0)

3.3.3 Even-ordered singularities occurring at punctures

We now show that if $\beta \in B_n$ fixes a puncture p and produces an even-ordered singularity that occurs at p then after filling in p the resulting braid $\beta' \in B_{n-1}$ is pseudo-Anosov and produces one less bad singularity than β.

Lemma 3.3.4. *Let $\beta \in B_n$ be a pseudo-Anosov braid that produces an even-ordered singularity at a puncture p and that p is fixed by β. Let D_{n-1} be the space obtained from D_n by filling in p. Define $\beta' \in B_{n-1}$ as the image of β after passing to D_{n-1}. Then β' is pseudo-Anosov.*

1. *If the singularity at* p *is order* 2 *then*

$$p(\beta', x) = p(\beta, x)/(x+1)$$
$$s(\beta', x) = s(\beta, x)$$
$$h(\beta', x) = h(\beta, x)/(x+1)$$

2. *Otherwise*

$$p(\beta', x) = p(\beta, x)/(x+1)$$
$$s(\beta', x) = s(\beta, x) \cdot (x+1)$$
$$h(\beta', x) = h(\beta, x)$$

Proof. Let k be the order of the singularity occurring at p. Let $g : G \to G$ be an efficient graph map that carries β with induced train track τ. Since $p \in D_n - \tau$ we can embed τ in D_{n-1} as defined above.

We first prove that β' is pseudo-Anosov. We separate the special case of $k = 2$ since the resulting braid has a slightly modified homology polynomial.

If $k = 2$ then τ' is not a train track since one of the components of $D_{n-1} - \tau'$ is a bigon as depicted in Figure 3.3. Let $\{e_1, \ldots, e_m\}$ be the edges of τ. We construct τ' by embedding a copy of τ and pushing e_j onto e_i across the bigon containing the filled in point p. We now define the map $g'_* : \tau' \to \tau'$ and argue it carries β' efficiently. For each edge e_k with $k \neq i, i+1$ define $g'_*(e_k)$ as the edge path $g_*(e_k)$ in τ with each occurrence of e_i or e_{i+1} replaced with e_i^*. Similarly define $g'_*(e_i^*)$ as the edgepath $g(e_i)$ with all occurrences of e_i or e_{i+1} replaced by e_i^*. Since β is carried by τ we can see that β' is carried by τ' since we can push β' along the same bigon we used when constructing τ'. Finally, the real transition matrix for $g'_* : \tau' \to \tau'$ is equal to the real transition matrix for $g_* : \tau \to \tau$ since e_i, e_{i+1}, and e_i^* are infintesimal edges of τ. Therefore it is Perron-Frobenius and β' is pseudo-Anosov.

Recall that $h(\beta', x)$ is the characteristic polynomial of the map $g_* : W(G, g) \to W(G, g)$. Recall further that $W(G, g) \approx Z \oplus W(G, g)/Z$ so that $p(x)$ and $s(x)$ represent g_* restricted to Z and $W(G,g)/Z$ respectively (see Theorem 2.4.1). Furthermore we know the generators of Z are the simple loops around punctures with an even numbers of corners when projected onto τ. The infintesimal edges e_i and e_j represented one of these generators since the loop has two corners. Since we assume all bad singularities are fixed by β the $g_*|Z = (x+1)^r$ where r is the number of even-ordered singularities occurring at punctures.

When p is filled we end up with the space Z' with $r-1$ generators, each fixed by β' which yealds $p(\beta',x) = p(\beta,x)/(x+1)$. Clearly $W(G',g')/Z' \approx W(G,g)/Z$ since G' is obtained from G by crushing a generator of Z to a point. Therefore $s(\beta',x) = s(\beta,x)$. Therefore

$$h(\beta',x) = p(\beta',x)s(\beta',x) = (p(\beta,x)/(x+1))s(\beta,x) = h(\beta,x)/(x+1).$$

Now, if k is even and $k > 2$ then $D_{n-1} - \tau$ is still a train track so $g'_* : \tau' \to \tau'$ carries β' with no modification from $g_* : \tau \to \tau$. Again, $p(\beta',x) = p(\beta,x)/(x+1)$ by the same reasoning as above. Furthermore, the transition matrix for β' acting on τ is the same as the transition matrix for β which means β' is pseudo-Anosov (again see Theorem 2.3.4.)

Suppose $W(G,g) \approx Z \oplus W(G,g)/Z$ is of dimension M. Let η_1 be the generator of Z associated to p. Let $\{\eta_1,\dots,\eta_r,\gamma_1,\dots,\gamma^{M-r}\}$ be a basis for $W(G,g)$ so that the $\{\eta_i\}$ generate Z. Since $g'_* : W(G',g') \to W(G',g')$ is unmodified from $g_* : W(G,g) \to W(G,g)$ $\{\eta_1,\dots,\eta_r,\gamma_1,\dots,\gamma_{M-r}\}$ is still a basis for $W(G',g')$. The generator η_1 is not longer an element of Z'. This leads to $p(\beta',x) = p(\beta,x)/(x+1)$. The element represented by η is now a generator of $W(G',g')/Z'$ fixed by β'. It follows that $s(\beta',x) = (x+1)s(\beta,x)$ and

$$h(\beta',x) = p(\beta',x)s(\beta',x) = (p(\beta,x)/(x+1))\,((x+1)\cdot s(\beta,x)) = h(\beta,x)$$

as desired. $\square$

3.3.4 Conclusion and proof

The following result from [3] gives a corollary that will be used in the proof of Theorem 1.0.2.

