“IT’S ALL CONNECTED”: MATHEMATICS TEACHERS’ OPENING CURRICULUM SPACES FOR STUDENTS’ MULTIPLE MATHEMATICAL KNOWLEDGE BASES

By

Sunyoung Park

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ABSTRACT

This dissertation study investigated secondary mathematics teachers' designing and enacting lessons to draw on students' multiple mathematical knowledge bases. The ultimate objective of incorporating open spaces in mathematics lessons is to support students from culturally and linguistically diverse backgrounds in developing mathematical understanding based on their prior knowledge and life experiences. Despite its potential, such open spaces do not explicitly appear in many commercially published curricula, and teachers are on their own with the responsibility to mobilize curriculum resources to ensure equitable learning opportunities.

Applying Skovsmose and Borba's (2004) model for participatory research in changing mathematics classroom practices, this study engaged two middle school mathematics teachers in collaborative planning and reflection meetings to seek practical ways to integrate students' multiple mathematical knowledge bases into teachers' usual interaction with the curriculum. The qualitative analysis examined both how the teachers made decisions on their curriculum use when incorporating open spaces in their daily lessons and how those decisions were influenced by their sense of teacher agency. The findings of this study revealed that as teachers set an instructional goal of providing students with opportunities to draw on their multiple mathematical knowledge bases, they decided to adhere to the curriculum in order to open spaces for mathematical thinking, whereas they adapted the curriculum in order to open spaces for life experiences. The teachers perceived that opening curriculum spaces can reinforce and be leveraged by building rapport with students through informal conversations. The findings of this study also suggest that a collaborative planning group with other teachers can enhance teachers' agentic engagement in designing and enacting open spaces.

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CHAPTER 1: INTRODUCTION

In the very first month of my Ph.D. program, for two hours of a potluck lunch with my new research team, I did not speak a single word out loud except to introduce myself. I can never forget the mixed feelings I went through on that day. It involved confusion, awkwardness, embarrassment, self-consciousness, depression, and desperate self-encouragement, all combined. When invited to the lunch as a new graduate assistant on the team, I was thrilled to get to know people in a movie-like setting! (I knew what a potluck lunch meant from watching Western movies, and yet I had never participated in it before.)

In reality, however, I literally felt like sitting in a 3D-movie theater for two hours, watching Western people talking. I could clearly understand most of their conversation, which means English as my second language was not the issue. The thing was that I was not able to get any of the cultural references that they kept bringing up. Famous characters and celebrities in a classic American TV show to explain the casts in a new movie (and the pronunciation of their names was different from what I used to know), locally and nationally popular ginger ale brands to vote for the best one, competing relationships among university sports teams, hilarious commercials, cool catchphrases as idioms, and signature dishes from different states to name a few. I nodded, nodded, nodded, with smiles on my face, and wondered, wondered, wondered, "Where in the world am I?"

Everyone seemed rejuvenated after non-work-related socializing over lunch, while I was feeling too dizzy to even stand up. Before leaving the room, one lovely girl on the team smiled at me and said, "Hey, you are so quiet today. Did you enjoy your lunch?" I smiled back at her and swallowed my frustration. It was yet too raw and premature to express, I thought. I needed to have private time to process.

When I decided to pursue a degree in the USA, my biggest concern was the fact that I had to communicate in English. I had questioned so many times if I would be able to overcome the language barrier and eloquently express my thoughts and emotions. Such fear urged me to take English courses at the university's language center during the summer prior to my grad program, which encouraged me to believe that I was ready to mingle. It did not take long to realize that learning and practicing English was helpful yet insufficient for me to be a part of a community in the USA.

Everyone in my research team wholeheartedly welcomed me and would probably be surprised if they knew about my frustration at our first potluck lunch. To some degree, I was relieved because no one put me on the spot to ask questions about my home country and cultural differences. It was perhaps them being thoughtful because I got the impression that people did not see me as someone different from them. I received an equal amount of eye contact while people were talking. At the same time, I recognized that it might be an implicit message that I was supposed to be already accustomed to *this* culture, thinking and (re-)acting like people in *this* culture, not based on my home culture. It was the very first moment that I learned about the necessity of ongoing (self-)negotiation about my cultural identity as a non-immigrant resident in the USA. And I am still questioning if I am solely responsible for developing the third identity between my home and American cultures while other people obliviously interact with only one side of me and ignore the other side.

My personal experiences as an international student in the USA have also influenced my view on educational research. As a secondary mathematics teacher, I used to advocate for *mathematics for all* (NCTM, 2000) to support every single student in my classroom in developing mathematical understanding. For a long time, my research interest centered around a

teacher's role in designing tasks that can connect students' classroom experiences with mathematical ideas. In other words, I focused on providing students with equal access to challenging tasks, and I did not necessarily consider how differently students might engage in those tasks depending on their personal attributes and prior experiences. Now that I understand that sitting at the same table does not guarantee equal contribution to the conversation, I have become more attentive to missed opportunities for students to develop mathematical power when the differences in their cultural background are not recognized and valued. I argue that those missed learning opportunities for underrepresented students are hard for educators to notice in the classroom, as I illustrate below in sharing one student's story that describes the subtlety and complexity of underrepresented students' missed learning opportunities.

At first, an Asian male student, SJ, in a culturally and ethnically diverse mathematics classroom, seemed quiet and focused. He did not involve himself much in conversations with classmates during group work, either about mathematics problems or non-mathematical chats. Instead, he seemed to independently focus on his work in the digital space and try to figure things out on his own. Not long after I, as a researcher, started watching his screen recording video, he typed "yes" to answer one problem that asked students to create mathematical representations. That was when I noticed that he was not comfortable speaking and thinking in English and might be from a recent immigrant family. There were constant deep sighs as if he was frustrated by something. It could have stemmed from the mathematics problems, communication in English, or the two combined.

During the whole class discussion, the teacher randomly called upon SJ and asked what his solution strategy was. SJ said, "I think he [another student] is right." The teacher again asked if his reasoning was any different to get the same answer, and SJ simply replied, "Same."

However, it was not true. On the screen recording, his digital workspace showed that he had a different, though correct answer, although he did not have his reasoning written. Soon after that moment, SJ turned his eyes to the laptop screen and started typing something in his personal workspace. It looked like a random array of alphabets that made no sense in English: "dkssudgktpdy wjsms [SJ]dlqslek dhsmfdms dkwn woaldlTsms gksrnrdjfmf tkdydgoqhfrjtdlqslek." However, these random letters do have meaning. When one types them in a Korean keyboard setting, it translates, "Hello, I am SJ. Today, I am going to speak in a very interesting and fun language, Korean."

This is one classroom episode that I randomly encountered while I was analyzing screen-recording video clips as a part of research on digital curriculum development.[1] Looking at SJ's private scribing, I had to stop playing the video to process what it would mean for him to learn mathematics in an American classroom. I could not help but wonder about how his engagement in problem-solving and discussion would be different if he was allowed to use his first language, "fun" Korean. In fact, the student sitting right next to SJ was also Korean. Rarely did they talk to each other, and they used English when they spoke.

Now, I bring back my first potluck experience to reflect on what could have been done differently. What if I was not introverted? What if I was not too shy to ask questions about cultural references? Can I change my personality? Do I need to? What if someone in the room asked me about my personal and professional experiences in two countries? Would they be even interested in my home country? What if there was another international student in the room? Can I assume they would feel the same? There was no single factor that made me uncomfortable

[1]The screen-recording video clips described in this introduction are obtained from data collected as part of work supported by the National Science Foundation under Grant No. DRL-1620934. Any opinions, findings, and conclusions or recommendations expressed in this dissertation are those of the author and do not necessarily reflect the views of the National Science Foundation.

participating in conversations, nor is there a single solution to improve it.

Similarly, I cannot simply recommend SJ develop confidence in speaking in English. I cannot simply hope for SJ to break out of his shell to actively participate in class discussions. I cannot simply expect the teacher to explicitly let him use his home language. (Should a teacher learn all the languages students use?) I cannot simply wait for the teacher to notice SJ and other students' struggles at some point. She asked him to share his solution strategy and reasoning. How could she know if SJ was not sharing his own ideas? More importantly and admittedly, I cannot argue with confidence that I would have noticed it if I was the teacher.

Recognizing the complexity of personal, relational, institutional, and societal influences on individual students' learning opportunities, which is perpetually marginalizing historically underrepresented populations within educational success, I am compelled to challenge the rhetoric of *mathematics for all* within a broader equity framework, as advocated by Martin (2003) and Pais (2012). At the same time, as a former mathematics teacher who always worked hard to help students enjoy learning mathematics, I am eager to understand what I could have done differently in order to *not* miss out on each and every student's learning opportunity. If I had a chance to reflect on the possibility of my students' missed learning opportunities, I believe that I would have put more effort into changing my teaching approach. In other words, I do not necessarily believe it is only outstanding and dedicated teachers (e.g., in Ladson-Billings' (2009) book, *The Dreamkeepers*) that can make such changes in their classrooms; instead, all mathematics teachers should be able to do this. As implicitly suggested in those expert teachers' practices, I am drawn to understand the roles of mathematics tasks in supporting teachers and students in pursuing equitable learning opportunities in their daily lessons.

Building on the foundation of theories for equitable teaching practices that call for

teachers' explicit attention to the cultural diversity that their students bring into mathematics classrooms (Civil, 2012; Gay, 2018; Ladson-Billings, 1995b; Moll et al., 1992; Turner et al., 2012), this dissertation study specifically examines how mathematics teachers utilize curriculum resources in order to provide students with *open spaces,* where students can bring their own life experiences and knowledge into their learning of mathematics. Moreover, this study seeks to understand under which context mathematics teachers can integrate open spaces into their daily mathematics lessons by focusing on dimensions and aspects of teacher agency (Priestley et al., 2015). The findings will provide empirical evidence of how teacher agency supports teachers' pedagogical design capacity. I hope that the findings of this study will help any mathematics teachers who strive to accommodate their students' meaningful engagement in learning mathematics to reflect on their own teaching contexts and leverage the resources available to them. Toward creating an equitable learning environment in schools, we all can take one step at a time starting from where we are.

CHAPTER 2: LITERATURE REVIEW

In the first section of this chapter, I draw from the work focusing on possibilities and challenges in achieving educational equity in association with the role of the curriculum. In the second section, I draw on work from teachers' use of curriculum resources in the pursuit of educational equity. Then, I present the research questions and the theoretical framework of this study.

