ABSTRACT HOMOMORPHISMS OF ALGEBRAIC GROUPS: RIGIDITY AND GROUP ACTIONS

By

Joshua Ruiter

A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

Mathematics—Doctor of Philosophy

2022

PUBLIC ABSTRACT

ABSTRACT HOMOMORPHISMS OF ALGEBRAIC GROUPS: RIGIDITY AND GROUP ACTIONS

By

Joshua Ruiter

By way of analogy, an algebraic group is like a houseboat. A houseboat is both house and boat, so construction involves solving engineering problems common to houses (e.g. plumbing) and problems common to boats (e.g. floating). Additional complications emerge from the interplay of the two structures – for instance, the presence of water may require more safeguards in the electrical system.

An algebraic group is a mathematical object with two structures: it is both an algebraic variety and a group. To study them, we use tools from from algebraic geometry and from group theory, as well as tools related to the interplay between the two structures. Mathematicians use *group homomorphisms* to describe relationships between different groups. To study algebraic groups, we usually use *algebraic group homomorphisms*, group homomorphisms that are compatible with the algebraic structure. In the 1970's, Armand Borel and Jacques Tits discovered that sometimes the "algebraic" assumption is superfluous. They proved that, in certain circumstances, an ordinary group homomorphism between two algebraic groups has to be almost algebraic.

After establishing their theorem, Borel and Tits conjectured that this was just one example of a more general phenomenon. Since then, mathematicians have made partial progress in confirming this, and this thesis extends that work by verifying the conjecture for a family of algebraic groups called special unitary groups. We hope that the methods developed here can later prove the conjecture for the larger family of quasi-split algebraic groups.

ABSTRACT

ABSTRACT HOMOMORPHISMS OF ALGEBRAIC GROUPS: RIGIDITY AND GROUP ACTIONS

By

Joshua Ruiter

We investigate two related problems involving abstract homomorphisms between the groups of rational points of algebraic groups. First, we show that under appropriate assumptions, abstract representations of quasi-split special unitary groups associated with quadratic extensions of the field of definition have standard descriptions, i.e. can be factored as a group homomorphism induced by a morphism of algebras, followed by a homomorphism arising from a morphism of algebraic groups. This establishes a new case of a longstanding conjecture of Borel and Tits. In the second part, we apply existing results on standard descriptions for abstract representations of Chevalley groups to study some rigidity properties of actions of elementary subgroups on algebraic varieties.

The thesis is organized as follows. To provide context for the study of abstract homomorphisms, in §1 we give a historical overview of key developments going back to Cartan's work on homomorphisms of Lie groups. In §2, we prove our rigidity result for special unitary groups, using a strategy inspired by [Rap11] which depends crucially on the construction of certain algebraic rings associated to abstract representations. In §3, we apply existing rigidity statements for representations of elementary subgroups of Chevalley groups to study rigidity properties of these groups acting on affine algebraic varieties and projective surfaces. We discuss some open questions and plans for future work in §4. In the appendices, we collect some relevant background material on algebraic rings, and also provide details on the computations of commutator relations needed for the constructions in §2.

ACKNOWLEDGMENTS

I am thankful to my advisor, Igor Rapinchuk, for advice in choosing a problem and guidance while working on it, as well as very clear exposition in his courses. I am also thankful to Rajesh Kulkarni for his excellent teaching in many courses and seminars, especially a summer course on algebraic groups. Thanks to Anastasia Stavrova for several helpful discussions about root systems and Steinberg groups. Finally, thanks to my parents, siblings, and friends for their support throughout my years at Michigan State University.

TABLE OF CONTENTS

LIST OF FIGURES

KEY TO SYMBOLS

A	Algebraic ring associated to an abstract representation
A_α	Algebraic ring associated to a particular root α
A_n	Root system of type A_n
$\mathbb{A}^n$	Affine space of dimension n
$\mathrm{AffAlgGp}_k$	Category of affine algebraic groups (representable functors $\mathrm{Alg}_k \to \mathrm{Gp}$)
Alg_k	Category of finitely-generated commutative k-algebras
α_i	Character which picks off the ith diagonal entry of the diagonal
α, β	(Relative) roots, elements of Φ_k
(α, β)	Positive integer linear combinations of roots α and β
$\mathrm{Aut}(X)$	Group of biregular automorphisms, identified with a subgroup of $\mathrm{Bir}(X)$
B_α	Algebraic variety from Lemma 2.4.2
$\mathrm{Bir}(X)$	Group of birational automorphisms of a variety X
$\mathbb{C}$	Complex numbers
$c_\alpha(u, v)$	Unitary Steinberg symbol
c_{ij}	1 if $i < j$, -1 if $i > j$
C_n	Root system of type C_n
conj_φ	Conjugation by φ
$\sqrt{d}$	Primitive element of the quadratic extension L/k
Δ	Finite-index subgroup of Γ
(D_n)	True if $\dim_K \mathrm{Der}^g(R, K) \leq n$ for all g
$\mathrm{Der}^g(R, K)$	Space of g-derivations of R into K
$(-)_\delta$	Identity if $\delta = 1$, conjugation if $\delta = -1$
$e_{ij}(x)$	Matrix with x in ij-th entry, ones on the diagonal, zeros elsewhere
$E_{ij}(x)$	Matrix with x in ij-th entry, zeros elsewhere
f	Ring homomorphism $k \to A$ with Zariski-dense image

f_α	Ring homomorphism $k \to A_\alpha$ with Zariski-dense image
F	Group homomorphism $G(k) \to G(A)$ induced by f
$\widetilde{F}$	Group homomorphism $\widetilde{G}(k) \to \widetilde{G}(A)$ induced by f
G	Special unitary group $\mathrm{SU}_{2n}(L, h)$
$\widetilde{G}$	Steinberg group of G
$G(k)$	Group of k-rational points of G
G_0	Split subgroup of a group G
$G_0(k)$	Group of k-rational points of G_0
G_A	Base change of G to A
$G(R)$	Group of points of $G = \mathrm{SU}_{2n}(L, h)$ over a k-algebra R
$\widetilde{G}(R)$	Steinberg group of $G(R)$
$G(A, J)$	Congruence subgroup of $G(A)$ for an ideal $J \subset A$
$\mathrm{Gal}(L/k)$	Galois group of L/k, order 2
$\Gamma, G^+, G(R)^+$	Classical elementary subgroup, generated by k-points of unipotent radicals of k-defined parabolic subgroups
$\Gamma.f$	Orbit of f for an action of the group Γ
$\mathbb{G}_a$	Additive group, $\mathbb{G}_a(k) = (k, +)$
$\mathbb{G}_m$	Multiplicative group, $\mathbb{G}_m(k) = (k^\times, \times)$
GL_n	General linear group
$\mathrm{GL}(V)$	General linear group of a vector space V
Gp	Category of groups
h	Skew-hermitian form on V with maximal Witt index
H	$H = \overline{\rho(G(k))}$
$\overline{H}$	Quotient group $H/Z(H)$
$h_\alpha(v)$	Weyl-group element
$\widetilde{h}_\alpha(v)$	Lift of $h_\alpha(v)$ to Steinberg group
inv	Inversion map from a group to itself, $G \mapsto G, g \mapsto g^{-1}$

J	Jacobson radical
k	Field
$k^\times$	Multiplicative group of a field k, $k^\times = k \setminus \{0\}$
$k[x]$	Ring of polynomials in the variable x with coefficients in k
K	Field, usually assumed $\operatorname{char} K = 0$ and algebraically closed
$K[X]$	Coordinate ring of a variety X
L	Quadratic extension of k, $L = k(\sqrt{d})$
LieGp	Category of real Lie groups
$M_n(R)$	$n \times n$ matrices with entries in a ring R
$N_{ij}^{\alpha\beta}$	Homogeneous polynomial map arising in Steinberg commutator formula
Nice pair	$2 \in R^\times$ if Φ contains a copy of B_2, and $2, 3 \in R^\times$ if Φ is type G_2
ν	Inverse map to π in Lemma 2.4.2
Φ	Absolute root system (with respect to a fixed maximal torus)
Φ_k	Relative root system (with respect to a fixed maximal k-split torus)
π	Compatibility isomorphism of Lemma 2.4.2
π_R	Canonical homomorphism $\widetilde{G}(R) \to G(R)$
ψ_α	Functions arising in Proposition 2.4.6
ρ	Abstract representation $G(k) \to \mathrm{GL}_m(K)$
ρ_0	Restriction of ρ to split subgroup $G_0(k)$
r	Group action $r : \Gamma \to \mathrm{Aut}(X)$
r^*	Associated action on coordinate ring, $r^* : \Gamma \to \mathrm{Aut}(K[X])$
R	Arbitrary k-algebra
R_L	Tensor product/extension of scalars $R \otimes_k L$
$R_u(H)$	Unipotent radical of H
$R_{A/k}$	Weil restriction/restriction of scalars functor
$r_{A/k}$	Canonical isomorphism associated with $R_{A/k}$
R_u	Unipotent radical

S	Maximal k-split torus (contained in T)
SL_n	Special linear group
Sp_{2n}	Symplectic group
$\mathrm{SU}_{2n}(L, h)$	Isometry group of hermitian form h
σ	Map constructed in Theorem 2.5.6
$\overline{\sigma}$	Map constructed in Proposition 2.5.2
$\widetilde{\sigma}$	Map constructed in Proposition 2.4.8
$\mathrm{Sym}(V)$	Symmetric algebra on a vector space V
(T)	Kazhdan's property (T)
T	Maximal torus (containing S)
τ	Unique nontrivial Galois automorphism in $\mathrm{Gal}(L/k)$
TopGp	Category of topological groups
Tr	Field trace map $L \to k$, or the extension $R_L \to R$
U_α	Root subgroup associated to α, $U_\alpha(R) = X_\alpha(V_\alpha(R))$
V	L^{2n}, viewed as an L-vector space
V_α	Vector space, dimension equal to dimension of root space for α
$w_\alpha\beta$	Reflection of β across the hyperplane perpendicular to α
$w_\alpha(v)$	Weyl-group element
$\widetilde{w}_\alpha(v)$	Lift of $w_\alpha(v)$ to Steinberg group
X_α	Root subgroup embedding $V_\alpha(R) \to G(R)$
$\widetilde{X}_\alpha(v)$	Generator of Steinberg group
$X^*(S)$	Character group $\mathrm{Hom}(S, \mathbb{G}_m)$
$[x, y]$	Commutator $xyx^{-1}y^{-1}$
(Z)	True if $Z(H) \cap U = \{e\}$
$Z(H)$	Center of H

Convention: All rings are assumed to be unital, associative, and commutative.

Chapter 1

Rigidity

In this chapter, we provide historical background on rigidity for algebraic groups, starting with Cartan's rigidity result for Lie groups and leading into the Borel-Tits conjecture. We discuss partial results towards the Borel-Tits conjecture, and outline how the current work fits into and extends known results.

One of the earliest forms of rigidity is Cartan's theorem regarding continuous group homomorphisms between Lie groups. In general, a continuous map $f : M \to N$ between smooth manifolds cannot be expected to be smooth. However, Cartan's theorem says that if M, N are Lie groups and f is a group homomorphism, then continuity implies smoothness[1]. In other words, group homomorphisms between Lie groups are much more rigid that they first appear.

Following Cartan, there was a considerable amount of work on rigidity properties of classical groups, by people including Schreier and van der Waerden [SVdW28], Dieudonné [Die80], and O'Meara [HO89]. However, all of these results were obtained in a somewhat *ad hoc* manner. The first organized approach was taken by Steinberg in [Ste16], who systematically studied all simply-connected Chevalley groups over fields, culminating in the rigidity result [Ste16, Theorem 30], which gives a structured factorization description for any abstract

[1] This goes back to [Car30] and [vN29], though Cartan's statement is not in these terms. Cartan proved the closed-subgroup theorem, of which this is a corollary. See [Lee13, Theorem 20.12] for the closed-subgroup theorem, and [Lee13, Exercise 20-11(b)] for the corollary as stated here.

automorphism of the group of points of a Chevalley group over a perfect field. Later work of Borel and Tits in [BT73] generalized Steinberg's results and focused the subject around a main conjecture. Before discussing this conjecture, we motivate the transition from Cartan's theorem to analogous statements about algebraic groups.

To translate Cartan's theorem into a conjecture for algebraic groups, we first rephrase it in more categorical terms: it is equivalent to the statement that the forgetful functor $\mathrm{LieGp} \to \mathrm{TopGp}$ is full. We replace Lie groups with the category $\mathrm{AffAlgGp}_k$ of affine algebraic groups over a field k (viewed as representable functors $\mathrm{Alg}_k \to \mathrm{Gp}$, where Alg_k is the category of finitely generated commutative k-algebras), and replace topological groups by the category Gp of abstract groups (groups with no additional structure). We replace the forgetful functor $\mathrm{LieGp} \to \mathrm{TopGp}$ with the functor

$$P : \mathrm{AffAlgGp}_k \to \mathrm{Gp}, \qquad G \mapsto G(k)$$

We think of this as a parallel to the forgetful functor in the sense that $G(k)$ "forgets" the algebraic group structure of G. Then Cartan's theorem suggests the question: is P full? We introduce some additional terminology to rephrase this more concretely.

Definition 1.0.1. Let G, G' be algebraic k-groups. An abstract group homomorphism $\rho : G(k) \to G'(k)$ is **algebraic** if there is a morphism of algebraic groups $\sigma : G \to G'$ such that ρ coincides with the map induced by σ on k-points. In other words, ρ is algebraic if it is in the image of the functor $P : \mathrm{AffAlgGp}_k \to \mathrm{Gp}$.

In the analogy between algebraic groups and Lie groups, algebraic group homomorphisms correspond to smooth group homomorphisms and abstract homomorphisms correspond to continuous homomorphisms. Our earlier question of whether $\mathrm{AffAlgGp}_k \to \mathrm{Gp}$ is full can now be rephrased as: is every abstract group homomorphism $\rho : G(k) \to G'(k)$ algebraic? Unfortunately, the answer is no. We give some counterexamples.

Example 1.0.2. Let $k = \mathbb{C}$ and $G = G' = \mathbb{G}_a$ be the additive group and let $\rho : \mathbb{C} \to \mathbb{C}$ be complex conjugation. All algebraic group homomorphisms $\mathbb{G}_a \to \mathbb{G}_a$ are given by (linear) polynomials, which ρ is not, so ρ is not algebraic. A similar counterexample is obtained by replacing $\mathbb{G}_a$ by $\mathbb{G}_m$. More generally, let $G = G'$ be an algebraic subgroup of GL_n and let $\rho : G(\mathbb{C}) \to G(\mathbb{C})$ be complex conjugation on each matrix entry; this is not algebraic.

Example 1.0.3. Let k be a perfect field of characteristic $p > 0$, let $G = G' = \mathbb{G}_a$ and let $\rho : k \to k, x \mapsto x^{1/p}$ be the inverse of the Frobenius map. Then $\rho : \mathbb{G}_a(k) \to \mathbb{G}_a(k)$ is not algebraic.

Example 1.0.4. Let k be a field, and let $G = G' = \mathrm{GL}_n$ over k. Let $\phi : k \to k$ be a field automorphism, which induces a map $\rho : \mathrm{GL}_n(k) \to \mathrm{GL}_n(k)$ applying ϕ to each matrix entry. If ϕ is not algebraic (as an abstract homomorphism $\mathbb{G}_a \to \mathbb{G}_a$), then ρ is also not algebraic.

As these examples demonstrate, field automorphisms are a major source of non-algebraic homomorphisms between linear algebraic groups. However, the general philosophy of rigidity is that maps arising from field automorphisms (and more generally, morphisms of algebras) are essentially the only obstruction to an abstract homomorphism being algebraic.

To explain the source of this philosophy, we formulate one of the main results of the landmark 1973 paper of Borel and Tits [BT73]. Let k, k' be infinite fields and let G, G' be algebraic groups over k, k' respectively. Suppose that G is absolutely almost simple k-isotropic, and G' is absolutely simple adjoint. Let G^+ denote the elementary subgroup of $G(k)$, the group generated by the k-points of unipotent radicals of k-defined parabolic subgroups, and let $\rho : G^+ \to G'(k')$ be an abstract homomorphism with Zariski-dense image. Then there exists a field embedding $f : k \to k'$ and a morphism of algebraic groups $\sigma : G \to G'$ such that $\rho = \sigma \circ F|_{G^+}$, where $F : G(k) \to G(k')$ is the group homomorphism induced by f. We depict this in the following commutative diagram.

$$\begin{tikzcd} G^+ \arrow[rr, "\rho"] \arrow[dr, "F"'] & & G'(k') \\ & G(k') \arrow[ur, "\sigma"'] & \end{tikzcd}$$

A factorization of ρ as above is called a **standard description**. Borel and Tits also extended of this result in [BT73, Theorem 8.16], where, roughly speaking, the hypothesis on G' is replaced by assuming that G' is reductive. Borel and Tits also gave the following example, which illustrates two aspects of the subject: (1) more abstract homomorphisms may have standard descriptions if we allow morphisms of algebras rather than just field embeddings, and (2) the image of a reductive group under an abstract homomorphism need not be reductive.

Example 1.0.5. Fix an infinite field k and let $G \subset \mathrm{GL}_n$ be a simple linear algebraic k-group, with Lie algebra $\mathfrak{g} \subset \mathrm{M}_n$. Let $\delta : k \to k$ be a nontrivial derivation, which induces a map $\Delta : G(k) \to G(k)$ by applying δ on each matrix entry[2]. Let $G' = \mathfrak{g} \ltimes G$, using the adjoint representation of G on $\mathfrak{g}$. Define

$$\rho : G(k) \to G'(k) \qquad g \mapsto \left(g^{-1} \cdot \Delta(g), g\right)$$

More conceptually, ρ can be described in the following way which explicitly shows how ρ factors as predicted by the result of Borel and Tits. Let $A = k[\varepsilon]$ with $\varepsilon^2 = 0$, and note that $G(A) \cong G'(k)$ via the algebraic isomorphism

$$\sigma : G(A) \to G'(k) \qquad X + Y\varepsilon \mapsto \left(X^{-1} \cdot Y, X\right)$$

[2]For concreteness, one can take $G = \mathrm{SL}_n$, $\mathfrak{g} = \mathfrak{sl}_n$, $k = \mathbb{C}(x)$, and $\delta = \frac{d}{dx}$ to be the differentiation operator.

Define $f : k \to A$ by $f(x) = x + \delta(x)\varepsilon$, which induces $F : G(k) \to G(A)$ by applying f to each matrix entry. It is straightforward to verify $\rho = \sigma \circ F$, i.e. ρ essentially arises from f.

$$\begin{tikzcd} G(k) \arrow[rr, "\rho"] \arrow[dr, "F"'] & & G'(k) \\ & G(A) \arrow[ur, "\sigma"'] & \end{tikzcd}$$

To summarize, ρ is a non-algebraic morphism between groups of k-points which essentially arises from a morphism of k-algebras $f : k \to k[\varepsilon]$. Also, ρ has Zariski-dense image in $\mathfrak{g} \ltimes G$. In particular, even if G is semisimple, the Zariski closure of the image $\overline{\rho(G(k))} = G'(k)$ is not reductive (it has nontrivial unipotent radical $\mathfrak{g}$). So in general, the image of a reductive group under an abstract morphism need not be reductive. Furthermore, the example demonstrates that in order to obtain a standard description for an abstract homomorphism, it may not be possible to factor using a map induced by a field embedding; it may be necessary to use a map induced by a morphism of k-algebras as F is in the example. This led them to formulate the following conjecture [BT73, 8.19].

Conjecture 1.0.6 (BT)**.** *Let G and G' be algebraic groups defined over infinite fields k and k', respectively. If $\rho\colon G(k) \to G'(k')$ is any abstract homomorphism such that $\rho(G^+)$ is Zariski-dense in $G'(k')$, then there exists a commutative finite-dimensional k'-algebra A and a ring homomorphism $f : k \to A$ such that*

$$\rho = \sigma \circ r_{A/k'} \circ F$$

where $F : G(k) \to G_A(A)$ is induced by f, $r_{A/k'} : G_A(A) \to R_{A/k'}(G_A)(k')$ is the canonical isomorphism, and σ is a rational k'-morphism of $R_{A/k'}(G_A)$ to G'.

In the conjecture, G_A is the group obtained by base change from k to A, and $R_{A/k'}$ is the functor of Weil restriction of scalars. Generalizing our previous usage, if an abstract homomorphism $\rho : G(k) \to G'(k')$ admits a factorization as in (BT), we will say that ρ has a **standard description**. We think of (BT) as a generalized, algebraic version of Cartan's theorem. Notably, it explains our previous counterexamples.

Example 1.0.7. We return to the setting of Example 1.0.2. Let $k = k' = \mathbb{C}$ and $G = G' \subset \mathrm{GL}_n$, and let $\rho : G(\mathbb{C}) \to G(\mathbb{C})$ be complex conjugation on each entry. Let $A = \mathbb{C}$, let $f : \mathbb{C} \to \mathbb{C}$ be complex conjugation, which induces $F = \rho : G(\mathbb{C}) \to G(\mathbb{C})$, and let $\sigma = \mathrm{Id}$. This gives a (somewhat vacuous) standard description of ρ.

$$\begin{tikzcd} G(\mathbb{C}) \arrow[rr, "\rho"] \arrow[dr, "F"'] & & G'(\mathbb{C}) \\ & G(\mathbb{C}) \arrow[ur, "\sigma=\mathrm{Id}"'] & \end{tikzcd}$$

Around the time Conjecture 1.0.6 was formulated, Tits sketched an argument in [Tit71, §4] for the case $k = k' = \mathbb{R}$. Later, Weisfeiler [Wei81] proved a case in which G is split by a quadratic extension and ρ is an abstract isomorphism. Seitz [Sei97] obtained a result when k is a perfect field of positive characteristic, and L. Lifschitz and A. Rapinchuk [LR01] gave (essentially) a proof of (BT) in the case where G is an absolutely simple, simply-connected Chevalley group, $\operatorname{char} k = 0$, and G' has commutative unipotent radical. Note that using the results of [CGP15, Chapter 9] (particularly Proposition 9.9.1), it is possible to give counterexamples to (BT) over any field k of characteristic 2 such that $[k : k^2] = 2$, using perfect and k-simple groups, so we do not expect (BT) to hold over fields of characteristic 2.

Most recently, I. Rapinchuk obtained significant results towards (BT) in [Rap11], [Rap13], and [Rap19], essentially resolving the conjecture for all split groups (avoiding characteristic 2 and 3), including results for groups over more general commutative rings. In [Rap11], he introduced a method for studying abstract representations of the elementary subgroups of simply-connected Chevalley groups over commutative rings based on the construction and analysis of certain algebraic rings. These techniques led to a general result on abstract representations that, in particular, yielded (BT) in the case where k is a field of characteristic $\neq 2$ or 3, $k' = K$ is an algebraically closed field of characteristic zero, and G is a split simply-connected k-group. Subsequently, this approach was extended in [Rap13] to confirm (BT) for abstract representations of groups of the form $\mathrm{SL}_{n,D}$, where D is a finite-dimensional central division algebra over a field of characteristic zero. We refer the reader to [Rap15] for a more extensive overview of work on (BT) and its connections to various classical forms of rigidity.

Except for results concerning the groups $\mathrm{SL}_{n,D}$, essentially all existing progress on (BT) has concerned split groups. Our main contribution is to extend the methods of [Rap11] to prove (BT) for a class of quasi-split special unitary groups (see §2). The primary obstacle in extending results to non-split groups is the fact that root spaces and root subgroups are no longer necessarily 1-dimensional. In the split case, analysis of abstract representations is aided by a construction involving SL_n going back to Kassabov and Sapir [KS09], where they put an algebraic ring structure on the closure of the image (under an abstract representation) of a 1-dimensional root subgroup. The generalization of this construction to more general Chevalley groups is the heart of the strategy of [Rap11] to prove (BT) for those groups. We have managed to extend this construction to a quasi-split group with 2-dimensional (relative) root subgroups. We hope that eventually the method can be further extended to more quasi-split groups.

Our second contribution to the study of rigidity of a more geometric nature. In §3, we apply rigidity statements in the vein of (BT) from [Rap19] to obtain rigidity statements for elementary subgroups of Chevalley groups acting on varieties. We think of these results as a kind of algebraic analog of the Zimmer program, where we replace diffeomorphism groups of smooth manifolds by biregular or birational automorphism groups of varieties.

Chapter 2

Abstract homomorphisms of special unitary groups

In this chapter, we prove that the conjecture of Borel and Tits holds for abstract representations of certain even-dimensional quasi-split special unitary groups, modulo an additional technical hypothesis on a certain unipotent radical. The precise statement is as follows.

Theorem 2.0.1. *Let $L = k(\sqrt{d})$ be a quadratic extension of a field k of characteristic zero, and for $n \geq 2$, set $G = \mathrm{SU}_{2n}(L, h)$ to be the special unitary group of a (skew-)hermitian form $h : L^{2n} \times L^{2n} \to L$ of maximal Witt index. Let K be an algebraically closed field of characteristic zero and consider an abstract representation*

$$\rho : G(k) \to \mathrm{GL}_m(K).$$

Set $H = \overline{\rho(G(k))}$ to be the Zariski closure of the image of ρ. Then if the unipotent radical $U = R_u(H)$ of H is commutative, there exists a commutative finite-dimensional K-algebra A, a ring homomorphism $f : k \to A$ with Zariski-dense image, and a morphism of algebraic K-groups $\sigma : G(A) \to H$ such that $\rho = \sigma \circ F$, where $F : G(k) \to G(A)$ is the group homomorphism induced by f.

$$\begin{tikzcd}
G(k) \arrow[rr, "\rho"] \arrow[dr, "F"'] & & \mathrm{GL}_m(K) \\
& G(A) \arrow[ur, "\sigma"'] &
\end{tikzcd}$$

In the statement of the theorem, we are using the functor of restriction of scalars to view $G(A)$ as an algebraic K-group. Namely, denoting by G_A the base change of G from k to A, restriction of scalars gives a natural isomorphism $r_{A/K}\colon G_A(A) \to R_{A/K}(G_A)(K)$. Since A is a finite-dimensional K-algebra, $R_{A/K}(G_A)(K)$ is an affine algebraic K-group, and the isomorphism $r_{A/K}$ allows us to endow $G(A) = G_A(A)$ with the structure of an algebraic K-group. Note that the group H appearing in the theorem is connected by Lemma 2.5.1. Also note that there is no meaningful difference between assuming h is hermitian or skew-hermitian, see Remark 2.1.2.

The proof of Theorem 2.0.1 spans this chapter, and proceeds as follows. First, since $G = \mathrm{SU}_{2n}(L, h)$ is a simply-connected k-group with relative root system Φ_k of type C_n, it follows that G contains a k-split simply connected k-group $G_0 = \mathrm{Sp}_{2n}$ of type C_n (see [BT65, Théorème 7.2] or [CGP15, Theorem C.2.30]). We describe one particular such split subgroup in Definition 2.2.5. Given an abstract representation $\rho : G(k) \to \mathrm{GL}_m(K)$, we consider the restriction of ρ to $G_0(k)$, and use the construction in [Rap11] to associate to $\rho|_{G_0(k)}$ an algebraic ring A, as well as a ring homomorphism $f : k \to A$ with Zariski-dense image. Since k and K are both fields of characteristic zero, A is a finite-dimensional K-algebra by [Rap11, Lemma 2.13(ii), Proposition 2.14]. See Appendix A for results we use concerning algebraic rings.

In the methodology of [Rap11], the algebraic ring A plays a central role in proving that $\rho|_{G_0(k)}$ has a standard description. We show here that A also suffices for the analysis of ρ. More precisely, following the general strategy of [Rap11] and [Rap13], we first show that ρ lifts to a representation $\widetilde{\sigma} : \widetilde{G}(A) \to \mathrm{GL}_m(K)$, where $\widetilde{G}(A)$ is the generalized Steinberg group introduced by Stavrova [Sta20] (which builds on an earlier construction of Deodhar [Deo78]). Then, using the fact that the kernel of the canonical map $\widetilde{G}(A) \to G(A)$ is central (which extends a result of Stavrova to the present situation), together with our assumption that the unipotent radical $R_u(H)$ is commutative, we establish the existence of the required algebraic representation $\sigma : G(A) \to \mathrm{GL}_m(K)$.

The structure of this chapter is as follows. We begin by describing the special unitary groups $\mathrm{SU}_{2n}(L,h)$ (§2.1), their elementary subgroups (§2.2), and associated Steinberg groups (§2.3). Then in §2.4 we describe an algebraic ring A associated with an abstract representation of $\mathrm{SU}_{2n}(L,h)$, describe how it is a commutative finite-dimensional K-algebra as claimed by the theorem, and show that ρ lifts to a representation $\widetilde{\sigma}$ of $\widetilde{G}(A)$. Finally, in §2.5, we descend $\widetilde{\sigma}$ to obtain the morphism σ and finish the proof of the theorem. The main results of this chapter also appear in more condensed form in the paper [RR22].

2.1 Special unitary group

We define the special unitary group $\mathrm{SU}_{2n}(L,h)$. Let k be a field of characteristic zero and let L/k be a quadratic extension, so $L = k(\sqrt{d})$ for some $d \in k$. We set $\tau : L \to L$ to be the nontrivial element of $\mathrm{Gal}(L/k)$ and note that for any commutative k-algebra R, the action of τ naturally extends to $R_L := R \otimes_k L$ via the second factor. More precisely, given a simple tensor $a \otimes b \in R \otimes_k L$ with $a \in R$ and $b \in L$, τ acts by

$$\tau(a \otimes b) = a \otimes \tau(b)$$

We write $\tau(x) = \overline{x}$ for $x \in R_L$. Next, fix an integer $n \geq 2$ and let $V = L^{2n}$ be a $2n$-dimensional L-vector space equipped with a (skew-)hermitian form $h \colon V \times V \to L$, and assume that h has maximal Witt index. Because of the Witt index assumption, with respect to a suitable basis of V the matrix of h is

$$H = \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -1 \\ & & & 1 & 0 \end{pmatrix}$$

Definition 2.1.1. With k, L, h as above, let $G = \mathrm{SU}_{2n}(L, h)$ be the isometry group of h. Explicitly in terms of matrices, for a commutative k-algebra R, we have

$$G(R) = \{X \in \mathrm{SL}_{2n}(R_L) \mid X^* H X = H\},$$

where for $X = (a_{ij})$, we let $X^* = (\overline{a}_{ji})$ denote the conjugate transpose matrix. Note in particular that the group of k-points $G(k)$ is a group of matrices with entries in L, not just in k. For the rest of this chapter, G denotes the special unitary group $\mathrm{SU}_{2n}(L, h)$. The choice of basis (or equivalently the choice of matrix representation for h) only affects G up to inner automorphism.

Remark 2.1.2. The special unitary groups considered here belong to a larger class of unitary groups described in [Bor91, V.23.9] (also see [Mil17, Theorem 24.44, Remark 24.46] and [PR94, Ch. 2, §2.3.3]). In a more general setting considered by Borel, L can be any division algebra over $k(\sqrt{d})$, τ is an involution of the second kind on L, and the form h may have arbitrary Witt index. As noted in [Bor91, V.23.8], "there is no essential difference" between hermitian and skew-hermitian when dealing with an involution of the second kind, in the sense that if the form h is τ-hermitian then a scalar multiple of h is τ-skew-hermitian. The group G also belongs to the class of unitary groups studied by Hahn and O'Meara in [HO89, Ch 5].

Remark 2.1.3. G is a k-form of SL_{2n}. More precisely, G becomes isomorphic to SL_{2n} over L, as we now describe. Let R be an L-algebra. Then we have the following sequence of group homomorphisms, all of which are functorial in R.

$$\mathrm{SU}_{2n}(R) \hookrightarrow \mathrm{SL}_{2n}(R_L) \to \mathrm{SL}_{2n}(R^2) \to \mathrm{SL}_{2n}(R)^2 \to \mathrm{SL}_{2n}(R)$$

The first map is inclusion, and the second is induced by the following isomorphism of L-algebras with involution[1].

$$R_L \cong R^2$$
$$a \otimes 1 + b \otimes \sqrt{d} \mapsto \left(a + b\sqrt{d}, a - b\sqrt{d}\right) \qquad a, b \in R$$

The third map takes matrices with ordered pair entries and turns it into an ordered pair of matrices, and the final map is projection onto the first copy of $\mathrm{SL}_{2n}(R)$. The entire composition gives an isomorphism $\mathrm{SU}_{2n}(R) \cong \mathrm{SL}_{2n}(R)$ which is functorial in R.

Let $\mathcal{G}$ be any quasi-split k-group and let L be a splitting field for $\mathcal{G}$, then extend if necessary so that L/k is Galois. Let $S \subset T \subset \mathcal{G}$ with S a maximal k-split torus and T a maximal torus (which splits over L). Let $\Phi = \Phi(\mathcal{G}, T)$ be the absolute root system and $\Phi_k = \Phi(\mathcal{G}, S)$ be the relative root system, viewing Φ as a subset of the character group $X^*(T)$ and Φ_k as a subset of the character group $X^*(S)$. The natural restriction map $X^*(T) \to X^*(S)$ takes Φ onto Φ_k, and the Galois group $\mathrm{Gal}(L/k)$ acts on Φ by inducing automorphisms of the associated Dynkin diagram, and restriction of characters from T to S gives a bijection

$$\{\mathrm{Gal}(L/k)\text{-orbits in } \Phi\} \xrightarrow{\cong} \Phi_k$$

[1] The involution on R^2 is $(x, y) \mapsto (y, x)$. In fact, this isomorphism the unique isomorphism $R_L \cong R^2$ of L-algebras with involution, up to multiplying by the scalar -1.

That is, two absolute roots in Φ restrict to the same relative root if and only if they lie in the same $\mathrm{Gal}(L/k)$-orbit.

In the particular case of $G = \mathrm{SU}_{2n}(L,h)$, Φ is type A_{2n-1} and Φ_k is type C_n. The Galois group is $\mathrm{Gal}(L/k) \cong \mathbb{Z}/2\mathbb{Z}\,\langle\tau\rangle$. The generator τ acts as on the A_{n-1} Dynkin diagram by reflection across the central node. This reflection is depicted in Figure 1 below.

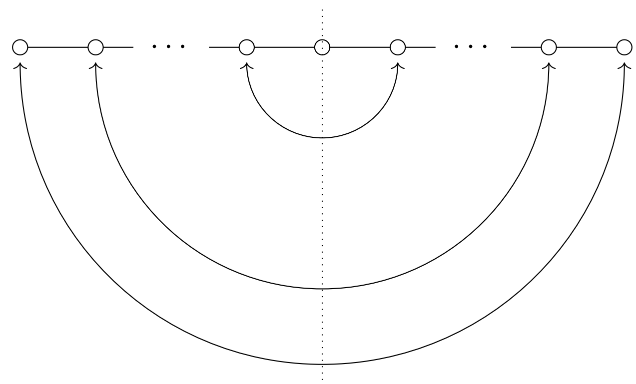


Figure 2.1: The nontrivial automorphism of the A_{2n-1} Dynkin diagram

There are n orbits; one singleton orbit and $(n-1)$ orbits each containing two nodes. As depicted below in Figure 2, the $(n-1)$ orbits with two nodes correspond to the $(n-1)$ short roots in a base for C_n, and the lone singleton orbit corresponds to the long root in a base for C_n.

