# WALL-CROSSING FOR TILT STABILITY

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**ABSTRACT**

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Recent techniques from Bridgeland Stability have given new, interesting results about stable vector bundles on surfaces. However, in higher dimensions the theory is substantially harder, so there has been significantly less progress in this case. In this thesis, we develop theory for a related notion—tilt stability— then apply this theory to stable vector bundles.

In the first part of this thesis, we recall and further develop the theory of tilt stability. This development culminates in a wall-crossing result for tilt stability.

In the second part of this thesis, we apply our wall-crossing result to study stable vector bundles. Our first application is a criterion for when the restriction of a slope stable bundle to an integral subvariety is still slope stable. Our second application is to the theory of Lazarsfeld-Mukai bundles. Specifically, we show the Lazarsfeld-Mukai bundle associated to a Gieseker stable bundle is slope stable, and that slope stability and slope semistability are equivalent for Lazarsfeld-Mukai bundles associated to ample line bundles.

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## KEY TO SYMBOLS

$X$ A smooth projective variety

$H$ An ample divisor on $X$

$\mathscr{E}$, $\mathscr{F}$, $\mathscr{G}$ Coherent sheaves on $X$

$D^b(X)$ The bounded derived category of coherent sheaves.

$E, F, G$ Objects in the bounded derived category

$\sigma_{\alpha,\beta}^{\text{tilt}}$ The tilt stability associated to $(\beta, \alpha) \in \mathbb{R} \times \mathbb{R}_{>0}$

$\mathrm{Coh}_H^{D+\beta H}(X)$ The heart associated to tilt stability

$Y$ A integral subscheme of $X$

$\mu_H(\cdot)$ The slope

$\overline{\Delta}_H(\cdot)$ The discriminant associated to $H$

$\mathrm{ch}_i(\cdot)$ the $i$-th Chern character

$\mathrm{c}_i(\cdot)$ the $i$-th Chern class

$\mathrm{ch}_i^D(\cdot)$ the $i$-th $D$-twisted Chern character

$\mathscr{M}$ A Lazarsfeld-Mukai sheaf

$\mathscr{H}^i$ Cohomology functors $\mathscr{H}^i : D^b(X) \to \mathrm{Coh}(X)$

$\mathcal{A}$ An abelian category

$A$, $B$, $C$ Objects in $\mathcal{A}$

$D$ An $\mathbb{R}$-divisor on $X$

$\sigma$ A very weak stability function

$A^k(X)$ The Chow ring of $X$

$G_H^D(\cdot)(t)$ The reduced $D$-twisted Hilbert polynomial

$K_0(\cdot)$ The Grothendieck group

$\mathfrak{I}(\cdot)$ The imaginary part of a complex number

$\mathfrak{R}(\cdot)$ The real part of a comple number

$\Lambda$ A finitely generated quotient of $K_0(X)$

$\mathrm{HN}(\cdot)$ The Harder-Narasimhan polygon.

$Q$ A quadratic form

$(\mathcal{T}, \mathcal{F})$ A torsion pair

$\mathrm{WStab}(X, \Lambda)$ The space of very weak stability conditions on $X$ with respect to $\Lambda$

$W$ A numerical wall in the $(H, D)$-slice

$\mathcal{T}_X$ The tangent bundle on $X$

# CHAPTER 1

# INTRODUCTION

A natural way to study a variety is to consider the collection of all vector bundles on it. For example, the Jacobian—the collection of all line bundles of degree 0—is our most important tool for studying curves. However, in higher dimensions and higher ranks, the collection of all vector bundles is not well-behaved (i.e. there is never a coarse moduli space). For this reason, we consider the smaller class of stable bundles.

Unlike vector bundles, there is a coarse moduli space of stable bundles. Furthermore, every vector bundle can be decomposed in an essentially unique way to stable bundles. For this reason, it suffices to study the moduli of stable bundles.

While there exists a moduli of stable bundles, very basic properties about this moduli are unknown. For example, even for surfaces, it is unknown exactly when the moduli of stable bundles (with fixed topological invariants) is non-empty. That is to say, it is difficult to construct stable bundles with fixed topological invariants. To illustrate, it is completely open whether there exists a rank 2 stable bundle on $\mathbb{P}^7$—projective space of dimension 7. In other words, for the most "basic" projective 7-fold, we do not know whether a certain moduli of stable bundles is non-empty.

A much easier problem is to construct semistable bundles with fixed topological invariants. Semistable bundles are essentially bundles that are in the closure of the moduli of stable bundes. It is easier to construct semistable bundles because many categorical constructions with semistable bundles are still semistable (e.g. extensions, kernels, tensor products, exterior products):

**Lemma 1.0.1** (Lemma 2.2.15)**.** *Assume $X$ is a smooth projective variety with ample class $H$. Consider a short exact sequence*

$$0 \to \mathscr{F} \to \mathscr{E} \to \mathscr{G} \to 0.$$

*If $\mathscr{F}$ and $\mathscr{G}$ are slope semistable of the same slope then $\mathscr{E}$ is also slope semistable.*

In particular, if $\mathscr{L}$ is a line bundle on $\mathbb{P}^7$ then $\mathscr{L} \oplus \mathscr{L}$ is a semistable bundle of rank 2 on $\mathbb{P}^7$.

These categorical constructions almost never extend to stable bundles. To wit, if

$$0 \to \mathscr{F} \to \mathscr{E} \to \mathscr{G} \to 0$$

is a short exact sequence with $\mathscr{F}$ and $\mathscr{G}$ stable of the same slope then $\mathscr{E}$ is never stable.

On the other hand, in the setting of Bridgeland stability many such categorical constructions do hold for stable objects. Bridgeland stability conditions are generalizations of slope stability from coherent sheaves to the bounded derived category, $D^b(X)$, due to Bridgeland [Bri07]. That is to say, Bridgeland stability conditions are notions of stability on certain abelian subcategories of the bounded derived category, called hearts, that satisfy a generalization of Harder-Narasimhan filtrations and Bogomolov's inequality.

While Bridgeland stability conditions do not have an associated Geometric Invariant Theory problem (in contrast to slope stability) they afford better deformation properties. Specifically, Bridgeland stability conditions are parameterized by a complex manifold. Moreover, for fixed object $E$, this complex manifold has a wall and chamber structure such that Bridgeland stability of $E$ only changes when crossing a wall. Each of these walls corresponds to a short exact sequence

$$0 \to F \to E \to G \to 0$$

in an abelian subcategory of $D^b(X)$.

In practice, we do not work with this whole complex manifold; instead, we consider certain 2-dimensional slices parameterized by $\mathbb{R} \times \mathbb{R}_{>0}$. In other words, for each $(\beta, \alpha) \in \mathbb{R} \times \mathbb{R}_{>0}$ there exists a Bridgeland stability condition $\sigma^{\text{tilt}}_{\alpha,\beta}$ on a heart $\text{Coh}_H^{D+\beta H}(X)$. Furthermore, for fixed object, the wall and chamber decomposition of these slice is very constrained—the walls must either be a unique vertical line or nested semicircles whose center lies on the horizontal axis.

In these slices, categorical constructions can often be deformed to give stable objects. One such construction is due to Bayer and Macrì which we call a wall-crossing result.

**Theorem 1.0.2** ([BM11, Lemma 5.9])**.** *Assume $X$ is a smooth projective surface. Let*

$$F \to E \to G \to F[1]$$

*be a distinguished triangle such that $F, G$ are $\sigma^{\mathrm{tilt}}_{\alpha,\beta}$-stable of the same slope for some $(\beta, \alpha) \in \mathbb{R} \times \mathbb{R}_{>0}$. If $E \neq F \oplus G$ then there exists an infinitesimal deformation $(\beta', \alpha')$ of $(\beta, \alpha)$ such that $E$ is $\sigma^{\mathrm{tilt}}_{\alpha,\beta}$-stable.*

In other words, an extension of Bridgeland stable objects of the same slope is stable up to infinitesimal deformation.

Many exciting results have used this construction to generalize previously known results about stable bundles. For example, [Bay18] generalizes Lazarsfeld's Brill-Noether theorem on surfaces while [Kop20] generalizes Bogomolov's restriction theorem. However, most such results have been only surfaces. This is, in part, because it is very difficult to construct Bridgeland stable conditions on higher dimensional varieties. Furthermore, even when Bridgeland stability conditions exist in higher dimensions, the wall and chamber structure is much more complicated.

To circumvent these problems in higher dimensions, we consider very weak stability conditions. Very weak stability conditions are weaker than Bridgeland stability conditions but are known to exist in higher dimensions. However, very weak stability conditions lose many of the properties we are accustomed to from Bridgeland stability (e.g. existence of Jordan-Hölder filtrations, uniqueness of Harder-Narasimhan filtrations, and Schur's Lemma all fail). Like Bridgeland stability, there are 2-dimensional families of very weak stability conditions parameterized by $\mathbb{R} \times \mathbb{R}_{>0}$. Furthermore, the wall and chamber structure is constrained like in the case of Bridgeland stability conditions.

Since very weak conditions lose many of the properties of Bridgeland stability conditions, Bayer and Macrì's wall-crossing argument does not hold. There have been some been a few wall-crossing results for very weak stability, but they do not fully generalize their result. For example, [Sch20, Theorem 6.1.4] gives a wall-crossing result for $\mathbb{P}^3$ and uses it to describe the two components of the Hilbert scheme of twisted cubics in $\mathbb{P}^3$ in more detail. Similarly, Feyzbakhsh [Fey21, Proposition 4.2 and Corollary 4.3] gives a wall-crossing result for torsion objects and uses it to generalize Bogomolov's restriction theorem to higher dimensions. These applications suggest that a general wall-crossing result will be a useful tool for moduli problems in higher dimensions. In this thesis we give such a result:

**Theorem 1.0.3** (Theorem 3.6.1)**.** *Assume $X$ is a smooth projective variety. Let*

$$F \to E \to G^{\oplus r} \to F[1]$$

*be a distinguished triangle such that $F$ and $G$ are weakly $\sigma^{\mathrm{tilt}}_{\alpha,\beta}$-stable of the same slope. If $G$ has good quotients and $\mathrm{Hom}_{D^b(X)}(G,E)=0$ then there exists an infinitesimal deformation $(\beta',\alpha')$ of $(\beta,\alpha)$ such that $E$ is weakly $\sigma^{\mathrm{tilt}}_{\alpha',\beta'}$-stable.*

Weak $\sigma^{\mathrm{tilt}}_{\alpha,\beta}$-stability is a slight variant of stability that only occurs for very weak stability conditions. Having good quotients is a notion that is vacuous for Bridgeland stability conditions and is equivalent to $\mathscr{H}^{-1}(G)=0$ in our case.

We give some applications of our wall-crossing result to slope stable sheaves. The first is a generalization of Bogomolov's restriction theorem to any integral subvariety. To explain, if $\mathscr{E}$ is slope stable, then $\mathscr{E}|_H$ may not be slope stable. However, Mehta and Ramanathan showed that $\mathscr{E}|_{aH}$ is slope stable for $a \gg 0$ [HL10, Theorem 7.2.8]. In the case of surfaces in characteristic 0, Bogomolov gave an explicit lower bound on $a$ [Bog93].

In a different thread, Kopper generalized Bogomolov's theorem to restrictions of integral curves (rather than just hyperplanes) [Kop20, Theorem 3.3]. We generalize Kopper's result to higher dimensions.

**Theorem 1.0.4** (Theorem 4.1.2)**.** *Let $\mathscr{E}$ be a reflexive, slope stable sheaf on $X$. Consider an integral subvariety $\iota : Y \to X$. If the following bounds are satisfied*

- $\mu_H(\mathscr{E}) - \dfrac{\overline{\Delta}_H(\mathscr{E})}{2} - \dfrac{1}{2\operatorname{rank}(\mathscr{E})^2} > \dfrac{H^{n-2}\cdot \operatorname{ch}_1(\mathscr{E})\cdot Y}{\operatorname{rank}(\mathscr{E})H^{n-1}\cdot Y} - \dfrac{H^{n-2}\cdot Y^2}{2H^{n-1}\cdot Y}$
- $\mu_H(\mathscr{E}(-Y)) + \dfrac{\overline{\Delta}_H(\mathscr{E}(-Y))}{2} + \dfrac{1}{2\operatorname{rank}(\mathscr{E})^2} < \dfrac{H^{n-2}\cdot \operatorname{ch}_1(\mathscr{E})\cdot Y}{\operatorname{rank}(\mathscr{E})H^{n-1}\cdot Y} - \dfrac{H^{n-2}\cdot Y^2}{2H^{n-1}\cdot Y}$

*then $\mathscr{E}|_Y$ is slope stable.*

We also give two applications to Lazarsfeld-Mukai bundles. If we have a surjection $\mathscr{E} \to \mathscr{F}$ between sufficiently positive stable bundles then it is natural to wonder whether the kernel is stable. Lazarsfeld-Mukai bundles are the most basic instance of this scenario.

If $\mathscr{E}$ is globally generated then there exists a natural surjection $H^0(\mathscr{E}) \otimes \mathscr{O}_X \to \mathscr{E} \to 0$ with kernel $\mathscr{M}$ which is called the Lazarsfeld-Mukai bundle assocaited to $\mathscr{E}$. On curves, stability of Lazarsfeld-Mukai bundles is well understood. In higher dimensions, much less is known. We list a some particularly noteworthy results:

- On $\mathbb{P}^n$, the Lazarsfeld-Mukai bundle associated to $\mathscr{O}_{\mathbb{P}^n(d)}$ for $d \geq 1$ is slope semistable [Fle84, Corollary 2.2].

- Assume $X$ is a smooth curve of genus $g$. Let $\mathscr{E}$ be a slope semistable bundle on $X$ with $\mu_H(\mathscr{E}) \geq 2g+1$. The Lazarsfeld-Mukai bundle associated to $\mathscr{E}$ is slope stable [But94, Theorem 1.2].

- Assume $X$ is a $K3$ surface. Let $\mathscr{L}$ be a globally generated, ample line bundle on $X$ The Lazarsfeld-Mukai bundle associated to $\mathscr{L}$ is slope stable [Cam12, Theorem 1]. If $X$ is an abelian surface and $\mathscr{L}$ additionally satisfies $\mathscr{L}^2 \geq 14$ then the same conclusion holds [Cam12, Theorem 2].

- Assume $X$ be a smooth projective surface. Let $\mathscr{L}$ be a globally generated line bundle on $X$. If $\deg_H(\mathscr{L}) \gg 0$ then the Lazarsfeld-Mukai bundle associated to $\mathscr{L}$ is slope stable [ELM13, Theorem A].

- Assume $X$ is a smooth projective variety of Picard rank 1. Let $\mathscr{L}$ be an ample line bundle on $X$. If $\deg_H(\mathscr{L}) \gg 0$ then the associated Lazarsfeld-Mukai bundle is slope stable [ELM13, Proposition C].

Our first result is for Del Pezzo surfaces:

**Theorem 1.0.5** (Theorem 4.2.4). *Assume $X$ is a smooth Del Pezzo surface over an algebraically closed field of arbitrary characteristic. For ease of notation, let $H = -K_X$ which is ample by definition. Consider a globally generated, slope stable, torsion-free sheaf $\mathscr{E}$ with associated Lazarsfeld-Mukai sheaf $\mathscr{M}$. If the following bounds are satisfied:*

- $0 < \deg_H(\mathscr{E}) \leq K_X^2(h^0(\mathscr{E}) - \mathrm{rank}(\mathscr{E}))$,
- $\mathrm{ch}_2(\mathscr{E}) > 0$,
- $2\dfrac{\mathrm{ch}_2(\mathscr{E})}{\deg_H(\mathscr{E})} + \dfrac{1}{\mathrm{rank}(\mathscr{E})^2} \geq \overline{\Delta}_H(\mathscr{E})$

*then $\mathscr{M}$ is $\mu_H$-stable.*

In fact, we show that this result holds with a weaker notion than slope stability. This is the first such result for higher rank sheaves on surfaces.

Our second result shows slope stability and semistability are equivalent for Lazarsfeld-Mukai bundles associated to ample bundles:

**Theorem 1.0.6** (Theorem 4.2.8). *Assume $X$ is a smooth projectiive variety equipped with ample divisor $H$. Let $\mathscr{M}$ be the Lazarsfeld-Mukai bundles associated to $\mathscr{O}_X(d)$ for $d \geq 1$. $\mathscr{M}$ is slope stable if and only if $\mathscr{M}$ is slope semistable.*

## 1.1 Notation and Assumptions

Assume $X$ is a regular projective variety over an algebraically closed field $k$. Unless stated otherwise, we assume $\dim(X) = n \geq 2$ and $k$ is of characteristic 0. We denote an integral ample divisor on $X$ by $H$.

We denote the bounded derived category of $\mathrm{Coh}(X)$ by $D^b(X)$ whose objects will be written as $E, F, G$. We write $\mathscr{H}^i$ for the cohomology functors $\mathscr{H}^i : D^b(X) \to \mathrm{Coh}(X)$. Coherent sheaves will be written as $\mathscr{E}$, $\mathscr{F}$, $\mathscr{G}$. Objects of a general abelian category $\mathcal{A}$ will be writen as $A, B, C$.

If $\mathscr{E} \in \mathrm{Coh}(X)$ then we will use the notation $\mathrm{codim}(\mathscr{E})$ to denote $\mathrm{codim}(\mathrm{Supp}(\mathscr{E}), X)$. If $E \in D^b(X)$, then we will write $\mathrm{codim}(E)$ to denote

$$\mathrm{codim}\left(\bigcup_{i\in\mathbb{Z}} \mathrm{Supp}(\mathscr{H}^i(E)), X\right).$$

# CHAPTER 2

# STABILITY

In this chapter, we provide background on stability for $\mathrm{Coh}(X)$, abelian categories, and triangulated categories.

The chapter is as follows:

§2.1 We introduce three notions of stability for coherent sheaves; $\mu_H$-stability, $(H, D)$-Gieseker stability, and $(H, D)$-twisted stability. All the results and definitions of this section are standard.

§2.2 We introduce very weak stability functions. A very weak stability function is essentially a generalization of $\mu_H$-stability to any abelian category (not just $\mathrm{Coh}(X)$). Most basic results about $\mu_H$-stability have equivalent results for a very weak stability functions. Unfortunately, this means that some of the "failures" of $\mu_H$-stability have corresponding "failures" for very weak stability functions. For example, Schur's lemma is false and Harder-Narasimhan filtrations are not unique.

To circumvent these failures, we introduce two classes of stable objects: $\sigma$-stable objects and weakly $\sigma$-stable objects. Weakly $\sigma$-stable objects are the natural generalization of $\mu_H$-stable sheaves to a very weak stability function. $\sigma$-stability is a stronger notion of stability that is a generalization of simple objects in the full subcategory of $\mathrm{Coh}(X)$ generated by $\mu_H$-semistable objects of fixed slope. Both of these notions are standard, but they are both called $\sigma$-stability in the literature. $\sigma$-stability is a partial fix for the failures mentioned above. Namely, Schur's lemma holds as expected and Harder-Narasimhan filtrations are unique for $\sigma$-stable objects. We compare $\sigma$-stability and weak $\sigma$-stability in Lemma 2.2.11.

We also define objects having good quotients and $\sigma$-pure objects. Having good quotients exactly determines when weakly $\sigma$-stability and $\sigma$-stability agree. $\sigma$-purity is a generalization of a pure coherent sheaf.

§2.3 We discuss additional properties of very weak stability functions including the Harder-Narasimhan property, the support property, and the existence/non-existence of Jordan-Hölder filtrations.

For a very weak stability function, a $\sigma$-semistable object may not contain a $\sigma$-stable subobject. Therefore, $\sigma$-semistable objects may not have Jordan-Hölder filtrations. Fortunately, we can show that under relatively mild assumptions all objects of finite slope have what we call a "weak Jordan-Hölder filtration" which is a sufficient replacement for usual Jordan-Hölder filtrations.

§2.4 We discuss very weak stability conditions on a triangulated category. The results and definitions of this subsection are standard.

## 2.1 Stability for Coherent Sheaves

In this section we introduce three different notions of a stable coherent sheaf. Two of these definitions are well known - $\mu_H$-stability (Definition 2.1.4) and $(H, D)$-Gieseker stability (Definition 2.1.6). The third notion is a newer notion due to Bridgeland called $(H, D)$-twisted stability (Definition 2.1.7). All three of these notions satisfy the seesaw inequality (Lemma 2.1.9). The strength of these three definitions is compared in Lemma 2.1.11.

We end this section by stating the weak Bogomolov inequality (Lemma 2.1.15) and some cases where it holds in positive characteristic (Remark 2.1.16).

**Definition 2.1.1.** *Let $\mathscr{E} \in \mathrm{Coh}(X)$, $E \in D^b(X)$, and $D$ be a $\mathbb{R}$-divisor.*

1. *We define the $D$-twisted Chern character*

$$\mathrm{ch}^D(\mathscr{E}) = \mathrm{ch}(\mathscr{E}) \cdot \exp(-D)$$

*where $\mathrm{ch}(\mathscr{E})$ denotes the usual Chern character and $\exp(-D)$ is formally a power series in $-D$ both viewed in the Chow ring with real coefficients $A^\bullet(X) \otimes \mathbb{R}$.*

*The first few terms of the $D$-twisted Chern character written in terms of Chern classes are*

$$\begin{aligned}
\mathrm{ch}^D(\mathscr{E}) &= \mathrm{ch}(\mathscr{E}) \cdot \exp(D) \\
&= \left(\mathrm{rank}(\mathscr{E}) + c_1(\mathscr{E}) + \frac{1}{2}(c_1(\mathscr{E})^2 - 2c_2(\mathscr{E})) + \cdots\right) \\
&\quad \cdot \left(1 - D + \frac{D^2}{2} - \frac{D^3}{6} + \cdots\right) \\
&= \mathrm{rank}(\mathscr{E}) + (c_1(\mathscr{E}) - \mathrm{rank}(\mathscr{E})D) \\
&\quad + \left(\frac{1}{2}(c_1(\mathscr{E})^2 - 2c_2(\mathscr{E})) - c_1(\mathscr{E}) \cdot D + \mathrm{rank}(\mathscr{E})\frac{D^2}{2}\right) + \cdots
\end{aligned}$$

*where $c_i(\mathscr{E})$ is the usual $i$-th Chern class of $\mathscr{E}$.*

2. *We denote the $k$-th coefficient of $\mathrm{ch}^D(\mathscr{E})$ by $\mathrm{ch}^D_k(\mathscr{E})$ which we view as an element of $A^k(X) \otimes \mathbb{R}$.*

3. *We define*

$$\mathrm{ch}^D_k(E) = \mathrm{ch}^D_k(\cdots \to E^{-1} \to E^0 \to E^1 \to \cdots) = \sum_{i \in \mathbb{Z}} (-1)\,\mathrm{ch}^D_k(\mathscr{H}^i(E)).$$

4. *If $D = 0$ we write $\mathrm{ch}^D_k(E) = \mathrm{ch}_k(E)$ which is the $k$-th coefficient of the usual Chern character $\mathrm{ch}(E)$.*

5. *For a nonnegative integer $k$ and $E \in D^b(X)$, we set*

$$H^{n-k} \cdot \mathrm{ch}^D_{\leq k}(E) = (H^n \cdot \mathrm{rank}^D(E), H^{n-1} \cdot \mathrm{ch}^D_1(E), \ldots, H^{n-k} \cdot \mathrm{ch}^D_k(E)) \in \mathbb{R}^{\oplus k}.$$

6. *For ease of notation, we set $\deg^D_H(E) = H^{n-1} \cdot \mathrm{ch}^D_1(E)$.*

Since $X$ is an integral smooth variety over an algebraically closed field, by the Hirzebruch-Riemann-Roch theorem, $\deg^D_H(E)$ and $\mathrm{rank}(E)$ agree with all usual definitions of the degree and rank. Furthermore, for the same reason, there is a group isomorphism $A^n(X) = \mathbb{Z}$ which we can make canonical by choosing the isomorphism where $H^n$ is positive. Therefore, we will identify objects in $A^n(X)$ with the corresponding integer.

**Example 2.1.2.** *If $E \in D^b(X)$ and $\mathrm{c}_k(E)$ denotes the $k$th Chern class (which is defined on $D^b(X)$ inductively using the Chern character) then*

- $\mathrm{ch}_0^D(E) = \mathrm{rank}(E)$.
- $\mathrm{ch}_1^D(E) = \mathrm{ch}_1(E) - D \cdot \mathrm{rank}(E) = \mathrm{c}_1(E) - D\,\mathrm{rank}(E)$.
- $\mathrm{ch}_2^D(E) = \mathrm{ch}_2(E) - D \cdot \mathrm{ch}_1(E) + \frac{D^2}{2} \cdot \mathrm{rank}(E)$
  $= \frac{1}{2}\left(\mathrm{c}_1(E)^2 - 2c_2(E)\right) - D \cdot \mathrm{c}_1(E) + \frac{D^2}{2}\,\mathrm{rank}(E)$.

The vanishing of high codimension Chern characters is controlled by the support of the coherent sheaf. This lemma is well known, but the author could not find it written in the following level of generality. However, by a series of reductions, the following result can be reduced to a result about vector bundles whose support is smooth, irreducible, and quasi-projective in which case we can apply [Ful98, Example 18.2.1 with a similar argument argument as Example 15.3.1]. These reductions are technically difficult, so we give a different proof.

**Lemma 2.1.3.** *Assume $\mathscr{E} \in \mathrm{Coh}(X)$. Let $\{Y_i\}_{i=1}^m$ be the top dimensional irreducible components of $\mathrm{Supp}(\mathscr{E})$. If $\mathrm{codim}(\mathscr{E}) = d$ then*

$$\mathrm{ch}_k^D(\mathscr{E}) = \begin{cases} 0 : & 0 \le k \le d-1 \\ \sum_{i=1}^m \mathrm{rank}(\mathscr{E}|_{Y_i})[Y_i] : & k = d \end{cases}.$$

*In particular, $H^{n-d} \cdot \mathrm{ch}_{\le d}^D(\mathscr{E}) = (0, 0, \ldots, +)$ if and only if $\mathscr{E}$ is supported in codimension $d$.*

*Proof.* Define $Y = \mathrm{Supp}(\mathscr{E})$ with the annihilator scheme structure (i.e. $\mathscr{O}_Y = \mathscr{K}er(\mathscr{O}_X \to \mathscr{E}nd(\mathscr{E})))$. For ease of notation, set $U = X \setminus Y$ with induced subscheme structure and $\iota : U \to X$ the associated open immersion. Since $X$ is regular, there is a finite locally free resolution in $\mathrm{Coh}(X)$:

$$0 \to \mathscr{F}_n \to \mathscr{F}_{n-1} \to \cdots \to \mathscr{F}_1 \to \mathscr{E} \to 0.$$

By construction, $\iota^*\mathscr{E} = 0$, so we have an exact sequence of locally free sheaves in $\mathrm{Coh}(U)$:

$$0 \to \iota^*\mathscr{F}_n \to \iota^*\mathscr{F}_{n-1} \to \cdots \to \iota^*\mathscr{F}_1 \to 0.$$

Since $\iota : U \to X$ is an open immersion (a fortiori flat), $\iota^*$ is exact, so, by additivity of the Chern character,

$$0 = \sum_{k=1}^{n} (-1)^k \operatorname{ch}(\iota^*\mathscr{F}_k) = \iota^* \sum_{k=1}^{n} (-1)^k \operatorname{ch}(\mathscr{F}_k) = -\iota^* \operatorname{ch}(\mathscr{E}).$$

In other words, $\iota^* \operatorname{ch}_k(\mathscr{E}) = 0$ for all $k$.

By excision, for all $k$ we have the following exact sequence of groups:

$$\mathrm{A}^k(Y) \otimes \mathbb{Q} \to \mathrm{A}^k(X) \otimes \mathbb{Q} \xrightarrow{\iota^*} \mathrm{A}^k(U) \otimes \mathbb{Q} \to 0.$$

Since $\iota^* \operatorname{ch}_k(\mathscr{E}) = 0$, by exactness, $\operatorname{ch}_k(\mathscr{E})$ is in the image of the inclusion $\mathrm{A}^k(Y) \otimes \mathbb{Q} \to \mathrm{A}^k(X) \otimes \mathbb{Q}$.

If $0 \le k \le d - 1$, since $\operatorname{codim}(Y, X) = d$, $\mathrm{A}^k(Y) = 0$, so $\operatorname{ch}_k(\mathscr{E}) = 0$, as needed.

If $k = d$ then $\mathrm{A}^d(Y)$ is the free abelian group generated by the top dimensional irreducible components, $\{Y_i\}_{i=1}^m$, of $Y$. Since $\operatorname{ch}_d(\mathscr{E})$ is in the image of $\mathrm{A}^d(Y) \otimes \mathbb{Q}$, $\operatorname{ch}_d(\mathscr{E}) = \sum_{i=1}^m a_i[Y_i]$ for rationals $a_i$. We claim $a_i = \operatorname{rank}(\mathscr{E}|_{Y_i})$. Let $V_{i_0}$ be the largest dense open subscheme of $Y_{i_0}$ such that $\mathscr{E}|_{V_{i_0}}$ is free (which exists by generic flatness). We have the following exact sequence

$$A^d(Y_{i_0} \setminus V_{i_0}) \to A^d(Y_{i_0}) \to A^d(V_{i_0}) \to 0.$$

Since $\dim(Y_{i_0} \setminus V_{i_0}) \le \dim(Y_{i_0})$, it follows that $A^d(Y_{i_0} \setminus V_{i_0}) = 0$. In other words, restriction gives an isomorphism $A^d(Y_{i_0}) \to A^d(V_{i_0})$. Thus, without loss of generality, we may assume $\mathscr{E}$ is free along $Y_{i_0}$.

Now, consider the closed immersion $j : Y_{i_0} \to X$ which induces a morphism $j^* : A^d(X) \to A^0(Y_{i_0})$. Since $\mathscr{E}$ is locally free along $Y_{i_0}$, we find that

$$\operatorname{rank}(\mathscr{E}|_{Y_{i_0}}) = \operatorname{rank}(\mathscr{E}|_{Y_{i_0}}) = j^* \operatorname{ch}_d(\mathscr{E}) = j^* \sum_{i=1}^m a_i[Y_i] = a_{i_0}[Y_{i_0}].$$

It follows that

$$\mathrm{ch}_d(\mathscr{E}) = \sum_{i=1}^{m} \mathrm{rank}(\mathscr{E}|_{Y_i})[Y_i],$$

as desired.

Last, since lower Chern characters vanish, by definition, $\mathrm{ch}_k^D(\mathscr{E}) = \mathrm{ch}_k(\mathscr{E})$ for all $0 \leq k \leq d$ and $\mathbb{R}$-divisors $D$. $\square$

We recall the definition of $\mu_H$-stability:

**Definition 2.1.4.** *Assume $\mathscr{E} \in \mathrm{Coh}(X)$, $E \in D^b(X)$, and $D$ is a $\mathbb{R}$-divisor.*

1. *We define*

$$\mu_H^D(E) = \begin{cases} \frac{\deg_H^D(E)}{\mathrm{rank}(E)} : & \mathrm{rank}(E) \neq 0 \\ +\infty : & \mathrm{rank}(E) = 0 \end{cases}$$

   *which we call the $(H, D)$-twisted slope of $E$.*

2. *For ease of notation, we also set $\mu_H(E) = \mu_H^0(E) = \frac{\deg_H(E)}{\mathrm{rank}(E)}$ which we call the usual slope of $E$.*

3. *We say that $\mathscr{E} \in \mathrm{Coh}(X)$ is $\mu_H$-(semi)stable if every proper nonzero subsheaf $0 \to \mathscr{F} \to \mathscr{E}$ satisfies $\mu_H^D(\mathscr{F}) \underset{(}{\leq}{}_{)} \mu_H^D(\mathscr{E}/\mathscr{F})$.*

   *The notation $\underset{(}{\leq}{}_{)}$ is short hand for "$<$ (resp. $\leq$)." Formally, we say that $+\infty = +\infty$ and any real number is strictly less than $+\infty$. We maintain this convention throughout this write-up.*

Our definition of $\mu_H$-stability differs slightly from the standard definition (e.g. [HL10, Definition 1.2.12]). If $\mathscr{E}$ is not torsion-free then Definition 2.1.4 is weaker than this standard notion. If $\mathscr{E}$ is torsion-free then our notion agrees with the standard notion (see Lemma 2.2.11 in view of Example 2.2.4).

We use Definition 2.1.4 because it allows us to compare stability of all coherent sheaves using the same slope function - rather than using a different slope function for torsion sheaves.

**Remark 2.1.5.** *We omit the divisor $D$ from the notation of $\mu_H$-stability because $D$ does not affect stability. Specifically, if $\mathscr{E}$ is supported everywhere, we find*

$$\begin{aligned}\mu_H^D(\mathscr{E}) &= \frac{H^{n-1}\cdot \mathrm{ch}_1^D(\mathscr{E})}{\mathrm{rank}(\mathscr{E})} \\ &= \frac{H^{n-1}\cdot \mathrm{ch}_1(\mathscr{E}) - H^{n-1}\cdot D\,\mathrm{rank}(\mathscr{E})}{\mathrm{rank}(\mathscr{E})} \\ &= \mu_H(\mathscr{E}) - H^{n-1}\cdot D\end{aligned}$$

*Note that second summand is independent of the sheaf $\mathscr{E}$. On the other hand, if $\mathscr{E}$ is torsion, then $\mu_H^D(\mathscr{E}) = +\infty = \mu_H(\mathscr{E})$.*

*Therefore, we find that $\mu_H^D(\mathscr{F}) \underset{(}{\le}{}_{)} \mu_H^D(\mathscr{E})$ if and only if $\mu_H(\mathscr{F}) \underset{(}{\le}{}_{)} \mu_H(\mathscr{E})$.*

$\mu_H$-stability is also called slope stability, Mumford stability, and Mumford-Takemoto stability.

Out of all notions of stability on $\mathrm{Coh}(X)$, $\mu_H$-stability is the best behaved with respect to standard sheaf operations (e.g. twisting by line bundle, pullbacks, restriction to subvarieties, symmetric products, etc.).

A weaker notion than $\mu_H$-stabiliy is Gieseker stability. In fact, we discuss a slight generalization of Gieseker stability due to Matsuki and Wentworth [MW97, Definition 3.2] called $(H,D)$-Gieseker stability.

**Definition 2.1.6.** *Assume $\mathscr{E}$ is torsion-free and $D$ is a $\mathbb{R}$-divisor.*

1. *We define the reduced twisted Hilbert polynomial of $\mathscr{E}$ to be*

$$G_H^D(\mathscr{E})(t) = \begin{cases} \frac{\chi(\mathscr{E}\otimes\mathscr{O}_X(tH+D))}{\mathrm{rank}(\mathscr{E})} : & \mathrm{rank}(\mathscr{E}) \neq 0 \\ +\infty : & \mathrm{rank}(\mathscr{E}) = 0 \end{cases}$$

   *where the Euler characteristic is defined formally as a polynomial in $t$ via Riemann-Roch (see Remark 2.1.10 for the first few terms).*

   *Note that $\mathscr{O}_X(tH+D)$ is an abuse of notation. Namely, since $D$ is a $\mathbb{R}$-divisor, $\mathscr{O}_X(tH+D)$ may not be a line bundle. However, we can still define $\chi(\mathscr{E}\otimes\mathscr{O}_X(tH+D))$ formally via the Grothendieck-Riemann-Roch theorem.*

2. *We say $\mathscr{E}$ is $(H,D)$-Gieseker (semi)stable if every nonzero proper subsheaf $\mathscr{F} \subseteq \mathscr{E}$ satisfies $G_H^D(\mathscr{F})(t) \underset{(}{\leq}_{)} G_H^D(\mathscr{E})(t)$ for all $t \gg 0$.*

3. *We say $\mathscr{E}$ is $H$-Gieseker (semi)stable if it is $(H,0)$-Gieseker (semi)stable.*

Unlike $\mu_H$-stability, $H$-Gieseker stability is not invariant under twists by a line bundle. $(H,D)$-Gieseker stability partially accounts for this failure. Namely, for an integral divisor $D$, $\mathscr{E}(D)$ is $H$-Gieseker (semi)stable if and only if $\mathscr{E}$ is $(H,D)$-Gieseker (semi)stable. If $\operatorname{Pic}(X)=1$ then $(H,D)$-Gieseker (semi)stability and $H$-Gieseker stability are equivalent.

Our last notion of stability on $\operatorname{Coh}(X)$ is a generalization of $(H,D)$-Gieseker stability due to Bridgeland ([Bri08, Definition 14.1]) which is called $(H,D)$-twisted stability. In short, twisted stability is weaker than $\mu_H$-stability but stronger than Gieseker stability (Lemma 2.1.11). On surfaces, twisted (semi)stability and Gieseker (semi)stability are equivalent Lemma 2.1.11.

Notation varies for $(H,D)$-twisted stability. Bridgeland originally called the notion twisted stable with respect to the pair $(D,H)$. Some authors call this definition polynomial stability because it arises as a polynomial stability condition in the sense of [Bay09]. Others call it limit stability because it naturally arises in the large volume limit (see Lemma 3.2.3).

**Definition 2.1.7.** *Assume $\mathscr{E} \in \operatorname{Coh}(X)$ and $E \in D^b(X)$.*

1. *Following* [Bri08, Definition 14.1] *we define*

$$\nu_H^D(E) = \begin{cases} \frac{H^{n-2}\cdot \operatorname{ch}_2^D(E)}{\operatorname{rank}(E)} : & \operatorname{rank}(E) \neq 0 \\ +\infty : & \operatorname{rank}(E) = 0 \end{cases}.$$

2. *We say $\mathscr{E} \in \operatorname{Coh}(X)$ is $(H,D)$-twisted (semi)stable if every proper nonzero subsheaf $0 \to \mathscr{F} \to \mathscr{E}$ satisfies either*

   - $\mu_H^D(\mathscr{F}) < \mu_H^D(\mathscr{E}/\mathscr{F})$ *or*

- $\mu_H^D(\mathscr{F}) = \mu_H^D(\mathscr{E}/\mathscr{F})$ *with* $\nu_H^D(\mathscr{F}) \underset{(}{\leq}{}_{)} \nu_H^D(\mathscr{E}/\mathscr{F})$.

The functions $\mu_H^D$, $\nu_H^D$, and $G_H^D(t)$ all satisfy Rudakov's seesaw inequality ([Rud97, Definition 1.1]): In a short exact sequence of torsion-free coherent sheaves then the slopes must be monontically ordered with respect to the sequence.

In the next subsection we work with a general abelian category, rather than just coherent sheaves, so we will prove the seesaw inequality in that generality. To this end, we recall the Grothendieck group of an abelian category.

**Definition 2.1.8.** *Assume $\mathcal{A}$ is an abelian category. We define the Grothendieck group of $\mathcal{A}$, denoted $K_0(\mathcal{A})$, to be the free abelian generated by objects in $\mathcal{A}$ with relations $[A] = [B] + [C]$ whenever there is a short exact sequence $0 \to B \to A \to C \to 0$.*

*We will write $K_0(X) = K_0(\mathrm{Coh}(X))$.*

One can define the Grothendieck group of a triangulated category similarly (Definition 2.4.3).

We now state and prove Rudakov's seesaw inequality following [Rud97, Lemma 3.2].

**Lemma 2.1.9** (Seesaw Inequality)**.** *Assume $\mathcal{A}$ is an abelian category and $d, r : K_0(X) \to \mathbb{R}$ are group homomorphisms. For ease of notation, for each $A \in \mathcal{A}$ set $\mu(A) = d(A)/r(A)$. If $0 \to B \to A \to C \to 0$ is a short exact sequence in $\mathcal{A}$ such that $r(A), r(B), r(C) \neq 0$ then one of the following inequalities must hold:*

1. $\mu(B) < \mu(A) < \mu(C)$,
2. $\mu(B) > \mu(A) > \mu(C)$, *or*
3. $\mu(B) = \mu(A) = \mu(C)$.

*In particular, if $0 \to \mathscr{F} \to \mathscr{E} \to \mathscr{G} \to 0$ is a short exact sequence of coherent sheaves supported on $X$ then one of the above inequalities must hold for $\mu_H^D(\cdot)$, $\nu_H^D(\cdot)$, and $G_H^D(\cdot)(t)$ for $t \gg 0$.*

*Proof.* Consider a short exact sequence $0 \to B \to A \to C \to 0$ in $\mathcal{A}$. Note that

$$\mu(A) - \mu(B) = \frac{1}{r(A)r(B)} \det \begin{bmatrix} r(B) & d(B) \\ r(A) & d(A) \end{bmatrix},$$

so $\mu(A) > \mu(B)$ (respectively $=$, $<$) if and only if the determinant of the above matrix is positive (respectively zero, negative). Since $r, d : K_0(\mathcal{A}) \to \mathbb{R}$ are group homomorphisms (i.e. $r$ and $d$ are additive in short exact sequences), we can rewrite this determinant as

$$\begin{aligned} \det \begin{bmatrix} r(B) & d(B) \\ r(A) & d(A) \end{bmatrix} &= \det \begin{bmatrix} r(B) & d(B) \\ r(B)+r(C) & d(B)+d(C) \end{bmatrix} \\ &= \det \begin{bmatrix} r(B) & d(B) \\ r(C) & d(C) \end{bmatrix} \\ &= \det \begin{bmatrix} r(B)+r(C) & d(B)+d(C) \\ r(C) & d(C) \end{bmatrix} \\ &= \det \begin{bmatrix} r(A) & d(A) \\ r(C) & d(C) \end{bmatrix}. \end{aligned}$$

Therefore, we find that $\mu(A) > \mu(B)$ (respectively $=$, $<$) if and only if $\mu(C) > \mu(A)$ (respectively $=$, $<$), as needed. $\square$

If we work in the generality of Lemma 2.1.9 and allow $r(\cdot) = 0$ then we have no control over the ordering of the slopes. However, by restricting the possible values of $r$ and $d$ in a natural way (e.g. generalizing Lemma 2.1.3), than we can obtain a version of the seesaw inequality that holds for objects with $r(A) = 0$ (see Lemma 2.2.6).

On a different note, the first few terms of reduced twisted Hilbert polynomial can be written solely in terms of $\mu_H^D$, $\nu_H$, and invariants of $X$. This will allow us to compare $\mu_H$-stability, $(H, D)$-Gieseker stability, and $(H, D)$-twisted stability.

**Remark 2.1.10.** *If $\mathscr{E}$ is a coherent sheaf that is supported everywhere then the first few terms of $G_H^D(\mathscr{E})$ can be written solely in terms of $\mu_H^{D+K_X/2}(\mathscr{E})$, $\nu_H^{D+K_X/2}(\mathscr{E})$ and invariants* *of $X$. Specifically, by the Hirzebruch-Riemann-Roch theorem*

$$\begin{aligned}
G_H^D(\mathscr{E})(t) &= \frac{1}{\operatorname{rank}(\mathscr{E})}\int_X \operatorname{ch}(\mathscr{E}(tH+D))\cdot \operatorname{td}(\mathscr{T}_X)\\
&= \frac{1}{\operatorname{rank}(\mathscr{E})}\int_X \left(\sum_{i=0}^{n}\operatorname{ch}_i^D(\mathscr{E})\right)\cdot\left(\sum_{i=0}^{n}\frac{(tH)^i}{i!}\right)\cdot\left(1+\frac{c_1}{2}+\frac{c_1^2+c_2}{12}+\frac{c_1c_2}{24}+\cdots\right)\\
&= \frac{\operatorname{rank}(\mathscr{E})\cdot H^n}{\operatorname{rank}(\mathscr{E})n!}t^n+\frac{(\operatorname{ch}_1^D(\mathscr{E})+\operatorname{rank}(\mathscr{E})\cdot c_1/2)\cdot H^{n-1}}{\operatorname{rank}(\mathscr{E})(n-1)!}t^{n-1}\\
&\quad+\frac{(\operatorname{ch}_2^D(\mathscr{E})+\operatorname{ch}_1^D(\mathscr{E})\cdot c_1/2+\operatorname{rank}(\mathscr{E})\cdot(c_1^2+c_2)/12)\cdot H^{n-2}}{\operatorname{rank}(\mathscr{E})(n-2)!}t^{n-2}+\cdots\\
&= \frac{H^nt^n}{n!}+\mu_H^{D+K_X/2}(\mathscr{E})\frac{t^{n-1}}{(n-1)!}\\
&\quad+\left(\nu_H^{D+K_X/2}(\mathscr{E})+\frac{(c_2-2K_X^2)\cdot H^{n-2}}{12}\right)\frac{t^{n-2}}{(n-2)!}+\cdots
\end{aligned}$$

Using this calculation of the reduced twisted Hilbert polynomial we can obtain implications among $\mu_H$-stability, Gieseker stability, and twisted stability.

**Lemma 2.1.11.** *Assume $\mathscr{E}\in\operatorname{Coh}(X)$ and $D$ is a $\mathbb{R}$-divisor. Each of the following statements implies the next*

*a. $\mathscr{E}$ is $\mu_H$-stable*

*b. $\mathscr{E}$ is $(H, D+\frac{K_X}{2})$-twisted stable.*

*c. $\mathscr{E}$ is $(H,D)$-Gieseker stable.*

*d. $\mathscr{E}$ is $(H,D)$-Gieseker semistable.*

*e. $\mathscr{E}$ is $(H, D+\frac{K_X}{2})$-twisted semistable.*

*f. $\mathscr{E}$ is $\mu_H$-semistable.*

*Furthermore, if $\dim(X)=2$ then $e\Rightarrow d$ and $c\Rightarrow b$.*

*Proof.* The implications $a\Rightarrow b\Rightarrow c\Rightarrow d\Rightarrow e\Rightarrow f$ all follow from definition and Remark 2.1.10.

If $\dim(X) = 2$ then

$$G_H^D(t) = \frac{H^2t^2}{2} + \mu_H^{D+K_X/2}(\mathscr{E})t + \left(\nu_H^{D+K_X/2}(\mathscr{E}) + \frac{(c_2 - 2K_X^2) \cdot H^{n-2}}{1}\right)$$

so the implications $e \Rightarrow d$ and $c \Rightarrow b$ immediately follow. $\square$

We end this subsection by defining the twisted discriminant and noting a weak form of Bogomolov's Inequality.

**Definition 2.1.12.** *Assume $E \in D^b(X)$ and $D$ is a $\mathbb{R}$-divisor.*

1. *We define the discriminant of $E$ as*

$$\Delta^D(E) = \mathrm{ch}_1^D(E)^2 - 2\,\mathrm{rank}(\mathscr{E})\,\mathrm{ch}_2^D(E).$$

*which is an element of $A^2(X) \otimes \mathbb{R}$.*

2. *We define the $(H, D)$-discriminant of $E$ as*

$$\overline{\Delta}_H^D(E) = \deg_H^D(E)^2 - 2\,\mathrm{rank}(\mathscr{E})H^{n-2} \cdot \mathrm{ch}_2^D(E).$$

*which is a real number.*

3. *If $\mathscr{E} \in \mathrm{Coh}(X)$ then we define $\Delta^D(\mathscr{E})$ and $\overline{\Delta}_H^D(\mathscr{E})$ by viewing $\mathscr{E}$ as a chain complex supported in degree $0$ in $D^b(X)$.*

The bar notation in the $(H, D)$-discriminant is meant to signify that $\overline{\Delta}_H^D(E)$ is a real number.

The following calculations allows us to compare these two variants of the discriminant.

**Lemma 2.1.13.** *Assume $D$ is a $\mathbb{R}$-divisor.*

1. *If $E \in D^b(X)$ then the disciminant $\Delta(E)$ is independent of the $\mathbb{R}$-divisor $D$ (as the notation suggests).*

2. *If $\mathscr{E} \in \mathrm{Coh}(X)$ then $\overline{\Delta}_H^D(\mathscr{E}) \geq H^{n-2} \cdot \Delta(\mathscr{E})$.*

*Proof.* Assume $D$ is a $\mathbb{R}$-divisor.

1. By definition,

$$\begin{aligned}\Delta(E) &= \mathrm{ch}_1^D(E)^2 - 2\,\mathrm{rank}(E)\,\mathrm{ch}_2^D(E)\\ &= (\mathrm{ch}_1(E) - D\,\mathrm{rank}(E))^2 - 2\,\mathrm{rank}(E)(\mathrm{ch}_2(E) - D\cdot \mathrm{ch}_1(E) + \frac{D^2}{2}\,\mathrm{rank}(E))\\ &= \mathrm{ch}_1(E)^2 - 2\,\mathrm{rank}(E)\cdot \mathrm{ch}_2(E),\end{aligned}$$

   as claimed.

2. By part 1, it suffices to show $\deg_H^D(E)^2 \geq H^{n-2}\cdot \mathrm{ch}_1^D(E)^2$, but this inequality holds by the Hodge Index Theorem [Laz04, Corollary 1.6.3 (i)]. We note [Laz04, Corollary 1.6.3 (i)] requires that both $H$ and $\mathrm{ch}_1^D(E)$ are nef, but the argument from [Laz04, Corollary 1.6.3 (i)] holds if $\mathrm{ch}_1^D(E)$ is not nef and $H$ is ample.

□

We note the analogous statment to Lemma 2.1.13.1 for $\overline{\Delta}_H^D$ is false. In other words, the $(H, D)$-discriminant truly depends on $D$:

**Example 2.1.14.** *Fix a very ample class $H$ on $X$, set $D = \alpha H$, and let $E \in D^b(X)$. By Example 2.1.2 we can rewrite*

$$\begin{aligned}\overline{\Delta}_H^D(E) &= \overline{\Delta}_H^{\alpha H}(E)\\ &= \deg_H(E)^2 - 2H^{n-2}\cdot \mathrm{ch}_2(E)\,\mathrm{rank}(E) + 2\deg_H(E)\,\mathrm{rank}(E)H^n\alpha\\ &\quad + H^n\left(H^n\cdot \mathrm{rank}(E)^2 - \mathrm{rank}(E)^3\right)\alpha^2\end{aligned}$$

*Therefore, we can always find an object $E \in D^b(X)$ such that $\overline{\Delta}_H^D$ depends on $D$, as claimed.*

**Lemma 2.1.15** (Weak Bogomolov Inequality)**.** *Assume* $\mathrm{char}(k) = 0$*. If $\mathscr{E}$ is torsion-free and $\mu_H$-semistable then $\overline{\Delta}_H^D(\mathscr{E}) \geq 0$.*

*Proof.* By Lemma 2.1.13, it suffices to show

$$H^{n-2} \cdot \mathrm{ch}_1(\mathscr{E})^2 - 2\,\mathrm{rank}(\mathscr{E}) H^{n-2} \cdot \mathrm{ch}_2(\mathscr{E})) \geq 0.$$

With this in mind, by Example 2.1.2, we can simplify the above expression as

$$\begin{aligned} H^{n-2} \cdot \mathrm{ch}_1(\mathscr{E})^2 - 2\,\mathrm{rank}(\mathscr{E}) H^{n-2} \cdot \mathrm{ch}_2(\mathscr{E}) & \\ &= H^{n-2} \cdot \mathrm{c}_1(\mathscr{E})^2 - \mathrm{rank}(\mathscr{E})(H^{n-2} \cdot \mathrm{c}_1(\mathscr{E})^2 - 2H^{n-2} \cdot \mathrm{c}_2(\mathscr{E})) \\ &= 2\,\mathrm{rank}(\mathscr{E}) H^{n-2} \cdot \mathrm{c}_2(\mathscr{E}) - (\mathrm{rank}(\mathscr{E}) - 1) H^{n-2} \cdot \mathrm{c}_1(\mathscr{E})^2 \end{aligned}$$

which is non-negative by the usual Bogomolov inequality [HL10, Theorem 7.3.1], as desired.