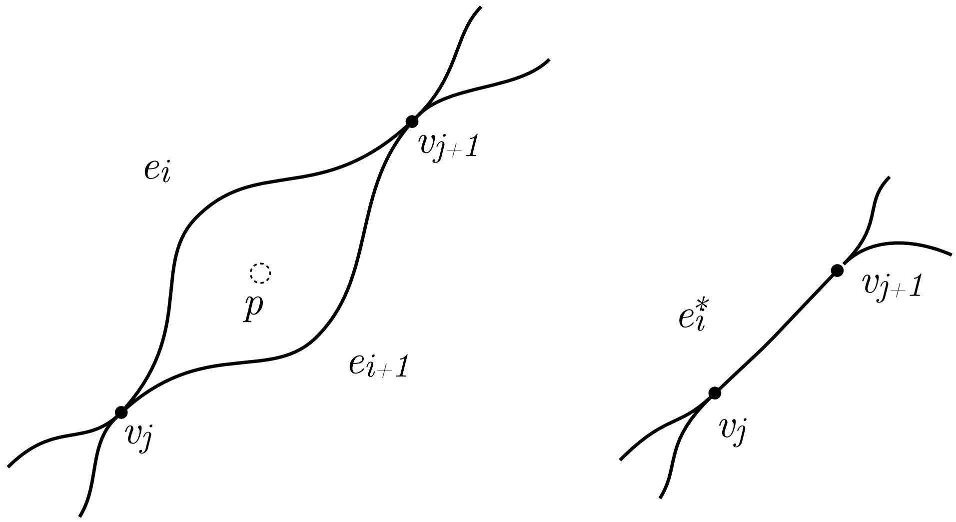

Figure 3.3: τ (left) and τ' (right) after filling in an order-2 singularity

Lemma 3.3.5 ([3], Corollary 4.5)**.** *Let $m > 0$. If $g : G \to G$ is a graph map representing a pseudo-Anosov mapping class β then $g^m : G \to G$ represents β^m. Suppose that the homology polynomial for β $h_\beta(x) = s_\beta(x)p_\beta(x)$ and that*

$$s_\beta(x) = \prod_i (x - z_i) \quad \textit{and} \quad p_\beta(x) = \prod_j (x - w_j), \quad z_i, w_j \in \mathbb{C}.$$

Then $h_{\beta^m}(x) = s_{\beta^m}(x)p_{\beta^m}(x)$ with

$$s_{\beta^m}(x) = \prod_i (x - z_i^m)$$

$$p_{\beta^m}(x) = \prod_j \left(x - w_j^m\right)$$

$$h_{\beta^m}(x) = \prod_i (x - z_i^m) \prod_j \left(x - w_j^m\right)$$

See [3], Corollary 4.5 for a proof.

We now prove Theorem 1.0.2

Proof of Theorem 1.0.2. Suppose β_0 produces q odd-ordered singularities at the interior points $s_1, \ldots, s_q$ and r even-ordered singularities at punctures $p_1, \ldots, p_r$ and that these $q + r$ are the only bad singularities produces by β. Suppose $\epsilon \geq 0$ of the r even-ordered singularities at the punctures of D_n are order-2. Choose k so that $\beta = \beta_0^k$ fixes all $q + r$ bad singularities.

Let $\widehat{\beta}$ be the braid obtained by applying Lemma 3.3.3 at each odd-ordered singularity in the interior of D_n. Then $h(\widehat{\beta},x) = h(\beta,x)$ and $wh\beta$ produces no odd-ordered singularities in the interior of D_n. Now let β' be the braid obtained from $\widehat{\beta}$ after applying Lemma 3.3.4 at each even-ordered singularity at a puncture. Then β' produces no bad singularities and

$$\chi(\beta') = h(\beta',x) = h(\widehat{\beta},x)/(x+1)^{\epsilon} = h(\beta,x)/(x+1)^{\epsilon}$$

By Lemma 3.3.5 if $h(\beta_0,x) = \prod_i (x - z_i)$ then

$$\begin{aligned} \chi(\beta') &= h(\beta',x) \\ &= \frac{h(\beta,x)}{(x+1)^{\epsilon}} &&= \frac{\prod_i (x - z_i^k)}{(x+1)^{\epsilon}} \end{aligned}$$

as desired. □

3.4 An algorithm for constructing β' from β

What follows is an algorithm for constructing β' for an arbitrary pseudo-Anosov β_0.

Let β_0 be pseudo-Anosov. If β_0 produces no bad singularities then $\chi(\beta_0) = h(\beta_0,x)$. If β_0 produces at least one bad singularity, we apply the following steps to obtain β'.

(1) Choose $k \geq 1$ so that $\beta = \beta_0^k$ fixes all bad singularities.

(2) If β produces an odd-ordered singularity at an interior point s of D_n puncture D_n at s and define $\widehat{\beta}_0$ as the image of β in $D_{n+1} = D_n - \{s\}$. Repeat until every interior point with an odd-ordered singularity is punctured. Let $\widehat{\beta}$ be the resulting braid in D_{n+q} where q is the number of interior points punctured.

(3) If $\widehat{\beta}$ produces an even-ordered singularity at a puncture P of D_{n+q} then fill in P and let $\widehat{\beta}'$ be the resulting braid in D_{n+q-1}. Repeat until every puncture with an even-ordered singularity is filled in. Let β' be the resulting braid.

By Theorem 1.0.2

$$\chi(\beta') = h(\beta',x)$$

and $h(\beta',x)$ is related to $h(\beta,x)$ as described in Theorem 1.0.2.