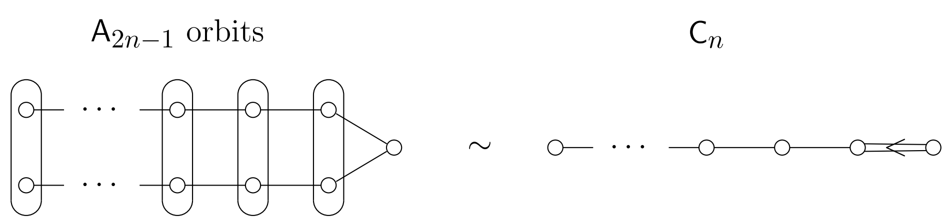


Figure 2.2: Orbits in A_{2n-1} corresponding to nodes in C_n

See §2.2 of [Deo78] and Table II of [Tit66] for more discussion on this correspondence for general quasi-split groups.

2.2 Elementary subgroup

We describe the elementary subgroup of $\mathrm{SU}_{2n}(L, h)$, first by recalling some aspects of the theory of elementary subgroups of isotropic reductive group schemes, developed by Petrov and Stavrova [PS08]. Suppose $\mathcal{G}$ is a reductive group scheme over a ring R that is isotropic of rank ≥ 1 (i.e. every semisimple normal R-subgroup of $\mathcal{G}$ contains a 1-dimensional split R-torus). Then $\mathcal{G}$ contains a pair of opposite parabolic R-subgroups P and P^- that intersect properly every semisimple normal R-subgroup of $\mathcal{G}$ (such subgroups are called *strictly parabolic*). In [PS08], the corresponding elementary subgroup $E_P(R)$ is then defined as the subgroup of $\mathcal{G}(R)$ generated by $U_P(R)$ and $U_{P-}(R)$, where U_P and U_{P^-} are the unipotent radicals of P and P^-, respectively (note that when $R = k$ is a field, then $E_P(k)$ coincides with the group G^+ appearing in the Borel-Tits Conjecture 1.0.6). The main result of [PS08] (see also [Sta14, Theorem 2.4]) is that if for any maximal ideal $\mathfrak{m} \subset R$, the group $\mathcal{G}_{R_\mathfrak{m}}$ is isotropic of rank ≥ 2, then $E_P(R)$ does not depend on the choice of a strictly parabolic subgroup P. This assumption is automatically satisfied in all situations considered here, so to simplify notations, we will denote the elementary subgroup simply by $E(R)$.

The elementary subgroup is functorial in R (i.e. a ring homomorphism $R_1 \to R_2$ gives rise to a group homomorphism $E(R_1) \to E(R_2)$) and it is compatible with finite products (i.e. $E(R_1 \times \cdots \times R_n) = E(R_1) \times \cdots \times E(R_n)$). Furthermore, since G is a quasi-split simply-connected k-group, this observation and [Sta20, Lemma 5.2] imply that $G(R) = E(R)$ for any ring R that is a finite product of local k-algebras. In particular $G(k) = E(k)$, and $G(A) = E(A)$ where A is the ring constructed later in Proposition 2.4.6 (note that while A may contain nontrivial idempotents, it is a product of local rings which do not).

In analogy with elementary subgroups of Chevalley groups, Petrov and Stavrova provide a description of $E(R)$ in terms of generators that satisfy certain generalized Steinberg commutator relations. The following statement collects the relevant parts of [PS08, Theorem 2] and [Sta20, Lemma 2.14] in the case of $G = \mathrm{SU}_{2n}(L, h)$ that will be needed for our analysis.

Theorem 2.2.1. *Let $G = \mathrm{SU}_{2n}(L, h)$, fix a maximal k-split torus $S \subset G$, and denote by Φ_k the corresponding relative root system (of type C_n), viewed as a subset of the character group $X^*(S)$. Then for every $\alpha \in \Phi_k$, there exists a vector k-group scheme V_α and a closed embedding of schemes $X_\alpha : V_\alpha \to G$ such that for any k-algebra R, we have the following:*

(1) For any $v, w \in V_\alpha(R)$,

$$X_\alpha(v) \cdot X_\alpha(w) = X_\alpha(v + w)$$

In particular, $X_\alpha(0) = 1$.

(2) For any $s \in S(R)$ and $v \in V_\alpha(R)$,

$$s \cdot X_\alpha(v) \cdot s^{-1} = X_\alpha\Big(\alpha(s)v\Big)$$

(3) (Steinberg commutator formula) For any $\alpha, \beta \in \Phi_k$ such that $\alpha \neq \pm\beta$, and for all $u \in V_\alpha(R), v \in V_\beta(R)$,

$$\Big[X_\alpha(u), X_\beta(v)\Big] = \prod_{\substack{i,j \geq 1 \\ i\alpha + j\beta \in \Phi_k}} X_{i\alpha+j\beta}\Big(N_{ij}^{\alpha\beta}(u, v)\Big)$$

for some polynomial maps $N_{ij}^{\alpha\beta} : V_\alpha(R) \times V_\beta(R) \to V_{i\alpha+j\beta}(R)$. The map $N_{ij}^{\alpha\beta}$ is homogeneous of degree i in the first variable and homogeneous of degree j in the second variable.

(4) The elementary subgroup $E(R)$ is generated by the elements $X_\alpha(v)$ for all $\alpha \in \Phi_k$ and all $v \in V_\alpha(R)$.

Remark 2.2.2. We make some remarks on the theorem.

1. The more general formulas in [PS08, Theorem 2] and [Sta20, Lemma 2.14] include complicated product terms in (1) and (2) above, but these terms are trivial in our case because Φ_k is reduced.

2. In view of the structure of the root system C_n, the product on the right hand side of (3) contains at most two terms with the possible values of i and j lying in $\{1, 2\}$. Moreover, when there are two terms, these commute with each other. In fact, all nontrivial commutator relations in Theorem 2.2.1 (3) arise from a copy of C_2 or A_2 sitting inside C_n. That is, if $\alpha, \beta \in \Phi_k$ and $\alpha + \beta \in \Phi_k$ so that the product on the right side is nontrivial, then $\left\{i\alpha + j\beta \in \Phi_k : i, j \in \mathbb{Z}_{\geq 0}\right\}$ sits inside a copy of C_2 or A_2. See Lemma B.1 for a more precise statement.

3. The dimension of V_α is the dimension of the relative root space associated to α. Thus there are two possibilities, distinguished by the two root lengths in C_n: if α is a long root, then $V_\alpha = \mathbb{G}_a$, so that $V_\alpha(R) = R$ for any k-algebra R. If α is a short root, then $V_\alpha \cong (\mathbb{G}_a)^2$, and we have $V_\alpha(R) = R \otimes_k L = R_L$ (to make the identification $(\mathbb{G}_a)^2(R) = R^2 \cong R_L$, we use the fact that $k^2 \cong L$ as a k-vector space).

For the calculations that we will carry out in subsequent sections, it will be useful to make the statement of Theorem 2.2.1 more explicit, as follows. The full diagonal subgroup $T \subset G$ is a non-split maximal torus, which contains the maximal k-split subtorus S consisting of elements of T fixed by conjugation.

$$T(R) = \left\{ \begin{pmatrix} s_1 & & & & \\ & \overline{s}_1^{-1} & & & \\ & & \ddots & & \\ & & & s_n & \\ & & & & \overline{s}_n^{-1} \end{pmatrix} : s_i \in R_L^\times \right\}$$

$$S(R) = \left\{ X \in T(R) : X = \overline{X} \right\} = \left\{ \begin{pmatrix} s_1 & & & & \\ & s_1^{-1} & & & \\ & & \ddots & & \\ & & & s_n & \\ & & & & s_n^{-1} \end{pmatrix} : s_i \in R^\times \right\}$$

Clearly T splits over L (this also follows from Remark 2.1.3). Let $\alpha_i : T \to \mathbb{G}_m$ be the character which picks off the ith diagonal entry. We abuse notation slightly and denote the restriction $\alpha_i|_S$ by α_i as well. The character group $X^*(S)$ is free abelian with basis $\{\alpha_1, \alpha_3, \ldots, \alpha_{2n-1}\}$. A computation shows that the relative root spaces of G with respect to S are

$$\Phi_k = \{\pm 2\alpha_i : i \text{ odd}\} \cup \left\{\pm\alpha_i \pm \alpha_j : i \neq j, \text{ both odd}\right\} \qquad 1 \leq i, j, \leq 2n$$

Roots of the form $\pm 2\alpha_i$ are long, and roots of the form $\pm\alpha_i \pm \alpha_j$ with $i \neq j$ are short. The root spaces for long roots are 1-dimensional, and root spaces for short roots are 2-dimensional.

Furthermore, the morphisms X_α of Theorem 2.2.1 look as follows. For a ring R, we denote by $E_{ij}(x) \in \mathrm{M}_{2n}(R)$ the matrix with x in the ij-th entry and 0 in all other entries. Recall that for a k-algebra R, the conjugation map $\tau(x) = \overline{x}$ extends to R_L. We denote matrix transpose by X^t. Then the root group morphisms for the long roots are

$$X_{2\alpha_i}(R) : R \to G(R) \qquad x \mapsto 1 + E_{i,i+1}(x)$$

$$X_{-2\alpha_i}(R) : R \to G(R) \qquad x \mapsto X_{2\alpha_i}(x)^t = 1 + E_{i+1,i}(x),$$

and for the short roots, the morphisms are

$$
\begin{aligned}
X_{\alpha_i-\alpha_j}(R) : R_L \to G(R) \qquad & x \mapsto 1 + E_{ij}(x) - E_{j+1,i+1}(\overline{x}) \\
X_{-\alpha_i+\alpha_j}(R) : R_L \to G(R) \qquad & x \mapsto X_{\alpha_i-\alpha_j}(x)^t = 1 + E_{ji}(x) - E_{i+1,j+1}(\overline{x}), \\
X_{\alpha_i+\alpha_j}(R) : R_L \to G(R) \qquad & x \mapsto 1 + E_{i',j'+1}(x) + E_{j',i'+1}(\overline{x}), \\
X_{-\alpha_i-\alpha_j}(R) : R_L \to G(R) \qquad & x \mapsto X_{\alpha_i+\alpha_j}(x)^t = 1 + E_{j'+1,i'}(x) + E_{i'+1,j'}(\overline{x}),
\end{aligned}
$$

where for a pair i, j, we set $i' = \min(i, j)$ and $j' = \max(i, j)$. Note that the definition of $X_{-\alpha_i+\alpha_j}$ is redundant (and consistent) with the definition of $X_{\alpha_i-\alpha_j}$. One can check by direct calculation that the maps X_α defined here have the properties asserted by Theorem 2.2.1. Immediately from the definitions, they have the additional property that negating a root α corresponds to taking the matrix transpose of $X_\alpha(x)$.

$$X_{-\alpha}(x) = X_\alpha(x)^t \qquad \forall \alpha \in \Phi_k$$

Example 2.2.3. We write out the maps X_α in the case $n = 2$.

$$X_{2\alpha_1}(v) = \begin{pmatrix} 1 & v & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \qquad X_{-2\alpha_1}(v) = \begin{pmatrix} 1 & & & \\ v & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$X_{2\alpha_3}(v) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & v \\ & & & 1 \end{pmatrix} \qquad X_{-2\alpha_3}(v) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & v & 1 \end{pmatrix}$$

$$X_{\alpha_1-\alpha_3}(v) = \begin{pmatrix} 1 & & v & \\ & 1 & & \\ & & 1 & \\ & -\overline{v} & & 1 \end{pmatrix} \qquad X_{-\alpha_1+\alpha_3}(v) = \begin{pmatrix} 1 & & & \\ & 1 & & -\overline{v} \\ v & & 1 & \\ & & & 1 \end{pmatrix}$$

$$X_{\alpha_1+\alpha_3}(v) = \begin{pmatrix} 1 & & & v \\ & 1 & & \\ & \overline{v} & 1 & \\ & & & 1 \end{pmatrix} \qquad X_{-\alpha_1-\alpha_3}(v) = \begin{pmatrix} 1 & & & \\ & 1 & \overline{v} & \\ & & 1 & \\ v & & & 1 \end{pmatrix}$$

By direct calculation, we can obtain explicit formulas for the polynomial maps $N_{ij}^{\alpha\beta}$ appearing in Theorem 2.2.1. To formulate the result, given a k-algebra R, we let

$$\mathrm{Tr} : R_L \to R \qquad a \mapsto a + \bar{a}$$

denote the extension of the usual trace map $\mathrm{Tr}_{L/k} : L \to k$ to R_L. Also, for $\delta = \pm 1$ and $v \in R_L$, we define

$$v_\delta = \begin{cases} v & \delta = 1 \\ \overline{v} & \delta = -1 \end{cases} \tag{2.1}$$

The notation is chosen so that $(\sqrt{d})_\delta = \delta\sqrt{d}$. The following lemma makes use of this notation to describe the maps $N_{ij}^{\alpha\beta}$ arising in all nontrivial Steinberg commutator relations for $\mathrm{SU}_{2n}(L, h)$.

Lemma 2.2.4. *Let $\alpha, \beta \in \Phi_k$ be relative roots such that $\alpha + \beta \in \Phi_k$, and let $u \in V_\alpha(R), v \in V_\beta(R)$.*

(1) Suppose α, β are both short and $\alpha + \beta$ is short. Then relabelling α, β if necessary we have $\alpha = \alpha_i - \alpha_j, \beta = \alpha_j - \alpha_\ell$ for distinct indices i, j, ℓ, and

$$N_{11}^{\alpha\beta}(u, v) = uv \qquad N_{11}^{\beta\alpha}(v, u) = -uv$$

(2) Suppose α, β are both short and $\alpha + \beta$ is long. Then relabelling α, β if necessary we have $\alpha = \varepsilon(\alpha_i - \alpha_j), \beta = \omega(\alpha_i + \alpha_j)$ for some $\varepsilon = \pm 1, \omega = \pm 1$, with $i < j$, and

$$N_{11}^{\alpha\beta}(u, v) = \omega\,\mathrm{Tr}(u_{-\varepsilon\omega} v) \qquad N_{11}^{\beta\alpha}(v, u) = -\omega\,\mathrm{Tr}(u_{-\varepsilon\omega} v)$$

(3) Suppose α is short and β long. Then we have $\alpha = \varepsilon\alpha_i + \omega\alpha_j$ and $\beta = -\varepsilon 2\alpha_i$ for some

$\varepsilon = \pm 1, \omega = \pm 1$ *and* $i \neq j$, *and*

$$N_{11}^{\alpha\beta}(u,v) = \omega u_{-c_{ij}} v \qquad N_{11}^{\beta\alpha}(v,u) = -\omega u_{-c_{ij}} v$$

$$N_{21}^{\alpha\beta}(u,v) = -\varepsilon\omega v u \overline{u} \qquad N_{12}^{\beta\alpha}(v,u) = \varepsilon\omega v u \overline{u}$$

where

$$c_{ij} = \begin{cases} 1 & i < j \\ -1 & i > j \end{cases}$$

In particular, whenever it is defined, the map $N_{11}^{\alpha\beta}$ *is surjective.*

Proof. See Appendix B, Lemma B.4. $\square$

A key step in the proof of Theorem 2.0.1 involves restricting a given representation $\rho : G(k) \to \mathrm{GL}_m(K)$ to a split subgroup $G_0(k) = \mathrm{Sp}_{2n}(k)$, so we now describe one such split subgroup explicitly.

Definition 2.2.5. Let H be the matrix from Definition 2.1.1. Let

$$G_0(k) = G(k) \cap \mathrm{GL}_{2n}(k) = \{X \in \mathrm{SL}_{2n}(k) \mid X^t H X = H\} = \mathrm{Sp}_{2n}(k)$$

Recall that if $\beta \in \Phi_k$ is a short root, then $V_\beta(k) = L = k \oplus k\sqrt{d}$. Our calculation then shows that $G_0(k)$ is the subgroup of $G(k) = E(k)$ generated by the images of the maps $X_\alpha(k) : V_\alpha(k) \to G(k)$ for long roots α and by the images of the maps $X_\beta(k) : V_\beta(k) \to G(k)$ restricted to the first component for long roots β.

Definition 2.2.6. For later use, following Steinberg (see [Ste16, Ch. 3]) we define the elements

$$w_\alpha(v) = X_\alpha(v) \cdot X_{-\alpha}(-v^{-1}) \cdot X_\alpha(v)$$

$$h_\alpha(v) = w_\alpha(v) \cdot w_\alpha(1)^{-1}$$

for $\alpha \in \Phi_k$ and $v \in V_\alpha(k)^\times$. Note that $w_\alpha(1), h_\alpha(1) \in G_0(k)$. We also have the following analog of Steinberg's relation (R7),

$$w_\alpha(1) \cdot X_\beta(v) \cdot w_\alpha(1)^{-1} = X_{w_\alpha\beta}(\varphi v), \tag{2.2}$$

where $\varphi(v) = \pm v_{\pm 1}$ (so that $\varphi^2 = \mathrm{Id}$) and $w_\alpha\beta$ denotes the action of the corresponding Weyl group element w_α on the root β. The element $w_\alpha(v)$ normalizes the split torus $S(k)$, and $w_\alpha(v)^2$ centralizes $S(k)$, so $w_\alpha(v)$ corresponds to an element of the Weyl group $N(S)/Z(S) = N(S)/T$ of order 2. In particular, $w_\alpha(1)$ corresponds to the element of the Weyl group of Φ_k which is reflection in the hyperplane orthogonal to α. In Appendix B, we do explicit computations to verify a case of equation (2.2) on the level of the Steinberg group (see §2.3 for the definition of the Steinberg group).

Example 2.2.7. We write out the matrices $w_\alpha(v)$ and $h_\alpha(v)$ in the case $n = 2$.

$$w_{2\alpha_1}(v) = \begin{pmatrix} & v & & \\ -v^{-1} & & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \qquad h_{2\alpha_1}(v) = \begin{pmatrix} v & & & \\ & v^{-1} & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$w_{-2\alpha_1}(v) = \begin{pmatrix} & -v^{-1} & & \\ v & & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \qquad h_{-2\alpha_1}(v) = \begin{pmatrix} v^{-1} & & & \\ & v & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$w_{2\alpha_3}(v) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & & v \\ & & -v^{-1} & \end{pmatrix} \qquad h_{2\alpha_3}(v) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & v & \\ & & & v^{-1} \end{pmatrix}$$

$$w_{-2\alpha_3}(v) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & & -v^{-1} \\ & & v & \end{pmatrix} \qquad h_{-2\alpha_3}(v) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & v^{-1} & \\ & & & v \end{pmatrix}$$

$$w_{\alpha_1-\alpha_3}(v) = \begin{pmatrix} & & v & \\ & & & \overline{v}^{-1} \\ -v^{-1} & & & \\ & -\overline{v} & & \end{pmatrix} \qquad h_{\alpha_1-\alpha_3}(v) = \begin{pmatrix} v & & & \\ & \overline{v}^{-1} & & \\ & & v^{-1} & \\ & & & \overline{v} \end{pmatrix}$$

$$w_{-\alpha_1+\alpha_3}(v) = \begin{pmatrix} & & -v^{-1} & \\ & & & -\overline{v} \\ v & & & \\ & \overline{v}^{-1} & & \end{pmatrix} \qquad h_{-\alpha_1+\alpha_3}(v) = \begin{pmatrix} v^{-1} & & & \\ & \overline{v} & & \\ & & v & \\ & & & \overline{v}^{-1} \end{pmatrix}$$

$$w_{\alpha_1+\alpha_3}(v) = \begin{pmatrix} & & & v \\ & & -\overline{v}^{-1} & \\ & \overline{v} & & \\ -\overline{v}^{-1} & & & \end{pmatrix} \qquad h_{\alpha_1+\alpha_3}(v) = \begin{pmatrix} v & & & \\ & \overline{v}^{-1} & & \\ & & \overline{v} & \\ & & & v^{-1} \end{pmatrix}$$

$$w_{-\alpha_1-\alpha_3}(v) = \begin{pmatrix} & & & -v^{-1} \\ & & \overline{v} & \\ & -\overline{v}^{-1} & & \\ v & & & \end{pmatrix} \qquad h_{-\alpha_1-\alpha_3}(v) = \begin{pmatrix} v^{-1} & & & \\ & \overline{v} & & \\ & & \overline{v}^{-1} & \\ & & & v \end{pmatrix}$$

Lemma 2.2.8. *Let R be a k-algebra, let $\alpha \in \Phi_k$, and let $u, v \in V_\alpha(R)^\times$. Then $h_\alpha(v)$ is a diagonal matrix with diagonal entries from $\{1, v^{\pm 1}, \overline{v}^{\pm 1}\}$, $w_\alpha(v)$ is a monomial matrix with nonzero entries from $\{\pm v^{\pm 1}, \pm\overline{v}^{\pm 1}\}$, and*

$$h_\alpha(v)^{-1} = h_\alpha(v^{-1}) \qquad h_\alpha(u) \cdot h_\alpha(v) \cdot h_\alpha(uv)^{-1} = 1$$

Proof. Obvious for $n = 2$ by inspection of the tables in Example 2.2.7. The general $n \geq 2$ case reduces to the $n = 2$ case because $h_\alpha(v), w_\alpha(v)$ are contained in the subgroup generated by $X_\alpha(v), X_{-\alpha}(v)$, which is contained in a copy of $\mathrm{SU}_4(L, h)$ sitting inside $\mathrm{SU}_{2n}(L, h)$, corresponding to a copy of C_2 sitting inside C_n. The relations follow immediately from the description of $h_\alpha(v)$ as a diagonal matrix. □

The elements $w_\alpha(v)$ correspond to similar elements described by Deodhar in [Deo78, Lemma 1.3] and also more explicitly in §2.32[2]. In particular, one can check that $w_\alpha(v) =$

[2] There is potential for confusion between the notation here and in Deodhar – in his notation, $w_\alpha(u)$ is defined for nontrivial $u \in U_\alpha(k)$, but our $w_\alpha(v)$ is defined for $v \in V_\alpha(k)^\times$. Since $X_\alpha : V_\alpha(k) \to U_\alpha(k)$ is an isomorphism, this is merely a notational difference.

$w_{-\alpha}\left(-v^{-1}\right)$, or more explicitly,

$$X_\alpha(v) \cdot X_{-\alpha}\left(-v^{-1}\right) \cdot X_\alpha(v) = X_{-\alpha}\left(-v^{-1}\right) \cdot X_\alpha(v) \cdot X_{-\alpha}\left(-v^{-1}\right)$$

It follows from this and the uniqueness aspect of [Deo78, Lemma 1.3] that the element $w_\alpha(v)$ we have defined agrees with Deodhar's definition of $w_\alpha(v)$.

2.3 Steinberg group

Next we recall Stavrova's generalization of the classical Steinberg group from [Sta20, Definition 3.1], noting that this is in turn inspired by a construction of Deodhar [Deo78, §1.9]. We establish a key centrality property for a unitary version of the group K_2 from algebraic K-theory (Proposition 2.3.2) which extends a centrality result of Stavrova [Sta20, Theorem 1.3]. Although Stavrova works in the general context of reductive group schemes, for the sake of concreteness, we will restrict ourselves to the case $G = \mathrm{SU}_{2n}(L, h)$.

As in the classical setting, for a k-algebra R, the generalized Steinberg group $\widetilde{G}(R)$ is the abstract group generated by symbols $\widetilde{X}_\alpha(v)$, for $\alpha \in \Phi_k$ and $v \in V_\alpha(R)$, subject to the relations (1) and (3) of Theorem 2.2.1 but replacing X_α with $\widetilde{X}_\alpha$. More precisely, it is defined as follows.

Definition 2.3.1. Let R be a k-algebra. The **Steinberg group** $\widetilde{G}(R)$ is the group generated by symbols $\widetilde{X}_\alpha(v)$, for all $\alpha \in \Phi_k$ and all $v \in V_\alpha(R)$, subject to the relations

(R1) For $\alpha \in \Phi_k$ and $v, w \in V_\alpha(R)$,

$$\widetilde{X}_\alpha(v) \cdot \widetilde{X}_\alpha(w) = \widetilde{X}_\alpha(v + w);$$

That is, $\widetilde{X}_\alpha$ is a group homomorphism $V_\alpha(R) \to \widetilde{G}(R)$.

(R2) For $\alpha, \beta \in \Phi_k$ such that $\alpha \neq \pm\beta$, and for all $u \in V_\alpha(R), v \in V_\beta(R)$,

$$\left[\widetilde{X}_\alpha(u), \widetilde{X}_\beta(v)\right] = \prod_{\substack{i,j \geq 1 \\ i\alpha + j\beta \in \Phi_k}} \widetilde{X}_{i\alpha+j\beta}\Big(N_{ij}^{\alpha\beta}(u,v)\Big);$$

where $N_{ij}^{\alpha\beta}$ are the same maps as in Theorem 2.2.1(3).

A morphism of k-algebras $\phi : R \to R'$ induces an additive group homomorphism $\phi : V_\alpha(R) \to V_\alpha(R)$ which then induces a group homomorphism

$$\widetilde{G}(R) \to \widetilde{G}(R') \qquad \widetilde{X}_\alpha(v), \mapsto \widetilde{X}_\alpha\big(\phi(v)\big).$$

This process of inducing maps on the Steinberg group is functorial in R. Furthermore, for every k-algebra R, we have a natural surjective homomorphism

$$\pi_R : \widetilde{G}(R) \to E(R), \qquad \widetilde{X}_\alpha(v) \mapsto X_\alpha(v).$$

The kernel of π_R is a unitary analog of the group $\mathrm{K}_2(R)$ from classical algebraic K-theory. The main result of this subsection is the following statement, which partially extends [Sta20, Theorem 1.3].

Proposition 2.3.2. *Suppose R is a k-algebra that is a finite product of local k-algebras. Then $\ker \pi_R$ is a central subgroup of $\widetilde{G}(R)$.*

Before proving this, we prove some lemmas.

Lemma 2.3.3. *Any generator $\widetilde{X}_\alpha(v)$ of $\widetilde{G}(R)$ can be written as a product of generators $\widetilde{X}_{\gamma_i}(u_i)$ with $\gamma_i \neq \alpha$ and some $u_i \in V_{\gamma_i}(R)$. Furthermore, $\widetilde{X}_\alpha(v)$ is contained in the commutator subgroup $[\widetilde{G}(R), \widetilde{G}(R)]$. In particular, $\widetilde{G}(R)$ and $E(R)$ are perfect groups, i.e. they coincide with their commutator subgroups*[3].

[3] See [LS11, Theorem 1] for a more general result on elementary subgroups of reductive groups being perfect.

Proof. First, suppose α is a long root. Then we can write $\alpha = \gamma_1 + \gamma_2$ for appropriate short roots γ_1 and γ_2, and it follows from (R2) that

$$\left[\widetilde{X}_{\gamma_1}(u_1), \widetilde{X}_{\gamma_2}(u_2)\right] = \widetilde{X}_\alpha\Big(N_{11}^{\gamma_1\gamma_2}(u_1, u_2)\Big)$$

Since $N_{11}^{\gamma_1\gamma_2}$ is surjective (Lemma 2.2.4), we can find $u_1 \in V_{\gamma_1}(R)$ and $u_2 \in V_{\gamma_2}(R)$ so that $N_{11}^{\gamma_1\gamma_2}(u_1, u_2) = v$, which yields our claim in this case.

Next, suppose α is short. Then we can write $\alpha = \gamma_1 + \gamma_2$ for an appropriate long root γ_1 and short root γ_2. Hence, by (R2) we have

$$\left[\widetilde{X}_{\gamma_1}(u_1), \widetilde{X}_{\gamma_2}(u_2)\right] = \widetilde{X}_\alpha\Big(N_{11}^{\gamma_1\gamma_2}(u_1, u_2)\Big) \cdot \widetilde{X}_{\gamma_1+2\gamma_2}\Big(N_{12}^{\gamma_1\gamma_2}(u_1, u_2)\Big)$$

Again by Lemma 2.2.4 we can choose u_1, u_2 so that $N_{11}^{\gamma_1\gamma_2}(u_1, u_2) = v$. Multiplying both sides by $\widetilde{X}_{\gamma_1+2\gamma_2}\Big(N_{12}^{\gamma_1\gamma_2}(u_1, u_2)\Big)^{-1}$ then yields our first claim. Furthermore, since $\gamma_1 + 2\gamma_2$ is a long root, the preceding case shows that $\widetilde{X}_{\gamma_1+2\gamma_2}\Big(N_{12}^{\gamma_1\gamma_2}(u_1, u_2)\Big)^{-1}$ is contained in the commutator subgroup, and hence so is $\widetilde{X}_\alpha(v)$.

Thus, since all generators of $\widetilde{G}(R)$ are contained in $[\widetilde{G}(R), \widetilde{G}(R)]$, it follows that $\widetilde{G}(R)$ is perfect. Since the natural map $\pi : \widetilde{G}(R) \to E(R)$ is surjective, we conclude that $E(R)$ is perfect as well. □

Lemma 2.3.4. *The Steinberg group commutes with finite products. That is, for any k-algebras $R_1, \dots, R_n$, we have $\widetilde{G}(R_1 \times \cdots \times R_n) = \widetilde{G}(R_1) \times \cdots \times \widetilde{G}(R_n)$.*

Proof. By induction, it suffices to show that $\widetilde{G}(A \times B) \cong \widetilde{G}(A) \times \widetilde{G}(B)$ for any k-algebras A and B. We do this by mimicking an argument of Stein [Ste73, Lemma 2.12]; note that our result is not a special case as Stein considers only split groups. First, we note that the projections of $A \times B$ onto its components induce a surjective group homomorphism

$$p : \widetilde{G}(A \times B) \to \widetilde{G}(A) \times \widetilde{G}(B), \qquad \widetilde{X}_\alpha(v) = \widetilde{X}_\alpha(v_1, v_2) \mapsto \Big(\widetilde{X}_\alpha(v_1), \widetilde{X}_\alpha(v_2)\Big)$$

Next we define a map in the reverse direction by

$$\begin{aligned} s : \widetilde{G}(A) \times \widetilde{G}(B) &\to \widetilde{G}(A \times B) \\ \left(\widetilde{X}_\alpha(v_1), 1\right) &\mapsto \widetilde{X}_\alpha(v_1, 0) \\ \left(1, \widetilde{X}_\alpha(v_2)\right) &\mapsto \widetilde{X}_\alpha(0, v_2) \end{aligned}$$

(This defines s on a set of generators for $\widetilde{G}(A) \times \widetilde{G}(B)$, and we then extend it to the whole group by multiplicativity.) It follows immediately from the definitions that $p \circ s$ and $s \circ p$ are the respective identity maps on generating sets. Thus, it remains to show that s is a homomorphism by verifying that s takes all the defining relations in $\widetilde{G}(A) \times \widetilde{G}(B)$ to relations in $\widetilde{G}(A \times B)$. We need to check that s preserves three kinds of relations:

(i) The defining relations of $\widetilde{G}(A)$ applied to the generators $\left(\widetilde{X}_\alpha(v_1), 1\right)$.

(ii) The defining relations of $\widetilde{G}(B)$ applied to the generators $\left(1, \widetilde{X}_\alpha(v_2)\right)$.

(iii) $\left[\left(\widetilde{X}_\alpha(a), 1\right), \left(1, \widetilde{X}_\beta(b)\right)\right] = 1$ for all $\alpha, \beta \in \Phi_k$ and all $a \in V_\alpha(A), b \in V_\beta(B)$.

It is clear that s preserves (i) and (ii), so it remains to show that s preserves (iii). First consider the case $\alpha \neq -\beta$. Let $a \in V_\alpha(A)$ and $b \in V_\beta$. Then

$$s\left[\left(\widetilde{X}_\alpha(a), 1\right), \left(1, \widetilde{X}_\beta(b)\right)\right] = \left[\widetilde{X}_\alpha(a, 0), \widetilde{X}_\beta(0, b)\right] = \prod_{\substack{i,j \geq 1 \\ i\alpha + j\beta \in \Phi_k}} X_{i\alpha + j\beta}\left(N_{ij}^{\alpha\beta}\big((a, 0), (0, b)\big)\right)$$

Since $N_{ij}^{\alpha\beta}$ is homogeneous of degree i in the first argument and of degree j in the second argument, it is at least linear in each argument. So each term of $N_{ij}^{\alpha\beta}\big((a, 0), (0, b)\big)$ includes a factor of $(a, 0) \cdot (0, b) = (0, 0) = 0 \in A \times B$. Hence $N_{ij}^{\alpha\beta}\big((a, 0), (0, b)\big) = 0$ and the product is trivial, so s preserves (iii) when $\alpha \neq -\beta$.

Now consider relation (iii) with $\alpha = -\beta$. For any $b \in B$, by Lemma 2.3.3, the element $\widetilde{X}_{-\alpha}(0,b) \in \widetilde{G}(0 \times B) \subset \widetilde{G}(A \times B)$ can be written as a product

$$\widetilde{X}_{-\alpha}(0,b) = \prod_i \widetilde{X}_{\gamma_i}(0,u_i)$$

where $\gamma_i \neq -\alpha$ for all i, with each $u_i \in B$. Thus, we have

$$s\left[\left(\widetilde{X}_\alpha(a),1\right),\left(1,\widetilde{X}_{-\alpha}(b)\right)\right] = \left[\widetilde{X}_\alpha(a,0),\widetilde{X}_{-\alpha}(0,b)\right] = \left[\widetilde{X}_\alpha(a,0),\prod_i \widetilde{X}_{\gamma_i}(0,u_i)\right]$$

Since $\gamma_i \neq -\alpha$, by the previous case, $\widetilde{X}_\alpha(a,0)$ commutes with each factor $\widetilde{X}_{\gamma_i}(0,u_i)$. Consequently, it commutes with the whole product, hence the last commutator vanishes. This shows that s preserves (iii) when $\alpha = -\beta$, completing the proof. $\square$

With the previous two lemmas in hand, we can proceed to the proof of Proposition 2.3.2.

Proof of Proposition 2.3.2. Suppose $R = R_1 \times \cdots \times R_n$, where $R_1, \ldots, R_n$ are local k-algebras. Since the elementary subgroup and the Steinberg group both commute with finite products (Lemma 2.3.4), it follows that

$$\pi_R = \prod_{i=1}^n \pi_{R_i},$$

and hence

$$\ker \pi_R = \prod_{i=1}^n \ker \pi_{R_i} \subset \prod_{i=1}^n \widetilde{G}(R_i) \cong \widetilde{G}(R).$$

Applying [Sta20, Theorem 1.3] to each local factor R_i, we see that $\ker \pi_{R_i}$ is central in $\widetilde{G}(R_i)$. Consequently, $\ker \pi_R$ is central in $\widetilde{G}(R)$, as claimed. $\square$

Our main application of Proposition 2.3.2 will be to the case where $R = A$ is the algebra obtained in Proposition 2.4.6. This application is used in Proposition 2.5.2.

2.4 Algebraic ring associated to an abstract representation

With our descriptions of the elementary and Steinberg groups of $\mathrm{SU}_{2n}(L,h)$, we now embark on the proof of Theorem 2.0.1, starting with the construction of A and f as in the statement of the theorem. In this section, we describe an algebraic ring A associated to an abstract represtation $\rho : G(k) \to \mathrm{GL}_m(K)$, then show how ρ can be lifted to a representation $\widetilde{\sigma}$ of the Steinberg group $\widetilde{G}(A)$. These techniques go back to a construction of Kassabov and Sapir [KS09], and more importantly extend the methods introduced in [Rap11] and [Rap13] to verify the Borel-Tits conjecure for split groups.

