STUDENTS’ TOOL USAGE, JUSTIFICATIONS, AND REPORTED CONFIDENCE WHEN USING DYNAMIC GEOMETRY ENVIRONMENTS

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ABSTRACT

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Abstract: Dynamic Geometry Environments (DGEs) are popular tools in the exploration of geometry. This research is designed to explore the confidence of undergraduate mathematics students as they make mathematical statements when completing geometric tasks using DGEs. Students completed two series of tasks in both Euclidean and hyperbolic geometry. The first series of tasks asked students about properties of parallel transports and the second series of tasks asked students about the existence of regular polygons. The ten students in this research used *Geometry Explorer*, a DGE which they had previous experience using in Euclidean geometry, but minimal experience using in hyperbolic geometry. Hyperbolic geometry tasks were included in this study because features of that geometry (e.g. curved lines and unexpected length measure) were expected to pose challenges for students' intuitive expectations. Because of this lack of intuition, students may use the features of DGEs (e.g. dragging and measurement) to make various justifications (e.g. authoritative, inductive, and deductive) of the mathematical claims they are making. Both the features of the DGE and students' justifications affect their confidence in the claims they make. This research explored the interaction between these three factors.

Analysis of the data showed that these two series of tasks elicited both dragging and measurement tool usage. During the parallel transport tasks, students used these tools in both in an exploratory mode looking for relationships and a validation mode confirming previous conjectures. During the regular polygon construction tasks, students mainly used the tools in a validation mode. Additionally, many students waited until the hyperbolic portion of the tasks to

begin using these tools. The tasks elicited a range of justifications, though students generally used inductive arguments. Deductive justifications, when used, were mainly for familiar tasks that took place within Euclidean geometry. Reported confidence was high across both series of tasks as well as across both Euclidean and hyperbolic geometry when working with the DGE. Reported confidence dropped when working on conjecturing or proof validation prompts that did not use the DGE.

This research suggests there is still much work to be done investigating how students use tools, make justifications, and report confidence when using DGEs in both Euclidean and non-Euclidean geometries. The researcher recommends further study including the exploration of additional tools within DGEs, the dynamics of working in partners within DGEs, and how students' expectations of justification affect their responses.

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This dissertation is dedicated to my wife and children.

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CHAPTER 1: INTRODUCTION

As students study mathematics in the classroom, they use tools to assist in their learning. At the most basic, students are given paper and a writing implement. While these tools may seem simplistic, they were the tools available when Euclid wrote *The Elements*, when al-Khwarizmi began the development of algebra, and when Newton and Leibniz both developed calculus. These tools still have an important place in the classroom, yet as technology has progressed, these basic tools have been augmented with a wide range of digital tools. These digital tools include such items as tablets, graphing calculators, spreadsheets, the internet, and more. These tools have changed the way students learn and engage in mathematics. As Pea (1985) wrote, some of these tools amplify the way students think about mathematics and allow them to perform mathematics more efficiently, while other tools reorganize the way students fundamentally think about mathematics.

One tool that has grown in popularity in the mathematics classroom is the Dynamic Geometry Environment (DGE). Dynamic geometry environments are specialized computer software that allow the user to create geometric constructions virtually, using a set of digital tools. Rather than having students use the traditional compass and straightedge, DGEs allow students to do all the same moves within a digital environment. At their core, DGEs provide tools that range from drawing circles (a compass) and straight lines (a straightedge) to more advanced tools that automatically construct perpendicular bisectors or construct circles given three points. Yet, those tools by themselves are just an amplification of what can be done by hand. Where DGEs shine is their ability to do things that are simply impossible by hand. For instance, DGEs allow students to measure any part of a figure quickly and accurately. While students could measure by hand, it is impossible to match the accuracy a computer is able to produce. Perhaps the biggest change of using DGEs is not simply that the computer is providing exceedingly accurate figures, but that these figures are *dynamic*. When using paper and pencil, the figures created are static and unchanging. DGEs, on the other hand, create a dynamic figure that can be modified (dragged) in real time. Because of this, the software can be used to discover, test, or even refute various relationships among the parts of the construction (Hadas et al., 2000). This allows DGEs to fundamentally reorganize the way we think about doing geometry. This advantage is one reason that DGEs are specifically suggested within the Common Core State Standards for Mathematics (2010) as a tool to use in mathematics classrooms.

Dynamic Geometry Environments

Dynamic Geometry Environments have been the focus of much research in the mathematics education community. Jones (2002) identified three main strands of research into the area of DGEs. The first strand is how students interact with the software. This can mostly be broken into two main categories. Some researchers look at the different ways that students use the measurement tool (Olivero & Robutti, 2007). Other researchers study how students interact with the dragging tools (Hölzl, 1996; Lopez-Real & Leung, 2006). The second strand identified by Jones is using DGEs to understand geometric concepts. For instance, researchers have looked at DGEs as a tool to learn about non-Euclidean geometries (Guven & Karatas, 2009; Hollebrands et al., 2010) or as a way to study geometric transformations (Denton, 2017). The third strand identified by Jones is how DGEs can be used to help students grasp the ideas of proof. For example, Leung & Lopez-Real (2002) showed how DGEs can be used to help students understand the role of contradiction in proof. Additionally, Mariotti (2012) shows how DGEs can be used to help students understand the structure and nature of theorems by making use of the dependencies that arise with DGEs. A decade later, Jones (2012) updated his original three themes by adding three emerging themes including research into how learners interact with pre-constructed DGE files, research into the challenges faced by teachers when learning DGEs, and research into the new developments within the software.

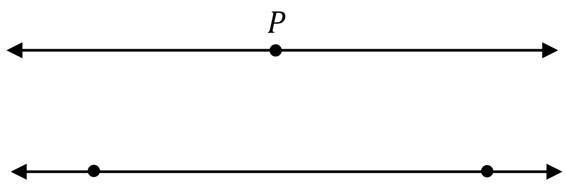
Euclidean and Non-Euclidean Geometry

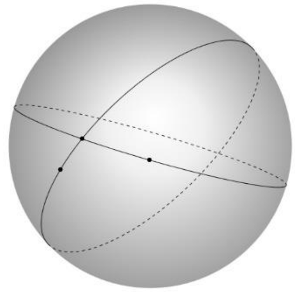
Students are exposed to geometry at all age levels. The Common Core State Standards for Mathematics (2010) devotes an entire domain to geometry, though most of this domain deals with what is commonly known as Euclidean geometry. Euclidean geometry is defined primarily by an axiom stating that given a line and a point, *P*, not on the line, there is a single parallel line through *P* parallel to the original line (see Figure 1.1). In layman's terms, this axiom defines a 'flat' geometry - the geometry studied by Euclid around 300 BC when he published *The Elements*. The importance of this geometry cannot be overstated, and we continue to teach it in primary and secondary education. And yet, despite its importance and relevance, mathematicians came to realize that not all geometry is 'Euclidean'. For instance, trying to describe the geometry of the Earth, an approximate sphere, proved impossible using Euclidean geometry. The basic axioms of Euclidean geometry no longer applied. On a sphere, the defining axiom of having exactly one parallel fails as there are no parallel lines on a sphere. All lines (defined as great circles) intersect (see Figure 1.1). This discrepancy eventually led to the development of non-Euclidean geometries such as elliptic geometry (positive curvature such as the surface of a sphere) and hyperbolic geometry (negative curvature such as the curvature of sea coral).

While most high school geometry curriculums cover Euclidean geometry in great detail, non-Euclidean geometries are noticeably missing (CCSSI, 2010). It is typically not until advanced geometry courses at the collegiate level that students are rigorously introduced to non-Euclidean geometries. These courses tend to be theoretical math courses with a strong reliance

Figure 1.1:

Lines in Euclidean and non-Euclidean Geometry





Single Parallel in Euclidean Geometry No Parallels in Elliptic Geometry

on proof. There is, of course, good reason for waiting to study non-Euclidean geometries until the post-secondary level. Non-Euclidean geometries use alternate axiomatic systems that students often struggle to understand. In the van Hiele model of geometric reasoning, non-Euclidean geometries are at the very highest level (van Hiele, 1986). That is not to say students cannot study these alternate geometries in high school, but it is not commonplace.

Whether in a high school class or in the post-secondary classroom, tools have been developed to help mathematicians and students study non-Euclidean geometries. For elliptical geometry, Lénárt introduced a toolset that allows geometers to do straightedge and compass constructions directly on the surface of a physical sphere (2003). This provides a direct non-Euclidean analogue of the traditional straightedge and compass. For hyperbolic geometry, no such tools exist. In terms of physical models, one possible suggestion for hyperbolic geometry put forth by Taimiņa (2009) is the use of crocheted corals. But this crocheted model brings about its own set of difficulties as you cannot easily draw on the surface of crochet. Instead of physical models for hyperbolic geometry, most mathematicians tend to project the geometry onto flat surfaces. In so doing, one must either lose the appearance of straight lines (e.g. the Poincaré model) or lose the appearance of right angles (e.g. the Klein model), neither of which is ideal.

These projections may mathematically work as models, but they are exceedingly challenging for students to use and understand. When lines are no longer straight and angles are distorted, it is difficult to gain an intuitive sense of things.

This difficulty is where DGEs can help. In their early days, DGEs were created specifically to model Euclidean geometry. Cabri Geometry (J. M. Laborde & Bellemain, 1995) and Geometer's Sketchpad (Jackiw, 2001), two of the earliest pieces of software, are still used in some schools. They have now been joined by the newer software, GeoGebra, a free and open source piece of software that is gaining traction (Jones, 2002). All three of these pieces of software focus on Euclidean geometry - the main type of geometry taught in primary and secondary education. In contrast, Geometry Explorer is a newer DGE that allows students to explore both Euclidean and non-Euclidean geometries with the same interface and toolset (Hvidsten, 2015). That is, students can perform the same style of virtual straightedge and compass constructions they perform in Euclidean geometry, but with Geometry Explorer they can do those constructions in either elliptic or hyperbolic geometry.

Proof and Argumentation

In addition to being introduced to non-Euclidean geometries, undergraduate mathematics students are introduced to the idea of formal proof and argumentation. While proof and argumentation is typically presented in high school geometry courses as two-column proofs (CCSSI, 2010; Herbst, 2002), at the undergraduate level students begin the transition to paragraph style proofs. This transition is known to be difficult for students (Moore, 1994; Pajela et al., 2020; Weber, 2002). Pajela et al. (2020) showed how students, even after taking an introduction to proofs class, struggle with not just the mathematical content being proved, but also the structure of proof itself. It is because of this difficulty that much research has been done into strategies designed to help students understand mathematical proof (e.g. Azrou & Khelladi, 2019; Laamena et al., 2018; Morrow, 2004).

Pajela et al. (2020) emphasized the importance of "sense-making" activities within the proving process. For instance, Pajela et al. (2020) showed how students considered similar, yet simpler activities or they created examples to explain things. Laamena (2018) also shows how examples are important to students as they can be used for exploration. These sense-making activities are important for understanding, but they do not always lead to proof. Indeed, as Azrou and Khelladi (2019) show, students struggle transitioning from the exploratory (sense-making) stage to the formal proof stage. Still, these sense making activities are an important part of the proving process. One such sense-making activity is the use of DGEs. As such, proof and DGEs have a long history of being studied in the literature (e.g. Kmetová et al., 2019; Mariotti, 2012; Unal & Hollebrands, 2021). As students in undergraduate mathematics begin their study of non-Euclidean geometries, the use of DGEs as a tool to assist with proof has also been studied (e.g. Guven & Karatas, 2009; Hollebrands et al., 2010).

Confidence When Using DGEs

As has been stated, proof in mathematics is a struggle for students (Moore, 1994). While DGEs have been shown to provide confidence for students in some situations (Guven & Karatas, 2009; Hollebrands et al., 2010), when working on proof related activities, students can express uncertainty about their mathematical ability or about the claims they are making. These issues can largely be classified under the topic of confidence – though confidence can have multiple meanings. At times, confidence can refer to one's self-efficacy, a students' belief in their ability to perform the task to produce specified outcomes (Bandura, 1997). Generally, self-efficacy has been interpreted to mean a general sense of one's ability, though some researchers such as

Parajes (1996) lament this generalized view of self-efficacy. Pajares (1996) writes that much educational literature “reflect[s] global or generalized attitudes about capabilities bearing slight or no resemblance to the criterial task with which they are compared” (p. 547). That is, while a student may indeed have a general mathematical self-efficacy, their belief in their ability to perform a specific task may relate to any number of current factors. For instance, a student may have a positive view of their ability to do mathematics until they are faced with doing a formal proof or a statistical analysis. Doing algebraic manipulation is different than doing logical deductions which is different than doing statistical analysis.

An alternative view of confidence is that of confidence as conviction (Segal, 1999). When students are working on proof related activities, they are often expected to make a conjecture and then justify that conjecture. When they do so, they do so with a certain amount of conviction – a sense their answer is correct (or incorrect). This view of confidence is not so much concerned about one’s overall mathematical ability, but about the specific task at hand and confidence that the claim being made is correct. This interpretation of confidence can be found within existing literature related to DGE use. For instance, when completing certain geometric tasks, Hollebrands et al. (2010) showed that DGEs can give students enough confidence in their claims they no longer feel the need to do deductive proof. Olivero & Robutti (2007) demonstrated instances where DGE gave students enough confidence the students dismissed valid proofs in favor of empirical evidence. The importance and role of deduction inside the math classroom can always be debated (e.g. Weber & Mejia-Ramos, 2015), but it is nevertheless important to understand how DGEs may affect the ways in which students gain confidence. Do students have confidence in their answer because of empirical methods, deductive methods, because an authority told them it was correct, or for a variety of other reasons?

Purpose

One of the goals of geometry courses is to introduce students to proof (CCSSI, 2010). In high school geometry classrooms, that means proofs within the realm of Euclidean geometry. In collegiate classrooms, this can additionally mean proofs in non-Euclidean geometries. It is well documented that students struggle with proof in mathematics classrooms at all levels (e.g. Mariotti, 2012; Moore, 1994). Teachers are constantly looking for new tools and techniques to help their students create and understand proofs. While geometric constructions have historically been, and continue to be, an avenue to help understand proof, dynamic geometry environments are being explored to see how they can further students understanding of proof and deductive reasoning (Hollebrands, 2007; Jones, 2012).

The use of non-Euclidean geometry at the collegiate level introduces students to geometric models with which they are typically unfamiliar. Taking students out of Euclidean geometry and having them explore geometry on a sphere or in a hyperbolic plane removes the intuition they have learned from living in a locally Euclidean world. Suddenly, the interior angles of triangles no longer sum to 180 degrees and rectangles no longer exist. Because of their lack of intuition about non-Euclidean geometry, students will often qualify the claims they make and rely heavily on software to give them confidence (Guven & Karatas, 2009; Hollebrands et al., 2010). In short, by removing students from the familiarity of Euclidean geometry, student confidence when making mathematical claims in non-Euclidean geometry appears to be affected.

Geometry Explorer gives students a multitude of tools to use in non-Euclidean geometry that simply do not exist or are extremely cumbersome to use without software. While there have been studies dealing with non-Euclidean geometries and DGEs (e.g. Hollebrands et al., 2010), there is still much research to be done about what role DGEs play in student understanding and confidence when exploring non-Euclidean geometries. This dissertation is designed to investigate how the use of DGEs affect students' confidence when working both in Euclidean and non-Euclidean geometries. It will investigate how students are using specific features of DGEs such as dragging and measurement, and how those uses may directly interact with student confidence. It will also include a focus on the justifications students are using when making claims. Specifically, this research will attempt to answer the following four research questions.

Research Questions

1. What are the features of DGEs that college geometry students use when making mathematical claims while completing DGE related tasks in Euclidean and hyperbolic geometry?
2. What are the justifications college geometry students use when making mathematical claims while completing DGE related tasks in Euclidean and hyperbolic geometry?
3. How do college geometry students self-report their confidence when making mathematical claims while completing DGE related tasks in Euclidean and hyperbolic geometry?
4. What are the relationships among the features of DGEs, the justifications students are making, and their confidence in making mathematical claims while completing DGE related tasks in Euclidean and hyperbolic geometry?

The next chapter will provide a literature review covering DGEs, proof, justifications, and confidence as they will relate to the research questions. Chapter 3 will explain the design of the study and how participants were chosen. Chapter 4 will give detailed summaries of student activity as they complete two series of tasks. Chapter 5 will then give the results of the study with a focus both on the overarching trends among the students as well as a focus on individual student behavior. Chapter 6 will be a discussion of the results followed lastly by a concluding chapter that will explore the implications and limitations of the study, as well as future directions for research.

CHAPTER 2: LITERATURE REVIEW

This chapter will outline several areas of research that are significant for understanding the purpose and objectives of this dissertation. The first area to be covered will be the use of DGEs, with a focus on their unique tools. The second area of research will focus on student proof schemes and justifications. The third area of research will focus on student confidence and what it means to be convinced of a claim.

Dynamic Geometry Environments

During the 1990s, as computers became more widespread in schools, more math classrooms were beginning to incorporate Dynamic Geometry Environments (DGEs) into their classrooms. Two of the more popular DGEs of the time were Geometers Sketchpad (Jackiw, 2001) and Cabri Geometry (J. M. Laborde & Bellemain, 1995). Over the next two decades as the importance of DGE was quickly recognized, the number of DGEs on the market expanded greatly and research into how best to use DGEs quickly became an important focus of mathematics education research.

Researchers have identified numerous advantages of using DGE within the classroom. These include increased student interest, ability to create complex constructions, and DGE's ability to help students transition from description to explanation. Numerous studies have shown that students tend to have higher interest in using DGE than traditional paper and pencil constructions (Barcelos et al., 2011; Pandiscio, 2002). Students have said using a DGE is "more interesting" and "more pleasant" (Barcelos et al., 2011, p. 260). Pre-service teachers have reported that DGE generates greater student interest and that even if students are struggling with an activity or trying to prove a conjecture, they are more willing to use a DGE (Pandiscio, 2002). DGEs also allow students to create and use much more complex constructions than they otherwise would be able to by hand. In Euclidean geometry, students can use built-in construction tools as a time saving feature. Most modern DGEs have tools built in that allow students to create perpendiculars, midpoints, bisectors, etc. with the click of a button. Some modern DGEs also allow students to create their own custom tools (macros) to record construction sequences to use in the future (Pratt & Ainley, 1997). Lastly, DGEs can assist students in the transition from description to explanation. As Habre (2009) wrote, DGEs can "bring together the construction process and the verification, thus leading to a formal proof/explanation" (p. 163). Through well thought out activities, students can discover relationships that are difficult to see when the geometry is static.

Because of the advantages DGEs have to offer, much research has been carried out regarding their use within mathematics education. In surveys of research on Dynamic Geometry Environments, Jones (2002, 2012) initially identified three strands of research into DGEs, later expanding these three strands into six. Jones's (2002) original three strands included how students interacted with DGEs, how DGEs helped students understand geometry concepts, and how DGEs helped students to understand the concepts of proof. Jones (2012) later added three emerging themes, including research into how learners interact with pre-constructed DGE files, research into how teachers collaborate with researchers to learn how to use DGEs, and research into the new developments within DGEs. Many of these themes are also touched upon by Laborde et al. (2006) in their more general review of all geometric software. As an example, in connecting the theme of how students interact with DGEs and how they understand proof, Laborde et al. (2006) discuss the use of the drag feature and its connection to the idea of geometric dependencies.

Two important aspects of DGE use, foundational to all areas of research, are the differences between drawing and figures, as well as the theoretical differences between the geometry of the DGE and theoretical geometry. For the first, Laborde (1993) highlights the differences between a drawing and a figure. A drawing is a physical representation of a geometrical object. A student may draw a square with approximately right angles and equal segments, but there is an understanding that it is not a theoretical square. Even in a DGE, a student may draw a square and use the measurement tool to ensure the sides and angles are congruent, but it is still a physical (albeit digital) representation of a theoretical square. Figures, on the other hand, refer to the theoretical object. When a student constructs a right angle with a straightedge and compass (digitally or on paper), there is an understanding that the picture they see is a drawing, but by using a straightedge and compass, they are representing the theoretical figure.

It is also important to highlight that geometry within a DGE is not a strict representation of the purely theoretical geometry it is representing. Jones (2000) underscores some of these differences by emphasizing how segments and angles can have orientation depending on the order the vertices were created. Angle measurements are a common issue for students as the order in which the students click the three defining points for the angle will give the student either the angle they are seeking or 360 degrees minus the angle they are seeking. Measurement issues can occur as rounding errors or screen resolution can prevent accuracy (Olivero & Robutti, 2007). Screens only have so many pixels compared to the theoretically infinite number of points on the Euclidean plane.

Common to all DGEs are the dragging tool and the measurement tools. The dragging tool is what gives *dynamic* geometry environments their name. Students can drag their figures and see the relationships that occur within those figures. Additionally, measurement has been added to modern DGEs to aid students in understanding their figures. With the click of a button, students can measure any angle or segment on their screen. These two features have fundamentally changed how students can interact with geometry. Understandably, much research has been carried out on these two tools.

Dragging Tool

The ability to drag points and see how these points affect other parts of a figure has been studied extensively in the literature (e.g. Baccaglini-Frank & Mariotti, 2010; Lopez-Real & Leung, 2006). Laborde et al. (2006) specifically list it as one of their strands of research into DGEs. This is a feature that simply cannot be done with paper and pencil. It allows students to see relationships, to generate infinite variations of a figure, to find the loci of figures, to trace objects, and more. One of the more immediate uses of the dragging tool is students' ability to use dragging to test the *robustness* of a construction. Geometric constructions are a staple of the geometry classroom. When completing geometric constructions with paper and pencil, constructions can often appear correct when they may contain flaws. When switching to DGEs, however, the dragging ability gives the user an immediate way to test whether the figure was constructed correctly. By dragging parts of the figure, the user can test whether a right angle is properly constructed to 90 degrees or just happens to look close to 90 degrees. The developers of CARMetal (Hakenholz et al., 2019), a relatively unknown DGE, recognized the need to test the robustness of geometric figures and included a novel 'Monkey' button that 'shakes' the figure by automatically and randomly dragging the independent points of the construction.

Moving beyond the basic ability to test the correctness of a construction, Baccaglini-Frank and Mariotti (2010) identified four ways that students use the drag feature of a DGE when doing construction based activities. The first of these four ways is wandering (random) dragging. Students using this method are dragging parts of their construction while looking for interesting features and relationships that may exist in their construction. Students in this modality are operating in a conjecturing mode. The second type of dragging is maintaining dragging where students drag a point along a perceived locus to test and see if the construction holds its properties. At this point, students are moving from conjecturing into testing a hypothesis. The third dragging modality is trace dragging which is when a student drags with the trace turned on to identify a locus by having it literally drawn (traced) on the screen. A student at this point is still in the conjecturing phase and may suspect a locus but may not know specifically what the locus is. Lastly, students can employ test dragging. If a student is convinced of a conjecture, they can drag to justify that a certain property holds.

Along with the dragging modalities, Mariotti (2012) also argues that dragging can serve an important function in teaching students about the nature of mathematical theorems. In a DGE, constructions and figures are built using dependencies. For instance, when two lines cross, a point may be added at their intersection. This point is now dependent on the two lines and cannot move independently. To move the point, the two lines must be moved as they are the independent objects. As the construction becomes more complex, the user must keep track of how these dependencies are layered. Mariotti (2012) suggests that this idea of dependency can be used to help students develop an understanding of what a theorem is. Independent points (or lines) become the premises of an argument, and dependent points (or lines) serve as the conclusion. Just as dragging independent points changes the figures, by changing the premises of an argument, the conclusions may or may not be valid anymore. By having students create dynamic figures with independent and dependent points, she showed students were able to better understand the principles of premises and conclusions.

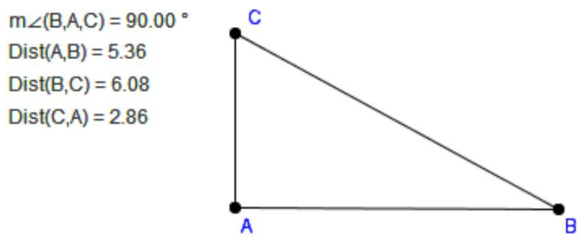
Measurement Tools

A second major feature of DGEs is the measurement tool. The measurement tool is typically two seperate tools that allow students to measure lengths and angles. Some DGEs even have measurement tools that can check specifically for parallelism and congruency. While students can make measurements using paper, pencil, and a ruler, the ease and accuracy with which the software can do it makes this tool markedly different than just using a ruler. When students are working with paper and pencil, measurements are always approximated. For instance, if a student incorrectly constructs a square but is at least close to a square, they may still measure with a ruler and think they have been successful. Any error is easily attributed to the inherent impreciseness of the ruler and compass. In DGEs, however, when students measure, the software will always give a reliable measurement. Mistakes cannot easily be ignored.

That is not to say that measurement within DGEs does not present its own potential issues. In Figure 2.1, we see a right triangle that has been constructed and measured in Geometry Explorer. If we attempt the Pythagorean Theorem, we get $\overline{\mathrm{AB}}^2 + \overline{\mathrm{AC}}^2 = 5.36^2 + 2.86^2 = 36.9092$ and $\overline{\mathrm{BC}}^2 = 6.08^2 = 36.9664$. Note that these values are not the same when they should be. Measurement in DGEs is dependent on a multitude of factors. In this case, rounding is causing problems as Geometry Explorer is rounding to two decimals. While Geometry Explorer has the option for more decimals, it defaults to two. There is also the issue that the pixelated screen is not the equivalent of a piece of paper. Geometric points can only be dragged to the physical points on a screen. This creates a mismatch between the theoretical and the physical that

Figure 2.1:

Example of a Right Triangle in Geometry Explorer



different DGEs handle in various ways. With that in mind, teachers need instruct their students that rounding errors and screen resolution affect the measurement tools.

Both Jones (2002, 2012), in his surveys of DGEs, and Laborde et al. (2006), in their literature review of teaching with geometry software, failed to mention the measurement tool as an area of research involving DGEs. Yet, there is research being done on the measurement tool. Specifically, a study by Olivero and Robutti (2007) gives a basic framework for how students use the measurement tool. Using categories reminiscent of Baccaglini-Frank and Mariotti (2010), Olivero and Robutti (2007) identified various modalities of measurement. The first of these is wandering measurement which is when students take measurements of various components of a figure until an interesting property appears to them. The second modality is guided measuring. In this modality, students are using the tools to turn their construction into a given image. For instance, students may be trying to create a parallelogram, and so they measure angles, and then use the measurements to guide drag an otherwise generic quadrilateral into a parallelogram. The third modality is perceptual measuring where students measure elements of the figure that they recognize may be important. Perhaps students think they see matching angles, or they think they see that segments are congruent. These first three categories all exist during the conjecturing stage. Students are looking for relationships and trying to form conjectures. The fourth modality is validation measuring where students already have a conjecture and are returning to verify that their claim is true. Lastly, there is proof measuring, where students already convinced of the truth take measurements to help with the development of a proof of their claim.

We see that there are four categories of dragging (Baccaglini-Frank & Mariotti, 2010) and five categories of measurement (Olivero & Robutti, 2007) that are outlined in the research. Hollebrands (2007) adds to this by defining two broad categories of strategies that can be applied to both measurement and dragging. Hollebrands (2007) differentiates between reactive and proactive strategies that students use. Reactive strategies refer to student use of DGEs when they do not quite know what to expect. Students are not "able to fully anticipate what will result in the action they perform prior to performing it" (Hollebrands, 2007, p. 184). Proactive strategies refer to strategies where students are expecting certain outcomes from the software. For instance, students may take validation measurements of angles because they want to confirm two angles are congruent. The various measurement and dragging modalities will fall into either of these two categories. For instance, wandering dragging, while primarily a reactive strategy, also has the potential to be used as a proactive strategy. A student who has formed a conjecture might start wandering dragging looking for an exception to the conjecture they formed. The student may not necessarily know what the exception is, but they know they should look for one and their wandering dragging is being done proactively.

In summary, DGE has been shown to have many advantages within geometry classrooms. While there are numerous avenues of research into DGE use (Jones, 2002, 2012), this dissertation will focus on tool usage, specifically, the use of dragging tools (Baccaglini-Frank & Mariotti, 2010) and measurement tools (Olivero & Robutti, 2007). The classification schemes presented in those studies will inform the framework for how student tool use is classified throughout this dissertation.

Proof and Justification

Mathematical proof is the process of using deductive arguments to connect a hypothesis to a conclusion. This definition of proof refers to using definitions, previous theorems, and logic to arrive at a conclusion. Proofs are traditionally introduced to students during their geometry course in high school, albeit in the style of a two-column proof (CCSSI, 2010; Herbst, 2002). At the post-secondary level, students pursuing the study of mathematics make the transition to the paragraph style proof of the professional mathematician (Moore, 1994). As students make this transition, proof remains a struggle for many students (Morrow, 2004; Senk, 1989; Weber, 2002).

Selden and Selden (2017) identified four related concepts in the literature about proof: proof comprehension, proof construction, proof validation, and proof evaluation. At the most basic level is proof comprehension. When students are presented with a proof, can they read the proof and understand what is being said? This involves knowing the key terms and definitions as well as understanding the structure of how proofs work. There is also the concept of proof construction. Can students successfully write a mathematically valid proof? The goal of many mathematics courses is to help students learn to deductively argue the truth of either a conjecture they have made themselves or a statement given to them by their instructor. The third related concept is that of proof validation. Students may be able to understand a proof, but can they verify that a given proof is correct? Rather than checking the "surface features" (Selden & Selden, 2017, p. 340) of the proof, mathematics educators are hoping students understand the actual argument being made. Last is the concept of proof evaluation. At this stage, proofs are being evaluated, not for correctness, but for clarity, elegance, conciseness. This current study will be focused both on proof construction and on proof validation. It will focus on how students are justifying their claims (proof construction) as well as how students understand a proof that has been given to them (proof validation).

Proof Schemes

One of the challenges that arises as instructors teach proof and deductive argumentation is that students often hold to a variety of proof schemes (Harel & Sowder, 1998). A student's proof scheme is "what constitutes ascertaining and persuading for that person" (Harel & Sowder, 1998, p. 244). That is, a proof scheme is the method by which a student is persuaded of the truth of a statement. Harel and Sowder (1998) identified three main proof schemes students use: external schemes, empirical schemes, and analytical schemes. Each of these main branches are then broken into smaller subcategories. Harel and Sowder's work, of course, is not the only way to categorize student thinking during the proof process. The same year Harel and Sowder published their proof schemes, DeVilliers (1998) offered a similar break down of student proof schemes that focused on positive versus negative justifications. However, for this study, the focus will be on Harel and Sowder's proof schemes as their conceptualizations are more thoroughly studied in the literature (Hadas et al., 2000; Weber et al., 2014).

In looking at the proof schemes of Harel and Sowder (1998), external schemes refer to students looking to outside sources like the teacher, the book, or even symbolic manipulation to give them a sense of certainty. With this conception, a claim is not true unless someone or something tells the student it is true. It is worth noting that Harel and Sowder (1998) include symbolic manipulation as an external proof scheme because performing a symbolic manipulation correctly can give students a sense of truth even if the students lack the understanding of what is being manipulated. It is the act of doing the manipulation that imparts a truthfulness to the claim being justified. Empirical proof schemes refer to students believing that inductive arguments or perceptual arguments (based on how figures, symbols, or numbers may initially appear) can be used to convey the truth of a claim. When working in this proof scheme, students believe that a few specific examples of a property is enough to then be generalized to all similar things. Lastly, analytic proof schemes refer either to axiomatic proof schemes or transformational proof schemes. Axiomatic proof schemes refer to a student believing that formal axiomatic proofs are essential for the truth of a statement. Transformational proof schemes refer to when students can generalize an example to a broader class by using established relationships they know about the example. They are not generalizing based on measurements, but deductively based on the relationships that exist and how those relationships can transform objects into each other. Lastly, it is worth mentioning that Harel and Sowder (1998) recognize that often there is overlap among these schemes and that students can quickly move from one scheme to another.

Harel and Sowder (1998) are careful not to present proof schemes as a hierarchy. That is, they do no present one proof scheme as necessarily better than any other. Context often plays an important role in what type of proof schemes students use. More recently, Weber et al. (2014) took a critical look at proof schemes and argued that it may not always be a correct goal to push students towards deductive proof in every situation. Historically, pushing students toward deductive reasoning has been a goal of mathematics instruction. This is one of the reasons that deduction is the highest level on the VanHiele levels of geometric reasoning (Senk, 1989). Part of the reason for this is the prevailing belief that all mathematicians operate under an axiomatic proof scheme and only accept deductive proof as convincing. Yet, Weber et al. (2014) present the case that, contrary to what many assume, practicing mathematicians often rely on proof schemes other than just deductive argument. For instance, Weber et al. (2014) argue that practicing mathematicians often rely on proof by authority whenever they read new articles in mathematics journals. The authors argue that practicing mathematicians do not have the time, nor the interest, to always double check every result, and often become convinced based on the reputation of the author or the publication. Weber et al. (2014) are not making this argument to downplay the importance of deductive argument, but simply to point out that context is important when looking at proof schemes. There are times and places where empirical evidence and authority may be all that is needed and desired.

Justifications

At this point, there is a need to differentiate between the proof scheme under which a student is operating and the justification a student uses when supporting a claim. These two concepts do not necessarily have to coincide. Justifications do not always have to be a formal proof. Segal (1999) approaches this issue by highlighting the difference between conviction and validity. Conviction is of an individual nature. It operates similarly to a proof scheme and is concerned with what it takes for the individual to be convinced of the truth of a claim. If the student is operating under an empirical proof scheme, a few examples demonstrating the claim may convince that student. In short, a justification can be any evidence (empirical, deductive, or authoritative) that convinces a student of the truth of a claim. Validity refers to the community aspect of what mathematicians or mathematics educators consider correct justifications. This generally refers to deductive justifications or a full proof of the claim being made. The same student who is operating with an empirical proof scheme and is convinced by examples may understand that inductive justifications do not fit the formal requirement a teacher requires. Thus, even though a student is operating with an empirical proof scheme, they may still go on to provide deductive justifications as they know that those justifications are what constitutes validity for the teacher. Because of this difference between conviction and validity, a given justification may or may not always match the proof scheme under which the student is operating.

It is also the case that justifications and proof schemes can switch from situation to situation (Harel & Sowder, 1998). A student's familiarity with a topic as well as the setting may have a strong influence on the type of proof schemes and justifications a student uses. That is, a student may choose to give deductive justifications within a geometry setting as they have had experience doing proofs in a previous course (e.g. high school geometry). That same student might also choose to give empirical evidence for a claim in number theory as they can see the claim is true for the first few values they check. Likewise, a student may give different justifications based on whether they are working on something for a course or investigating something on their own time. In summary, students will approach proof from different places and with different ideas of what proof means to them.

In summary, proof and deductive arguments are a large part of mathematics education. Students operate and think about deductive argumentation using internal personal proof schemes but are often asked to give external justifications for why claims are true. These justifications are of interest as they can change from situation to situation depending on numerous factors, including expectations, previous experience, and internal proof schemes. As such, this research will investigate how students are choosing to justify their claims when using DGE.

Confidence

Confidence is a term that can take on many meanings. People can express confidence in themselves. They may be confident in a certain area of study or confident in their job. People can express confidence in skills they possess. They can be confident in their ability to shoot a free throw or factor a quadratic equation. Confidence can also refer to one's belief in something. A person can express confidence that something is true. Perhaps someone is confident it will rain tomorrow or confident that the Reimann hypothesis is true. Different aspects of confidence have been studied in the existing literature.

Bandura (1997) coined the term self-efficacy. This refers to a person's own judgements about their capabilities at a task. This is at times used interchangeably with the term self-confidence. Bandura (1997) distinguished these concepts based on scope. Self-confidence refers to one's overall confidence in themselves, where self-efficacy refers to one's confidence in their ability at the task at hand. Yet even this smaller focus of self-efficacy is often not enough. Pajares (1996) argues that much mathematical education literature looking at self-efficacy is still too broadly focused. A student's mathematical self-efficacy is still too broad as students' confidence can change even from task to task within mathematics. A student may have strong self-efficacy with geometry, but weak self-efficacy within a calculus course. Or even within geometry, a student may have strong self-efficacy when working on discovering conjectures, but low self-efficacy when attempting to prove conjectures.

Beyond this generalized view of confidence in oneself, the term confidence is also used to refer to ones' belief in the truth of a statement. When students are asked how confident they are in an answer they just gave, or in a discovery they just made, they may express varying levels of confidence. This understanding of confidence is similar to the concept of conviction (Segal, 1999). This is not a belief in oneself, but rather a belief about an external claim. Weber and Mejia-Ramos (2015) took a critical look at conviction and examined what it means for someone to have confidence (conviction) in a claim. As referenced earlier, a proof scheme may be *how* a person becomes convinced of a claim, but it does not specifically say *when* a person becomes convinced of a claim. For some students operating under an empirical proof scheme, perhaps one example is enough to convince them. For others, perhaps three or five examples are needed for someone to be convinced. Likewise for students operating under an external proof scheme, they may be convinced of a claim because they saw an online video about the claim. Others, however, may want to read the claim in a book to be convinced. Perhaps not all types of authorities provide the same level of confidence (conviction) in a claim.

Weber and Mejia-Ramos (2015) argue that the term 'convinced' contains an inherent ambiguity and should be broken into two categories. Specifically, they argue that mathematicians can be either absolutely or relatively convinced of a claim. Absolute conviction refers to when a person "has a stable psychological feeling of indubitability about that claim" (Weber & Mejia-Ramos, 2015, p. 16). The authors give the example that they believe most mathematicians are absolutely convinced that $2 + 2 = 4$. For the typical mathematician, there is no doubt this claim is true.

In contrast, relative conviction refers to when the "subjective level of probability that one attributes to that claim being true exceeds a certain threshold to provide a warrant for some future actions" (Weber & Mejia-Ramos, 2015, p. 16). For instance, consider the four-color theorem. The initial proof was done in 1976 with the aid of a computer that evaluated a vast number of cases. This use of a computer, while *relatively* convincing, was not enough to *absolutely* convince the mathematical community (Gonthier, 2008). While many mathematicians believed the theorem to be true because they believed the software algorithms to be correctly implemented, there was still a hint of doubt until a formal proof without the aid of a computer could be accomplished.

In looking at proof schemes through the lens of absolute and relative certainty, Weber and Mejia-Ramos (2015) make the argument that students can operate within certain proof schemes while acknowledging that the proof scheme does not provide absolute conviction. Weber and Mejia-Ramos (2015) give an example of a student who operates within an empirical proof scheme and *is convinced* by such a scheme, but also acknowledges that there are potentially more convincing arguments available. In other words, Weber and Mejia-Ramos are claiming a student can make a mathematical claim using a proof scheme without being absolutely convinced of what they are claiming.

Related to this concept of conviction is the distinction of *who* is being convinced. A student may be confident in a claim, and have justifications that correspond to their proof scheme, but still understand that the justifications they are giving are not mathematically valid justifications. That is, internally the student is convinced (conviction), but externally the justifications are not deemed valid. Segal (1999) clarifies this as the distinction between conviction and validity. In the previous example from Weber and Mejia-Ramos, the student with an empirical proof scheme is *convinced* by the empirical evidence but recognizes that the empirical evidence is not a valid mathematical proof.

In summary, there seems to be two main ways confidence is used. A view of confidence based in self-efficacy (Bandura, 1997; Pajares, 1996) is confidence in oneself to perform certain tasks. A view of confidence based in conviction (Segal, 1999; Weber et al., 2014) is confidence in a claim that is being made. This study will mainly be looking at this second type of confidence. Students will be making claims, providing justifications for those claims, and then rating their confidence (conviction) in those claims. This confidence can come from several factors. These factors include the proof scheme under which they are operating (Harel & Sowder, 1998), their beliefs in their given justifications (Segal, 1999), the current situation in which they are working (Weber et al., 2014), and even their general mathematical self-efficacy (Bandura, 1997). In short, there are many factors that can affect one's confidence when making mathematical claims. For math educators, it is important to understand these factors so that supports can be put in place to give students confidence.

Summary

This literature provides the background for the three research questions being asked. The first section examined the role of DGEs in the geometry classroom and the tool usage it affords. Specifically, it looked at two main tools – the dragging tool and the measurement tool. Both tools can affect the way students interact with geometry. Both these tools have been studied in the literature and have classifications for the different modalities in which students interact with the tools (Baccaglini-Frank & Mariotti, 2010; Olivero & Robutti, 2007). Additionally, research has been done showing that general tool usage within DGEs can be classified into proactive and reactive strategies (Hollebrands, 2007). This current research will build off this base to investigate how students are using these tools within DGEs. Euclidean and hyperbolic geometry behave differently, and this research will investigate if students use the tools the same in both geometries. This research will examine how students use the tools differently depending on the type of task they are given. For instance, tasks that focus on transformational geometry may generate different tool usage than tasks that focus on geometric constructions. Additionally, novel tasks may correspond to different tool usage than familiar tasks.

For the second question, this review discussed the proof schemes students hold when working in mathematics. Specifically, student proof schemes can be categorized into three broad categories of analytic, empirical, and external (Harel & Sowder, 1998). However, just because students operate under a specific proof scheme does not necessarily mean their given justifications correspond to that proof scheme. That is, there is a difference between how a student is convinced internally of an argument, and the justifications they give externally for the sake of the teacher or their peers (Segal, 1999; Weber & Mejia-Ramos, 2015). Despite this distinction, the broad categorizations of proof schemes provide a framework for classifying the types of justifications students make. This research will explore what types of justifications (e.g. full/partial proof, inductive arguments, or appeals to authority) students give when completing tasks. Because students have more experience in Euclidean geometry than they do hyperbolic geometry, this research will examine if the justifications students provide are different in each geometry. Likewise, this research will also investigate how the justifications vary across different types of tasks.

For the third research question, this review discussed the different interpretations of confidence. On one hand, confidence is often used to refer to a person's self-efficacy (Bandura, 1997; Pajares & Miller, 1994). On the other hand, confidence can refer to a person's conviction in the truth of a statement (Segal, 1999; Weber & Mejia-Ramos, 2015). Research has shown that there are differences between being relatively convinced of a claim and absolutely convinced of a claim (Weber et al., 2014; Weber & Mejia-Ramos, 2015). When mathematicians or students make a claim, their confidence in that claim is on a scale and can be affected by many different factors. This research will investigate how confidence varies when working on a series of geometric tasks within two different geometries. It is expected that confidence in Euclidean geometry will be different than confidence in hyperbolic geometry. Confidence may also vary on different tasks. It is possible the nature of the tasks themselves may have an influence on the confidence of the students.