Refining 'Math for All'

The Dominant vs. the Critical

In the introduction, I called attention to SJ's unseen struggle in order to claim that inequitable learning opportunities do not only fall into the achievement gap but also unintentionally arise in daily classroom activities (Flores, 2007; Ladson-Billings, 2009; White et al., 2012). Although the school reform toward '*mathematics for all'* has foregrounded the importance of equitable learning opportunities (NCTM, 2014), culturally nondominant students still deal with more layers of challenges in pursuing academic success compared to their counterpart students. It is due in part to the educational focus heavily on providing equitable access to rigorous mathematics without considering different students' cultural identities and epistemologies (Civil, 2012; Gutiérrez, 2007; Martin, 2003; Tate, 1995). Given that learning mathematics involves the creative exploration of ideas through interacting with others (Louie et al., 2021), it is critical for students to perceive themselves as agents in mathematical sense-making (Langer-Osuna & Nasir, 2016), which cannot be isolated from their cultural identities (Ladson-Billings, 2009). When the ultimate goal of mathematics education is to support students in using mathematics in their lives and developing critical citizenship (Boaler & Staples, 2008; Gutstein, 2003), mathematics education reform should take the reality and needs of culturally

nondominant students into account beyond providing equal access to the mathematics content (Gay, 2018; Tate, 1995).

Pointing out the ambiguity of its meaning, Gutiérrez (2007) explicated multifaceted aspects of educational equity: *access* (i.e., availability of resources to participate in learning mathematics), *achievement* (i.e., learning outcomes such as advanced course-taking or standardized test scores), *identity* (i.e., being able to become a better person in their own eyes), and *power* (i.e., opportunities to use mathematics as a tool to analyze society). All four dimensions are necessary and complementary to resolve the issues around inequitable learning opportunities, and any one of these dimensions can take precedence over the others depending on particular contexts (Gutiérrez, 2007). For true equity, however, Gutiérrez (2012) claimed that identity and power dimensions, which she referred to as the *critical axis*, should not be dismissed because the access and achievement dimensions (the *dominant axis*) can only serve to maintain the status quo.

Mathematics educators cannot merely expect those students, whose identity and power are systemically oppressed, to use mathematics as a tool to get higher education and solve problems in their lives (Freire, 2018) without providing support. Identity and power dimensions can have a wide range of influences on the spectrum of students' journeys from access to achievement. As long as the dimensions of identity and power are ignored, *mathematics for all* narratives will only perpetuate the marginalization of adequate support for underrepresented groups of students (Martin, 2015). In other words, being disconnected from their identities and lives, culturally nondominant students would only assimilate into the dominant culture in order to succeed in schooling. Thus, in the pursuit of *mathematics for all*, it is critical for mathematics educators to pay attention to underrepresented students' learning opportunities (Civil, 2012) that

develop their identities as human beings and mathematical power in their lives.

To interrogate and mitigate the tension between dominant and nondominant cultures at play within mathematics education, Civil (2012) identified three important elements for culturally nondominant students' learning opportunities: *the nature of the mathematics problem* (Whose mathematics problem is it?)*, the language(s) involved* (Whose language gets privileged?)*,* and *the valorization of knowledge* (Whose knowledge gets privileged?). When a mathematics problem involves real-life contexts, it can be interpreted and solved in different ways depending on students' cultural experiences and home languages. If a curriculum and a classroom environment only accept a single perspective to interpret and solve a contextualized mathematics problem, students who are not familiar with the culture will find school mathematics disconnected from their lives.

Tate (1994) provided one example of how African American students got the 'wrong' answer to a real-world related problem. The problem asks, "It costs $1.50 each way to ride the bus between home and work. A weekly pass is $16. Which is the better deal, paying the daily fare or buying the weekly pass?" (Tate, 1994, p. 480). It is assumed that people work five days per week and have only one job, which makes 'daily fare' the correct answer. However, many African American students in the study answered 'weekly pass' is better because people in their communities have several jobs not only on weekdays but also on weekends. Thus, in order for culturally nondominant students to have meaningful learning opportunities, they should be encouraged to interpret problems based on their experiences and utilize their home cultures. Also, their own ways of doing mathematics should be respected and regarded as one of many correct ways to solve a problem. Tate (1994) cautioned that the opportunity-to-learn framework could be a barrier to mathematics education reform if it would not take the realities and needs of

nondominant students into account.

Although scholars have been using slightly different terminologies—e.g., culturally appropriate (Au & Jordan, 1981), culturally relevant (Ladson-Billings, 1995b), culturally responsive (Gay, 2018), or culturally sustaining (Paris, 2012)—all of them highlight the criticality of both recognizing differences between home and school experiences and incorporating students' cultural backgrounds and experiences into classroom instruction. For example, Ladson-Billings (2009) conducted an ethnographic case study and documented how some expert teachers across various subjects supported their African American students to be successful in schools. With a caring commitment to providing African American students with opportunities for high-quality mathematics instruction, she believes that culture is closely interrelated to teaching and learning contexts, and thus, students' culture should be considered an important part of teachers' knowledge about students. Reflecting on her own education as an African American female student, Ladson-Billings (2009) attended to educational inequity issues and challenged a widely held perception that African American students lack the abilities to succeed in the school system. With cultural referents embedded into instructions, she witnessed that African American students could demonstrate a high level of academic achievement and self-efficacy. The expert teachers in her case study adapted school-mandated curricula in order to integrate cultural referents, which afforded accessible problem contexts for the students to build new knowledge. When using cultural referents, it is important to consider that they should not be deemed as a vehicle for "explaining the dominant culture; they are aspects of the curriculum in their own right" (Ladson-Billings, 2009, p. 20). In other words, curriculum content is to serve as a "mirror" for nondominant students to see their own cultures as they engage in learning mathematics rather than a "window" to see others' cultures (Style, 1996).

For *culturally relevant pedagogy*, Ladson-Billings (1995b) defined that it "not only addresses student achievement but also helps students to accept and affirm their cultural identity while developing critical perspectives that challenge inequities that schools (and other institutions) perpetuate" (p. 469). Furthermore, from her interviews with expert teachers and classroom observations, she offered three criteria for culturally relevant pedagogy: (a) an ability to develop students academically, (b) a willingness to nurture and support cultural competence, and (c) the development of sociopolitical or critical consciousness. She deliberately selected the term *pedagogy* because empowering students intellectually, emotionally, socially, and politically cannot be attained through simply employing some teaching strategies. Rather, it takes a holistic approach that necessitates not only beliefs and knowledge of teaching, as described by Shulman (1986), but also caring relationships with students and parents (Bartell, 2011).

While Ladson-Billings' (1995a, 1995b, 2009) work explicitly focused on African American students' learning opportunities, Gay (2018) pointed out that more studies on educational inequity needed to focus on other students of color. As immigrant populations in the United States increase, school classrooms have become heterogeneous in terms of students' ethnicity, race, and home language. Attending to such diversity within school systems, Gay (2002) defined *culturally responsive teaching* as "using the cultural characteristics, experiences, and perspectives of ethnically diverse students as conduits for teaching them more effectively" (p. 106). That is, culturally responsive teachers create a classroom environment where ethnically and linguistically diverse students can connect their own experiences and prior knowledge with mathematics learning. Also, Gay (2002, 2018) placed a great emphasis on the development of teaching practices, outlining the essential practices for culturally responsive teaching: developing a knowledge base about cultural diversity, including ethnic and cultural diversity content in the

curriculum, demonstrating caring and building learning communities, communicating with ethnically diverse students, and responding to ethnic diversity in the delivery of instruction.

Culturally relevant pedagogy proposed by Ladson-Billings (1995a, 1995b, 2009), culturally responsive teaching by Gay (2002, 2018), and other culturally relevant education (e.g., Aronson & Laughter, 2016) share the goal that all students, including those from non-dominant cultural backgrounds, can and should have opportunities to make a meaningful connection between learning mathematics and their lives. Research results have provided evidence that culturally underrepresented students can cultivate mathematical understanding in their own meaningful ways when they have opportunities to connect mathematics with their personal life (Civil & Khan, 2001); community issues (Gutstein, 2003; Lipka et al., 2005; Tate, 1995); or families' cultural knowledge and experiences (Andrews et al., 2005; Planas & Civil, 2009; Moll et al., 1992). In these studies, the valorization of students' cultural knowledge and everyday experiences encouraged students to develop interest and confidence in learning mathematics, which consequently advanced students' academic achievement and mathematical identities. Despite its promising influence on students' equitable learning experiences, however, transforming schools toward integrating culturally responsive teaching practice into daily lessons has been slow due to the political nature of education (Delpit, 1988; Gay, 2018; Martin, 2015; Sleeter, 2012). In part, it is because, as Gay (2018) pointed out, textbooks, as the most common source of curriculum content, give "too little attention to different groups of color; […]; to race, racism, and other forms of oppression; to conflict; and to experiences and interactions that are different from mainstream norms and standards" (p. 145).

Gay (2018) highlighted the critical role of curriculum content in ethnically diverse students' learning opportunities. She used the term *curriculum* in a broader sense to refer to all

resources used in classrooms, which includes *formal* plans for instruction (e.g., the school-adopted standards and/or textbooks), *symbolic curriculum* (e.g., multiple representations on classroom bulletin boards), and *societal curriculum* (e.g., information in the mass media). With the influence of symbolic and societal curriculum, culturally responsive teaching involves critical analyses of formal textbook contents and making them meaningful to students from diverse backgrounds. For example, in a curriculum analysis focusing on how mathematics textbooks represent different races, gender, and ethnicity, Piatek-Jimenez et al. (2014) revealed that three sets of widely adopted mathematics curriculum materials in the USA (one traditional, one reform-based, and one in between) did not sufficiently offer all students the opportunity to learn about different cultures. In their analysis, even though images in the textbooks reflected some diversity in race, white people appeared in mathematical careers 42 times, whereas black and Hispanic people combined were only seen three times in mathematics careers. This might hamper students' identity development in that the perceptions and stereotypes about minority ethnic groups could be perpetuated through the curriculum rather than being challenged and changed (Piatek-Jimenez et al., 2014). Also, curriculum content reflecting the dominant European American culture can impede multiple ways of being and knowing from flourishing in the classroom discourse (Tate, 1995). This influence has an additional impact that teachers and students might hold the incorrect perception that different ways of solving problems are “nice but [students] need to learn to do things the U.S. way” (Civil, 2016, p. 220). For this reason, Gay (2018) argued that a curriculum should not be a fixed set of content, but its transformation process is essential. In addition, since conceptions and forms of textbooks keep evolving to include digital resources (Pepin et al., 2017), symbolic and societal curriculum contents have become more accessible than ever. In such an environment, teachers would need to develop

skills to appropriate various resources flexibly and effectively to promote cultural relevancy.

Moreover, official curricula, such as Common Core State Standards Mathematics (CCSSM) (Common Core State Standards Initiative, 2010), reflect the dominant dimensions of equity. In a research commentary, Bartell et al. (2017) called for mathematics educators' attention to making explicit connections between the CCSSM and equitable teaching practices, asserting that the standards do not explicitly address issues about inequity and even reify the status quo. Situated within larger social and political contexts, the Mathematical Practices Standards in CCSSM (e.g., make sense of problems and persevere in solving them, reason abstractly and quantitatively, model with mathematics, etc.) do not consider the plurality of cultures and epistemologies; and thus, it implicitly sends a message that attending to nondominant students' learning opportunities is optional and extra effort as opposed to necessary work. By providing an analysis tool to connect the Mathematical Practices Standards in CCSSM with equitable teaching practices, Bartell et al. (2017) suggested researchers and teachers develop a critical awareness of the dominant power and impact of the standards on student learning opportunities.