□

**Remark 2.1.16** (Stability in Positive Characteristic)**.** *All results in this subsection except the weak Bogomolov inequality (Lemma 2.1.15) remain valid over algebraically closed fields of positive characteristic.*

*We provide a sketch a that the weak Bogomolov inequality fails over Raynaud surfaces. Using the same argument as* [Gie79]*, one can show that if the weak Bogomolov inequality holds on $X$ then Kodaira vanishing also holds. By* [Ray78] *for any algebraically closed field of positive characteristic, there is a smooth surface where Kodaira vanishing fails. Therefore, the weak Bogomolov inequality also fails for such surfaces.*

*Surprisingly, the weak Bogomolov inequality holds for every surface that is neither quasi-elliptic nor of general type* [Lan16, Theorem 7.1]*. This result result can be extended to higher dimensions by Mehta-Ramanathan's Restriction Theorem* [HL10, Theorem 7.2.1] *and Lemma 2.1.13 to give the following result valid in positive characteristic:*

> *Assume $X$ is a smooth projective variety of dimension at least 2 (with very ample class $H$) over an algebraically closed field of positive characteristic $p$. Let $H_1, H_2, \ldots, H_{n-2}$ be general hyperplanes in the linear system of $H$ and set $Y = H_1 \cap H_2 \cap \cdots \cap H_{n-2}$. Also, assume the Kodaiira dimension of $Y$ is at*

*most* 1 *and* $Y$ *is not quasi-elliptic. If* $\mathscr{E}$ *is torsion-free and* $\mu_H$*-semistable then* $\overline{\Delta}_H^D(\mathscr{E}) \geq 0$.

## 2.2 Stability Functions on an Abelian Category

In this section we introduce stability functions and very weak stability functions on any abelian category $\mathcal{A}$ (Definition 2.2.1). For a very weak stability function, we introduce two different notions of stable objects (Definition 2.2.3). For a stability function these two notions are equivalent (Lemma 2.2.11 in view of Lemma 2.2.8).

The rest of this subsection consists of generalizing knnown results about stability functions to very weak stability functions. For example, we have variants of the seesaw inequality (Lemma 2.2.6), Schur's Lemma (Lemma 2.2.13), and the extension of semistable objects of the same slope is semistable (Lemma 2.2.15).

Broadly, stability functions and very weak stability functions are two generalizations of $\mu_H$-stability to a general abelian category $\mathcal{A}$. Stability functions have a theory similar to $\mu_H$-stability on curves. Very weak stability functions have a theory similar to $\mu_H$-stability on higher dimensional varieties. One goal of this subsection is to understand exactly which results hold for stability functions that do not hold for very weak stability functions.

Stability functions are a variant of Rudakov's stability structures [Rud97, Definition 1.1] due to Bridgeland [Bri07, Definition 2.1]. Very weak stability functions are a newer notion implicit in Bayer, Macrì, and Stellari [BMS16, Definition B.1] which are a variant of Toda's weak stability functions [Tod10, Definition 2.7].

**Definition 2.2.1.** *Assume* $\mathcal{A}$ *is an abelian category. Fix a finitely generated free abelian group* $\Lambda$ *and a group homomorphism* $K_0(\mathcal{A}) \to \Lambda$.

1. *A stability function on* $\mathcal{A}$ *is a group homomorphism* $Z : \Lambda \to \mathbb{C}$ *satisfying the positivity property:*

   *For every* $A \in \mathcal{A}$, $\Im Z(A) \geq 0$ *and if* $\Im Z(A) = 0$ *then* $\Re Z(A) < 0$.

*Note that $Z(E)$ is an abuse of notation which denotes the following construction. If $E \in \mathcal{A}$, there is a corresponding element in $K_0(\mathcal{A})$ and so in $\Lambda$. The image of this element in $\Lambda$ under $Z$ is what we call $Z(E)$.*

2. *A very weak stability function on $\mathcal{A}$, denoted $(Z : \Lambda \to \mathbb{C}, \mathcal{A})$ is a group homomorphism $Z : \Lambda \to \mathbb{C}$ satisfying the weak positivity property:*

   *For every $A \in \mathcal{A}$, $\Im Z(A) \geq 0$ and if $\Im Z(A) = 0$ then $\Re Z(A) \leq 0$.*

3. *We will denote a very weak stability function on $\mathcal{A}$ as a pair $\sigma = (Z : \Lambda \to \mathbb{C}, \mathcal{A})$. If $\Lambda$ is clear from context, we will just write $\sigma = (Z, \mathcal{A})$.*

The positivity and weak positivity properties should be thought of as a generalization of Lemma 2.1.3.

Even though it is not suggested by the notation $(Z : \Lambda \to \mathbb{C}, \mathcal{A})$, a very weak stability function also depends on the group homomorphism $K_0(X) \to \Lambda$.

Since $Z : \Lambda \to \mathbb{C}$ is a group homomorphism and there is a group homomorphism $K_0(\mathcal{A}) \to \Lambda$, $Z$ is additive in short exact sequences of objects in $\mathcal{A}$. Therefore, $\Im Z$ and $\Re Z$ are also additive in short exact sequences of objects in $\mathcal{A}$ where $\Im Z(E)$ and $\Re Z(E)$ denote the imaginary and real part of $Z(E)$ respectively.

**Definition 2.2.2.** *Assume $\sigma = (Z, \mathcal{A})$ is a very weak stability function and $A \in \mathcal{A}$. We define the slope of $A$ as*

$$\mu_\sigma(A) = \begin{cases} -\frac{\Re Z(A)}{\Im Z(A)} : & \Im Z(A) \neq 0 \\ +\infty : & \Im Z(A) = 0 \end{cases}$$

*where $\Im Z(A)$ and $\Re Z(A)$ denote the imaginary and real parts of $Z(A)$ respectively.*

We have defined the slope of a very weak stability function, so we can essentially copy Definition 2.1.4 to obtain a notion of stability. We will call this notion weak $\sigma$-stability. However, there is also a stronger notion which we call $\sigma$-stability.

The definitions of $\sigma$-stability and weak $\sigma$-stability appear in [Bri07, Definition 2.2] and [Tod10, Definition 2.9] respectively. However, in both articles the definition is called $\sigma$-stable. For this reason, we have introduced the title of weak $\sigma$-stability.

**Definition 2.2.3.** *Assume $\sigma = (Z, \mathcal{A})$ is a very weak stability function and $A \in \mathcal{A}$ is nonzero.*

1. *We say that $A$ is $\sigma$-(semi)stable if every nonzero proper subobject $0 \to B \to A$ in $\mathcal{A}$ satisfies $\mu_\sigma(B) \underset{(}{\leq}{}_{)} \mu_\sigma(A)$.*

2. *We say that $A$ is weakly $\sigma$-(semi)stable if every nonzero proper subobject $0 \to B \to A$ in $\mathcal{A}$ satisfies $\mu_\sigma(B) \underset{(}{\leq}{}_{)} \mu_\sigma(A/B)$.*

We consider some examples and non-examples of very weak stability functions.

**Example 2.2.4.** *Consider the function $Z_{\mu_H^D} : K_0(\mathrm{Coh}(X)) \to \mathbb{C}$ defined by $Z_{\mu_H^D}(\mathscr{E}) = -\deg_H^D(\mathscr{E}) + \sqrt{-1}\,\mathrm{rank}(\mathscr{E})$. By Lemma 2.1.3 $Z^D_{\mu_H^D}$ is a very weak stability function when $\dim(X) \geq 1$ and a stability function exactly when $\dim(X) = 1$.*

*Moreover, $\Im Z_{\mu_H^D}(\mathscr{E}) = \mathrm{rank}(\mathscr{E})$ and $-\Re Z_{\mu_H^D}(\mathscr{E}) = \deg_H^D(\mathscr{E})$, so $\mu_{\sigma_{\mu_H^D}} = \mu_H^D$. For this reason, if $(Z, \mathcal{A})$ is a very weak stability function then $\Im Z$ is called the generalized rank while $-\Re Z$ is called the generalized degree. Weakly $\sigma$-(semi)stable objects are exactly $\mu_H$-(semi)stable objects. In contrast, we will see in Example 2.2.9 that $\sigma$-stable objects are exactly skyscraper sheaves.*

*We denote this very weak stability function as $\sigma_{\mu_H^D} = (Z_{\mu_H^D}, \mathrm{Coh}(X))$ and call it the very weak stability function associated to $\mu_H$-stability.*

A slightly more general example arises from the generalization of $\mu_H$-stability to coherent sheaves supported on a variety of codimension at least $d$ [HL10, Definition and Corollary 1.6.9]:

**Example 2.2.5.** *Consider the full abelian subcategory* $\mathrm{Coh}_d(X)$ *of* $\mathrm{Coh}(X)$ *generated by coherent sheaves supported in codimension greater than or equal to d. We will define a very weak stability function on* $\mathrm{Coh}_d(X)$*.*

*We define the very weak stability function* $\sigma_{d,\mu_H^D} = (Z_{d,\mu_H^D} : K_0(\mathrm{Coh}_d(X)) \to \mathbb{C}, \mathrm{Coh}_d(X))$ *where*

$$Z_{d,\mu_H^D}(\mathscr{E}) = -H^{d+1} \cdot \mathrm{ch}_{d+1}^D(\mathscr{E}) + \sqrt{-1} H^d \cdot \mathrm{ch}_d^D(\mathscr{E}).$$

*Note that* $\sigma_{0,\mu_H^D} = \sigma_{\mu_H^D}$ *from Example 2.2.4. Moreover, using a similar argument to that example, we find that* $\sigma_{d,\mu_H^D}$ *is a very weak stability function whenever* $0 \leq d \leq n-1 = \dim(X)-1$*.*

For now, these are our only examples of very weak stability functions. In the next section, we will see a class of very weak stability functions called tilt stability.

For technical reasons, $(H,D)$-Gieseker and $(H,D)$-twisted stability are not very weak stability functions. To explain, $(H,D)$-twisted stability involves three different topological invariants (rank, $\deg_H^D$, and $H^{n-2} \cdot \mathrm{ch}_2^D$) which cannot be written as a group homomorphism $K_0(X) \to \mathbb{C}$. Instead, the putative group homomorphism should be defined via $K_0(X) \to \mathbb{R}^{\oplus 3}$ or $K_0(X) \to \mathbb{C}[t]$.

Similarly, $(H,D)$-Gieseker stability involves $\dim(X)$ different topological invariants. In this case, the putative group homomorphism should be defined as a group homomorphism $K_0(X) \to \mathbb{R}^{\oplus \dim(X)}$ or $K_0(X) \to \mathbb{C}[t]$.

Allowing for group homomorphisms $K_0(X) \to \mathbb{R}^{\oplus k}$ is essentially the definition of Toda's weak stability function [Tod10, Definition 2.11]. Allowing for group homomorphisms $K_0(X) \to \mathbb{C}[t]$ is essentially the definition of Bayer's polynomial stability [Bay09, Definition 2.3.1].

$\nu_H$ is also a non-example of a stability function. For example, any sheaf supported in codimension 1 with negative second Chern character shows that $\nu_H$ does not satisfy the required positivity property.

The remainder of this subsection is about generalizing results about $\mu_H$-stability to a very weak stability function or noting when the generalization fails.

Our first such result is a variant of the seesaw inequality (Lemma 2.1.9) that holds for objects with generalized rank 0. This result can also be thought of as a refinement of a variant of [Tod10, Comments after Definition 2.7]. A weaker form of this inequality appears in [BM11, Comments before Remark 3.1.1].

**Lemma 2.2.6** (Weak Seesaw Inequality)**.** *Let $\sigma = (Z : \Lambda \to \mathbb{C}, \mathcal{A})$ be a very weak stability function. If $0 \to B \to A \to C \to 0$ is exact in $\mathcal{A}$ then one of the following inequalities must hold:*

- $\mu_\sigma(B) = \mu_\sigma(A) = \mu_\sigma(C)$
- $\mu_\sigma(B) < \mu_\sigma(A) < \mu_\sigma(C)$
- $\mu_\sigma(B) > \mu_\sigma(A) > \mu_\sigma(C)$
- $\mu_\sigma(B) > \mu_\sigma(A) = \mu_\sigma(C)$ *with* $Z(B) = 0$
- $\mu_\sigma(B) = \mu_\sigma(A) < \mu_\sigma(C)$ *with* $Z(C) = 0$

*Proof.* If $\Im Z(B), \Im Z(A), \Im Z(C) \neq 0$ then this result follows by Lemma 2.1.9.

Therefore, we may assume that at least one of $\Im Z(B)$, $\Im Z(A)$, $\Im Z(C)$ is zero. If two of these values is zero, then by additivity of $Z$, the third must also be zero. In this case, we find that $\mu_\sigma(B) = \mu_\sigma(A) = \mu_\sigma(C) = +\infty$, as needed. Therefore, we may further assume that exactly one of $\Im Z(B)$, $\Im Z(A)$, $\Im Z(C)$ is zero.

If $\Im Z(A) = 0$, by additivity and nonnegativity of $\Im Z$, $\Im Z(B) = \Im Z(C) = 0$ which is the case above.

If $\Im Z(B) = 0$, by definition of a very weak stability function, $\Re Z(B) \leq 0$. Therefore, by additivity of $Z$, $\Im Z(C) = \Im Z(A)$ and $\Re Z(C) \leq \Re Z(A)$ (with equality exactly when $\Re Z(B) = 0$). Since $\Im Z \geq 0$, it follows that $\mu_\sigma(C) \leq \mu_\sigma(A)$ (with equality exactly when $\Re Z(B) = 0$). Therefore, we have shown that either $\mu_\sigma(B) = +\infty > \mu_\sigma(A) > \mu_\sigma(C)$ or

$\mu_\sigma(B) = +\infty > \mu_\sigma(A) = \mu_\sigma(C)$ with $Z(A) = 0$, as needed. Note that $\mu_\sigma(A) < +\infty$ for $\Im Z(A) \neq 0$.

If $\Im Z(C) = 0$ then a similar argument to the $\Im Z(B) = 0$ case holds. $\square$

We now introduce necessary and sufficient conditions (see Proposition 2.2.12 and Lemma 2.2.11) where $\sigma$-stability and weak $\sigma$-stability agree.

**Definition 2.2.7.** *Assume $\sigma = (Z, \mathcal{A})$ is a very weak stability function. We say that $A$ has good quotients (with respect to $\sigma$) if every nonzero subobject $0 \to B \to A$ satisfies $Z(A/B) \neq 0$.*

In particular, if $A$ has good quotients then $Z(A) \neq 0$.

In fact, $\sigma = (Z, \mathcal{A})$ is a stability function if and only if every object in $\mathcal{A}$ has good quotients. Similarly, if an object $A$ does not have good quotients then it is not $\sigma$-stable. In particular, weakly $\sigma$-stable objects that are not $\sigma$-stable never have good quotients (Proposition 2.2.12).

**Lemma 2.2.8.** *Assume $\sigma = (Z, \mathcal{A})$ is a very weak stability function.*

1. *$\sigma$ is a stability function if and only if every object in $\mathcal{A}$ has good quotients.*

2. *Assume $A \in \mathcal{A}$ is $\sigma$-stable then $A$ has good quotients.*

*Proof.* Assume $\sigma = (Z, \mathcal{A})$ is a very weak stability function.

1. Assume $\sigma$ is a stability function. Therefore, by definition, every nonzero object $A \in \mathcal{A}$ satisfies $Z(A) \neq 0$. In particular, every proper subobject $0 \to B \to A$ satisfies $Z(A/B) \neq 0$, as needed.

   Conversely, assume that $\sigma$ is not a stability function. Therefore, we can find a nonzero object $A \in \mathcal{A}$ such that $Z(A) = 0$. Consider the short exact sequence $0 \to A \to A^{\oplus 2} \to A \to 0$. Since $0 \to A \to A^{\oplus 2}$ is a proper nonzero subobject and $Z(A^{\oplus 2}/A) = Z(A) = 0$,

$A^{\oplus 2}$ does not have good quotients. Thus, we have shown that there are objects of $\mathcal{A}$ that do not have good quotients, as needed.

2. Assume $A \in \mathcal{A}$ is $\sigma$-stable. Therefore, we know that every nonzero proper subobject $0 \to B \to A$ satisfies $\mu_\sigma(B) < \mu_\sigma(A)$. By the generalized seesaw inequality, it follows that $Z(A/B) \neq 0$, as desired.

□

In general, for a very weak stability function we should not expect objects to have good quotients. For instance, with respect to $\mu_H$-stability, the only objects with good quotients on $X$ when $\dim(X) \geq 2$ are skyscraper sheaves. Moreover, this example shows that most objects in $\mathrm{Coh}(X)$ do not contain a nonzero $\sigma$-stable subobject.

**Example 2.2.9.** *Assume* $\dim(X) \geq 2$ *and consider the very weak stability condition* $\sigma_{\mu_H^D}$ *associated to* $\mu_H^D$*. We will show* $\sigma$*-stable objects are exactly skyscraper sheaves.*

*With this in mind, consider a nonzero coherent sheaf* $\mathscr{E}$ *on* $X$*, and a closed point* $x \in X$ *with associated short exact sequence*

$$0 \to \mathscr{I}_x \to \mathscr{O}_X \to \iota_* \mathscr{O}_x \to 0.$$

*We obtain a surjection* $\mathscr{E} \to \mathscr{E} \otimes \iota_* \mathscr{O}_x \to 0$ *which induces the following short exact sequence in* $\mathrm{Coh}(X)$*:*

$$0 \to \mathscr{C} \to \mathscr{E} \to \mathscr{E} \otimes \iota_* \mathscr{O}_x \to 0$$

*By definition of the twisted Chern character, we find that*

$$H^{n-1} \cdot \mathrm{ch}_{\leq 1}^D(\mathscr{E} \otimes \iota_* \mathscr{O}_x) = (H^{n-1} \cdot \mathrm{ch}_{\leq 1}^D(\mathscr{E}))(H^{n-1} \cdot \mathrm{ch}_{\leq 1}^D(\iota_* \mathscr{O}_x)) = (\mathrm{rank}(\mathscr{E}) \cdot 0, \deg_H^D(\mathscr{E}) \cdot 0)$$

*for* $H^{n-1} \cdot \mathrm{ch}_1^D(\iota_* \mathscr{O}_x) = (0,0)$ *by Lemma 2.1.3. By additivity of* $Z_{\mu_H^D}$*, it follows that* $Z_{\mu_H^D}(\mathscr{E} \otimes \iota_* \mathscr{O}_x) = Z_{\mu_H^D}(\mathscr{E})$*. In other words, if the surjection* $\mathscr{E} \to \mathscr{E} \otimes \mathscr{I}_x$ *is not an isomorphism then* $\mathscr{E}$ *does not have good quotients.*

*If $\mathscr{E} \to \mathscr{E} \otimes \iota_* \mathscr{O}_x$ is an isomorphism then $\mathscr{E}$ is supported in dimension $0$. In other words, $\mathscr{E} \cong \mathscr{O}_x^{\oplus k}$ for some $k \geq 0$. In this case, $\mathscr{E}$ has good quotients if and only if $k = 0, 1$.*

*In all, we have shown that $\mathscr{E} \in \mathrm{Coh}(X)$ has good quotients if and only if $\mathscr{E}$ is a skyscraper sheaf or $0$. By Lemma 2.2.8, it follows that $\sigma$-stable objects are exactly skyscraper sheaves.*

Example 2.2.9 may suggest that if $\sigma$ is a very weak stability function then objects with good quotients are exactly simple objects and 0—this is not the case! For example, in Lemma 3.1.13 we show that if $\mathscr{E} \in \mathrm{Coh}(X)$ then $\mathscr{E}[1]$ has good quotients with respect to tilt stability.

We also introduce a notion which is a generalization of a pure coherent sheaf which we call $\sigma$-purity. This is a necessary condition for the following lemma, but, more importantly, we will eventually see that $\sigma$-pure objects have Jordan-Hölder filtrations (Lemma 2.3.14).

**Definition 2.2.10.** *Assume $\sigma = (Z, \mathcal{A})$ is a nonnegative stability condition. We say that a nonzero object $E \in \mathcal{A}$ is $\sigma$-pure if for every nonzero subobject $0 \to F \to E$ satisfies $\Im Z(F) \neq 0$.*

In particular, if $A$ is $\sigma$-pure then $\Im Z(A) \neq 0$.

We now compare our two notions of stability plus a third that is similar to the usual definition of $\mu_H$-stability (e.g. [HL10, Definition 1.2.12]). The author has not seen the following implications explicitly written together, but it is certain that most are well known to experts. For example, [BMT14, Remark 3.1.1] is essentially part 2 of the following lemma while [PT19, Remark 2.2] notes that the implication $f \Rightarrow d$ is strict in general.

**Lemma 2.2.11.** *Assume $\sigma = (Z, \mathcal{A})$ is a very weak stability function and let $A \in \mathcal{A}$. Consider the following statements*

*a. For every subobject $0 \to B \to A$ in $\mathcal{A}$ satisfying $0 < \Im Z(B) < \Im Z(A)$, we have $\mu_\sigma(B) \leq \mu_\sigma(A)$.*

*b. For every subobject $0 \to B \to A$ in $\mathcal{A}$ satisfying $0 < \Im Z(B) < \Im Z(A)$, we have $\mu_\sigma(B) < \mu_\sigma(A)$.*

*c. $A$ is weakly $\sigma$-semistable.*

*d. $A$ is weakly $\sigma$-stable.*

*e. $A$ is $\sigma$-semistable.*

*f. $A$ is $\sigma$-stable.*

*The following results hold:*

1. *There are implications*

$$\begin{array}{ccccc} a & \Longleftarrow & c & \Longleftrightarrow & e \\ \Uparrow & & \Uparrow & & \Uparrow \\ b & \Longleftarrow & d & \Longleftarrow & f \end{array}$$

2. *If $A$ is $\sigma$-pure then $b \Rightarrow d$ and $a \Rightarrow c$.*

3. *If $A$ has good quotients with respect to $\sigma$ then $d \Rightarrow f$.*

*Proof.* Assume $\sigma = (Z, \mathcal{A})$ is a very weak stability function and $A \in \mathcal{A}$.

1. $f \Rightarrow e$: This follows from definition.

   $d \Rightarrow c$: This follows from definition.

   $b \Rightarrow a$: This follows from definition.

   $f \Rightarrow d$: Assume $A$ is $\sigma$-stable and assume $0 \to B \to A$ is a proper nonzero subobject. Since $A$ is $\sigma$-stable, $\mu_\sigma(B) < \mu_\sigma(A)$. By the generalized seesaw inequality, it follows that $\mu_\sigma(B) < \mu_\sigma(A) < \mu_\sigma(A/B)$, as needed.

   $e \Rightarrow c$: A similar argument to $f \Rightarrow d$ holds.

   $d \Rightarrow b$: Assume $A$ is weakly $\sigma$-stable assume $0 \to B \to A$ is a proper nonzero subobject satisfying $0 < \Im Z(B) < \Im Z(A)$. Since $A$ is weakly $\sigma$-stable, $\mu_\sigma(B) < \mu_\sigma(A/B)$. Moreover, since $\Im Z(A/B) = \Im Z(A) - \Im Z(B) > 0$, we know that $Z(A/B) \neq 0$. Since $\mu_\sigma(B) < \mu_\sigma(A/B)$ and $Z(A/B) \neq 0$, by the seesaw inequality, $\mu_\sigma(B) < \mu_\sigma(A)$, as needed.

$c \Rightarrow a$: A similar argument to $d \Rightarrow b$ holds.

$c \Rightarrow e$: Assume $A$ is weakly $\sigma$-semistable and let $0 \to B \to A$ be a nonzero proper subobject. Since $A$ is weakly $\sigma$-semistable, we know that $\mu_\sigma(B) \leq \mu_\sigma(A/B)$. By the generalized seesaw inequality, it follows that $\mu_\sigma(B) \leq \mu_\sigma(A)$, as needed.

2. Assume $A$ is $\sigma$-pure.

$b \Rightarrow d$: Assume that for every subobject $0 \to B \to A$ in $\mathcal{A}$ satisfying $0 < \Im Z(B) < \Im Z(A)$ we have $\mu_\sigma(B) < \mu_\sigma(A)$. Consider a nonzero proper subobject $0 \to B \to A$. By assumption, $\Im Z(B) \neq 0$.

If $0 < \Im Z(B) < \Im Z(A)$ then $\mu_\sigma(B) < \mu_\sigma(A)$ by assumption. By the generalized seesaw inequality it follows that $\mu_\sigma(B) < \mu_\sigma(A/B)$, as needed. If $\Im Z(B) = \Im Z(A)$ then we find that $\Im Z(A/B) = 0$. Since $\Im Z(B) = \Im Z(A) \neq 0$, $\mu_\sigma(B) < +\infty = \mu_\sigma(A/B)$, as needed.

$a \Rightarrow c$: A similar argument to $b \Rightarrow d$ holds.

3. Assume $A$ has good quotients with respect to $\sigma$.

$d \Rightarrow f$ : Assume $A$ is weakly $\sigma$-stable. Let $0 \to B \to A$ be a nonzero proper subobject. Since $A$ is weakly $\sigma$-stable, $\mu_\sigma(B) < \mu_\sigma(A/B)$. Furthermore, since $A$ has good quotients, $Z(A/B) \neq 0$. Therefore, the generalized seesaw inequality tells us that $\mu_\sigma(B) < \mu_\sigma(A)$, as needed.

□

In view of Lemma 2.2.11.1 we no longer distinguish between weak $\sigma$-semistability and $\sigma$-semistability - we will just say an object is $\sigma$-semistable.

Having good quotients is necessary to obtain the implication $d \Rightarrow f$ above. Similarly, $\sigma$-purity is necessary to obtain the implications $b \Rightarrow d$ and $a \Rightarrow c$:

**Proposition 2.2.12.** *Assume that $\sigma = (Z, \mathcal{A})$ is a very weak stability function and $A \in \mathcal{A}$.*

*1. If $A$ is weakly $\sigma$-stable but not $\sigma$-stable then $A$ does not have good quotients.*

*2. If for every subobject $0 \to B \to A$ in $\mathcal{A}$ satisfying $0 < \Im Z(B) < \Im Z(A)$ we have $\mu_\sigma(B) \;_{(}\!\leq_{)}\; \mu_\sigma(A)$ but $A$ is not weakly $\sigma$-(semi)stable then $A$ is not $\sigma$-pure.*

*Proof.* Assume $\sigma = (Z, \mathcal{A})$ is a very weak stability function and $A \in \mathcal{A}$.

1. Assume $A$ is weakly $\sigma$-stable but not $\sigma$-stable. Therefore, we can proper nonzero subobject $0 \to B \to A$ such that $\mu_\sigma(B) = \mu_\sigma(A)$. By the generalized seesaw inequality either $\mu_\sigma(B) = \mu_\sigma(A) = \mu_\sigma(A/B)$ or $Z(A/B) = 0$. Since $A$ is weakly $\sigma$-stable, we know that $\mu_\sigma(B) < \mu_\sigma(A/B)$ so $Z(A/B) = 0$. In other words, $A$ does not have good quotients, as desired.

2. Assume that for every subobject $0 \to B \to A$ in $\mathcal{A}$ satisfying $0 < \Im Z(B) < \Im Z(A)$ we have $\mu_\sigma(B) < \mu_\sigma(A)$ but $A$ is not weakly $\sigma$-stable. Since $A$ is not weakly $\sigma$-stable, we can choose a proper nonzero subobject $0 \to B \to A$ such that $\mu_\sigma(B) \geq \mu_\sigma(A/B)$. By the generalized seesaw inequality we find that $\mu_\sigma(B) \geq \mu_\sigma(A)$. Therefore, by assumption, either $\Im Z(B) = 0$ or $\Im Z(B) = \Im Z(A)$. If $\Im Z(B) = 0$ then we are done, so assume $\Im Z(B) = \Im Z(A)$. It follows that $\Im Z(A/B) = 0$ so $\mu_\sigma(B) \geq \mu_\sigma(A) \geq \mu_\sigma(A/B) = +\infty$. In other words, $\Im Z(B) = \Im Z(A) = 0$. Thus, $A$ is not $\sigma$-pure, as desired.

   A similar argument works for the statement involving $\sigma$-semistability.

$\square$

We now prove an analogue of Schur's Lemma. Schur's lemma holds as expected for $\sigma$-stable objects, but the results are weaker for weakly $\sigma$-stable objects. However, we truly need the weaker assumption of weakly $\sigma$-stable in order for our applications. This is because, as we saw in Example 2.2.9 objects of interest are generally not $\sigma$-stable—only weakly $\sigma$-stable.

**Lemma 2.2.13** (Schur's Lemma)**.** *Let $\sigma = (Z, \mathcal{A})$ be a very weak stability function. Assume $f : A \to B$ is a morphism of $\sigma$-semistable objects.*

*1. If $\mu_\sigma(A) > \mu_\sigma(B)$ then $f = 0$.*

*2. Assume $A$ is weakly $\sigma$-stable. If $\mu_\sigma(A) = \mu_\sigma(B)$ then $f$ is an injection or $0$.*

*3. Assume $B$ is $\sigma$-stable. If $\mu_\sigma(A) = \mu_\sigma(B)$ then $f$ is a surjection or $0$.*

*4. Assume $A$ and $B$ are $\sigma$-stable. If $\mu_\sigma(A) = \mu_\sigma(B)$ then $f$ is an isomorphism or $0$.*

*Proof.* Assume $\sigma = (Z, \mathcal{A})$ is a very weak stability function and $f : A \to B$ is a morphism of $\sigma$-semistable objects.

1. Assume that $f \neq 0$, so $\mathrm{Im}(f)$ is a nonzero quotient of $A$.

   If $\mathrm{Ker}(f) = 0$ then $A = \mathrm{Im}(f)$ so $\mu_\sigma(A) = \mu_\sigma(\mathrm{Im}(f))$. If $\mathrm{Ker}(f) \neq 0$, since $A$ is $\sigma$-semistable, then $\mu_\sigma(\mathrm{Ker}(f)) \leq \mu_\sigma(\mathrm{Im}(f))$. By the generalized seesaw inequality, we find that $\mu_\sigma(A) \leq \mu_\sigma(\mathrm{Im}(f))$. In either case, we find that $\mu_\sigma(A) \leq \mu_\sigma(\mathrm{Im}(f))$.

   On the other hand, if $\mathrm{Im}(f) = B$ then we find that $\mu_\sigma(\mathrm{Im}(f)) = \mu_\sigma(B)$. If $\mathrm{Im}(f) \neq B$, since $B$ is $\sigma$-semistable, we know that $\mu_\sigma(\mathrm{Im}(f)) \leq \mu_\sigma(B/\mathrm{Im}(f))$ (for $\mathrm{Im}(f) \neq 0$). By the generalized seesaw inequality, we find that $\mu_\sigma(\mathrm{Im}(f)) \leq \mu_\sigma(B)$. In either case, we find that $\mu_\sigma(\mathrm{Im}(f)) \leq \mu_\sigma(B)$.

   By combining the above two inequalities, we find that $\mu_\sigma(A) \leq \mu_\sigma(\mathrm{Im}(f)) \leq \mu_\sigma(B)$, as desired.

2. Assume $A$ is weakly $\sigma$-stable. Also assume that $f \neq 0$ and $f$ is not an injection. Therefore, we find that $\mathrm{Ker}(f)$ is a nonzero proper subobject of $A$.

   Since $A$ is weakly $\sigma$-stable, it follows that $\mu_\sigma(\mathrm{Ker}(f)) < \mu_\sigma(\mathrm{Im}(f))$. By the generalized seesaw inequality, it follows that $\mu_\sigma(A) < \mu_\sigma(\mathrm{Im}(f))$.

   Since $B$ is $\sigma$-semistable, by the same argument as part 1, $\mu_\sigma(\mathrm{Im}(f)) \leq \mu_\sigma(B)$.

   Combining our two inequalities, we find that $\mu_\sigma(A) < \mu_\sigma(\mathrm{Im}(f)) \leq \mu_\sigma(B)$. Therefore, $\mu_\sigma(E) \neq \mu_\sigma(F)$, as needed.

3. Assume $A$ is $\sigma$-stable. Also assume that $f$ is neither 0 nor a surjection. Therefore, we know that $\mathrm{Im}(f)$ is a proper nonzero subobject of $B$.

   Since $B$ is $\sigma$-stable, it follows that $\mu_\sigma(\mathrm{Im}(f)) < \mu_\sigma(B)$.

   Since $A$ is $\sigma$-semistable, by the same argument as part 1, $\mu_\sigma(A) \leq \mu_\sigma(\mathrm{Im}(f))$.

   Combining these two inequalities, we find that $\mu_\sigma(A) \leq \mu_\sigma(\mathrm{Im}(f)) < \mu_\sigma(B)$. In particular, $\mu_\sigma(A) \neq \mu_\sigma(B)$, as desired.

4. Since $A$ is $\sigma$-stable, by Lemma 2.2.11, $A$ is weakly $\sigma$-stable. Therefore, by part 2, $f$ is injective or 0.

   Since $B$ is $\sigma$-stable, by part 3, $f$ is surjective or 0.

   In all, this shows that $f$ is either both surjective and injective or 0. Since $\mathcal{A}$ is abelian, $f$ is both surjective and injective if and only if $f$ is an isomorphism, as desired.

□

The following result shows that $\sigma$-stability is necessary for Schur's Lemma.

**Proposition 2.2.14.** *If $B$ is weakly $\sigma$-stable and satisfies the following property:*

> *For every $\sigma$-semistable object $A$ with $\mu_\sigma(A) = \mu_\sigma(B)$ any morphism $f : A \to B$ is surjective or* 0.

*then $B$ is $\sigma$-stable.*

*Proof.* Consider a subobject $0 \to A \to B$ in $\mathcal{A}$. Since $B$ is weakly $\sigma$-stable, by Lemma 2.2.11, $B$ is $\sigma$-semistable. Therefore, $\mu_\sigma(A) \leq \mu_\sigma(B)$.

Assume that $\mu_\sigma(A) = \mu_\sigma(B)$ Since $B$ is $\sigma$-semistable, $A$ is also $\sigma$-semistable. Therefore, by assumption, the inclusion $0 \to A \to B$ is surjective or 0. Thus, $A \to B$ is an isomorphism or $A = 0$. In either case, this shows that $B$ is $\sigma$-stable, as desired. □

We also describe the failure of Schur's lemma for weakly $\sigma$-stable objects from a purely categorical view in Proposition 2.2.17.

In analogy to $\mu_H$-stability any extension of $\mu_\sigma$-semistable objects of the same slope is also $\mu_\sigma$-semistable. The argument for a very weak stability function is essentially the same as for $\mu_H$-stability.

**Lemma 2.2.15.** *Assume $\sigma = (Z, \mathcal{A})$ is a very weak stability function and $0 \to B \to A \to C \to 0$ is a short exact sequence with both $B$ and $C$ $\sigma$-semistable. If $\mu_\sigma(B) = \mu_\sigma(A)$ or $\mu_\sigma(A) = \mu_\sigma(C)$ then $A$ is $\sigma$-semistable.*

*Proof.* Consider a short exact sequence $0 \to B' \to A \to C' \to 0$ in $\mathcal{A}$. For ease of notation, define $B \cap B' \to A$ to be kernel of the natural morphism $A \to (A/B) \oplus (A/B')$. Similarly, define $0 \to B + B' \to A$ to be the image of the natural morphism $B \oplus B' \to A$. By the universal property of the kernel, there is an injection $B \cap B' \to A$. On the other hand, by the Second Isomorphism Theorem, $B'/(B \cap B') \cong (B + B')/B$ and there is an injection $(B+B')/B \to A/B \cong C$. Since $B$ and $C$ are $\sigma$-semistable, we find that $\mu_\sigma(B \cap B') \leq \mu_\sigma(B)$ and $\mu_\sigma(B'/(B \cap B') \leq \mu_\sigma(C)$.

Without loss of generality, assume $\mu_\sigma(B) = \mu_\sigma(A)$. Therefore, by the Generalized Seesaw Inequality either $Z_\sigma(C) = 0$ or $\mu_\sigma(C) = \mu_\sigma(A)$.

If $\mu_\sigma(A) = \mu_\sigma(C)$ then, by the inequalities above, $\mu_\sigma(B \cap B') \leq \mu_\sigma(B) = \mu_\sigma(A)$ and $\mu_\sigma(B/(B \cap B')) \leq \mu_\sigma(C) = \mu_\sigma(A)$. It follows by the Generalized Seesaw Inequality applied to $0 \to B \cap B' \to B' \to B/(B \cap B') \to 0$ that $\mu_\sigma(B') \leq \mu_\sigma(A)$, as needed.

If $Z(C) = 0$ then $Z(B/(B \cap B')) = 0$. It follows by the Generalized Seesaw Inequality that $\mu_\sigma(B \cap B') = \mu_\sigma(B')$, so $\mu_\sigma(B') = \mu_\sigma(B \cap B') \leq \mu_\sigma(B) = \mu_\sigma(A)$. $\square$

The naive generalization of Lemma 2.2.15. to $\sigma$-stable objects is false. Specifically, if $B$ and $C$ are $\sigma$-stable of the same slope, then $A$ will never be $\sigma$-stable (for $\mu_\sigma(B) = \mu_\sigma(A) = \mu_\sigma(C)$). The same issue happens with weakly $\sigma$-stable objects. However, this generalization is morally true up to deformation of the stability function within a parameter space of stability conditions. Specifically, Theorem 3.6.1 essentially says that if $B$ and $C$ are weakly $\sigma$-stable and $\sigma$-stable respectively of the same slope and $A \neq B \oplus C$ then $A$ is weakly $\sigma'$-stable for a deformation $\sigma'$ of $\sigma$.

Lemma 2.2.15 essentially says the full subcategory of $\mathcal{A}$ generated by $\sigma$-semistable objects of a fixed slope is closed under extensions. This statement is not technically correct because the full subcategory of $\mathcal{A}$ generated by $\sigma$-semistable object of fixed slope is not abelian. We make this statement precise below and clarify the failure of Schur's Lemma for weakly $\sigma$-stable objects.

**Definition 2.2.16.** *Let $\sigma = (Z, \mathcal{A})$ be a very weak stability function and fix $\phi \in \mathbb{R} \cup \{+\infty\}$.*

1. *We define $\mathcal{A}(\phi)$ to be the full subcategory of $\mathcal{A}$ generated by objects $E \in \mathcal{A}$ satisfying either $E$ is $\sigma$-semistable with $\mu_\sigma(E) = \phi$ or $Z(E) = 0$.*

2. *We also define $\mathcal{A}_0$ to be the full subcategory of $\mathcal{A}$ generated by objects $E \in \mathcal{A}$ satisfying $Z(E) = 0$.*

**Proposition 2.2.17.** *Let $\sigma = (Z, \mathcal{A})$ be a very weak stability function and fix $\phi \in \mathbb{R} \cup \{+\infty\}$.*

1. *$\mathcal{A}(\phi)$ is a full extension closed abelian subcategory of $\mathcal{A}$.*

2. *$\mathcal{A}_0$ is a full Serre subcategory of $\mathcal{A}$ (i.e. whenever $0 \to B \to A \to C \to 0$ is exact in $\mathcal{A}$, $A \in \mathcal{A}_0$ if and only if $B, C \in \mathcal{A}_0$).*

3. *$A \in \mathcal{A}(\phi)$ is $\sigma$-stable if and only if $A$ is simple in $\mathcal{A}(\phi)$. Recall that an object in an abelian category is simple if there are no proper nonzero subobjects.*

4. *$A \in \mathcal{A}(\phi)$ is weakly $\sigma$-stable if and only if $\pi(A)$ is simple in $\mathcal{A}(\phi)/\mathcal{A}_0$ where $\pi : \mathcal{A}(\phi) \to \mathcal{A}(\phi)/\mathcal{A}_0$ is the quotient functor.*

*Proof.* Assume $\sigma = (Z, \mathcal{A})$ is a very weak stability function.

1. We will first show that $\mathcal{A}(\phi)$ is closed under extensions. Therefore, assume we have a short exact sequence $0 \to B \to A \to C \to 0$ in $\mathcal{A}$ with $B, C \in \mathcal{A}(\phi)$. If $Z(B) =$

$Z(C) = 0$ then $Z(A) = 0$ so we are done. If $Z(B) \neq 0$ with $Z(C) = 0$ then the generalized seesaw inequality tells us that $\mu_\sigma(A) = \mu_\sigma(B)$. Similarly, if $Z(C) \neq 0$ with $Z(B) = 0$ the same argument gives $\mu_\sigma(A) = \mu_\sigma(C)$. In either case, by Lemma 2.2.15, $E$ is $\sigma$-semistable. If $Z(B) \neq 0$ and $Z(C) \neq 0$ then the generalized seesaw inequality tells us that $\mu_\sigma(B) = \mu_\sigma(A) = \mu_\sigma(C)$ so by Lemma 2.2.15, $E$ is $\sigma$-semistable. Thus, $\mathcal{A}(\phi)$ is extension closed.

Since $Z(0) = 0$ and $\mathcal{A}(\phi)$ is an extension closed full subcategory of an abelian category (and so additive category) $\mathcal{A}$, $\mathcal{A}(\phi)$ is an additive category. Furthermore, since $\mathcal{A}$ is abelian, it remains to show that $\mathcal{A}(\phi)$ has all kernels and cokernels. With this in mind, consider a morphism $f : A \to B$ in $\mathcal{A}(\phi)$. We have the following short exact sequences in $\mathcal{A}$:

$$0 \to \mathrm{Ker}(f) \to A \to \mathrm{Im}(f) \to 0$$

$$0 \to \mathrm{Im}(f) \to B \to \mathrm{Coker}(f) \to 0.$$

First, assume $Z(\mathrm{Ker}(f)) = 0$ then

$$\mu_\sigma(A) = \mu_\sigma(\mathrm{Im}(f)) \leq \mu_\sigma(B) = \mu_\sigma(A),$$

so $\mu_\sigma(\mathrm{Im}(f)) = \phi$. By the seesaw inequality, it follows that $Z(\mathrm{Coker}(f)) = 0$ or $\mu_\sigma(\mathrm{Coker}(f)) = \phi$. If $Z(\mathrm{Coker}(f)) = 0$ then we are done. If $\mu_\sigma(\mathrm{Coker}(f)) = \phi$ then $\mathrm{Coker}(f)$ is a quotient of a $\sigma$-semistable object of the same slope, so $\mathrm{Coker}(f)$ is $\sigma$-semistable. In all, we have shown that if $Z(\mathrm{Ker}(f)) = 0$ then $\mathrm{Ker}(f), \mathrm{Coker}(f) \in \mathcal{A}(\phi)$. A similar argument works in the case that $Z(\mathrm{Coker}(f)) = 0$.

Thus, we may assume that $Z(\mathrm{Ker}(f)) \neq 0$ and $Z(\mathrm{Coker}(f)) \neq 0$. Since $A$ and $B$ are $\sigma$-semistable, we find that $\phi = \mu_\sigma(A) \leq \mu_\sigma(\mathrm{Im}(f)) \leq \mu_\sigma(B) = \phi$, so $\mu_\sigma(\mathrm{Im}(f)) = \phi$. Since $Z(\mathrm{Ker}(f)) \neq 0$ and $Z(\mathrm{Coker}(f)) \neq 0$, it follows by the seesaw inequality that $\mu_\sigma(\mathrm{Ker}(f)) = \phi$ and $\mu_\sigma(\mathrm{Coker}(f)) = \phi$. Since $\mathrm{Ker}(f)$ is a subobject of a $\sigma$-semistable object of the same slope, $\mathrm{Ker}(f)$ is $\sigma$-semistable. Similarly, $\mathrm{Coker}(f)$ is $\sigma$-semistable, as desired.

2. Assume $0 \to B \to A \to C \to 0$ is exact in $\mathcal{A}(\phi)$.

   If $B, C \in \mathcal{A}_0$, by definition, $Z(B) = Z(C) = 0$. By additivity of $Z$, it follows that $Z(A) = 0$ so $A \in \mathcal{A}_0$, as needed.

   Conversely, assume $A \in \mathcal{A}_0$ so $Z(A) = 0$. By additivity of $Z$, $\mathfrak{I}Z(B) + \mathfrak{I}Z(C) = 0$ and similarly for $\mathfrak{R}Z$. Since the image of $Z$ lies in $\in \mathbb{R} \times \mathbb{R}^+_{>0} \cup \{0\}$, it follows that $\mathfrak{I}Z(B) = 0$ and $\mathfrak{I}Z(C) = 0$. Moreover, we know that $\mathfrak{R}Z(B) \leq 0$ and $\mathfrak{R}Z(C) \leq 0$ so we find that $\mathfrak{R}Z(B) = \mathfrak{Z}(C) = 0$ as well. In other words, $B, C \in \mathcal{A}_0$, as needed.

3. First assume $A \in \mathcal{A}(\phi)$ is $\sigma$-stable. Therefore, by definition, we know that any proper nonzero subobject $0 \to B \to A$ in $\mathcal{A}$ must satisfying $\mu_\sigma(B) < \mu_\sigma(A) = \phi$. Since $\mathcal{A}(\phi)$ is a full subcategory of $\mathcal{A}$, it follows that $A$ has no proper nonzero subobjects in $\mathcal{A}(\phi)$.

   Second, assume that $A \in \mathcal{A}(\phi)$ is simple. Consider a proper nonzero subobject $0 \to B \to A$ in $\mathcal{A}$. Since $A$ is simple in $\mathcal{A}(\phi)$, we find that either $\mu_\sigma(B) \leq \mu_\sigma(A)$ or $\mu_\sigma(B) = \mu_\sigma(A)$ with $B$ not $\sigma$-semistable. However, since $A$ is $\sigma$-semistable if $\mu_\sigma(B) = \mu_\sigma(A)$, we find that $B$ must be $\sigma$-semistable. Thus, we find that $\mu_\sigma(B) < \mu_\sigma(A)$, so $A$ is $\sigma$-stable, as desired.

4. Let $\pi : \mathcal{A}(\phi) \to \mathcal{A}(\phi)/\mathcal{A}_0$ be the quotient functor. We know that $\pi$ is essentially surjective, full, and exact. Furthermore, $\pi(C) = 0$ if and only if $Z(C) = 0$.

   First, assume that $A \in \mathcal{A}(\phi)$ is weakly $\sigma$-stable. Consider a subobject $0 \to B' \to \pi(A)$ in $\mathcal{A}(\phi)/\mathcal{A}_0$. Since $\pi : \mathcal{A}(\phi) \to \mathcal{A}(\phi)/\mathcal{A}_0$ is full and essentially surjective, the injection $0 \to B' \to \pi(A)$ can be written as $0 \to \pi(B) \xrightarrow{\pi(f)} \pi(A)$ where $f : B \to A$ is a morphism in $\mathcal{A}(\phi)$. Note that $f$ is not necessarily injective. [1]

   If $Z(B) = 0$ then $\pi(B) = 0$ and we are done, so assume $Z(B) \neq 0$. By definition of $\mathcal{A}(\phi)$, it follows that $\mu_\sigma(A) = \mu_\sigma(B) = \phi$. Moreover, since $\pi(f) : \pi(B) \to \pi(A)$ is injective and $Z(B) \neq 0$, $\pi(\mathrm{Im}(f)) = \mathrm{Im}(\pi(f)) \neq 0$. In other words, $Z(\mathrm{Im}(f)) \neq 0$

[1] In fact, it is not difficult to show that if $\sigma$ is not a stability function then $\pi : \mathcal{A}(\phi) \to \mathcal{A}(\phi)/\mathcal{A}_0$ reflects injections (i.e. if $\pi(f) : \pi(B) \to \pi(A)$ is injective then $f : B \to A$ is injective) if and only if $\phi \neq +\infty$.

so by definition of $\mathcal{A}(\phi)$, $\mu_\sigma(\mathrm{Im}(f)) = \mu_\sigma(A)$. On the other hand, since $A$ is weakly $\sigma$-stable, $\mu_\sigma(\mathrm{Im}(f)) < \mu_\sigma(\mathrm{Coker}(f))$ or $\mathrm{Im}(f) = A$.

Assume $\mu_\sigma(\mathrm{Im}(f)) < \mu_\sigma(\mathrm{Coker}(f))$ Since $\mu_\sigma(\mathrm{Im}(f)) = \mu_\sigma(A)$, it follows by the generalized seesaw inequality that $Z(\mathrm{Coker}(f)) = 0$. Equivalently, $0 = \pi(\mathrm{Coker}(f)) = \mathrm{Coker}(\pi(f))$. Therefore, $\pi(f) : \pi(B) \to \pi(A)$ is an isomorphism, as needed.

For the other case, assume that $\mathrm{Im}(f) = A$. It follows that $\pi(f) : \pi(B) \to \pi(A)$ is an isomorphism, as needed.

Second, assume that $\pi(A)$ is simple. Consider a proper nonzero subobject $0 \to B \to A \to C \to 0$. Since $\pi$ is exact, we obtain an exact sequence

$$0 \to \pi(B) \to \pi(A) \to \pi(C) \to 0$$

in $\mathcal{A}(\phi)/\mathcal{A}_0$. Since $\pi(A)$ is simple, it follows that $\pi(B) = 0$ or $\pi(B) = \pi(A)$.

If $\pi(B) = \pi(A)$ then $\pi(C) = 0$, so $Z(C) = 0$. It follows by the generalized seesaw inequality that $\mu_\sigma(B) = \mu_\sigma(A) < +\infty = \mu_\sigma(C)$, as needed. If $\pi(B) = 0$ then $Z(B) = 0$, so $\mu_\sigma(B) = +\infty > \mu_\sigma(A)$. Since $A$ is $\sigma$-semistable, it follows that $B = 0$. In either case, we find that $E$ is weakly $\sigma$-stable, as desired.

$\square$

The failure of Schur's lemma for weakly $\sigma$-stable objects can be seen through the lens of the categories introduced above. Schur's lemma for an abelian category states that a morphism between simple objects is either the zero morphism or an isomorphism. This result applied to $\mathcal{A}(\phi)$ gives us a different proof of Lemma 2.2.13.4. On the other hand, since weakly $\sigma$-stable objects correspond to simple objects in $\mathcal{A}(\phi)/\mathcal{A}_0$, and $\mathcal{A}(\phi)/\mathcal{A}_0$ is not a full subcategory of $\mathcal{A}$. However, one can use Proposition 2.2.17.4 to obtain a different proof of Lemma 2.2.13.2. [2]

[2] Here is a sketch of the proof. We may reduce to the case that $\mu_\sigma(A) \neq +\infty$. As noted in the earlier footnote, $\pi : \mathcal{A}(\phi) \to \mathcal{A}(\phi)/\mathcal{A}_0$ reflects injections if and only if $\phi \neq +\infty$ or $\sigma$ is a stability function. Therefore, by Schur's lemma for an abelian category, we find that any morphism $\pi(A) \to \pi(B)$ must be an isomorphism or 0 Since $\pi$ is full, essentially surjective and reflects injections; we find that any morphism $A \to B$ is an injection or 0.

## 2.3 Additional Properties of Very Weak Stability Functions

In general, the notion of a stability function is too weak. For this reason, we consider stability functions that satisfy the Harder-Narasimhan and support properties. These can be thought of as generalizations of Harder-Narasimhan filtrations and Bogomolov's Inequality for $\mu_H$-stability respectively. In this subsection, we will define these notions. We will also discuss Jordan-Hölder filtrations.

The new results in this subsection are about the existence/non-existence of Jordan-Hölder filtrations for very weak stability functions. If a stability function satisfies the Harder-Narasimhan and support properties then every $\sigma$-semistable object with good quotients has a unique Jordan-Hölder filtration.

In contrast, there most very weak stability functions satisfying the Harder-Narasimhan and support properties have $\sigma$-semistable objects which do not have a Jordan-Hölder filtration (Example 2.3.15) To this end, we introduce the notion of a weak Jordan-Hölder filtration which generalizes Jordan-Hölder filtrations.

In contrast to Jordan-Hölder filtrations, for "real-life" very weak stability functions any $\sigma$-semistable object with finite slope has a weak Jordan-Hölder filtration (Lemma 2.3.14). Unfortunately, the trade-off for the existence is that weak Jordan-Hölder filtrations are not unique (Proposition 2.3.11).