For the fourth research question, this study will explore the interactions between these three factors - tool usage, justifications, and confidence. For instance, we know student confidence can and will vary as students complete tasks. How do tool usage and types of justification affect that confidence? Are students more confident when they can provide numerous examples and make inductive justifications, or might having deductive justifications instill more confidence? As students are using DGEs, what role does the DGE play in those relationships? Hollebrands et al. (2010) and Guven and Karatas (2009) have both shown that students can gain confidence from the use of DGEs. But DGEs can provide confidence in multiple ways. DGEs can provide students with multiple examples leading to inductive arguments. DGEs can help students recognize relationships leading to deductive justifications. DGEs can also be a form of authority for students.

Further, how are these relationships affected by the geometry in which a student is operating? Olivero and Robutti (2007) and Pandiscio (2002) both show instances of students dismissing deductive justifications because of evidence from a DGE. But this evidence is in Euclidean geometry where the evidence confirms pre-existing notions of how geometry behaves. In hyperbolic geometry, where geometry behaves in unanticipated ways, will students still give inductive justifications when the empirical evidence may appear to contract their pre-existing notions. Might the uncertainty prompt students to look for deductive justifications?

Chapter 3 will provide a description of this study using the existing research to inform the study. Students will be given two different series of geometric tasks to complete within a DGE. As they complete these tasks, they will be making mathematical claims. As they make these claims, they will give justifications for their claims and rate their confidence in their claims. Additionally, the researcher will be observing their tool usage as they go about these tasks.

Narrative descriptions of this student activity will be presented in Chapter 4 focusing on pairs of students. Chapter 5 will present an analysis of the data focusing both on overarching trends among the students as well as individual observations about specific students and tasks. This will be followed by a discussion in Chapter 6, followed by the conclusions, limitations, and next plans in Chapter 7.

CHAPTER 3: METHODS

The purpose of this chapter is to describe the study. This chapter will start with a description of the site and the participants. A detailed account will also be given of the participants' involvement in a previously taken geometry course. This will provide context about the experience students had in completing similar tasks, primarily in Euclidean and hyperbolic geometry. Following this will be a full description of the tasks the students were asked to complete and why those tasks were chosen. The data collection will then be described in detail, highlighting the procedures under which the data was collected. A section will then follow explaining the analysis of the data with subsections on each of the main factors of analysis. This study and its analysis should help to answer the following research questions:

1. What are the features of DGEs that college geometry students are using when making mathematical claims while completing DGE related tasks in Euclidean and hyperbolic geometry?
2. What are the justifications college geometry students are using when making mathematical claims while completing DGE related tasks in Euclidean and hyperbolic geometry?
3. How do college geometry students self-report their confidence when making mathematical claims while completing DGE related tasks in Euclidean and hyperbolic geometry?
4. What are the relationships among the features of DGEs, the justifications students are making, and their confidence in making mathematical claims while completing DGE related tasks in Euclidean and hyperbolic geometry?

Site and Participants

The participants for this study were ten students from a small liberal arts college in mid-Michigan. Prior to this study, the researcher taught a course in Euclidean and non-Euclidean geometry with thirteen students at this college. After the completion of the semester, a letter went out to the students of that class for volunteers to participate in this study. The students were told that participating in the study would involve working in pairs on geometry related tasks using a dynamic geometry environment (Geometry Explorer). The students in the class were also told the goal of the study was to better understand student justifications, student use of DGE features, and student confidence as students use DGEs. Lastly, the students were told that as compensation for participating in the study they would be gifted a $50 gift card.

There are several reasons for choosing students at this college. First, the researcher had easy access to the students as the researcher had just finished teaching these students in a geometry course. Second, because of the familiarity with these students and the work they did for the course, the researcher had a general idea of what they did or did not know and their overall ability levels. This allowed the tasks to be tailored to these students in such a way that the tasks would be both familiar enough to be completed, but also present new challenges. Third, because these students had all taken the same geometry course, they were familiar with the DGE chosen for this research. These students used Geometry Explorer extensively throughout the semester working in both Euclidean and non-Euclidean geometry.

The geometry course the students took was a mix of Euclidean and non-Euclidean geometry. Major topics included axiomatic systems, Euclidean geometry, constructions, analytic geometry, and non-Euclidean geometries. A weekly summary of the course can be found in Table 3.1. The coursework consisted of weekly homework sets with a focus on geometric proofs. Homework was completed individually, but students were encouraged to work on it in groups. Additionally, there were seven projects throughout the semester. These projects were done as partners and involved heavy DGE usage as well as writing short papers that explained what they

Table 3.1:

Week-by-Week Summary of Students' Geometry Course

Week	Topic
Week 1	History of Geometry and Axiomatic Systems
Week 2	Axiomatic Systems
Week 3	Basics of Euclidean Geometry
Week 4	Triangle Congruencies Theorems
Week 5	Constructions and Constructability
Week 6	Analytic and Vector Geometry
Week 7	Intro to Transformational Geometry
Week 8	Isometries and Transformations
Week 9	Models of Non-Euclidean Geometry
Week 10	Hyperbolic Geometry
Week 11	Parallels, Omega Points, and Triangles
Week 12	Hyperbolic Quadrilaterals
Week 13	Fractal Geometry
Week 14	Similarity and Dimension
Week 15	Students Presentations on a Variety of Topics

explored in the DGE for their project. Lastly, the students ended the course by working in pairs to present a geometric topic of interest to them that was not covered in the course.

Even before the course, these students had many years of exposure to Euclidean geometry. In addition to their experience living in a locally Euclidean world, they all took high school geometry that focused primarily on Euclidean geometry. Hyperbolic geometry was less familiar to the students, having studied it for approximately three weeks during their college geometry course. Students were familiar with the axiomatic definition of hyperbolic geometry and had done some proofs and constructions in hyperbolic geometry. In general, this would not have been enough time in the geometry to gain an intuitive understanding of how hyperbolic geometry behaves.

During the course, the students spent time doing standard constructions in Euclidean geometry, as well as specific constructions to help with more complicated proofs. They did a few constructions in hyperbolic geometry, but is was minimal. These students also spent time studying Euclidean transformations. Specifically, as will relate to the first series of tasks, they studied translations, spending time in class and outside of class working with transformations both in Geometry Explorer and on paper doing proofs. The students had minimal experience doing translations in hyperbolic geometry. Their experience in translations in hyperbolic geometry was limited to a few proofs in their course that were dependent on translations. These proofs were done strictly on paper and the translations were sketched by hand as approximations.

The targeted number of participants for this study was ten of the thirteen students in the geometry course. Of those thirteen students, exactly ten students volunteered to participate. A list of the ten students can be found in Table 3.2. These students were a mix of juniors and seniors, math majors and minors, and mostly teacher education students. The decision to partner students was made to foster communication between students so that their justifications were more easily identified (e.g. Hollebrands, 2007; Olivero & Robutti, 2007). That is, while students were encouraged to write justifications on their worksheets, it helped to have a verbal record that went into more detail than what they wrote. The students were paired based on previous experience working together throughout their geometry course. This gave the students familiarity with each other that encouraged conversation.

Tasks

The students were asked to work in pairs to complete two series of tasks involving both Euclidean and hyperbolic geometry. The first series of tasks was to complete parallel transports in both Euclidean and hyperbolic geometry. This task was heavily focused on geometric transformations – how objects move within the geometry. The second series of tasks was a series of geometric constructions in both Euclidean and hyperbolic geometry. Each series of tasks was

Table 3.2:

List of Participants

Student	Junior/Senior	Teacher Ed	Major/Minor
Ann	Senior	X	Major
Beth	Senior	X	Major
Carl	Senior		Major
Dan	Junior	X	Minor
Eve	Senior	X	Major
Fran	Senior	X	Minor
Gray	Junior	X	Major
Hal	Senior	X	Major
Ivy	Senior	X	Minor
John	Junior	X	Minor

Note: Names have been changed for privacy reasons.

broken into smaller pieces for the students. For these smaller pieces, the students were given a short task to complete within the DGE. They were then asked to make a claim, give a justification for their claim, rate their confidence in that claim, and provide a statement about what may have raised their confidence.

For these tasks, students were using a relatively new DGE called Geometry Explorer (Hvidsten, 2015). Geometry Explorer is a DGE that combines both Euclidean and non-Euclidean geometry into one single interface. Traditional DGEs such as Geometer's Sketchpad (Jackiw, 2001) and Cabri Geometry (J. M. Laborde & Bellemain, 1995) only focus on Euclidean geometry. Conversely, NonEuclid (Castellanos et al., 2009), which has been used for existing research (e.g. Hollebrands et al., 2010), is a DGE that focuses solely on non-Euclidean geometry. The advantage of Geometry Explorer is that it has multiple geometries built into it using the same interface. By having the same interface, students did not have to struggle to learn two pieces of software or try to remember which software does what. Within Geometry Explorer, the students completed the tasks both in both the Euclidean plane and the hyperbolic Poincaré disk. While Geometry Explorer supports multiple hyperbolic non-Euclidean models, including the Klein disc and the upper half-plane model, the students in this study were most familiar with the Poincaré disk.

The decision to have two series of tasks was to provide multiple contexts in which the students could use the software and make justifications. The first series of tasks on parallel transports was heavily focused on geometric transformations. In particular, the focus was on translations. It was anticipated that this task would be routine in Euclidean geometry, but novel in hyperbolic geometry. As such, after students completed the parallel transport in Euclidean geometry, this series of tasks included a subtask asking students to predict what would happen in the hyperbolic case before they carried out the remainder of the task. This subtask was included to see how students would respond to making a conjecture without having the DGE readily available to them. Additionally, this series of tasks ended with a subtask asking students to read a proof. This subtask was unique in that students were validating a given proof. This subtask was included to see how students would respond to reading a proof of a conjecture they had just spent time exploring inductively. It was anticipated that many students would gloss over the proof and rely on the inductive justifications they had previously given.

The second series of tasks was focused on traditional geometric construction. This series of tasks was included as it provided students a chance to make deductive justifications as they had commonly done throughout their geometry course. This series of tasks started with routine subtasks in Euclidean geometry, asking students to construct a regular triangle and quadrilateral. While it was expected that the construction would be completed correctly, it was anticipated that many students would forego the formal proof and instead use inductive justifications. After completing the constructions in Euclidean geometry, the students were asked to complete the same constructions in hyperbolic geometry. This took a routine task and changed it to a non-routine task. As such, it was anticipated that students would move from a more deductive approach in Euclidean geometry to a more inductive approach in hyperbolic geometry.

In both series of tasks, the students worked in Euclidean and hyperbolic geometry. The students had more experience with Euclidean geometry and had an intuitive understanding of how Euclidean geometry behaves. On the other hand, the students had much less experience and minimal intuition regarding how hyperbolic geometry behaves. Because of these differences in experiences, it was anticipated that student confidence would decrease when working in hyperbolic geometry and that students would prefer to use inductive justifications. In both series of tasks, the students performed the tasks first in Euclidean geometry and then in hyperbolic geometry. This was done because performing the task in Euclidean geometry gave a baseline for how students would perform a routine task. I anticipated that most students would have a modicum of success in Euclidean geometry, that they would rate themselves with high confidence, that they would be more likely to use deductive justifications, and that their tool usage would be minimal. Repeating the same tasks, but in hyperbolic geometry, would then move students to a situation with which they had less familiarity. What was initially a routine task then become non-routine.

The Parallel Transport Series of Tasks

The first series of tasks examined parallel transport in both geometries. The students first completed a transport in Euclidean geometry. The first part of this task had students constructing a triangle of any configuration they wanted (acute, obtuse, etc.). They then added a segment protruding outward from vertex A of their triangle. Next, they were directed to transport, via the translation tool, the segment from vertex A to vertex B (Figure 3.1). The students were then asked three questions:

- What do you and your partner notice about the segment?
- On a scale of 1 to 5, how confident are you in your response to the previous question? Why?
- If you did not rate yourself a 5, what would it take to raise your confidence level to a 5?

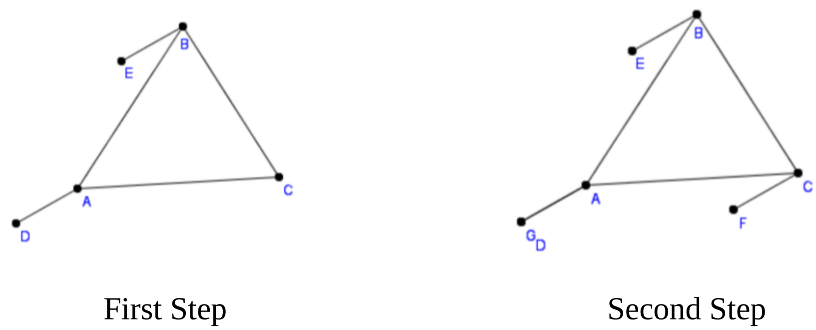
The first of these three questions is unique to this part of the task, while the second two follow-up questions about confidence will be asked after every part of the task. In terms of the first question, the transported segment remains congruent and parallel to the original segment. It was anticipated that students would notice this and justify this either by using their knowledge of translations or by using the measurement tools within the software.

The second part of this series of tasks had students complete the transport by transporting the segment around the triangle (Figure 3.1). Successful completion of this step resulted in a new segment (segment $\overline{AG}$) landing directly on top of the original segment (segment $\overline{AD}$). The students were then asked, "What do you and your partner notice about the final segment compared to the first segment?" This was followed up with the same two confidence questions as they saw previously. It was anticipated that students would notice the segments were coincident and justify it either purely visually or using their knowledge of how translations work.

For the third part, students were asked, "If we complete the same activity as before, but in hyperbolic geometry, what do you and your partner think will happen in terms of the final segment compared to the first segment?" This is the first of the conjecturing sub-tasks the students encountered. For this task, students were not able to use the DGE and were expected to

Figure 3.1:

Euclidean Parallel Transport

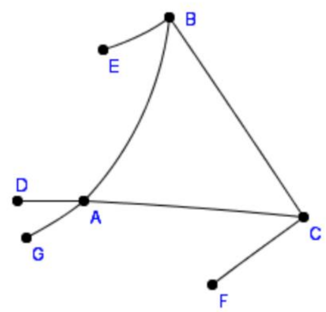


make arguments based on their knowledge of translations and hyperbolic geometry or based on the task they just completed. While students could not use the DGE, they did have paper and pencil available if they wanted to make sketches. Because of the switch to hyperbolic geometry, and because they no longer had access to the DGE, it was anticipated that confidence would decrease when students got to this sub-task. As is typical, the students were also asked the two follow-up questions about confidence.

For the fourth part of the task, students were asked to complete the parallel transport in hyperbolic geometry (Figure 3.2). After completing the transport, students were asked, “What do you and your partner notice about the final segment compared to the first segment?” Again, this was followed by the two questions about confidence. As is noticeable in Figure 3.2, in hyperbolic geometry, the transported segment (segment $\overline{AG}$) will no longer be coincident with the original segment (segment $\overline{AD}$). The segments themselves are congruent with each other as they are translations, though the distances can appear distorted in hyperbolic geometry. While it was anticipated that students might argue deductively about segment lengths in Euclidean geometry, it was anticipated that students would measure these segments to test whether they stayed the same length. It was also anticipated that because of how hyperbolic geometry appears

Figure 3.2:

Hyperbolic Parallel Transport



to distort lengths and angles, students would drag the image to see if their triangle configuration had any effect on the segments.

It turns out the angle between the first and final segment will always be equal to the defect of the triangle (180 degrees minus the sum of all the interior angles). While students were expected to notice the segments were not coincident, it was not anticipated that students would immediately realize the angle was equal to the defect of the triangle. Instead, it was anticipated that students would measure this angle and drag the image to investigate how that angle may or may not change.

For the fifth part of the task, students were asked to find the defect of the triangle. Students could do this either by measuring individual angles and using a calculator (in the DGE), or by using the defect tool in the DGE. After finding the defect, students were asked, "What do you and your partner notice about the defect of the triangle?" Again, they were also asked the two confidence questions. As was mentioned previously, the defect of the triangle will match the angle between the original and transported segment. Students may or may not have measured this angle in the previous part of the task. Thus, this prompt had multiple expectations. It was feasible students had measured the angle between segments in the previous task and by measuring the defect, students would immediately see they match. If students had not previously measured the angle, the expectation was that students would then use the measurement tools and the dragging tool to see if this angle had any relation to the rest of the figure.

The sixth part of the task was quite different from the earlier parts. The sixth part of the task presented students with a proof that the difference in angles (or holonomy) of the triangle will always match the defect of the triangle. The proof had an accompanying diagram. Because of the diagram, it was not strictly necessary for students to use the DGE, but it remained an option if they wanted to use it for any reason. This part of the task did not have students make a claim, but rather to verify the proof and then rate their confidence in the claim being made by the proof. Specifically, the students were asked, "How confident are you that the defect of the triangle will always match the holonomy? Why?" This was followed up with the standard question asking what it would take to raise their confidence to a 5. As most students were expected to have found this claim inductively in the earlier step (via measurement), this part of the task was designed to see how validating a proof would have an impact on their confidence. It was also designed to see what type of justifications students would use after reading a proof.

Lastly, this series of tasks ended by asking students to make a sketch of their confidence throughout all the sections of this series of tasks. This was designed to give students an opportunity to express their confidence in a cumulative manner after having finished the full series of tasks, rather than on individual tasks. A graph was provided with the axes labeled for the students (Figure 3.3). The horizontal axis measures their time within the activity and the vertical axis measures their confidence.

The Regular Polygon Series of Tasks

The second series of tasks examined regular polygons. The students were asked to construct regular triangles and regular quadrilaterals first in Euclidean geometry, and then in

Figure 3.3:

Confidence Graph

hyperbolic geometry. The first part of this series of tasks asked students to construct a regular triangle in Euclidean geometry. This was a routine construction the students had previously done in their geometry course. It was expected the students would all use two circles to complete their construction as in Figure 3.4. The students were then asked, "Have you and your partner successfully constructed a regular triangle? Why?" This was followed by the two standard confidence questions. It was anticipated that students would be able to complete the construction and provide deductive justifications as they had previously done this in their geometry course.

The second part of the task asked students to construct a regular quadrilateral in Euclidean geometry. While this is typically called a square in Euclidean geometry, the question referred to a regular quadrilateral so that it matched the upcoming question in hyperbolic geometry where a regular quadrilateral is not called a square. While this construction was not actually done in the students' geometry course using a DGE, the general method was discussed during class. That method was to start with a segment and then add two perpendiculars at the end points (either using a perpendicular tool or by constructing them). You can then either use two circles to mark the heights on the perpendiculars for the last two points, or you can use a single circle and another perpendicular to mark the last two points (Figure 3.5). While these are not the

Figure 3.4:

Construction of a Regular Triangle in Euclidean Geometry

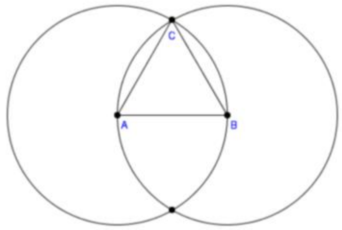
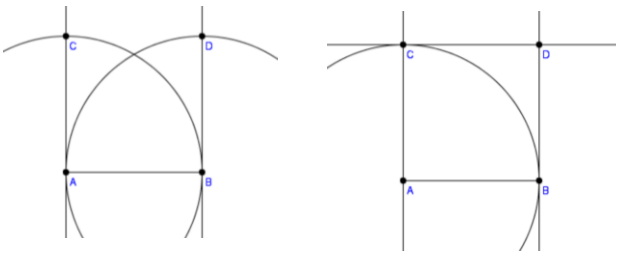


Figure 3.5:

Two Options for Constructing a Euclidean Regular Quadrilateral



only ways to construct a regular quadrilateral in Euclidean geometry, they are what was discussed during class. Students were then asked, "Have you and your partner successfully constructed a regular quadrilateral? Why?" Again, this was followed by the two confidence questions. While the students may not have actually done the construction during class, this was still considered a routine task and it was anticipated they would have a fair degree of confidence. It was also anticipated that students would have varied justifications for this construction. While it was expected that some may use deductive justifications, it was also anticipated that others would rely on inductive justifications. Dragging and measurement were also anticipated as tool usages to support the inductive justifications.

The third part of the series of tasks asked students to construct a regular triangle in hyperbolic geometry. While this was a routine construction in Euclidean geometry, completing the triangle in hyperbolic geometry presents a new set of challenges. Despite being done in hyperbolic geometry, the construction is identical to the construction of a regular triangle in Euclidean geometry. However, the results may look distinctly different (Figure 3.6). The three angles will be congruent but will be an angle strictly between 0 and 60 degrees depending on location within the Poincaré disc and the triangle's size. After completing the construction, the students were asked, "Have you and your partner correctly constructed a regular triangle? Why?" Again, this was followed by the two confidence questions. While it was anticipated the students would successfully construct the regular triangle, there was not an expectation that students would immediately trust their construction worked. Because of the odd way these triangles can appear, it was anticipated that students would either use measurements to justify the triangle was in fact regular or try to make a deductive argument to convince themselves.

The fourth part of the task asked students if they believed regular quadrilaterals exist in hyperbolic geometry. This is the second of the conjecturing sub-tasks the students encountered. Specifically, they were asked, "Do you and your partner believe that regular quadrilaterals exist in hyperbolic geometry? Why or why not?" Again, this was followed by the two confidence questions. While students were not allowed to use the DGE, students could make the argument deductively, visually based on their previous activities, or they could use paper and pencil to make sketches. The students had just seen that regular triangles exist, so there was an expectation that some would build on this previous claim to argue that regular quadrilaterals exist as well.

The fifth part of the task asked students to construct a regular quadrilateral in hyperbolic geometry. As compared to the regular triangle, the standard expected construction does not translate to hyperbolic geometry. The construction in Euclidean was dependent on right angles. However, in hyperbolic geometry, the regular quadrilateral will not have right angles. The four angles will be congruent but strictly between 0 and 90 degrees. See Figure 3.7 for an example of

Figure 3.6:

Regular Triangle in Hyperbolic Geometry

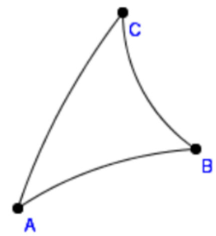
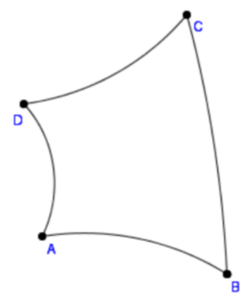


Figure 3.7:

Regular Quadrilateral in Hyperbolic Geometry



a regular quadrilateral in hyperbolic geometry. Because of the ways distances and angles work in hyperbolic geometry, it may not appear regular to our Euclidean eyes, but it is in fact regular within the hyperbolic plane. Figure 3.8 shows what happens if the standard construction from Euclidean geometry is attempted. Notice the right angles at the base of the figure, but the acute angles at the top. It was anticipated that students would attempt the construction using the same method as in Euclidean geometry, but then use the measurement tools to realize that this method does not work. Two correct methods of construction are shown in Figure 3.9. While it was anticipated some students may not find success, it was anticipated that the first option would be the construction students used. It uses congruent triangles to get the angles and sides of the quadrilateral congruent. The second option is a more unusual option that uses a Lambert quadrilateral and reflections to construct the regular quadrilateral. While the students had studied Lambert quadrilaterals, it was not expected students would use them to answer this question.

Figure 3.8:

Potential Construction of a Regular Quadrilateral in Hyperbolic Geometry

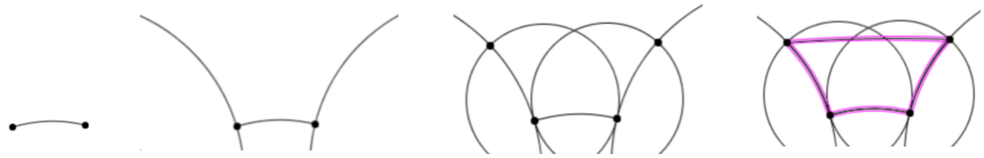
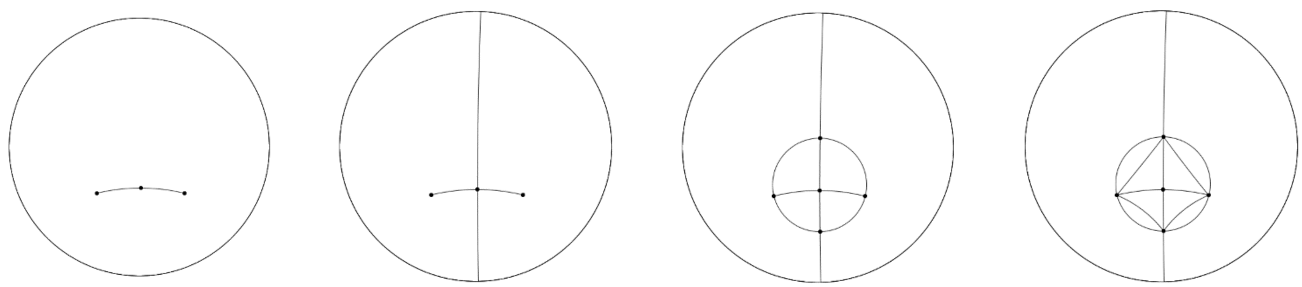


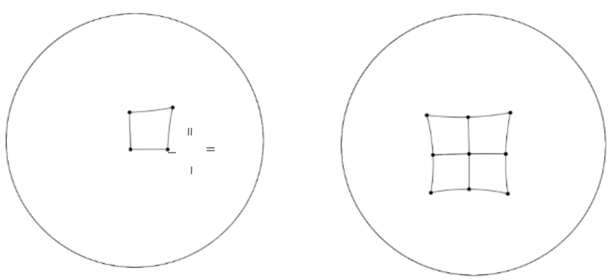
Figure 3.9:

Two Options for Construction of a Regular Quadrilateral in Hyperbolic Geometry

Option 1



Option 2



After working on their constructions, the students were asked, "Have you and your partner successfully constructed a regular quadrilateral? Why?"

It was anticipated that students would approach this task very differently. As mentioned, it was expected some students would attempt the same construction as in the Euclidean case. On the other hand, it was also anticipated that students' measurements in the previous task may have prevented students from completing this same construction. For students that measured and found the regular triangle had interior angles less than 60 degrees, it was anticipated some students may recognize the angles in the regular quadrilateral should be less than 90 degrees. This would have ruled out the traditional construction. It was also anticipated that some students would begin by attempting to ‘draw’ the figure in the DGE by just making arbitrary segments and dragging them into a ‘squarish’ shape just to see what a regular quadrilateral might look like.

Lastly, as in the previous series of tasks, students were asked to make a graph of their confidence throughout all the sections of this series of tasks. A graph was provided with the axes labeled for the students (Figure 3.3). Again, this was designed to give students an opportunity to express their confidence cumulatively at the end of the series of tasks, rather than individually for all the subtasks.

Summary

The full series of tasks with student instructions can be found in Appendix A. The list of the prompts the students are responding to can be found in Table 3.3. For the parallel transport series of tasks, there were six prompts in total with two prompts in Euclidean geometry, and four prompts in hyperbolic geometry. For the regular polygon series of tasks, there were five prompts in total with two prompts in Euclidean geometry, and three prompts in hyperbolic geometry. For each prompt, students would finish by writing a claim, giving a justification for their claim, rating their confidence, and then writing what could have raised their confidence if they did not rate themselves highly on confidence. There was also a conjecturing prompt for each series of tasks asking students to make a conjecture about what might happen in hyperbolic geometry. Lastly, there was a prompt asking students to read a proof and comment on how confident they were after reading the proof.

For the parallel transport series of tasks, it was expected that students would be able to predict what would happen in the Euclidean case. They had the most experience with Euclidean

Table 3.3:

Lists of Prompts to Which the Students Responded

Prompt ID	Prompt	Euclidean or Hyperbolic
P1	What do you and your partner notice about the segment?	Euclidean
P2	What do you and your partner notice about the final segment compared to the first segment?	Euclidean
P3	If we complete the same activity as Task 1, but in hyperbolic geometry, what do you and your partner think will happen in terms of the final segment compared to the first segment?	Hyperbolic
P4	What do you and your partner notice about the final segment compared to the first segment?	Hyperbolic
P5	What do you and your partner notice about the defect of the triangle?	Hyperbolic
P6	On a scale of 1 to 5, how confident are you that the defect of the triangle will always match the holonomy? Why?	Hyperbolic
R1	Have you and your partner successfully constructed a regular triangle? Why?	Euclidean
R2	Have you and your partner successfully constructed a regular quadrilateral? Why?	Euclidean
R3	Have you and your partner correctly constructed a regular triangle? Why?	Hyperbolic
R4	Do you and your partner believe that regular quadrilaterals exist in hyperbolic geometry? Why or why not?	Hyperbolic
R5	Have you and your partner successfully constructed a regular quadrilateral? Why?	Hyperbolic

Note: After every prompt except P6, the students were asked the following two confidence questions. For P6, the students were only additionally asked the second confidence question.

- On a scale of 1 to 5, how confident are you in your response to the previous question? Why?
- If you did not rate yourself a 5, what would it take to raise your confidence level to a 5?

geometry, and most of their study of transformational geometry had taken place within Euclidean geometry. For instance, students had learned that transformations preserve segment length and angles - two theorems that are directly applicable to a parallel transport. In hyperbolic geometry, it was not expected that students would predict what would happen to the final segment. The students had learned about triangle defect and that the interior angles of a triangle sum to less than 180. They also knew that hyperbolic geometry has multiple parallels through a point. Lastly, they had experience seeing how segments can change in apparent curvature as they move from the center of the triangle (where the geometry appears more Euclidean) to the edge of the Poincaré disk (where the geometry appears dramatically more curvy). It was anticipated that there would be minimal deductive justifications being used during this series of tasks. Students had previously not had much experience doing proofs relying on Euclidean transformations. Because of this it was anticipated that students would make extensive use of the measurement tools as well as the dragging tools to provide inductive justifications.

For the second series of tasks, the students were aware of regular polygons in Euclidean geometry. The students had frequently constructed regular triangles as the construction of regular triangles is instrumental in more complicated constructions. The construction of a square had been discussed as to the general procedure but not physically done in Geometry Explorer. In hyperbolic geometry, students had neither constructed regular triangles or regular quadrilaterals. Rather, by the end of the geometry course, students had spent some time constructing Saccheri and Lambert quadrilaterals, shapes unique to hyperbolic geometry. Saccheri quadrilaterals have two right angles at the base and two congruent and acute summit angles. Lambert quadrilaterals have three right angles and an acute fourth angle. In particular, Lambert quadrilaterals hint at the fact that if regular quadrilaterals exist, then regular quadrilaterals cannot have four 90 degree angles. Lambert quadrilaterals also had the potential to be useful in the construction of a regular quadrilateral.

The construction of regular triangles and quadrilaterals in Euclidean geometry was expected to go smoothly for the students as these were routine tasks. The regular triangle in hyperbolic was also expected to go well for students as the construction uses the same steps as in Euclidean geometry. Additionally, the justification for why the construction of the hyperbolic regular triangle works is the same argument that is made in Euclidean geometry. Because of this, it was anticipated students would provide deductive justifications.

The regular quadrilateral in hyperbolic geometry, however, was expected to be a challenge. It was anticipated students would use the standard construction for a Euclidean square, only to realize, by using the measurement tools, that this construction does not work. Despite this construction not working, there are multiple constructions students could use to create a regular quadrilateral in hyperbolic geometry (Figure 3.9). Justifications were mostly anticipated to be inductive.

Procedures

This next section will describe the procedures of the study. It will be broken into two pieces. First will be a short section on how the data was collected. The second, longer section, will be descriptions of the unit analysis. It will contain descriptions of how tool usages, justifications, and confidence were coded. It will also describe how the paired relationships were analyzed.

Data Collection

The participants participated in two sessions to complete two series of tasks. They worked in pairs for roughly 30-50 minutes on each series of tasks. The exact amount of time each pair worked on each series of tasks can be found in Table 3.4. In every pair except Gary and Hal, the students spent more time working on the parallel transport series of tasks than they did the regular polygon series of tasks. Also of note is that the pairs of Eve and Fran, and Ivy and John took considerably longer than the other three pairs.

Table 3.4:

Time Spent by Each Pair on Each Series of Tasks

	Time Spent on Each Task in Minutes		
	Parallel Transport	Regular Polygons	Total Time
Ann and Beth	46	25	71
Carl and Dan	43	31	74
Eve and Fran	68	45	113
Gary and Hal	39	51	90
Ivy and John	78	52	130

The pairs were videotaped while they worked. This video allowed the audio to be transcribed and provided visual data for where the students were looking as they commented on their work. Additionally, screen-recording software captured their activity within Geometry Explorer. As the students completed the tasks, they recorded their claims on the given worksheets (Appendix A), giving their justifications and rating their confidence as they progressed. They also used the worksheets to occasionally make quick sketches to support their claims or as part of thinking about their justifications.

The students worked with minimal interactions with the researcher. However, there were instances where the researcher stepped in and asked the students for clarification. Specifically, when students made general comments, the researcher stepped in and asked students to "think longer about what they might notice." An example of a general response would a response like, "the segments are different." While this may be true, it does not say very much. Are the segments different because they are a different length or at a different angle or in a different location? The goal of this additional question was not to get a correct answer, but rather a more detailed answer. The student could then respond by commenting on the segment's length, angle, parallelism, position, etc. The researcher did not step in until after the student had finished writing their claim and rating their confidence. When answering this additional prompt, the students would write a new claim, and again rate their confidence. This extra prompt was only needed after prompts P4 and P5 of the parallel transport series of tasks.

Data Analysis

The results of the study are presented in two chapters. Chapter 4 provides detailed descriptions of student activity. Focus is given in these descriptions to specific student activity that pertains to the research questions. Attention is given to shifts that occur between the subtasks, between the series of tasks, and between the two different geometries. Chapter 5 then highlights the important observations from Chapter 4, summarizing their relation to the four research questions.

Analysis for Research Question 1

The first research question posed was, *"What are the features of DGEs that college geometry students are using when making mathematical claims while completing DGE related tasks in Euclidean and hyperbolic geometry?"* There are a multitude of tools available within Geometry Explorer. However, the three main tools as studied in the literature are the dragging tool and the two measurement tools that measure angle and length. As such, this analysis limited itself to looking at these three tools. As mentioned in Chapter 2, there are various modalities of using the dragging tools and the measurement tools (Baccaglini-Frank & Mariotti, 2010; Olivero & Robutti, 2007), though these modalities were not a one-to-one match between dragging and measurement. Likewise Hollebrands (2007) categorized general tool usage into the broad categories or proactive and reactive strategies. However, not every measuring or dragging modality fit nicely into one or the other of those general strategies.

For this research, the different modalities of measuring and dragging were condensed into two broad categories of *wandering* and *validation* tool usage. Wandering tool usage includes tool usage where the student is unsure of the outcome or does not display a clear purpose or goal in their measuring or dragging. It includes such modalities as wander, maintaining, and trace dragging and wander and guided measuring. In all these modalities, students are looking for relationships and patterns. Validation tool usage includes tool usage where the student displays clear intention before they start using the tool. This includes such modalities as the dragging test as well as perception, validation, and proof measuring. In all these modalities, the students are validating conjectures they had already formed about the figures on the screen. In some cases, such as the dragging test or perception measurement, students are validating relationships they see on the screen before a conjecture is fully formed. In other cases, such as validation and proof measuring, students are validating fully formed conjectures.

Video analysis showed exactly when students were using each of the three tools. Each prompt acted as its own unit of analysis. For each prompt, a tally took place on which of the three tools was used to answer that prompt and how the tool was used by the student. Counts were not taken on *how often* the tool was used. That is, on any given prompt, if a student measured five angles, it was only recorded that the student used the angle measurement tool, not that the student used the tool five times. It was also recorded how the student used the measurement tool. One purpose of the pairs was to encourage discussion between the students so that *how* the tool was being used could more easily be determined. It was expected that during any given prompt, the tool usage may be used in either or both types of usage, for wandering and validation purposes.

Anticipated Results for Research Question 1

It was anticipated that tool usage would be greater within hyperbolic geometry. That is, when students reached the hyperbolic portion of each series of tasks, there would be an expected shift upward in the amount of tool usage. Because of the nature of how hyperbolic geometry appears and the perceived difficulty of hyperbolic geometry, it was anticipated that students would resort to more tool usage as they worked to answer the prompts related to hyperbolic geometry. Previous research has shown that uncertainty causes students to look for justification (C. Laborde, 2000) and the DGE tools are one way to provide justification. Conversely, because Euclidean geometry generally behaves in ways that students expect, it was anticipated that tool usage would be minimal (Olivero & Robutti, 2007; Weber & Mejia-Ramos, 2015).

It was also anticipated there would be a shift from mixed tool usage (wandering and validation) to mostly validation tool usage when students began the regular polygon series of tasks. Specifically, when students were working on constructing the regular polygons, it was anticipated most of the tool usage would be validating whether their construction was either correct or incorrect. This shift was expected to continue even through the regular hyperbolic triangle. Despite how different a regular triangle could appear, it was anticipated that because of the relative simplicity of the construction students would still have an idea before they measured or dragged to test their constructions. It was when students began the regular quadrilateral that an anticipated shift back to mixed tool usage would occur. The regular quadrilateral construction is more complex, and it was expected students would switch between the two modes of tool usage.

The reporting of data in Chapters 4 and 5 highlights how students used the tools. Chapter 4 descriptively shows how individual students were using the tools to answer each prompt. A focus is given on how tool usage varied across the series of tasks, highlighting shifts that occur in how students used the tools. That is, special attention was given to instances where students switched between types of tool usage or where students used the tools in ways that were not anticipated. Chapter 5 provides counts of which tools were being used for each prompt and how the tools were being used for each prompt. Additionally, Chapter 5 provides summaries of where shifts occurred in how the students used the tools.

Analysis for Research Question 2

The second research question posed was, "*What are the justifications college geometry students are using when making mathematical claims while completing DGE related tasks in Euclidean and hyperbolic geometry?*" As a framework for classifying justifications, this research categorizes justifications into five categories: comparison to Euclidean geometry, previous knowledge, single case inductive, multiple case inductive, and deductive justifications. These categories roughly parallel the proof schemes given by Harel and Sowder (1998). That is, comparison to Euclidean geometry and previous knowledge justifications parallel an external proof scheme. Single case inductive and multiple case inductive justifications parallel an empirical proof scheme. Lastly, deductive justifications parallel analytical proof schemes. Note that this research is specifically looking at the justifications the students gave and not the proof schemes under which they were operating. That is, while a student may give a deductive justification for a task, that same student may be operating under an inductive proof scheme and only giving a deductive justification because they believe that is what the researcher wants.

The first two categories of justifications are *comparison to Euclidean geometry* and *previous knowledge*. A justification by comparison to Euclidean geometry is when a student claimed something was true in hyperbolic geometry because the analogue was true in Euclidean geometry. The second category is previous knowledge. This justification is when a student remembered a specific piece of knowledge that told them what they needed to know. In both cases, these arguments were being made by appealing to an authority. In the first case, the authority was what the student knew about Euclidean geometry. In the second case, the authority was what they remembered from a previous class.

As an example of using a comparison to Euclidean geometry to justify a claim, when Carl justified his construction of a regular hyperbolic triangle, he wrote, “We did the same thing as Euclidean, but we measured all the lines and angles to be sure.” There were two justifications happening here. The first justification is that by repeating what he did in the Euclidean case, this gave him a correct construction. Despite this justification, the student also followed that justification with a second justification to offer more support to his claim. The second justification was an inductive justification based upon the specific triangle he constructed.

As an example of using previous knowledge to justify a claim, John justified his construction of a regular Euclidean triangle by writing it was correct “because we used the two-circle method.” This was the standard method the students used during their geometry course (Figure 3.4). Note that John did not get into the specifics of why this method works, but simply wrote that the method works. He was recalling previous knowledge about how to do this task. To contrast this, his partner Ivy justified the congruent sides of the triangle by mentioning that the circles share common radii. In this case, her justification was categorized as a deductive justification.

The next two categories are *single case inductive* and *multiple case inductive justifications*. Rather than grouping all empirical justifications together, the use of a DGE was good reason to split this category into two parts. Dragging is a key feature of DGE use and it easily allows one to see multiple examples of a figure. When a student constructed a figure with a certain property, dragging allowed them to see multiple versions of that figure and make a claim based not on a single static figure, but on a multitude of related figures. Dragging a figure, therefore, led to multiple case inductive arguments by its very nature. That is not to say that all multiple case inductive arguments corresponded to dragging. There were some students who constructed multiple figures. There were also students who, as they were working in pairs, specifically used each other's figures as evidence of multiple cases in their justifications.

Lastly, *deductive* justifications will refer to any justification that used deductive arguments, whether correct or incorrect. This justification did not need to be a complete proof, but rather that an attempt was made to justify their claim deductively.

Justifications were examined at the level of the prompt. While working on the activities, students made many claims and gave many justifications as they were working. For this research, only the students' final written claim and the justifications given for that claim were counted. Note that there were instances where more than one type of justification was given. For instance, Carl used two justifications, a comparison to Euclidean as well as an inductive justification, when he justified his construction of a hyperbolic regular triangle. In that case, both justifications were recorded. Also, there were instances where justifications were not always written down. In those instances, video and transcript analysis was used to determine ambiguous justifications. For instance, when working on the Euclidean parallel transport, Carl and Dan were translating their first segment. During this translation, Carl specifically commented verbally he was going to construct his triangle differently than Dan. He also took noticeable time to look at Dan's screen before writing his response that the original and transported segments were parallel. Because of his verbal comment and his purposeful look at Dan's screen, Carl was recorded as using multiple inductive justification when he claimed the segments were parallel. On the other hand, as Dan had given no indication of paying attention to Carl's screen either verbally or visually, Dan was recorded as a single case inductive justification as he appeared to only use his screen when making the argument.