Let $\rho\colon G(k) \to \mathrm{GL}_m(K)$ be an abstract representation (with K an algebraically closed field of characteristic zero). We set $H = \overline{\rho(G(k))}$ to be the Zariski closure of the image of ρ. Furthermore, we let $\rho_0\colon G_0(k) \to \mathrm{GL}_m(K)$ denote the restriction $\rho|_{G_0(k)}$ (see Definition 2.2.5). By [Rap11, Theorem 3.1], we can associate to ρ_0 an algebraic ring A, together with a ring homomorphism $f\colon k \to A$ with Zariski-dense image, as follows. Recall that if $\alpha \in \Phi_k$ is a long root, then $V_\alpha(k) = k$, whereas if α is short, then $V_\alpha(k) = L = k \oplus k\sqrt{d}$.

Definition 2.4.1. For $\alpha \in \Phi_k$, define $A_\alpha = \overline{\rho(X_\alpha(k))}$ and

$$f_\alpha\colon k \to A_\alpha, \qquad u \mapsto \rho_0(X_\alpha(u)).$$

Note that for α short, the root subgroup $X_\alpha(V_\alpha(k))$ has dimension 2 (over k), and in this case, A_α is the closure of the image of the 1-dimensional subgroup $X_\alpha(k) \subset X_\alpha(V_\alpha(k))$, arising from the natural embedding $k \hookrightarrow L = V_\alpha(k)$.

As shown in [Rap11, Theorem 3.1], each A_α has the structure of an algebraic ring (see Appendix A for some background on algebraic rings) — we recall that the addition operation is obtained simply by restricting matrix multiplication in H to A_α, whereas the multiplication operation is defined using the Steinberg commutator relations. Moreover, for any $\alpha, \beta \in \Phi_k$, there exists an isomorphism $\pi_{\alpha\beta}\colon A_\alpha \to A_\beta$ of algebraic rings such that $\pi_{\alpha\beta} \circ f_\alpha = f_\beta$ ([Rap11, Lemma 3.3]). We denote this common algebraic ring by A, and, for each $\alpha \in \Phi_k$, we fix an isomorphism of algebraic rings $\pi_\alpha\colon A \to A_\alpha$ such that $\pi_{\alpha\beta} \circ \pi_\alpha = \pi_\beta$. So we have a ring homomorphism $f\colon k \to A$ with Zariski-dense image such that $\pi_\alpha \circ f = f_\alpha$ for all $\alpha \in \Phi_k$. The following commutative diagram depicts the situation.

$$\begin{tikzcd}
 & k \arrow[d, "f"] \arrow[ddl, bend right, "f_\alpha"'] \arrow[ddr, bend left, "f_\beta"] & \\
 & A \arrow[dl, "\pi_\alpha"', "\cong"] \arrow[dr, "\pi_\beta", "\cong"'] & \\
A_\alpha \arrow[rr, "\cong", "\pi_{\alpha\beta}"'] & & A_\beta
\end{tikzcd}$$

Also from [Rap11, Theorem 3.1], we have (injective) regular maps $\psi_\alpha^1 : A \to H$ satisfying $(\psi_\alpha^1 \circ f)(u) = (\rho_0 \circ X_\alpha)(u)$ for all $u \in k$. In other words, the following diagram commutes.

$$\begin{tikzcd}
k \arrow[r, "X_\alpha"] \arrow[d, "f"] & G_0(k) \arrow[d, "\rho_0"] \\
A \arrow[r, "\psi_\alpha^1"] & H
\end{tikzcd}$$

The algebraic ring A plays a pivotal role in the proof of the main results of [Rap11], i.e. obtaining a standard description of ρ_0. It turns out that A also suffices for the analysis of the representation ρ. The precise statement that is needed in our context will be given in Proposition 2.4.6 below. First, we make the following construction. Given a short root $\alpha \in \Phi_k$, let us define a subset $B_\alpha \subset \mathrm{GL}_m(K)$.

$$B_\alpha = \overline{\rho\left(X_\alpha\left(k\sqrt{d}\right)\right)}$$

We also define the following map:

$$g_\alpha : k \to B_\alpha, \qquad u \mapsto \rho\left(X_\alpha\left(u\sqrt{d}\right)\right).$$

Our next step is to give an isomorphism analogous to $\pi_{\alpha\beta}$ to identify A_β and B_α for some relative roots α, β. Rapinchuk constructs $\pi_{\alpha\beta}$ in [Rap11, Lemma 3.3] as follows: for a particular pair of roots $\alpha, \beta \in \mathsf{B}_2$ with α short, β long, and $\beta - \alpha \in \Phi_k$, $\pi_{\alpha\beta}$ is given by

$$\pi_{\alpha\beta}(x) = \left[x, f_{\beta-\alpha}\left(\frac{1}{2}\right)\right] \tag{2.3}$$

After translating through an identification $\mathsf{B}_2 \cong \mathsf{C}_2$, for such a pair α, β as above, they span a copy of C_2 sitting inside Φ_k consisting of the roots $\{\pm\alpha, \pm\beta, \pm(\beta-\alpha), \pm(2\alpha-\beta)\}$, where $\pm\alpha$ and $\pm(\beta-\alpha)$ are short and the others are long. All of the Steinberg commutator relations in the following lemma involve only this copy of C_2 inside Φ_k.

Lemma 2.4.2. *Take the short root $\alpha = \alpha_1 - \alpha_3$ and the long root $\beta = -2\alpha_3$. Then there is an isomorphism of algebraic varieties $\pi\colon B_\alpha \to A_\beta$ such that $\pi \circ g_\alpha = f_\beta$.*

Proof. We perform a similar computation to that in [Rap11, Lemma 3.3]. Define a regular map $\pi\colon B_\alpha \to H$ by

$$\pi(x) = \left[x, g_{\beta-\alpha}\left(\frac{-1}{2d}\right)\right].$$

(Compare with equation (2.3).) Note that $2d$ is invertible as $\operatorname{char} k = 0$. This commutator occurs inside $\mathrm{GL}_m(K)$ and multiplication is regular, so π is regular. By Lemma 2.2.4 (2), we have $N_{11}^{\alpha,\beta-\alpha}(u,v) = -\operatorname{Tr}(uv)$. Now let $s \in k$. Then

$$\begin{aligned}\pi \circ g_\alpha(s) &= \left[\rho \circ X_\alpha\left(s\sqrt{d}\right), \rho \circ X_{\beta-\alpha}\left(\frac{-\sqrt{d}}{2d}\right)\right] = \rho\left[X_\alpha\left(s\sqrt{d}\right), X_{\beta-\alpha}\left(\frac{-1}{2\sqrt{d}}\right)\right] \\ &= \rho \circ X_\beta\left(N_{11}^{\alpha,\beta-\alpha}\left(s\sqrt{d}, \frac{-1}{2\sqrt{d}}\right)\right) = \rho \circ X_\beta\left(-\operatorname{Tr}\left(\frac{-s}{2}\right)\right) = \rho\left(X_\beta(s)\right) = f_\beta(s)\end{aligned}$$

This shows that $\pi \circ g_\alpha = f_\beta$, and that π maps $g_\alpha(k)$ into $f_\beta(k)$. Since π is regular, it follows that $\pi(B_\alpha) \subset A_\beta$. It remains to show that π is invertible. First, using Lemma 2.2.4 (3), we obtain

$$N_{11}^{\beta,\alpha-\beta}(v,u) = -uv$$
$$N_{12}^{\beta,\alpha-\beta}(v,u) = vu\overline{u}$$

Let $h = h_{2\alpha-\beta}(1/2) = h_{2\alpha_1}(1/2)$ be the element introduced in Definition 2.2.6 and define

$$\nu : A_\beta \to B_\alpha, \qquad \nu(y) = \rho(h) \cdot \big[y, g_{\alpha-\beta}(-1)\big] \cdot \big[y, g_{\alpha-\beta}(1)\big]^{-1} \cdot \rho(h)^{-1}.$$

It is clear that ν is a regular map; we claim it is an inverse for π. Let $t \in k$. Using the commutator relation in Theorem 2.2.1(3), we have

$$\begin{aligned}\Big[X_\beta(t), X_{\alpha-\beta}\left(-\sqrt{d}\right)\Big] &= X_\alpha\Big(N_{11}^{\beta,\alpha-\beta}\left(t, -\sqrt{d}\right)\Big) \cdot X_{2\alpha-\beta}\Big(N_{12}^{\beta,\alpha-\beta}\left(t, -\sqrt{d}\right)\Big)\\ &= X_\alpha\Big(t\sqrt{d}\Big) \cdot X_{2\alpha-\beta}\Big(-td\Big).\end{aligned}$$

Thus

$$\begin{aligned}&\Big[X_\beta(t), X_{\alpha-\beta}\left(-\sqrt{d}\right)\Big] \cdot \Big[X_\beta(t), X_{\alpha-\beta}\left(\sqrt{d}\right)\Big]^{-1}\\ &= X_\alpha\Big(t\sqrt{d}\Big) \cdot \cancel{X_{2\alpha-\beta}\Big(-td\Big)} \cdot \cancel{X_{2\alpha-\beta}\Big(-td\Big)^{-1}} \cdot X_\alpha\Big(-t\sqrt{d}\Big)^{-1} = X_\alpha\left(2t\sqrt{d}\right).\end{aligned}$$

We also have the relation

$$h \cdot X_\alpha(2v) \cdot h^{-1} = X_\alpha\left(v\right)$$

for all $v \in L$. Putting everything together, we obtain

$$\begin{aligned}\nu \circ f_\beta(t) &= \rho\left(h \cdot \Big[X_\beta(t), X_{\alpha-\beta}\left(-\sqrt{d}\right)\Big] \cdot \Big[X_\beta(t), X_{\alpha-\beta}\left(\sqrt{d}\right)\Big]^{-1} \cdot h^{-1}\right)\\ &= \rho\left(h \cdot X_\alpha\left(2t\sqrt{d}\right) \cdot h^{-1}\right) = \rho\left(X_\alpha\left(t\sqrt{d}\right)\right) = g_\alpha(t).\end{aligned}$$

Thus, $\nu \circ \pi$ and $\pi \circ \nu$ are the respective identity maps on dense subsets of A_β and B_α. Since they are regular, it follows that they are the respective identities on the whole space, so ν is the inverse of π as claimed. $\square$

Remark 2.4.3. Although we fixed the roots α and β in the preceding argument for the sake of concreteness, essentially the same calculations can be carried for any short root α and long root β such that $\alpha - \beta$ is a root (with appropriate modifications to the definitions of π and ν, depending on the signs arising in computing N_{11} and N_{12}). We carry this out in Lemma B.9.

Remark 2.4.4. In Appendix A, we describe an algebraic ring structure on $B_{\alpha_1 - \alpha_3}$ and show that π is not just an isomorphism of varieties, but an isomorphism of algebraic rings.

Next, we make the following observation, which will streamline the proof of Proposition 2.4.6 by allowing us to consider just a single root of each length. In the statement, we refer to the group schemes V_α introduced in Theorem 2.2.1. Given a k-algebra homomorphism $f\colon k \to A$, we denote by $V_\alpha(f)\colon V_\alpha(k) \to V_\alpha(A)$ the associated group homomorphism. Note that if α is a long root, then $V_\alpha(f)$ can be identified with f, and if α is short, then $V_\alpha(f)$ is the map $a + b\sqrt{d} \mapsto f(a) + f(b)\sqrt{d}$ — in particular, if α and β have the same length, then the homomorphisms $V_\alpha(f)$ and $V_\beta(f)$ coincide.

Lemma 2.4.5. *Let $\alpha, \beta \in \Phi_k$ be roots of the same length. Then*

(1) There exists $w \in E(k)$ such that for all $v \in V_\alpha(k)$, we have $w \cdot X_\alpha(v) \cdot w^{-1} = X_\beta(\varphi v)$, where $\varphi v = \pm v_{\pm 1}$.

(2) Let $f\colon k \to A$ be a k-algebra homomorphism. Suppose there exists a regular map $\psi_\alpha\colon V_\alpha(A) \to H$ such that $\psi_\alpha \circ V_\alpha(f) = \rho \circ X_\alpha$. Let w, φ be as in part (1), and define $\psi_\beta\colon V_\beta(A) \to H$ by

$$\psi_\beta(v) = \rho(w) \cdot \psi_\alpha(\varphi v) \cdot \rho(w)^{-1}$$

Then ψ_β is regular and satisfes $\psi_\beta \circ V_\beta(f) = \rho \circ X_\beta$.

Proof. (1) It is well-known that the Weyl group W acts transitively on roots of the same length, so there exists $\widetilde{w} \in W$ such that $\widetilde{w}\alpha = \beta$. Write $\widetilde{w}$ as a product of simple reflections, $\widetilde{w} = w_{\gamma_1} \cdots w_{\gamma_n}$ for roots $\gamma_i \in \Phi_k$. Using the relation (2.2), we have

$$w_{\gamma_i}(1) \cdot X_\alpha(v) \cdot w_{\gamma_i}(1)^{-1} = X_{w_{\gamma_i}\alpha}(\varphi_i v)$$

where $\varphi_i v = \pm v_{\pm 1}$. Let $w = w_{\gamma_1}(1) \cdots w_{\gamma_n}(1)$ and $\varphi = \varphi_1 \cdots \varphi_n$. Then repeatedly applying the above relation yields

$$w \cdot X_\alpha(v) \cdot w^{-1} = X_{w_{\gamma_1} \cdots w_{\gamma_n}\alpha}(\varphi_1 \cdots \varphi_n v) = X_{\widetilde{w}\alpha}(\varphi v) = X_\beta(\varphi v)$$

Since each φ_i is a composition of negation and conjugation, $\varphi v = \pm v_{\pm 1}$.

(2) This is proved by a direct calculation using the result of part (1), the fact that $\varphi^2 = \mathrm{Id}$, and the assumption that $\psi_\alpha \circ V_\alpha(f) = \rho \circ X_\alpha$. Namely, let $v \in V_\beta(A)$. Then

$$\begin{aligned}
\psi_\beta \circ V_\beta(f)(v) &= \rho(w) \cdot \psi_\alpha\Big(\varphi \circ V_\beta(f)(v)\Big) \cdot \rho(w)^{-1} \\
&= \rho(w) \cdot \Big(\psi_\alpha \circ V_\alpha(f)(\varphi v)\Big) \cdot \rho(w)^{-1} \\
&= \rho(w) \cdot \Big(\rho \circ X_\alpha(\varphi v)\Big) \cdot \rho(w)^{-1} \\
&= \rho\Big(w \cdot X_\alpha(\varphi v) \cdot w^{-1}\Big) \\
&= \rho\Big(X_\beta(\varphi^2 v)\Big) \\
&= \rho \circ X_\beta(v)
\end{aligned}$$

$\square$

We now come to one of the main statements of this section.

Proposition 2.4.6. *Let $G = \mathrm{SU}_{2n}(L, h)$ and let $\rho : G(k) \to \mathrm{GL}_m(K)$ be an abstract representation, with K an algebraically closed field of characteristic zero. Set $H = \overline{\rho(G(k))}$.*

There exists a finite-dimensional K-algebra A, a ring homomorphism $f\colon k \to A$ with Zariski-dense image, and for each $\alpha \in \Phi_k$, a regular map $\psi_\alpha\colon V_\alpha(A) \to H$ such that $\psi_\alpha \circ V_\alpha(f) = \rho \circ X_\alpha$.

$$\begin{array}{ccc} V_\alpha(k) & \xrightarrow{X_\alpha} & G(k) \\ {\scriptstyle V_\alpha(f)}\big\downarrow & & \big\downarrow{\scriptstyle \rho} \\ V_\alpha(A) & \xrightarrow{\psi_\alpha} & H \end{array}$$

Proof. We take A and $f\colon k \to A$ to be the algebraic ring and ring homomorphism constructed from the restriction $\rho|_{G_0(k)}$ using [Rap11, Theorem 3.1], as described at the beginning of this section. Recall that for each $\alpha \in \Phi_k$, there is a regular map $\psi_\alpha^1 : A_\alpha \to H$, and an isomorphism $\pi_\alpha\colon A \to A_\alpha$ such that $\pi_\alpha \circ f = f_\alpha$ and $\psi_\alpha^1 \circ V_\alpha(f)|_k = \rho \circ X_\alpha|_k$. By Lemma 2.4.5, it suffices to construct $\psi_\alpha\colon V_\alpha(A) \to H$ satisfying $\psi_\alpha \circ V_\alpha(f) = \rho \circ X_\alpha$ for a single root of each length. If α is a long root, then, since the corresponding root space is 1-dimensional, we can simply set $\psi_\alpha = \psi_\alpha^1$.

Now let us consider the short root $\alpha = \alpha_1 - \alpha_3$. Then, taking $\beta = -2\alpha_3$, Lemma 2.4.2 yields an isomorphism of algebraic varieties $\pi\colon B_\alpha \to A_\beta$ such that $\pi^{-1} \circ f_\beta = g_\alpha$. Define

$$\psi_\alpha^2\colon A \to H, \qquad \psi_\alpha^2 = \iota_B \circ \pi^{-1} \circ \pi_\beta,$$

where $\pi_\beta\colon A \to A_\beta$ is the previously fixed isomorphism satisfying $f_\beta = \pi_\beta \circ f$, and $\iota_B\colon B_\alpha \hookrightarrow H$ is the natural inclusion. Using the identification $V_\alpha(A) = A_L \simeq A^2$, $v = v_1 + v_2\sqrt{d} \mapsto (v_1, v_2)$, we define

$$\psi_\alpha : V_\alpha(A) \to H, \qquad v = (v_1, v_2) \mapsto \psi_\alpha^1(v_1) \cdot \psi_\alpha^2(v_2).$$

Note that for any $u \in k$, we have

$$\begin{aligned} (\psi_\alpha^2 \circ f)(u) &= (\iota_B \circ \pi^{-1} \circ \pi_\beta \circ f)(u) = (\iota_B \circ \pi^{-1} \circ f_\beta)(u) \\ &= (\iota_B \circ g_\alpha)(u) = (\rho \circ X_\alpha)\left(u\sqrt{d}\right) \end{aligned}$$

Hence, for $v = (v_1, v_2) \in V_\alpha(k) = L = k \oplus k\sqrt{d}$, we have

$$\begin{aligned}\Big(\psi_\alpha \circ V_\alpha(f)\Big)(v) = \psi_\alpha\Big(f(v_1), f(v_2)\Big) &= \Big(\psi_\alpha^1 \circ f(v_1)\Big) \cdot \Big(\psi_\alpha^2 \circ f(v_2)\Big) = \\ &= \Big(\rho \circ X_\alpha(v_1)\Big) \cdot \Big(\rho \circ X_\alpha\left(v_1\sqrt{d}\right)\Big) = \\ &= \rho \circ X_\alpha\left(v_1 + v_2\sqrt{d}\right) = (\rho \circ X_\alpha)(v)\end{aligned}$$

Thus, $\psi_\alpha \circ V_\alpha(f) = \rho \circ X_\alpha$, as needed. $\square$

Remark 2.4.7. The algebraic ring $A \cong A_\beta \cong B_\alpha$ above is a connected algebraic ring over the algebraically closed field K of characteristic zero, so it is a finite-dimensional K-algebra (see [Gre64, Proposition 5.1] or [Rap11, Lemma 2.13, Proposition 2.14], or Appendix A). By [Rap11, Lemma 2.9], A is artinian so it is a finite product of local rings, and each of those rings is a k-algebra. In particular, A is connected and satisfies the hypotheses of Proposition 2.3.2. This is a key input for Proposition 2.5.2.

Note that since $f : k \to A$ has Zariski-dense image in A, for any root α the map $V_\alpha(f) : V_\alpha(k) \to V_\alpha(A)$ has Zariski-dense image in $V_\alpha(A)$. If α is long this is vacuous, and if α is short this just says that applying f to both components of k^2 has dense image in A^2.

To conclude this section, we show the representation ρ can be lifted to a representation $\widetilde{\sigma}$ of the Steinberg group $\widetilde{G}(A)$. The precise statement is as follows.

Proposition 2.4.8. *Let $\rho\colon G(k) \to \mathrm{GL}_m(K)$ be an abstract representation, with K an algebraically closed field of characteristic zero. Set $H = \overline{\rho(G(k))}$, and let A, f, ψ_α be as in Proposition 2.4.6. Let $\widetilde{F} : \widetilde{G}(k) \to \widetilde{G}(A)$ be the homomorphism induced by f and $\pi_k\colon \widetilde{G}(k) \to G(k)$ be the natural map. Then there exists a group homomorphism $\widetilde{\sigma} : \widetilde{G}(A) \to H$ such that $\widetilde{\sigma} \circ \widetilde{F} = \rho \circ \pi_k$ and $\widetilde{\sigma} \circ \widetilde{X}_\alpha = \psi_\alpha$ for all $\alpha \in \Phi_k$; i.e. $\widetilde{\sigma}$ makes the following diagrams commute.*

$$
\begin{tikzcd}
V_\alpha(A) \arrow[rr, "\widetilde{X}_\alpha"] \arrow[dr, "\psi_\alpha"'] & & \widetilde{G}(A) \arrow[dl, dotted, "\widetilde{\sigma}"] \\
& H &
\end{tikzcd}
\qquad
\begin{tikzcd}
\widetilde{G}(k) \arrow[r, "\pi_k"] \arrow[d, "\widetilde{F}"] & G(k) \arrow[d, "\rho"] \\
\widetilde{G}(A) \arrow[r, dotted, "\widetilde{\sigma}"] & H
\end{tikzcd}
$$

Proof. In order for the relation $\widetilde{\sigma} \circ \widetilde{X}_\alpha = \psi_\alpha$ to hold, we must define $\widetilde{\sigma}$ on the generators of $\widetilde{G}(A)$ by

$$\widetilde{\sigma}\Big(\widetilde{X}_\alpha(v)\Big) := \psi_\alpha(v).$$

To show that $\widetilde{\sigma}$ is well defined, we need to verify that the relations (R1) and (R2) hold, replacing $\widetilde{X}_\alpha$ with ψ_α. For this, we imitate the proof of [Rap11, Proposition 4.2], starting with (R1). To make the notation less burdensome, let us set $\tilde{f} = V_\alpha(f)$. Let $a, b \in \tilde{f}(V_\alpha(k))$, and choose $v, w \in V_\alpha(k)$ such that $\tilde{f}(v) = a$ and $\tilde{f}(w) = b$. Then

$$
\begin{aligned}
\psi_\alpha(a) \cdot \psi_\alpha(b) &= \left(\psi_\alpha \circ \tilde{f}\right)(v) \cdot \left(\psi_\alpha \circ \tilde{f}\right)(w) = (\rho \circ X_\alpha)(v) \cdot (\rho \circ X_\alpha)(w) \\
&= \rho\Big(X_\alpha(v) \cdot X_\alpha(w)\Big) = (\rho \circ X_\alpha)(v + w) = \left(\psi_\alpha \circ \tilde{f}\right)(v + w) \\
&= \psi_\alpha\left(\tilde{f}(v) + \tilde{f}(w)\right) = \psi_\alpha(a + b).
\end{aligned}
$$

Thus, we have two regular maps $V_\alpha(A) \times V_\alpha(A) \to H$ given by

$$(a, b) \mapsto \psi_\alpha(a) \cdot \psi_\alpha(b) \quad \text{and} \quad (a, b) \mapsto \psi_\alpha(a + b)$$

that agree on the Zariski-dense subset $\tilde{f}(V_\alpha(k)) \subset V_\alpha(A)$. So, they must coincide on all of $V_\alpha(A)$. This verifies (R1). By a similar calculation, for any $\alpha, \beta \in \Phi_k$ with $\alpha \neq \pm\beta$, we have two regular maps $V_\alpha(A) \times V_\beta(A) \to H$ given by

$$(a, b) \mapsto \Big[\psi_\alpha(a), \psi_\beta(b)\Big] \quad \text{and} \quad (a, b) \mapsto \prod_{i,j>0} \psi_{i\alpha + j\beta}\Big(N_{ij}^{\alpha\beta}(a, b)\Big)$$

that agree on the Zariski-dense subset $\tilde{f}(V_\alpha(k))$, and hence on all of $V_\alpha(A)$. Thus, (R2) holds as well. This shows that $\widetilde{\sigma}$ is well defined, and, by construction, satisfies $\widetilde{\sigma} \circ \widetilde{X}_\alpha = \psi_\alpha$

for all $\alpha \in \Phi_k$. Finally, for any $\alpha \in \Phi_k$ and $v \in V_\alpha(k)$, we have

$$(\widetilde{\sigma} \circ \widetilde{F})\Big(\widetilde{X}_\alpha(v)\Big) = \widetilde{\sigma}\Big(\widetilde{X}_\alpha\left(\tilde{f}(v)\right)\Big) = \psi_\alpha\left(\tilde{f}(v)\right) = (\rho \circ X_\alpha)(v) = (\rho \circ \pi_k)\Big(\widetilde{X}_\alpha(v)\Big),$$

from which it follows that $\widetilde{\sigma} \circ \widetilde{F} = \rho \circ \pi_k$. $\square$

2.5 Rationality of σ

We retain the notations of the previous section. Namely, we let $G = \mathrm{SU}_{2n}(L, h)$, consider an abstract representation $\rho\colon G(k) \to \mathrm{GL}_m(K)$, with K an algebraically closed field of characteristic zero, and set $H = \overline{\rho(G(k))}$. By Proposition 2.4.6, one can associate to ρ an algebraic ring A together with a ring homomorphism $f\colon k \to A$ with Zariski-dense image. Then in Proposition 2.4.8, we constructed a group homomorphism $\widetilde{\sigma}\colon \widetilde{G}(A) \to H$ that lifts ρ to a representation of the Steinberg group $\widetilde{G}(A)$. More precisely, the diagrams formed by the solid arrows below commute.

$$\begin{tikzcd}
\widetilde{G}(k) \arrow[r, "\widetilde{F}"] \arrow[d, "\pi_k"'] & \widetilde{G}(A) \arrow[d, "\pi_A"] \arrow[ddr, bend left, "\widetilde{\sigma}"] & \\
G(k) \arrow[r, "F"] \arrow[drr, bend right, "\rho"'] & G(A) \arrow[dr, dotted, "\sigma"] & \\
& & H
\end{tikzcd} \tag{2.4}$$

(Here F and $\widetilde{F}$ denote the group homomorphisms induced by f.) To complete the proof of Theorem 2.0.1, it remains to show the existence, under the assumptions of the theorem, of a morphism of algebraic groups $\sigma\colon G(A) \to H$ as indicated in the diagram. Note that before this section, we have not made use of the assumption that $R_u H$ is commutative; it appears in Lemma 2.5.5. Before addressing the algebraicity of σ, we show that H is connected and perfect. In the following lemma, H° denotes the connected component of the identity of H.

Lemma 2.5.1. *The homomorphism $\widetilde{\sigma} : \widetilde{G}(A) \to H$ is surjective and the algebraic group H is connected and perfect. That is, $H = H^\circ = \widetilde{\sigma}\left(\widetilde{G}(A)\right) = [H^\circ, H^\circ] = [H, H]$.*

Proof. Let $\mathcal{H} \subset H$ be the (abstract) subgroup generated by the elements $\psi_\alpha(v) = \widetilde{\sigma} \circ \widetilde{X}_\alpha(v)$ for all $\alpha \in \Phi_k$ and $v \in V_\alpha(A)$, where $\psi_\alpha \colon V_\alpha(A) \to H$ are the regular maps introduced in Proposition 2.4.6.

$$\mathcal{H} = \langle \psi_\alpha(v) : v \in V_\alpha(A), \alpha \in \Phi_k \rangle$$

Since, by definition, the $\widetilde{X}_\alpha(v)$ generate $\widetilde{G}(A)$, their images generate $\widetilde{\sigma}\left(\widetilde{G}(A)\right)$, so $\mathcal{H} = \widetilde{\sigma}(\widetilde{G}(A))$. Now, A is connected by Remark 2.4.7, so $V_\alpha(A)$ is connected, and hence $\psi_\alpha(V_\alpha(A))$ is connected. Thus, it follows from [Bor91, Proposition 2.2] that $\mathcal{H}$ is Zariski-closed and connected, so $\mathcal{H} \subset H^\circ$. On the other hand, $\mathcal{H}$ contains $\rho(E(k))$, which is Zariski-dense in H. So, $\mathcal{H}$ is Zariski-dense in H, and since $\mathcal{H}$ is closed, we see that $\mathcal{H} = H$. This shows that $\mathcal{H} = H = H^\circ$. Furthermore, by Lemma 2.3.3, $\widetilde{G}(A)$ is equal to its commutator subgroup, so the same is true for $\mathcal{H} = \widetilde{\sigma}(\widetilde{G}(A))$. □

In the remainder of the this section, we will complete the proof of Theorem 2.0.1 using a strategy inspired by that of [Rap11, §5.6]. Namely, let $Z(H)$ be the center of H, set $\overline{H} = H/Z(H)$, and denote by $\nu \colon H \to \overline{H}$ the corresponding quotient map. We first show that $\widetilde{\sigma}$ gives rise to a group homomorphism $\overline{\sigma} \colon G(A) \to \overline{H}$ satisfying $\overline{\sigma} \circ \pi_A = \nu \circ \widetilde{\sigma}$, and verify that $\overline{\sigma}$ is in fact a morphism of algebraic groups. Then, using the assumption that the unipotent radical $U = R_u(H)$ is commutative (together with our standing hypothesis that $\operatorname{char} K = 0$), we lift $\overline{\sigma}$ to the required morphism of algebraic groups $\sigma \colon G(A) \to H$.

Proposition 2.5.2. *There exists a group homomorphism $\overline{\sigma} \colon G(A) \to \overline{H}$ such that $\overline{\sigma} \circ \pi_A = \nu \circ \widetilde{\sigma}$, where $\pi_A \colon \widetilde{G}(A) \to G(A)$ is the canonical map.*

$$\begin{array}{ccc} \widetilde{G}(A) & \xrightarrow{\pi_A} & G(A) \\ \Big\downarrow \widetilde{\sigma} & & \Big\downarrow \overline{\sigma} \\ H & \xrightarrow[\nu]{} & \overline{H} \end{array}$$

Proof. First, it follows from Remark 2.4.7 that $G(A) = E(A)$. Moreover, since π_A is surjective, we have $E(A) \cong \widetilde{G}(A)/\ker \pi_A$. Now, according to Proposition 2.3.2, $\ker \pi_A$ is central in $\widetilde{G}(A)$, and by Lemma 2.5.1, $\widetilde{\sigma}\colon \widetilde{G}(A) \to H$ is surjective. Consequently, we have $\widetilde{\sigma}(\ker \pi_A) \subset Z(H)$, and hence $\widetilde{\sigma}$ induces a map $\overline{\sigma}\colon G(A) \to \overline{H}$ on the quotients satisfying $\overline{\sigma} \circ \pi_A = \nu \circ \widetilde{\sigma}$. The following commutative diagram with exact rows depicts the situation.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \ker \pi_A & \hookrightarrow & \widetilde{G}(A) & \xrightarrow{\pi_A} & G(A) & \longrightarrow & 1 \\ & & \Big\downarrow \widetilde{\sigma} & & \Big\downarrow \widetilde{\sigma} & & \vdots\, \overline{\sigma} & & \\ 1 & \longrightarrow & Z(H) & \hookrightarrow & H & \xrightarrow{\nu} & \overline{H} & \longrightarrow & 1 \end{array}$$

□

Next we show that $\overline{\sigma}$ is algebraic.

Proposition 2.5.3. *The group homomorphism $\overline{\sigma}\colon G(A) \to \overline{H}$ from Proposition 2.5.2 is a morphism of algebraic groups.*

Proof. Using [Bor91, Proposition 2.2] or [Sta20, Lemma 2.14(iv)], we can write $G(A) = E(A)$ as a product

$$G(A) = \prod_{\alpha \in \Phi_k} U_\alpha^{e_\alpha} = \prod_{i=1}^{m} U_{\alpha_i}^{e_i}$$

for some sequence of roots $\{\alpha_1, \dots, \alpha_m\} \subset \Phi_k$, where $U_{\alpha_i} = X_{\alpha_i}(V_{\alpha_i}(A))$ is the root subgroup associated with $\alpha \in \Phi_k$ and each $e_i = \pm 1$. Let $X = \prod_{i=1}^{m} A_{\alpha_i}$ be a product of copies of A indexed by the α_i, and define a regular map $s\colon X \to G(A)$ by

$$s(a_1, \dots, a_m) = X_{\alpha_1}(a_1)^{e_1} \cdots X_{\alpha_m}(a_m)^{e_m}.$$

The maps X_{α_i} are all regular, so s is regular. Let us also define a regular map $t' \colon X \to H$ by

$$t'(a_1, \dots, a_m) = \psi_{\alpha_1}(a_1)^{e_1} \cdots \psi_{\alpha_m}(a_m)^{e_m},$$

where the ψ_{α_i} are the morphisms from Proposition 2.4.6. Since each ψ_{α_i} is regular, t' is regular. Set $t = \nu \circ t'$. One easily checks that $\overline{\sigma} \circ s = t$. In particular, for any $x_1, x_2 \in X$, the condition $s(x_1) = s(x_2)$ implies $t(x_1) = t(x_2)$. So, by [Rap13, Lemma 3.10], $\overline{\sigma}$ is a rational map. Hence, there is an open subset of $G(A)$ on which $\overline{\sigma}$ is regular. Then, applying [Rap13, Lemma 3.12], we conclude that $\overline{\sigma}$ is a morphism of algebraic groups. □

To conclude the argument, we will show that $\overline{\sigma}$ can be lifted to a morphism $\sigma \colon G(A) \to H$. For this, we first discuss several preliminary statements, which are analogues in the present setting of results established in [Rap11, §§5,6].

Let A be the algebraic ring associated with the representation ρ, and let $J \subset A$ be its Jacobson radical. As previously noted, A is a finite-dimensional K-algebra; in particular, A is artinian, and hence $J^d = \{0\}$ for some $d \geq 1$ (see [AM16, Proposition 8.4]). Moreover, by the Wedderburn-Malcev theorem (see [Pie82, Theorem 11.6]), there exists a semisimple subalgebra $\overline{A} \subset A$ such that $A = \overline{A} \oplus J$ as K-vector spaces and $\overline{A} \cong A/J$ as K-algebras. We note that [Rap11, Proposition 2.20] implies that $\overline{A} \simeq K \times \cdots \times K$ (r copies). Since $G = \mathrm{SU}_{2n}(L, h)$ is K-isomorphic to SL_{2n}, it follows that $G(\overline{A})$ is a connected, simply connected, semisimple algebraic group. For the next statement, we consider the canonical homomorphism $A \to A/J$ and set

$$G(A, J) = \ker(G(A) \to G(A/J))$$

to be the corresponding congruence subgroup. We then have the following.

Lemma 2.5.4. *Retain the notation above.*

(1) The congruence subgroup $G(A,J)$ is nilpotent.

(2) We have a Levi decomposition $G(A) = G(A,J) \ltimes G(\overline{A})$.