CHAPTER 4

EXAMPLES AND APPLICATIONS

In this chapter we give examples and applications of Theorem 1.0.2. This result is only useful if we can use it to avoid finding train tracks and singularity types. Otherwise we may as well compute $h(x)$ from definition. With this in mind we first introduce a large family of braids for which Theorem 1.0.2 can be used to compute $h(x)$ from the braid word alone. As a simple example we first show that $h(\beta, x) = |xI - \psi_\beta(-1)|$ for any pseudo-Anosov $\beta \in B_3$.

When applying the Nielsen-Thurston Classification to a mapping class on a surface with boundary we first attack punctured disks to each boundary component. In particular, we attach a punctured disk to the boundary component of D_n to consider $\beta \in B_n$ as an element of $\mathrm{Mod}(S_{n+1})$ where S_{n+1} is the sphere with $n+1$ points removed.

Proposition 4.0.1. *If $\beta \in B_3$ is a pseudo-Anosov braid, then*

$$h(\beta, x) = |xI - \psi_\beta(-1)]|$$

Proof. Suppose $\beta \in B_3$ is pseudo-Anosov carried by a singular foliation $\mathcal{F}$ on S_4. Let $x_1, \dots, x_k$ be the singular points (possibly occurring at a puncture) of $\mathcal{F}$. Let $P_i \geq 1$ denote the order of x_i. Recall that $P_i \geq 3$ if x_i is in the interior and $P_i \geq 1$ if x_i occurs at a puncture.

According to Theorem 2.4.2,

$$4 = \sum_{i=1}^{k} (2 - P_i)$$

For convenience assume $x_1, x_2, x, 3, x_4$ occur at the punctures of S_4. Then $P_i \geq 1$ for $1 \leq i \leq 4$ and $P_i \geq 3$ for $i > 4$. Therefore

$$\begin{aligned} 4 &= \sum_{i=1}^{4} (2 - P_i) + \sum_{i=5}^{k} (2 - P_i) \\ &\geq 4 + \sum_{i=5}^{k} (-1) \end{aligned}$$

Therefore there can be no singularities in the interior of D_3. The above application of the Euler-Poincarè-Hopf formula also implies that $P_i = 1$ for $i = 1, 2, 3, 4$.

Let $\beta \in B_3$ be pseudo-Anosov. By the above the foliations associated to β have exactly four singularities. Each is odd-ordered and each occurs at a puncture. By Theorem 1.0.1

$$h(\beta, x) = |xI - \psi_\beta(-1)|.$$

□

4.1 Application of Theorem 1.0.2 to a large family of braids

The family of braids presented in this section and the methods for studying them are an extension of the methods of [7].

We first define two "building blocks" for constructing the elements.

Definition 4.1.1. For any integers $m, p \geq 1$ we define two elements of B_m

$$\beta_{(m,p)} = (\sigma_1 \sigma_2 \ldots \sigma_{m-1})^p \quad \text{and} \quad \beta_{(-m,p)} = (\sigma_1^{-1} \sigma_2^{-1} \ldots \sigma_{m-1}^{-1})^p$$

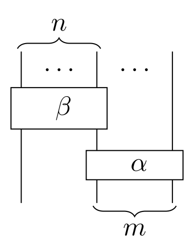

Figure 4.1: $\beta \star \alpha$

An illustration of $\beta_{(3,2)}$ is given in Figure 4.3 (left). Elements of $\mathcal{B}$ are constructed from the above with a modified form of concatination.

Definition 4.1.2. Let $\beta \in B_n$ and $\alpha \in B_m$. Let $\beta' \in B_{n+m-1}$ be the image of β under the usual inclusion map $\sigma_i \mapsto \sigma_i$ and let α' be the shifted image of α under the map $\sigma_i \mapsto \sigma_{n+i-1}$. Then $\beta \star \alpha = \beta' \alpha' \in B_{n+m-1}$. See Figure 4.1.

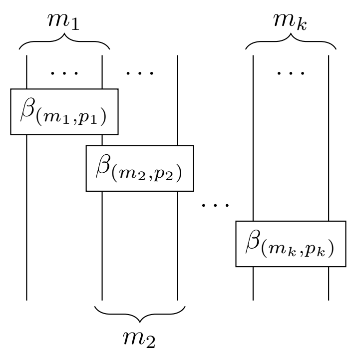

Figure 4.2: $\beta_{(m_1,p_1),\ldots,(m_k,p_k)}$

Definition 4.1.3. A sequence of ordered pairs $\{(m_i,p_i)\}_{i=1}^k$ is a *pA-sequence* if

1. $|m_i|, p_i > 0$ for all i

2. $|m_i|$ and p_i are relatively prime for all i

3. The sequence $m_1,\ldots,m_k$ is alternating.

We define

$$\mathcal{B} = \{\beta_{\{(m_i,p_i)\}_{i=1}^k} \ \mid\ \{(m_i,p_i)\}_{i=1}^k \text{ is a pA-sequence}\}.$$

where

$$\beta_{\{(m_i,p_i)\}_{i=1}^k} = \beta_{(m_1,p_1)} \star \beta_{(m_2,p_2)} \star \cdots \star \beta_{(m_k,p_k)}.$$

An illustration of $\beta_{(m,p)}$ is given in Figure 4.3 (left).