Anticipated Results for Research Question 2

It was anticipated that students would mostly be giving inductive justifications. Previous research has shown that if students have evidence from DGE, they are not always interested in continuing to find deductive justifications (Hollebrands et al., 2010; Olivero & Robutti, 2007). For the Euclidean tasks, these inductive justifications were anticipated to be based on a single case as dragging was expected to be minimum. It was the tasks in hyperbolic geometry where multiple case inductive justifications were anticipated as that is where students were expected to be dragging. Thus, in both series of tasks, there was an expected shift from single case to multiple case inductive justifications as students switched from the Euclidean to the hyperbolic tasks.

Deductive justifications, if they occurred, were anticipated to appear most often in the Euclidean construction tasks as those were the most routine tasks for the students. While it was possible deductive justifications would continue to the hyperbolic constructions, this was not expected.

There were also two conjecturing tasks, one in each series of tasks. As these conjecturing tasks mostly precluded the use of the DGE, inductive justifications were anticipated to be minimal as the students no longer had access to creating examples. It was anticipated that students answering the conjecture prompts would answer by using either a comparison to the Euclidean case they had just completed or by using previous knowledge they might remember from class.

Analysis for Research Question 3

The third research question posed was, "*How do college geometry students self-report their confidence when making mathematical claims while completing DGE related tasks in Euclidean and hyperbolic geometry?"* The series of tasks included questions throughout that gave the students opportunities to make mathematical claims. Each question was followed by an opportunity for the student to rate their confidence level as they answered the question. Weber and Mejia-Ramos (2015) argued that mathematicians can have different levels of conviction. Mathematicians can be absolutely convinced or relatively convinced. That is, students were not expected to always be fully confident in the claims they were making. It was not a simple binary between confident and not confident. As such, students were asked to report their confidence on a Likert scale with 1 representing *not confident*, 2 representing *somewhat not confident,* 3 representing *neutral,* 4 representing *somewhat confident,* and 5 representing *very confident*.

In addition to rating their confidence, students were asked why they gave their specific confidence level. While this gave the students a chance to give a justification for their earlier claim, this question also provided insight on how students were thinking about confidence and how their interpretation of confidence changed throughout the tasks. There emerged two main interpretations of confidence. These two interpretations of confidence most closely align with Segel's definition of conviction (1999) in that students were rating their confidence on whether they personally found their claim to be correct. The first interpretation of confidence was confidence that the student believed they recorded *a* correct (true) statement. A student would rate themselves high because they were convinced that what they wrote down was a true statement. The second interpretation of confidence was confidence that that the student believed they record *the* correct statement. A student would rate themselves high because they were convinced they wrote down the ‘right’ answer to the question. In both interpretations, confidence was referring to confidence in the individual claims being made, and whether they were “correct.” In analyzing the data, a third interpretation of confidence appeared unique to Ivy. There are many instances where Ivy’s interpretation of confidence refers, not to the claim itself, but rather her overall confidence in her ability to do mathematics. Ivy’s interpretation can most closely be associated to the concept of self-efficacy (Bandura, 1997). These different interpretations will be explored more in Chapter 6.

There were two other pieces of data gathered on confidence in this research. The first was a question for each prompt asking students what would raise their confidence to a 5 if their reported confidence was less than 5. It was anticipated students would not rate themselves as very confident on every prompt. This question was included to see from a student’s perspective what could be done to give them more confidence. Anticipated responses include such things as “being told I’m right”, “a formal proof”, “reading it in a book”, or “more examples.” However, due to the large number of prompts with a rating of 5, this question did not yield significant insight on what might raise their confidence. The exception was the prompt asking students to read a proof (P6). Due to the uniqueness of this prompt, it will be discussed in Chapter 6.

The last piece of data gathered on confidence was a question at the end of each series of tasks asking students to draw a graph that represented their confidence throughout the series of tasks. Reporting on this data will not be given as it did not provide meaningful insights beyond the confidence already reported for the individual prompts. The reason for this is most students referred to their confidence ratings and simply graphed those ratings. This meant the graphs essentially gave a duplicate form of the data.

The reporting of data in Chapters 4 and 5 highlights the confidence being reported by students when responding to the prompts. Chapter 4 describes the confidence of individual students as they respond to each prompt. Special attention was given to instances where student confidence changed dramatically or was different from what was anticipated. Chapter 5 provides summarized data showing how confidence shifts throughout the series of tasks. A focus is given on how student confidence varied across the series of tasks highlighting specific instances where shifts happened in how students rated their confidence or where reported confidence was different than expected.

Anticipated Results for Research Question 3

It was anticipated that confidence during the parallel transport series of tasks would be lower than during the regular polygon series of tasks. Of the two series of tasks, the parallel transport was least routine. That is, students did not have as much experience doing translations as they had constructions. The students had even less experience doing proofs related to translations. Thus, there was an anticipated shift upward in reported confidence when students moved from the parallel transport series of tasks to the regular polygon series of tasks.

It was also anticipated that reported student confidence would shift slightly downward as students moved from the Euclidean parts of the tasks to the hyperbolic parts of the tasks. We know hyperbolic geometry is generally more difficult for students (Senk, 1989) so reported confidence was expected to fall. However, we have also seen that DGE use can boost student confidence (Guven & Karatas, 2009; Hollebrands et al., 2010). Despite this gain, it was anticipated that students' perceptions of hyperbolic geometry would still result in reporting lower confidences. It was anticipated that students would report their lowest confidence when attempting to answer the conjecturing prompts in hyperbolic geometry as they did not have the DGE as a support.

Lastly, with regard to the last prompt in the parallel transport series of tasks asking students to read a proof (P6), it was anticipated that reported confidence would shift upwards. That is, while we know that students do not always see proof as necessary when using a DGE (Olivero & Robutti, 2007; Weber & Mejia-Ramos, 2015), it was anticipated that seeing a proof of what they had discovered in the previous prompt would reinforce their claims and give the students greater confidence.

The reporting of data in Chapters 4 and 5 highlights the different justifications the students used when responding to the prompts. Chapter 4 gives descriptions of which justifications individual students were using to respond to each prompt. Special attentional was given to instances where students switched between types of justifications or used justifications in ways that were not anticipated. Chapter 5 provides counts of which justifications were being used for each of the prompts. Additionally, summaries will be given highlighting instances where shifts happened in how students chose to justify their responses.

Analysis for Research Question 4

The fourth research question posed was, "*What are the relationships among the features of DGEs, the justifications students are making, and their confidence in making mathematical claims while completing DGE related tasks in Euclidean and hyperbolic geometry?*" This section focuses on relationships that occur among the three different factors being analyzed. Tool usages, justifications, and confidence are matched pairwise to highlight important relationships and correspondences between the three factors. Additionally, there will be a section that focuses on relationships among all three factors. This analysis will be based on the anticipated relationships described below.

In a quantitative approach, responses to individual prompts were categorized and grouped by both justifications and tool usages to investigate how they corresponded to reported confidence. Additionally, responses were categorized by justifications and by tool usage to see how these two factors interacted with each other bi-directionally. In a qualitative approach, instances were identified where students behaved in notable ways. Focus was given to areas of individual student movement as well as instances when students switched between Euclidean and hyperbolic geometry or when students switched from one series of tasks to the other series of tasks. These responses were then analyzed individually to see how shifts in tool usages or justification corresponded to shifts in confidence as well as how shifts in tool usage may corresponded with shifts in justifications and vice versa.

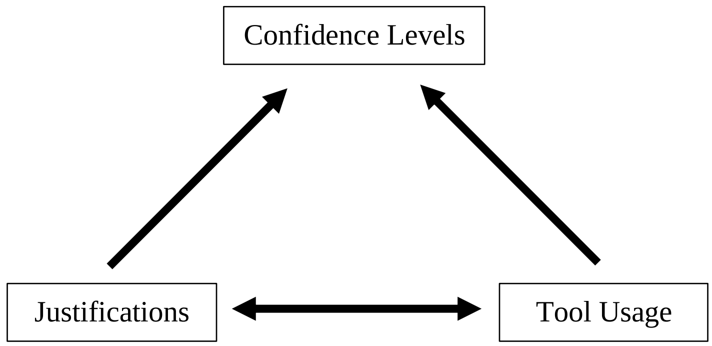
Anticipated Results for Research Question 4

It was anticipated the relationship between tool usage and confidence would be a directional relationship (Figure 3.10). That is, it was expected that high tool usage would correspond with higher confidence. As students used the tools, they would be generating evidence for claims they make, and this evidence would provide confidence. We have seen this previously where higher tool usage is associated with higher confidences (Guven & Karatas, 2009; Hollebrands et al., 2010). It was also anticipated that this relationship would be especially noticeable within hyperbolic geometry where students are otherwise known to struggle (Senk, 1989).

It was anticipated the relationship between deductive arguments and confidence would also be a directional relationship (Figure 3.10). That is, it was expected that deductive

Figure 3.10:

Relationships for Analysis



justifications would correspond to higher confidences. As students are known to struggle with proof (Mariotti, 2012; Weber, 2002), if a student felt comfortable giving a proof, it was anticipated they would also report higher confidence. However, it was also anticipated that there would be many instances of high confidence that do not involve deductive arguments. For instance, it was anticipated that many students would use the DGE to provide inductive justification that would also lead to high confidence (Hollebrands et al., 2010).

The last pairwise relationship was anticipated to be a bi-directional relationship between the justifications the students are using and the features of the DGE students are using (Figure 3.10). That is, certain tool usages are strongly associated with certain types of justifications. For instance, dragging, by its very nature, is anticipated to be associated with multiple case inductive justification as dragging is useful for creating a multitude of examples. Conversely, certain types of justifications are anticipated to be strongly associated with certain types of tool usage. For instance, deductive justifications are anticipated to be associated with validation tool usage. Likewise, using previous knowledge or comparison to Euclidean geometry is anticipated to be associated with minimal tool usage.

Lastly, it was anticipated that among all three relationships, the combination of deductive justifications and high tool usage will correspond to high confidences. As individually, it was anticipated that justifications and high tool usage would each correspond to high confidence, their combination is anticipated to gives students the most confidence. Alternatively, it is anticipated that when students report low confidence, there will be minimal tool usage and some form of non-deductive justification taking place.

The reporting of data in Chapters 4 and 5 highlights these anticipated relationships that appear as students are responding to the prompts. Chapter 4 highlights these anticipated relationships as they occur among the individual students. Chapter 5 will summarize these relationships as they occurred for multiple students. Special attention will be given in Chapter 5 to instances where these anticipated relationships did not occur, as well as to other relationships that were found during the analysis process.

CHAPTER 4: DESCRIPTIONS OF STUDENT ACTIVITY

This chapter provides detailed descriptions of how each pair of students completed the two series of tasks. These descriptions cannot fully encapsulate all student activity, but they should provide insight into how students interpreted and carried out the tasks. Focus will be given in the descriptions to specific student activity that pertains to the research questions. Attention will also be given as students shift between the two different geometries as well as when students shift between the two series of tasks. Screenshots from Geometry Explorer and excerpts of dialogue will be presented as needed to help clarify student behavior. The chapter is broken into five sections with each section devoted to one pair of students. This chapter is presented as an introduction to Chapter 5. Chapter 5 will summarize this student activity, paying special attention to relationships anticipated relationships, as well as to other relationships that were found during the analysis process.

Students Ann and Beth

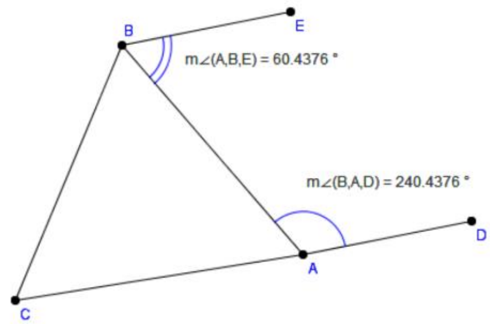
The first two students to be discussed are Ann and Beth. Ann and Beth were both majoring in mathematics education and had worked together previously for multiple projects in their college geometry course. Ann earned an A in her geometry course and was a quiet student during class. Beth also earned an A in her geometry course, and was one of the more vocal students during class, both answering questions from the instructor and asking her own clarifying questions.

Parallel Transport Series of Tasks

Ann and Beth began with the parallel transport series of tasks in Euclidean geometry. They began by constructing a triangle, an initial segment, and then transporting that segment (segment AD) from point A to point B (Figure 4.1). As they were working on this, Ann noticed

Figure 4.1:

Ann and Beth – Parallel Segments



that she and Beth had constructed their initial triangles and segments in roughly the same arrangement. Both students decided to leave their triangles in this arrangement. In response to Prompt P1, both students wrote they noticed the new segment was parallel to the original segment. Beth additionally wrote the segments were the same length, saying it out loud as she wrote it. Upon hearing it, Ann also decided to write down that observation. While thinking about how to justify their response, Ann suggested using a "parallel checking tool," before remembering this software does not contain a parallel checking tool. The pair settled upon measuring the angles (Figure 4.1). Beth suggested that if the angles were supplementary, it would prove parallelism. When measuring the angles, both students measured angle BAD in reverse giving 240.4376 degrees rather than 119.5624 degrees, but they both quickly realized the mistake. They fixed their mistake when measuring the second angle. Doing a bit of mental arithmetic, the students decided the angles were supplementary, and therefore parallel. Both students rated themselves a 5 in terms of their confidence. Neither student offered a justification for why the segments would be the same length.

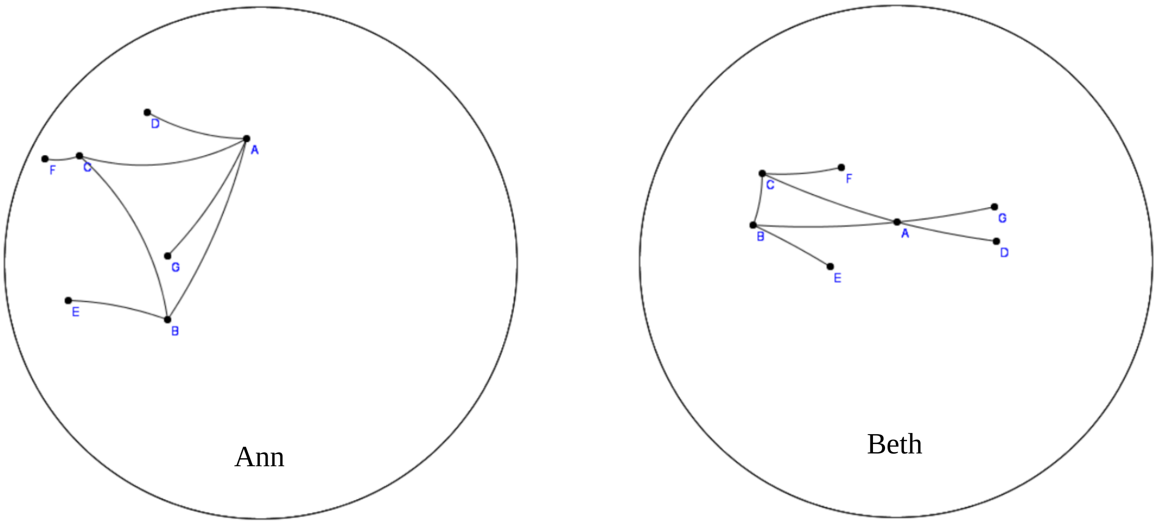
The students then finished transporting the segment around the triangle. Beth voiced the segments "were the same," declaring she was "pretty confident they are the same." Almost immediately, she hedged her answer as if she thought it may be a trick saying, “math is weird, maybe they’re a little off.” Despite this hesitation, both students wrote the segments were the “exact same” in response to Prompt P2. Both students gave themselves a 4 for confidence, a lower rating than their responses for Prompt P1. Beth indicated that she would have liked a way to quickly zoom in on the diagram to increase her confidence the segments were the same. Ann wrote she wanted to “know how to prove it” to raise her confidence.

For prompt P3, Ann and Beth were asked to predict what would happen for this same task when done in hyperbolic geometry. After a short discussion about the “weirdness” of hyperbolic geometry, Ann decided that the first and last segments will be the same, while Beth said they may be the same or they may be different. Ann commented specifically that it did work in Euclidean geometry, so it may also work in hyperbolic geometry. Neither student was confident, and they both gave themselves a 2. This is a noticeable shift in confidence from their previous responses with confidences of 4 (for P1) and 5 (for P2). To raise confidence, Ann wrote that she wanted to “see it done.” Beth wrote she wanted to “remember more about hyperbolic geometry.”

Ann and Beth then proceeded to repeat the transported segment in hyperbolic geometry. As Ann was drawing the first segment of the triangle, Ann suggested to Beth they make different triangles “so they can see if it works.” This is a shift in strategy from Euclidean geometry where the students noticed their triangle configurations essentially matched and decided to leave the triangles matching. This is possibly an indication Ann recognized a single example is not sufficient to prove mathematical claims. It is also possibly an indication Ann was less confident in hyperbolic geometry and sought more examples. Beth agreed and modified her triangle so it was significantly different (Figure 4.2). Ann was the first student to complete the transport, and,

Figure 4.2:

Ann and Beth – Construction for a Hyperbolic Transport



in her case, the first three segments (segments AD, BE, and CF) were somewhat parallel looking, but the last segment (segment AG) was distinctly not parallel. After Beth finished hers, she expressed frustration saying she felt “like I’m doing it wrong.” Beth then began to drag her triangle, though the intent of this drag is unknown. It is worth noting this was the first time she dragged, and it followed a verbal expression of self-doubt.

When thinking about the angle between the first and last segment (angle GAD in Figure 4.2), Ann wondered if her angle matched that of Beth’s. They both measured and found the angles to be different. Beth wondered if the angles would change when they dragged the image prompting Ann to reply, “that’s smart!” Beth tried proactively dragging her point A around the screen trying to get the first and last segments to “touch” (coincide). After failing to get the segments to coincide, Beth was ready to move on when Ann interrupted and suggested measuring the length of the segments. Ann proceeded to measure lengths, while Beth did not. Through a quirk of her construction, the software did not report Ann’s figure having equal segment lengths. When she had done the translation, she had translated just the segments, and not the endpoints. To measure the lengths, she put points on the ends of the segment and used these new points to take her measurement with the point distance tool. As Ann had attached her own points on the new segments, they were not truly at the end of the segments, so her lengths were slightly different. Just as Beth was starting to explain why the lengths ought to be the same theoretically, Ann declared "they are different." This declaration prompted Beth to finally measure her segments. Beth, who also did not translate her points, measured her lengths using the segment length tool. This gave her exact measurements and Beth saw the lengths were the same. Beth declared that matching lengths make sense "since they're all translated lines." Ann still expressed hesitation questioning that perhaps she did it wrong. Ann briefly considered redoing her construction, but finally decided to use Beth's construction.

In answering Prompt P4, Ann wrote that the segments were "the same length." In rating her confidence to that answer, she wrote that she feels "like there's something more we're missing." She rated herself a 3 on confidence and wrote that she needed to "know more about hyperbolic [geometry]". Beth wrote the segments were "the same length and possibly parallel". She justified the claim about length in two ways. First, she said verbally the lengths were the same because they are translations. Second, she wrote the lengths are the same "by the program telling me." She gave no explicit justification for parallelism. Ann, who had previously rated herself a 2 for Prompt P3 and wanted to "see it done" rated her confidence only slightly higher with a 3. Beth, who had previously rated herself a 2 for Prompt P3 and wanted more information about hyperbolic geometry rated herself a 2 again. Note that the shift in confidence that occurred when responding to Prompt P3 continued for this prompt as their reported confidence levels remained low.

Ann and Beth were then asked to find the defect of the triangle and to write down what they noticed about the defect. Both students found the defect by measuring the three internal angles and using the calculator tool to subtract the three angles from 180 degrees. They each had different defects as their triangles were different, and Ann immediately asked if the defect changes when you "move it [the triangle] around." Prompted by that comment, Beth used a validation drag to verify this claim by dragging her figure around to check that the defect does change depending on the configuration of the triangle. Both students proceeded to write down their current triangle defect.

While Beth was still writing down her defect, Ann began to wander drag her image on the screen thinking there must be something to notice. The following transcript shows how Ann switched from a wander drag to a validation drag.

Ann: [*Wander dragging her triangle on the screen.]*

Beth: [*Writing down her answer about defect.]*

2 seconds pass while Ann wander drags her triangle.

Ann: "Oh wait, isn't it something like… the closer you get to the center…"

Beth: [*Looking up at Ann for a second*] "The closer you get to the sides, the smaller the…"

Ann: [D*ragging points back to the center of the disc]*

Ann: "The bigger the defect is. But the smaller you make it…"

Ann: [*Looking towards Beth's computers*] "Can you make it super small?"

Beth proceeds to drag her triangle to the center of the disc making it small as well.

At the beginning of the scenario, Ann was wander dragging her image. After 2 seconds, Ann made the claim "the closer you get to the center… the bigger the defect is." As she made this claim verbally, she dragged her triangle to the center to validate the claim. In the process, she noticed her triangle was getting smaller and that the defect was getting smaller. She then formed a new conjecture that defect is related to size and asked Beth to verify that a small triangle will have a small defect. Beth then used a validation drag to verify Ann's conjecture. Both students wrote down this claim that smaller triangles will have smaller defects.

Beth gave herself a 3 for confidence. As justification, Beth wrote the angles "show this *[defect]* shrinking on the screen," but Beth also wrote that she needs more hyperbolic knowledge. This is reflective of her previous statements. Beth remained consistent with a low confidence in hyperbolic repeatedly saying she wanted more information about hyperbolic geometry. Ann, however, gave herself a 5 for confidence. This was a shift in her confidence. Ann had written for Prompt P3 that she wanted to "see it done." Yet, for this prompt she rated herself a 5 while for Prompt P4 she only rated herself a 3. One difference between her work to answer Prompt P4 and her work to answer Prompt P5 was the issue in Prompt P4 where the DGE did not report equal segments lengths based on how Ann had measured them. Ann had relied on Beth's measurements for justification. For this prompt, Ann's measurements were accurate, and Ann was able to use her own measurements to justify her answer.

Both students were then prompted to think longer about what they might notice, and specifically told to think about how defect might relate to the parallel transport activity they had just completed. The students began with a discussion before they returned to using the DGE. Beth began by talking about the angle sum in hyperbolic geometry versus angle sum in Euclidean. She stated angle sums are constant in Euclidean space and vary in hyperbolic space and that this difference can explain the change in defect based on the location of the triangle. While speaking, Beth began to validate drag her image around the screen to show that defect does change based on location. While watching Beth, Ann suggested that if they move the triangle closer to the center (where they had previously conjectured the defect shrinks) the first and last transported segments will get closer together. Beth then dragged toward the middle and both observed how the segments became closer together. Ann, watching Beth, formalized this claim, writing it down, and rating herself a 4 because "it did it on Geometry Explorer."

Again, both students were prompted to think longer about what they might notice. The students again began with a discussion while looking at their screens and talking through what they had already noticed. After 2 minutes, Ann took a guess that perhaps the defect and the angle between the segments might match. Beth then measured the angles, and they both saw the angles were the same. After seeing the angles matched, Ann and Beth went directly to writing their claim, not taking time to drag her image around the screen. Halfway through writing her claim, Ann paused and the following bit of dialogue took place:

Ann: Wait, can you move it around a little bit to see if it stays the same?

Beth: You think it's just a random coincidence?

Both: *[start laughing]*

Ann was concerned about basing a claim on a single image. To justify this claim, she wanted a multiple case inductive argument. The dragging they did was enough to give Ann a 5 for confidence, but Beth only gave herself a 4 for confidence as she wrote she was "not confident that there aren't exceptions."

When reading the formal proof after the activity, there was a noticeable shift in confidence. For the previous prompt, Ann gave herself a 5 and Beth gave herself a 4. After reading the proof, Ann rated herself a 2 and Beth rated herself a 3. They both wrote that defect should match the holonomy, but that they did not fully understand the proof.

Regular Polygon Series of Tasks

For the regular polygon series of tasks, Ann and Beth were asked to construct a regular triangle within Euclidean geometry (Figure 4.3). Both students proceeded to create an equilateral triangle using two circles that share a radius. Though Beth did the construction from memory, she initially expressed doubt saying, "Is this equilateral?" Ann leaned over and walked Beth through how the radii all match so it must be equilateral. Despite Ann's valid verbal proof, Beth asked "Should we measure it so we have high confidence?" Ann was ready to use a deductive justification, but Beth wanted to use the DGE to give her confidence. Both students proceeded to measure both the sides and the angles for validation. When reporting their confidence for Prompt R1, both students gave themselves a 5 and both students mentioned both the measurements and the deductive arguments in their justification.

The students were then asked to create a regular quadrilateral in Euclidean geometry. Ann proceeded to construct a square using three perpendiculars (using the perpendicular tool) and a circle (Figure 4.4). Beth struggled for a bit as she was convinced the construction was similar to the construction for the equilateral triangle (left image in Figure 4.5). After watching Beth struggle, Ann suggested that Beth may be thinking of how to construct a perpendicular, the first step of creating a square. Ann had skipped these steps as she used the perpendicular tool. Ann's comment prompted Beth to add the perpendicular. After the perpendicular was added, both students stared at Beth's screen for 10 seconds until Ann suggested using four existing points to make a square. After hearing the suggestion, Beth laughed saying, "it's a sideways square." Beth's construction process can be seen in Figure 4.5. As Beth began to place the segments for the square, Ann questioned her own claim about Beth's square, asking "do we know it's a square, though?"

Figure 4.3:

Ann and Beth – Construction for a Regular Euclidean Triangle

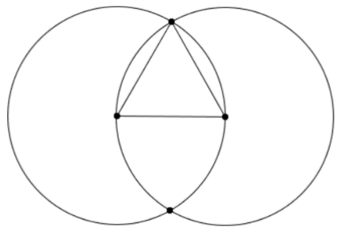


Figure 4.4:

Ann – Construction of a Regular Euclidean Quadrilateral

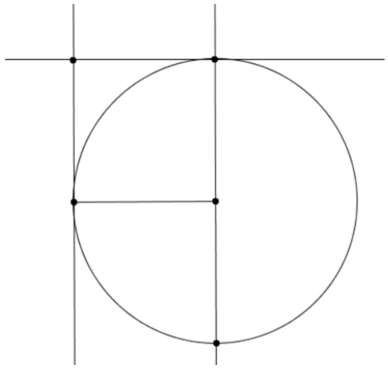
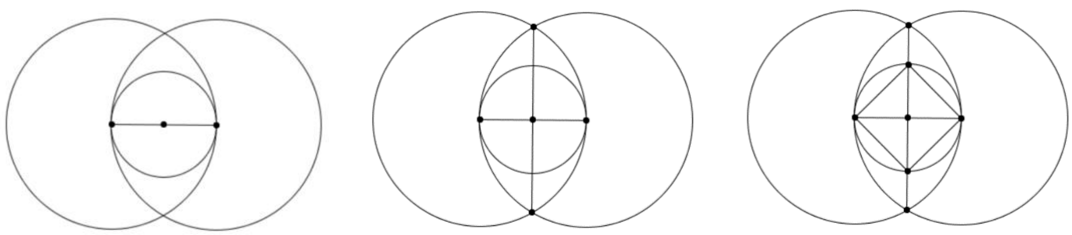


Figure 4.5:

Beth – Construction of a Regular Euclidean Quadrilateral



Ann proceeded to validate measure the side lengths of her own square. Beth, however, proceeded to drag her square so that it was oriented with a flat base. After rotating her square, Beth measured all the sides and the angles to validate her construction. Both students gave themselves a 5 for their confidence rating. They both justified it per the measurements, but Ann also gave some verbal deductive arguments about her perpendiculars being necessary for the square.

Ann and Beth were then asked to construct a regular triangle in hyperbolic geometry. Ann and Beth both started the same construction (Figure 4.6) as in Euclidean geometry with Beth finishing first. Beth declared, “I don’t know if it’s right” and Ann responded, “We could measure it.” Ann proceeded to measure just the lengths and Beth proceeded to measure the lengths and the angles. Both students gave themselves a 5 for their confidence referencing the software measurements. Neither student dragged their image and neither student made a deductive argument. This is a shift in justification from the Euclidean case where the students gave deductive arguments for both the triangle and the square. Also, note that in the parallel transport, Ann and Beth’s confidence dropped immediately when switching to hyperbolic. For the regular polygons, their confidence remained high after this initial hyperbolic task.

Ann and Beth were then asked if they believed regular quadrilaterals were possible in hyperbolic geometry. Both students said they do not exist. They recalled from their geometry course that if you have three right angles in a quadrilateral, the fourth angle will always be acute in hyperbolic geometry. While this is true, it presumes that regular quadrilaterals have 90-degree angles. Ann rated herself a 3 for confidence and Beth gave herself a 4.

For the last part of this series of tasks, the students were asked to construct a regular quadrilateral in hyperbolic geometry. After some brief discussion, they decided they would each use their construction method from Euclidean geometry. Their constructions can be seen in Figure 4.7. On the left is Ann’s diagram with three right angles and B, C, and D. Notice the perpendicular at D does not connect with A. Beth’s construction appears square, and she immediately measured all the angles and found they were the same, though they were less than

Figure 4.6:

Ann and Beth - Construction of a Regular Hyperbolic Triangle

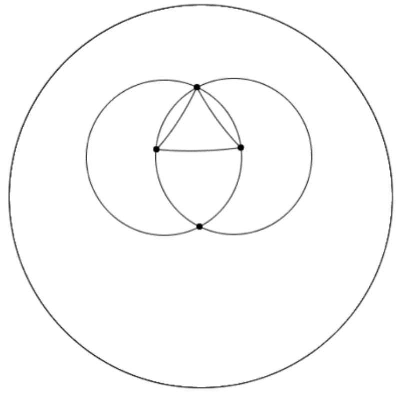
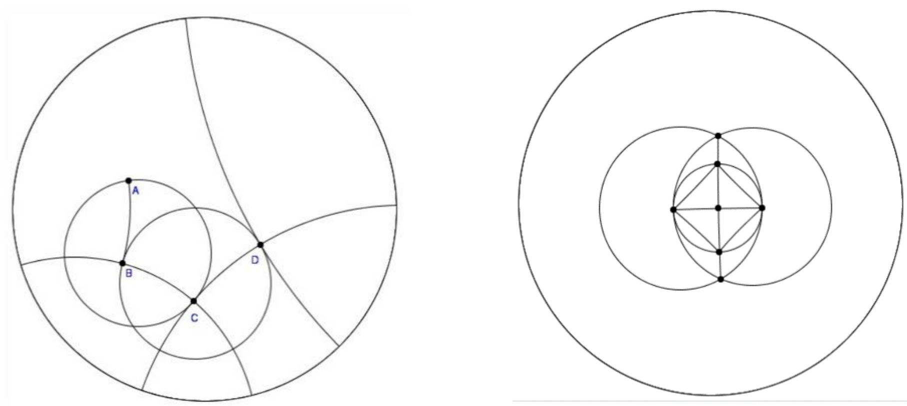


Figure 4.7:

Ann and Beth - Construction of a Regular Hyperbolic Quadrilateral



Ann Beth

90 degrees. Ann looked over and saw Beth's apparent success and asked Beth to move it around using a purposeful dragging. They both commented the angles were less than 90 and Beth argued that this makes sense since triangles have less than 180 degrees for their angle sum. Ann was still unconvinced and asked Beth to measure the sides as well. The sides all measured the same. Despite asking for a validation drag for the angles, Ann did not ask for a validation drag for the side lengths. Seeing that Beth's method worked, Ann did not attempt to correct her figure. Rather, she erased her construction and recreated Beth's construction with Beth's help. She did not do any dragging or measurement of her own construction once it was complete.

Both students felt they created the regular quadrilateral successfully and rated themselves a 4 for confidence citing the measurements as their justifications. Beth did comment specifically that despite the measurements, she remained skeptical it was correct. When answering what it would take to raise her confidence to a 5, Beth wrote that it would take “better confidence in hyperbolic geometry” to raise her confidence. This is an indication that Beth’s lower confidence ratings for hyperbolic geometry, at least in part, stem from a lower general confidence in hyperbolic geometry, rather than a task specific lower confidence in geometry.

Observed Shifts with Ann and Beth

There was a noticeable drop in reported confidence when the two students moved from Euclidean geometry to hyperbolic geometry during the parallel transport task. This shift was noticed both in the reported confidences being lower as well as an increase in dragging that took place when working in hyperbolic geometry. Beth’s confidence remained low throughout the remainder of the parallel transport task demonstrated by repeatedly writing that she wished she knew more about hyperbolic geometry. Ann, who had written that she wanted to “see it done,” saw her confidence shift slightly higher when she was able to use the DGE to provide evidence for her claims. Ann’s confidence lowered again when she got to the proof task, as she was no longer able to “see” the activity. There was also a shift in how students were using the drag tool when switching from Euclidean to hyperbolic geometry. Neither student used the dragging tool in Euclidean geometry during the parallel transport, though they both used it extensively when working in hyperbolic geometry. Lastly, there was a shift in how Ann and Beth constructed their figures between Euclidean and hyperbolic geometry. In Euclidean geometry, the students made a conscious choice to keep their figures looking alike. However, in hyperbolic geometry, they very deliberately created different figures.

For the regular polygon task, both students again expressed high confidence during the Euclidean tasks. However, as compared to the parallel transport task, this confidence remained high when they began work in hyperbolic geometry making the hyperbolic regular triangle. There was, however, a shift in the justifications being made. In Euclidean geometry, the students, especially Ann, showed a willingness to make deductive arguments. When moving to hyperbolic geometry with the regular triangle, neither student gave deductive arguments. Rather, both students relied on inductive arguments supported by the measurement tools. This shift continued as they worked on the regular quadrilateral.

In the parallel transport tasks, Ann and Beth used the tools in both wandering and validation modes. However, when switching to the regular polygon tasks, both students shifted to using the tools exclusively in a validation mode. Additionally, in the parallel transport task, we saw confidence vary between high and low confidences as they worked, especially in Ann's case. This is different from when they were working on the regular polygon tasks where both students had high confidence (4 or 5) throughout the task.

Students Carl and Dan

The second pair of students in the study were Carl and Dan. Carl was a senior math major and was the only student in the study who was not in teacher education. Dan was a junior math minor in elementary education. Carl earned the highest grade in the geometry course and scored near perfect marks on almost every assignment. He excelled at making deductive arguments in his homework. Dan was also a good student, showing strong knowledge of geometry during class sessions. He excelled at making deductive arguments verbally in group settings during class, but often struggled when writing those arguments for his homework. Carl and Dan worked together frequently on projects throughout the course.

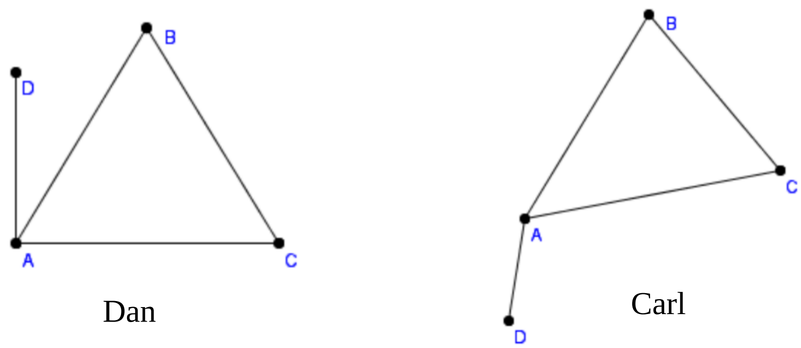
Parallel Transport Series of Tasks

Carl and Dan started the first task of the parallel transport series of tasks by each creating a triangle. The instructions then asked the students to create the segment to be transported. Dan created his segment first and drew his segment pointing up (Figure 4.8). Two seconds later as Carl was adding his segment, Carl stated, “going up,” but as he drew the segment, he looked over to Dan’s screen and noticed Dan had already drawn his segment pointing up. He then said, “oh wait, gotta go down now” and changed his segment to point down (Figure 4.8). This indicates that Carl was looking to create multiple examples. Rather than duplicating Dan’s triangle and segment, Carl chose to create a new example. Both students then proceeded to transport the segment from vertex A to vertex B. For Prompt P1, both students wrote the segments were parallel and congruent. Additionally, Carl voiced that the segments were pointed in the same direction, and so they both decided to write that the new segment was oriented the same as the first. Carl rated himself a 4 and Dan rated himself a 5 for confidence. Neither wrote a justification, but Carl reiterated that while his observation was correct, he was “not sure if we’re missing any key information.” When responding to how to raise his confidence, Carl wrote that he would like a list of what to be looking for.

After finishing the transport, Dan decided to drag point A declaring, “they [the segments] are the same.” Carl also dragged his image and agreed with Dan. Both students wrote down the segments were the same and they both rated themselves a 5 for confidence. In giving himself a 5, Dan wrote that he “expected the segments to be on top of the other, and that is what happened.” This indicates that his previous drag was a validation drag. He was confirming that the segments were behaving how he expected. Carl wrote as justification, “I moved some points around and the two segments lay on top of each other.”

Figure 4.8:

Dan and Carl – Initial Layout for the Euclidean Parallel Transport



Carl and Dan were then asked to think about parallel transport in hyperbolic geometry. Carl suggested the intermediate steps would not "appear parallel," but that the last segment would still be coincident with the first. Carl said he remembered that the triangle would curve inward and that this would affect angles, but he was "guessing" that they would be coincident once the segment completed the transport. After Carl brought up the notion of parallel being "a little different," Dan mentioned that hyperbolic can be visualized in two different ways. Specifically, lines appear curvy and angles appear correct in the Poincare disc, and lines appear straight and angles appear distorted in the Klein disc. Dan decided to agree with Carl, and both wrote down that the last segment will coincide with the original. Carl, who first voiced the idea, gave himself a 2 for confidence, and Dan give himself a 3. This is a downward shift in confidence from both students as they had previously given themselves 4s and 5s for confidence. Carl wrote, "there's logical justification," but that he's not positive of how "the hypothetical concept of translation translates to hyperbolic." From his geometry course, Carl knew about Euclidean translations, but was unsure if the same theorems and rules applied to hyperbolic translations. Carl was looking for a theoretical justification, but recognized he lacked the necessary information. This was confirmed when he wrote that "learning more about the strict definition of translation" would raise his confidence. Dan wrote that "doing the problem in Geometry Explorer and measuring the angles and distances" would allow him to feel more confident. This indicates that for hyperbolic geometry Dan was anticipating his confidence would come from visually seeing what would happen and making measurements rather than forming a deductive argument.

Carl and Dan then proceeded to do the parallel transport in hyperbolic geometry. They each constructed their triangles and added their segment to be transported. Again, like in Euclidean geometry, the students purposely added their initial segments going in opposite directions. As was the case in Euclidean, the students were planning to have multiple cases. After placing the first segment, Carl started thinking ahead to what the first transported segment would look like saying, "I know for sure this is going to look weird because if it looked exactly the same it would fall off the triangle." He was anticipating one of the translations as shown in Figure 4.9. To keep the segment visually the same, it could no longer be attached at the point, or it would have to go outside the boundary of the circle. This was an instance of Carl using the accuracy of the software's diagrams as a basis for his justification.

At the same time Carl was thinking about the segments not being congruent, Dan was transporting his first segment. Dan was surprised after doing the transport as the results were not what he was anticipating. His surprise was enough that he erased his segment and repeated this step getting the same result. Dan had created his triangle and segment in such a way that the transported segment landed perfectly on one side of the triangle (Figure 4.10). Dan's segment AD was transported to segment BE which landed perfectly on segment BC. After seeing the result for a second time, and pondering out loud, "Why did it do that," he decided to move on and continue the construction. This was an instance of the construction tools giving a student low

Figure 4.9:

Carl – Anticipation of a Hyperbolic Parallel Transport

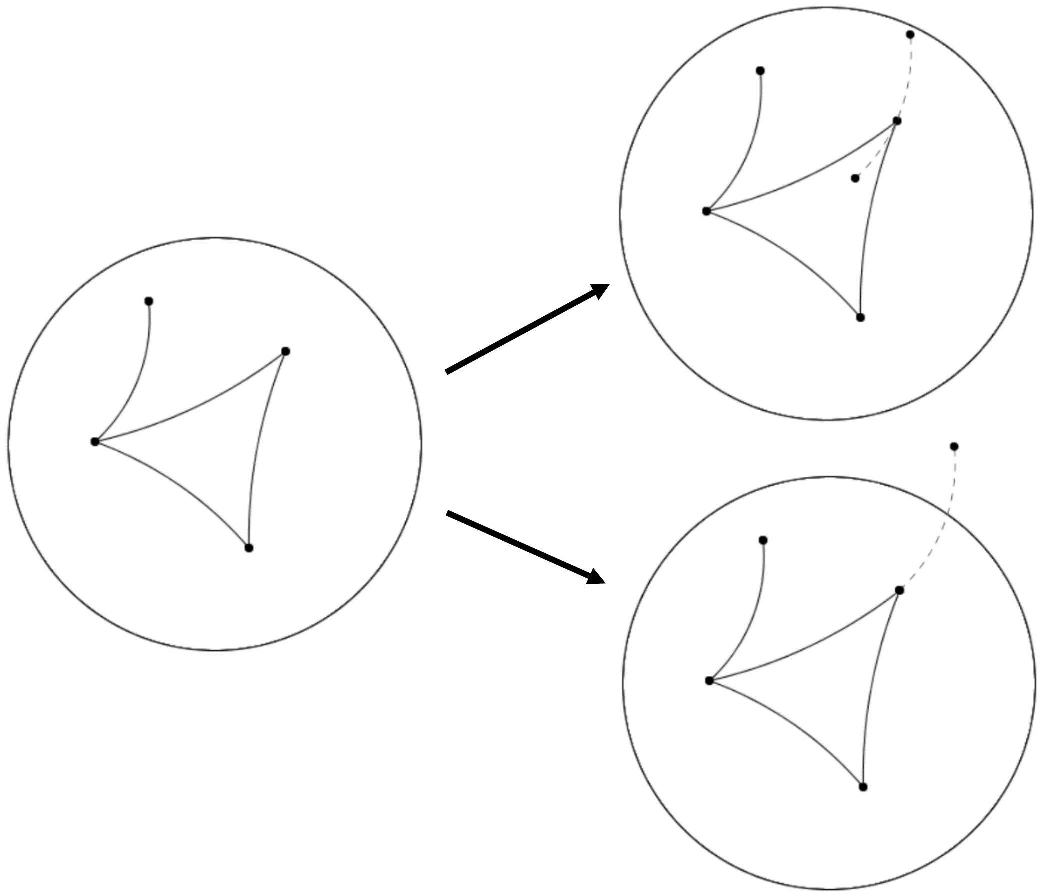
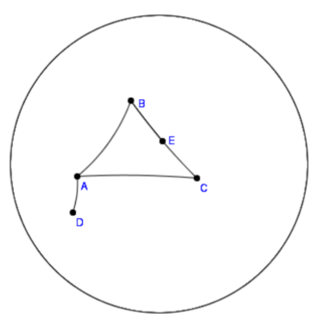


Figure 4.10:

Dan - First Segment of Hyperbolic Parallel Transport



confidence. Dan's confidence decreased when he the construction tool did something Dan was not expecting. This caused Dan to believe he had done something wrong, so he repeated it. But after redoing it, and getting the same result, he trusted the tool and moved on.