**Definition 2.3.1.** *Assume $\sigma = (Z, \mathcal{A})$ is a very weak stability function.*

1. *Let $A \in \mathcal{A}$. We say that $A$ has a Harder-Narasimhan filtration (with respect to $\sigma$) if there is a filtration of the form*

$$0 = A_0 \to A_1 \to \cdots \to A_{m-1} \to A_m = A$$

*such that $A_i/A_{i-1}$ is $\sigma$-semistable for all $i = 1, 2 \ldots, m-1, m$ and*

$$\mu_\sigma(A_1) > \mu_\sigma(A_2/A_1) > \cdots > \mu_\sigma(A_{m-1}/A_{m-2}) > \mu_\sigma(A/A_{m-1}).$$

2. *We say that $\sigma$ satisfies the Harder-Narasimhan property if every object in $\mathcal{A}$ has a Harder-Narasimhan filtration with respect to $\sigma$.*

**Definition 2.3.2.** *Assume that $\sigma = (Z, \mathcal{A})$ is a very weak stability function and*

$$0 = A_0 \to A_1 \to \cdots \to A_{m-1} \to A_m = A$$

*is a Harder-Narasimhan filtration of $A$.*

1. *We define $\mu_\sigma^+(A) = \mu_\sigma(A_1)$.*

2. *We $\mu_\sigma^-(A) = \mu_\sigma(A/A_{m-1})$.*

3. *We define the mass of $A$ to be*

$$m_\sigma(A) = \sum_{i=1}^{m} |Z(A_i/A_{i-1})|.$$

If $\sigma$ is a very weak stability function then a Harder-Narasimhan filtration is not necessarily unique. However, the length of the filtration and the slopes of the consecutive quotients are well-defined. In particular, $\mu_\sigma^+(A)$, $\mu_\sigma^-(A)$, and $m_\sigma(A)$ are well-defined. Furthermore, the objects $A_1$ and $A/A_{m-1}$ are a maximal destabilizing subobject and minimal destabilizing quotient respectively.

**Lemma 2.3.3.** *Assume $\sigma = (Z, \mathcal{A})$ be a very weak stability function and $A \in \mathcal{A}$ has a Harder-Narasimhan filtration given by*

$$0 = A_0 \to A_1 \to \cdots \to A_{m-1} \to A_m = A.$$

1. *If*

$$0 = B_0 \to B_1 \to \cdots \to B_{n-1} \to B_n = A$$

*is another Harder-Narasimhan filtration of $A$ with respect to $\sigma$ then*

a) *$m = n$,*

*b)* $Z(A_i) = Z(B_i)$ *for all* $i = 1, \ldots, m$. *In particular,* $\mu_\sigma(A_i/A_{i+1}) = \mu_\sigma(B_i/B_{i+1})$ *for all* $i = 1, \ldots, m-1$.

*c) There are injections* $A_i \to B_i$ *that make the following diagram commute:*

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \longrightarrow & \cdots & \longrightarrow & A_{m-1} & \longrightarrow & A_m = A \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \| \\ 0 & \longrightarrow & B_1 & \longrightarrow & B_2 & \longrightarrow & \cdots & \longrightarrow & B_{m-1} & \longrightarrow & B_m = A \end{array}.$$

*Furthermore, if* $\sigma$ *is a stability function then the injections* $A_i \to B_i$ *are isomorphisms.*

2. $A_1$ *is a maximal destabilizing subobject of* $A$. *In other words, every nonzero subobject* $0 \to B \to A$ *satisfies* $\mu_\sigma(B) \leq \mu_\sigma(A_1)$ *and if equal then* $0 \to B \to A$ *factors through* $0 \to B \to A_1$.

3. $A/A_{m-1}$ *is a minimal destabilizing quotient of* $A$. *In other words, every nonzero quotient* $A \to C \to 0$ *satisfies* $\mu_\sigma(C) \geq \mu_\sigma(A/A_{m-1})$ *and if equal then* $A \to A/A_{m-1} \to 0$ *factors through* $A \to C \to 0$.

*Proof.* Assume $\sigma$ is a very weak stability function and $A \in \mathcal{A}$ has a Harder-Narasimhan filtration.

1. Our proof essentially follows [HL10, Paragraphs before Theorem 1.3.7].

   Consider two Harder-Narasimhan filtrations of $E$:

$$0 = A_0 \to A_1 \to \cdots \to A_{m-1} \to A_m = A$$
$$0 = B_0 \to B_1 \to \cdots \to B_{n-1} \to B_n = A.$$

   Without loss of generality, assume $\mu_\sigma(A_1) \leq \mu_\sigma(B_1)$. Let $i$ be the smallest integer such that $B_1 \to A$ factors through $B_i \to A$ (which exists because $i = m$ satisfies this property). By minimality of $i$, the composition $B_1 \to A_i \to A_i/A_{i-1}$ is nonzero. Thus, by Schur's Lemma (Lemma 2.2.13), $\mu_\sigma(B_1) \leq \mu_\sigma(B_i/B_{i-1})$. By combining the above inequality with the assumption $\mu_\sigma(A_1) \leq \mu_\sigma(B_1)$, we find that

$$\mu_\sigma(A_1) \leq \mu_\sigma(B_1) \leq \mu_\sigma(A_i/A_{i-1})$$

It follows by definition of a Harder-Narasimhan filtration that $i = 1$, so the injection $B_1 \to E$ factors through an injection $B_1 \to A_1$, as needed.

Since $A_1$ is $\sigma$-semistable, it follows that $\mu_\sigma(B_1) \leq \mu_\sigma(A_1)$ so $\mu_\sigma(B_1) = \mu_\sigma(A_1)$. Furthermore, since $\sigma$ is a very weak stability function, $\Im Z(B_1) \leq \Im Z(A_1)$. By using the same argument as above with the roles of $B_1$ and $A_1$ switched (which we can do because $\mu_\sigma(F_1) = \mu_\sigma(E_1)$), we find that $\Im Z(A_1) \leq \Im Z(B_1)$. This shows that $\Im Z(A_1) = \Im Z(B_1)$.

If $\Im Z(A_1) \neq 0$, since $\mu_\sigma(A_1) = \mu_\sigma(B_1)$, it follows that $\Re Z(A_1) = \Re Z(B_1)$ so $Z(A_1) = Z(B_1)$. If $\Im Z(A_1) = 0 = \Im Z(B_1)$, we know

$$\Re Z(A_1), \Re Z(B_1), \Re Z(A_1/B_1), \Re Z(B_1/A_1) \leq 0.$$

Since $\Re Z$ is additive in short exact sequences, it follows that $\Re Z(A_1) - \Re Z(B_1) \leq 0$ and $\Re Z(B_1) - \Re Z(A_1) \leq 0$. In other words, $\Re Z(B_1) = \Re Z(A_1)$ so $Z(B_1) = Z(A_1)$, as needed.

Now, consider the following Harder-Narasimhan filtrations of $A/A_1$:

$$0 \to A_2/A_1 \to \cdots \to A_{m-1}/A_1 \to A_m/A_1 = A/A_1$$
$$0 \to B_2/A_1 \to \cdots \to B_{n-1}/A_1 \to B_n/A_1 = A/A_1.$$

Note that the above filtrations truly are Harder-Narasimhan filtrations by the third isomorphism theorem. By induction on $m$, we find that

- $m - 1 = n - 1$,
- $Z(A_{i+1}/A_1) = Z(B_{i+1}/B_1)$ for all $i = 1, \ldots, m-1$. Therefore, $Z(A_{i+1}) = Z(B_{i+1})$ for all $i = 1, \ldots, m - 1$.

- There are injections $A_i/A_1 \to B_i/A_1$ that make the following diagram commute:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & A_2/A_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & A_{m-1}/A_1 \arrow[r] \arrow[d] & A_m/A_1 = A/A_1 \arrow[d, equal] \\
0 \arrow[r] & B_2/A_1 \arrow[r] & \cdots \arrow[r] & B_{m-1}/A_1 \arrow[r] & B_m/A_1 = A/A_1
\end{tikzcd} .$$

We already saw that $Z(A_1) = Z(B_1)$, so we have proven parts a and b. It remains to show part (c).

By the second isomorphism theorem, we can lift the injections $A_i/A_1 \to B_i/A_1$ to injections $A_i \to B_i$ that makes the following diagram commute:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & A_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & A_{m-1} \arrow[r] \arrow[d] & A_m = A \arrow[d, equal] \\
0 \arrow[r] & B_2 \arrow[r] & \cdots \arrow[r] & B_{m-1} \arrow[r] & B_m = A
\end{tikzcd} .$$

As we saw above, $0 \to B_1 \to A$ factors through $0 \to A_1 \to A$, so we obtain the desired commutative diagram.

Last, assume that $\sigma$ is a stability function. Since $Z(A_i) = Z(B_i)$, we find that $Z(B_i/A_i) = 0$. Since $\sigma$ is a stability function, it follows that $B_i/A_i = 0$ so the injections $A_i \to B_i$ are isomorphisms, as desired.

2. Assume $0 \to A_1 \to A$ is a maximal destabilizing subobject of $A$. By definition of a Harder-Narasimhan filtration we know that $B$ is $\sigma$-semistable.

   Now, assume that $0 \to B \to A$ is a proper nonzero subobject. By definition of a maximal destabilizing subobject, there is a Harder-Narasimhan filtration

$$0 \to A_1 \to A_2 \to \cdots \to A_{m-1} \to A$$

   We proceed by induction of the smallest integer $i$ such that $0 \to B \to A$ factors through $0 \to A_i \to A$.

   If $i = 1$ then $0 \to B \to A$ factors through $0 \to A_1 \to A$, as needed. Also, since $A_1$ is $\sigma$-semistable $\mu_\sigma(B) \le \mu_\sigma(A_1)$.

Now, assume $k$ is the smallest integer such that $0 \to B \to A$ factors through $0 \to A_k \to A$. By minimality, there is a nonzero morphism $f : B \to A_k/A_{k-1}$. By definition of a Harder-Narasimhan filtration, $A_k/A_{k-1}$ is $\sigma$-semistable so $\mu_\sigma(\mathrm{Im}(f)) \leq \mu_\sigma(A_k/A_{k-1})$. In addition, there is an injection $\mathrm{Ker}(f) \to A_{k-1}$. Thus, by the inductive hypothesis we find that $\mu_\sigma(\mathrm{Ker}(f)) \leq \mu_\sigma(A_1)$ with equality exactly if the composition $0 \to \mathrm{Ker}(f) \to A_{k-1} \to A$ factors through $0 \to A_1 \to A$.

Since $\mu_\sigma(\mathrm{Ker}(f)) \leq \mu_\sigma(A_1)$ and $\mu_\sigma(\mathrm{Im}(f)) \leq \mu_\sigma(A_k/A_{k-1}) \leq \mu_\sigma(A_1)$, by the generalized seesaw inequality, $\mu_\sigma(B) \leq \mu_\sigma(A_1)$. Now, assume that $\mu_\sigma(B) = \mu_\sigma(A_1)$. Therefore, by the generalized seesaw inequality, $\mu_\sigma(A_k/A_{k-1}) \geq \mu_\sigma(\mathrm{Im}(f)) \geq \mu_\sigma(B) = \mu_\sigma(A_1)$. By definition of a Harder-Narasimhan filtration, it follows that $k = 1$. Thus, $0 \to B \to A$ factors through $0 \to A_1 \to A$, as desired.

3. The dual argument to part 2 holds.

□

We can use the above result to obtain a generalization of Lemma 2.2.13.1. Alternatively, the following lemma can be thought of as a natural generalization of [HL10, Lemma 1.3.3] from $\mu_H$-stability to any very weak stability function.

**Lemma 2.3.4.** *Assume that $\sigma = (Z, \mathcal{A})$ is a very weak stability function and consider $A, B \in \mathcal{A}$. Additionally assume $A$ and $B$ have Harder-Narasimhan filtrations with respect to $\sigma$. If $\mu_\sigma^-(A) > \mu_\sigma^+(B)$ then $f = 0$.*

*Proof.* For ease of notation, let

$$0 \to A_1 \to \cdots \to A_{m-1} \to A$$

$$0 \to B_1 \to \cdots \to B_{n-1} \to B$$

be Harder-Narasimhan filtrations of $A$ and $B$. Also, assume that $f : A \to B$ is nonzero. Let $i$ be the smallest nonzero integer such that the composition $A_i \to A \to B$ is nonzero.

By minimality of $i$, we obtain a nonzero morphism $A_i/A_{i-1} \to B$. Let $j$ be the smallest nonzero integer such that $A_i/A_{i-1} \to B$ factors through $B_j \to B$. By composition, we obtain a morphism $A_i/A_{i-1} \to B_j \to B_j/B_{j-1}$, and, by minimality of $j$, the morphism $A_i/A_{i-1} \to B_j/B_{j-1}$ is nonzero.

By definition of a Harder-Narasimhan filtration, $A_i/A_{i-1}$ and $B_j/B_{j-1}$ are $\sigma$-semistable. Therefore, by Schur's Lemma (Lemma 2.2.13.1), we find that $\mu_\sigma(A_i/A_{i-1}) \leq \mu_\sigma(B_j/B_{j-1})$. By Lemma 2.3.3, it follows that

$$\mu_\sigma^-(A) \leq \mu_\sigma(A_i/A_{i-1}) \leq \mu_\sigma(B_j/B_{j-1}) \leq \mu_\sigma(B)^-),$$

as desired. □

We also note that under natural assumptions, a very weak stability function satisfies the Harder-Narasimhan property. This result first appears in [BM11, Proposition B.2] for stability functions. The corresponding result for very weak stability functions first appears in [BMS16, Proposition B.2]. A simpler proof for stability functions is given in [Bay19, Section 3] which can be extended to very weak stability conditions by [MS17, Remark 4.14]. We reproduce this proof below.

We need two technical definitions for proving the following lemma. The first definition is a natural generalization of [Sha77, Middle of Page 173 in view of Theorem 2] to very weak stability functions due to Bayer (Definition [Bay19, Definition 3.1]). The second definition first appears in [Bay19, Definition 3.2].

**Definition 2.3.5.** *Assume $\sigma = (Z, \mathcal{A})$ is a very weak stability function and $A \in \mathcal{A}$.*

1. *We define the Harder-Narasimhan polygon of $A$ (with respect to $\sigma$), written* $\mathrm{HN}(A)$*, to be the convex hull of the set*

$$\{Z(B) \mid 0 \to B \to A \text{ is injective}\} \subseteq \mathbb{C}.$$

2. *We say that* $\mathrm{HN}(A)$ *is polyhedral on the left if the region in* $\mathrm{HN}(A)$ *to the left of the line connecting* $0$ *to* $Z(E)$ *is a polygon with finitely many vertices (see Figure 2.3).*

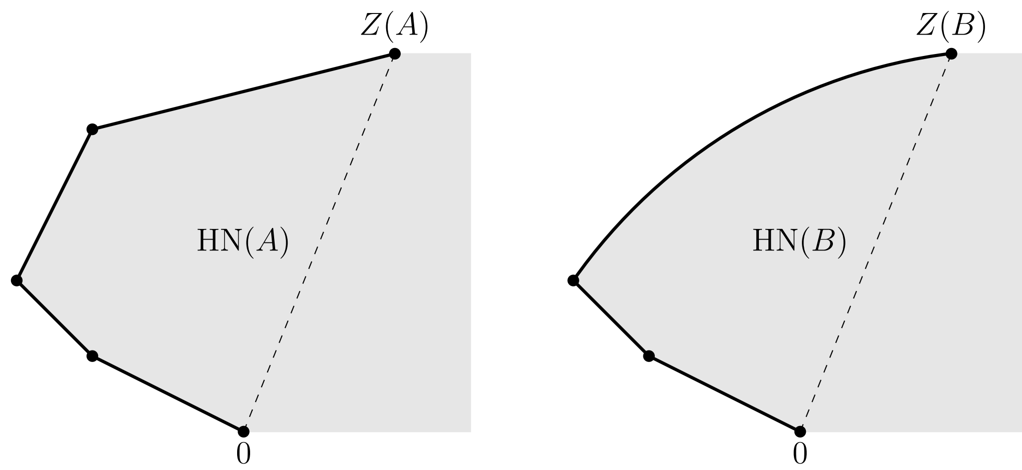


Figure 1 Polyhedral on the Left: HN($A$) is polyhedral on the left while HN($B$) is not.

Note that if $A$ has a Harder-Narasimhan filtration with respect to a very weak stability function $\sigma = (Z, \mathcal{A})$ then the mass, $m_\sigma$, is the length of the left boundary curve connecting 0 and $Z(A)$ in HN($A$).

**Lemma 2.3.6.** *Assume $\sigma = (Z, \mathcal{A})$ is a very weak stability condition with $A \in \mathcal{A}$. If $HN(A)$ is polyhedral on the left and either of the following conditions holds*

- *$\sigma = (Z, \mathcal{A})$ is a stability function or*
- *$A$ is Noetherian (i.e. $A$ satisfies the ascending chain condition)*

*then $A$ has a Harder-Narasimhan filtration with respect to $\sigma$.*

*Proof.* Assume $HN(A)$ is polyhedral on the left and either of the above assumptions holds. Let $0 = z_0, z_1, z_2, \ldots, z_{m-1}, z_m = Z(A)$ be the vertices of the left boundary curve of $HN(A)$ written with increasing $y$-coordinate. Since $z_i$ is a vertex of $HN(A)$, $z_i$ cannot be written as a finite linear combination of elements in $HN(A) \setminus \{z_i\}$. Therefore, by Carathéodory's theorem there exists $A_i \in \mathcal{A}$ such that $Z(A_i) = z_i$. For fixed $i$, consider the set

$$\mathcal{C} = \{B \in \mathcal{A} \mid 0 \to B \to A \text{ is injective and } Z(B) = z_i\}$$

with the ordering $B \leq B'$ if $0 \to B \to A$ factors through $0 \to B' \to A$. If $A$ is Noetherian then choose $A_i$ to be a maximal element in the above poset. If $\sigma$ is a stability function, choose arbitrary $A_i$.

We first claim the injections $0 \to A_i \to A$ factor through $0 \to A_{i+1} \to A$ for all $i$. There is a short exact sequence

$$0 \to A_i \cap A_{i+1} \to A_i \oplus A_{i+1} \to A_i + A_{i+1} \to 0.$$

By additivity of $Z$, we find

$$\frac{Z(A_i \cap A_{i+1}) + Z(A_i + A_{i+1})}{2} = \frac{Z(A_i) + Z(A_{i+1})}{2}.$$

In other words, the midpoint of the line segment connecting $Z(A_i) = z_i$ to $Z(A_{i+1}) = z_{i+1}$ is equal to the midpoint of the line segment connecting $Z(A_i \cap A_{i+1})$ to $Z(A_i + A_{i+1})$. By convexity of $HN(A)$, it follows that $Z(A_i \cap A_{i+1})$ lies on the line segment connecting $Z(A_i)$ to $Z(A_{i+1})$. This implies that

$$\Im Z(A_i) \leq \Im Z(A_i \cap A_{i+1}), \Im Z(A_i + A_{i+1}) \leq \Im Z(A_{i+1}).$$

There are natural injections $A_i \cap A_{i+1} \to A_i$ and $A_{i+1} \to A_i + A_{i+1}$ so we also find that $\Im Z(A_i) \geq \Im Z(A_i \cap A_{i+1})$ and $\Im Z(A_{i+1}) \leq \Im Z(A_i + A_{i+1})$. This shows $\Im Z(A_i \cap A_{i+1}) = \Im Z(A_i)$ and $\Im Z(A_i + A_{i+1}) = \Im Z(A_{i+1})$. A similar argument gives that $\Re Z(A_i \cap A_{i+1}) = \Re Z(A_i)$ and $\Re Z(A_i + A_{i+1}) = \Re Z(A_{i+1})$. In all, we have shown $Z(A_i \cap A_{i+1}) = Z(A_i)$ and $Z(A_i + A_{i+1}) = Z(A_{i+1})$.

If $A$ is Noetherian, by maximality of $A_i$, we find that the natural injection $A_{i+1} \to A_i + A_{i+1}$ is an isomorphism. If $\sigma$ is a stability condition, then $A_i/(A_i \cap A_{i+1}) = 0$ so $A_i \cap A_{i+1} \to A_i$ is an isomorphism. Thus, we we find either $A_i \cap A_{i+1} \to A_i$ or $A_{i+1} \to A_i + A_{i+1}$ is an isomorphism. In other words, $A_i \to A$ factors through $A_{i+1} \to A$, as desired.

We now show $A_{i+1}/A_i$ is $\sigma$-semistable. With this in mind, consider a subobject $B/A_i$ induced by injections $0 \to A_i \to B \to A_{i+1}$. It follows that $\Im Z(A_i) \leq \Im Z(B) \leq \Im Z(A_{i+1})$. Furthermore, by convexity of $HN(A)$, $Z(B)$ must lie to the right of the line connecting $Z(A_i)$

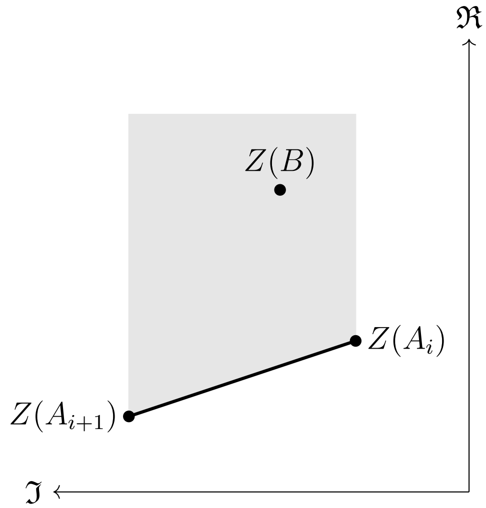


Figure 2 The Harder-Narasimhan Polygon: Position of $Z(B)$ relative to $Z(A_i)$ and $Z(A_{i+1})$. Note the complex plane is rotated 90 degrees. In this orientation, the slope of the line between two points agrees with the slope of the quotient. In other words, with this orientation the slope of $\overline{Z(B)Z(A_i)}$ is $\mu_\sigma(B/A_i)$.

to $Z(A_{i+1})$. Therefore, if $m$ is the slope of $\overline{Z(A_i)Z(A_{i+1})}$ and $m'$ is the slope of $\overline{Z(A_i)Z(B)}$ then $-1/m' \leq -1/m$. (see Figure 2.3). In other words

$$\mu_\sigma(B/A_i) = -\frac{\mathfrak{R}Z(B) - \mathfrak{R}Z(A_i)}{\mathfrak{I}Z(B) - \mathfrak{I}Z(A_i)} \leq -\frac{\mathfrak{R}Z(A_{i+1}) - \mathfrak{R}Z(A_i)}{\mathfrak{I}Z(A_{i+1}) - \mathfrak{I}Z(A_i)} = \mu_\sigma(A_{i+1}/A_i),$$

as needed. Therefore, $A_{i+1}/A_i$ is $\sigma$-semistable, as claimed.

Last, since $HN(A)$ is convex and $\mu_\sigma(\cdot)$ has the same ordering as $\arg(Z(\cdot))$, $\mu_\sigma(A_2/A_1) > \cdots > \mu_\sigma(A/A_{m-1})$ $\square$

If $\mathfrak{I}Z : \Lambda \to \mathbb{R}$ is discrete then HN$(A)$ will always be polyhedral on the left.

A converse to Lemma 2.3.6 holds. Specifically, by the same argument as [Bay19, Proposition 3.3], if $A$ has a Harder-Narasimhan filtration with respect to $\sigma$ then $HN(A)$ is polyhedral on the left. In particular, we do not need additional assumptions on $\mathcal{A}$ nor $\sigma$.

The next property we consider is the support property The support property first appears in [KS08, Section 2.1 Support Property]. By [BM11, Proposition B.4], in practice, the support property is equivalent to Bridgeland's notion of locally finite [Bri07, Definition 5.7].

The support property can be thought of as a generalized Bogomolov inequality. For our purposes, the most important consequence of the support property is that it induces a useful wall and chamber structure on the parameter spaces of stability conditions.

From a different point of view, if $\sigma$ is a stability function satisfying the support property then every $\sigma$-semistable object has a Jordan-Hölder filtration with unique factors up to permutation (see Proposition 2.3.11 and Proposition 2.3.10). Unfortunately, there is no similar result for very weak stability functions.

**Definition 2.3.7.** *We say that a very weak stability function $(Z : \Lambda \to \mathbb{C}, \mathcal{A})$ satisfies the support property if either of the equivalent conditions are satisfied:*

1. *There exists a quadratic form $Q$ on $\Lambda_{\mathbb{R}}$ such that*

   a) *The $Q$ is negative definite with respect to the kernel of $Z$ (i.e. if $Z(A) = 0$ then $Q(A) < 0$), and*

   b) *If $A \in \mathcal{A}$ is semistable with respect to $Z$, then $Q(A) \geq 0$ (where $A$ is identified with the corresponding object in $\Lambda_{\mathbb{R}}$).*

2. *For any norm $||\cdot||$ on $\Lambda_{\mathbb{R}}$,*

$$\inf\left\{\frac{|Z(A)|}{||A||} : A \in \mathcal{A} \text{ is } \sigma - semistable\right\} > 0$$

To see that these two notions are equivalent consider the quadratic form

$$Q(A) = |Z(A)|^2/C - ||A||^2$$

where $||\cdot||$ is the standard norm on $\mathbb{R}^{\oplus \operatorname{rank}(\Lambda)}$ and $C$ is a sufficiently positive real number. Since $\Lambda$ is a finitely generated free abelian group, we know $\Lambda_{\mathbb{R}}$ is a finite dimensional real vector space. Therefore, any two norms on $\Lambda_{\mathbb{R}}$ are equivalent, so the second definition can be checked on a single norm (such as the Euclidean norm above).

We note a linear algebra lemma about the quadratic form in short exact sequences

**Lemma 2.3.8** ([Sch20, Lemma 2.7])**.** *Consider a very weak stability function $\sigma = (Z, \mathcal{A})$ satisfying the support property with respect to $Q$. If $0 \to B \to A \to C \to 0$ is a short exact sequence in $\mathcal{A}$ satisfying the following:*

- $\mu_\sigma(B) = \mu_\sigma(A)$ *or* $\mu_\sigma(A) = \mu_\sigma(C)$,
- $Q(B), Q(C) \geq 0$

*then* $Q(A) \geq 0$.

We now define (weak) Jordan-Hölder filtrations and relate them to the support property.

**Definition 2.3.9.** *Assume $\sigma = (Z, \mathcal{A})$ is a very weak stability function and $A \in \mathcal{A}$ is $\sigma$-semistable.*

1. *We say that $A$ has a (weak) Jordan-Hölder filtration if there exists a filtration:*

$$0 = A_0 \to A_1 \to A_2 \to \cdots \to A_m = A$$

   *such that $A_i/A_{i-1}$ is (weakly) $\sigma$-stable and $\mu_\sigma(A_i/A_{i-1}) = \mu_\sigma(A)$ for all $i = 1, \ldots, m$.*

2. *For a (weak) Jordan-Hölder filtrations, the consecutive quotients $A_i/A_{i-1}$ are called (weak) Jordan-Hölder factors.*

**Proposition 2.3.10.** *Assume $\sigma = (Z, \mathcal{A})$ is a stability function and fix $\phi \in \mathbb{R} \cup \{+\infty\}$. If $\sigma$ satisfies the support property then the abelian category $\mathcal{A}(\phi)$ is both Noetherian and Artinian (recall that $\mathcal{A}(\phi)$ is defined in Definition 2.2.16).*

*In particular, if $\sigma$ satisfies the support property then every $\sigma$-semistable object in $\mathcal{A}$ has a Jordan-Hölder filtration.*

*Proof.* We will first show that $\mathcal{A}(\phi)$ is Noetherian. Thus, fix an object $A \in \mathcal{A}(\phi)$ and consider an ascending chain

$$0 = A_0 \to A_1 \to A_2 \to \cdots \to A$$

in $\mathcal{A}(\phi)$. By additivity of $Z$, we find that

$$\begin{aligned}
Z(A) &= Z(A_1/A_0) + Z(A/A_1) \\
&= Z(A_2/A_1) + Z(A_1/A_0) - Z(A_2/A_1) + Z(A/A_1) \\
&= Z(A_2/A_1) + Z(A_1/A_0) + Z(A/A_2) \\
&= \cdots \\
&= Z(A/A_k) + \sum_{i=1}^{k} Z(A_i/A_{i-1}).
\end{aligned}$$

for all integers $k \geq 1$. Since $A_i \in \mathcal{A}(\phi)$ for all $i$, we know that $\arg(Z(A/A_i) = \arg(Z(A)) = \arg(Z(A_i))$ for all $i$. In other words, $Z(A/A_i)$ and $Z(A/A_i)$ lie on the same ray in $\mathbb{R}\times\mathbb{R}^{+}_{>0}\cup\{0\}$ for all $i$. Thus, the above equality gives us

$$|Z(A)| = \left|Z(A/A_k) + \sum_{i=1}^{k} Z(A_i/A_{i-1})\right| = |Z(A/A_k)| + \sum_{i=1}^{k} |Z(A_i/A_{i-1})| \geq \sum_{i=1}^{k} |Z(A_i/A_{i-1})|$$

for all $k$.

In short, we have shown that $\left\{\sum_{i=1}^{k} |Z(A_i/A_{i-1})|\right\}_{k=1}^{\infty}$ is bounded above. Therefore $\sum_{i=1}^{\infty} |Z(A_i/A_{i-1})|$ converges and so $\lim_{i\to\infty} |Z(A_i/A_{i-1})| = 0$. By definition of the support property, it follows that $Z(A_i/A_{i-1}) = 0$ for all $i \gg 0$. Since $\sigma$ is a stability function, it follows that $A_i/A_{i-1} = 0$ for all $i \gg 0$. In other words, the ascending chain eventually stabilizes, as needed.

The argument that $\mathcal{A}(\phi)$ is Artinian is essentially the same.

Consider an object $A \in \mathcal{A}$ with $\mu_\sigma(A) = \phi$. Since $\mathcal{A}(\phi)$ is both Noetherian and Artinian, any object in $\mathcal{A}(\phi)$ has a Jordan-Hölder filtration. In other words, there is a filtration $0 \to A_1 \to A_2 \to \cdots A_{m-1} \to A$ such that $A_i/A_{i-1}$ is simple in $\mathcal{A}(\phi)$. By proposition 2.2.17.3, simple in $\mathcal{A}(\phi)$ is equivalent to $\sigma$-stable. Therefore, we have shown that there is a filtration $0 \to A_1 \to A_2 \to \cdots \to A_{m-1} \to A$ in $\mathcal{A}$ (for $\mathcal{A}(\phi)$ is a full subcategory) such that $A_i/A_{i-1}$ is $\sigma$-stable of slope $\phi = \mu_\sigma(A)$, as desired. □

Unfortunately, Proposition 2.3.10 fails for very weak stability conditions. For example, we saw that in Example 2.2.9 that if $\sigma_{\mu_H^D}$ is the very weak stability function associated to

$\mu_H^D$ then the only objects with a $\sigma_{\mu_H^D}$-stable subobjects are direct sums of skyscraper sheaves. Consequently, the only $\sigma_{\mu_H^D}$-semistable objects with Jordan-Hölder filtrations are direct sums of skyscraper sheaves.

To avoid this issue, we will work with weak Jordan-Hölder filtrations. However, in contrast to Jordan-Hölder filtrations, weak Jordan-Hölder factors may not be unique. We only obtain the following:

**Proposition 2.3.11.** *Assume $\sigma = (Z, \mathcal{A})$ is a very weak stability function and $A \in \mathcal{A}$ is $\sigma$-semistable. Given two weak Jordan-Hölder filtrations*

$$0 = A_0 \to A_1 \to A_2 \to \cdots \to A_l = A \text{ and } 0 = B_0 \to B_1 \to B_2 \to \cdots \to B_m = A$$

*then $l = m$ and there exists a permutation $\sigma \in S_m$ such that $A_i/A_{i-1}$ is a subobject of $B_{\sigma(i)}/B_{\sigma(i-1)}$ for all $i$.*

*Moreover, if $A_\bullet \to A$ is additionally a Jordan-Hölder filtration then the injections*

$$A_i/A_{i-1} \to B_{\sigma(i)}/B_{\sigma(i-1)}$$

*are isomorphisms for all $i$. In other words, if $A$ has a Jordan-Hölder filtration then every weak Jordan-Hölder is actually a Jordan-Hölder, and, in this case, the Jordan-Hölder factors are unique up to permutation.*

*Proof.* Choose the smallest $i$ such that $A_1$ is a subobject of $B_i$. By minimality of $i$, we obtain a nonzero morphism $A_1 \to B_i/Bi - 1$. By assumption, both $A_1$ and $B_i/B_{i-1}$ are weakly $\sigma$-stable, so the nonzero morphism $A_1 \to B_i/B_{i-1}$ must be injective by Schur's Lemma (Lemma 2.2.13.2). The result follows by induction on the weak Jordan-Hölder filtrations:

$$0 \to A_2/A_1 \to A_3/A_1 \to \cdots \to A_l/A_1 = A/A_1$$

and

$$0 \to (B_1 + A_1)/A_1 \to \cdots \to (B_{i-1} + A_1)/A_1 \to B_i/A_1 \to B_{i+1}/A_1 \to \cdots \to B_m/A_1 = A/A_1.$$

If $A_\bullet \to A$ is a Jordan-Hölder filtration then the injections $A_i/A_{i-1} \to B_{\sigma(i)}/B_{\sigma(i-1)}$ is an isomorphism by Schur's lemma (2.2.13.4). In particular, $B_{\sigma(i)}/B_{\sigma(i-1)}$ is $\sigma$-stable so $B_\bullet \to B$ is a Jordan-Hölder filtration. $\square$

Even though weak Jordan-Hölder filtrations are not afforded the standard uniqueness properties, they exist in much more general scenarios. Specifically, we will show that under relatively weak assumptions, $\sigma$-semistable objects of finite slope have weak Jordan-Hölder filtrations (Lemma 2.3.14). We first need some basic facts about $\sigma$-pure objects. We also provide some examples to elucidate $\sigma$-purity. Recall that an object $A \in \mathcal{A}$ is $\sigma$-pure if every nonzero subobject $0 \to B \to A$ satisfies $\Im Z(B) \neq 0$ (Definition 2.2.10).

**Example 2.3.12.** *Consider $\sigma_{\mu_H^D}$ - the very weak stability condition corresponding to $\mu_H^D$. Assume $\mathscr{E}$ is $\sigma_{\mu_H^D}$-pure. Therefore, every nonzero subsheaf of $\mathscr{E}$ has positive rank. In other words, every nonzero subsheaf is supported everywhere, so $\mathscr{E}$ has no torsion subsheaves - i.e. $\mathscr{E}$ is torsion-free or* 0.

*Conversely, if $\mathscr{E}$ is torsion-free or* 0 *then every nonzero subsheaf is supported everywhere so every nonzero subsheaf has positive rank. Thus, $\mathscr{E}$ is $\sigma_{\mu_H^D}$-pure.*

*Recall that $\mathscr{E}$ is torsion-free if and only if $\mathscr{E}$ is pure and supported everywhere (*[HL10, Definition 1.1.2]*). In other words, we have shown that $\mathscr{E}$ is $\sigma_{\mu_H^D}$-pure if and only if $\mathscr{E} = 0$ or $\mathscr{E}$ is pure and supported everywhere.*

*More generally, consider $\sigma_{d,\mu_H^D}$ from Example 2.2.5. Assume $\mathscr{E} \in \mathrm{Coh}_d(X)$ is $\sigma_{d,\mu_H^d}$-pure and consider a nonzero subsheaf $0 \to \mathscr{F} \to \mathscr{E}$. By definition of $Z_{d,\mu_H^D}$ and $\sigma_{d,\mu_H^D}$-purity, $H^d \cdot \mathrm{ch}_d^D(\mathscr{F}) \neq 0$, so $\mathscr{F}$ is supported in codimension at most $d$ by Lemma 2.1.3. On the other hand, since $\mathscr{F} \in \mathrm{Coh}_d(X)$, $\mathscr{F}$ is supported in codimension at least $d$. Therefore, every nonzero subsheaf of $\mathscr{E}$ is supported in codimension $d$. In other words, $\mathscr{E}$ is pure of dimension $d$ (*[HL10, Definition 1.1.2]*).*

*By Lemma 2.1.3 we also find that if $\mathscr{E}$ is pure of codimension $d$ then $\mathscr{E}$ is $\sigma_{d,\mu_H^D}$-pure.*

Here are some easy facts about $\sigma$-purity. Each of these results have a corresponding result for $\mu_H$-stability and torsion-free sheaves that is well known.

**Lemma 2.3.13.** *Assume $\sigma = (Z, \mathcal{A})$ is a very weak stability function and $A \in \mathcal{A}$.*

1. *If $A$ is $\sigma$-pure then any nonzero subobject $0 \to B \to A$ is $\sigma$-pure.*

2. *Assume $A$ has a Harder-Narsimhan filtration with maximal destabilizing subobject $0 \to A_1 \to A$. $A$ is $\sigma$-pure if and only if $\Im Z(A_1) \neq 0$.*

   *In particular, a $\sigma$-semistable object $A$ is $\sigma$-pure if and only if $\Im Z(A) \neq 0$.*

3. *If $0 \to B \to A \to C \to 0$ is a short exact sequence such that $B$ and $C$ are $\sigma$-pure then $A$ is $\sigma$-pure.*

4. *If $A$ is $\sigma$-pure and $0 \to A_1 \to A$ is a maximal destabilizing subobject then $A/A_1$ is $0$ or $\sigma$-pure.*

*Proof.* Assume $\sigma = (Z, \mathcal{A})$ is a very weak stability function and $A \in \mathcal{A}$.

1. Assume $A$ is $\sigma$-pure and $0 \to B \to A$ is a nonzero subobject. Consider a nonzero subobject $0 \to C \to B$. Since $C$ is a nonzero subobject of $A$, $\Im Z(C) \neq 0$, as desired.

2. Assume $A$ is $\sigma$-pure. In particular, $\Im Z(A_1) \neq 0$, as desired.

   Conversely, assume $\Im Z(A_1) \neq 0$ and consider a nonzero subobject $0 \to B \to A$. By Lemma 2.3.3, $\mu_\sigma(B) \leq \mu_\sigma(A_1)$. Since $\Im Z(A_1) \neq 0$, $\mu_\sigma(B) \leq \mu_\sigma(A_1) < +\infty$. In particular, $\Im Z(B) \neq 0$, as desired.

3. Consider a subobject $0 \to B' \to A$ such that $\Im Z(B') = 0$. Consider the composition $f : B' \to A \to C$. By additivity of $\Im Z$, $\Im Z(\operatorname{Im}(f)) = 0$. Since $C$ is $\sigma$-pure, it follows that $\operatorname{Im}(f) = 0$. Therefore, the injection $0 \to B' \to A$ factors through the injection $0 \to B \to A$. However, since $B$ is $\sigma$-pure and $\Im Z(B') = 0$, it follows that $B = 0$, as desired.

4. Assume $A/A_1 \neq 0$ and consider a subobject $0 \to B \to A/A_1$ satisfying $\Im Z(B) = 0$. We can find a subobject $0 \to B' \to A$ such that $B = B'/A_1$. Since $\Im Z(B) = 0$, by additivity, $\Im Z(B') = \Im Z(A_1)$. If $\mu_\sigma(B') = \mu_\sigma(A_1)$, by definition of a maximal destabilizing subobject, $B' = A_1$ and so $B = 0$. Therefore, we may assume $\mu_\sigma(B') < \mu_\sigma(A_1)$, so $-\Re Z(B') < -\Re Z(A_1)$. It follows that $\Im Z(B) = 0$ and $\Re Z(B) = \Re Z(B') - \Re Z(A_1) > 0$ which contradicts the definition of a very weak stability function. Therefore, $A_1 = B'$ and $B = 0$, as claimed.

□

We can now give an existence result for weak Jordan-Hölder filtrations for a very weak stability function:

**Lemma 2.3.14.** *Assume $\sigma = (Z : \Lambda \to \mathbb{R}, \mathcal{A})$ is a very weak stability function and the image of $\Im Z : \Lambda \to \mathbb{R}$ is discrete. If $A \in \mathcal{A}$ is $\sigma$-semistable, Noetherian, and satisfies $\Im Z(A) \neq 0$ then $A$ has a Jordan-Hölder filtration.*

*Proof.* Since $\Im Z(A) \neq 0$ and $A$ is $\sigma$-semistable, by Lemma 2.3.13.2, $A$ is $\sigma$-pure.

Consider the set $\mathcal{C}_1$ consisting of subobjects $0 \to B \to A$ such that $\mu_\sigma(B) = \mu_\sigma(A)$. We define an order on $\mathcal{C}_1$ where $0 \to B_1 \to A < 0 \to B_2 \to A$ if $\Im Z(B_1) < \Im Z(B_2)$. The identity $0 \to A \to A$ is in $\mathcal{C}$, so $\mathcal{C}$ is nonempty. Furthermore, since $\Im Z : \Lambda \to \mathbb{R}$ is discrete, any chain in $\mathcal{C}$ has a minimal object. Let $\mathcal{D}_1$ be the collection of all minimal objects in $\mathcal{C}_1$. We define a partial order on $\mathcal{D}_1$ where $0 \to B_1 \to A \leq 0 \to B_2 \to A$ if $0 \to B_1 \to A$ factors through $0 \to B_2 \to A$. Since $A$ is Noetherian, $\mathcal{D}_1$ has a maximal object. Let $0 \to A_1 \to A$ be the maximal object of $\mathcal{D}_1$. Since $A$ is $\sigma$-pure, $A_1$ is $\sigma$-pure by Lemma 2.3.13.

We claim that $A_1$ is weakly $\sigma$-stable. Since $A_1$ is $\sigma$-pure, by Lemma 2.2.11, it suffices to show that for every subobject $0 \to B \to A_1$ in $\mathcal{A}$ satisfying $0 < \Im Z(B) < \Im Z(A_1)$ we have $\mu_\sigma(B) < \mu_\sigma(A_1)$. However, by minimality of $A_1$ in $\mathcal{C}_1$, every subobject $0 < \Im Z(B) < \Im Z(A_1)$ satisfies $\mu_\sigma(B) \neq \mu_\sigma(A_1)$. Since $A$ is $\sigma$-semistable, it follows that $\mu_\sigma(B) < \mu_\sigma(A) = \mu_\sigma(A_1)$. Therefore, $A_1$ is weakly $\sigma$-stable, as desired.

If $\Im Z(A_1) = \Im Z(A)$ then, by definition of $\mathcal{C}_1$, $A = A_1$. In this case, we find that a Jordan-Hölder filtration of $A$ is given by $0 \to A$. Thus, we may assume that $\Im Z(A_1) < \Im Z(A)$. Define $\mathcal{C}_2$ to be the set of all subobjects $0 \to B \to A$ satisfying the following

- $0 \to A_1 \to A$ factors through $0 \to B \to A$,
- $\mu_\sigma(B) = \mu_\sigma(A)$, and
- $\Im Z(B) > \Im Z(A_1)$.

We define an order on $\mathcal{C}_2$ where $0 \to B_1 \to A < 0 \to B_2 \to A$ if $\Im Z(B_1) < \Im Z(B_2)$. Since $\Im Z(A_1) < \Im Z(A)$, the morphism $0 \to A_1 \to A$ is contained in $\mathcal{C}_2$, so $\mathcal{C}_2$ is non-empty. Furthermore, since $\Im Z : \Lambda \to \mathbb{R}$ is discrete, any chain in $\mathcal{C}_2$ has a minimal object. Let $\mathcal{D}_2$ be the set of all minimal objects in $\mathcal{C}_2$. We define a partial order on $\mathcal{D}_2$ where $0 \to B_1 \to A \leq \ 0 \to B_2 \to A$ if $0 \to B_1 \to A$ factors through $0 \to B_2 \to A$. Since $A$ is Noetherian, $\mathcal{D}_2$ has a maximal object. Let $0 \to A_2 \to A$ be a maximal object in $\mathcal{D}_2$.

Note that $A_2/A_1$ is well-defined because $0 \to A_1 \to A$ factors through $0 \to A_2 \to A$. We claim that $A_2/A_1$ is weakly $\sigma$-stable and $\mu_\sigma(A_2/A_1) = \mu_\sigma(A)$. We first show $\mu_\sigma(A_2/A_1) = \mu_\sigma(A)$. By construction, $\mu_\sigma(A_2) = \mu_\sigma(A_1)$, so, by the weak seesaw inequality (Lemma 2.2.6), $\mu_\sigma(A_2) \leq \mu_\sigma(A_2/A_1)$ with equality exactly when $Z(A_2/A_1) \neq 0$. Since $\Im Z(A_2) > \Im Z(A_1)$, $\Im Z(A_2/A_1) > 0$, so $Z(A_2/A_1) \neq 0$. Therefore, $\mu_\sigma(A) = \mu_\sigma(A_2) = \mu_\sigma(A_2/A_1)$, as needed. We now show $A_2/A_1$ is weakly $\sigma$-stable. Assume $0 \to B \to A_2/A_1$ is a subobject. We can write $B = B'/A_1$ where $B'$ is a subobject of $A_2$, so there is a short exact sequence

$$0 \to A_1 \to B' \to B \to 0$$

Since $B'$ is a subobject of $A_2$ and $0 \to A_1 \to A$ factors through the composition $0 \to B' \to A_2 \to A$, by minimality of $A_2$ in $\mathcal{C}_2$, one of the following must hold:

- $B' = A_2$,
- $\mu_\sigma(B') \neq \mu_\sigma(A)$, or

- $\Im Z(B') \leq \Im Z(A_1)$.

First, if $B' = A_2$ then $B = A_2/A_1$, so $B$ is not a proper subobject of $A_2/A_1$.

Second, if $\mu_\sigma(B') \neq \mu_\sigma(A)$, since $A$ is $\sigma$-semistable, $\mu_\sigma(B') < \mu_\sigma(A) = \mu_\sigma(A_1)$. By the weak seesaw inequality (Lemma 2.2.6), it follows that $\mu_\sigma(A_2/A_1) = \mu_\sigma(A_1) > \mu_\sigma(B') > \mu_\sigma(B)$, as needed.

Third, assume that $\Im Z(B') \leq \Im Z(A_1)$. We may additionally assume that $\mu_\sigma(B') = \mu_\sigma(A)$. By minimality of $A_1$ in $\mathcal{C}_1$, it follows that $\Im Z(B') = \Im Z(A_1)$. Therefore, $B'$ is a minimal object of $\mathcal{C}_1$. In other words, $B'$ is an object in $\mathcal{D}_1$. Since $A_1$ is a maximal object in $\mathcal{D}_1$ and $0 \to A_1 \to A$ factors through $0 \to B' \to A$, we know that $B' = A_1$. It follows that $B = B'/A_1 = 0$, so $B$ is not a nonzero subobject of $A_2/A_1$.

All cases considered, we find that $A_2/A_1$ is weakly $\sigma$-stable, as desired.

If $\Im Z(A_2) = \Im Z(A)$ then $A = A_2$ by definition of $\mathcal{C}_2$ and the filtration $0 \to A_1 \to A$ is a Jordan-Hölder filtration. If $\Im Z(A_2) < \Im Z(A)$ then we can continue this process inductively. Since $\Im Z : \Lambda \to \mathbb{R}$ is discrete, this process will stop in finitely many steps to give us a Jordan-Hölder filtration. $\square$

Later, we will use a deformation argument to extend this result to tilt stability—where $\Im Z$ is not necessarily discrete and $\mathcal{A}$ is not necessarily Noetherian (Lemma 3.3.2).

Lemma 2.3.14 does not extend to $\sigma$-semistable objects with $\Im Z(A) = 0$ as shown in the following example. In fact, in this example we completely classify coherent sheaves that have Jordan-Hölder filtration with respect to $\sigma_{\mu_H^D}$ to illustrate the theory.

**Example 2.3.15.** *We first deduce some general facts. If $\Im Z(A) = 0$ then any short exact sequence $0 \to B \to A \to C \to 0$ satisfies $\Im Z(B), \Im Z(C) = 0$. In this case, $\mu_\sigma(B) = \mu_\sigma(A) = \mu_\sigma(C) = +\infty$ Therefore, an object $A$ satisfying $\Im Z(A) = 0$ is weakly $\sigma$-stable if and only if $A$ contains no nonzero proper subobjects. In other words, and object satisfying $\Im Z(A) = 0$ is weakly $\sigma$-stable if and only if $A$ is simple.*

*With this in mind, consider the very weak stability condition associated to $\mu_H$-stability: $\sigma_{\mu_H^D}$. The only nonzero coherent sheaves with no nonzero proper subobjects are skyscraper sheaves. In other words, the weakly $\sigma$-stable objects with $\Im Z(\cdot) = 0$ are skyscraper sheaves. As we saw in Example 2.2.9, skyscraper sheaves are in fact $\sigma$-stable.*

*Assume $\mathscr{E}$ is $\sigma_{\mu_H}^D$-semistable.*

- *If $\dim(X) = 1$ then $\sigma_{\mu_H^D}$ is a stability function so $\mathscr{E}$ has a Jordan-Hölder filtratioon.*
- *Assume $\dim(X) \geq 2$ and $\mathscr{E} \cong \oplus_{i=1}^m \mathscr{O}_{x_i}$. Then $\mathscr{E}$ has a Jordan-Hölder filtration whose Jordan-Hölder factors are $\{\mathscr{O}_{x_i}\}_{i=1}^m$.*
- *Assume $\dim(X) \geq 2$ and $0 < \dim(\mathscr{E}) < \dim(X)$. Every subsheaf and quotient of $\mathscr{E}$ satisfies $\Im Z_{\mu_H}^D(\cdot) = 0$. Suppose, for contradiction, there is a weak Jordan-Hölder fiiltration of $\mathscr{E}$:*

  $$0 = \mathscr{E}_0 \to \mathscr{E}_1 \to \mathscr{E}_2 \to \cdots \to \mathscr{E}_{m-1} \to \mathscr{E}_m = \mathscr{E}.$$

  *By definition $\mathscr{E}_i/\mathscr{E}_{i-1}$ is weakly $\sigma_{\mu_H^D}$-stable for all $i = 1, \ldots, m$. Therefore, $\mathscr{E}_i/\mathscr{E}_{i-1}$ is a skycraper sheaf, so $\dim(\mathscr{E}_i/\mathscr{E}_{i-1}) = 0$ for all $i$. By induction, we find that $0 < \dim(\mathscr{E}) = \dim(\mathscr{E}_1)$. Thus, $\mathscr{E}_1$ is a weakly $\sigma_{\mu_H^D}$-stable object with $\Im Z_{\mu_H^D}(\mathscr{E}_1) = 0$ and $\dim(\mathscr{E}_1) > 0$, a contradiction.*
- *If $\dim(X) \geq 2$ and $\dim(\mathscr{E}) = \dim(X)$ (i.e. $\Im Z_{\mu_H^D}(\mathscr{E}) = \operatorname{rank}(\mathscr{E}) > 0$) then $\mathscr{E}$ has a Jordan-Hölder filtration by Lemma 2.3.14.*

  *By definition of a Jordan-Hölder filtration each Jordan-Hölder factor must be supported everywhere. In particular, by Example 2.2.9, each Jordan-Hölder factor is weakly $\sigma_{\mu_H^D}$-stable but not $\sigma_{\mu_H^D}$-stable.*

Note that the very weak stability function associated to $\mu_H^D$ satisfies both the Harder-Narasimhan and support properties.

**Example 2.3.16.** *The category* $\mathrm{Coh}(X)$ *is Noetherian and* $\operatorname{rank} : K_0(\mathrm{Coh}(X)) \to \mathbb{R}$ *is discrete. Therefore, $\sigma_{\mu_H^D}$ satisfies the Harder-Narasimhan property by Corollary 2.3.6.*

*By the weak Bogomolov inequality (Lemma 2.1.15), $\sigma_{\mu_H^D}$ satisfies the support property with respect to $\overline{\Delta}_H^D$.*

We end this subsection by presenting an important consequence of the support property.

