LEVEL STRUCTURES ON FINITE GROUP SCHEMES AND APPLICATIONS

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ABSTRACT

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The notion of level structures originates from the study of the moduli of elliptic curves. In this thesis, we consider generalizing the notion of level structures and make explicit calculations on different moduli spaces.

The first moduli space we consider is the moduli of finite flat (commutative) group schemes. We give a definition of $\Gamma(p)$-level structure (also called the "full level structure") over group schemes of the form $G \times G$, where G is a group scheme or rank p over a $\mathbb{Z}_p$-scheme. The full level structure over $G \times G$ is flat over the base of rank $|\operatorname{GL}_2(\mathbb{F}_p)|$. We also observe that there is no natural notion of full level structures over the stack of all finite flat commutative group schemes.

The second moduli space we consider is the moduli of principally polarized abelian surfaces in characteristic $p > 0$ with symplectic level-n structure ($n \geq 3$), which is known as the Siegel threefold. By decomposing the Siegel threefold using the Ekedahl–Oort stratification, we analyze the p-torsion group scheme of the universal abelian surface over each stratum. To do this, we establish a machinery to produce group schemes from their Dieudonné modules using a version of Dieudonné theory due to de Jong. By using this machinery, we give explicit local equations of the Hopf algebras over the superspecial locus, the supersingular locus and ordinary locus. Using these local equations, we calculate explicit equations of the $\Gamma_1(p)$-covers over these strata using Kottwitz–Wake primitive elements. These equations can be used to prove geometric and arithmetic properties of the $\Gamma_1(p)$-cover over the Siegel threefold. In particular, we prove that the $\Gamma_1(p)$-cover over the Siegel threefold is not normal.

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CHAPTER 1

INTRODUCTION

Let $\mathcal{H}$ be the complex upper half plane. The special linear group $\mathrm{SL}_2(\mathbb{Z})$ and its subgroups act on $\mathcal{H}$ by Möbius transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az+b}{cz+d}. \tag{1.1}$$

Let $N \geq 1$ be an integer. Let $\Gamma_0(N), \Gamma_1(N)$ and $\Gamma(N)$ be subgroups of $\mathrm{SL}_2(\mathbb{Z})$ defined as follows:

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \mod N \right\}, \tag{1.2}$$

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \mod N \right\}, \tag{1.3}$$

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\}. \tag{1.4}$$

In general, we call a subgroup Γ a congruence subgroup (of $\mathrm{SL}_2(\mathbb{Z})$) if Γ contains $\Gamma(N)$ for some integer $N \geq 1$. The subgroups $\Gamma_0(N), \Gamma_1(N)$ and $\Gamma(N)$ are the most interesting congruence subgroups. In particular, the quotients of the upper half plane by these congruence subgroups have the following moduli interpretations:

$$\Gamma_0(N)\backslash\mathcal{H} = \left\{ \begin{array}{c} \text{isomorphism class of pairs } (E, G)\text{, where } E \text{ is an elliptic curve} \\ \text{over } \mathbb{C} \text{ and } G \subset E \text{ a subgroup of order } N \end{array} \right\},$$

$$\Gamma_1(N)\backslash\mathcal{H} = \left\{ \begin{array}{c} \text{isomorphism class of pairs } (E, P)\text{, where } E \text{ is an elliptic curve} \\ \text{over } \mathbb{C} \text{ and } P \in E \text{ generates a subgroup of order } N \end{array} \right\},$$

$$\Gamma(N)\backslash\mathcal{H} = \left\{ \begin{array}{c} \text{isomorphism class of triples } (E, P, Q)\text{, where } E \text{ is an elliptic} \\ \text{curve over } \mathbb{C} \text{ and } P, Q \in E \text{ generate the } N\text{-torsion points } E[N] \\ \text{with } \langle P, Q\rangle = e^{2\pi i/N} \text{ for the Weil pairing } \langle\cdot,\cdot\rangle \end{array} \right\}.$$

These quotients are denoted by $Y_0(N)$ (resp. $Y_1(N)$, $Y(N)$). The modular curves $X_0(N)$ (resp. $X_1(N)$, $X(N)$) are constructed by compactifying $Y_0(N)$ (resp. $Y_1(N)$, $Y(N)$). These modular curves are known to admit smooth models over $\mathbb{Z}[1/N]$.

As we can see above, modular curves arise as the moduli spaces of elliptic curves with some extra structures. These extra structures are called level structures. In particular, the extra structures that appear in the moduli description of $Y_0(N)$ (resp. $Y_1(N)$, $Y(N)$) are called $\Gamma_0(N)$ (resp. $\Gamma_1(N)$, $\Gamma(N)$)-level structures.

The first systematic study of integral models of modular curves over $\mathbb{Z}$ was done by Deligne–Rapoport [8], who construct models of $X_0(p)$ and $X_1(p)$ over the p-adic integers $\mathbb{Z}_p$. In [19], Katz and Mazur construct integral models of $X_0(N), X_1(N)$ and $X(N)$, by carefully defining the moduli problems of elliptic curves with level structures. For example, following an idea of Drinfeld in [10], Katz and Mazur define a set of sections $\{P_1, \ldots, P_{N^2}\}$ of $E[N]$ to be a "full set of sections", if the points generate the group scheme $E[N]$ as Cartier divisors. Using this notion, the $\Gamma(N)$-level structure on $E[N]$, also called "full level structure", is defined to be the maps in $\mathrm{Hom}((\mathbb{Z}/N\mathbb{Z})^2, E[N])$ whose images form a full set of sections. Katz and Mazur use this notion of full level structure to construct integral models of $X(N)$.

In Chapter 2, we review some preliminaries such as the Oort–Tate and Raynaud theory of group schemes of order p, the Ekedahl–Oort stratification and the definition of primitive elements of group schemes due to Kottwitz–Wake. Throughout the rest of this thesis, the central questions we consider are to generalize these notions of level structures, and make explicit calculations on different moduli spaces. Particularly, we consider $\Gamma(p)$-level structures on moduli of finite flat group schemes and $\Gamma_1(p)$-level structures on the moduli of principally polarized abelian surfaces with symplectic level structure. For each of the case, we consider the following questions:

(A) How to define a good notion of level structures over the moduli space?

(B) Given a good notion of level structure over the moduli, find equations that describe the universal covers, at first locally.

(C) What arithmetic/geometric properties can we obtain from the above descriptions?

First we consider the moduli of finite flat group schemes (over $\mathbb{Z}_p$) and consider Question (A) for $\Gamma(p)$-level structures (also called "full level structure") on it. Note that although there is no "moduli space" of finite flat group schemes, we can still consider the "moduli stack" instead. Here by a stack, we simply mean a category fibered in groupoids over $\mathbf{Sch}_{\mathbb{Z}_p}$ as in [6]. More precisely, let $\mathbf{C}$ be a stack of group schemes G/S of certain type (for example, finite flat commutative so that $G[1/p]$ is étale locally isomorphic to $(\mathbb{Z}/p^n\mathbb{Z})^g$) over $\mathbf{Sch}_{\mathbb{Z}_p}$. The objects in $\mathbf{C}$ are group schemes G/S of the fixed type and the morphisms are Cartesian squares. By a "good" $\Gamma(p)$-level structure over $\mathbf{C}$, we mean a fibered functor $\mathcal{F}\colon \mathbf{C} \to \mathbf{Sch}$, such that $\mathcal{F}(G/S)$ is a closed subscheme of $\underline{\mathrm{Hom}}_S((\mathbb{Z}/p^n\mathbb{Z})^g, G)$ and such that for $f\colon G/S \to G'/S'$, the morphism $\mathcal{F}(f)\colon \mathcal{F}(G/S) \to \mathcal{F}(G'/S')$ is the restriction of the morphism $\underline{\mathrm{Hom}}_S((\mathbb{Z}/p^n\mathbb{Z})^g, G) \to \underline{\mathrm{Hom}}_{S'}((\mathbb{Z}/p^n\mathbb{Z})^g, G')$ induced by f, and satisfies the following conditions:

(1) $\mathcal{F}(G/S)$ is flat over S and of rank $|\mathrm{GL}_g(\mathbb{Z}/p^n\mathbb{Z})|$.

(2) $\mathcal{F}(G/S)$ is invariant under the right $\mathrm{GL}_g(\mathbb{Z}/p^n\mathbb{Z})$-action on $\underline{\mathrm{Hom}}_S((\mathbb{Z}/p^n\mathbb{Z})^g, G)$. When inverting p, we have an identification

$$\mathcal{F}(G[\tfrac{1}{p}]/S[\tfrac{1}{p}]) = \mathrm{Isom}_{S[\frac{1}{p}]}((\mathbb{Z}/p^n\mathbb{Z})^g, G[\tfrac{1}{p}])$$

as closed subschemes of $\underline{\mathrm{Hom}}_{S[\frac{1}{p}]}((\mathbb{Z}/p^n\mathbb{Z})^g, G[\frac{1}{p}])$.

(3) When identifying

$$\underline{\mathrm{Hom}}_S((\mathbb{Z}/p^n\mathbb{Z})^g, G) \times_S T = \underline{\mathrm{Hom}}_T((\mathbb{Z}/p^n\mathbb{Z})^g, G_T)$$

in the natural way, we have

$$\mathcal{F}(G/S) \times_S T = \mathcal{F}(G_T/T)$$

as closed subschemes, for any S-scheme T.

(4) For any group scheme isomorphism $G \xrightarrow{\sim} G'$, the induced isomorphism

$$\underline{\mathrm{Hom}}_S((\mathbb{Z}/p^n\mathbb{Z})^g, G) \xrightarrow{\sim} \underline{\mathrm{Hom}}_S((\mathbb{Z}/p^n\mathbb{Z})^g, G')$$

restricts to an isomorphism

$$\mathcal{F}(G/S) \xrightarrow{\sim} \mathcal{F}(G'/S).$$

The condition (4) is automatic from being a functor.

There have been many attempts by other mathematicians to give such a construction. In [19], Katz and Mazur suggest a construction of full level structures for general group schemes. Unfortunately, this is shown to be badly behaved (for example, not flat) by Chai and Norman in [3]. In [42], Wake gives a good notion of full level structure for $\mu_p \times \mu_p$. In his paper, Wake also gives an alternative description of his full level structure on $\mu_p \times \mu_p$ that can be defined for general group schemes. However, this alternative description still fails to behave well for general group schemes.

In [21], Kottwitz and Wake construct a general notion of $\Gamma_1(p)$-level structure on finite flat group schemes, given by so-called primitive elements (see Section 2.3). In [13], we extend the result of Wake and give a definition of full level structure on all group schemes of the form $G \times G$ using the notion of Kottwitz-Wake primitive elements. Here is the main theorem in Chapter 3:

Theorem 1.0.1. *Let S be a $\mathbb{Z}_p$-scheme and let G be a finite flat commutative group scheme of rank p over S. The full level structure on $G \times G$ defined in Definition 3.1.2 satisfies condition (1)-(3).*

Back to the language of stacks, Theorem 1.0.1 implies that we defined a well-behaved notion of full level structure over the stack OT, whose objects are group schemes of the form $G \times G$ where G is an Oort–Tate scheme (see Section 2.1) over a $\mathbb{Z}_p$-scheme S, and morphisms are diagonal group scheme isomorphisms $G \times G \to G' \times G'$ induced by two identical isomorphisms of Oort–Tate group schemes $G \to G'$.

One might hope to extend this result over the stack of finite flat commutative group schemes. Unfortunately, we record the following negative result in Chapter 4:

Theorem 1.0.2. *There is no "natural" notion of full level structure over group schemes which is flat of the expected rank over the base.*

Here, by "natural", we mean one is actually defined over the stack, i.e. satisfying condition (4) as above. In fact, we observe that there is no good notion of full level structure even over the substack $OT \times OT$, whose objects are $G \times G'$ where G, G' are Oort–Tate group schemes. We show that the full level structure defined in Theorem 1.0.1 is the only possible definition that gives a flat model. The full level structure we define is preserved by the diagonal group scheme isomorphisms $G \times G \to G' \times G'$ induced by two identical isomorphisms of Oort–Tate group schemes $G \to G'$. However, there exists group schemes for which the full level structure scheme is not preserved by all automorphisms. It is notable that the counterexamples come from non-p-divisible groups. So there is still a possibility of a positive result for truncated p-divisible groups.

Next, we consider $\Gamma_1(p)$-level structures over the moduli of principally polarized abelian surface with symplectic level-N structure. For $N \geq 3$, this is a fine moduli space by [28]. This moduli space is called the Siegel threefold and is denoted by $\mathcal{A} := \mathcal{A}_{2,1,N}$. We are particularly interested in the level structures over the special fiber $\bar{\mathcal{A}} := \mathcal{A} \otimes \mathbb{F}_p$. There is already a well-behaved notion of $\Gamma_1(p)$-level structure defined by taking Kottwitz–Wake primitive elements on the p-torsion of the universal abelian surface $\mathcal{X}$. The $\Gamma_1(p)$-cover is then given by $\mathcal{X}^\times[p] := (\mathcal{X}[p])^\times$ over $\mathcal{A}$. (Some mathematicians use the name "$\Gamma_1(p)$-cover" differently. For example, Haines and Rapoport use "$\Gamma_1(p)$-cover" for the pro-p Iwahori structure in [16].) So in this case, Question (A) has already been resolved.

Consider Question (B). In [30], Oort defines a stratification of $\mathcal{A}$ (for general dimension g), now called the Ekedahl–Oort stratification. The Ekedahl–Oort stratification is parametrized by "elementary sequences". For each elementary sequence φ, we denote the associated stratum by S_φ. We want to give the local equations of the $\Gamma_1(p)$-cover $\mathcal{X}^\times[p]$ over each stratum S_φ. To do this, we need some machinery to systematically produce group schemes.

This machinery is built in Chapter 4. The tool we use is a version of Dieudonné theory due to de Jong [4]. We review this version of Dieudonné theory and compare it with the crystalline Dieudonné theory. Although this version of Dieudonné theory does not establish an antiequivalence of categories, there are still bijections between morphisms and extensions. Using these bijections, we construct group schemes killed by p whose Dieudonné modules have nilpotent Verschiebung. These group schemes are constructed by taking consecutive extensions of group schemes with trivial Verschiebung, and their Hopf algebras are given explicitly in terms of their Dieudonné modules. Here is a sample of such result:

Theorem 1.0.3. *Let $S = \operatorname{Spec} R$, where R is a local ring in characteristic p. Assume that S and its Frobenius lift modulo p^2. (This is true, for example, when S is smooth.) Let G/S be a (finite flat commutative) group scheme of rank p^4 and killed by p. Suppose that the Frobenius and Verschiebung on the Dieudonné module are given by matrices of the following form:*

$$F = \begin{pmatrix} 0 & a_1 & a_2 & c \\ 0 & 0 & 0 & e_1 \\ 0 & 0 & 0 & e_2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, V = \begin{pmatrix} 0 & b_1 & b_2 & d \\ 0 & 0 & 0 & f_1 \\ 0 & 0 & 0 & f_2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{1.5}$$

where $a_1, a_2, b_1, b_2, c, d, e_1, e_2, f_1, f_2 \in R$. Then

$$G \cong \operatorname{Spec} R[x, y_1, y_2, z] / \left(x^p, y_1^p - a_1 x, y_2^p - a_2 x, z^p - cx - e_1 y_1 - e_2 y_2\right) \tag{1.6}$$

and there are also explicit formulas for the coalgebra structure (see Equation (4.12)).

In Chapter 6, we give some explicit calculations of the $\Gamma_1(p)$-cover $\mathcal{X}^\times[p]$ over each stratum S_φ of the Siegel threefold $\bar{\mathcal{A}}$. There are 4 Ekedahl–Oort strata. They are: the superspecial locus, the supersingular (but not superspecial) locus, the p-rank-1 locus and the ordinary locus. The loci have dimensions 0,1,2,3 respectively. Over each stratum, there is a canonical group scheme filtration of the p-torsion of the universal abelian surface $\mathcal{X}$.

Let $\mathcal{X}_\varphi$ be the restriction of $\mathcal{X}$ over S_φ. The $\Gamma_1(p)$-cover $\mathcal{X}^\times[p] \to \mathcal{A}$ restricts to $\mathcal{X}_\varphi^\times[p] := (\mathcal{X}_\varphi[p])^\times$ over S_φ. We want to calculate explicit descriptions of the $\Gamma_1(p)$-cover

$\mathcal{X}_\varphi^\times[p]/S_\varphi$ by finding (local) equations. To do this, we first want to get a (local) description of the Hopf algebras of $\mathcal{X}_\varphi[p]/S_\varphi$ for each stratum.

The superspecial locus is a union of discrete points corresponding to products of supersingular elliptic curves. The Hopf algebra of $\mathcal{X}_\varphi[p]$ over this locus can be easily calculated using classical Dieudonné theory over perfect fields.

Now consider the supersingular locus. Theorem 1.0.3 applies in this situation and it shows that the group scheme $\mathcal{X}_\varphi[p]$ over S_φ, where S_φ is the supersingular locus, is (Zariski-locally) of the form (1.6).

To make the Hopf algebra description of $\mathcal{X}_\varphi[p]/S_\varphi$ more precise, we will use some specific constructions of the supersingular locus. Following an idea of Moret-Bailly [25] and Oort [32], one can form families of supersingular abelian surfaces $\mathcal{Y}$ over $\mathbb{P}^1$. It is shown in [18] that for any irreducible component W of the supersingular locus, $\mathcal{X}_W/W$ pulls back to $\mathcal{Y}/\mathbb{P}^1$ via some surjective morphism $\mathbb{P}^1 \to W$ and the morphism is generically an immersion. In [22], Kudla and Rapoport give nice descriptions of the Dieudonné modules of $\mathcal{Y}/\mathbb{P}^1$, whose F and V modulo p are of the shape (1.5) in Theorem 1.0.3. Using the construction in Theorem 1.0.3, we can explicitly write down the Hopf algebra of $\mathcal{Y}[p]$.

For the p-rank-1 locus, we are not able to obtain explicit descriptions of the Hopf algebras. However, we give some partial results using the theory of mixed extensions in Theorem 5.3.7 due to Grothendieck. In particular, we explicitly calculate all extensions and ext groups that are ingredients of the theory of mixed extensions. The only obstruction is to construct an explicit mixed extension using the calculated data. Once we have one explicit mixed extension, we can get all mixed extensions by applying all calculated ingredients to Proposition 5.3.2.

For the ordinary locus, Serre–Tate theory applies and we can get explicit expressions of the Hopf algebras using Serre–Tate coordinates.

All these group schemes that we construct over the three Ekedahl–Oort strata are written explicitly as complete intersections. In this case, one can obtain explicit generators of the defining ideal of Kottwitz-Wake primitive elements as certain determinants. Putting all these results together, we have the following result:

Theorem 1.0.4. *Over each Ekedahl–Oort stratum S_φ, the $\Gamma_1(p)$-cover $\mathcal{X}^\times_\varphi[p]/S_\varphi$ has the following description:*

1. *Let S_φ be the superspecial locus. Over each point of S_φ, the $\Gamma_1(p)$-cover $\mathcal{X}^\times_\varphi[p]/S_\varphi$ is given by*
$$\operatorname{Spec}\bar{\mathbb{F}}_p[x,y]/(x^{p^2}, y^{p^2}, x^{p^2-1}y^{p^2-1})$$
over $\operatorname{Spec}\bar{\mathbb{F}}_p$.

2. *Let S_φ be the supersingular stratum and let W be an irreducible component of S_φ. The $\Gamma_1(p)$-cover $\bar{\mathcal{X}}^\times_W[p]/W$ is the pullback of $\mathcal{Y}^\times[p]/\mathbb{P}^1_{\bar{\mathbb{F}}_p}$ via some open immersion $W \to \mathbb{P}^1_{\bar{\mathbb{F}}_p}$. Over each affine chart of the standard cover $\mathbb{P}^1_{\bar{\mathbb{F}}_p} = \mathbb{A}^1_0 \cup \mathbb{A}^1_\infty$, the restricted $\Gamma_1(p)$-cover $\mathcal{Y}^\times[p]|_{\mathbb{A}^1_{\bar{\mathbb{F}}_p}}/\mathbb{A}^1_{\bar{\mathbb{F}}_p}$ is isomorphic to*
$$\operatorname{Spec}\bar{\mathbb{F}}_p[\mu,x,y]/(x^{p^2}, y^{p^2}-(\mu^p-\mu)x^p, x^{p^2-1}y^{p^2-1})$$
over $\operatorname{Spec}\left(\bar{\mathbb{F}}_p[\mu]\right)$.

3. *Let S_φ be the ordinary locus. Let S_φ be the ordinary locus and x be a closed point of S_φ. Let $\hat{\mathcal{O}}_{S_\varphi,x}$ be the completion of the local ring of S_φ at x. Then the base change of $\mathcal{X}^\times_\varphi[p]/S_\varphi$ to $\operatorname{Spec}\hat{\mathcal{O}}_{S_\varphi,x}$ is isomorphic to*
$$\operatorname{Spec}\bar{\mathbb{F}}_p[\![t_1,t_2,t_3]\!][x_1,x_2,y_1,y_2]\Big/\begin{pmatrix} x_1^p-P_1(y_1,y_2), x_2^p-P_2(y_1,y_2), \\ y_1^p-y_1, y_2^p-y_2, \\ (y_1^{p-1}-1)(y_2^{p-1}-1)\Phi_p(x_1)\Phi_p(x_2) \end{pmatrix}$$
over $\operatorname{Spec}\bar{\mathbb{F}}_p[\![t_1,t_2,t_3]\!]$. *Here, Φ_p denotes the cyclotomic polynomial, the polynomials $P_1, P_2 \in \bar{\mathbb{F}}_p[\![t_1,t_2,t_3]\!][y_1,y_2]$ are certain interpolation polynomials and the variables t_1,t_2,t_3 are the Serre-Tate coordinates.*

Next, we consider Question (C), the geometric and arithmetic properties of the $\Gamma_1(p)$-cover over each stratum S_φ and over the whole moduli space $\mathcal{A}_{2,1,N}$ in mixed characteristics.

The first property we consider is the normality. Using the descriptions given in Theorem (1.0.4), we prove that the $\Gamma_1(p)$-cover $\mathcal{X}_\varphi^\times[p]$ is not normal in all three cases of Theorem 1.0.4 and the whole integral model $\mathcal{X}^\times[p]/\mathcal{A}_{2,1,N}$ is not normal as well.

Next, we consider the regularity properties. The whole integral model $\mathcal{X}^\times[p]$ is Cohen–Macaulay since it is finite flat over a Cohen–Macaulay base. Over the superspecial and ordinary locus, we prove that $\mathcal{X}_\varphi^\times[p]$ is Cohen–Macaulay, but not Gorenstein using the Hopf algebra descriptions. Using computer programs like Macaulay2, we can check the same properties hold over the supersingular locus for fixed primes. In particular, over the supersingular locus, the $\Gamma_1(p)$-cover $\mathcal{X}_\varphi^\times[p]$ is also Cohen–Macaulay, but not Gorenstein. We summarize these results in the following table:

Table 1.1 Regularity Properties of the $\Gamma_1(p)$-cover over the Siegel Threefold

	superspecial locus	supersingular locuss	ordinary locus	whole integral model
Normal	No	No	No	No
Cohen–Macaulay	Yes	Yes (for fixed primes)	Yes	Yes
Gorenstein	No	No (for fixed primes)	No	

CHAPTER 2

PRELIMINARIES

In this chapter, we will review some important tools that will be used later.

2.1 Group Schemes and Classification Theorems

Let S be a scheme. A group scheme G over S is a representable functor from the category of S-schemes to the category of groups. Equivalently, a group scheme G is an S-scheme together with scheme morphisms $m : G \times_S G \to G$, $\mathrm{inv} : G \to G$ and $\epsilon : S \to G$ so that the following diagrams commute:

$$\begin{array}{ccc} G \underset{S}{\times} G \underset{S}{\times} G & \xrightarrow{(m,\mathrm{Id})} & G \underset{S}{\times} G \\ {\scriptstyle (\mathrm{Id},m)}\downarrow & & \downarrow{\scriptstyle m} \\ G \underset{S}{\times} G & \xrightarrow{\quad m \quad} & G \end{array} \tag{2.1}$$

$$\begin{array}{ccc} S \underset{S}{\times} G & \xrightarrow{(\epsilon,\mathrm{Id})} & G \underset{S}{\times} G \\ {\scriptstyle \mathrm{pr}_2}\downarrow & & \downarrow{\scriptstyle m} \\ G & = & G \end{array} \tag{2.2}$$

$$\begin{array}{ccc} G \underset{S}{\times} G & \xrightarrow{(\mathrm{inv},\mathrm{Id})} & G \underset{S}{\times} G \\ \downarrow & & \downarrow{\scriptstyle m} \\ S & \xrightarrow{\quad \epsilon \quad} & G \end{array} \tag{2.3}$$

A group scheme G/S is called commutative if the points $G(T)$ are commutative groups, for all S-schemes T. Equivalently, this means the diagram

$$\begin{array}{ccc} G \underset{S}{\times} G & \xrightarrow{(\mathrm{pr}_2,\mathrm{pr}_1)} & G \underset{S}{\times} G \\ & {\scriptstyle m}\searrow \quad \swarrow{\scriptstyle m} & \\ & G & \end{array} \tag{2.4}$$

commutes.

We say a group scheme G/S is locally free (resp. flat, finite) if $G \to S$ is a locally free (resp. flat, finite) morphism. For a finite, locally free group scheme G/S, we call the rank of $\mathcal{O}_G$ as a locally free $\mathcal{O}_S$-module "the rank of the group scheme G/S" (suppose that S is connected).

Throughout this paper, all base schemes are noetherian and all group schemes are assumed to be commutative and flat over the base. Note that being flat and locally free are the same when the morphism is locally of finite presentation. Therefore all group schemes are locally free and commutative over the base.