Definition 4.1.4. Define

$$\gamma_{(m,p)} = \begin{cases} \beta_{(m,p)} & \text{if } m \text{ even} \\ \left(\sigma_2 \cdot \beta_{(m+1,1)}\right)^p & \text{if } m \text{ is odd} \end{cases}$$

and

$$\gamma_{\{(m_i,p_i)\}_{i=1}^k} = \gamma_{(m_1,p_1)} \star \gamma_{(m_2,p_2)} \star \cdots \star \gamma_{(m_k,p_k)}$$

If $\beta \in \mathcal{B}$ then $\beta = \beta_{\{(m_i,p_i)\}_{i=1}^k}$ for some pA-sequence $(m_i,p_i)_{i=1}^k$. In this case we define

$$\begin{aligned}\gamma(\beta) &= \gamma(\beta_{\{(m_i,p_i)\}}) \\ &= \gamma_{\{(m_i,p_i)\}})\end{aligned}$$

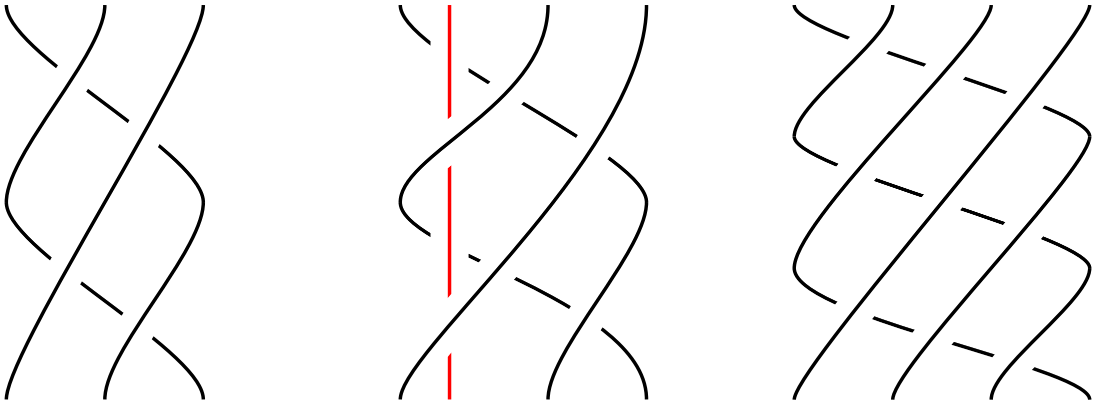

Figure 4.3: Comparison of $\beta_{m,p}$ and $\gamma_{m,p}$

Note that $\gamma_{(m,p)}$ is always a braid on an even number of strands. A comparison of $\beta_{(m,p)}$ and $\gamma_{(m,p)}$ is given in Figure 4.3.

We will show the following:

Theorem 4.1.5. *Let $\beta \in \mathcal{B}$. Then*

1. *β and $\gamma(\beta)$ are pseudo-Anosov*

2. *$h(\beta, x) = h(\gamma(\beta), x)$*

3. *$h(\gamma(\beta)), x) = |xI - \psi_{\gamma(\beta)}(-1)|$*

To prove these braids are pseudo-Anosov we use *combined tree maps* [7].

Definition 4.1.6. For any $m \geq 1$ let $\mathcal{T}_m^+$ and $\mathcal{T}_m^-$ be trees of star type shown in Figure 4.4. Each has m valence-1 vertices and 1 valence-m vertex. See Figure 4.4. $\mathcal{T}_0$ is the trivial tree consisting of exactly one vertex.

Given a sequence

$$S = \{(m_1, p_1), \dots, (m_k, p_k)\}$$

Define

$$T_S = \left(\bigcup_{i=1}^{k} \mathcal{T}_{m_i}^{(-1)^{i+1}} \right) / (r_i \sim l_{i+1})$$

Label the edges of $\mathcal{T}_{m_1}^{+}$ $e_1, \ldots, e_{m_1}$ in the order indicated in Figure 4.4. Then the edges of $\mathcal{T}_{m_2}^{-}$ are $e_{m_1+1}, \ldots, e_{m_1+m_2}$ and so forth. For $1 \le j \le k$, define $g_j : \mathcal{T}_S \to \mathcal{T}_S$ by

$$e_i \mapsto e_i \quad i < m_1 + \cdots + m_{j-1} - 1$$

$$e_i \mapsto \overline{e}$$

Then $g_S : \mathcal{T}_S \to \mathcal{T}_S$ is given by

$$g_S = g_k^{p_k} \circ \cdots \circ g_1^{p_1}.$$

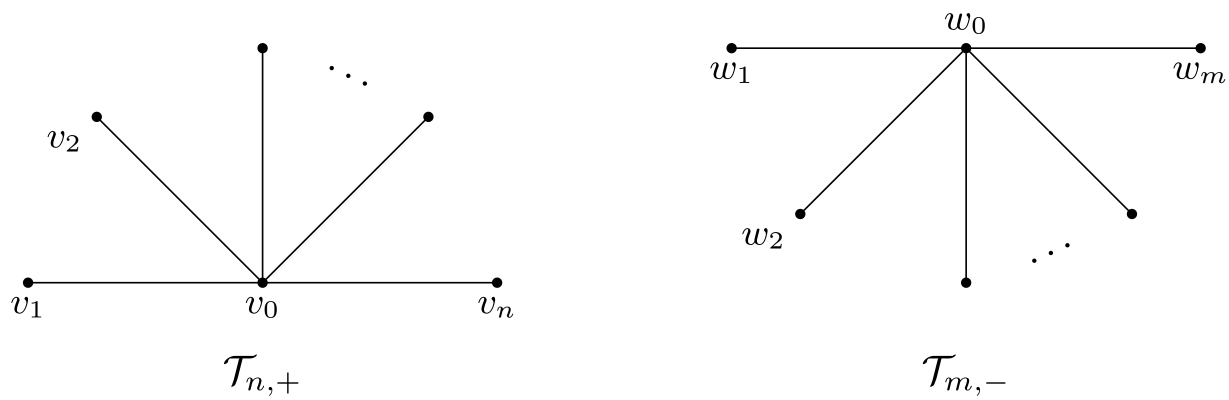

Figure 4.4: Trees of star type

Example 4.1.7. Let $S = \{(3,2),(4,1)\}$. The combined tree map $g_S : \mathcal{T}_S \to \mathcal{T}_S$ is shown in Figure 4.5.