**Definition 2.3.17.** *Assume $\sigma = (Z, \mathcal{A})$ is a very weak stability function satisfying the Harder-Narasimhan property. We say that a set of objects in $S \subseteq \mathcal{A}$ has bounded mass if*

$$\sup\{m_\sigma(A) \mid A \in S\} < +\infty.$$

*Recall that $m_\sigma$ is the mass of $A$ with respect to $\sigma$ (Definition 2.3.2).*

It is clear that any finite collection of objects in $D^b(X)$ has bounded mass. The converse is false; if $S$ has bounded mass then $S$ may not be finite. However, if $S$ has bounded mass and $\sigma = (Z, \mathcal{A})$ is a very weak stability function satisfying the Harder-Narasimhan property and the support property (plus a technical assumption) then the image in $\Lambda$ is finite. This result was first shown for $K3$ surfaces in [Bri08, Lemma 9.2].

**Lemma 2.3.18.** *Assume $\sigma = (Z : \Lambda \to \mathbb{C}, \mathcal{A})$ is a very weak stability function satisfying the Harder-Narasimhan property and the support property with respect to the quadratic form $Q$ on $\Lambda_\mathbb{R}$. If $Q$ has signature $(2, \operatorname{rank}(\Lambda) - 2)$ then every set of bounded mass has finite image in $\Lambda$.*

*Proof.* Assume that $S \subseteq \mathcal{A}$ has bounded mass. It suffices to show that $\{||A||^2 \mid A \in S\}$ is bounded in $\Lambda_\mathbb{R}$ for some norm $||\cdot||$ on $\Lambda_\mathbb{R}$. Let $M = \sup\{m_\sigma(A) \mid A \in S\}$.

Since $Q$ has signature $(2, \dim_\mathbb{R}(\Lambda_\mathbb{R}) - 2)$ we can find a basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_r\}$ of $\Lambda_\mathbb{R}$ such that

$$Q(A) = Q\left(\sum_{i=1}^k a_i \mathbf{e}_i\right) = a_1^2 + a_2^2 - a_3^2 - \cdots - a_r^2.$$

Since $Q$ is negative definite with respect to the kernel of $Z$, by definition,

$$\operatorname{Ker}(Z) \otimes \mathbb{R} \subseteq \operatorname{span}\{\mathbf{e}_3, \mathbf{e}_4, \ldots, \mathbf{e}_r\}.$$

On the other hand, by the fundamental theorem of linear algebra, $\dim_{\mathbb{R}} \operatorname{Ker}(Z) \geq \operatorname{rank}(\Lambda) - \dim_{\mathbb{R}}(\mathbb{C}) = r - 2$, so

$$\operatorname{Ker}(Z) \otimes \mathbb{R} = \operatorname{span}\{\mathbf{e}_3, \mathbf{e}_4, \dots, \mathbf{e}_r\}$$

Thus, we can choose another basis $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}_3, \dots, \mathbf{e}_r\}$ of $\Lambda_{\mathbb{R}}$ such that

$$Q\left(a_1\mathbf{e}'_1 + a_2\mathbf{e}'_2 + a_3\mathbf{e}_3 + \cdots + a_r\mathbf{e}_r\right) = |Z(a_1\mathbf{e}'_1 + a_2\mathbf{e}'_2)| - a_3^2 - \cdots - a_r^2.$$

where $Z$ is defined on $\Lambda_{\mathbb{R}}$ by linearity. Also, define a norm $||\cdot||$ on $\Lambda_{\mathbb{R}}$ by

$$||a_1\mathbf{e}'_1 + a_2\mathbf{e}'_2 + a_3\mathbf{e}_3 + \cdots + a_r\mathbf{e}_r|| = |Z(a_1\mathbf{e}'_1 + a_2\mathbf{e}'_2)| + a_3^2 + \cdots + a_r^2.$$

By definition of mass, if $A_i/A_{i-1}$ is a $\sigma$-semistable factor of $A \in S$ then $m_\sigma(A_i/A_{i-1}) < m_\sigma(A)$. Furthermore, we know that $A = \sum_{i=1}^m A_i/A_{i-1}$ in $\Lambda$. Therefore, if the set

$$S' = \{A_i/A_{i-1} \in \mathcal{A} \mid A_i/A_{i-1} \text{ is a } \sigma - \text{semistable factor of some } A \in S\}.$$

is finite in $\Lambda$ then $S$ is finite in $\Lambda$ as well. Thus, without loss of generality, we may assume that every object $A \in S$ is $\sigma$-semistable.

Let $A \in S$ and write $A = a_1\mathbf{e}'_1 + a_2\mathbf{e}'_2 + a_3\mathbf{e}_3 + \cdots + a_r\mathbf{e}_r$ in $\Lambda_{\mathbb{R}}$. Since $A$ is $\sigma$-semistable, by the support propery,

$$0 \leq Q(A) = Q\left(a_1\mathbf{e}'_1 + a_2\mathbf{e}'_2 + a_3\mathbf{e}_3 + \cdots + a_r\mathbf{e}_r\right) = |Z(A)| - a_3^2 - \cdots - a_r^2.$$

In other words,

$$a_3^2 + a_4^2 + \cdots + a_r^2 \leq |Z(A)|.$$

It follows that

$$||A|| = |Z(A)| + a_3^2 + \cdots + a_r^2 \leq 2|Z(A)| \leq 2M.$$

Hence, $S$ is bounded in $\Lambda_{\mathbb{R}}$ and so must be finite, as desired. □

While the assumption on the signature of $Q$ in Lemma 2.3.18 may seem restrictive, in most cases we can reduce to this case using [Bay19, Lemma 7.2].

## 2.4 Stability Conditions on a Triangulated Category

In this section we recall very weak stability conditions on a triangulated category $\mathcal{D}$. All the definitions and results of this subsection are standard.

Broadly, a very weak stability condition on a triangulated category $\mathcal{D}$ is a very weak stability function satisfying the Harder-Narasimhan and support properties such that the associated abelian category embeds "nicely" into $\mathcal{D}$. The prototypical example is $\mu_H$-stability on $\mathrm{Coh}(X)$ with the natural embedding into $D^b(X)$.

Surprisingly, the collection of stability conditions on $\mathcal{D}$ satisfy a deformation property - the collection is locally homeomorphic to $\mathbb{C}^k$. In other words, there is a topology on the collection of stability condition on $\mathcal{D}$ such that each connected component is a complex manifold. For very weak stability conditions on $\mathcal{D}$ the deformation property is much weaker. Namely, there is a topology on the collection of very weak stability conditions on $\mathcal{D}$ that locally injects into $\mathbb{C}^k$.

We begin this subsection by introducing the heart of a bounded $t$-structure on $\mathcal{D}$. These are abelian categories that embed "nicely" into $\mathcal{D}$.

**Definition 2.4.1.** *Assume $\mathcal{D}$ is a triangulated category. We say that an full additive subcategory $\mathcal{A}$ of $\mathcal{D}$ is the heart of a bounded $t$-structure if both of the following conditions are satisfied.*

- *If $k_1 > k_2$ and $A, B \in \mathcal{A}$ then $\mathrm{Hom}_{\mathcal{D}}(A[k_1], B[k_2]) = 0$.*
- *For every nonzero object $E \in \mathcal{D}$ there exists integers $k_1 > k_2 > \cdots > k_m$, objects $A_1, A_2, \ldots, A_m \in \mathcal{A}$, and objects $E_1, \ldots, E_{m-1} \in \mathcal{D}$ that fit into the following collection of distinguished triangles*

$$\begin{tikzcd}
0 \arrow[r] & E_1 \arrow[r] \arrow[d] & E_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & E_{m-1} \arrow[r] \arrow[d] & E \arrow[d] \\
& A_1[k_1] \arrow[ul, dashed] & A_2[k_2] \arrow[ul, dashed] & \cdots \arrow[ul, dashed] & A_{m-1}[k_{m-1}] \arrow[ul, dashed] & A_m[k_m] \arrow[ul, dashed]
\end{tikzcd}.$$

*where*

$$\begin{tikzcd} E_i \arrow[r] & E_{i+1} \arrow[d] \\ & A_{i+1} \arrow[ul, dashed] \end{tikzcd}$$

*denotes the distinguished triangle* $E_i \to E_{i+1} \to A_{i+1} \to E_i[1]$

Traditionally, one defines $t$-structures then defines its heart. These notions first appear in [BBD82, Definition 1.3.1]. If we only consider bounded $t$-structures then there is a canonical bijection between $t$-structures in the sense of [BBD82, Definition 1.3.1] and Definition 2.4.1 by [Bri08, Lemma 3.1]. For this reason, we avoid introducing the theory of $t$-structures.

The following lemma notes some standard facts about hearts of bounded $t$-structures. We will use these facts tacitly throughout.

**Lemma 2.4.2** ([BBD82])**.** *Assume $\mathcal{A}$ is the heart of a bounded $t$-structure on $\mathcal{D}$.*

1. *$\mathcal{A}$ is an abelian category.*
2. *Assume $A, B, C \in \mathcal{A}$. Then $0 \to B \to A \to C \to 0$ is exact in $\mathcal{A}$ if and only if $B \to A \to C \to B[1]$ is a distinguished triangle in $\mathcal{D}$.*
3. *$\mathcal{A}$ is stable under extensions. In other words, if $B, C \in \mathcal{A}$ and $B \to A \to C \to B[1]$ is a distinguished triangle in $\mathcal{D}$ then $A \in \mathcal{A}$.*

If $\mathcal{A}$ is an abelian category such that there is an equivalence of triangulated categories $D^b(\mathcal{A}) = \mathcal{D}$ then $\mathcal{A}$ is the heart of a bounded $t$-structure on $\mathcal{D}$. However, the converse is false. Specifically, if $\mathcal{A}$ is the heart of a bounded $t$-structure on $\mathcal{D}$ then there may not be an equivalence of triangulated categories between $D^b(\mathcal{A})$ and $\mathcal{D}$. A counter example is described in [MS17, Exercise 5.3] for $D^b(\mathbb{P}^1)$. However, if $\mathcal{A}$ is the heart of a bounded $t$-structure on $\mathcal{D}$ then there is a natural isomorphism of Grothendieck groups $K_0(D^b(\mathcal{A})) = K_0(\mathcal{A}) = K_0(\mathcal{D})$.

We defined the Grothendieck group of an abelian category in Definition 2.1.8. One can define the Grothendieck group of a triangulated category similarly.

**Definition 2.4.3.** *We define the Grothendieck group of a triangulated category $\mathcal{D}$, written $K_0(\mathcal{D})$, to be the free abelian group generated by objects in $\mathcal{D}$ with relations $E = F + G$ whenever there is a distinguished triangle $F \to E \to G \to F[1]$.*

By definition of a triangulated category, for any object $E \in \mathcal{D}$ there is a distinguished triangle $E \to 0 \to E[1] \to E[1]$. Therefore, by definition of $K_0(\mathcal{D})$, we know that $-[E] = [E[1]]$. More generally, if $k \in \mathbb{Z}$ then $(-1)^k[E] = [E[k]]$.

**Lemma 2.4.4.** *If $\mathcal{A}$ is the heart of a bounded $t$-structure on $\mathcal{A}$ then there is a natural isomorphism $K_0(\mathcal{A}) = K_0(\mathcal{D})$.*

*Proof.* The inclusion $\mathcal{A} \to \mathcal{D}$ induces a morphism $\iota : K_0(\mathcal{A}) \to K_0(\mathcal{D})$ which is well-defined by Lemma 2.4.2.3. We will construct an inverse $\phi : K_0(\mathcal{D}) \to K_0(\mathcal{A})$.

Assume $E \in \mathcal{D}$. By definition of the heart of a bounded $t$-structure we can find a collection of distinguished triangles:

$$\begin{tikzcd}
0 \arrow[r] & E_1 \arrow[r] \arrow[d] & E_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & E_{m-1} \arrow[r] \arrow[d] & E \arrow[d] \\
& A_1[k_1] \arrow[ul, dashed] & A_2[k_2] \arrow[ul, dashed] & \cdots \arrow[ul, dashed] & A_{m-1}[k_{m-1}] \arrow[ul, dashed] & A_m[k_m] \arrow[ul, dashed]
\end{tikzcd}$$

appearing in the definition of the heart of a bounded $t$-structure. We define $\phi : K_0(\mathcal{D}) \to K_0(\mathcal{A})$ by $\phi([E]) = \sum_{i=1}^{m} [A_i](-1)^{k_m}$.

We first claim that $\phi$ is well-defined. By uniqueness of this collection of distinguished triangles (Lemma 2.4.2.5), it suffices to show that this construction is additive in distinguished triangles. Assume $F \to E \to G \to F[1]$ is a distinguished triangle. We proceed by induction on $m$ appearing in Definition 2.4.2.b (i.e. the number of distinguished triangles in the collection for $E$).

If $m = 1$ then $E = A[k]$ for some $A \in \mathcal{A}$ and $k \in \mathbb{Z}$. Therefore, we have a distinguished triangle $F \to A[k] \to G \to F[1]$. Thus, if

$$\begin{tikzcd}
0 \arrow[r] & G_1 \arrow[r] \arrow[d] & G_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & G_{n-1} \arrow[r] \arrow[d] & G \arrow[d] \\
& C_1[l_1] \arrow[ul, dashed] & C_2[l_2] \arrow[ul, dashed] & \cdots \arrow[ul, dashed] & C_{n-1}[l_{n-1}] \arrow[ul, dashed] & C_n[l_n] \arrow[ul, dashed]
\end{tikzcd}$$

is a collection of distinguished triangles appearing in Definition 2.4.2.b, then

$$\begin{tikzcd}
0 \arrow[r] & G_1[-1] \arrow[r] \arrow[d] & \cdots \arrow[r] & G_{n-1}[-1] \arrow[r] \arrow[d] & G[-1] \arrow[r] \arrow[d] & F \arrow[d] \\
 & C_1[l_1-1] \arrow[ul, dashed] & \cdots \arrow[ul, dashed] & C_{n-1}[l_{n-1}-1] \arrow[ul, dashed] & C_n[l_n-1] \arrow[ul, dashed] & A[k] \arrow[ul, dashed]
\end{tikzcd}$$

is such a collection for $F$. Therefore

$$\phi(F)+\phi(G)=\left(\sum_{i=1}^{n}[C_n](-1)^{l_i}\right)+\left([A](-1)^k+\sum_{i=1}^{n}[C_n](-1)^{l_i-1}\right)=[A](-1)^k=\phi(E),$$

as desired.

Consider the collection of distinguished triangles:

$$\begin{tikzcd}
0 \arrow[r] & E_1 \arrow[r] \arrow[d] & E_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & E_{m-1} \arrow[r] \arrow[d] & E \arrow[d] \\
 & A_1[k_1] \arrow[ul, dashed] & A_2[k_2] \arrow[ul, dashed] & \cdots \arrow[ul, dashed] & A_{m-1}[k_{m-1}] \arrow[ul, dashed] & A_m[k_m] \arrow[ul, dashed]
\end{tikzcd}.$$

The composition $E_{m-1} \to E \to G$ gives us a distinguished triangle

$$E_{m-1} \to G \to \mathrm{Cone}(E_{m-1} \to G).$$

By applying the octohedral axiom to the distinguished triangles:

$$E_{m-1} \to E \to A_m[k_m]$$

$$E_{m-1} \to G \to \mathrm{Cone}(E_{m-1} \to G)$$

$$E \to G \to F[1]$$

we obtain a distinguished triangle $A_m[k_m] \to \mathrm{Cone}(E_{m-1} \to G) \to F[1]$. The inductive hypothesis applies to $A_m[k_m]$ and $E_{m-1}$, so by considering the distinguished triangles

$$\mathrm{Cone}(E_{m-1} \to G)[-1] \to E_{m-1} \to G$$

$$F \to A_m[k_m] \to \mathrm{Cone}(E_{m-1} \to G).$$

we find that

$$\phi(E_{m-1}) = \phi(G) - \phi(\mathrm{Cone}(E_{m-1} \to G))$$

$$\phi(A_m[k_m]) = \phi(F) + \phi(\mathrm{Cone}(E_{m-1} \to G)).$$

It follows that

$$\phi(E) = \phi(E_{m-1}) + \phi(A_m[k_m]) = \phi(G) + \phi(F),$$

as claimed. Thus, we have shown that $\phi : K_0(\mathcal{D}) \to K_0(\mathcal{A})$ is well-defined.

By construction, $\phi$ and $\iota$ are inverses. $\square$

**Remark 2.4.5.** *Assume that $\sigma = (Z : \Lambda \to \mathbb{C}, \mathcal{A})$ is a very weak stability function and $\mathcal{A}$ is the heart of a bounded $t$-structure. If $E \in \mathcal{D}$, we can identify $E$ with its image in $K_0(\mathcal{D})$ and so an element in $K_0(\mathcal{A})$ by Lemma 2.4.4. Thus, we obtain an element in $\Lambda$ associated with $E$. We will define $Z(E)$ to be $Z$ applied to this element in $E$. In particular, if $E \in \mathcal{D}$ then we can define the slope $\mu_\sigma(E)$.*

*On the other hand, if we have a group homomorphism $K_0(\mathcal{D}) \to \Lambda$ then we naturally obtain a group homomorphism $K_0(\mathcal{A}) = K_0(\mathcal{D}) \to \Lambda$ by Lemma 2.4.4. In short, fixing a group homomorphism $K_0(\mathcal{D}) \to \Lambda$ is equivalent to fixing a group homomorphism $K_0(\mathcal{A}) \to \Lambda$.*

The definition of the heart of a bounded $t$-structure seems especially restrictive, but there are a few standard techniques to construct them.

The first technique is to give an equivalence of triangulated categories $D^b(\mathcal{A}) = \mathcal{D}$. We will eventually restrict our attention to $D^b(X) = D^b(\mathrm{Coh}(X))$ so we have the "standard" heart $\mathrm{Coh}(X)$. Surprisingly, if $X$ has a full strong exceptional collection and $Q$ is the quiver associated to this collection then there is equivalence of triangulated categories $D^b(X) = D^b(\mathrm{Rep}_k(Q))$ (see [Bon90, Theorem 6.2 and definitions within]). Therefore, $\mathrm{Rep}_k(Q)$ is the heart of a bounded $t$-structure on $D^b(X)$. However, since full strong exceptional collections are difficult to construct, this surprising fact is not sufficiently general for our applications. For example, a folklore conjecture is that a smooth projective surface admitting a full strong exceptional collection is rational.

The second technique is to to find a semiorthogonal decomposition of $\mathcal{D}$ such that the semiorthogonal subcategories each admit the heart of a bounded $t$-structure. If each of the semiorthogonal subcategories is sufficiently "nice" (i.e. they satisfy a six functor formalism) then we can glue the hearts of bounded $t$-structure of each subcategory to obtain a heart of a bounded $t$-structure on $\mathcal{D}$ ([BBD82, Theorem 1.4.10]).

The third technique for constructing the heart of a bounded $t$-structure is by starting with the heart of a bounded $t$-structure and tilting with respect to a torsion pair. This method is due to [HRS96]. This is the only construction that we describe in more detail.

**Definition 2.4.6.** *Assume $\mathcal{A}$ is an abelian category. We say that a pair of full additive subcategories $(\mathcal{T}, \mathcal{F})$ of $\mathcal{A}$ is a torsion pair if the following two conditions hold:*

- *If $T \in \mathcal{T}$ and $F \in \mathcal{F}$ then $\mathrm{Hom}_{\mathcal{A}}(T, F) = 0$,*
- *If $A \in \mathcal{A}$ then there exists a short exact sequence*

$$0 \to T \to A \to F \to 0$$

  *with $T \in \mathcal{T}$ and $F \in \mathcal{F}$.*

*We call $\mathcal{T}$ a torsion class and $\mathcal{F}$ a torsion-free class.*

The short exact sequence appearing in the definition above is unique up to isomorphism of short exact sequences.

The prototypical example of a torsion pair is given by the full additive subcategory of $\mathrm{Coh}(X)$ generated by torsion sheaves and the full additive subcategory generated by torsion-free sheaves (forming the torsion and torsion-free classes respectively).

Torsion pairs first appear in [Dic66, Bottom of Page 224] where they are called a torsion theory. Dickson shows that torsion pairs are easy to construct on categories with all coproducts or products [Dic66, Theorem 2.3]. A similar argument gives a construction for Noetherian or Artinian categories.

Using a stability function to construct a torsion pair was first used in [Bri08, Lemma 6.1]. The following is a natural generalization of that result to very weak stability functions.

**Lemma 2.4.7.** *Assume that $\sigma = (Z, \mathcal{A})$ is a very weak stability satisfying the Harder-Narasimhan property. The full additive subcategories of $\mathcal{A}$ generated by*

$$\mathcal{T} = \{A \in \mathcal{A} \mid \textit{Any nonzero quotient } A \to C \to 0 \textit{ satisfies } \mu_\sigma(C) > 0\}$$

*and*

$$\mathcal{F} = \{A \in \mathcal{A} \mid \textit{Any nonzero subobject } 0 \to B \to A \textit{ satisfies } \mu_\sigma(B) \leq 0\}$$

*form a torsion pair.*

*Proof.* Let $T \in \mathcal{T}$ and $F \in \mathcal{F}$. By definition of $\mathcal{T}$ and $\mathcal{F}$, we know that $\mu_\sigma^-(T) > 0$ and $\mu_\sigma^+(F) \leq 0$. It follows by Lemma 2.3.4 that $\operatorname{Hom}_{\mathcal{A}}(T, F) = 0$, as needed.

Consider $A \in \mathcal{A}$ with a Harder-Narasimhan filtration

$$0 \to A_1 \to \cdots \to A_{m-1} \to A.$$

If $\mu_\sigma^-(A) > 0$ then $A \in \mathcal{T}$ and a short exact sequence of the necessary form is given by $0 \to A \to A \to 0 \to 0$, as needed. Therefore, we may assume that $\mu_\sigma^-(A) \leq 0$. If $\mu_\sigma^+(A) \leq 0$ then $A \in \mathcal{F}$ and a short exact sequence of the necessary form is given by $0 \to 0 \to A \to A \to 0$, as needed. Therefore, we may also assume that $\mu_\sigma^+(A) > 0$.

Choose the largest integer $i$ such that $\mu_\sigma(A_i/A_{i-1}) > 0$ (which exists because $\mu_\sigma^+(A) > 0$). Since $A_i/A_{i-1}$ is the minimal destabilizing quotient of $A_i$, by Lemma 2.3.3.3, $A_i \in \mathcal{T}$. It remains to show that $A/A_i \in \mathcal{F}$. Note that

$$0 \to A_{i+1}/A_i \to A_{i+2}/A_i \to \cdots \to A_{m-1}/A_i \to A/A_i$$

is a Harder-Narasimhan filtration of $A/A_i$. In particular, $A_{i+1}/A_i$ is a maximal destabilizing suboject of $A/A_i$. Note that $i + 1 \leq m$ by the assumption $\mu_\sigma^{-1}(A) \leq 0$, so $A_{i+1}$ is defined. By maximality of $i$, we find that $0 \geq \mu_\sigma(A_{i+1}/A_i) = \mu_\sigma^+(A/A_i)$. On the other hand, since $T \in \mathcal{T}$, we know that $\mu_\sigma^-(T) > 0$. It follows that $A/A_i \in \mathcal{F}$, as desired. $\square$

We give two more examples of torsion-pairs in Lemma 3.2.1

If we have a torsion pair, then the following result allows us to construct a heart of a bounded $t$-structure on the derived category.

**Lemma 2.4.8** ([HRS96, Proposition 2.1])**.** *Assume $\mathcal{A}$ is an abelian category. Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $\mathcal{A}$.*

1. *The full additive subcategory of $D^b(\mathcal{A})$ generated by*

$$\mathcal{A}^\sharp = \{E \in D^b(\mathcal{A}) \mid \mathcal{H}^0(E) \in \mathcal{T}, \mathcal{H}^{-1}(X) \in \mathcal{F}, \mathcal{H}^i(X) = 0 \text{ otherwise}\}$$

   *is the heart of a bounded $t$-structure on $D^b(\mathcal{A})$.*

   *In this case, we say that $\mathcal{A}^\sharp$ is obtained from $\mathcal{A}$ by tilting with respect to $(\mathcal{T}, \mathcal{F})$.*

2. *$(\mathcal{F}[1], \mathcal{T})$ is a torsion pair in $\mathcal{A}^\sharp$.*

Tilting the heart of a bounded $t$-structure generally does not preserve categorical properties. For example, $\mathcal{A}$ can be Noetherian while $\mathcal{A}^\sharp$ is not (and vice versa).

We now define stability conditions on a triangulated category $\mathcal{D}$.

**Definition 2.4.9.** *Fix a group finite rank lattice $\Lambda$ and a group homomorphism $K_0(\mathcal{D}) \to \Lambda$.*

1. *A very weak stability condition on $\mathcal{D}$ is a very weak stability function $\sigma = (Z : \Lambda \to \mathbb{C}, \mathcal{A})$ such that*

   - *$\mathcal{A}$ is the heart of a bounded $t$-structure on $\mathcal{D}$ and*
   - *$\sigma$ satisfies both the Harder-Narasimhan and support properties.*

2. *A Bridgeland stability condition (also called a stability condition) is a very weak stability condition $\sigma = (Z : \Lambda \to \mathbb{C}, \mathcal{A})$ such that $\sigma$ is a stability function (i.e. $Z$ satisfies the positivity property of Definition 2.2.1).*

3. *Assume $\sigma = (Z, \mathcal{A})$ is a very weak stability condition on $\mathcal{D}$ and $A \in \mathcal{A}$ is nonzero. We say that $A$ is $\sigma$-(semi)stable (resp. weakly $\sigma$-(semi)stable) if $A$ is $\sigma$-(semi)stable (resp. weakly $\sigma$-(semi)stable) in the sense Definition 2.2.3. In particular, if $E \in \mathcal{D}$ but $E \notin \mathcal{A}$ then we will not discuss stability of $E$.*

The surprising result of [Bri07] is that stability conditions on $D^b(X)$ with $X$ a smooth variety actually form a topological space where each connected component is a complex manifold! Bridgeland's argument can be generalized to include very weak stability conditions but we only obtain obtain a local embedding into $\mathbb{C}^n$.

**Proposition 2.4.10** ([Bri07])**.** *Fix a finite rank quotient* $K_0(X) \to \Lambda$ *and let* $\mathrm{WStab}(X, \Lambda)$ *be the set of all very weak stability conditions* $\sigma = (Z : \Lambda \to \mathbb{C}, \mathcal{A})$*. We give* $\mathrm{WStab}(X, \Lambda)$ *the coarsest topology such that*

- $\mathrm{WStab}(X, \Lambda) \to \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{C})$ *given by* $(Z, \mathcal{A}) \mapsto Z$ *is continuous.*
- $\mathrm{WStab}(X, \Lambda) \to \mathbb{R} \cup \{+\infty\}$ *given by* $\sigma \mapsto \mu_\sigma^+(E)$ *is continuous for all* $E \in D^b(X)$*.*
- $\mathrm{WStab}(X, \Lambda) \to \mathbb{R} \cup \{+\infty\}$ *given by* $\sigma \mapsto \mu_\sigma^-(E)$ *is continuous for all* $E \in D^b(X)$*.*

*Recall* $\mu_\sigma^+$ *and* $\mu_\sigma^-$ *are the slopes of the maximal destabilizing and minimal destabilizing objects respectively- see Definition 2.3.2 The morphism* $\mathrm{WStab}(X, \Lambda) \to \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{C})$ *given by* $(Z, \mathcal{A}) \mapsto Z$ *is a continuous local injection.*

*Furthermore, let* $\mathrm{Stab}(X, \Lambda)$ *is the subset of* $\mathrm{WStab}(X, \Lambda)$ *consisting of stability conditions with the subspace topology. The induced continuous function* $\mathrm{Stab}(X, \Lambda) \to \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{C})$ *is a local homeomorphism. In other words, each connected component of* $\mathrm{Stab}(X, \Lambda)$ *is a complex manifold of dimension* $\mathrm{rank}(\Lambda)$*.*

# CHAPTER 3

# TILT STABILITY

In this chapter, we recall the construction of tilt stability. In addition, we describe the wall structure of the $(H, D)$-slice. This includes our new wall-crossing theorem. The chapter is structured as follows.

§3.1 We recall the construction of tilt stability which is a family $\sigma^{\text{tilt}}_{\alpha,\beta}$ of very weak stability functions parameterized by $(\beta, \alpha) \in \mathbb{R} \times \mathbb{R}_{>0}$, ample divisor $H$, and $\mathbb{Q}$-divisor $D$ (culumatively called an $(H, D)$-slice). We also completely characterize objects with good quoitients and discuss a deformation property of $\sigma^{\text{tilt}}_{\alpha,\beta}$-pure objects.

§3.2 We show $\sigma^{\text{tilt}}_{\alpha,\beta}$ is a very weak stability condition when $\beta \in \mathbb{Q}$. In addition, we prove a general form of the Large Volume Limit—which is a tool to compare $\sigma^{\text{tilt}}_{\alpha,\beta}$-stable objects with $\mu_H$-stable sheaves.

§3.3 We show $\sigma^{\text{tilt}}_{\alpha,\beta}$ is a very weak stability condition when $\beta \in \mathbb{R}$. Using the theory of pure objects we developed, this argument is much simpler than previous arguments. In addition, we show that any $\sigma^{\text{tilt}}_{\alpha,\beta}$-semistable object, $E$, with $\Im Z^{\text{tilt}}_{\alpha,\beta}(E) \neq 0$ has a weak Jordan-Hölder filtration.

§3.4 We discuss the wall and chamber structure of the $(H, D)$-slice.

§3.5 We give explicit bounds on the largest wall in the $(H, D)$-slice.

§3.6 We prove the main technical theorem of this thesis: our wall-crossing result for tilt stability.

## 3.1 Tilt Stability as a Very Weak Stability Function

In this subsection, we introduce tilt stability. For each $\beta \in \mathbb{R}$, there is a heart of a bounded $t$-structure $\mathrm{Coh}^{D+\beta H}_H(X)$ on $D^b(X)$. The general theory first appeared in [Bri08] for $K3$

surfaces, [AB13] for any surface, and [BMT14] for any variety.

We also show that for each positive real number $\alpha > 0$ there is a very weak stability function $\sigma^{\text{tilt}}_{\alpha,\beta}$ with heart $\mathrm{Coh}^{D+\beta H}_{H}(X)$. We end this subsection by investigating $\sigma^{\text{tilt}}_{\alpha,\beta}$-pure objects and objects with good quotients. We can completely classify objects in $\mathrm{Coh}^{D+\beta H}_{H}(X)$ that have good quotients and, surprisingly, it contains more than just simple objects (Lemma 3.1.13). We also show $\sigma^{\text{tilt}}_{\alpha,\beta}$-pure objects have especially nice deformation properties (Lemma 3.1.10).

**Definition 3.1.1.** *Fix a $\mathbb{Q}$-divisor $D$ and an ample divisor $H$ on $X$. For each $\beta \in \mathbb{R}$ we define*

- $\mathcal{T}^{D+\beta H}_{H}(X)$ *to be the full subcategory of* $\mathrm{Coh}(X)$ *generated by*

$$\{\mathscr{E} \in \mathrm{Coh}(X) \mid \text{ Any nonzero quotient } \mathscr{E} \to \mathscr{G} \to 0 \text{ satisfies } \mu^{D+\beta H}_{H}(\mathscr{G}) > 0\}$$

- $\mathcal{F}^{D+\beta H}_{H}(X)$ *to be the full subcategory of* $\mathrm{Coh}(X)$ *generated by*

$$\{\mathscr{E} \in \mathrm{Coh}(X) \mid \text{ Any nonzero subsheaf } 0 \to \mathscr{F} \to \mathscr{E} \text{ satisfies } \mu^{D+\beta H}_{H}(\mathscr{F}) \leq 0\}.$$

- $\mathrm{Coh}^{D+\beta H}_{H}(X)$ *to be the full subcategory of* $D^b(X)$ *generated by*

$$\{E \in D^b(X) \mid \mathscr{H}^{-1}(E) \in \mathcal{F}^{D+\beta H}_{H}(X), \mathscr{H}^{0}(E) \in \mathcal{T}^{D+\beta H}_{H}(X), \mathscr{H}^{i}(E) = 0 \text{ if } i \neq -1, 0\}$$

**Remark 3.1.2.** *Notice that if* $\mathrm{rank}(\mathscr{E}) \neq 0$ *then*

$$\begin{aligned}
\mu^{D+\beta H}_{H}(\mathscr{E}) &= \frac{H^{n-1} \cdot \mathrm{ch}^{D+\beta H}_{1}(\mathscr{E})}{\mathrm{rank}(\mathscr{E})} \\
&= \frac{H^{n-1} \cdot \mathrm{ch}_{1}(\mathscr{E}) - H^{n-1} \cdot (D + \beta H) \cdot \mathrm{rank}(\mathscr{E})}{\mathrm{rank}(\mathscr{E})} \\
&= \frac{H^{n-1} \cdot \mathrm{ch}^{D}_{1}(\mathscr{E}) - \beta H^{n} \cdot \mathrm{rank}(\mathscr{E})}{\mathrm{rank}(\mathscr{E})} \\
&= \mu^{D}_{H}(\mathscr{E}) - \beta H^{n} \\
&= \mu^{D}_{H}(\mathscr{E}) - \beta H^{n}
\end{aligned}$$

*Therefore,* $\mu_H^{D+\beta H}(\mathscr{E}) > 0$ *if and only if* $\mu_H^D(\mathscr{E}) > \beta H^n$ *(and similarly for* $\leq$*).*

*In short,* $\mathcal{T}_H^{D+\beta H}(X)$ *can also be written as the full subcategory of* $\mathrm{Coh}(X)$ *generated by*

$$\{\mathscr{E} \in \mathrm{Coh}(X) \mid \text{ Any nonzero quotient } \mathscr{E} \to \mathscr{G} \text{ satisfies } \mu_H^D(\mathscr{G}) > \beta H^n\}$$

*while* $\mathcal{F}_H^{D+\beta H}(X)$ *can also be written as the full subcategory of* $\mathrm{Coh}(X)$ *generated by*

$$\{\mathscr{E} \in \mathrm{Coh}(X) \mid \text{ Any nonzero subsheaf } \mathscr{F} \to \mathscr{E} \text{ satisfies } \mu_H^D(\mathscr{F}) \leq \beta H^n\}$$

An injection of coherent sheaves does not induce an injection in $\mathrm{Coh}_H^{D+\beta H}(X)$. However, a maximal $\mu_H$-destabilizing subsheaf does induce an injection in $\mathrm{Coh}_H^{D+\beta H}(X)$:

**Lemma 3.1.3.** *Let* $\mathscr{E} \in \mathrm{Coh}(X)$*.*

1. *Assume* $\mathscr{E}$ *is not* $\mu_H$*-semistable. If* $0 \to \mathscr{F} \to \mathscr{E}$ *is a maximal* $\mu_H$*-destabilizing subobject then* $\mu_H^+(\mathscr{E}/\mathscr{F}) \leq \mu_H^+(\mathscr{F})$*.*

   *In particular, in this case,* $0 \to \mathscr{F}[1] \to \mathscr{E}[1]$ *is injective in* $\mathrm{Coh}_H^{D+\beta H}(X)$ *for all* $\beta \geq \mu_H^{D+}(\mathscr{E})$*.*

2. *Assume* $\mathscr{E}$ *is* $\mu_H$*-semistable. If* $0 \to \mathscr{F} \to \mathscr{E}$ *is a coherent subsheaf satisfying* $\mu_H(\mathscr{F}) = \mu_H(\mathscr{E}) = \mu_H(\mathscr{E}/\mathscr{F})$ *then* $0 \to \mathscr{F}[1] \to \mathscr{E}[1]$ *is injective in* $\mathrm{Coh}_H^{D+\beta H}(X)$ *for all* $\beta \geq \mu_H(\mathscr{E})$*.*

*Proof.* Let $\mathscr{E} \in \mathrm{Coh}(X)$.

1. We proceed by induction on the length of the Harder-Narasimhann filtration of $\mathscr{E}$. If $\mathscr{E}$ has a Harder-Narasimhan filtration of length 2 then $\mathscr{E}/\mathscr{F}$ is $\mu_H$-stable with $\mu_H(\mathscr{E}/\mathscr{F}) < \mu_H(\mathscr{F})$, as claimed.

   Consider a Harder-Narasimhan filtration

$$0 = \mathscr{E}_0 \to \mathscr{E}_1 \to \cdots \to \mathscr{E}_{m-1} \to \mathscr{E}_m = \mathscr{E}.$$

By the inductive hypothesis, we know $\mu_H^+(\mathscr{E}_{m-1}/\mathscr{E}_1) < \mu_H(\mathscr{E}_1)$. We have the following short exact sequence

$$0 \to \mathscr{E}_{m-1}/\mathscr{E}_1 \to \mathscr{E}/\mathscr{E}_1 \to \mathscr{E}/\mathscr{E}_{m-1} \to 0.$$

Therefore

$$\mu_H^+(\mathscr{E}/\mathscr{E}_1) \leq \max\{\mu_H^+(\mathscr{E}/\mathscr{E}_{m-1}), \mu_H^+(\mathscr{E}_{m-1}/\mathscr{E}_1)\}.$$

However, $\mu_H^+(\mathscr{E}/\mathscr{E}_{m-1}) = \mu_H(\mathscr{E}/\mathscr{E}_{m-1}) < \mu_H(\mathscr{E}_1)$ by definition of a Harder-Narasimhan filtration and $\mu_H^+(\mathscr{E}_{m-1}/\mathscr{E}_1) \leq \mu_H(\mathscr{E}_1)$ by the inductive hypothesis, so we obtain the desired inequality.

It follows that $\mathscr{F}[1], \mathscr{E}[1], \mathscr{E}/\mathscr{F}[1] \in \mathrm{Coh}_H^{D+\beta H}(X)$ and so $0 \to \mathscr{F}[1] \to \mathscr{E}[1] \to \mathscr{E}/\mathscr{F}[1] \to 0$ is exact in $\mathrm{Coh}_H^{D+\beta H}(X)$, as claimed.

2. Since $\mathscr{E}$ is $\mu_H$-semistable, $\mathscr{F}$ and $\mathscr{E}/\mathscr{F}$ are also $\mu_H$-semistable of slope $\mu_H(\mathscr{E})$. In particular, $\mathscr{F}[1], \mathscr{E}[1], \mathscr{E}/\mathscr{F}[1] \in \mathrm{Coh}_H^{D+\beta H}(X)$ and so $0 \to \mathscr{F}[1] \to \mathscr{E}[1] \to \mathscr{E}/\mathscr{F}[1] \to 0$ is exact in $\mathrm{Coh}_H^{D+\beta H}(X)$, as claimed.

$\square$

All coherent sheaves and their shift by [1] lie in $\mathrm{Coh}_H^{D+\beta H}(X)$ for some $\beta \in \mathbb{R}$ depending on $\mu_H^{D-}(\mathscr{E})$ and $\mu_H^{D+}(\mathscr{E})$. Recall that $\mu_H^{D-}(\mathscr{E})$ (resp. $\mu_H^{D+}(\mathscr{E})$) is the slope of a minimal destabilizing quotient (resp. maximal destabilizing subobject) see Definition 2.3.2.

**Lemma 3.1.4.** *Assume $\mathscr{E} \in \mathrm{Coh}(X)$ is nonzero.*

1. *$\beta < \mu_H^{D-}(\mathscr{E})$ if and only if $\mathscr{E} \in \mathcal{T}_H^{D+\beta H}(X)$. In particular, $\mathscr{E}$ is a torsion sheaf if and only if $\mathscr{E} \in \mathcal{T}_H^{D+\beta H}(X)$ for all $\beta \in \mathbb{R}$.*

2. *$\beta \geq \mu_H^{D+}(\mathscr{E})$ if and only if $\mathscr{E} \in \mathcal{F}_H^{D+\beta H}(X)$ (and so $\mathscr{E}[1] \in \mathrm{Coh}_H^{D+\beta H}(X)$).*

3. *$E \in \mathrm{Coh}_H^{D+\beta H}(X)$ if and only if $\mu_H^{D+}(\mathscr{H}^{-1}(E)) \leq \beta < \mu_H^{D-}(\mathscr{H}^0(E))$.*

*Proof.* Assume $\mathscr{E} \in \mathrm{Coh}(X)$ is nonzero.

1. Assume $\beta < \mu_H^{D-}(\mathscr{E})$. Therefore, by Lemma 2.3.3.3, every quotient $\mathscr{E} \to \mathscr{G} \to 0$ satisfies $\beta < \mu_H^{D-}(\mathscr{E}) \leq \mu_H^D(\mathscr{G})$ so $\mathscr{E} \in \mathcal{T}_H^{D+\beta H}(X)$, as needed.

   Conversely, assume that $\mathscr{E} \in \mathcal{T}_H^{D+\beta H}(X)$. Therefore, every nonzero quotient $\mathscr{E} \to \mathscr{G} \to 0$ satisfies $\mu_H^D(\mathscr{G}) > \beta$. In particular, if $\mathscr{E} \to \mathscr{E}/\mathscr{E}_{m-1} \to 0$ is a minimal destabilizing quotient then

   $$\beta < \mu_H^D(\mathscr{E}/\mathscr{E}_{m-1}) = \mu_H^{D-}(\mathscr{E}),$$

   as desired.

   If $\mathscr{E}$ is a torsion sheaf then $\mathscr{E}$ is $\mu_H$-semistable of slope $\mu_H^{D-}(\mathscr{E}) = \mu_H^D(\mathscr{E}) = +\infty$. Therefore, $\mathscr{E} \in \mathcal{T}_H^{D+\beta H}(X)$ for all $\beta \in \mathbb{R}$.

   Conversely, if $\mathscr{E} \in \mathcal{T}_H^{D+\beta H}(X)$ for all $\beta \in \mathbb{R}$ then $\beta < \mu_H^{D-}(\mathscr{E})$ for all $\beta \in \mathbb{R}$. By Lemma 2.3.3.3, $\beta < \mu_H^{D-}(\mathscr{E}) \leq \mu_H^D(\mathscr{E})$, so $\mu_H^D(\mathscr{E}) = +\infty$. In other words, $\operatorname{rank}(\mathscr{E}) = 0$ so $\mathscr{E}$ is a torsion sheaf, as desired.

2. Assume $\beta > \mu_H^{D+}(\mathscr{E})$. By Lemma 2.3.3.2, every subsheaf $0 \to \mathscr{F} \to \mathscr{E}$ satisfies $\beta \geq \mu_H^{D+}(\mathscr{E}) \geq \mu_H^D(\mathscr{F})$. Therefore, $\mathscr{E} \in \mathcal{F}_H^{D+\beta H}(X)$, as needed.

   Conversely, assume $\mathscr{E} \in \mathcal{F}_H^{D+\beta H}(X)$. By definition every nonzero subobject $0 \to \mathscr{F} \to \mathscr{E}$ satisfies $\mu_H^D(\mathscr{F}) \leq \beta$. In particular, if $0 \to \mathscr{E}_1 \to \mathscr{E}$ is a maximal destabilizing subobject then

   $$\beta \geq \mu_H^D(\mathscr{E}_1) = \mu_H^{D+}(\mathscr{E}),$$

   as desired.

3. This result follows from parts 1 and 2.

□

The following lemma shows if $E \in \mathrm{Coh}_H^{D+\beta H}(X)$ then $E$ will still be in the heart for an infinitesimal deformations of $\beta$ to the right. We need an additional assumption if we want to deform $\beta$ to the left. We will see in Lemma 3.1.10 that this additional assumption is essentially purity with respect to tilt stability.

**Corollary 3.1.5.** *Assume $E \in \mathrm{Coh}_H^{D+\beta_0 H}(X)$.*

1. *There exists $\varepsilon > 0$ such that $E \in \mathrm{Coh}_H^{D+\beta H}(X)$ for all $\beta \in [\beta_0, \beta_0 + \varepsilon)$.*

2. *There exists $\varepsilon > 0$ such that $E \in \mathrm{Coh}_H^{D+\beta H}(X)$ for all $\beta \in (\beta_0 - \varepsilon, \beta_0]$ if and only if $\mu_H^{D+}(\mathscr{H}^{-1}(E)) \neq \beta_0$ or $\mathscr{H}^{-1}(E) = 0$.*

*Proof.* Both of these results follow from Lemma 3.1.4.3. □

Deformations of the heart are generally well-behaved with respect to subobjects and quotients:

**Lemma 3.1.6.** *Assume $E \in \mathrm{Coh}_H^{D+\beta_0 H}(X)$ and $E \in \mathrm{Coh}_H^{D+\beta_1 H}(X)$ for real numbers $\beta_0 < \beta_1$.*

1. *If $0 \to F \to E$ is a subobject in $\mathrm{Coh}_H^{D+\beta_1 H}(X)$ then $F \in \mathrm{Coh}_H^{D+\beta_0 H}(X)$.*

2. *If $E \to G \to 0$ is a quotient object in $\mathrm{Coh}_H^{D+\beta_0 H}(X)$ then $G \in \mathrm{Coh}_H^{D+\beta_1 H}(X)$.*

*Proof.*

1. Since $\beta_0 < \beta_1$, we know that $\mathscr{H}^0(F) \in \mathcal{T}_H^{D+\beta_0 H}(X)$. Therefore, it suffices to show $\mathscr{H}^{-1}(F) \in \mathcal{F}_H^{D+\beta_0 H}(X)$. If $F \notin \mathrm{Coh}_H^{D+\beta_0 H}(X)$ then we can find a subsheaf $0 \to \mathscr{F} \to \mathscr{H}^{-1}(F)$ such that $\mu_H^{D+\beta_0 H}(\mathscr{F}) > 0$. It follows that $0 \to \mathscr{F} \to \mathscr{H}^{-1}(F) \to \mathscr{H}^{-1}(E)$ is a subsheaf satisfying $\mu_H^{D+\beta_0 H}(\mathscr{F}) > 0$ so $E \notin \mathrm{Coh}_H^{D+\beta_0 H}(X)$, as claimed.

2. The dual argument holds.

□

For all positive real numbers $\alpha$ we define a very weak stability function on the heart $\mathrm{Coh}_H^{D+\beta H}(X)$:

**Definition 3.1.7.** *Define $\Lambda_H^D$ to be the image of $H^{n-2} \cdot \mathrm{ch}_{\leq 2} : K_0(D^b(X)) \to \mathbb{Q}^{\oplus 3}$ (the image lies in $\mathbb{Q}^{\oplus 3}$ because $D$ is a $\mathbb{Q}$-divisor).*

*For all $\alpha \in \mathbb{R}_{>0}$ and $\beta \in \mathbb{R}$, we define the function $\sigma_{\alpha,\beta}^{\mathrm{tilt}} = (Z_{\alpha,\beta}^{\mathrm{tilt}} : \Lambda \to \mathbb{C}, \mathrm{Coh}_H^{D+\beta H}(X))$ where*

$$Z_{\alpha,\beta}^{\mathrm{tilt}}(E) = -H^{n-2} \cdot \mathrm{ch}_2^{D+\beta H}(E) + \frac{\alpha^2 H^n}{2} \mathrm{rank}^{D+\beta H}(E) + \sqrt{-1} H^{n-1} \cdot \mathrm{ch}_1^{D+\beta H}(E).$$

*We denote the associated slope by*

$$\mu_{\alpha,\beta}^{\mathrm{tilt}}(E) = -\frac{-H^{n-2} \cdot \mathrm{ch}_2^{D+\beta H}(E) + \frac{\alpha^2}{2} H^n \cdot \mathrm{rank}(E)}{H^{n-1} \cdot \mathrm{ch}_1^{D+\beta H}(E)}.$$

To see that $Z_{\alpha,\beta}^{\mathrm{tilt}}$ is actually a group homomorphism $\Lambda_H^D \to \mathbb{C}$, notice that

$$\begin{aligned}\mathfrak{R} Z_{\alpha,\beta}^{\mathrm{tilt}}(E) &= -H^{n-2} \cdot \mathrm{ch}_2^{D+\beta H}(E) + \frac{\alpha^2}{2} H^n \cdot \mathrm{rank}^{D+\beta H}(E) \\ &= -H^{n-2} \cdot \mathrm{ch}_2^{D}(E) + \beta \deg_H^D(E) - \left(\frac{\beta^2}{2} - \frac{\alpha^2}{2}\right) H^n \cdot \mathrm{rank}(E)\end{aligned}$$

and

$$\begin{aligned}\mathfrak{I} Z_{\alpha,\beta}^{\mathrm{tilt}}(E) &= H^{n-1} \cdot \mathrm{ch}_1^{D+\beta H}(E) \\ &= \deg_H^D(E) - \beta H^n \cdot \mathrm{rank}(E).\end{aligned}$$

**Remark 3.1.8.** *In fact, the calculation above shows that if* $\mathrm{rank}(E) \neq 0$ *then we can rewrite* $\mu_{\alpha,\beta}^{\mathrm{tilt}}$ *in terms of* $\mu_H^D$, $\nu_H^D$, $\alpha$, *and* $\beta$*:*

$$\begin{aligned}\mu_{\alpha,\beta}^{\mathrm{tilt}}(E) &= -\frac{\mathfrak{R} Z_{\alpha,\beta}^{\mathrm{tilt}}(E)}{\mathfrak{I} Z_{\alpha,\beta}^{\mathrm{tilt}}(E)} \\ &= \frac{H^{n-2} \cdot \mathrm{ch}_2^D(E) - \beta \deg_H^D(E) + \left(\frac{\beta^2}{2} - \frac{\alpha^2}{2}\right) H^n \cdot \mathrm{rank}(E)}{\deg_H^D(E) - \beta H^n \cdot \mathrm{rank}(E)} \\ &= \frac{\nu_H^D(E) - \beta \mu_H^D(E) + \left(\frac{\beta^2}{2} - \frac{\alpha^2}{2}\right) H^n}{\mu_H^D(E) - \beta H^n}.\end{aligned}$$

*In particular, we find that*

$$\lim_{\alpha \to \infty} \frac{2}{\alpha^2} \mu_{\alpha,\beta}^{\mathrm{tilt}}(E) = -\frac{1}{\mu_H^D(E) - \beta H^n} = -\frac{1}{\mu_H^{D+\beta H}(E)}.$$

**Lemma 3.1.9.** *Assume* $(\beta, \alpha) \in \mathbb{R} \times \mathbb{R}_{>0}$ *and* $E \in \mathrm{Coh}_H^{D+\beta H}(X)$*.*

1. $\mathfrak{I} Z_{\alpha,\beta}^{\mathrm{tilt}}(E) \geq 0$.

2. *If* $\mathfrak{I} Z_{\alpha,\beta}^{\mathrm{tilt}}(E) = 0$ *then*

$$H^{n-2} \cdot \mathrm{ch}_{\leq 2}^{D+\beta H}(\mathscr{H}^0(E)) = (0, 0, \geq 0)$$

*and*

$$H^{n-2} \cdot \mathrm{ch}_{\leq 2}^{D+\beta H}(\mathscr{H}^{-1}(E)) = (\geq 0, 0, \leq 0).$$

*In particular,* $\sigma_{\alpha,\beta}^{\mathrm{tilt}} = (Z_{\alpha,\beta}^{\mathrm{tilt}} : \Lambda_H^D \to \mathbb{C}, \mathrm{Coh}_H^{D+\beta H}(X))$ *is a very weak stability function (and a stability function when* $\dim(X) = 2$*).*

3. $\Im Z_{\alpha,\beta}^{\mathrm{tilt}}(E) = 0$ *if and only if* $\mathrm{codim}(E) \geq 2$ *or* $\beta = \mu_H^D(E)$*.*

4. $Z_{\alpha,\beta}^{\mathrm{tilt}}(E) = 0$ *if and only if* $\mathrm{codim}(E) \geq 3$*.*

*Proof.* Assume $(\beta, \alpha) \in \mathbb{R} \times \mathbb{R}_{>0}$ and $E \in \mathrm{Coh}_H^{D+\beta H}(X)$.