Proof. (1) Fixing an embedding $G(A) \hookrightarrow \mathrm{GL}_{2n}(A_L)$, it is straightforward to show that

$$\big[G(A,J^a), G(A,J^b)\big] \subset G(A,J^{a+b})$$

for any $a,b \in \mathbb{Z}_{\geq 1}$. Since J is a nilpotent ideal, our claim follows.

(2) Using the fact that $G(A)$ is perfect (see Lemma 2.3.3), this statement is proved by the same argument as [Rap11, Proposition 6.5]. □

Next, we note the following analogue of [Rap11, Proposition 5.5] — this result is proved exactly as in that context, employing Lemma 2.5.1 in place of the corresponding statements in *loc. cit.*. Following the discussion there, we say that H **satisfies condition (Z)** if $Z(H) \cap U = \{e\}$. As previously noted, the following lemma is is the first and only place where the assumption on the commutativity of the unipotent radical $U = R_u(H)$ is used.

Lemma 2.5.5. *Let ρ, H be as above, and let U be the unipotent radical of H.*

(1) If $U = R_u(H)$ is commutative and $\operatorname{char} K = 0$, then H satisfies (Z).

(2) If H satisfies (Z), then $Z(H)$ is finite. If additionally $\operatorname{char} K = 0$, then $Z(H)$ is contained in any Levi subgroup of H.

Proof. (1) Let $H = U \ltimes S$ be a Levi decomposition of H. Since $\operatorname{char} K = 0$, by [Bor91, Remark II.7.3] $U \cong (K^m, +)$ where $m = \dim U$, and the action of S on U gives a rational representation of S on K^m. By Weyl's Theorem, this representation is completely reducible (see [Hum72, Theorem 6.3] and [Hum75, Theorem 13.2], and note that these rely on char $K = 0$). Since $H = [H, H]$ (Lemma 2.5.1) the representation cannot contain the trivial representation, so U has no nonzero elements fixed by the S-action, so $Z(H) \cap U = \{e\}$.

(2) Consider the quotient H/U, which is a reductive algebraic group that coincides with its commutator. So by [Bor91, Corollary 14.2], $Z(H/U)$ is finite. Since $Z(H)\cap U = \{e\}$, the restriction of the quotient map $H \to H/U$ is injective, so $Z(H)$ is finite.

Now suppose char $K = 0$. Let S be a Levi subgroup of H, so $H = U \ltimes S$. Since $Z(H)$ is a finite abelian group, it is reductive, so some conjugate of it is contained in S (see [Mos56]). Since $Z(H)$ is central, it is itself contained in S. $\square$

The next statement completes the proof of Theorem 2.0.1.

Theorem 2.5.6. *Assume that* $U = R_u(H)$ *is commutative and* char $K = 0$*. Then there exists a morphism of algebraic groups* $\sigma\colon G(A) \to H$ *making the diagram below (same as 2.4) commute.*

$$\begin{tikzcd}
\widetilde{G}(k) \arrow[r, "\widetilde{F}"] \arrow[d, "\pi_k"'] & \widetilde{G}(A) \arrow[d, "\pi_A"] \arrow[ddr, bend left, "\widetilde{\sigma}"] & \\
G(k) \arrow[r, "F"] \arrow[drr, bend right, "\rho"'] & G(A) \arrow[dr, dotted, "\sigma"] & \\
& & H
\end{tikzcd} \tag{2.5}$$

Proof. (cf. [Rap11, Proposition 6.6]) Let $G(A) = G(A, J) \ltimes G(\overline{A})$ be the Levi decomposition from Lemma 2.5.4, and set

$$\overline{U} = \overline{\sigma}(G(A, J)), \quad \overline{S} = \overline{\sigma}(G(\overline{A})), \quad \text{and} \quad S = (\nu^{-1}(\overline{S}))^{\circ},$$

where $\nu\colon H \to \overline{H}$ is the quotient map. Then $\overline{H} = \overline{U} \ltimes \overline{S}$ and $H = U \ltimes S$ are also Levi decompositions. By Lemma 2.5.5, we have $Z(H) \subset S$. Consequently, $\overline{S} = S/Z(H)$ and the restriction $\nu|_U : U \to \overline{U}$ is an isomorphism.

Now, since the quotient map $\nu\colon H \to \overline{H}$ is a central isogeny, and, as we observed above, $G(\overline{A})$ is simply connected, it follows from [BT72, Proposition 2.24(i)] that there exists a morphism of algebraic groups $\sigma_S\colon G(\overline{A}) \to S$ such that $\nu|_S \circ \sigma_S = \overline{\sigma}|_{G(\overline{A})}$.

$$\begin{tikzcd} & S \subset H \arrow[d, "\nu|_S"] \\ G(\overline{A}) \arrow[ur, dotted, "\sigma_S"] \arrow[r, "\overline{\sigma}"'] & \overline{S} \subset \overline{H} \end{tikzcd}$$

Define $\sigma_U = \nu|_U^{-1} \circ \left(\overline{\sigma}|_{G(A,J)}\right)$. Then

$$\sigma = (\sigma_U, \sigma_S)\colon G(A) \to H$$

is a morphism of algebraic groups such that $\nu \circ \sigma = \overline{\sigma}$. It remains to show that σ makes the diagram (2.4) commute. Define

$$\chi : \widetilde{G}(A) \to H, \qquad g \mapsto \widetilde{\sigma}(g)^{-1} \cdot (\sigma \circ \pi_A)(g).$$

Since $\widetilde{\sigma}$ and $\sigma \circ \pi_A$ are group homomorphisms, so is χ. Also, by Proposition 2.5.2, we have $\overline{\sigma} \circ \pi_A = \nu \circ \widetilde{\sigma}$, which yields $\nu \circ \sigma \circ \pi_A = \nu \circ \widetilde{\sigma}$. Thus, the image of χ is contained in $\ker \nu = Z(H)$. Since $\widetilde{G}(A)$ coincides with its commutator subgroup, if the image is central in H then we conclude that χ is trivial, and hence $\sigma \circ \pi_A = \widetilde{\sigma}$. Finally, the equality $\sigma \circ F = \rho$ follows from the commutativity of the rest of the diagram and the surjectivity of π_k. $\square$

Chapter 3

Group actions on varieties

In this chapter, we discuss applications of rigidity statements for Chevalley groups to obtain rigidity statements for actions of elementary groups on algebraic varieties. We think of this as an algebraic analog of the Zimmer program (see [Fis11]). Throughout the chapter, K will denote an algebraically closed field of characteristic zero, and all algebraic groups and varieties will be assumed to be K-defined. As in the preceeding chapter, we take a classical approach of identifying algebraic groups and varieties with their K-points. An overline such as $\overline{H}$ denotes the Zariski closure of H. For a quasi-projective variety X, we denote by $\mathbf{Bir}\,(\boldsymbol{X})$ the group of birational morphisms $X \dashrightarrow X$, and $\mathbf{Aut}\,(\boldsymbol{X})$ the subgroup of biregular maps. For a K-algebra A, $\mathbf{Aut}\,(\boldsymbol{A})$ is its group of automorphisms. If X is affine, the groups $\mathrm{Aut}(K[X])$ and $\mathrm{Aut}(X)$ are naturally (anti-)isomorphic (see §3.1).

Definition 3.0.1. Let Γ be an abstract group and X an affine variety. An **abstract action** of Γ on X is given by a group homomorphism $\rho : G \to \mathrm{Aut}(X)$. Equivalently, there is a map $\Gamma \times X \to X$ such that for every $\gamma \in \Gamma$, the map $X \to X, x \mapsto \rho(\gamma)(x)$ is biregular.

Definition 3.0.2. Let $\mathcal{G}$ be an algebraic group and X an affine variety. An abstract action of $\mathcal{G}$ on X is **algebraic** if the map $\mathcal{G} \times X \to X$ is a morphism of varieties ($\mathcal{G} \times X$ is the product variety)[1]. An action is **almost algebraic** if $\mathcal{G}$ contains a finite-index subgroup Δ

[1] It is tempting to characterize an algebraic action as one in which the associated group homomorphism $\rho : \mathcal{G} \to \mathrm{Aut}(X)$ is algebraic, but this does not work because the automorphism group $\mathrm{Aut}(X)$ is "too big" to be an algebraic group. Instead, $\mathrm{Aut}(X)$ has the structure of an ind-group, see [Sha81].

such that $\rho|_\Delta$ gives an algebraic action of Δ on X.

In this chapter, we study a different but related form of ridigity to that in Conjecture 1.0.6. We prove rigidity statements of the form: under some hypotheses, an abstract action of Γ on X agrees with an algebraic action of some algebraic group $\mathcal{G}$ on X, at least on a finite-index subgroup $\Delta \subset \Gamma$ and after passing through some abstract homomorphism $F : \Gamma \to \mathcal{G}$, which may or may not be an embedding. We obtain two main results of this type, both of which rely critically on [Rap19, Theorem 1.1], a rigidity result for Chevalley groups over commutative rings in the spirit of the Borel-Tits conjecture.

We set some notation and terminology. Let Φ be a reduced irreducible root system of rank ≥ 2 and let R be a commutative ring. The pair (Φ, R) is **nice** if $2 \in R^\times$ whenever Φ contains a subsystem of type B_2 and $2, 3 \in R^\times$ if Φ is of type G_2. Let G be the corresponding universal Chevalley-Demazure group scheme over $\mathbb{Z}$, and let $\Gamma = G(R)^+$ be the elementary subgroup of G, the group generated by R-points of unipotent radicals of R-defined parabolic subgroups. Let $g : R \to K$ be a ring homomorphism. A ***g*-derivation** is map $\delta : R \to K$ such that for any $a, b \in R$ we have

$$\delta(a+b) = \delta(a) + \delta(b) \qquad \delta(ab) = \delta(a)g(b) + g(a)\delta(b)$$

The set of all g-derivations is denoted $\mathrm{Der}^g(R, K)$, and has a natural K-vector space structure. We say that R has **property (D_n)** if $\dim_K \mathrm{Der}^g(R, K) \leq n$ for every ring homorphism $g : R \to K$. We focus on the case $n = 1$.

Example 3.0.3. A ring R of S-integers in a number field has (D_1). See [Rap19, Lemma 6.1] and [Rap13, Lemma 4.7] for more general statements.

We recall and rephrase the notion of a standard description from §1. Let $\rho : \Gamma \to \mathrm{GL}_m(K)$ be an abstract representation, where $\Gamma = G(R)^+$ as above. We say ρ has a

standard description if there exists a finite-dimensional commutative K-algebra A, a ring homomorphism $f : R \to A$ with Zariski-dense image, a morphism of algebraic groups $\sigma : G(A) \to \mathrm{GL}_m(K)$, and a finite-index subgroup $\Delta \subset \Gamma$ such that $\rho|_\Delta = (\sigma \circ F)|_\Delta$, where $F : G(R) \to G(A)$ is induced by f. Using this terminology, [Rap19, Theorem 1.1] says that if G is the universal Chevalley group of a reduced irreducible root system Φ of rank ≥ 2, R is a commutative ring with (D_1), (Φ, R) is a nice pair, and $\Gamma = G(R)^+$ then any abstract homomorphism $\rho : \Gamma \to \mathrm{GL}_m(K)$ has a standard description.

3.1 Elementary groups acting on affine varieties

In this section we give two closely related rigidity results for elementary subgroups of Chevalley groups acting on affine varieties. We begin by recalling the correspondence between actions on a variety and on its coordinate ring. For an affine variety X defined over K, there is a natural anti-automorphism $\eta_X : \mathrm{Aut}(X) \cong \mathrm{Aut}(K[X])$ which takes $\phi \in \mathrm{Aut}(X)$ to $\phi^* \in \mathrm{Aut}(K[X])$, where $\phi^*(f) = f \circ \phi$. Let Γ be a group acting on X via $r : \Gamma \to \mathrm{Aut}(X)$. Note that r induces an action $r^* : \Gamma \to \mathrm{Aut}(K[X])$ of Γ on $\mathrm{Aut}(K[X])$ characterized by

$$\Big(r^*(\gamma) f\Big) x = f\Big(r\big(\gamma^{-1}\big) x\Big)$$

for all $x \in X$ and $f \in K[X]$. That is, $r^* = \eta_X \circ r \circ \mathrm{inv}$, where inv is the inversion map $\gamma \mapsto \gamma^{-1}$. We will refer to r^* as the **associated action** on $K[X]$, or the **coaction** induced by r. If we suppress r from the notation (as in e.g. Springer [Spr09, 2.3.5]) and denote $r(\gamma)x = \gamma x$ and $r^*(\gamma)x = \gamma^* x$, this can be rewritten as $(\gamma^* f)(x) = f(\gamma^{-1} x)$. For a comorphism $\psi \in \mathrm{Aut}(K[X])$, denote the associated morphism by $\psi^\vee = \eta_X^{-1}(\psi)$. Clearly $(\phi^*)^\vee = \phi$ and $(\psi^\vee)^* = \psi$. Given an action $\rho : \Gamma \to \mathrm{Aut}(K[X])$, we define $\rho^\vee = \eta_X^{-1} \circ \rho \circ \mathrm{inv}$ to obtain an action $\rho^\vee : \Gamma \to \mathrm{Aut}(X)$, given explicitly by $\rho^\vee(\gamma) = \big(\rho\left(\gamma^{-1}\right)\big)^\vee$. It is clear that $(r^*)^\vee = r$ and $(\rho^\vee)^* = \rho$.

For a group Γ acting on a K-algebra A, we will say the action is **locally finite-dimensional** if the orbit of any single element spans a finite-dimensional K-subspace of A. Recall the following classical result (see [Spr09, Proposition 2.3.6] or [Bor91, Proposition I.1.9]).

Proposition 3.1.1. *If $\mathcal{G}$ is an affine algebraic group acting algebraically on an affine variety X, the associated action on $K[X]$ is locally finite-dimensional.*

We think of the main rigidity statements from this section as a partial converses to this statement. To formulate them, we set some notation. Let K be an algebraically closed field of characteristic zero. Let Φ be a reduced, irreducible root system of rank ≥ 2 and G be the corresponding universal Chevalley-Demazure group scheme over $\mathbb{Z}$. Let R be a commutative ring with (D_1), such that (Φ, R) is a nice pair. Let $\Gamma = G(R)^+$ be the elementary subgroup of G.

Theorem 3.1.2. *Let K, Φ, G, R, Γ be as above. Let X be an affine K-variety, and let Γ act abstractly on X via $r : \Gamma \to \mathrm{Aut}(X)$, such that the associated action on $K[X]$ is locally finite-dimensional. Then there exists an algebraic group $\mathcal{G}$, an algebraic action $\widetilde{r} : \mathcal{G} \to \mathrm{Aut}(X)$, and a finite-index subgroup $\Delta \subset \Gamma$ such that $r|_\Delta = (\widetilde{r} \circ F)|_\Delta$, where $F : G(R) \to G(A)$ is an abstract group homomorphism arising from a ring homomorphism $f : R \to A$ with Zariski-dense image.*

Note that in the proof we will define $\mathcal{G} = \overline{F(\Gamma)}$, so the composition $\widetilde{r} \circ F$ is well-defined. Conceptually, the theorem describes how the abstract action of Γ is rigidified by agreeing with an algebraic action of $\mathcal{G}$. More precisely, the equality $r|_\Delta = (\widetilde{r} \circ F)|_\Delta$ says that the abstract action of Γ on X agrees with the algebraic action of $\mathcal{G}$ on X, after passing through F and restricting to the finite-index subgroup Δ.

We explain how the theorem is a partial converse to Proposition 3.1.1. Suppose an abstract group Γ acts abstractly on an affine K-variety X via r, an algebraic group $\mathcal{G}$ acts algebraically on X via $\widetilde{r}$, and there is a group homomorphism $F : \Gamma \to \mathcal{G}$ such that $r|_\Delta = (\widetilde{r} \circ F)|_\Delta$ for some finite-index subgroup $\Delta \subset \Gamma$. By Proposition 3.1.1, the coaction of $\mathcal{G}$ on $K[X]$ is locally finite-dimensional; we claim it follows that that the coaction of Γ on $K[X]$ is also locally finite-dimensional. Indeed, fix a set $\{\gamma_1, \ldots, \gamma_n\}$ of left coset representatives for Δ in Γ. Then for any $f \in K[X]$ the orbit $\Gamma.f$ can be written

$$\begin{aligned}
\Gamma.f &= \bigcup_{i=1}^{n} \{r(\gamma)f : \gamma \in \gamma_i\Delta\} = \bigcup_{i=1}^{n} \{r(\gamma_i\delta)f : \delta \in \Delta\} \\
&= \bigcup_{i=1}^{n} r(\gamma_i)\{r(\delta)f : \delta \in \Delta\} = \bigcup_{i=1}^{n} r(\gamma_i)\{\widetilde{r}(F(\delta))\, f : \delta \in \Delta\}
\end{aligned}$$

Observe that $\{\widetilde{r}(F(\delta))\, f : \delta \in \Gamma\} \subset \{\widetilde{r}(g)f : g \in \mathcal{G}\}$ and the right hand set is finite-dimensional. Thus the orbit $\Gamma.f$ is a finite union of translations (via K-linear automorphisms $r(\gamma_i)$) of these finite-dimensional subsets, so it is finite-dimensional.

Because of Example 3.0.3, Theorem 3.1.2 applies when R is a ring of S-integers in a number field. However, we can also obtain this case more directly with stronger conclusions, as detailed in the next theorem.

Theorem 3.1.3. *Under the same hypotheses as Theorem 3.1.2, assume in addition that R is a ring of S-integers in a number field. Then there exists an algebraic group $\mathcal{G}$ containing Γ, an algebraic action $\widetilde{r} : \mathcal{G} \to \mathrm{Aut}(X)$, and a finite-index subgroup $\Delta \subset \Gamma$ such that $\widetilde{r}|_\Delta = r|_\Delta$.*

The remainder of this section is occupied with proving Theorems 3.1.2 and 3.1.3. We begin with some remarks and lemmas.

Remark 3.1.4. Let A be a finitely generated K-algebra with a locally finite-dimensional action of a group Γ. Then A contains a finite-dimensional Γ-invariant subspace V which generates A as a K-algebra. Concretely, take a finite generating set $\{f_1, \ldots, f_n\}$ of A and let V be the sum of the K-spans of the orbits of the f_i. By assumption each orbit is finite-dimensional, so V has the desired properties. In particular, this implies that if X is an affine variety and Γ acts on X via r such that the coaction r^* on $K[X]$ is locally finite-dimensional, then the image $r^*(\Gamma) \subset \mathrm{Aut}(K[X])$ is contained in $\mathrm{GL}(V)$. As previously noted, $\mathrm{Aut}(X)$ and $\mathrm{Aut}(K[X])$ are not algebraic groups, but we avoid complications of ind-groups by ensuring that everything happens inside the algebraic group $\mathrm{GL}(V)$.

Remark 3.1.5. Let Γ be a group and V a finite-dimensional K-vector space. In this situation, an action of Γ on V via $\rho : \Gamma \to \mathrm{GL}(V)$ is called a representation, and the coaction is called the corepresentation. Every K-linear automorphism of V extends uniquely to a K-algebra automorphism of the symmetric algebra $\mathrm{Sym}(V)$; this gives an embedding $\mathrm{GL}(V) \hookrightarrow \mathrm{Aut}(\mathrm{Sym}(V))$. Hence an abstract representation $\rho : \Gamma \to \mathrm{GL}(V)$ uniquely extends to an abstract representation $\rho : \Gamma \to \mathrm{Aut}(\mathrm{Sym}(V))$ making the following diagram commute.

$$\begin{tikzcd} & \mathrm{Aut}(\mathrm{Sym}(V)) \\ \Gamma \arrow[ur, dotted, "\rho"] \arrow[r, "\rho"'] & \mathrm{GL}(V) \arrow[u, hook] \end{tikzcd}$$

Next we prove a lemma which establishes that the corepresentation of an algebraic representation is algebraic.

Lemma 3.1.6. *Let $\mathcal{G}$ be an algebraic group, V a K-vector space of dimension $d < \infty$, and $\sigma : \mathcal{G} \to \mathrm{GL}(V)$ an algebraic representation. Extend σ to a representation $\sigma : \mathcal{G} \to \mathrm{Aut}(B)$, where $B = \mathrm{Sym}(V) = K[Y]$ and $Y = \mathbb{A}^d_K$. Then the associated representation $\sigma^\vee : \mathcal{G} \to \mathrm{Aut}(Y)$ is algebraic.*

Proof. Remark 3.1.5 ensures that such an extension of σ exists. Fix an isomorphism of varieties $\phi : Y = \mathbb{A}^d_K \to V$, and consider the following diagram.

$$\begin{array}{ccc} \mathcal{G} \times Y & \xrightarrow{\sigma^\vee} & Y \\ {\scriptstyle \mathrm{inv}\times\phi}\downarrow & & \downarrow{\scriptstyle \phi} \\ \mathcal{G} \times V & \xrightarrow{\sigma} & V \end{array}$$

We have abused notation slightly by having $\sigma, \sigma^\vee$ refer to maps $\mathcal{G} \to \mathrm{GL}(V), \mathcal{G} \to \mathrm{Aut}(Y)$ and also to the associated maps $\mathcal{G} \times V \to V, \mathcal{G} \times Y \to Y$. For $(g, y) \in \mathcal{G} \times Y$ we have

$$\sigma(g^{-1})(\phi(y)) = \left(\sigma^\vee(g)\right)^* (\phi(y)) = \phi\left(\sigma^\vee(g)(y)\right)$$

so the diagram commutes. Since σ is algebraic, $\mathcal{G} \times V \xrightarrow{\sigma} V$ is regular, so we have written $\sigma^\vee$ as the composition $\phi^{-1} \circ \sigma \circ (\mathrm{inv} \times \phi)$ of three regular maps, so it is regular, i.e. $\mathcal{G}$ acts algebraically on Y. □

Lemma 3.1.7. *Let Γ be an abstract group acting on affine varieties X, Y, and let $\theta : X \to Y$ be regular. Then θ is Γ-equivariant if and only if the comorphism θ^* is Γ-equivariant. More precisely, let Γ act on X via r_X and on Y via r_Y, and fix $\gamma \in \Gamma$. Then one of the following diagrams commutes if and only if the other also commutes.*

$$\begin{array}{ccc} X & \xrightarrow{\theta} & Y \\ {\scriptstyle r_X(\gamma^{-1})}\downarrow & & \downarrow{\scriptstyle r_Y(\gamma^{-1})} \\ X & \xrightarrow{\theta} & Y \end{array} \qquad\qquad \begin{array}{ccc} K[X] & \xleftarrow{\theta^*} & K[Y] \\ {\scriptstyle r_X^*(\gamma)}\downarrow & & \downarrow{\scriptstyle r_Y^*(\gamma)} \\ K[X] & \xleftarrow{\theta^*} & K[Y] \end{array}$$

In the diagram on the right, r_X^*, r_Y^* are the respective coactions of r_X, r_Y.

Proof. Straightforward calculation utilizing separatedness of X and Y. □

Remark 3.1.8. Suppose $\mathcal{G}$ is an algebraic group with an abstract subgroup H and H' is a finite-index abstract subgroup of H. Then $\overline{H}$ (Zariski closure) is an algebraic subgroup of

$\mathcal{G}$, and $\overline{H'}$ is a finite-index[2] algebraic subgroup of $\overline{H}$. Furthermore, if $\mathcal{G}$ is connected and $\overline{H} = \mathcal{G}$, then $\overline{H}' = \mathcal{G}$ (use [Bor91, Proposition I.1.2]).

We are now ready to prove Theorem 3.1.2. We start with an overview. First, we use [Rap19, Theorem 1.1] to obtain A, f, F, σ, Δ as in the theorem, and define $\mathcal{G} = \overline{F(\Gamma)}$. Then we construct an algebraic action $\widetilde{r} = \sigma^* : \mathcal{G} \to \mathrm{Aut}(Y)$, where $Y = \mathbb{A}^d_K$. Then we describe a closed Γ-equivariant embedding $\theta : X \hookrightarrow Y$, and use it to show that $r|_\Delta = (\widetilde{r} \circ F)|_\Delta$. Finally, we show that $\widetilde{r} : \mathcal{G} \to \mathrm{Aut}(Y)$ leaves X invariant, so that we can think of $\widetilde{r}$ as an algebraic action $\widetilde{r} : \mathcal{G} \to \mathrm{Aut}(X)$.

Proof. Let $V \subset K[X]$ be a finite-dimensional Γ-invariant K-subspace (Remark 3.1.4), let $d = \dim_K V$, and let ρ be the restriction of the corepresentation r^* to V.

$$\rho : \Gamma \to \mathrm{GL}(V) \qquad \rho(\gamma) = r^*(\gamma)$$

By [Rap19, Theorem 1.1], ρ has a standard description. That is, there exists a finite-dimensional K-algebra A, a ring homomorphism $f : R \to A$ with Zariski-dense image, an algebraic representation $\sigma : G(A) \to \mathrm{GL}(V)$, and a finite-index subgroup $\Delta \subset \Gamma$ such that $\rho|_\Delta = (\sigma \circ F)|_\Delta$, where $F : G(R) \to G(A)$ is the (abstract) homomorphism induced by f.

Define $\mathcal{G} = \overline{F(\Gamma)}$ (Zariski closure) and $\widetilde{\rho} = \sigma|_{\mathcal{G}}$. The diagrams below depict the situation. The left diagram is commutative, and the outer triangle of the right diagram commutes.

$$\begin{tikzcd}
G(R) \arrow[r, "F"] & G(A) \arrow[r, "\sigma"] & \mathrm{GL}(V) \\
\Gamma \arrow[u, hook] \arrow[r, "F"] & \mathcal{G} \arrow[u, hook] \arrow[ur, "\widetilde{\rho}"'] & \\
\Delta \arrow[u, hook] \arrow[uurr, bend right, "\rho|_\Delta"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
G(R) \arrow[r, "F"] & G(A) \arrow[r, "\sigma"] & \mathrm{GL}(V) \\
\Gamma \arrow[u, hook] \arrow[urr, bend right, "\rho = r^*"] & & \\
\Delta \arrow[u, hook] \arrow[uurr, bend right, "\rho|_\Delta = (\sigma\circ F)|_\Delta = (\widetilde{\rho}\circ F)|_\Delta"'] & &
\end{tikzcd}$$

[2] $\overline{H}$ is the union (not necessarily disjoint) of closures of cosets of H' in H, so $\left[\overline{H} : \overline{H}'\right] \leq \left[H : H'\right]$.

Let $Y = \mathbb{A}^d_K$ and $B = \mathrm{Sym}(V) = K[Y]$. Next we construct an algebraic action $\widetilde{r} : \mathcal{G} \to \mathrm{Aut}(Y)$. Using Remark 3.1.5, the abstract action $\rho : \Gamma \to \mathrm{GL}(V)$ and algebraic action $\widetilde{\rho} : \mathcal{G} \to \mathrm{GL}(V)$ extend to actions on B.

$$\rho : \Gamma \to \mathrm{Aut}(B)$$

$$\widetilde{\rho} : \mathcal{G} \to \mathrm{Aut}(B)$$

These actions on the coordinate ring $B = K[Y]$ induce associated actions on Y.

$$\rho^\vee : \Gamma \to \mathrm{Aut}(Y)$$

$$(\widetilde{\rho})^\vee : \mathcal{G} \to \mathrm{Aut}(Y)$$

We set $\widetilde{r} = (\widetilde{\rho})^\vee : \mathcal{G} \to \mathrm{Aut}(Y)$. Because $\widetilde{\rho} = \sigma$ is algebraic, $\widetilde{r}$ is also algebraic by Lemma 3.1.6. Next we describe a Γ-equivariant closed embedding $\theta : X \hookrightarrow Y$. The inclusion $V \hookrightarrow B$ induces a K-algebra homomorphism $\theta^* : B \to K[X]$ as follows. Fix a K-basis $\{t_1, \ldots, t_d\}$ of V, and set θ^* to be the map

$$\theta^* : B \to K[X] \qquad t_i \mapsto t_i$$

Since B is generated as a K-algebra by the t_i, this defines a K-algebra homomorphism. We claim that θ^* is Γ-equivariant; that is, the follow diagram commutes for all $\gamma \in \Gamma$.

$$\begin{array}{ccc} B & \xrightarrow{\theta^*} & K[X] \\ {\scriptstyle \rho(\gamma)}\downarrow & & \downarrow{\scriptstyle r^*(\gamma)} \\ B & \xrightarrow{\theta^*} & K[X] \end{array}$$

It is clear that this commutes starting with a generator $t_i \in B$, since by definition $\rho(\gamma)$ just acts as $r^*(\gamma)$ on V, and then since all the maps in the diagram are K-algebra homomorphisms, they agree on all of B. As θ^* is a surjective K-algebra homomorphism $K[Y] \to K[X]$, it is the comorphism of a closed embedding $\theta : X \hookrightarrow Y$. Since θ^* is Γ-equivariant, so is θ by

Lemma 3.1.7. Note that Γ acts on $\theta(X) \subset Y$ via $\rho^\vee = (r^*)^\vee = r$, so after identifying X with its image $\theta(X)$ this is just the original action of Γ on X via r. In particular, $\rho^\vee = r$ as maps $\Gamma \to \mathrm{Aut}(X)$[3].

Next, we show that $r|_\Delta = (\widetilde{r} \circ F)|_\Delta$. We have $\rho|_\Delta = (\widetilde{\rho} \circ F)|_\Delta$, so taking coactions we obtain $r|_\Delta = \rho^\vee|_\Delta = (\widetilde{\rho} \circ F)^\vee|_\Delta$. Working through the definitions and a short calculation shows that for all $\gamma \in \Gamma, y \in Y$, and $\alpha \in K[Y]$,

$$\alpha\Big(\big[(\widetilde{\rho} \circ F)^\vee(\gamma)\big](y)\Big) = \alpha\Big(\big[\,(\widetilde{r} \circ F)\,(\gamma)\big](y)\Big)$$

Since Y is separated, it follows that $(\widetilde{\rho} \circ F)^\vee|_\Delta = (\widetilde{r} \circ F)|_\Delta$, so we obtain $r|_\Delta = (\widetilde{r} \circ F)|_\Delta$ as desired.

We are not quite done, because $\widetilde{r}$ maps $\mathcal{G}$ to $\mathrm{Aut}(Y)$, not to $\mathrm{Aut}(X)$ as claimed. To finish the proof, we need to show that $\mathcal{G}$ acting on Y via $\widetilde{r}$ leaves X invariant. Consider the restriction of the action of $\mathcal{G}$ on Y using $\widetilde{r}$ to X.

$$\mathcal{G} \times X \to Y \qquad (g, x) \mapsto \widetilde{r}(g)(x)$$

We need to verify that $\widetilde{r}(g)(x) \in X$. We know that if $g \in F(\Delta)$ so that $g = F(\delta)$ for some $\delta \in \Delta$, then $\widetilde{r}(g) = r(\delta)$, and when Δ acts on X via r, the image lands back in X. For a fixed $x \in X$, since $\mathcal{G}$ acts algebraically, we get a regular (hence continuous) map

$$\ell_x : \mathcal{G} \to Y \qquad g \mapsto \widetilde{r}(g)(x)$$

Note that $\mathcal{G}$ is connected by [Bor91, Proposition I.2.2], and since $[F(\Gamma) : F(\Delta)] < [\Gamma : \Delta] < \infty$, it follows from Remark 3.1.8 that $F(\Delta)$ is dense in $\mathcal{G}$. Consider the image of $\mathcal{G}$ under ℓ_x. Since ℓ_x is continuous, $\ell_x(\mathcal{G}) = \ell_x\left(\overline{F(\Delta)}\right) = \overline{\ell_x(F(\Delta))}$. Then because Δ acting on X

[3] A potential point of confusion here is that $\rho^\vee$ can be either a map $\Gamma \to \mathrm{Aut}(X)$ or a map $\Gamma \to \mathrm{Aut}(Y)$. These are really the same map; the alternative interpretations just come from the fact that Γ leaves X (identified with $\theta(X)$) invariant inside Y.

via $\widetilde{r} \circ F = r$ leaves X invariant, $\ell_x(\mathcal{G}) = \overline{\ell_x(F(\Delta))} \subset \overline{X} = X$. Thus $\mathcal{G}$ acting on Y via $\widetilde{r}$ leaves X invariant, so we can view $\widetilde{r}$ as a map $\mathcal{G} \to \mathrm{Aut}(X)$. $\square$

Using a very similar argument, we now prove Theorem 3.1.3. The only significant difference is that we substitute the use of [Rap19, Theorem 1.1] by the refined statement [Rap13, Proposition 5.1] which gives more detailed information about the algebraic ring A and ring homomorphism f obtained in the standard description of ρ, coming from the fact that R is a ring of S-integers.

Proof. Let $V \subset K[X]$ be a finite-dimensional Γ-invariant subspace, let $d = \dim_K V$, and let ρ be the restriction of the corepresentation r^* to V. Fix a K-basis $\{t_1, \ldots, t_d\}$ of V, and with it fix an isomorphism $\mathrm{GL}(V) \cong \mathrm{GL}_d(K)$ in the usual way, so we think of ρ as a map $\Gamma \to \mathrm{GL}_d(K)$.

By [Rap13, Proposition 5.1], ρ has a special standard description, where A is a product of copies of K, and the ring homomorphism $f : R \to A$ has the form $f = (f^{(1)}, \ldots, f^{(m)})$ with each $f^{(i)} : R \to K$ is the restriction of a (distinct) embedding $L \hookrightarrow K$, and we have $\rho|_\Delta = \sigma|_\Delta$ where $\sigma : G(A) \to \mathrm{GL}_d(K)$ is a morphism of algebraic groups and Δ is a finite-index subgroup of Γ. In particular, each $f^{(i)}$ is injective, so the induced map $F : G(R) \to G(A)$ is injective, allowing us to identify Γ with its image in $G(A)$. Let $\mathcal{G} = \overline{\Gamma}$ (technically $\overline{F(\Gamma)}$), which is an algebraic subgroup of $G(A)$ (Remark 3.1.8).

From this point, the same arguments as in the proof of Theorem 3.1.2 suffice to complete the proof. We let $Y = \mathbb{A}^d_K$ and $B = \mathrm{Sym}(V) = K[Y]$, and let $\widetilde{r} : \mathcal{G} \to \mathrm{Aut}(Y)$ be the coaction of $\sigma|_{\mathcal{G}}$, which is algebraic because σ is algebraic (Lemma 3.1.6). The obvious map $\theta^* : B = K[Y] \to K[X]$ is a Γ-equivariant surjection of K-algebras, so it is the comorphism of a closed Γ-equivariant embedding of affine varieties $\theta : X \hookrightarrow Y$ (Lemma 3.1.7). Then one can show that $r|_\Delta = \widetilde{r}|_\Delta$, and that $\widetilde{r}$ leaves (the image under θ of) X invariant in the same way as in the proof of Theorem 3.1.2, so $\widetilde{r}$ can be viewed as an algebraic action $\widetilde{r} : \mathcal{G} \to \mathrm{Aut}(X)$. $\square$

The following diagram depicts the situation of Theorem 3.1.3.

$$\begin{tikzcd}
\mathcal{G} \arrow[dr, bend left, "\widetilde{r}"] & \\
\Gamma \arrow[u, hook] \arrow[r, bend left, "\widetilde{r}|_\Gamma"] \arrow[r, bend right, "r"'] & \mathrm{Aut}(X) \\
\Delta \arrow[u, hook] \arrow[ur, bend right, "r|_\Delta = \widetilde{r}|_\Delta"'] &
\end{tikzcd}$$

The upper and lower triangles obviously commute (simply function restriction), as does the triangle involving $\widetilde{r}|_\Gamma$ and $\widetilde{r}|_\Delta$. The content of Theorem 3.1.3 is that the outer triangle commutes, though we note that the theorem makes no claim regarding equality of $\widetilde{r}|_\Gamma$ and r as maps on Γ, only that they agree on the subgroup Δ.