Let S be a scheme over $\operatorname{Spec} \mathbb{Z}_p$ and let G be a finite flat commutative group scheme over S of rank p. In [33], Oort and Tate give a classification theorem for all group schemes G/S of this type. They define an anti-equivalence of categories as following:

$$\left\{\begin{array}{l}\text{triples } (\mathcal{L}, u, v), \text{ where } \mathcal{L} \text{ is a line bundle over} \\ S,\ u \in \Gamma(S, \mathcal{L}^{\otimes(p-1)}),\ v \in \Gamma(S, \mathcal{L}^{\otimes(1-p)}) \text{ so} \\ \text{that } u \otimes v = w_p\end{array}\right\} \longrightarrow \left\{\begin{array}{l}G/S, \text{ finite flat commutative} \\ \text{group schemes of rank } p\end{array}\right\}$$

Here w_p is a constant in $p\mathbb{Z}_p^\times \subset \mathbb{Z}_p$. Specifically, when $S = \operatorname{Spec} R$ where R is a local ring, the line bundle $\mathcal{L}$ is trivial. Therefore to give a rank p group scheme over such S, it suffices to give two elements $u, v \in R$ satisfying $uv = w_p$. For such a pair (u, v), the corresponding Hopf algebra is

$$\operatorname{Spec} R[x]/(x^p - ux),$$

where the coalgebra operations are given by

$$\begin{aligned} m^*(x) &= 1 \otimes x + x \otimes 1 + \frac{1}{1-p} \sum_{i=1}^{p-1} \frac{v x^i \otimes x^{p-i}}{w_i w_{p-i}}, \\ \operatorname{inv}^*(x) &= -x, \\ \epsilon^*(x) &= 0. \end{aligned}$$

Here w_i's are also constants in $\mathbb{Z}_p$ with $w_1, \dots, w_{p-1} \in \mathbb{Z}_p^\times$ and $w_p = p w_{p-1}$. The constants $w_1, \dots, w_{p-1}$ satisfy that $w_i \equiv i! \mod p$. For more details on the w_i's, see [33, page 10].

Haines and Rapoport express this result using stack language in [16, Theorem 3.3.1]. For convenience, we give the result here:

Theorem 2.1.1 ([16])**.** *The $\mathbb{Z}_p$-stack OT of finite flat commutative group schemes of rank p, satisfies the following properties:*

(i) OT is an Artin stack isomorphic to

$$[(\operatorname{Spec} \mathbb{Z}_p[s,t]/(st - w_p))/\mathbb{G}_m].$$

The action of $\mathbb{G}_m$ is given by $\lambda \cdot (s,t) = (\lambda^{p-1}s, \lambda^{1-p}t)$ with $w_p \in p\mathbb{Z}_p^\times$ as above.

(ii) The universal group scheme $\mathcal{G}$ over OT is

$$\mathcal{G} = [(\mathcal{S}pec_{OT}\, \mathcal{O}[x]/(x^p - tx))/\mathbb{G}_m].$$

The action of $\mathbb{G}_m$ is given by $\lambda \cdot x = \lambda x$.

In [38], Raynaud generalizes the notion of Oort–Tate group schemes to higher ranks. In particular, let $\mathbb{F}_q$ be a finite field, where $q = p^n$. Raynaud consider $\mathbb{F}_q$-vector space schemes of rank 1, which are the same as group schemes of rank $q = p^n$ together with a $\mathbb{F}_q$-action on it. Let $\mathbb{Q}_q := \mathbb{Q}_p(\zeta_{q-1})$ be the unique unramified extension of $\mathbb{Q}_p$ of degree n, and let $\mathbb{Z}_q$ be the ring of integers of $\mathbb{Q}_q$. The character group of $\mathbb{F}_q^\times$ is a cyclic group of order $q-1$. Let $\chi_1 : \mathbb{F}_q^\times \to \mathbb{Z}_q$ be the generator of the character group and we let $\chi_i := \chi_1^{p^{i-1}}$. Therefore $\chi_{n+1} = \chi_1$ and any character χ can be written as $\prod_{i=1}^n \chi_i^{e_i}$ with $0 \le e_i < p$. A character χ acts on the vector space scheme by

$$[\chi] := \frac{1}{q-1} \sum_{\lambda \in \mathbb{F}_q^\times} \chi^{-1}(\lambda)[\lambda].$$

Let the base scheme S be a $\mathbb{Z}_q$-scheme. Raynaud shows that there is an anti-equivalence of categories

$$\left\{\begin{array}{l} \text{n triples } (\mathcal{L}_i, u_i, v_i)\text{, where } \mathcal{L}_i\text{'s are line bun-} \\ \text{dles over } S,\ u_i : \mathcal{L}_{i+1} \to \mathcal{L}_i^{\otimes p},\ v_i : \mathcal{L}_i^{\otimes p} \to \\ \mathcal{L}_{i+1} \text{ so that } u \otimes v = w \end{array}\right\} \longrightarrow \left\{\begin{array}{l} G/S\text{, finite flat } \mathbb{F}_q\text{-vector space} \\ \text{schemes whose eigenspaces of} \\ \chi_i \text{ are of rank } 1 \end{array}\right\}$$

Here $w \in \mathbb{Z}_q$ is a constant.

As in the Oort–Tate case, when $S = \operatorname{Spec} R$ with R a local ring, the line bundles $\mathcal{L}_i$ are trivial and u_i, v_i are given by elements in R. Given n pairs (u_i, v_i) with $u_i v_i = w$, the associated $\mathbb{F}_q$-vector space scheme is given by

$$\operatorname{Spec} R[x_1, \dots, x_n] \big/ (x_1^p - u_1 x_2, x_2^p - u_2 x_3, \dots, x_n^p - u_n x_1)$$

with coalgebra operations

$$m^*(x_i) = x_i \otimes 1 + 1 \otimes x_i + \sum_{\chi' \chi'' = \chi_i} \frac{v_{i-h} \cdots v_{i-1}}{w_{\chi'} w_{\chi''}} \left(\prod_{j=1}^{r} x_j^{e'_j} \right) \otimes \left(\prod_{j=1}^{r} x_j^{e''_j} \right),$$

$$\operatorname{inv}^*(x_i) = -x_i,$$

$$\epsilon^*(x_i) = 0.$$

Here we write $\chi' = \prod_{i=1}^{r} \chi_i^{e'_i}$ and $\chi'' = \prod_{i=1}^{r} \chi_i^{e''_i}$ and $w_\chi \in \mathbb{Z}_q$ are constants. The index $0 < h \leq n$ is uniquely characterized by $e'_{i-h} + e''_{i-h} = p$.

Oort–Tate group schemes and Raynaud group schemes have explicit descriptions that are locally quotients of polynomials rings as complete intersections. This fact holds for group schemes more generally: for any group scheme G/S with S Noetherian, the morphism $G \to S$ is a local complete intersection morphism, i.e. all fibers are locally complete intersections (see [23, Lemma 31.14]). When further assuming that $S = \operatorname{Spec} R$ with R a local complete Noetherian ring, the group schemes G/S have the following form:

Proposition 2.1.2. *([41, Page 28, Corollary]) Let (R, m) be a complete local noetherian ring with the residue field R/m perfect of characteristic p. Let $G = \operatorname{Spec} A$ be a local group scheme over $\operatorname{Spec} R$. Then*

$$A \cong R[\![x_1, \dots, x_n]\!]/(f_1, \dots, f_n)$$

where $f_i = x_i^{p^{e_i}} + g_i$ and g_i's are polynomials with coefficients in m and degree $< p^{e_i}$. In particular, when R is a perfect field in characteristic p, then

$$A \cong R[x_1, \dots, x_n]/(x_1^{p^{e_1}}, \dots, x_n^{p^{e_n}}).$$

2.2 Siegel Modular Varieties and Ekedahl–Oort Stratification

Siegel modular varieties are generalizations of modular curves. In particular, over the complex numbers $\mathbb{C}$, the Siegel modular variety $\mathcal{A}_{g,1,N}(\mathbb{C})$ is the moduli spaces of isomorphism classes of triples (A, λ, η), where A is an abelian variety over $\mathbb{C}$, $\lambda : A \to A^\vee$ is a principal polarization on A, and $\eta : A[N] \to (\mathbb{Z}/N\mathbb{Z})^{2g}$ is an isomorphism compatible with the Weil pairing on $A[N]$ and the symplectic pairing on $(\mathbb{Z}/N\mathbb{Z})^{2g}$. In [28], Mumford shows that $\mathcal{A}_{g,1,N}$ is a fine moduli space when $N \geq 3$. Similar to the case of modular curves, the Siegel modular variety $\mathcal{A}_{g,1,N}$ also has a "model" over $\mathbb{Z}[1/N]$. This is in fact a smooth scheme of relative dimension $g(g+1)/2$ over $\mathbb{Z}[1/N]$ (see [28]).

We are particularly interested in the geometry of Siegel modular variety in characteristic p, i.e. the scheme $\mathcal{A} := \mathcal{A}_{g,1,N} \times_{\operatorname{Spec} \mathbb{Z}[1/N]} \operatorname{Spec} \mathbb{F}_p$ where p is prime to N. In [30], Oort defines a stratification on $\mathcal{A}$ by the isomorphism class of the p-torsion $(A, \lambda)[p]$. This stratification is now called the "Ekedahl–Oort stratification". The stratification is constructed as follows.

Let (A, λ) be an principally polarized abelian variety of dimension g over a base scheme S of characteristic $p > 0$. Consider the p-torsion group scheme $G := A[p]$. Let $F_G \colon G \to G^{(p)}$ and $V_G \colon G^{(p)} \to G$ be the Frobenius and Verschiebung respectively. Note that G is a truncated p-divisible group. Therefore $\ker F_G = \operatorname{Im} V_G$. The principal polarization on A gives a group scheme isomorphism $\zeta : G \to G^D$ of G with its Cartier dual. For any subgroup scheme H of G, we define $V(H) := \operatorname{Im} V_H$ and $F^{-1}(H)$ as the fiber product:

$$\begin{array}{ccc} F^{-1}(H) & \longrightarrow & H^{(p)} \\ \big\downarrow & \square & \big\downarrow \\ G & \xrightarrow[F_G]{} & G^{(p)} \end{array}$$

where the bottom arrow is the Frobenius. We also define

$$H^\perp := \ker(G \xrightarrow{\zeta} G^D \xrightarrow{i^D} H^D).$$

Note that by identifying G with G^D via ζ, we get a non-degenerate pairing $\langle \cdot, \cdot \rangle_G : G \times G \to \mu_p$. The notation "$\perp$" comes from the fact that

$$x \in H^{\perp}(S) \iff \langle x, y \rangle_G = 0$$

for all $y \in H$.

Consider the points (A, λ, η) of $\mathcal{A}$. The Ekedahl–Oort stratification of $\mathcal{A}$ is defined by the isomorphism classes of $(A, \lambda)[p]$. For each isomorphism class of $(A, \lambda)[p]$, one can associate it with an "elementary sequence" φ, which is an increasing sequence of integers of length $g+1$ with initial term 0 and increments less than or equal to 1, i.e. $\varphi(i) \leq \varphi(i+1) \leq \varphi(i) + 1$. These sequences are called elementary sequences and it is easy to see that there are 2^g elementary sequences. The Ekedahl–Oort strata are parametrized by elementary sequences; we denote the stratum corresponding to φ by S_φ. The stratification has the following properties:

(i) Every stratum S_φ is non-empty, smooth, quasi-affine and equi-dimensional of dimension $\sum_{i=0}^{g} \varphi(i)$.

(ii) Let $\mathcal{X}$ be the universal abelian surface over $\mathcal{A}$ and let $\mathcal{X}_\varphi := \mathcal{X} \times_{\mathcal{A}} S_\varphi$. Over each strata S_φ, there is a so-called "canonical filtration" of group schemes

$$0 = G_0 \subset \cdots \subset G_r = V(G) \subset \cdots \subset G_{2r} = G,$$

of $G = \mathcal{X}_\varphi[p]$ satisfying

$$\operatorname{Rank} G_i = p^{\rho(i)} \tag{2.5}$$

$$V(G_i) = G_{v(i)} \tag{2.6}$$

$$F^{-1}(G_i) = G_{f(i)} \tag{2.7}$$

$$G_i^{\perp} = G_{2r-i} \tag{2.8}$$

$$(G_i/G_j)^D \cong G_{2r-j}/G_{2r-i} \tag{2.9}$$

Here the index r and maps $\rho : \{0, \dots, 2r\} \to \mathbb{Z}$, $v : \{0, \dots, 2r\} \to \{0, \dots, r\}$, $f : \{0, \dots, 2r\} \to \{r, \dots, 2r\}$ are determined and can be easily calculated from φ. For a given φ, set $\rho(0) = 0$ and define $\rho(i)$ to be the maximum index $> \rho(i-1)$ so that $\varphi(\rho(i)) = \varphi(\rho(i-1))$ or $\varphi(\rho(i)) - \varphi(\rho(i-1)) = \rho(i) - \rho(i-1)$. In other words, $\rho(i)$ is the maximal index after $\rho(i-1)$ so that φ keeps increasing or stationary from $\rho(i-1)$ to $\rho(i)$. In this way, we form a $\rho : \{0, \dots, r\} \to \mathbb{Z}$ (note that r is the highest index in ρ). We define $v : \{0, \dots, r\} \to \mathbb{Z}$ and $f : \{0, \dots, r\} \to \mathbb{Z}$ by setting $v(0) = 0$, $f(0) = r$ and define $v(i)$ and $f(i)$ by

$$v(i) = v(i-1) \Leftrightarrow f(i) = f(i-1) + 1 \Leftrightarrow \varphi(\rho(i)) = \varphi(\rho(i-1)),$$
$$v(i) = v(i-1) + 1 \Leftrightarrow f(i) = f(i-1) \Leftrightarrow \varphi(\rho(i)) - \varphi(\rho(i-1)) = \rho(i) - \rho(i-1).$$

We then expand ρ, v, f to $\{0, \dots, 2r\}$ by defining

$$\rho(r+i) = 2\rho(r) - \rho(r-i)$$
$$v(r+i) = 2r - f(r-i)$$
$$f(r+i) = 2r - v(r-i)$$

for all $1 \le i \le r$.

Example 2.2.1. *Let $g = 2$. We consider pricipally polarized abelian surfaces. We have 4 elementary sequences. For each φ, let $\mathcal{X}_\varphi$ be the universal abelian surface over the associated stratum S_φ and $G = \mathcal{X}_\varphi[p]$ as before. For any geometric point (A, λ, η) of S_φ, we define the p-rank k and a-number of A by*

$$A[p](\bar{\mathbb{F}}_p) = (\mathbb{Z}/p\mathbb{Z})^k \tag{2.10}$$
$$a(A) := \dim_{\bar{\mathbb{F}}_p} \mathrm{Hom}(\alpha_p, A) \tag{2.11}$$

For each elementary sequence φ, the stratum S_φ has the following data:

(i) Let $\varphi = (0, 0, 0)$. In this case, the canonical filtration is

$$0 = G_0 \subset G_1 \subset G_2 = G.$$

The corresponding canonical type is $\rho = (0, 2, 4)$, $v = (0, 0, 1)$ *and* $f = (1, 2, 2)$. *In this case,* A *is of* p*-rank 0 and* a*-number* 2, *corresponding to superspecial abelian surfaces.*

(ii) *Let* $\varphi = (0, 0, 1)$. *In this case, the canonical filtration is*

$$0 = G_0 \subset G_1 \subset G_2 \subset G_3 \subset G_4 = G.$$

The corresponding canonical type is $\rho = (0, 1, 2, 3, 4)$, $v = (0, 0, 1, 1, 2)$ *and* $f = (2, 3, 3, 4, 4)$. *In this case,* A *is of* p*-rank 0 and* a*-number* 1, *corresponding to supersingular but not superspecial (sometimes called supergeneral) abelian surfaces.*

(iii) *Let* $\varphi = (0, 1, 1)$. *In this case, the canonical filtration is*

$$0 = G_0 \subset G_1 \subset G_2 \subset G_3 \subset G_4 = G.$$

The corresponding canonical type is $\rho = (0, 1, 2, 3, 4)$, $v = (0, 1, 1, 2, 2)$ *and* $f = (2, 2, 3, 3, 4)$. *In this case* A *is of* p*-rank 1 and* a*-number* 1.

(iv) *Let* $\varphi = (0, 1, 2)$. *In this case, the canonical filtration is*

$$0 = G_0 \subset G_1 \subset G_2 = G.$$

The corresponding canonical type is $\rho = (0, 2, 4)$, $v = (0, 1, 1)$ *and* $f = (1, 1, 2)$. *In this case* A *is of* p*-rank 2 and* a*-number* 0. *This corresponds to ordinary abelian surfaces.*

2.3 Kottwitz-Wake Primitive Elements

Recall that all base schemes are assumed to be locally of finite presentation and all group schemes are assumed to be commutative and flat over the base. Let G/S be a finite group scheme. In [21], Kottwitz and Wake define a subscheme of "non-nullity" of G. This subscheme is denoted by $G^\times$.

Definition 2.3.1 ([21])**.** *Let* G *be a finite group scheme over a base scheme* S. *Let* $\mathcal{I} \subset \mathcal{O}_G$ *be the augmentation ideal of* G. *We define the scheme of "non-null" elements* $G^\times$ *to be the* *closed subscheme of G with the defining ideal sheaf given by* $\mathrm{Ann}(\mathcal{I})$*, the annihilator of the augmentation ideal sheaf.*

A key property of "non-nullity" is that the scheme of "non-null" elements $G^\times$ is locally free of rank $|G| - 1$ over S (see [21]).

This notion of "non-nullity" is used to describe the subscheme of "non-zero" points. But one needs to be careful with this analogy. When G/S is not étale, the group scheme G may have no points other than the unit, but $G^\times$ can still have points. Here is a concrete example:

Example 2.3.2. *Consider the additive group* $\mathbb{G}_a/\mathbb{F}_p$*. Let* α_p *be the kernel of the Frobenius of* $\mathbb{G}_a/\mathbb{F}_p$*. In particular, the group scheme* α_p *can be written as* $\operatorname{Spec}\mathbb{F}_p[x]/(x^p)$ *with augmentation ideal* (x) *and additive group operation. The annihilator of the augmentation ideal* (x) *is generated by* x^{p-1} *and therefore* $\alpha_p^\times = \operatorname{Spec}\mathbb{F}_p[x]/(x^{p-1})$*.*

Note that for any reduced ring R *in characteristic* p*, we have* $\alpha_p(R) = \{0\}$ *and* $\alpha_p^\times(R) = \{0\}$*. The unit section can still be non-null.*

Using "non-nullity", Kottwitz and Wake define primitive elements for truncated p-divisible groups. More precisely, let $\mathcal{G}/S$ be a truncated p-divisible group of height h and level r (i.e. the smallest exponent such that p^r kills $\mathcal{G}$). This happens, for example, when $\mathcal{G}/S$ is the p^n-torsion of a p-divisible group over S. In this case, the (scheme of) primitive elements $\mathcal{G}^\times$ is defined to be the fiber product

$$\begin{array}{ccc} \mathcal{G}^\times & \longrightarrow & (\mathcal{G}[p])^\times \\ \big\downarrow & \square & \big\downarrow \\ \mathcal{G} & \xrightarrow[p^{r-1}]{} & \mathcal{G}[p] \end{array}$$

where $(\mathcal{G}[p])^\times$ is the subscheme of "non-nullity". The scheme of primitive elements $\mathcal{G}^\times/S$ is flat of rank $p^{r-1}(p^h - 1)$ and describes the points "of exact order p^r". Note that for truncated p-divisible groups of level 1, primitive elements are the same as "non-null" elements. For simplicity, on general group schemes killed by p, we will abuse the name of primitive elements for "non-null" elements.

In general, it may not be easy to calculate an annihilator ideal in a ring. However, by Proposition 2.1.2, many group schemes are quotients of polynomial rings or power series rings as complete intersections. In these cases, we have the following lemma:

Lemma 2.3.3. *Let R be a Noetherian ring and let (R', ϵ) be an augmented R-algebra, i.e. an R-algebra R' together with a ring homomorphism $\epsilon : R' \to R$ such that the composition $R \to R' \xrightarrow{\epsilon} R$ is identity. Assume that the augmentation ideal $I := \ker \epsilon = (x_1, \dots, x_n)$ is generated by a regular sequence $x_1, \dots, x_n$. Let $J = (f_1, \dots, f_n) \subset I$ be a subideal generated by a regular sequence $f_1, \dots, f_n$ in R'. Write*

$$f_i = M_{i1}x_1 + \cdots + M_{in}x_n, i = 1, \dots, n$$

with $M_{ij} \in R'$ and set $M = (M_{ij}) \in \mathrm{Mat}_{n\times n}(R')$. Then we have

$$(J : I) = (\det(M)) + J \tag{2.12}$$

where $(J : I) := \{x \in R' | xI \subset J\}$. Let $A := R'/J$. Let $I_A = I/J$ be the corresponding ideal of A and let d be the image of $\det(M)$ in A. Then we have

$$\mathrm{Ann}(I_A) := (d) \tag{2.13}$$

Proof. The proof follows from [9, Proposition 2.1]. We sketch the argument here.

The equivalence of Equation (2.12) and Equation (2.13) is immediate from the definition. We will prove Equation (2.13).

Consider the Koszul resolutions $K((x_i), R')$ and $K((f_i), R')$. They are complexes of R'-modules defined by $K_m(-, R') = \mathrm{Hom}_{R'}(\wedge^m_{R'}(R')^n, R')$. For $\phi \in K_m((x_i), R')$, the boundary map is defined by $d\phi(y) = \phi(\underline{x} \wedge y)$ and $K_m((f_i), R')$ is similarly defined. It is a standard fact that $K((x_i), R')$ and $K((f_i), R')$ are exact from the regularity of the sequences (see [24, Section 6]). There is a map $K((f_i), R') \to K((x_i), R')$ given as follows:

$$\begin{array}{ccccccccccccc} K((f_i), R') : 0 & \longrightarrow & R' & \xrightarrow{\underline{f}^t} & (R')^n & \longrightarrow \cdots \longrightarrow & (R')^n & \xrightarrow{\underline{f}} & R' & \longrightarrow & A = R'/J & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \det(M)} & & \big\downarrow & & \big\downarrow{\scriptstyle (M_{ij})} & & \big\| & & \big\downarrow{\scriptstyle \pi} & & \\ K((x_i), R') : 0 & \longrightarrow & R' & \xrightarrow{\underline{x}^t} & (R')^n & \longrightarrow \cdots \longrightarrow & (R')^n & \xrightarrow{\underline{x}} & R' & \longrightarrow & R = R'/I & \longrightarrow & 0 \end{array} \tag{2.14}$$

Note that the exact rows are free resolutions of A and R respectively. By tensoring the diagram with A over R' and consider the n-th homology, we get a commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Tor}^n_{R'}(A,A) & \longrightarrow & A & \xrightarrow{0} & A^n \\ & & \downarrow \pi_* & & \downarrow d & & \downarrow \\ 0 & \longrightarrow & \mathrm{Tor}^n_{R'}(R,A) & \longrightarrow & A & \xrightarrow{\underline{x}} & A^n \end{array} \tag{2.15}$$

From the bottom row, we have $\mathrm{Tor}^n(R,A) = \mathrm{Ann}(I_A)$. On the other hand, consider $K((f_i),R') \otimes_{R'} A \to K((f_i),R') \otimes_{R'} R$ given by $A = R'/J \xrightarrow{\pi} R = R'/I$ as in the right colomn of Diagram (2.14). We get another commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Tor}^n(A,A) & \longrightarrow & A & \xrightarrow{0} & A^n \\ & & \downarrow \pi_* & & \downarrow \pi & & \downarrow \\ 0 & \longrightarrow & \mathrm{Tor}^n(R,A) & \longrightarrow & R & \xrightarrow{0} & R^n \end{array} \tag{2.16}$$

This implies that π_* is surjective and therefore by Equation (2.15), the image $(d) \subset A$ coincides with $\mathrm{Tor}^n(R,A) = \mathrm{Ann}(I_A)$, as claimed. $\square$

Lemma 2.3.3 is a very powerful tool to calculate primitive elements on finite group schemes. In many cases, a group scheme killed by p over $\operatorname{Spec} R$ can be written in the form

$$G \cong \operatorname{Spec} R[\underline{x}]/(\underline{x}^p - A \cdot \underline{x}) := R[x_1, \dots, x_n]/\left(\left\{x_i^p - \sum_{j=1}^n a_{ij} x_j\right\}_{1 \le i \le n}\right), \tag{2.17}$$

where $\underline{x} = (x_1, \dots, x_n)^t$ and A is an $n \times n$ matrix over R. Pappas gives the following lemma in [36]:

Lemma 2.3.4. *Let G be a group scheme over $\operatorname{Spec} R$ killed by p as given in Equation (2.17) with augmentation ideal $(\underline{x}) = (x_1, \dots, x_n)$. Then the defining ideal of the primitive elements $G^\times$ is generated by*

$$\det(D(\underline{x}^{p-1}) - A) = \sum_{J \subset \{1,\dots,n\}} (-1)^{|J^c|} \det(A_{J^c \times J^c}) \prod_{j \in J} x_j^{p-1}. \tag{2.18}$$

Here the $A_{J^c \times J^c}$ is the minor of A with rows and columns in J^c, the complement of J in $\{1, \dots, n\}$.

Proof. By Lemma 2.3.3, the generator of the primitive elements in G is generated by $\det(D(\underline{x}^{p-1}) - A)$. Expand the determinant $\det(D(\underline{x}^{p-1}) - A)$ and consider the coefficient of $\prod_{j\in J} x_j^{p-1}$. Note that the sign $\mathrm{sgn}(\sigma)$ of a permutation σ on $\{1, \dots, n\}$ fixing a subset $J \subset \{1, \dots, n\}$ is the same as the sign $\mathrm{sgn}(\bar{\sigma})$ of the induced permutation $\bar{\sigma}$ on J^c. Therefore the coefficient of $\prod_{j\in J} x_j^{p-1}$ in $\det(D(\underline{x}^{p-1}) - A)$ is the determinant $\det(-A_{J^c\times J^c})$, which is equal to $(-1)^{|J^c|}\det(A_{J^c\times J^c})$ as claimed in Equation (2.18). □

We can apply Lemma 2.3.4 to the Oort–Tate group schemes and Raynaud group schemes in Section 2.1:

Example 2.3.5. *Let $S = \operatorname{Spec} R$ where R is a local ring. Consider an Oort–Tate group scheme given by $\operatorname{Spec} R[x]/(x^p - ux)$ with augmentation ideal given generated by (x). In this case, the 1×1 matrix in Lemma 2.3.3 is given by $x^{p-1} - u$, implying the primitive elements are defined by the ideal $(x^{p-1} - u)$.*

The Raynaud case is slightly more complicated. Using the notation in Equation (2.17), the Raynaud group scheme over S is given by

$$G = \operatorname{Spec} R[\underline{x}]/(\underline{x}^p - U \cdot \underline{x})$$

where $\underline{x} = (x_1, \dots, x_n)^t$ and

$$U = \begin{pmatrix} 0 & \cdots & 0 & u_r \\ u_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & u_{r-1} & 0 \end{pmatrix},$$

with augmentation ideal given by $(x_1, \dots, x_n)$. Note that $\det(U) = (-1)^{r-1}u_1 \cdots u_r$ and the only principal minor of U with nonzero determinant is the whole matrix U. Therefore by Lemma 2.3.4, the primitive elements $G^\times \subset G$ is defined by $(x_1^{p-1} \cdots x_r^{p-1} - u_1 \cdots u_r)$.