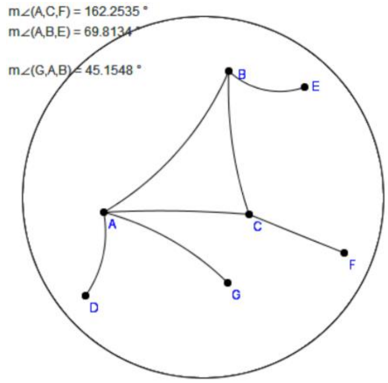
Dan was the first to complete the entire transport, and when he saw the final segment was different from the first, he again expressed doubt in his first transport. The final segment being different reinforced his doubt about the first segment. He went back one more time and transported the initial segment. It again landed on the edge of his triangle. After looking at Carl's figure he saw none of Carl's segments landed on the triangle. Dan then decided to drag his initial segment ending with a configuration in which none of the segments landed on the triangle. After seeing a figure that looked closer to his expectation, he then moved to answer the question on what he noticed about the new segment.

Carl and Dan both wrote, "The two segments are not coincident." They both rated themselves a 5 and Carl specifically said he was confident because he "performed all the steps correctly." He was acknowledging that he was trusting the translation tool to work correctly. His confidence was coming from both his skill in using the tool as well as the accuracy of the tool. Carl had not dragged his image at all. Dan dragged it enough to force his second segment not to be coincident with the triangle. We see a shift upward in confidence from the previous prompt after being able to use the software.

After writing their responses, the students were asked to think longer about what they might notice. The two students then began different strategies. Carl began by talking about how hyperbolic geometry is defined, going back to the definition of hyperbolic geometry. Dan began by measuring angles using a wandering technique. He was looking for a relationship or pattern but did not yet have a conjecture (Figure 4.11). Carl decided to shift from his theoretical discussion to using the software. Specifically, he began adding extra lines to his diagram. In adding these lines, he eventually became lost due to complexity and began anew with a new document. Starting again, Carl used hyperbolic lines, rather than hyperbolic segments when he

Figure 4.11:

Dan – Initial Wandering Measurement of a Hyperbolic Parallel Transport



completed the parallel transport. His figure is shown in Figure 4.12 with his transported lines shown as dotted lines for clarity. Carl is the only student in the study who extended the segments into lines. As he created this new version, he said to Dan, "I don't know if this is actually going to do anything, but you know…"

After finishing the construction with the lines, Carl began *wander* dragging the original line (segment) attempting to get the lines coincident saying out loud, "Is there even a way to get those to be, like, kind of coincident?" He had a goal, but he remained unsure if it was possible to get the lines coincident. As he dragged, he noticed that the angle between the first and last line (segment) visually stayed the same. This caused him to make a conjecture the two angles would always be equal. To validate this conjecture, he decided to measure the angle between the first line and the last transported line. He then dragged the initial line again to validate the angle stayed constant and noticed that the angle remained constant at 36.1045 degrees (Figure 4.13). This caused Carl to decide the lines will never be coincident. Notice the progression here. He began with *wander* dragging, not having a clear goal in mind. From his wander dragging, he formed the conjecture the angle will stay the same. He then measured to test that conjecture. It was not enough to *see* visually the angle does not change, he needed to see numerically that the

Figure 4.12:

Carl – Hyperbolic Transported Segment with Extended Lines

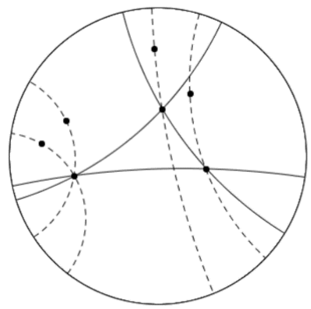
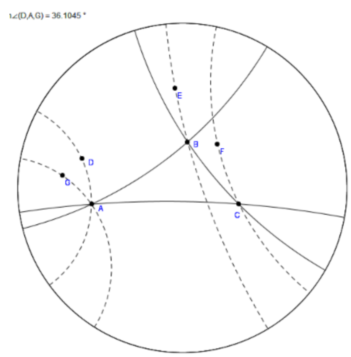


Figure 4.13:

Carl – Hyperbolic Transported Segment with Extended Lines and a Single Measurement

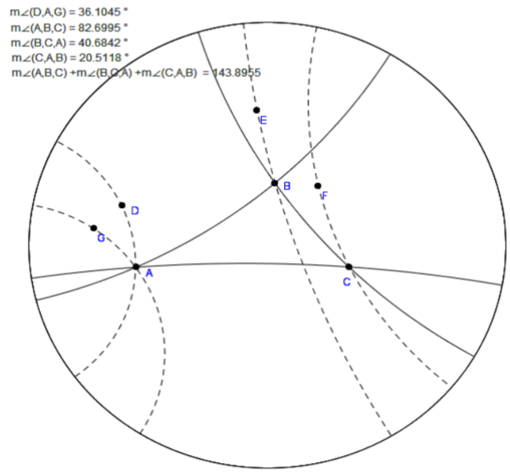


angle does not change. He needed to *validate* his conjecture via a measurement and a validation drag. Once he saw the number was constant, it was enough for him to declare the angle is constant.

After realizing the angle does not change, Carl conjectured that the angle between the segments was directly related to the interior angle sum of the triangle. He proceeded to measure and add the three interior angles to *validate* his conjecture (Figure 4.14). Carl saw that the angle sum and the previously measured angle between lines (segments) added to 180 degrees. He proceeded to write down this observation. As Carl wrote down this claim, he did stop and go back to the software to do a *validation* drag before rating his confidence level. For this validation drag, Carl decided to change the triangle itself and not just the initial line (segment). Seeing the

Figure 4.14:

Carl – Hyperbolic Transported Segment with Extended Lines and Multiple Measurements



numbers matched, he proceeded to rate himself a 5 for his confidence. Dan, who had been watching Carl, measured his own angles and rated himself a 4 stating that "I don't know why it works, but it seems to hold true." Dan indicated that he trusted their conjecture even though he did not have a logical reason for it to be true.

For the next part of the activity, Carl and Dan were asked to find the defect of the triangle and write what they noticed. Carl, realizing he had essentially done this, joked saying, "Jokes on you, we already did so." He proceeded to rewrite his previous claim, only changing the wording to include the concept of defect, rather than angle sum. He again rated himself a 5. Dan, meanwhile, proceeded to use the software to measure the defect and then proceeded to *wander* drag his triangle while watching the defect change. He appeared to be looking for something additional to notice about the defect. When writing his claim, he wrote only about how the defect changes as the size of the triangle changes and rated himself a 5 using the software measurement as justification. He did not repeat his previous claim.

Carl and Dan were then asked to read the proof that the angle between segments matched the transported segment. Carl rated himself a 5 after reading the proof but wrote his confidence came from "modeling it in the software and dragging the points around." Dan gave himself a 4 writing the "formula [at the end of the proof] inspired great confidence," but that the proof was a little difficult to follow.

Regular Polygon Series of Tasks

Carl and Dan then proceeded to the regular polygon activity. In the first task, the students constructed the equilateral triangles in Euclidean space. Neither student measured, though Carl did a quick *validation* drag after the construction. Both rated themselves a 5 on confidence. Both students referred to the use of circles as part of their justification, with Dan specifically mentioning the circles were responsible for the segment lengths being congruent. Additionally, both students mentioned having learned this construction in class.

When constructing the regular quadrilaterals, both students constructed it by making three right angles and a circle to keep the segment lengths congruent (the same way Ann did earlier). After Carl finished his construction, he looked over to Dan and saw Dan also had finished. Dan had finished first and had erased his circle. Carl expressed concerned about Dan's lack of circles and asked Dan how he guaranteed his segment lengths were the same. Dan proceeded to reassure Carl that he had used a circle to have congruent sides. When they reported their confidences, both students rated themselves a 5 with Dan mentioning his circles guaranteed segments of equal lengths and perpendicular lines guaranteed congruent angles. Carl just referenced his construction steps.

When they switched to hyperbolic geometry, both Carl and Dan created a hyperbolic triangle using the same construction as they did in Euclidean geometry. As he was adding the last segment, Carl said “Since I do not, um, you know what, we’re just going to do this” as he began to measure the angles in his triangle. Carl was expressing doubt about his construction, so he turned to measuring angles. At the same time, Dan was also measuring his angles. Carl finished after Dan and suggested to Dan that they also measure their lengths as well. Both students rated themselves a 4, and both referenced the measurements from the software. Carl noted he would have been less confidant than a 4, but measuring the lines and angles made him “feel better.” Carl stated that being told his construction was right would raise his confidence. Dan said having angles adding up to 180 degrees would make him feel more confident. There was a shift in this activity as the students began taking measurements. For the Euclidean triangle, neither student measured angles or lengths. But in hyperbolic geometry, both students measured both angles and lengths.

The students were then asked if regular quadrilaterals exist in hyperbolic geometry. After a discussion, both students claimed they do not exist. In his written response, Carl referenced his geometry course where he learned about Lambert quadrilaterals having 3 right angles and an acute angle. This told him that four right angles would be impossible, but Carl hedged his claim by making a verbal claim that if they did exist, they would not have right angles. Carl rated himself a 2 on confidence, stating that “testing it out” would raise his confidence. Dan rated himself a 3 because “parallel lines and perpendiculars” are “distorted” in hyperbolic geometry. Like Carl, Dan wrote that attempting a construction would raise his confidence. As expected their confidence shifted downward for this prompt.

Carl and Dan were then asked to create a regular quadrilateral in hyperbolic geometry. Carl began by stating they should not be using the perpendicular tool as the quadrilateral cannot have right angles. Dan agreed, and after thinking for a few moments, suggested making a regular triangle and then stacking another on top of it. Carl informed him this would make a rhombus and Dan abandoned his idea. Both students started constructing some circles. Both students started with a segment AB and added two circles creating the construction as seen in the left side of Figure 4.15. They both then add segment CD by connecting the points (on the right in Figure 4.15). Despite telling themselves not to use the perpendicular tools, they inadvertently created perpendiculars, though neither had verbalized that fact yet. Carl then decided to add a midpoint to segment CD. When the midpoint appeared, it lined up perfectly with the intersection of segment AB and segment CD and Carl exclaimed, “Woah… that’s not where I was expecting that to be.” He then used a circle tool with the center at the intersection E to check whether it really was a midpoint (Figure 4.16). Satisfied it was a midpoint, Carl thought for a bit, before stating he had an idea. He extended segment AB into a line, creating E and G at the same time. He gave himself four points (D,C,E, and G) that were equidistant from the circle center E. He did not say he knew this would work, but his construction became purposeful as it was based on his unverbalized idea. As he began connecting those points, he said, “that does not look equilateral, but you know, sometimes geometry is weird sometimes.”

While Carl created his quadrilateral, Dan was working independently adding multiple circles and segments in a kind of *wandering* construction. Not making progress, he eventually reverted to just two circles and two segments as in Figure 4.15. Like Carl, but not knowing Carl had done it, Dan decided to add a midpoint to segment CD just as Carl had. They shared this exchange when the midpoint appeared a

Dan: “What? That’s also the midpoint?”

Carl: “I know that struck me to my core…”

Dan: “Yeah?

Figure 4.15:

Carl and Dan – Beginning Attempt of a Regular Hyperbolic Quadrilateral

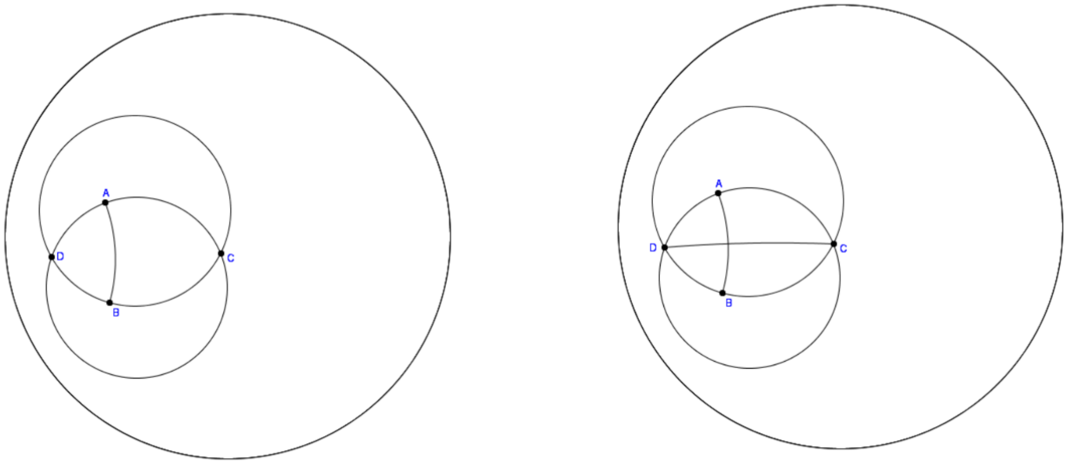
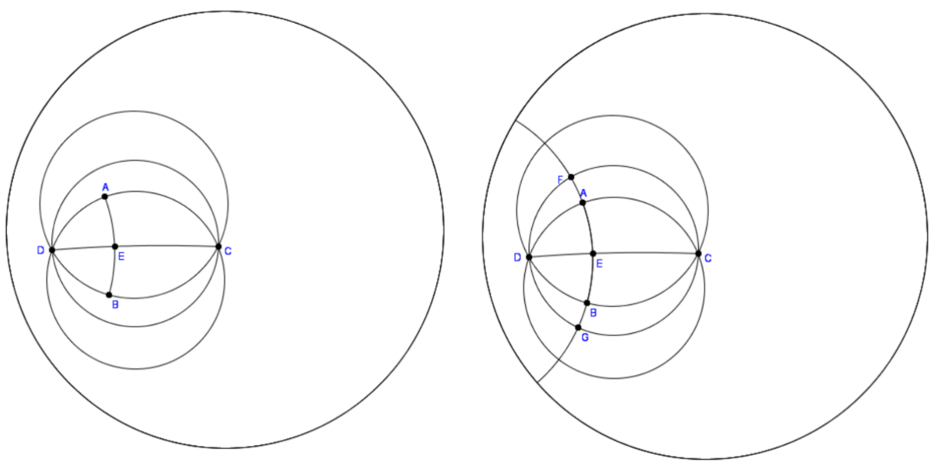


Figure 4.16:

Carl and Dan – Later Attempts of a Regular Hyperbolic Quadrilateral



Dan then proceeds differently than Carl. Carl, wanting to test the midpoint, had added a final circle to make sure segment ED was congruent to segment EC. Dan used a measurement tool to check they were the same. As Dan did not have this additional circle, Dan used the points A, C, B, and D to construct a quadrilateral. Both students recognized the importance of the midpoint and used if for their construction. This implies a deductive rationale for at least part of their construction.

As both students now had quadrilaterals, both students began to measure their constructions. Dan began with angles and quickly realized the angles were different. Carl leaned over and suggested Dan add in the final circle that Carl had previously added. Carl finished measuring all the angles and all the sides and then proceeded to drag the construction all around the Poincaré disc doing a validation drag. Dan finished his new quadrilateral, removed the extra lines, but did not measure anything else. Instead, he did a short drag until the construction landed in the middle of the Poincare disc where it looked relatively square. Carl and Dan had two different approaches to their drag test. Carl dragged his construction into just about every configuration possible while making sure the measures stayed equal. Dan, who had not measured, dragged his image into a single configuration that looked square.

Both students gave themselves a 5 in their confidence. They both cited their measurments of both angles and segments as justification their construction was correct. Dan verbally acknowledged he was using Carl's measurements. Carl mentioned the use of the midpoint as important to his construction, but did not write specifically about why that was important, just that it was. After writing his response, Carl returned to Geometry Explorer to attempt a new simpler construction knowing the midpoint appears to be the important piece of the construction. However, his hestitancy to use the perpendicular tool prevented him from discovering a simpler version of his construction (the version Beth had created).

Observed Shifts with Carl and Dan

Carl and Dan maintained high confidences throughout both series of tasks with the exception of prompts P3 and R4. These two prompts were the conjecturing questions. When Carl and Dan were asked to think about what might happen, both students reported low confidence.

However, as soon as the students were able to work within the DGE, they once again reported high confidence levels.

Despite their reported high confidences, there is evidence that their confidences shifted throughout the individual tasks themselves. For instance, when responding to prompt P4, Dan rated himself a 5 for what he noticed about the last segment. However, it is evident he had instances of low confidence when working on the task. When Dan was transporting the initial segment, he had an idea in his head of where the segment should land, and the initial transport did not land there. Further, he had predicted when he finished the transport that the first and last segments would be coincident, and they were not. This caused Dan to question whether he was doing the construction correctly. Because of this doubt, Dan went back and repeated the construction twice. In all instances, the first transported segment landed on the side of his triangle, and he expressed verbal hesitation whether he was doing his construction correctly. That doubt was overcome once Dan started dragging his image. Dragging his original segment moved all the segments off the triangle into a figure that more closely matched his expectation. This dragging gave him the confidence to accept that the segments were not coincident, and his construction was correct.

As with Ann and Beth, there was also a noticeable shift in the type of justifications given by Carl and Dan when working in Euclidean versus hyperbolic geometry. When working on the regular polygons, both students gave deductive justifications for both the triangle and the square when working in Euclidean geometry. Yet, when the students moved to hyperbolic geometry, neither student offered deductive justifications. Their arguments shifted to purely inductive arguments based on their measurements taken within the software.

Lastly, there was also a shift in when Carl and Dan chose to use the tools. When working on the parallel transport in hyperbolic geometry, the students were asked what they noticed about the final segment. In answering this question, neither student used a measurement tool. They wrote what they saw, and they saw the segments were not coincident. This seemed to satisfy the two students. However, they were asked to think longer about what they might notice. Only after being asked to think longer did the students begin to use the measurement tools.

Students Eve and Fran

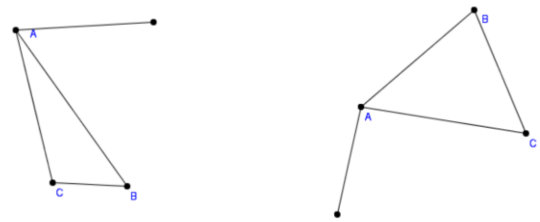
The third pair of students in this study were Eve and Fran. Eve was a senior math major while Fran was a senior math minor. Both students were going into education and both students had worked together previously for multiple projects in their geometry course. Eve was a hard worker who asked lots of questions during class. She was always excited to be in class, but she struggled with writing proofs. Fran was the quietest student in the geometry course. She had good ideas when she felt confident to speak, but she often struggled getting those ideas written down.

Parallel Transport Series of Tasks

For the first task of the parallel transport activities, Eve and Fran created two differently configured triangles and attached their segments going in different directions (Figure 4.17). They were both looking at their own screens and there was no verbal indication the different triangles were created purposely. After translating the segment, Eve commented verbally that the transported segment and the original segment "looked parallel," and Fran gave a verbal assent. When they wrote their answers, Eve did not write about parallelism, but did write the segments were the same length. Eve rated herself a 4 noting that "Geometry Explorer did the work for us with accuracy." To raise her confidence, Eve wrote she wanted "the correct answer given." Fran

Figure 4.17:

Eve and Fran – Initial Layout for Euclidean Parallel Transport



wrote the transported segment was translated. She did not mention length or parallelism, only that it had moved. She rated herself a 5.

For the second part of the parallel transport activity, after Eve and Fran finished translating the segment, they both wrote that the first and last segment "are the same." As justification, Fran wrote they look the same and Eve wrote, "the directions were clear, and it seems right to me." Eve gave herself a 5 for confidence, and Fran gave herself a 4. Fran wrote her confidence would increase if she had measured the "angles and lengths to make sure they were the same."

For the third task, Eve and Fran were asked to predict what would happen with a parallel transport in hyperbolic geometry. The following conversation took place:

Eve: "I think they'll be the same. Well, I remember that in class it would look different… It might not look like….

Fran: It would look different, but it would still be the same?

Eve: Because like for that, you could tell right away because they look parallel. But in hyperbolic geometry…

Fran: But do you think they will still overlap? Like they are exactly on top of each other here. [points to screen in Euclidean geometry] Will they still in hyperbolic?

Eve: What do you think?

Fran: I get nervous when I disagree with you, but I don't think that they will still be in the same location.

Eve: Well, I'm just trying… do you remember the activity we did in class similar to this? I think it will be the same.

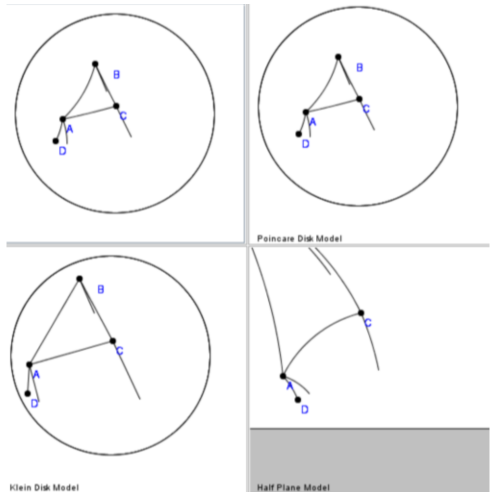
From this conversation, Eve used previous knowledge, an external justification, for her claim. Eve was convinced that an activity from class was similar enough to this activity and that she learned from this class activity that the segments would be the same, even though they might "look different." Fran was convinced the segments would be different but does not specify a reason verbally. She did specify that she gets nervous disagreeing with Eve. When writing their claims, Eve rated herself a 4 for confidence and referred to the similar class activity that she remembered. Fran rated herself a 2 and wrote that "hyperbolic changes a lot." She saw the segments matched in Euclidean geometry and used this to reason that they would then be different in hyperbolic as hyperbolic would change how the segments move. This indicates Fran was using previous knowledge and a comparison to Euclidean geometry, but then made the opposite claim Eve made. To raise her confidence, Eve wanted to go back and look at the activity she had previously done in class. Fran wanted to see the figure "drawn in hyperbolic." While Fran's confidence shifted to a lower confidence, Eve's reported confidence stayed high with a 4. Eve was one of only two students whose confidence remained above a 3 when responding to this prompt.

Even and Fran then completed the transport in hyperbolic geometry. After she completed the transport, Eve expressed frustration seeing the segments were not coincident saying, "Aww man….I should have known I was wrong. You know what, it's probably trying to remember the activity in class, it was probably a different shape. I might be wrong." She was still holding firm to the external justification, just rationalizing that her memory of the shape was wrong. Fran finished after Eve and agreed the segments were not the same. When writing down what they noticed, both students wrote the segments were not the same. Eve mentioned specifically that they do not overlap “according to Geometry Explorer.” She is using the construction tools of the software as a justification. Both rated themselves a 5 on this observation.

After they wrote their claims, the researcher stepped in and asked them if they could think longer about what they might notice. Both students sat for a full minute before Eve asked Fran if she had any ideas. Fran shrugged and they both returned to the DGE. Fran then decided to measure the segments in a wandering manner. Meanwhile, Eve started exploring the menu system looking for an idea of what to do. After another minute, Fran had measured the first and last segment. Eve expressed frustration to Fran that she wished she could change the system back to Euclidean. Fran mumbled an assent, “mmm..hmm,” but did not volunteer her measurements with Eve. Eve continued exploring the menu, choosing an option to display “All Models” which gave her three different hyperbolic models, none of which were the Euclidean model she wanted or was hoping for (Figure 4.18). She switched back to the standard model after 30 seconds. Fran, in the meantime, had *wander* dragged the vertices of the triangle around the screen but still had not expressed either verbally or in writing her observation that the segment lengths match. Eve finally said, “I know they are the same length, but that they don’t overlap.” They both proceeded to write this claim. Eve rated herself a 4 stating that Geometry Explorer shows they are the same length. It is worth noting that Eve did not measure her segments. She may have been relying on Fran’s measurements, but it is not clear. Fran rated herself a 5 writing that her justification was that she measured them in Geometry Explorer.

Figure 4.18:

Eve and Fran – Multiple Models of a Parallel Transport



The students then moved on to the next task by finding the defect of the triangle. After using the software to measure the defect, they noted their defect numbers were different. They wrote this down, and Eve also commented her defect was a small number. She rated her confidence a 2 stating that she does not “know the purpose of it [the defect].” Fran rated herself a 5 but did not verbalize any reasons. This is the first prompt where Eve gave herself a low confidence.

The researcher stepped in and asked the students to think longer about the question and how it might relate to the previous question. The students spent the next few minutes in discussion with each other rather than using their computer. After four minutes they decided that if the triangle defect is zero, like in Euclidean space, then the segments will overlap. As Fran began to write this down, she decided to test the idea in Geometry Explorer. She returned to the DGE and dragged the triangle to make the defect close to zero. She dragged to *validate* a claim she had made. After dragging the triangle so the defect was almost zero, she saw the segments overlapping. She rated herself a 3 for confidence and specified that Geometry Explorer showed her the segments overlap. Eve made the same claim with the same confidence level but justified her answer by referring to prior knowledge.

Both students continued thinking about this question for another 10 minutes. At this point the students had been working for over 50 minutes, longer than any other group up until that point. As this was the last prompt before students were shown a proof that the defect matches the angle between the segments, the researcher made the decision to direct the students to focus on the angles, and specifically the measurement of the angles. This prompted Eve to start *wander* measuring all the angles looking for any pattern she could find while Fran watched her. After measuring all the angles, Eve noticed the angle between the segments matched the defect. Eve rated herself a 3 without giving a specific justification, though it was clear her justification was the relationship she saw when measuring the angles. Fran rated herself a 4 referencing Eve's measurements as justification.

After reading the proof, Fran gave herself a 1 for confidence, writing that it was hard for her to visualize so she did not "really understand what the proof was saying." Eve gave herself a 4 writing that the proof told her "she was right" and that the "argument helps my confidence". She wrote that doing it again in Geometry Explorer would raise her confidence.

Regular Polygon Series of Tasks

Eve and Fran then began the regular polygons series of tasks. The first task was to construct an equilateral triangle in Euclidean geometry. As has been mentioned, this is a construction that was done multiple times in their geometry course. Both Eve and Fran initially expressed frustration at forgetting how to do it. Eve expressed this frustration aloud saying, "I can't believe that I forgot it." Both students began with a segment, but then appeared to get stuck. After thinking for a full two minutes, Fran remembered, "Wait, is this the one where you had to do the circles?" This prompted their memories, and both proceeded independently to construct their triangles correctly using two circles with matching radii (Figure 4.19). Once they saw what appeared to be a regular triangle, they both proceeded to record their responses.

Eve rated herself a 5, writing that she constructed it "using circles, which is used as a proof for regular triangles." She also stated she had "previous knowledge about constructing regular triangles." Her justification was a combination of deductive proof and external authority (previous knowledge). Fran rated herself a 4 and wrote about using circles "like we learned in class." She continued writing, "it's been a long time since she's done it." Fran was also relying on external authority (previous knowledge). She noted that measuring the side and angles would have increased her confidence to a 5.

Eve and Fran were then asked to construct a regular quadrilateral in Euclidean geometry. Both students began by repeating the beginning of the construction for the equilateral triangle (Figure 4.19). Eve added to this figure by starting a circle with center at A, but then never finished the new circle as she realized using B, C, or D as a radius point would just give her the same circle. She then decided to create segment CD, followed by a circle with center C and radius AB, finally shaking her head, and undoing the last circle (Figure 4.20). Meanwhile, Fran added a vertical segment, and then four additional segments to create a rhombus. As Fran finished her rhombus, Eve looked over and expressed excitement before realizing the angles in Fran's figure were not all congruent. Fran agreed with Eve and then deleted the segments that made the rhombus.

Eve went back to work on her screen and had a thought that maybe she could rotate her original segment AB by 90 degrees. She then attempted this via the menu, but after a few minutes of trying could not figure out how to get the software to do what she wanted. This was

Figure 4.19:

Eve and Fran – Beginnings of a Regular Euclidean Triangle

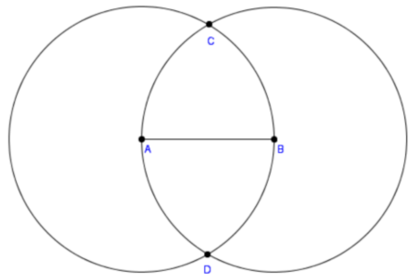
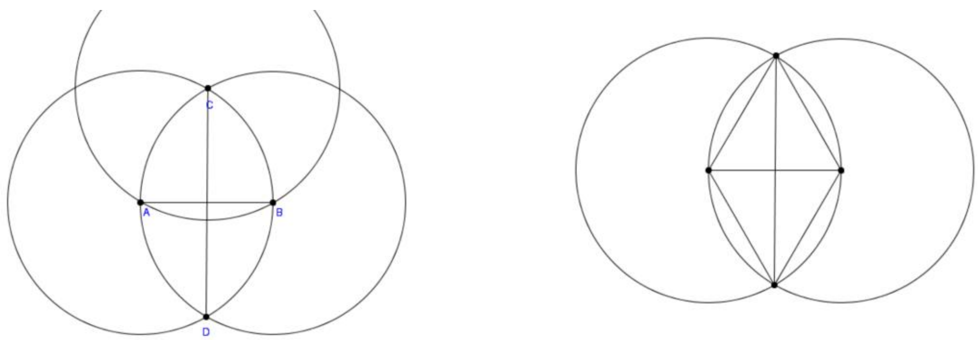


Figure 4.20:

Eve and Fran – Attempt at a Regular Euclidean Quadrilateral



Eve Fran

the only instance where the software hindered an intentional act of the student because the student could not figure out how to use the software. Because she never figured out how to use the tool, Eve leaned to Fran and told Fran she wished she could have had her book.

Despite Eve not using the tool correctly here, there was a shift in strategy that is unique among the students in the study. Despite this being a construction task, Eve was attempting to use transformational geometry to construct her square. That is, rather than using the equivalent of the straightedge and compass, Eve was trying to use a rotation tool. This was not a typical construction tool, but it was an affordance of the software one does not have when using paper and pencil.

While Eve struggled with the rotation tool, Fran slowly added elements to her construction as shown in Figure 4.21. Eve looked over and started laughing, saying, “that’s not right. I remember we didn’t do anything like that.” Eve very specifically said it is “not right” because it was not like what she did in class. Eve did not mention the angles, which were clearly not 90 degrees, or even the side lengths, which were clearly not equal. To Eve, this image was more complicated than what she remembered, and therefore “not right”. It was a justification by authority. Fran then reverted her construction all the way back to the original two circles and a segment.

After she critiqued Fran’s figure, Eve looked back at her own screen, and moments later, declared, “Oooh, I have an idea. OK, I remember, do you remember, this line, and we can make…I’m so happy…I think I’m getting there.” As she said this, Eve used the perpendicular tool to add perpendiculars at A and B giving her the Figure 4.22. Eve then slowed down as she thought how to finish her square. She finally settled on making a segment though the top point parallel to AB declaring “Yes, I got it.” Hearing excitement, Fran looked over to see what Eve was doing. Upon seeing what she believed to be Eve’s success, Fran asked how she did it. Eve responded by saying she did not have a formal proof, but “Geometry Explorer helps.” Eve started measuring to convince both Eve and Fran that her construction worked. She began by measuring the segment lengths. She measured both the top segment and the left segment and realized they did not match. The top segment was just a bit longer than the sides.

They decided to stop and move on. They both wrote down they had not been able to construct a regular quadrilateral, and both rated themselves a 5 for their confidence. Both gave the same reason that the software showed them the sides were different lengths. This is the first instance where students rated themselves a high confidence in a negative outcome. That is, the

Figure 4.21:

Fran – Further Attempt at a Euclidean Regular Quadrilateral

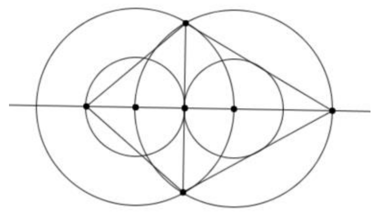
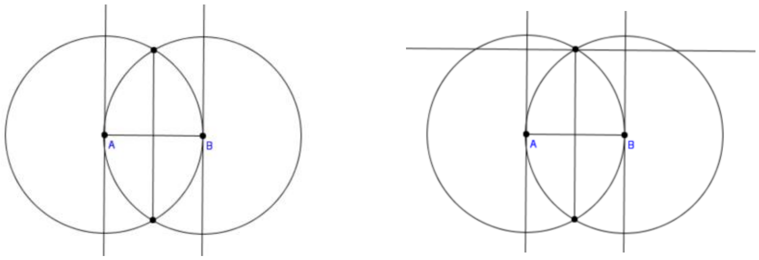


Figure 4.22:

Eve – Further Attempt at a Euclidean Regular Quadrilateral



students were asked to construct a regular quadrilateral and they were unable to do so. Their 5 represents confidence that they knew their construction was not correct, and they were sure their construction is not correct.

Both students then moved to the construction of a hyperbolic regular triangle. As expected, they both decided to try the same construction as in the Euclidean regular triangle. As soon as they finished their constructions, they both measured the side lengths and angles to validate their constructions were correct. Eve rated herself a 5 referring to the measurements from Geometry Explorer. Fran rated herself a 4 stating she was unsure if the definition of "regular" is the same in hyperbolic geometry. She was confident the angles and sides matched, just unsure if she understood the terminology completely.

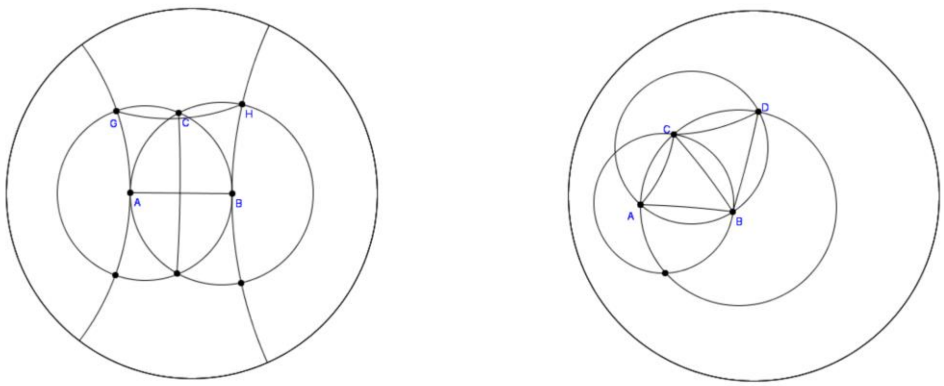
The students were then asked if they thought regular quadrilaterals exist in hyperbolic geometry. Eve mentioned "prior knowledge" specifically mentioning the parallel transport where these two students had discussed triangles having angle sums less than 180 degrees. She then used that point to argue the interior angle sum of a regular quadrilateral would be less than 360 degrees. Based on that, she concluded regular quadrilaterals could not exist. Fran suggested the angles could not be the same "because the lines are all curvy." Both Eve and Fran rated themselves a 3, writing the same reasons they stated out loud. Eve wrote her confidence could be raised if she had "more prior knowledge." Fran wrote her confidence could be raised by "working it out."

Eve and Fran both then proceeded to attempt the construction. It should be noted this is the only group that proceeded to the construction of the hyperbolic regular quadrilateral without first having successfully constructed the Euclidean regular quadrilateral. Eve and Fran took two different approaches to this construction. Eve attempted to copy her construction from Euclidean geometry, even though, as she stated out loud, she "knows it's wrong." That is, she started with segment AB, and created two perpendicular lines at points A and B (Figure 4.23). She also used circles to guarantee her vertical segments matched the base. However, she did the last step slightly differently. Referring to the left side of Figure 4.23, she added her last segment across the top by connecting G and H. In the Euclidean version, she instead created a parallel to AB through point C. Interestingly, had Eve connected G and H in the Euclidean version, she would have successfully created a square. It is unknown if this change was purposeful or not. However, in the hyperbolic, this construction still did not work, which Eve learned when she measured the segment lengths. The top segment was longer than the sides.

After accepting her construction did not work, Eve attempted a second construction, ending with the right side of Figure 4.23. She measured the angles CAB and ACD and saw that they were different. She then proceeded to proactively drag her figure to get matching angles by

Figure 4.23:

Eve – Attempts at a Regular Hyperbolic Quadrilateral



moving the highlighted circle seen in Figure 4.24. Her figure started with A and B as free points. Had she dragged either of those points, the angles would have changed as she was intending. By dragging the circle, a dependent object, she moved her figure, but the angles stayed constant. After she saw the angle remain constant, she moved her image back to its starting position. She finished by measuring the sides and saw that they matched.

Fran went in a different direction. She shifted to a strategy that no other student had yet done. Rather than attempt a formal construction, Fran began by 'constructing' her regular quadrilateral by just drawing four segments in a rough quadrilateral shape (Figure 4.25). After making the quadrilateral, she dragged A toward the edge of the disc and back, but then erased her figure. It is unclear what she was looking for when she dragged A towards the edge of the circle. She quickly gave up on that approach and proceeded to construct a figure similar to what Eve eventually settled on in Figure 4.23. Unlike Eve, Fran did not measure any sides or angles.

At this point, the researcher stepped in and told them it was OK to not successfully construct the quadrilateral. The two students had again been working longer than any other pair up until that point. They wrote down they could not construct a regular quadrilateral. Fran rated herself a 5 that her construction is not correct writing, "I'm sure I don't know how." Eve rated

Figure 4.24:

Eve – Attempts to Make Angles Match in a Regular Hyperbolic Quadrilateral

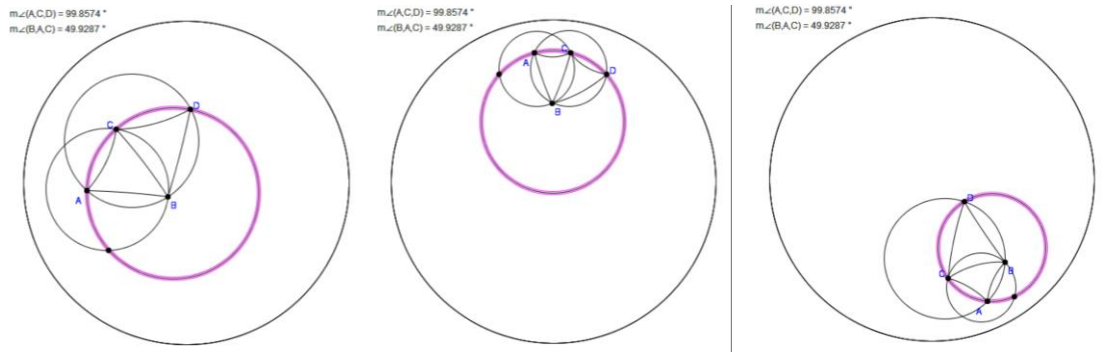
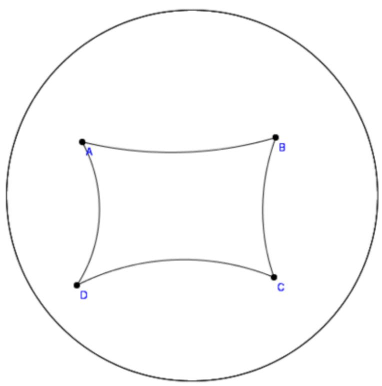


Figure 4.25:

Fran – Drawing of a Regular Hyperbolic Quadrilateral



herself a 3 writing that she could not get the lengths the same. To raise her confidence, Eve wrote that she wished she could remember from her geometry course how to do it.

Observed Shifts with Eve and Fran

When working on the regular polygon task, we saw a shift in strategy from Fran as she attempted to construct the regular quadrilateral in hyperbolic. For the previous three constructions, she approached the tasks in a comparable manner. She used the construction tools to create a constructed figure. But with the regular quadrilateral, Fran spent a few minutes seeing if she could draw a quadrilateral rather than construct a quadrilateral. In other words, before trying to construct the figure, Fran spent a few moments exploring what the figure might actually look like. This was an expected strategy for the students to try, but Fran was the only student who tried it. Despite measurement tools, Fran did not measure sides or angles, so it remains unclear exactly how Fran used her drawing to help her.

There was also a shift in tool usage that occurred in the regular polygon series of tasks. For the first three constructions, neither student dragged their constructions. They either based their answers on looking at the static figure or measuring the static figure. Even in the case of the regular quadrilateral in Euclidean geometry, when they saw they had not been successful, neither student dragged their image. It was not until creating the hyperbolic regular quadrilateral that both students dragged. Eve, for instance, saw that her angles were not congruent and decided to drag her figure to see if she could get congruent angles. Comparatively, when her sides were not congruent in Euclidean geometry, she did not use the dragging tool.

Throughout the regular transport task, Eve reported high confidences of 4 and 5 until she got to the last prompt asking what she noticed about the defect. Even when she responded to the prediction prompt (P3), Eve reported a high confidence. Yet when she reported on the defect (prompt P5), she reported a 2 for confidence. The claim Eve was making was that her triangle defect was different than Fran's triangle defect. Having measured the defect, Eve observed this difference clearly, and she still rated herself quite low for confidence. This indicates her interpretation of confidence had shifted from her confidence in the correctness of what she wrote to an interpretation of confidence about whether what she wrote was what the question was looking for.

Students Gary and Hal

The next pair of students was Gary and Hal. Gary was a senior and Hal was a junior. Both students were math majors going into education. This pair, more than any of the others, worked most closely together throughout the semester. They were athletes together, and so they spent much time studying together as well as working on projects together. Gary was typically quiet during class, whereas Hal was more willing to take risks and ask questions during class.

Parallel Transport Series of Tasks

Gary and Hal started with the parallel transport activity in Euclidean geometry. After they both translated the first segment, Gary read prompt P1 asking what the students noticed about the segment, and Hal responded, "They are parallel and the same length." Gary rated himself a 4 writing "the angles may be different." To raise himself to a five, Gary suggested "extending the lines to see if they get any closer together." In other words, Gary knew parallel lines should not cross, but the short segment did not present enough length to see if they would cross. That Gary wants to 'see' whether the lines cross implies that Gary is thinking inductively based on his single image. Hal rated himself a 5 writing, "It's a perfect translation, so my answer would make sense." Rather than Gary's inductive approach, Hal is focused on translations and the properties they preserve – parallelism and length.