In sum, for all students to learn and use mathematics, attending to equitable access and achievement is insufficient, and equitable opportunities to develop identity and power should also be considered. Although research has emphasized that culturally underrepresented students need to have learning opportunities that validate their cultural identities and combat the societal issues that impact their lives, culturally responsive teaching practices have not widely been adopted in schools partly because the curriculum and the standards implicitly and explicitly reside within the dominant culture. The urgency of attending to the critical dimensions of equity has been shared among educators to provide collegial awareness of the current issues pertinent to

culturally and linguistically nondominant students' learning opportunities. However, as Gutiérrez (2012) mentioned, it is not to claim that one is more important than the other; rather, harmonizing the two dimensions is of essential importance.

The Dominant and the Critical in Harmony

In her book, *The Dreamkeepers*, Ladson-Billings (2009) introduced expert teachers across subject matters who holistically helped African American students' successful schooling. One of the things that these teachers shared in common was that they created or adapted tasks to support underrepresented students' academic success, although the author did not analyze the direct influence of such task adaptations. Their expertise, as culturally relevant educators, may be interpreted as such work is difficult for the majority of teachers in their daily lessons. Limited, if not absent, suggestions in the curriculum for teachers to incorporate students' funds of knowledge in daily lessons might result in slow changes toward equity-oriented instructional practices.

Focusing on the critical axis of equitable learning (i.e., identity and power dimensions), some researchers have developed mathematics curriculum materials specifically contextualized in the local communities. Remillard et al. (2014), for instance, collaboratively developed a locally relevant curriculum by adapting *Mathematics in Context* (Romberg, 1997-8) to incorporate their local contexts, such as the correlation between attendance and graduation rates in local high schools. The students explored and discussed not only what the correlation between the two rates means but also why the neighborhood high school that most of the students will attend by default has lower attendance and graduation rates than 'special admission' schools. Remillard et al. (2014) argued that the curriculum could support students to understand the world around them using mathematics and further provide students with access to mathematical

achievement and critical thinking. Although this curriculum was developed using contexts within a specific region in the USA, the authors suggested that teachers in other areas accordingly modify it to reflect local contexts in which the teachers and students live.

For another example, *Math in a Cultural Context* (MCC) is supplemental curriculum material designed for Alaskan students to develop problem-solving competency drawing on their Yup'ik cultures and local contexts (Lipka et al., 2005). Tasks in this curriculum are not only contextualized within Yup'ik culture (e.g., building a fish rack to dry salmon) but also reflect cultural ways of problem-solving (e.g., eyeballing the space first before making adjustments or using rope to measure and indicate rectangular space) (Lipka et al., 2005). Collaborative endeavors among teachers, schools, and Yup'ik elders made it possible to include authentic mathematics problems relevant to the community. The MCC developers believe that a culturally based curriculum can open up a third space (Moje et al., 2004) in which teachers and students bring cultural experiences and prior knowledge into the classroom discussion (Lipka et al., 2005). Both Native Alaskan and non-Native students could benefit from learning through the MCC curriculum in terms of mathematical achievement (Kisker et al., 2012).

It is worth emphasizing that the two aforementioned curricula were developed on the foundation of student-centered, inquiry-oriented teaching approaches. The culturally based curriculum, MCC, underscores the importance of student participation in mathematics inquiry through open-ended questions (Kisker et al., 2012; Lipka et al., 2005). The authors deliberately developed the curriculum to elicit students' mathematical understanding based on their own cultural references. Similarly, a locally relevant curriculum (Remillard et al., 2014) was developed drawing on *Mathematics in Context* (Romberg, 1997-8) that foregrounded students' conceptual understanding. Thus, it is equally important for students to make sense of

mathematics by drawing on both their mathematical thinking (Carpenter et al., 1989) and their family and community's funds of knowledge (Planas & Civil, 2009; González et al., 2005; Leonard, 2008; Lipka et al., 2005).

Turner et al. (2012) encapsulated it as *children's multiple mathematical knowledge bases*, "the understandings and experiences that have the potential to shape and support children's mathematics learning—including children's mathematical thinking, and children's cultural, home, and community-based knowledge" (p. 68). The concept of children's multiple mathematical knowledge bases lends itself to harmonizing both the dominant and critical axes for true educational equity (Gutiérrez, 2007). It emphasizes the importance of teachers' understanding of students' mathematical thinking; and it also calls for teachers' more explicit understanding of students' *funds of knowledge*, "the diverse cultural and linguistic knowledge, skills, and experiences found in children's homes and communities" (Turner et al., 2012, p. 68). Thus, incorporating students' multiple mathematical knowledge bases into lessons means that in their teaching practices, teachers draw on students' mathematical thinking and their cultural, home, and community knowledge.

Despite its importance, attending to both the dominant and critical axes of equitable learning and teaching is not straightforward (Cahnmann & Remillard, 2002; Clark et al., 2013; Rubel, 2017; Turner & Drake, 2016). Cahnmann and Remillard (2002) and Clark et al. (2013) reported that teachers in their case studies, who were dedicated to supporting students' development of identity and power, did not emphasize mathematical understanding. While utilizing students' home language and cultural referents as learning resources, the teachers shifted lessons to direct instructions to demonstrate problem-solving procedures when they perceived tasks to be complicated (Cahnmann & Remillard, 2002), or they encouraged students

to memorize mathematical concepts and procedures so as to become successful in the society (Clark et al., 2013).

Conversely, Rubel's (2017) case study, applying Gutiérrez's (2007) distinction between the dominant and critical axes, illustrated that three mathematics teachers found it challenging to orient critical teaching practices while effectively using equity-directed practices within the dominant axis. For example, the teachers in Rubel's (2017) study successfully facilitated students' sense-making and reasoning through multiple mathematical representations and participation modes (individually, in pairs, in groups, and as a whole class). When they attempted to create or adapt tasks to make them relevant to their students, on the other hand, they could not draw on students' and communities' funds of knowledge. Rubel (2017) explained that it might be because teachers have little chance to learn about what culturally nondominant students would experience outside of school.

While Rubel's (2017) study revealed the challenges for teachers to attend to both dominant and critical aspects of equity, it also implied how curriculum materials could support teachers in developing equitable teaching practices. As they used standards-based curricula, the teachers enhanced both content and pedagogical knowledge that could help students develop conceptual understanding and mathematical reasoning. Even without professional development support, teachers can develop teaching practices that align with the vision of standards-based curricula as they interact with the curriculum materials' educative features for teacher learning (Collopy, 2003; Davis & Krajcik, 2005). In contrast, since published curriculum materials tend not to be as explicit about culturally responsive teaching practices as teaching for mathematical understanding (Gay, 2018), teachers have little chance to learn about, from the curriculum, what it means to connect mathematics with students' cultural experiences.

One group of researchers, the TEACH Math project (Drake et al., 2015a), has focused on the potential of curriculum materials as a starting point to make adaptations toward equitable teaching and learning. The authors developed teacher preparation program modules that support prospective teachers in analyzing and adapting curriculum materials to incorporate students' multiple mathematical knowledge bases into daily lessons. Utilizing a lesson analysis tool (Amador & Earnest, 2019; Aguirre & Zavala, 2013) and curriculum adaptation strategies (Drake et al., 2015b), prospective teachers could develop noticing skills to connect students' learning with prior knowledge and/or community experiences (Amador & Earnest, 2019; Harper et al., 2018). In addition, Land et al.'s (2019) curriculum analysis has provided evidence that there exist spaces within curriculum materials in which students can draw on their prior knowledge and experiences.

In sum, despite the importance of conscious integration and orchestration of students' multiple mathematical knowledge bases that encompass both mathematical understanding and cultural funds of knowledge, we as a field know little about how mathematics teachers can learn and develop critical awareness and related practices (Abdulahim & Orosco, 2020; Enyedy & Mukhopadhyay, 2007; Thomas & Berry III, 2019; Turner & Drake, 2016). Moreover, prior research has predominantly focused on elementary prospective teachers' learning about children's multiple mathematical knowledge bases compared to secondary in-service teachers' practices. Drawing on what we know from the prior research, it is important to expand our understanding of secondary mathematics teachers' perceptions and utilization of multiple mathematical knowledge bases given the situative nature of teacher learning (Borko & Putnam, 1996). From a situative perspective, teachers' interaction with the curriculum and teacher learning about teaching practice are influenced by the mathematics content and the teaching and

learning situation (Greeno, 2003); thus, we cannot generalize our understanding of teacher learning from one context (elementary level or preservice teachers' practices) to another (secondary level or in-service teachers' practices) (Peressini et al., 2004). As Drake et al. (2015b) claimed that making *small adjustments* to curriculum materials can have a huge impact on student learning, more empirical studies can provide teachers with explicit suggestions of how to use curriculum resources in a way to harmonize both dominant and critical dimensions of educational equity.

Re-Sourcing Curriculum Resources

In recent decades, curriculum studies have largely focused on understanding different ways in which teachers use curriculum materials. Such study results demonstrated that teachers' different orientations toward curriculum materials or prior teaching experiences impact their different implementations (Choppin, 2011; Remillard & Bryans, 2004; Remillard & Reinke, 2012; Sherin & Drake, 2009), teachers develop a certain curriculum orientation as they interact with the features of curriculum materials (Drake & Sherin, 2009; Roth McDuffie et al., 2018), and teachers' intended curriculum (or planned curriculum) is influenced by the perspectives of curriculum materials on learning mathematics (Choppin et al., 2021). While the results shed light on the complex influences on teachers' varying relationships with curriculum materials (Remillard, 2005; Stein et al., 2007), these studies are mostly describing contexts in which "instructional materials become part of the official curriculum" in the form of designated textbooks (Remillard & Heck, 2014, p. 710).

Although *instructional materials* are defined broadly as "resources that are designed to support or supplement instruction" (Remillard & Heck, 2014, p. 707), Remillard and Heck (2014) also indicate that "[t]extbooks and curriculum guides are the most common form of

instructional materials used throughout the world and continue to play a critical role in national education systems" (p. 707). Yet, little research has explored how mathematics teachers interact with other instructional materials than the official curriculum. This may imply that teachers' interaction with other resources is left to the teachers' choice and optional.

Given that drawing on students' multiple mathematical knowledge bases calls for an understanding of students' life experiences (Drake et al., 2015a), the scope of resources is not limited to material objects, such as textbooks or manipulatives, but also includes both human and cultural resources (Adler, 2000). In order to highlight the interactive relationships between teachers and resources, Adler (2000) called educators to "think about resource as the verb *re-source*, to source again or differently" (p. 207). That is, teachers, as human resources, play a critical role to supplement or replace curriculum materials with other resources, given the scarcity of attention to students' cultural and linguistic diversity within the official curriculum and published curriculum materials (Bartell et al., 2017; Gay, 2018; Land et al., 2019).