We will use the following special case of Lemma 2.3.4 later:

Example 2.3.6. *Let $G/\operatorname{Spec} R$ be a group scheme as given in Equation (2.17). Suppose the matrix A is upper triangular, i.e. $a_{ij} = 0$ if $i > j$. Then $D(\underline{x}^{p-1}) - A$ is an upper triangular* *matrix with diagonal elements* $x_i^{p-1} - a_{ii}$*. Therefore by Lemma 2.3.4, the defining ideal of* $G^\times \subset G$ *is generated by* $\prod_{i=1}^n (x_i^{p-1} - a_{ii})$*.*

In particular, suppose A *is strictly upper triangular, i.e. assume further that* $a_{ii} = 0$*. Then the defining ideal of* $G^\times \subset G$ *is generated by* $x_1^{p-1} \cdots x_n^{p-1}$*.*

CHAPTER 3

FULL LEVEL STRUCTURE ON FINITE GROUP SCHEMES

In this chapter, we consider the $\Gamma(p)$-level structure, also called "full level structure", on finite flat commutative group schemes.

Suppose that H is annihilated by p^r and $H[\frac{1}{p}] := H \times_S S[\frac{1}{p}]$ is étale-locally isomorphic to the constant group scheme $(\underline{\mathbb{Z}/p^r\mathbb{Z}})^g$ for some r and g. This happens, for example, when H is the p^r-torsion of some abelian variety of dimension $g/2$.

Let $\mathrm{Hom}_S((\mathbb{Z}/p^r\mathbb{Z})^g, H)$ be the functor from the category of S-schemes $\mathbf{Sch}_S$ to the category of abelian groups $\mathbf{Ab}$, defined by

$$\mathrm{Hom}_S((\mathbb{Z}/p^r\mathbb{Z})^g, H)(T) := \mathrm{Hom}_{gp}((\mathbb{Z}/p^r\mathbb{Z})^g, H(T)).$$

We will use $\underline{\mathrm{Hom}}_S((\mathbb{Z}/p^r\mathbb{Z})^g, H)$ to denote the representing scheme. Since H is annihilated by p^r, the representing scheme is just H^g. The general linear group $\mathrm{GL}_g(\mathbb{Z}/p^r\mathbb{Z})$ has a natural right action on $\underline{\mathrm{Hom}}_S((\mathbb{Z}/p^r\mathbb{Z})^g, H)$ by acting on $(\mathbb{Z}/p^r\mathbb{Z})^g$ by precomposition.

The problem we consider is to give a notion of full level structure on H. We expect it to be a closed subscheme of $\underline{\mathrm{Hom}}_S((\mathbb{Z}/p^r\mathbb{Z})^g, H)$, which we denote by $\underline{\mathrm{Hom}}^*_S((\mathbb{Z}/p^r\mathbb{Z})^g, H)$, satisfying:

1. $\underline{\mathrm{Hom}}^*_S((\mathbb{Z}/p^r\mathbb{Z})^g, H)$ is flat over S and of rank $|\mathrm{GL}_g(\mathbb{Z}/p^r\mathbb{Z})|$.

2. $\underline{\mathrm{Hom}}^*_S((\mathbb{Z}/p^r\mathbb{Z})^g, H)$ is $\mathrm{GL}_g(\mathbb{Z}/p^r\mathbb{Z})$-invariant under the right $\mathrm{GL}_g(\mathbb{Z}/p^r\mathbb{Z})$-action on $\underline{\mathrm{Hom}}_S((\mathbb{Z}/p^r\mathbb{Z})^g, H)$. When inverting p, we have an identification
$$\underline{\mathrm{Hom}}^*_{S[\frac{1}{p}]}((\mathbb{Z}/p^r\mathbb{Z})^g, H[\frac{1}{p}]) = \mathrm{Isom}_{S[\frac{1}{p}]}((\mathbb{Z}/p^r\mathbb{Z})^g, H[\frac{1}{p}])$$
as closed subschemes of $\underline{\mathrm{Hom}}_{S[\frac{1}{p}]}((\mathbb{Z}/p^r\mathbb{Z})^g, H[\frac{1}{p}])$.

3. When identifying $\underline{\mathrm{Hom}}_S((\mathbb{Z}/p^r\mathbb{Z})^g, H) \times_S T$ with $\underline{\mathrm{Hom}}_T((\mathbb{Z}/p^r\mathbb{Z})^g, H_T)$ in the natural way, we have $\underline{\mathrm{Hom}}^*_S((\mathbb{Z}/p^r\mathbb{Z})^g, H) \times_S T = \underline{\mathrm{Hom}}^*_T((\mathbb{Z}/p^r\mathbb{Z})^g, H_T)$ as closed subschemes, for any S-scheme T.

We also expect our definition to coincide with the intuitive definition for some familiar group schemes. For example, for $H = \mu_{p^r}$, we expect $\underline{\mathrm{Hom}}^*_{\mathbb{Z}}(\mathbb{Z}/p^r\mathbb{Z}, H)$ to be the closed subscheme of μ_{p^r} defined by the cyclotomic polynomial

$$\Phi_{p^r}(x) := \frac{x^{p^r} - 1}{x^{p^{r-1}} - 1} = x^{(p-1)p^{r-1}} + x^{(p-2)p^{r-1}} + \cdots + 1.$$

When H is the constant group scheme $(\underline{\mathbb{Z}/p^r\mathbb{Z}})^g$, the resulting full level structure on H given by $\underline{\mathrm{Hom}}^*_{\mathbb{Z}}((\mathbb{Z}/p^r\mathbb{Z})^g, H)$ should be $\underline{\mathrm{GL}_g(\mathbb{Z}/p^r\mathbb{Z})} \subset \underline{\mathrm{Mat}_g(\mathbb{Z}/p^r\mathbb{Z})}$.

The motivation for giving a well-behaved notion of full level structure comes from the study of integral models of Shimura varieties. For example, for modular curves, finding an integral model of the modular curve $X(p^r)$ essentially amounts to finding a flat model of full level structure on the p^r-torsion of elliptic curves. This is done by Katz and Mazur in their book [19]: following an idea of Drinfeld in [10], Katz and Mazur consider the case when H can be embedded into a curve. In this case a set of sections $\{P_1, \dots, P_n\}$ of H is defined to be a "full set of sections", if the points generate the group H as Cartier divisors. Using this notion, the full level structure on H is defined to be the maps in $\mathrm{Hom}_S((\mathbb{Z}/p^r\mathbb{Z})^g, H)$ whose image forms a full set of sections. As a scheme, $\underline{\mathrm{Hom}}^*_S((\mathbb{Z}/p^r\mathbb{Z})^g, H)$ can be also described as the closed subscheme of $\underline{\mathrm{Hom}}_S((\mathbb{Z}/p^r\mathbb{Z})^g, H)$ cut out by the Cartier divisor equation

$$H = \sum_{x \in (\mathbb{Z}/p^r\mathbb{Z})^g} [h(x)]$$

where h is the universal homomorphism. Katz and Mazur's construction, for example, gives a definition of full level structure on $\underline{\mathbb{Z}/p\mathbb{Z}} \times \mu_p$, as it is the p-torsion of an ordinary elliptic curve. They also suggest a natural generalization of their construction, given by "$\times$-homomorphisms" [19, Appendix of Chapter 1], that can be defined for general group schemes. Unfortunately, the notion of $\times$-homomorphisms is deficient because the resulting closed subscheme is generally not flat over the base. Such a negative result has been observed by Chai and Norman in [3, Appendix 2]. For example, the nonflatness for $\times$-homomorphisms even happens on $\mu_p \times \mu_p$.

As an improvement, Wake gives in [42] a good definition in the case of $H = \mu_p \times \mu_p$ over $\operatorname{Spec} \mathbb{Z}$. By using a notion of "primitive elements", he defines the full level structure, called "scheme of full homomorphisms", to be cut out by the condition that all nontrivial linear combinations of rows and columns of the universal homomorphism are primitive. Alternatively, Wake also gives another level structure, called "KM+D" level structure, short for Katz-Mazur + Dual. The notion of KM+D level structure is defined by requiring both universal homomorphism and its dual being $\times$-homomorphisms as defined by Katz and Mazur. Wake proves that in the case $\mu_p \times \mu_p$, the KM+D level structure coincides with his original notion of full homomorphisms. Unfortunately, in general the "KM+D" level structure does not give a flat scheme over the base. For example, it is observed in [42, Example 4.8] that $\underline{\mathrm{Hom}}^{\mathrm{KM+D}}_{\mathbb{F}_2}((\mathbb{Z}/2\mathbb{Z})^2, (\alpha_2)^2)$ has larger rank than expected.

In this chapter, we give a definition of full level structure for H of the form $H = G \times G$, where G is a rank p group scheme over a $\mathbb{Z}_p$-scheme S. When G is μ_p, our definition coincides with the one in [42]. The idea of our construction is to generalize Wake's "rows-and-columns" construction to a general group scheme G using Kottwitz-Wake's notion of primitive elements [21]. In [21] the authors give a notion of primitive elements which is well-behaved, even for general p-divisible groups. Using this notion, our full level structure will be cut out by the condition that rows and columns of the universal homomorphism are linearly independent, as in Wake's construction. The precise description and properties are discussed in Section 3. The main point is that this construction gives a flat model. We show this by using Oort–Tate theory to reduce to Wake's result.

One might also expect the following naturality condition:

4. For any group scheme isomorphism $H \xrightarrow{\sim} H'$, the induced isomorphism

$$\underline{\mathrm{Hom}}_S((\mathbb{Z}/p^r\mathbb{Z})^g, H) \to \underline{\mathrm{Hom}}_S((\mathbb{Z}/p^r\mathbb{Z})^g, H')$$

restricts to an isomorphism $\underline{\mathrm{Hom}}^*_S((\mathbb{Z}/p^r\mathbb{Z})^g, H) \to \underline{\mathrm{Hom}}^*_S((\mathbb{Z}/p^r\mathbb{Z})^g, H')$.

This condition (4) can be interpreted as saying the notion of full level structure is defined over the stack. Unfortunately, it turns out that in general there is no level structure on H satisfying all conditions $(1)-(4)$. Wake pointed out to us that the construction cannot extend to the stack. We discuss this negative result in Section 3.3 and include their example there. We thank Wake for the communication.

3.1 Full level structure on $G \times G$

The fundamental tool in defining the full level structure on $G \times G$ is the notion of "primitive elements" (see Section 2.3). An important example of the primitive elements is the Oort–Tate group scheme $G = \operatorname{Spec} A[x]/(x^p - tx)$, where A is a $\mathbb{Z}_p$-algebra. The augmentation ideal is (x). Thus $G^\times$ is defined by the ideal $(x^{p-1} - t)$, coinciding with the scheme of generators defined in [16]. Another example is $G \times G$. Its underlying algebra is $A[x, y]/(x^p - tx, y^p - ty)$ with the augmentation ideal (x, y). By a direct calculation or using Lemma 2.3.3, we can see that the scheme of primitive elements in G^2 is

$$(G^2)^\times = \operatorname{Spec} A[x, y]/\left((x^{p-1} - t)(y^{p-1} - t)\right).$$

See also [21, Section 3.8].

Now we consider the operation on the points of $\operatorname{Hom}_S((\mathbb{Z}/p\mathbb{Z})^2, G^2) = G^4$ (as functors). We will identify $G^4(T)$ with $\operatorname{Mat}_2(G(T))$, the additive group of 2×2 matrices with entries in $G(T)$. On each entry $\operatorname{Hom}_S(\mathbb{Z}/p\mathbb{Z}, G)(T) = G(T)$, there is a natural addition arising from the group structure of G. We denote this addition by $\dot{+}$, to distinguish it from the addition on $\mathcal{O}_G$. For simplicity, for any $f \in G(T)$, let $[m]f$ be $f \dot{+} f \dot{+} \cdots \dot{+} f$, the sum of m copies of f. Since the Oort–Tate Group is annihilated by p, the operation $[m]$ only depends on m modulo p.

Example 3.1.1. *Let $S = \operatorname{Spec} \mathbb{Z}_p$ and $G = \operatorname{Spec} \mathbb{Z}_p[x]/(x^p - x)$ with comultiplication*

$$m^*(x) = 1 \otimes x + x \otimes 1 + \frac{1}{1-p} \sum_{i=1}^{p-1} \frac{w_p x^i \otimes x^{p-i}}{w_i w_{p-i}}.$$

This is obtained by taking $u = 1$ *and* $v = w_p$ *from Section 2. Let* $T = \operatorname{Spec} \mathbb{Z}_p$*. In* $G(T)$*, let* $\chi(j) \in \operatorname{Hom}_S(\mathbb{Z}/p\mathbb{Z}, G)(T) = G(T)$ *be the map sending* x *to* $\chi(j)$*, where* χ *is the Teichmüller character and let* $\chi(0) = 0$*. Since the elements in* $G(T)$ *are closed under the group action, we have* $([j](1))^p - [j](1) = 0$*. On the other hand, by the definition of the comultiplication of* G*, we have* $[j](1) \equiv j \mod p$*. Therefore* $[j](1) = \chi(j)$*. From* $[j](1) \dot{+} [k](1) = [j+k](1)$*, we get a useful equation:*

$$\chi(j+k) = \chi(j) + \chi(k) + \frac{1}{1-p} \sum_{i=1}^{p-1} \frac{w_p \chi(j^i) \chi(k^{p-i})}{w_i w_{p-i}}. \tag{3.1}$$

In fact, G *is isomorphic to the constant group scheme* $\underline{\mathbb{Z}/p\mathbb{Z}}_S = \operatorname{Spec} \mathbb{Z}_p^{\mathbb{Z}/p\mathbb{Z}}$*. The Hopf algebra isomorphism between* $\mathbb{Z}_p[x]/(x^p - x)$ *and* $\mathbb{Z}_p^{\mathbb{Z}/p\mathbb{Z}}$ *is given by* $x \mapsto \sum \chi(i) e_i$ *and* $e_i \mapsto \lambda(i) \prod_{j \neq i} (x - \chi(j))$*, where* $\lambda(0) = -1$ *and* $\lambda(i) = \frac{1}{p-1}$ *otherwise. To see this, we first easily observe that the maps give algebra isomorphisms. To see that it preserves the comultiplication, we can check straightforwardly using Equation (3.1). We will skip the detailed calculation here.*

Now we define $\operatorname{Hom}^*_S((\mathbb{Z}/p\mathbb{Z})^2, G^2)$ to be the subfunctor of $\operatorname{Hom}_S((\mathbb{Z}/p\mathbb{Z})^2, G^2)$ which is given as follows:

Definition 3.1.2. *Define* $\operatorname{Hom}^*_S((\mathbb{Z}/p\mathbb{Z})^2, G^2)$ *to be the functor whose* T*-valued points are the elements in* $\operatorname{Hom}_S((\mathbb{Z}/p\mathbb{Z})^2, G^2)(T) = \operatorname{Mat}_2(G(T))$ *so that all nonzero* $\mathbb{F}_p$*-linear combinations of rows and columns are in* $(G^2)^\times(T)$*. For nonzero* $\mathbb{F}_p$*-linear combinations, we mean elements like* $[m]f \dot{+} [n]g$ *where* m *and* n *are not both zero in* $\mathbb{F}_p$*.*

Remark 3.1.3. *It is easy to see that the functor* $\operatorname{Hom}^*_S((\mathbb{Z}/p\mathbb{Z})^2, G^2)$ *we defined above is representable. Indeed, each linear combination being primitive is a closed condition and thus gives a subscheme of* $\underline{\operatorname{Hom}}^*_S((\mathbb{Z}/p\mathbb{Z})^2, G^2) = G^4$*. Therefore the functor* $\operatorname{Hom}^*_S((\mathbb{Z}/p\mathbb{Z})^2, G^2)$ *is represented by the scheme-theoretical intersection of those subschemes. We will use* $\underline{\operatorname{Hom}}^*_S((\mathbb{Z}/p\mathbb{Z})^2, G^2)$ *for the representing scheme.*

Here are some elementary properties of $\underline{\operatorname{Hom}}^*_S((\mathbb{Z}/p\mathbb{Z})^2, G^2)$:

Proposition 3.1.4. *Let S be a $\mathbb{Z}_p$-scheme and let G, G' be finite flat commutative group schemes of rank p over S. Let $\mathrm{GL}_2(\mathbb{F}_p)$ act on $\underline{\mathrm{Hom}}_S((\mathbb{Z}/p\mathbb{Z})^2, G^2)$ by acting on $(\mathbb{Z}/p\mathbb{Z})^2$ by precomposition. Then $\underline{\mathrm{Hom}}^*_S((\mathbb{Z}/p\mathbb{Z})^2, G^2)$ satisfies:*

(i) By identifying $\underline{\mathrm{Hom}}_S((\mathbb{Z}/p\mathbb{Z})^2, G^2) \times_S T = \underline{\mathrm{Hom}}_T((\mathbb{Z}/p\mathbb{Z})^2, G^2_T)$ for any S-scheme T, we have

$$\underline{\mathrm{Hom}}^*_S((\mathbb{Z}/p\mathbb{Z})^2, G^2) \times_S T = \underline{\mathrm{Hom}}^*_T((\mathbb{Z}/p\mathbb{Z})^2, G^2_T)$$

as closed subschemes.

(ii) The full level structure $\underline{\mathrm{Hom}}^*_S((\mathbb{Z}/p\mathbb{Z})^2, G^2)$ is $\mathrm{GL}_2(\mathbb{F}_p)$-invariant. Away from characteristic p, we have

$$\underline{\mathrm{Hom}}^*_{S[\frac{1}{p}]}((\mathbb{Z}/p\mathbb{Z})^2, G[\tfrac{1}{p}]^2) = \mathrm{Isom}_{S[\frac{1}{p}]}((\mathbb{Z}/p\mathbb{Z})^2, G[\tfrac{1}{p}]^2)$$

as closed subschemes of $\underline{\mathrm{Hom}}_{S[\frac{1}{p}]}((\mathbb{Z}/p\mathbb{Z})^2, G[\frac{1}{p}]^2)$.

(iii) Let $\phi : G \to G'$ be an isomorphism and let $\Phi : G^2 \to (G')^2$ be the isomorphism given by $\begin{pmatrix} \phi & 0 \\ 0 & \phi \end{pmatrix}$. Then the isomorphism $\underline{\mathrm{Hom}}_S((\mathbb{Z}/p\mathbb{Z})^2, G^2) \to \underline{\mathrm{Hom}}_S((\mathbb{Z}/p\mathbb{Z})^2, (G')^2)$ induced by Φ restricts to an isomorphism on the full level structures $\underline{\mathrm{Hom}}^*_S((\mathbb{Z}/p\mathbb{Z})^2, G^2) \to \underline{\mathrm{Hom}}^*_S((\mathbb{Z}/p\mathbb{Z})^2, (G')^2)$.

Proof.

(i) It follows straightforwardly from Definition 3.1.2 and the fact that the notion of primitive elements is compatible with base change [21, 3.5].

(ii) Let $f \in \underline{\mathrm{Hom}}^*_S((\mathbb{Z}/p\mathbb{Z})^2, G^2)$, regarded as a 2×2 matrix in $G(S)$. Let $g \in \mathrm{GL}_2(\mathbb{F}_p)$. Then g acts on f by $f \mapsto g^t f$, where the scalar multiplication is $[\cdot]$ and the addition is $\dot{+}$. By an elementary calculation, one can see that it suffices to show that if $(u, v) \in (G^2)^\times$ then $(u, v)g \in (G^2)^\times$ for all $g \in \mathrm{GL}_2(\mathbb{F}_p)$. Note that since G is annihilated by p, every $m \in \mathbb{F}_p$ defines an endomorphism of G and therefore every 2×2 matrix over $\mathbb{F}_p$ defines an endomorphism of G^2 and invertible matrices induce automorphisms of G^2. In fact,

$(u, v)g$ is the image of (u, v) under the automorphism induced by g. Since group scheme automorphisms preserve the augmentation ideal sheaf, they also preserve the primitive elements by Definition 2.3.1. Therefore $(u, v)g \in (G^2)^\times$ and we are done.

For the second half of (ii), note that $G[\frac{1}{p}]$ is étale locally isomorphic to $\underline{\mathbb{Z}/p\mathbb{Z}}$ and by definition $\underline{\mathrm{Hom}}^*((\mathbb{Z}/p\mathbb{Z})^2, (\underline{\mathbb{Z}/p\mathbb{Z}})^2) = \mathrm{Isom}((\mathbb{Z}/p\mathbb{Z})^2, (\underline{\mathbb{Z}/p\mathbb{Z}})^2)$. Then the statement is an immediate result of (i).

(iii) As in (ii), since every group scheme isomorphism preserves the augmentation ideal sheaf, by Definition 2.3.1 it also preserves the primitive elements. Then it is straightforward to check that (iii) holds by Definition 3.1.2.

□

Now here is the main theorem in this chapter:

Theorem 3.1.5. *Let S be a $\mathbb{Z}_p$-scheme and let G be a finite flat commutative group scheme of rank p over S. Let $\underline{\mathrm{Hom}}^*_S((\mathbb{Z}/p\mathbb{Z})^2, G^2)$ be as defined in Definition 3.1.2. Then the full level structure $\underline{\mathrm{Hom}}^*_S((\mathbb{Z}/p\mathbb{Z})^2, G^2)$ is flat over S of rank $|\mathrm{GL}_2(\mathbb{F}_p)|$.*

3.2 Proof of Theorem 3.1.5

By Proposition 3.1.4 (i), since being flat is a local property, we can reduce to the case where $S = \operatorname{Spec} A$ with A being a local $\mathbb{Z}_p$-algebra. Recall from Section 2 that the group scheme G/S is determined by a triple $(\mathcal{L}, u, v)$. Since A is local, the line bundle $\mathcal{L}$ on S is trivial. Let $\mathcal{A} = \mathbb{Z}_p[s, t]/(st - w_p)$ and $\mathcal{S} = \operatorname{Spec} \mathcal{A}$. Let $\mathcal{G} = \operatorname{Spec} \mathcal{A}[x]/(x^p - tx)$ be the group scheme over $\mathcal{S}$ with comultiplication

$$m^*(x) = 1 \otimes x + x \otimes 1 + \frac{1}{1-p} \sum_{i=1}^{p-1} \frac{sx^i \otimes x^{p-i}}{w_i w_{p-i}}. \tag{3.2}$$

Then G/S will be the pull back of $\mathcal{G}/\mathcal{S}$ through a morphism $S \to \mathcal{S}$ determined by u and v. Applying Proposition 3.1.4 (i) again, we can see that it suffices to show the flatness of the full level structure for $\mathcal{G}^2/\mathcal{S}$.

We first look at $\underline{\mathrm{Hom}}^*_{\mathcal{S}}((\mathbb{Z}/p\mathbb{Z})^2, \mathcal{G}^2)$ over the two open subschemes $\operatorname{Spec}\mathbb{Z}_p[s, s^{-1}]$ and $\operatorname{Spec}\mathbb{Z}_p[t, t^{-1}]$ of $\mathcal{S}$. It is easy to check that after applying étale base changes by adding the $p-$ 1th root of s, s^{-1}, t, t^{-1}, we get $\mathcal{G}\times_{\mathcal{S}}\operatorname{Spec}\mathbb{Z}_p[s^{\frac{1}{p-1}}, s^{-\frac{1}{p-1}}] \cong \mu_p$ and $\mathcal{G}\times_{\mathcal{S}}\operatorname{Spec}\mathbb{Z}_p[t^{\frac{1}{p-1}}, t^{-\frac{1}{p-1}}] \cong \underline{\mathbb{Z}/p\mathbb{Z}}$. In these cases, the following lemma is as expected:

Lemma 3.2.1. *We have the following two isomorphisms of group schemes:*

(i) $\underline{\mathrm{Hom}}^*_{\mathcal{S}}((\mathbb{Z}/p\mathbb{Z})^2, \mathcal{G}^2) \times_{\mathcal{S}} \operatorname{Spec}\mathbb{Z}_p[s^{\frac{1}{p-1}}, s^{-\frac{1}{p-1}}] \cong \underline{\mathrm{Hom}}^{\mathrm{full}}_{\operatorname{Spec}\mathbb{Z}_p[s^{\frac{1}{p-1}}, s^{-\frac{1}{p-1}}]}((\mathbb{Z}/p\mathbb{Z})^2, \mu_p^2)$.

Here the $\underline{\mathrm{Hom}}^{\mathrm{full}}$ *is the full level structure for* $\mu_p \times \mu_p$ *defined by Wake in [42].*

(ii) $\underline{\mathrm{Hom}}^*_{\mathcal{S}}((\mathbb{Z}/p\mathbb{Z})^2, \mathcal{G}^2) \times_{\mathcal{S}} \operatorname{Spec}\mathbb{Z}_p[t^{\frac{1}{p-1}}, t^{-\frac{1}{p-1}}] \cong \underline{\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})}$.