Example 3.1.9. We give a somewhat trivial example in which the locally finite-dimensional hypothesis of Theorem 3.1.2 and Theorem 3.1.3 holds. Let $X = \mathbb{A}^1_K$ be the affine line, whose coordinate ring is the polynomial ring in one variable $K[x]$. Any K-algebra automorphism of $K[x]$ is a linear transformation, i.e. of the form $x \mapsto ax + b$ for some nonzero $a \in K^\times$ and some $b \in K$. Hence for any abstract group Γ acting on $K[x]$, the Γ-orbits are finite-dimensional. Then we can apply Theorem 3.1.3 with $\Gamma = \mathrm{SL}_n(\mathbb{Z})$ $(n \geq 3)$ and conclude that any action of $\mathrm{SL}_n(\mathbb{Z})$ on $\mathbb{A}^1_K$ coincides on a finite-index subgroup Δ with an algebraic action of an algebraic group $\mathcal{G}$ containing Γ.

Remark 3.1.10. It was pointed out to us by Friedrich Knop that in general, an action of any abstract group Γ on an affine variety X which has locally finite-dimensional coaction is close to being algebraic in the following sense. As in the arguments above, let $V \subset K[X]$ be a finite-dimensional Γ-invariant subspace which generates $K[X]$ as an algebra, and consider the corresponding closed embedding $\theta : X \hookrightarrow V^*$. Then $\mathcal{G} = \{g \in \mathrm{GL}(V^*) : g(X) = X\}$ is a Zariski-closed subgroup of $\mathrm{GL}(V^*)$ acting algebraically on V^*, leaving X invariant, and in particular the action on X agrees with the original action of Γ on X.

3.2 Elementary groups acting on projective surfaces

Let Γ be an abstract group and X be a projective variety with $\mathrm{Bir}(X)$ the group of birational automorphisms. In this section we study abstract homomorphisms $\alpha : \Gamma \to \mathrm{Bir}(X)$. Because an element of $\mathrm{Bir}(X)$ generally has an indeterminacy locus, such a map does not give a true action of Γ on X; nevertheless, in such a situation we say that Γ **acts birationally** on X.

Let $d \geq 2$ be an integer which is not a perfect square. In [CdC19, Theorem 9.1], Cantat and de Cornulier give a rigidity result for $\mathrm{SL}_2\left(\mathbb{Z}[\sqrt{d}]\right)$ acting by birational maps on projective surfaces, showing that such an action is in some sense close to being an algebraic action by biregular maps. More precisely, they prove that if

- Γ is a finite-index subgroup of $\mathrm{SL}_2\left(\mathbb{Z}[\sqrt{d}]\right)$, and
- X is an irreducible projective surface over an algebraically closed field K, and
- $\alpha : \Gamma \to \mathrm{Bir}(X)$ is an (abstract) group homomorphism with infinite image,

then $\operatorname{char} K = 0$ and there exists a finite-index subgroup $\Delta \subset \Gamma$ and a birational map $\varphi : Y \dashrightarrow X$ such that

- Y is one of $\mathbb{P}^2$, the ruled surface $\mathbb{F}_m$, or $C \times \mathbb{P}^1$ for some curve C, and
- conj_φ maps $\alpha(\Gamma)$ into $\mathrm{Aut}(Y)$, where conj_φ is the group isomorphism

$$\mathrm{conj}_\varphi : \mathrm{Bir}(X) \to \mathrm{Bir}(Y) \qquad \psi \mapsto \varphi^{-1} \circ \psi \circ \varphi$$

- There is a (unique) algebraic homomorphism $\beta : G \to \mathrm{Aut}(Y)$ such that $\beta \circ j|_\Delta = \mathrm{conj}_\varphi \circ \alpha|_\Delta$, where $G = \mathrm{SL}_2(K) \times \mathrm{SL}_2(K)$ and $j : \mathrm{SL}_2\left(\mathbb{Z}[\sqrt{d}]\right) \hookrightarrow G$ is the embedding arising from the two distinct embeddings $\mathbb{Q}(\sqrt{d}) \hookrightarrow K$. That is, the following diagram commutes.

$$\begin{tikzcd} \Delta \arrow[r, "j"] \arrow[d, "\alpha"'] & G \arrow[d, "\beta"] \\ \alpha(\Gamma) \arrow[r, "\mathrm{conj}_\varphi"'] & \mathrm{Aut}(Y) \end{tikzcd} \tag{3.1}$$

We focus on the rigidity implications of the second and third points. The second says that, after replacing X by a birational equivalent Y and conjugating by an appropriate φ, the (abstract) birational action of Γ is actually an (abstract) biregular action. The third says that this action by biregular maps is essentially algebraic, after passing to a finite index subgroup Δ.

Cantant and de Cornulier's proof has two main ingredients: [CdC19, Theorem 2] which gives a criterion for the image of conj_φ to land in $\mathrm{Aut}(Y)$, and [CdC19, Lemma 9.2] which is a ridigity statement based on Margulis' superrigidity. More precisely, the criterion in Theorem 2 is the so-called property (FW); for our purposes, it is sufficient that Kazhdan's property (T) implies property (FW) ([CdC19, Remark 3.2]). In this section, we prove a result parallel to the above rigidity result. Our approach combines the (FW) criterion with a more algebraic rigidity statement, namely [Rap19, Theorem 1.1].

We also replace the ring $\mathbb{Z}[\sqrt{d}]$ with the class of rings R with property (D_1) (note that $\mathbb{Z}[\sqrt{d}]$ is such a ring by Example 3.0.3), and replace the group $\mathrm{SL}_2\left(\mathbb{Z}[\sqrt{d}]\right)$ with the elementary subgroup $\Gamma = G(R)^+$ of a universal Chevalley group with root system Φ of rank ≥ 2 ($\mathrm{SL}_2(R)$ is the elementary subgroup of the Chevalley group SL_2 with root system $\Phi = \mathsf{A}_1$ of rank 1). Ours is not a generalization, since we assume $\mathrm{char}\, K = 0$, and more importantly we require $\mathrm{rank}\Phi \geq 2$ which excludes SL_2. Our new result is as follows.

Theorem 3.2.1. *Let K, Φ, R, G, Γ be as in Theorem 3.1.2, and assume R is finitely generated as a ring. Let X be an irreducible projective surface over K, and let $\alpha : \Gamma \to \mathrm{Bir}(X)$ be an abstract homomorphism with infinite image. Then there exists a birational map $\varphi : Y \dashrightarrow X$, a finite-dimensional K-algebra A, a ring homomorphism $f : R \to A$ with Zariski-dense image, a morphism of algebraic groups $\sigma : G(A) \to \mathrm{GL}_m(K)$, and a finite-index subgroup $\Delta \subset \Gamma$ such that*

- *Y is one of $\mathbb{P}^2$, the ruled surface $\mathbb{F}_m$, or $C \times \mathbb{P}^1$ for some curve C, and*
- *conj_φ maps $\alpha(\Gamma)$ into $\mathrm{Aut}(Y)$, and*
- *$(\mathrm{conj}_\varphi \circ \alpha)|_\Delta = (\sigma \circ F)|_\Delta$, where $F : G(R) \to G(A)$ is induced by $f : R \to A$. That is, the following diagram commutes (compare with diagram (3.1).).*

$$\begin{tikzcd}
\Delta \arrow[rr, "F"] \arrow[d, "\alpha"'] & & G(A)^+ \arrow[d, "\sigma"] \\
\alpha(\Gamma) \arrow[r, "\mathrm{conj}_\varphi"'] & \mathrm{Aut}(Y) \arrow[r, hook] & \mathrm{GL}_m(K)
\end{tikzcd}$$

Conceptually, Γ acts abstractly on X by birational maps, but if we replace X with a suitable birational equivalent Y, then the abstract action of Γ on Y coincides with an algebraic action of an algebraic group $\mathcal{G}$, after restricting to a finite-index subgroup Δ and passing through an abstract homomorphism F.

Proof. Since R is finitely generated, Γ has Kazhdan's property (T) (by [EJZK17, Theorem 1.1]), so $\Gamma' = \alpha(\Gamma)$ also has (T), so Γ' has (FW). Thus we can apply [CdC19, Theorem 2] to $\Gamma' \subset \mathrm{Bir}(X)$. This gives a birational map $\varphi : Y \dashrightarrow X$ where Y is one of $\mathbb{P}^2, \mathbb{F}_m, C \times \mathbb{P}^1$, and $\mathrm{conj}_\varphi \circ \alpha(\Gamma) \subset \mathrm{Aut}(Y)$. Because the possibilities for Y are sufficiently limited, $\mathrm{Aut}(Y)$ is a linear algebraic K-group (see the discussion following [CdC19, Theorem 9.1]). Now we have an abstract group action

$$r := \mathrm{conj}_\varphi \circ \alpha : \Gamma \to \mathrm{Aut}(Y) \qquad \gamma \mapsto r(\gamma) = \varphi^{-1} \circ \alpha(\gamma) \circ \varphi$$

Since $\mathrm{Aut}(Y)$ is linear, composing with an inclusion map we get a linear (abstract) representation $r : \Gamma \to \mathrm{GL}_m(K)$. Then using [Rap19, Theorem 1.1], r has a standard description, so there is a finite dimensional K-algebra A, a ring homomorphism $f : R \to A$ with Zariski-dense image, a morphism of algebraic groups $\sigma : G(A) \to \mathrm{GL}_m(K)$, and a finite-index subgroup $\Delta \subset \Gamma$ such that $(\mathrm{conj}_\varphi \circ \alpha)|_\Delta = (\sigma \circ F)|_\Delta$, where $F : \Gamma \to G(A)$ is the map induced by f. $\square$

We depict the resulting equality of the previous result some commutative diagrams. Philosophically, we would like to factor the abstract homomorphism α as a composition $\sigma \circ F$.

$$\begin{tikzcd}
\Gamma \arrow[rr, "\alpha"] \arrow[dr, "F"'] & & \mathrm{Bir}(X) \\
& G(A) \arrow[ur, dotted, "?"'] &
\end{tikzcd}$$

However, we cannot quite do this. Instead, we pass to a birational equivalent Y via conjugation by φ, and restrict to the finite-index subgroup Δ, extending the above picture.

$$\begin{tikzcd}
\Delta \arrow[r, hook] \arrow[drr, bend right, "F|_\Delta"'] & \Gamma \arrow[rr, "\alpha"] \arrow[dr, "F"] & & \mathrm{Bir}(X) \arrow[r, dotted, "\mathrm{conj}_\varphi"] & \mathrm{Aut}(Y) \arrow[r, hook] & \mathrm{GL}_m(K) \\
& & G(A) \arrow[ur, dotted, "?"'] \arrow[urrr, bend right=15, "\sigma"'] & & &
\end{tikzcd}$$

We can also depict this with the diagrams below. All arrows are group homomorphisms, though only σ is algebraic.

$$\begin{tikzcd}
\Delta \arrow[r, "F"] \arrow[d, hook] & G(A)^+ \arrow[r, "\sigma"] & \mathrm{GL}_m(K) \\
\Gamma \arrow[r, "r"] \arrow[d, "\alpha"] & \mathrm{Aut}(Y) \arrow[ur, hook] \arrow[d, hook] & \\
\mathrm{Bir}(X) \arrow[r, "\mathrm{conj}_\varphi"] & \mathrm{Bir}(Y) &
\end{tikzcd}$$

$$\begin{tikzcd}
\Delta \arrow[r, hook] \arrow[d, hook] & \Gamma \arrow[r, "F"] & G(A) \arrow[r, "\sigma"] & \mathrm{GL}_m(K) \\
\Gamma \arrow[r, "\alpha"] & \mathrm{Bir}(X) \arrow[r, "\mathrm{conj}_\varphi"] & \mathrm{Bir}(Y) & \mathrm{Aut}(Y) \arrow[l, hook'] \arrow[u, hook]
\end{tikzcd}$$

To emphasize, the composition $(\sigma \circ F)|_\Delta$ lands in the subgroup $\mathrm{Aut}(Y) \subset \mathrm{GL}_m(K)$, and the composition $r|_\Delta = (\mathrm{conj}_\varphi \circ \alpha)|_\Delta$ also lands the subgroup $\mathrm{Aut}(Y) \subset \mathrm{Bir}(Y)$, so the equality $r|_\Delta = (\sigma \circ F)|_\Delta$ is an equality of maps $\Delta \to \mathrm{Aut}(Y)$.

Chapter 4

Future directions

In this chapter, we discuss future directions for extending the methods of §2 and their potential as related to the Borel-Tits conjecture. In particular, we hope that eventually our techniques can resolve the Borel-Tits conjecture for all quasi-split groups of isotropic rank ≥ 2.

In §4.1, we discuss the history and generalizations of a foundational aspect of the methods of §2, namely the construction of the algebraic ring A associated to an abstract representation and its role in the analysis of the abstract representation. The main difficulty, we which have resolved for the special unitary groups considered in §2, is how to handle multi-dimensional root subgroups.

In §4.2, we revisit the ideas of §2.3, especially those related to centrality of the kernel $\ker \pi_A$ and generalizations of Proposition 2.3.2. We describe an approach to establishing rationality of σ via different methods than that in §2.5. In particular, we bypass the technical hypothesis on $R_u(H)$ required for Lemma 2.5.5, but replace it by the conjectural hypothesis that $\widetilde{G}(A)$ is generated by Steinberg symbols, so we also discuss reasons to expect that $\widetilde{G}(A)$ is generated by symbols.

4.1 Algebraic rings associated to abstract representations

In this section, we discuss future research directions for algebraic rings associated to abstract representations. In §2.4, we described an algebraic ring associated to an abstract representation $\rho : \mathrm{SU}_{2n}(L,h)(k) \to \mathrm{GL}_m(K)$, essentially by imitating the methods of [Rap11] for associating an algebraic ring to an abstract representation of a Chevalley group. We provide more historical background on this construction and discuss aspects of this construction which may generalize to other groups.

The earliest form of this construction appeared in a 2009 note of Kassabov and Sapir [KS09]. Let R be an associative ring with unity, and $\mathrm{EL}_n(R)$ the classical elementary subgroup of $\mathrm{GL}_n(R)$ with $n \geq 3$. Let $\rho : \mathrm{EL}_n(R) \to \mathrm{GL}_m(K)$ be an abstract representation, where K is an algebraically closed field. Let $x_{ij}(r) \in \mathrm{EL}_n(R)$ be the elementary matrix with r in the (i,j) entry, 1's on the diagonal, and zeros elsewhere. Then consider the root subgroup $U_{13} = x_{13}(R)$, and let $V = \overline{\rho(U_{13})}$ be the Zariski closure of $\rho(U_{13})$. It is clear that V is an abelian group under matrix multiplication. Less obviously, one can put an algebraic ring structure on V, with the multiplication defined by

$$u_1 \times u_2 := \left[w_{23}u_1w_{23}^{-1}, w_{12}u_2w_{12}^{-1}\right]$$

where w_{12}, w_{23} are the following matrices representing elements of the Weyl group of $\mathrm{GL}_n(R)$ (if $n \geq 3$ extend w_{12}, w_{23} with 1's on the diagonal as needed).

$$w_{12} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad w_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

Verifying that the multiplication as defined above gives a map $V \times V \to V$ utilizes the Steinberg commutator relation $[x_{12}(r), x_{23}(s)] = x_{13}(rs)$ and additional relations describing how one can conjugate by w_{ij} to move between root subgroups, such as the relation $w_{12}x_{13}(r)w_{12}^{-1} = x_{23}(r)$.

We summarize this construction more conceptually. Given an abstract representation ρ of $\mathrm{EL}_n(R)$, consider the 1-parameter subgroup $\rho(U_{13})$ inside $\mathrm{GL}_m(K)$. After taking the Zariski closure to obtain V, we put an algebraic ring structure on V in which addition is matrix multiplication, and multiplication is obtained by the action of the Weyl group and the Steinberg commutator relations.

Later, Igor Rapinchuk generalized this construction by replacing $\mathrm{EL}_n(R)$ with the elementary subgroup of any simply-connected Chevalley group. Let Φ be a reduced, irreducible root system and G be the associated simply-connected Chevalley group, and let $G^+ \subset G(R)$ be the elementary subgroup. Assume that (Φ, R) is a nice pair, and let $\rho : G^+ \to \mathrm{GL}_m(K)$ be an abstract representation. Then one can associate an algebraic ring to ρ in much the same way as the construction of Kassabov and Sapir: choose a root α[1], take the root subgroup $U_\alpha \subset G^+$, and let $A = \overline{\rho(U_\alpha)}$ be the Zariski closure of $\rho(U_\alpha)$. As before, addition in A is given by matrix multiplication in $\mathrm{GL}_m(K)$. Then define multiplication in A using Steinberg commutator relations and Weyl group conjugations in G^+. If Φ is a type A root system, the same relations used by Kassabov and Sapir suffice. If Φ is type B or C, the assumption that (Φ, R) is a nice pair allows for use of 2 as a denominator in the relevant calculations[2], and similarly if Φ is type G_2 both 2 and 3 appear as denominators. After analyzing abstract representations of split groups, Rapinchuk further extended this method

[1] The construction appears to depend on an arbitrary choice of root α, but the symmetry of Φ implies that the computations should work equally well for any choice of root.

[2] See Equation 2.3 and the map π in Lemma 2.4.2 for an example of 2 appearing as a denominator.

to analyze groups of the form $\mathrm{SL}_{n,D}$ where D is a division algebra ([Rap13]), essentially reusing the construction for type A root systems.

Our main contribution in extending this construction is to consider abstract representations of a group with 2-dimensional root subgroups. In particular, we have analyzed $G = \mathrm{SU}_{2n}(L, h)$ which has a mix of 1-dimensional and 2-dimensional root subgroups, and described how to associate an algebraic ring to an abstract representation of $G(k)$. Our construction is based on the following general idea: according to the structure theory of reductive groups, a group G contains a split subgroup G_0 whose root system has the same type as the relative root system of G ([CGP15, Theorem C.2.30]). For our arguments, we exhibited such a subgroup $G_0(k)$ explicitly (2.2.5). Restricting an abstract representation $\rho : G(k) \to \mathrm{GL}_m(K)$ to $G_0(k)$, we then associate an algebraic ring A to $\rho|_{G_0(k)}$ using the construction summarized above. Through explicit computations, we are then able to incorporate the additional components in the 2-dimensional root subgroups into the picture, which ultimately shows that A is sufficient for the analysis of the representation ρ on the whole group $G(k)$.

This strategy naturally leads to the following question, whose resolution will significantly expand the scope of the techniques developed in this thesis: given a k-isotropic group G of k-rank ≥ 2 and an abstract representation $\rho : G(k) \to \mathrm{GL}_m(K)$, can the additional components in the root groups of dimension ≥ 2 be incorporated in such a way that the algebraic ring A constructed from the restriction $\rho|_{G_0(k)}$ to a split subgroup suffice for the analysis of ρ?

4.2 Steinberg symbols

Let $G = \mathrm{SU}_{2n}(L, h)$ as in §2, and let R be a k-algebra. We discuss the Steinberg group $\widetilde{G}(R)$ in more detail, particularly the canonical map $\pi_R : \widetilde{G}(R) \to G(R)$ and its kernel. We showed in Proposition 2.3.2 that under suitable hypotheses on R, the kernel is central in $\widetilde{G}(R)$. Although the argument depended on certain specific features of the Steinberg commutator relations of $\mathrm{SU}_{2n}(L, h)$, we are confident that this argument can be carried out in more generality, or at least in many other specific cases. In particular, we conjecture that if G is an isotropic quasi-split reductive group with associated Steinberg group $\widetilde{G}$ (defined by Stavrova [Sta20]), and R is a product of local rings, then $\ker \pi_R$ is central in $\widetilde{G}(R)$. We anticipate this being relevant for future analysis of rigidity for other quasi-split groups.

Knowing that $\ker \pi_R$ is central was enough for §2, but we would like to understand $\ker \pi_R$ in terms of generators. For split groups over fields, the answer is given by Matsumoto's theorem, which says that the kernel, known as $\mathrm{K}_2(R)$ in this context, is generated by symbols with only a short list of relations needed to give a presentation. This generating set is critical for the methods of [Rap11], [Rap19] in verifying the Borel-Tits conjecture for split groups. The work of Deodhar [Deo78] extends the ideas of Matsumoto's theorem to quasi-split groups over fields; that is, Deodhar shows that the general quasi-split version of $\ker \pi_R$ is generated by symbols for a field R. We believe methods of Deodhar may extend to show that $\ker \pi_R$ is generated by symbols when G is quasi-split and R is a local ring (or product of local rings). In particular, if Deodhar's main result on generation of $\ker \pi_R$ could be extended from fields to products of local rings, then we could obtain Theorem 2.0.1 in a more direct way, bypassing the hypothesis on commutativity of the unipotent radical. We describe this more direct method in the remainder of this section.

We begin by explicitly describing Steinberg symbols for $\mathrm{SU}_{2n}(L,h)$. Let R be a k-algebra. For $\alpha \in \Phi_k$, we have a vector k-group scheme V_α, so $V_\alpha(R)$ has the structure of an additive group. Furthermore, we know the dimension is one for long roots and two for short roots. Recall that $L = k(\sqrt{d})$ and write $V_\alpha(R)$ as

$$V_\alpha(R) = \begin{cases} R & \alpha \text{ short} \\ R \oplus R\sqrt{d} & \alpha \text{ long} \end{cases}$$

This allows us to view $V_\alpha(R)$ as not just an additive group, but as ring. This agrees with our previous identification of $V_\alpha(k) \cong k \oplus k\sqrt{d} \cong L$ in the case where $R = k$ and α is long. Now that we have a ring structure on $V_\alpha(R)$, we can consider its group of units $V_\alpha(R)^\times$, and define

$$\begin{aligned} w_\alpha(v) &= X_\alpha(v) \cdot X_{-\alpha}\left(-v^{-1}\right) \cdot X_\alpha(v) \\ h_\alpha(v) &= w_\alpha(v) \cdot w_\alpha(1)^{-1} \end{aligned}$$

for $v \in V_\alpha(R)^\times$. Note that Lemma 2.2.8 still applies to these more generally defined $w_\alpha(v)$ and $h_\alpha(v)$ by the same arguments. We also generalize these elements to the Steinberg group $\widetilde{G}(R)$ by setting

$$\begin{aligned} \widetilde{w}_\alpha(v) &= \widetilde{X}_\alpha(v) \cdot \widetilde{X}_{-\alpha}\left(-v^{-1}\right) \cdot \widetilde{X}_\alpha(v) \\ \widetilde{h}_\alpha(v) &= \widetilde{w}_\alpha(v) \cdot \widetilde{w}_\alpha(1)^{-1} \end{aligned}$$

Let $\pi_R : \widetilde{G}(R) \to G(R)$ be the canonical map, and let $\alpha \in \Phi_k$. It is clear that $\pi_R\big(\widetilde{w}_\alpha(v)\big) = w_\alpha(v)$ and $\pi_R\left(\widetilde{h}_\alpha(v)\right) = h_\alpha(v)$. It is immediate from the definitions that

$$\widetilde{w}_\alpha(v) \cdot \widetilde{w}_\alpha(-v) = 1 \qquad \widetilde{w}_\alpha(v)^{-1} = \widetilde{w}_\alpha(-v) \qquad \widetilde{h}_\alpha(1) = 1$$

Applying π_R, versions of the above equations also hold in $G(R)$. Finally, we define

$$c_\alpha(u,v) = \widetilde{h}_\alpha(u) \cdot \widetilde{h}_\alpha(v) \cdot \widetilde{h}_\alpha(uv)^{-1}$$

for $u, v \in V_\alpha(R)^\times$. The elements $c_\alpha(u, v)$ are called **Steinberg symbols**. As an immediate consequence of Lemma 2.2.8, $c_\alpha(u, v) \in \ker \pi_R$. Our definition of Steinberg symbols agrees with Deodhar's description of elements $b_\beta(\lambda, \mu)$ in [Deo78, §2.32]. One can also compare with the unitary Steinberg symbols in Hahn and O'Meara in [HO89, §5.5A, §5.5F, §5.6A].

Deodhar [Deo78] shows that, in the case where $R = k$ is a field (and G is any quasi-split group), $\pi_k : \widetilde{G}(k) \to G(k)$ is the universal central extension (Theorem 1.9) and the kernel of π_k is generated by symbols $c_\alpha(u, v)$ where α is a single fixed long root and $u, v \in k^\times$ (Theorem 2.1). Also see [HO89, 5.6.4] for a result of similar flavor, regarding generation of a unitary Steinberg group by unitary Steinberg symbols. This leads us to formulate the following conjecture concerning the Steinberg group of $\mathrm{SU}_{2n}(L, h)$.

Conjecture 4.2.1. *Let R be a k-algebra which is a product of local rings (e.g. the ring A from 2.4.6). Then $\ker \pi_R$ is generated by symbols $c_\alpha(u, v)$ for $\alpha \in \Phi_k$ and $u, v \in V_\alpha(R)^\times$.*

A stronger version of the conjecture might assert that $\ker \pi_R$ is generated by symbols $c_\alpha(u, v)$ for some particular fixed long root α, but this stronger statement is unnecessary for our applications. More generally, one might conjecture that for G a quasi-split group and R as in the conjecture that $\ker \pi_R$ is generated by Steinberg symbols. In the remainder of this section, we show how to obtain a similar result to Theorem 2.0.1 in which we remove the technical hypothesis on commutativity of the unipotent radical $R_u(H)$ but replace it by the assumption that $\ker \pi_R$ is generated by symbols. The result is as follows.

Theorem 4.2.2. *Let $L = k(\sqrt{d})$ be a quadratic extension of a field k of characteristic zero, and for $n \geq 2$, set $G = \mathrm{SU}_{2n}(L, h)$ to be the special unitary group of a (skew-)hermitian form $h : L^{2n} \times L^{2n} \to L$ of maximal Witt index. Let K be an algebraically closed field of* *characteristic zero and consider an abstract representation*

$$\rho : G(k) \to \mathrm{GL}_m(K).$$

Set $H = \overline{\rho(G(k))}$ to be the Zariski closure of the image of ρ. If Conjecture 4.2.1 holds, then there exists a commutative finite-dimensional K-algebra A, a ring homomorphism $f : k \to A$ with Zariski-dense image, and a morphism of algebraic K-groups $\sigma : G(A) \to H$ such that $\rho = \sigma \circ F$, where $F : G(k) \to G(A)$ is the group homomorphism induced by f.

To prove Theorem 4.2.2, we reuse results used for Theorem 2.0.1 through the end of §2.4. Then we use Conjecture 4.2.1 to descend $\widetilde{\sigma}$ to a map on $G(A)$. The arguments for rationality of σ are essentially the same as in Proposition 2.5.3. We recall the notation and setup of §2.1-§2.4. We have the special unitary group $G = \mathrm{SU}_{2n}(L, h)$ with root subgroup maps X_α, an algebraically closed field K of characteristic zero, an abstract homomorphism $\rho : G(k) \to \mathrm{GL}_m(K)$, and set $H = \overline{\rho(G(k))}$. Associated to G is the Steinberg group $\widetilde{G}$ and its generators $\widetilde{X}_\alpha(v)$. In Proposition 2.4.6, we constructed a K-algebra A, a ring homomorphism $f : k \to A$, and regular maps $\psi_\alpha : V_\alpha(A) \to H$, and in Proposition 2.4.8 we constructed a group homomorphism $\widetilde{\sigma} : \widetilde{G}(A) \to H$ making the following diagram commute.

$$\begin{array}{ccc} \widetilde{G}(k) & \xrightarrow{\pi_k} & G(k) \\ \downarrow{\scriptstyle \widetilde{F}} & & \downarrow{\scriptstyle \rho} \\ \widetilde{G}(A) & \xrightarrow{\widetilde{\sigma}} & H \end{array} \tag{4.1}$$

The map $\widetilde{F}$ is induced by f; explicitly $\widetilde{F}$ is given by $\widetilde{X}_\alpha(v) \mapsto \widetilde{X}_\alpha(f(v))$.

Lemma 4.2.3. *Retain the notation above and let $a, b \in V_\alpha(A)^\times$. Then $c_\alpha(a, b) \in \ker \widetilde{\sigma}$.*

Proof. As in the proof of Proposition 2.4.8, we denote $V_\alpha(f) = \tilde{f}$. Consider $u, v \in V_\alpha(k)^\times$, and write out $c_\alpha\left(\tilde{f}u, \tilde{f}v\right)$ as a product of factors of the form $\widetilde{X}_\alpha\left(\tilde{f}u\right), \widetilde{X}_\alpha\left(\tilde{f}v\right)$, then apply the definition of $\widetilde{F}$ to obtain $c_\alpha\left(\tilde{f}u, \tilde{f}v\right) = \widetilde{F}\left(c_\alpha(u,v)\right)$. Applying the commutative square (4.1), we obtain

$$\widetilde{\sigma}\Big(c_\alpha(fu, fv)\Big) = \widetilde{\tau} \circ \widetilde{F}\Big(c_\alpha(u,v)\Big) = \rho \circ \pi_k\Big(c_\alpha(u,v)\Big) = \rho(1) = 1$$

Thus $c_\alpha\left(\tilde{f}u, \tilde{f}v\right) \in \ker \widetilde{\sigma}$. Now consider the regular map

$$\theta : V_\alpha(A^\times) \times V_\alpha(A^\times) \to H \qquad (a,b) \mapsto \widetilde{\sigma}\Big(c_\alpha(a,b)\Big)$$

Since θ vanishes on the dense open subset $f(V_\alpha(k)^\times) \times f(V_\alpha(k)^\times) \subset V_\alpha(A)^\times \times V_\alpha(A)^\times$ and is regular, θ vanishes everywhere, hence $\widetilde{\sigma}$ vanishes on symbols $c_\alpha(a,b)$ for all $a, b \in V_\alpha(A^\times)$. ☐

Note that the previous lemma does not depend on Conjecture 4.2.1. However, combined with the conjecture, it allows us to prove the following, which we view as a unitary version of [Rap13, Proposition 3.7].

Proposition 4.2.4. *Retain the setup above, and assume Conjecture 4.2.1 holds. Then there exists an abstract group homomorphism $\sigma : G(A) \to H$ making the diagram commute.*

$$\begin{tikzcd}
\widetilde{G}(k) \arrow[r, "\widetilde{F}"] \arrow[d, "\pi_k"'] & \widetilde{G}(A) \arrow[d, "\pi_A"] \arrow[ddr, bend left, "\widetilde{\sigma}"] & \\
G(k) \arrow[r, "F"] \arrow[drr, bend right, "\rho"'] & G(A) \arrow[dr, dotted, "\sigma"] & \\
 & & H
\end{tikzcd}$$

Proof. By Lemma 4.2.3, $\widetilde{\sigma}$ vanishes on unitary Steinberg symbols $c_\alpha(u,v)$ and by Conjecture 4.2.1, these symbols generate $\widetilde{G}(A)$. So $\widetilde{\sigma}$ vanishes on all of $\ker \pi_A$, and hence induces a map on the quotient $\widetilde{G}(A)/\ker \pi_A \to H$. Since π_A is surjective, the quotient is exactly $G(A)$, and we obtain the claimed map σ. Commutativity of the lower triangle is a consequence of the fact that the rest of the diagram commutes and that π_k is surjective, just as in the proof of Theorem 2.5.6. $\square$

To complete the proof of Theorem 4.2.2, we use similar arguments as in Proposition 2.5.3 to show that σ is algebraic.

Proposition 4.2.5. *The map σ of Proposition 4.2.4 is a morphism of algebraic groups.*

Proof. We write $G(A)$ as a product of root subgroups

$$G(A) = \prod_{i=1}^{m} U_i^{e_i}$$

with $e_i = \pm 1$, and let

$$X = \prod_{i=1}^{m} A_i$$

Then define s exactly as in Proposition 2.5.3, and define

$$t : X \to H \qquad t(a_1, \ldots, a_m) = \psi_{\alpha_1}(a_1)^{e_1} \cdots \psi_{\alpha_m}(a_m)^{e_m}$$

where ψ_{α_i} are the maps from Proposition 2.4.6. It is easy to verify that $\sigma \circ s = t$; then apply [Rap11, Lemma 3.10] with $Y = G(A), Z = H$ to conclude that σ is rational. Finally, apply [Rap11, Lemma 3.12] to conclude that σ is a morphism of algebraic groups. $\square$

APPENDICES

Appendix A

Algebraic rings

In this appendix, we briefly summarize the theory of algebraic rings necessary for our analysis in §2.4 and §2.5. We also include some material expanding the discussion of the algebraic ring B_α introduced in §2.4. Historically, algebraic rings were first studied systematically by Greenberg [Gre64]. Much of the notation and terminology here draws from the later development by I. Rapinchuk [Rap11, §2]. For this whole section, K denotes an algebraically closed field.

An **algebraic ring** is an affine K-variety A with a ring structure, such that addition and multiplication are regular maps[1]. In particular, $(A, +)$ is a commutative algebraic group. A **homomorphism of algebraic rings** is a ring homomorphism which is also a regular morphism of varieties. If the source and target are algebraic rings with identity, we require a homomorphism to send the identity to the identity.

Example A.1. Let B be a finite-dimensional K-algebra. Giving B the Zariski topology on K^n (where $n = \dim_K B$), addition and multiplication in B are regular so B is an algebraic ring. (One culmination of the theory is that in characteristic zero that all connected algebraic rings are of this form.)

An algebraic ring is **commutative** if multiplication is commutative. When we want to emphasize that a particular ring lacks an algebraic ring structure or we wish to temporarily

[1] The affine assumption can be omitted, but this adds very little to the theory, since by [Rap11, Theorem 2.21] any irreducible variety with a regular ring structure must be affine.

ignore the algebraic structure on a given algebraic ring, we will call it an **abstract ring**. By [Rap11, Lemma 2.8], every commutative algebraic ring is semilocal as an abstract ring.

A commutative algebraic ring A is **connected** if it is connected as a variety[2]. By [Rap11, Lemma 2.9], in a connected algebraic ring A, every right and left ideal is connected and Zariski-closed, hence A is artinian as an abstract ring. We denote the connected component of $0 \in A$ by A°, and note that this is an ideal of A. Following [Rap11], we denote the following property by (FG).

$$\text{(FG) } A^\circ \text{ is finitely generated as an ideal of } A$$

If A satisfies (FG), then A° is an artinian algebraic ring with identity, and A decomposes as a direct sum of algebraic rings $A = A^\circ \oplus C$ where C is a finite ring isomorphic to A/A° ([Rap11, Lemma 2.12]). We also have the following criterion not depending on the characteristic: if R is an abstract Noetherian ring and $f : R \to A$ is an abstract ring homomorphism such that $\overline{f(R)} = A$, then A satisfies (FG). In the same situation, if R is not only a Noetherian ring but an infinite field, then the finite ring C in the decomposition of A must be trivial; in other words, if R is an infinite field then $A = A^\circ$ is connected.