After finishing the parallel transport around the triangle, both students wrote the segments were the same. Gary wrote, "They line up perfectly" and Hal wrote, the segment "moved right back to the original exactly." Both rated themselves a 5.

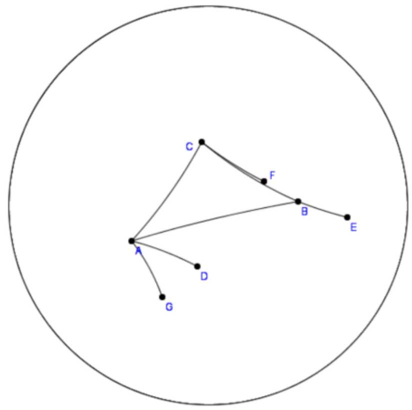
Both students were then asked to think about what would happen to the transported segment in hyperbolic geometry. They both decided the new segment would look different from the original segment. Gary mentioned how segments look different depending on where the segment is within the circle, and Hal followed by implying the length stays the same despite how it looked. Gary mentioned, “it’s different the further in [the circle] you go.” Gary was referring to how segments lengths appear to change size as you move them from the edge to the center of the Poincaré disc. Hal agreed, pointing to his Euclidean triangle saying, “This line here [*the segment*] would look different over here.” Gary, who first suggested the idea the segments will appear to change size, rated himself a 5. Hal rated himself a 2, writing, “There is a possibility the segment would revert back to its original shape and angle.” Hal decided that while the segment may appear to change size as it moves, there is a chance it will return completely to its original size once it is back where it started. Hal wrote that “doing it on the computer” would increase his confidence.

Both Gary and Hal then proceeded to complete the transport in hyperbolic geometry. After completing the transport, both noticed the segment was “different.” Gary added that the “angle has increased from the original segment and the new segment appears to be slightly longer.” As seen in Figure 4.26, the angles to which he was referring are angle BAD and angle BAG. Hal wrote that the new segment was in a different position. Both rated themselves a 5 and gave single case inductive justifications for their responses. Gary wrote that “when you look at them, they do not match up” and Hal wrote “I can see that it is different.”

The students were then asked to find the defect of the triangle and write down what they noticed. After measuring to find the defect, Gary’s triangle had a defect of 41 degrees and Hal’s triangle had a defect of 27 degrees. Hal commented verbally and wrote that this is a big defect for a triangle, though he also wrote, “I am not quite sure what would qualify as a ‘large’ defect.” He rated himself a 2 and wrote that he would like more info on hyperbolic geometry. This is a shift in Hal’s reported confidence. For the previous prompt in hyperbolic geometry, Hal reported

Figure 4.26:

Gary – Hyperbolic Parallel Transport



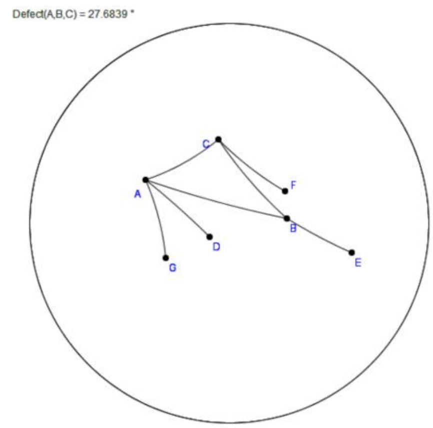
a 5 for confidence. Yet, his confidence dropped dramatically on this prompt. It is especially noticeable as his partner's confidence did not drop. Gary wrote that the "big" defect shows the "triangle is much different than its Euclidean counterpart." Gary rated himself a 4 writing that he wanted to "prove what the defect means" in relation to this problem.

Before moving onto the proof, the researcher asked the students to think longer about what they might notice, and specifically to think about how the defect might relate to the parallel transport activity they had just completed. Hal, within a few seconds of the being asked to think longer, said, "I have a guess. I'm gonna see if, like, this angle [angle GAD] (Figure 4.27) is the same as the defect. It looks like it might be 27 degrees." Hal then proceeded to *validate* measure his diagram. Gary watched him measure, saw the conjecture was correct, and then proceeded to measure his own diagram. Other than measuring the defect, this was the first instance that either of these students used the measurement tool. They had previously made their claims visually based on how the figure looked. Both students rated themselves a 5 for confidence writing that they measured their figures and the angles were equal.

Gary and Hal then proceeded to read the proof. After reading, they both rated their confidence a 4. Gary commented the proof is "full of information" but that parts "seem a little

Figure 4.27:

Hal – Hyperbolic Parallel Transport



confusing." He wrote that his confidence would be raised if it was "dumbed down" and the "algebra explained more." Hal wrote that "it seemed correct, but that there might be exceptions." He wrote his confidence would be raised by "seeing a formal proof" and "more examples."

Regular Polygon Series of Tasks

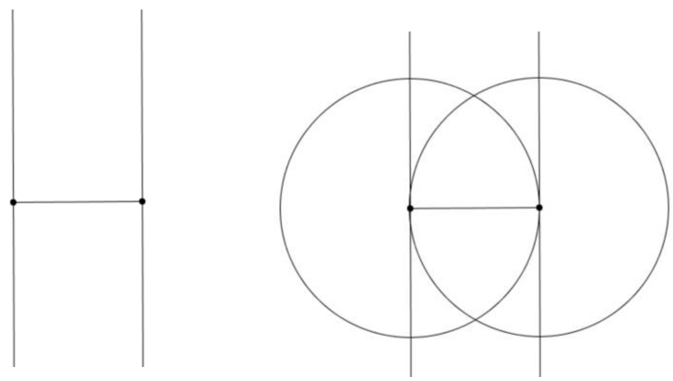
Gary and Hal then proceeded to the regular polygon series of tasks. As they were starting the regular triangle in Euclidean, Hal declared, "I remember how to do this," and proceeded to construct an equilateral triangle. After watching Hal construct his first circle, Gary turned to his own computer and completed the construction as he remembered it. Hal finished first and started dragging the vertices of his construction to make it repeatedly bigger and smaller saying, "Here's how we know. You know how we make these bigger?" He was doing *validation* drags to verify that it was indeed a correct construction. Gary finished and immediately measured his three sides as well as a single interior angle. His sides were congruent, and his measured angle was 60 degrees. Hal leaned over and asked Gary, "Does yours do that?" This prompted Gary to do a drag test as well. When they wrote their responses, Gary mentioned his measurements of the lengths and angle. He also gave a partial deductive justification as well, mentioning the “overlapping circles which shared the same length [radii]”. Gary mentioned his measurements as well as his memory of doing the construction in their geometry course. Both rated themselves a 5 for confidence.

Gary and Hal then began constructing a regular quadrilateral. Both students started by creating a segment and placing circles at the end of the segments. Hal said verbally that he “needs to drop a perpendicular”, but he could not recall how. That verbal prompt caused Gary to erase his circles and use the perpendicular tools to create two perpendicular lines at the end of his segment. Hal looked over and saw the perpendiculars, causing him to ask Gary for help making the perpendiculars. After Gary helped him, Hal said, “Now we make the circles,” and proceeded to make circles at the ends of his segments (Figure 4.28). Hal then finished his regular quadrilateral with a segment across the top and did a *verification* drag to test his construction. Gary followed Hal’s lead and finished his construction though Gary chose to measure the left and top segments of his square to see that they were equal. He also did a verification drag. Gary rated himself a 5, writing a deductive justification that they used circle radii for equal sides and perpendiculars for right angles. Hal rated himself a 5 as well. Hal referred to both deductive justifications (common radii and constructing perpendiculars) as well as previous knowledge that he remembered from constructing the square in his geometry course.

Both students then proceeded to the construction of the regular triangle in hyperbolic geometry. As expected, both students used the same construction from Euclidean geometry. Gary then proceeded to measure his construction by measuring two sides and two angles, whereas Hal decided to drag his image around the screen. This was similar to the Euclidean case where Gary measured and Hal dragged. After he dragged, Hal mentioned he was not sure of the interior angles. He had used congruent radii for congruent lengths so he knew the sides were

Figure 4.28:

Gary and Hal – Beginning of Regular Quadrilateral in Euclidean Geometry

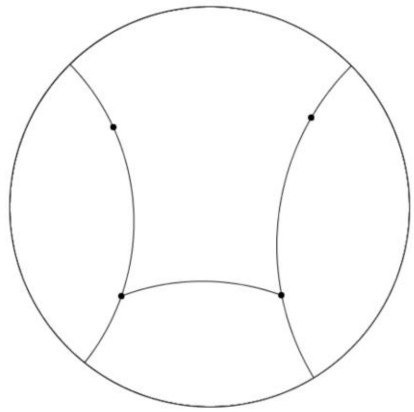


congruent, but he was unsure equal side lengths implied equal angles. This was a shift for Hal. When doing the same construction for Euclidean, Hal did not mention the angles even though the construction steps were the same. Hal was about to return to the worksheet to check the definition of regular shapes, but as he reached for his packet he noticed Gary has measured his angles and this confirmed for Hal the angles were congruent. Hal stopped looking for the definition of regular polygons and proceeded to measure his own three angles to verify they were the same. Both students rated themselves a 5 mentioning the use of circles to create congruent segments and their measurements as justification for the angles.

The students were then asked about whether regular quadrilaterals exist in hyperbolic geometry. After a brief discussion, they both decided that regular quadrilaterals would not exist. Hal led the conversation making the argument that if you have two perpendiculars to the bottom, the top angles would be “way different.” As he made this argument, he did not use the software, but he did point at the screen and draw with his finger what he thought it may look like. Figure 4.29 is an approximate representation of what he drew with his finger. His mental image convinced him it was impossible. Gary agreed with him. Hal rated himself a 4 writing down as his justification that the top angles would be less than 90 degrees. To raise his confidence, he

Figure 4.29:

Hal – Mental Image of What a Regular Quadrilateral Looks Like in Hyperbolic Geometry



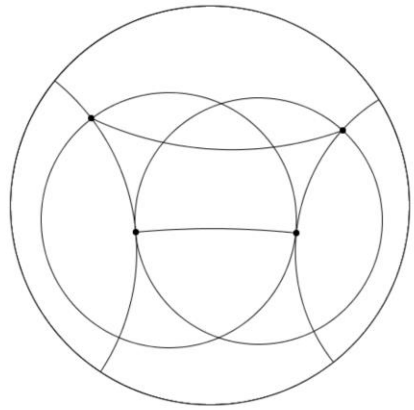
wrote he would either like to try the construction or see a formal proof. Gary rated himself a 2 for confidence writing that he was "unsure if this is correct." Like Hal, he wrote that he would like to be able to prove or disprove it.

Gary and Hal then started on the construction of a regular quadrilateral. They began by repeating the construction they had used in the Euclidean case. Hal declared, "Woah, that's wild" after adding his top line, clearly not expecting the top line to look the way it did (Figure 4.30). He immediately began to drag his image. Gary finished up, and he began to drag his image around as well. While Hal's dragging had no discernable pattern, Gary seemed specifically trying to "square up" his image. He was trying to visually get all four sides to have the same length. This resulted in him 'centering' his image in the Poincaré disc. Gary was interrupted when Hal declared they should measure their figures. Gary then measured all the sides and all the angles of his figure. Hal proceeded to measure just the sides. In both bases, their measurements told them the sides and angles were not congruent. Rather than undo steps of their construction, they both decided to start a new document.

Hal's second construction was based on an alternate idea for constructing the square. Rather than using two perpendiculars and two circles as they had just done. Hal decided to start

Figure 4.30:

Hal – Attempted Hyperbolic Regular Quadrilateral

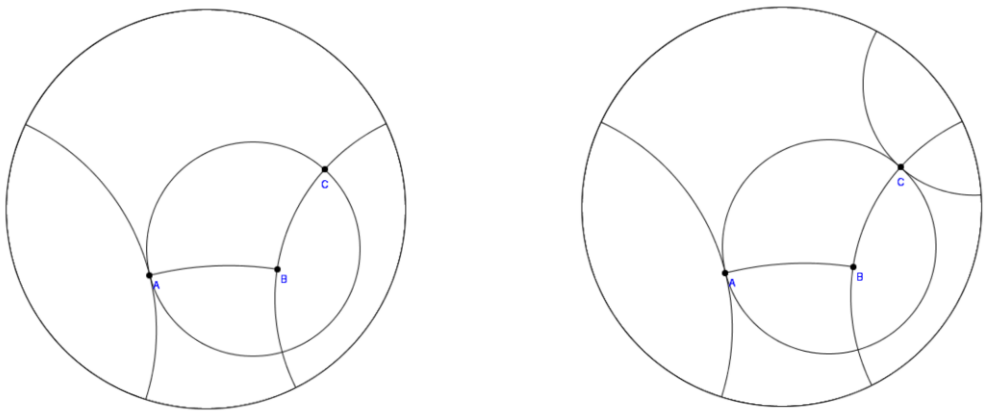


with three perpendiculars and a single circle. Right before he constructed his third perpendicular (left side of Figure 4.31), he said out loud, “I don’t think they’re gonna touch.” He saw his prediction was correct after constructing the third perpendicular (right side of Figure 4.31). As with his previous attempt, this would have been a correct construction in Euclidean geometry, but in hyperbolic it does not work. Hal did begin to wander drag and found that by moving point B he was able to close his quadrilateral (Figure 4.32). However, he immediately recognized the new angle was less than 90 degrees. He used this to make the claim that you can only have three 90 degree angles. Before writing this down, he did check this with a purposeful drag to see if he could make the fourth angle 90 degrees. He found that shrinking his figure got the angle close but not equal to 90 degrees.

Meanwhile Gary had been working on his second attempt. He eventually ended with a construction equivalent to his first construction but rotated 90 degrees. It is unclear if Gary realized his new figure was the same construction as his last. After two attempts, they agreed to write down that they had not constructed a regular quadrilateral, citing their measurements as

Figure 4.31:

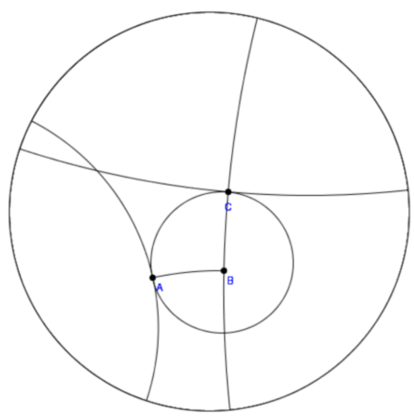
Hal – Second Attempted Hyperbolic Regular Quadrilateral



Before Third Perpendicular　　　　After Third Perpendicular

Figure 4.32:

Hal – Closing His Hyperbolic Quadrilateral



justification their figures were not regular. They both rated themselves a 5 knowing their quadrilaterals were not regular.

Observed Shifts with Gary and Hal

Gary had a noticeable shift between the two tasks in how he used the tools. When working on the parallel transport series of tasks, Gary did not use either of the measurement tools or the dragging tool. Gary made claims about lengths and angles, but these claims, in both Euclidean and hyperbolic, were either all made visually based on perception or made deductively. This changed when Gary started the regular transport task. For the regular transport, Gary measured and dragged more than any other student except for Beth. Even if Gary made an initial conjecture deductively or based on perception, Gary chose to measure before he wrote down his final claim.

Hal had a noticeable shift in confidence between the two tasks. When working on the parallel transport tasks, there were two tasks where he reported low confidence. He reported low confidence when making a predication and then when he was asked to notice something about the defect. In the second instance, he made a claim, but his confidence reflected unsureness about whether it was the correct claim to be making. For the regular polygon series of tasks, however, Hal reported high confidence throughout the task. Even when he was not able to construct a regular polygon in hyperbolic geometry, Hal still reported a high confidence. He knew the answer was wrong, but he was confident it was wrong.

Also, when working on the regular polygon series of tasks, there was a noticeable shift in how Hal justified his claims. In Euclidean geometry, Hal did validation drags of each of his shapes, but he did not take any measurements. It was not until Hal was working in hyperbolic geometry that he began taking measurements. It is worth noticing what Hal chose to measure. As per his written response, Hal knew the sides were correct as he used congruent radii. Because of that, he did not measure the sides. Rather he only measured the angles. This shift is more noticeable as his partner Gary was careful to take measurements of both sides and angles, even in the Euclidean constructions.

Students Ivy and John

The last pair of students were Ivy and John. This was the only pair where both students were math minors. Ivy was a senior and John was a junior and both students were going into elementary education. John struggled throughout the geometry course but was a hard worker and would often come for help. Ivy was always a strong participant in class, though she also struggled. These students had worked together previously.

Parallel Transport Series of Tasks

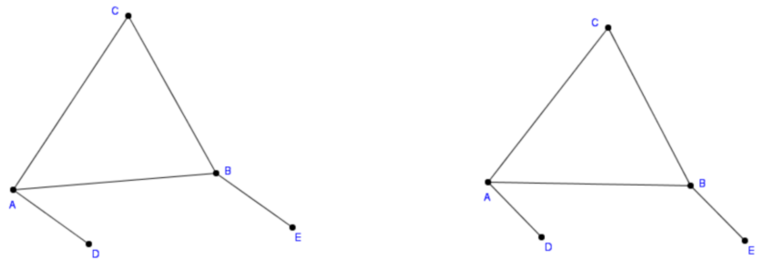
Ivy and John began the first part of the parallel transport task by creating two triangles in Euclidean geometry. As John added the segment to be transported, he leaned over to Ivy asking "Which direction do you want to go?" John was ensuring their on-screen figures were roughly similar. Ivy agreed with this and made her figure in a similar manner (Figure 4.33). Both students wrote the segments were the same length and the angle was the same as well. John rated himself a 5 writing that "translations do not change the length or shape of something" and that "it just moves the object in a direction." Ivy rated herself a 5 stating she was confident because "there's no right or wrong answer about your noticings."

Both students then proceeded to the next part of the task which was to finish the parallel transport. After seeing the segment return to its original position, both students commented the first and last segments were the exact same. John wrote the segments were the same length and angle, rating himself a 5, and again commenting that translations do not change shapes or angles. Ivy did not write about length or angle, only writing they were "the exact same." She rated herself a 5 writing she was very confident because she "wrote what I saw."

Ivy and John were then asked what they thought would happen to the transported segment in hyperbolic geometry. John began by discussing how the transported segments would be curvy. Ivy reminded him the triangle segments would be curvy as well. John "drew" the image on the screen with his finger to visualize what might happen. Ivy leaned over and added to

Figure 4.33:

Ivy and John – Similar Triangles for Parallel Transport



Ivy *John*

his image talking about how the curves might change direction as they transport. John eventually drew their combined mental image onto his worksheet (Figure 4.34).

Both students wrote the transported image would not "land on itself" and that it would "curve in the opposite direction." They both rated themselves a 3. Ivy wrote her confidence was low because it had been a while since she had done hyperbolic geometry and her confidence could have increased with "more practice." John also rated himself a 3 writing he thought he "had the concepts of hyperbolic geometry down", but that he would "have a hard time picturing which way the segment will curve." He wrote his confidence would be raised by working with the software.

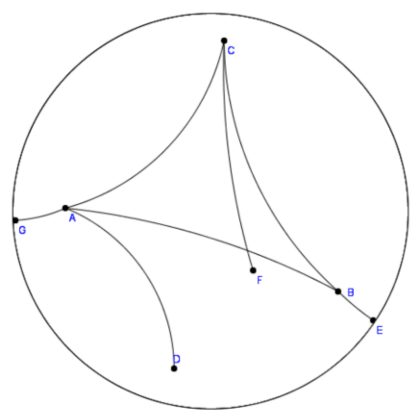
John and Ivy then were asked to try the parallel transport in hyperbolic geometry. As compared to the Euclidean case where John and Ivy purposely created similar triangles, this time there was no evidence they attempted to create similar figures. Ivy finished first, and while John was finishing his figure, Ivy began to drag her figure. She quickly tried making a bigger triangle, placing the three vertices near the edges. As she did this, John finished up and looked over to see what Ivy was doing. Ivy had her figure arranged as in Figure 4.35. She commented that the first and last segment were in fact curving in the same direction. John surmised that if the "triangle is

Figure 4.34:

Ivy and John – Prediction of Hyperbolic Parallel Transport

Figure 4.35:

Ivy – Hyperbolic Parallel Transport

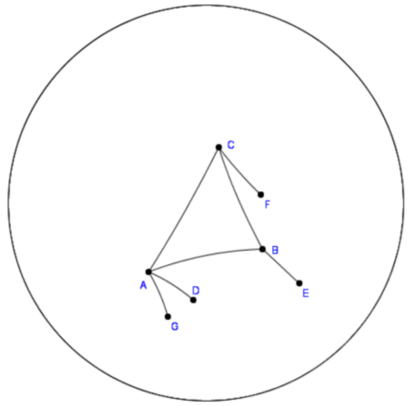


the same distance from the ends, it's gonna change the angle and distance." He saw that AD and AG looked like very different lengths. Ivy commented that this still does not explain why they are curving in the same direction. They then wrote down that their "prediction is wrong." Before writing though, John made one more observation based on his image (Figure 4.36). Their short discussion is below:

> John: "Mine is at this weird point where mine looked like the same length, but if I move this point (A) they're gonna change."
>
> Ivy: "But they are, like, technically the same length."
>
> John: "It just looks like they're not the same length cuz hyperbolic is weird."

Figure 4.36:

John – Hyperbolic Parallel Transport



John saw in his image that the first and last segment appeared to be the same length. But he had also seen earlier that if the triangle vertices are near the edge of the disc, their lengths appear differently. Ivy reassured him that length stays the same and John accepted this. Ivy's knowledge that transformations do not change length overrode John's initial thought that length would change. Both students wrote that the segment stayed the same length, though the curve switched direction. They also both rated themselves a 5 with John mentioning his figure in Geometry Explorer and Ivy mentioning "it's what she saw."

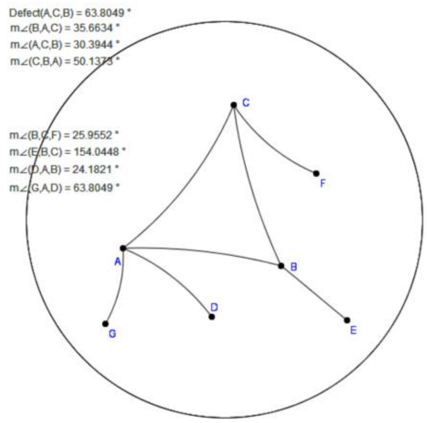
The students were then asked to find the defect of the triangle. They both used the 'Defect' menu option to measure the defect. John noticed and commented they have different defects, but also explained that was because they have different triangles. Ivy then started to measure the individual angles in the triangle and John asked, "Wait, are you going to do them all separate?" Ivy responded, "I kind of want to". She was hesitant about using the 'Defect' menu option. This was likely because during her geometry course when discussing defect, we had measured the angles individually rather than using the tool. She was validating that the 'Defect' menu tool does what she thought it would. Both she and John proceeded to measure the individual angles and computed the defect by hand.

After checking that the measurement from the ‘Defect’ menu option matched their computation, John asked Ivy what they noticed. Ivy suggested that if you flatten the triangle, the defect goes to zero showing John how this worked in a validation drag. She noted that she cannot shrink the defect past 0.02 as the software will not let her make the triangle completely flat. They both wrote how the defect got closer to zero as the triangle got closer to a line. Both rated themselves 3. Ivy wrote that her confidence was lower because she did not remember much about defect. John wrote that he had “little knowledge of why the defect would go down.” To raise their confidence, both wrote they would like to learn more about defect.

Before the students read the proof, the researcher asked the students to think longer about what they might notice, and specifically about how defect might relate to the parallel transport activity they had just completed. John began to *wander* measure various angles around the outside of the triangle. He started by measuring angles clockwise around the triangle (Figure 4.37). He eventually measured angle GAD and exclaimed, “Woah! That’s cool” as he noticed the angle matches the defect. He shared that info with Ivy saying, “See what all that measuring did!” Ivy then measured her angle to verify that the relation held true for her figure as well. Both proceeded to write down that observation. Trying to relate her answer to her previous claim, in addition to writing the defect matches the angle, Ivy also pointed out how the segments coincide as the defect goes to 0. Both also drew images of their triangles on the paper labeling the defect and the angle measurement. They both rated themselves a 4 citing their measurements, and they both wrote they need to learn more about defects to raise their confidence.

Figure 4.37:

John – Wander Measures on Hyperbolic Parallel Transport



John and Ivy then proceeded to read the proof. John read the entire proof out loud. When he was done, John decided to substitute his specific angles into the equations at the end of the proof to see if those equations held true. When he rated his confidence, he rated himself a 4 citing that plugging the numbers into the formula and seeing it 'work' gave him confidence. Ivy rated herself a 1 writing that "parts of the proof are confusing to me" and that it "would help if I got to practice the proof on my own."

Regular Polygon Series of Tasks

John and Ivy then started the series of regular polygon tasks. The first activity in this task was to create a regular triangle in Euclidean geometry. Ivy remembered the "circle trick" from her geometry course and quickly created the construction. John heard her mention the circle trick and then constructed his own equilateral triangle. They both wrote that they successfully created the regular triangle using the circle method. They also both rated themselves a 5 writing that they had used that technique in class.

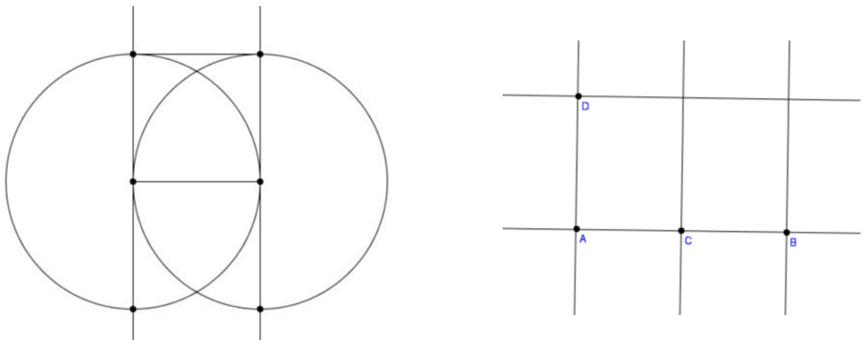
The two students then proceeded to start the construction of a regular quadrilateral in Euclidean geometry. Ivy used a construction with two perpendicular lines and two circles so that each side of the square would match the base (left side of Figure 4.38). After she finished her construction, Ivy measured the two top angles to ensure they were 90 degrees. She did not measure the bottom angles, indicating she trusted they were perpendicular as she used the perpendicular tool. She also measured the side lengths and the top to ensure they were equal. This indicates she already knew that the sides matched the base. Meanwhile, John constructed his figure (right side of Figure 4.38). John constructed point C as the midpoint of A and B. Then perpendiculars were created at A, B and C. Finally, he placed point D on the perpendicular through A guessing on the vertical distance. When Ivy finished her construction, she saw what John was doing and suggested he start over. Before he started over, he claimed he was “close.” She then walked him through the construction she made. On the worksheet, they both claimed their constructions were correct. John mentioned using the perpendicular tool and circles. Ivy mentioned the measurements that she took. They both rated themselves a 5. John mentioned they “practiced with this in class” and Ivy wrote that she was confident “because this is fun to me.”

John and Ivy were then asked to create a regular triangle in hyperbolic geometry. As expected, both students recreated their construction from Euclidean geometry. Without discussing it, John immediately measured the three side lengths and Ivy immediately measured the three angles. Ivy then wrote that she was able to construct a regular triangle, rating her confidence a 5 stating she was confident because she measured. John was about to write his answer, but then decided to also measure two of the three angles to be sure. He also rated himself a 5. He wrote he measured to “double check” the angles and segments were the same – a clear indication it was *validation* measurement.

The two students were then asked about whether regular quadrilaterals exist in hyperbolic geometry. Ivy initially responded verbally that she vaguely remembered this from her geometry

Figure 4.38:

Ivy and John – Constructions of Regular Euclidean Quadrilaterals

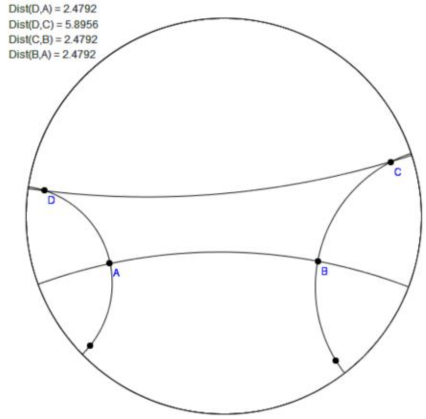


course. John agreed and claimed to remember writing about it for one of his projects. Though these students did not study regular quadrilaterals in their course, this mistaken memory was enough to give them confidence that regular quadrilaterals exist. John rated himself a 5 and Ivy rated herself a 4.

Both students then attempted to recreate their construction from Euclidean geometry. In doing so, they both inadvertently created Saccheri quadrilaterals, rather than regular quadrilaterals (Figure 4.39). They had three sides that matched and two perpendiculars at the base, but the top angles were not right angles, and the top segment was longer. After they made their figures, both students measured all four sides, after which Ivy declared it "didn't work." John tried dragging different vertices of his figure to see if that would have any effect on the side lengths, and while the numbers changed, it was always three equal sides and one that was different. While John dragged, Ivy decided to measure a single base angle, one of the perpendiculars she had constructed. Both of their measurements and John's dragging were enough for the students to decide hyperbolic regular quadrilaterals do not exist. John wrote that only three sides were equal, rating himself a 5 that they do not exist. Ivy wrote that "all the

Figure 4.39:

Ivy – First Attempt at Regular Hyperbolic Quadrilateral



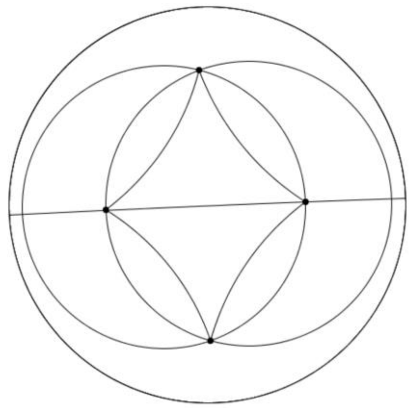
angles were the same, but one of the lines was longer than the other three." She also rated herself a 5.

The researcher then prompted the students to think longer about the question, suggesting they go back to the previous activity about regular triangles to see if there was something they might learn from that. Ivy went back to the triangle she had constructed and stated, "there is nothing left to measure." She asked John, "What's interesting?" Not giving John a chance to answer, she answered her own question stating, "Is it that the angles don't add up to 180 degrees? Because we already knew that." She then realized that the regular quadrilateral "might not be 360." John then agreed the angle sum cannot be 360 and points out this means they cannot use perpendiculars. They both then try to think back to class, convinced they had previously done this during their class with John saying, "There was a way to do it, and we learned it in class."

Ivy's next attempt was to create two regular triangles on top of each other as in Figure 4.40. After she saw her construction, she *validated* measured all four sides and proudly declared, "I figured it out!" Despite her declaration, she proceeded to measure the angles and realized the

Figure 4.40:

Ivy – Second Attempt at Hyperbolic Regular Quadrilateral



angles did not match. John, meanwhile, decided to keep adding circles and segments until he found something that looked roughly like a regular quadrilateral (Figure 4.41). At various times, he measured sides and angles to check, but he remained unsuccessful. Eventually, John and Ivy decided to maintain their original written claim that regular quadrilaterals do not exist.

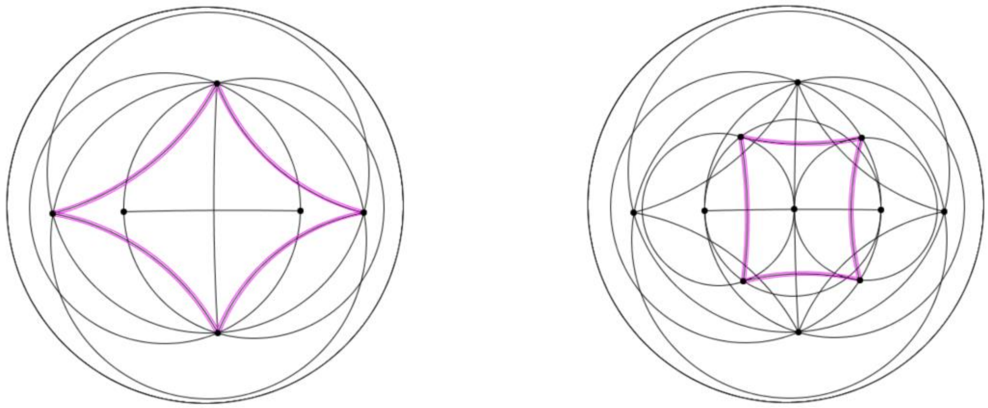
Observed Shifts with Ivy and John

Ivy was unique among the students with her interpretation of confidence. For the parallel transport activity, when Ivy was in Euclidean geometry, she rated herself 5 for confidence and wrote she was confident "because there is no right or wrong answer about your noticings." She also wrote she was confident because she "wrote what she saw." However, when she switched to a hyperbolic parallel transport, her confidence and comments shifted. Now she started reporting 2s and 3s writing, "I haven't done anything in hyperbolic geometry" and "I just learned what a defect is." Despite her earlier writings that there is no right or wrong answer when "noticing", in hyperbolic she rated herself lower even when the question asked her to notice.

When working on the parallel transport task, Ivy's reported confidence remained relatively high with 4s and 5s, but there was a shift in the type of written comments she made. When working in Euclidean geometry, she wrote statements saying she was confident because "I

Figure 4.41:

John – Continued Attempts at Hyperbolic Regular Quadrilateral



like to make triangles" and "this is fun to me." Once Ivy switched to hyperbolic geometry, however, she wrote she was confident because "I measured" and because "I know the definition of a regular shape." Her reasons for confidence shifted from a more general self-efficacy to specific confidence about the claim itself that she was making.

John had a noticeable shift in the types of justifications he made between the two tasks. When working on the parallel transport, John made the most deductive arguments of all the students in the study. This is especially noticeable as this series of tasks had the least number of deductive arguments being made. However, when John switched to the regular polygon series of tasks, John's arguments switched away from deductive. This is opposite from most other students. When working on the regular polygons, John did not give any deductive justifications either verbally or on paper. Rather his deductions were mainly single case inductive arguments or arguments based on previous knowledge. This is not to say John did not know any deductive justifications, only that he did not express those justifications.

CHAPTER 5: RESULTS

This chapter presents on the findings from the data as they relate to the research questions. The first half of the chapter presents the data in a tabular method as it relates to the research questions. The second half of the chapter presents individual and pairwise observations that focus on the individual students. In both halves, focus will be given to what features of the DGE students are using, what justifications students are using, and how students are self-reporting their confidence when completing DGE related tasks in Euclidean and hyperbolic geometry. Additionally, these three factors are matched pairwise to answer the fourth research question about relationships that exist among the three factors being studied.

The students responded to a series of 11 prompts across two series of tasks. Six of those prompts were from the parallel transport task and five of those prompts were from the regular polygons task. For two of these prompts (P4 and P5), some of the pairs were asked to think longer about the questions, and to provide an additional, secondary response to the original prompt. This occurred when the students recorded a basic response to the question. As an example of what might be considered a basic response, consider the response given by Eve and Fran in the parallel transport task, responding to what they noticed about the transported segment in hyperbolic geometry (prompt P5). Eve and Fran both responded that the transported segment is “not the same” as the original segment. After Eve and Fran wrote their response for this prompt, they were asked to think longer about this question so that they might give a more specific answer. Because of these interventions, this resulted in some pairs responding to a total of 13 prompts, compared to the original 11. When counting total responses (including the additional responses), this resulted in the 10 students giving a total of 120 responses.

Research Question One - Tool Usage of DGE When Completing DGE Related Tasks

The three main features of DGE observed in this study include the dragging tool, the angle measurement tool, and the length measurement tool. While DGE has a wide variety of available tools, the two measurement tools and dragging tool are the three main tools that distinguish DGE use from that of traditional paper-based geometrical activities. These tools reorganize, rather than amplify, what students can do on paper (Pea, 1987). Of the thirteen total prompts, there were ten prompts, including the two 'think longer' prompts, (P4.5 and P5.5) that reasonably allowed for students to make use of the measurement and dragging tools. Of the original ten prompts, there were two prompts that asked students to make a conjecture about what they think might happen when they use the software. As such, these questions did not warrant the use of the DGE. Additionally, there was one prompt asking students to read a proof. This prompt also did not immediately warrant the use of DGE, though it did not specifically rule it out. Removing those three prompts, and including the two 'think longer' prompts, the prompts provided 90 responses where students had potential reason to use the tools of the DGE. Fifty of those responses were for the parallel transport task and forty were for the regular polygon task.

For the following counts, the number of times a student used a given tool was not taken into consideration. Rather it was whether the student used the tool at least once when responding to the prompt. If a student measured one or five lengths, it was still only counted once to indicate the student used that tool. Likewise, if a student was dragging, the number of drags was not being counted. Rather it was recorded that dragging took place.

For the parallel transport task, the results showed the angle measurement tool was used to help answer 12 of the 50 responses, and the length measurement tool was used to help answer 3 of the 50 responses (Table 5.1). From this, we saw that during this task students more often used

Table 5.1:

Number of Responses in Which Students Used the Measurement Tools

Angle Measurement in Parallel Transport Task

	Wandering	Validation	Wandering & Validation	Total
Euclidean	0	2	0	2
Hyperbolic	5	5	0	10

Angle Measurement in Regular Polygon Task

	Wandering	Validation	Wandering & Validation	Total
Euclidean	0	6	0	6
Hyperbolic	0	16	1	17

Length Measurement in Parallel Transport Task

	Wandering	Validation	Wandering & Validation	Total
Euclidean	0	0	0	0
Hyperbolic	0	3	0	3

Length Measurement in Regular Polygon Task

	Wandering	Validation	Wandering & Validation	Total
Euclidean	0	8	0	8
Hyperbolic	1	12	3	16

Note: 'Wandering', 'Validation', and 'Wandering & Validation' are distinct categories. In other words, 'Wandering' would mean the student in that response *only* used a wandering measurement, whereas 'Wandering & Validation' would mean the student used both types of usage.

the angle measurement tool than they did the length measurement tool. There are two reasons for this. The first reason for greater angle measurement was the nature of the task itself. By the end of the task, students were being asked to find the defect of the triangle and see what they noticed. As defect is related to angles, this prompted the students to turn their attention to the angles in the figures which naturally caused the students to measure the angles more.

The second reason for the greater number of angle measurements was students' understanding of how transformations work. Students learned in their geometry class that transformations in Euclidean geometry preserve lengths and angles. When completing the task, students saw they were transporting a single *segment*. As such, some students correctly assumed the length of the segment would be a preserved property and did not have incentive to measure the length of the transported segment. As an example, when Ivy was asked what she noticed about the final segment in hyperbolic geometry (prompt P4), she argued that even though the lengths looked different, translations preserved length. Because she knew this, Ivy did not have a need to measure. Thus, even before students were later asked about defect, these students were measuring angles at a greater rate than lengths.

For the regular polygon task, the results showed the angle measurement tool was used to help answer 23 of the 40 responses and the length measurement tool was used to help answer 24 of the 40 responses (Table 5.1). Despite fewer prompts, the regular polygon series of tasks had more prompts with instances of measurement tool usage than the parallel transport series of tasks. The angle and length measurement tools were roughly equivalent in their usage. There was, however, a large difference between Euclidean geometry and hyperbolic geometry in the number of prompts where students used measurement tools. Of the responses where students were using a measurement tool, 33 of the instances were in hyperbolic geometry, while only 14 were in Euclidean geometry. This difference was to be expected based on how hyperbolic geometry appears to distort angles and distances causing uncertainty within the students.

From Table 5.2, the results showed the dragging tool was used to help answer 15 of the 50 prompts in the parallel transport task and 18 of the 40 prompts in the regular polygon task. Consistent with the measurement tools, there were more prompts in the regular polygon series of

Table 5.2:

Number of Responses in Which Students Used the Dragging Tool

Dragging in Parallel Transport Task

	Wandering	Validation	Wandering & Validation	Total
Euclidean	0	1	0	1
Hyperbolic	5	3	6	14

Dragging in Regular Polygon Task

	Wandering	Validation	Wandering & Validation	Total
Euclidean	0	5	0	5
Hyperbolic	6	7	0	13

Note: ‘Wandering’, ‘Validation’, and ‘Wandering & Validation’ are distinct categories. In other words, ‘Wandering’ would mean the student in that response *only* used a wandering measurement, whereas ‘Wandering & Validation’ would mean the student used both types of usage.

tasks where students dragged, despite the regular polygon series of tasks having fewer prompts than the parallel transport task. Also consistent with the measurement tools, of the 33 responses where students used dragging, 27 of those responses occurred within hyperbolic geometry, with only six responses in Euclidean geometry.

In addition to looking at how often students used the three tools, tallies were also completed to see how the students used the three tools. While research has been done dividing dragging and measurement modalities into a multitude of smaller categories (Baccaglini-Frank & Mariotti, 2010; Olivero & Robutti, 2007), this research simplified those categories into two broad categories of wandering and validation usage. While there were instances of students using the tools in both types of usage, the results showed most tool usages were for validation purposes (Table 5.3). This was especially true in Euclidean geometry where there were no instances of

Table 5.3:

Number of Responses in Which Students Used Validation and Wandering Tool Usages

	All Tool Usage			
	Wandering	Validation	Wandering & Validation	Total
Euclidean	0	22	0	22
Hyperbolic	17	46	10	73
Total	17	68	10	95

Note: 'Wandering', 'Validation', and 'Wandering & Validation' are distinct categories. In other words, 'Wandering' would mean the student in that response *only* used a wandering measurement, whereas 'Wandering & Validation' would mean the student used both types of usage.

wandering tool usage. As students were more familiar with Euclidean geometry, it was expected wandering tool usage would be minimal. Another factor for validation tool usages to outnumber wandering tool usages was due to the follow-up nature of wandering dragging. In the instances where students did wandering dragging, this was often followed up with validation measurements or drags to validate their conjecture. Because most wandering drags were paired with a validation tool usage, this follow-up approach contributed to a high number of validation usages.