Proof. Note that from the definition of $\underline{\mathrm{Hom}}^*$, we have

$$\underline{\mathrm{Hom}}^*((\mathbb{Z}/p\mathbb{Z})^2, \mu_p^2) = \underline{\mathrm{Hom}}^{\mathrm{full}}((\mathbb{Z}/p\mathbb{Z})^2, \mu_p^2)$$

as they are defined in the same way. For the étale part, note that sections of constant group schemes being primitive exactly means being nonzero. So $\underline{\mathrm{Hom}}^*((\mathbb{Z}/p\mathbb{Z})^2, (\underline{\mathbb{Z}/p\mathbb{Z}})^2)$ consists of the matrices satisfying that nonzero linear combinations of rows and columns are nonzero, thus invertible matrices. Hence $\underline{\mathrm{Hom}}^*((\mathbb{Z}/p\mathbb{Z})^2, (\underline{\mathbb{Z}/p\mathbb{Z}})^2) = \underline{\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})}$ and the claim is immediate from Proposition 3.1.4 (i). □

To make the full level structure explicit for $\mathcal{G}^2/\mathcal{S}$, it is helpful to use the universal homomorphism for description. Consider the universal base $\mathcal{S}^{\mathrm{univ}} = \operatorname{Spec}\mathcal{A}^{\mathrm{univ}}$ where $\mathcal{A}^{\mathrm{univ}} = \mathcal{A}[a, b, c, d]/(a^p - ta, b^p - tb, c^p - tc, d^p - td)$. Then we have $\mathcal{S}^{\mathrm{univ}} = \underline{\mathrm{Hom}}_{\mathcal{S}}((\mathbb{Z}/p\mathbb{Z})^2, \mathcal{G}^2)$. Let $h \in \mathrm{Hom}_{\mathcal{S}^{\mathrm{univ}}}((\mathbb{Z}/p\mathbb{Z})^2, \mathcal{G}^2_{\mathcal{S}^{\mathrm{univ}}})$ be the universal homomorphism defined over $\mathcal{S}^{\mathrm{univ}}$, given by $(1, 0) \mapsto (a, b)$, $(0, 1) \mapsto (c, d)$. Then $\underline{\mathrm{Hom}}^*_{\mathcal{S}}((\mathbb{Z}/p\mathbb{Z})^2, \mathcal{G}^2)$, as a subscheme of the universal base $\mathcal{S}^{\mathrm{univ}}$, is cut out by the condition $h \in \mathrm{Hom}^*_{\mathcal{S}^{\mathrm{univ}}}((\mathbb{Z}/p\mathbb{Z})^2, \mathcal{G}^2_{\mathcal{S}^{\mathrm{univ}}})$. Therefore, by definition, $\underline{\mathrm{Hom}}^*_{\mathcal{S}}((\mathbb{Z}/p\mathbb{Z})^2, \mathcal{G}^2)$ is given by the ideal $I \subset \mathcal{A}^{\mathrm{univ}}$ generated by

$$\Big\{ \Big(([m]a \dot{+} [n]b)^{p-1} - t \Big) \Big(([m]c \dot{+} [n]d)^{p-1} - t \Big), \\ \Big(([m]a \dot{+} [n]c)^{p-1} - t \Big) \Big(([m]b \dot{+} [n]d)^{p-1} - t \Big) \Big\}_{(m,n)\in\mathbb{F}_p^2 \ \{(0,0)\}}.$$

Recall that in the notion $[m]a \dot{+} [n]b$, we are regarding a, b, c, d as elements in $\mathcal{G}(\mathcal{S}^{\mathrm{univ}})$, corresponding to the homomorphisms

$$A[x]/(x^p - tx) \to \mathcal{A}[a, b, c, d]/(a^p - ta, b^p - tb, c^p - tc, d^p - td)$$

sending x to a, b, c, d. As an abstract group, $\mathcal{G}(\mathcal{S}^{\mathrm{univ}})$ is given by

$$\{x \in \mathcal{A}[a, b, c, d]/(a^p - ta, b^p - tb, c^p - tc, d^p - td) \big| x^p = tx\}$$

with the group structure given by $x \dot{+} y = x + y + \frac{1}{1-p} \sum_{i=1}^{p-1} \frac{s x^i y^{p-i}}{w_i w_{p-i}}$. Therefore

$$[2]a = 2a + \frac{1}{1-p} \sum_{i=1}^{p-1} \frac{s a^p}{w_i w_{p-i}} = 2a + \frac{1}{1-p} \sum_{i=1}^{p-1} \frac{s t a}{w_i w_{p-i}} = \left(2 + \frac{1}{1-p} \sum_{i=1}^{p-1} \frac{w_p}{w_i w_{p-i}}\right) a.$$

Using Equation (3.1), we get $[2]a = \chi(2)a$ and in general by induction we have $[m]a = \chi(m)a$. Therefore the full level structure on $\mathcal{G}^2/\mathcal{S}$ has the following expression:

$$\begin{aligned} &\underline{\mathrm{Hom}}_{\mathcal{S}}^*((\mathbb{Z}/p\mathbb{Z})^2, \mathcal{G}^2) \\ &\cong \operatorname{Spec} \mathbb{Z}_p[s, t, a, b, c, d] \Big/ \left(\begin{matrix} st - w_p, a^p - ta, b^p - tb, c^p - tc, d^p - td, \\ \left\{ \left((\chi(m)a \dot{+} \chi(n)b)^{p-1} - t \right) \left((\chi(m)c \dot{+} \chi(n)d)^{p-1} - t \right), \right. \\ \left. \left((\chi(m)a \dot{+} \chi(n)c)^{p-1} - t \right) \left((\chi(m)b \dot{+} \chi(n)d)^{p-1} - t \right) \right\} \end{matrix} \right). \end{aligned} \tag{3.3}$$

Having all these set up, we will prove the flatness of $\underline{\mathrm{Hom}}_{\mathcal{S}}^*((\mathbb{Z}/p\mathbb{Z})^2, \mathcal{G}^2)$ over $\mathcal{S}$ using the lemma below:

Lemma 3.2.2 ([27] Page 51 Lemma 1)**.** *Let Y be a reduced scheme and $\mathcal{F}$ a coherent sheaf on Y such that $\dim_{k(y)} \mathcal{F} \otimes_{\mathcal{O}_y} k(y) = r$, for all $y \in Y$. Then $\mathcal{F}$ is a locally free of rank r on Y.*

Apply Lemma 3.2.2 to $Y = \mathcal{S}$. Note that for $y \in \operatorname{Spec} \mathbb{Z}_p[t, t^{-1}]$, we know that

$$\dim_{k(y)} \left(\mathcal{O}_{\underline{\mathrm{Hom}}_{\mathcal{S}}^*} \otimes_{\mathcal{O}_y} k(y) \right) = |\mathrm{GL}_2(\mathbb{F}_p)|$$

from Lemma 3.2.1 (ii) and étale descent. For $y \in \operatorname{Spec} \mathbb{Z}_p[s, s^{-1}]$, we can get

$$\dim_{k(y)} \left(\mathcal{O}_{\underline{\mathrm{Hom}}_{\mathcal{S}}^*} \otimes_{\mathcal{O}_y} k(y) \right) = |\mathrm{GL}_2(\mathbb{F}_p)|$$

by combining Lemma 3.2.1 (i) together with Wake's result on $\underline{\mathrm{Hom}}^{\mathrm{full}}$ and étale descent. The only remaining point is y_0 for $s = t = p = 0$.

Consider $\mathcal{G}/\mathcal{S}$ modulo p, denoted by $\bar{\mathcal{G}}/\bar{\mathcal{S}}$. The underlying base scheme, which is given by $\bar{\mathcal{S}} = \operatorname{Spec} \mathbb{F}_p[s,t]/(st)$, is a union of two affine lines and the concerning point y_0 is the origin of $\bar{\mathcal{S}}$. Note that $\chi(m) \equiv m \bmod p$. Therefore, by setting $p = 0$ from Equation (1.6), we get

$$\begin{aligned} &\underline{\mathrm{Hom}}^*_{\bar{\mathcal{S}}}((\mathbb{Z}/p\mathbb{Z})^2, \bar{\mathcal{G}}^2) \\ &\cong \mathcal{S}pec\, \mathcal{O}_{\bar{\mathcal{S}}}[a,b,c,d] \Big/ \left(\begin{smallmatrix} a^p - ta, b^p - tb, c^p - tc, d^p - td, \\ \{ \left((ma\dot{+}nb)^{p-1} - t\right)\left((mc\dot{+}nd)^{p-1} - t\right), \\ \left((ma\dot{+}nc)^{p-1} - t\right)\left((mb\dot{+}nd)^{p-1} - t\right) \} \end{smallmatrix} \right). \end{aligned} \tag{3.4}$$

Here the "$\dot{+}$" operation is given as $x\dot{+}y = x + y + \sum_{i=1}^{p-1} \frac{sx^i y^{p-i}}{i!\,(p-i)!}$ (recall that $w_i \equiv i! \mod p$ from Section 2). Now we have a key observation on Equation (3.4).

Theorem 3.2.3. *Let $\bar{\mathcal{G}}/\bar{\mathcal{S}}$ be the "universal" Oort–Tate group scheme in characteristic p as above. Then the ideal defining the full level structure $\underline{\mathrm{Hom}}^*_{\bar{\mathcal{S}}}((\mathbb{Z}/p\mathbb{Z})^2, \bar{\mathcal{G}}^2)$ as a closed subscheme of $\bar{\mathcal{G}}^4$ is generated by elements which do not involve the parameter s.*

Proof. We claim that in the coordinate ring (3.4), we have

$$(ma\dot{+}nb)^{p-1} - t = u\left((ma+nb)^{p-1} - t\right)$$

for some unit u. Then it follows that

$$\begin{aligned} &\underline{\mathrm{Hom}}^*_{\bar{\mathcal{S}}}((\mathbb{Z}/p\mathbb{Z})^2, \bar{\mathcal{G}}^2) \\ &\cong \operatorname{Spec} \mathcal{O}_{\bar{\mathcal{S}}}[a,b,c,d] \Big/ \left(\begin{smallmatrix} a^p - ta, b^p - tb, c^p - tc, d^p - td, \\ \{ \left((ma+nb)^{p-1} - t\right)\left((mc+nd)^{p-1} - t\right), \\ \left((ma+nc)^{p-1} - t\right)\left((mb+nd)^{p-1} - t\right) \} \end{smallmatrix} \right). \end{aligned} \tag{3.5}$$

and we are done.

When $p = 2$, since $st = 0$ and $a^2 = ta$, we simply have

$$ma\dot{+}nb = ma + nb + smnab = (ma+nb)(1+sma).$$

Here $1 + sma$ is a unit as $(1+sma)^2 = 1 + s^2m^2a^2 = 1 + s^2m^2at = 1$.

Now suppose that $p > 2$. Let $g(x,y) = \sum_{i=1}^{p-1} \frac{x^i y^{p-i}}{i!\,(p-i)!}$ be a polynomial in $\mathbb{F}_p[x,y]$. Note this polynomial $g(x,y)$ is divisible by $x+y$ as $g(x,-x) = 0$ (note that p is odd). Assume $g(x,y) = (x+y)g'(x,y)$. Then $ma \dot{+} nb = (ma+nb)(1+sg'(ma,nb))$. Note that g' has no constant term and $st = 0$. So we have

$$(1+sg'(ma,nb))^p = 1 + s^p g'(m^p at, n^p bt) = 1.$$

Therefore $1 + sg'(ma,nb)$ is a unit and we have

$$(1+sg'(ma,nb))^{p-1}\left((ma+nb)^{p-1} - t\right) = (ma \dot{+} nb)^{p-1} - t$$

as claimed. □

As a consequence of Theorem 3.2.3, for any point $y \in \bar{\mathcal{S}}$ away from y_0, we have

$$\dim_{k(y_0)} \mathcal{O}_{\underline{\mathrm{Hom}}^*_{\mathcal{S}}} \otimes_{\mathcal{O}_{y_0}} k(y_0) = \dim_{k(y)} \mathcal{O}_{\underline{\mathrm{Hom}}^*_{\mathcal{S}}} \otimes_{\mathcal{O}_y} k(y) = |\mathrm{GL}_2(\mathbb{F}_p)|.$$

Applying Lemma 3.2.2, we finish proving the flatness.

3.3 Nonexistence of full level structure over the stack

Let $\mathbf{C}$ be a stack of group schemes of certain type over $\mathbf{Sch}_{\mathbb{Z}_\mathbf{p}}$. (By a stack here we simply mean a category fibered in groupoids over $\mathbf{Sch}_{\mathbb{Z}_\mathbf{p}}$ as in [6].) So, we assume that the objects in $\mathbf{C}$ are group schemes G/S of certain fixed type (for example, finite flat commutative and of certain rank) and the morphisms are Cartesian squares.

Definition 3.3.1. *Let $\mathbf{C}$ be a stack of group schemes as above. By a full level structure over $\mathbf{C}$, we mean a fibered functor $\mathcal{F}\colon \mathbf{C} \to \mathbf{Sch}$, such that*

(1) For any G/S, the scheme $\mathcal{F}(G/S)$ is a closed subscheme of $\underline{\mathrm{Hom}}_S((\mathbb{Z}/p^r\mathbb{Z})^g, G)$.

(2) For any $f\colon G/S \to G'/S'$, the morphism $\mathcal{F}(f)\colon \mathcal{F}(G/S) \to \mathcal{F}(G'/S')$ is the restriction of the induced morphism $\underline{\mathrm{Hom}}_S((\mathbb{Z}/p^r\mathbb{Z})^g, G) \to \underline{\mathrm{Hom}}_{S'}((\mathbb{Z}/p^r\mathbb{Z})^g, G')$.

(3) The scheme $\mathcal{F}(G/S)$ satisfies the conditions (1)-(3) in the beginning of this chapter.

Note that the condition (4) is automatically satisfied since $\mathcal{F}$ is a functor.

Using this terminology of full level structure over the stack, we may briefly summarize the results in Section 3.1 as that we define a well-behaved notion of full level structure on the stack OT, whose objects are group schemes of the form $G \times G$ where G is an Oort–Tate scheme over a $\mathbb{Z}_p$-scheme S, and morphisms are group scheme isomorphisms $G \times G \to G' \times G'$ of the form $\Phi = \begin{pmatrix} \phi & 0 \\ 0 & \phi \end{pmatrix}$ as in Proposition 3.1.4.

However, this full level structure on OT cannot be extended to the stack of finite flat commutative group schemes. In fact, consider the substack $OT \times OT$, whose objects are $G \times G'$ where G, G' are Oort–Tate schemes with morphisms be arbitrary group scheme isomorphisms. We will see that even on $OT \times OT$, there is no good notion of full level structure:

Theorem 3.3.2. *There is no notion of full level structure over the stack $OT \times OT$ in the sense of Definition 3.3.1*

Proof. Let $\mathcal{G}/\mathcal{S}$ be as in Section 3. Assume there is a full level structure on $OT \times OT$ satisfying (1)-(4). Then the full level structure on $\mathcal{G}^2/\mathcal{S}$ must be the one we defined. In fact over the generic fiber of $\mathcal{S}$, the full level structure is given by the condition (2). Therefore the only way to satisfy condition (1) is defining the full level structure over S as the Zariski closure of the corresponding scheme over the generic fiber. Note that any group scheme of rank p over a local ring can be obtained from $\mathcal{G}/\mathcal{S}$ by base change. Because of condition (3), the full level structure on $G \times G$ over a local base must be the one we defined above. However, this only possible structure is not preserved under all group scheme automorphisms. Here is one example communicated to the author by Wake:

Consider the full level structure on $\alpha_p \times \alpha_p$ over $\bar{\mathbb{F}}_p$ with $p > 2$. By our definition and Theorem 3.2.3, we have

$$\underline{\mathrm{Hom}}^*_{\bar{\mathbb{F}}_p}((\mathbb{Z}/p\mathbb{Z})^2, \alpha_p^2) \cong \operatorname{Spec} \bar{\mathbb{F}}_p[a,b,c,d] \Bigg/ \left(\begin{smallmatrix} a^p, b^p, c^p, d^p \\ \{(ma+nb)^{p-1}(mc+nd)^{p-1}, \\ (ma+nc)^{p-1}(mb+nd)^{p-1}\} \end{smallmatrix} \right).$$

Note that $\mathrm{Aut}_{\bar{\mathbb{F}}_p}(\alpha_p^2) = \mathrm{GL}_2(\bar{\mathbb{F}}_p)$, with the action given by multiplying $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ by elements of $\mathrm{GL}_2(\bar{\mathbb{F}}_p)$ from the right. Since $(m, n) \in \mathbb{F}_p^2 \ \ \{(0,0)\}$, it is not hard to see that the ideal is not invariant under the action of $\mathrm{GL}_2(\bar{\mathbb{F}}_p)$. □

Remark 3.3.3. *Although as shown, a good notion of full level structure on the stack of all finite group schemes does not exist, one might still hope to define a full level structure on truncated p-divisible groups. However, some new idea is needed.*

3.4 Full level structure on $G \times G \times G$

A natural question we may ask is whether we can have some similar results for group schemes of the form G^n, where G is an Oort–Tate group scheme. We record some partial results here. However, a full answer to this question requires some new idea.

Let us take $G = \mu_p$ over $\mathrm{Spec}\,\mathbb{Z}$. One intermediate step towards defining a full level structure on G^3 is defining a "partial level structure" as a subscheme of $\underline{\mathrm{Hom}}_{\mathbb{Z}}((\mathbb{Z}/p\mathbb{Z})^2, (\mu_p)^3)$. We will still require that the resulting scheme is flat over the base and when inverting p we want $\underline{\mathrm{Hom}}^*_{\mathbb{Z}}((\mathbb{Z}/p\mathbb{Z})^2, (\mu_p)^3) \cong \underline{\mathrm{Mat}^*_{2\times 3}(\mathbb{F}_p)}$, where $\mathrm{Mat}^*_{2\times 3}$ denote the set of all 2×3 matrices of rank 2. It turns out that this can be done using our result in this paper. Let h be the universal homomorphism. Then $\underline{\mathrm{Hom}}^*_{\mathbb{Z}}((\mathbb{Z}/p\mathbb{Z})^2, (\mu_p)^3)$ is cut out by the following conditions:

(i) All nonzero linear combinations of rows and columns are primitive.

(ii) After applying any left $\mathrm{GL}_2(\mathbb{F}_p)$-action and right $\mathrm{GL}_3(\mathbb{F}_p)$-action to h, one of the three 2×2 blocks of the resulting homomorphism lies in the full level structure $\underline{\mathrm{Hom}}^*_{\mathbb{Z}}((\mathbb{Z}/p\mathbb{Z})^2, (\mu_p)^2)$.

Let us make (ii) clear here. Let

$$h = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

be the universal homomorphism. Let I_1, resp. I_2, I_3, be the ideal defined by requiring that

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad \text{resp. } \begin{pmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{pmatrix}, \begin{pmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{pmatrix},$$

lies in the full level structure subscheme $\underline{\mathrm{Hom}}^*((\mathbb{Z}/p\mathbb{Z})^2, (\mu_p)^2)$. Then the ideal defining "one of the three 2×2 blocks lies the full level structure" is the ideal $I_1I_2 \cap I_1I_3 \cap I_2I_3$. The closed subscheme $\underline{\mathrm{Hom}}^*_{\mathbb{Z}}((\mathbb{Z}/p\mathbb{Z})^2, (\mu_p)^3)$ cut out by these conditions is flat of rank $|\mathrm{Mat}^*_{2\times 3}(\mathbb{F}_p)|$ over the base. This result of "partial level structure" $\underline{\mathrm{Hom}}^*_{\mathbb{Z}}((\mathbb{Z}/p\mathbb{Z})^2, (\mu_p)^3)$ can be extended to $\underline{\mathrm{Hom}}^*((\mathbb{Z}/p\mathbb{Z})^2, G^3)$.

One might hope to define $\underline{\mathrm{Hom}}^*_{\mathbb{Z}}((\mathbb{Z}/p\mathbb{Z})^3, (\mu_p)^3)$ using the "partial level structure" above, by requiring that after applying the left and right $\mathrm{GL}_3(\mathbb{F}_p)$-action and possibly Cartier dual to the universal homomorphism, the resulting homomorphism is such that any 2×3 block is giving a "partial level structure". It turns out that this condition is very close to what we want, but still not enough. Here are some numerical results. Consider μ_p over $\mathbb{F}_p$. For $p = 2$, the above condition will give a closed subscheme of rank 169 over $\mathbb{F}_p$, while $|\mathrm{GL}_3(\mathbb{F}_2)| = 168$. For $p = 3$, the obtained subscheme has rank 11473 over $\mathbb{F}_p$, while $|\mathrm{GL}_3(\mathbb{F}_3)| = 11232$ (comparing with $3^9 = 19683$). So, some further conditions need to be discovered.

CHAPTER 4

CONSTRUCTIONS OF GROUP SCHEMES USING DIEUDONNÉ MODULES

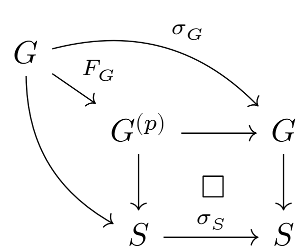
Dieudonné theory is a powerful tool for studying group schemes, p-divisible groups and abelian varieties. The classical Dieudonné theory works over perfect fields in characteristic $p > 0$. There has been many variations of Dieudonné theory over difference bases. In this chapter, we will use a version of Dieudonné theory due to de Jong to construct group schemes.

Let G/S be a finite group scheme. (Recall that all group schemes are assumed to be commutative and flat over the base.) We define its Cartier dual G^D by

$$G^D(T) = \underline{\mathrm{Hom}}(G_T, \mathbb{G}_{m,T}).$$

This is a priori a functor but can be shown to be representable. This makes G^D a finite group scheme over S and there is a canonical isomorphism $G \xrightarrow{\sim} (G^D)^D$. Furthermore, for any group scheme homomorphism $f : G \to H$, we have an induced homomorphism $f^D : H^D \to G^D$ such that $(f^D)^D = f$ under the canonical isomorphism $G = (G^D)^D$.

Let G/S be a group scheme (not necessarily finite) with the $p\mathcal{O}_S = 0$. Then there are Frobenius morphisms $\sigma_G : G \to G$ and $\sigma_S : S \to S$ that are defined by Frobenius maps $f_G : \mathcal{O}_G \to \mathcal{O}_G$ and $f_S : \mathcal{O}_S \to \mathcal{O}_S$. This induces a morphism $F_G : G \to G^{(p)} := G \times_{S,\sigma_S} S$, which turns out to be a homomorphism of group schemes:



We can define another group scheme homomorphism $V_G : G^{(p)} \to G$. Consider the following diagram:

$$\begin{array}{ccccc} \mathcal{O}_G & \longrightarrow & (\mathcal{O}_G^{\otimes p})^{S_p} & \twoheadrightarrow & \mathcal{O}_G \otimes_{\mathcal{O}_S, f_S} \mathcal{O}_S \\ & \searrow^{comult} & \big\downarrow & & \\ & & \mathcal{O}_G^{\otimes p} & & \end{array}$$

Here $(\mathcal{O}_G^{\otimes p})^{S_p}$ is the S_p-invariant elements of $\mathcal{O}_G^{\otimes p}$. Since G is commutative, the comultiplication factors through $(\mathcal{O}_G^{\otimes p})^{S_p}$. The map $(\mathcal{O}_G^{\otimes p})^{S_p} \to \mathcal{O}_G \otimes_{\mathcal{O}_S, f_S} \mathcal{O}_S$ is the unique homomorphism such that $a(x \otimes \cdots \otimes x) \mapsto x \otimes a$. In this way, we get a morphism $V_G : G \to G^{(p)}$ which also turns out to be a homomorphism of group schemes. This group scheme homomorphism is called the Verschiebung of G. When the group scheme G/S is finite, the Verschiebung V_G can also be defined as $(F_{G^D})^D : ((G^D)^{(p)})^D = G^{(p)} \to (G^D)^D = G$. It is a basic fact that $F_G \circ V_G = p \cdot \mathrm{Id}_{G^{(p)}}$ and $V_G \circ F_G = p \cdot \mathrm{Id}_G$ (see [5, II]).

4.1 A Version of Dieudonné Theory by De Jong

Let W be the Witt ring (over $\mathbb{Z}$) as in [5, III]. It is a ring scheme over $\operatorname{Spec} \mathbb{Z}$. Let k be a perfect field in characteristic $p > 0$. The k-points of W are called Witt vectors and $W(k)$ is the ring of Witt vectors. The ring $W(k)$ is a complete discrete valuation ring in mixed characteristics $(0, p)$ with uniformizer p and residue field k (see [5, III. 3]).

Moreover, there is a ring homomorphism $\sigma : W(k) \to W(k)$ which lifts the Frobenius map on the residue field k. Let $D_k := W(k)\{F, V\}/(FV - p)$ be the Dieudonné ring, where $W(k)\{F, V\}$ is the non-commutative polynomial ring in variables F, V with relations

$$FV = VF,$$

$$Fa = \sigma(a)F,$$

$$Va = \sigma^{-1}(a)V.$$

The left D_k-modules are called Dieudonné modules. Equivalently, one can think of Dieudonné modules as $W(k)$-modules with actions of F and V that are subject to the relations above.

Dieudonné modules are powerful tools to study group schemes, p-divisible groups and abelian varieties. (Recall that all group schemes are assumed to be commutative and flat over the base.) The classical (contravariant) Dieudonné theory gives an anti-equivalence between the category of finite group schemes over a perfect field k with positive characteristics and the category of Dieudonné modules over $W(k)$ with finite length (see [5, III. 7]):

$$\left\{ \begin{array}{c} \text{finite group schemes} \\ G/\operatorname{Spec} k \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{Dieudonné modules with} \\ \text{finite } W(k)\text{-length} \end{array} \right\}.$$

The classical Dieudonné theory also gives an similar anti-equivalence for p-divisible groups over a perfect field (see [5, III. 8]):

$$\left\{ \begin{array}{c} p\text{-divisible groups} \\ \mathcal{G}/\operatorname{Spec} k \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{Dieudonné modules that are} \\ \text{finitely generated free} \\ W(k)\text{-modules} \end{array} \right\}.$$

Starting from the classical Dieudonné theory, there has been variations of Dieudonné theory for different base schemes. Let $S = \operatorname{Spec} R$ with R a ring of characteristic $p > 0$ and let $\Sigma := \operatorname{Spec} \mathbb{Z}_p$. In [2], using the notion of crystals over the crystalline site, Berthelot–Breen–Messing define the crystalline Dieudonné functor $\mathbb{D}$ from the category of p-divisible groups over S to the category of F-crystals over the crystalline site $\mathrm{CRIS}(S/\Sigma)$, given by

$$\mathbb{D}(\mathcal{G}) := \mathcal{E}xt^1_{S/\Sigma}\left(\mathcal{G}, \mathcal{O}_{S/\Sigma}\right). \tag{4.1}$$

Here $\mathcal{O}_{S/\Sigma}$ is the structure sheaf defined by

$$\mathcal{O}_{S/\Sigma}(U, T, \delta) := \mathcal{O}_T. \tag{4.2}$$

and $\mathcal{G}$ is regarded as an abelian sheaf on $\mathrm{CRIS}(S/\Sigma)$. It is shown in [1] and [2] that when R is a perfect field or more generally a perfect valuation ring, the Dieudonné functor $\mathbb{D}$ defines an anti-equivalence of categories. Moreover, let $f : A \to \operatorname{Spec} R$ be an abelian scheme, then we have

$$\mathbb{D}(A[p^\infty])_S = R^1 f_* \mathcal{O}_{A/\Sigma}.$$

In [4], de Jong established a version of Dieudonné theory for group schemes G/S satisfying $p\mathcal{O}_S = 0$ and S and its Frobenius f_S lift modulo p^2. More precisely, let $S = \operatorname{Spec} R$ be as above. Assume that S admits a lift to $\mathbb{Z}/p^2\mathbb{Z}$, i.e. a scheme S' flat over $\operatorname{Spec} \mathbb{Z}/p^2\mathbb{Z}$ with $S' \times_{\operatorname{Spec} \mathbb{Z}/p^2\mathbb{Z}} \operatorname{Spec} \mathbb{F}_p \cong S$:

$$\begin{array}{ccc} S & \longrightarrow & S' \\ \downarrow & \square & \downarrow \\ \operatorname{Spec} \mathbb{F}_p & \longrightarrow & \operatorname{Spec} \mathbb{Z}/p^2\mathbb{Z} \end{array}$$

Then the divided power on $\mathbb{Z}_p$ extends to a unique divided power γ on $p\mathcal{O}_{S'}$. In this way, the triple (S, S', γ) forms an object in the Crystalline site $\mathrm{CRIS}(S/\Sigma)$.