Furthermore, Remillard (2019) emphasized that sequencing and mapping student learning over time is the core activity in curriculum design because it is "curricular in nature" (p. 179). She cautioned that many online resources available to teachers in this digitalized society tend to be presented as discrete activities without coherent connections (Remillard, 2016). By accounting for both the importance of teachers' re-sourcing and the curricular nature, she defined *curriculum resources* as "print or digital artifacts designed to support a program of instruction and student learning over time" that consists of curriculum materials (both student textbooks and teacher's guides) and teachers' documents of lesson design (Remillard, 2019, p. 179). Teachers' documentation processes to design lessons (Gueudet & Trouche, 2009) highlight how teachers act as agents to interact with resources beyond merely implementing a provided curriculum set.

Thus, the conception of curriculum resources expands the meaning of instructional materials (Remillard & Heck, 2014) to include teachers' re-sourcing activities.

Acknowledging that not all teachers can have curriculum materials that are specifically relevant to their local community, some scholars have provided strategies to create or adapt mathematics tasks to foreground students' diverse cultural and linguistic backgrounds. For example, Matthews and colleagues (2022) defined *culturally relevant mathematics tasks* as "tasks (1) with high cognitive *demand*, (2) where culture and community are the source of mathematics inquiry (*relevance*), and (3) where individual and collective *agency* is the intentional outcomes" (p. 31). Consistent with the claim that students need learning opportunities to build mathematical understanding based on their cultural experiences, these authors argued that culturally relevant mathematics tasks can engage students in "a space where learners thrive and find voice and meaning in the mathematics that they do" (p. 20).

Chval and colleagues (2021) focused on multilingual students' learning of mathematics and highlighted the importance of maintaining mathematical rigor when creating or adapting curriculum materials to draw on students' community and cultural knowledge. It is because "[i]t is not enough to simply insert a real-life context into a problem for the context may not be meaningful or familiar to students" (p. 63). These authors further suggested strategies for teachers to consider as they create or adapt mathematics tasks drawing on multilingual students' cultural experiences, including reflection questions, "What were the meaningful and attractive elements of this context?" and "What was the potential of this context for creating mathematics tasks/problems?" (p. 67).

While culturally relevant mathematics tasks can open a space for students to draw on their multiple mathematical knowledge bases, creating and adapting such tasks require teachers

to continuously learn and develop their re-sourcing capacity. Turner and colleagues (2012) conjectured teacher learning trajectory to illustrate how prospective teachers develop their practices to provide students with learning opportunities drawing on their multiple mathematical knowledge bases. Considering that teachers develop teaching practices over time as they participate in various communities of practice (Wenger, 1998), Turner et al. (2012) viewed prospective teachers' learning about integrating students' multiple mathematical knowledge bases in practice progresses across three phases: *Initial Practices*, *Making Connections*, and *Incorporating*. During the *Initial Practices* phase, teachers *attend* to, become *aware* of, and *elicit* students' multiple mathematical knowledge bases. In their case study, prospective teachers tended to *attend* to either students' mathematical thinking or their funds of knowledge. Some teachers expressed their *awareness* of the scarcity of culturally relevant tasks provided in standard curricula. Also, they applied questioning strategies to *elicit* students' mathematical thinking and interacted with students and their families to elicit their cultural, home, and community-based knowledge.

Moving to the *Making Connections* phase, teachers "consider how to make connections among those knowledge bases in instruction" (Turner et al., 2012, p. 76). Teachers are likely to begin by making superficial connections with either mathematical thinking or culture, home, or community knowledge (*emergent connections*). Examples include a case where teachers utilized cultural referents or community settings, and yet they did not consider the embedded mathematics in the problem context. As they develop an understanding of students' multiple mathematical knowledge bases, teachers can *make meaningful connections* by "designing mathematically rich problem-solving experiences that incorporated knowledge of children's mathematical thinking and knowledge about children's use of mathematical ideas in their homes

and communities" (p. 77). Then, teachers continue working on the *purposeful incorporation of multiple mathematical knowledge bases* (the *Incorporating* phase).

Aguirre et al.'s (2013) study provided examples of prospective teachers' practices to incorporate students' multiple mathematical knowledge bases in practices. As a part of the course assignment, the prospective teachers in their study chose a neighborhood market as a task context and interviewed the owner in order to understand what mathematics is used at the market. Not only did the prospective teachers make the task context familiar to students' out-of-school experiences by presenting photos of a local store and prices for items, but they also made the task more authentic by considering how local people use mathematics in their work. Based on this connection to students' funds of knowledge, the prospective teachers encouraged students to generate multiple strategies and supported students' reasoning. They particularly focused on providing students with multiple entry points while maintaining tasks' cognitive demand levels.

On the one hand, this example demonstrates the possibility for teachers to attend to students' multiple mathematical knowledge bases through the teacher learning opportunity. On the other hand, it also provides evidence that it can be hard for teachers to make such adaptations; among all tasks analyzed in their study, only one-fifth of them made *meaningful connections* to both mathematical thinking and funds of knowledge, while many prospective teachers made *emergent connections* (Aguirre et al., 2013). As Turner and Drake (2016) revealed in their systematic review, only a small number of research studies have investigated how teachers connect teaching and learning mathematics with their students' multiple mathematical knowledge bases (e.g., Brenner, 1998; Gutstein et al., 1997; Tate, 1995; Turner et al., 2009).

The challenges for teachers to plan mathematics lessons connecting with students' multiple mathematical knowledge bases might stem from the lack of teacher learning

opportunities. In order to navigate resources from students' and communities' cultural referents and appropriate them with learning goals, teachers need to critically analyze the affordances and constraints of curriculum materials at hand and further make adaptations to connect them with other resources. Research on teachers' curriculum adaptations suggests that teacher learning opportunity is critical in such endeavors. For example, in Wyatt (2014)'s case study, participating teachers had typically used scripted curriculum programs, which "speak *through* teachers" by guiding their actions (Remillard, 2000, p. 347). During the professional development program provided by the researcher, the teachers were introduced to the principles of flexibly using curriculum materials for diverse student populations—the CREDE model (Center for Research on Education, Diversity, and Excellence Hawai'i Project, Wyatt, 2014). Without having specific steps to follow, teachers were encouraged to utilize the principles as they made adjustments to connect mathematics with their students. Wyatt (2014) argued that when provided such learning opportunities, teachers are able to be attuned to adapting a scripted curriculum to connect with students' out-of-school experiences.

In addition, even if teachers' official curriculum is designed for a particular group of students to utilize their cultural referents, teachers still need learning opportunities to appropriate the curriculum for their own students (Remillard et al., 2014). For example, despite its specificity toward the Yup'ik culture, *Math in a Cultural Context* (MCC) strives to support teachers in implementing the curriculum adaptively, encouraging teachers to "make accommodations based on her or his mathematical knowledge, pedagogical knowledge, and familiarity with the local culture, language, and history" (Lipka et al., 2005, p. 368). Although the teachers gained, from the MCC curriculum materials, both content and pedagogical knowledge to facilitate student inquiry, it does not mean they used the curriculum with implementational fidelity. Applying

Taylor's (2016) distinction, MCC offers guidance for teachers to develop *curriculum-proof* capacity rather than position teachers to implement a *teacher-proof* curriculum in a vacuum.

Given that curriculum materials have a strong influence on individual teachers' planning and teaching (Brown & Edelson, 2003; Gay, 2018), the ways that they were written (i.e., textbook voice) can also impact how teachers make adaptations for the purpose of incorporating students' multiple mathematical knowledge bases. When textbooks position students as performers of activities rather than thinkers by repeatedly using (exclusive) imperatives in the problems, students and teachers would be less likely to play agentic roles in their teaching and learning (Herbel-Eisenmann, 2004). Considering that teachers also learn about teaching as they use curriculum materials over time (Sherin & Drake, 2009), such textbook positioning would hinder teachers from allowing students to draw on their mathematical thinking. Moreover, if a textbook uses verbs and modalities with uncertainty when making connections with real-life context while using them with certainty for mathematical understanding, it might send a message that connections with mathematical thinking would be more important than real-life connections (Herbel-Eisenmann & Wagner, 2007). Analyzing and adapting curriculum materials to incorporate students' multiple mathematical knowledge bases require teachers to develop a critical awareness of ways the textbooks perpetuate the hegemony of the dominant culture.

There is also a tension between teaching mathematics for understanding (the dominant axis) and for social justice (the critical axis). When planning lessons, in Bartell's (2013) case study, no matter how much they understood the meaning of teaching mathematics for social justice, teachers found it overwhelming to decide to what extent they could and should attend to the discussion for social justice while being obligated to implement a required curriculum. It means that even if teachers develop awareness and capacity to incorporate students' multiple

mathematical knowledge bases, "work on issues of culturally sustaining pedagogy and social justice has been similarly constrained, focusing most often on teachers' beliefs, socio-political commitments, and curricular choices without addressing the specific moves which teachers can and should enact in the classroom" (Richmond et al., 2017, p. 432).

In sum, no matter whether teachers use culturally relevant or scripted curricula, it is important and yet challenging to make adaptations to connect mathematics with students' prior knowledge and cultural experiences. Rather than being left alone to take the authority to re-source curriculum resources, teachers should have opportunities to develop perspectives and practices pertaining to culturally nondominant students' learning opportunities. The rest of this chapter describes three possible strategies that teachers can utilize as they re-source curriculum resources for integrating students' multiple mathematical knowledge bases into designing their daily lessons: productive adaptations, curricular noticing, and opening curriculum spaces. In addition, one important yet implicitly assumed concept that underlies teachers' curriculum adaptations, *teacher agency,* is discussed.

Productive Adaptations

Published curriculum materials, including culturally based curricula, are written in a way to support student learning in generic classrooms (Drake et al., 2015b; Goldsmith et al., 2014). This means that curriculum materials are subject to be modified by teachers according to a particular teaching and learning context. For that, as Debarger et al. (2013, p. 299) pointed out, teachers need to have more explicit support for curriculum adaptation:

> In order for curriculum materials to promote student learning and interest in a domain, both the design of curriculum materials and their enactment by teachers need to be responsive to the peculiarities of the implementation contexts,

> including teachers' and students' cultural and linguistic backgrounds. The responsibility of implementing curriculum materials in a responsive way often falls solely to teachers, who receive little support for this kind of adaptation.

Even though they do not specifically draw from critical theories for mathematics education, Debarger et al. (2013) advocated for the necessity of curriculum adaptation in order to provide all students with relevant learning experiences. It requires teachers to have "flexibility of mathematical understanding, particularly of mathematical structure and practice" and "the ability to recognize as mathematical a range of specific activities, reasoning processes, and ways of representing" (Hoover et al., 2016, p. 26). Considering that "few studies focused directly on specific practices of equitable mathematics teaching or knowledge for equitable mathematics teaching" (Hoover et al., 2016, p. 27), criteria for *productive adaptations* (Debarger et al., 2013) can shed light on the support mechanism for teachers to draw on students' multiple mathematical knowledge bases when they plan lessons.