In summary, the results showed that students used all three tools for both series of tasks, but that students used the three tools to answer more prompts in the regular polygon series of tasks than they did the parallel transport series of tasks. The data also showed, as expected, students used the three tools to answer more prompts in hyperbolic geometry than they did Euclidean geometry. Lastly, for these series of tasks, students were using the three tools more often in a validation mode as compared to a wandering mode.

Research Question Two – Justifications Used When Completing DGE Related Tasks

Of the eleven initial prompts that were posed to students, one of them did not require a justification. Specifically, the parallel transport task P6 asked students how confident they were after reading a proof. As the task had students reading a proof as justification, students would not be giving their own justification. Removing that prompt gives 100 responses where students were asked to give a justification for the claims they were making. Additionally, there were two of these eleven prompts that asked students to make a prediction about what may happen. Specifically, prompt P3 asked students what they thought would happen to the final segment in a hyperbolic parallel transport and prompt R4 asked students if they thought a regular quadrilateral was possible in hyperbolic geometry. By the nature of these two questions, students were not using the DGE and would not have provided inductive justifications. As such, these questions were removed from the analysis. Taking out these two prompts leaves 80 prompts where students could have reasonably given any of the five justifications. Lastly, there were instances where students' original responses to a prompt were either overly general or did not ready the student for the next part of the task. For instance, in the parallel transport task, when responding to prompt P4 asking what they noticed after a hyperbolic parallel transport, Eve responded by noting "the first and last segment are not the same line." This prompted the researcher to ask the students to think longer about the question. With additional prompts by the researcher, there were a total of 90 responses.

As specified in Chapter 3, justifications were coded into five categories: previous knowledge, Euclidean comparison, single case inductive, multiple case inductive, and deductive arguments. Many of the responses contained more than one type of justification, either written or given orally as students discussed their work. From Table 5.4, of the justifications given for the

Table 5.4:

Justifications Given by Students for Each Prompt

Prompt	Geometry	Single Case Inductive	Multiple Case Inductive	Deductive	Compare to Euclidean	Previous Knowledge
P1	Euclidean	8	2	4	0	0
P2	Euclidean	9	1	1	0	0
P4	Hyperbolic	4	6	2	0	0
P4.5	Hyperbolic	2	2	0	0	0
P5	Hyperbolic	3	5	0	0	1
P5.5	Hyperbolic	7	1	0	0	0
R1	Euclidean	2	2	8	0	8
R2	Euclidean	5	1	5	0	2
R3	Hyperbolic	7	3	1	1	0
R5	Hyperbolic	1	9	0	0	0
Total		48	32	21	1	11

Note: There are more justifications than responses as some responses included multiple justifications.

90 responses, 80 of those responses contained inductive justifications, with 48 single case inductive justifications and 32 multiple case inductive justifications. Because students were using a DGE, the high number of inductive justifications was expected. However, since students had the dragging tool available, it was not expected to see single case inductive arguments more prevalent than multiple case inductive arguments. Of the 80 responses with inductive arguments, 12 of those responses also included a deductive justification, 5 referenced previous knowledge, and 1 response had a specific Euclidean comparison. So, while there were some instances of multiple justifications when making inductive arguments, most inductive arguments were made in isolation.

There were 10 responses that did not include inductive justifications. Of those 10 responses, 4 included only deductive justifications, 1 referenced previous knowledge, and 5 were a combination of deductive justifications while also referencing previous knowledge. Overall, 21 of the responses included a deductive argument of some sort. Of those 21 responses, 18 of those deductive justifications occurred within Euclidean geometry. Moreover, 13 of these 21 deductive arguments occurred when constructing both the regular triangle and regular quadrilateral in Euclidean geometry. In hyperbolic geometry, the construction for the regular triangle is the same and the deductive reasoning is the same, yet only one student mentioned a deductive justification for their construction. While it is possible students may not have recognized the similarity and not determined any deductive justifications, it is also possible students thought the similar reasoning was not worth repeating from the Euclidean case.

In summary, these students overwhelmingly used inductive arguments when completing DGE related tasks. We also saw that even with dragging as a tool, students more often relied on single case inductive arguments. There were also very few comparisons to Euclidean geometry or references to previous knowledge. Lastly, students gave more deductive justifications in Euclidean geometry than they did in hyperbolic geometry.

Research Question Three – Students' Self-Reported Confidence When Completing DGE Related Tasks

When responding to each prompt, students were asked how confident they were in their response to the prompt by rating themselves on a five-point Likert scale with 1 = "not confident," 2 = "somewhat not confident," 3 = "neutral," 4 = "somewhat confident" and 5 = "very confident." As in the previous sections, the prompts discussed here are limited to the prompts that involve DGE use. In total, that narrows the list to the 90 prompts where students reported confidence related to their justification when using DGE. Of the 10 students, one student, Ivy, had a unique interpretation of confidence. She interpreted confidence to mean her general confidence about mathematics rather than confidence about the specific prompt to which she was responding. For instance, when working on the regular triangle, she rated herself a 5 on one prompt because “it is fun to me” and a 5 on a different prompt because “I like to make triangles”. While her interpretation was interesting, these responses did not fall in line with the other 9 students who were rating their confidence in relation to their answer and their justification for that answer. Removing Ivy’s responses gave 81 responses for which this analysis will be based.

Of these remaining responses, the raw numbers are reported in Table 5.5. The results showed that 90% of responses were “somewhat” and “very” confident. In Euclidean geometry, 100% of response were either “somewhat” or “very” confident. In hyperbolic geometry, the results showed that confidence had decreased slightly. Only 82% of responses were “somewhat” or “very” confident. There was also a difference between the two tasks themselves. In the parallel transport task, 84% of the responses were “somewhat” or “very” confident compared to 97% of responses being “somewhat” or “very” confident in the regular polygon task.

The difference between Euclidean and non-Euclidean geometry was not surprising. As has been mentioned, we know from the VanHiele levels (Burger & Shaughnessy, 1986) that students struggle with reasoning in non-Euclidean axiomatic systems. We also know that the nature of non-Euclidean geometry provides its own challenges in how angles and segments appear. Because of this, there was a general hesitancy among students when in hyperbolic geometry. For instance, when working on the parallel transport task in hyperbolic geometry, Ann used the software and measured the initial and final segment, noting they were the same per the measurements given by the software. Yet, when rating her confidence, she rated herself a 3

Table 5.5:

Reported Student Confidence Categorized by Type of Geometry and by Task

	Overall (n = 81)	Euclidean (n = 36)	Hyperbolic (n = 45)	Parallel Transport (n = 45)	Regular Polygon (n = 36)
Very Confident (5)	67%	81%	56%	56%	81%
Somewhat Confident (4)	24%	19%	27%	29%	17%
Neutral (3)	6%	0%	11%	9%	3%
Somewhat Not Confident (2)	4%	0%	7%	7%	0%
Not Confident (1)	0%	0%	0%	0%	0%

Note: These 81 responses do not include Ivy's responses.

writing that "she *thinks* [emphasis added] they are the same." Comparatively, in Euclidean geometry, after transporting the initial segment and without measuring, Ann wrote they were the same length, and rated herself a 5 without measuring. In hyperbolic, Ann had software telling her they were the same and her confidence was reported as a 3. In Euclidean, without software confirming her response, she rated herself a 5. Furthermore, when working on her hyperbolic regular quadrilateral, Ann even said specifically, "the numbers tell me it is correct, but I am skeptical."

The results also showed the reported confidence was much higher in the regular polygon task than it was in the parallel transport task. This most likely stems from the tasks themselves. The construction tasks had very clear goals – that of making a figure. It was straightforward for students to verify whether their construction was regular or not – especially when having tools available to make measurements. As such, students were able to report their confidence in both the positive and the negative based on those measurements. For instance, Fran reported a confidence level of 5 on constructing the regular polygon in hyperbolic geometry noting specifically that she did not create one. On the other hand, the parallel transport task was more open ended. When asking students “What do you notice about…,” students often gave a statement that was justified through software measurement and/or dragging but was accompanied with a low rating for confidence. For instance, when looking at defect in hyperbolic geometry, John noticed that as the defect approaches 0, the first and last segment “fall onto each other” (i.e., the triangle collapse) and he rated himself a 3. It is not clear from his written response or his conversation whether his rating was due to hesitancy about the correctness of his observation itself or hesitancy about whether it was the correct observation. This issue of two interpretations of confidence will be discussed further in Chapter 6.

In summary, students generally reported high confidence throughout both activities. However, their reported confidences were higher when working in Euclidean geometry as well as when they were working on the regular polygon task. Additionally, there are instances where students appeared to be using slightly different interpretations of what confidence means.

Research Question Four - Relationships

This next section addresses the fourth research question – the relationships that exist among the three previously discussed factors. This section is broken into three sub-sections which will match the factors pairwise. There will also be a final section where all three factors will be considered at one time.

Justification and Confidence

The first relationship to be addressed is that of confidence and justifications. As stated in the previous section on confidence, Ivy had a unique misinterpretation of confidence, so for this section as well, her results will be omitted. It was reported in the previous section that students overwhelmingly gave inductive justifications. In particular, 80 out of 90 justifications as originally reported included inductive justifications, or 72 out of 81 once Ivy is removed. For these 72 remaining inductive justifications, the average reported confidence was a 4.4. In comparison, the overall average confidence across all responses was a 4.5. Students' use of inductive justifications corresponded to a slightly lower confidence rating.

In contrast to the inductive justifications, when students made deductive or partial deductive arguments, those corresponded to slightly higher confidence levels being reported. Of the 19 instances (out of 81 prompts) where students gave deductive or partial deductive responses, the average confidence rating was a 4.8, slightly higher than the overall average of 4.5. To be specific, of the 19 instances with deductive justifications, students rated themselves a 5 in 17 of those responses. Though this data sample is small, there is some directionality in this correlation. While students making a deductive argument corresponds with a higher confidence, a high confidence did not always correspond to making deductive arguments. Also, for discussion in Chapter 6, students *making their own deductive argument* corresponded to higher confidence, but when reading a supplied deductive argument, their confidence was much lower.

As stated earlier, out of the 81 responses, 90% of the responses included a 4 or 5 rating for confidence. Of the other 10% of responses, those with ratings of 1, 2, or 3, there were 4 students who used a multiple case inductive justification, 1 student who used both a multiple case inductive justification and a partial deductive justification, and 3 students who used a single case inductive argument. In this small sub-sample of low confidence ratings, multiple case inducive justifications were most common. This was contrary to the overall justifications where single case inductive arguments were more common. Overall, 53% (43 of 81) of responses had single case inductive justifications while 36% (29 of 81) of responses had multiple case inductive justifications. In the small sub-sample of low confidence responses, 63% (5 of 8) had multiple case inductive arguments, while 38% (3 of 8) had single case inductive arguments. This result was opposite of what would be expected. In general, more cases would seem to imply students should have stronger confidence, not less confidence. A potential reason for this apparent discrepancy (beyond the small sample size) was the nature of the task itself. Seven of the 8 responses were in the parallel transport task in hyperbolic geometry, with 5 responses all for the same prompt (P5). This prompt was an open-ended question, with students spending much time dragging looking for a relationship. This gave them evidence for multiple case inductive justifications, but the open-ended nature of the question still gave them hesitancy and they scored themselves low. These low prompts, and this disparity, will be discussed in more detail in Chapter 6.

Tool Usage and Confidence

The next set of relationships is between confidence and tool usage. As shown in Table 5.6, the results showed the prompts with the two highest observed tool usages were the regular polygon tasks in hyperbolic geometry (R5 and R3). These two prompts had average reported confidence levels of 4.6 and 4.7. This put them in the upper mid-range of average confidence levels reported for all the prompts. It is also worth noting that these two prompts had over twice the observed tool usage as any other prompt. The prompts with the two lowest observed tool usages were the prompts asking what students noticed about the segments in a Euclidean parallel transport (P1 and P2). These had average reported confidence levels of 4.7 for both. Again, this put them in the upper mid-range of the confidence levels reported.

While overall tool usage did not correspond in a direct relationship with confidence levels, there was a relationship once the prompts were split into each geometry. Looking at Table

Table 5.6:

Observed Tool Usages and Average Reported Confidence Levels Sorted by Total Usages

Prompt ID	Prompt	Tool Usages Observed	Average Reported Confidence Levels (n=9)
R5	Did you create a regular quadrilateral (hyper)?	22	4.6
R3	Did you create a regular triangle (hyper)?	19	4.7
R1	Did you create a regular triangle?	9	4.9
P5.5	Think longer about this question (defect).	8	4.1
R2	Did you create a regular quad?	8	5.0
P4	What did you notice happens to final segment in hyperbolic?	7	4.4
P4.5	Think longer about this question (final segment).	6	4.5
P5	What do you notice about defect?	3	3.4
P1	What do you and your partner notice about first segment?	2	4.7
P2	What do you and your partner notice about final segment?	1	4.7

Note: This table does not include Ivy's responses.

5.7, the results showed that within each geometry, higher tool usage generally, though not perfectly, corresponded with higher reported confidence levels. That is, just looking at the Euclidean tasks, higher tool usage corresponded to higher reported confidence levels. The two prompts with most tool usage corresponded to the two prompts with highest average reported confidence. The relationship was not quite as defined in hyperbolic geometry. The results did show the two prompts with the highest tool usage corresponded to the highest reported confidence. Additionally, the prompt with the lowest tool usage corresponded with the lowest reported confidence. While this was a small sample, it is worth exploring in a future study to see if this holds for larger samples.

In addition to looking at total tool usage, this section will also look at how the tools were being used. As stated, tool usages (dragging, angle measurements, and length measurements)

Table 5.7:

Observed Tool Usages and Average Reported Confidence Levels Split by Type of Geometry

Prompt ID	Prompt	Tool Usages Observed	Average Reported Confidence Levels
	Euclidean Geometry		
R1	Did you create a regular triangle?	9	4.89
R2	Did you create a regular quadrilateral?	8	5
P1	What do you and your partner notice about first segment?	2	4.67
P2	What do you and your partner notice about final segment?	1	4.67
	Hyperbolic Geometry		
R5	Did you create a regular quadrilateral?	22	4.56
R3	Did you create a regular triangle?	19	4.67
P5.5	Think longer about this question (defect).	8	4.14
P4	What did you notice happens to final segment in hyperbolic?	7	4.44
P4.5	Think longer about this question (final segment).	6	4.5
P5	What do you notice about defect?	3	3.43

Note: This table does not include Ivy's responses.

were categorized into wandering usages and validation usages. From Table 5.8, for any prompt where the average confidence was above 4.6, the observed tool usages were 100% validation usages. This occurred in 5 of the 10 prompts. Once average confidence fell below 4.6, validation usages dropped substantially, and wandering usages became more common. This was to be expected. If students were only using validation usages, this validation measurement or dragging confirmed their conjecture and gave them confidence. If students were wandering, this implied they were unsure of their conjecture and the students' reported confidence confirmed this.

It is also worth highlighting that of the five prompts with 100% validation measurement, four of those prompts were the four prompts in Euclidean geometry. That is, in Euclidean

Table 5.8:

Students' Reported Confidence Paired with Validation and Wandering Tool Usage

Prompt ID	Prompt	Average Confidence (n = 81)	Number of Validation Usages	Number of Wandering Usages	Percent of usages that were Validation Usages
P1	What do you and your partner notice about first segment?	4.67	2	0	100%
P2	What do you and your partner notice about final segment?	4.67	1	0	100%
P4	What did you notice happens to final segment in hyperbolic?	4.44	4	4	50%
P4.5	Think longer about this question.	4.5	3	4	43%
P5	What do you notice about defect?	3.43	2	3	40%
P5.5	Think longer about this question.	4.14	6	3	67%
R1	Did you create a regular Euclidean triangle?	4.89	9	0	100%
R2	Did you create a regular Euclidean quadrilateral?	5	8	0	100%
R3	Did you create a regular hyperbolic triangle?	4.67	19	0	100%
R5	Did you create a regular hyperbolic quadrilateral?	4.56	16	9	64%

Note: This table does not include Ivy's responses.
Note: Because this table differentiates between validation and wandering usages, usage tallies appear higher in this table than in Tables 5.6 and 5.7 which did not differentiate between uses.

geometry, students were only using the tools to validate what they had done. For instance, even though the parallel transport task was a new task to the students, because the students were operating in Euclidean geometry, they were still able to predict the outcomes and use the tool to validate those predictions. The conjecturing and exploring with wandering tool usage were not needed. The exception to this was the construction of the regular triangle in hyperbolic geometry. That was the sole prompt where students were in hyperbolic geometry, had a high confidence, and only used validation measurements. This is attributable to the fact that the regular triangle construction in hyperbolic geometry is virtually identical to the regular triangle construction in Euclidean geometry. Because of this, students did not need to use the tools in a wandering method.

Tool Usage and Justification

When looking at tool usage and justification, each series of tasks will be looked at individually. In Table 5.9, the results showed that for both tasks, low tool usage corresponded with higher levels of deductive justifications. For the parallel transport series of tasks, the results showed the prompts with the lowest tool usages, P1 and P2, contain five of the seven instances of deductive justifications for that task. Likewise, for the regular polygon task, the results showed the prompts with the lowest tool usages, R1 and R2, contain thirteen of the fourteen deductive justifications for that task. However, it was also noticeable that all the prompts with low tool usage and high deductive arguments (P1, P2, R1, and R2) also happened to be based in Euclidean geometry rather than hyperbolic geometry. Thus, this relationship between low tool usage and high rate of deductive justification may in fact be more indicative of the differences between Euclidean and hyperbolic geometry. If all the Euclidean prompts are grouped together and sorted by tool usage or all the hyperbolic prompts are grouped together and sorted by tool usage, the correspondence all but disappeared. Also, if all the prompts are grouped as one group and sorted by tool usage, this relationship again disappeared. Thus, this research can only say that when working in Euclidean geometry, these students had low tool usage and a higher rate of

Table 5.9:

Tool Usage Paired with Student Justifications

Prompt	Total Tool Usage	Geometry	Single Case Inductive	Multiple Case Inductive	Deductive	Compare to Euclidean	Previous Knowledge
			Parallel Transport Task				
P1	2	Euclidean	8	2	4	0	0
P2	1	Euclidean	9	1	1	0	0
P4	8	Hyperbolic	4	6	2	0	0
P4.5	6	Hyperbolic	2	2	0	0	0
P5	4	Hyperbolic	3	5	0	0	1
P5.5	9	Hyperbolic	7	1	0	0	0
			Regular Polygon Task				
R1	9	Euclidean	2	2	8	0	8
R2	10	Euclidean	5	1	5	0	2
R3	21	Hyperbolic	7	3	1	1	0
R5	25	Hyperbolic	1	9	0	0	0
Total	95		48	32	21	1	11

Note: This includes Ivy's responses.

making deductive justifications.

Switching to looking at justifications, the results showed that when making multiple case inductive justifications, the students were using the widest variety of tool usages (Table 5.10). There were 32 responses with students making multiple case inductive justifications which correspond with 66 instances of tool usage. This does not imply students were using the tools *more*, but rather that they were using a wider variety of tools. Recall the number of usages is not how many times a tool is used, but the number of distinct types of usages being recorded. These numbers mean that on average, these students were using two different tools when making multiple case inductive justification. Because multiple case inductive justifications typically involved dragging, it was not surprising to see the higher variety of tool usage. Comparatively, in

Table 5.10:

Justifications Paired with Tool Usages (Wandering or Validation)

Justification	Number of Responses with Justification	Number of Validation Usages	Number of Wandering Usages	Total Observed Tool Usages	No tool Usage
Single Case Inductive	48	32	4	36	26
Multiple Case Inductive	32	43	23	66	5
Deductive	21	17	3	20	10
Comparison to Euclidean	1	2	0	2	0
Previous Knowledge	11	9	1	10	5

Note: This table includes Ivy's responses. The numbers in the first column represent how many responses used that type of justification. The numbers in the next two columns represent how many of each type of tool usages were observed with that type of justification.

the 48 instances where students gave single case inductive justifications, there were only 36 observed tool usages. Also, note that for single case inductive justification, for 26 of the 48 responses, students did not use any of the three tools. Taking that into consideration, this implies that when students gave single case inductive justifications, they were either not using the tools or using multiple tools.

A second aspect of Table 5.10 worth mentioning is the difference between validation and wandering tool usage. Single case inductive, deductive, and previous knowledge justifications corresponded with high validation usage compared to wandering usage (8 times as much, 5 times as much, and 9 times as much, respectively). When making multiple case inductive arguments, students still favored validation usages, but not by the same margin. Note, however, it was not that validation usages were less for multiple case inductive arguments, but rather multiple case inductive arguments corresponded to an *increase* of wandering measurements. Much of this can be explained when looking at how the students used the tools in a wandering manner. Of the 27 instances where wandering usage occurred, 17 of those instances were dragging. This increase in dragging correlated with a larger number of students using multiple case inductive arguments than they would have without dragging.

The justifications were also broken down by the type of tool usage observed (Table 5.11). One noticeable relationship was the large number of instances of dragging associated with multiple case induction. This was not a surprise. One of the main features of DGE is the ability to drag and provide students with multiple cases to observe. Not every multiple case justification was because of dragging, however. As an example, there were two instances where Carl looked at Dan's screen in the beginning stages of a construction and commented to Dan that he was going to construct his figure in a different arrangement than Dan. There was recognition from Carl that certain figure arrangements can imply relationships that do not exist.

Additionally, multiple case inductive justifications correspond to high tool usage with angle and length measurement tools as well. Note in Table 5.11 how multiple case inductive justification had a low number of non-tool usages. Of the 32 responses, only 5 responses did not use a tool. When making multiple case inductive arguments, students were overwhelmingly relying on tool usage. As has already been stated, much of that tool usage came about because of dragging; but from Table 5.11, angle measures and length measures also appeared frequently as well.

Summary of Pairwise Relationships

In summary, the results showed many pairwise relationships. In looking at confidence and justifications, deductive justifications corresponded to slightly higher confidence levels, but a high confidence did not always mean a deductive justification. In other words, if students were

Table 5.11:

Justifications Paired with Type of Tool Usage

	Number of occurrences of Justification	Number of Responses with: Dragging	Angle Measure	Length Measure	No Tool Usage
Single Case Inductive	48	4	19	13	26
Multiple Case Inductive	32	33	17	16	5
Deductive	21	8	7	5	10
Comparison to Euclidean	1	0	1	1	0
Previous Knowledge	11	4	3	3	5

Note: This table includes Ivy's responses.

making an inductive argument, they were confident in their answer, more so than just with the software itself as their evidence. We also saw that when students reported low confidence, they were most often using multiple case inductive arguments. As multiple cases give one more evidence of the truth of a claim, this relationship was somewhat surprising.

In looking at confidence and tool usage, the results showed that within each geometry, higher tool usage corresponded with higher confidence. This was to be expected as the DGE can provide strong empirical evidence for the truth of a claim. The results also showed that high confidence corresponded with a high percentage of validation tool usages, while low confidence corresponded with a more balanced split of validation and wandering tool usages.

Lastly, in looking at tool usage and justifications, the results showed that in Euclidean geometry, students had low tool usage and gave a high number of deductive justifications. The results also showed that multiple case inductive arguments, as expected, corresponded to a wide range of tool usage. This was expected due to the dragging often associated with multiple case inductive arguments, but also to the corresponding measurements students make that inform the multiple cases. Also, due to dragging, multiple case inductive arguments corresponded to a balance of validation and wandering tool usages.

Confidence, Justifications, and Tool Usage

With this set of activities and prompts, the two prompts with the highest average reported confidence were the Euclidean construction of the triangle (R1) and the Euclidean construction of the square (R2), with confidence levels of 4.9 and 5 respectively. In responding to these two prompts, students used the tools in a strictly validation usage rather than wandering usage. These prompts also had the widest range of types of justifications used given that each of these prompts had four different types of justifications used. In summary, in this small set of activities, the two prompts with the highest confidence corresponded to responses with only validation tool usage and responses with the widest range of justifications.

However, these two prompts also happened to be the *Euclidean constructions* – the prompts the students are the most familiar with. Removing either the *Euclidean* aspect of those tasks or the *construction* aspect of the tasks, the data shows the average reported confidence and the average number of types of justification decreases, though the percent of validation measurement remains consistently high. In particular, prompts R3 and R5 were constructions, but were in hyperbolic geometry rather than Euclidean geometry. Prompts P1 and P2 were Euclidean, but they were parallel transports. From Table 5.12, the results showed the confidence for these four tasks decreased from the Euclidean constructions which were 4.9 and 5. The results also showed the number of types of justifications given in the prompts decreased from the Euclidean constructions prompts which each had 4 types. Lastly from the table, the results showed validation usages remained high with three of the prompts at 100% validation usages. Euclidean constructions then appeared to be unique in their combination of high confidence,

Table 5.12:

Non-Euclidean Constructions and Euclidean Parallel Transport

Prompt ID	Prompt	Average Confidence (n=9)	Percent of Usages that were Validation	Types of Justifications Given
P1	What do you and your partner notice about first segment?	4.67	100%	3
P2	What do you and your partner notice about final segment?	4.67	100%	3
R3	Did you create a regular triangle (hyper)?	4.67	100%	4
R5	Did you create a regular quad (hyper)?	4.56	64%	2

Note: This table does not include data from Ivy.

wide use of justifications, and exclusive use of validation measurements.

Individual and Pairwise Observations Organized by Research Question

This portion of the analysis will take a qualitative look at the data highlighting instances where students behaved in notable ways. It will emphasize instances that inform the research questions. Focus will be given on areas of movement. That is, special attention will be given to instances where students are moving from Euclidean tasks to non-Euclidean tasks and when students are switching from the parallel transport series of tasks to the regular polygon series of tasks. Additionally, special attention will be given to the two conjecturing tasks (P3 and R4), as well as the proof task (P6). The results are organized around which research question the observation will inform. Summaries of the observations can be found in the tables that follow each section.

Observations Related to Research Question 1 – Tool Usage Tool

Tool usage was an essential part of completing the tasks. As was anticipated, the students were comfortable using the three main tools (dragging, angle measurement, and length measurement) within the DGE and the tasks were designed to give students varied approaches in how they incorporated the tools to respond to the prompts. For both series of tasks, it was observed that tool usage increased when switching from Euclidean to hyperbolic geometry. This was most apparent when students were completing the parallel transport series of tasks. Three students (Ann, Beth, and Dan) used the tools throughout both the Euclidean and hyperbolic portions of the parallel transport. In Euclidean, they used the tools in a validation mode, but in hyperbolic these students increased their tool usage by using the tools in a combination of validation and wandering. For the parallel transport series of tasks, seven of the students did not use the tools at all in responding to the prompts in Euclidean geometry, but when working on the parallel transport in hyperbolic geometry, those same seven students started using the tools. Ivy, for instance, did not use the tools at all when working on the Euclidean parallel transport (prompts P1 and P2), but while was working on the hyperbolic parallel transport (P4 and P5), Ivy began to both drag her image and measure angles as she was looking for a relationship. This was not something she had done during the Euclidean portion of the series of tasks. Similarly, Fran also did not begin dragging her figure until she started working in hyperbolic geometry. Fran, however, measured lengths and not angles.

For the regular polygon series of tasks, the increase in tool usage when students switched from Euclidean to hyperbolic geometry was not as noticeable, but still occurred. On this series of tasks, seven students used the tools throughout both types of geometry. But again, there was a shift when students were working in hyperbolic geometry. For the last prompt in hyperbolic geometry, constructing the hyperbolic quadrilateral, students began using the tools in a wandering fashion. Gary, for instance, validated measured angles and segments for both the Euclidean and hyperbolic regular triangle, and validated measured the segments for the

Euclidean regular quadrilateral. However, when working on the hyperbolic regular quadrilateral, Gary began both wander dragging and wander measurement of segments.

There were also three students (Dan, Fran, and John) who did not use the dragging or measurement tools for the Euclidean portion of the regular polygon series of tasks. However, these three students did begin to use those tools when they began the hyperbolic constructions, first in validation modes for the hyperbolic triangle and then in a combination of wandering and validation for the hyperbolic quadrilateral.

For both series of tasks, there were some students who did not begin to use the tools until the hyperbolic portions of the tasks and there were some students who used the tools throughout and then added wandering tool usage when they were working in hyperbolic geometry. Despite this difference, it is clear that the students were able to make use of the tools to help them complete the tasks.

There were also instances where students wanted additional types of tool usage that the DGE did not provide. Ann, for instance, wanted a parallel checking tool to check for parallelism when working on the parallel transport. It is not clear if she thought this tool existed in Geometry Explorer and she could not find it, or if it was just a wish for a tool that she would find useful. Beth also desired a tool that Geometry Explorer lacks. When working on the Euclidean parallel transport, Beth wanted to be able to zoom in on the first and last segment to visually check if they really were perfectly coincident. Many other modern DGEs offer zooming via the scroll wheel. However, Geometry Explorer only offers zooming via a menu where you enter a scale factor. Because of that, Beth was unable to zoom in and check as she wanted.

Lastly, when working on the construction tasks, Eve looked for a way to rotate a segment around a point. This, in fact, is possible in Geometry Explorer, though Eve could not recall how

Table 5.13:

Observations Related to Research Question 1 – Tool Usage

Behaviors Exhibited by Multiple Individuals:
• During the parallel transport series of tasks, seven students did not begin to use the tools until the hyperbolic tasks (P4 and P5) • During the regular polygon series of tasks, three students did not begin to use the tools until the hyperbolic tasks (R3 and R5).
Behaviors Exhibited by Single Individuals:
• In working on the regular hyperbolic quadrilateral (R5), Fran displayed the only instance of a student using the tools to draw rather than construct a figure. • There were instances of individual students wanting additional tools. Beth wanted to be able to zoom (P2). Ann wanted a parallel checking tool (P1). Eve wanted a rotation tool.

to accomplish the rotation. In all three instances, students were finding the three main tools (dragging, measuring sides, and measuring angles) insufficient and wanting a wider range of tools within the DGE.

There is one last instance of tool usage that is worth highlighting. Fran used the tools in a unique way that no other student in the study did. When Fran began working on the regular polygon task, Fran did not immediately begin by doing a construction. Fran took a few moments and drew an approximate quadrilateral just using four segments. Fran spent a few minutes dragging this figure to see what a quadrilateral might look like. She quickly abandoned this strategy, however, as she saw her partner working with the normal construction tools (e.g. circles and perpendiculars). In addition to noting Fran’s unique tool usage, this observation fits with the theme that tool usage became essential within the hyperbolic tasks. Whether it was a general increase in tool usage, or the addition of wandering tool usage, or Fran’s drawing, there was a shift to additional tool usage within hyperbolic geometry.

Observations Related to Research Question 2 – Justifications

From the data it was clear that inductive arguments (single case and multiple case) were essential to students as they were responding to the prompts. Yet, despite the heavy reliance on inductive arguments, there are many instances where students chose to give other types of arguments. In particular, there were two locations in the series of tasks where students were seen introducing other types of arguments. The first location, as discussed below, were the Euclidean constructions (R1 and R2) as evidenced by students including a large number of deductive justifications. The second location was the conjecturing tasks (P3 and R4). For the conjecturing tasks, as was expected, we saw multiple students referring both to previous knowledge and comparison to Euclidean.

As students worked on the Euclidean regular constructions, there was an increased number of deductive arguments being made. That is not to say these deductive arguments replaced inductive arguments, but rather these deductive arguments supplemented inductive arguments. Eight students gave deductive justifications for at least one of the constructions in Euclidean geometry. Of those eight students, seven students no longer gave deductive arguments when they switched to hyperbolic geometry. Ann, for instance, argued deductively about why her Euclidean regular triangle and quadrilateral constructions were correct. As was typical with most students in this study, she argued deductively exclusively about the side lengths, ignoring the angles. When she switched to hyperbolic, however, her justifications switched to exclusive multiple case inductive justification. The eighth student who started with deductive justifications for his Euclidean constructions was Hal. Unlike the other seven, Hal continued to give deductive justifications for his hyperbolic regular triangle. It was only when Hal began working on the hyperbolic regular quadrilateral he ceased giving deductive justifications. This was due mainly to not being able to correctly construct the quadrilateral. Instead, he offered a counter example to say that it could not be done.

In both series of tasks, after the students finished the Euclidean portion of the tasks, there was a conjecturing task asking the students to conjecture about what might happen if they repeated the tasks in hyperbolic geometry. As these conjecturing tasks did not involve the use of DGE, students did not use inductive justifications to answer these prompts. As students had just previously completed the task in Euclidean geometry, it was anticipated that students would use the Euclidean case to argue for what might happen in the hyperbolic case. Of the ten students, however, there were only two instances during the conjecturing tasks where students specifically referred to what they had just done in the Euclidean case. Both Ann and Fran referred to the Euclidean parallel transport when they conjectured about the hyperbolic parallel transport (prompt P3). Ann used the Euclidean transport to argue the hyperbolic transport would behave similarly, while Fran used the Euclidean transport to argue the hyperbolic transport would act differently. It is noteworthy that both of these students used comparison to Euclidean during the parallel transport and not the regular polygon conjecturing task (R4). The only other comparison to Euclidean justification in both series of tasks was given by Carl when he justified his hyperbolic regular triangle. He made a claim that the construction was correct because it was a repeat of what they did in the Euclidean case. While all the other students copied the Euclidean triangle construction for the hyperbolic triangle construction, only Carl made an explicit claim relating to the Euclidean case.

Additionally, during the conjecturing tasks, students made use of previous knowledge to justify their constructions. Ann and Beth, for instance, both argued that regular quadrilaterals cannot exist in hyperbolic geometry because they recalled you cannot have four right angles in a

Table 5.14:

Observations Related to Research Question 2 – Justifications

Behaviors Exhibited by Multiple Individuals:
• During the regular polygon series of tasks, eight students included deductive justifications in addition to inductive justifications when justifying Euclidean regular polygons (R1 and R2). Of those eight, seven of them shifted to purely inductive arguments for hyperbolic regular polygons (R3 and R5).
• When responding to the hyperbolic conjecturing tasks, students made frequent use of comparison to Euclidean justifications and previous knowledge.
Behaviors Exhibited by Single Individuals:
• Hal was the only student to give a deductive argument for correctness of the regular hyperbolic triangle.

quadrilateral. Carl used the same piece of previous knowledge, but then went one step further to argue that if regular quadrilaterals did exist in hyperbolic geometry, they would not have right angles. Eve used previous knowledge on her conjecture about the hyperbolic parallel transport arguing the segments "would be the same" as she remembered doing a similar activity in class, though it remains unclear what activity she was referring to.

In summary, we have seen the use of inductive justifications was essential to students throughout both series of tasks, though students were seen to include additional justifications as they were able. In particular, students included deductive justifications when working the regular constructions and included previous knowledge and comparisons to Euclidean when they responding to the conjecturing subtasks.

Observations Related to Research Question 3 – Confidence

As we saw earlier in the chapter, students reported high confidence throughout both series of tasks. Even as students were working on the hyperbolic portions of the tasks, the students still reported relatively high confidence. That is not to say all students maintained high confidence. Beth, for instance, reported a large drop in confidence as she switched from the

Euclidean hyperbolic parallel transport to the Euclidean hyperbolic parallel transport. When reporting her confidence for the Euclidean parallel prompts (P1 and P2), Beth rated her confidence a 5 and 4, respectively. Once she switched to the hyperbolic case, Beth reported a 2 for P3, a 2 for P4, and a 3 for P5. For the very first hyperbolic prompt (P3), Beth mentioned the "weirdness" of hyperbolic geometry. As she began to use the software to respond to prompt P4, Ann and Beth purposely made distinct figures and Beth used all three tools for the first time. When responding to how to raise her confidence, Beth repeatedly wrote that she wished she knew and remembered more information about hyperbolic geometry. Beth did report a slightly higher confidence after thinking longer about what relationship might exist. Beth's increased tool usage eventually helped her find the relationship between defect and the angle, and having that relationship, Beth reported a 4 for confidence. Interestingly, as Beth worked on the regular polygon series of tasks, there was not a noticeable drop in reported confidence as she switched to the hyperbolic portion of the series of tasks. Beth reported 4s and 5s throughout that entire series of tasks.

The first of two places the loss of confidence was most apparent for the majority of students was when students were working on the conjecturing tasks. For the conjecturing tasks, students were not using the DGE to answer the questions. When responding to the conjecturing task for the parallel transport, seven students reported at least two levels lower in their confidence than in the Euclidean tasks. Beth, for instance, reported a 2 for prompt P3 as compared to a 4 for prompt P2 writing that she wished she knew more about hyperbolic geometry. Carl, as another example, reported a 5 on the two Euclidean prompts, but then reported a 2 for confidence when conjecturing. He wrote that he was not sure how translations work in hyperbolic geometry. When responding to the conjecturing task for regular polygons, only three students reported at least two levels lower in their confidence. Carl reported a 4 for confidence on his regular polygon, but then dropped to a 2 when answering prompt R4. Carl remembered from class that you cannot have four right angles but expressed doubt about whether this precludes their existence. Gary also reported a drop in confidence when answering prompt P4. Gary reported a confidence of 2 for this prompt whereas he reported a confidence of 4 or 5 for every other prompt in both series of tasks. When thinking about whether hyperbolic regular quadrilaterals exist, his partner Hal traced a potential shape on the screen using his finger. Gary used this undrawn figure to convince himself it was impossible, but wrote he was "unsure if this is correct" and that he wanted to be able to prove it (as he had for the regular Euclidean polygons).

The second of two places where students reported loss of confidence was as they completed the proof task. Students were asked to read a proof that the defect matched the angle between segments of a hyperbolic parallel transport. They then had to rate their confidence in that statement. Every pair found the relationship during the previous prompt, and at least one partner in each pair verified the relationship using the angle measurement tool. As students responded to the given proof, some students reported how the given proof helped validate their previous claims. For instance, Eve wrote, "Now I know that I was right, and the argument helps my confidence." After reading the prompt, her confidence increased from a 3 when she made the claim on the previous prompt, to a 4 having now had her claim verified. For three students (Ann, Fran, and Ivy), their reported confidence showed a change from somewhat confident or very confident to somewhat not or not confident. From the previous prompt where they had validated measured to responding to the proof, Ann dropped from a reported 5 to a 2, Fran from a 4 to a 1, and Ivy from a 4 to a 1. Ann and Ivy both wrote they did not fully understand the proof. Fran wrote that it was difficult to understand the proof "without actually doing it."

This drop in confidence highlights an important aspect of the analysis. Confidence took on different meanings to students as they worked on the tasks. For instance, when Fran reported a 1 after reading the proof, did that 1 signify she was not confident about the relationship she found and which the proof confirmed, or did her reporting of a 1 represent something different? It is feasible the 1 simply represented her general understanding of hyperbolic geometry at that moment. As she did not "really understand what this page [was] saying", it may not be that she lacked confidence in the claim, but rather she lacked confidence in her understanding of hyperbolic geometry. In a similar manner, Ivy rated her confidence at times, not because of her justifications, but "because this is fun to me." That is, Ivy's confidence was not because she could deductive prove something, or because she had measurements to back up her claim, but rather she felt good about it. It was making sense. Yet, at times, these interpretations of confidence switched. For instance, Ivy wrote about her confidence being based on fun on prompt R2 when she constructed a Euclidean square. On the very next prompt, she rated herself a 5, saying, "I'm confident because I measured." From one prompt to the next, her interpretation of confidence changed.

In summary, while students generally reported high confidence, there were instances throughout both series of tasks where students reported drops in confidence. Notably, these instances occurred most when students did not have access to the DGE. In particular, this occurred most often during the conjecturing task as well as the proof validation task. Additionally, there was evidence that the term "confidence" was being interpreted by the

Table 5.15:

Observations Related to Research Question 3 – Confidence

Behaviors Exhibited by Multiple Individuals:
• Seven students reported a drop in confidence on the parallel transport conjecture prompt (P3). Three students reported a drop in confidence for regular polygon conjecture prompt (R4). • Three students showed a drop in confidence after the proof task (P6).

Behaviors Exhibited by Single Individuals:
• Students had different interpretations of confidence – Eve and Fran confidence in incorrect construction. Ivy – Confidence based on fun. • Gary reported high confidence for every prompt, except the conjecturing task for regular hyperbolic quadrilaterals (R4) • Beth reported a drop in confidence when she began the hyperbolic parallel transport.

students differently from task to task. These different interpretations will be discussed in Chapter 6.

Observations Related to Research Question 4 – Relationships

The last part of this section will look at observations that inform the relationships that occur between the three factors being considered. The first relationship to be considered is that of justifications and tool usage. As students were working in hyperbolic geometry, students can be seen making more inductive arguments, and with those inductive arguments came an increased use of multiple figures. Many of these inductive arguments became multiple case inductive arguments because of students' increased usage of the dragging tool within hyperbolic geometry. Yet, dragging was not the only way that these students were able to argue inductively with multiple cases. As these students were working with partners, they also had their partner's figures upon which to rely. For instance, when working on the Euclidean parallel transport (P1 and P2), Ann and Beth purposely made the figures in roughly the same layout. They made similar triangles and began with the first segment angled approximately the same as each other.

However, when they began working on their hyperbolic transport, Ann told Beth that she was going to specifically make her figure distinct from Beth's figure so they would have multiple figures to examine. In the Euclidean case, Ann and Beth, who did not drag, were content making their argument based on a single inductive case. However, switching to hyperbolic, Ann felt the need to begin with multiple cases. In a similar fashion, Carl and Dan also made different figures so they would have multiple cases to consider. As compared to Ann and Beth, Carl and Dan created different figures for both Euclidean and hyperbolic geometry.

Also, with regards to justification and tool usage, it is worth noting there were times where students already had a justification but chose to use the tools to confirm their previous justification. When working on the Euclidean triangle (R1 and R2), both Ann and Beth were able to quickly construct the regular triangle. Ann then began explaining this construction to Beth going into detail about how the radii match as they used congruent circles. However, Beth (and eventually Ann) still chose to measure the segments to confirm that they matched. Ann had provided a deductive justification, but they both decided to use the tools to add an inductive justification to go along with it.

With regard to the second relationship justification and confidence, the first observation to be highlighted is the proof task (P6) asking students to rate their confidence after reading a proof. As was mentioned earlier, three of the students reported a large drop in confidence after reading the proof. All three of these students (Ann, Fran, and Ivy) had previously made deductive arguments themselves and reported high confidence as they made their deductive arguments. One key difference here is that for this prompt, they were given the deductive proof rather than constructing their own deductive proof.