The following is a consequence of [7] (section 3):

Proposition 4.1.8. *Let $g_S : \mathcal{T}_S \to \mathcal{T}_S$ be a combined tree map for a pA-sequence S and let M_S be the transition matrix of g_S. Then M is Perron-Frobenius.*

Given $\mathcal{T}_S$ as above, we can produce a train track in the punctured disk:

1. Replace each valence 1 vertex with a 1-gon bounding a punctured disk.

2. Each valence-2 vertex shared by two trees is replaced by a 1-gon bounding a punctured disk as indicated in Figure 4.6.

3. The valence-m vertex in each $\mathcal{T}_m^{\pm}$ is replaced with an m-gon bounding a disk.

Finally, we extend g_S to $\phi_S : \tau_S \to \tau_S$ by permuting the infintensimal edges to match the rotation of $\mathcal{T}_m^{\pm}$. The following is a result of [7].

Proposition 4.1.9. *Let S be a pA-sequence and let $g_S : \mathcal{T}_S \to \mathcal{T}_S$ be the induced combined tree map.*

1. *The transition matrix M_S is Perron-Frobenius*
2. *The induced train track τ_S carries β_S.*
3. *M_S is the transition matrix for the real edges of τ_S.*

See [7] Section 4. The assumption that each pair (m, p) are relatively prime is needed for M_S to be Perron-Frobenius.

Proof of Theorem 4.1.5. By Proposition 4.1.9 the transition matrix for the real edges of β_S is Perron-Frobenius. Then by Theorem 2.3.4, β_S is pseudo-Anosov.

If $\beta_S \in B_n$ then $\gamma_S \in B_{n+n'}$ where n' is the number of pairs (m_i, p_i) in S with m_i odd. However, by construction, γ_S is carried by a copy of τ_S embedded in $D_{n+n'}$ with the same train track map representing γ_S. Therefore the transition matrix for the real edges of a train track invariant under γ_S is Perron-Frobenius and γ_S is pseudo-Anosov.

Using combined tree maps we can predict the singularity types produced by β_S. Specifically, if $S = \{(m_i, p_i)\}_{i=1}^k$, β_S produces an order-1 singularity at each puncture and an order-m_i singularity in the interior of S_4 for each $m_i \geq 3$. The braid γ_S is β_S after applying Theorem 1.0.2 at each odd-ordered singularity in the interior. Then by Theorem 1.0.2 we have