The term *productive adaptations* are used to describe "curriculum adaptations in which teachers effectively respond to the dilemma of being faithful to the intentions of curriculum developers and policymakers while also being responsive to particular students and to the characteristics and resources found in local contexts" (Debarger et al., 2013, p. 300-301). Since it is challenging for teachers to navigate spaces that can accommodate both needs in a particular classroom context and the instructional vision of the curriculum, explicit criteria can guide them to include curriculum adaptations into their daily teaching practices so as to be responsive to students and learning contexts. First, productive adaptations consider multiple stakeholders, such as students, parents, and their communities, while maintaining the content and approach articulated in curricula (*students' needs*). Also, productive adaptations allow students to make

cultural and discursive connections between classrooms and home communities (*discursive connection*). Lastly, productive adaptations maintain or enhance the task complexity so that students can engage in cognitively demanding practices (*task complexity*).

In mathematics education, Choppin's (2008, as cited in Debarger et al., 2013) case study illustrated how teachers engaged in productive adaptation practices as they used a standard-based middle school curriculum over time. The teachers in his study adapted the mathematics curriculum to add a whole-class discussion in order to address emerging conjectures about integers when they noticed the curriculum materials did not provide such opportunities. Debarger et al. (2013) explained this adaptation as an example of discursive connection. Also, when the teachers noticed a grouping strategy for subtraction as described in the curriculum was not as useful for their students as it was for addition problems, they adapted to asking students to create their own grouping categories (task complexity). In addition, they adapted to include more deliberate activities to develop algorithms when they noticed their students exhibited incoherent and inaccurate algorithms (students' needs).

Choppin's (2008) study highlighted the possibility that teachers can purposefully observe student thinking and accordingly adapt the curriculum, which was supported by the engagement in the co-design process where teachers and curriculum developers communicated with one another about the adaptations. Even though Choppin's (2008) study did not specifically focus on underrepresented students' learning needs, criteria for students' needs and discursive connections call for attention to students' funds of knowledge, which can provide those students with meaningful learning experiences. It is important to note that, in Choppin's (2008) study, teachers made adaptations based on what they noticed regarding student learning. It means that productive adaptations implicitly require teachers to notice opportunities for student learning

within curriculum resources. Without noticing what opportunities exist and what is missing, teachers would be less likely to make changes in task and lesson designs. Thus, curricular noticing skills can serve as a strategy for re-sourcing curriculum resources.

Curricular Noticing

The concept of teachers' *professional noticing* has been used to describe how teachers re-source student thinking during lessons. It involves not only teachers' interpretations of student discourse but also their decisions about how to respond to them (Jacob et al, 2010; Sherin & van Es, 2009). In order for teachers to focus on student thinking in the classroom, Jacob et al. (2010) underscored the importance of teachers' task selection during lesson planning. If selected tasks engage students in memorizing procedures, it is less likely for teachers to elicit students' conceptual understanding than to assess the accuracy to follow provided procedures. That is, teachers' professional noticing during a lesson can possibly be influenced by the ways teachers make sense of curriculum materials prior to the lesson. This notion of teachers' noticing student learning opportunities within curriculum materials is referred to as *curricular noticing* (Amador et al., 2017; Dietiker et al., 2018).

Curricular noticing means "the set of skills that constitute curricular work of mathematics teaching", and these skills include three interrelated phases: *curricular attending* (the skills to view aspects of curriculum materials that include reading, looking, and searching for curriculum materials), *curricular interpreting* (the skills to make sense of what teachers have attended to), and *curricular responding* (the skills to decide how to respond to their interpretation of curriculum materials) (Dietiker et al., 2018, p. 524). Research has revealed that what teachers notice within curriculum materials can be different from one another influenced by personal experiences, preferences, or knowledge (Amador & Earnest, 2019); curriculum features and

formats can influence teachers' curricular noticing (Males & Setniker, 2019; Roth McDuffie et al., 2018); and teachers can develop curricular noticing skills when they utilize a particular tool that informs what to notice in the curriculum materials (Amador & Earnest, 2019).

In Amador and Earnest's (2019) case study, prospective mathematics teachers analyzed and adapted curriculum materials using the Curriculum Spaces Analysis Tool. This tool was developed specifically to support teachers' curricular noticing of students' multiple mathematical knowledge bases. As teachers considered the analysis prompts in the tool, such as "Where are the opportunities for activating or connecting to family/cultural/community knowledge?", "Do students have opportunities to make their own [real-world] connections?", "Where are the opportunities for students to make sense of the mathematics and develop/use their own solution strategies and approaches?" (p. 129), they could practice what to attend to, how to interpret, and how to adapt in order to draw on students' multiple mathematical knowledge bases. That is, the Curriculum Spaces Analysis Tool can serve as a teacher-learning mechanism because "in order for teachers to draw upon community resources, they must first treat them as integral to the practice of teaching mathematics and must notice opportunities in the curriculum for said connections as they interact with materials" (Amador & Earnest, 2019, p. 127-128).

With a particular emphasis on students' multiple mathematical knowledge bases, curricular noticing skills can potentially be used to *open* curriculum spaces. While a curriculum may or may not include open spaces for student learning, it is always open for teacher learning as long as the teacher can notice student learning opportunities within the curriculum. In other words, if the teacher has the tools at their disposal to analyze and adapt curriculum materials, the opportunity to create an open space for students to bring their multiple mathematical knowledge bases can always be available.

Opening Curriculum Spaces

Land et al. (2019) viewed curriculum materials as a place for teachers to develop equitable teaching practices. Drawing from the concept of third space (Moje et al., 2004), where students can bring multiple funds of knowledge to connect with mathematical content that disproportionately reflects the dominant culture, the authors examined the third space within curriculum materials, called *curriculum space*. More specifically, the authors argued that curriculum materials can offer students opportunities to *make connections to the real world*, *develop multiple strategies*, and *explain their own strategies*. Based on the analysis of existing curriculum sets, this study highlighted both affordances of curriculum materials that can leverage student understanding by offering *open space* and constraints that can limit learning opportunities by including *closed space* or *conflicting space*.

For example, when curriculum materials pose a real-world related problem, it can *open* a space for students to bring their own relevant experiences to understand the problem context, investigate multiple strategies, or justify their solutions in different ways. Conversely, curriculum materials can *close* the space by jumping straight into problem-solving without engaging students in interpreting problem contexts based on personal experiences, by introducing only one particular strategy (usually preferred by people from the dominant culture), or by asking students to explain the strategy given in the curriculum in their own words. In addition, curriculum materials can provide a *conflicting space* where students are encouraged to come up with their own strategies (*open*), immediately followed by introductions to a particular strategy (*closed*). Although students might be able to consider their prior knowledge or experiences within a conflicting space, the authors claimed that a conflicting space is likely to turn into a closed space during instruction.

Acknowledging that the enactment of curriculum materials can be different from the curriculum designers' vision, influenced by various ways teachers interact with them, Land et al. (2019) also argued that curriculum materials could support teachers in learning about how to value and utilize students' multiple mathematical knowledge bases. The authors further suggested strategies to open curriculum spaces even if curriculum materials seem to be closed or conflicting: frontload problem-solving, cut components, adjust numbers or offer choices, encourage multiple representations and strategies, and make authentic connections (Drake et al., 2015b). These strategies are practical in that, whichever curriculum materials they use, teachers can facilitate all students' sense-making of mathematical ideas by making "small adjustments" (p. 348).

As revealed in Harper et al.'s (2018) study, however, it is challenging yet not impossible for teachers to make modifications to curriculum materials that consider students' multiple mathematical knowledge bases. In particular, the teachers in Harper et al.'s (2018) study made fewer connections with students' and families' funds of knowledge than to multiple strategies and reasoning opportunities. This result resonates with others' assertion that it is not straightforward for teachers to attend to both the dominant and critical axes of equitable teaching and learning (Gutiérrez, 2007; Rubel, 2017). Given the potential of the lesson analysis tools (Aguirre & Zavala, 2013) and the practical strategies (Drake et al., 2015b) for teacher learning opportunities, more empirical studies can expand our understanding of how mathematics teachers make use of these tools under particular conditions of their teaching contexts. Furthermore, based on the promising results from the studies that focused on prospective teachers' learning about opening curriculum spaces, more studies are needed to support practicing teachers in learning about how to utilize multiple mathematical knowledge bases as

assets for student learning.

Put together, curriculum studies in mathematics education have evidenced the possibility that teachers can adapt curriculum resources to provide all students with relevant learning experiences. Even though integrating students' funds of knowledge is more challenging than focusing on students' mathematical understanding, studies have shown that teachers can gradually develop the capacity to design a lesson that meaningfully connects with students' multiple mathematical knowledge bases. What is not explicit in the literature on teachers' curriculum adaptations is the underlying assumption about teachers' sense of agency. Prior research implicitly suggests that teachers' sense of agency would play an important role in curriculum adaptation, and yet little is known about when and how teachers can perceive themselves as agents of curriculum adaptation. The following section summarizes research findings on teacher agency with regard to the interaction with curriculum resources.

Teacher Agency in the Curriculum Use

In the literature on curriculum adaptations, teachers are viewed as active participants in interacting with curriculum resources rather than implementers with fidelity to scripted curricula (Brown, 2009; Remillard, 2005; Taylor, 2016). In their participatory relationships with curriculum resources (Remillard, 2005), teachers make decisions on ways in which they utilize resources for the purpose of achieving instructional goals (Brown, 2009). While features in a curriculum can influence teachers' decision-making (Collopy, 2003; Davis & Krajcik, 2005; Roth McDuffie et al., 2018), different teachers can perceive affordances and constraints of the curriculum in different ways depending on their capacity to mobilize resources (Choppin, 2011). This individual teacher's capacity is referred to as *pedagogical design capacity* (Brown, 2009), and it provides a lens to understand how teachers manipulate resources in order to accomplish

their instructional goals. Participatory relationships with curriculum resources call for teachers' perceptions of themselves as lesson designers, which implies that teachers' sense of agency in their curriculum use determines their enactment of pedagogical design capacity (Priestley et al., 2015).

When defining teachers' sense of agency, scholars consider not only individuals' personal aspects (e.g., beliefs, perceptions, experiences, etc.) but also the relational aspects that involve particular situations in which the teachers work. For example, Pietarinen et al. (2016) described the professional agency as a teacher's capacity to both "prepare the way for the intentional and responsible management of new learning, both at the personal level and in the school community" and "construct the context for the learning" (p. 114). Focusing on this relational nature of professional agency within a professional learning community, Pietarinen et al. (2016) revealed the correlation between teacher agency in a professional learning community and teacher agency in the classroom. They argued that those teachers, who were engaged in agentic activities within a professional learning community, such as seeking or giving feedback, tended to be active in reflecting on their teaching practices to better support student learning in their lessons. Their findings highlight that teachers' professional agency can change depending on the context and further suggest that the professional learning community can catalyze teacher agency in their learning about teaching.

Similarly, Priestley et al. (2015) emphasized the ecological nature of teacher agency which is not individual teachers' ability but transactional engagement within particular contexts. By viewing teacher agency as something teachers can achieve by interacting with the context they are in, this perspective of agency allows researchers to investigate factors that influence teachers to enact agency rather than focusing on assessing individuals' fixed ability of teaching.