A second observation regarding justification and confidence is Eve's and John's use of previous knowledge and the confidence it provides them. They are not the only students to use previous knowledge, but they are unique in that their recollection of previous knowledge appeared to refer to course activities the researcher has no recollection of. When Eve was responding to the parallel transport hyperbolic conjecture, Eve was convinced the segments would be the same because she remembered an "activity in class similar to this." This memory was strong enough that Eve was one of only two people to express confidence in her answer for this prompt. Likewise, when responding to the regular quadrilateral conjecture task, John claimed they existed because he remembered writing about them for his geometry course. Like Eve, the researcher found no evidence of John writing about regular quadrilaterals. It remains unclear what their recollections are based on, but in both cases, these perceived memories allowed the students to argue based on previous knowledge and provided them the confidence to do so.

For tool usage and confidence, the first two observations relate to how tool usage can negatively affect student confidence. There were many instances, as expected, where students were able to gain confidence, but at times tool usage was seen to lower students' confidence. The first instance was when Dan worked on the hyperbolic parallel transport (P4). When Dan transported his first segment, it happened to land almost perfectly on one edge of his triangle. This caught Dan off guard and made him question whether he did the task correctly. He repeated the initial transport. Upon seeing the same result, he continued and completed the transport. Yet, when the transport was done, Dan again expressed doubt in whether he did the task correctly. This doubt was strong enough, he started over and repeated the entire parallel transport. In both cases, the software caused Dan to lose confidence.

Beth also had an interesting incident where the software appeared to cause her to lose confidence. When Beth was working in Euclidean geometry on the parallel transport (P1), she saw the first two segments were parallel and measured angles to double check. This gave her confidence and she rated herself a 5. On the next part of the task when she completed the transport (P2), Beth saw the segments were on top of each other, but then expressed doubt. Despite saying she was confident they were the same, she also said, “math is weird, maybe they are a little off.” She then rated herself a 4 and expressed a desire to zoom in on her image. Her comment and her rating showed that she doubted whether the two segments really were coincident. This had nothing to do with the DGE’s ability to render the figure, but rather that Beth had no easy way to verify the two segments were coincident. Because of this, her confidence was lower than on the previous task where she could measure. Thus, like Dan, her confidence faltered because of the DGE.

The last observation related to tool usage and confidence is an observation about the importance of the DGEs’ figures in giving students confidence. When the students were working on the conjecturing tasks, students did not have the DGE available for them to use. Because of this, the students commented repeatedly about their desire for the DGE to help with justification. For instance, when answering the conjecturing prompt for the parallel transport (P3), many students wrote about how they wished they could “see” the transport in the DGE. John resorted to drawing sketches on their worksheets to help with their conjecture and John wrote that he wanted to “work it out on the software.” It is interesting to note that often it was the figure itself that gave students confidence and not the tool usage. Gary, for instance, rated himself somewhat or very confident for every prompt in the parallel transport series of tasks having not dragged or

Table 5.16:

Observations Related to Research Question 4 – Relationships

Tool Usage and Justification
• Two pairs of students specifically used different figures rather than just dragging to give themselves multiple cases for multiple case inductive justifications. • Ann and Beth gave deductive justifications for the Euclidean constructions but additionally chose to measure their figures.

Justification and Confidence
• Multiple students reported a drop in confidence after reading the proof. • Eve and John both reported high confidence because of previous knowledge, though the knowledge to which they refer was not covered in their geometry course.

Tool Usage and Confidence
• Though the Euclidean parallel transport (P2) behaved as Beth expected, Beth's confidence dropped as she did not have a way to use the DGE to confirm her conjecture. • Dan's hyperbolic parallel transport (P4) caused him to lose confidence as it did not behave as expected. • When conjecturing for prompt P4, John attempted to draw the figure using paper and pencil as the DGE was not available.

measured for any prompt. Having an accurate figure was enough to provide him confidence for the claims he was making.

Summary

In this chapter, I have presented the data in both a tabular and qualitative manner as it relates to the four research questions. It has been observed that students made extensive use of the tools to complete the activities, but that tool usage varied between Euclidean and hyperbolic tasks, and between the regular polygon series of tasks and the parallel transport series of tasks. It has also been observed that students used inductive justifications for the majority of the prompts, but that there were specific instances where students chose to use other types of justifications (e.g. during the conjecturing prompts or the Euclidean constructions). Additionally, the students reported high confidence throughout both series of tasks, but there were specific instances where students chose to report a lower confidence. Additionally, it was observed students have multiple and changing views of what the term confidence means. Additionally, individual and pairwise observations were presented in the second half of this chapter (Tables 5.13, 5.14, 5.15, and 5.16). These observations will be further discussed in Chapter 6 as they relate to the existing research base.

CHAPTER 6: DISCUSSION OF FINDINGS

This chapter will take a closer look at the findings from Chapter 5. This chapter will be broken into three sections related to confidence, justifications, and tool usage. The section on confidence will focus on the different interpretations of confidence that we saw from the students as well as highlighting specific instances of high and low confidence. The section on justification will look at individual instances where students did or did not use specific types of justifications and potential reasons they may have used the justifications they chose. The third section will focus on issues related to tool usage. Time will be spent looking at individual instances of tool usage where students used or attempted to use the tools in novel ways that were not expected.

Confidence

This section will begin with different interpretations of confidence. As we saw in Chapter 5, students used different interpretations of confidence, and would even change their interpretation from prompt to prompt. The second part of this section will highlight where individual students displayed instances of high confidence and low confidence and the potential reasons for this.

Confidence as Self-Efficacy

The first area worthy of discussion is the students' interpretation of confidence. As reported in the previous chapter, students reported they were 'somewhat confident' or 'very confident' in over 90% of responses when using the DGE. Even narrowing this to prompts related to hyperbolic geometry, students reported they were 'somewhat confident' or 'very confident' in 82% of responses when using the DGE. These were much higher numbers than were expected. We know from previous research that students struggle when making deductive arguments in non-Euclidean geometries (Guven & Karatas, 2009; Hollebrands et al., 2010) and that deduction in non-Euclidean geometries is rated as the highest level of geometric thinking (van Hiele, 1986). Because of this, it was surprising to see such high confidence numbers.

As such, a discussion on how students *interpreted* the term "confidence" is warranted. Every prompt in both series of tasks asked students, "how confident are you in your response to the previous question?" As discussed in Chapter 2, the term confidence is a term that already has many interpretations. On one hand, confidence can be closely associated with self-efficacy (Bandura, 1997; Pajares & Miller, 1994). That is, confidence refers to a student's feelings about themselves, either their overall confidence or topic level confidence (Pajares & Miller, 1994; Parsons et al., 2009). On the other hand, confidence is also closely associated with the concept of conviction (Weber & Mejia-Ramos, 2015). That is, a person is confident (convinced) of a certain outcome or belief.

In looking at students' responses to the confidence prompt, the students generally responded based on how confident they were in their previous claim based on a conviction view of confidence rather than a self-efficacy view of confidence. Hal, as a typical response, wrote he was confident in his construction of the regular triangle in Euclidean geometry because he "measured the angles and side lengths." It is clear he was referencing this specific task. He felt he answered the prompt correctly, he had evidence for it, and he gave himself a high confidence.

Ivy, however, often interpreted confidence through a self-efficacy viewpoint. Ivy consistently reported confidences of 5 with reasons such as "because this is fun to me" and "because I like to make triangles." This interpretation of confidence is closely related to the concept of mathematical self-efficacy (Bandura, 1997; Pajares & Miller, 1994). She was not reporting her confidence in the correctness of her answer. In some cases, she does not even reference her previous answer. Rather, she was reporting a high confidence because she felt good either about math in general (overall confidence) or at least the current task she was working on (topic level confidence) (Parsons et al., 2009). It was making sense to her, and because of that she had high confidence.

There were times, however, where Ivy's interpretation of confidence appeared to switch to the conviction interpretation of confidence. For instance, when Ivy began her first hyperbolic regular construction, she still reported she was "very confident," but her reasoning changed. Rather than writing about her enjoyment, she wrote that she was confident because she measured the angles and segments of her construction. This is a stark contrast from her earlier comments. It is worth noting this was not the first time Ivy measured. When working on the construction of the Euclidean quadrilateral, Ivy measured the angles and lengths as well. In that instance, her confidence was "because it was fun." Yet in this case, doing a similar activity and making similar measurements, her reason for her confidence was "because I measured."

A key difference was the switch from Euclidean geometry to hyperbolic geometry. One possible reason for this switch of interpretation is that in Euclidean geometry, Ivy may have perceived the construction and its correctness as obvious. The diagram itself on the screen served as her warrant (Hollebrands et al., 2010) and so there was no reason to think about and then give a detailed justification. In this case, the measurements she made may have been done for external validity rather than internal conviction (Segal, 1999; Weber & Mejia-Ramos, 2015). That is, she measured because she felt that is what she ought to do even if she felt it unnecessary. Then when she was asked to give a reason for her confidence, she commented on her self-efficacy in that she was having fun. The measurements did not actually serve a purpose in giving her confidence. Then in hyperbolic, when the shape did not look as expected, Ivy felt a justification was necessary both for external validity and for her internal conviction (Hollebrands et al., 2010).

This led her to change her interpretation of confidence from the self-efficacy interpretation to the interpretation of conviction.

Another possible reason for Ivy's switched confidence interpretation is that rather than a shift from an "obvious" answer to a "not obvious" answer, the task in hyperbolic geometry suddenly became "not fun" to Ivy. We know reasoning in hyperbolic geometry can be difficult for students (van Hiele, 1986), and perhaps when switching from Euclidean to hyperbolic geometry, Ivy really did stop having "fun." However, this seems unlikely as video evidence showed she and her partner were still enjoying themselves.

It is not clear from Ivy that she was conscious of her different interpretations as she switched between them. Despite the confidence prompt remaining the same, we see Ivy switching between a conviction view of confidence and a self-efficacy view of confidence. In short, confidence was being interpreted by the student in ways not originally intended by the researcher. So when claims are made about DGE giving confidence to students, consideration must be given to how students are interpreting confidence. Is the DGE giving confidence to the claims students are making in the moment (conviction view) or is the DGE giving confidence to a student's self-efficacy?

Confidence as Conviction

While Ivy may have interpreted confidence using a self-efficacy view, the other students tended to interpret confidence as conviction - as confidence in the claims they were making (Weber et al., 2014; Weber & Mejia-Ramos, 2015). Students were responding to the confidence prompts giving reasons that directly related to the answer they gave. Students reported being confident because they had measured angles, or because they dragged, or because they had a deductive argument, etc. Notice that these responses refer to the statements *the students* *themselves made* and not a statement provided by a teacher or researcher. Because of this, there are two further interpretations of confidence that are worth exploring.

Researchers have previously drawn a distinction between conviction and validity (Segal, 1999). That is, students can be personally convinced of a statement (internally), but not have valid (external) reasons. Or perhaps they do have mathematically valid external reasons (such as a deductive proof), but find they are convinced by empirical evidence. Weber et al. (2014) showed that even professional mathematicians do not always rely on "valid" mathematical deduction to become convinced of the truth of a statement. That is, even mathematicians may be confident of a statement because of overwhelming inductive evidence, rather than deductive justifications. One important point in the work of Weber and Meija-Ramos (2015), Weber et al. (2014), and Segal (1999) is that conviction and validity generally reference statements given by the researchers. That is, when subjects were responding how confident they were, it was in reference to a specific statement and the justifications for that statement.

For this current study, however, there was an added dimension. The students themselves were providing their own claims. Because of this, when students were rating their confidence, there were instances where confidence responses were not specifically referring to students' confidence in the mathematical correctness of a statement, but rather whether the statement being made was the statement the question was looking for.

This first interpretation to the question was for students to rate how confident they were in whether they *reported* a mathematically accurate answer. That is, students using this view of confidence ratted how confident they were the statement they wrote down was true. This most closely aligns with previous research on conviction (Segal, 1999; Weber et al., 2014). In this view of confidence, students were concerned with either their internal conviction a statement was true or an external justification for why the statement was true. In other words, a student with this view was not necessarily concerned about getting the right answer or completing the task correctly, but rather that they were confident they reported a true statement.

The second interpretation of confidence is when students rated how confident they were in the *correctness* of the claim they just made. That is, students using this view rated how confident they were that their answer was what they thought the prompt was 'looking for'. For the regular polygon series of tasks, this was if the student successfully constructed regular polygons. For the parallel transport series of tasks, this was if the student noticed what they thought the question wanted them to notice.

As an example of this difference between these views, we will look at the difference in how partners Eve and Fran responded to prompt R5 asking them to construct a regular quadrilateral in hyperbolic geometry. Neither student was ultimately successful in their attempts and they both answered 'No' indicating they did not construct a regular quadrilateral. But in rating their confidence for that answer, they gave very different ratings. Eve reported a 3 for her confidence and Fran reported a 5 for her confidence.

In rating herself a 3, Eve commented that she could not get the lengths to be the same. She had measured the lengths and the DGE showed they were clearly different lengths. Having been told by the DGE that the lengths were different, it would seem clear she knew her construction was not completed correctly. In responding that her construction was not correct, what was the reason Eve gave herself a 3 for confidence? Using the *correctness* view of confidence, Eve knew from the DGE she did not successfully complete the construction and was therefore rating herself a 3 as her answer was "wrong". She was not concerned with the truthfulness of her claim, or about whether she believed her claim. Rather her confidence was in response to her belief that she did not successfully complete the construction. Because she could not give the "correct" answer, she gave herself a low confidence.

Is it possible Eve rated herself a 3 because she thought her construction was correct and the DGE was wrong? This view implies that she did not view the DGE as an authority. While this is a possibility, her previous comments discount this possibility. In the parallel transport task, she wrote comments such as "Geometry Explorer did the work for us *with accuracy* [emphasis added]", "Geometry Explorer *helps a lot* [emphasis added]", and "*According to* [emphasis added] Geometry Explorer….". Then for this task, she specifically said "I'm just doing the same thing [as a previous attempt], even though I know it's wrong". It is clear she trusted the DGE and recognized she had not constructed a regular polygon. Her trust of the DGE matched previous research showing students view DGE with authority (Guven & Karatas, 2009; Hollebrands et al., 2010).

To contrast Eve's view of confidence in correctness, consider her partner Fran. Fran rated herself a 5 and wrote "My confidence that I don't know how to make a regular quadrilateral is high because I'm sure I don't know how to make it." Fran's view of confidence was confidence in her reporting. Fran went through the steps of making a construction and then she and her partner measured and dragged their constructions. The measurement tool in the DGE showed the line segments were different lengths. Because of this, Fran recognized she was unable to figure out the correct construction and was confidently acknowledging that her construction was incorrect. Her rating of 5 was not a rating that she did the construction correctly, but rather that she was confident of the statement that she recorded. That is, Fran knew her statement was true. It was not what the question was looking for (a correct construction), but it was at least a true statement. Both students recognized their construction was incorrect, but their response to it was different. These differences in interpretation highlight that in future research, when students make their own claims, a distinction will need to be made of the difference between confidence in the correctness of a claim and confidence in the reporting of the answer.

In summary, classifying confidence between conviction (Weber & Mejia-Ramos, 2015) and self-efficacy (Bandura, 1997; Pajares & Miller, 1994) is not enough to distinguish types of confidence students have. When students create claims themselves, the conviction view of confidence can be broken into a *correctness* view of confidence and a *reporting* view of confidence.

Low Confidence When Using a DGE

Regardless of which of the interpretations of confidence students were using, the students reported overwhelming high confidences when completing the tasks using the DGE. They were either extremely confident in their correctness, in their reporting, or in the case of Ivy, confident in her overall mathematical ability. As noted in the results section, excluding Ivy, 90% of the responses for confidence were a 4 or a 5. Because of this large percentage of high confidence levels, it is worth investigating specific incidents when students rated themselves lower.

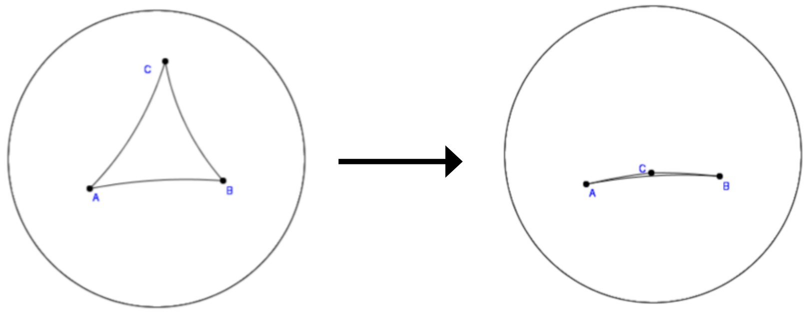
When looking at low confidence ratings, none of the students reported a 1 for confidence for any prompt that *involved the DGE.* This is not terribly surprising as DGE has been shown to provide confidence (Guven et al., 2010; Hollebrands et al., 2010; Weber et al., 2014). Knowing there were no 1's reported for confidence when using the software, we will investigate where the 2's and 3's were reported. As reported in Chapter 5, there were eight instances of students using the software where their reported confidence level was either a 2 or 3. Of those eight responses, five of them occurred on one specific question: Prompt P5 asking what students notice about the defect of the triangle in hyperbolic geometry. There were at least two potential aspects of this question that led to it having such low confidence levels. The first aspect of the question was the open-ended nature of the question (Silver, 1995). The second aspect of the question was its focus on a singular feature of the diagram.

In terms of the question being open-ended, Prompt P5 asked students, ‘what do you notice?’ The students knew they were looking for a relationship, but they were unsure of what a ‘correct’ relationship would be or if there even is a ‘correct’ relationship. There were multiple acceptable (true) answers, and there were multiple ways of discovering these answers (Silver, 1995). John, for instance, wrote about how if you collapse the triangle by moving the third point, C, onto segment AB, the defect goes to 0 (see Figure 6.1). He was correct that the defect goes to zero, and he used the DGE to measure the defect to confirm that it goes to zero. And yet, despite writing down a true statement that is confirmed by the DGE, he only rated himself a 3. He wrote that he is unsure “what its purpose is or why it does what it does.” After completing multiple tasks related to parallel transport, he may have recognized that his response about the defect going to zero is unrelated to what came before. Therefore, while he may have been confident the defect went to zero, he was not confident this response is what the question was asking. This caused a drop in confidence for two potential reasons. It is possible he was responding to confidence using a *correctness* interpretation of confidence, and he was not confident his observation was the ‘correct’ observation. It is also possible that because he did not know if it was the correct observation, he was responding using a self-efficacy interpretation of confidence (Bandura, 1997; Pajares & Miller, 1994).

Prompt P5 was not the only open-ended question that was asked. Prompt P4 was also an open-ended question asking students what they noticed about the first and last segment. However, as compared to the five students that rated themselves substantially lower for P5, only

Figure 6.1:

John Showing Defect Going to Zero



one student, Beth, rated herself substantially lower for prompt P4. This implies that there was a second aspect of P5 causing low confidence other than just the open-ended nature of the question. This second aspect of prompt P5, the triangle defect question, that potentially led to low scores was its focus on a *single feature* of the diagram.

Prompt P4 very specifically asked students to compare two parts of a diagram. When students were able to make a comparison, regardless of the implied importance, they reported higher confidence as their answer connected the two parts the question referenced. So even though they had switched to hyperbolic geometry where students are known to have less confidence (Hollebrands et al., 2010), the relationship gave them confidence their answer was the ‘correct’ answer. Prompt P5, however, only asked students to write what they notice about the defect of the triangle. This put a focus on a single aspect of the diagram with nothing to compare it too. As such, students appeared unsure how to answer that prompt. The expectation of the researcher was that students would engage in wandering angle measurements looking for a relationship to what they had previously been doing. However, none of the pairs used the measurement tool to measure any angles beyond what was needed for the defect. Three of the groups engaged in dragging to see how the defect changes, but given the nature of hyperbolic geometry, none of the students visually saw that the defect matched the angle between the segments.

It is possible that P5 had a lower reported confidence simply because students were unsure of the concept of defect. John, for instance, had written he was unsure of its purpose. It is feasible that the lower scores reported for P5 were simply a result of students' feelings of confusion about defect as a concept. That is, the lower reported confidences could have resulted as students' general self-efficacy (Pajares, 1996) lowered because of their perceived lack of understanding of defect or its purpose. However, after writing their response, the researcher stepped in asking each group how this prompt might relate to the previous prompts and to think longer about their answer. By encouraging the students to think how it related to the previous prompts, it gave the students a second focus. They had an implied relationship to look for at this point. After students found that relationship, they showed an increase in their confidence. Focusing on John again, his reported confidence increased from a 3 to a 4. He did not suddenly understand defect, but he found a relationship that was most likely not a coincidence. This allowed him to increase his reported confidence. From his comments, this confidence is confidence in his correctness that he found the "right" answer. That is, his self-efficacy about his knowledge of hyperbolic geometry or defect did not increase. It was not suddenly making more sense. He even wrote that he still did not have a "specific why" and that he would still "need a lesson" on defect. Rather, he was confident he found he found the answer the question was looking for.

In summary, there appears to be at least two factors that corresponded with low confidence when using DGEs. First, students reported low confidence on the open-ended questions. Second, students also reported low confidence when the question only had one focus, rather than asking students to look for a relationship.

Low Confidence Without Using a DGE

As mentioned earlier, reported confidence was generally high throughout the tasks when using the DGEs (Guven et al., 2010; Hollebrands et al., 2010). However, when students did not have access to the DGE, confidence was noticeably lower. Specifically, this occurred during the proof prompt (P6) as well as the conjecturing prompts (P3 and R4).

The Proof Prompt

The proof prompt saw noticeable decreases in reported confidence from the students. This was the only prompt in the whole study where any student reported a 1 for confidence. Specifically, Fran and Ivy both reported 1's with Fran saying, "I can't picture what this would look like without actually doing it." In other words, Fran was indicating that at least part of the confidence she has been reporting was due to 'doing it' in the software. The abstract nature of the proof was not convincing for her. When Ivy rated herself a 1, she wrote "It would help if I could practice the proof." It is unclear what 'practice' means. Does 'practice' mean to read and write the proof multiple times? It could be that Ivy is working on proof comprehension (Selden & Selden, 2017) and simply trying to understand the proof. Does it mean to work through the proof in software? This aligns with previous research on proof validation (Selden & Selden, 2003) that suggests students like to check the surface features (e.g. diagrams and equations) of the proof to see if it works.

Notice that either interpretation shows Ivy's confidence switched from rating her confidence in the statement to her confidence in the proof itself. She does not appear to be rating whether she was confident that the statement was true or false, but rather how confident she was in the proof itself. This appears to show that Ivy was still using a self-efficacy mode of confidence. Her low confidence was caused by a lack of understanding, rather than a lack of belief. That would match much of her previous interpretations.

It should also be noted that it was not just Fran and Ivy who rated the proof low. From Chapter 5, Ann also reported a drop in confidence when she worked on the proof task. She had rated herself a 5 on confidence when she discovered the relationship in the previous prompt, but only rated herself a 2 on confidence when she read the proof. While these three students saw the biggest drop in confidence, three other students reported small decreases in confidence after reading the proof. These low ratings are especially interesting as students had reported high ratings on other prompts involving deductive proof (notably the Euclidean constructions). A key difference with the constructions is that students were constructing their own arguments. For this proof, students were attempting to read and understand a given argument. Students were approaching this either as proof comprehension or as proof validation (Selden & Selden, 2017). When students were constructing their own proofs for the Euclidean constructions, students were engaging in proof construction (Selden & Selden, 2017).

We know from research, such as Knuth (2014), that students do not find proofs infallible. Knuth showed that even when students read and understand a proof, they still expressed doubt about the conclusion or thought there might still be counterexamples. For this research, while it was to be expected that some students may not fully understand the presented proof or doubt the generality of proofs (as Knuth showed), it was somewhat surprising that for six of ten students in this study, their reported confidences *decreased* after reading the proof.

The expectation for this prompt was perhaps best illustrated by Eve. In using the software on the previous prompts, Eve discovered the relationship between the defect and the angle and reported a 3 for confidence after having used the software to measure the angles and see that they matched. Upon reading the proof, Eve then reported a confidence of 4 writing that "now I know that I was right, and the argument helps my confidence." Eve was showing that having a deductive argument strengthened her confidence. It is unclear whether Eve was arguing based on an axiomatic proof scheme (Harel & Sowder, 1998) and found the argument convincing or an authoritarian proof scheme where the very existence of a proof from an authority figure (the researcher) implied correctness. She started with "now I know that I was right" which initially points to a belief that simply having a proof constitutes authority, but then followed it with "and the argument helps my confidence" which points to the actual argument itself doing the convincing. Either interpretation was an expected interpretation for the task.

To contrast this, consider Beth. On the previous prompt, Beth correctly concluded that the defect of the triangle matched the angle between segments. She gave herself a 4 for confidence and wrote that "it looks to be correct everywhere we move it [the figure], but I am not confident there are not exceptions." Beth measured the angles, and then dragged her figure around testing various configurations. Despite her testing, Beth still expressed doubt in her answer. Upon reading the proof, she wrote she "doesn't totally follow it" but "it seems like it [the angles] should [match] based on the proof." She then gave herself a 3 for confidence, a lower score than she previously reported.

Beth was not an isolated incident. Six of the ten students had similar responses to the proof. Why then are students' reported confidences decreasing? The most likely explanation is that students switched their interpretations of confidence. It is feasible that students interpreted the question as "How confident are you in the proof you just read that the defect matches the holonomy?" as compared to "How confident are you that the defect matches the holonomy?" If students interpreted the question in the first way, then confusion on the logic or interpretation of the proof would naturally cause their confidence to decrease. Put another way, by engaging in proof comprehension and validation, these students also switched from a conviction interpretation of confidence to a self-efficacy view of confidence. Fran, for instance, was quite explicit writing “I don’t really understand what this page is saying” implying her confidence decreased because of a lack of understanding the proof, rather than her hesitancy about the claim itself.

Having seen their confidence decrease, it was worth looking at the follow-up question to see what students reported as to what would raise their confidence. Of the nine students who did not rate themselves a 5 for this prompt, four students reported a general sense of wanting to better understand the proof with one student specifically asking for a better diagram. These four students responded to the prompt with proof comprehension (Selden & Selden, 2017). Three students thought ‘doing the proof’ in Geometry Explorer would raise their confidence. Lastly two students wanted some way to see if it worked ‘in all cases.’ The first case of students wanting a better understanding was expected. Students came into this project with different levels of understanding proof. There were going to be some who struggled to understand the proof or were not interested in taking the time to understand the proof.

The last two types of responses to the proof validation question show how software can potentially aid students with deductive reasoning. ‘Doing the proof’ in Geometry Explorer suggests either these students were wanting to use the software to aid in their understanding of the proof or to use the software to confirm various aspects of the proof. John, for instance, wrote specifically, that he substituted his angle measures into some of the equations in the proof to “confirm” the proof. For him, it appeared the proof was valid because the proof aligned with the measurements the software provided him. This is similar to how Selden and Selden (2017) describe students checking "surface level features" of a proof to validate a proof (p. 340). The second type of response asking if the proof worked "in all cases" implies that students may not fully accept the proof as covering all cases. Students may still feel that a proof, while potentially correct, has exceptions. Again, this falls into a type of proof validation because often students do not believe that proofs are infallible (Knuth, 2014; Weber & Mejia-Ramos, 2015).

The Conjecturing Prompts

The second area where students showed low confidence were the two conjecturing prompts. As reported in Chapter 5, seven students reported a decrease in confidence when responding to the parallel transport conjecture prompt (P3) and three students reported a drop in confidence when responding to the regular polygon conjecture prompt (R4). As students did not have access to the DGE to work on this prompt, their lower confidence was not unexpected. It was interesting, however, to see the difference between the two prompts. For the parallel transport conjecturing prompt, there were many more students who expressed a lower confidence than for the regular polygon prompt.

There are several potential reasons for this. The first is task familiarity. The parallel transport task was a novel task for these students. However, students previously had some experience with constructions. This difference is noticeable in the comments made by students. When responding to prompt R4, many students referenced doing similar constructions in class. Having done similar constructions in class, their general self-efficacy was most likely higher and this may have influenced their confidence if they were responding with a self-efficacy interpretation of confidence. In some instances (such as John), students claimed to remember this exact question from class. This allowed those students to answer the question confidently using a conviction interpretation of confidence. On the other hand, the parallel transport task was novel. Students made comments such as Ann writing, "It's mostly a prediction" or Beth writing, "I'm not knowledgeable about hyperbolic geometry." As it was a novel task, these students were highlighting their hesitancy in making a claim. As such it appears their confidence rating switched at least partly from a conviction view in their previous prompts to a general self-efficacy view in their later prompts.

A second reason for the lower confidences reported for the parallel transport task is that the parallel transport task was an open question (Silver, 1995). As we saw earlier, students reported lower confidences when working on open ended questions. A similar thing may have happenned here. The parallel transport question asked students what they thought would "happen in terms of the final segment compared to the first." There were numerous potential answers to this prompt. Thus, in addition to confidence in the truthfulness of their conjecture, students may also have been rating confidence in whether their conjecture was the 'correct' conjecture. In comparison, the regular polygon conjecturing task was a closed question. It was either yes or no.

Summary of Confidence Discussion

In summary, student confidence can be classified by either conviction in the statements they made (Weber & Mejia-Ramos, 2015) or by their general self-efficacy (Bandura, 1997; Pajares & Miller, 1994). The results indicate students' interpretation of confidence switched as they moved from task to task. Additionally, when students created claims themselves, the conviction view of confidence was broken into a correctness view of confidence and a reporting view of confidence. In general, students had high confidence throughout this study, but there were instances of low confidence both when using the DGE and when not using the DGE. When using the DGE, students showed low confidence on open-ended questions and especially when those open-ended questions only had a single focus. When students were not using the DGE (the conjecturing tasks and the proof validation tasks), students reported lower confidences, with the proof validation task being the only task where any students reported a 1 for their confidence.

Justifications

As was reported in Chapter 5, students overwhelmingly used inductive arguments to justify their statements. Previous research already suggested students tend to rely on inductive argumentation when working with DGE. For instance, Hollebrands, Conner, and Smith (2010) wrote students using DGE "reduced their uncertainty to the point at which they did not see a need for a formal proof" (p. 348). When framing the tasks, I specifically did not use the word "proof." I wanted students to have the option to choose whether to attempt a proof. Students in this study all had experience writing proofs. They had all taken courses involving mathematical proofs (a proofs course, a geometry course, and an abstract algebra course). Moreover, the students had all taken these courses with the researcher as the instructor. As is typical of those courses, much of their homework was proof based so the students were used to turning in proofs. For this study, however, these students participated in these tasks outside of any class and any final grades had been turned in. As such, the students knew they did not necessarily have the same expectation for justifications as they did in a classroom. This may have had an impact on the overwhelming number of inductive arguments being made. As such, this section will be looking at instances where students did not make inductive justifications and examining the reasons why these other justifications may have occurred.

Deductive Justifications

Per the results in Chapter 5, when using the DGE the students only gave deductive justifications, either partial or full, in 21 of the 90 prompts to which they responded. Much like

Hollebrands, Conner, and Smith (2010) found, the software appeared to negate the students' need for a proof. The software generally provided students with enough evidence to be confident in their responses. What then was the reason why students gave deductive justifications? Two potential reasons are given below. The first reason is familiarity with the task and the second is lack of measurements.

As has been stated, the students in this project had previously taken a geometry course from the researcher. During this course, students completed many constructions, including the Euclidean regular triangle. The regular triangle was used as a foundational move for more advanced construction techniques as the same steps to make a right triangle are also useful for making perpendicular lines. In the students' geometry class, this basic construction of making perpendicular lines was referred to as the 'two circle trick.' Because of this familiarity, it was no surprise when all the students were able to complete the Euclidean regular triangle construction. It was also no surprise that eight students gave at least a partial deductive proof for why their construction worked. Likewise, the Euclidean regular quadrilateral (square) was a familiar shape to the students. While the students had not formally done a proof of its construction during the course, the general method had at least been discussed. This familiarity was likely one of the reasons that another five of the deductive arguments occurred during this task.

It is worth noting that for these tasks, students were not required to give a proof. They simply had to "justify" their answer. And yet, for the Euclidean constructions, students were choosing to justify their answers at least partially with deductive justifications. This is some indication that students are aware of the importance of public validity (Segal, 1999). That is, while students may have been internally convinced their constructions were correct (either by measurement or remembrance), they recognized that "justify" has a public aspect to it. It is not simply to justify to themselves it is correct (conviction), but to justify it to others. In this case the public aspect was that of the researcher (their former instructor).

When giving these deductive arguments, there was tension in what the students decided to write down. None of the students wrote down formal proofs. Two students gave full verbal proofs, but their written responses often resorted to inductive justifications. For instance, Ann gave a full verbal proof to her partner that the triangle was regular, but then proceeded to write that it was regular because “the measurements are equal.” In this case, she knew deductively how to do it, but still referred to the measurements in her written response to the question. Again, this highlights the difference between conviction and validity (Segal, 1999). While Ann may have recognized that a proof is considered a “valid” justification, she also demonstrated in her final written answer that the measurements were more personally convincing. It was a familiar task, and the measurements supported her answer, so writing the full proof was unnecessary. It is also possible as Hollebrands, Conner, and Smith (2010) found that Ann simply did not see the need as the software showed them it was correct.

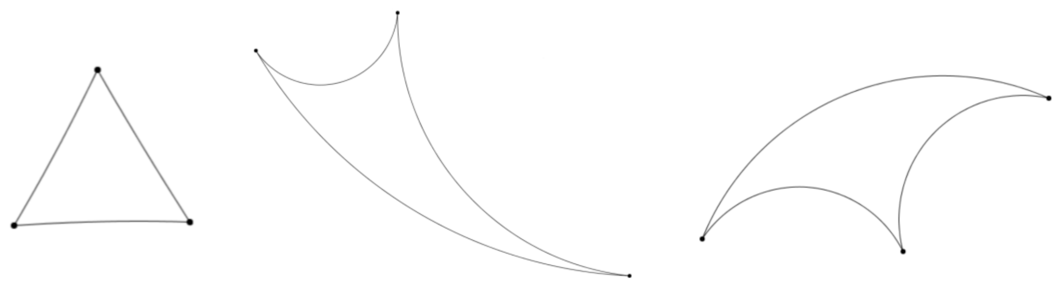
Comparing this to the construction of the regular triangle in hyperbolic geometry, only one student gave deductive justifications. Unlike the regular triangle in Euclidean geometry, the students had not previously constructed a regular triangle in hyperbolic geometry. However, both the construction and the proof are exactly the same as in Euclidean. Yet, for this task, there was much less deduction taking place. It could be that students did not know how to justify their construction deductively. Seeing as the proof was the same as they had just done minutes earlier in Euclidean geometry, this seems unlikely. It could also be that students simply did not feel the need to do a proof they had previously done. They may have found it redundant. A third possibility could be the difference in how students measured between the two tasks.

This difference in measurement arises in part because of the difference between Euclidean and hyperbolic regular triangles. For the Euclidean construction of a triangle, students were able to complete the construction and visually tell whether their construction was correct. Students knew what a Euclidean regular triangle looked like. Because of that, not all students chose to take measurements. Overall, there were 9 tool usages recorded for this task in Euclidean geometry. If we contrast that to the hyperbolic triangle, there were 21 recorded tool usages in hyperbolic. Students were able to correctly construct a regular hyperbolic triangle, but due to the nature of hyperbolic geometry, they were unsure if their construction was correct. As we can see in Figure 6.2, hyperbolic regular triangles can look drastically different. Because of these differences, students used the measurement tools more. These measurements then provided students with data they were able to use when making their justifications. They did not have a reason to use a deductive justification as the software already told them it was correct (Hollebrands et al., 2010; Olivero & Robutti, 2007). Students were convinced; they had evidence; and they no longer felt the need to give deductive justifications.

In summary, it appears at least two factors came together for students to give deductive justifications for the regular triangle. First, students were more likely to use a deductive argument when they had a familiarity with the task. Even if students were unsure of the deductive argument, they were used to the type of arguments being made and were more willing to attempt that type of argument. Second, if students had not used the tools extensively, they were more likely to turn to a deductive argument as they otherwise had no data with which to base their argument.

Figure 6.2:

Variety of Regular Triangles in Hyperbolic Geometry.



Other Justifications

When using the DGE, there were very few instances of justifications that were not inductive or deductive. There were eleven instances of justification by previous knowledge and one instance of comparison to Euclidean. Of the eleven instances of justification using previous knowledge, they mostly occurred during the Euclidean constructions. Students had a memory of performing the constructions and reiterated they had done the construction during their previous class, so they knew it was correct. However, with only one exception, previous knowledge was partnered with at least one other type of justification. When justifying her Euclidean regular triangle, Eve, for instance, wrote about how she used "prior knowledge by taking Modern Geometry class," but then followed that with some deductive justifications. There is a recognition that claiming previous knowledge is not enough of a justification. Again, this highlights the concept of validity (Segal, 1999). While Eve may have been convinced by her previous knowledge, she recognized the need to go further than that as she recognizes her personal conviction is not necessarily mathematically valid.

For comparison to Euclidean, there was only one instance of a student using the software and making a direct comparison to Euclidean geometry. When making his hyperbolic regular triangle, Carl specifically mentioned his construction was correct as it had worked in the Euclidean case. However, Carl, also measured the lines and angles “just to make sure.” That is, in the one instance where a student used a comparison to Euclidean, the student still expressed hesitation and measured to double check.

It was somewhat surprising that only one student used a comparison to Euclidean justification. Having designed the series of tasks such that the hyperbolic tasks mirror the Euclidean tasks, it was expected that more students would have answered as Carl did. Even including the non-DGE tasks, only Ann and Fran used comparison to Euclidean once each in answering the prompts for task P3, predicting what would happen for the hyperbolic parallel transport. It is possible that students disassociated the Euclidean tasks from the hyperbolic tasks such that any comparison was unwarranted. As we saw in Chapter 4, multiple pairs had discussions about how “weird” hyperbolic geometry is. Thus, even when completing such similar tasks as the Euclidean regular triangle compared to the hyperbolic regular triangle, the stark difference between geometries prevented any type of comparison. Per many of the written comments for P3, the hyperbolic parallel transport conjecture, students know hyperbolic geometry is bound to be different, but they are unsure in what way.

Summary of Justification Discussion

As stated in chapter 5, students overwhelmingly used inductive justifications. Deductive justifications occurred when students were previously familiar with the activity or when students did not take any measurements. Prior knowledge and comparisons to Euclidean geometry were relatively rare. When justifications of prior knowledge were made, they were almost always accompanied by other justifications implying that students felt justification by previous knowledge was not sufficient by itself. Lastly, comparisons to Euclidean geometry were only made three times highlighting the distinction students made between Euclidean and hyperbolic geometry.

Tool Usage

As was reported in Chapter 5, tool usage (dragging and measurement) was much more prominent when students were working in hyperbolic geometry. This was to be expected based on previous research (Guven & Karatas, 2009; Hollebrands et al., 2010). As hyperbolic geometry is known to be more difficult for students (van Hiele, 1986), it was expected that students would look to the DGE to provide evidence for them as deductive justifications would be more difficult to use (Guven & Karatas, 2009; Olivero & Robutti, 2007). In many instances, students did not use the dragging or measurement tools at all until hyperbolic geometry. That is not to say students did not use the DGE in Euclidean geometry. Much of the tool usage in Euclidean geometry was using the DGE as an accurate way to construct figures (C. Laborde, 1993). That is, the perceived accuracy of the construction done by the DGE was enough evidence for students to make claims. For instance, on the Euclidean parallel transport, the transported segment by the DGE appeared parallel and the students trusted the DGE to be displaying it properly. Hollebrands et al. (2010) described this phenomenon as students finding the software "reliable." Hollebrands et al. (2010) found that students trust the software to display accurate images and will base their arguments on those images.

When switching to hyperbolic, students still found the software reliable, but used the measurement and dragging tools to either check their own work or to look for relationships. For the parallel transport, students used the tools to find relationships to respond to the prompts. Measurement tools were used to find those relationships, and then dragging often followed to show those relationships held for multiple configurations of the figure. That is, wandering measurement and dragging increased in hyperbolic geometry as students were looking for relationships that were difficult to see due to the nature of hyperbolic geometry. For the Euclidean constructions, validation measurement and dragging increased as students were using the measurement tools to verify they had constructed their figures correctly. They were not necessarily doubting whether the DGE had drawn their figure correctly, but rather, they needed the measurement tools to verify that they had constructed their figure properly. Seeing a figure was not enough justification. Having measurements was a necessity for having confidence in their constructions.

Novel Instances of Tool Usage

While this study primarily examined where students were using the dragging and measurement tools (Baccaglini-Frank & Mariotti, 2010; Olivero & Robutti, 2007), there were a few instances where students used the tools in ways fundamental ways that were not originally expected or looked for when designing the study. This section will highlight a few of those instances.

The first instance to be highlighted is Fran's use of the DGE to *draw* a figure. Fran began her construction of a hyperbolic regular quadrilateral, not by starting with a construction, but by exploring what a hyperbolic regular quadrilateral might look like. That is, Fran put four segments on the screen and began dragging them into a shape that looked roughly regular (Figure 4.25). This is the difference between drawing and figure (C. Laborde, 1993). As this shape was new to Fran, she was drawing a proposed version of the figure that may later point her to what the actual figure looks like. Perhaps unfortunately, Fran did not pursue this avenue very long as she got sidetracked by her partner into attempting direct constructions of the figure. It is worth highlighting the potential importance of what Fran initially attempted. If Fran had added measurements to her angles and sides, it is feasible Fran could have discovered the crucial property that hyperbolic regular quadrilaterals do not use 90 degree angles; they must only have congruent acute angles. This type of proactive guided measurement (Hollebrands, 2007; Olivero & Robutti, 2007) is a use of the measurement and dragging tools that could have helped many of the pairs.

The other instances of novel tool usages are actually instances where students attempted to use the tools but were hampered either by software limitations or lack of how to use the software. When working on the Euclidean parallel transport, Beth commented specifically that she wanted to have a way to zoom in on the figure. Beth was concerned that the first and last segments were not perfectly coincident. While the figure may look coincident, Beth seemed concerned about the difference between the graphical representation and the theoretical objects (C. Laborde et al., 2006). Beth supposed that if she could zoom in further, she could test whether the segments truly were coincident. While the Geometry Explorer does have a zoom function, it is difficult to use and these students did not have much experience using it. This lack of zooming was a reason Beth only rated herself a 4 for confidence.