$$h(\beta_S, x) = h(\gamma_S, x) = |xI - \psi_{\gamma_S}(-1)|$$

as desired. □

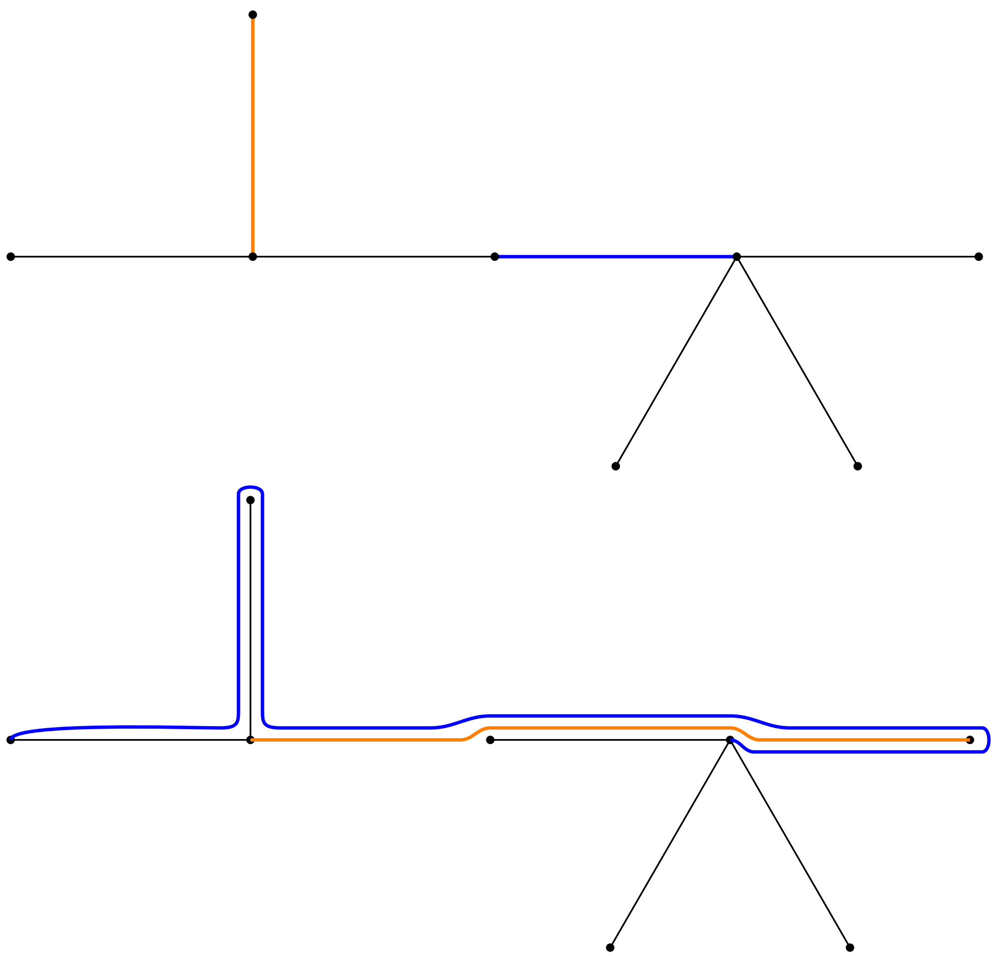

Figure 4.5: The tree map $g_S : \mathcal{T}_S \to \mathcal{T}_S$ for $S = \{(3,2),(4,1))\}$.

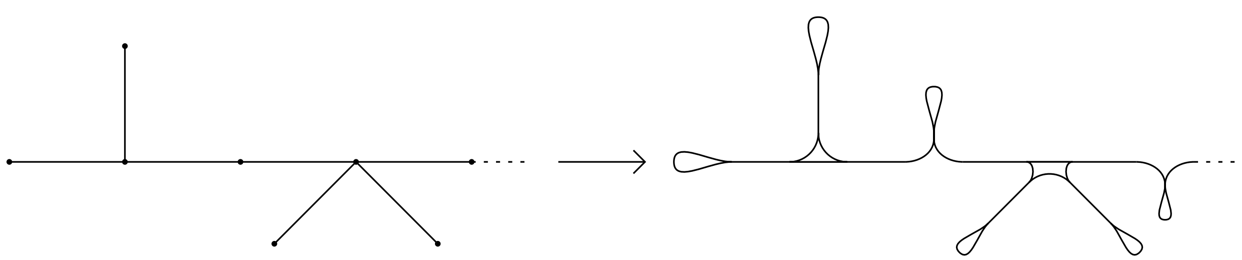

Figure 4.6: Constructing τ from $\mathcal{T}_S$

Proposition 4.1.10. *Let $S = \{(m_i, p_i)\}_{i=1}^k$ be a pA-sequence. Then*

$$\beta = \beta_S = \beta_{(m_1,p_1)} \star \cdots \star \beta_{(m_k,p_k)}$$

is pseudo-Anosov.

Proposition 4.1.11. *Let $\{(m_i, p_i)\}_{i=1}^k$ be a pA-sequence. Let*

$$\beta_\star = \beta_{\{(m_i,p_i)\}_{i=1}^k}$$

and

$$\gamma_\star = \gamma_{\{(m_i,p_i)\}_{i=1}^k}$$

Then

1. *$\gamma_\star$ is pseudo-Anosov*

2. *The characteristic polynomial of the Burau matrix for $\gamma_\star$ is the homology polynomial:*

$$h(\gamma_\star, x) = |xI - \psi[\gamma_\star, -1]|$$

3. *The homology polynomials for $\beta_\star$ and $\gamma_\star$ are equal*

$$h(\beta_\star, x) = h(\gamma_\star, x)$$

4.2 An example comparing the computation of $h(x)$ from definition and computing $h(x)$ using Theorem 1.0.2

In this section we will compute the homology polynomial for a pseudo-Anosov braid from definition and then using Theorem 1.0.2 The efficient graphs used in the following examples were determined with the help of [10].

4.2.1 An odd ordered singularity in the interior of the disk

Let $\beta = \sigma_1\sigma_2\sigma_3^{-1}$ denote the braid in B_4 represented as a mapping class in the 4-punctured sphere. We will find an efficient graph and graph map that carries β and construct the corresponding train track. After this we will find the homology polynomial for βusing both the definition given in Section 2.4 and by using Theorem 1.0.2.

The graph map, transition matrix, and train track

Let G be the graph as depicted in Figure 4.7. The edges and vertices are labeled and will be refered to throughout this example. Also depicted is the graph maph $g : G \to G$. An orientation is given for convenience.

Recall that if G has k edges, the transition matrix of $g : G \to G$ is the $k \times k$ matrix with ij-th entry equal to the number of times the edgepath $g(e_j)$ passes through e_j in either direction. The transition matrix for the map constructed above is

$$T = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \\ e_8 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

To construct a train track τ that carries β we need to determine the gates at each vertex. Consider edges e_3 and e_4 emanating from v_4. As is shown the edgepaths $g(e_3)$ and $g(e_4)$ have the same initial segment. Therefore they belong to the same gate. The peripheral edges are permuted, and the two ends are never sent to the same initial segment, so at v_4 we have three distinct gates. See Figure 4.8 for a visual representation of τ. The enlarged dashed circles represent the gates and infintesimal edges that replace the vertices of G.

The vertex v_2 is odd. To see this, let U be a neighborhood of v_2 so that $U \cap G$ consists of three open-ended arcs emanating from v_2. Since v_2 is fixed by g and the arcs are permuted, we see that for any $p > 0$ the edgepaths $g^p(e_1)$, $g^p(e_2)$, and $g^p(e_3)$ never coincide and belong to distinct gates.