1. There is a short exact sequence

$$0 \to \mathscr{H}^{-1}(E)[1] \to E \to \mathscr{H}^0(E) \to 0$$

with $\mathscr{H}^{-1}(E) \in \mathcal{F}_H^{D+\beta H}(X)$ and $\mathscr{H}^0(E) \in \mathcal{T}_H^{D+\beta H}(X)$. By additivity,

$$\begin{aligned} \Im Z_{\alpha,\beta}^{\mathrm{tilt}}(E) &= \Im Z_{\alpha,\beta}^{\mathrm{tilt}}(\mathscr{H}^0(E)) - \Im Z_{\alpha,\beta}^{\mathrm{tilt}}(\mathscr{H}^{-1}(E)) \\ &= \deg_H^{D+\beta H}(\mathscr{H}^0(E)) - \deg_H^{D+\beta H}(\mathscr{H}^{-1}(E)). \end{aligned}$$

Since $\mathscr{H}^0(E) \in \mathcal{T}_H^{D+\beta H}(X)$, by definition, $\mu_H^{D+\beta H}(\mathscr{H}^0(E)) > 0$. If $\mathrm{rank}(\mathscr{H}^0(E)) = 0$ then by Lemma 2.1.3, $\deg_H^{D+\beta H}(\mathscr{H}^0(E)) \geq 0$. Otherwise, since $\mu_H^{D+\beta H}(\mathscr{H}^0(E)) > 0$, $\deg_H^{D+\beta H}(\mathscr{H}^0(E)) > 0$. In either case, $\deg_H^{D+\beta H}(\mathscr{H}^0(E)) \geq 0$. On the other hand, since $\mathscr{H}^{-1}(E) \in \mathcal{F}_H^{D+\beta H}(X)$, $\mu_H^{D+\beta H}(\mathscr{H}^{-1}(E)) \leq 0$ or $\mathscr{H}^{-1}(E) = 0$. Either way, we find that $\deg_H^{D+\beta H}(\mathscr{H}^{-1}(E)) \leq 0$. In all, we have shown that

$$\Im Z_{\alpha,\beta}^{\mathrm{tilt}}(E) = \deg_H^{D+\beta H}(\mathscr{H}^0(E)) - \deg_H^{D+\beta H}(\mathscr{H}^{-1}(E)) \geq 0,$$

as desired.

2. Assume $\Im Z_{\alpha,\beta}^{\mathrm{tilt}}(E) = 0$ so $\deg_H^{D+\beta H}(\mathscr{H}^0(E)) = \deg_H^{D+\beta H}(\mathscr{H}^{-1}(E))$. Therefore, by the same argument as part 1,

$$0 \leq \deg_H^{D+\beta H}(\mathscr{H}^0(E)) = \deg_H^{D+\beta H}(\mathscr{H}^{-1}(E)) \geq 0.$$

Since $\deg_H^{D+\beta H}(\mathscr{H}^0(E)) = 0$ and $\mathscr{H}^0(E) \in \mathcal{T}_H^{D+\beta H}(X)$, $\operatorname{rank}(\mathscr{H}^0(E)) = 0$. Since $H^{n-1}\cdot \operatorname{ch}_{\leq 1}^{D+\beta H}(\mathscr{H}^0(E)) = (0,0)$, by Lemma 2.1.3, $H^{n-1}\cdot \operatorname{ch}_{\leq 2}^{D+\beta H}(\mathscr{H}^0(E)) = (0,0,\geq 0)$.

Since $\deg_H^{D+\beta H}(\mathscr{H}^{-1}(E)) = 0$ and $\mathscr{H}^{-1}(E) \in \mathcal{T}_H^{D+\beta H}(X)$, $\mathscr{H}^{-1}(E)$ is $\mu_H$-semistable (or 0). Therefore, by Bogomolov inequality,

$$\begin{aligned}\deg_H^{D+\beta H}(\mathscr{H}^{-1}(E))^2 - 2\operatorname{rank}(\mathscr{H}^{-1}(E))H^{n-2}\cdot \operatorname{ch}_2^{D+\beta H}(\mathscr{H}^{-1}(E)) &= \overline{\Delta}_H^{D+\beta H}(\mathscr{H}^{-1}(E)) \\ &\geq 0\end{aligned}$$

Since $H^{n-1}\cdot \operatorname{ch}_{\leq 1}^{D+\beta H}(\geq 0, 0)$, by rearranging the above inequality, $H^{n-2}\cdot \operatorname{ch}_{\leq 2}^{D+\beta H}(\geq 0, 0, \leq 0)$, as claimed.

By Lemma 2.4.7 and Lemma 2.4.8, we know that $\operatorname{Coh}_H^{D+\beta H}(X)$ is the heart of a bounded $t$-structure on $D^b(X)$. Furthermore, by part 1, $\Im Z_{\alpha,\beta}^{\text{tilt}}(E) \geq 0$ for all $E \in D^b(X)$. Moreover, if $\Im Z_{\alpha,\beta}^{\text{tilt}}(E) = 0$, by the argument above,

$$\begin{aligned}\Re Z_{\alpha,\beta}^{\text{tilt}}(E) &= \Re Z_{\alpha,\beta}^{\text{tilt}}(\mathscr{H}^0(E)) - \Re Z_{\alpha,\beta}^{\text{tilt}}(\mathscr{H}^{-1}(E)) \\ &= -H^{n-2}\cdot \operatorname{ch}_2^{D+\beta H}(\mathscr{H}^0(E)) + \frac{\alpha^2}{2}\operatorname{rank}(\mathscr{H}^0(E)) \\ &\quad + H^{n-2}\cdot \operatorname{ch}_2^{D+\beta H}(\mathscr{H}^{-1}(E)) - \frac{\alpha^2}{2}\operatorname{rank}(\mathscr{H}^{-1}(E)) \\ &\leq 0,\end{aligned}$$

as needed. Hence, $\sigma_{\alpha,\beta}^{\text{tilt}}$ is a very weak stability condition, as desired.

3. First assume $\Im Z_{\alpha,\beta}^{\text{tilt}}(E) = 0$. Therefore, $\deg_H^D(E) - \beta\operatorname{rank}(E) = 0$. If $\operatorname{rank}(E) \neq 0$ then $\beta = \deg_H^D(E)/\operatorname{rank}(E) = \mu_H^D(E)$, as needed. Therefore, assume $\operatorname{rank}(E) = 0$. It follows by part 2 that

$$H^{n-2}\cdot \operatorname{ch}_{\leq 2}^{D+\beta H}(\mathscr{H}^0(E)) = (0,0,\geq 0)$$

and

$$H^{n-2}\cdot \operatorname{ch}_{\leq 2}^{D+\beta H}(\mathscr{H}^{-1}(E)) = (0,0,\leq 0).$$

Therefore, $\operatorname{codim}(E) \geq 2$ by Lemma 2.1.3, as needed.

Second, assume $\beta = \mu_H^D(E)$ or $\operatorname{codim}(E) \geq 2$. In the first case, the result follows by direct computation. In the second case, the result follows by Lemma 2.1.3.

4. First assume $Z_{\alpha,\beta}^{\text{tilt}}(E) = 0$. In particular, $\Re Z_{\alpha,\beta}^{\text{tilt}}(E) = 0$ so

$$\begin{aligned} 0 &= \Re Z_{\alpha,\beta}^{\text{tilt}} \\ &= -H^{n-2} \cdot \operatorname{ch}_2^{D+\beta H}(\mathscr{H}^0(E)) + \frac{\alpha^2}{2} \operatorname{rank}(\mathscr{H}^0(E)) \\ &\quad + H^{n-2} \cdot \operatorname{ch}_2^{D+\beta H}(\mathscr{H}^{-1}(E)) - \frac{\alpha^2}{2} \operatorname{rank}(\mathscr{H}^{-1}(E)) \end{aligned}$$

However, $\Im Z_{\alpha,\beta}^{\text{tilt}}(E) = 0$ as well, by part 2,

$$H^{n-2} \cdot \operatorname{ch}_{\leq 2}^{D+\beta H}(\mathscr{H}^0(E)) = (0, 0, \geq 0)$$

and

$$H^{n-2} \cdot \operatorname{ch}_{\leq 2}^{D+\beta H}(\mathscr{H}^{-1}(E)) = (\geq 0, 0, \leq 0).$$

Using these inequalities and the equality involving $\Re Z_{\alpha,\beta}^{\text{tilt}}$, it follows that

$$H^{n-2} \cdot \operatorname{ch}_{\leq 2}^{D+\beta H}(\mathscr{H}^0(E)) = (0, 0, 0)$$

and

$$H^{n-2} \cdot \operatorname{ch}_{\leq 2}^{D+\beta H}(\mathscr{H}^{-1}(E)) = (0, 0, 0).$$

In other words, by Lemma 2.1.3, $\operatorname{codim}(E) \geq 3$, as desired.

$\square$

We note $\sigma_{\alpha,\beta}^{\text{tilt}}$-pure objects can be deformed to the left in $\operatorname{Coh}_H^{D+\beta H}(X)$:

**Lemma 3.1.10.** *Assume $E \in \operatorname{Coh}_H^{D+\beta H}(X)$ is nonzero. If $E$ is $\sigma_{\alpha,\beta}^{\text{tilt}}$-pure for some $\alpha > 0$ then there exists $\varepsilon > 0$ such that $E \in \operatorname{Coh}_H^{D+\beta' H}(X)$ for all $\beta' \in (\beta - \varepsilon, \beta]$.*

*Proof.* It suffices to show $\beta_0 \neq \mu_H^{D+}(\mathscr{H}^{-1}(E))$. With this in mind, consider a maimxal $\mu_H$-destabilizing subobject $0 \to \mathscr{F} \to \mathscr{H}^{-1}(E)$. By Lemma 3.1.3, $0 \to \mathscr{F}[1] \to \mathscr{H}^{-1}(E)[1]$ is injective in $\operatorname{Coh}_H^{D+\beta H}(X)$. Furthermore, by definition, there is an injection $0 \to \mathscr{H}^{-1}(E)[1] \to$

$E$ in $\mathrm{Coh}_H^{D+\beta H}(X)$. Thus, $\mathscr{F}[1]$ is a subobject of $E$. Since $E$ is $\sigma_{\alpha,\beta}^{\mathrm{tilt}}$-pure, $\Im Z_{\alpha,\beta}^{\mathrm{tilt}}(\mathscr{F}[1]) \neq 0$. In other words,

$$\mu_H^{D+}(\mathscr{H}^{-1}(E)) = \mu_H^D(\mathscr{F}) \neq \beta,$$

as desired. $\square$

We end this section by discussing reflexive sheaves and then classifying objects with good quotients with respect to $\sigma_{\alpha,\beta}^{\mathrm{tilt}}$.

**Definition 3.1.11.** *A nonzero coherent sheaf $\mathscr{E}$ on $X$ is said to be reflexive if any of the equivalent conditions are satisfied:*

- *The natural morphism $\mathscr{E} \to \mathscr{E}^{\vee\vee}$ is an isomorphism.*
- *$\mathscr{E}$ is torsion-free and $S_2$ (i.e. if $x \in X$ satisfies $\dim(\mathscr{O}_{X,x}) \geq 2$ then $\operatorname{depth}_x \mathscr{E}_x \geq 2$).*
- *$\mathscr{E}$ is torsion-free and satisfies the following property*

  *If $0 \to \mathscr{E} \to \mathscr{E}' \to \mathscr{G} \to 0$ is a short exact sequence with $\mathrm{codim}(\mathrm{Supp}(\mathscr{G})) \geq 2$ then the short exact sequence splits.*

The equivalence of the first and second statement is [Har80, Proposition 1.3]. The equivalence of the second and third statements is a local cohomology calculation in view of [Har80, Proposition 1.6]. Furthermore, by definition, a reflexive sheaf is torsion-free, and every locally free sheaf is reflexive.

**Lemma 3.1.12.** *Assume $\mathscr{E} \in \mathrm{Coh}(X)$.*

1. *Assume $\beta \geq \mu_H^{D+}(\mathscr{E})$. $\mathscr{E}$ is reflexive if and only if the following property holds*

   *If $0 \to F \to \mathscr{E}[1] \to G \to 0$ is a short exact sequence in $\mathrm{Coh}_H^{D+\beta H}(X)$ with $F \neq 0$ then $\mathrm{codim}(F) \leq 1$.*

2. *Assume $\beta < \mu_H^{D-}(\mathscr{E})$. Every nonzero subsheaf of $\mathscr{E}$ is supported in codimension at most 1 (e.g. $\mathscr{E}$ is pure of codimension at most 1) if and only if the following property holds*

   *If $0 \to F \to \mathscr{E} \to G \to 0$ is a short exact sequence in $\mathrm{Coh}_H^{D+\beta H}(X)$ with $F \neq 0$ then $\mathrm{codim}(F) \leq 1$.*

*Proof.* Assume $\mathscr{E} \in \mathrm{Coh}(X)$.

1. First assume that $\mathscr{E}$ is reflexive. We have the induced exact sequence arising from cohomology:

$$0 \to \mathscr{H}^{-1}(F) \to \mathscr{E} \to \mathscr{H}^{-1}(G) \to \mathscr{H}^{0}(F) \to 0.$$

   We know that $\mathscr{H}^{-1}(F)$ is torsion-free, so if $\mathscr{H}^{-1}(F) \neq 0$ we are done. If $\mathscr{H}^{-1}(F) = 0$ then we have the following short exact sequence

$$0 \to \mathscr{E} \to \mathscr{H}^{-1}(G) \to \mathscr{H}^{0}(F) \to 0.$$

   Conversely, consider the short exact sequence

$$0 \to \mathscr{E} \to \mathscr{E}^{\vee\vee} \to \mathscr{E}^{\vee\vee}/\mathscr{E} \to 0.$$

   $\mathscr{E}^{\vee\vee}$ is torsion-free, so $\mathscr{E}^{\vee\vee} \in \mathcal{F}_H^D(X)$. Similarly, $\mathrm{codim}(\mathscr{E}^{\vee\vee}/\mathscr{E}) \geq 2$, so $\mathscr{E}^{\vee\vee}/\mathscr{E} \in \mathcal{T}_H^D(X)$. In all, this shows that this exact sequence is of the form given in the lemma statement, so $\mathrm{codim}(\mathscr{E}^{\vee\vee}/\mathscr{E}) \leq 1$. However, as noted above, $\mathrm{codim}(\mathscr{E}^{\vee\vee}/\mathscr{E}) \geq 2$ Therefore, we find that $\mathscr{E}^{\vee\vee}/\mathscr{E} = 0$, so $\mathscr{E}$ is reflexive, as desired.

2. First assume every subsheaf of $\mathscr{E}$ is supported in codimension at most 1. Let $0 \to F \to \mathscr{E} \to G \to 0$ be a short exact sequence in $\mathrm{Coh}_H^{D+\beta H}(X)$ with $F \neq 0$. We have the following exact sequence from cohomology:

$$0 \to \mathscr{H}^{-1}(G) \to \mathscr{H}^{0}(F) \xrightarrow{f} \mathscr{E} \to \mathscr{H}^{0}(G) \to 0.$$

Therefore, we have the following short exact sequence

$$0 \to \mathscr{H}^{-1}(G) \to \mathscr{H}^0(F) \to \mathscr{I}m(f) \to 0$$

If $\mathscr{H}^{-1}(G) = 0$ then $\mathscr{H}^0(F) = \mathscr{I}m(f)$, so $1 \geq \operatorname{codim}(\mathscr{H}^0(F)) = \operatorname{codim}(F)$ since every subsheaf of $\mathscr{E}$ is supported in codimension at most 1. If $\mathscr{H}^{-1}(G) \neq 0$ then $\mathscr{H}^{-1}(G)$ is torsion-free so $\mathscr{H}^0(F)$ is supported in codimension 0, as needed.

The converse argument is similar to part 1.

□

**Lemma 3.1.13.** *Assume $E \in \mathrm{Coh}_H^{D+\beta H}(X)$. $E$ has good quotients with respect to $\sigma_{\alpha,\beta}^{\mathrm{tilt}}$ if and only if $\mathscr{H}^0(E) = 0$ or $\dim(X) = 2$.*

*Proof.* Assume $\dim(X) \geq 3$ and $\mathscr{H}^0(E) \neq 0$. Consider a closed point $\iota : x \to X$ with surjection $\mathscr{H}^0(E) \to \mathscr{H}^0(E) \otimes \iota_* \mathscr{O}_x \to 0$. Since $\dim(X) \geq 3$, by Lemma 2.1.3,

$$Z_{\alpha,\beta}^{\mathrm{tilt}}(\mathscr{H}^0(E) \otimes \iota_* \mathscr{O}_x) = 0$$

Furthermore, by construction, there is a surjection $E \to \mathscr{H}^0(E) \otimes \iota_* \mathscr{O}_x \to 0$. Since $\mathscr{H}^0(E) \neq 0$, $\mathscr{H}^0(E) \otimes \iota_* \mathscr{O}_x$ is a nonzero quotient of $E$ satisfying $Z_{\alpha,\beta}^{\mathrm{tilt}}(\mathscr{H}^0(E) \otimes \iota_* \mathscr{O}_x) = 0$. In other words, $E$ does not have good quotients with respect to $\sigma_{\alpha,\beta}^{\mathrm{tilt}}$, as claimed.

For the converse, assume $\dim(X) = 2$ or $\mathscr{H}^0(E) = 0$. If $\dim(X) = 2$ then $\sigma_{\alpha,\beta}^{\mathrm{tilt}}$ is a Bridgeland stability condition, so the result follows by Lemma 2.3.13.5. Thus, we may assume $\dim(X) \geq 3$ and $\mathscr{H}^0(E) = 0$. Therefore, consider a quotient $E \to G \to 0$ in $\mathrm{Coh}_H^{D+\beta H}(X)$ satisfying $Z_{\alpha,\beta}^{\mathrm{tilt}}(G) = 0$. Since $Z_{\alpha,\beta}^{\mathrm{tilt}}(G) = 0$, by Lemma 3.1.9, $\operatorname{codim}(G) = 0$. In particular, since $\mathscr{H}^{-1}(G)$ is torsion-free, $\mathscr{H}^{-1}(G) = 0$. Furthermore, since $\mathscr{H}^0(E) = 0$, $\mathscr{H}^0(G) = 0$. Hence, $G = 0$ and so $E$ has good quotients, as claimed. □

## 3.2 Tilt Stability is a Very Weak Stability Condition for Rational $\beta$

We prove that if $\beta \in \mathbb{Q}$ then $\sigma^{\text{tilt}}_{\alpha,\beta}$ satisfies the Harder-Narasimhan property. Using a similar argument, we also show that the collection of all $\sigma^{\text{tilt}}_{\alpha,\beta}$-pure objects form a torsion-free class in $\text{Coh}^{D+\beta H}_H(X)$. In order to prove the support property, we also discuss the Large Volume Limit which is independently useful.

Parts 1 and 2 of the following lemma were first shown in [BMT14, Lemma 3.2.4]. Parts 3 and 4 are new.

**Lemma 3.2.1.** *Assume $D$ is a $\mathbb{Q}$-divisor and $H$ an ample divisor.*

1. *If $0 \to E_1 \to E_2 \to \cdots \to E$ is an ascending chain in $\text{Coh}^{D+\beta H}_H(X)$ with $\Im Z^{\text{tilt}}_{\alpha,\beta}(E_i) = \Im Z^{\text{tilt}}_{\alpha,\beta}(E_{i+1})$ for all $i \gg 0$ then $E_i = E_{i+1}$ for all $i \gg 0$.*

2. *If $(\beta, \alpha) \in \mathbb{Q} \times \mathbb{R}_{>0}$ then $\text{Coh}^{D+\beta H}_H(X)$ is Noetherian and $\Im Z^{\text{tilt}}_{\alpha,\beta} : \Lambda^D_H(X) \to \mathbb{R}$ is discrete.*

   *In particular, if $(\beta, \alpha) \in \mathbb{Q} \times \mathbb{R}_{>0}$ then $\sigma^{\text{tilt}}_{\alpha,\beta}$ satisfies the Harder-Narasimhan property.*

3. *Define $\mathcal{T}$ and $\mathcal{F}$ to be the full additive subcategories of $\text{Coh}^{D+\beta H}_H(X)$ generated by*

$$\mathcal{T} = \{E \in \text{Coh}^{D+\beta H}_H(X) \mid \Im Z_{\alpha,\beta}(E) = 0\}$$
$$\mathcal{F} = \{E \in \text{Coh}^{D+\beta H}_H(X) \mid E \textit{ is } \sigma^{\text{tilt}}_{\alpha,\beta} - \textit{pure}\}$$

   *respectively. If $(\beta, \alpha) \in \mathbb{R} \times \mathbb{R}_{>0}$ then $(\mathcal{T}, \mathcal{F})$ is a torsion pair in $\text{Coh}^{D+\beta H}_H(X)$.*

4. *Define $\mathcal{T}$ and $\mathcal{F}$ to be the full additive subcategories of $\text{Coh}^{D+\beta H}_H(X)$ generated by*

$$\mathcal{T} = \{E \in \text{Coh}^{D+\beta H}_H(X) \mid E \textit{ has good quotients with respect to } \sigma^{\text{tilt}}_{\alpha,\beta}\}$$
$$\mathcal{F} = \{E \in \text{Coh}^{D+\beta H}_H(X) \mid Z^{\text{tilt}}_{\alpha,\beta}(E) = 0\}$$

   *respectively. If $(\beta, \alpha) \in \mathbb{R} \times \mathbb{R}_{>0}$ then $(\mathcal{T}, \mathcal{F})$ is a torsion pair in $\text{Coh}^{D+\beta H}_H(X)$.*

*Proof.* Assume $D$ is a $\mathbb{Q}$-divisor and $H$ an ample divisor.

1. Assume $0 \to E_1 \to E_2 \to \cdots \to E$ is an ascending chain in $\mathrm{Coh}_H^{D+\beta H}(X)$ satisfying $\Im Z^{\mathrm{tilt}}_{\alpha,\beta}(E_i) = \Im Z^{\mathrm{tilt}}_{\alpha,\beta}(E_{i+1})$ for all $i \gg 0$. Without loss of generality, we may assume that this equality holds for all $i > 0$.

   Since $\Im Z^{\mathrm{tilt}}_{\alpha,\beta}(E_i) = \Im Z^{\mathrm{tilt}}_{\alpha,\beta}(E_{i+1})$, so by additivity, $\Im Z^{\mathrm{tilt}}_{\alpha,\beta}(E_i/E_{i+1}) = 0$. By Lemma 3.1.9.2, $\mathrm{rank}(E_i/E_{i+1}) \geq 0$ for all $i \gg 0$, so by additivity $\mathrm{rank}(E_i) \geq \mathrm{rank}(E_{i+1})$ for all $i \gg 0$. Since $E_i$ is a subobject of $E$, there is an injection $0 \to \mathscr{H}^{-1}(E_i) \to \mathscr{H}^{-1}(E)$ and so

$$\mathrm{rank}(E_i) = \mathrm{rank}(\mathscr{H}^0(E_i)) - \mathrm{rank}(\mathscr{H}^{-1}(E_i)) \geq 0 - \mathrm{rank}(\mathscr{H}^{-1}(E)).$$

   Thus, $\mathrm{rank}(E_i)$ is discrete, bounded below by $-\mathrm{rank}(\mathscr{H}^{-1}(E))$, and decreasing; so $\mathrm{rank}(E_i) = \mathrm{rank}(E_{i+1})$ for all $i \gg 0$. By additivity, $\mathrm{rank}(E_i/E_{i+1}) = 0$ for all $i \gg 0$ and so $\mathrm{codim}(E_i/E_{i+1}) \geq 2$ by Lemma 3.1.9.3. In particular, $\mathscr{H}^{-1}(E_i/E_{i+1}) = 0$ for all $i \gg 0$. Without loss of generality, we may assume $\mathscr{H}^{-1}(E_i/E_{i+1}) = 0$ for all $i > 0$.

   On the other hand, there is a chain of surjections

$$E \to E/E_1 \to E/E_2 \to \cdots$$

   in $\mathrm{Coh}_H^{D+\beta H}(X)$ which induces a chain of surjections in $\mathrm{Coh}(X)$:

$$\mathscr{H}^0(E) \to \mathscr{H}^0(E/E_1) \to \mathscr{H}^0(E/E_2) \to \cdots .$$

   Since $\mathrm{Coh}(X)$ is Noetherian, the above chain of surjections eventually stabilizes, so $\mathscr{H}^0(E/E_i) = \mathscr{H}^0(E/E_{i+1})$ for all $i \gg 0$. Without loss of generality, we may assume that this isomorphism holds for $i > 0$.

   There is a short exact sequence

$$0 \to E_{i+1}/E_i \to E/E_i \to E/E_{i+1} \to 0$$

   in $\mathrm{Coh}_H^{D+\beta H}(X)$ which induces the following exact sequence in $\mathrm{Coh}(X)$:

$$\begin{aligned} 0 \longrightarrow \mathscr{H}^{-1}(E_{i+1}/E_i) \longrightarrow \mathscr{H}^{-1}(E/E_i) \longrightarrow \mathscr{H}^{-1}(E/E_{i+1}) \\ \longrightarrow \mathscr{H}^{0}(E_{i+1}/E_i) \longrightarrow \mathscr{H}^{0}(E/E_i) \longrightarrow \mathscr{H}^{0}(E/E_{i+1}) \longrightarrow 0 \end{aligned}$$

Using the vanishing of $\mathscr{H}^{-1}(E_i/E_{i+1})$ and the isomorphisms above, we have the following short exact sequence in $\mathrm{Coh}(X)$:

$$0 \to \mathscr{H}^{-1}(E/E_i) \to \mathscr{H}^{-1}(E/E_{i+1}) \to \mathscr{H}^0(E_{i+1}/E_i) \to 0$$

Taking the dual, we obtain the following exact sequence in $\mathrm{Coh}(X)$:

$$\begin{array}{l} 0 \longrightarrow \mathscr{H}^0(E_{i+1}/E_i)^\vee \longrightarrow \mathscr{H}^{-1}(E/E_{i+1})^\vee \longrightarrow \mathscr{H}^{-1}(E/E_i)^\vee \\ \longrightarrow \mathscr{E}xt^1(\mathscr{H}^0(E_{i+1}/E_1), \mathscr{O}_X) \longrightarrow \mathscr{E}xt^1(\mathscr{H}^{-1}(E/E_{i+1}), \mathscr{O}_X) \longrightarrow \cdots \end{array}$$

As stated above, $\mathrm{codim}(\mathscr{H}^0(E_i/E_{i+1})) \geq 2$ so $\mathscr{H}^0(E_i/E_{i+1})^\vee = 0$ and

$$\mathscr{E}xt^1(\mathscr{H}^0(E_{i+1}/E_1), \mathscr{O}_X) = 0.$$

Therefore, we have an isomorphism $\mathscr{H}^{-1}(E/E_i)^\vee = \mathscr{H}^{-1}(E/E_{i+1})^\vee$ for all $i > 0$. Since $\mathscr{H}^{-1}(E/E_i)$ is torsion-free or 0, there is a natural injection

$$\mathscr{H}^{-1}(E/E_i) \to \mathscr{H}^{-1}(E/E_i)^{\vee\vee} = \mathscr{H}^{-1}(E/E_1)^{\vee\vee}.$$

for all $j > 0$. Therefore, we have the following ascending chain in $\mathrm{Coh}(X)$:

$$0 \to \mathscr{H}^{-1}(E/E_1) \to \mathscr{H}^{-1}(E/E_2) \to \cdots \to \mathscr{H}^{-1}(E/E_1)^{\vee\vee}.$$

Since $\mathrm{Coh}(X)$ is Noetherian, $\mathscr{H}^{-1}(E/E_i) = \mathscr{H}^{-1}(E/E_{i+1})$ for all $i \gg 0$.

Hence, we have shown $\mathscr{H}^j(E/E_i) = \mathscr{H}^j(E/E_{i+1})$ via the natural morphisms for all $i \gg 0$ and $j \in \mathbb{Z}$. In other words, $E/E_i = E/E_{i+1}$ for all $i \gg 0$ and so $E_i = E_{i+1}$ for all $i \gg 0$, as desired.

2. Fix $\beta \in \mathbb{Q}$. Note that

$$\Im Z^{\mathrm{tilt}}_{\alpha,\beta}(E) = \deg_H(E) - H^{n-1} \cdot (D + \beta H) \cdot \mathrm{rank}(E)$$

is discrete because $D$ is a $\mathbb{Q}$-divisor and $\beta \in \mathbb{Q}$.

Consider an ascending chain

$$0 \to E_1 \to E_2 \to \cdots \to E.$$

in $\mathrm{Coh}_H^{D+\beta H}(X)$. Since $\Im Z_{\alpha,\beta}^{\mathrm{tilt}}$ is additive in short exact sequences and non-negative, $\Im Z_{\alpha,\beta}^{\mathrm{tilt}}(E_i) \le \Im Z_{\alpha,\beta}^{\mathrm{tilt}}(E_{i+1}) \le \Im Z_{\alpha,\beta}^{\mathrm{tilt}}(E)$. Therefore, since $\Im Z_{\alpha,\beta}^{\mathrm{tilt}}(E_i)$ is discrete, increasing, and bounded above by $\Im Z_{\alpha,\beta}^{\mathrm{tilt}}(E)$, $\Im Z_{\alpha,\beta}^{\mathrm{tilt}}(E_i) = \Im Z_{\alpha,\beta}^{\mathrm{tilt}}(E_{i+1})$ for all $i \gg 0$. By applying part 1, we find that $\mathrm{Coh}_H^{D+\beta H}(X)$ is Noetherian, as claimed.

Since $\mathrm{Coh}_H^{D+\beta H}(X)$ is Noetherian and $\Im Z_{\alpha,\beta}^{\mathrm{tilt}} : \Lambda_H^D \to \mathbb{R}$ is discrete, by Lemma 2.3.6, $\sigma_{\alpha,\beta}^{\mathrm{tilt}}$ satisfies the Harder-Narasimhan property.

3. Assume $T \in \mathcal{T}$ and $F \in \mathcal{F}$. Consider a morphism $f : T \to F$. By additivity of $\Im Z_{\alpha,\beta}^{\mathrm{tilt}}$, $\Im Z_{\alpha,\beta}^{\mathrm{tilt}}(\mathrm{Im}(f)) = 0$. Since $F$ is $\sigma_{\alpha,\beta}^{\mathrm{tilt}}$-pure, $\mathrm{Im}(f) = 0$, as needed.

   Now, let $A \in \mathrm{Coh}_H^{D+\beta H}(X)$. Consider the collection

   $$\mathcal{C} = \{T \in \mathcal{T} \mid \text{there is an injection } 0 \to T \to A\}.$$

   which we give the natural poset structure. We know $\mathcal{C} \neq \varnothing$ because $0 \in \mathcal{C}$. Consider a chain

   $$0 \to T_1 \to T_2 \to \cdots \to A$$

   in $\mathcal{C}$. Since $\Im Z_{\alpha,\beta}^{\mathrm{tilt}}(T_i) = 0 = \Im Z_{\alpha,\beta}^{\mathrm{tilt}}(T_{i+1})$, by part 1, $T_i = T_{i+1}$ for all $i \gg 0$. Therefore, by Zorn's lemma, $\mathcal{C}$ has a maximal element.

   Let $T$ be a maximal element of $\mathcal{C}$. We claim that $A/T \in \mathcal{F}$. Assume that $0 \to B \to A/T$ is a subobject satisfying $\Im Z_{\alpha,\beta}^{\mathrm{tilt}}(B) = 0$. Therefore, we can write $B = B'/T$ and by additivity of $\Im Z_{\alpha,\beta}^{\mathrm{tilt}}$, $\Im Z_{\alpha,\beta}^{\mathrm{tilt}}(B') = 0$. However, by maximality of $T$, it follows that $B' = T$ and so $B = 0$, as desired.

   Therefore, $A/T$ is $\sigma_{\alpha,\beta}^{\mathrm{tilt}}$-pure and so $A/T \in \mathcal{F}$, as desired.

4. The "dual" argument to part 3 holds.

□

We need some more theory before proving the support property for $\sigma_{\alpha,\beta}^{\mathrm{tilt}}$. In particular, we need a variant of the large volume limit for $\sigma_{\alpha,\beta}^{\mathrm{tilt}}$-stability Loosely, the large volume limit gives an equivalence between $\sigma_{\alpha,\beta}^{\mathrm{tilt}}$-stable objects for $\alpha \gg 0$ with $(H, D)$-twisted stable objects.

The following lemma is a preliminary step to the large volume limit.

**Lemma 3.2.2.** *Assume $\mathscr{E} \in \mathrm{Coh}(X)$ is torsion-free and nonzero*

1. *Let $0 \to F \to \mathscr{E} \to G \to 0$ be exact in $\mathrm{Coh}_H^D(X)$ with $F \neq 0$.*

   *a) If $\mathscr{E}$ is $\mu_H$-semistable then $\mu_H^D(F) \leq \mu_H^D(\mathscr{E})$.*

   *b) If $\mathscr{E}$ is $(H, D)$-twisted semistable then either*

   - $\mu_H^D(F) < \mu_H^D(\mathscr{E})$ *or*
   - $\mu_H^D(F) = \mu_H^D(\mathscr{E})$ *with* $\nu_H^D(F) \leq \nu_H^D(G)$

   *c) If $\mathscr{E}$ is $(H, D)$-twisted stable and $F \neq \mathscr{E}$ then either*

   - $\mu_H^D(F) < \mu_H^D(\mathscr{E})$ *or*
   - $\mu_H^D(F) = \mu_H^D(\mathscr{E})$ *with* $\nu_H^D(F) < \nu_H^D(G)$

2. *Let $0 \to F \to \mathscr{E}[1] \to G \to 0$ be exact in $\mathrm{Coh}_H^D(X)$ with $G \neq 0$.*

   *a) If $\mathscr{E}$ is $\mu_H$-semistable then $\mu_H^D(\mathscr{E}[1]) \leq \mu_H^D(G)$.*

   *b) If $\mathscr{E}$ is $\mu_H$-stable, $\mathrm{codim}(F) \leq 1$, and $G \neq \mathscr{E}[1]$ then $\mu_H^D(\mathscr{E}[1]) < \mu_H^D(G)$.*

*Proof.* Assume $\mathscr{E} \in \mathrm{Coh}(X)$ is nonzero and torsion-free.

1. Let $0 \to F \to \mathscr{E} \to G \to 0$ be exact in $\mathrm{Coh}_H^D(X)$ with $F \neq 0$. The induced long exact sequence in $\mathrm{Coh}(X)$ is

$$0 \to \mathscr{H}^{-1}(F) \to 0 \to \mathscr{H}^{-1}(G) \to \mathscr{H}^0(F) \xrightarrow{f} \mathscr{E} \to \mathscr{H}^0(G) \to 0.$$

   Since $\mathscr{H}^{-1}(F) = 0$, we find that $\mu_H^D(\mathscr{H}^0(F)) = \mu_H^D(F)$ and similarly for $\nu_H^D$. The above exact sequence induces two short exact sequences in $\mathrm{Coh}(X)$:

$$0 \to \mathscr{H}^{-1}(G) \to \mathscr{H}^0(F) \to \mathscr{I}m(f) \to 0$$

$$0 \to \mathscr{I}m(f) \to \mathscr{E} \to \mathscr{H}^0(G) \to 0.$$

Since $\mathscr{H}^{-1}(G) \in \mathcal{F}_H^D(X)$, $\mathscr{H}^{-1}(G)$ is torsion-free or 0. Furthermore since $\mathscr{E}$ is torsion-free, $\mathscr{I}m(f)$ is 0 or torsion-free, It follows that $\mathscr{H}^0(F)$ is also torsion-free or 0. By assumption, $0 \neq F = \mathscr{H}^0(F)$ so $\mathscr{H}^0(F)$ is torsion-free.

Since $\mathscr{H}^{-1}(G) \in \mathcal{F}_H^{D+\beta H}(X)$ and $\mathscr{H}^0(G) \in \mathcal{T}_H^{D+\beta H}(X)$, $\mathscr{H}^{-1}(G) \cong \mathscr{H}^0(F)$ if and only $\mathscr{H}^0(F) = 0$. However, $F = \mathscr{H}^0(F)$ is nonzero, so $\mathscr{H}^{-1}(G) \neq \mathscr{H}^0(F)$ and so $\mathscr{I}m(f) \neq 0$.

We have two cases, either $\mathscr{H}^{-1}(G) = 0$ or $\mathscr{H}^{-1}(G) \neq 0$. If $\mathscr{H}^{-1}(G) = 0$ then $\mathscr{H}^0(F) = \mathscr{I}m(f)$, so $\mu_H^D(\mathscr{H}^0(F)) = \mu_H^D(\mathscr{I}m(f))$. If $\mathscr{H}^{-1}(G) \neq 0$, by definition of $\mathcal{F}_H^{D+\beta H}(X)$ and $\mathcal{T}_H^{D+\beta H}(X)$, we find that $\mu_H^D(\mathscr{H}^{-1}(G)) < 0 \leq \mu_H^D(\mathscr{H}^0(F))$. It follows by the seesaw inequality that $\mu_H^D(\mathscr{H}^0(F)) < \mu_H^D(\mathscr{I}m(f))$ (we obtain a strict inequality because $\mathscr{I}m(f)$ is torsion-free). All cases considered, we find that $\mu_H^D(\mathscr{H}^0(F)) \leq \mu_H^D(\mathscr{I}m(f))$ with equality exactly if $\mathscr{H}^{-1}(G) = 0$.

a) Assume $\mathscr{E}$ is $\mu_H$-semistable. Since $\mathscr{I}m(f) \neq 0$, we know that $\mu_H^D(\mathscr{I}m(f)) \leq \mu_H^D(\mathscr{E})$. As we saw above, $\mu_H^D(\mathscr{H}^0(F)) \leq \mu_H^D(\mathscr{I}m(f))$, so

$$\mu_H^D(F) = \mu_H^D(\mathscr{H}^0(F)) \leq \mu_H^D(\mathscr{E}),$$

as needed.

b) Assume $\mathscr{E}$ is $(H, D)$-twisted semistable. By Lemma 2.1.11 and part $1a$, $\mu_H^D(F) \leq \mu_H^D(\mathscr{E})$. Therefore, it suffices to show that if $\mu_H^D(F) = \mu_H^D(\mathscr{E})$ then $\nu_H^D(F) \leq \nu_H^D(G)$.

Assume $\mu_H^D(F) = \mu_H^D(\mathscr{E})$. Since $\mathscr{E}$ is $\mu_H$-semistable, $\mu_H^D(\mathscr{I}m(f)) \leq \mu_H^D(\mathscr{E})$. It follows that $\mu_H^D(F) = \mu_H^D(\mathscr{I}m(f))$, so we find that $\mathscr{H}^{-1}(G) = 0$. It follows that we have the following sequence is exact in $\mathrm{Coh}(X)$:

$$0 \to \mathscr{H}^0(F) \to \mathscr{E} \to \mathscr{H}^0(G) \to 0.$$

If $\mathrm{codim}(\mathscr{H}^0(G), X) \leq 1$ then $H^{n-2} \cdot \mathrm{ch}_{\leq 1}^D(\mathscr{H}^0(G)) \neq (0, 0)$ by Lemma 3.1.9 so $\mu_H^D(\mathscr{H}^0(F)) = \mu_H^D(\mathscr{E}) = \mu_H^D(\mathscr{H}^0(G))$ by the generalized seesaw inequality. By definition of $(H, D)$-twisted semistability, it follows that $\nu_H^D(\mathscr{H}^0(F)) \leq \nu_H(\mathscr{H}^0(G))$ as needed. If $\mathrm{codim}(\mathscr{H}^0(G), X) \geq 2$ then we find that

$$\nu_H^D(F) = \nu_H^D(\mathscr{H}^0(F)) < +\infty = \nu_H^D(\mathscr{H}^0(G)) = \nu_H^D(G),$$

as needed.

c) The same argument as part 1b holds.

2. Assume $0 \to F \to \mathscr{E}[1] \to G \to 0$ is exact in $\mathrm{Coh}_H^D(X)$ with $F \neq 0$. The induced long exact sequence in $\mathrm{Coh}(X)$ is:

$$0 \to \mathscr{H}^{-1}(F) \to \mathscr{E} \xrightarrow{g} \mathscr{H}^{-1}(G) \to \mathscr{H}^0(F) \to 0 \to \mathscr{H}^0(G) \to 0.$$

Since $\mathscr{H}^0(G) = 0$, we find that $\mu_H^D(\mathscr{H}^0(G)) = \mu_H^D(G)$ and similarly for $\nu_H^D$. The above exact sequence induces the following two short exact sequences in $\mathrm{Coh}(X)$:

$$0 \to \mathscr{H}^{-1}(F) \to \mathscr{E} \to \mathscr{I}m(g) \to 0$$
$$0 \to \mathscr{I}m(g) \to \mathscr{H}^{-1}(G) \to \mathscr{H}^0(F) \to 0.$$

Since $\mathscr{I}m(g)$ is a subsheaf of torsion-free $\mathscr{H}^{-1}(G)$, $\mathscr{I}m(g)$ is torsion-free or 0.

Since $\mathscr{H}^{-1}(G) \in \mathcal{F}_H^{D+\beta H}(X)$ and $\mathscr{H}^0(F) \in \mathcal{T}_H^{D+\beta H}(X)$, $\mathscr{H}^{-1}(G) = \mathscr{H}^0(F)$ if and only $\mathscr{H}^{-1}(G) = 0$. However, $0 \neq G = \mathscr{H}^{-1}(G)$ by assumption so $\mathscr{I}m(g) \neq 0$, as claimed.

Since $\mathscr{H}^{-1}(G) \in \mathcal{F}_H^{D+\beta H}(X)$, $\mathscr{H}^{-1}(G) \neq 0$, and $\mathscr{H}^0(F) \in \mathcal{F}_H^{D+\beta H}(X)$,

$$\mu_H^D(G) = \mu_H^D(\mathscr{H}^{-1}(G)) \leq 0 < \mu_H^D(\mathscr{H}^0(F)).$$

Therefore, by the generalized seesaw inequality, $\mu_H^D(\mathscr{I}m(g)) \leq \mu_H^D(G)$ with equality exactly if $\mathrm{codim}(\mathscr{H}^0(F)) \geq 2$.

a) Assume $\mathscr{E}$ is $\mu_H$-semistable. If $\mathscr{H}^{-1}(F) = 0$ then $\mathscr{E} = \mathscr{I}m(g)$ and so

$$\mu_H^D(\mathscr{E}[1]) = \mu_H^D(\mathscr{E}) = \mu_H^D(\mathscr{I}m(g)) \leq \mu_H^D(G)$$

by the inequality above, as needed.

If $\mathscr{H}^{-1}(F) \neq 0$, since $\mathscr{E}$ is $\mu_H$-semistable and $\mathscr{I}m(g) \neq 0$, we find that $\mu_H^D(\mathscr{E}) \leq \mu_H^D(\mathscr{I}m(g))$. Moreover, as noted above, $\mu_H^D(\mathscr{I}m(g)) \leq \mu_H^D(\mathscr{H}^{-1}(G)) = \mu_H^D(G)$. Thus, we find that

$$\mu_H^D(\mathscr{E}[1]) = \mu_H^D(\mathscr{E}) \leq \mu_H^D(\mathscr{I}m(g)) \leq \mu_H^D(G).$$

In either case, $\mu_H^D(\mathscr{E}[1]) \leq \mu_H^D(G)$, as desired.

b) Assume $\mathscr{E}$ is $\mu_H$-stable, $\mathscr{E}[1] \neq G$, and $\operatorname{codim}(F, X) \leq 1$. Since $\operatorname{codim}(F, X) \leq 1$, $\mathscr{H}^{-1}(F) \neq 0$ or $\mathscr{H}^{-1}(F) = 0$ with $\operatorname{codim}(\mathscr{H}^0(F)) \leq 1$. First, assume that $\mathscr{H}^{-1}(F) \neq 0$. Since $\mathscr{I}m(g) \neq 0$, we know that $\mathscr{H}^{-1}(F) \neq \mathscr{E}$. Moreover, since $\mathscr{H}^{-1}(F) \neq 0$ and $\mathscr{H}^{-1}(F) \neq \mathscr{E}$, since $\mathscr{E}$ is $\mu_H$-stable, we find that $\mu_H^D(\mathscr{H}^{-1}(F)) < \mu_H^D(\mathscr{I}m(g))$. By the generalized seesaw inequality $\mu_H^D(\mathscr{E}) < \mu_H^D(\mathscr{I}m(g))$. Using the inequality $\mu_H^D(\mathscr{I}m(g)) \leq \mu_H^D(G)$ from above, we find that

$$\mu_H^D(\mathscr{E}[1]) = \mu_H^D(\mathscr{E}) < \mu_H^D(\mathscr{I}m(g)) \leq \mu_H^D(G),$$

as needed.

Second, assume that $\mathscr{H}^{-1}(F) = 0$ with $\operatorname{codim}(\mathscr{H}^0(F), X) \leq 1$. As stated above, in this case, $\mu_H^D(\mathscr{I}m(g)) < \mu_H^D(G)$. Therefore, since $\mathscr{E} = \mathscr{I}m(g)$,

$$\mu_H^D(\mathscr{E}[1]) = \mu_H^D(\mathscr{E}) = \mu_H^D(\mathscr{I}m(g)) < \mu_H^D(G),$$

as desired.

$\square$

Recall by Lemma 3.1.12.1 that the codimension assumption above is satisfied for every subobject $0 \to F \to \mathscr{E}[1]$ exactly if $\mathscr{E}$ is reflexive.

Furthermore, note that the inequalities appearing in Lemma 3.2.2 are not the naive generalization of the inequalities in the definition of $\mu_H$-(semi)stability nor $(H, D)$-twisted (semi)stability.

We now prove the Large Volume Limit. The terminology comes from physics, but the idea is that $\sigma^{\text{tilt}}_{\alpha,\beta}$-stability for $\alpha \gg 0$ is equivalent to notions of stability on $\mathrm{Coh}(X)$. The Large Volume Limit is one of only a few techniques to detect $\mu_H$-stability in the $(H, D)$-slice. The other techniques are to show $\sigma^{\text{tilt}}_{\alpha,\beta}$-stability when $\beta = \mu^D_H(\mathscr{E})$ (see Proposition 3.2.6), or to show that a torsion object is $\sigma^{\text{tilt}}_{\alpha,\beta}$-stable (see Lemma 4.1.1).

This lemma originally appeared in [Bri08, Section 14] for stability functions on a $K3$ surface. Part 2 is most comonly written with the weaker conclusion that $\mathscr{E}$ is $\mu_H$-semistable. Lemma 3.2.4 is the technical tool that allows us to strengthen this conclusion.

**Lemma 3.2.3** (Large Volume Limit)**.** *Assume $\mathscr{E} \in \mathrm{Coh}(X)$ is torsion-free.*

1. *$\mathscr{E}$ is $(H, D)$-twisted stable if and only if $\mathscr{E}$ is weakly $\sigma^{\text{tilt}}_{\alpha,\beta}$-stable for all $\alpha \gg 0$ and all $\beta < \mu^D_H(\mathscr{E})$.*
2. *If $\mathscr{E}[1]$ is weakly $\sigma^{\text{tilt}}_{\alpha,\beta}$-stable (equivalently $\sigma^{\text{tilt}}_{\alpha,\beta}$-stable by Lemma 3.1.13) for all $\beta > \mu^D_H(\mathscr{E})$ then $\mathscr{E}$ is $\mu_H$-stable.*
3. *If $\mathscr{E}$ is reflexive and $\mu_H$-stable then $\mathscr{E}[1]$ is weakly $\sigma^{\text{tilt}}_{\alpha,\beta}$-stable ((equivalently $\sigma^{\text{tilt}}_{\alpha,\beta}$-stable by Lemma 3.1.13) for all $\alpha \gg 0$ and all $\beta > \mu^D_H(\mathscr{E})$.*

*Proof.* Assume $\mathscr{E}$ is torsion-free.

1. First, assume that $\mathscr{E}$ is $(H, D)$-twisted stable. Let $\beta < \mu^D_H(\mathscr{E})$, so by Lemma 3.1.4, $\mathscr{E} \in \mathrm{Coh}^{D+\beta H}_H(X)$. Consider a short exact sequence $0 \to F \to \mathscr{E} \to G \to 0$ in $\mathrm{Coh}^{D+\beta H}_H(X)$ with $F \neq 0$ and $F \neq \mathscr{E}$. By writing out the long exact sequence in cohomology we find that $\mathscr{H}^{-1}(F) = 0$. Therefore, $\mu^D_H(F) = \mu^D_H(\mathscr{H}^{-1}(F)) > \beta$ where the inequality holds by Lemma 3.1.4. Since $\mathscr{E}$ is $(H, D)$-twisted stable, by Lemma 3.2.2.1c, either
   - $\mu^D_H(F) < \mu^D_H(\mathscr{E})$ or
   - $\mu^D_H(F) = \mu^D_H(\mathscr{E})$ with $\nu^D_H(F) < \nu^D_H(G)$.

We will consider each case.

Assume $\mu_H^D(F) < \mu_H^D(\mathscr{E})$. Since $\beta < \mu_H^D(F) < \mu_H^D(\mathscr{E})$, we know that

$$-1/(\mu_H^D(F) - \beta) < -1/(\mu_H^D(E) - \beta).$$

Furthermore, $\mu_H^D(F) < \mu_H^D(\mathscr{E})$ and $\mathscr{E}$ is torsion-free and nonzero, so $\operatorname{rank}(F) > 0$ and $\operatorname{rank}(\mathscr{E}) > 0$. Therefore Remark 3.1.8 applies and we find that

$$\lim_{\alpha\to\infty} \frac{2}{\alpha^2}\mu_{\alpha,\beta}^{\mathrm{tilt}}(F) = -\frac{1}{\mu_H^D(F) - \beta} < -\frac{1}{\mu_H^D(E) - \beta} = \lim_{\alpha\to\infty} \frac{2}{\alpha^2}\mu_{\alpha,\beta}^{\mathrm{tilt}}(\mathscr{E})$$

In other words, $\mu_{\alpha,\beta}^{\mathrm{tilt}}(F) < \mu_{\alpha,\beta}^{\mathrm{tilt}}(\mathscr{E})$ for all $\alpha \gg 0$. By the generalized seesaw inequality, it follows that $\mu_{\alpha,\beta}^{\mathrm{tilt}}(F) < \mu_{\alpha,\beta}^{\mathrm{tilt}}(G)$ for all $\alpha \gg 0$, as desired.