Assume further that there is a morphism $f_{S'} : S' \to S'$ which lifts the Frobenius map on S. Let $C(1)$ denote the category of group schemes over S killed by p and let $M(1)$ denote the category of triples (M, F, V), where M is a finite locally free $\mathcal{O}_S$-module and $F : f^*M \to M$, $V : M \to f^*M$ are $\mathcal{O}_S$-linear maps such that $V \circ F = p \cdot \mathrm{Id}_{f^*M}$ and $F \circ V = p \cdot \mathrm{Id}_M$.

Consider the functor $M_S : C(1) \to M(1)$ defined by

$$M_S(G) := \mathcal{E}xt^1_{S/\Sigma}\left(G, \mathcal{I}_{S/\Sigma}\right)_{(S,S',\gamma)} \tag{4.3}$$

where $\underline{G}$ is regarded as an abelian sheaf on $\mathrm{CRIS}(S/\Sigma)$ and $\mathcal{I}_{S/\Sigma}$ is the abelian sheaf on $\mathrm{CRIS}(S/\Sigma)$ defined by

$$\mathcal{I}_{S/\Sigma}(U, T, \delta) := \ker(\mathcal{O}_T \to \mathcal{O}_U).$$

Proposition 4.1.1. *(de Jong [4, Proposition 8.6]) The functor $M_S : C(1) \to M(1)$ induces the following isomorphisms when $V_G = 0$ or $F_H = 0$:*

$$\mathrm{Hom}_{C(1)}(H, G) \xrightarrow{\sim} \mathrm{Hom}_{M(1)}(M_S(G), M_S(H))$$

and

$$\mathrm{Ext}^1_{C(1)}(H, G) \xrightarrow{\sim} \mathrm{Ext}^1_{M(1)}(M_S(G), M_S(H)).$$

Remark 4.1.2. *The conditions in Proposition 4.1.1 are satisfied when S is affine and smooth. In fact, when S is smooth, the obstruction class lifting S together with the Frobenius lies in certain cohomology $H^1(S, T_S \otimes B^1_S)$ (see [26, Appendix]). When S is affine, the cohomology vanishes. In this case, the lift of S together with the Frobenius exists automatically.*

Note that this version of Dieudonné theory by de Jong is slightly different from the crystalline Dieudonné theory of finite group schemes or abelian schemes as in the definitions

(4.1) and (4.3). In the rest of this section, we will compare these two versions of Dieudonné theory.

Let $\mathbb{G}_a$ be the abelian sheaf on CRIS(S/Σ) defined by

$$\mathbb{G}_a(U,T,\delta) := \mathcal{O}_U.$$

Therefore there is a exact sequence of abelian sheaves

$$0 \to \mathcal{J}_{S/\Sigma} \to \mathcal{O}_{S/\Sigma} \to \mathbb{G}_a \to 0.$$

Let A/S (resp. G/S) be an abelian scheme (resp. a finite group scheme). We define the crystalline Dieudonné crystal of A/S (resp. G/S) as $\mathbb{D}(-) := \mathcal{E}xt^1_{S/\Sigma}\left(-, \mathcal{O}_{S/\Sigma}\right)$. As a standard result of crystalline Dieudonné theory, we have the following proposition:

Proposition 4.1.3. *([2, Proposition 2.5.8]) Let A/S be an abelian scheme. Then we have*

$$\mathcal{E}xt^i_{S/\Sigma}\left(A, \mathcal{J}_{S/\Sigma}\right) = \mathcal{E}xt^i_{S/\Sigma}\left(A, \mathbb{G}_a\right) = 0$$

for $i = 0$ or $i = 2$. In particular, when evaluating at S, we have the following commutative diagram that connects with De Rham cohomology:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E}xt^1_{S/\Sigma}\left(A, \mathcal{J}_{S/\Sigma}\right)_S & \longrightarrow & \mathbb{D}(A)_S & \longrightarrow & \mathcal{E}xt^1_{S/\Sigma}\left(A, \mathbb{G}_a\right)_S & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\ 0 & \longrightarrow & \omega_A & \longrightarrow & H^1_{DR}(A/S) & \longrightarrow & R^1 f_*(\mathcal{O}_A) & \longrightarrow & 0 \end{array}$$

Here w_A is the pullback of $\Omega^1_{A/S}$ along the unit section, and $H^1_{DR}(A/S)$ is the first De Rham cohomology.

Remark 4.1.4. *The identification in Proposition 4.1.3 can also be realized as following:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V(\mathbb{D}(A)_S) & \longrightarrow & \mathbb{D}(A)_S & \longrightarrow & \mathbb{D}(A)_S/V(\mathbb{D}(A)_S) & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\ 0 & \longrightarrow & \omega_A & \longrightarrow & H^1_{DR}(A/S) & \longrightarrow & R^1 f_*(\mathcal{O}_A) & \longrightarrow & 0 \end{array}$$

where V is the induced action of Verschibung on $\mathbb{D}(A)_S$. In particular, $\mathcal{E}xt^1_{S/\Sigma}\left(A, \mathcal{J}_{S/\Sigma}\right)_S \subset \mathbb{D}(A)_S$ identifies with $V\mathbb{D}(A)_S$. Similarly, $\mathcal{E}xt^1_{S/\Sigma}\left(A, \mathcal{J}_{S/\Sigma}\right)_{(S,S',\gamma)} \subset \mathbb{D}(A)_{(S,S',\gamma)}$ identifies with $V\mathbb{D}(A)_{(S,S',\gamma)}$. This follows from the observation over a field in [29, Corollary 5.11] and the fact that the Dieudonné crystal functor commutes with base change.

Now we give the following comparison lemma:

Lemma 4.1.5. *Let $0 \to G \to A \to B \to 0$ be an exact sequence, where G is a finite group scheme killed by p and A, B are abelian varieties over S. Then we have*

$$M_S(G) \cong \operatorname{Coker}\left(V\mathbb{D}(B)_S \to V\mathbb{D}(A)_S\right). \tag{4.4}$$

In particular, we have

$$M_S(A[p]) \cong \operatorname{Coker}\left(V\mathbb{D}(A)_S / pV\mathbb{D}(A)_S\right). \tag{4.5}$$

Proof. By Proposition 4.1.3, we can get a commutative diagram

$$\mathcal{E}xt^1_{S/\Sigma}\left(B, \mathcal{J}_{S/\Sigma}\right)_{(S,S',\gamma)} \to \mathcal{E}xt^1_{S/\Sigma}\left(A, \mathcal{J}_{S/\Sigma}\right)_{(S,S',\gamma)} \to M_S(G) \to 0.$$

From Remark 4.1.4, we have

$$M_S(G) \cong \operatorname{Coker}\left(V\mathbb{D}(B)_{(S,S',\gamma)} \to V\mathbb{D}(A)_{(S,S',\gamma)}\right). \tag{4.6}$$

Note that G is killed by p. Therefore $M_S(G)$ is also annihilated by p. Therefore we may modulo p before taking the cokernel in Equation (4.6). Note that the base $S \subset S'$ is defined by the ideal (p). Therefore we have $\mathbb{D}(A)_{(S,S',\gamma)} \otimes \mathbb{F}_p = \mathbb{D}(A)_S$ and the statement follows. □

Assume that S, S' are spectra of local rings. From Proposition 4.1.3, by evaluating the first row at (S, S', γ), we get that

$$0 \to \mathcal{E}xt^1_{S/\Sigma}\left(A, \mathcal{J}_{S/\Sigma}\right)_{(S,S',\gamma)} \to \mathbb{D}(A)_{(S,S',\gamma)} \to i_*(R^1 f_*(\mathcal{O}_A)) \to 0.$$

where $i : S \to S'$ is the embedding. Let $s_1, \dots, s_{2g} \in \mathbb{D}(A)_{(S,S',\gamma)}$ be a basis so that the images $\bar{s}_1, \dots, \bar{s}_g \in \mathbb{D}(A)_S$ generate ω_A and $\bar{s}_{g+1}, \dots, \bar{s}_{2g} \in \mathbb{D}(A)_S$ generates $R^1 f_*(\mathcal{O}_A)$.

Then $\mathcal{E}xt^1_{S/\Sigma}\left(A, \mathcal{J}_{S/\Sigma}\right)_{(S,S',\gamma)} = V\mathbb{D}(A)_{(S,S',\gamma)}$ is generated by $s_1, \dots, s_g, ps_{g+1}, \dots, ps_{2g}$. The Dieudonné module $M_S(A[p])$, by Equation (4.5), is generated by

$$s_1, \dots, s_g, ps_{g+1}, \dots, ps_{2g} \mod \langle ps_1, \dots, ps_g, p^2 s_{g+1}, \dots, p^2 s_{2g} \rangle.$$

4.2 Group Schemes Annihilated by V

Let G/S be a group scheme as in Section 4.1. Suppose the rank of G/S is r. From Section 4.1, the Dieudonné module $M := M_S(G)$ is a locally free $\mathcal{O}_S$-module of rank r. Let $f : \mathcal{O}_S \to \mathcal{O}_S$ be the Frobenius map. Then $f^*M = M \otimes_{\mathcal{O}_S, f} \mathcal{O}_S$ is also a locally free $\mathcal{O}_S$-module of rank r. Let F_G, V_G be the Frobenius and Verschiebung on G respectively. The Frobenius F_G and Verschiebung V_G induce two linear maps $F_M : f^*M \to M$ and $V_M : M \to f^*M$, so that $F_M \circ V_M = 0$ and $V_M \circ F_M = 0$.

The goal of the rest of this chapter is to determine the group scheme G when we are given its Dieudonné module $M_S(G)$. Upon a choice of bases of M and f^*M, this is equivalent to the two matrices F, V with $FV = VF = 0$. The first theorem of this type is the following result of Grothendieck in [12, Exposé VII, Theorem 7.4]:

Proposition 4.2.1. *Let S be a base scheme over $\mathbb{F}_p$. We have the following anti-equivalence of categories:*

$$\left\{ \begin{array}{c} G/S, \textit{ finite (flat commutative)} \\ \textit{group schemes killed by } p \textit{ with} \\ V_G = 0 \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \textit{pairs } (M, F), \textit{ where } M \textit{ is a locally free} \\ \mathcal{O}_S\textit{-module, } F_M : f^*M \to M \textit{ is a} \\ \textit{homomorphism of } \mathcal{O}_S\textit{-modules} \end{array} \right\}$$

In particular, when $S = \operatorname{Spec} R$ where R is a local ring, the Dieudonné module M is a free R-module. Let $x_1, \dots, x_r$ be a basis of M. Then $x_1 \otimes 1, \dots, x_r \otimes 1$ form a basis of f^*M. We write the linear map $F_M : f^*M \to M$ as

$$F_M\left(x_1 \otimes 1, \dots, x_r \otimes 1\right) = (x_1, \dots, x_r) \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rr} \end{pmatrix}.$$

Then the corresponding group scheme G/S is given by

$$G = \operatorname{Spec} R[x_1, \dots, x_r] / \left(\left\{ x_i^p - \textstyle\sum_{j=1}^{r} a_{ji} x_j \right\}_{1 \le i \le r} \right),$$

with additive coalgebra operations

$$\begin{aligned} m^*(x_i) &= 1 \otimes x_i + x_i \otimes 1, \\ \operatorname{inv}^*(x_i) &= -x_i, \\ \epsilon^*(x_i) &= 0. \end{aligned}$$

4.3 Group Schemes Annihilated by V^2

Let G/S be a group scheme as in Section 4.1. Suppose that $V_G^2 = 0$ and all images and coimages of Verschiebung are flat over S. Let $G_1 := G/\operatorname{Im} V_G$ and $G_2 := \operatorname{Im} V_G$. Then we have an exact sequence of group schemes:

$$0 \to G_2 \to G \to G_1 \to 0.$$

By our assumption, G_1 and G_2 are both annihilated by the Verschiebung, i.e. $V_{G_1} = 0$ and $V_{G_2} = 0$. According to Proposition 4.2.1, we may write

$$G_1 = \operatorname{Spec} R[x_1, \dots, x_n] / \left(\left\{ x_i^p - \textstyle\sum_{j=1}^{n} a_{ji} x_j \right\}_{1 \le i \le n} \right)$$

and

$$G_2 = \operatorname{Spec} R[y_1, \dots, y_m] / \left(\left\{ y_i^p - \textstyle\sum_{j=1}^{m} b_{ji} y_j \right\}_{1 \le i \le m} \right)$$

The Dieudonné module of G_1 (resp. G_2) is given by $M_1 = \bigoplus_{i=1}^{n} R x_i$ (resp. $M_2 = \bigoplus_{i=1}^{m} R y_i$) with Frobenius matrix $A = (a_{ij})_{n \times n}$ (resp. $B = (b_{ij})_{m \times m}$) and Verschibung matrix $0_{n \times n}$ (resp. $0_{m \times m}$). Let M be a Dieudonné module that is an extension of M_2 by M_1:

$$0 \to M_1 \to M \to M_2 \to 0.$$

Therefore M is a free R-module of rank $m+n$ with basis $x_1, \dots, x_n, y_1, \dots, y_n$ and the Frobenius and Verschiebung of M has the description

$$\begin{aligned} F\left(x_1 \otimes 1, \dots, x_n \otimes 1, y_1 \otimes 1, \dots, y_m \otimes 1\right) &= \left(x_1, \dots, x_r, y_1, \dots, y_m\right) \begin{pmatrix} A_{n\times n} & C_{n\times m} \\ 0_{m\times n} & B_{m\times m} \end{pmatrix}, \\ V\left(x_1, \dots, x_r, y_1, \dots, y_m\right) &= \left(x_1 \otimes 1, \dots, x_n \otimes 1, y_1 \otimes 1, \dots, y_m \otimes 1\right) \begin{pmatrix} 0_{n\times n} & D_{n\times m} \\ 0_{m\times n} & 0_{m\times m} \end{pmatrix}. \end{aligned} \tag{4.7}$$

so that $F \circ V = V \circ F = 0$. This is equivalent to $AD = DB = 0$.

By Proposition 4.1.1, there is a bijection between the extensions G of G_1 by G_2 and the extensions M of M_1 by M_2. Therefore, to get all group schemes G that are extensions of G_1 by G_2, we only need to construct a group scheme of G_1 by G_2 that has F and V as in Equation (4.7). Here is the theorem:

Theorem 4.3.1. *Let G/S be a group scheme as in Section 4.1. Suppose that $S = \operatorname{Spec} R$ where R is a local ring. Assume that $\operatorname{Im} V$ and $G/\operatorname{Im} V_G$ are flat over S. Then G can be written as*

$$\operatorname{Spec} R[x_1, \dots, x_n, y_1, \dots, y_m] / \left(\left\{x_i^p - \textstyle\sum_{j=1}^n a_{ji} x_j\right\}, \left\{y_i^p - \textstyle\sum_{j=1}^m b_{ji} y_j - \textstyle\sum_{j=1}^n c_{ji} x_j\right\}\right)$$

with coalgebra structure given by

$$\begin{aligned} &m_G^*(x_i) = 1 \otimes x_i + x_i \otimes 1, && m_G^*(y_i) = 1 \otimes y_i + y_i \otimes 1 + \sum_{k=1}^{p-1} \sum_{j=1}^{n} \frac{d_{ji} x_j^k \otimes x_j^{p-k}}{k!\,(p-k)!}, \\ &\epsilon_G^*(x_i) = 0, && \epsilon_G^*(y_i) = 0, \\ &\operatorname{inv}_G^*(x_i) = -x_i, && \operatorname{inv}_G^*(y_i) = -y_i. \end{aligned}$$

Proof. Note that the coordinate ring of G is a free module of rank p^{m+n} generated by $x_1^{e_1} \cdots x_n^{e_n} y_1^{f_1} \cdots y_m^{f_m}$ for $0 \le e_i, f_i < p$. Therefore $G \to S$ is flat. The only thing to check is that these operations give a group scheme. This is similar to the proof of Theorem 4.4.1 in the next section but simpler. We will skip the direct calculations here. □

4.4 Group Schemes Annihilated by V^3

Now we consider the next case. Let G/S be a finite group scheme that is killed by p and such that $V_G^3 = 0$. We will also assume that all kernels and cokernels of V_G are flat group schemes over the base S. In this case, we have exact sequences

$$0 \to \operatorname{Im} V_G / \operatorname{Im} V_G^2 \to G / \operatorname{Im} V_G^2 \to G / \operatorname{Im} V_G \to 0$$

and

$$0 \to \operatorname{Im} V_G^2 \to G \to G / \operatorname{Im} V_G^2 \to 0.$$

Both left terms in these two exact sequences are annihilated by V_G to satisfy the condition in Proposition 4.1.1. The idea to obtain the explicit expression of G is to use the first exact sequence to get a description of $G/\operatorname{Im} V_G^2$ using the result in Section 5.1. Then we will use the second exact sequence to construct the group scheme G.

Since the notation and calculation get very complicated immediately, we will only state the result in a specific case that we need in Section 6.2. However, note that the theorem can be stated with more generality.

Theorem 4.4.1. *Let G/S be a group scheme as in Section 4.1. Assume $S = \operatorname{Spec} R$ with R a local ring. Assume the rank of G is p^4 and suppose that G admits a filtration*

$$0 = G_0 \subset G_1 \subset G_3 \subset G_4 = G$$

such that

$$\operatorname{Rank} G_i = p^i, \tag{4.8}$$

$$V(G_i) = G_{v(i)}, \tag{4.9}$$

$$F^{-1}(G_i) = G_{f(i)}, \tag{4.10}$$

$$(G_i/G_j)^D \cong G_{4-j}/G_{4-i}, \tag{4.11}$$

where $v = (0, 0, 1, 2)$ and $f = (2, 3, 4, 4)$. Then there exists a 10-tuple

$$(a_1, a_2, b_1, b_2, c, d, e_1, e_2, f_1, f_2)$$

over R such that

$$G = G_4 \cong \operatorname{Spec} R[x, y_1, y_2, z]/\left(x^p, y_1^p - a_1 x, y_2^p - a_2 x, z^p - cx - e_1 y_1 - e_2 y_2\right)$$

where coalgebra operations given by

$$\begin{aligned}
&\epsilon^*(x) = 0; \quad \mathrm{inv}^*(x) = -x; \qquad m^*(x) = 1 \otimes x + x \otimes 1, \\
&\epsilon^*(y_i) = 0; \quad \mathrm{inv}^*(y_i) = -y_i; \quad m^*(y_i) = 1 \otimes y_i + y_i \otimes 1 + \sum_{k=1}^{p-1} \frac{b_i x^k \otimes x^{p-k}}{k!\,(p-k)!}, \qquad (4.12) \\
&\epsilon^*(z) = 0; \quad \mathrm{inv}^*(z) = -z; \qquad m^*(z) = 1 \otimes z + z \otimes 1 + \sum_{k=1}^{p-1} \frac{d x^k \otimes x^{p-k}}{k!\,(p-k)!} \\
&\qquad + \sum_{i=1}^{2} \sum_{k=1}^{p-1} \frac{f_i y_i^k \otimes y_i^{p-k}}{k!\,(p-k)!} - \sum_{i=1}^{2} \sum_{k=1}^{p-1} \frac{f_i b_i \left(1 \otimes y_i + y_i \otimes 1\right)^{p-1} \left(x^k \otimes x^{p-k}\right)}{k!\,(p-k)!}.
\end{aligned}$$

Proof. We start by analyzing G_3/G_1. Note that we have $F^{-1}(G_1) = G_3$ and $V(G_3) = G_1$. Therefore we have $V_{G_3/G_1} = 0$ and $F_{G_3/G_1} = 0$. By Proposition 4.2.1, we have $G_3/G_1 \cong \alpha_p \times \alpha_p = \operatorname{Spec} R[y_1, y_2]/(y_1^p, y_2^p)$.

Now consider the exact sequence

$$0 \to G_3/G_1 \to G_4/G_1 \to G_4/G_3 \to 0.$$

Similarly, we have $F_{G_4/G_3} = 0$ and $V_{G_4/G_3} = 0$. Thereofore we can write $G_4/G_3 \cong \alpha_p = \operatorname{Spec} R[x]/(x^p)$. By Theorem 4.3.1, we have

$$G_4/G_1 = \operatorname{Spec} R[x, y_1, y_2]/\left(x^p, y_1^p - a_1 x, y_2^p - a_2 x\right)$$

with

$$\begin{aligned}
m^*_{G_4/G_1}(x) &= 1 \otimes x + x \otimes 1 \\
m^*_{G_4/G_1}(y_i) &= 1 \otimes y_i + y_i \otimes 1 + \sum_{k=1}^{p-1} \frac{b_i x^k \otimes x^{p-k}}{k!\,(p-k)!}.
\end{aligned}$$

The Frobenius and Verschebung acts on $M_S(G_4/G_1)$ by

$$\begin{aligned}
F(x \otimes 1, y_1 \otimes 1, y_2 \otimes 1) &= (x, y_1, y_2) \begin{pmatrix} 0 & a_1 & a_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
V(x, y_1, y_2) &= (x \otimes 1, y_1 \otimes 1, y_2 \otimes 1) \begin{pmatrix} 0 & b_1 & b_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\end{aligned} \qquad (4.13)$$

With the Hopf algebra of G_4/G_1, now we consider the exact sequence

$$0 \to G_1 \to G_4 \to G_4/G_1 \to 0.$$

Similarly, we have $G_1 \cong \alpha_p = \operatorname{Spec} R[z]/(z^p)$. Consider the extension of Dieudonné modules

$$0 \to M_S(G_4/G_1) \to M_S(G_4) \to M_S(G_1) \to 0.$$

As an R module, $M_S(G) \cong R^4$. Using the matrices of the Dieudonné modules in Equations (4.13), we can calculate that the Frobenius of $M_S(G_4)$ acts by

$$F(x \otimes 1, y_1 \otimes 1, y_2 \otimes 1, z \otimes 1) = (x, y_1, y_2, z) \begin{pmatrix} 0 & a_1 & a_2 & c \\ 0 & 0 & 0 & e_1 \\ 0 & 0 & 0 & e_2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.14}$$

and the Verschiebung acts by

$$V(x, y_1, y_2, z) = (x \otimes 1, y_1 \otimes 1, y_2 \otimes 1, z \otimes 1) \begin{pmatrix} 0 & b_1 & b_2 & d \\ 0 & 0 & 0 & f_1 \\ 0 & 0 & 0 & f_2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.15}$$

so that $FV = VF = 0$. Equivalently, we have the following equations:

$$a_1 f_1 + a_2 f_2 = 0 \tag{4.16}$$

$$b_1 e_1 + b_2 e_2 = 0 \tag{4.17}$$

By Proposition 4.1.1, it suffices to construct a group scheme with Frobenius and Verschiebung in forms of (4.14) and (4.15). We construct this group scheme explicitly as stated in Theorem 4.4.1. The rest of this proof is devoted to checking that the scheme G with operations above defines a group scheme. We will check the following conditions:

(I) The comultiplication, counit, coinverse defined in Equation (4.12) give well-defined algebra homomorphisms.

(II) The Hopf algebra axioms are satisfied, i.e.

$$\begin{aligned} (\operatorname{Id} \otimes m^*) \circ m^* &= (m^* \otimes \operatorname{Id}) \circ m^*, \\ (\operatorname{Id} \otimes \epsilon^*) \circ m^* &= \operatorname{Id}, \\ (\operatorname{Id}, \operatorname{inv}^*) \circ m^* &= \epsilon^*. \end{aligned} \tag{4.18}$$

We first check condition (I). We will check that the comultiplication gives a well-defined algebra homomorphism $m^*\colon \mathcal{O}_G \to \mathcal{O}_G \otimes \mathcal{O}_G$. The well-definedness of the counit and coinverse are easy to check.

The well-definedness of the comultiplication amounts to checking that

$$(m^*(z))^p - cm^*(x) - e_1 m^*(y_1) - e_2 m^*(y_2) = 0. \tag{4.19}$$

Note that

$$\begin{aligned} (m^*(z))^p &= 1 \otimes z^p + z^p \otimes 1 + \sum_{i=1}^{2} \sum_{k=1}^{p-1} \frac{f_i^p (a_i x)^k \otimes (a_i x)^{p-k}}{k!\,(p-k)!} \\ &= 1 \otimes z^p + z^p \otimes 1 + \sum_{k=1}^{p-1} \left(\sum_{i=1}^{2} f_i^p a_i^p \right) \frac{x^k \otimes x^{p-k}}{k!\,(p-k)!}. \end{aligned} \tag{4.20}$$

Note that we have $a_1 f_1 + a_2 f_2 = 0$ from Equation (4.16). So we have $a_1^p f_1^p + a_2^p f_2^p = 0$ and therefore

$$\begin{aligned} (m^*(z))^p &= 1 \otimes z^p + z^p \otimes 1 \\ &= c\,(1 \otimes x + x \otimes 1) + e_1\,(1 \otimes y_1 + y_1 \otimes 1) + e_2\,(1 \otimes y_2 + y_2 \otimes 1)\,. \end{aligned} \tag{4.21}$$

On the other hand, we have

$$\begin{aligned} cm^*(x) + e_1 m^*(y_1) + e_2 m^*(y_2) &= c\,(1 \otimes x + x \otimes 1) \\ &+ e_1\,(1 \otimes y_1 + y_1 \otimes 1) + e_2\,(1 \otimes y_2 + y_2 \otimes 1) + \sum_{k=1}^{p-1} \left(\sum_{i=1}^{2} e_i^p b_i^p \right) \frac{x^k \otimes x^{p-k}}{k!\,(p-k)!}. \end{aligned} \tag{4.22}$$

Similarly, note that $e_1 b_1 + e_2 b_2 = 0$ from Equation (4.16) and therefore $e_1^p b_1^p + e_2^p b_2^p = 0$. Hence we have

$$\begin{aligned} & cm^*(x) + e_1 m^*(y_1 + e_2 m^*(y_2) \\ =& c\,(1 \otimes x + x \otimes 1) + e_1\,(1 \otimes y_1 + y_1 \otimes 1) + e_2\,(1 \otimes y_2 + y_2 \otimes 1)\,. \end{aligned} \tag{4.23}$$

By combining Equation (4.21) and (4.23), Equation (4.19) holds. It follows that the algebra homomorphism $m^* : \mathcal{O}_G \to \mathcal{O}_G \otimes \mathcal{O}_G$ is well-defined.