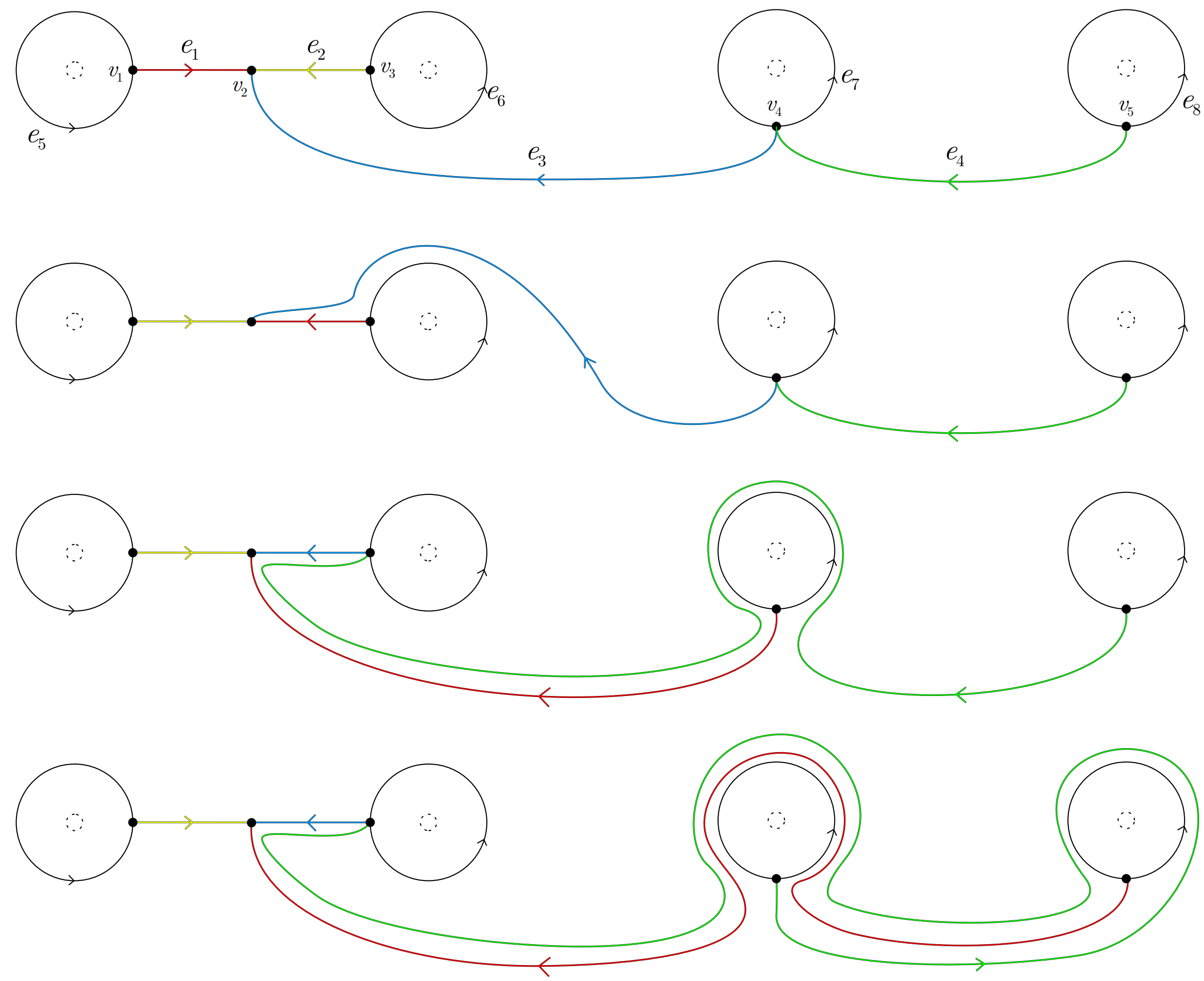

Figure 4.7: From top to bottom: G, $\sigma_1(G)$, $\sigma_1\sigma_2(G)$, and $\sigma_1\sigma_2\sigma_3{}^{-1}(G)$

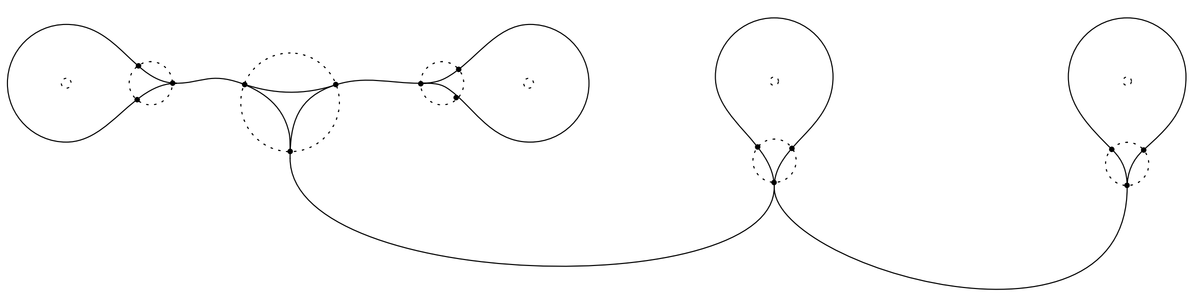

Figure 4.8: The train track induced by the graph map shown in Figure 4.7.

The homology polynomial from $W(G, g)$

As seen above v_2 is an odd vertex and all others are non-odd.