Second, assume that $\mu_H^D(F) = \mu_H^D(E)$ with $\nu_H^D(F) < \nu_H^D(G)$. We shall consider two subcases: $\operatorname{rank}(G) = 0$ or $\operatorname{rank}(G) \neq 0$. If $\operatorname{rank}(G) = 0$, since $\mu_H^D(F) = \mu_H^D(\mathscr{E})$, we find that $\deg_H^D(F) = \deg_H^D(\mathscr{E})$ so $\deg_H(G) = 0$. By Lemma 2.1.3, it follows that $H^{n-2}\cdot \mathrm{ch}_2^D(G) \geq 0$. Since $H^{n-2}\cdot \mathrm{ch}_2^D(G) \geq 0$ and $\operatorname{rank}(F) = \operatorname{rank}(\mathscr{E}) > 0$, by additivity of the Chern characters, $\nu_H^D(F) \leq \nu_H^D(\mathscr{E})$ with equality exactly if $H^{n-2}\cdot \mathrm{ch}_2^D(G) = 0$. By Remark 3.1.8, we find that

$$\begin{aligned}
\mu_{\alpha,\beta}^{\mathrm{tilt}}(F) &= \frac{\nu_H^D(F) - \beta\mu_H^D(F) + \left(\frac{\beta^2}{2} - \frac{\alpha^2}{2}\right)}{\mu_H^D(F) - \beta} \\
&= \frac{\nu_H^D(F) - \beta\mu_H^D(\mathscr{E}) + \left(\frac{\beta^2}{2} - \frac{\alpha^2}{2}\right)}{\mu_H^D(\mathscr{E}) - \beta} \\
&\leq \frac{\nu_H^D(\mathscr{E}) - \beta\mu_H^D(\mathscr{E}) + \left(\frac{\beta^2}{2} - \frac{\alpha^2}{2}\right)}{\mu_H^D(\mathscr{E}) - \beta} \\
&\mu_{\alpha,\beta}^{\mathrm{tilt}}(\mathscr{E})
\end{aligned}$$

with equality exactly if $H^{n-2}\cdot \mathrm{ch}_2^D(G) = 0$. If $H^{n-2}\cdot \mathrm{ch}_2^D(G) = 0$ then $\mu_{\alpha,\beta}^{\mathrm{tilt}}(G) = +\infty > \mu_{\alpha,\beta}^{\mathrm{tilt}}(\mathscr{E})$ where $\mathfrak{I}Z_{\alpha,\beta}^{\mathrm{tilt}}(\mathscr{E}) = \deg_H^{D+\beta H}(\mathscr{E})$ is nonzero because $\operatorname{rank}(\mathscr{E}) \neq 0$ and $\mu_H^{D+\beta H}(\mathscr{E}) > \beta$. If $H^{n-2}\cdot \mathrm{ch}_2^D(G) \neq 0$, $\mu_{\alpha,\beta}^{\mathrm{tilt}}(F) < \mu_{\alpha,\beta}^{\mathrm{tilt}}(\mathscr{E}) < \mu_{\alpha,\beta}^{\mathrm{tilt}}(G)$ by the generalized seesaw inequality. In either scenario, we find that $\mu_{\alpha,\beta}^{\mathrm{tilt}}(F) \leq \mu_{\alpha,\beta}^{\mathrm{tilt}}(\mathscr{E}) < \mu_{\alpha,\beta}^{\mathrm{tilt}}(G)$ for all $\alpha > 0$, as needed.

If $\operatorname{rank}(G) \neq 0$ then by the seesaw inequality, we find that $\mu_H^D(F) = \mu_H^D(\mathscr{E}) = \mu_H^D(G)$. Thus, $\mu_H^D(F) = \mu_H^D(G)$ and $\nu_H^D(F) < \nu_H^D(G)$, so by Remark 3.1.8,

$$\begin{aligned}
\mu_{\alpha,\beta}^{\text{tilt}}(F) &= \frac{\nu_H^D(F) - \beta\mu_H^D(F) + \left(\frac{\beta^2}{2} - \frac{\alpha^2}{2}\right)}{\mu_H^D(F) - \beta} \\
&= \frac{\nu_H^D(F) - \beta\mu_H^D(G) + \left(\frac{\beta^2}{2} - \frac{\alpha^2}{2}\right)}{\mu_H^D(G) - \beta} \\
&< \frac{\nu_H^D(G) - \beta\mu_H^D(G) + \left(\frac{\beta^2}{2} - \frac{\alpha^2}{2}\right)}{\mu_H^D(G) - \beta} \\
&\mu_{\alpha,\beta}^{\text{tilt}}(G)
\end{aligned}$$

for all $\alpha > 0$.

In every case, we have shown $\mu_{\alpha,\beta}^{\text{tilt}}(F) < \mu_{\alpha,\beta}^{\text{tilt}}(G)$ for all $\alpha \gg 0$, so $\mathscr{E}$ is weakly $\sigma_{\alpha,\beta}^{\text{tilt}}$-stable for all $\alpha \gg 0$, as desired.

For the converse, suppose that $\mathscr{E}$ is weakly $\sigma_{\alpha,\beta}^{\text{tilt}}$-stable for all $\alpha \gg 0$ and $\beta < \mu_H^D(\mathscr{E})$. In particular, $\mathscr{E} \in \mathcal{T}_H^{D+\beta H}(X)$ for all $\beta < \mu_H^D(\mathscr{E})$. Taking the limit as $\beta$ approaches $\mu_H^D(\mathscr{E})$ from the left, we find that nonzero every quotient of $\mathscr{E}$ in $\operatorname{Coh}(X)$ has slope at least $\mu_H^D(\mathscr{E})$. In other words, $\mathscr{E}$ is $\mu_H$-semistable.

Consider a short exact sequence $0 \to \mathscr{F} \to \mathscr{E} \to \mathscr{G} \to 0$ in $\operatorname{Coh}(X)$. We already know that $\mathscr{E}$ is $\mu_H$-semistable, so assume $\mu_H^D(\mathscr{F}) = \mu_H^D(\mathscr{G})$. We must show that $\nu_H^D(\mathscr{F}) < \nu_H^D(\mathscr{G})$. Since $\mathscr{E} \in \operatorname{Coh}_H^{D+\beta H}(X)$ and $\mathscr{G}$ is a quotient of $\mathscr{E}$, we find that $\mathscr{G} \in \operatorname{Coh}_H^{D+\beta H}(X)$ for all $\beta < \mu_H^D(\mathscr{E})$. Thus, we have a short exact sequence in $\operatorname{Coh}_H^{D+\beta H}(X)$: $0 \to F \to \mathscr{E} \to \mathscr{G} \to 0$. Since $\mathscr{E}$ is weakly $\sigma_{\alpha,\beta}^{\text{tilt}}$-stable, by definition, $\mu_{\alpha,\beta}^{\text{tilt}}(F) < \mu_{\alpha,\beta}^{\text{tilt}}(\mathscr{G})$ for all $\alpha \gg 0$. By the generalized seesaw inequality, we find that

$\mu^{\mathrm{tilt}}_{\alpha,\beta}(\mathscr{E}) < \mu^{\mathrm{tilt}}_{\alpha,\beta}(\mathscr{G})$ for all $\alpha \gg 0$. Therefore, by Remark 3.1.8,

$$\begin{aligned}
\frac{\nu^D_H(\mathscr{E}) - \beta\mu^D_H(\mathscr{E}) + \left(\frac{\beta^2}{2} - \frac{\alpha^2}{2}\right)}{\mu^D_H(\mathscr{E}) - \beta} &= \mu^{\mathrm{tilt}}_{\alpha,\beta}(\mathscr{E}) \\
&< \mu^{\mathrm{tilt}}_{\alpha,\beta}(\mathscr{G}) \\
&= \frac{\nu^D_H(\mathscr{G}) - \beta\mu^D_H(\mathscr{G}) + \left(\frac{\beta^2}{2} - \frac{\alpha^2}{2}\right)}{\mu^D_H(\mathscr{G}) - \beta} \\
&= \frac{\nu^D_H(\mathscr{G}) - \beta\mu^D_H(\mathscr{E}) + \left(\frac{\beta^2}{2} - \frac{\alpha^2}{2}\right)}{\mu^D_H(\mathscr{E}) - \beta}
\end{aligned}$$

for all $\alpha \gg 0$. Solving for $\nu^D_H(\mathscr{G})$ on the right, we find that $\nu^D_H(\mathscr{E}) < \nu^D_H(\mathscr{G})$. By the seesaw inequality we find that $\nu^D_H(\mathscr{F}) < \nu^D_H(\mathscr{G})$, as desired.

2. Assume $\mathscr{E}[1]$ is weakly $\sigma^{\mathrm{tilt}}_{\alpha,\beta}$-stable for all $\alpha \gg 0$ and $\beta > \mu^D_H(\mathscr{E})$. In particular, $\mathscr{E} \in \mathcal{F}^{D+\beta H}_H(X)$ for all $\beta > \mu^D_H(\mathscr{E})$. If we take the limit as $\beta$ approaches $\mu^D_H(\mathscr{E})$ from the right, we find that every subsheaf of $\mathscr{E}$ (in $\mathrm{Coh}(X)$) has slope at most $\mu^D_H(\mathscr{E})$. In other words, $\mathscr{E}$ is $\mu_H$-semistable.

   Consider a short exact sequence $0 \to \mathscr{F} \to \mathscr{E} \to \mathscr{G} \to 0$ in $\mathrm{Coh}(X)$ with $\mathscr{F} \neq 0, \mathscr{E}$. We already know that $\mathscr{E}$ is $\mu_H$-semistable, so assume $\mu^D_H(\mathscr{F}) = \mu^D_H(\mathscr{G})$. We must show that $\nu^D_H(\mathscr{F}) > \nu^D_H(\mathscr{E})$. Since $\mu^D_H(\mathscr{F}) = \mu^D_H(\mathscr{G})$, by the seesaw inequality, $\mu^D_H(\mathscr{F}) = \mu^D_H(\mathscr{E})$. Furthermore, since $\mathscr{E}$ is $\mu_H$-semistable, $\mathscr{F}[1] \in \mathrm{Coh}^{D+\beta H}_H(X)$ for all $\beta > \mu^D_H(\mathscr{E})$. Thus, we have a short exact sequence $0 \to \mathscr{F}[1] \to \mathscr{E}[1] \to G \to 0$ in $\mathrm{Coh}^{D+\beta H}_H(X)$. Since $\mathscr{E}[1]$ is weakly $\sigma^{\mathrm{tilt}}_{\alpha,\beta}$-stable and, has good quotients (by Lemma 3.1.13), by Lemma 2.2.11 we find that $\mathscr{E}$ is $\sigma^{\mathrm{tilt}}_{\alpha,\beta}$-stable. Since $\mathscr{F}$ is a proper nonzero subobject of $\mathscr{E}$, we find that $\mu^{\mathrm{tilt}}_{\alpha,\beta}(\mathscr{F}) = \mu^{\mathrm{tilt}}_{\alpha,\beta}(\mathscr{F}[1]) < \mu^{\mathrm{tilt}}_{\alpha,\beta}(\mathscr{E}[1]) = \mu^{\mathrm{tilt}}_{\alpha,\beta}(\mathscr{E})$. It follows by remark 3.1.8 that for

$\alpha \gg 0$ and $\beta > \mu_H^D(\mathscr{E})$ that

$$\begin{aligned}
\frac{\nu_H^D(\mathscr{F}) - \beta\mu_H^D(\mathscr{F}) + \left(\frac{\beta^2}{2} - \frac{\alpha^2}{2}\right)}{\mu_H^D(\mathscr{F}) - \beta} &= \mu_{\alpha,\beta}^{\mathrm{tilt}}(\mathscr{F}) \\
&< \mu_{\alpha,\beta}^{\mathrm{tilt}}(\mathscr{E}) \\
&= \frac{\nu_H^D(\mathscr{E}) - \beta\mu_H^D(\mathscr{E}) + \left(\frac{\beta^2}{2} - \frac{\alpha^2}{2}\right)}{\mu_H^D(\mathscr{E}) - \beta} \\
&= \frac{\nu_H^D(\mathscr{E}) - \beta\mu_H^D(\mathscr{F}) + \left(\frac{\beta^2}{2} - \frac{\alpha^2}{2}\right)}{\mu_H^D(\mathscr{F}) - \beta}
\end{aligned}$$

Since $\mu_H^D(\mathscr{F}) < \beta$, we can simplify the above inequality to find $\nu_H^D(\mathscr{F}) > \nu_H^D(\mathscr{E})$.

By assumption, $\mathscr{F}$ is nonzero, torsion-free (because it is a subsheaf of $\mathscr{E}$ which is torsion-free), and $\mu_H^D(\mathscr{F}) = \mu_H^D(\mathscr{E}) = \mu_H^D(\mathscr{G})$. Therefore, $\mathscr{F}$, $\mathscr{E}$, and $\mathscr{G}$ all have nonzero rank, so the seesaw inequality (with respect to $-H^{n-2} \cdot \mathrm{ch}_2^D(\cdot) + \sqrt{-1}\,\mathrm{rank}(\cdot)$) holds. Since $\nu_H^D(\mathscr{F}) > \nu_H^D(\mathscr{E})$, it follows that $\nu_H^D(\mathscr{F}) > \nu_H^D(\mathscr{G})$. In all, we have shown that every proper nonzero subsheaf $0 \to \mathscr{F} \to \mathscr{E}$ satisfies either

- $\mu_H^D(\mathscr{F}) < \mu_H^D(\mathscr{E}/\mathscr{F})$ or
- $\mu_H^D(\mathscr{F}) = \mu_H^D(\mathscr{E}/\mathscr{F})$ with $\nu_H^D(\mathscr{F}) > \nu_H^D(\mathscr{E}/\mathscr{F})$.

By Lemma 3.2.4, it follows that $\mathscr{E}$ is $\mu_H$-stable, as desired.

3. Assume $\mathscr{E}$ is reflexive and $\mu_H$-stable. Therefore, $\mathscr{E}[1] \in \mathrm{Coh}_H^{D+\beta H}(X)$ for all $\beta > \mu_H^D(\mathscr{E})$. Consider a short exact sequence $0 \to F \to \mathscr{E}[1] \to G \to 0$ in $\mathrm{Coh}_H^{D+\beta H}(X)$ with $F \neq 0, \mathscr{E}$. By Lemma 3.2.2 in view of Lemma 3.1.12, we find that $\mu_H^D(\mathscr{E}[1]) < \mu_H^D(G)$. By considering the long exact sequence on cohomology, we find that $\mu_H^D(G) = \mu_H^D(\mathscr{H}^{-1}(G)) \leq \beta$ because $\mathscr{H}^{-1}(G) \in \mathcal{F}_H^{D+\beta H}(X)$. We already saw that $\mu_H^D(\mathscr{E}) < \beta$, so by Remark 3.1.8, it follows that

$$\lim_{\alpha\to\infty} \frac{2}{\alpha^2}\mu_{\alpha,\beta}^{\mathrm{tilt}}(\mathscr{E}[1]) = -\frac{1}{\mu_H^D(\mathscr{E}[1]) - \beta} < -\frac{1}{\mu_H^D(G) - \beta} = \lim_{\alpha\to\infty} \frac{2}{\alpha^2}\mu_{\alpha,\beta}^{\mathrm{tilt}}(G).$$

Note that this inequality still holds in the case that $\beta = \mu_H^D(G)$ if we formally say $-1/(\mu_H^D(G) - \beta) = +\infty$. It follows that $\mu_{\alpha,\beta}^{\mathrm{tilt}}(\mathscr{E}[1]) < \mu_{\alpha,\beta}^{\mathrm{tilt}}(G)$ for all $\alpha \gg 0$ $\beta > \mu_H^D(\mathscr{E})$.

Since $\mu^{\text{tilt}}_{\alpha,\beta}(\mathscr{E}[1]) < \mu^{\text{tilt}}_{\alpha,\beta}$, by the seesaw inequality, $\mu^{\text{tilt}}_{\alpha,\beta}(F) < \mu^{\text{tilt}}_{\alpha,\beta}(G)$ for all $\alpha \gg 0$ and $\beta > \mu^D_H(\mathscr{E})$. In other words, $\mathscr{E}[1]$ is weakly $\sigma^{\text{tilt}}_{\alpha,\beta}$-stable for all $\alpha \gg 0$ and all $\beta > \mu^D_H(\mathscr{E})$. By Lemma 3.1.13, $\mathscr{E}[1]$ has good quotients, so by Lemma 2.2.11, we find that $\mathscr{E}[1]$ is $\sigma^{\text{tilt}}_{\alpha,\beta}$-stable for all $\alpha \gg 0$ and all $\beta > \mu^D_H(\mathscr{E})$, as desired.

□

**Lemma 3.2.4.** *Assume $\mathscr{E}$ is a torsion-free sheaf. If $\mathscr{E}$ satisfies the following property*

*for every proper nonzero subsheaf $0 \to \mathscr{F} \to \mathscr{E}$ one of the following holds:*

- $\mu^D_H(\mathscr{F}) < \mu^D_H(\mathscr{E}/\mathscr{F})$ *or*
- $\mu^D_H(\mathscr{F}) = \mu^D_H(\mathscr{E}/\mathscr{F})$ *with* $\nu^D_H(\mathscr{F}) \geq \nu^D_H(\mathscr{E}/\mathscr{F})$

*then $\mathscr{E}$ is $\mu_H$-stable.*

*Proof.* We will prove the contrapositive, so assume that $\mathscr{E}$ is not $\mu_H$-stable. We will first show that $\mathscr{E}^{\oplus 2}$ does not satisfy the property in the lemma statement then show that $\mathscr{E}$ does not satisfy the property.

If $\mathscr{E}$ is not $\mu_H$-semistable then we are done, so assume $\mathscr{E}$ is $\mu_H$-semistable. Therefore, by definition, we can choose a proper nonzero subsheaf $0 \to \mathscr{F} \to \mathscr{E}$ such that $\mu^D_H(\mathscr{F}) = \mu^D_H(\mathscr{E}/\mathscr{F})$. On the other hand, let $Y$ be a closed subvariety of $X$ of codimension 2 satisfying

$$H^{n-2} \cdot Y \geq \frac{H^{n-2} \cdot \operatorname{ch}^D_2(\mathscr{F}) \operatorname{rank}(\mathscr{E}) - H^{n-2} \cdot \operatorname{ch}^D_2(\mathscr{E}) \operatorname{rank}(\mathscr{F})}{\operatorname{rank}(\mathscr{E})^2}. \tag{3.1}$$

We have a surjection $\mathscr{O}_X \to \iota_* \mathscr{O}_Y \to 0$ and so we have a short exact sequence

$$0 \to \mathscr{K} \to \mathscr{E} \to \mathscr{E} \otimes \iota_* \mathscr{O}_Y \to 0.$$

Since $Y$ is supported in codimension 2,

$$H^{n-2} \cdot \operatorname{ch}^D_2(\mathscr{E} \otimes \iota_* \mathscr{O}_Y) = \operatorname{rank}(\mathscr{E}) H^{n-2} \cdot Y.$$

It follows that

$$H^{n-2} \cdot \operatorname{ch}^D_2(\mathscr{K}) = H^{n-2} \cdot \operatorname{ch}^D_2(\mathscr{E}) - \operatorname{rank}(\mathscr{E}) H^{n-2} \cdot Y.$$

In all, we have constructed a short exact sequence

$$0 \to \mathscr{F} \oplus \mathscr{K} \to \mathscr{E}^{\oplus 2} \to (\mathscr{E}/\mathscr{F}) \oplus (\mathscr{E} \otimes \iota_* \mathscr{O}_Y) \to 0.$$

Therefore, by additivity of the Chern character,

$$\nu_H^D(\mathscr{F} \oplus \mathscr{K}) = \frac{H^{n-2} \cdot \mathrm{ch}_2^D(\mathscr{F}) + H^{n-2} \cdot \mathrm{ch}_2^D(\mathscr{E}) - H^{n-2} \cdot Y \,\mathrm{rank}(\mathscr{E})}{\mathrm{rank}(\mathscr{F}) + \mathrm{rank}(\mathscr{E})}.$$

Note that $\mathrm{rank}(\mathscr{E}) = \mathrm{rank}(\mathscr{K})$ because $\mathscr{E} \otimes \iota_* \mathscr{O}_Y$ is supported in codimension 2. On the other hand,

$$\nu_H^D(\mathscr{E}^{\oplus 2}) = \frac{H^{n-2} \cdot \mathrm{ch}_2^D(\mathscr{E})}{\mathrm{rank}(\mathscr{E})}.$$

By using Equation 3.1, we find that $\nu_H^D(\mathscr{F} \oplus \mathscr{K}) \leq \nu_H^D(\mathscr{E}^{\oplus 2})$. By the seesaw inequality, we find that $\nu_H^D(\mathscr{F} \oplus \mathscr{K}) \leq \nu_H^D((\mathscr{E}/\mathscr{F}) \oplus (\mathscr{E} \otimes \iota_* \mathscr{O}_Y))$. Also, by construction,

$$\mu_H^D(\mathscr{F} \oplus \mathscr{K}) = \mu_H^D(\mathscr{E}) = \mu_H^D((\mathscr{E}/\mathscr{F}) \oplus (\mathscr{E} \otimes \iota_* \mathscr{O}_Y)).$$

In all, $\mathscr{E}^{\oplus 2}$ does not satisfy the property in the lemma statement.

We will now show that $\mathscr{E}$ does not satisfy the given property. Consider the short exact sequence

$$0 \to \mathscr{E} \cap (\mathscr{F} \oplus \mathscr{K}) \to \mathscr{F} \oplus \mathscr{K} \to (\mathscr{F} \oplus \mathscr{K})/(\mathscr{E} \cap (\mathscr{F} \oplus \mathscr{K})) \to 0$$

where the intersection is with respect to the inclusion $0 \to \mathscr{E} \to \mathscr{E}^{\oplus 2}$ into the first component.

Since $\mathrm{rank}(\mathscr{F} \oplus \mathscr{K}) > \mathrm{rank}(\mathscr{E})$, we know that $\mathscr{E} \cap (\mathscr{F} \oplus \mathscr{K}) \neq 0, \mathscr{E}$. It follows that $(\mathscr{F} \oplus \mathscr{K})/(\mathscr{E} \cap (\mathscr{F} \oplus \mathscr{K})) \neq 0, \mathscr{E}$ as well. Using a similar argument to Lemma 2.2.15, $\mathscr{E} \cap (\mathscr{F} \oplus \mathscr{K})$ and $(\mathscr{F} \oplus \mathscr{K})/(\mathscr{E} \cap (\mathscr{F} \oplus \mathscr{K}))$ are subsheaves of $\mathscr{E}$. Moreover, that same argument tells us that $\mu_H^D(\mathscr{E} \cap (\mathscr{F} \oplus \mathscr{K})) = \mu_H^D(\mathscr{E})$ and similarly for $(\mathscr{F} \oplus \mathscr{K})/(\mathscr{E} \cap (\mathscr{F} \oplus \mathscr{K}))$.

Furthermore, as we saw above, $\nu_H^D(\mathscr{F} \oplus \mathscr{K}) \leq \nu_H^D(\mathscr{E}^{\oplus 2}) = \nu_H^D(\mathscr{E})$. Therefore, by the seesaw inequality, either $\nu_H^D(\mathscr{E} \cap (\mathscr{F} \oplus \mathscr{K})) \leq \nu_H^D(\mathscr{E})$ or $\nu_H^D((\mathscr{F} \oplus \mathscr{K})/(\mathscr{E} \cap (\mathscr{F} \oplus \mathscr{K})) \leq \nu_H^D(\mathscr{E})$. In other words, $\mathscr{E}$ does not satisfy the proper in the lemma statement, as desired. □

There is also a variant of the Large Volume Limit for $\sigma$-semistability which is essentially as one would expect. Interestingly, in the semistable case, we do not need the reflexive assumption.

There is also a variant of the Large Volume Limit for chain complexes.

**Lemma 3.2.5** ([Bri08, Proposition 14.2])**.** *If $E$ is $\sigma^{\text{tilt}}_{\alpha,\beta}$-semistable for all $\alpha \gg 0$ then one of the following holds:*

- $\mathscr{H}^{-1}(E) = 0$ *and* $\mathscr{H}^0(E)$ *is* $\mu_H$*-semistable.*
- $\mathscr{H}^{-1}(E)$ *is* $\mu_H$*-semistable and* $\operatorname{codim}(\mathscr{H}^0(E)) \geq 2$*.*

The following cultural aside notes that $\mu_H$-stability can also be detected along the vertical wall in the $(H, D)$-slice (we will discuss walls in the next subsection). Note that we do not need the reflexive assumption in this case. This proposition can be proven using the same techniques as the Large Volume Limit.

**Proposition 3.2.6.** *Assume $\mathscr{E}$ is torsion-free and for ease of notation set $\beta_0 = \mu^D_H(\mathscr{E})$. $\mathscr{E}$ is $\mu_H$-stable if and only if $\mathscr{E}[1]$ is $\sigma_{\alpha,\beta_0}$-stable for all $\alpha > 0$.*

We will eventually see that $\sigma^{\text{tilt}}_{\alpha,\beta}$ satisfies the support property with respect to $\overline{\Delta}^D_H$. For now, we will only prove this result when $\beta \in \mathbb{Q}$. We use a similar argument to [BMT14, Section 7.3].

**Lemma 3.2.7.** *If $(\beta, \alpha) \in \mathbb{Q} \times \mathbb{R}_{>0}$ then $\sigma^{\text{tilt}}_{\alpha,\beta}$ satisfies the support property with respect to $\overline{\Delta}^D_H$. In particular, if $(\beta, \alpha) \in \mathbb{Q} \times \mathbb{R}_{>0}$ then $\sigma^{\text{tilt}}_{\alpha,\beta}$ is a very weak stability condition.*

*Proof.* Since $\beta \in \mathbb{Q}$, $\Im Z^{\text{tilt}}_{\alpha,\beta} : \Lambda \to \mathbb{R}$ is discrete so we can proceed by induction on $\Im Z^{\text{tilt}}_{\alpha,\beta}$.

First, assume that $\Im Z^{\text{tilt}}_{\alpha,\beta}(E) = 0$. By definition $\Im Z^{\text{tilt}}_{\alpha,\beta}(E) = \deg^{D+\beta H}_H(E)$ which is independent of $\alpha$, so $\mu^{\text{tilt}}_{\alpha,\beta}(E) = +\infty$ for all $\alpha > 0$. In other words, $E$ is $\sigma^{\text{tilt}}_{\alpha,\beta}$-semistable for all $\alpha \gg 0$. By Lemma 3.2.5, one of the following must hold:

- $\mathscr{H}^{-1}(E) = 0$ and $\mathscr{H}^0(E)$ is $\mu_H$-semistable.

- $\operatorname{codim}(\mathscr{H}^0(E)) \geq 2$ and $\mathscr{H}^{-1}(E)$ is $\mu_H$-semistable.

In the first case, by Bogomolov's Inequality (Lemma 2.1.15)

$$\overline{\Delta}_H^D(E,E) = \overline{\Delta}_H^D(\mathscr{H}^0(E), \mathscr{H}^0(E)) \geq 0.$$

In the second case, by Lemma 2.1.3,

$$H^{n-2} \cdot \mathrm{ch}_{\leq 2}^D(E) = -H^{n-2} \cdot \mathrm{ch}_{\leq 2}^D(\mathscr{H}^{-1}(E)) + (0,0,d)$$

with $d \geq 0$. It follows by the calculation above and Bogomolov's Inequality (Lemma 2.1.15) that

$$\begin{aligned}
\overline{\Delta}_H^D(E,E) &= \deg_H^D(E)^2 - 2\,\mathrm{rank}(E)(H^{n-2} \cdot \mathrm{ch}_2^D(E)) \\
&= \deg_H^D(\mathscr{H}^{-1}(E))^2 + 2\,\mathrm{rank}(\mathscr{H}^{-1}(E))(d' - H^{n-2} \cdot \mathrm{ch}_2^D(\mathscr{H}^{-1}(E))) \\
&\geq \deg_H^D(\mathscr{H}^{-1}(E))^2 - 2\,\mathrm{rank}(\mathscr{H}^{-1}(E))(H^{n-2} \cdot \mathrm{ch}_2^D(\mathscr{H}^{-1}(E))) \\
&\geq 0
\end{aligned}$$

This completes the base case.

Second, assume that $\Im Z_{\alpha,\beta}(E) > 0$. If $E$ is $\sigma_{\alpha,\beta}$-semistable for all $\alpha \gg 0$ then $\overline{\Delta}_H^D(E,E) \geq 0$ by the same argument as above. Therefore, we may assume that $E$ is not $\sigma_{\alpha',\beta}^{\mathrm{tilt}}$-semistable for some $\alpha' > 0$. Since $\mu_{\alpha,\beta}^{\mathrm{tilt}}$ is continuous in $\alpha$, we can find $\alpha_{1/2} \in [\alpha_0, \alpha_1]$ such that $E$ has a nonzero proper subobject $0 \to F \to E$ satisfying $\mu_{\alpha_{1/2},\beta_0}(F) = \mu_{\alpha_{1/2},\beta_0}(E)$.

Since $E$ is not $\sigma_{\alpha,\beta_0}^{\mathrm{tilt}}$-semistable for all $\alpha > 0$ we know $\Im Z^{\mathrm{tilt}}(E) \neq 0$. Since $E$ is not weakly $\sigma_{\alpha_{1/2},\beta_0}^{\mathrm{tilt}}$-semistable and $\Im Z_{\alpha_{1/2},\beta_0}(E) \neq 0$, by the same argument as Lemma 2.3.14, we can find a $\sigma_{\alpha_{1/2},\beta_0}^{\mathrm{tilt}}$-stable subobject $0 \to \hat{E} \to E$ satisfying $\mu_{\alpha_{1/2},\beta_0}^{\mathrm{tilt}}(\hat{E}) = \mu_{\alpha_{1/2},\beta_0}^{\mathrm{tilt}}(E)$ and $\Im Z_{\alpha_{1/2},\beta_0}^{\mathrm{tilt}}(\hat{E}) < \Im Z_{\alpha_{1/2},\beta_0}^{\mathrm{tilt}}(E)$. In short, we have a short exact sequence $0 \to \hat{E} \to E \to E/\hat{E} \to 0$ all of the same slope at $(\beta_0, \alpha_{1/2})$ with $\hat{E}$ and $E/\hat{E}$ both $\sigma_{\alpha_{1/2},\beta_0}$-semistable.

Since $\Im Z_{\alpha_{1/2},\beta_0}^{\mathrm{tilt}}(\hat{E}) < \Im Z_{\alpha_{1/2},\beta_0}^{\mathrm{tilt}}(E)$, we know that $\Im Z_{\alpha_{1/2},\beta_0}^{\mathrm{tilt}}(E/\hat{E}) < \Im Z_{\alpha_{1/2},\beta_0}^{\mathrm{tilt}}(E)$, so by the inductive hypothesis $\overline{\Delta}_H^D(\hat{E}) \geq 0$ and $\overline{\Delta}_H^D(E/\hat{E}) \geq 0$. Therefore, by Lemma 2.3.8, we find $\overline{\Delta}_H^D(E) \geq 0$, as needed.

By Lemma 3.1.9, $\mathrm{Ker}(Z_{\alpha_0,\beta_0})^{\mathrm{tilt}} = \{(0,0,0)\}$ and so $\overline{\Delta}_H^D$ is vacuously negative definite.

Last, we find $\sigma^{\mathrm{tilt}}_{\alpha_0,\beta_0}$ satisfies the Harder-Narasimhan property by Lemma 2.3.6. $\square$

## 3.3 Tilt Stability is a Very Weak Stability Condition for Real Beta

In this subsection we show that $\sigma^{\mathrm{tilt}}_{\alpha,\beta}$ is a very weak stability condition for all $(\beta,\alpha) \in \mathbb{R} \times \mathbb{R}_{>0}$. Furthermore, we show that the induced morphism $\mathbb{R} \times \mathbb{R}_{>0} \to \mathrm{WStab}(X)$ is a homeomorphism onto its image. For this reason, we will say that tilt stability is a continuous family.

There are currently two techniques to extend the results from the previous section to all $\beta \in \mathbb{R}$.

The first technique, from [BMS16, Appendix B], is to show that $\sigma^{\mathrm{tilt}}_{\alpha,\beta}$ for $\beta \in \mathbb{R} \setminus \mathbb{Q}$ must correspond to a point of $\mathrm{WStab}(X)$ that is close to $\sigma^{\mathrm{tilt}}_{\alpha,\beta_0}$ for $\beta_0 \in \mathbb{Q}$. This method is useful because it automatically shows tilt stability forms a continuous family. However, this method is technically difficult because it requires a detailed understanding of $\mathrm{WStab}(X)$ (which is done by essentially reducing to arguments about $\mathrm{Stab}(X)$).

The second technique, from [Fey18, Section 2.3], is to show $\sigma^{\mathrm{tilt}}_{\alpha,\beta}$ satisfies the Harder-Narasimhan and support properties directly then show that tilt stability forms a continuous family. This is method is useful because it avoids an analysis of $\mathrm{WStab}(X)$. However, it is technically difficult because the argument requires a detailed understanding $\mu^{\mathrm{tilt}+}_{\alpha,\beta}(E)$ as a function of $\beta$.

We introduce a third technique which is to extend our results to $\beta \in \mathbb{R}$ for $\sigma^{\mathrm{tilt}}_{\alpha,\beta}$-pure objects then to extend to all objects using that pure objects form a torsion-free class. Since $\sigma^{\mathrm{tilt}}_{\alpha,\beta}$-pure have nice deformation properties, the existence of Harder-Narasimhan filtrations and the support property are easy to show. Furthermore, this new technique allows us to construct weak Jordan-Hölder filtrations (which were not previously known to exist).

**Lemma 3.3.1.** *Assume* $E \in \mathrm{Coh}_H^{D+\beta H}(X)$ *satisfies* $\Im Z_{\alpha,\beta}^{\mathrm{tilt}}(E) \neq 0$. *Define* $\mathcal{C}$ *to be the collection:*

$$\{F \in \mathrm{Coh}_H^{D+\beta H}(X) \mid 0 \to F \to E \text{ is a } \sigma_{\alpha,\beta}^{\mathrm{tilt}} - \text{semistable and } \mu_{\alpha,\beta}^{\mathrm{tilt}}(F) \geq \mu_{\alpha,\beta}^{\mathrm{tilt}}(E)\}.$$

*The collection* $\{H^{n-1} \cdot \mathrm{ch}_{\leq 1}^D(F)\}_{F \in \mathcal{C}}$ *is finite.*

*Proof.* We first claim $\{\mathrm{rank}(F)\}_{F\in\mathcal{C}}$ is finite. Since $0 \to F \to E$ is a subobject,

$$\mathrm{rank}(F) = \mathrm{rank}(\mathscr{H}^0(F)) - \mathrm{rank}(\mathscr{H}^{-1}(F)) \geq -\mathrm{rank}(\mathscr{H}^{-1}(E)).$$

In other words, $\{\mathrm{rank}(F)\}_{F\in\mathcal{C}}$ is bounded below. If $\mathrm{rank}(F) \leq 0$ then we are done, so we may assume $\mathrm{rank}(F) > 0$. By direct computation,

$$\Re Z_{\alpha,\beta}^{\mathrm{tilt}}(F) = -\frac{\Im Z_{\alpha,\beta}^{\mathrm{tilt}}(F)^2}{2\,\mathrm{rank}(F)} + \frac{\overline{\Delta}_H^D(F)}{2\,\mathrm{rank}(F)} + \frac{\alpha^2}{2}\,\mathrm{rank}(F).$$

Therefore, by definition of $\mathcal{C}$,

$$\begin{aligned}
|\mu_{\alpha,\beta}^{\mathrm{tilt}}(E)|\Im Z_{\alpha,\beta}^{\mathrm{tilt}}(E) &\geq -\mu_{\alpha,\beta}^{\mathrm{tilt}}(E)\Im Z_{\alpha,\beta}^{\mathrm{tilt}}(F) \\
&\geq \Re Z_{\alpha,\beta}^{\mathrm{tilt}}(F) \\
&\geq -\frac{\Im Z_{\alpha,\beta}^{\mathrm{tilt}}(E)^2}{2\,\mathrm{rank}(F)} + 0 + \frac{\alpha^2}{2}\,\mathrm{rank}(F) \\
&\geq -\frac{\Im Z_{\alpha,\beta}^{\mathrm{tilt}}(E)^2}{2} + \frac{\alpha^2}{2}\,\mathrm{rank}(F).
\end{aligned}$$

In other words,

$$\frac{2}{\alpha^2}\left(|\Re Z_{\alpha,\beta}^{\mathrm{tilt}}(E)| + \frac{\Im Z_{\alpha,\beta}^{\mathrm{tilt}}(E)^2}{2}\right) \geq \mathrm{rank}(F)$$

so $\{\mathrm{rank}(F)\}_{F\in\mathcal{C}}$ is bounded above. Therefore, the set $\{\mathrm{rank}(F)\}_{F\in\mathcal{C}}$ is finite.

By definition of $\mathcal{C}$,

$$0 \leq \Im Z_{\alpha,\beta}^{\mathrm{tilt}}(F) \leq \Im Z_{\alpha,\beta}^{\mathrm{tilt}}(E).$$

In other words,

$$0 \leq \deg_H^D(F) - \beta\,\mathrm{rank}(F) \leq \Im Z_{\alpha,\beta}^{\mathrm{tilt}}(E).$$

Since $\{\mathrm{rank}(F)\}_{F\in\mathcal{C}}$ is finite, $\{H^{n-1} \cdot \mathrm{ch}_{\leq 1}^D(F)\}_{F\in\mathcal{C}}$ is finite as well, as claimed. $\square$

**Lemma 3.3.2.** *Assume $E \in \mathrm{Coh}_H^{D+\beta H}(X)$ is $\sigma_{\alpha,\beta}^{\mathrm{tilt}}$-pure.*

1. *There exists a Harder-Narasimhan filtration*

$$0 \to E_1 \to E_2 \to \cdots \to E_{m-1} \to E$$

*with respect to $\sigma_{\alpha,\beta}^{\mathrm{tilt}}$.*

2. *If $E$ is $\sigma_{\alpha,\beta}^{\mathrm{tilt}}$-semistable then there exists is a weak Jordan-Hölder filtration*

$$0 \to E_1 \to E_2 \to \cdots \to E_{m-1} \to E.$$

*with respect to $\sigma_{\alpha,\beta}^{\mathrm{tilt}}$.*

*Proof.* 1. For ease of notation, let $\mathcal{C}$ be the collection of all $\sigma_{\alpha,\beta}^{\mathrm{tilt}}$-destabilizing subobjects of $E$. By Lemma 3.3.1, $\{\Im Z_{\alpha,\beta}^{\mathrm{tilt}}(F)\}_{F\in\mathcal{C}}$ is finite. Therefore, by Lemma 3.2.1.1 any ascending chain in $\mathcal{C}$ must stabilize. Therefore, let $E_1$ be a maximal object of an ascending chain. If $E_1 = E$ we are done, so assume $E \neq E_1$. Furthermore, if $E/E_1$ is $\sigma_{\alpha,\beta}^{\mathrm{tilt}}$-semistable then

$$\mu_{\alpha,\beta}^{\mathrm{tilt}}(E_1) > \mu_{\alpha,\beta}^{\mathrm{tilt}}(E) > \mu_{\alpha,\beta}^{\mathrm{tilt}}(E/E_1)$$

by the seesaw inequality (note that $Z_{\alpha,\beta}^{\mathrm{tilt}}(E_1)$ since $E$ is $\sigma_{\alpha,\beta}^{\mathrm{tilt}}$-pure). In this case, $0- > E_1- > E$ is a Harder-Narasimhan filtration with respect to $\sigma_{\alpha,\beta}^{\mathrm{tilt}}$. Threfore, we may assume $E/E_1$ is not $\sigma_{\alpha,\beta}^{\mathrm{tilt}}$-semistable. By Lemma 2.3.13.4, either $E/E_1$ is either $\sigma_{\alpha,\beta}^{\mathrm{tilt}}$-pure or 0. Since $E \neq E_1$, $E/E_1 \neq 0$ so $E/E_1$ is $\sigma_{\alpha,\beta}^{\mathrm{tilt}}$-pure. Since $E/E_1$ is $\sigma_{\alpha,\beta}^{\mathrm{tilt}}$-pure, by the same argument as above, we can choose a maximal $\sigma_{\alpha,\beta}^{\mathrm{tilt}}$-subobject $0- > E_2- > E_1$.

We can continue the argument above to obtain a filtration

$$0 \to E_1 \to E_2 \to \cdots \to E$$

such that $E_{i+1}/E_i$ is $\sigma$-semistable for all $i$ and

$$\mu_\sigma(E_1) > \mu_\sigma(E_2/E_1) > \mu_\sigma(E_3/E_2) > \cdots .$$

Furthermore, by induction, $\mu^{\mathrm{tilt}}_{\alpha,\beta}(E_i) > \mu^{\mathrm{tilt}}_{\alpha,\beta}(E)$ for all $i$. Therefore, by Lemma 3.3.1, $\{\Im Z^{\mathrm{tilt}}_{\alpha,\beta}(E_i)\}_{i\geq 1}$ is finite. Thus, by Lemma3.2.1, $E_i = E_{i+1}$ for all $i \gg 0$. Hence, we have constructed a Harder-Narasimhan filtration of $E$ with respect to $\sigma^{\mathrm{tilt}}_{\alpha,\beta}$, as desired.

2. By the same argument as part 1, any ascending chain of $\sigma^{\mathrm{tilt}}_{\alpha,\beta}$-destabilizing subobjects must stabilize and $\Im Z^{\mathrm{tilt}}_{\alpha,\beta}$ takes only finitely many values over all destabilizing subobjects. Therefore, Lemma 2.3.14 gives us a Jordan-Hölder filtration.

□

**Remark 3.3.3.** *Note that the same argument as Lemma 3.3.1 and Lemma 3.3.2 allow for infinitesimal deformations of $(\beta, \alpha)$. Specifically, by Lemma 3.1.10 if $E$ is $\sigma^{\mathrm{tilt}}_{\alpha,\beta}$-pure then there exists $\varepsilon > 0$ such that $E \in \mathrm{Coh}^{D+\beta' H}_H(X)$ for all $\beta' \in (\beta - \varepsilon, \beta]$. The rest of the arguments go through as expected.*

*In particular, if $E$ is $\sigma^{\mathrm{tilt}}_{\alpha,\beta}$-pure then there exists $\varepsilon > 0$ and a Harder-Narasimhan filtration*

$$0 \to E_1 \to E_2 \to \cdots \to E_{m-1} \to E$$

*with respect to $\sigma^{\mathrm{tilt}}_{\alpha,\beta'}$ for all $\beta' \in (\beta - \varepsilon, \beta + \varepsilon)$.*

**Lemma 3.3.4.** *Assume $(\beta, \alpha) \in \mathbb{R} \times \mathbb{R}_{>0}$.*

*1. $\sigma^{\mathrm{tilt}}_{\alpha,\beta}$ satisfies the Harder-Narasimhan property*

*2. $\sigma^{\mathrm{tilt}}_{\alpha,\beta}$ satisfies the support property with respect to $\overline{\Delta}^D_H$.*

*In other words, for all $(\beta, \alpha) \in \mathbb{R} \times \mathbb{R}_{>0}$, $\sigma^{\mathrm{tilt}}_{\alpha,\beta}$ is a very weak stability condition.*

*Proof.* Assume $E \in \mathrm{Coh}^{D+\beta H}_H(X)$. By Lemma 3.2.1.3 there is a short exact sequence

$$0 \to T \to E \to E/T \to 0$$

where $E/T$ is $\sigma^{\mathrm{tilt}}_{\alpha,\beta}$-pure and $\Im Z^{\mathrm{tilt}}_{\alpha,\beta}(T) = 0$. By Lemma 3.3.2.2, there is a Harder-Narasimhan filtration

$$0 \to E_1/T \to E_2/T \to \cdots \to E_{m-1}/T \to E/T$$

of $E/T$. We claim that

$$0 \to T \to E_1 \to \cdots \to E_{m-1} \to E$$

is a Harder-Narasimhan filtration of $E$. It remains to show that $T$ is $\sigma^{\mathrm{tilt}}_{\alpha,\beta}$-semistable and $\mu^{\mathrm{tilt}}_{\alpha,\beta}(T) > \mu^{\mathrm{tilt}}_{\alpha,\beta}(E_1/T)$. However, since $\Im Z^{\mathrm{tilt}}_{\alpha,\beta}(T) = 0$, $\mu^{\mathrm{tilt}}_{\alpha,\beta}(T) = +\infty$ so $T$ is $\sigma^{\mathrm{tilt}}_{\alpha,\beta}$-semistable. Moreover, since $E/T$ is $\sigma^{\mathrm{tilt}}_{\alpha,\beta}$-pure, $\Im Z^{\mathrm{tilt}}_{\alpha,\beta}(E_1/T) \neq 0$ so $\mu^{\mathrm{tilt}}_{\alpha,\beta}(T) > \mu^{\mathrm{tilt}}_{\alpha,\beta}(E_1/T)$, as desired.

By Lemma 3.1.9, $\overline{\Delta}^D_H$ is vacuously negative definite. Therefore, assume $E \in \mathrm{Coh}^{D+\beta H}_H(X)$ is $\sigma^{\mathrm{tilt}}_{\alpha,\beta}$-semistable.

If $\Im Z^{\mathrm{tilt}}_{\alpha,\beta}(E) = 0$, by Lemma 3.1.9, either $\beta = \mu^D_H(E)$ or $\mathrm{codim}(E) \geq 2$. In the first case, since $D$ is a $\mathbb{Q}$-divisor, $\beta \in \mathbb{Q}$, $\overline{\Delta}^D_H(E) \geq 0$ by Lemma 3.2.7. In the second, case, $\overline{\Delta}^D_H(E) = 0$ by direct calculation.

Thus, we may assume $\Im Z^{\mathrm{tilt}}_{\alpha,\beta}(E) \neq 0$. In particular, by Lemma 2.3.13, $E$ is $\sigma^{\mathrm{tilt}}_{\alpha,\beta}$-pure. By Remark 3.3.3, we can choose $\beta' < \beta$ such that $\beta' \in \mathbb{Q}$ and $E$ is $\sigma^{\mathrm{tilt}}_{\alpha,\beta'}$-semistable. The result follows by Lemma 3.2.7. $\square$

We now show that if we vary $(\beta, \alpha)$ continuously then $\sigma^{\mathrm{tilt}}_{\alpha,\beta}$ varies continuously with respect to the topology on $\mathrm{WStab}(X, \Lambda)$.

**Lemma 3.3.5.** *There is a continuous injection* $\mathbb{R} \times \mathbb{R}_{>0} \to \mathrm{WStab}(X)$ *(where* $\mathbb{R} \times \mathbb{R}_{>0}$ *has the usual Euclidean topology) given by* $(\beta, \alpha) \mapsto \sigma^{\mathrm{tilt}}_{\alpha,\beta}$.

*Proof.* By Lemma 3.3.2.2 there is a short exact sequence

$$0 \to T \to E \to E/T \to 0$$

where $E/T$ is $\sigma^{\mathrm{tilt}}_{\alpha,\beta}$-pure and $\Im Z^{\mathrm{tilt}}_{\alpha,\beta}(T) = 0$. If $E$ is $\sigma^{\mathrm{tilt}}_{\alpha,\beta}$-pure (i.e. $T = 0$) then the result follows by Remark 3.3.3. Suppose $E$ is not $\sigma^{\mathrm{tilt}}_{\alpha,\beta}$-pure, so $\mu^{\mathrm{tilt}+}_{\alpha,\beta}(E) = +\infty$ and $T$ is a maximal $\sigma^{\mathrm{tilt}}_{\alpha,\beta}$-destabilizing subobject of $E$. By Lemma 3.1.9 either $\beta = \mu^D_H(T)$ or $\mathrm{codim}(T) \geq 2$. In the first case, there exists $\varepsilon > 0$ such that $0 \to T \to E$ ois a subobject in $\mathrm{Coh}^{D+\beta' H}_H(X)$ for all $\beta' \in (\beta - \varepsilon, \beta]$ or $\beta' \in [\beta, \beta + \varepsilon)$. Therefore, $\lim_{\beta \to \mu^D_H(T)} \mu^{\mathrm{tilt}}_{\alpha,\beta}(T) = +\infty$, so $\mu^{\mathrm{tilt}+}_{\alpha,\beta}(E)$ is continuous. If $\mathrm{codim}(T) \geq 2$ then $\mu^{\mathrm{tilt}+}_{\alpha,\beta}(E) = +\infty$ for all $(\beta, \alpha) \in \mathbb{R} \times \mathbb{R}_{>0}$, as needed.

The argument for continuity of $\mu^{\mathrm{tilt}-}_{\alpha,\beta}(E)$ is nearly identical. $\square$

**Definition 3.3.6.** *We define the $(H, D)$-slice to be the image of $\mathbb{R} \times \mathbb{R}_{>0}$ in* $\mathrm{WStab}(X)$ *under the continuous injection above.*

## 3.4 Walls for Tilt Stability

In this subsection we will see that the $(H, D)$-slice has a locally finite collection of real codimension submanifolds (called walls) such that stability only changes when crossing over a wall. Furthermore, the possible walls are either nested semicircles with center along the line $\alpha = 0$ or a unique vertical wall. It is well-known that there is a largest semicircular wall, but we provide an explicit bound on the radius of this wall.

We first describe "numerical" walls. Numerical walls are class of real codimenson 1 submanifolds that include all "actual" walls. Even though most numerical walls are not actual walls, they are relatively easy to describe and restrict the possible actual walls.

**Definition 3.4.1.** *A numerical wall associated $E$ is the set of points $(\beta, \alpha)$ in the $(H, D)$-slice of the form:*

$$W(E, F) = \{(\beta, \alpha) \mid \mu^{\mathrm{tilt}}_{\alpha,\beta}(E) = \mu^{\mathrm{tilt}}_{\alpha,\beta}(F)\} \subseteq \mathbb{R} \times \mathbb{R}_{>0}$$

*for some object $F \in D^b(X)$ (not necessarily in the heart $\mathrm{Coh}^{D+\beta H}_H(X)$).*

**Remark 3.4.2.** *A numerical wall $W(E, F)$ can equivalently be defined by*

$$W' = \{(\beta, \alpha) \mid \Re Z^{\mathrm{tilt}}_{\alpha,\beta}(E) \Im Z^{\mathrm{tilt}}_{\alpha,\beta}(F) = \Re Z^{\mathrm{tilt}}_{\alpha,\beta}(F) \Im Z^{\mathrm{tilt}}_{\alpha,\beta}(E)\}.$$

*To see this, first note that $W(E, F) \subseteq W'$.*

*Now, assume that $(\beta, \alpha) \in W'$. If $\Im Z^{\mathrm{tilt}}_{\alpha,\beta}(F) \neq 0$ and $\Im Z^{\mathrm{tilt}}_{\alpha,\beta}(E) \neq 0$ then it is clear that $(\beta, \alpha) \in W(E, F)$. Thus, assume that $\Im Z^{\mathrm{tilt}}_{\alpha,\beta}(F) = 0$ or $\Im Z^{\mathrm{tilt}}_{\alpha,\beta}(E) = 0$. Without loss of generality, we may assume that $\Im Z^{\mathrm{tilt}}_{\alpha,\beta}(F) = 0$. It follows that $0 = \Re Z^{\mathrm{tilt}}_{\alpha,\beta}(F) \Im Z^{\mathrm{tilt}}_{\alpha,\beta}(E)$ so $\Re Z^{\mathrm{tilt}}_{\alpha,\beta}(F) = 0$ or $\Im Z^{\mathrm{tilt}}_{\alpha,\beta}(E) = 0$. We claim that $\Re Z^{\mathrm{tilt}}_{\alpha,\beta}(F) \neq 0$.*

*Since* $\Im Z^{\text{tilt}}_{\alpha,\beta}(F) = 0$, *$F$ is* $\sigma^{\text{tilt}}_{\alpha,\beta}$*-semistable so, by Lemma 3.3.4,* $\overline{\Delta}^D_H(F) \geq 0$. *We also find that* $\beta = \mu^D_H(F)$. *If* $\Re Z^{\text{tilt}}_{\alpha,\beta}(F) = 0$, *we can solve for* $\alpha$ *to find*

$$\alpha = \pm \frac{\sqrt{-\overline{\Delta}^D_H(F)}}{\operatorname{rank}(F)}.$$

*Since* $\overline{\Delta}^D_H(F) \geq \alpha \notin \mathbb{R}_{>0}$, *so* $\Re Z^{\text{tilt}}_{\alpha,\beta}(F) \neq 0$. *Therefore,* $\Im Z^{\text{tilt}}_{\alpha,\beta}(E) = 0$ *as needed.*

**Definition 3.4.3.** *We say a numerical wall $W$ is an actual wall associated to $E$ if for some* $(\beta, \alpha) \in W$ *there exists a short exact sequence*

$$0 \to F \to E \to G \to 0$$

*in* $\operatorname{Coh}^{D+\beta H}_H(X)$ *such that*

- $W = W(E, G)$ *or* $W = W(E, F)$,
- *$F$ and $G$ are* $\sigma^{\text{tilt}}_{\alpha,\beta}$*-semistable (in particular,* $F, G \in \operatorname{Coh}^{D+\beta H}_H(X)$*),*
- $\mu^{\text{tilt}}_{\alpha,\beta}(F) = \mu_{\alpha,\beta}(F)$,
- $W \neq \mathbb{R} \times \mathbb{R}_{>0}$ *(i.e. $W$ is* 1*-dimensional).*

*We call* $0 \to F \to E \to G \to 0$ *a destabilizing sequence associated to $W$ at* $(\beta, \alpha)$.