Now we check condition (II). We will check the equation

$$(\mathrm{Id} \otimes m^*) \circ m^*(z) = (m^* \otimes \mathrm{Id}) \circ m^*(z). \tag{4.24}$$

The calculation for the other two equations of (4.18) is straightforward and therefore omitted.

We first fix some notations. Let R' be an R-algebra and let $\underline{x}, \underline{y}, \underline{z}$ be points of $G(R')$. A point $\underline{x} \in G(R')$ is given by $\underline{x} = (x_0, x_1^{(1)}, x_1^{(2)}, x_2)$, a quadruple of elements in R' satisfying $x_0^p = 0, (x_1^{(i)})^p = a_i x_0$ and $x_2^p = cx_0 + e_1 x_1^{(1)} + e_2 x_1^{(2)}$. Similar conditions hold for $\underline{y} = (y_0, y_1^{(1)}, y_1^{(2)}, y_2)$ and $\underline{z} = (z_0, z_1^{(1)}, z_1^{(2)}, z_2)$. For any $a, b, c \in R'$, we define $P(a, b) \in R'$ as

$$P(a,b) := \sum_{k=1}^{p-1} \frac{a^k b^{p-k}}{k!\,(p-k)!} \tag{4.25}$$

and define

$$P(a,b,c) := \sum_{\substack{0 \le k,l,m \le p-1 \\ k+l+m=p}} \frac{a^k b^l c^m}{k!\,l!\,m!}. \tag{4.26}$$

One basic property of this expression is

$$P(a,b,c) = P(a,b) + P(a+b,c) = P(a,b+c) + P(b,c). \tag{4.27}$$

Now we define $\underline{s}(\underline{x}, \underline{y})$ to be the sum of $\underline{x}$ and $\underline{y}$ under the group operation defined by the coalgebra operators in Equation (4.12). More explicitly, using the notation $P(a, b)$, we define $\underline{s}(\underline{x}, \underline{y}) := (s_0, s_1^{(1)}, s_1^{(2)}, s_2)$, where

$$\begin{aligned}
s_0(\underline{x}, \underline{y}) &:= x_0 + y_0 \\
s_1^{(i)}(\underline{x}, \underline{y}) &:= x_1^{(i)} + y_1^{(i)} + b_i P(x_0, y_0) \\
s_2(\underline{x}, \underline{y}) &:= x_2 + y_2 + dP(x_0, y_0) + \sum_{i=1}^{2} f_i P(x_1^{(i)}, y_1^{(i)}) - \sum_{i=1}^{2} f_i b_i (x_1^{(i)} + y_1^{(i)})^{p-1} P(x_0, y_0)
\end{aligned}$$

By Yoneda's Lemma, Equation (4.24) is equivalent to the associativity of s, i.e.

$$\underline{s}(\underline{x}, \underline{s}(\underline{y}, \underline{z})) = \underline{s}(\underline{s}(\underline{x}, \underline{y}), \underline{z})$$

for any $\underline{x}, \underline{y}, \underline{z} \in G(R')$. Note that $s_0(\underline{x}, \underline{s}(\underline{y}, \underline{z})) = s_0(\underline{s}(\underline{x}, \underline{y}), \underline{z})$ is trivial and $s_1^{(i)}(\underline{x}, \underline{s}(\underline{y}, \underline{z})) = s_1^{(i)}(\underline{s}(\underline{x}, \underline{y}), \underline{z})$ follows from Equation (4.27). The difficult part is checking $s_2(\underline{x}, \underline{s}(\underline{y}, \underline{z})) = s_2(\underline{s}(\underline{x}, \underline{y}), \underline{z})$.

We start with the left side. Denote $\underline{w} = (w_0, w_1^{(1)}, w_1^{(2)}, w_2) = s(\underline{x}, \underline{y})$. Therefore

$$\begin{aligned}
&s_2(\underline{s}(\underline{x}, \underline{y}), \underline{z}) \\
=&w_2 + z_2 + dP(w_0, z_0) + \sum_{i=1}^{2} f_i P(w_1^{(i)}, z_1^{(i)}) - \sum_{i=1}^{2} f_i b_i (w_1^{(i)} + z_1^{(i)})^{p-1} P(w_0, z_0) \\
=&x_2 + y_2 + dP(x_0, y_0) + \sum_{i=1}^{2} f_i P(x_1^{(i)}, y_1^{(i)}) - \sum_{i=1}^{2} f_i b_i (x_1^{(i)} + y_1^{(i)})^{p-1} P(x_0, y_0) \qquad (4.28)\\
&+ z_2 + dP(x_0 + y_0, z_0) + \sum_{i=1}^{2} f_i P(x_1^{(i)} + y_1^{(i)} + b_i P(x_0, y_0), z_1^{(i)}) \\
&- \sum_{i=1}^{2} f_i b_i (x_1^{(i)} + y_1^{(i)} + b_i P(x_0, y_0) + z_1^{(i)})^{p-1} P(x_0 + y_0, z_0)
\end{aligned}$$

Note that $x_0^p = y_0^p = 0$. By definition of P in (4.25), we have $(P(x_0, y_0))^2 = 0$. Therefore

$$\begin{aligned}
P(x_1^{(i)} + y_1^{(i)} + b_i P(x_0, y_0), z_1^{(i)}) =& \sum_{k=1}^{p-1} \frac{(x_1^{(i)} + y_1^{(i)} + b_i P(x_0, y_0))^k (z_1^{(i)})^{p-k}}{k!\,(p-k)!} \\
=& \sum_{k=1}^{p-1} \frac{\left((x_1^{(i)} + y_1^{(i)})^k + b_i k (x_1^{(i)} + y_1^{(i)})^{k-1} P(x_0, y_0)\right)(z_1^{(i)})^{p-k}}{k!\,(p-k)!} \\
=& P(x_1^{(i)} + y_1^{(i)}, z_1^{(i)}) + \sum_{k=1}^{p-1} \frac{b_i (x_1^{(i)} + y_1^{(i)})^{k-1} (z_1^{(i)})^{p-k} P(x_0, y_0)}{(k-1)!\,(p-k)!}
\end{aligned}$$

Plugging this into (4.28), we get

$$\begin{aligned}
&s_2(\underline{s}(\underline{x}, \underline{y}), \underline{z}) = \\
&x_2 + y_2 + z_2 + dP(x_0, y_0, z_0) + \sum_{i=1}^{2} f_i P(x_1^{(i)}, y_1^{(i)}) - \sum_{i=1}^{2} f_i b_i (x_1^{(i)} + y_1^{(i)})^{p-1} P(x_0, y_0) \\
&+ \sum_{i=1}^{2} f_i \left(P(x_1^{(i)} + y_1^{(i)}, z_1^{(i)}) + \sum_{k=1}^{p-1} \frac{b_i (x_1^{(i)} + y_1^{(i)})^{k-1} (z_1^{(i)})^{p-k} P(x_0, y_0)}{(k-1)!\,(p-k)!} \right) \\
&- \sum_{i=1}^{2} f_i b_i \left((x_1^{(i)} + y_1^{(i)} + z_1^{(i)})^{p-1} - b_i P(x_0, y_0)(x_1^{(i)} + y_1^{(i)} + z_1^{(i)})^{p-2} \right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \cdot (P(x_0, y_0, z_0) - P(x_0, y_0))
\end{aligned}$$

After rearranging and simplifying, we get

$$
\begin{aligned}
&s_2(\underline{s}(\underline{x},\underline{y}),\underline{z}) = \\
&x_2 + y_2 + z_2 + dP(x_0,y_0,z_0) + \sum_{i=1}^{2} f_i P(x_1^{(i)}, y_1^{(i)}, z_1^{(i)}) \qquad (4.29)\\
&-\sum_{i=1}^{2} f_i b_i (x_1^{(i)} + y_1^{(i)})^{p-1} P(x_0,y_0) + \sum_{i=1}^{2}\sum_{k=1}^{p-1} \frac{f_i b_i (x_1^{(i)} + y_1^{(i)})^{k-1} (z_1^{(i)})^{p-k} P(x_0,y_0)}{(k-1)!\,(p-k)!}\\
&-\sum_{i=1}^{2} f_i b_i \left(x_1^{(i)} + y_1^{(i)} + z_1^{(i)}\right)^{p-1} P(x_0,y_0,z_0) + \sum_{i=1}^{2} f_i b_i \left(x_1^{(i)} + y_1^{(i)} + z_1^{(i)}\right)^{p-1} P(x_0,y_0)\\
&+\sum_{i=1}^{2} f_i b_i^2 (x_1^{(i)} + y_1^{(i)} + z_1^{(i)})^{p-2} P(x_0,y_0,z_0) P(x_0,y_0)
\end{aligned}
$$

Now we calculate the term $P(x_0,y_0,z_0)P(x_0,y_0)$. Note that $x_0^p = y_0^p = z_0^p = 0$. Therefore we have

$$
\begin{aligned}
P(x_0,y_0,z_0)P(x_0,y_0) &= \sum_{\substack{0\le k_1,l_1,m_1\le p-1\\ k_1+l_1+m_1=p}} \frac{x_0^{k_1} y_0^{l_1} z_0^{m_1}}{k_1!\,l_1!\,m_1!} \cdot \sum_{\substack{1\le k_2,l_2\le p-1\\ k_2+l_2=p}} \frac{x_0^{k_2} y_0^{l_2}}{k_2!\,l_2!}\\
&= \sum_{\substack{2\le k,l,m\le p-1\\ k+l+m=2p}} \left(\sum_{l_1=0}^{p-m} \frac{1}{(p-l_1-m)!\,l_1!\,m!\,(p-l+l_1)!\,(l-l_1)!} \right) x_0^k y_0^l z_0^m .
\end{aligned}
$$

Here $m_1 = m, l_2 = l - l_1, k_2 = p - l_2 = p - l + l_1, k_1 = p - l_1 - m$. Note that all k,l,m are at least 2. Otherwise at least one of the other 2 variables will be greater than or equal to p. The bounds for l_1 is from 0 to $p-m$ since $l-p-m > 0$ from $k \le p-1$ and $k+l+m = 2p$. So $l_2 = l - l_1 > 0$, as we want. Now we want to calculate the constant in the parenthesis, which is given by the following lemma:

Lemma 4.4.2. *For any $2 \le k,l,m \le p-1$ such that $k+l+m = 2p$, we have*

$$
\sum_{l_1=0}^{p-m} \frac{1}{(p-l_1-m)!\,l_1!\,m!\,(p-l+l_1)!\,(l-l_1)!} \equiv \frac{1}{k!\,l!\,m!} \mod p.
$$

Proof. Note that

$$
\sum_{l_1=0}^{p-m} \frac{1}{(p-l_1-m)!\,l_1!\,m!\,(p-l+l_1)!\,(l-l_1)!} = \frac{\sum_{l_1=0}^{p-m} \binom{l}{l_1}\binom{2p-l-m}{p-m-l_1}}{m!\,l!\,(2p-l-m)!} = \frac{\binom{2p-m}{p-m}}{k!\,l!\,m!}.
$$

Now observe that modulo p, we have

$$\binom{2p-m}{p-m} = \frac{(p+p-m)\cdot(p+p-m-1)\cdots(p+1)}{1\cdot 2\cdots(p-m)} \equiv \frac{(p-m)(p-m-1)\cdots 1}{1\cdot 2\cdots(p-m)} = 1.$$

This finishes the proof of the lemma. □

From Lemma 4.4.2, we get

$$P(x_0,y_0,z_0)P(x_0,y_0) = \sum_{\substack{2\le k,l,m\le p-1\\ k+l+m=2p}} \frac{1}{k!\,l!\,m!} x_0^k y_0^l z_0^m. \tag{4.30}$$

Plug it into Equation (4.29), we get

$$\begin{aligned}
&s_2(\underline{s}(\underline{x},\underline{y}),\underline{z}) = \\
&x_2+y_2+z_2+dP(x_0,y_0,z_0)+\sum_{i=1}^{2} f_i P(x_1^{(i)},y_1^{(i)},z_1^{(i)}) \\
&-\sum_{i=1}^{2} f_i b_i \left(x_1^{(i)}+y_1^{(i)}+z_1^{(i)}\right)^{p-1} P(x_0,y_0,z_0) \\
&+\sum_{i=1}^{2} \sum_{\substack{2\le k,l,m\le p-1\\ k+l+m=2p}} \frac{f_i b_i^2}{k!\,l!\,m!}(x_1^{(i)}+y_1^{(i)}+z_1^{(i)})^{p-2} x_0^k y_0^l z_0^m \\
&-\sum_{i=1}^{2} f_i b_i (x_1^{(i)}+y_1^{(i)})^{p-1} P(x_0,y_0) + \sum_{i=1}^{2}\sum_{k=1}^{p-1} \frac{f_i b_i (x_1^{(i)}+y_1^{(i)})^{k-1}(z_1^{(i)})^{p-k} P(x_0,y_0)}{(k-1)!\,(p-k)!} \\
&+\sum_{i=1}^{2} f_i b_i \left(x_1^{(i)}+y_1^{(i)}+z_1^{(i)}\right)^{p-1} P(x_0,y_0)
\end{aligned} \tag{4.31}$$

Now we observe that the last three terms cancel. To see this, we just need to expand the last term using the binomial expansion:

$$\begin{aligned}
&\sum_{i=1}^{2} f_i b_i \left(x_1^{(i)}+y_1^{(i)}+z_1^{(i)}\right)^{p-1} P(x_0,y_0) \\
&=\sum_{i=1}^{2} f_i b_i \left(x_1^{(i)}+y_1^{(i)}\right)^{p-1} P(x_0,y_0) + \sum_{i=1}^{2}\sum_{k=0}^{p-2} \frac{f_i b_i (x_1^{(i)}+y_1^{(i)})^{k}(z_1^{(i)})^{p-k} P(x_0,y_0)}{(p-1)!\,(k-1)!\,(p-k)!} \\
&=\sum_{i=1}^{2} f_i b_i (x_1^{(i)}+y_1^{(i)})^{p-1} P(x_0,y_0) - \sum_{i=1}^{2}\sum_{k=1}^{p-1} \frac{f_i b_i (x_1^{(i)}+y_1^{(i)})^{k-1}(z_1^{(i)})^{p-k} P(x_0,y_0)}{(k-1)!\,(p-k)!}
\end{aligned} \tag{4.32}$$

Here, the last step uses $(p-1)! \equiv -1 \mod p$. After canceling the last three terms, we get

$$\begin{aligned} s_2(\underline{s}(\underline{x}, \underline{y}), \underline{z}) =& x_2 + y_2 + z_2 + dP(x_0, y_0, z_0) + \sum_{i=1}^{2} f_i P(x_1^{(i)}, y_1^{(i)}, z_1^{(i)}) \\ & - \sum_{i=1}^{2} f_i b_i \left(x_1^{(i)} + y_1^{(i)} + z_1^{(i)}\right)^{p-1} P(x_0, y_0, z_0) \\ & + \sum_{i=1}^{2} \sum_{\substack{2 \le k,l,m \le p-1 \\ k+l+m=2p}} \frac{f_i b_i^2}{k!\, l!\, m!} (x_1^{(i)} + y_1^{(i)} + z_1^{(i)})^{p-2} x_0^k y_0^l z_0^m \end{aligned} \tag{4.33}$$

By the symmetry of this expression, we can conclude that $s_2(\underline{s}(\underline{x}, \underline{y}), \underline{z}) = s_2(\underline{x}, \underline{s}(\underline{y}, \underline{z}))$. This completes the proof that the G we constructed in Theorem 4.4.1 is a group scheme. $\square$

CHAPTER 5

$\Gamma_1(p)$ -COVER OVER THE SIEGEL THREEFOLD

Let $\mathcal{A} := \mathcal{A}_{2,1,N}$, $N \geq 3$ be the Siegel threefold, which is the fine moduli scheme of principally polarized abelian surfaces with symplectic level-N structure as in Section 2.3. For the existence of $\mathcal{A}$, see [28]. In this chapter, we consider the special fiber $\bar{\mathcal{A}} := \mathcal{A} \times \operatorname{Spec} \mathbb{F}_p$. As in Example 2.2.1, there are 4 Ekedahl–Oort strata of $\bar{\mathcal{A}}$, corresponding to the superspecial locus, supersingular (but not superspecial) locus, p-rank-1 locus and ordinary locus. The loci have dimensions 0,1,2,3 respectively. On each stratum, there is a canonical group scheme filtration of the p-torsion of the universal abelian surface as in Example 2.2.1.

Let $\mathcal{X}$ be the universal abelian surface over $\mathcal{A}$ and $\bar{\mathcal{X}} := \mathcal{X} \times_{\mathcal{A}} \bar{\mathcal{A}}$. Let $\bar{\mathcal{X}}[p]$ be the p-torsion group scheme of $\bar{\mathcal{X}}$ and let $\bar{\mathcal{X}}^{\times}[p] := (\bar{\mathcal{X}}[p])^{\times}$ be its subscheme of primitive elements. We will call the morphism $\bar{\mathcal{X}}^{\times}[p] \to \bar{\mathcal{A}}$ "the $\Gamma_1(p)$-cover" of $\bar{\mathcal{A}}$. (Note that the name "$\Gamma_1(p)$-cover" has different meaning in various papers. For example, Haines and Rapoport use "$\Gamma_1(p)$-cover" for the pro-p Iwahori structure in [16].)

Let S_φ be an Ekedahl–Oort stratum of $\bar{\mathcal{A}}$. Let $\bar{\mathcal{X}}_\varphi$ be the restriction of $\bar{\mathcal{X}}$ over S_φ and let $\bar{\mathcal{X}}_\varphi^{\times}[p] \to S_\varphi$ be the restriction of the $\Gamma_1(p)$-cover. We want to study the geometry of the $\Gamma_1(p)$-cover $\bar{\mathcal{X}}/S_\varphi$ on each Ekedahl–Oort stratum by calculating the (local) eqautions. The main tool in this chapter is the machinery of constructing group schemes from their Dieudonné modules built in Chapter 4 and Lemma 2.3.3 (particularly Example 2.3.6) for calculating primitive elements.

5.1 $\Gamma_1(p)$ -cover over the Superspecial Locus

The superspecial locus consists of discrete points. Over each point $\operatorname{Spec} \bar{\mathbb{F}}_p$, the universal abelian surface $\bar{\mathcal{X}}_\varphi$ is a product of supersingular elliptic curves. Let E be a supersingular elliptic curve over $\operatorname{Spec} \bar{\mathbb{F}}_p$. Then $\bar{\mathcal{X}}_\varphi \cong E \times E$ by a result due to Deligne ([40, Theorem 3.5]). Therefore $\bar{\mathcal{X}}_\varphi[p] \cong E[p] \times E[p]$.

Note that $E[p]$ is a self-dual group scheme of rank p^2, killed by p, of local-local type and has nonzero Frobenius and Verschiebung. By classical Dieudonné theory over perfect fields, there is a unique group scheme with these properties, given by

$$\operatorname{Spec} \bar{\mathbb{F}}_p[x]/(x^{p^2})$$

with coalgebra operations

$$\begin{aligned} m^*(x) &= 1 \otimes x + x \otimes 1 + \sum_{k=1}^{p-1} \frac{x^{pk} \otimes x^{p(p-k)}}{k!\,(p-k)!}, \\ \epsilon^*(x) &= 0, \\ \operatorname{inv}^*(x) &= -x. \end{aligned}$$

Therefore, we have $\bar{\mathcal{X}}_\varphi[p] \cong E[p] \times E[p] \cong \bar{\mathbb{F}}_p[x, y]/(x^{p^2}, y^{p^2})$ with coalgebra operations as above. The augmentation ideal is given by (x, y). By Lemma 2.3.3, we can see that $\bar{\mathcal{X}}_\varphi^\times[p] \subset \bar{\mathcal{X}}_\varphi[p]$ is defined by ideal $(x^{p^2-1}y^{p^2-1})$. To conclude, we have the following result: for the $\Gamma_1(p)$-cover $\bar{\mathcal{X}}_\varphi^\times[p]/S_\varphi$ over the superspecial locus:

Theorem 5.1.1. *Let S_φ be the superspecial locus of the Sigel threefold $\bar{\mathcal{A}}$. Over each point of S_φ, the $\Gamma_1(p)$-cover $\bar{\mathcal{X}}_\varphi^\times[p]/S_\varphi$ is given by*

$$\operatorname{Spec} \bar{\mathbb{F}}_p[x, y]/(x^{p^2}, y^{p^2}, x^{p^2-1}y^{p^2-1})$$

over $\operatorname{Spec} \bar{\mathbb{F}}_p$. *In particular, the scheme $\bar{\mathcal{X}}_\varphi^\times$ is Cohen–Macaulay, but not Gorenstein.*

Proof. It only remains to prove the last statement. Note that the expression of $\mathcal{O}_{\bar{\mathcal{X}}_\varphi^\times[p]}$ in Theorem 5.1.1 is an Artin ring. Therefore $\bar{\mathcal{X}}_\varphi^\times[p]$ is automatically Cohen–Macaulay. Also it is easy to see that the socle of $\bar{\mathbb{F}}_p[x, y]/(x^{p^2}, y^{p^2}, x^{p^2-1}y^{p^2-1})$ has dimension 1 as an $\bar{\mathbb{F}}_p$-vector space, spanned by $x^{p^2-1}y^{p^2-1}$, while $\mathcal{O}_{\bar{\mathcal{X}}_\varphi^\times[p]}$ has dimension 0. Therefore $\bar{\mathcal{X}}_\varphi^\times[p]$ is not Gorenstein. $\square$

5.2 $\Gamma_1(p)$ -cover over the Supersingular Locus

In this section, let S_φ be the supersingular locus of the Siegel threefold and $\bar{\mathcal{X}}_\varphi$ be the restriction of the universal abelian surface on S_φ. Let $G := \bar{\mathcal{X}}_\varphi[p]$ be the p-torsion of $\bar{\mathcal{X}}_\varphi$. Recall that from Example 2.2.1, the canonical filtration of G has the form

$$0 = G_0 \subset G_1 \subset G_2 \subset G_3 \subset G_4 = G.$$

The corresponding canonical type is $\rho = (0, 1, 2, 3, 4)$, $v = (0, 0, 1, 1, 2)$ and $f = (2, 3, 3, 4, 4)$. Recall that this means the following:

$$\operatorname{Rank} G_i = p^{\rho(i)}, \tag{5.1}$$

$$V(G_i) = G_{v(i)}, \tag{5.2}$$

$$F^{-1}(G_i) = G_{f(i)}, \tag{5.3}$$

$$G_i^{\perp} = G_{4-i}, \tag{5.4}$$

$$(G_i/G_j)^D \cong G_{4-j}/G_{4-i}. \tag{5.5}$$

We want to give an explicit description of $G = G_4$. This is precisely the situation in Theorem 4.4.1. From this, we have the following result:

Theorem 5.2.1. *Let $\bar{\mathcal{X}}_\varphi/S_\varphi$ be the universal abelian surface over the supersingular locus of the Siegel threefold. Let $x \in S_\varphi$ be a point and let $R := \mathcal{O}_{x,S_\varphi}$ be the local ring at $x \in S_\varphi$. Let $S = \operatorname{Spec} R$ and let $\bar{\mathcal{X}}_S$ be the pullback of $\bar{\mathcal{X}}_\varphi$ to S. Consider its p-torsion $G := \bar{\mathcal{X}}_S[p]$. Then there exists a 10-tuple $(a_1, a_2, b_1, b_2, c, d, e_1, e_2, f_1, f_2)$ with entries in R, so that*

$$G = G_4 \cong \operatorname{Spec} R[x, y_1, y_2, z]/\left(x^p, y_i^p - a_i x, z^p - cx - e_1 y_1 - e_2 y_2\right)$$

where coalgebra operations given by

$$\begin{aligned}
m^*(x) &:= 1\otimes x + x\otimes 1,\\
m^*(y_i) &:= 1\otimes y_i + y_i\otimes 1 + \sum_{k=1}^{p-1}\frac{b_i x^k\otimes x^{p-k}}{k!\,(p-k)!},\\
m^*(z) &:= 1\otimes z + z\otimes 1 + \sum_{k=1}^{p-1}\frac{d x^k\otimes x^{p-k}}{k!\,(p-k)!} + \sum_{i=1}^{2}\sum_{k=1}^{p-1}\frac{f_i y_i^k\otimes y_i^{p-k}}{k!\,(p-k)!}\\
&\quad - \sum_{i=1}^{2}\sum_{k=1}^{p-1}\frac{f_i b_i\,(1\otimes y_i + y_i\otimes 1)^{p-1}\,(x^k\otimes x^{p-k})}{k!\,(p-k)!}.
\end{aligned}$$

The primitive elements of G is given by

$$G^\times \cong \operatorname{Spec} R[x,y_1,y_2,z]/\left(x^p, y_i^p - a_i x, z^p - cx - e_1y_1 - e_2y_2, x^{p-1}y_1^{p-1}y_2^{p-1}z^{p-1}\right)$$

Proof. This is an immediate corollary of Theorem 4.4.1 and Example 2.3.6. □

In Theorem 5.2.1, the only property we used about $\bar{\mathcal{X}}_S[p]/S$ is that it allows a canonical filtration that satisfies properties (5.2)-(5.5). It does not use the properties of the base scheme, i.e. the supersingular locus of the Siegel threefold. To make a more precise description of the group scheme $\bar{\mathcal{X}}_S[p]/S$, we can use a construction of the supersingular locus by Moret-Bailly [25] and Oort [32]. This construction is also studied in [18].

Let E be a supersingular elliptic curve over $\bar{\mathbb{F}}_p$ and consider $E\times E$. Note that the kernel of Frobenius on $E\times E$ is $\alpha_p\times\alpha_p$. For each $\mu\in\bar{\mathbb{F}}_p$, we define a group scheme morphism $\mu^*:\alpha_p\to\alpha_p$ by sending $a\mapsto\mu a$. Let $(\mu,\nu)\subset\bar{\mathbb{F}}_p^2-\{0,0\}$ and consider the embedding $i_{\mu,\nu}:\alpha_p\xrightarrow{(\mu,\nu)}\alpha_p\times\alpha_p\subset E\times E$. Note that for any $\lambda\in\bar{\mathbb{F}}_p^\times$, the image $i_{\lambda\mu,\lambda\nu}(\alpha_p)$ is equal to the image $i_{\mu,\nu}(\alpha_p)$ (though the maps are not the same).