When working on her construction of the Euclidean regular quadrilateral, Eve attempted to make use of a rotation to give herself a right angle. Eve had initially struggled to construct right angles in a way that preserved segment lengths. That is, Eve had a segment representing the base of her regular quadrilateral but was struggling with how to construct the sides. Eve decided one approach would be using the rotation tool. These students spent a week during their course on Euclidean transformations, so deciding to use the rotation tool was an understandable option for Eve to try. Unfortunately, Eve struggled with remembering how to use the rotation tool. In this case, Eve's difficulty in using the software directly hampered a valid approach to constructing a square.

In both of these instances, Eve and Fran had strategies for using the tools that would have helped them both, but they were hindered by the DGE itself. This points to the importance of both knowledge of the software and design of the software itself. In both cases, it was theoretically possible to do what the students wanted. Yet, the design of the software hampered both students. Had the researcher stepped in, it is possible Beth would have gained her confidence and Eve would have constructed a regular quadrilateral. This points to the importance of mediation as students are using DGE (Barcelos et al., 2011). Dynamic Geometry Environments are only useful if students know how to use them in productive ways (Olivero & Robutti, 2007).

Summary of Tool Usage Discussion

As expected, student tool usage was increased during hyperbolic geometry. For the parallel transport series of tasks, students found wandering measurement and dragging an important aspect of finding relationships in hyperbolic geometry that were otherwise difficult to see. For the regular polygon series of tasks, students found validation measurement and dragging an important aspect of showing their constructions were made correctly. There were also instances throughout the study of students using, or attempting to use, tools in ways that were not originally considered. This includes using the DGE as a tool to draw as compared to construct, using the zoom features to validate conjectures, and using the rotational tool as a type of construction tool.

Task Design

One important consideration of this research is the tasks themselves. There were two main series of tasks. The first of these series of tasks was the parallel transport series of tasks. This series of tasks consisted of open-ended prompts with which the students were less familiar. The second series of task was the regular polygon task. This series of tasks consisted of prompts which were not open-ended and with which the students had some familiarity. Additionally, the tasks included both a Euclidean geometry component and a hyperbolic geometry component. As is the case with research, the results are entirely dependent on the tasks chosen. Because of this, it is worth discussing how the tasks may have impacted the data.

The regular polygon task was the most familiar to the students. They had done numerous construction-based activities during their geometry class - many of which included a proof component. As a result of this, the students were familiar with making construction-related arguments using such theorems as the triangle congruence theorems. On the other hand, the parallel transport task was unfamiliar to students. They had done transformations in their geometry course, and had studied the theoretical underpinnings of transformational geometry, but they had minimal experience doing proofs on their own. Most of the proofs involving transformational geometry had been done together as a class either during lecture or during in-class group work time.

A second difference was the type of question being asked. The parallel transport task included open-ended prompts (Silver, 1995) asking what students noticed. Because of the open-ended nature of the question, there was a wide range of answers that students could write down. For instance, on the very first prompt asking what students noticed about the transported segment, some students commented on the segment’s length, others talked the segment being parallel to the original segment, and others talked about angles. From the students' perspective, there was not necessarily a "right" answer. This is a defining feature of open questions (Silver, 1995). On the other hand, the regular polygon task included closed prompts asking students whether they were successful in constructing a given shape. Students were able to judge whether they were successful. This relates to the multiple interpretations of confidence. For the open-ended questions, students switched between confidence in writing down a true statement or in giving the "correct" answer. Comparatively, for the closed questions of the construction, this distinction was mostly removed by virtue of the fact that students either did or did not create the construction correctly. This meant while the construction process and the justifications varied greatly, the actual answers themselves were standard with either a yes or no.

These two differences (familiarity and open-endedness) undoubtably accounted for some of the disparities that have been reported so far. For instance, both familiarity and closed questions may have contributed to the disparity between the reported confidence for the two series of tasks. While both tasks reported high confidence, the more familiar task, the regular polygon task, had much higher reported confidences. Is it a result of familiarity or a result of the type of confidence they were reporting? Likewise, these differences may also account for some of the disparity in the use of deductive arguments between the tasks. There were many more deductive justifications given during the regular polygon series of tasks. This is likely the result of students' familiarity with the tasks.

Additionally, the difference between open and closed questions may have contributed to how students used the tools during the tasks. During the regular polygon task, students used the tools overwhelmingly for validation and during the parallel transport task the students were

Table 6.1:

Number of Instances of Wandering and Validation Tool Usages for Each Task

	Wandering	Validation	Both
Parallel Transport Task	10	14	6
Regular Polygon Task	7	54	4

much more balanced between wandering and validation. This disparity is seen in the Table 6.1. When Olivero and Roberto originally defined wandering measurement, the very first phrase they used was “When students do not have any precise ideas….” (2007, p. 141). This idea lends itself to open-ended questions. The students do not necessarily know what they are looking for, and so they begin to wander. Then, once they find something, they can use the tools to verify what they found. Comparatively, when working on the regular polygon task, students were using the tools to confirm they constructed the polygon correctly or to confirm they did not. By using the software, the students had an accurate picture and they were typically able to judge before they measured whether their regular polygon was regular. Because of this, we see most tool usages as validation usages.

Within the tasks themselves, there were also individual aspects that could have affected the outcomes. For instance, in the parallel transport task, the first four prompts asked students what they noticed about “the segment”. It was not until the fifth prompt asking about the defect that students were told to start thinking about angles. That is not to say that students waited until the fifth prompt to think about angles, but the task itself did not ask students to specifically think about the angles until prompt five. To be precise, 8 of the 10 students wrote about parallelism for the first prompt in the parallel transport, though only two used the angle measurement tool. Notice the focus in their response is on parallelism, and not the angles themselves. The two students who did not mention parallelism, Eve and Fran, only commented on the segment staying the same length. They made no mention of parallelism or angles. Continuing with this task, Eve and Fran used the angle measurement tool the least, as did Gary and Hal who used it the same amount as Eve and Fran. This suggests that had Eve and Fran thought about parallelism or angles during the first prompt, as did the other groups, they may have been more inclined to use the angle measurement tools throughout the remainder of that task. To summarize, just because tools were available did not mean students chose to make use of them. If a teacher or researcher wants students to make use of DGE tools, the language they use may influence whether the students actually use the tools (Barcelos et al., 2011).

Summary

This discussion took a deeper look at confidence. Did confidence measure what it was proposed to measure? There seems to be two main interpretations students had for confidence. Students either were using a conviction view of confidence (Weber & Mejia-Ramos, 2015) or a self-efficacy view of confidence (Bandura, 1997; Pajares & Miller, 1994). Further, this study identified that when dealing with a conviction view of confidence, there were two types of confidence being reported. At times, confidence referred to the *correctness* of students' answers and whether they felt they gave the "right" answer. At other times, confidence referred to their reporting of a true statement. Additionally, this study confirmed that students generally report high confidence when using DGE, though there were instances of low confidence reported. When using a DGE, students reported lower confidence when answering open-ended questions, and even more so, when the open-ended questions only had a single focus.

This discussion also highlighted the times that students used non-inductive arguments when using the DGE. This study found that deductive justifications occurred when students were previously familiar with the activity and when students did not use any measurements.

Justification by prior knowledge was extremely rare and happened almost exclusively in conjunction with other justifications. Lastly, justifications by comparison to Euclidean geometry happened extremely rarely highlighting the distinction students are making between the two types of geometry.

Tool usage increased as expected during the hyperbolic portion of the tasks. In particular, students used both the measurement and dragging tools in a wandering aspect during the parallel transport series of tasks. This was mainly due to the challenge of finding difficult to see relationships within hyperbolic geometry. For the regular polygon series of tasks, students increased their validation use of the tools as they were mainly using the tools to verify whether they had successfully completed the constructions. There were also instances where students were attempting to use the tools in novel ways that were not originally considered. This highlights the importance of the lesser studied features of DGE.

Lastly, this discussion looked at task design. The regular polygon series of tasks was both familiar and open-ended, and it corresponded to having a much higher confidence level with much more frequent use of deductive justifications. Additionally, because of the open-ended nature of the parallel transport task, we also saw that students tended to have a more balanced usage of validation and wandering tool usages.

CHAPTER 7: CONCLUSIONS, LIMITATIONS, AND FURTHER STUDY

The key findings from Chapter 5 and 6 related to the research questions are as follows:

- The tasks elicited substantial measurement and dragging tool usage across both series of tasks. However, many students only began using these tools during the hyperbolic portion of the tasks.
- The tasks elicited multiple types of justifications, though students generally used inductive justifications throughout both series of tasks. Deductive justifications, when they occurred, most often accompanied tasks that were in both Euclidean geometry and familiar to students.
- Reported confidence was generally high across both Euclidean and non-Euclidean tasks when using the DGE. The lowest reported confidences occurred for prompts not involving the DGE (e.g. the conjecturing prompts).

Recommendations

This study has implications for at least three groups. First, this study can inform further research as it relates to confidence, justification, and tool usages. Second, this study has implications for educators. As this was a task-based research project, how students completed the tasks and how they justified their response can inform how teachers create tasks in the future. Lastly, this study can inform software developers as they continue to develop DGEs for students to use.

Recommendations for Researchers as Related to the Research Questions

As was seen, reported confidence when using a DGE was generally high - across both Euclidean and hyperbolic geometry. Previous research has generally focused on only one type of geometry. For instance, Hollebrands et al. (2010) describe instances where DGEs can help provide confidence as conviction when students are working in hyperbolic geometry. Similarly, Guven and Karatas (2009) described instances of students gaining confidence as conviction within elliptic geometry. This dissertation furthers that by showing that confidence when using DGE remained relatively high as students completed similar tasks across different geometries.

There were instances in this study, however, where students did report low confidence when using the DGE. These mostly occurred in hyperbolic geometry for open-ended questions that lacked multiple foci. As the visual nature of hyperbolic geometry tends to obscure relationships, is it common for students to report low confidence when questions are open-ended in hyperbolic geometry or was the low confidence because of this specific task? Due to the limited research that has taken place within non-Euclidean geometries, further research is necessary to explore a wider variety of tasks within hyperbolic geometry - including tasks with both open and closed questions and tasks that can be replicated within both geometries.

Additionally, though confidence has currently been looked at from both a conviction view (Weber et al., 2014) and a self-efficacy view (Bandura, 1997; Pajares & Miller, 1994), there is room to expand on how students interpret the term confidence. As we saw, even within the conviction view of confidence students held two different views. At times, their confidence referred to the correctness of the answer (what the instructor was looking for) and at other times their confidence referred to reporting a mathematically true statement (and not necessarily what the instructor was looking for). Note that this is only an issue when a student on their own is making a conjecture. Existing research on confidence as conviction (Segal, 1999; Weber et al., 2014) give participants claims and justifications to decide on their confidence in the existing claim or justification. Yet in this study, students were rating their own conjectures. More research is needed on confidence as conviction when students are reporting confidence on the claim they are making. Is confidence higher when justifying an existing conjecture or when creating and justifying one's own conjecture?

As was expected, this study saw an overwhelming number of inductive justifications being made. This was not unexpected as previous research has shown that students will often use the evidence the DGEs provides as justification (Hollebrands et al., 2010; Olivero & Robutti, 2007). Yet, there is also research showing that DGEs can be a motivation for students to pursue deductive justifications (Guven & Karatas, 2009). Additionally, Kmetová et al. (2019) shows students using DGEs to form conjectures, but acknowledging that inductive arguments are not sufficient. Why then do some students, such as those in this study, primarily use inductive justifications, while other studies show that DGEs can provide motivations to reach for deductive justifications? For this study, students gave deductive justifications when completing the Euclidean constructions for regular triangles and quadrilaterals but did not give deductive justifications when completing the hyperbolic constructions – even when the justifications were identical. If students had started with the hyperbolic constructions before Euclidean constructions, would they have given deductive justifications for the hyperbolic constructions?

As deductive reasoning is often a goal of mathematics education, more research is needed into how to implement DGEs in ways that can lead students toward deductive justification. Hadas et al. (2000) showed how contradiction and uncertainty can be used to guide students toward deductive justification. Their study involved students working in Euclidean geometry. Moving away from Euclidean into hyperbolic, as was done in this dissertation, would seem to introduce uncertainty and yet the students in this dissertation still reported high confidence without feeling the need to move to deductive justifications. If deduction is a goal, then more research is needed to explore what types of tasks, be they Euclidean or non-Euclidean, lead students toward deductive justification. For instance, these tasks took place after the semester had ended. Might repeating the tasks during the semester as students are in the midst of studying hyperbolic geometry make a difference?

Additionally, as Kmetová et al. (2019) discussed, expectations may have contributed to when students chose to give deductive justifications. This study purposefully did not specify to students how they had to justify their statements. Repeating these same series of tasks with language implying students should “justify” or “prove” would undoubtably lead to an increase in deductive justifications. But would students’ confidence remain as high as the ten students in this study? That is, research is needed investing how requiring certain justifications may impact student confidence.

In this study, there were situations where students used the tools in wandering modes and in validation modes. Much of the research into tool usage modalities (Baccaglini-Frank & Mariotti, 2010; Olivero & Robutti, 2007) has focused on activities within Euclidean geometry. Making the switch to hyperbolic geometry was expected to result in a greater amount of wandering tool usage as students were unfamiliar with that type of geometry. While that occurred, it was mainly seen as students worked on the hyperbolic parallel transport. This task was both open-ended and novel. Were these two aspects of the tasks the reasons for the increase of wandering measurements? Research into task design within non-Euclidean geometry is needed to see how varying the types of tasks affects how students are using the tools.

In summary, this research points to a multitude of opportunities for future research. This study showed differences in how students used tools, made justifications, and reported confidence as they moved not only across the tasks, but within the tasks. How might different tasks affect the results? How does task design (e.g. open-ended questions, novel questions, singular focus questions) affect how students interact with DGE? How are students interpreting confidence? In this study, students interpreted confidence in several ways. Specifically, confidence as conviction was interpreted two different ways. Further study can explore why students switch between these two different views.

Other Recommendations for Researchers

An area of interest not directly related to any one research question is the need to study the impact of partner work when studying tool usages, justifications, and confidence as students complete activities involving DGEs. Pairs were chosen specifically to foster dialogue between the students. This is a common procedure used in numerous studies (e.g., Baccaglini-Frank & Mariotti, 2010; Hollebrands, 2007). The expectation was that this choice would lead to sustained dialogue between students because they would be comfortable with each other. While this worked well for some pairs, for other pairs (e.g. Eve and Fran), there were long gaps where students worked independently with minimal interaction.

Research into the effects of partner work is one avenue that can be explored. For instance, the issue of dominant partners. Beth and Ann were a prime example of this imbalance between the partners. Beth directed much of the conversation and activity within their work. Beth, as a dominant partner, was often able to steer Ann away from deductive justifications toward inductive justifications. There were several instances where Ann appeared ready to attempt deductive justification, but Beth jumped in and sidetracked Ann with inductive evidence Beth had gained from the software. Including partners in this study, influenced how students justified their claims. Ann may have given many more deductive justifications had Beth not been there.

These two issues related to partner work are at odds. On the one hand, partners generally encouraged discussion between the students. There were many instances of partners working in tandem to complete tasks or create justifications. On the other hand, having partners led to power imbalances where one partner had strong influence over the other. There is some research exploring issues of students working in pairs (Evens & Houssart, 2007; Wilson et al., 2016). Yet, the use of DGEs alongside paired activities undoubtedly plays a large part of how students interact and think about the geometry. For instance, there were instances where partners like Carl and Dan specifically used their separate machines to make different examples. There were other instances where one student discovered a construction, and the other student deferred to that construction. In the case of Eve and Fran, Fran began a productive approach using drawing rather than construction but was pulled away by Eve's constructions. When working with DGEs and partners, how might only using one computer affect tool usage?

A last area of research that needs exploring is a look at other tools of the DGE beyond the dragging and measurement tools. As we saw, students used, or attempted to use, tools beyond just the dragging and measurement tools. For instance, Beth, wanted an easy way to zoom in on an image. When working on the parallel transport in Euclidean geometry, Beth did not trust that the segments really were on top of each other. Being able to zoom would have alleviated her concerns. Much like the dragging tool, zooming is a fundamental difference between the physical world of paper and pencil and the digital domain of DGEs.

Recommendations for Educators

As was seen in this study, students made overwhelmingly inductive justifications. If an educator is wanting their students to make deductive arguments, the teacher must be conscious about teaching students how to use DGE in a way that supports deductive argument. Educators need to focus on the exploration and discovery aspects of DGE (DeVilliers, 1998), and leverage those aspects to the students' advantage. While the DGE may be enough to give students data to justify their conjectures and give them confidence, it does not always translate to students constructing a proof. Some of this does relate to how students are taught about what is or is not a valid justification (Segal, 1999). Kmetová et al. (2019) give examples of students using DGE to explore and investigate, but also recognize that even though they may be convinced, the DGE does not mathematically justify their conjectures.

This reaffirms the suggestion from Olivero and Robutti (2007) that educators need to teach students "how to use them [the tools] in a productive way" (p. 154). For instance, in the regular polygon series of tasks, we saw students verify that they had successfully created regular polygons, but those validation measurements rarely translated to deductive justifications. In hyperbolic geometry, only Hal gave a deductive justification for the regular triangle. Others undoubtably could have but may not have felt the need. Olivero and Robutti (2007) suggest this is where instruction in the tool use itself can become important because tool usage can be used to lead students towards deduction. Teaching students productive tool usages can also give students reassurance that they can spend time on the exploration phase without feeling the pressure of getting an answer right away.

A good example of how to use the DGE productively can be seen by looking at Fran. She was the only student in the study to use the tools to draw, rather than to construct. Rather than begin with a construction, Fran put four segments on the screen as a drawing in an approximate square shape and attempted to drag it into what she thought was a regular polygon. She was attempting to see if they even existed; and if so, what they may look like. Had Fran been taught how to use the tools in a guided wandering method (Olivero & Robutti, 2007), she may have taken the next steps to measure the sides and angles and noticed the crucial detail that the corner angles were not right angles. This is a crucial understanding that is needed to be productive in the construction. This example reinforces how students need to be taught the various modalities and how they can be used both for exploration and as a guide toward deduction.

Another recommendation for educators is to teach students how to use DGE to support proof comprehension, validation, and evaluation (Selden & Selden, 2017). When reading the proof for the parallel transport task, reported student confidence decreased. A common comment regarding how to raise their confidence was to "do the proof" in Geometry Explorer and to see if it worked in all cases. The students all had access to Geometry Explorer while they read the proof, but none made use of it despite thinking it might raise their confidence. This suggests that although they want to use DGE to help support their proof, they may not know how DGE can be used to support the proof.

We know that students do not always gain conviction from reading proofs (Knuth, 2014) and so teachers can help students use proof for explanation and discovery (DeVilliers, 1998). The proof in the parallel transport series of tasks was a good example of how this could be done. In basic terms, the proof follows an angle as it makes its' way around a triangle. This gives students an opportunity to 'do' the proof in DGE by constructing the angle and following it along with measurements as the proof proceeds. Students can then see the proof step by step. But if students think of DGE and proof as separate activities, this link may not occur. Thus, both teachers and students should be encouraged to use a DGE to help clarify and explain the individual steps of a proof as they are being taken.

A last recommendation for teachers is to discuss appropriate use of justifications. We know mathematics instructors often will espouse a belief in the importance of deduction. We also know mathematics instructors will use DGEs because they increase student interest (Barcelos et al., 2011; Pandiscio, 2002). However, we see that use of a DGE, while giving students confidence and increased student interest, can lead students away from deductive arguments in favor of inductive arguments. Students who otherwise may have given deductive arguments for such proofs as the construction of a square, instead gave inductive arguments. Educators must be clear about what expectations are for students. That is, educators can focus on the distinction of validity and conviction (Segal, 1999) and how DGE can support both.

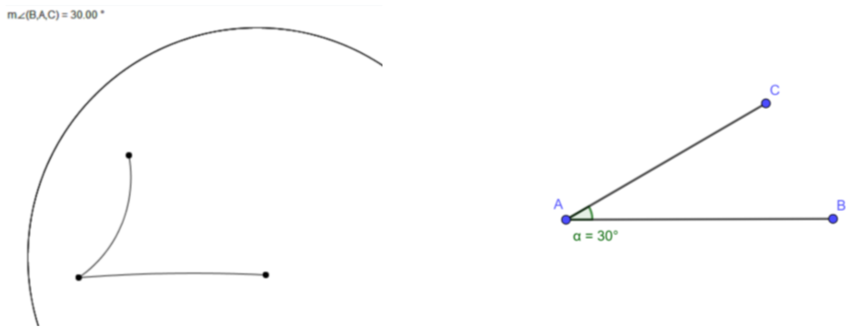
Recommendations for Software Developers

Geometry Explorer was used during this research because it was available software that could model hyperbolic geometry using the same interface as Euclidean geometry. This is not an endorsement of Geometry Explorer, only a recognition that it succeeds in giving users a unified experience across geometries. There are other DGE available that have many features that exceed Geometry Explorer at the expense of only focusing on one type of geometry. GeoGebra is a DGE that has emerged over the last decade that is quickly becoming a standard in high school classrooms, and is starting to find its way into newer research involving DGE (e.g. Albaladejo et al., 2015; Kutluca, 2013). As a newer piece of software with an active development team, it offers many features that Geometry Explorer does not offer. For instance, GeoGebra offers easy zooming via the mouse, automatic labeling of points, and angle symbols any time an angle measurement is taken.

As developers keep creating software, it is important to include as many of these little features as possible. For instance, the GeoGebra angle symbols are an important feature that may affect how students use the angle measurement tool in hyperbolic geometry. In Figure 7.1, we see two 30-degree angles. On the left is a hyperbolic 30-degree angle as measured in Geometry

Figure 7.1:

Differences Between Angle Measurements in Geometry Explorer and GeoGebra



Explorer and on the right is a Euclidean 30-degree angle in GeoGebra. While both DGEs show the same information, in Geometry Explorer the angle measurement is placed far from the angle, and the angle has no indication it is the specific angle being measured. On the right in GeoGebra, we see the measurement next to the angle with a shaded area to highlight the angle. As diagrams move beyond a single angle and as they get dragged around, it becomes easy to lose track of the 30-degree angle in Geometry Explorer. Putting measurements next to the angles, using angle symbols, and automatically labeling points has the potential to make the tool much more useful to the student. When students are engaging in wandering dragging, these small things may have a big difference in what students notice.

This is not meant to be an attack on the developers of Geometry Explorer. Being designed specifically to support non-Euclidean geometry, which is typically only studied at the collegiate level, guarantees the user base and the development team are much smaller than software such as GeoGebra. Developers only have so much time and effort they can devote. My critique is simply to point out to developers that small decisions may have large impacts on how students use the tools. One thing Geometry Explorer does well is that it allows for tabbed windows within the program itself. This was noticeable in this study as students had the ability to easily reference previous work. When completing similar tasks, students were not forced to save, close, and reopen previous work. Rather, the students could simply switch to the previous tab to find what they were working on.

Limitations

As with any research study, there were limitations to the study. Some of those limitations were known ahead of time, while others arose as the research was being analyzed. Student population, limited questions, and how tool usage was counted were all limitations that were expected and planned for. Students' interpretations of confidence and their brief responses, however, were not planned for.

Student Population

As this is a small study consisting of only ten students, the claims being made are only directly applicable to those ten students. Additionally, those ten students were all from the same school and took multiple math classes together, including the same collegiate geometry course. Because of this, it is to be expected that there is some homogeneity with how students approached the tasks. For instance, only one student, Fran, ever used the DGE to draw as compared to construct. The other nine students all began directly with construction. With a larger group of students from a more diverse mathematical background, it is likely more students would have engaged in that type of behavior. This is an area where a course instructor may find it useful to model such a skill to their students. Because of this small sample size and the similarity of the students, these results should not be generalized to larger groups of students. Instead, future research can explore whether the trends seen with these students generalize to more diverse populations.

Limited Student Activity

A second limitation of this study is the variety and number of prompts to which students responded. The first series of tasks involved transformational geometry and parallel transports. The second series involved geometric constructions. These two tasks are limited in scope. There are a wide range of other types of questions that can be asked and should be asked of students. For instance, questions from analytic geometry or vector geometry could expand on the type of problems. Might measurement tools become an even greater source of justification in analytic geometry because of analytic geometry's greater focus on numbers and coordinates? Including a wider variety of constructions beyond regular polygons or other types of transformations could also show if the increased validation tool usage we saw with these students carries over to other constructions. This study took place after students had taken a course in geometry. Having students complete a variety of tasks throughout a geometry course could provide a wider range of tasks. Completing tasks throughout a course could also bring insight into if students learn wider varieties of tool usage as they proceed through a geometry course.

Counting of Instances of Tool Usage Versus Counting of Tool Usage Types

Another limitation of this research is how tool usage was being counted. For this study, tool usage counts were based on whether the student used the tool in a given modality at least once during each prompt. A student may have made multiple validation measurements, but it would only have been counted once. There are both advantages and disadvantages to this approach. By only counting a tool usage once, the amount of wandering tool usages was more balanced with validation tool usages. By the nature of wandering usages, there is an expectation that wandering usages will far outnumber validation usages. For instance, a student validating a conjecture only needs to measure once. Comparatively, a student who is making wandering measurements may make numerous measurements until they find a relationship. Counting only whether a tool had been used allowed for a greater focus on *how* the tool was used rather than *how often* the tool was used.

With that said, only counting types of tool usages has its drawbacks. First, it limits some types of comparisons among the prompts. While two prompts may be listed as having the same amount of wandering measurement, this research does not report on how many instances of wandering measurement took place. This is a potentially useful piece of information; not simply to know that two prompts both had wandering dragging, but to know which prompt had more wandering dragging.

Various Understandings of Confidence

Another limitation of this study is how students chose to understand the term confidence. As discussed in Chapter 6, students used multiple interpretations of confidence – confidence as self-efficacy and confidence as conviction. More so, confidence as conviction was also broken into two types of confidence. At times it was clear that students were rating their confidence in the correctness of their answer, and at times it was clear students were rating their confidence in the accurate reporting of their answer. It is feasible that these multiple ways of understanding confidence influenced how confidence was reported. Having a unified understanding of confidence, using either understanding, will certainly change the results.

Brevity of Students Responses

One last limitation of the study was the brevity of student responses. There were instances throughout the study where students' justifications were short and non-descriptive. Having video evidence to support those justifications proved essential. At times, students verbalized their justification, and then only provided a short summary of that justification. Ann, for instance, gave a verbal proof that her Euclidean regular triangle was correct, but only wrote a few brief comments. In future studies, using these same tasks but with differently worded prompts may encourage students to spend more time on their written responses yielding more accurate reporting of justifications and allowing for a more precise categorization of those justifications.

Next Plans

This section will discuss future plans for this research and changes that can be made in future research for greater clarity. First, this research would benefit from taking place over an extended period of time with a wider variety of questions. Conducting this research throughout a semester course would allow the researcher to see how tool use changes in a wider variety of activities. As the current results are highly dependent on the two series of tasks, it would be an interesting question to see how the results change when the tasks change. For instance, in both series of tasks, students were asked to create their own diagrams. In the parallel transport series of tasks, the diagram creation was guided. In the construction tasks, students were left to create their own diagrams. Recently, Unal and Hollebrands (2021) investigated how diagrams affect student interactions. They found that providing diagrams led students to be more likely to find solutions to the given geometric problems, but also corresponded to mostly empirical justification. Kmetová et al. (2019) conducted a small survey to see in what parts of a task students found DGE use to be most helpful. The most popular answer was the use of DGE in finding relationships. If that is indeed how students see the use of DGE, can we design tasks around finding relationships? Or can tasks be designed to show students other important features of DGE such as finding locus of points or using it to help construct proofs.

Second, this research would also benefit from a broader range of students completing the activities. The participants in this study were a relatively homogenous group of students. Recruiting students from multiple mathematics programs would predictably change the results. Students who have used different textbooks or have different instructors would naturally approach these tasks in different ways. It is possible more students would have approached the regular polygons as Fran did by beginning with a drawing rather than a figure. It is also possible that based on classroom norms, students would have given different justifications based on what had been considered appropriate justifications.

Third, repeating some of these same activities but with different word choice may also prove enlightening. In the activities for this dissertation, the prompts avoided using the words "justify" and "prove". Rather, the prompts asked the students, "why?", allowing students to justify their responses as they wished. Future studies could experiment with word choice to see how word choice affects student responses. The differences between these words may impact whether students are responding based on what convinces them compared to what they know to be a valid deductive argument (Segal, 1999).

Fourth, for this current dissertation, students completed the series of tasks with minimal interaction from the researcher. Repeating this study in the future with greater researcher interaction could prove beneficial. Incorporating ideas from current research on paired interviews (e.g. Wilson et al., 2016) could lead to greater clarity in student responses. For instance, greater interaction from the researcher could provide clarification of student justifications and reasoning. Asking questions such as "Can you expand on that justification?" or "Can you clarify what you mean?" may prompt students to think more deeply about their responses. Likewise, interaction from the researcher may help to alleviate power imbalances among partners. Issues that arose in this study where one partner deferred to the other could have been alleviated by researcher interaction.

In summary, there are at least four avenues for future research. First, a wider variety of tasks can provide additional insight into how students use the tools, justify their conjectures, and rate their confidence. Secondly, and similar to the first, a wider variety of students would introduce a wider range of tool usage and justifications. Third, word choice within the tasks themselves may lead to certain types of justifications and tool usage. Repeating the study with different choices of words may impact how students are thinking about the tasks. Lastly, the amount of researcher interaction has the potential to dig deeper into how students are thinking as well as affect how students interact with each other.

Future Questions

As with all research, questions arise from the analysis and conclusions. Specific questions include:

- Beyond the major tools of DGEs (dragging and measurement), how do other tools aid in student understanding and student confidence? This research focused on the major tools (dragging and measurement), but there are tools like zooming that may affect how students use the software.
- What is the role of previous expectations as students complete DGE activities? How do students' previous understandings of what it means to be mathematically valid affect the justifications they provide or the confidence they report?
- How does the language of the prompts affect the justifications students use when completing similar activities using DGEs?

- How do partner dynamics affect how students think about geometry while completing activities using a DGE?
- How is confidence affected by being given a statement to justify compared to having to conjecture and then justify one's own statement?

APPENDICES

APPENDIX A: DESIGNED TASKS - PARALLEL TRANSPORTS

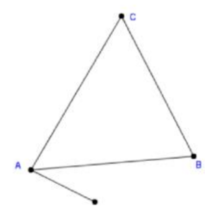
Figure A.1:

Geometry Tasks – Parallel Transports – Page 1

Geometry Tasks – Parallel Transports

As you work through the following tasks, please complete each question before moving on to the next question, and each page before moving on to the next page.

Task 1: Parallel Transports in Euclidean Geometry



Each of you should complete the following task on your own computer, working together as you go.

Create a new document with Euclidean geometry and create an arbitrary triangle, ΔABC, within the document.

From vertex A, create a segment pointing out from the triangle in any direction you like.

Step 1:

We will now transport this segment around all three vertices.

Highlight vertex A, and then B, and then click 'Mark' and then 'Vector' to tell GEX that we want to use $\overrightarrow{AB}$ as a translation vector.

Highlight the segment and the new point and then hit the 'Translate' button to translate the segment and the point to vertex B.

<u>1a) What do you and your partner notice about the segment?</u>

Answer the following 2 questions on your own.

<u>1b) On a scale of 1 to 5, how confident are you in your response to the previous question? Why?</u>

1 = Not Confident, 2 = Somewhat Not Confident, 3 = Neutral, 4 = Somewhat Confident, 5 = Very Confident

<u>1c) If you did not rate yourself a 5, what would it take to raise your confidence level to a 5?</u>

Figure A.2:

Geometry Tasks – Parallel Transports – Page 2

Step 2:

Repeat Step 1 with the new segment, but this time translate the segment to vertex C using vector $\overrightarrow{BC}$.

Step 3:

Repeat Step 1 again with the newest segment, but this time, translate the segment back to A using vector $\overrightarrow{CA}$.

2a) What do you and your partner notice about the final segment compared to the first segment?

Answer the following 2 questions on your own.

2b) On a scale of 1 to 5, how confident are you in your response to the previous question? Why?

1 = Not Confident, 2 = Somewhat Not Confident, 3 = Neutral, 4 = Somewhat Confident, 5 = Very Confident

2c) If you did not rate yourself a 5, what would it take to raise your confidence level to a 5?

Figure A.3:

Geometry Tasks – Parallel Transports – Page 3

Task 2: Thinking about Parallel Transports in Hyperbolic Geometry

Answer the following questions as they relate to hyperbolic geometry:

3a) If we complete the same activity as Task 1, but in hyperbolic geometry, <u>what do you and your partner think will happen in terms of the final segment compared to the first segment?</u>

Answer the following 2 questions on your own.

<u>3b) On a scale of 1 to 5, how confident are you in your response to the previous question? Why?</u>

1 = Not Confident, 2 = Somewhat Not Confident, 3 = Neutral, 4 = Somewhat Confident, 5 = Very Confident

<u>3c) If you did not rate yourself a 5, what would it take to raise your confidence level to a 5?</u>

Figure A.4:

Geometry Tasks – Parallel Transports – Page 4

Task 3: Parallel Transports in Hyperbolic Geometry

Now, repeat the activity from Task 1 with your partner, but within hyperbolic geometry and with your own computer, and then answer the following questions.

4a) What do you and your partner notice about the final segment compared to the first segment?

Answer the following 2 questions on your own.

4b) On a scale of 1 to 5, how confident are you in your response to the previous question? Why?

1 = Not Confident, 2 = Somewhat Not Confident, 3 = Neutral, 4 = Somewhat Confident, 5 = Very Confident

4c) If you did not rate yourself a 5, what would it take to raise your confidence level to a 5?

Figure A.5:

Geometry Tasks – Parallel Transports – Page 5

Task 4: Defect of Triangles in Hyperbolic Geometry

5a) If you haven't already done so, find the *defect* of the triangle you created. What do you and your partner notice about the defect of the triangle?

Answer the following 2 questions on your own.

5b) On a scale of 1 to 5, how confident are you in your response to the previous question? Why?

1 = Not Confident, 2 = Somewhat Not Confident, 3 = Neutral, 4 = Somewhat Confident, 5 = Very Confident

5c) If you did not rate yourself a 5, what would it take to raise your confidence level to a 5?

Figure A.6:

Geometry Tasks – Parallel Transports – Page 6

Task 5: Parallel Transports in Hyperbolic Geometry

Consider the following proof that the defect of the triangle will match the angle between the original segment and the transported segment.

Let β_1, β_2 and β_3 be the interior angles of the triangle and α_1, α_2, and α_3 be the exterior angles of the same triangle as in the diagram to the right. Note that while the diagram to the right was for an elliptic triangle, all the angle names match this proof.

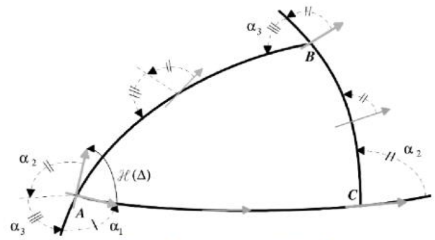


Figure 5.2. Holonomy of a geodesic triangle.

Assume our original segment lies parallel to $\overline{AC}$.

After the first transport the segment remains parallel to $\overline{AB}$.

As we move along the second transport, notice the segment now forms angle α_2 with $\overline{BC}$ by definition of how exterior angles are defined.

Notice when the segment arrives at B, the segment forms angle $\alpha_2 + \alpha_3$ with $\overline{AB}$.

Finally, after the third transport, when the segment arrives back at A, the angle is now $\alpha_1 + \alpha_2 + \alpha_3$.

Thus, the difference in angles (or *holonomy*) of the triangle is $360 - (\alpha_1 + \alpha_2 + \alpha_3)$.

Note that this is negative as $\alpha_1 + \alpha_2 + \alpha_3 > 180°$.

But we also know that $\alpha_i + \beta_i = 180$ for each angle. Thus, we can say

$$\text{Holonomy} = 360 - (\alpha_1 + \alpha_2 + \alpha_3)$$
$$= 360 - (180 - \beta_1 + 180 - \beta_2 + 180 - \beta_3)$$
$$= (\beta_1 + \beta_2 + \beta_3) - 180$$
$$= \text{Defect}$$

Answer the following 2 questions on your own.

6a) On a scale of 1 to 5, how confident are you that the defect of the triangle will always match the holonomy? Why?

1 = Not Confident, 2 = Somewhat Not Confident, 3 = Neutral, 4 = Somewhat Confident, 5 = Very Confident

6b) If you did not rate yourself a 5, what would it take to raise your confidence level to a 5?

Figure A.7:

Geometry Tasks – Parallel Transports – Page 7

Task 6: Drawing a Confidence Graph

At various times throughout these tasks, your confidence in your claims was probably high and your confidence was probably low. Fill in the graph below with a line that represents your confidence level throughout the activity. Label any significant points throughtout the activity that contributed postively or negatively to your confidence level.

Fill in this graph on your own.

APPENDIX B: DESIGNED TASKS - PARALLEL TRANSPORTS

Figure B.1:

Geometry Tasks – Regular Polygons – Page 1

Geometry Tasks – Regular Polygons

As you work through the following tasks, please complete each question before moving on to the next question, and each page before moving on to the next page.

Task 1: Regular Triangles in Euclidean Geometry

Recall the definition of regular polygons. They are polygons whose sides are all congruent and whose interior angles are also all congruent.

With your partner, create a new document with Euclidean geometry and construct a regular triangle in Euclidean geometry.

1a) Have you and your partner successfully constructed a regular triangle? Why?

Answer the following 2 questions on your own.

1b) On a scale of 1 to 5, how confident are you in your response to the previous question? Why?

1 = Not Confident, 2 = Somewhat Not Confident, 3 = Neutral, 4 = Somewhat Confident, 5 = Very Confident

1c) If you did not rate yourself a 5, what would it take to raise your confidence level to a 5?

Figure B.2:

Geometry Tasks – Regular Polygons – Page 2

Task 2: Regular Quadrilaterals in Euclidean Geometry

With your partner, create a new document with Euclidean geometry and construct a regular quadrilateral in Euclidean geometry.

2a) Have you and your partner successfully constructed a regular quadrilateral? Why?

Answer the following 2 questions on your own.

2b) On a scale of 1 to 5, how confident are you in your response to the previous question? Why?

1 = Not Confident, 2 = Somewhat Not Confident, 3 = Neutral, 4 = Somewhat Confident, 5 = Very Confident

2c) If you did not rate yourself a 5, what would it take to raise your confidence level to a 5?

Figure B.3:

Geometry Tasks – Regular Polygons – Page 3

Task 3: Regular Triangles in Hyperbolic Geometry

With your partner, create a new document with Hyperbolic geometry and construct a regular triangle in hyperbolic geometry.

<u>3a) Have you and your partner correctly constructed a regular triangle? Why?</u>

Answer the following 2 questions on your own.

<u>3b) On a scale of 1 to 5, how confident are you in your response to the previous question? Why?</u>

1 = Not Confident, 2 = Somewhat Not Confident, 3 = Neutral, 4 = Somewhat Confident, 5 = Very Confident

<u>3c) If you did not rate yourself a 5, what would it take to raise your confidence level to a 5?</u>

Figure B.4:

Geometry Tasks – Regular Polygons – Page 4

Task 4: Regular Quadrilaterals in Hyperbolic Geometry

You have seen that regular triangles exist in hyperbolic geometry.

4a) Do you and your partner believe that regular quadrilaterals exist in hyperbolic geometry? Why or why not?

Answer the following 2 questions on your own.

4b) On a scale of 1 to 5, how confident are you in your response to the previous question? Why?

1 = Not Confident, 2 = Somewhat Not Confident, 3 = Neutral, 4 = Somewhat Confident, 5 = Very Confident

4c) If you did not rate yourself a 5, what would it take to raise your confidence level to a 5?

Figure B.5:

Geometry Tasks – Regular Polygons – Page 5

Task 5: Constructing Regular Quadrilaterals in Hyperbolic Geometry

Create a new document with Hyperbolic geometry and construct a regular quadrilateral in hyperbolic geometry.

5a) Have you and your partner successfully constructed a regular quadrilateral? Why?

Answer the following 2 questions on your own.

5b) On a scale of 1 to 5, how confident are you in your response to the previous question? Why?

1 = Not Confident, 2 = Somewhat Not Confident, 3 = Neutral, 4 = Somewhat Confident, 5 = Very Confident

5c) If you did not rate yourself a 5, what would it take to raise your confidence level to a 5?

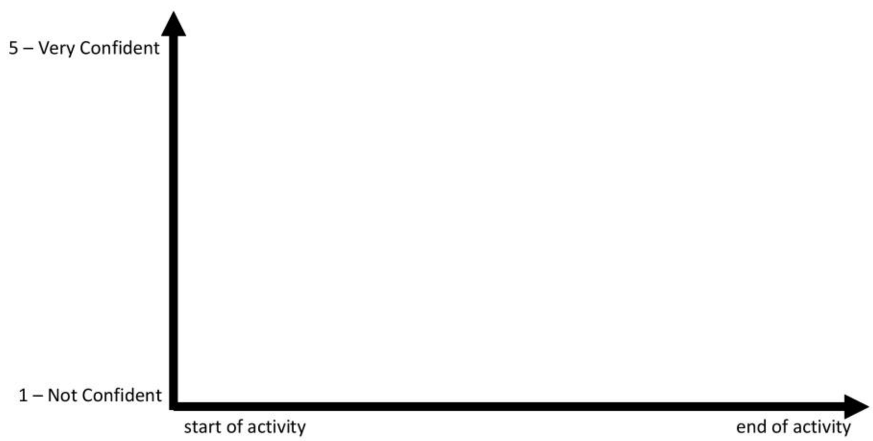
Figure B.6:

Geometry Tasks – Regular Polygons – Page 6

Task 6: Drawing a Confidence Graph

At various times throughout these tasks, your confidence in your claims was probably high and your confidence was probably low. Fill in the graph below with a line that represents your confidence level throughout the activity. Label any significant points throughtout the activity that contributed postively or negatively to your confidence level.

Fill in this graph on your own.



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