For $\boldsymbol{w} = (w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8) \in W(G, g)$, w_i denotes the weight assigned to e_i,

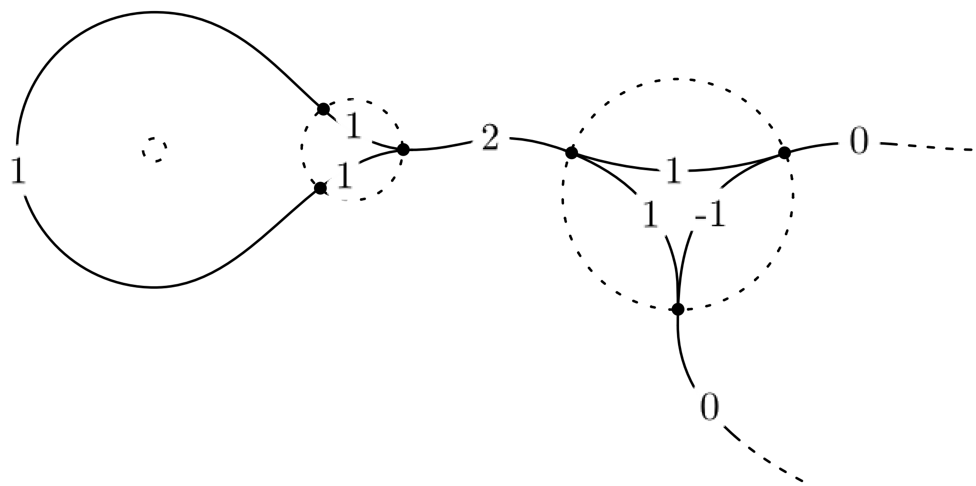

Figure 4.9: The basis element η_1. All other edges are assigned a weight of 0.

$i = 1, \ldots, 8$. Let $\mathcal{B} = \{\eta_1, \eta_2, \eta_3, \eta_4\}$ where

$$\eta_1 = (2, 0, 0, 0, 1, 0, 0, 0),$$

$$\eta_2 = (0, 2, 0, 0, 0, 1, 0, 0),$$

$$\eta_3 = (0, 0, 2, 0, 0, 0, 1, 0),$$

$$\eta_4 = (0, 0, -2, 2, 0, 0, 0, 1)$$

Recall that an element $\boldsymbol{w} \in V(G)$ belongs to $W(G, g)$ if at each non-odd vertex v_i the alternating sum of the weights at incident gates is zero (see Lemma 2.4.8). Thus $\mathcal{B} \subset W(G, g)$. According to Lemma 2.4.9, the dimension of $W(G, g)$ is

$$\#(\text{Edges}) - \#(\text{Non-odd vertices}) = 8 - 4 = 4$$

which implies $\mathcal{B}$ is a basis for $W(G, g)$. See Figure 4.9 for a depiction of η_1.

Let T be the transition matrix given above and let P denote the matrix for the map $\mathbb{R}^8 \to \mathbb{R}^4$ which projects onto the first four coordinates. Let

$$Q = \begin{pmatrix} \eta_1^T & \eta_2^T & \eta_3^T & \eta_4^T \end{pmatrix}$$

denote the 8×4 matrix with column vectors equal to the elements of $\mathcal{B}$.

Then the homology polynomial for β is

$$h(x) = |xI - PTQ| = x^4 - 2x^3 - 2x + 1$$

The homology polynomial from the Burau representation

The image of β under the Burau representation is

$$\Psi_\beta(t) = \begin{pmatrix} 0 & t & 0 \\ 1-t & 1-t & 1 \\ -\frac{1}{t} & -\frac{1}{t} & -\frac{1}{t} \end{pmatrix}$$

By Proposition 3.2.2 if

$$\chi(\Psi_\beta(\nu)) = |xI - \Psi_\beta(\nu)|$$

is equal to the homology polynomial then $\nu = -1$. However this will not hold for β because of the bad singularity occuring at v_2 above. In fact $\chi(\Psi_\beta(-1)) = (1-x)^3$.

Following the strategy outlined in Section 3.3.2 we will "add a strand" by declaring the bad singularity a new puncture. The resulting braid is

$$\overline{\beta} = \sigma_2\sigma_1\sigma_2\sigma_3{\sigma_4}^{-1} \in B_5$$

which is shown in Figure 4.10. We now have

$$\Psi_{\overline{\beta}}(-1) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

and

$$|xI - \Psi_{\overline{\beta}}(-1)| = x^4 - 2x^3 - 2x + 1 = h(x)$$

as expected.

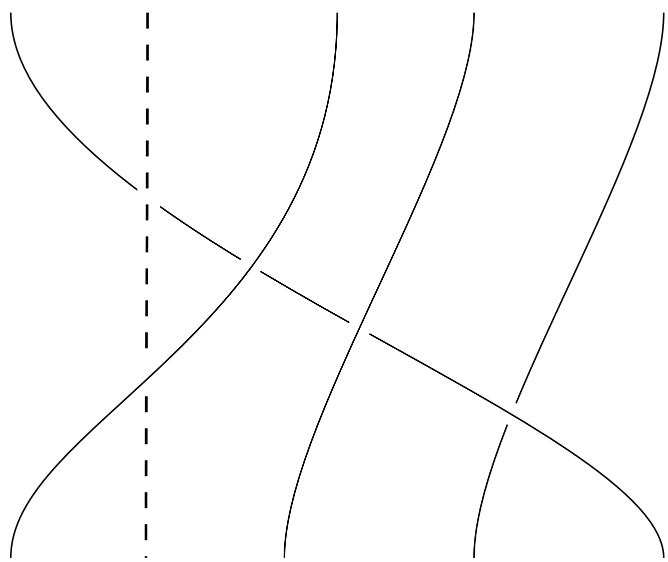

Figure 4.10: The braid $\sigma_1\sigma_2\sigma_3{}^{-1}$. The dashed line represents the additional strand after declaring the singularity a new puncture resulting in the 5-braid $\sigma_1\sigma_2\sigma_1\sigma_3\sigma_4{}^{-1}$

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