Consider $\mathbb{P}^1=\operatorname{Proj}\left(\bar{\mathbb{F}}_p[\mu,\nu]\right)$ and write $\alpha_p\times\alpha_p\times\mathbb{P}^1=\mathcal{S}pec\,\mathcal{O}_{\mathbb{P}^1}[x,y]/(x^p,y^p)$. We define $H\subset\alpha_p\times\alpha_p\times\mathbb{P}^1\subset E\times E\times\mathbb{P}^1$ to be the subgroup scheme defined by $\nu x-\mu y=0$. In particular, at each point $[\mu,\nu]\in\mathbb{P}^1$, the restriction $H_{\mu,\nu}\subset\alpha_p\times\alpha_p$ is the image $i_{\mu,\nu}(\alpha_p)$. Over any affine chart $U\subset\mathbb{P}^1$, the restriction of H on U satisfies that $H_U\cong\alpha_{p,U}$.

Now let $\mathcal{Y}:=(E\times E\times\mathbb{P}^1)/H$ to be the quotient abelian surface over $\mathbb{P}^1$. This abelian surface $\mathcal{Y}/\mathbb{P}^1$ has the following property:

Proposition 5.2.2. *Let $\bar{\mathcal{A}} = \mathcal{A}_{2,1,N} \times \operatorname{Spec} \bar{\mathbb{F}}_p$ be the Siegel threefold in characteristic p with $N \geq 3$. Consider the supersingular locus $\bar{\mathcal{A}}^{ss}$, which is the union of superspecial stratum and supersingular (but not superspecial) stratum. Then we have*

(1) The singular points of $\bar{\mathcal{A}}^{ss}$ are exactly the points in the superspecial stratum.

(2) Each irreducible component of $\bar{\mathcal{A}}^{ss}$ is isomorphic to $\mathbb{P}^1$ and there are exactly $(p+1)$ branches of $\mathbb{P}^1$ intersecting transversally at each superspecial point.

(3) Let $V \subset \bar{\mathcal{A}}^{ss}$ be an irreducible component and let $\bar{\mathcal{X}}_V$ be the restriction of the universal abelian surface on V. Then there is an isomorphism $\phi : \mathbb{P}^1 \to V$ so that $\mathcal{Y} \cong \bar{\mathcal{X}}_V \times_V \mathbb{P}^1$.

Proof. See [20, Page 193] and [18, Section 2]. □

Let S_φ be the supersingular stratum and $\bar{\mathcal{X}}_\varphi$ be the universal abelian surface over S_φ. By Proposition 5.2.2, we have

$$\bar{\mathcal{X}}_\varphi \cong \bigsqcup_i \mathcal{Y} \times_{\mathbb{P}^1} U_i$$

where $U_i \subset \mathbb{P}^1$ are open subschemes. Therefore, we would want to have a description of $\mathcal{Y}[p]^\times/\mathbb{P}^1$ to study the $\Gamma_1(p)$-cover.

Consider the exact sequence $0 \to H \to E \times E \to \mathcal{Y} \to 0$ and restrict it to $U = \operatorname{Spec} \bar{\mathbb{F}}_p[\mu] \subset \operatorname{Proj} \bar{\mathbb{F}}_p[\mu, \nu]$. In this case, we have $H_U \cong \alpha_p \xrightarrow{(\mu,1)} \alpha_p \times \alpha_p \hookrightarrow E \times E$. The exact sequence above yields an exact sequence of Dieudonné modules

$$\mathbb{D}(\mathcal{Y})_U \to \mathbb{D}(E \times E)_U \to \mathbb{D}(H)_U \to 0.$$

Note that H and $E \times E$ are base changed from $\operatorname{Spec} \bar{\mathbb{F}}_p$ and $\mathbb{D}$ is a crystal. Therefore the Dieudonné module of $\mathbb{D}(H)_U$ and $\mathbb{D}(E \times E)_U$ can be directly obtained from the Dieudonné modules of E and α_p over $\bar{\mathbb{F}}_p$. In particular, we have that

$$\mathbb{D}(H)_U = \bar{\mathbb{F}}_p \otimes_{W(\bar{\mathbb{F}}_p)} \mathcal{O}_U = \mathcal{O}_U$$

and

$$\mathbb{D}(E \times E)_U = \bigoplus_{i=1}^{2} W(\bar{\mathbb{F}}_p)\{F, V\}/(F - V) \otimes \mathcal{O}_U.$$

From this, we write $\mathbb{D}(E \times E)_U$ as

$$L := \mathbb{D}(E \times E)_U = W(\bar{\mathbb{F}}_p)[\mu]e_1 \oplus W(\bar{\mathbb{F}}_p)[\mu]e_2 \oplus W(\bar{\mathbb{F}}_p)[\mu]e_3 \oplus W(\bar{\mathbb{F}}_p)[\mu]e_4,$$

where $Fe_1 = e_3, Fe_2 = e_4, Fe_3 = pe_1, Fe_4 = pe_2$, and same for V.

Now we want to identify $\mathbb{D}(\mathcal{Y})_U \subset \mathbb{D}(E \times E)_U$, which is the kernel of the map $\phi : \mathbb{D}(E\times E)_U \to \mathbb{D}(H)_U$. Since H is killed by F, the kernel $\ker \phi$ contains FL, and $\mathbb{D}(\mathcal{Y})_U$ can be identified with a line $\ell \subset L/FL$ which generated by $\mu e_1 + e_2$. To sum up, $\mathbb{D}(\mathcal{Y})_U \subset \mathbb{D}(E\times E)_U$ is generated by $e_3, e_4, pe_1, pe_2, \mu e_1 + e_2$. Note that $pe_2 = p(\mu e_1 + e_2) - \mu(pe_1)$. Therefore, we have

$$\mathbb{D}(\mathcal{Y})_U = W(\bar{\mathbb{F}}_p)[\mu]pe_1 \oplus W(\bar{\mathbb{F}}_p)[\mu]e_3 \oplus W(\bar{\mathbb{F}}_p)[\mu]e_4 \oplus W(\bar{\mathbb{F}}_p)[\mu](\mu e_1 + e_2),$$

with Frobenius given by

$$F(pe_1 \otimes 1, e_3 \otimes 1, e_4 \otimes 1, (\mu e_1 + e_2) \otimes 1) = (pe_1, e_3, e_4, \mu e_1 + e_2) \begin{pmatrix} 0 & 1 & -\mu & 0 \\ 0 & 0 & 0 & \mu \\ p & 0 & 0 & 1 \\ 0 & 0 & p & 0 \end{pmatrix}, \quad (5.6)$$

and the Verschiebung given by

$$V(pe_1, e_3, e_4, \mu e_1 + e_2) = (pe_1 \otimes 1, e_3 \otimes 1, e_4 \otimes 1, (\mu e_1 + e_2) \otimes 1) \begin{pmatrix} 0 & 1 & -\mu & 0 \\ 0 & 0 & 0 & \mu \\ p & 0 & 0 & 1 \\ 0 & 0 & p & 0 \end{pmatrix}. \quad (5.7)$$

Now we are ready for the following result:

Theorem 5.2.3. *Let S_φ be the supersingular stratum and let W be an irreducible component of S_φ. The $\Gamma_1(p)$-cover $\bar{\mathcal{X}}^\times_W[p]/W$ is the pullback of $\mathcal{Y}^\times[p]/\mathbb{P}^1_{\bar{\mathbb{F}}_p}$ via some open immersion $W \to \mathbb{P}^1_{\bar{\mathbb{F}}_p}$. Over each affine chart of the standard cover $\mathbb{P}^1_{\bar{\mathbb{F}}_p} = \mathbb{A}^1_0 \cup \mathbb{A}^1_\infty$, the restricted $\Gamma_1(p)$-cover $\mathcal{Y}^\times[p]|_{\mathbb{A}^1_{\bar{\mathbb{F}}_p}}/\mathbb{A}^1_{\bar{\mathbb{F}}_p}$ is isomorphic to*

$$\operatorname{Spec} \bar{\mathbb{F}}_p[\mu, x, y]/(x^{p^2}, y^{p^2} - (\mu^p - \mu)x^p, x^{p^2-1}y^{p^2-1}) \quad (5.8)$$

over $\operatorname{Spec}(\bar{\mathbb{F}}_p[\mu])$.

Proof. Without loss of generality, consider $U = \operatorname{Spec}\bar{\mathbb{F}}_p[\mu] \subset \mathbb{P}^1$ as above. Other affine charts are similar. Write $G = \mathcal{Y}[p]$. By Section 2.4, we have $M_S(G) = V\mathbb{D}(\mathcal{Y})_U/pV\mathbb{D}(\mathcal{Y})_U$. Following the calculations in (5.6) and (5.7), we can see that $V\mathbb{D}(\mathcal{Y})_U/pV\mathbb{D}(\mathcal{Y})_U$ is a free $\mathcal{O}_U$-module with basis $pe_3, pe_1, pe_2, \mu e_3 + e_4$ with Frobenius and Verschiebung given by

$$F(pe_3 \otimes 1, pe_1 \otimes 1, pe_2 \otimes 1, (\mu e_3 + e_4) \otimes 1) = (pe_3, pe_1, pe_2, \mu e_3 + e_4) \begin{pmatrix} 0 & 1 & -\mu & 0 \\ 0 & 0 & 0 & \mu \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$V(pe_3, pe_1, pe_2, \mu e_3 + e_4) = (pe_3 \otimes 1, pe_1 \otimes 1, pe_2 \otimes 1, (\mu e_3 + e_4) \otimes 1) \begin{pmatrix} 0 & 1 & -\mu & 0 \\ 0 & 0 & 0 & \mu \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

By Theorem 4.4.1, the group scheme G can be written as

$$\operatorname{Spec}\bar{\mathbb{F}}_p[\mu, x, y_1, y_2, z]/(x^p, y_1^p - x, y_2^p + \mu x, z^p - \mu y_1 - y_2). \tag{5.9}$$

After substituting $x = y_1^p$ and $y_2 = z^p - \mu y_1$ and changing the variables, we have the expression (5.8).

On the other chart $T = \operatorname{Spec}\bar{\mathbb{F}}_p[\nu] \subset \mathbb{P}^1$, we have $V\mathbb{D}(\mathcal{Y})_T/pV\mathbb{D}(\mathcal{Y})_T$ a free $\mathcal{O}_T$-module with basis $pe_4, pe_1, pe_2, e_3 + \nu e_4$. A similar calculation follows and we have the group scheme as

$$\operatorname{Spec}\bar{\mathbb{F}}_p[\nu, x, y_1, y_2, z]/(x^p, y_1^p - \nu x, y_2^p - x, z^p - y_1 - \nu y_2). \tag{5.10}$$

Note that (5.9) and (5.10) are isomorphic by sending $\mu \mapsto \nu$ and swapping y_1 and y_2.

The calculation of primitive elements is immediate from Example 2.3.6. □

5.3 $\Gamma_1(p)$ -cover over p-rank-1 Locus

Let S_φ be the p-rank-1 stratum and let x be a point of S_φ. We set $R = \mathcal{O}^{sh}_{S_\varphi, \bar{x}}$, the strict henselization of the local coordinate ring $\mathcal{O}_{S_\varphi, \bar{x}}$ and set $S = \operatorname{Spec} R$. Recall that in this case, the canonical filtration is

$$0 = G_0 \subset G_1 \subset G_2 \subset G_3 \subset G_4 = G.$$

The corresponding canonical type is $\rho = (0, 1, 2, 3, 4)$, $v = (0, 1, 1, 2, 2)$ and $f = (2, 2, 3, 3, 4)$.

We will use the notion mixed extensions in this case, established by Grothendieck. First, we will give a quick review of this theory. For more details, see [11, Exposé IX, 9.3] and [3, 4.2].

5.3.1 Mixed extensions

Let $\mathcal{C}$ be an abelian category. Suppose that there are two extensions in $\mathcal{C}$:

$$(F)\colon 0 \to P \to F \to R \to 0, \tag{5.11}$$

$$(E)\colon 0 \to R \to E \to Q \to 0. \tag{5.12}$$

Definition 5.3.1. *A mixed extension (extension panachée) of E by F is an object x in $\mathcal{C}$ together with a filtration*

$$0 \subset X^2 \subset X^1 \subset X$$

such that

$$0 \to X^2 \to X^2 \to X^1/X^2 \to 0 \cong 0 \to P \to F \to R \to 0, \tag{5.13}$$

$$0 \to X^1/X^2 \to X/X^2 \to X/X^1 \to 0 \cong 0 \to R \to E \to Q \to 0. \tag{5.14}$$

Let $\mathrm{Extpan}(E, F)$ be the category of all mixed extensions of E by F and let $\mathrm{Ext}(Q, P)$ be the category of all extensions of Q by P. We define a functor $w : \mathrm{Ext}(Q, P) \times \mathrm{Extpan}(E, F) \to \mathrm{Extpan}(E, F)$ as follows: Let X be a mixed extension of E by F. Regard X as an ordinary extension of Q by F. Let G be an extension of Q by P. Let $\bar{G}$ be the induced extension of Q by F via the injection $P \hookrightarrow F$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P & \longrightarrow & G & \longrightarrow & Q & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\| & & \\ 0 & \longrightarrow & F & \longrightarrow & \bar{G} & \longrightarrow & Q & \longrightarrow & 0 \end{array}$$

Then we define $w(G,X) := \bar{G} \wedge X$. Here $\wedge$ denotes the Baer sum in the Ext group. Let $Y = w(G,X)$. We need to define the filtration on Y to get an element in $\mathrm{Extpan}(E,F)$. By the definition of Y, we have an extension in $\mathcal{C}$:

$$0 \to F \to Y \to Q \to 0.$$

Let Y^1 be the image of F under the inclusion. Push this exact sequence out along $F \twoheadrightarrow R$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \longrightarrow & Y & \longrightarrow & Q & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & R & \longrightarrow & \tilde{Y} & \longrightarrow & Q & \longrightarrow & 0 \end{array}$$

By $Y = w(G,X)$, one can get $\tilde{Y} \cong E$. We take $Y^2 = \ker(Y \to \tilde{Y})$. In this way we form the filtration $0 \subset Y^2 \subset Y^1 \subset Y$.

The main result of the theory of mixed extensions is the following proposition:

Proposition 5.3.2. *Let E and F be two extensions as above. Consider the category* $\mathrm{Extpan}(E,F)$ *of mixed extensions of E by F. Let* $\mathrm{Ext}(Q,P)$ *be the category of all extensions of Q by P. Then the set of all isomorphism classes of objects in* $\mathrm{Extpan}(E,F)$ *is either empty, or it is a torsor under* $\mathrm{Ext}^1(Q,P)$ *(the group of isomorphism classes of extensions) via the functor*

$$w : \mathrm{Ext}(Q,P) \times \mathrm{Extpan}(E,F) \to \mathrm{Extpan}(E,F).$$

defined above.

5.3.2 Calculations on extensions

Apply the theory of mixed extensions to the filtration $0 \subset G_1 \subset G_3 \subset G_4$. We need to understand the extensions

$$\begin{aligned} (F) &: 0 \to G_1 \to G_3 \to G_3/G_1 \to 0 \\ (E) &: 0 \to G_3/G_1 \to G_4/G_1 \to G_4/G_3 \to 0 \end{aligned} \tag{5.15}$$

and the Ext group $\mathrm{Ext}(G_4/G_3, G_1)$. We will analyze these groups one by one. We will analyze (E) first.

First, we consider G_3/G_1. Consider the p-divisible group $\bar{\mathcal{X}}_S[p^\infty]$. There is a short exact sequence

$$0 \to \bar{\mathcal{X}}_S[p^\infty]^0 \to \bar{\mathcal{X}}_S[p^\infty] \to \bar{\mathcal{X}}_S[p^\infty]^{et} \to 0, \tag{5.16}$$

where 0 denotes the identity component and *et* denotes the maximal étale quotient. Let $\vee$ denote the Cartier dual of p-divisible groups, defined by

$$\bar{\mathcal{X}}_S[p^\infty]^\vee := (\bar{\mathcal{X}}_S[p^n]^D)_n,$$

where D denotes the Cartier dual of finite group schemes. Apply the same exact sequence to $(\bar{\mathcal{X}}_S[p^\infty]^0)^\vee$ and take Cartier dual again: We get

$$0 \to \bar{\mathcal{X}}_S[p^\infty]^{mul} \to \bar{\mathcal{X}}_S[p^\infty]^0 \to (\bar{\mathcal{X}}_S[p^\infty]^0)^{uni} \to 0. \tag{5.17}$$

Consider $\mathcal{G} = (\bar{\mathcal{X}}_S[p^\infty]^0)^{uni}$. By the construction above, we have $\mathcal{G}[p] = G_3/G_1$ and the Newton polygon of each point of $\mathcal{G}$ is of slope $(1/2, 1/2)$. Now we need the following result by Oort and Zink from [34]:

Proposition 5.3.3. *([4, Proposition 8.6]) Let R be a strictly henselian reduced local ring over $\bar{\mathbb{F}}_p$. Let $\mathcal{G}$ be an isoclinic p-divisible group over $S = \operatorname{Spec} R$. Then there is a p-divisible group $\mathcal{G}_0$ over $\mathbb{F}_p$ with an isogeny $\mathcal{G}_0 \times_{\operatorname{Spec} \bar{\mathbb{F}}_p} S \to \mathcal{G}$.*

Applying Proposition 5.3.3 to $\mathcal{G} = (\bar{\mathcal{X}}_S[p^\infty]^0)^{uni}$, we obtain an isogeny $\mathcal{G}_0 \times_{\operatorname{Spec} \bar{\mathbb{F}}_p} S \to \mathcal{G}$. By classical Dieudonné theory ([5, III. 8]), there is a unique p-divisible group of dimension 1 and height 2 over an algebraically closed field, which is the p-divisible group associated to a supersingular elliptic curve. Therefore we assume $\mathcal{G}_0 = E[p^\infty]$ for some supersingular elliptic curve over $\bar{\mathbb{F}}_p$ and we get an isogeny

$$E[p^\infty] \times_{\operatorname{Spec} \bar{\mathbb{F}}_p} S \to \mathcal{G}. \tag{5.18}$$

Lemma 5.3.4. *The isogeny* $E[p^\infty] \times_{\operatorname{Spec} \bar{\mathbb{F}}_p} S \to \mathcal{G}$ *is an isomorphism.*

Proof. Here we use the tool of Rapoport–Zink space $\mathcal{M}$ of $E[p^\infty]$. For an $\bar{\mathbb{F}}_p$-scheme T, a T-point of $\mathcal{M}$ is given by a pair $(\mathcal{G}, \rho)$, where $\mathcal{G}$ is a p-divisible group over T and $\rho : E[p^\infty] \times_{\operatorname{Spec} \bar{\mathbb{F}}_p} T \to \mathcal{G}$ is a quasi-isogeny of height 0. Two points $(\mathcal{G}_1, \rho_1)$ and $(\mathcal{G}_2, \rho_2)$ are identified if they are isomorphic. The fundamental result of $\mathcal{M}$ is that it is represented by the formal scheme $\operatorname{Spf} W(\bar{\mathbb{F}}_p)[\![t]\!]$. For more details of this Rapoport–Zink space, see [37, 3.78, 3.79].

The isogeny in (5.18) gives an S-point of $\mathcal{M}$, which corresponds to a morphism $S \to \operatorname{Spf} W(\bar{\mathbb{F}}_p)[\![t]\!]$. Note that the only possible map $S \to \operatorname{Spf} W(\bar{\mathbb{F}}_p)[\![t]\!]$ is by sending $t \mapsto 0$ since S is smooth (thus integral). So $\mathcal{M}(S)$ has only one point and therefore the isogeny $E[p^\infty] \times_{\operatorname{Spec} \bar{\mathbb{F}}_p} S \to \mathcal{G}$ is in fact an isomorphism. □

By Lemma 5.3.4 above, we see that G_3/G_1 is isomorphic to the p-torsion of a supersingular elliptic curve base changed from a field. Therefore we have

$$G_3/G_1 \cong E[p] \cong \operatorname{Spec} R[y_1, y_2]/(y_1^p, y_2^p - y_1) \tag{5.19}$$

with

$$\begin{aligned} m^*_{G_3/G_1}(y_1) &= 1 \otimes y_1 + y_1 \otimes 1, \\ m^*_{G_3/G_1}(y_2) &= 1 \otimes y_2 + y_2 \otimes 1 + \sum_{k=1}^{p-1} \frac{y_1^k \otimes y_1^{p-k}}{k!\,(p-k)!} \end{aligned}$$

Now consider G_4/G_3. From the canonical type of G, we get that F_{G_4/G_3} is an isomorphism and $V_{G_4/G_3} = 0$. Since the base $\operatorname{Spec} R$ is strictly henselian, we have $G_4/G_3 \cong \mathbb{Z}/p\mathbb{Z}$. Dually, we also have $G_1 \cong \mu_p$. We write them as

$$\begin{aligned} &G_4/G_3 \cong \mathbb{Z}/p\mathbb{Z} \cong \operatorname{Spec} R[x]/(x^p - x), \qquad m^*_{G_4/G_3}(x) = 1 \otimes x + x \otimes 1, \\ &G_1 \cong \mu_p \cong \operatorname{Spec} R[z]/(z^p), \qquad m^*_{G_1}(z) = 1 \otimes z + z \otimes 1 + \sum_{i=1}^{p-1} \frac{z^i \otimes z^{p-i}}{i!\,(p-i)!}. \end{aligned} \tag{5.20}$$

From the discussion above, the two extensions in (5.15) are now given by

$$\begin{aligned}(F) &: 0 \to \mu_p \to G_3 \to E[p] \to 0 \\ (E) &: 0 \to E[p] \to G_4/G_1 \to \mathbb{Z}/p\mathbb{Z} \to 0\end{aligned} \tag{5.21}$$

Now we want to give the expression of G_4/G_1 in (E). Consider G_3/G_2. By the canonical type, we can see that $V_{G_3/G_2} = 0$ and $F_{G_3/G_2} = 0$. Then $G_3/G_2 \cong \alpha_p$ and we write G_3/G_2 as

$$G_3/G_2 \cong \operatorname{Spec} R[y_1]/(y_1^p)$$

with

$$m^*_{G_3/G_2}(y_1) = 1 \otimes y_1 + y_1 \otimes 1.$$

Note that we have G_4/G_3 from (5.20). Consider the extension

$$0 \to G_3/G_2 \to G_4/G_2 \to G_4/G_3 \to 0$$

By Theorem 4.3.1, G_4/G_2 is of the form

$$G_4/G_2 \cong \operatorname{Spec} R[x, y_1]/(x^p - x, y_1^p - ax) \tag{5.22}$$

with

$$m^*_{G_4/G_2}(x) = 1 \otimes x + x \otimes 1,\ m^*_{G_4/G_2}(y_1) = 1 \otimes y_1 + y_1 \otimes 1.$$

Then we consider the extension

$$0 \to G_2/G_1 \to G_4/G_1 \to G_4/G_2 \to 0$$

Here again $G_2/G_1 \cong \alpha_p$. We write it as

$$G_2/G_1 \cong \operatorname{Spec} R[y_2]/(y_2^p)$$

with

$$m^*_{G_2/G_1}(y_2) = 1 \otimes y_2 + y_2 \otimes 1.$$

Again by Theorem 4.3.1, G_4/G_1 is of the form

$$G_4/G_1 \cong \operatorname{Spec} R[x, y_1, y_2]/(x^p - x, y_1^p - ax, y_2^p - bx - cy_1) \tag{5.23}$$

with

$$m^*_{G_4/G_1}(x) = 1 \otimes x + x \otimes 1,$$
$$m^*_{G_4/G_1}(y_1) = 1 \otimes y_1 + y_1 \otimes 1,$$
$$m^*_{G_4/G_1}(y_2) = 1 \otimes y_2 + y_2 \otimes 1 + \sum_{k=1}^{p-1} \frac{-adx^k \otimes x^{p-k}}{k!\,(p-k)!} + \sum_{k=1}^{p-1} \frac{dy_1^k \otimes y_1^{p-k}}{k!\,(p-k)!}.$$

Note that $G_3/G_1 = \ker F^2_{G_4/G_1}$. Comparing with (5.19), we get $c = d = 1$ and therefore we have the expression of G_4/G_1 as

$$G_4/G_1 \cong \operatorname{Spec} R[x, y_1, y_2]/(x^p - x, y_1^p - ax, y_2^p - bx - y_1) \tag{5.24}$$

with

$$m^*_{G_4/G_1}(x) = 1 \otimes x + x \otimes 1,$$
$$m^*_{G_4/G_1}(y_1) = 1 \otimes y_1 + y_1 \otimes 1,$$
$$m^*_{G_4/G_1}(y_2) = 1 \otimes y_2 + y_2 \otimes 1 + \sum_{k=1}^{p-1} \frac{-ax^k \otimes x^{p-k}}{k!\,(p-k)!} + \sum_{k=1}^{p-1} \frac{y_1^k \otimes y_1^{p-k}}{k!\,(p-k)!}$$

This expression of G_4/G_1 gives the description of all terms in (E). Now we want to analyze (F). Note that (F) is in fact the Cartier dual of E from the information of the canonical type. However, the explicit expressions of the group schemes are complicated and difficult to directly write down the algebras and coalgebras. Therefore, we will also analyse (F) using another approach based on the fact that G_3 is an extension of the p-torsion of supersingular elliptic curves by μ_p. The result is as follows:

Lemma 5.3.5. *The extension group* $\operatorname{Ext}^1_{fppf,S}(E[p], \mu_p)$ *is isomorphic to* $E(S)/pE(S)$.

Proof. We will be using fppf topology throughout this proof. It is a classical result that for any group scheme G/S, we have an isomorphism $H^1(S, G) \cong \operatorname{Ext}^1_S(G^D, \mathbb{G}_m)$. This isomorphism is explained in many places, for example [43, Theorem 2, Theorem 3]. Apply this isomorphism to $E[p]$. Note that $E[p]$ is self-dual. Therefore we get an isomorphism $H^1(S, E[p]) \cong \operatorname{Ext}^1_S(E[p], \mathbb{G}_m)$

For $H^1(S, E[p])$, consider the long exact sequence associated to

$$0 \to E[p] \to E \xrightarrow{p} E \to 0. \tag{5.25}$$

We get

$$E(S) \xrightarrow{p} E(S) \to H^1(S, E[p]) \to H^1(S, E). \tag{5.26}$$

Note that $H^1(S, E) = 0$ since E is smooth over S and S is strictly henselian. Therefore we have $E(S)/pE(S) \cong H^1(S, E[p])$. Given any point $S \to E$, the $E[p]$-torsor over S is obtained by pulling back the short exact sequence (5.25) along $S \to E$.