We also have a more general decomposition statement, not depending on (FG). Any algebraic ring A is a direct sum $A' \oplus C$, where A' is an algebraic subring of A consisting of all unipotent elements in $(A, +)$ and C is a finite ring consisting of semisimple elements ([Rap11, Lemma 2.15]). In particular, this decomposition shows that if A is connected, it consists entirely of unipotent elements.

We already mentioned the simplest examples of algebraic rings – finite dimensional K-algebras with the usual Zariski topology on K^n. An algebraic ring A is said to **come from**

[2] Note that in some contexts, a ring A is called connected if spec A with the Zariski topology is connected, or equivalently A contains no nontrivial idempotents. This notion of connectedness is not equivalent to the one we use.

an algebra if there exists a finite-dimensional K-algebra B and an isomorphism of algebraic rings $A \cong B$. More generally, an algebraic ring A **virtually comes from an algebra** if there is a finite-dimensional K-algebra B and a finite ring C such that $A \cong B \oplus C$. If $\operatorname{char} K = 0$ (as always, algebraically closed), then every algebraic ring A virtually comes from an algebra ([Rap11, Proposition 2.14]).

Let A be a connected algebraic ring over an algebraically closed field K of characteristic zero, so A virtually comes from an algebra. Since A is connected, the finite ring summand C in the decomposition $A \cong A' \oplus C$ must be trivial, so in fact A comes from an algebra. In other words, there is a equivalence of categories

$$\left\{\begin{array}{l} \mathit{Obj}\text{: connected algebraic rings over } K \\ \mathit{Mor}\text{: homomorphisms of algebraic rings} \end{array}\right\} \cong \left\{\begin{array}{l} \mathit{Obj}\text{: finite-dimensional } K\text{-algebras} \\ \mathit{Mor}\text{: } K\text{-algebra homomorphisms} \end{array}\right\}$$

This categorical description is due to Greenberg [Gre64, Proposition 5.1].

To finish this discussion, we mention another statement which uses an abstract ring homomorphism $f : R \to A$ to gain information about A, coming from [Rap11, Lemma 3.2]. Let A be an affine variety with two regular maps $\alpha : A \times A \to A$ and $\mu : A \times A \to U$, such that (A, α) is a commutative algebraic group. Let R be an abstract commutative unital ring and $f : R \to A$ a map such that $\overline{f(R)} = A$ and

$$f(t_1 + t_2) = \alpha\Big(f(t_1), f(t_2)\Big) \qquad f(t_1 t_2) = \mu\Big(f(t_1), f(t_2)\Big)$$

for all $t_1, t_2 \in R$. In other words, f is an additive group homomorphism $(R, +) \to (A, \alpha)$ and a multiplicative map $(R, \times) \to (A, \mu)$. If such f exists, then (A, α, μ) is a commutative algebraic ring with identity.

Agreement of algebraic ring structures

In the remainder of this section, we return to the situation of §2.4, immediately following Lemma 2.4.2. We describe an algebraic ring structure on $B_{\alpha_1-\alpha_3}$ and verify that the map π from the lemma is an isomorphism of algebraic rings.

The setup is as follows: $L/k = k(\sqrt{d})/k$ is a quadratic extension, G is the special unitary group $\mathrm{SU}_{2n}(L,h)$ with relative root system $\Phi_k = \mathsf{C}_n$, and $\rho : G(k) \to \mathrm{GL}_m(K)$ is an abstract representation with K algebraically closed of characteristic zero. Let $H = \overline{\rho(G(k))}$ be the Zariski closure. For a short root $\alpha \in \Phi_k$, the root space is $V_\alpha(k) = k^2 \cong k \oplus k\sqrt{d} \cong L$ and the root subgroup $X_\alpha(V_\alpha(k))$ is 2-dimensional (over k). We define

$$B_\alpha = \overline{\rho(X_\alpha(k\sqrt{d}))}$$

and a map $g_\alpha : k \to B_\alpha$ by $g_\alpha(u) = \rho(X_\alpha(u\sqrt{d}))$. We focus on the particular short root $\alpha = \alpha_1 - \alpha_3$ and denote $B = B_\alpha$. Let $\beta = -2\alpha_3$, and recall that in Lemma 2.4.2 we gave an isomorphism of varieties $\pi : B \to A_\beta$ such that $\pi \circ g_\alpha = f_\beta$. We wish to make B into an algebraic ring. Addition in B is straightforward to define; it is just matrix multiplication inside $\mathrm{GL}_m(K)$. More precisely, define

$$\mathrm{add} : B \times B \to H \qquad \mathrm{add}(x,y) = xy$$

where xy is matrix multiplication, and note that for all $s, t \in k$ we have

$$\mathrm{add}\Big(g_\alpha(s), g_\alpha(t)\Big) = g_\alpha(s+t)$$

Since $g_\alpha(k)$ is Zariski-dense in B, this shows that $\mathrm{add}(B \times B) \subset B$ hence (B, add) is an abelian algebraic group. It remains to describe multiplication in B. Let

$$h = h_{2\alpha-\beta}\left(\frac{1}{2}\right) \qquad h' = h_{2\alpha-\beta}\left(-\frac{1}{2d}\right) \qquad w = w_{2\alpha-\beta}(1)$$

For $b \in B$, set

$$b' = \rho(h') \cdot b \cdot \rho(h')^{-1}$$

$$b'' = \rho(w) \cdot b' \cdot \rho(w)^{-1}$$

Then define

$$\text{mult} : B \times B \to H \qquad \text{mult}(a, b) = \nu\big(\left[a, b''\right] \big)$$

where $\nu = \pi^{-1}$ is the inverse map of π from Lemma 2.4.2.

$$\nu : A_\beta \to B, \qquad \nu(y) = \rho(h) \cdot \left[y, g_{\alpha-\beta}(-1)\right] \cdot \left[y, g_{\alpha-\beta}(1)\right]^{-1} \cdot \rho(h)^{-1}$$

We claim that for all $s, t \in k$,

$$\text{mult}\Big(g_\alpha(s), g_\alpha(t)\Big) = g_\alpha(st) \tag{A.1}$$

Assuming this holds, it follows that $\text{mult}(g_\alpha(k) \times g_\alpha(k)) \subset g_\alpha(k)$, and since $g_\alpha(k)$ is Zariski-dense in B_α, it further follows that $\text{mult}(B \times B) \subset B$, so mult is a regular map $B \times B \to B$. Then applying [Rap11, Lemma 3.2] to $g_\alpha : k \to B$ makes $(B, \text{add}, \text{mult})$ into an algebraic ring. So to complete the description of the algebraic ring structure on B, it suffices to verify equation (A.1).

Let $s, t \in k$ and $u, v \in L$. We want to simplify $\text{mult}\big(g_\alpha(s), g_\alpha(t)\big) = \nu\big(\left[g_\alpha(s), g_\alpha(t)''\right] \big)$, so we begin with $g_\alpha(t)''$. We have the following relation, which appeared in a less general form in the proof of Lemma 2.4.2.

$$h_{2\alpha-\beta}(u) \cdot X_\alpha(2v) \cdot h_{2\alpha-\beta}(u)^{-1} = X_\alpha(2vu)$$

In that lemma, we used the case $u = 1$ which gives

$$h \cdot X_\alpha(2v) \cdot h^{-1} = X_\alpha(v) \tag{A.2}$$

We will need this relation again, and also the case $u = -\frac{1}{2d}$ which gives

$$h' \cdot X_\alpha\left(t\sqrt{d}\right) \cdot (h')^{-1} = X_\alpha\left(-\frac{t\sqrt{d}}{2d}\right) \tag{A.3}$$

One particular instance of equation (2.2) is

$$w \cdot X_\alpha(v) \cdot w^{-1} = X_{\beta-\alpha}(-\overline{v}) \tag{A.4}$$

Combining equations (A.3) and (A.4), we obtain

$$\begin{aligned} g_\alpha(t)'' &= \rho(w) \cdot \rho(h') \cdot \rho\left(X_\alpha(t\sqrt{d})\right) \cdot \rho(h')^{-1} \cdot \rho(w)^{-1} \\ &= \rho\Big(w \cdot h' \cdot X_\alpha(t\sqrt{d}) \cdot (h')^{-1} \cdot w^{-1}\Big) \\ &= \rho\left(X_{\beta-\alpha}\left(-\frac{t\sqrt{d}}{2d}\right)\right) \end{aligned}$$

Our next goal is to simplify the commutator $\left[g_\alpha(s), g_\alpha(t)''\right]$. From Lemma 2.2.4 (2), we have the commutator relation

$$\Big[X_\alpha(u), X_{\beta-\alpha}(v)\Big] = X_\beta\big(-\operatorname{Tr}(uv)\big)$$

for $u, v \in L$. Note that

$$\operatorname{Tr}\left(s\sqrt{d}, -\frac{t\sqrt{d}}{2d}\right) = -st$$

so

$$\left[X_\alpha\left(s\sqrt{d}\right), X_{\beta-\alpha}\left(-\frac{t\sqrt{d}}{2d}\right)\right] = X_\beta(st) \tag{A.5}$$

Using equation (A.5), we can simplify $\left[g_\alpha(s), g_\alpha(t)''\right]$.

$$\begin{aligned} \left[g_\alpha(s), g_\alpha(t)''\right] &= \left[\rho\left(X_\alpha\left(s\sqrt{d}\right)\right), \rho\left(X_{\beta-\alpha}\left(-\frac{t\sqrt{d}}{2d}\right)\right)\right] \\ &= \rho\left[X_\alpha\left(s\sqrt{d}\right), X_{\beta-\alpha}\left(-\frac{t\sqrt{d}}{2d}\right)\right] \\ &= \rho\left(X_\beta(st)\right) \end{aligned}$$

Using the above expression for $\left[g_\alpha(s), g_\alpha(t)''\right]$, we can write mult $\big(g_\alpha(s), g_\alpha(t)\big)$ as

$$
\begin{aligned}
&= \nu\Big(\left[g_\alpha(s), g_\alpha(t)''\right] \Big) \\
&= \nu\Big(\rho\left(X_\beta(st)\right)\Big) \\
&= \rho(h)\cdot\left[\rho\left(X_\beta(st)\right), g_{\alpha-\beta}(-1)\right]\cdot\left[\rho\left(X_\beta(st)\right), g_{\alpha-\beta}(1)\right]^{-1}\cdot\rho(h)^{-1} \\
&= \rho\left(h\right)\cdot\left[\rho\left(X_\beta(st)\right), \rho\left(X_{\alpha-\beta}(-\sqrt{d})\right)\right]\cdot\left[\rho\left(X_\beta(st)\right), \rho\left(X_{\alpha-\beta}(\sqrt{d})\right)\right]^{-1}\cdot\rho\left(h^{-1}\right) \\
&= \rho\Big(h\cdot\left[X_\beta(st), X_{\alpha-\beta}(-\sqrt{d})\right]\cdot\left[X_\beta(st), X_{\alpha-\beta}(\sqrt{d})\right]^{-1}\cdot h^{-1}\Big)
\end{aligned}
$$

The product of two commutators appearing in this expression simplifies via the following relation from the proof of Lemma 2.4.2.

$$
\left[X_\beta(st), X_{\alpha-\beta}\left(-\sqrt{d}\right)\right]\cdot\left[X_\beta(st), X_{\alpha-\beta}\left(\sqrt{d}\right)\right]^{-1} = X_\alpha\left(2st\sqrt{d}\right) \tag{A.6}
$$

Using equations (A.6) and (A.2) we can complete the calculation.

$$
\text{mult}\,\big(g_\alpha(s), g_\alpha(t)\big) = \rho\left(h\cdot X_\alpha(2st\sqrt{d})\cdot h^{-1}\right) = \rho\Big(X_\alpha(st\sqrt{d})\Big) = g_\alpha(st)
$$

This completes the description of the algebraic ring structure on $B = B_\alpha = B_{\alpha_1-\alpha_3}$.

Remark A.2. In §2.4, we used the results of [Rap11] for the split group $G_0(k)$ to put an algebraic ring structure on $A_\beta = A_{-2\alpha_1}$, and the ring structure on B_α was not relevant for any of the results of that section. However, this raises the natural question: what is the relationship between the ring structures on A_β and B_α, in view of the isomorphism of varieties in Lemma 2.4.2?

Denote addition and multiplication in B_α by $\underset{B_\alpha}{+}, \underset{B_\alpha}{\times}$ respectively and similarly denote the operations in A_β by $\underset{A_\beta}{+}, \underset{A_\beta}{\times}$. We claim that π is, in fact, an isomorphism of algebraic rings $\pi : \left(B_\alpha, \underset{B_\alpha}{+}, \underset{B_\alpha}{\times}\right) \to \left(A_\beta, \underset{A_\beta}{+}, \underset{A_\beta}{\times}\right)$. Recall that for $s, t \in k$,

$$g_\alpha(s) \underset{B_\alpha}{+} g_\alpha(t) = g_\alpha(s+t) \qquad f_\beta(s) \underset{A_\beta}{+} f_\beta(t) = f_\beta(s+t)$$

$$g_\alpha(s) \underset{B_\alpha}{\times} g_\alpha(t) = g_\alpha(st) \qquad f_\beta(s) \underset{A_\beta}{\times} f_\beta(t) = f_\beta(st)$$

Also recall that $\pi(g_\alpha(s)) = f_\beta(s)$ for all $s \in k$. Consider the following two regular maps.

$$\sigma_1 : B_\alpha \times B_\alpha \to A_\beta \qquad \sigma_1(x, y) = \pi(x \underset{B_\alpha}{+} y)$$

$$\sigma_2 : B_\alpha \times B_\alpha \to A_\beta \qquad \sigma_2(x, y) = \pi(x) \underset{A_\beta}{+} \pi(y)$$

Then

$$\sigma_1\Big(g_\alpha(s), g_\alpha(t)\Big) = \pi\Big(g_\alpha(s) \underset{B_\alpha}{+} g_\alpha(t)\Big) = \pi\Big(g_\alpha(s+t)\Big) = f_\beta(s+t)$$

$$\sigma_2\Big(g_\alpha(s), g_\alpha(t)\Big) = \pi\Big(g_\alpha(s)\Big) \underset{A_\beta}{+} \pi\Big(g_\alpha(t)\Big) = f_\beta(s) \underset{A_\beta}{+} f_\beta(t) = f_\beta(s+t)$$

Thus σ_1 and σ_2 coincide on $g_\alpha(k) \times g_\alpha(k)$, which is a dense subset of $B_\alpha \times B_\alpha$, so $\sigma_1 = \sigma_2$. In other words, π is compatible with the addition operations. Similarly, consider regular maps

$$\theta_1 : B_\alpha \times B_\alpha \to A_\beta \qquad \theta_1(x, y) = \pi(x \underset{B_\alpha}{\times} y)$$

$$\theta_2 : B_\alpha \times B_\alpha \to A_\beta \qquad \theta_2(x, y) = \pi(x) \underset{A_\beta}{\times} \pi(y)$$

Then

$$\theta_1\Big(g_\alpha(s), g_\alpha(t)\Big) = \pi\Big(g_\alpha(s) \underset{B_\alpha}{\times} g_\alpha(t)\Big) = \pi\Big(g_\alpha(st)\Big) = f_\beta(st)$$

$$\theta_2\Big(g_\alpha(s), g_\alpha(t)\Big) = \pi\Big(g_\alpha(s)\Big) \underset{A_\beta}{\times} \pi\Big(g_\alpha(t)\Big) = f_\beta(s) \underset{A_\beta}{\times} f_\beta(t) = f_\beta(st)$$

So θ_1, θ_2 coincide on the dense subset $g_\alpha(k) \times g_\alpha(k) \subset B_\alpha \times B_\alpha$ and we conclude that $\theta_1 = \theta_2$, so π is compatible with the multiplication operations. Hence π is an isomorphism of algebraic rings.

Appendix B

Computations

This appendix contains various technical computations involving the group $\mathrm{SU}_{2n}(L, h)$ and its Steinberg group.

Commutator coefficients $N_{ij}^{\alpha\beta}(u, v)$

This section contains the computations for Lemma 2.2.4. Throughout, Φ_k is the root system of type C_n and R is a fixed k-algebra. Given two roots $\alpha, \beta \in \Phi_k$, following [PS08] we denote by (α, β) the set of all roots consisting of positive integral linear combinations of α and β.

$$(\alpha, \beta) = \left\{ i\alpha + j\beta \in \Phi_k : i, j \in \mathbb{Z}_{\geq 0} \right\}$$

This set serves as the indexing set for the right hand side of the Steinberg commutator relation given in Theorem 2.2.1(3), i.e. for $\alpha \neq \pm\beta$,

$$\left[X_\alpha(u), X_\beta(v) \right] = \prod_{(\alpha,\beta)} X_{i\alpha+j\beta} \left(N_{ij}^{\alpha\beta}(u, v) \right)$$

We wish to describe the maps $N_{ij}^{\alpha\beta}(u, v)$ as concretely as possible. First, we begin with a lemma which completely describes the possibilities for (α, β) in the C_n root system. In any reduced root system, $(\alpha, \beta) = \emptyset$ if $\alpha = \pm\beta$, so we ignore this case (the commutator formula does not apply when $\alpha = \pm\beta$ anyway).

Lemma B.1. *Let $\alpha, \beta \in \Phi_k$ and assume $\alpha \neq \pm\beta$.*

(1) If α, β are both long roots, then $(\alpha, \beta) = \emptyset$.

(2) Suppose α, β are short roots.

(2a) If $\alpha+\beta$ is a short root, then $\{\alpha, \beta\} = \big\{\alpha_i - \alpha_j, \alpha_j - \alpha_\ell\big\}$ for three distinct indices i, j, ℓ.

(2b) If $\alpha + \beta$ is a long root, then $\{\alpha, \beta\} = \big\{\varepsilon(\alpha_i + \alpha_j), \omega(\alpha_i - \alpha_j)\big\}$ for some $i < j$ and signs $\varepsilon = \pm 1, \omega = \pm 1$.

In particular, if $\alpha + \beta \in \Phi_k$, then $(\alpha, \beta) = \{\alpha + \beta\}$.

(3) Suppose α is a short root and β a long root. The following are equivalent.

(3a) $\alpha + \beta$ is a root.

(3b) $\alpha + \beta$ is a short root.

(3c) $2\alpha + \beta$ is a root.

(3d) $2\alpha + \beta$ is a long root.

(3e) $(\alpha, \beta) = \{\alpha + \beta, 2\alpha + \beta\}$

(3f) $\alpha = \varepsilon\alpha_i + \omega\alpha_j$ and $\beta = -\varepsilon 2\alpha_i$ with $i \neq j$ and independent signs $\varepsilon = \pm 1, \omega = \pm 1$.

Proof.

(1) It's clear that if α, β are long roots, then $\alpha+\beta$ is not a root, nor is any linear combination with larger coefficients.

(2) Part (a) is clear from structure of the type A_n root system. For part (b), if two short roots $\pm\alpha_i \pm \alpha_j$ and $\pm\alpha_k \pm \alpha_\ell$ add up to a long root $\pm 2\alpha_m$, then there cannot be three distinct indices, so $\{i, j\} = \{k, \ell\}$ and then $\alpha \neq \pm\beta$ forces $\{\alpha, \beta\}$ to be as claimed.

(3) The following implications are immediate.

$$\begin{tikzcd}
(3d) \arrow[d, Rightarrow] & (3f) \arrow[l, Rightarrow] \arrow[r, Rightarrow] \arrow[dl, Rightarrow] \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & (3b) \arrow[d, Rightarrow] \\
(3c) \arrow[r, Rightarrow] & (3e) \arrow[r, Rightarrow] & (3a)
\end{tikzcd}$$

To complete the equivalence it suffices to show $(3a) \implies (3f)$ and $(3c) \implies (3f)$.

$(3a) \implies (3f)$ We can write $\alpha = \varepsilon\alpha_i + \omega\alpha_j$ and $\beta = \delta 2\alpha_k$ with $\varepsilon = \pm 1, \omega = \pm 1, \delta = \pm 1$. Since $\alpha + \beta \in \Phi_k$, we must have $k \in \{i, j\}$. The sign δ must be $-\varepsilon$ or $-\omega$ since otherwise $\alpha + \beta$ would have a $3\alpha_k$ term, so relabelling if necessary we can make $k = i$ and $\beta = -\varepsilon 2\alpha_i$.

$(3c) \implies (3f)$ Again write $\alpha = \varepsilon\alpha_i + \omega\alpha_j$ and $\beta = \delta 2\alpha_k$. Since $2\alpha + \beta \in \Phi_k$, as before $k \in \{i, j\}$. Again δ must be $-\varepsilon$ or $-\omega$, since otherwise $2\alpha + \beta$ would have a $4\alpha_k$ term, and we can relabel to write α, β as claimed. $\square$

Let $\alpha, \beta \in \Phi_k$ and $u \in V_\alpha(R), v \in V_\beta(R)$. Lemma B.1 tells us what the commutator formula looks like in all possible cases.

1. If α, β are both long, then $[X_\alpha(u), X_\beta(v)] = 1$.

2. If α, β are both short and $\alpha + \beta \in \Phi_k$, then

$$\Big[X_\alpha(u), X_\beta(v)\Big] = X_{\alpha+\beta}\left(N_{11}^{\alpha\beta}(u, v)\right)$$

3. If α, β are different lengths, assume α is short by relabelling if necessary, and then

$$\Big[X_\alpha(u), X_\beta(v)\Big] = X_{\alpha+\beta}\left(N_{11}^{\alpha\beta}(u, v)\right) \cdot X_{2\alpha+\beta}\left(N_{21}^{\alpha\beta}(u, v)\right)$$

   Note that the two factors on the right hand side commute because $(\alpha + \beta, 2\alpha + \beta) = \emptyset$.

Lemma B.2. *Let $\alpha, \beta \in \Phi_k$ be short roots such that $\alpha + \beta \in \Phi_k$, and let $u \in V_\alpha(R), v \in V_\beta(R)$.*

(1) Interchanging α, β changes N_{11} by a sign[1]. That is, $N_{11}^{\beta\alpha}(v, u) = -N_{11}^{\alpha\beta}(u, v)$.

[1] This interchange also reverses the order of the arguments, but eventually we show that N_{11} is symmetric so this is irrelevant.

(2) Negating both roots changes N_{11} by a sign. That is, $N_{11}^{-\alpha,-\beta}(u,v) = -N_{11}^{\alpha\beta}(u,v)$.

Proof.

(1) We compute the commutator of $X_\beta(v)$ and $X_\alpha(u)$ two different ways.

$$[X_\beta(v), X_\alpha(u)] = X_{\alpha+\beta}\Big(N_{11}^{\beta\alpha}(v,u)\Big)$$
$$[X_\beta(v), X_\alpha(u)] = [X_\alpha(u), X_\beta(v)]^{-1} = X_{\alpha+\beta}\Big(N_{11}^{\alpha\beta}(u,v)\Big)^{-1} = X_{\alpha+\beta}\Big(-N_{11}^{\alpha\beta}(u,v)\Big)$$

Since $X_{\alpha+\beta}$ is injective, the claimed equality follows.

(2) Recall that $X_{-\alpha}(u) = X_\alpha(u)^t$ by 2.2. We have the relation $[x^t, y^t] = [y^{-1}, x^{-1}]^t$. Now we calculate the commutator of $X_{-\alpha}(u)$ and $X_{-\beta}(v)$ in two different ways.

$$[X_{-\alpha}(u), X_{-\beta}(v)] = X_{-\alpha-\beta}\Big(N_{11}^{-\alpha,-\beta}(u,v)\Big)$$
$$\begin{aligned}[X_{-\alpha}(u), X_{-\beta}(v)] &= [X_\alpha(u)^t, X_\beta(v)^t] = [X_\beta(v)^{-1}, X_\alpha(u)^{-1}]^t \\ &= [X_\beta(-v), X_\alpha(-u)]^t = X_{\alpha+\beta}\Big(N_{11}^{\beta\alpha}(-v,-u)\Big)^t \\ &= X_{-\alpha-\beta}\Big(N_{11}^{\beta\alpha}(-v,-u)\Big)\end{aligned}$$

By injectivity of $X_{-\alpha-\beta}$, we get $N_{11}^{-\alpha,-\beta}(u,v) = N_{11}^{\beta\alpha}(-v,-u)$. Since N_{11} is linear in both variables, $N_{11}^{\beta\alpha}(-v,-u) = N_{11}^{\beta\alpha}(v,u)$, and from part (1) we have $N_{11}^{\beta\alpha}(v,u) = -N_{11}^{\alpha\beta}(u,v)$. Combining these we get the claimed equality. □

Lemma B.3. *Let $\alpha, \beta \in \Phi_k$ be roots with α short, β long, and $\alpha+\beta \in \Phi_k$, and let $u \in V_\alpha(R), v \in V_\beta(R)$.*

1. *Interchanging α, β changes N_{11} and N_{21} by a sign.*

$$N_{11}^{\beta\alpha}(v,u) = -N_{11}^{\alpha\beta}(u,v)$$
$$N_{12}^{\beta\alpha}(v,u) = -N_{21}^{\alpha\beta}(u,v)$$

2. Negating both α, β changes N_{11} by a sign and does not change N_{21}.

$$N_{11}^{-\alpha,-\beta}(u,v) = -N_{11}^{\alpha\beta}(u,v)$$

$$N_{21}^{-\alpha,-\beta}(u,v) = N_{21}^{\alpha\beta}(u,v)$$

Proof.

(1) This is the same as the argument as for Lemma B.2(1). We compute the commutator of $X_\alpha(u)$ and $X_\beta(v)$ two different ways.

$$[X_\alpha(u), X_\beta(v)] = X_{\alpha+\beta}\Big(N_{11}^{\alpha\beta}(u,v)\Big) \cdot X_{2\alpha+\beta}\Big(N_{21}^{\alpha\beta}(u,v)\Big)$$

$$\begin{aligned}
[X_\alpha(u), X_\beta(v)] = [X_\beta(v), X_\alpha(u)]^{-1} &= \Bigg(X_{\alpha+\beta}\Big(N_{11}^{\beta\alpha}(v,u)\Big) \cdot X_{2\alpha+\beta}\Big(N_{12}^{\beta\alpha}(v,u)\Big)\Bigg)^{-1} \\
&= X_{2\alpha+\beta}\Big(N_{12}^{\beta\alpha}(v,u)\Big)^{-1} \cdot X_{\alpha+\beta}\Big(N_{11}^{\beta\alpha}(v,u)\Big)^{-1} \\
&= X_{2\alpha+\beta}\Big(-N_{12}^{\beta\alpha}(v,u)\Big) \cdot X_{\alpha+\beta}\Big(-N_{11}^{\beta\alpha}(v,u)\Big) \\
&= X_{\alpha+\beta}\Big(-N_{11}^{\beta\alpha}(v,u)\Big) \cdot X_{2\alpha+\beta}\Big(-N_{12}^{\beta\alpha}(v,u)\Big)
\end{aligned}$$

So we conclude that

$$X_{\alpha+\beta}\Big(N_{11}^{\alpha\beta}(u,v)\Big) \cdot X_{2\alpha+\beta}\Big(N_{21}^{\alpha\beta}(u,v)\Big) = X_{\alpha+\beta}\Big(-N_{11}^{\beta\alpha}(v,u)\Big) \cdot X_{2\alpha+\beta}\Big(-N_{12}^{\beta\alpha}(v,u)\Big)$$

We can rearrange this to

$$X_{\alpha+\beta}\Big(N_{11}^{\alpha\beta}(u,v) + N_{11}^{\beta\alpha}(v,u)\Big) = X_{2\alpha+\beta}\Big(-N_{21}^{\alpha\beta}(u,v) - N_{12}^{\beta\alpha}(v,u)\Big)$$

Since $\alpha+\beta$ and $2\alpha+\beta$ are distinct roots, these lie in distinct root subgroups, so equality is only possible if both are the identity. Since both $X_{\alpha+\beta}, X_{2\alpha+\beta}$ are injective, this implies that both inputs are zero. Hence

$$N_{11}^{\beta\alpha}(v,u) = -N_{11}^{\alpha\beta}(u,v)$$

$$N_{21}^{\beta\alpha}(v,u) = -N_{12}^{\alpha\beta}(u,v)$$

(2) This is the same argument as for Lemma B.2(2). We compute the commutator of $X_{-\alpha}(u)$ and $X_{-\beta}(v)$ two different ways.

$$
\begin{aligned}
[X_{-\alpha}(u), X_{-\beta}(v)] &= X_{-\alpha-\beta}\Big(N_{11}^{-\alpha,-\beta}(u,v)\Big) \cdot X_{-2\alpha-\beta}\Big(N_{21}^{-\alpha,-\beta}(u,v)\Big) \\
[X_{-\alpha}(u), X_{-\beta}(v)] &= \Big[X_\alpha(u)^t, X_\beta(v)^t\Big] = \Big[X_\beta(v)^{-1}, X_\alpha(u)^{-1}\Big]^t = \big[X_\beta(-v), X_\alpha(-u)\big]^t \\
&= \Bigg(X_{\alpha+\beta}\Big(N_{11}^{\beta\alpha}(-v,-u)\Big) \cdot X_{2\alpha+\beta}\Big(N_{12}^{\beta\alpha}(-v,-u)\Big)\Bigg)^t \\
&= X_{2\alpha+\beta}\Big(N_{12}^{\beta\alpha}(-v,-u)\Big)^t \cdot X_{\alpha+\beta}\Big(N_{11}^{\beta\alpha}(-v,-u)\Big)^t \\
&= X_{-2\alpha-\beta}\Big(N_{12}^{\beta\alpha}(-v,-u)\Big) \cdot X_{-\alpha-\beta}\Big(N_{11}^{\beta\alpha}(-v,-u)\Big) \\
&= X_{-\alpha-\beta}\Big(N_{11}^{\beta\alpha}(-v,-u)\Big) \cdot X_{-2\alpha-\beta}\Big(N_{12}^{\beta\alpha}(-v,-u)\Big)
\end{aligned}
$$

so we conclude

$$
\begin{aligned}
&X_{-\alpha-\beta}\Big(N_{11}^{-\alpha,-\beta}(u,v)\Big) \cdot X_{-2\alpha-\beta}\Big(N_{21}^{-\alpha,-\beta}(u,v)\Big) \\
&\qquad = X_{-\alpha-\beta}\Big(N_{11}^{\beta\alpha}(-v,-u)\Big) \cdot X_{-2\alpha-\beta}\Big(N_{12}^{\beta\alpha}(-v,-u)\Big)
\end{aligned}
$$

By the same kind of argument in (1), we can conclude that the respective inputs are equal.

$$
\begin{aligned}
N_{11}^{-\alpha,-\beta}(u,v) &= N_{11}^{\beta\alpha}(-v,-u) \\
N_{21}^{-\alpha,-\beta}(u,v) &= N_{12}^{\beta\alpha}(-v,-u)
\end{aligned}
$$

Finally, we do some rearranging using part (1), along with the fact that N_{11} is linear in both arguments while N_{12} is linear in the first argument and quadratic in the second.

$$
\begin{aligned}
N_{11}^{-\alpha,-\beta}(u,v) &= N_{11}^{\beta\alpha}(-v,-u) = N_{11}^{\beta\alpha}(v,u) = -N_{11}^{\alpha\beta}(u,v) \\
N_{21}^{-\alpha,-\beta}(u,v) &= N_{12}^{\beta\alpha}(-v,-u) = -N_{12}^{\beta\alpha}(v,u) = N_{21}^{\alpha\beta}(u,v)
\end{aligned}
$$

□

Lemma B.4 (Repeat of Lemma 2.2.4). *Let $\alpha, \beta \in \Phi_k$ be relative roots such that $\alpha+\beta \in \Phi_k$, and let $u \in V_\alpha(R), v \in V_\beta(R)$.*

(1) Suppose α, β are both short and $\alpha+\beta$ is short. Then by Lemma B.1 (2a) we have $\alpha = \alpha_i - \alpha_j, \beta = \alpha_j - \alpha_\ell$ for distinct indices i, j, ℓ (relabelling α, β if necessary), and

$$N_{11}^{\alpha\beta}(u,v) = uv \qquad N_{11}^{\beta\alpha}(u,v) = -uv$$

(2) Suppose α, β are both short and $\alpha+\beta$ is long. Then by Lemma B.1 (2b), relabelling α, β if necessary we have $\alpha = \varepsilon(\alpha_i - \alpha_j), \beta = \omega(\alpha_i + \alpha_j)$ for some $\varepsilon = \pm 1, \omega = \pm 1$, with $i < j$, and

$$N_{11}^{\alpha\beta}(u,v) = \omega \operatorname{Tr}(u_{-\varepsilon\omega} v) \qquad N_{11}^{\beta\alpha}(v,u) = -\omega \operatorname{Tr}(u_{-\varepsilon\omega} v)$$

(3) Suppose α is short and β long. Then by Lemma B.1 (3) we have $\alpha = \varepsilon\alpha_i + \omega\alpha_j$ and $\beta = -\varepsilon 2\alpha_i$ for some $\varepsilon = \pm 1, \omega = \pm 1$ and $i \neq j$, and

$$N_{11}^{\alpha\beta}(u,v) = \omega u_{-c_{ij}} v \qquad N_{11}^{\beta\alpha}(v,u) = -\omega u_{-c_{ij}} v$$
$$N_{21}^{\alpha\beta}(u,v) = -\varepsilon\omega v u \overline{u} \qquad N_{12}^{\beta\alpha}(v,u) = \varepsilon\omega v u \overline{u}$$

where

$$c_{ij} = \begin{cases} 1 & i < j \\ -1 & i > j \end{cases}$$

In particular, whenever it is defined, the map $N_{11}^{\alpha\beta}$ is surjective.

Proof.

(1) This is known from the classical Steinberg relations for type A_n.

(2) The second equation follows from the first using Lemma B.2 (1), so we only need to prove the formula for $N_{11}^{\alpha\beta}$. By part (2) of the same lemma, it suffices to verify the formula for

$N_{11}^{\alpha\beta}(u,v)$ in the two cases $\varepsilon=\omega=1$ and $\varepsilon=1,\omega=-1$. First we verify the case $\varepsilon=\omega=1$. By a direct computation (Example B.5) we verify that

$$[X_{\alpha_i-\alpha_j}(u), X_{\alpha_i+\alpha_j}(v)] = X_{2\alpha_i}(\overline{u}v+u\overline{v})$$

so $N_{11}^{\alpha\beta}(u,v)=\overline{u}v+u\overline{v}=\omega\operatorname{Tr}(\overline{u}v)=\omega\operatorname{Tr}(u_{-\varepsilon\omega}v)$ as claimed in the first case. Now we verify the case $\varepsilon=1,\omega=-1$, again by direct computation verifying that

$$[X_{\alpha_i+\alpha_j}(u), X_{-\alpha_i-\alpha_j}(v)] = X_{-2\alpha_i}(-uv-\overline{uv})$$

so $N_{11}^{\alpha\beta}(u,v)=-uv-\overline{uv}=-\operatorname{Tr}(uv)=\omega\operatorname{Tr}(u_{-\varepsilon\omega}v)$ as claimed.

(3) The strategy is similar to that in part (1). By Lemma B.3(2), it suffices to verify just the formulas for $N_{11}^{\alpha\beta}$ and $N_{21}^{\alpha\beta}$ in the following four cases.