For example, Kneen et al. (2023) investigated teachers' agentic engagement in the district's curriculum development work, drawing on Priestley et al.'s (2015) three dimensions of teacher agency (i.e., the *iterational*, *projective*, and *practical-evaluative* dimensions). Kneen et al. (2023) considered three levels of contexts: *Micro-level* (individual level: the *iterational* dimension), *Macro-level* (national and potentially lasting significance level: the *projective* dimension), and *Meso-level* (local, institutional level: the *practical-evaluative* dimension) as they explained how teachers deployed their sense of agency in a curriculum development project. They highlighted that "achieving agency at meso-level was less clear-cut" (p. 261) as it involved all three dimensions, particularly influenced by structural and cultural factors in schools. They argued that the school management (structural factor) and the responses of teacher peers (cultural factor) played a critical role in these expert teachers' agentic engagement in curriculum development. Their findings suggest that teachers can perceive themselves as agents of changes in their curriculum use when they are in an institutional context supportive of such changes.

Attending to multiple factors influencing teachers' perceptions of their agency, Jenkins (2019) particularly examined different agency modes that secondary school home economics teachers demonstrated as they were given a new official curriculum. *The proactive agency* can motivate teachers to change the curriculum to accommodate their school and students' specific needs, which can help teachers develop confidence in their planning and teaching. When teachers perceive that they are forced to change their teaching based on the new curriculum, they could enact *reactive agency* to minimize negative impacts entangled with the changes. Teachers could demonstrate *passive agency* to decide to make only minimal or no changes. Acknowledging that these agency modes were not static or consistent over time and context, Jenkins (2019) highlighted the importance of the contextual factor—teachers' relationships with

their school leaders—in teachers' enactment of agency toward the effective use of the effective curriculum.

Given the impact of instructional goals on teachers' pedagogical design capacity (Amador, 2016; Matic, 2019), it is important to understand how explicit instructional goals can influence teachers' sense of agency. Felton and Koestler (2015) investigated how elementary mathematics teachers in a teacher education program developed their perceptions of teaching throughout one semester. The course was organized to enhance the prospective teachers' understanding of both student-centered teaching and social analysis teaching in mathematics class. As they were engaged in the course activities and reflections deliberately designed by the instructor, the prospective teachers shifted their views on teaching from teacher-centered to student-centered. Moreover, the authors argued that the prospective teachers developed an understanding of teachers' agentic roles in connecting school mathematics to the real world and the socio-political world so that they can incorporate them in their future lessons if they want. The authors highlighted the critical role of reflections as a consistent course assignment in the teachers' development of the agency.

Taken together, in the literature on teacher agency focusing on curriculum use, teachers' explicit learning opportunities and support from the institution and colleagues play important roles in the achievement of teacher agency. For such teacher learning opportunities, Rycroft-Smith and Macey (2021) suggested that evidence-based professional development considers the important role of teacher agency in their learning. Despite its critical role in teachers' pedagogical design capacity (Priestley et al., 2015) and teacher learning (Pietarinen et al., 2016), little is known about how secondary mathematics teachers enact their agency as they design lessons to accommodate culturally and linguistically diverse students' interests and needs. Given

that one teacher's pedagogical design capacity can be exhibited differently depending on contexts (Leshota & Adler, 2018), and prior research has largely focused on elementary-level prospective teachers' learning about children's mathematical knowledge bases, more empirical studies can help the field understand contextual factors that contribute to in-service teachers' agentic engagement while they navigate curriculum resources and negotiate challenges within their particular school system.

Research Questions

The purpose of this study is to investigate how secondary mathematics teachers interact with curriculum resources to draw on students' multiple mathematical knowledge bases. For this purpose, this study is guided by the following research question: How do mathematics teachers open curriculum spaces in order to incorporate students' multiple mathematical knowledge bases into daily lessons? Specifically, two sub-questions will be answered:

1. In what ways do mathematics teachers use curriculum resources in their daily lessons drawing on students' multiple mathematical knowledge bases?
2. How does teacher agency, when supported in a conducive context, allow for teachers' capacity to open curriculum spaces?

Theoretical Framework

This study draws upon the conceptions of both *pedagogical design capacity* (Brown, 2009) and *teacher agency* (Priestley et al., 2015) in order to elaborate on mathematics teachers' capacity to *open curriculum spaces* for student learning opportunities. I contend that pedagogical design capacity plays a critical role in opening curriculum spaces in that it explains how teachers make decisions about using curriculum resources when they have an explicit instructional goal—providing students with open spaces for multiple mathematical knowledge bases. Moreover,

since teachers' pedagogical design capacity cannot be enacted without their sense of agency (Priestley et al., 2015), attending to dimensions of teacher agency will explain how teacher agency influences teachers' mobilization of curriculum resources.

Pedagogical Design Capacity

I take a perspective on teaching as a design activity by which teachers "must perceive and interpret existing resources, evaluate the constraints of the classroom setting, balance tradeoffs, and devise strategies—all in the pursuit of their instructional goals" (Brown, 2009, p. 18). As a designer of instruction, a teacher is to be capable of "perceiving the affordances and constraints of curriculum resources and making decisions on how to use them in order to craft instructional episodes that achieve [their] goals" (p. 29), which Brown (2009) referred to as *pedagogical design capacity*. I argue that teachers' pedagogical design capacity plays a critical role in opening curriculum spaces. When curriculum materials contain mixed spaces—open, closed, and conflicting— teachers as lesson designers will open curriculum spaces by including opportunities for students to make connections to their life experiences, develop their own strategies, and explain their own strategies. Such adaptations are likely to vary depending on teachers' pedagogical design capacity because teachers perceive and mobilize curriculum resources in different ways even if they have similar knowledge, skills, and commitments. Also, the extent to which teachers exhibit pedagogical design capacity is influenced by individual teachers' beliefs, personal and professional experiences, and schools' structural environment. By drawing on the concept of pedagogical design capacity and the design process, this study focuses on illustrating how teachers adapt curriculum resources focusing on providing students with open spaces for multiple mathematical knowledge bases.

Brown (2009) specified three types of curriculum use—*offloading, adapting,* and

improvising—and claimed that teachers make decisions as to what extent they need to offload, adapt, or improvise parts of curriculum resources based on their instructional goals. In the design process, teachers utilize curriculum resources as design artifacts: *selecting* resources, *interpreting* them, *reconciling* the intended goals with their own goals, *accommodating* their students' experiences, interests, and limitations, and *modifying* them. This relationship between a teacher and curriculum resources is dynamic and participatory (Remillard, 2005); while curriculum resources provide affordances for instructions, the teacher mobilizes the curriculum resources based on their perceptions and decisions (Brown, 2009).

Brown (2009) illustrated teachers who possess "high" pedagogical design capacities are "able to deconstruct curriculum materials, recognize their essential elements, and reconstruct them in order to suit [students'] needs" (p. 31). While it accounts for teachers' complex decision-making processes about how they use curriculum resources in order to reach instructional goals, pedagogical design capacity should not be attributed to individual teachers' fixed abilities. As opposed to assessing the degrees to which teachers possess such capacities, it is important to understand how the pedagogical design capacity is flexibly mobilized, within particular situations, in order for teachers to not only create instructional artifacts but also develop professional expertise (Pepin et al., 2019). Through collaborative design work among teachers, Pepin et al. (2019) claimed that teachers can develop professional practices and further view themselves as curriculum designers. Thus, this study strives to avoid using the language of "high" pedagogical design capacity as in Brown's (2009) description and focus on illustrating the relationships between teachers' instructional decisions and their situations (Pepin et al., 2019; Rezat et al., 2018).

Teacher Agency

Opening curriculum spaces can become viable when "efforts toward justice and equity are not seen as curricular add-ons or as separate from practice, but instead, as the lens through which teachers engage in core daily practices" (Bartell et al., 2019, p. 303). Although teachers' perceptions about daily teaching practices involve their sense of agency, especially within collaborative teacher networks (Bartell et al., 2019), teacher agency is implicitly assumed within the pedagogical design capacity framework. When teachers open a curriculum space for their own students, they need to take account of students' knowledge and experience, learning contexts, community cultures, and social issues. This particular consideration in designing a lesson calls for a teacher's role to be a researcher rather than a transmitter of well-organized, scripted content in the curriculum. Stenhouse (1975) underscored the importance of teachers' critical awareness of their roles as researchers by arguing that there is no curriculum development without teacher development. Stenhouse (1975) suggested moving away from the *objectives model* to use curriculum toward the *process model* of curriculum design. From the perspective of the process model, a curriculum is not a mere set of objectives that can be used to evaluate students' learning outcomes at the end of each lesson. Instead, curriculum design is a process of deepening the understanding of teaching accommodated by collaborative learning among teachers and students. Teachers as researchers are to inquire about their classrooms and schools to challenge and change emerging issues therein, which calls for their sense of agency.

To conceptualize teacher agency, Priestley et al. (2015) adopted Emirbayer and Mische's (1998) definition of human agency, "the temporally constructed engagement by actors of different structural environments—the temporal-relational contexts of action—which, through the interplay of habit, imagination, and judgment, both reproduces and transforms those

structures in interactive response to the problems posed by changing historical situations" (p. 970). This highlights the temporal and relational nature of agency that should not be understood "as an individual's capacity, but should always be understood in transactional terms, that is, as a quality of the engagement of actors with temporal-relational contexts of action" (Biesta & Tedder, 2006, p. 18). As Priestly et al. (2015) argue that teacher agency is "[t]he main distinctive factor" in curriculum design that involves "intentionality, the capacity to formulate possibilities for action, active consideration of such possibilities and the exercise of choice" (p. 23), attending to teacher agency will provide insights into the understanding of how teachers navigate possibilities to make changes to their curriculum resources.