For $\operatorname{Ext}^1_S(E[p], \mathbb{G}_m)$, consider the long exact sequence associated to the Kummer sequence

$$0 \to \mu_p \to \mathbb{G}_m \to \mathbb{G}_m \to 0. \tag{5.27}$$

We get

$$\operatorname{Hom}(E[p], \mathbb{G}_m) \to \operatorname{Ext}(E[p], \mu_p) \to \operatorname{Ext}(E[p], \mathbb{G}_m) \xrightarrow{p} \operatorname{Ext}(E[p], \mathbb{G}_m) \tag{5.28}$$

Since $\mathscr{H}om(E[p], \mathbb{G}_m)$ is a sheaf, we get an exact sequence

$$0 \to \operatorname{Hom}(E[p], \mathbb{G}_m) \to \mathscr{H}om(E[p], \mathbb{G}_m)(S). \tag{5.29}$$

Note that $\mathscr{H}om(E[p], \mathbb{G}_m)(S) = E[p](S)$, which vanishes since E is supersingular and S is integral. Therefore the first term $\operatorname{Hom}(E[p], \mathbb{G}_m)$ in (5.28) vanishes. Note that since $E[p]$ is annihilated by p, the map $\operatorname{Ext}(E[p], \mathbb{G}_m) \xrightarrow{p} \operatorname{Ext}(E[p], \mathbb{G}_m)$ in (5.28) is the zero map. Therefore we have $\operatorname{Ext}(E[p], \mu_p) \cong \operatorname{Ext}(E[p], \mathbb{G}_m)$.

Combining all the above results, we get $\operatorname{Ext}^1_S(E[p], \mu_p) \cong E(S)/pE(S)$. □

The last ingredient from Proposition 5.3.2 is the Ext group $\operatorname{Ext}(\mathbb{Z}/p\mathbb{Z}, \mu_p)$. For this, we have the following lemma:

Proposition 5.3.6. *The Ext group* $\operatorname{Ext}^1_S(\mathbb{Z}/p\mathbb{Z}, \mu_p)$ *is isomorphic to* $H^1(S, \mu_p) \cong \mathcal{O}_S^* / (\mathcal{O}_S^*)^p$.

Proof. Consider the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$. This gives a long exact sequence:

$$\mathrm{Hom}(\mathbb{Z}, \mu_p) \to \mathrm{Ext}^1(\mathbb{Z}/p\mathbb{Z}, \mu_p) \to \mathrm{Ext}^1(\mathbb{Z}, \mu_p) \to 0.$$

Note that S is reduced and of characteristic p. Therefore $\mathcal{O}_S$ has no non-trivial p-th root of unity and $\mathrm{Hom}(\mathbb{Z}, \mu_p) = 0$. On the other hand, it is easy to see that the extensions of $\mathbb{Z}$ by μ_p are completely freely determined by the μ_p-torsor over $1 \in \mathbb{Z}$. Therefore

$$\mathrm{Ext}^1_S(\mathbb{Z}/p\mathbb{Z}, \mu_p) \cong \mathrm{Ext}^1(\mathbb{Z}, \mu_p) \cong H^1(S, \mu_p).$$

The isomorphism $\mathcal{O}_S^*/\left(\mathcal{O}_S^*\right)^p \cong H^1(S, \mu_p)$ is a classical result of Kummer theory. Consider the Kummer exact sequence $1 \to \mu_p \to \mathbb{G}_m \xrightarrow{p} \mathbb{G}_m \to 1$. Then we have the associated long exact sequence

$$H^0(S, \mathbb{G}_m) \xrightarrow{p} H^0(S, \mathbb{G}_m) \to H^1(S, \mu_p) \to H^1(S, \mathbb{G}_m).$$

Since S is local, we have that $H^1(S, \mathbb{G}_m) = \mathrm{Pic}(S) = 0$. Note that $H^0(S, \mathbb{G}_m) = \mathcal{O}_S^*$, we have that $H^1(S, \mu_p) \cong \mathcal{O}_S^*/\left(\mathcal{O}_S^*\right)^p$. □

As a conclusion, we record all previous results as follows:

Theorem 5.3.7. *The group scheme $G = G_4$ is a mixed extension of*

$$(F) : 0 \to \mu_p \to G_3 \to E[p] \to 0$$
$$(E) : 0 \to E[p] \to G_4/G_1 \to \mathbb{Z}/p\mathbb{Z} \to 0$$

as in (5.21). Here, the extension (E) is explicitly given by Equation (5.19), (5.20) and (5.23). The extension (F) is the Cartier dual of (E) and can alternatively obtained by Lemma 5.3.5. All mixed extensions form a $\mathrm{Ext}^1_S(\mathbb{Z}/p\mathbb{Z}, \mu_p)$-torsor

$$w : \mathrm{Ext}^1_S(\mathbb{Z}/p\mathbb{Z}, \mu_p) \times \mathrm{Extpan}(E, F) \to \mathrm{Extpan}(E, F),$$

where the ext group $\mathrm{Ext}^1_S(\mathbb{Z}/p\mathbb{Z}, \mu_p) \cong \mathcal{O}_S^*/\left(\mathcal{O}_S^*\right)^p$ and the map w is described in Proposition 5.3.2.

5.4 $\Gamma_1(p)$ -cover over the Ordinary Locus

Now, let S_φ be the ordinary locus. The universal abelian surface $\bar{\mathcal{X}}_\varphi$ is ordinary. In this case, we have a powerful tool, namely the Serre–Tate theory, to help with analyzing the $\Gamma_1(p)$-cover. For details about Serre–Tate local moduli, see [17].

Let $x \in S_\varphi$ be a geometric point which corresponds to a principally polarized abelian variety X over $\bar{\mathbb{F}}_p$. Serre–Tate theory states that the deformation space of X is canonically pro-represented by $\operatorname{Spf} \bar{\mathbb{F}}_p[\![t_1, t_2, t_3]\!]$. Alternatively, this means there is a compatible system of universal principally polarized abelian schemes $X_n / \operatorname{Spec} \bar{\mathbb{F}}_p[t_1, t_2, t_3]/(t_1, t_2, t_3)^n$. By the definition of the Siegel threefold, this gives canonical isomorphisms $\hat{\mathcal{O}}_{x,S_\varphi}/\hat{m}^n \to \bar{\mathbb{F}}_p[t_1, t_2, t_3]/(t_1, t_2, t_3)^n$ where $\hat{\mathcal{O}}_{x,S_\varphi}$ is the completion of the local coordinate ring at x and $\hat{m}$ is the maximal ideal of $\hat{\mathcal{O}}_{x,S_\varphi}$. This induces a canonical isomorphism $\bar{\mathbb{F}}_p[\![t_1, t_2, t_3]\!] \cong \hat{\mathcal{O}}_{x,S_\varphi}$.

Consider $\hat{\mathcal{X}} := \bar{\mathcal{X}}_\varphi \times_{S_\varphi} \operatorname{Spec} \hat{\mathcal{O}}_{x,S_\varphi}$. It is the universal abelian surface over the Serre–Tate local moduli. The goal of this section is to give explicit description of $\hat{\mathcal{X}}[p] / \operatorname{Spec} \hat{\mathcal{O}}_{x,S_\varphi}$. To do this, we will calculate the universal extension over the Serre–Tate local moduli.

We first sketch the idea of Serre–Tate theory. Let X be the principally polarized ordinary abelian variety corresponding to $x \in S_\varphi$. Let $T_pX(k)$ be the Tate module of X. By choosing a basis, $T_pX(k) \cong \mathbb{Z}_p^2$. Let X^t be the dual abelian variety of X. We also have $T_pX(k) \cong \mathbb{Z}_p^2$ after a choice of basis.

The first result is that the deformation theory of X is the same as the deformation theory of the associated p-divisible group $X[p^\infty]$ (see [17, Theorem 1.2.1]). Therefore we only need to work with $X[p^\infty]$. Let $\mathbb{X}[p^\infty]$ be a deformation of $X[p^\infty]$ over an Artin local ring R. Let X^{mul} the maximal toroidal subgroup of $X[p^\infty]$ and let $\mathbb{X}^{mul}$ be the unique lift of X^{mul}. Then there is the canonical decomposition

$$0 \to \mathbb{X}^{mul} \to \mathbb{X}[p^\infty] \to T_pX(k) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to 0. \tag{5.30}$$

It turns out that the extension (5.30) can be obtained from the basic extension

$$0 \to T_pX(k) \to T_pX(k) \otimes \mathbb{Q}_p \to T_pX(k) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to 0. \tag{5.31}$$

by pushing out along a unique homomorphism $\phi_{\mathbb{X}/R}\colon T_pX(k)\to\mathbb{X}^{mul}$. On the other hand, there is a pairing of group schemes over R

$$E_{\mathbb{X}}:\ \mathbb{X}^{mul}\times T_pX^t(k)\to\hat{\mathbb{G}}_m, \tag{5.32}$$

which is the unique lift of the paring of group schemes over k

$$E_X:\ X^{mul}\times T_pX^t(k)\to\hat{\mathbb{G}}_m \tag{5.33}$$

that is induced from the Weil pairing $X^{mul}[p]\times T_pX^t[p](k)\to\mu_{p^n}$. By composing the map $\phi_{\mathbb{X}/R}\colon T_pX(k)\to\mathbb{X}^{mul}$ with the pairing (5.32), we get a pairing

$$q(\mathbb{X}/R;-,-)\colon T_pX(k)\otimes T_pX^t(k)\to\hat{\mathbb{G}}_m(R). \tag{5.34}$$

It turns out that this q contains all the information of the deformation $\mathbb{X}$:

Proposition 5.4.1. *([17, Theorem 2.1]) The construction*

$$\mathbb{X}/R\mapsto q(\mathbb{X}/R;-,-)$$

gives a bijection of the set of isomorphism classes of deformations of X and the group $\mathrm{Hom}_{\mathbb{Z}_p}(T_pX(k)\otimes T_pX^t(k),\hat{\mathbb{G}}_m(R))$.

Now we will calculate the universal extension. Take e_1,e_2 as a basis for $T_pX(k)$ and f_1,f_2 for $T_pX^t(k)$. Over the universal base $S=\mathrm{Spf}\,\bar{\mathbb{F}}_p[\![t_{11},t_{12},t_{21},t_{22}]\!]$, let

$$\phi\in\mathrm{Hom}_S(T_pX(k)\otimes T_pX^t(k),\hat{\mathbb{G}}_m)=\mathrm{Hom}_S(T_pX(k),\mathrm{Hom}(T_pX^t(k),\hat{\mathbb{G}}_m))$$

be the universal homomorphism given by $e_i\otimes f_j\mapsto t_{ij}$. By the Serre–Tate theory above, the universal extension is the pushout of the basic extension (5.31) along ϕ:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_pX(k) & \longrightarrow & T_pX(k)\otimes\mathbb{Q}_p & \longrightarrow & T_pX(k)\otimes\mathbb{Q}_p/\mathbb{Z}_p & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle\phi} & & \downarrow & & \Downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}(T_pX^t(k),\hat{\mathbb{G}}_m) & \longrightarrow & \hat{X} & \longrightarrow & T_pX(k)\otimes\mathbb{Q}_p/\mathbb{Z}_p & \longrightarrow & 0
\end{array}$$

Note that $\mathrm{Hom}(T_pX^t(k), \hat{\mathbb{G}}_m) \cong \hat{\mathbb{G}}_m \times \hat{\mathbb{G}}_m$ by taking the images of f_1 and f_2. We will denote the elements in $\mathrm{Hom}(T_pX^t(k), \hat{\mathbb{G}}_m)$ by their images under this isomorphism. In this way, the generators of $\mathrm{Hom}(T_pX^t(k), \hat{\mathbb{G}}_m)$ are denoted by $(1+t_{11}, 1+t_{12})$ and $(1+t_{21}, 1+t_{22})$ and a general element $ae_1+be_2 \in T_pX(k)$ is mapped to $((1+t_{11})^a(1+t_{21})^b, (1+t_{12})^a(1+t_{22})^b)$ by ϕ.

Now we consider the p-torsion $\hat{X}[p]$ of the fiber coproduct

$$\hat{X} = \left(T_pX(k) \otimes \mathbb{Q}_p\right) \sqcup_{T_pX(k)} \mathrm{Hom}(T_pX^t(k), \hat{\mathbb{G}}_m).$$

The points of $\hat{X}[p]$ are given by

$$\hat{X}[p](A) = \{(a, b, x, y) | a, b \in \{0, \dots, p-1\}, x^p = (1+t_{11})^a(1+t_{21})^b, y^p = (1+t_{12})^a(1+t_{22})^b\} \tag{5.35}$$

with group multiplication defined by

$$(a_1, b_1, x_1, y_1)\dot{+}(a_2, b_2, x_2, y_2) =$$
$$\begin{cases} (a_1+a_2, b_1+b_2, x_1x_2, y_1y_2), & \text{if } a_1+a_2, b_1+b_2 < p, \\ \left(a_1+a_2-p, b_1+b_2, \dfrac{x_1x_2}{(1+t_{11})^p}, \dfrac{y_1y_2}{(1+t_{12})^p}\right), & \text{if } b_1+b_2 < p \leq a_1+a_2, \\ \left(a_1+a_2, b_1+b_2-p, \dfrac{x_1x_2}{(1+t_{21})^p}, \dfrac{y_1y_2}{(1+t_{22})^p}\right), & \text{if } a_1+a_2 < p \leq b_1+b_2, \\ \left(a_1+a_2-p, b_1+b_2-p, \dfrac{x_1x_2}{(1+t_{11})^p(1+t_{21})^p}, \dfrac{y_1y_2}{(1+t_{12})^p(1+t_{22})^p}\right), & \\ & \text{if } p \leq a_1+a_2, b_1+b_2. \end{cases} \tag{5.36}$$

Note that we haven't used the polarization structure on A yet and hence the dimension of the formal moduli is 4. Now we consider the polarization. From [17, Theorem 21. (4)], the principal polarization $\lambda\colon X \to X^t$ lifts to $\mathbb{X} \to \mathbb{X}^t$ if and only if

$$q(\mathbb{X}/R; \alpha, \lambda^t(\beta)) = q(\mathbb{X}^t/R; \lambda(\alpha), \beta) \tag{5.37}$$

for all $\alpha \in T_pX(k)$ and $\beta \in T_pX^{tt}(k) = T_pX(k)$. From the symmetry formula in [17, Theorem 21. (3)], we have

$$q(\mathbb{X}^t/R; \lambda(\alpha), \beta) = q(\mathbb{X}/R; \beta, \lambda(\alpha)). \tag{5.38}$$

By combining Equation (5.37), (5.38) and since $\lambda = \lambda^t$, we have

$$q(\mathbb{X}/R; \alpha, \lambda(\beta)) = q(\mathbb{X}/R; \beta, \lambda(\alpha)). \tag{5.39}$$

We choose a basis e_1, e_2 for $T_pX(k)$ and let $f_i = \lambda(e_i), i = 1, 2$ be a basis of $T_pX^t(k)$. Then Equation (5.39) is equivalent to $q(\mathbb{X}/R; e_i, f_j) = q(\mathbb{X}/R; e_j, f_i)$. Recall that t_{ij} is the image of $e_i \otimes f_j$ under the universal homomorphism $\phi \in \mathrm{Hom}_S(T_pX(k) \otimes T_pX^t(k), \widehat{\mathbb{G}}_m)$. Therefore the Serre–Tate local coordinates for principally polarized abelian surfaces are given by $t_{11}, t_{12} = t_{21}, t_{22}$.

Now we denote $t_1 := t_{11}, t_2 := t_{12} = t_{21}$ and $t_3 := t_{22}$. After changing the variables, we are now ready to state the final result:

Theorem 5.4.2. *Let S_φ be the ordinary locus. Let S_φ be the ordinary locus and x be a closed point of S_φ. Let $\hat{\mathcal{O}}_{S_\varphi,x}$ be the completion of the local ring of S_φ at x. Then the base change of $\mathcal{X}^\times_\varphi[p]/S_\varphi$ to* $\mathrm{Spec}\, \hat{\mathcal{O}}_{S_\varphi,x}$ *is isomorphic to*

$$\mathrm{Spec}\, \bar{\mathbb{F}}_p[\![t_1, t_2, t_3]\!][x_1, x_2, y_1, y_2] \Big/ \begin{pmatrix} x_1^p - P_1(y_1, y_2), x_2^p - P_2(y_1, y_2), \\ y_1^p - y_1, y_2^p - y_2, \\ (y_1^{p-1} - 1)(y_2^{p-1} - 1)\Phi_p(x_1)\Phi_p(x_2) \end{pmatrix}$$

over $\mathrm{Spec}\, \bar{\mathbb{F}}_p[\![t_1, t_2, t_3]\!]$. *Here, Φ_p denotes the cyclotomic polynomial and the polynomials $P_1, P_2 \in \mathbb{F}_p[\![t_1, t_2, t_3]\!][y_1, y_2]$ are interpolation polynomials characterized by*

$$P_1(i, j) = (1 + t_1)^i (1 + t_2)^j,$$
$$P_2(i, j) = (1 + t_2)^i (1 + t_3)^j$$

for $0 \le i, j \le p - 1$. The variables t_1, t_2, t_3 are the Serre-Tate coordinates.

Proof. From the description of points of $\hat{X}[p]$, we can write

$$\hat{X}[p] = \bigsqcup_{a,b=0,\dots,p-1} \hat{X}_{a,b}$$

where

$$\hat{X}_{a,b} = \mathrm{Spec}\, R[x, y]/(x^p - (1 + t_1)^a (1 + t_2)^b, y^p - (1 + t_2)^a (1 + t_3)^b).$$

Note that in particular, $\hat{X}_{0,0} \cong \mu_p \times \mu_p$.

We want to modify the description of $\hat{X}[p]$ to make it consistent with the forms in the previous cases. Note that the constant group scheme can be written as $\mathbb{Z}/p\mathbb{Z} \cong \operatorname{Spec} R[x]/(x^p - x)$. Using this for the indices $a, b \in 0, \ldots, p-1$, we can write $\hat{X}[p]$ as

$$\hat{X}[p] \cong \operatorname{Spec} \bar{\mathbb{F}}[\![t_1, t_2, t_3]\!][x_1, x_2, y_1, y_2] / \left(x_1^p - x_1, x_2^p - x_2, y_1^p - P_1(x_1, x_2), y_2^p - P_2(x_1, x_2)\right).$$

Here, the polynomials $P_1, P_2 \in \mathbb{F}_p[\![t_1, t_2, t_3]\!][x_1, x_2]$ are defined by

$$P_1(x_1, x_2) = \sum_{0 \le i, j \le p-1} (1 + t_{11})^i (1 + t_{21})^j \left(\prod_{k \ne i, l \ne j} \frac{(x_1 - k)(x_2 - l)}{(i - k)(j - l)} \right),$$

$$P_2(x_1, x_2) = \sum_{0 \le i, j \le p-1} (1 + t_{12})^i (1 + t_{22})^j \left(\prod_{k \ne i, l \ne j} \frac{(x_1 - k)(x_2 - l)}{(i - k)(j - l)} \right).$$

They are interpolation polynomials so that we have $P_1(i, j) = (1+t_1)^i (1+t_2)^j$ and $P_2(i, j) = (1 + t_2)^i (1 + t_3)^j$ for $0 \le i, j \le p - 1$.

For the primitive elements $\hat{X}^\times[p]$, we will use Lemma 2.3.3 again. Note that in this case, the augmentation ideal is generated by $x_1, x_2, y_1 - 1, y_2 - 1$. Note that the constant terms of $P_i(x_1, x_2)$ are equal to 0 since we have $P(0, 0) = 1$ from interpolation conditions. Denote $\Phi_p(x) := \dfrac{x^p - 1}{x - 1}$ for the cyclotomic polynomial. Therefore, using the notation in Lemma 2.3.3, the matrix M with respect to generators $x_1, x_2, y_1 - 1, y_2 - 1$ is

$$M = \begin{pmatrix} x_1^{p-1} - 1 & 0 & * & * \\ 0 & x_2^{p-1} - 1 & * & * \\ 0 & 0 & \Phi_p(y_1) & 0 \\ 0 & 0 & 0 & \Phi_p(y_2) \end{pmatrix}$$

Therefore the primitive elements $\hat{X}^\times[p] \subset \hat{X}[p]$ are defined by $(\det M) = (x_1^{p-1} - 1)(x_2^{p-1} - 1)\Phi_p(y_1)\Phi_p(y_2)$. □

5.5 Applications

In the previous sections, we use the explicit descriptions of the $\Gamma_1(p)$-cover to prove some geometric properties of the $\Gamma_1(p)$-cover over each stratum. In fact, these descriptions can also be used to show some geometric properties of the whole integral model $\mathcal{A}$ in mixed characteristics. In this section, we will use the descriptions in Section 5.4 to prove that the whole $\Gamma_1(p)$-cover in mixed characteristics is not normal. More precisely, consider the Siegel threefold $\mathcal{A} = \mathcal{A}_{2,1,N}$ in mixed characteristics and let $\mathcal{X}$ be the universal abelian surface over $\mathcal{A}$. We will prove:

Theorem 5.5.1. *The universal $\Gamma_1(p)$-cover $(\mathcal{X}[p])^\times$ over the Siegel threefold $\mathcal{A}$ in mixed characteristic is not normal.*

Proof. By Serre's criterion for normality, it is enough to prove that $(\mathcal{X}[p])^\times$ does not satisfy the condition R_1, which says $\mathcal{O}_{\bar{\mathcal{X}}^{un},x}$ is regular for any $x \in \bar{\mathcal{X}}^{un}$ with codimension ≤ 1. In fact, we will show that the local ring at the generic point of the special fiber, i.e. with respect to the ideal (p), is not regular.

Recall our notation that $\mathcal{A}$ is the Siegel threefold over $\operatorname{Spec}\mathbb{Z}_p$ and $\bar{\mathcal{A}} = \mathcal{A} \times_{\operatorname{Spec}\mathbb{Z}_p} \operatorname{Spec}\mathbb{F}_p$ is the special fiber. Let ξ be the generic point of the special fiber $\mathcal{A}$ and let $\mathcal{O}_{\mathcal{A},\xi}$ to be the local ring of ξ in $\mathcal{A}$. Let $G_\xi := \mathcal{X}[p] \times_{\mathcal{A}} \operatorname{Spec}\mathcal{O}_{\mathcal{A},\xi}$. Then $G_\xi^\times$ is the universal $\Gamma_1(p)$-cover over $\operatorname{Spec}\mathcal{O}_{\mathcal{A},\xi}$. By Serre's criterion, it suffices to prove that $\mathcal{O}_{G_\xi^\times}$ is not regular.

Let $k(\xi) := \mathcal{O}_{\mathcal{A},\xi}/p\mathcal{O}_{\mathcal{A},\xi}$ be the residue field of $\mathcal{O}_{\mathcal{A},\xi}$. Note that we also have $k(\xi) = \operatorname{Frac}(\mathcal{O}_{\bar{\mathcal{A}}})$. On the other hand, let $x \in \bar{\mathcal{A}}$ be a geometric ordinary point. Let $\mathcal{O}_{\bar{\mathcal{A}},x}$ be the local ring of $\bar{\mathcal{A}}$ at x and let $\hat{\mathcal{O}}_{\bar{\mathcal{A}},x}$ be its completion. Then we have an inclusion

$$\mathcal{O}_{\bar{\mathcal{A}}} \subset \mathcal{O}_{\bar{\mathcal{A}},x} \subset \hat{\mathcal{O}}_{\bar{\mathcal{A}},x},$$

which induces a field extension

$$k(\xi) = \operatorname{Frac}(\mathcal{O}_{\bar{\mathcal{A}}}) \subset \hat{K} := \operatorname{Frac}(\hat{\mathcal{O}}_{\bar{\mathcal{A}},x}).$$

Note that by Serre–Tate theory, we have a canonical isomorphism $\hat{\mathcal{O}}_{\bar{\mathcal{A}},x} \cong \bar{\mathbb{F}}_p[\![t_1, t_2, t_3]\!]$.

Now consider the following Cartesian diagrams

$$
\begin{array}{ccccccc}
\hat{G}_\xi^\times & \longrightarrow & \bar{G}_\xi^\times & \longrightarrow & G_\xi^\times & \longrightarrow & \mathcal{X}^\times[p] \\
\xi''' & \longmapsto & \xi'' & \longmapsto & \xi' & & \\
\downarrow & \Box & \downarrow & \Box & \downarrow \;\; \xi' \mapsto \xi & \Box & \downarrow \\
\operatorname{Spec} \hat{K} & \longrightarrow & \operatorname{Spec} k(\xi) & \longrightarrow & \operatorname{Spec} \mathcal{O}_{\mathcal{A},\xi} & \longrightarrow & \mathcal{A}
\end{array}
\tag{5.40}
$$

By construction, $\hat{G}_\xi^\times / \operatorname{Spec} \hat{K}$ is the generic fiber of the group scheme in Theorem 5.4.2, given by

$$
\hat{G}_\xi^\times \cong \operatorname{Spec} \hat{K}[x_1, x_2, y_1, y_2] \Big/ \begin{pmatrix} x_1^p - x_1, x_2^p - x_2, \\ y_1^p - P_1(x_1, x_2), y_2^p - P_2(x_1, x_2), \\ (x_1^{p-1} - 1)(x_2^{p-1} - 1)\Phi_p(y_1)\Phi_p(y_2) \end{pmatrix}. \tag{5.41}
$$

Let ξ''' be the point of $\hat{G}_\xi^\times$ in (5.41) corresponding to the maximal ideal $m_{\xi'''} := (x_1, x_2, y_1 - 1, y_2 - 1)$. One can check directly from (5.41) that $m_{\xi'''}/m_{\xi'''}^2$ is a $\hat{K}$-vector space generated by $y_1 - 1, y_2 - 1$. This proves

$$
\dim_{\hat{K}} m_{\xi'''}/m_{\xi'''}^2 = 2. \tag{5.42}
$$

Let ξ', ξ'' be the images of ξ''' as in (5.40). Then we have

$$
\begin{aligned}
\dim_{k(\xi')} m_{\xi'}/m_{\xi'}^2 &\geq \dim_{k(\xi'')} \big((p) + m_{\xi'}\big) / \big((p) + m_{\xi'}^2\big) \\
&= \dim_{k(\xi'')} m_{\xi''}/m_{\xi''}^2 \\
&= dim_{\hat{K}} m_{\xi'''}/m_{\xi'''}^2 = 2
\end{aligned}
$$

Therefore $\mathcal{O}_{G_\xi^\times}$ is not regular and we finish the proof by Serre's criterion. $\Box$

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