(a) $\varepsilon=\omega=c_{ij}=1$

(b) $\varepsilon=c_{ij}=1$ and $\omega=-1$

(c) $\varepsilon=\omega=1$ and $c_{ij}=-1$

(d) $\varepsilon=1$ and $\omega=c_{ij}=-1$

Each of these cases is verified by direct computation. For example, case (a) requires verifying that when $i<j$ we have

$$[X_{\alpha_i+\alpha_j}(u), X_{-2\alpha_i}(v)] = X_{-\alpha_i+\alpha_j}(\overline{u}v)\cdot X_{2\alpha_j}(-vu\overline{u})$$

$\square$

Example B.5. To illustrate, we work through one of the direct computations asserted in part (2) of the previous lemma. We will verify the relation

$$[X_{\alpha_i-\alpha_j}(u), X_{\alpha_i+\alpha_j}(v)] = X_{2\alpha_i}(\overline{u}v+u\overline{v})$$

in the case $i < j$. The relevant root subgroup maps are

$$X_{\alpha_i - \alpha_j}(u) = 1 + E_{ij}(u) - E_{j+1,i+1}(\overline{u})$$
$$X_{\alpha_i + \alpha_j}(v) = 1 + E_{i,j+1}(v) - E_{j,i+1}(\overline{v})$$

Using $X_\alpha(u)^{-1} = X_\alpha(-u)$, we expand the commutator bracket.

$$\begin{aligned}
[X_{\alpha_i-\alpha_j}(u), X_{\alpha_i+\alpha_j}(v)] = &\Big(1 + E_{ij}(u) - E_{j+1,i+1}(\overline{u})\Big) \cdot \Big(1 + E_{i,j+1}(v) - E_{j,i+1}(\overline{v})\Big) \\
&\cdot\Big(1 - E_{ij}(u) + E_{j+1,i+1}(\overline{u})\Big) \cdot \Big(1 - E_{i,j+1}(v) + E_{j,i+1}(\overline{v})\Big)
\end{aligned}$$

Recall the identity $E_{ij}(x)E_{k\ell}(y) = \delta_{jk}E_{i\ell}(xy)$ (δ_{jk} is the Kronecker delta function). In partilar, whenever $j = k$, the product vanishes. So when we distribute the products on the right side, multiple terms vanish.

$$\begin{aligned}
= &\Big(1 + E_{i,j+1}(v) - E_{j,i+1}(\overline{v}) + E_{ij}(u) + \cancel{E_{ij}(u)E_{i,j+1}(v)} - E_{ij}(u)E_{j,i+1}(\overline{v}) \\
&\quad - E_{j+1,i+1}(\overline{u}) - \cancel{E_{j+1,i+1}(\overline{u})E_{i,j+1}(v)} + \cancel{E_{j+1,i+1}(\overline{u})E_{j,i+1}(\overline{v})}\Big)\cdot \\
&\cdot \Big(1 - E_{i,j+1}(v) + E_{j,i+1}(\overline{v}) - E_{ij}(u) - \cancel{E_{ij}(u)E_{i,j+1}(v)} + E_{ij}(u)E_{j,i+1}(\overline{v}) \\
&\quad + E_{j+1,i+1}(\overline{u}) - \cancel{E_{j+1,i+1}(\overline{u})E_{i,j+1}(v)} - \cancel{E_{j+1,i+1}(\overline{u})E_{j,i+1}(\overline{v})}\Big) \\
= &\Big(1 + E_{i,j+1}(v) - E_{j,i+1}(\overline{v}) + E_{ij}(u) - E_{i,i+1}(u\overline{v}) - E_{j+1,i+1}(\overline{u})\Big) \\
&\cdot \Big(1 - E_{i,j+1}(v) + E_{j,i+1}(\overline{v}) - E_{ij}(u) + E_{i,i+1}(u\overline{v}) + E_{j+1,i+1}(\overline{u})\Big)
\end{aligned}$$

Then we distribute again, and again many terms vanish.

$$
\begin{aligned}
&= \Big(1 - E_{i,j+1}(v) + E_{j,i+1}(\overline{v}) - E_{ij}(u) + E_{i,i+1}(u\overline{v}) + E_{j+1,i+1}(\overline{u})\Big) \\
&\quad + E_{i,j+1}(v)\Big(1 - \cancel{E_{i,j+1}(v)} + \cancel{E_{j,i+1}(\overline{v})} - \cancel{E_{ij}(u)} + \cancel{E_{i,i+1}(u\overline{v})} + E_{j+1,i+1}(\overline{u})\Big) \\
&\quad - E_{j,i+1}(\overline{v})\Big(1 - \cancel{E_{i,j+1}(v)} + \cancel{E_{j,i+1}(\overline{v})} - \cancel{E_{ij}(u)} + \cancel{E_{i,i+1}(u\overline{v})} + \cancel{E_{j+1,i+1}(\overline{u})}\Big) \\
&\quad + E_{ij}(u)\Big(1 - \cancel{E_{i,j+1}(v)} + E_{j,i+1}(\overline{v}) - \cancel{E_{ij}(u)} + \cancel{E_{i,i+1}(u\overline{v})} + \cancel{E_{j+1,i+1}(\overline{u})}\Big) \\
&\quad - E_{i,i+1}(u\overline{v})\Big(1 - \cancel{E_{i,j+1}(v)} + \cancel{E_{j,i+1}(\overline{v})} - \cancel{E_{ij}(u)} + \cancel{E_{i,i+1}(u\overline{v})} + \cancel{E_{j+1,i+1}(\overline{u})}\Big) \\
&\quad - E_{j+1,i+1}(\overline{u})\Big(1 - \cancel{E_{i,j+1}(v)} + \cancel{E_{j,i+1}(\overline{v})} - \cancel{E_{ij}(u)} + \cancel{E_{i,i+1}(u\overline{v})} + \cancel{E_{j+1,i+1}(\overline{u})}\Big) \\
&= 1 - E_{i,j+1}(v) + E_{j,i+1}(\overline{v}) - E_{ij}(u) + E_{i,i+1}(u\overline{v}) + E_{j+1,i+1}(\overline{u}) \\
&\quad + E_{i,j+1}(v) + E_{i,j+1}(v)E_{j+1,i+1}(\overline{u}) - E_{j,i+1}(\overline{v}) \\
&\quad + E_{ij}(u) + E_{ij}(u)E_{j,i+1}(\overline{v}) - E_{i,i+1}(u\overline{v}) - E_{j+1,i+1}(\overline{u})
\end{aligned}
$$

From here, it is just a matter of reorganizing the terms and recognizing various cancellations in pairs.

$$
\begin{aligned}
&= 1 - \cancel{E_{i,j+1}(v)} + \cancel{E_{j,i+1}(\overline{v})} - \cancel{E_{ij}(u)} + E_{i,i+1}(u\overline{v}) + \cancel{E_{j+1,i+1}(\overline{u})} \\
&\quad + \cancel{E_{i,j+1}(v)} + E_{i,i+1}(v\overline{u}) - \cancel{E_{j,i+1}(\overline{v})} \\
&\quad + \cancel{E_{ij}(u)} + \cancel{E_{i,i+1}(u\overline{v})} - \cancel{E_{i,i+1}(u\overline{v})} - \cancel{E_{j+1,i+1}(\overline{u})} \\
&= 1 + E_{i,i+1}(\overline{u}v + u\overline{v}) \\
&= X_{2\alpha_i}(\overline{u}v + u\overline{v})
\end{aligned}
$$

Thus we have the claimed equality.

$$[X_{\alpha_i-\alpha_j}(u), X_{\alpha_i+\alpha_j}(v)] = X_{2\alpha_i}(\overline{u}v + u\overline{v})$$

Conjugation by $\widetilde{w}_\alpha(1)$

As in §2, let L/k be a quadratic extension in characteristic zero, with nontrivial Galois automorphism $\tau(x) = \overline{x}$. We denote

$$v_\delta = \begin{cases} v & \delta = 1 \\ \overline{v} & \delta = -1 \end{cases}$$

Let R be a k-algebra, and denote $R_L = R \otimes_k L$, and extend τ to R_L by acting on the L part. Let Φ_k be the root system of type C_n, which we write as

$$\Phi_k = \left\{ \pm\beta_i \pm \beta_j : 1 \leq i, j \leq n \right\} \setminus \{0\}$$

For $\alpha, \beta \in \Phi_k$, let

$$(\alpha, \beta) = \left\{ i\alpha + j\beta \in \Phi_k : i, j \in \mathbb{Z}_{\geq 1} \right\}$$

We believe that the relation (2.2) from §2.2 should lift to the Steinberg group, and that this should be possible to prove directly using the Steinberg relations and Lemma 2.2.4. However, we have not yet worked this out. More precisely, our conjecture is the following.

Conjecture B.6. *Let $\alpha, \beta \in \Phi_k$ such that $\alpha + \beta \in \Phi_k$. Let $v \in V_\alpha(R)$. Then*

$$\widetilde{w}_\beta(1) \cdot \widetilde{X}_\alpha(v) \cdot \widetilde{w}_\beta(1)^{-1} = \widetilde{X}_{w_\beta \alpha}(\varphi v)$$

where $\varphi : R_L \to R_L$ is a function of the form $v \mapsto \pm v_{\pm 1}$.

It is not clear exactly what relationship this bears to [Deo78, Proposition 1.11, Corollary 1.12]. It is possible we have misunderstood Deodhar, but Deodhar's result does not appear imply the above. Regardless, we have the following computation which covers at least one case of the conjecture above. Note that in the situation of the next lemma, $w_\beta\alpha = \alpha + \beta$.

Lemma B.7. *Let $\alpha, \beta \in \Phi_k$ with α short and β long, such that $\alpha + \beta \in \Phi_k$. Let $v \in V_\alpha(R)$. Then*

$$\widetilde{w}_\beta(1) \cdot \widetilde{X}_\alpha(v) \cdot \widetilde{w}_\beta(1)^{-1} = \widetilde{X}_{\alpha+\beta}(\varphi v)$$

where $\varphi : R_L \to R_L$ is a function of the form $v \mapsto \pm v_{\pm 1}$.

Proof. This is simply a long calculation involving the Steinberg relations (R1), (R2) from Definition 2.3.1, followed by application of Lemma 2.2.4. First, for α short and β long with $\alpha + \beta \in \Phi_k$,

$$(\alpha, \beta) = \{\beta + \alpha, \beta + 2\alpha\}$$

$$(-\beta, \beta + \alpha) = \{\alpha, \beta + 2\alpha\}$$

$$(\alpha, \beta + \alpha) = \{\beta + 2\alpha\}$$

We repeatedly apply (R1) and (R2) in a long calculation, the end result of which is

$$\widetilde{w}_\beta(1) \cdot \widetilde{X}_\alpha(v) \cdot \widetilde{w}_\beta(1)^{-1} = \widetilde{X}_{\beta+2\alpha}(N_6 + N_7 + N_4 + 2N_2) \cdot \widetilde{X}_{\beta+\alpha}(N_5 + 2N_1) \cdot \widetilde{X}_\alpha(N_3 + v)$$

where N_i are various commutator coefficients that arise along the way. To complete the proof, we show that

$$N_6 + N_7 + N_4 + 2N_2 = 0$$

$$N_5 + 2N_1 = \pm v_{\pm 1}$$

$$N_3 + v = 0$$

Now we do the calculation. Let $w = \widetilde{w}_\beta(1)$ and $X = \widetilde{X}_\alpha(v)$. First we expand the left hand side.

$$wXw^{-1} = \widetilde{X}_\beta(1) \cdot \widetilde{X}_{-\beta}(-1) \cdot \widetilde{X}_\beta(1) \cdot \widetilde{X}_\alpha(v) \cdot \widetilde{X}_\beta(-1) \cdot \widetilde{X}_{-\beta}(1) \cdot \widetilde{X}_\beta(-1)$$

Apply the commutator formula for $\widetilde{X}_\beta(1)$ and $\widetilde{X}_\alpha(v)$. There are two new terms because $(\beta, \alpha) = \{\beta+\alpha, \beta+2\alpha\}$.

$$\begin{aligned}
wXw^{-1} &= \widetilde{X}_\beta(1) \cdot \widetilde{X}_{-\beta}(-1) \cdot \widetilde{X}_\beta(1) \cdot \widetilde{X}_\alpha(v) \cdot \widetilde{X}_\beta(-1) \cdot \widetilde{X}_{-\beta}(1) \cdot \widetilde{X}_\beta(-1) \\
&= \widetilde{X}_\beta(1) \cdot \widetilde{X}_{-\beta}(-1) \cdot \widetilde{X}_{\beta+\alpha}(N_1) \cdot \widetilde{X}_{\beta+2\alpha}(N_2) \cdot \widetilde{X}_\alpha(v) \cdot \widetilde{X}_\beta(1) \\
&\qquad \cdot \widetilde{X}_\beta(-1) \cdot \widetilde{X}_{-\beta}(1) \cdot \widetilde{X}_\beta(-1)
\end{aligned}$$

where

$$N_1 = N_{1,1}^{\beta,\alpha}(1, v)$$

$$N_2 = N_{1,2}^{\beta,\alpha}(1, v)$$

The adjacent factors $\widetilde{X}_\beta(1)$ and $\widetilde{X}_\beta(-1)$ cancel.

$$\begin{aligned}
wXw^{-1} &= \widetilde{X}_\beta(1) \cdot \widetilde{X}_{-\beta}(-1) \cdot \widetilde{X}_{\beta+\alpha}(N_1) \cdot \widetilde{X}_{\beta+2\alpha}(N_2) \\
&\qquad \cdot \widetilde{X}_\alpha(v) \cdot \cancel{\widetilde{X}_\beta(1) \cdot \widetilde{X}_\beta(-1)} \cdot \widetilde{X}_{-\beta}(1) \cdot \widetilde{X}_\beta(-1) \\
&= \widetilde{X}_\beta(1) \cdot \widetilde{X}_{-\beta}(-1) \cdot \widetilde{X}_{\beta+\alpha}(N_1) \cdot \widetilde{X}_{\beta+2\alpha}(N_2) \cdot \widetilde{X}_\alpha(v) \cdot \widetilde{X}_{-\beta}(1) \cdot \widetilde{X}_\beta(-1)
\end{aligned}$$

Apply the commutator formula for $\widetilde{X}_{-\beta}(-1)$ and $\widetilde{X}_{\beta+\alpha}(N_1)$. There are two new terms because $(-\beta, \beta+\alpha) = \{\alpha, \beta+2\alpha\}$.

$$\begin{aligned}
wXw^{-1} &= \widetilde{X}_\beta(1) \cdot \widetilde{X}_{-\beta}(-1) \cdot \widetilde{X}_{\beta+\alpha}(N_1) \cdot \widetilde{X}_{\beta+2\alpha}(N_2) \cdot \widetilde{X}_\alpha(v) \cdot \widetilde{X}_{-\beta}(1) \cdot \widetilde{X}_\beta(-1) \\
&= \widetilde{X}_\beta(1) \cdot \widetilde{X}_\alpha(N_3) \cdot \widetilde{X}_{\beta+2\alpha}(N_4) \cdot \widetilde{X}_{\beta+\alpha}(N_1) \cdot \widetilde{X}_{-\beta}(-1) \\
&\qquad \cdot \widetilde{X}_{\beta+2\alpha}(N_2) \cdot \widetilde{X}_\alpha(v) \cdot \widetilde{X}_{-\beta}(1) \cdot \widetilde{X}_\beta(-1)
\end{aligned}$$

where

$$N_3 = N_{1,1}^{-\beta,\beta+\alpha}(-1, N_1)$$

$$N_4 = N_{1,2}^{-\beta,\beta+\alpha}(-1, N_1)$$

Both $-\beta$ and $\beta+2\alpha$ are long roots so $\widetilde{X}_{-\beta}(-1)$ and $\widetilde{X}_{\beta+2\alpha}(N_1)$ commute. Also $-\beta+\alpha$ is not a root, so $\widetilde{X}_{-\beta}(-1)$ also commutes with $\widetilde{X}_{\alpha}(v)$.

$$\begin{aligned} wXw^{-1} &= \widetilde{X}_{\beta}(1)\cdot\widetilde{X}_{\alpha}(N_3)\cdot\widetilde{X}_{\beta+2\alpha}(N_4)\cdot\widetilde{X}_{\beta+\alpha}(N_1) \\ &\qquad \cdot\widetilde{X}_{-\beta}(-1)\cdot\widetilde{X}_{\beta+2\alpha}(N_2)\cdot\widetilde{X}_{\alpha}(v)\cdot\widetilde{X}_{-\beta}(1)\cdot\widetilde{X}_{\beta}(-1) \\ &= \widetilde{X}_{\beta}(1)\cdot\widetilde{X}_{\alpha}(N_3)\cdot\widetilde{X}_{\beta+2\alpha}(N_4)\cdot\widetilde{X}_{\beta+\alpha}(N_1) \\ &\qquad \cdot\widetilde{X}_{\beta+2\alpha}(N_2)\cdot\widetilde{X}_{\alpha}(v)\cdot\widetilde{X}_{-\beta}(-1)\cdot\widetilde{X}_{-\beta}(1)\cdot\widetilde{X}_{\beta}(-1) \end{aligned}$$

The adjacent factors $\widetilde{X}_{-\beta}(-1)$ and $\widetilde{X}_{-\beta}(1)$ cancel.

$$\begin{aligned} wXw^{-1} &= \widetilde{X}_{\beta}(1)\cdot\widetilde{X}_{\alpha}(N_3)\cdot\widetilde{X}_{\beta+2\alpha}(N_4)\cdot\widetilde{X}_{\beta+\alpha}(N_1) \\ &\qquad \cdot\widetilde{X}_{\beta+2\alpha}(N_2)\cdot\widetilde{X}_{\alpha}(v)\cdot\cancel{\widetilde{X}_{-\beta}(-1)\cdot\widetilde{X}_{-\beta}(1)}\cdot\widetilde{X}_{\beta}(-1) \\ &= \widetilde{X}_{\beta}(1)\cdot\widetilde{X}_{\alpha}(N_3)\cdot\widetilde{X}_{\beta+2\alpha}(N_4)\cdot\widetilde{X}_{\beta+\alpha}(N_1)\cdot\widetilde{X}_{\beta+2\alpha}(N_2)\cdot\widetilde{X}_{\alpha}(v)\cdot\widetilde{X}_{\beta}(-1) \end{aligned}$$

Since $(\beta+2\alpha)+(\beta+\alpha)=2\beta+3\alpha$ is not a root, the $\widetilde{X}_{\beta+\alpha}(N_1)$ and $\widetilde{X}_{\beta+2\alpha}(N_4)$ terms commute.

$$\begin{aligned} wXw^{-1} &= \widetilde{X}_{\beta}(1)\cdot\widetilde{X}_{\alpha}(N_3)\cdot\widetilde{X}_{\beta+2\alpha}(N_4)\cdot\widetilde{X}_{\beta+\alpha}(N_1)\cdot\widetilde{X}_{\beta+2\alpha}(N_2)\cdot\widetilde{X}_{\alpha}(v)\cdot\widetilde{X}_{\beta}(-1) \\ &= \widetilde{X}_{\beta}(1)\cdot\widetilde{X}_{\alpha}(N_3)\cdot\widetilde{X}_{\beta+\alpha}(N_1)\cdot\widetilde{X}_{\beta+2\alpha}(N_4)\cdot\widetilde{X}_{\beta+2\alpha}(N_2)\cdot\widetilde{X}_{\alpha}(v)\cdot\widetilde{X}_{\beta}(-1) \end{aligned}$$

We can combine the adjacent $\widetilde{X}_{\beta+2\alpha}$ terms.

$$\begin{aligned} wXw^{-1} &= \widetilde{X}_{\beta}(1)\cdot\widetilde{X}_{\alpha}(N_3)\cdot\widetilde{X}_{\beta+\alpha}(N_1)\cdot\widetilde{X}_{\beta+2\alpha}(N_4)\cdot\widetilde{X}_{\beta+2\alpha}(N_2)\cdot\widetilde{X}_{\alpha}(v)\cdot\widetilde{X}_{\beta}(-1) \\ &= \widetilde{X}_{\beta}(1)\cdot\widetilde{X}_{\alpha}(N_3)\cdot\widetilde{X}_{\beta+\alpha}(N_1)\cdot\widetilde{X}_{\beta+2\alpha}(N_4+N_2)\cdot\widetilde{X}_{\alpha}(v)\cdot\widetilde{X}_{\beta}(-1) \end{aligned}$$

Apply the commutator formula for $\widetilde{X}_{\beta}(1)$ and $\widetilde{X}_{\alpha}(N_3)$. There are two factors introduced because $(\beta,\alpha)=\{\beta+\alpha,\beta+2\alpha\}$.

$$\begin{aligned}
wXw^{-1} &= \widetilde{X}_\beta(1)\cdot\widetilde{X}_\alpha(N_3)\cdot\widetilde{X}_{\beta+\alpha}(N_1)\cdot\widetilde{X}_{\beta+2\alpha}(N_4+N_2)\cdot\widetilde{X}_\alpha(v)\cdot\widetilde{X}_\beta(-1)\\
&= \widetilde{X}_{\beta+\alpha}(N_5)\cdot\widetilde{X}_{\beta+2\alpha}(N_6)\cdot\widetilde{X}_\alpha(N_3)\cdot\widetilde{X}_\beta(1)\\
&\quad\cdot\widetilde{X}_{\beta+\alpha}(N_1)\cdot\widetilde{X}_{\beta+2\alpha}(N_4+N_2)\cdot\widetilde{X}_\alpha(v)\cdot\widetilde{X}_\beta(-1)
\end{aligned}$$

where

$$N_5 = N_{1,1}^{\beta,\alpha}(1,N_3)$$
$$N_6 = N_{1,2}^{\beta,\alpha}(1,N_3)$$

Since $2\beta+\alpha$ and $2\beta+2\alpha$ are not roots, $\widetilde{X}_\beta(1)$ commutes with $\widetilde{X}_{\beta+\alpha}(N_1)$ and $\widetilde{X}_{\beta+2\alpha}(N_4+N_2)$.

$$\begin{aligned}
wXw^{-1} &= \widetilde{X}_{\beta+\alpha}(N_5)\cdot\widetilde{X}_{\beta+2\alpha}(N_6)\cdot\widetilde{X}_\alpha(N_3)\cdot\widetilde{X}_\beta(1)\\
&\quad\cdot\widetilde{X}_{\beta+\alpha}(N_1)\cdot\widetilde{X}_{\beta+2\alpha}(N_4+N_2)\cdot\widetilde{X}_\alpha(v)\cdot\widetilde{X}_\beta(-1)\\
&= \widetilde{X}_{\beta+\alpha}(N_5)\cdot\widetilde{X}_{\beta+2\alpha}(N_6)\cdot\widetilde{X}_\alpha(N_3)\\
&\quad\cdot\widetilde{X}_{\beta+\alpha}(N_1)\cdot\widetilde{X}_{\beta+2\alpha}(N_4+N_2)\cdot\widetilde{X}_\beta(1)\cdot\widetilde{X}_\alpha(v)\cdot\widetilde{X}_\beta(-1)
\end{aligned}$$

Apply the commutator lemma again for $\widetilde{X}_\beta(1)$ and $\widetilde{X}_\alpha(v)$. We get the same commutator coefficients N_1, N_2 as before.

$$\begin{aligned}
w\widetilde{X}w^{-1} &= \widetilde{X}_{\beta+\alpha}(N_5)\cdot\widetilde{X}_{\beta+2\alpha}(N_6)\cdot\widetilde{X}_\alpha(N_3)\cdot\widetilde{X}_{\beta+\alpha}(N_1)\\
&\quad\cdot\widetilde{X}_{\beta+2\alpha}(N_4+N_2)\cdot\widetilde{X}_\beta(1)\cdot\widetilde{X}_\alpha(v)\cdot\widetilde{X}_\beta(-1)\\
&= \widetilde{X}_{\beta+\alpha}(N_5)\cdot\widetilde{X}_{\beta+2\alpha}(N_6)\cdot\widetilde{X}_\alpha(N_3)\cdot\widetilde{X}_{\beta+\alpha}(N_1)\cdot\widetilde{X}_{\beta+2\alpha}(N_4+N_2)\\
&\quad\cdot\widetilde{X}_{\beta+\alpha}(N_1)\cdot\widetilde{X}_{\beta+2\alpha}(N_2)\cdot\widetilde{X}_\alpha(v)\cdot\widetilde{X}_\beta(1)\cdot\widetilde{X}_\beta(-1)
\end{aligned}$$

Then the $\widetilde{X}_\beta$ terms cancel.

$$\begin{aligned} wXw^{-1} &= \widetilde{X}_{\beta+\alpha}(N_5)\cdot\widetilde{X}_{\beta+2\alpha}(N_6)\cdot\widetilde{X}_{\alpha}(N_3)\cdot\widetilde{X}_{\beta+\alpha}(N_1)\cdot\widetilde{X}_{\beta+2\alpha}(N_4+N_2)\\ &\qquad\cdot\widetilde{X}_{\beta+\alpha}(N_1)\cdot\widetilde{X}_{\beta+2\alpha}(N_2)\cdot\widetilde{X}_{\alpha}(v)\cdot\cancel{\widetilde{X}_{\beta}(1)\cdot\widetilde{X}_{\beta}(-1)}\\ &= \widetilde{X}_{\beta+\alpha}(N_5)\cdot\widetilde{X}_{\beta+2\alpha}(N_6)\cdot\widetilde{X}_{\alpha}(N_3)\cdot\widetilde{X}_{\beta+\alpha}(N_1)\\ &\qquad\cdot\widetilde{X}_{\beta+2\alpha}(N_4+N_2)\cdot\widetilde{X}_{\beta+\alpha}(N_1)\cdot\widetilde{X}_{\beta+2\alpha}(N_2)\cdot\widetilde{X}_{\alpha}(v)\end{aligned}$$

Since $(\beta+\alpha)+(\beta+2\alpha)=2\beta+3\alpha$ is not a root, we can commute these terms.

$$\begin{aligned} wXw^{-1} &= \widetilde{X}_{\beta+\alpha}(N_5)\cdot\widetilde{X}_{\beta+2\alpha}(N_6)\cdot\widetilde{X}_{\alpha}(N_3)\cdot\widetilde{X}_{\beta+\alpha}(N_1)\\ &\qquad\cdot\widetilde{X}_{\beta+2\alpha}(N_4+N_2)\cdot\widetilde{X}_{\beta+\alpha}(N_1)\cdot\widetilde{X}_{\beta+2\alpha}(N_2)\cdot\widetilde{X}_{\alpha}(v)\\ &= \widetilde{X}_{\beta+\alpha}(N_5)\cdot\widetilde{X}_{\beta+2\alpha}(N_6)\cdot\widetilde{X}_{\alpha}(N_3)\cdot\widetilde{X}_{\beta+\alpha}(N_1)\\ &\qquad\cdot\widetilde{X}_{\beta+\alpha}(N_1)\cdot\widetilde{X}_{\beta+2\alpha}(N_4+N_2)\cdot\widetilde{X}_{\beta+2\alpha}(N_2)\cdot\widetilde{X}_{\alpha}(v)\end{aligned}$$

We can combine the adjacent $\widetilde{X}_{\beta+\alpha}$ terms. Similarly, we can combine the adjacent $\widetilde{X}_{\beta+2\alpha}$ terms.

$$\begin{aligned} wXw^{-1} &= \widetilde{X}_{\beta+\alpha}(N_5)\cdot\widetilde{X}_{\beta+2\alpha}(N_6)\cdot\widetilde{X}_{\alpha}(N_3)\cdot\widetilde{X}_{\beta+\alpha}(N_1)\cdot\widetilde{X}_{\beta+\alpha}(N_1)\\ &\qquad\cdot\widetilde{X}_{\beta+2\alpha}(N_4+N_2)\cdot\widetilde{X}_{\beta+2\alpha}(N_2)\cdot\widetilde{X}_{\alpha}(v)\\ &= \widetilde{X}_{\beta+\alpha}(N_5)\cdot\widetilde{X}_{\beta+2\alpha}(N_6)\cdot\widetilde{X}_{\alpha}(N_3)\cdot\widetilde{X}_{\beta+\alpha}(2N_1)\cdot\widetilde{X}_{\beta+2\alpha}(N_4+2N_2)\cdot\widetilde{X}_{\alpha}(v)\end{aligned}$$

Since $(\beta+2\alpha)+\alpha=\beta+3\alpha$ is not a root, we can commute the last two terms.

$$\begin{aligned} wXw^{-1} &= \widetilde{X}_{\beta+\alpha}(N_5)\cdot\widetilde{X}_{\beta+2\alpha}(N_6)\cdot\widetilde{X}_{\alpha}(N_3)\cdot\widetilde{X}_{\beta+\alpha}(2N_1)\cdot\widetilde{X}_{\beta+2\alpha}(N_4+2N_2)\cdot\widetilde{X}_{\alpha}(v)\\ &= \widetilde{X}_{\beta+\alpha}(N_5)\cdot\widetilde{X}_{\beta+2\alpha}(N_6)\cdot\widetilde{X}_{\alpha}(N_3)\cdot\widetilde{X}_{\beta+\alpha}(2N_1)\cdot\widetilde{X}_{\alpha}(v)\cdot\widetilde{X}_{\beta+2\alpha}(N_4+2N_2)\end{aligned}$$

Now $(\alpha,\beta+\alpha)=(\beta+2\alpha)$, so when we apply the commutator formula for $\widetilde{X}_\alpha(v)$ and

$\widetilde{X}_{\beta+\alpha}(2N_1)$ there is one new factor introduced.

$$\begin{aligned} wXw^{-1} &= \widetilde{X}_{\beta+\alpha}(N_5) \cdot \widetilde{X}_{\beta+2\alpha}(N_6) \cdot \widetilde{X}_{\alpha}(N_3) \cdot \widetilde{X}_{\beta+\alpha}(2N_1) \cdot \widetilde{X}_{\alpha}(v) \cdot \widetilde{X}_{\beta+2\alpha}(N_4 + 2N_2) \\ &= \widetilde{X}_{\beta+\alpha}(N_5) \cdot \widetilde{X}_{\beta+2\alpha}(N_6) \cdot \widetilde{X}_{\beta+2\alpha}(N_7) \cdot \widetilde{X}_{\beta+\alpha}(2N_1) \cdot \widetilde{X}_{\alpha}(N_3) \\ &\quad \cdot \widetilde{X}_{\alpha}(v) \cdot \widetilde{X}_{\beta+2\alpha}(N_4 + 2N_2) \end{aligned}$$

where

$$N_7 = N_{1,1}^{\alpha,\beta+\alpha}(N_3, 2N_1)$$

We combine the adjacent $\widetilde{X}_{\alpha}$ terms into a single term. Similarly, we combine the adjacent $\widetilde{X}_{\beta+2\alpha}$ terms.

$$\begin{aligned} wXw^{-1} &= \widetilde{X}_{\beta+\alpha}(N_5) \cdot \widetilde{X}_{\beta+2\alpha}(N_6) \cdot \widetilde{X}_{\beta+2\alpha}(N_7) \cdot \widetilde{X}_{\beta+\alpha}(2N_1) \\ &\quad \cdot \widetilde{X}_{\alpha}(N_3) \cdot \widetilde{X}_{\alpha}(v) \cdot \widetilde{X}_{\beta+2\alpha}(N_4 + 2N_2) \\ &= \widetilde{X}_{\beta+\alpha}(N_5) \cdot \widetilde{X}_{\beta+2\alpha}(N_6 + N_7) \cdot \widetilde{X}_{\beta+\alpha}(2N_1) \cdot \widetilde{X}_{\alpha}(N_3 + v) \cdot \widetilde{X}_{\beta+2\alpha}(N_4 + 2N_2) \end{aligned}$$

Since $(\beta + \alpha) + (\beta + 2\alpha)$ is not a root, we can commute these terms.

$$\begin{aligned} wXw^{-1} &= \widetilde{X}_{\beta+\alpha}(N_5) \cdot \widetilde{X}_{\beta+2\alpha}(N_6 + N_7) \cdot \widetilde{X}_{\beta+\alpha}(2N_1) \cdot \widetilde{X}_{\alpha}(N_3 + v) \cdot \widetilde{X}_{\beta+2\alpha}(N_4 + 2N_2) \\ &= \widetilde{X}_{\beta+2\alpha}(N_6 + N_7) \cdot \widetilde{X}_{\beta+\alpha}(N_5) \cdot \widetilde{X}_{\beta+\alpha}(2N_1) \cdot \widetilde{X}_{\alpha}(N_3 + v) \cdot \widetilde{X}_{\beta+2\alpha}(N_4 + 2N_2) \end{aligned}$$

We combine the adjacent $\widetilde{X}_{\beta+\alpha}$ terms.

$$\begin{aligned} wXw^{-1} &= \widetilde{X}_{\beta+2\alpha}(N_6 + N_7) \cdot \widetilde{X}_{\beta+\alpha}(N_5) \cdot \widetilde{X}_{\beta+\alpha}(2N_1) \cdot \widetilde{X}_{\alpha}(N_3 + v) \cdot \widetilde{X}_{\beta+2\alpha}(N_4 + 2N_2) \\ &= \widetilde{X}_{\beta+2\alpha}(N_6 + N_7) \cdot \widetilde{X}_{\beta+\alpha}(N_5 + 2N_1) \cdot \widetilde{X}_{\alpha}(N_3 + v) \cdot \widetilde{X}_{\beta+2\alpha}(N_4 + 2N_2) \end{aligned}$$

Since $(\beta+2\alpha)+\alpha$ and $(\beta+2\alpha)+(\beta+\alpha)$ are both not roots, we can move the $\widetilde{X}_{\beta+2\alpha}(N_4+2N_2)$ term on the far right through the two terms to its left.

$$\begin{aligned} wXw^{-1} &= \widetilde{X}_{\beta+2\alpha}(N_6 + N_7) \cdot \widetilde{X}_{\beta+\alpha}(N_5 + 2N_1) \cdot \widetilde{X}_{\alpha}(N_3 + v) \cdot \widetilde{X}_{\beta+2\alpha}(N_4 + 2N_2) \\ &= \widetilde{X}_{\beta+2\alpha}(N_6 + N_7) \cdot \widetilde{X}_{\beta+2\alpha}(N_4 + 2N_2) \cdot \widetilde{X}_{\beta+\alpha}(N_5 + 2N_1) \cdot \widetilde{X}_{\alpha}(N_3 + v) \end{aligned}$$

We combine the adjacent $\widetilde{X}_{\beta+2\alpha}$ terms.

$$\begin{aligned} wXw^{-1} &= \widetilde{X}_{\beta+2\alpha}(N_6+N_7)\cdot\widetilde{X}_{\beta+2\alpha}(N_4+2N_2)\cdot\widetilde{X}_{\beta+\alpha}(N_5+2N_1)\cdot\widetilde{X}_{\alpha}(N_3+v) \\ &= \widetilde{X}_{\beta+2\alpha}(N_6+N_7+N_4+2N_2)\cdot\widetilde{X}_{\beta+\alpha}(N_5+2N_1)\cdot\widetilde{X}_{\alpha}(N_3+v) \end{aligned}$$

In summary, we showed that

$$\widetilde{w}_\beta(1)\cdot\widetilde{X}_\alpha(v)\cdot\widetilde{w}_\beta(1)^{-1} = \widetilde{X}_{\beta+2\alpha}(N_6+N_7+N_4+2N_2)\cdot\widetilde{X}_{\beta+\alpha}(N_5+2N_1)\cdot\widetilde{X}_{\alpha}(N_3+v)$$

To complete the proof, we need to show that

$$N_6+N_7+N_4+2N_2=0$$

$$N_5+2N_1=\pm v_{\pm 1}$$

$$N_3+v=0$$

This is done in Lemma B.8 below. $\square$

Lemma B.8. *Let $\alpha,\beta\in\Phi_k$ with α a short root and β a long root, such that $\alpha+\beta\in\Phi_k$. Let $v\in V_\alpha(R)=R_L$. Using Lemma 2.2.4 as $\beta=\varepsilon 2\beta_i$ and $\alpha=-\varepsilon\beta_i+\omega\beta_j$ using independent signs ε,ω, and let c_{ij} be as in Lemma 2.2.4. Then*

$$\begin{aligned} N_1 &= N_{1,1}^{\beta,\alpha}(1,v) = -\omega v_{c_{ij}} \\ N_2 &= N_{1,2}^{\beta,\alpha}(1,v) = -\varepsilon\omega v\overline{v} \\ N_3 &= N_{1,1}^{-\beta,\beta+\alpha}(-1,N_1) = -v \\ N_4 &= N_{1,2}^{-\beta,\beta+\alpha}(-1,N_1) = -\varepsilon\omega v\overline{v} \\ N_5 &= N_{1,1}^{\beta,\alpha}(1,N_3) = \omega v_{c_{ij}} \\ N_6 &= N_{1,2}^{\beta,\alpha}(1,N_3) = -\varepsilon\omega v\overline{v} \\ N_7 &= N_{1,1}^{\alpha,\beta+\alpha}(N_3,2N_1) = 4\varepsilon\omega v\overline{v} \end{aligned}$$

In particular,

$$N_6 + N_7 + N_4 + 2N_2 = 0$$

$$N_5 + 2N_1 = -\omega v_\delta \in \{\pm v, \pm \overline{v}\}$$

$$N_3 + v = 0$$

Proof. Using Lemma 2.2.4 we can write the N_1 through N_6 in terms of $\varepsilon, \omega, \delta$ and v_δ.

$$N_1 = N_{1,1}^{\beta,\alpha}(1, v) = -\omega v_\delta$$

$$N_2 = N_{1,2}^{\beta,\alpha}(1, v) = -\varepsilon\omega v\overline{v}$$

To calculate N_3 and N_4, temporarily denote

$$\beta' = -\beta = \varepsilon' 2\beta_i \qquad \alpha' = \beta + \alpha = -\varepsilon'\beta_i + \omega'\beta_j$$

To calculate N_3 and N_4 using the corollary, we use the signs $\varepsilon' = -\varepsilon, \omega' = \omega, \delta' = \delta$. We get

$$(N_1)_{\delta'} = (-\omega v_\delta)_\delta = -\omega v$$

$$N_1\overline{N}_1 = (-\omega v_\delta)\overline{(-\omega v_\delta)} = v_\delta \overline{v_\delta} = v\overline{v}$$

$$N_3 = N_{1,1}^{-\beta,\beta+\alpha}(-1, N_1) = N_{1,1}^{\beta',\alpha'}(-1, N_1) = -\omega(-1)(-\omega v) = -\omega^2 v = -v$$

$$N_4 = N_{1,2}^{-\beta,\beta+\alpha}(-1, N_1) = N_{1,2}^{\beta',\alpha'}(-1, N_1) = -\varepsilon'\omega'(-1)N_1\overline{N}_1 = -\varepsilon\omega v\overline{v}$$

Now we calculate N_5 and N_6.

$$N_5 = N_{1,1}^{\beta,\alpha}(1, N_3) = -\omega(N_3)_\delta = -\omega(-v)_\delta = \omega v_\delta$$

$$N_6 = N_{1,2}^{\beta,\alpha}(1, N_3) = -\varepsilon\omega N_3\overline{N_3} = -\varepsilon\omega(-v)(-\overline{v}) = -\varepsilon\omega v\overline{v}$$

To calculate N_7, we use Lemma 2.2.4. The roots in question are

$$\alpha = -\varepsilon\beta_i + \omega\beta_j \qquad \beta + \alpha = \varepsilon\beta_i + \omega\beta_j$$

To apply the lemma, we need to know which one of these roots has the form $\pm(\beta_i + \beta_j)$ and which has the form $\pm(\beta_i - \beta_j)$, but these depend on whether $\varepsilon = \omega$ or $\varepsilon = -\omega$. So we consider these cases separately.