By definition and Lemma 3.3.5, actual walls control $\sigma^{\text{tilt}}_{\alpha,\beta}$-(semi)stability. Local finiteness of actual walls follows by [Bri08, Section 9] for surfaces and [BMS16, Proposition B.5] in general.

**Lemma 3.4.4.** *Assume* $E \in D^b(X)$ *is nonzero.*

1. *Let $\mathcal{W}$ be the collection of all actual walls associated to $E$ in the $(H, D)$-slice. If $E$ is weakly* $\sigma^{\text{tilt}}_{\alpha_0,\beta_0}$*-stable for some* $(\beta_0, \alpha_0)$ *in the complement of $\mathcal{W}$ then $E$ is weakly* $\sigma^{\text{tilt}}_{\alpha,\beta}$*-stable for all* $(\beta, \alpha)$ *in the connected component in the complement of $\mathcal{W}$ containing* $(\beta_0, \alpha_0)$.

   *In other words, weak $\sigma$-stability is constant in chambers.*

*2. There exists a locally finite collection of walls, $\mathcal{W}$, in the $(H, D)$-slice such that if*

$$0 \to E_1 \to E_2 \to \cdots \to E_{m-1} \to E$$

*is a Harder-Narasimhan filtration of $E$ with respect to $\sigma^{\mathrm{tilt}}_{\alpha_0,\beta_0}$ for some $(\beta_0, \alpha_0)$ then*

$$0 \to E_1 \to E_2 \to \cdots \to E_{m-1} \to E$$

*is a Harder-Narasimhan filtration of $E$ with respect to $\sigma_{\alpha,\beta}$ for all $(\beta, \alpha)$ in the connected component in the complement of $\mathcal{W}$ containing $(\beta_0, \alpha_0)$. In other words, Harder-Narasimhan filtrations are constant in chambers.*

*3. There exists a locally finite collection of walls, $\mathcal{W}$, in the $(H, D)$-slice such that Jordan-Hölder filtrations of each semistable factor of $E$ are constant.*

We will first introduce some constraints on numerical walls (and thus on actual walls). The following standard result, first appearing in [Mac14] for surfaces, gives initial restrictions on the class of numerical walls.

**Lemma 3.4.5.** *Fix objects $E, F \in D^b(X)$.*

*1. $W(E, F)$ is given by the equation:*

$$x\alpha^2 + x\beta^2 - 2y\beta + 2z = 0$$

*where*

$$\begin{aligned}
x &= \deg_H^D(E)\,\mathrm{rank}(F) - \deg_H^D(F)\,\mathrm{rank}(E) \\
y &= (H^{n-2} \cdot \mathrm{ch}_2^D(E))\,\mathrm{rank}(F) - (H^{n-2} \cdot \mathrm{ch}_2^D(F))\,\mathrm{rank}(E) \\
z &= (H^{n-2} \cdot \mathrm{ch}_2^D(E))\deg_H^D(F) - (H^{n-2} \cdot \mathrm{ch}_2^D(F))\deg_H^D(E)
\end{aligned}$$

*In particular,*

- *If $x \neq 0$ then $W(E, F)$ is a semicircle centered at $(y/x, 0)$ with radius squared $y^2/x^2 - 2z/x$, so $W(E, F) = \varnothing$ if $y^2/x^2 - 2z/x \leq 0$.*

- *If* $x = 0$ *and* $y \neq 0$ *then* $W(E,F)$ *is a vertical line given by the equation* $\beta = -\frac{z}{y}$.
- *If* $x, y = 0$, *and* $z \neq 0$ *then* $W(E,F) = 0$.
- *If* $x, y, z = 0$ *then* $W(E,F) = \mathbb{R} \times \mathbb{R}_{>0}$.

2. *If* $\operatorname{rank}(E) \neq 0$ *and* $x \neq 0$ *then* $W(E,F)$ *is a semicircle with radius squared*

$$\rho^2 = (\mu_H^D(E) - c)^2 - \frac{\overline{\Delta}_H^D(E)}{\operatorname{rank}(E)^2}$$

*where* $(c, 0)$ *is the center.*

*If, in addition,* $\operatorname{rank}(F) \neq 0$ *then*

$$c = \frac{\nu_H^D(E) - \nu_H^D(F)}{\mu_H^D(E) - \mu_H^D(F)}.$$

3. *If* $\operatorname{rank}(E) \neq 0$ *then there is a unique vertical numerical wall given by the equation* $\beta = \mu_H^D(E)$.

*The shapes of the numerical walls are shown in Figure 3.4.*

*Proof.* 1. Consider the wall $W(E,F) = \{(\beta, \alpha) \in \mathbb{R} \times \mathbb{R}_{>0} \mid \mu_{\alpha,\beta}^{\text{tilt}}(E) = \mu_{\alpha,\beta}^{\text{tilt}}(F)\}$.

Equivalently, by Remark 3.4.2, $W(E,F)$ is all points $(\beta, \alpha)$ such that

$$\mathfrak{R} Z_{\alpha,\beta}^{\text{tilt}}(E) \mathfrak{I} Z_{\alpha,\beta}^{\text{tilt}}(F) = \mathfrak{R} Z_{\alpha,\beta}^{\text{tilt}}(F) \mathfrak{I} Z_{\alpha,\beta}^{\text{tilt}}(E)$$

which simplifies to

$$\begin{aligned}
0 = &\frac{\alpha^2}{2}\Big(\deg_H^D(E)\operatorname{rank}(F) - \deg_H^D(F)\operatorname{rank}(E)\Big) \\
&+ \frac{\beta^2}{2}\Big(\deg_H^D(E)\operatorname{rank}(F) - \deg_H^D(F)\operatorname{rank}(E)\Big) \\
&- \beta\Big((H^{n-2} \cdot \operatorname{ch}_2^D(E))\operatorname{rank}(F) - (H^{n-2} \cdot \operatorname{ch}_2^D(F))\operatorname{rank}(E)\Big) \\
&+ (H^{n-2} \cdot \operatorname{ch}_2^D(E))\deg_H^D(F) - (H^{n-2} \cdot \operatorname{ch}_2^D(F))\deg_H^D(E).
\end{aligned}$$

If we multiply each side of this equation by 2 and set

$$x = \deg_H^D(E)\operatorname{rank}(F) - \deg_H^D(F)\operatorname{rank}(E)$$

$$y = (H^{n-2}\cdot \mathrm{ch}_2^D(E))\operatorname{rank}(F) - (H^{n-2}\cdot \mathrm{ch}_2^D(F))\operatorname{rank}(E)$$

$$z = (H^{n-2}\cdot \mathrm{ch}_2^D(E))\deg_H^D(F) - (H^{n-2}\cdot \mathrm{ch}_2^D(F))\deg_H^D(E),$$

we obtain the desired result.

Assume that $x \neq 0$, so we can divide each side of the above equation by $x$ then complete the square to find that

$$\alpha^2 + (\beta - \frac{y}{x})^2 = \frac{y^2}{x^2} - 2\frac{z}{x}.$$

In other words, $W(E,F)$ is a semicircle centered at $\left(\frac{y}{x}, 0\right)$ with radius squared $\frac{y^2}{x^2} - 2\frac{z}{x}$, as needed.

Assume that $x = 0$ and $y \neq 0$, so the above equation simplifies to $\beta = -\frac{z}{y}$.

Assume that $x, y = 0$ then $W(E,F)$ is given by the equation $2z = 0$. Thus, if $z = 0$ then $W(E,F) = \mathbb{R}\times\mathbb{R}_{>0}$, and if $z \neq 0$ then $W(E,F) = \varnothing$.

2. Assume $\operatorname{rank}(E) \neq 0$ and $x \neq 0$ We will consider two cases. Either $\operatorname{rank}(F) = 0$ or $\operatorname{rank}(F) \neq 0$.

   If $\operatorname{rank}(F) = 0$ then it we find that $x = -\deg_H^D(F)\operatorname{rank}(E)$ and $y = -(H^{n-2}\cdot \mathrm{ch}_2^D(F))\operatorname{rank}(E)$. It follows that $y/x = (H^{n-2}\cdot \mathrm{ch}_2^D(F))/\deg_H^D(F)$. Similarly, we find that

$$\begin{aligned}\frac{z}{x} &= \frac{(H^{n-2}\cdot \mathrm{ch}_2^D(E))\deg_H^D(F) - (H^{n-2}\cdot \mathrm{ch}_2^D(F))\deg_H^D(E)}{-\deg_H^D(F)\operatorname{rank}(E)} \\ &= -\nu_H^D(E) + \mu_H^D(E)\frac{(H^{n-2}\cdot \mathrm{ch}_2^D(F))}{\deg_H^D(F)}.\end{aligned}$$

In all, we find that

$$\begin{aligned}(\mu_H^D(E) - \frac{y}{x})^2 - \frac{\overline{\Delta}_H^D(E)}{\operatorname{rank}(E)^2} =& \mu_H^D(E)^2 - 2\mu_H^D(E)\frac{y}{x} + \frac{y^2}{x^2} \\ &- \frac{\deg_H^D(E)^2 - 2\operatorname{rank}(E)(H^{n-2}\cdot \operatorname{ch}_2^D(E))}{\operatorname{rank}(E)^2} \\ =& \frac{y^2}{x^2} - 2\mu_H^D(E)\frac{H^{n-2}\cdot \operatorname{ch}_2^D(F)}{\deg_H^D(F)} + 2\nu_H^D(E) \\ =& \frac{y^2}{x^2} - 2\frac{z}{x} \\ =& \rho^2,\end{aligned}$$

as needed.

If $\operatorname{rank}(F) \neq 0$ then, by Remark 3.1.8, $W(E,F)$ is given by the solutions $(\beta,\alpha)$ of

$$\begin{aligned}&\left(\nu_H^D(E) - \beta\mu_H^D(E) + \left(\frac{\beta^2}{2} - \frac{\alpha^2}{2}\right)\right)(\mu_H^D(F) - \beta) \\ &\qquad = \left(\nu_H^D(F) - \beta\mu_H^D(F) + \left(\frac{\beta^2}{2} - \frac{\alpha^2}{2}\right)\right)(\mu_H^D(E) - \beta).\end{aligned}$$

Which simplifies to

$$\begin{aligned}0 =& \frac{\alpha^2}{2}(\mu_H^D(E) - \mu_H^D(F)) + \frac{\beta^2}{2}(\mu_H^D(E) - \mu_H^D(F)) \\ &+ \beta(\nu_H^D(F) - \nu_H^D(E)) + \nu_H^D(E)\mu_H^D(F) - \nu_H^D(F)\mu_H^D(E)\end{aligned}$$

Since $\operatorname{rank}(E), \operatorname{rank}(F) \neq 0$ and $x \neq 0$, we find that $\mu_H^D(E) \neq \mu_H^D(F)$. Therefore, we can multiply the above equation by 2, divide each side by $\mu_H^D(E) - \mu_H^D(F)$, and complete the square to obtain

$$\begin{aligned}\alpha^2 + \left(\beta - \frac{\nu_H^D(E) - \nu_H^D(F)}{\mu_H^D(E) - \mu_H^D(F)}\right)^2 =& \left(\frac{\nu_H^D(E) - \nu_H^D(F)}{\mu_H^D(E) - \mu_H^D(F)}\right)^2 \\ &- 2\frac{\nu_H^D(E)\mu_H^D(F) - \nu_H^D(F)\mu_H^D(E)}{\mu_H^D(E) - \mu_H^D(F)}.\end{aligned}$$

In other words, $W(E,F)$ is a semicircle centered at $(c,0)$ where

$$c = \frac{\nu_H^D(F) - \nu_H^D(E)}{\mu_H^D(E) - \mu_H^D(F)}.$$

It remains to show that

$$\rho^2 = (\mu_H^D(E) - c)^2 - \frac{\overline{\Delta}_H^D(E)}{\operatorname{rank}(E)^2}.$$

However,

$$\begin{aligned}
(\mu_H^D(E) - c)^2 - \frac{\overline{\Delta}_H^D(E)}{\operatorname{rank}(E)^2} &= c^2 - 2\mu_H^D(E)c + \mu_H^D(E)^2 - \mu_H^D(E)^2 + 2\nu_H^D(E) \\
&= c^2 - 2\mu_H^D(E)c + 2\nu_H^D(E).
\end{aligned}$$

and

$$\begin{aligned}
-2\mu_H^D(E)c + 2\nu_H^D(E) &= -2\mu_H^D(E)\frac{\mu_H^D(E)\nu_H^D(E) - \mu_H^D(E)\nu_H^D(F)}{\mu_H^D(E) - \mu_H^D(F)} \\
&\quad + 2\nu_H^D(E)\frac{\nu_H^D(E)\mu_H^D(E) - \nu_H^D(E)\mu_H^D(F)}{\mu_H^D(E) - \mu_H^D(F)} \\
&= -2\frac{\nu_H^D(E)\mu_H^D(F) - \nu_H^D(F)\mu_H^D(E)}{\mu_H^D(E) - \mu_H^D(F)}.
\end{aligned}$$

In all, we have shown that

$$(c - \mu_H^D(E))^2 - \frac{\overline{\Delta}_H^D(E)}{\operatorname{rank}(E)^2} = \left(\frac{\nu_H^D(E) - \nu_H^D(F)}{\mu_H^D(E) - \mu_H^D(F)}\right)^2 - 2\frac{\nu_H^D(E)\mu_H^D(F) - \nu_H^D(F)\mu_H^D(E)}{\mu_H^D(E) - \mu_H^D(F)} = \rho^2,$$

as desired.

3. Assume $\operatorname{rank}(E) \neq 0$. By part 1, it suffices to show $-z/y = \mu_H^D(E)$ for all $F$ with $x = 0$ and $y \neq 0$.

   Since $x = 0$, we know $\deg_H^D(F)\operatorname{rank}(E) = \deg_H^D(E)\operatorname{rank}(F)$. It follows that

$$\begin{aligned}
-\frac{z}{y} &= -\frac{(H^{n-2}\cdot \operatorname{ch}_2^D(E))\deg_H^D(F) - (H^{n-2}\cdot \operatorname{ch}_2^D(F))\deg_H^D(E)}{(H^{n-2}\cdot \operatorname{ch}_2^D(F))\operatorname{rank}(E) - (H^{n-2}\cdot \operatorname{ch}_2^D(E))\operatorname{rank}(F)} \\
&= \frac{-1}{\operatorname{rank}(E)}\frac{(H^{n-2}\cdot \operatorname{ch}_2^D(E))\deg_H^D(F)\operatorname{rank}(E) - (H^{n-2}\cdot \operatorname{ch}_2^D(F))\deg_H^D(E)\operatorname{rank}(E)}{(H^{n-2}\cdot \operatorname{ch}_2^D(F))\operatorname{rank}(E) - (H^{n-2}\cdot \operatorname{ch}_2^D(E))\operatorname{rank}(F)} \\
&= \frac{-1}{\operatorname{rank}(E)}\frac{(H^{n-2}\cdot \operatorname{ch}_2^D(E))\deg_H^D(E)\operatorname{rank}(F) - (H^{n-2}\cdot \operatorname{ch}_2^D(F))\deg_H^D(E)\operatorname{rank}(E)}{(H^{n-2}\cdot \operatorname{ch}_2^D(F))\operatorname{rank}(E) - (H^{n-2}\cdot \operatorname{ch}_2^D(E))\operatorname{rank}(F)} \\
&= -\mu_H^D(E)\frac{(H^{n-2}\cdot \operatorname{ch}_2^D(E))\operatorname{rank}(F) - (H^{n-2}\cdot \operatorname{ch}_2^D(F))\operatorname{rank}(E)}{(H^{n-2}\cdot \operatorname{ch}_2^D(F))\operatorname{rank}(E) - (H^{n-2}\cdot \operatorname{ch}_2^D(E))\operatorname{rank}(F)} \\
&= \mu_H^D(E),
\end{aligned}$$

as needed.

□

We refine Lemma 3.4.5.4 to show that if $\deg_H^D(E) \geq 0$ and $\operatorname{rank}(E) \neq 0$ then semicircular walls are nested. Note that these assumptions are not at all restrictive. Specifically, by the support property, if $\overline{\Delta}_H^D(E) < 0$ then for all $(\beta, \alpha) \in \mathbb{R} \times \mathbb{R}_{>0}$, $E$ is not $\sigma_{\alpha,\beta}^{\text{tilt}}$-semistable. In particular, there are no actual walls associated to $E$. Also, if $\operatorname{rank}(E) = 0$ then by Lemma 3.4.5, all semicircular numerical walls associated to $E$ have center $(-(H^{n-2} \cdot \operatorname{ch}_2^D(E))/\deg_H^D(E), 0)$ and so must be nested.

**Lemma 3.4.6.** *Assume* $\operatorname{rank}(E) \neq 0$ *and* $\overline{\Delta}_H^D(E) \geq 0$. *For all* $c \in \mathbb{R}$ *satisfying* $|c - \mu_H^D(E)| > \sqrt{\overline{\Delta}_H^D(E)/\operatorname{rank}(E)^2}$ *define* $B_c$ *to be the semicircle in* $\mathbb{R} \times \mathbb{R}_{>0}$ *with center* $c$ *and and radius squared*

$$\rho_c^2 = (\mu_H^D(E) - c)^2 - \frac{\overline{\Delta}_H^D(E)}{\operatorname{rank}(E)^2}.$$

1. *$B_c$ does not intersect the vertical line $\beta = \mu_H^D(E)$.*

2. *If $B_{c_1}$ and $B_{c_2}$ are on the same side of the vertical line $\beta = \mu_H^D(E)$ (i.e. both $c_1, c_2 < \mu_H^D(E)$ or $c_1, c_2 > \mu_H^D(E)$) then one of the following must hold:*

    - *$B_{c_1} = B_{c_2}$,*

    - *$B_{c_1}$ is nested within $B_{c_2}$ (i.e. $B_{c_1}$ is a subset of the region strictly between $B_{c_2}$ and the real axis), or*

    - *$B_{c_2}$ is nested within $B_{c_1}$.*

3. *Numerical walls associated to $E$ in the $(H, D)$-slice are all disjoint. On each side of the unique vertical wall, numerical walls are nested semicircles.*

*The relation between numerical walls is represented in Figure 3.4.*

*Proof.* Assume $\operatorname{rank}(E) \neq 0$ and $\overline{\Delta}_H^D(E) \geq 0$. Define $B_c$ as in the lemma statement.

1. Without loss of generality, assume $c < \mu_H^D(E)$. Since $\overline{\Delta}_H^D(E) \geq 0$, by definition of $B_c$,

$$\rho_c^2 = (\mu_H^D(E) - c)^2 - \frac{\overline{\Delta}_H^D(E)}{\operatorname{rank}(E)^2} \leq (\mu_H^D(E) - c)^2.$$

   Since $c < \mu_H^D(E)$ and $\rho_c > 0$, it follows that $\rho_c < \mu_H^D(E) - c$ so $\rho_c + c < \mu_H^D(E)$. In other words, the right endpoint of $B_c$ lies to the left of the vertical line $\beta = \mu_H^D$, as claimed.

2. Without loss of generality, assume that $c_1, c_2 < \mu_H^D(E)$. If $c_1 = c_2$ then $B_{c_1} = B_{c_2}$, so assume, without loss of generality, $c_1 < c_2$. We will show that $c_1 + \rho_{c_1} > c_2 + \rho_{c_2}$ and $c_1 - \rho_{c_1} < c_2 - \rho_{c_2}$.

   Since $c_1 < c_2 < \mu_H^D(E)$,

$$\rho_{c_2}^2 = (\mu_H^D(E) - c_2)^2 - \frac{\overline{\Delta}_H^D(E)}{\operatorname{rank}(E)^2} \leq (\mu_H^D(E) - c_1)^2 - \frac{\overline{\Delta}_H^D(E)}{\operatorname{rank}(E)^2} = \rho_{c_1}^2$$

   and so $\rho_{c_2} < \rho_{c_1}$. Since $c_1 < c_2$, it follows that $c_1 - \rho_{c_1} < c_2 - \rho_{c_1} < c_2 - \rho_{c_2}$, as needed.

   By part 1, neither $B_{c_1}$ nor $B_{c_2}$ intersect the line $\beta = \mu_H^D(E)$, so $\rho_{c_1} \leq \mu_H^D(E) - c_1$ and $\rho_{c_2} \leq \mu_H^D(E) - c_2$. It follows that

$$\rho_{c_1} + \rho_{c_2} \leq (\mu_H^D(E) - c_1) + (\mu_H^D(E) - c_2).$$

   Moreover, since $c_1 < c_2$, we find that

$$\begin{aligned}
(c_2 - c_1)(\rho_{c_1} + \rho_{c_2}) &\leq (c_2 - c_1)((\mu_H^D(E) - c_1) + (\mu_H^D(E) - c_2)) \\
&= (\mu_H^D(E) - c_1)^2 - (\mu_H^D(E) - c_2)^2 \\
&= \rho_{c_1}^2 - \rho_{c_2}^2 \\
&= (\rho_{c_1} - \rho_{c_2})(\rho_{c_1} + \rho_{c_2}).
\end{aligned}$$

   Since $\rho_{c_1} + \rho_{c_2} > 0$, we can divide each side of the above inequality by $\rho_{c_1} + \rho_{c_2}$ to find that $c_2 - c_1 \leq \rho_{c_1} - \rho_{c_2}$. Equivalently, we find that $c_2 + \rho_{c_2} \leq c_1 + \rho_{c_1}$, as needed.

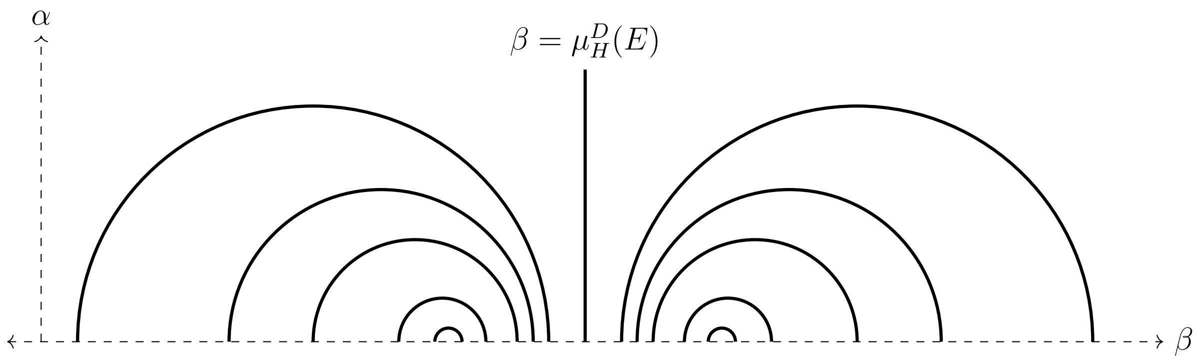


Figure 3 Shapes and Orientation of Walls: The solid lines represent numerical walls for fixed $E \in D^b(X)$ satisfying $\operatorname{rank}(E) \neq 0$ and $\overline{\Delta}_H(E) \geq 0$. The dashed lines represent the $\alpha$ and $\beta$ axes in $\mathbb{R} \times \mathbb{R}_{>0}$.

3. By Lemma 3.4.5, all numerical walls associated to $E$ are either the unique vertical wall or semicircles of the form

$$\rho^2 = (\mu_H^D(E) - c)^2 - \frac{\overline{\Delta}_H^D(E)}{\operatorname{rank}(E)^2}.$$

where $\rho$ is the radius and $c$ is the center. The result the follows by part 1 and part 2.

□

**Remark 3.4.7.** *Assume that $B_c$ is a semicircle appearing in Lemma 3.4.6. If $s$ is an intersection point of $B_c$ with the line $\alpha = 0$ (i.e. $s = c - \rho$ or $s = c + \rho$) then*

$$c = \frac{\mu_H^D(E) + s}{2} - \frac{\overline{\Delta}_H^D(E)}{2\operatorname{rank}(E)^2(\mu_H^D(E) - s)}.$$

*It follows by Lemma 3.4.5.2 that we can write the radius squared in terms of the endpoints of $B_c$ as*

$$\rho^2 = \frac{(\overline{\Delta}_H^D(E) - \operatorname{rank}(E)^2(\mu_H^D(E) - s)^2)^2}{4\operatorname{rank}(E)^2(\mu_H^D(E) - s)^2}$$

The following is a consequence of the fact that numerical walls do not intersect. We follow the same proof as [ABCH13, Lemma 6.3].

**Lemma 3.4.8.** *If $0 \to F \to E \to G \to 0$ is a destabilizing sequence for $W(E,F)$ at $(\beta_0, \alpha_0)$ then $F, E, G \in \operatorname{Coh}_H^{D+\beta H}(X)$ for all $(\beta, \alpha) \in W(E,F)$.*

*Proof.* Assume $0 \to F \to E \to G \to 0$ is a destabilizing sequence in $\mathrm{Coh}_H^{D+\beta_0 H}(X)$. We first claim that $F, E, G \in \mathrm{Coh}_H^{D+\beta H}(X)$ for all $(\beta,\alpha) \in W(E,F)$. We will just show that $E \in \mathrm{Coh}_H^{D+\beta H}(X)$ for all $(\beta,\alpha) \in W(E,F)$. The same argument holds for $F$ and $G$. If $W(E,F)$ is a vertical wall, then the result follows because $\mathrm{Coh}_H^{D+\beta H}(X)$ is independent of $\alpha$. Therefore, assume that $W(E,F)$ is a semicircular wall.

If $\beta > \beta_0$ then $\mathscr{H}^{-1}(E) \in \mathcal{F}_H^{D+\beta H}(X)$, so we just need to show that $\mathscr{H}^0(E) \in \mathcal{T}_H^{D+\beta H}(X)$. Assume for contradiction that $\mathscr{H}^0(E) \notin \mathcal{T}_H^{D+\beta H}(X)$ for some $(\beta,\alpha) \in W(E,F)$. Choose a smallest such $\beta$ which we call $\beta_1$ such that $\mathscr{H}^0(E) \notin \mathcal{T}_H^{D+\beta_1 H}(X)$ and $(\beta_1,\alpha_1) \in W(E,F)$. Thus, we have a quotient $\mathscr{H}^0(E) \to \mathscr{G} \to 0$ in $\mathrm{Coh}(X)$ such that $\mu_H^D(\mathscr{G}) = \beta_1$. By minimality of $\beta_1$, $\mathscr{G}$ is $\mu_H$-semistable and for all $\beta \in [\beta_0, \beta_1)$ we have a short exact sequence $0 \to K \to E \to \mathscr{G} \to 0$ in $\mathrm{Coh}_H^{D+\beta H}(X)$. Since $\mathscr{G}$ is $\mu_H$-semistable, by Bogomolov's inequality, we know that $\nu_H^D - \mu_H^D(\mathscr{G})^2/2 = -\overline{\Delta}_H^D(\mathscr{G}) \le 0$, so

$$
\begin{aligned}
\lim_{\beta \to \beta_1^-} \mu^{\mathrm{tilt}}_{\alpha_1,\beta}(\mathscr{G}) &= \lim_{\beta \to \beta_1^-} \frac{\nu_H^D(\mathscr{G}) - \beta\mu_H^D(\mathscr{G}) + \left(\frac{\beta^2}{2} - \frac{\alpha_1^2}{2}\right)}{\mu_H^D(\mathscr{G}) - \beta} \\
&= \lim_{\beta \to \beta_1^-} \frac{\nu_H^D(\mathscr{G}) - \mu_H^D(\mathscr{G})^2 + \left(\frac{\mu_H^D(\mathscr{G})^2}{2} - \frac{\alpha_1^2}{2}\right)}{\mu_H^D(\mathscr{G}) - \beta} \\
&= \lim_{\beta \to \beta_1^-} \frac{-\overline{\Delta}_H^D(\mathscr{G}) - \frac{\alpha_1^2}{2}}{\mu_H^D(\mathscr{G}) - \beta} \\
&= -\infty
\end{aligned}
$$

because $\alpha_1 > 0$.

By Lemma 3.4.6, no two numerical walls can interesect, so we know that $\beta = \mu_H^D(E)$ does not intersect $W(E,F)$. Therefore, we know that $\mu^{\mathrm{tilt}}_{\alpha,\beta} > -\infty$ for all $(\beta,\alpha) \in W(F,G)$ and since $\lim_{\beta \to \beta_1^-} \mu^{\mathrm{tilt}}_{\alpha,\beta}(\mathscr{G}) = -\infty$, we know that there is a region on $W(F,G)$ such that $\mu^{\mathrm{tilt}}_{\alpha,\beta}(E) > \mu^{\mathrm{tilt}}_{\alpha,\beta}(\mathscr{G})$. However, by Lemma 3.4.6 no two numerical walls can intersect, so $\mu^{\mathrm{tilt}}_{\alpha,\beta}(E) > \mu^{\mathrm{tilt}}_{\alpha,\beta}(\mathscr{G}))$ for all $(\beta,\alpha) \in W(E,F)$. In particular, $\mu^{\mathrm{tilt}}_{\alpha_0,\beta_0}(E) > \mu^{\mathrm{tilt}}_{\alpha_0,\beta_0}(\mathscr{G})$. By the seesaw inequality, we find that $\mu^{\mathrm{tilt}}_{\alpha_0,\beta_0}(K) > \mu^{\mathrm{tilt}}_{\alpha_0,\beta_0}(E)$ which tells us that $K = 0$ or $E$ is not $\sigma$-semistable. If $E$ is not $\sigma^{\mathrm{tilt}}_{\alpha_0,\beta_0}$-semistable, then $0 \to F \to E \to G \to 0$ is not a destabilizing sequence at $(\beta_0, \alpha_0)$, a contradiction. If $K = 0$ then $E = \mathscr{G}$ which contradicts that $\mu^{\text{tilt}}_{\alpha_0,\beta_0}(E) > \mu^{\text{tilt}}_{\alpha_0,\beta_0}(\mathscr{G})$. Hence, we find that $\mathscr{H}^0(E)$ for all $(\beta, \alpha) \in W(E, F)$.

Now, assume that $\beta < \beta_0$ then $\mathscr{H}^{-1}(E) \in \mathcal{T}_H^{D+\beta H}(X)$, so we just need to show that $\mathscr{H}^{-1}(E) \in \mathcal{F}_H^{D+\beta H}(X)$. Assume for contradiction that $\mathscr{H}^0(E) \notin \mathcal{T}_H^{D+\beta H}(X)$ for some $(\beta, \alpha) \in W(E, F)$. It is clear that if $\mathscr{H}^{-1}(E) \notin \mathcal{T}_H^{D+\beta H}(X)$ for some $(\beta_1, \alpha_1) \in W(E, F)$ then $\mathscr{H}^{-1}(E) \notin \mathcal{T}_H^{D+\beta H}(X)$ for all $\beta \leq \beta_1$. Therefore, we can choose the largest $\beta$, which we call $\beta_1$, such that $\mathscr{H}^{-1}(E) \in \mathcal{T}_H^{D+\beta H}(X)$ and $(\beta_1, \alpha_1) \in W(E, F)$. Therefore, we have a nonzero subobject $0 \to \mathscr{F} \to \mathscr{H}^{-1}(E)$ in $\mathrm{Coh}(X)$ such that $\mu_H^D(\mathscr{F}) = \beta_1$. By maximality of $\beta_1$, we know that $\mathscr{H}^0(E)$ is $\mu_H$-semistable and we have a short exact sequence $\mathscr{F} \to E \to C \to 0$ in $\mathrm{Coh}_H^{D+\beta H}(X)$ for all $\beta \in [\beta_1, \beta_0]$. Since $\mathscr{F}$ is $\mu_H$-semistable, we know that

$$\nu_H^D(\mathscr{F}) - \mu_H^D(\mathscr{F})^2/2 = -\overline{\Delta}_H^D(\mathscr{F}) \leq 0,$$

so

$$\begin{aligned}
\lim_{\beta \to \beta_1^+} \mu^{\text{tilt}}_{\alpha_1,\beta}(\mathscr{F}) &= \lim_{\beta \to \beta_1^+} \frac{\nu_H^D(\mathscr{F}) - \beta\mu_H^D(\mathscr{F}) + \left(\frac{\beta^2}{2} - \frac{\alpha_1^2}{2}\right)}{\mu_H^D(\mathscr{F}) - \beta} \\
&= \lim_{\beta \to \beta_1^+} \frac{\nu_H^D(\mathscr{F}) - \mu_H^D(\mathscr{F})^2 + \left(\frac{\mu_H^D(\mathscr{F})^2}{2} - \frac{\alpha_1^2}{2}\right)}{\mu_H^D(\mathscr{F}) - \beta} \\
&= \lim_{\beta \to \beta_1^+} \frac{-\overline{\Delta}_H^D(\mathscr{F}) - \frac{\alpha_1^2}{2}}{\mu_H^D(\mathscr{F}) - \beta} \\
&= +\infty
\end{aligned}$$

because $\alpha_1 > 0$.

By Lemma 3.4.6, no two numerical walls can interesect, so we know that $\beta = \mu_H^D(E)$ does not intersect $W(E, F)$. Therefore, we know that $\mu^{\text{tilt}}_{\alpha,\beta}(E) < +\infty$ for all $(\beta, \alpha) \in W(F, G)$ and since $\lim_{\beta \to \beta_1^+} \mu^{\text{tilt}}_{\alpha,\beta}(\mathscr{F}) = +\infty$, we know that there is a region on $W(F, E)$ such that $\mu^{\text{tilt}}_{\alpha,\beta}(\mathscr{F}) > \mu^{\text{tilt}}_{\alpha,\beta}(E)$. However, by Lemma 3.4.6 no two numerical walls can intersect, so $\mu^{\text{tilt}}_{\alpha,\beta}(\mathscr{F}) > \mu^{\text{tilt}}_{\alpha,\beta}(E))$ for all $(\beta, \alpha) \in W(E, F)$. In particular, $\mu^{\text{tilt}}_{\alpha_0,\beta_0}(\mathscr{F}) > \mu^{\text{tilt}}_{\alpha_0,\beta_0}(E)$. In other words, either $\mathscr{F} = 0$ or $E$ is not $\sigma^{\text{tilt}}_{\alpha_0,\beta_0}$-semistable. If $E$ is not $\sigma^{\text{tilt}}_{\alpha_0,\beta_0}$-semistable, then $0 \to F \to E \to G \to 0$ is not a destabilizing sequence at $(\beta_0, \alpha_0)$, a contradiction. Also, $\mathscr{F} \neq 0$

by assumption, a contradiction. Hence $\mathscr{H}^{-1}(E) \in \mathcal{F}_H^{D+\beta H}(X)$ for all $(\beta, \alpha) \in W(E, F)$, as needed.

All cases considered, we find that $E \in \mathrm{Coh}_H^{D+\beta H}(X)$ for all $(\beta, \alpha) \in W(E, F)$, as desired. The same argument shows that $F, G \in \mathrm{Coh}_H^{D+\beta H}(X)$ for all $(\beta, \alpha) \in W(E, F)$ as well.

□

By a direct calculation, it immediately follows that destabilizing sequences do not depend on the given point:

**Lemma 3.4.9.** *Assume $\mathscr{E}$ is a torsion-free sheaf.*

1. *If $0 \to F \to \mathscr{E} \to G \to 0$ is a destabilizing sequence for $W(E, F)$ at $(\beta_0, \alpha_0)$ then $0 \to F \to \mathscr{E} \to G \to 0$ is a destabilizing sequence at all $(\beta, \alpha) \in W(E, F)$.*

2. *If $0 \to F \to \mathscr{E}[1] \to G \to 0$ is a destabilizing sequence for $W(E, F)$ at $(\beta_0, \alpha_0)$ then $0 \to F \to \mathscr{E}[1] \to G \to 0$ is a destabilizing sequence at all $(\beta, \alpha) \in W(E, F)$.*

## 3.5 Bounding Actual Walls

In this subsection, we will give some explicit bounds on the radius of actual walls associated to torsion-free sheaves in the $(H, D)$-slice. These result cans be thought of as effective versions of the large volume limit.

The following lemma first appeared in [ABCH13, Lemma 6.4] for surfaces.

**Lemma 3.5.1.** *Fix $E \in D^b(X)$ and assume $W$ is an actual semicircular wall associated to $E$ with radius $\rho$ and center $(c, 0)$. If $0 \to F \to E \to G \to 0$ is a destabilizing sequence associated to $W$ then*

- *$\mu_H^D(\mathscr{H}^{-1}(F)), \mu_H^D(\mathscr{H}^{-1}(E)), \mu_H^D(\mathscr{H}^{-1}(G)) \leq c - \rho$ and*

- *$\mu_H^D(\mathscr{H}^{0}(F)), \mu_H^D(\mathscr{H}^{0}(E)), \mu_H^D(\mathscr{H}^{0}(G)) \geq c + \rho$.*

*Proof.* Since $W$ is an actual wall, we know that $0 \to F \to E \to G \to 0$ is a stabilizing sequence at $(\beta_0, \alpha_0)$. By Lemma 3.4.9; $F, E, G \in \mathrm{Coh}_H^{D+\beta H}(X)$ for $c - \rho < \beta < c + \rho$. Therefore, if we let $\beta$ approach $c - \rho$, we obtain the first result. If we let $\beta$ approach $c + \rho$ we obtain the second result. $\square$

If $\mathscr{E} \in \mathrm{Coh}(X)$ then Lemma 3.5.1 shows that any actual wall of $\mathscr{E}$ must lie entirely to the left of the unique vertical wall $\beta = \mu_H^D(\mathscr{E})$. Similarly, any actual wall of $\mathscr{E}[1]$ must lie entirely to the right of the unique vertical wall $\beta = \mu_H^D(\mathscr{E})$.

This is provided by the following lemma. Part 1 uses essentially the same argument as [CH16, Proposition 8.3]. Part 2 uses a similar argument to [Kop20, Theorem 3.3].

**Lemma 3.5.2.** *Let $\mathscr{E}$ be a torsion-free sheaf satisfying $\overline{\Delta}_H^D(\mathscr{E}) \geq 0$. Assume $W$ is an actual semicircular wall associated to $\mathscr{E}$ (resp. $\mathscr{E}[1]$) with associated destabilizing sequence $0 \to F \to \mathscr{E} \to G \to 0$ (resp. $0 \to F \to \mathscr{E}[1] \to G \to 0$). Let $\rho$ be the radius of $W$.*

1. *If $\mathscr{H}^{-1}(G) \neq 0$ (resp. $\mathrm{rank}(\mathscr{H}^0(F)) \neq 0$) then*
$$\rho \leq \sqrt{\frac{\overline{\Delta}_H^D(\mathscr{E})}{4(\mathrm{rank}(\mathscr{E}) + 1)}}.$$
*In particular, $W$ is empty if $\overline{\Delta}_H^D(\mathscr{E}) = 0$.*

2. *If $\mathscr{H}^{-1}(G) = 0$ (resp. $\mathrm{rank}(\mathscr{H}^0(F)) = 0$) then*
$$\rho \leq \frac{1}{2}\left(\overline{\Delta}_H^D(\mathscr{E}) - \frac{1}{\mathrm{rank}(\mathscr{E})^2}\right).$$
*In particular, in this case, $W$ is empty if either $\overline{\Delta}_H^D(\mathscr{E}) = 0$ or $\overline{\Delta}_H^D(\mathscr{E}) = 1$ with* $\mathrm{rank}(\mathscr{E}) = 1$.

*Proof.* We will only consider the case of $\mathscr{E}$. The arguments for $\mathscr{E}[1]$ are essentially the same. Furthermore, if $\overline{\Delta}_H(\mathscr{E}) = 0$ then $W$ is empty by [BMS16, Proposition A.8]. Thus, we may assume $\overline{\Delta}_H^D(\mathscr{E}) > 0$. Taking cohomology of the destabilizing sequence gives us the following exact sequence of coherent sheaves:

$$0 \to \mathscr{H}^{-1}(G) \to \mathscr{H}^0(F) \to \mathscr{E} \to \mathscr{H}^0(G) \to 0.$$

1. Assume $\mathscr{H}^{-1}(G) \neq 0$, so $\mathscr{H}^0(F)$ is torsion-free; in particular, $\mathscr{H}^0(F)$ has non-zero rank. By Lemma 3.5.1, $c + \rho \leq \mu_H^D(\mathscr{H}^0(F))$. Since $\mathscr{H}^0(F)$ has nonzero rank, by additivity,

$$\begin{aligned}(c+\rho)\operatorname{rank}(\mathscr{H}^0(F)) &\leq \deg_H^D(\mathscr{H}^0(F)) \\ &= \deg_H^D(\mathscr{H}^{-1}(G)) + \deg_H^D(\mathscr{E}) - \deg_H^D(\mathscr{H}^0(G)) \\ &= \mu_H^D(\mathscr{H}^{-1}(G))\operatorname{rank}(\mathscr{H}^{-1}(G)) + \deg_H^D(\mathscr{E}) - \deg_H^D(\mathscr{H}^0(G))\end{aligned}$$

Using the bounds from Lemma 3.5.1, it follows that

$$\begin{aligned}(c+\rho)\operatorname{rank}(\mathscr{H}^0(F)) &\leq \mu_H^D(\mathscr{H}^{-1}(G))\operatorname{rank}(\mathscr{H}^{-1}(G)) + \deg_H^D(\mathscr{E}) - \deg_H^D(\mathscr{H}^0(G)) \\ &\leq (c-\rho)\operatorname{rank}(\mathscr{H}^{-1}(G)) + \deg_H^D(\mathscr{E}) - \deg_H^D(\mathscr{H}^0(G)).\end{aligned}$$

Either $\mathscr{H}^0(G)$ is torsion or has positive rank. If $\mathscr{H}^0(G)$ is torsion, by Lemma 2.1.3, then $\deg_H^D(\mathscr{H}^0(G)) \geq 0$. Therefore,

$$\rho(\operatorname{rank}(\mathscr{H}^0(F)) + \operatorname{rank}(\mathscr{H}^{-1}(G))) \leq c(\operatorname{rank}(\mathscr{H}^{-1}(G)) - \operatorname{rank}(\mathscr{H}^{-1}(F))) + \deg_H^D(\mathscr{E})$$

and so

$$\begin{aligned}\rho(\operatorname{rank}(\mathscr{H}^0(F)) + \operatorname{rank}(\mathscr{H}^{-1}(G))) &\leq -c\operatorname{rank}(\mathscr{E}) + \deg_H^D(\mathscr{E}) \\ &= (\mu_H^D(\mathscr{E}) - c)\operatorname{rank}(\mathscr{E})\end{aligned}$$

On the other hand, by additivity,

$$\operatorname{rank}(\mathscr{E}) + 2 \leq \operatorname{rank}(\mathscr{E}) + 2\operatorname{rank}(\mathscr{H}^{-1}(G)) = \operatorname{rank}(\mathscr{H}^0(F)) + \operatorname{rank}(\mathscr{H}^{-1}(G)),$$

so

$$\rho(\operatorname{rank}(\mathscr{E}) + 2) \leq (\mu_H^D(\mathscr{E}) - c)\operatorname{rank}(\mathscr{E}).$$

If $\operatorname{rank}(\mathscr{H}^0(G)(\neq 0$, by Lemma 3.5.1, then

$$\begin{aligned}&(c+\rho)(\operatorname{rank}(\mathscr{H}^0(F)) + \operatorname{rank}(\mathscr{H}^0(G))) \\ &\leq (c+\rho)\operatorname{rank}(\mathscr{H}^0(F)) + \mu_H^D(\mathscr{H}^0(G))\operatorname{rank}(\mathscr{H}^0(G)) \\ &\leq (c-\rho)\operatorname{rank}(\mathscr{H}^{-1}(G)) + \deg_H^D(\mathscr{E})\end{aligned}$$

In other words, by additivity,

$$
\begin{aligned}
&\rho(\operatorname{rank}(\mathscr{H}^0(G)) + \operatorname{rank}(\mathscr{H}^0(F)) + \operatorname{rank}(\mathscr{H}^{-1}(G))) \\
&\leq c(\operatorname{rank}(\mathscr{H}^{-1}(G)) - \operatorname{rank}(\mathscr{H}^0(F)) - \operatorname{rank}(\mathscr{H}^0(G))) + \deg_H^D(\mathscr{E}) \\
&= -c\operatorname{rank}(\mathscr{E}) + \mu_H^D(\mathscr{E})\operatorname{rank}(\mathscr{E}) \\
&= (\mu_H^D(\mathscr{E}) - c)\operatorname{rank}(\mathscr{E}).
\end{aligned}
$$

On the other hand, by additivity,

$$
\begin{aligned}
\operatorname{rank}(\mathscr{H}^0(G) + \operatorname{rank}(\mathscr{H}^0(F)) + \operatorname{rank}(\mathscr{H}^{-1}(G)) &= \operatorname{rank}(\mathscr{E}) + 2\operatorname{rank}(\mathscr{H}^0(G) \\
&\geq \operatorname{rank}(\mathscr{E}) + 2.
\end{aligned}
$$

Thus, we have found that

$$\rho(\operatorname{rank}(\mathscr{E}) + 2) \leq (\mu_H^D(\mathscr{E}) - c)\operatorname{rank}(\mathscr{E})$$

in both cases. By Lemma 3.5.2, both sides of the above inequality are positive, so

$$\rho^2(\operatorname{rank}(\mathscr{E}) + 2)^2 \leq (\mu_H^D(\mathscr{E}) - c)^2\operatorname{rank}(\mathscr{E})^2.$$

By Lemma 3.4.5,

$$\rho^2 = (\mu_H^D(\mathscr{E}) - c)^2 - \frac{\overline{\Delta}_H^D(\mathscr{E})}{\operatorname{rank}(\mathscr{E})^2},$$

so we find that

$$\rho^2(\operatorname{rank}(\mathscr{E}) + 2)^2 \leq \left(\rho^2 + \frac{\overline{\Delta}_H^D(\mathscr{E})}{\operatorname{rank}(\mathscr{E})^2}\right)\operatorname{rank}(\mathscr{E})^2 = \rho^2\operatorname{rank}(\mathscr{E})^2 + \overline{\Delta}_H^D(\mathscr{E})$$

Equivalently,

$$\rho^2(4\operatorname{rank}(\mathscr{E}) + 4) \leq \overline{\Delta}_H^D(\mathscr{E})$$

so

$$\rho^2 \leq \frac{\overline{\Delta}_H^D(\mathscr{E})}{4(\operatorname{rank}(\mathscr{E}) + 1)}$$

2. Since $\mathscr{H}^{-1}(G) = 0$, we have the following exact sequence in $\mathrm{Coh}(X)$:

$$0 \to \mathscr{H}^0(F) \to \mathscr{E} \to \mathscr{H}^0(G) \to 0.$$

By weak the seesaw inequality (Lemma 2.2.6), one of the following inequalities must hold:

- $\mu_H^D(\mathscr{H}^0(F)) \leq \mu_H^D(\mathscr{E})$
- $\mu_H^D(\mathscr{H}^0(G))\mu_H^D(\mathscr{E})$.

First suppose $\mu_H^D(\mathscr{H}^0(F)) \leq \mu_H^D(\mathscr{E})$. Since $W$ is a semicircular wall, by Lemma 3.4.5.1, $\mu_H^D(\mathscr{H}^0(F)) \neq \mu_H^D(\mathscr{E})$. In other words, $\mu_H^D(\mathscr{H}^0(F)) < \mu_H^D(\mathscr{E})$ so

$$\mu_H^D(\mathscr{H}^0(F)) + \frac{1}{\mathrm{rank}(\mathscr{E})^2} \leq \mu_H^D(\mathscr{E}).$$

Therefore, by Lemma 3.5.1,

$$c + \rho \leq \mu_H^D(\mathscr{H}^0(F)) \leq \mu_H^D(\mathscr{E}) - \frac{1}{\mathrm{rank}(\mathscr{E})^2}$$

We obtain the same inequality in the case of $\mu_H^D(\mathscr{H}^0(G)) \leq \mu_H^D(\mathscr{E})$.

In other words, $W$ is contained in the semicircle with right endpoint $\mu_H^D(\mathscr{E}) - \frac{1}{\mathrm{rank}(\mathscr{E})^2}$ whose center and radius satisfy the equation of Lemma 3.4.5.2. In other words,

$$\rho \leq \frac{1}{2}\left|\overline{\Delta}_H^D(\mathscr{E}) - \frac{1}{\mathrm{rank}(\mathscr{E})^2}\right|,$$

as claimed.

$\square$

The results of this section and the previous section are illustrated in Figure 3.5.

## 3.6 A Wall-Crossing Theorem

If $W$ is an actual semicircular wall associated to $E$ and $E$ is weakly $\sigma_{\alpha,\beta}^{\mathrm{tilt}}$-stable on one side of the semicircular wall then $E$ is $\sigma_{\alpha,\beta}^{\mathrm{tilt}}$-unstable on the other side of the semicircular wall.

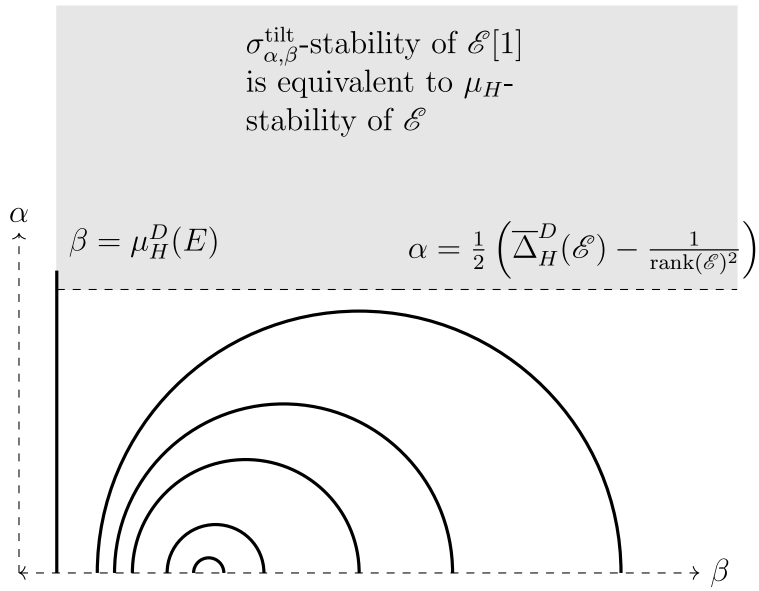


Figure 4 Bound on the Largest Actual Wall: The solid lines represent actual walls of $\mathscr{E}[1]$ when $\mathscr{E}$ is torsion-free. The horizontal dashed line represents our bound on the largest actual wall. The shaded region is the large volume limit. The actual walls of $\mathscr{E}$ is the same picture mirrored over $\beta = \mu_H^D(\mathscr{E})$ expect that weak $\sigma^{\text{tilt}}_{\alpha,\beta}$-stability of $\mathscr{E}$ is equivalent to $(H, D)$-twisted stability of $\mathscr{E}$.

The converse is false, $E$ may be $\sigma^{\text{tilt}}_{\alpha,\beta}$-unstable on both sides of an actual semicircular wall. When $\dim(X) = 2$, [BM11, Lemma 5.9] gives a partial converse.

There have been a few related results to higher dimensions, but they do not fully generalize Bayer and Macrì's result. For example, in the case of $X = \mathbb{P}^3$, [Sch20, Theorem 6.1.4] gives a partial converse. Similarly, in the case of destabilizing sequences $0 \to F \to E \to G \to 0$ satisfying $\mathscr{H}^{-1}(E) = 0$, $\operatorname{codim}(\mathscr{H}^0(E)) = 1$, and $\mathscr{H}^0(G) = 0$; [Fey21, Proposition 4.2 and Corollary 4.3] gives a partial converse. Using the theory developed in the previous sections, we can give a higher dimensional generalization.