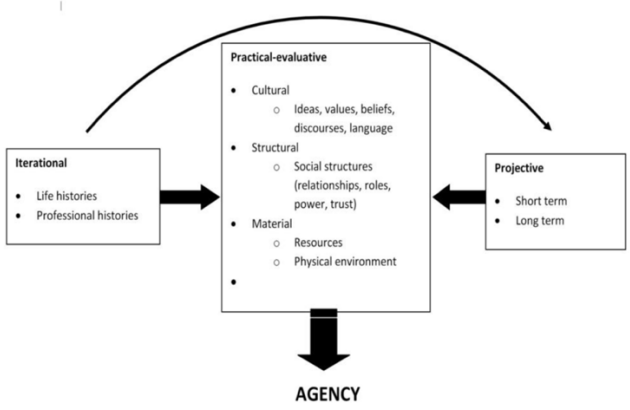
By drawing on Priestley et al.'s (2015) three elements of teacher agency (Figure 1), this study illustrates how teacher agency influences pedagogical design capacity in the process of designing lessons to draw on students' multiple mathematical knowledge bases. Priestley et al. (2015) argued that teachers achieve agency by reflecting on their past experiences (the *iterational* dimension) and by exploring possibilities for future trajectories (the *projective* dimension) in order to make decisions in the present teaching and learning contexts (the *practical-evaluative* dimension). More specifically, the *iterational dimension of agency* can involve how and why mathematics teachers selectively attend to curriculum spaces in association with their prior experiences. Priestley et al. (2015) highlighted that teachers' personal skills and knowledge, beliefs, and values rooted in their past experiences serve as resources for current and future actions. Also, the *projective dimension* of teacher agency is associated with teachers' broader instructional goals as they design lessons to open curriculum spaces, which is a prerequisite for deploying pedagogical design capacity. When it comes to teachers' curriculum use, the expansive trajectories can include teachers' curriculum vision—i.e., understanding

where the curriculum goes mathematically (Drake & Sherin, 2009) or developing the big mathematical ideas over time (Edson et al., 2019). In addition, the *practical-evaluative dimension of agency* involves how teachers navigate possibilities and negotiate conflicts and tensions within the school system as they strive to open curriculum spaces. Teachers' work involves daily decision-making that is sometimes unavoidable to facing conflicting pressures or dilemmas. By engaging in reflection and professional dialogues with colleagues, teachers will be able to achieve agency to utilize resources to make decisions (Priestley et al., 2015).

The practical-evaluative dimension of teacher agency deals with the present, drawing on past experiences and orienting toward future objectives. It entails various contextual aspects that can potentially influence teachers' decisions on making changes in their curriculum use. Some examples include the material aspect (resources, physical environment), the structural aspect (social structures such as relationships, roles, power, and trust), and the cultural aspect (ideas, values, beliefs, discourses, and language) (Priestley et al., 2015). By attending to these aspects in the practical-evaluative dimension of teacher agency, this study illustrates how teachers navigate practical ways to open curriculum spaces by leveraging the contextual aspects and negotiating the challenges therein.

Figure 1.

Three Dimensions of Teacher Agency (Priestley, Biesta, & Robinson, 2015, p. 4)



CHAPTER 3: METHODOLOGY

I locate this study within an interpretive case study approach, where close investigations of a case can "provide insight into an issue or refinement of theory" (Stake, 1994, p. 237). This study focused on a case of practicing mathematics teachers' design processes to *open* curriculum spaces in the context of collaborative lesson planning. Investigating the process of teachers' work to *open* curriculum spaces, which is intrinsically intertwined with both the pedagogical design capacity and teacher agency, will provide practical suggestions for mathematics teachers to use curriculum resources in a way that ensures students have equitable learning opportunities. The analysis of this study was exploratory about the affordances and constraints of a particular context (Yin, 2014), where mathematics teachers collaboratively participate in the process of teacher inquiry for the purpose of utilizing students' multiple mathematical knowledge bases as assets for learning mathematics.

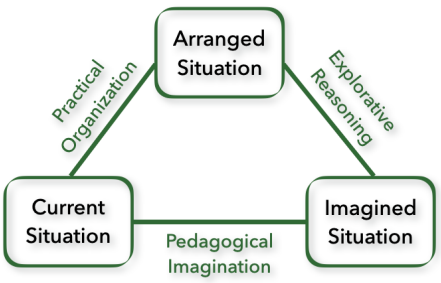
Also, I took a participatory approach to design and conduct this study. Given the critical role of teacher agency to account for pedagogical design capacity (Brown, 2009; Pepin et al., 2019; Priestley et al., 2015), a participatory approach can engage teachers to act as agents that change classroom practices to ensure their students' equitable learning opportunities rather than merely implementing teaching strategies generated by university-based research that may be disconnected from their classroom practice (Cochran-Smith & Lytle, 1999; Skovsmose & Borba, 2004; Wright, 2021). The participatory action research views teachers as knowers and thinkers who can participate in reflective discussions to create pedagogical knowledge grounded on their own practice; it legitimizes practical knowledge created through teacher inquiry which blurs the boundary between theory and practice (Cochran-Smith & Lytle, 1999; Nolen & Putten, 2007; Somekh & Zeichner, 2009). Teacher inquiry in participatory action research does not simply

refer to a set of strategies or methods. Rather, as a stance, it calls for teachers' continuous and iterative reflections in and out of teaching practice in order to provoke changes (Cochran-Smith & Lytle, 1999, Fine, 2010). Rather than doing research *on* teachers about what they do regarding curriculum spaces, I took a participatory approach to research *with* teachers about what we can do to change learning environments to be more equitable for students from diverse cultures (Skovsmose & Borba, 2004; Stenhouse, 1975).

The design of this study is guided by Skovsmose and Borba's (2004) critical research model of participatory action research, which provides a methodology for critical mathematics education focusing on classroom changes. Skovsmose and Borba (2004) categorized a pattern of critical research with regard to three situations (*current, imagined*, and *arranged*) and three qualities of the process (*pedagogical imagination, practical organization*, and *explorative reasoning*) (Figure 2). Provided that a participatory action research methodology resonates with critical mathematical pedagogy, Wright (2021) advocated for Skovsmose and Borba's (2004) model for critical research as a means to conduct participatory research to transform mathematics teaching practices. I will describe each component in the model and how it is specifically related to this study's objective. The data collection section later in this chapter will provide more details about how I applied the process to this study.

Figure 2.

Model of Critical Research (Skovsmose & Borba, 2004, p. 216)



The issues within the *current situation* can be revealed by teachers (as researchers) through curriculum analyses and classroom observations. The *imagined situation* is constructed through a process of *pedagogical imagination*, which involves conceptual explorations of what could have been done differently while drawing from educational research findings. Based on this vision of ideal alternatives to the current situation, researchers collaboratively develop an *arranged situation* that is more realistic for a particular context than other alternatives through the process of *practical organization*. The evidence regarding the original issues and changes gathered in the arranged situation is analyzed through *explorative reasoning* in order to see the feasibility of the imagined situation.

Although critical mathematics education research has emphasized the importance and promises of student identity and social justice in learning mathematics (i.e., the critical axis of educational equity), there are relatively few studies that focus on how mathematics teachers can navigate possibilities for criticality while using a school-designated curriculum set. This study aims to support mathematics teachers in deploying pedagogical design capacity in order to use curriculum resources in a way that could support their students' learning. In this case, the *current situation* represents mathematics teachers' use of curriculum resources that do not necessarily consider drawing on students' multiple mathematical knowledge bases. The *imagined situation* represents teachers' capacity to *open* curriculum spaces—providing students with opportunities to connect with real-life experiences, make sense of mathematics through multiple strategies, and explain their strategies (Land et al., 2019). The *arranged situation* represents teachers' collaborative lesson design and implementation by which they exercise to re-source curriculum resources, including both the official curriculum and other resources from students and communities.

Through *pedagogical imagination*, teachers have critical conversations about the current curriculum resource use, curriculum spaces for students' learning opportunities, students' multiple mathematical knowledge bases, and examples of open spaces. As external stimuli, researchers provide relevant research articles that can facilitate critical discussion. Through a *practical organization*, teachers engage in designing lessons to provide students with open spaces by considering their own students' multiple mathematical knowledge bases. In their design process—select, interpret, reconcile, accommodate, and modify resources (Brown, 2009)—teachers also navigate resources from students' and families' funds of knowledge. Through *explorative reasoning*, teachers reflect on the lessons that they have designed and enacted in order to obtain insights into further design ideas. By writing reflection journals and participating in group reflection discussions, teachers evaluate how the curriculum spaces worked out in their lessons based on the student work evidence they have collected. The reflection can inform planning for the subsequent lessons, which would help teachers initiate their next iteration of inquiry.

Participants

Two mathematics teachers joined this study through recruitment emails. Ms. Mer and Ms. Jay teach at the same middle school in a suburban town in the Midwest. As I worked on the research projects in the *Connected Mathematics Project* (CMP, Lappan et al., 2014) curriculum development team, I used to go to Ms. Mer's classroom for data collection. When I reached out to mathematics teachers in the area in order to recruit participants in this study, she was the very first person who responded to my email to indicate her interest. After having an informal introductory meeting with her, she connected me with other mathematics teachers in her school and district. It was because she hoped to have a teacher collaboration group within the same

district that would enable teachers to share area-specific resources for culturally relevant problem contexts. Among those teachers, Ms. Jay showed her interest in this collaboration while other teachers gently expressed that they did not have sufficient time to commit to participating in the study.

Ms. Mer

Ms. Mer identified herself as a mother, a teacher, and a friend. When I reached out to her, it was soon after she came back to school from her maternity leave. Becoming a mother of a newborn baby, she expressed the joy that the baby brought into her life. As a mathematics teacher for more than ten years, being a teacher is "such a huge part of [her] identity," which is "so hard not to be." She said, "We are one of the only professions that define who we are by our profession." She pointed out that teachers and nurses wear graphic shirts showing their professions which salesmen or human resource representatives would not. That the teaching profession is a big part of their identity makes her somewhat sad because it is inseparably intertwined with her personal life, and yet she feels societal responsibility and expectations imposed on teachers. As a friend, she values connections with people including her extended family.

During her participation in this study, she was teaching 7th-grade mathematics using the school's official curriculum, *Connected Mathematics* (CMP). Although it was her 8th year of teaching the same grade at the school, she said that it felt different every year. She believes that mathematics lessons through a problem-based curriculum in an inquiry-based classroom work out differently depending on students' different personalities, characters, needs, classroom dynamics, etc. She participated in this study because she hoped to work with other teachers on equitable teaching practices with a specific goal to learn about making mathematics problems

more culturally relevant to her students. She mentioned that the concepts of students' multiple mathematical knowledge bases and open spaces resonate with her teaching philosophy and her district's mission. Based on the book club readings with other teachers in her district-level workshops, she found that it was not straightforward to make practical changes in her own teaching although she was eager to do so. Thus, she wanted to be able to turn theoretical ideas about equitable teaching into her own practices.

Ms. Jay

Ms. Jay identified herself as a mom, a teacher, and a cook/chef. She has two daughters, living close to her parents and her younger brother. She loved showing photos of her little girls and sharing cute episodes. Every time, her face was beaming with happy smiles. Similar to Ms. Mer, she also found that being a teacher is a big part of her identity. She also recognized teachers' responsibility to be a good role model for students. For example, she pointed out that "when things going wrong on the news, it says, 'Teacher suspected of whatever.' And it is not for like businessman, [it'd rather be] man suspected of, or woman." In addition, she likes cooking her family's traditional food and trying new recipes. Since her parents are from Middle Eastern countries, she can speak Arabic with her family and cook Middle Eastern food using her mother's recipes.

During her participation in this study, she was teaching 8th-grade mathematics using the CMP curriculum. It was her first year working at the school after teaching middle school mathematics in a different school district for four years. In her previous school, she used a different curriculum, *Illustrative Mathematics*, and she said that she was able to recycle her first year's lesson plans over time because her lessons were focused on students' practicing problem-solving skills followed by the teacher's demonstration. Since she joined the new school, she

immediately recognized that the new curriculum required teachers to approach differently from what she was used to doing. As she felt like she was learning about teaching the CMP curriculum along with her students, she expressed that she would become more confident facilitating group work and discussion next year.