(Case 1, $\omega = \varepsilon$) Let $\alpha = \beta + \alpha = \varepsilon(\beta_i + \beta_j) = \varepsilon'(\beta_i + \beta_j)$, and let $\beta = \alpha = -\varepsilon(\beta_i - \beta_j) = \omega'(\beta_i - \beta_j)$. So our signs are $\varepsilon' = \varepsilon$ and $\omega' = -\varepsilon$. We still have $\delta' = \delta = \delta^{ij}$. By Lemma 2.2.4 (2),

$$\begin{aligned} N_7 &= N_{1,1}^{\alpha,\beta+\alpha}(N_3, 2N_1) = N_{1,1}^{\beta,\alpha}(N_3, 2N_1) = -N_{1,1}^{\alpha,\beta}(N_3, 2N_1) = \varepsilon' \operatorname{Tr}\left((N_3)_{\varepsilon'\omega'\delta'} \cdot 2N_1\right) \\ &= \varepsilon \operatorname{Tr}\left((-v)_{-\varepsilon^2\delta} \cdot (-2\omega v_\delta)\right) = \varepsilon\omega \operatorname{Tr}\left(2v_{-\delta}v_\delta\right) = 2\varepsilon\omega \operatorname{Tr}(v\overline{v}) = 4\varepsilon\omega v\overline{v} \end{aligned}$$

(Case 2, $\omega = -\varepsilon$) Let $\alpha = \alpha = -\varepsilon(\beta_i + \beta_j) = \varepsilon'(\beta_i + \beta_j)$ and let $\beta = \beta + \alpha = \varepsilon(\beta_i - \beta_j) = \omega'(\beta_i - \beta_j)$, so our signs are $\varepsilon' = -\varepsilon$ and $\omega' = \varepsilon$. We still use the same $\delta' = \delta = \delta^{ij}$. By Lemma 2.2.4 (2),

$$\begin{aligned} N_7 &= N_{1,1}^{\alpha,\beta+\alpha}(N_3, 2N_1) = N_{1,1}^{\alpha,\beta}(N_3, 2N_1) = -\varepsilon' \operatorname{Tr}\left((N_3)_{\varepsilon'\omega'\delta'} \cdot 2N_1\right) \\ &= \varepsilon \operatorname{Tr}\left((-v)_{-\varepsilon^2\delta} \cdot (-2\omega v_\delta)\right) = 2\varepsilon\omega \operatorname{Tr}\left(v_{-\delta}v_\delta\right) = 2\varepsilon\omega \operatorname{Tr}(v\overline{v}) = 4\varepsilon\omega v\overline{v} \end{aligned}$$

So in both cases we can write N_7 as $4\varepsilon\omega v\overline{v}$. This completes our calculation of the N values. The relations follow immediately. $\square$

Generalization of Lemma 2.4.2

In Remark 2.4.3, we noted that the calculations in Lemma 2.4.2 can be done in slightly more generality, so we do this here.

Lemma B.9. *Let $\alpha, \beta \in \Phi_k$ with α short, β long, and such that $\beta - \alpha \in \Phi_k$. Then there is an isomorphism of algebraic varieties $\pi\colon B_\alpha \to A_\beta$ such that $\pi \circ g_\alpha = f_\beta$.*

Proof. This is just a slight generalization of the computation for Lemma 2.4.2. By Lemma 2.2.4 (2), we can write α and $\beta-\alpha$ as

$$\alpha = \varepsilon(\alpha_i - \alpha_j) \qquad \beta - \alpha = \omega(\alpha_i + \alpha_j)$$

for some signs $\varepsilon = \pm 1, \omega = \pm 1$ and indicies i, j with $i < j$, and futhermore

$$N_{11}^{\alpha,\beta-\alpha}(u,v) = \omega \operatorname{Tr}\left(u_{-\varepsilon\omega} v\right)$$

Now define $\pi : B_\alpha \to H$ by

$$\pi(x) = \left[x, g_{\beta-\alpha}\left(\frac{\omega}{2d}\right)\right]$$

Note that $2d$ is invertible because $\operatorname{char} k = 0$. This commutator occurs inside $\mathrm{GL}_m(K)$ and multiplication is regular, so π is regular. Now let $s \in k$. Then

$$\begin{aligned}\pi\big(g_\alpha(s)\big) &= \left[\rho\left(X_\alpha\left(s\sqrt{d}\right)\right), \rho\left(X_{\beta-\alpha}\left(\frac{\omega\sqrt{d}}{2d}\right)\right)\right] = \rho\left(\left[X_\alpha\left(s\sqrt{d}\right), X_{\beta-\alpha}\left(\frac{\omega}{2\sqrt{d}}\right)\right]\right) \\ &= \rho\left(X_\beta\left(N_{11}^{\alpha,\beta-\alpha}\left(s\sqrt{d}, \frac{\omega}{2\sqrt{d}}\right)\right)\right) = \rho\left(X_\beta\left(\omega \operatorname{Tr}\left(\frac{\omega s}{2}\right)\right)\right) = \rho\left(X_\beta(s)\right) = f_\beta(s)\end{aligned}$$

This shows that $\pi \circ g_\alpha = f_\beta$, and that π maps $g_\alpha(k)$ into $f_\beta(k)$. Since π is regular, it follows that $\pi(B_\alpha) \subset A_\beta$. It remains to show that π is invertible (with regular inverse). By Lemma 2.2.4(3), we can write

$$\alpha - \beta = \varepsilon'\alpha_{i'} + \omega'\alpha_{j'} \qquad \beta = -\varepsilon' 2\alpha_{i'}$$

for some signs $\varepsilon' = \pm 1, \omega' = \pm 1$, and indices i', j', and furthermore

$$N_{11}^{\beta,\alpha-\beta}(v,u) = -\omega' u_{-c_{i'j'}} v \qquad N_{12}^{\beta,\alpha-\beta}(v,u) = \varepsilon'\omega' v u \overline{u}$$

Let $c = -c_{i'j'}$ and let $h = h_{2\alpha-\beta}(1/2)$ be the element introduced in Definition 2.2.6. Then define

$$\nu : A_\beta \to B_\alpha, \qquad \nu(y) = \rho(h) \cdot \left[y, g_{\alpha-\beta}(-\omega' c)\right] \cdot \left[y, g_{\alpha-\beta}(\omega' c)\right]^{-1} \cdot \rho(h)^{-1},$$

It is clear that ν is a regular map; we claim it is an inverse for π. Let $s, t \in k$, and note that $\left(\sqrt{d}\right)_c = c\sqrt{d}$. Using the commutator relation from Theorem 2.2.1, we have

$$\begin{aligned}
\left[X_\beta(t), X_{\alpha-\beta}\left(s\sqrt{d}\right)\right] &= X_\alpha\Big(N_{11}^{\beta,\alpha-\beta}\left(t, s\sqrt{d}\right)\Big) \cdot X_{2\alpha-\beta}\Big(N_{12}^{\beta,\alpha-\beta}\left(t, s\sqrt{d}\right)\Big) \\
&= X_\alpha\Big(-\omega' t\left(s\sqrt{d}\right)_c\Big) \cdot X_{2\alpha-\beta}\left(\varepsilon'\omega' t\left(s\sqrt{d}\right)\overline{\left(s\sqrt{d}\right)}\right) \\
&= X_\alpha\Big(-\omega' cts\sqrt{d}\Big) \cdot X_{2\alpha-\beta}\Big(-\varepsilon'\omega' ts^2 d\Big).
\end{aligned}$$

Specializing the above to the case $s = -\omega' c$ and noting that $(\omega' c)^2 = 1$, we get

$$\begin{aligned}
&\left[X_\beta(t), X_{\alpha-\beta}\left(-\omega' c\sqrt{d}\right)\right] \cdot \left[X_\beta(t), X_{\alpha-\beta}\left(\omega' c\sqrt{d}\right)\right]^{-1} \\
&= X_\alpha\Big(t\sqrt{d}\Big) \cdot \cancel{X_{2\alpha-\beta}\Big(-\varepsilon'\omega' td\Big)} \cdot \cancel{X_{2\alpha-\beta}\Big(-\varepsilon'\omega' td\Big)^{-1}} \cdot X_\alpha\Big(-t\sqrt{d}\Big)^{-1} = X_\alpha\left(2t\sqrt{d}\right).
\end{aligned}$$

We also have the relation

$$h \cdot X_\alpha(2v) \cdot h^{-1} = X_\alpha(v)$$

for all $v \in L$. Putting everything together, we obtain

$$\begin{aligned}
\nu\big(f_\beta(t)\big) &= \rho\left(h \cdot \left[X_\beta(t), X_{\alpha-\beta}\left(-\omega' c\sqrt{d}\right)\right] \cdot \left[X_\beta(t), X_{\alpha-\beta}\left(\omega' c\sqrt{d}\right)\right]^{-1} \cdot h^{-1}\right) \\
&= \rho\left(h \cdot X_\alpha\left(2t\sqrt{d}\right) \cdot h^{-1}\right) = \rho\left(X_\alpha\left(t\sqrt{d}\right)\right) = g_\alpha(t).
\end{aligned}$$

Thus, $\nu \circ \pi$ and $\pi \circ \nu$ are the respective identity maps on dense subsets of A_β and B_α. Since they are regular, it follows that they are the identity on the whole space, so ν is the inverse of π as claimed. $\square$

Lemma 2.4.2 is the special case of Lemma B.9 where $\alpha = \alpha_1 - \alpha_3$ and $\beta = -2\alpha_3$. In this case, the various signs and indices are

$i = 1$	$j = 3$	$\varepsilon = 1$	$\omega = -1$
$i' = 3$	$j' = 1$	$\varepsilon' = 1$	$\omega' = 1$

Appendix C

Logical dependency chart

In this appendix we depict logical dependencies among various lemmas, propositions, and theorems from this document. Arrows in these diagrams do not indicate logical implication, only that the proof of the target depends on the source. Key results from outside sources are included as well.

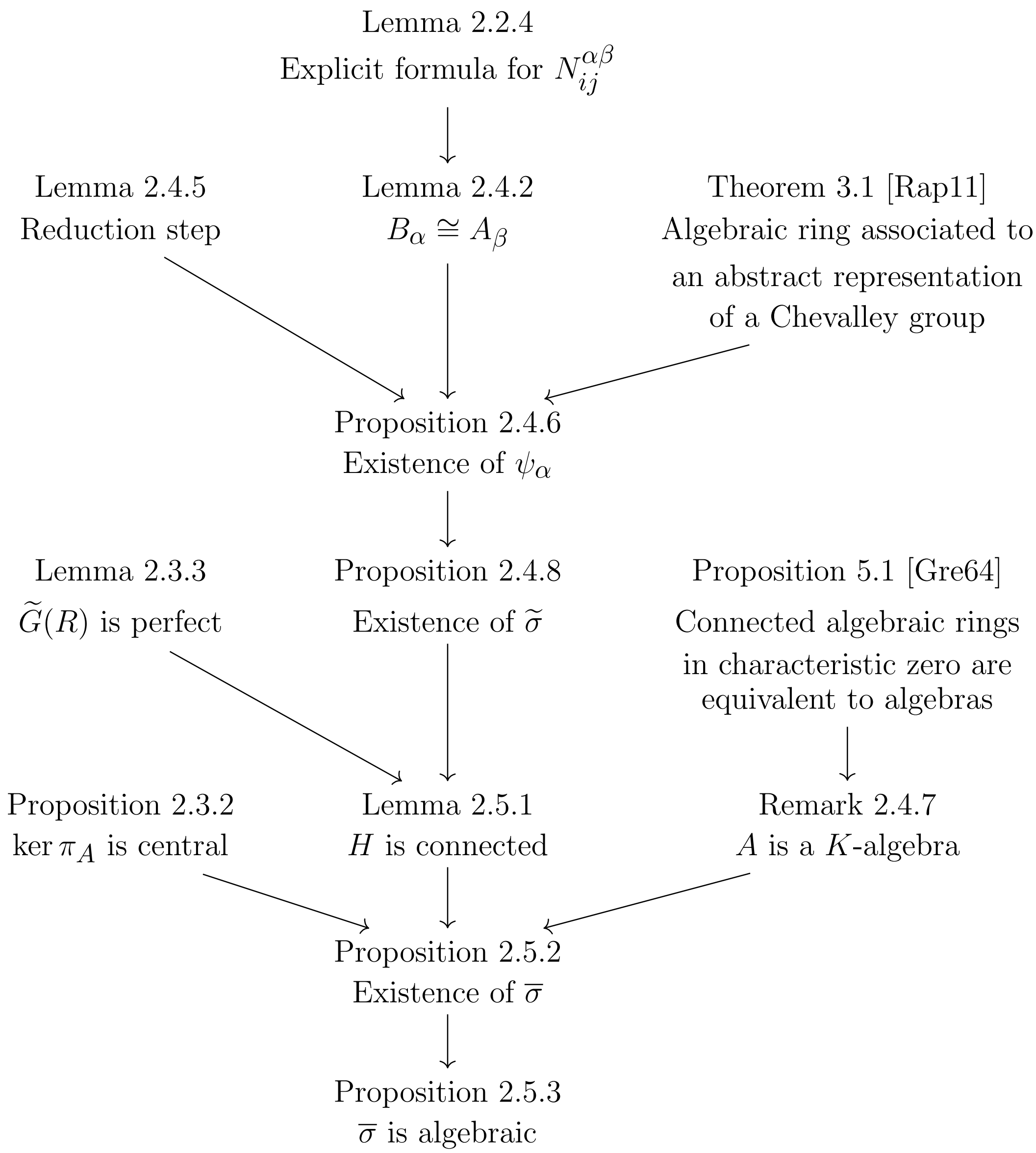
Section 2.2 and **Appendix** B

$$\begin{tikzcd}
& \begin{matrix}\text{Theorem 2 [PS08]} \\ \text{Relative root subschemes}\end{matrix} \arrow[d] & \\
& \begin{matrix}\text{Theorem 2.2.1} \\ \text{Relative root subgroups for } \mathrm{SU}_{2n}(L,h)\end{matrix} \arrow[d] & \\
& \begin{matrix}\text{Lemmas B.1, B.2, B.3, B.5} \\ \text{Various properties of maps } N_{ij}^{\alpha\beta}\end{matrix} \arrow[d] & \\
& \begin{matrix}\text{Lemma 2.2.4/}B\text{.4} \\ \text{Explicit formula for } N_{ij}^{\alpha\beta}\end{matrix} \arrow[dl] \arrow[dr] & \\
\begin{matrix}\text{Lemma 2.3.3} \\ \widetilde{G}(R) \text{ is perfect}\end{matrix} & & \begin{matrix}\text{Lemma 2.4.2} \\ B_\alpha \cong A_\beta\end{matrix}
\end{tikzcd}$$

Section 2.3

$$
\begin{tikzcd}
 & & \begin{array}{c}\text{Theorem 1.3 [Sta20]}\\ \ker\pi_A \text{ is central for local } A\end{array} \arrow[d] \\
\begin{array}{c}\text{Lemma 2.3.3}\\ \widetilde{G}(R) \text{ is perfect}\end{array} \arrow[r] & \begin{array}{c}\text{Lemma 2.3.4}\\ \widetilde{G} \text{ respects products}\end{array} \arrow[r] & \begin{array}{c}\text{Proposition 2.3.2}\\ \ker\pi_A \text{ is central for semilocal } A\end{array} \arrow[d] \\
 & & \begin{array}{c}\text{Proposition 2.5.2}\\ \text{Existence of } \overline{\sigma}\end{array}
\end{tikzcd}
$$

Section 2.4


Lemma 2.2.4
Explicit formula for $N_{ij}^{\alpha\beta}$
Lemma 2.4.5
Reduction step
Lemma 2.4.2
$B_\alpha \cong A_\beta$
Theorem 3.1 [Rap11]
Algebraic ring associated to
an abstract representation
of a Chevalley group
Proposition 2.4.6
Existence of ψ_α
Lemma 2.3.3
$\widetilde{G}(R)$ is perfect
Proposition 2.4.8
Existence of $\widetilde{\sigma}$
Proposition 5.1 [Gre64]
Connected algebraic rings
in characteristic zero are
equivalent to algebras
Proposition 2.3.2
$\ker \pi_A$ is central
Lemma 2.5.1
H is connected
Remark 2.4.7
A is a K-algebra
Proposition 2.5.2
Existence of $\overline{\sigma}$
Proposition 2.5.3
$\overline{\sigma}$ is algebraic

Section 2.5

$$
\begin{tikzcd}[column sep=small]
 & \begin{array}{c}\text{Proposition 2.4.6}\\ \text{Existence of } \psi_\alpha\end{array} \arrow[d] & & \\
\begin{array}{c}\text{Proposition 2.3.2}\\ \ker \pi_A \text{ is central}\end{array} \arrow[dr] & \begin{array}{c}\text{Lemma 2.5.1}\\ H \text{ is connected}\end{array} \arrow[d] \arrow[ddr] & & \\
\begin{array}{c}\text{Lemma 3.10 [Rap13]}\\ \text{Lemma 3.12 [Rap13]}\end{array} \arrow[dr] & \begin{array}{c}\text{Proposition 2.5.2}\\ \text{Existence of } \overline{\sigma}\end{array} \arrow[d] & \begin{array}{c}\text{Theorem 6.3 [Hum72]}\\ \text{Weyl's theorem}\end{array} \arrow[d] & \begin{array}{c}\text{Lemma 2.3.3}\\ \widetilde{G}(R) \text{ is perfect}\end{array} \arrow[d] \\
 & \begin{array}{c}\text{Proposition 2.5.3}\\ \overline{\sigma} \text{ is algebraic}\end{array} \arrow[dr] & \begin{array}{c}\text{Lemma 2.5.5}\\ \text{Property (Z)}\end{array} \arrow[d] & \begin{array}{c}\text{Lemma 2.5.4}\\ \text{Levi decomposition}\\ \text{of } G(A)\end{array} \arrow[dl] \\
 & & \begin{array}{c}\text{Theorem 2.0.1/2.5.6}\\ \text{(BT) conjecture}\\ \text{for } \mathrm{SU}_{2n}(L,h)\end{array} &
\end{tikzcd}
$$

Section 3.1

$$
\begin{tikzcd}[column sep=small]
\begin{array}{c}\text{Theorem 1.1 [Rap19]}\\ \text{(BT) conjecture for}\\ \text{elementary groups}\\ \text{over rings with } (D_1)\end{array} \arrow[d] & \begin{array}{c}\text{Remarks 3.1.4, 3.1.5, 3.1.8}\\ \text{Lemmas 3.1.6, 3.1.7}\end{array} \arrow[dl] \arrow[dr] & \begin{array}{c}\text{Proposition 5.1[Rap13]}\\ \text{Refined (BT) conjecture for}\\ \text{elementary groups}\\ \text{over rings of } S\text{-integers}\end{array} \arrow[d] \\
\begin{array}{c}\text{Theorem 3.1.2}\\ \text{Rigidity for biregular action}\\ \text{of elementary groups}\\ \text{over rings with } (D_1)\\ \text{on affine varieties}\end{array} & & \begin{array}{c}\text{Theorem 3.1.3}\\ \text{Rigidity for biregular action}\\ \text{of elementary groups}\\ \text{over rings of } S\text{-integers}\\ \text{on affine varieties}\end{array}
\end{tikzcd}
$$

Section 3.2

$$
\begin{tikzcd}
\begin{array}{c}\text{Theorem 2 [CdC19]}\\ \text{Criterion for}\\ \mathrm{conj}_{\varphi}(\Gamma)\subset \mathrm{Aut}(Y)\end{array} \arrow[dr] & \begin{array}{c}\text{Theorem 1.1 [EJZK17]}\\ \text{Elementary group has}\\ \text{Kazhdan's property (T)}\end{array} \arrow[d] & \begin{array}{c}\text{Theorem 1.1 [Rap19]}\\ \text{(BT) conjecture for}\\ \text{elementary groups}\\ \text{over rings with } (D_1)\end{array} \arrow[dl] \\
 & \begin{array}{c}\text{Theorem 3.2.1}\\ \text{Rigidity for birational action}\\ \text{of elementary groups}\\ \text{over rings with } (D_1)\\ \text{on projective surfaces}\end{array} &
\end{tikzcd}
$$

Section 4.2

$$
\begin{tikzcd}
\begin{array}{c}\text{Lemma 2.2.8}\\ \text{Symbols in } \ker \pi_A\end{array} \arrow[d] & \begin{array}{c}\text{Proposition 2.4.8}\\ \text{Existence of } \widetilde{\sigma}\end{array} \arrow[d] \\
\begin{array}{c}\text{Conjecture 4.2.1}\\ \ker \pi_A \text{ generated by symbols}\end{array} \arrow[dr] & \begin{array}{c}\text{Lemma 4.2.3}\\ \text{Symbols in } \ker \widetilde{\sigma}\end{array} \arrow[d] \\
\begin{array}{c}\text{Lemma 3.10 [Rap13]}\\ \text{Lemma 3.12 [Rap13]}\end{array} \arrow[dr] & \begin{array}{c}\text{Proposition 4.2.4}\\ \text{Existence of } \sigma\end{array} \arrow[d] \\
 & \begin{array}{c}\text{Proposition 4.2.5}\\ \sigma \text{ is algebraic}\end{array} \arrow[d] \\
 & \begin{array}{c}\text{Theorem 4.2.2}\\ \text{(BT) conjecture}\\ \text{for } \mathrm{SU}_{2n}(L,h)\end{array}
\end{tikzcd}
$$

BIBLIOGRAPHY

BIBLIOGRAPHY

[AM16] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra*. Addison-Wesley Series in Mathematics. Westview Press, Boulder, CO, economy edition, 2016.

[BM66] Armand Borel and George D. Mostow, editors. *Proceedings of Symposia in Pure Mathematics. Vol. IX: Algebraic groups and discontinuous subgroups*. American Mathematical Society, Providence, R.I., 1966.

[Bor91] Armand Borel. *Linear algebraic groups*, volume 126 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.

[BR16] Mitya Boyarchenko and Igor A. Rapinchuk. On abstract representations of the groups of rational points of algebraic groups in positive characteristic. *Arch. Math. (Basel)*, 107(6):569–580, 2016.

[BT65] Armand Borel and Jacques Tits. Groupes réductifs. *Inst. Hautes Études Sci. Publ. Math.*, (27):55–150, 1965.

[BT72] Armand Borel and Jacques Tits. Compléments à l'article: "Groupes réductifs". *Inst. Hautes Études Sci. Publ. Math.*, (41):253–276, 1972.

[BT73] Armand Borel and Jacques Tits. Homomorphismes "abstraits" de groupes algébriques simples. *Ann. of Math. (2)*, 97:499–571, 1973.

[Car30] Elie Cartan. Sur les représentations linéaires des groupes clos. *Comment. Math. Helv.*, 2(1):269–283, 1930.

[CdC19] Serge Cantat and Yves de Cornulier. Commensurating actions of birational groups and groups of pseudo-automorphisms. *J. Éc. polytech. Math.*, 6:767–809, 2019.

[CGP15] Brian Conrad, Ofer Gabber, and Gopal Prasad. *Pseudo-reductive groups*, volume 26 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, second edition, 2015.

[Che05] Claude Chevalley. *Classification des groupes algébriques semi-simples*. Springer-Verlag, Berlin, 2005. Collected works. Vol. 3, Edited and with a preface by P. Cartier, with the collaboration of Cartier, A. Grothendieck and M. Lazard.

[CX18] Serge Cantat and Junyi Xie. Algebraic actions of discrete groups: the p-adic method. *Acta Math.*, 220(2):239–295, 2018.

[Deo78] Vinay V. Deodhar. On central extensions of rational points of algebraic groups. *Amer. J. Math.*, 100(2):303–386, 1978.

[Die80] Jean Dieudonné. *On the automorphisms of the classical groups*. Memoirs of the American Mathematical Society, No. 2. American Mathematical Society, Providence, R.I., 1980. With a supplement by Loo Keng Hua, reprint of the 1951 original.

[EJZK17] Mikhail Ershov, Andrei Jaikin-Zapirain, and Martin Kassabov. Property (T) for groups graded by root systems. *Mem. Amer. Math. Soc.*, 249(1186):v+135, 2017.

[Fis11] David Fisher. Groups acting on manifolds: around the Zimmer program. In *Geometry, rigidity, and group actions*, Chicago Lectures in Math., pages 72–157. Univ. Chicago Press, Chicago, IL, 2011.

[Gre64] Marvin J. Greenberg. Algebraic rings. *Trans. Amer. Math. Soc.*, 111:472–481, 1964.

[Hah86] Alexander J. Hahn. Algebraic K-theory, Morita theory, and the classical groups. In *Group theory, Beijing 1984*, volume 1185 of *Lecture Notes in Math.*, pages 88–117. Springer, Berlin, 1986.

[HO89] Alexander J. Hahn and O. Timothy O'Meara. *The classical groups and K-theory*, volume 291 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1989. With a foreword by J. Dieudonné.

[Hum72] James E. Humphreys. *Introduction to Lie algebras and representation theory*. Graduate Texts in Mathematics, Vol. 9. Springer-Verlag, New York-Berlin, 1972.

[Hum75] James E. Humphreys. *Linear algebraic groups*. Graduate Texts in Mathematics, No. 21. Springer-Verlag, New York-Heidelberg, 1975.

[KS09] Martin Kassabov and Mark V. Sapir. Nonlinearity of matrix groups. *J. Topol. Anal.*, 1(3):251–260, 2009.

[Lee13] John M. Lee. *Introduction to smooth manifolds*, volume 218 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2013.

[LR01] L. Lifschitz and A. Rapinchuk. On abstract homomorphisms of Chevalley groups with nonreductive image. I. *J. Algebra*, 242(1):374–399, 2001.

[LS11] A. Yu. Luzgarev and A. A. Stavrova. The elementary subgroup of an isotropic reductive group is perfect. *Algebra i Analiz*, 23(5):140–154, 2011.

[LSV21] Andrei Lavrenov, Sergey Sinchuk, and Egor Voronetsky. Centrality of K_2 for chevalley groups: a pro-group approach, 2021.

[Mat69] Hideya Matsumoto. Sur les sous-groupes arithmétiques des groupes semi-simples déployés. *Ann. Sci. École Norm. Sup. (4)*, 2:1–62, 1969.

[Mil71] John Milnor. *Introduction to algebraic K-theory*. Annals of Mathematics Studies, No. 72. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1971.

[Mil17] J. S. Milne. *Algebraic groups*, volume 170 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2017. The theory of group schemes of finite type over a field.

[Mos56] G. D. Mostow. Fully reducible subgroups of algebraic groups. *Amer. J. Math.*, 78:200–221, 1956.

[Pie82] Richard S. Pierce. *Associative algebras*, volume 9 of *Studies in the History of Modern Science*. Springer-Verlag, New York-Berlin, 1982.

[PR94] Vladimir Platonov and Andrei Rapinchuk. *Algebraic groups and number theory*, volume 139 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1994. Translated from the 1991 Russian original by Rachel Rowen.

[PS08] V. A. Petrov and A. K. Stavrova. Elementary subgroups in isotropic reductive groups. *Algebra i Analiz*, 20(4):160–188, 2008.

[PS11] Victor Petrov and Anastasia Stavrova. The Tits indices over semilocal rings. *Transform. Groups*, 16(1):193–217, 2011.

[Rap11] Igor A. Rapinchuk. On linear representations of Chevalley groups over commutative rings. *Proc. Lond. Math. Soc. (3)*, 102(5):951–983, 2011.

[Rap13] Igor A. Rapinchuk. On abstract representations of the groups of rational points of algebraic groups and their deformations. *Algebra Number Theory*, 7(7):1685–1723, 2013.

[Rap15] Igor A. Rapinchuk. Abstract homomorphisms of algebraic groups and applications. In *Handbook of group actions. Vol. II*, volume 32 of *Adv. Lect. Math. (ALM)*, pages 397–447. Int. Press, Somerville, MA, 2015.

[Rap19] Igor A. Rapinchuk. On abstract homomorphisms of Chevalley groups over the coordinate rings of affine curves. *Transform. Groups*, 24(4):1241–1259, 2019.

[Ros94] Jonathan Rosenberg. *Algebraic K-theory and its applications*, volume 147 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994.

[RR11] Andrei S. Rapinchuk and Igor A. Rapinchuk. Centrality of the congruence kernel for elementary subgroups of Chevalley groups of rank > 1 over Noetherian rings. *Proc. Amer. Math. Soc.*, 139(9):3099–3113, 2011.

[RR22] Igor A. Rapinchuk and Joshua Ruiter. On abstract homomorphisms of some special unitary groups, 2022.

[Sei97] Gary M. Seitz. Abstract homomorphisms of algebraic groups. *J. London Math. Soc. (2)*, 56(1):104–124, 1997.

[sem70] *Seminar on Algebraic Groups and Related Finite Groups. (Held at The Institute for Advanced Study, Princeton, N. J., 1968/69)*. Lecture Notes in Mathematics, Vol. 131. Springer-Verlag, Berlin-New York, 1970.

[Sha81] I. R. Shafarevich. On some infinite-dimensional groups. II. *Izv. Akad. Nauk SSSR Ser. Mat.*, 45(1):214–226, 240, 1981.

[Spr09] T. A. Springer. *Linear algebraic groups*. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, second edition, 2009.

[Sta14] A. Stavrova. Homotopy invariance of non-stable K_1-functors. *J. K-Theory*, 13(2):199–248, 2014.

[Sta20] A. Stavrova. On the congruence kernel of isotropic groups over rings. *Trans. Amer. Math. Soc.*, 373(7):4585–4626, 2020.

[Ste73] Michael R. Stein. Surjective stability in dimension 0 for K_2 and related functors. *Trans. Amer. Math. Soc.*, 178:165–191, 1973.

[Ste16] Robert Steinberg. *Lectures on Chevalley groups*, volume 66 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2016. Notes prepared by John Faulkner and Robert Wilson, revised and corrected edition of the 1968 original, with a foreword by Robert R. Snapp.

[SVdW28] O. Schreier and B. L. Van der Waerden. Die Automorphismon der projektiven Gruppen. *Abh. Math. Sem. Univ. Hamburg*, 6(1):303–322, 1928.

[Tit66] J. Tits. Classification of algebraic semisimple groups. In *Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965)*, pages 33–62. Amer. Math. Soc., Providence, R.I., 1966, 1966.

[Tit71] J. Tits. Homomorphismes et automorphismes “abstraits” de groupes algébriques et arithmétiques. In *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2*, pages 349–355. 1971.

[vN29] J. v. Neumann. Über die analytischen Eigenschaften von Gruppen linearer Transformationen und ihrer Darstellungen. *Math. Z.*, 30(1):3–42, 1929.

[Wei81] B. Weisfeiler. Abstract isomorphisms of simple algebraic groups split by quadratic extensions. *J. Algebra*, 68(2):335–368, 1981.

[Wei82] Boris Weisfeiler. Abstract homomorphisms of big subgroups of algebraic groups. In *Topics in the theory of algebraic groups*, volume 10 of *Notre Dame Math. Lectures*, pages 135–181. Univ. Notre Dame Press, South Bend, Ind.-London, 1982.