**Theorem 3.6.1.** *Assume $F, G \in D^b(X)$, $G$ has good quotients, and there exists $(\beta_0, \alpha_0) \in W(F, G)$ such that $F$ and $G$ are weakly $\sigma^{\text{tilt}}_{\alpha_0,\beta_0}$-stable. Let $0 \to F \to E \to G^{\oplus r} \to 0$ be a short exact sequence in $\operatorname{Coh}_H^{D+\beta_0 H}(X)$ for some positive integer $r$. If the following conditions are satisfied*

- $\operatorname{Hom}_{D^b(X)}(G, E) = 0$,

- *there exists $\varepsilon > 0$ such that $\mu^{\mathrm{tilt}}_{\alpha,\beta_0}(F) < \mu^{\mathrm{tilt}}_{\alpha,\beta}(G^{\oplus r})$ for all $\alpha \in (\alpha_0, \alpha_0 + \varepsilon)$.*

*then there exists $\delta > 0$ such that $E$ is weakly $\sigma^{\mathrm{tilt}}_{\alpha,\beta_0}$-stable for all $\alpha \in (\alpha_0, \alpha_0 + \delta)$.*

Before giving the proof, we remark on the assumptions.

The requirement that $G$ has good quotients allows us to use the full strength of Schur's Lemma (Lemma 2.2.13).

The assumption $\mathrm{Hom}_{D^b(X)}(G, E) = 0$ is essentially avoiding the case $E = F \oplus G$. In fact, if $r = 1$ then it suffices to assume $E \neq F \oplus G$.

Last, since $W$ is an actual wall, there exists $\varepsilon > 0$ such that either $\mu^{\mathrm{tilt}}_{\alpha,\beta_0}(F) < \mu^{\mathrm{tilt}}_{\alpha,\beta_0}(G)$ for all $\alpha \in (\alpha_0, \alpha_0 + \varepsilon)$ or $\alpha \in (\alpha_0, \alpha_0 - \varepsilon)$. The first case is considered in Theorem 3.6.1. The same proof will give a similar result in the second case.

By Lemma 2.3.13 and Lemma 3.1.9, the good quotients assumption is vacuous when $\dim(X) = 2$. Therefore, Theorem 3.6.1 truly generalizes [BM11, Lemma 5.9] in the case of tilt stability.

*Proof.* Choose $\varepsilon > 0$ as in the theorem statement. By Lemma 3.4.4, and decreasing $\varepsilon$ if necessary, we may assume that $F$ and $G$ are weakly $\sigma^{\mathrm{tilt}}_{\alpha,\beta_0}$-stable for all $\alpha \in [\alpha_0, \alpha_0 + \varepsilon)$, Harder-Narasimhan filtrations of $E$ are constant in the region $\{\beta_0\} \times (\alpha_0, \alpha_0 + \varepsilon)$, and Jordan-Hölder filtrations of semistable factors of $E$ are also constant in that region. For some $\alpha \in (\alpha_0, \alpha_0 + \varepsilon)$ consider a Harder-Narasimhan filtration of $E$:

$$0 \to E_1 \to E_2 \to \cdots \to E_{m-1} \to E.$$

with respect to $\sigma^{\mathrm{tilt}}_{\alpha,\beta_0}$. By choice of $\varepsilon$ above, this filtration is a Harder-Narasimhan filtration of $E$ with respect to $\sigma^{\mathrm{tilt}}_{\alpha,\beta_0}$ for all $\alpha \in (\alpha_0, \alpha_0 + \varepsilon)$. By Lemma 2.3.3, $E \to E/E_{m-1} \to 0$ is a minimal destabilizing quotient with respect to $\sigma^{\mathrm{tilt}}_{\alpha,\beta_0}$.

Since $E/E_{m-1}$ is a minimal destabilizing quotient of $E$ and $\Im Z_{\alpha_0,\beta_0}(E) \neq 0$, we find $\Im Z^{\mathrm{tilt}}_{\alpha_0,\beta_0}(E/E_{m-1}) \neq 0$ as well. In particular, by Lemma 3.3.2 and choice of $\varepsilon$, $E/E_{m-1}$ has a weak Jordan-Hölder filtration

$$0 \to \hat{E}_1 \to \hat{E}_2 \to \cdots \to \hat{E}_{l-1} \to E/E_{m-1}$$

with respect to $\sigma^{\text{tilt}}_{\alpha,\beta_0}$ for all $\alpha \in (\alpha_0, \alpha_0 + \varepsilon)$. In particular,

$$E \to (E/E_{m-1}) \to (E/E_{m-1})/E_{l-1} \to 0$$

is a weakly $\sigma^{\text{tilt}}_{\alpha,\beta_0}$-stable quotient of minimal slope for all $\alpha \in (\alpha_0, \alpha_0+\varepsilon)$. For ease of notation, rewrite this quotient as $E \to \hat{E} \to 0$.

Since $E$ is $\sigma^{\text{tilt}}_{\alpha_0,\beta_0}$-semistable, it follows that $\mu^{\text{tilt}}_{\alpha_0,\beta_0}(\hat{E}) = \mu^{\text{tilt}}_{\alpha_0,\beta_0}(E)$ and so $\hat{E}$ is $\sigma^{\text{tilt}}_{\alpha_0,\beta_0}$-semistable. Therefore, by Schur's Lemma (Lemma 2.2.13), the composition $F \to E \to \hat{E}$ must be an injection or 0. First assume the composition $F \to E \to \hat{E}$ is 0. Therefore, by exactness of the short exact sequence $0 \to F \to E \to G^{\oplus r} \to 0$, $E \to G^{\oplus r} \to 0$ factors through $E \to \hat{E} \to 0$. In particular, there is a surjection $G^{\oplus r} \to \hat{E} \to 0$. Since $\hat{E}$ and $G$ are weakly $\sigma^{\text{tilt}}_{\alpha,\beta_0}$-stable for all $\alpha \in (\alpha_0, \alpha_0 + \varepsilon)$, by Schur's Lemma (Lemma 2.2.13), $\mu^{\text{tilt}}_{\alpha,\beta_0}(\hat{E}) \geq \mu^{\text{tilt}}_{\alpha,\beta_0}(G^{\oplus r})$. Moreover, by construction, $\mu^{\text{tilt}}_{\alpha,\beta_0}(\hat{E}) \leq \mu^{\text{tilt}}_{\alpha,\beta_0}(E)$ for all $\alpha \in (\alpha_0, \alpha_0 + \varepsilon)$. By assumption, $\mu^{\text{tilt}}_{\alpha,\beta_0}(F) < \mu^{\text{tilt}}_{\alpha,\beta_0}(G^{\oplus r})$ for all $\alpha \in (\alpha_0, \alpha_0 + \varepsilon)$ so, by the weak seesaw inequality, $\mu^{\text{tilt}}_{\alpha,\beta_0}(E) < \mu^{\text{tilt}}_{\alpha,\beta_0}(G^{\oplus r})$ (we obtain strict inequality because $W(F,G)$ is semicircular and so $Z^{\text{tilt}}_{\alpha,\beta_0}(G) \neq 0$). Combining these inequalities, we find

$$\mu^{\text{tilt}}_{\alpha,\beta_0}(G^{\oplus r}) \leq \mu^{\text{tilt}}_{\alpha,\beta_0}(\hat{E}) \leq \mu^{\text{tilt}}_{\alpha,\beta_0}(E) < \mu^{\text{tilt}}_{\alpha,\beta_0}(G^{\oplus r}),$$

for all $\alpha \in (\alpha_0, \alpha_0 + \varepsilon)$ which is not possible. Therefore, the composition $F \to E \to \hat{E}$ is injective.

Since $F \to E \to \hat{E}$ is injective, we have a short exact sequence

$$0 \to F \to \hat{E} \to Q \to 0$$

with a surjection $G^{\oplus r} \to Q \to 0$. We claim $Q = G^{\oplus s}$ for $0 \leq s \leq r$. Since $G$ has good quotients, by Lemma 2.3.13, $G^{\oplus r}$ also has good quotients. Therefore, if $Z^{\text{tilt}}_{\alpha_0,\beta_0}(Q) = 0$ then $Q = 0$ or $Q = G^{\oplus r}$, as claimed. If $Z^{\text{tilt}}_{\alpha_0,\beta_0}(Q) \neq 0$ then, by the weak seesaw inequality (Lemma 2.2.6) $\mu^{\text{tilt}}_{\alpha_0,\beta_0}(F) = \mu^{\text{tilt}}_{\alpha_0,\beta_0}(\hat{E}) = \mu^{\text{tilt}}_{\alpha_0,\beta_0}(Q)$. In particular, $Q$ is $\sigma^{\text{tilt}}_{\alpha_0,\beta_0}$-semistable. Thus, by Schur's Lemma (Lemma 2.2.13), the surjection $G^{\oplus r} \to Q \to 0$ induces an isomorphism $G^{\oplus s} = Q$ for some $0 \leq s \leq r$.

Thus, we have shown we have a short exact sequence

$$0 \to F \to \hat{E} \to G^{\oplus s} \to 0.$$

By the nine lemma, the kernel of the surjection $E \to \hat{E}$ is $G^{\oplus r-s}$. If $s \neq r$ then we obtain a nonzero morphism $\mathrm{Hom}_{D^b(X)}(G, E)$ which is a contradiction. Therefore, we must have $s = r$ which means that $E \to \hat{E}$ is an isomorphism. In particular, $E$ is weakly $\sigma^{\mathrm{tilt}}_{\alpha,\beta_0}$-stable for all $\alpha \in (\alpha_0, \alpha_0 + \varepsilon)$, as desired. ☐

There is also a dual version of Theorem 3.6.1 corresponding to walls of the form

$$0 \to F^{\oplus r} \to E \to G \to 0.$$

**Remark 3.6.2.** *If $X$ satisfies Bogomolov's Inequality then the results of this chapter hold over an algebraically closed field of characteristic $p > 0$. In particular, the results hold for Del Pezzo surfaces (see Remark 2.1.16).*

# CHAPTER 4

# APPLICATIONS

In this chapter, we apply our wall-crossing result in two scenarios. The first scenario is a restriction theorem for $\mu_H$-stable sheaves. The second scenario is $\mu_H$-stability of Lazarsfeld-Mukai sheaves.

## 4.1 A Generalization of Bogomolov's Restriction Theorem

Bogomolov's Restriction theorem states that on a smooth projective surface, $\mu_H$-stable bundles remain stable when restricted a general divisor $D \in |aH|$ for $a \gg 0$ [Bog93]. In fact, Bogomolov is even able to give an effective bound on $a$. John Kopper gives a generalization of Flenner's theorem for surfaces [Kop20, Theorem 3.3]. Specifically, Kopper proved that $\mu_H$-stability on a smooth projective surface is preserved when restricting to any divisor of sufficiently high degree (with an effective bound on the degree) - not just a divisor in some multiple of the very ample class. We generalize this result to higher dimensional varieties.

A preliminary lemma shows a codimension 1 sheaf is $\mu_H$-stable on its support iff $\mathscr{E}$ is weakly $\sigma^{\text{tilt}}_{\alpha,\beta}$-stable on the ambient space. This result is now standard.

**Lemma 4.1.1.** *Consider an integral subvariety $\iota : Y \to X$. Let $\mathscr{E}$ be a torsion-free sheaf on $Y$. If $\mathscr{E}$ is $\sigma^{\text{tilt}}_{\alpha,\beta}$-(semi)stable for some $(\beta, \alpha) \in \mathbb{R} \times \mathbb{R}_{>0}$ in the $(H, D)$-slice then $\mathscr{E}$ is $\mu_{H|_Y}$-(semi)stable.*

*Proof.* Since $\operatorname{codim}(\iota_*\mathscr{E}) = 1$, $\operatorname{rank}(\iota_*\mathscr{E}) = 0$ so

$$\begin{aligned}\mu^{\text{tilt}}_{\alpha,\beta}(\iota_*\mathscr{E}) &= \frac{H^{n-2} \cdot \operatorname{ch}_2^{D+\beta H}(\iota_*\mathscr{E})}{\deg_H^{D+\beta H}(\iota_*\mathscr{E})} \\ &= \frac{H^{n-2} \cdot \operatorname{ch}_2(\iota_*\mathscr{E}) - H^{n-2} \cdot (D + \beta H) \cdot \operatorname{ch}_1(\iota_*\mathscr{E})}{\deg_H(\iota_*\mathscr{E})}\end{aligned}$$

By a similar argument to Lemma 2.1.3, we find

$$H^{n-2} \cdot \operatorname{ch}_2(\iota_*\mathscr{E}) = H^{n-2} \cdot \operatorname{ch}_1(\mathscr{E}) - \frac{\operatorname{rank}(\mathscr{E}) H^{n-2} \cdot Y}{2}.$$

Therefore, since $\mathrm{ch}_1(\iota_*\mathscr{E}) = \mathrm{rank}(\mathscr{E})$,

$$\mu^{\mathrm{tilt}}_{\alpha,\beta}(\iota_*\mathscr{E}) = \frac{\deg_H(\mathscr{E})}{\mathrm{rank}(\mathscr{E})} - \frac{H^{n-2}\cdot Y}{2}$$

where $\deg_H(\mathscr{E})$ is the degree of $\mathscr{E}$ on $Y$ with respect to the ample divisor induced by $H$.

Now, consider a proper, nonzero subsheaf $0 \to \mathscr{F} \to \mathscr{E}$ in $\mathrm{Coh}(Y)$. In particular, we obtain an injection $0 \to \iota_*\mathscr{F} \to \iota_*\mathscr{E}$ in $\mathrm{Coh}^{D+\beta H}_H(X)$ for all $\beta \in \mathbb{R}$. In particular, since $\iota_*\mathscr{E}$ is $\sigma^{\mathrm{tilt}}_{\alpha,\beta}$-(semi)stable,

$$\mu_{H|_Y}(\mathscr{F}) - \frac{H^{n-2}\cdot Y}{2} = \mu^{\mathrm{tilt}}_{\alpha,\beta}(\iota_*\mathscr{F}) \underset{(-)}{<} \mu^{\mathrm{tilt}}_{\alpha,\beta}(\iota_*\mathscr{E}) = \mu_{H|_Y}(\mathscr{E}) - \frac{H^{n-2}\cdot Y}{2}.$$

Therefore, $\mathscr{E}$ is $\mu_{H|_Y}$-(semi)stable, as desired. □

**Theorem 4.1.2.** *Consider a reflexive, $\mu_H$-stable sheaf $\mathscr{E}$ on $X$. Consider an integral subvariety $\iota : Y \to X$. If the following bounds are satisfied*

1. $\mu_H(\mathscr{E}) - \dfrac{\overline{\Delta}_H(\mathscr{E})}{2} - \dfrac{1}{2\,\mathrm{rank}(\mathscr{E})^2} > \dfrac{H^{n-2}\cdot \mathrm{ch}_1(\mathscr{E})\cdot Y}{\mathrm{rank}(\mathscr{E})H^{n-1}\cdot Y} - \dfrac{H^{n-2}\cdot Y^2}{2H^{n-1}\cdot Y}$
2. $\mu_H(\mathscr{E}(-Y)) + \dfrac{\overline{\Delta}_H(\mathscr{E}(-Y))}{2} + \dfrac{1}{2\,\mathrm{rank}(\mathscr{E})^2} < \dfrac{H^{n-2}\cdot \mathrm{ch}_1(\mathscr{E})\cdot Y}{\mathrm{rank}(\mathscr{E})H^{n-1}\cdot Y} - \dfrac{H^{n-2}\cdot Y^2}{2H^{n-1}\cdot Y}$

*then $\mathscr{E}|_Y$ is $\mu_{H|_Y}$-stable.*

*Proof.* Since $\mathscr{E}$ is reflexive, we have a short exact sequence

$$0 \to \mathscr{E}(-Y) \to \mathscr{E} \to \mathscr{E}|_Y \to 0.$$

This induces a distinguished triangle

$$\mathscr{E} \to \mathscr{E}|_Y \to \mathscr{E}(-Y)[1] \to \mathscr{E}[1].$$

We claim that this distinguished triangle lies in $\mathrm{Coh}^{\beta H}_H(X)$ for some $\beta$.

Since $\mathscr{E}$ is $\mu_H$-stable, $\mathscr{E}(-Y)$ is $\mu_H$-stable as well. Since $\mathscr{E}$ and $\mathscr{E}(-Y)$ are $\mu_H$-stable, by the Large Volume Limit (Lemma 3.2.3), $\mathscr{E}$ is weakly $\sigma^{\mathrm{tilt}}_{\alpha,\beta}$-stable and $\mathscr{E}(-Y)[1]$ is $\sigma^{\mathrm{tilt}}_{\alpha,\beta}$-stable for all $\alpha \gg 0$ and $-\mu^D_H(\mathscr{E}) + \deg_H(-Y) < \beta < \mu^D_H(\mathscr{E})$.

On the other hand, by additivity

$$\mathrm{ch}_{\leq 2}(\mathscr{O}_X(-Y)) = \left(1, -Y, \frac{Y^2}{2}\right).$$

Therefore, by multiplicativity of the Chern character,

$$\deg_H(\mathscr{E}(-Y)) = \deg_H(\mathscr{E}) - \mathrm{rank}(\mathscr{E})H^{n-1}\cdot Y$$
$$H^{n-2}\cdot \mathrm{ch}_2(\mathscr{E}(-Y)) = H^{n-2}\cdot \mathrm{ch}_2(\mathscr{E}) - H^{n-2}\cdot \mathrm{ch}_1(\mathscr{E})\cdot Y + \mathrm{rank}(\mathscr{E})\frac{H^{n-2}\cdot Y^2}{2}.$$

Therefore, by Lemma 3.4.5, the center of $W(\mathscr{E}, \mathscr{E}(-Y)[1])$ is

$$\frac{H^{n-2}\cdot \mathrm{ch}_1(\mathscr{E})\cdot Y}{\mathrm{rank}(\mathscr{E})H^{n-1}\cdot Y} - \frac{H^{n-2}\cdot Y^2}{2H^{n-1}\cdot Y}.$$

The assumed inequalities guarentee $W(\mathscr{E}, \mathscr{E}(-Y)[1])$ lies between $\mu_H(\mathscr{E}(-Y)[1])$ and $\mu_H(\mathscr{E})$.

Furthermore, by Lemma 3.5.2 and the assumed inequalities, $W(\mathscr{E}, \mathscr{E}(-Y)[1])$ is larger than every actual wall of $\mathscr{E}$ and $\mathscr{E}(-Y)[1]$. In particular, $\mathscr{E}$ is weakly $\sigma^{\mathrm{tilt}}_{\alpha,\beta}$-stable and $\mathscr{E}(-Y)[1]$ is $\sigma^{\mathrm{tilt}}_{\alpha,\beta}$-stable along $W(\mathscr{E}, \mathscr{E}(-Y)[1])$.

Note

$$\mathrm{Hom}_{D^b(X)}(\mathscr{E}(-Y)[1], \mathscr{E}|_Y) = \mathrm{Ext}^{-1}_{D^b(X)}(\mathscr{E}(-Y), \mathscr{E}|_Y) = 0.$$

Furthermore, since $H\cdot Y \neq 0$, $H^{n-2}\cdot\mathrm{ch}_{\leq 2}(\mathscr{E})$ and $H^{n-2}\cdot\mathrm{ch}_{\leq 2}(\mathscr{E}(-Y))$ are not scalar multiples. In particular, $\mu^{\mathrm{tilt}}_{\alpha,\beta}(\mathscr{E}(-Y)) < \mu^{\mathrm{tilt}}_{\alpha,\beta}(\mathscr{E})$ in either the chamber directly above or directly below $W(\mathscr{E}, \mathscr{E}(-Y)[1])$. In either case, by Theorem 3.6.1, $\mathscr{E}|_Y$ is weakly $\sigma^{\mathrm{tilt}}_{\alpha,\beta}$-stable in either the chamber directly above or below the actual wall $W(\mathscr{E}, \mathscr{E}(-Y)[1])$. The desired result follows by Lemma 4.1.1. □

In the case that $Y$ is a multiple of the ample divisor $H$ then the inequalities in Theorem 4.1.2 simplifies to:

**Corollary 4.1.3.** *Assume $\mathscr{E}$ is a reflexive $\mu_H$-stable sheaf and $H$ is ample divisor with $H^n = 1$. If*

$$a > \frac{1}{\mathrm{rank}(\mathscr{E})^2} + \overline{\Delta}_H(\mathscr{E})$$

*then $\mathscr{E}|_{aH}$ is $\mu_H$-stable.*

This result first appeared (via a similar argument) in [Fey21, Theorem 1]. The asymptotic form of this corollary (i.e. $\alpha \gg 0$ rather than an explicit bound) is the Mehta-Ramanathan theorem [HL10, Theorem 7.2.8]. A slightly more general form of this corollary first appeared in [Lan04]

## 4.2 Stability of Lazarsfeld-Mukai Sheaves

In this section, we apply our wall-crossing theorem to study Lazarsfeld-Mukai bundles associated to sufficiently positive, stable bundles.

**Definition 4.2.1.** *Assume $\mathscr{E}$ is a globally generated, torsion-free sheaf. In other words, there is a short exact sequence*

$$0 \to \mathscr{M}_{\mathscr{E}} \to H^0(\mathscr{E}) \otimes \mathscr{O}_X \to \mathscr{E} \to 0. \tag{4.1}$$

*We call $\mathscr{M}_{\mathscr{E}}$ the Lazarsfeld-Mukai sheaf associated to $\mathscr{E}$. If $\mathscr{E}$ is clear from context, we just write $\mathscr{M}$ instead of $\mathscr{M}_{\mathscr{E}}$. The Lazarsfeld-Mukai sheaf is also called the kernel or syzygy sheaf.*

As short hand, we will write $\mathscr{O}_X^{\oplus h^0(\mathscr{E})}$ instead of $H^0(\mathscr{E}) \otimes \mathscr{O}_X$.

In this section we show the Lazarsfeld-Mukai bundle associated to a sufficiently positive stable bundle is also stable. Broadly, by shifting Equation 4.1 we obtain the following distinguished triangle:

$$0 \to \mathscr{E} \to \mathscr{M}[1] \to \mathscr{O}_X^{\oplus h^0(\mathscr{E})}[1] \to 0.$$

If $\mathscr{E}$ is stable and sufficiently positive then $\mathscr{M}[1]$ is $\sigma^{\text{tilt}}_{\alpha,\beta}$-stable by Theorem 3.6.1. This stability enforces severe constraints on the first few Chern characters of a maximal $\mu_H$-destabilizing subsheaf of $\mathscr{M}$. In many cases, we can use these constraints to show that $\mathscr{M}$ is stable.

This general method of proof has been used in many recent results (e.g. [Bay18],[Kop20], and [Fey21]). Our case differs from these because we are working with torsion-free sheaves—while each of these recent works only considers torsion sheaves. For a torsion sheaf, $\sigma^{\text{tilt}}_{\alpha,\beta}$-

stability is equivalent to $\mu_H$-stability along its support ([Kop20, Lemma 2.6]). For torsion-free sheaves this result is false, so more work is needed to show $\mu_H$-stability.

**Lemma 4.2.2.** *Let $\mathscr{E}$ be a globally generated, torsion-free, $H$-twisted stable sheaf on $X$ with associated Lazarsfeld-Mukai sheaf $\mathscr{M}$. If the following bounds are satisfied:*

- $\deg_H(\mathscr{E}) > 0$,
- $H^{n-2} \cdot \mathrm{ch}_2(\mathscr{E}) > 0$, *and*
- $2\frac{H^{n-2}\cdot \mathrm{ch}_2(\mathscr{E})}{\deg_H(\mathscr{E})} + \frac{1}{\mathrm{rank}(\mathscr{E})^2} > \overline{\Delta}_H(\mathscr{E})$

*then $W(\mathscr{O}_X^{\oplus h^0(\mathscr{E})}[1], \mathscr{E})$ is an actual wall associated to $\mathscr{M}[1]$ in the $(H,0)$-slice, and $\mathscr{M}[1]$ is $\sigma^{\mathrm{tilt}}_{\alpha,\beta}$-stable in the chamber directly above this wall.*

*Proof.* There is an exact sequence in $\mathrm{Coh}(X)$:

$$0 \to \mathscr{M} \to \mathscr{O}_X^{\oplus h^0(\mathscr{E})} \to \mathscr{E} \to 0$$

which induces the following exact sequence in $\mathrm{Coh}^{\beta H}_H(X)$ for all $\beta \in [0, \mu_H(\mathscr{E}))$:

$$0 \to \mathscr{E} \to \mathscr{M}[1] \to \mathscr{O}_X^{\oplus h^0(\mathscr{E})}[1] \to 0.$$

Since $\mathscr{O}_X$ is $\mu_H$-stable and $\mathscr{E}$ is $H$-twisted stable, by the Large Volume Limit (Lemma 3.2.3), $\mathscr{O}_X[1]$ and $\mathscr{E}$ are weakly $\sigma^{\mathrm{tilt}}_{\alpha,\beta}$-stable for all $\beta \in [0, \mu_H(\mathscr{E}))$ and $\alpha \gg 0$. In fact, since $\overline{\Delta}_H(\mathscr{O}_X) = 0$, $\mathscr{O}_X[1]$ is weakly $\sigma^{\mathrm{tilt}}_{\alpha,\beta}$-stable for all $\beta \geq 0$ and all $\alpha > 0$. Furthermore, since $2\frac{H^{n-2}\cdot \mathrm{ch}_2(\mathscr{E})}{\deg_H(\mathscr{E})} + \frac{1}{\mathrm{rank}(\mathscr{E})^2} > \overline{\Delta}_H(\mathscr{E})$,

$$\max\left\{\sqrt{\frac{\overline{\Delta}_H(\mathscr{E})}{4(\mathrm{rank}(\mathscr{E})+1)}}, \frac{1}{2}\left|\overline{\Delta}_H(\mathscr{E}) - \frac{1}{\mathrm{rank}(\mathscr{E})^2}\right|\right\} < \frac{H^{n-2}\cdot \mathrm{ch}_2(\mathscr{E})}{\deg_H(\mathscr{E})}.$$

On the other hand, by direct calculation and Lemma 3.4.5.1, the radius of $W(\mathscr{O}_X^{\oplus h^0(\mathscr{E})}[1], \mathscr{E})$ is $H^{n-2}\cdot \mathrm{ch}_2(\mathscr{E})/\deg_H(\mathscr{E})$. Therefore, $\mathscr{E}$ is weakly $\sigma^{\mathrm{tilt}}_{\alpha,\beta}$-stable for all $(\beta,\alpha)$ lying above the wall $W(\mathscr{O}_X^{\oplus h^0(\mathscr{E})}[1], \mathscr{E})$. This shows that $W(\mathscr{O}_X^{\oplus h^0(\mathscr{E})}[1], \mathscr{E})$ is an actual wall associated to $\mathscr{M}[1]$ in the $(H,0)$-slice.

It remains to show $\mathscr{M}[1]$ is $\sigma^{\text{tilt}}_{\alpha,\beta}$-stable for all $(\beta,\alpha)$ in the chamber directly above $W(\mathscr{O}_X^{\oplus h^0(\mathscr{E})}[1],\mathscr{E})$. By Lemma 3.1.13, $\mathscr{O}_X^{\oplus h^0(\mathscr{E})}[1]$ has good quotients. Furthermore, by definition of a Lazarsfeld-Mukai sheaf,

$$\operatorname{Hom}_{D^b(X)}(\mathscr{O}_X[1],\mathscr{M}[1]) = H^0(\mathscr{M}) = 0.$$

For ease of notation, set

$$(\beta_0,\alpha_0) = \left(\frac{H^{n-2}\cdot \operatorname{ch}_2(\mathscr{E})}{\deg_H(\mathscr{E})}, \frac{H^{n-2}\cdot \operatorname{ch}_2(\mathscr{E})}{\deg_H(\mathscr{E})}\right) \in W(\mathscr{O}_X^{\oplus h^0(\mathscr{E})}[1],\mathscr{E}).$$

By direct computation,

$$\mu^{\text{tilt}}_{\alpha_0+\varepsilon,\beta_0}(\mathscr{O}_X^{\oplus h^0(\mathscr{E})}[1]) - \mu^{\text{tilt}}_{\alpha_0+\varepsilon,\beta_0}(\mathscr{E}) = \frac{\deg_H(\mathscr{E})^2\varepsilon(2H^{n-2}\cdot \operatorname{ch}_2(\mathscr{E}) + \deg_H(\mathscr{E})\varepsilon)}{2H^{n-2}\cdot \operatorname{ch}_2(\mathscr{E})(\deg_H(\mathscr{E})^2 - \operatorname{rank}(\mathscr{E})H^{n-2}\cdot \operatorname{ch}_2(\mathscr{E}))}$$

which is positive as long as

$$\deg_H(\mathscr{E})^2 - \operatorname{rank}(\mathscr{E})H^{n-2}\cdot \operatorname{ch}_2(\mathscr{E}) > 0.$$

However, since $\mathscr{E}$ is $H$-twisted stable (a fortiori $\mu_H$-semistable by Lemma 2.1.11), this inequality follows from Bogomolov's Inequality.

Hence, by Theorem 3.6.1, $\mathscr{M}[1]$ is weakly $\sigma^{\text{tilt}}_{\alpha,\beta}$-stable for $(\beta,\alpha)$ in the chamber directly above $W$. In fact, by Lemma 3.1.13 and Lemma 2.2.11, $\mathscr{M}[1]$ is $\sigma^{\text{tilt}}_{\alpha,\beta}$-stable for all such $(\beta,\alpha)$, as desired. □

**Lemma 4.2.3.** *Let $\mathscr{M}$ be the Lazarsfeld-Mukai sheaf associated $\mathscr{E}$ satisfying the assumptions of Lemma 4.2.2.*

1. *If $\mathscr{M}$ is not $\mu_H$-semistable and $0 \to \mathscr{N} \to \mathscr{M}$ is a maximal $\mu_H$-destabilizing subsheaf then*
$$\frac{H^{n-2}\cdot \operatorname{ch}_2(\mathscr{N})}{\deg_H(\mathscr{N})} \le \frac{H^{n-2}\cdot \operatorname{ch}_2(\mathscr{M})}{\deg_H(\mathscr{M})}.$$
2. *If $\mathscr{M}$ is $\mu_H$-semistable and $0 \to \mathscr{N} \to \mathscr{M}$ is a subsheaf satisfying $\mu_H(\mathscr{N}) = \mu_H(\mathscr{M})$ and $\operatorname{rank}(\mathscr{N}) < \operatorname{rank}(\mathscr{M})$ then*
$$\frac{H^{n-2}\cdot \operatorname{ch}_2(\mathscr{N})}{\deg_H(\mathscr{N})} < \frac{H^{n-2}\cdot \operatorname{ch}_2(\mathscr{M})}{\deg_H(\mathscr{M})}$$

*Proof.* Assume $\mathscr{M}$ is the Lazarsfeld-Mukai sheaf associated to $\mathscr{E}$.

1. Since $W(\mathscr{E}, \mathscr{O}_X^{\oplus h^0(\mathscr{E})}[1])$ has endpoints $(0,0)$ and $(2H^{n-2}\cdot \mathrm{ch}_2(\mathscr{E})/\deg_H(\mathscr{E}), 0)$ and walls are locally finite, by Lemma 4.2.2, there exists $\alpha_0 > 0$ such that $\mathscr{M}[1]$ is $\sigma^{\mathrm{tilt}}_{\alpha,0}$-stable for all $\alpha \in (0, \alpha_0)$. By Lemma 3.1.3, $0 \to \mathscr{N}[1] \to \mathscr{M}[1]$ is a subobject in $\mathrm{Coh}^{0H}_H(X)$ and so

$$\begin{aligned}\frac{H^{n-2}\cdot \mathrm{ch}_2(\mathscr{N}[1]) - \frac{\alpha^2}{2}\mathrm{rank}(\mathscr{N}[1])}{\deg_H(\mathscr{N}[1])} &= \mu^{\mathrm{tilt}}_{\alpha,0}(\mathscr{N}[1]) \\ &< \mu^{\mathrm{tilt}}_{\alpha,0}(\mathscr{M}[1]) \\ &= \frac{H^{n-2}\cdot \mathrm{ch}_2(\mathscr{M}[1]) - \frac{\alpha^2}{2}\mathrm{rank}(\mathscr{M}[1])}{\deg_H(\mathscr{M}[1])}\end{aligned}$$

   for all $\alpha \in (0, \alpha_0)$. Taking the limit as $\alpha$ approaches 0 gives

$$\frac{H^{n-2}\cdot \mathrm{ch}_2(\mathscr{N})}{\deg_H(\mathscr{N})} \leq \frac{H^{n-2}\cdot \mathrm{ch}_2(\mathscr{M})}{\deg_H(\mathscr{M})},$$

   as claimed.

2. By the same argument as part 1, we find

$$\frac{H^{n-2}\cdot \mathrm{ch}_2(\mathscr{N})}{\deg_H(\mathscr{N})} \leq \frac{H^{n-2}\cdot \mathrm{ch}_2(\mathscr{M})}{\deg_H(\mathscr{N})}.$$

   If we have equality, by Remark 3.1.8, $\mu^{\mathrm{tilt}}_{\alpha,\beta}(\mathscr{N}[1]) = \mu^{\mathrm{tilt}}_{\alpha,\beta}(\mathscr{M}[1])$ for all $(\beta,\alpha) \in \mathbb{R}\times\mathbb{R}_{>0}$. In particular, $\mathscr{M}[1]$ is not $\sigma^{\mathrm{tilt}}_{\alpha,\beta}$-stable for $\beta > \mu_H(\mathscr{M}[1])$. This contradicts Lemma 4.2.2, so we must have

$$\frac{H^{n-2}\cdot \mathrm{ch}_2(\mathscr{N})}{\deg_H(\mathscr{N})} < \frac{H^{n-2}\cdot \mathrm{ch}_2(\mathscr{M})}{\deg_H(\mathscr{M})}$$

   as claimed.

□

**Theorem 4.2.4.** *Assume $X$ is a smooth Del Pezzo surface over an algebraically closed field of arbitrary characteristic. For ease of notation, let $H = -K_X$ which is ample by definition. Let $\mathscr{E}$ be a globally generated, torsion-free, $(H, \frac{H}{2})$-Gieseker stable sheaf on $X$ with associated Lazarsfeld-Mukai bundle $\mathscr{M}$. If the following bounds are satisfied:*

- $0 < \deg_H(\mathscr{E}) \leq K_X^2(h^0(\mathscr{E}) - \operatorname{rank}(\mathscr{E}))$,
- $\operatorname{ch}_2(\mathscr{E}) > 0$,
- $2\dfrac{H^{n-2}\cdot \operatorname{ch}_2(\mathscr{E})}{\deg_H(\mathscr{E})} + \dfrac{1}{\operatorname{rank}(\mathscr{E})^2} \geq \overline{\Delta}_H(\mathscr{E})$

*then $\mathscr{M}$ is $\mu_H$-stable.*

*Proof.* Consider a maximal $\mu_H$-destabilizing subsheaf $0 \to \mathscr{N} \to \mathscr{M}$. Since $H = -K_X$, by the Hirzebruch-Riemann-Roch theorem,

$$\frac{\chi(\mathscr{N})}{\deg_H(\mathscr{N})} = \frac{\operatorname{ch}_2(\mathscr{N})}{\deg_H(\mathscr{N})} - \frac{1}{2} + \frac{\operatorname{rank}(\mathscr{N})}{\deg_H(\mathscr{N})}.$$

Since $0 \to \mathscr{N} \to \mathscr{M}$ is a maximal $\mu_H$-destabilizing subsheaf, by definition and Lemma 4.2.3,

$$\frac{\chi(\mathscr{N})}{\deg_H(\mathscr{N})} = \frac{\operatorname{ch}_2(\mathscr{N})}{\deg_H(\mathscr{N})} - \frac{1}{2} + \frac{\operatorname{rank}(\mathscr{N})}{\deg_H(\mathscr{N})} < \frac{\operatorname{ch}_2(\mathscr{M})}{\deg_H(\mathscr{M})} - \frac{1}{2} + \frac{\operatorname{rank}(\mathscr{M})}{\deg_H(\mathscr{M})} = \frac{\chi(\mathscr{M})}{\deg_H(\mathscr{M})}.$$

Since $X$ is a Del Pezzo surface,

$$\frac{\chi(\mathscr{N})}{\deg_H(\mathscr{N})} < \frac{\chi(\mathscr{M})}{\deg_H(\mathscr{M})} = \frac{h^1(\mathscr{E})}{\deg_H(\mathscr{M})} \leq 0.$$

Since $\mathscr{M}$ is a Lazarsfeld-Mukai bundle, $h^0(\mathscr{M}) = 0$ and so $h^0(\mathscr{N}) = 0$. We claim $h^2(\mathscr{N}) = 0$ as well. Since $\mathscr{N}$ is torsion-free, there is a natural injection $0 \to \mathscr{N} \to \mathscr{N}^{\vee\vee}$ whose cokernel is supported in dimension 2. Furthermore, $\mathscr{N}^{\vee}$ is reflexive and so locally free [Har80, Corollary 1.4]. Thus, by Serre duality,

$$h^2(\mathscr{N}) = h^2(\mathscr{N}^{\vee\vee}) = h^0(\mathscr{N}^{\vee} \otimes \omega_X).$$

However, since $\mathscr{N}$ is $\mu_H$-semistable,

$$\mu_H^+(\mathscr{N}^{\vee} \otimes \omega_X) = -\mu_H(\mathscr{N}) + \deg_H(\omega_X) < -\mu_H(\mathscr{M}) + \deg_H(\omega_X) < 0,$$

where the last inequality follows from assumed bound: $\deg_H(\mathscr{E}) \leq K_X^2(h^0(\mathscr{E}) - \operatorname{rank}(\mathscr{E}))$. Since $\mu_H^+(\mathscr{N}^{\vee} \otimes \omega_X) < 0$,

$$h^2(\mathscr{N}) = h^0(\mathscr{N}^{\vee} \otimes \omega_X) = 0,$$

as claimed. Therefore, we have shown

$$0 \leq -\frac{h^1(\mathscr{N})}{\deg_H(\mathscr{N})} = \frac{\chi(\mathscr{N})}{\deg_H(\mathscr{N})} < 0,$$

a contradiction. Hence, $\mathscr{M}$ must be $\mu_H$-semistable.

We now claim $\mathscr{M}$ is $\mu_H$-stable, so consider a subsheaf $0 \to \mathscr{N} \to \mathscr{M}$ with $\mu_H(\mathscr{N}) = \mu_H(\mathscr{M})$. Furthermore, by Lemma 2.2.11, we may assume $\operatorname{rank}(\mathscr{N}) < \operatorname{rank}(\mathscr{M})$. By Lemma 4.2.3 and the Hirzebruch-Riemann-Roch theorem,

$$\frac{\chi(\mathscr{N})}{\deg_H(\mathscr{N})} = \frac{\operatorname{ch}_2(\mathscr{N})}{\deg_H(\mathscr{N})} - \frac{1}{2} + \frac{\operatorname{rank}(\mathscr{N})}{\deg_H(\mathscr{N})} < \frac{\operatorname{ch}_2(\mathscr{M})}{\deg_H(\mathscr{M})} - \frac{1}{2} + \frac{\operatorname{rank}(\mathscr{M})}{\deg_H(\mathscr{M})} = \frac{\chi(\mathscr{M})}{\deg_H(\mathscr{M})}.$$

By the same argument as above, $\chi(\mathscr{N}) < 0$ and $\chi(\mathscr{M}) > 0$ so we find

$$0 = \frac{\chi(\mathscr{N})}{\deg_H(\mathscr{N})} < \frac{\chi(\mathscr{M})}{\deg_H(\mathscr{M})}$$

which is a contradiction. Thus, $\mathscr{M}$ is $\mu_H$-stable, as claimed.

Last, by remark 3.6.2, this argument holds in arbitrary characteristic. $\square$

In the case of smooth Del Pezzo surfaces and $H = -K_X$, Theorem 4.2.4 completely generalizes the best previously known result [TLZ21, Theorem 3.7].

**Example 4.2.5.** *We note Theorem 4.2.4 cannot be weakened to just the inequalities* $0 < \deg_H(\mathscr{E})$ *and* $0 < \operatorname{ch}_2(\mathscr{E})$.

*Consider the tangent bundle* $\mathscr{T}_{\mathbb{P}^2}$ *on* $\mathbb{P}^2$. *There is a short exact sequence*

$$0 \to \Omega_{\mathbb{P}^2}(1)^{\oplus 3} \to H^0(\mathscr{T}_{\mathbb{P}^2}) \otimes \mathscr{O}_{\mathbb{P}^2} \to \mathscr{T}_{\mathbb{P}^2} \to 0.$$

*In particular, the Lazarsfeld-Mukai bundle associated to* $\mathscr{T}_{\mathbb{P}^2}$ *is not* $\mu_H$*-stable even though* $\mathscr{T}_{\mathbb{P}^2}$ *is* $\mu_H$*-stable (*[OSS80, Chapter 2 Theorem 1.3.2]*) with* $\deg_H(\mathscr{T}_{\mathbb{P}^2}), \operatorname{ch}_2(\mathscr{T}_{\mathbb{P}^2}) > 0$. *We note that*

$$2\frac{\operatorname{ch}_2(\mathscr{E})}{\deg_H(\mathscr{E})} + \frac{1}{\operatorname{rank}(\mathscr{E})^2} = 1 + \frac{1}{4} < 3 = \overline{\Delta}_H(\mathscr{T}_{\mathbb{P}^2}),$$

*so* $\mathscr{T}_{\mathbb{P}^2}$ *does not satisfy the positivity properties of Theorem 4.2.4.*

*Interestingly,* $\mathscr{T}_{\mathbb{P}^2}(2)$ *does satisfy the required positivity properties, so we find that the Lazarsfeld-Mukai bundle associated to* $\mathscr{T}_{\mathbb{P}^2}(2)$ *is* $\mu_H$*-stable.*

We note that any sufficiently high twist of a torsion-free $\mu_H$-stable sheaf satisfies the assumptions of Theorem 4.2.4.

**Corollary 4.2.6.** *Assume $X$ is a smooth Del Pezzo surface. For ease of notation, let $H = -K_X$ and* $\mathrm{reg}(\mathscr{E})$ *the Castelnuovo-Mumford regularity of $\mathscr{E}$* [Laz04, Definition 1.8.1]. *Assume $\mathscr{E}$ is a $\mu_H$-stable, torsion-free sheaf. If*

$$d \geq \max\left\{\sqrt{\frac{4\overline{\Delta}_H(\mathscr{E})}{\mathrm{rank}(\mathscr{E})^2}+1} - \mu_H(\mathscr{E}) + \frac{1}{2}, \overline{\Delta}_H(\mathscr{E}) - \mu_H(\mathscr{E}), \mathrm{reg}(\mathscr{E}), 0\right\}$$

*then the Lazarsfeld-Mukai bundle associated to $\mathscr{E}(d)$ is $\mu_H$-stable.*

*Proof.* By Mumford's theorem ([Laz04, Theorem 1.8.3]), $\mathscr{E}(d)$ is globally-generated, so $\mathscr{M}$ is well-defined.

By definition of the Castelnuovo-Mumford regularity, $h^1(\mathscr{E}(d)), h^2(\mathscr{E}(d)) = 0$. Therefore, by the Hirzebruch-Riemann-Roch theorem and direct computation, $\mathscr{E}(d)$ satisfies the assumptions of Theorem 4.2.4. The result follows. ☐

In particular, the Lazarsfeld-Mukai associated to $\mathscr{O}_X(-dK_X)$ is $\mu_H$-stable for $d \geq 1$.

**Remark 4.2.7.** *We expect Theorem 4.2.4 to extend to smooth, projective surfaces such that $h^1(\mathscr{O}_X), h^2(\mathscr{O}_X) = 0$ and either $K_X$ is ample or numerically trivial. Specifically, the argument from Theorem 4.2.4 goes through in this setting (at least in characteristic $0$) except, possibly, the vanishing of $h^2(\mathscr{N})$.*

We end by showing $\mu_H$-semistability and $\mu_H$-stability are equivalent for Lazarsfeld-Mukai bundles associated to ample line bundles.

**Theorem 4.2.8.** *Assume $X$ is a smooth projective variety equipped with ample divisor $H$. Let $\mathscr{M}$ be the Lazarsfeld-Mukai bundle associated to $\mathscr{O}_X(d)$ for $d \geq 1$. The following are equivalent*

1. *$\mathscr{M}$ is $\mu_H$-stable.*

*2.* $\mathscr{M}$ *is* $\mu_H$*-semistable.*

*3. Every actual wall to the right of* $\mu_H(\mathscr{M})$ *associated to* $\mathscr{M}[1]$ *in the* $(H,0)$*-slice is nexted within the wall* $W(\mathscr{O}_X^{\oplus h^0(\mathscr{O}_X(d))}[1], \mathscr{O}_X(d))$.

*Proof.* $1 \Rightarrow 2$: This follows from definition.

$2 \Rightarrow 3$: Suppose $W(\mathscr{O}_X^{\oplus h^0(\mathscr{O}_X(d))}[1], \mathscr{O}_X(d))$ is not the largest wall associated to $\mathscr{M}[1]$ in the $(H,0)$-slice. Therefore, we can choose a larger actual semicircular wall $W$ with destabilizing sequence $0 \to F \to \mathscr{M}[1] \to G \to 0$. Since $\mathscr{M}$ is the Lazarsfeld-Mukai bundle associated to $\mathscr{O}_X(d)$,

$$\sqrt{\frac{\overline{\Delta}_H(\mathscr{M})}{4(\operatorname{rank}(\mathscr{M})+1)}} = \frac{d}{2} = \frac{H^{n-2}\cdot \operatorname{ch}_2(\mathscr{M})}{\deg_H(\mathscr{M})}.$$

Therefore, by Lemma 3.5.2, $\operatorname{rank}(\mathscr{H}^0(F)) = 0$. Furthermore, by Lemma 3.4.5.1, $W$ intersects the line $\beta = 0$ at

$$0 < \alpha = 2\frac{-(H^{n-2}\cdot \operatorname{ch}_2(\mathscr{M}[1]) \deg_H(F) - H^{n-2}\cdot \operatorname{ch}_2(F)\deg_H(\mathscr{M}[1]))}{\deg_H(\mathscr{M}[1])\operatorname{rank}(F) - \deg_H(F)\operatorname{rank}(\mathscr{M}[1])}.$$

By the same argument as Lemma 4.2.3, the numerator of this expression is negative. Therefore,

$$\deg_H(\mathscr{M}[1])\operatorname{rank}(F) - \deg_H(F)\operatorname{rank}(\mathscr{M}[1]) < 0$$

or, since $\operatorname{rank}(\mathscr{H}^0(F)) = 0$, $\mu_H(\mathscr{M}[1]) < \mu_H(F)$. Furthermore, since $\operatorname{rank}(\mathscr{H}^0(F)) = 0$, by Lemma 2.1.3, $\deg_H(\mathscr{H}^0(F)) \geq 0$. It follows that

$$\begin{aligned}
\mu_H(\mathscr{H}^{-1}(F)) &= \frac{\deg_H(\mathscr{H}^{-1}(F))}{\operatorname{rank}(\mathscr{H}^{-1}(F))} \\
&> \frac{\deg_H(\mathscr{H}^{-1}(F)) - \deg_H(\mathscr{H}^0(F)}{\operatorname{rank}(\mathscr{H}^{-1}(F))} \\
&= \mu_H(F) \\
&> \mu_H(\mathscr{M}[1]) \\
&= \mu_H(\mathscr{M}).
\end{aligned}$$

In particular, taking cohomology of $0 \to F \to \mathscr{M}[1] \to G \to 0$ gives us an injection $0 \to \mathscr{H}^{-1}(F) \to \mathscr{M}$ with $\mu_H(\mathscr{H}^{-1}(F)) > \mu_H(\mathscr{M})$. In other words, $\mathscr{M}$ is not $\mu_H$-semistable, as claimed.

$3 \Rightarrow 1$: This follows from Lemma 4.2.2 and the Large Volume Limit (Lemma 3.2.3). $\square$

We note that Theorem 5.8 and Lemma 4.2.2 gives us an explicit description of the largest actual wall of $\mathscr{M}[1]$ in infinitely many cases. In practice, such a description seems difficult to find unless $\deg_H(\mathscr{M}[1]) \leq 0$ or $\operatorname{rank}(\mathscr{M}[1]) = 0$. In fact, as far as the author knows, the only other example outside of these cases is spherical bundles on $K3$ surfaces.

# CHAPTER 5

# FUTURE WORK

In this chapter, we discuss more possible applications of our wall-crossing result and further investigations into the applications above.

## 5.1 Slope Stable Bundles on Rational Varieties

As stated in the introduction, it is unknown whether there exists a slope stable bundle of rank 2 on $\mathbb{P}^7$. In fact, in characteristic 0, it is unknown whether there exists a slope stable bundle of rank 2 on $\mathbb{P}^n$ for any $n \geq 5$. On $\mathbb{P}^4$ over $\mathbb{C}$, all known examples come from the Horrocks-Mumford bundle construction. This bundle is constructed by considering certain representations of $\mathbb{Z}/5\mathbb{Z}$. This construction suggests that low rank indecomposable vector bundles on $\mathbb{P}^n$ for $n$ sufficiently large tend to rise via representation theoretic considerations rather than geometric ones.

On the other hand, if $X$ is equipped with an exceptional collection (e.g. many rational varieties), then there exists a quiver $Q$ and a triangulated equivalence $D^b(X) = D^b(\mathrm{Rep}(Q))$ where $\mathrm{Rep}(Q)$ is the representations of $Q$ [Bon90]. Therefore, we obtain a heart $\mathrm{Rep}(Q)$ in $D^b(X)$. Furthermore, there is a notion of stability on $\mathrm{Rep}(Q)$ due to [Kin94], and this induces a very weak stability condition on the heart $\mathrm{Rep}(Q)$. We can often find stable objects in the $(H, D)$-slice that correspond to stable representations of $Q$. This has already been done for $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, and $\mathbb{P}^3$ in [AB13], [AM17], and [Sch20] respectively. Moreover, with sufficient understanding of the walls in the $(H, D)$-slice, we hope to use Theorem 3.6.1 to relate slope stable vector bundles to stable representations of $Q$. Further, it is very easy to calculate whether there exists a stable representation with given fixed invariants, so we hope to use this putative correspondence to construct slope stable bundles.

## 5.2 Lazarsfeld-Mukai Bundles on Other Varieties

Theorem 4.2.4 shows that Lazarsfeld-Mukai bundles associated to sufficiently positive stable bundles on Del Pezzo surfaces are slope stable. As noted in Remark 4.2.7, the same argument would work on any surface with $h^1(\mathscr{O}_X), h^2(\mathscr{O}_X) = 0$ granted $h^2(\mathscr{N}) = 0$. Therefore, in future work, we hope to investigate this vanishing more thoroughly.

Moreover, in future work we aim to show Theorem 4.2.4 extends to higher dimensional Fano varieties. One possible argument would be to show $W(\mathscr{O}_X^{\oplus h^0(\mathscr{E})}[1], \mathscr{E})$ is the largest actual wall associated to $\mathscr{M}_{\mathscr{E}}$. This has been done when $\mathscr{E}$ is an ample line bundle and $X$ is a $K3$ suface. However, we believe this method is too hopeful in general. Instead, we expect we would need to use a similar method to Theorem 4.2.4.

Specifically, the same argument as Theorem 4.2.4 shows that if $\mathscr{M}$ is the Lazarsfeld-Mukai sheaf associated to sufficiently positive stable sheaf $\mathscr{E}$ then a maximal $\mu_{-K_X}$-destabilizing subsheaf $0 \to \mathscr{N} \to \mathscr{M}$ satisfies:

$$\frac{H^{n-2} \cdot \mathrm{ch}_2(\mathscr{N})}{\deg_H(\mathscr{N})} \leq \frac{H^{n-2} \cdot \mathrm{ch}_2(\mathscr{M})}{\deg_H(\mathscr{M})}.$$

This puts severe constrains on the second Chern character of $\mathscr{N}$ which induce constrains on the cohomology of $\mathscr{N}$. In fact, this method can be used on $\mathbb{P}^n$ to show that the Lazarsfeld-Mukai bundle associated to $\mathscr{O}_X(d)$ with $d \geq 1$ is slope stable.

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