Her classroom was located right next to Ms. Mer's room until Ms. Mer moved to a different room in the same building in the middle of the second semester. Sometimes they had lunch together in her room and had conversations about kids, students, teaching, and school policies. She wanted to participate in this study because she perceived it as a learning opportunity for developing strategies to use a new curriculum by collaborating with other teachers. She particularly appreciated that this study would attend to students' understanding and experiences because she found it both important and challenging.

Researcher Positionality

In this participatory case study, I positioned myself to be a facilitator. I organized meetings, pursued to create a comfortable and inviting environment, shared resources from educational research, facilitated discussions, posed questions, visited teachers' classrooms to observe lessons, and had conversations about teaching and learning mathematics. First, it was important to establish trustworthy relationships among members. I had previous interactions with Ms. Mer through the data collection for the CMP research projects, which helped both Ms. Mer and me to feel comfortable initiating new collaborative work. On the other hand, it was my first time working with Ms. Jay. Since the two teachers had already been sharing personal and professional stories whenever they had a chance, I did not necessarily need to plan for connecting the two teachers, to which I would have paid special attention otherwise. Instead, I focused on developing solidarity among the three of us by incorporating activities and questions

into our weekly meetings. At the beginning of the first meeting, I shared my own stories about my relationship with students, teaching and learning episodes, teacher learning community experiences, and my transition from a secondary mathematics teacher to a mathematics educator/researcher. Then, I asked us to share three hashtags that could show who we are. All of us had *#teacher* in common, and both teachers put *#mom* as the first item. When I visited their classrooms after school, I asked if they had new stories and photos of their kids to share, while having refreshments and snacks. It helped us start our weekly meetings with warm smiles and lightened moods. When we had our meetings through video conference calls, their children occasionally were sitting next to their mom, so we had a chance to wave at each other. Local restaurants and recipes were also popular topics for our brief chats before discussions since we learned that we all love cooking and enjoy trying new restaurants. Sometimes, the teachers talked to each other about administrative work, such as the exam schedules or the principal's class observation and evaluation plans. I waited for them to take time so as to get their pressure or wonders out of their chest before we started our discussion.

Second, I provided evidence-based teaching practice examples relevant to our discussions as external stimuli. I selected resources based on the topics and used them to amplify our discussions about understanding students' backgrounds and connecting mathematics with students' life. The teachers viewed me to be knowledgeable with research literature and showed their trust in my resources. I carefully phrased discussion questions only to open a space for conversations about the readings and reflections on their own teaching practices rather than imposing my perspectives or the main argument from the literature. My previous experiences serving multiple roles, such as a mathematics teacher, a mentor teacher, and a professional learning community leader, allowed me to have conversations with the teachers as a teaching

colleague. This was important because the teachers should not feel as if they were assessed on their planning and teaching by a researcher during our discussions. Given that the goal was to conduct research *with* teachers, I constantly reflected on my roles during the meetings so that the teachers would not feel pressured to say the *right things* that researchers would want to hear. In addition, at the end of each meeting, I co-planned the next meeting agenda with the teachers. We also flexibly chose the day (either Monday or Tuesday for most meetings), time (either 3 PM or 4 PM), and place (either in their classrooms or online) depending on our life circumstances.

Third, I ensured that my affiliation with the CMP research team would not influence the teachers' genuine interaction with the curriculum. Because the discussions and activities inevitably involved their interpretation of the CMP curriculum, I constantly made it explicit that our analysis of the curriculum resources could result in providing both positive and negative feedback, which should not be an issue because both aspects are critical for curriculum development. Rather than merely applauding or criticizing the curriculum, I facilitated the discussion to be identifying the current situation and brainstorming possible changes. Since I could not divorce myself from a graduate research assistant role in the CMP team, I instead set the discussion norm to be sharing constructive feedback for the next revision of the curriculum materials (both student textbooks and the teacher's guide), so that the teachers could feel comfortable explaining their ideas and suggestions without hesitations.

Lastly, I was conscious of the influence of my identity as a Korean living in the USA on our discussions. I openly shared that I had no teaching experience in America but in South Korea, and that I had not lived abroad before joining my doctoral program. I proposed a pre-agreement that we could pose follow-up questions if we could notice cultural or institutional differences. I honestly shared that my English as a second language might cause confusion or

misunderstandings, in which cases I would either ask for or welcome clarification questions. Both teachers were very understanding of my English as a second language and appreciated my perspectives and experiences in a different school context and culture from their own. Also, we as a group could utilize my ability to read and write Korean as a means to enhance our discussion. When I brought a mathematics problem written in Korean excerpted from the literature, my intention was for us to experience how it would feel to learn mathematics using a foreign language. The teachers appreciated that I could listen to and/or communicate with their Korean students. In addition to this collegial effort, I was very careful when I wanted to share my international perspective with the teachers. I focused on using my knowledge of Asian cultures only to bring diverse perspectives into our conversations, and I did not want to imply that I was pointing out what they could have considered.

Data Collection

After meeting with the participating teachers individually for the recruitment and the introduction to the study, the two teachers and I met once a week for planning, teaching, and reflection. The teachers suggested weekly meetings rather than bi-weekly, and our meetings took place after school either in one of their classrooms or on a video conference platform. Although each meeting had agenda that was guided by Skovsmose and Borba's (2004) model of critical research, we did not rush to get through planned activities or discussion topics. As we sometimes needed to continue the discussion, each week's agenda was informed by the previous week's conversation. Then, we collectively decided when to move to the next phase (i.e., from pedagogical imagination to practical organization). In addition to video and audio recordings of our weekly meeting, we shared one Google Drive folder to keep our meeting notes, reflection journals, curriculum analysis tables, and digital copies of student work. Also, when I visited the

school for classroom observation, I videotaped seven lessons in total from the two teachers' classes. Recordings of weekly meetings were first transcribed using online transcription tools (mainly Microsoft Word Transcribe) and stored on the MAXQDA 2022 software after I manually edited the transcripts. Table 1 summarizes data collection and sources. All data sources were used for the qualitative data analysis. In the following, I will describe the data collection process separating the pedagogical imagination phase, the practical organization phase, and the explorative reasoning phase.

Table 1.

Data Collection Phases and Data Sources

Phase	Meeting Schedule	Discussion Topic	Data Sources (In addition to video recordings and memos during meetings)
Pedagogical Imagination	Step 1 (Nov. 15. 2022)	Culturally relevant math tasks	Notes in the shared document
	Step 2 (Nov. 21. 2022)	Math tasks for multilingual students	Notes in the shared document
	Step 3 (Nov. 28. 2022)	Curriculum spaces for students' multiple mathematical knowledge bases	Curriculum spaces analysis tables
	Step 4 (Dec. 6. 2022)	Open spaces in the textbook	
Practical Organization	Step 5 (Dec. 12. 2022)	Planning activities to learn about students' multiple mathematical knowledge bases	Notes in the shared document
	Step 6 (Dec. 19. 2022)	Reflecting on the advisory hour activities	Student work examples
	Step 7 (Jan. 10. 2023)	Strategies for opening curriculum spaces	
	Step 8 (Jan. 17. 2023)	Designing a lesson to integrate open spaces	
	Step 9 (Jan. 26, 30. 2023)	Enacting the planned lesson and gathering information about student engagement	Video recordings of lessons
Explorative Reasoning	Step 10 (Jan. 31. 2023)	Reflecting on the enactment	Reflection journals, student work samples, and notes in the shared document

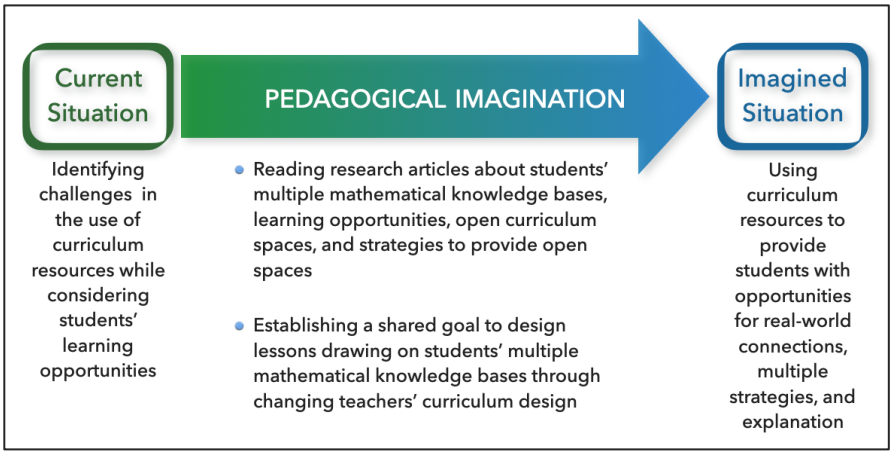
The Pedagogical Imagination Phase (Figure 3)

From the first meeting to the fourth meeting, we focused on identifying the current situation and processing the pedagogical imagination. As a researcher in this study, I had explicit

research questions and goals going into the meetings. The teachers understood what I was hoping to learn from this collaborative study since they had a chance to read through the research goals, anticipated outcomes, and expectations for the participants before they signed a consent form. Also, they were briefly introduced to the concepts of students' mathematical knowledge bases and open curriculum spaces before they agreed with the participation. In this regard, this study did not ask participating teachers to identify their own topics of teacher inquiry as the participatory action research approach typically suggests.

Figure 3.

The Data Collection Phase 1: The Pedagogical Imagination



Nonetheless, I wanted to draw from the reflection on their current teaching practices in order to establish a shared goal. I posed questions about their use of curriculum resources in general, perspectives and experiences of equitable teaching practices, and interactions with their students throughout the first four meetings. As we discussed culturally relevant mathematics tasks (Matthews et al., 2022) and mathematics lessons for multilingual students (Chval et al., 2021), we continued identifying the challenges in their current teaching practices and curriculum use. This kind of iterative conversation between the imagined situation and the current situation allowed them to specify the changes they wanted to make through our collaboration in this study.

Moreover, during these early weeks' meetings, I posed questions about our daily experiences (both as a teacher and as a person) so we could learn about one another and develop trust among us.

During the pedagogical imagination phase, we utilized several resources to facilitate our discussions about the teachers' current and imagined situations. First, we looked into culturally relevant mathematics tasks suggested in the book, *Engaging in Culturally Relevant Math Tasks* (Matthews et al., 2022), and reflected on our own teaching experiences individually and collectively in order to identify what we were already doing and what we hoped to focus more on. While we discussed the potential of mathematically rich and culturally relevant tasks, the teachers also shared the struggles and uncertainties regarding the limited time and the pressure of pacing guides and exams. For example, they mentioned it was challenging to decide how much time students were to spend talking about problem situations and how much time teachers were to spend creating (or adapting) tasks. Second, we investigated example mathematics problems in the book, *Teaching Math to Multilingual Students Grade K-8* (Chval et al., 2021). We attempted to solve mathematics problems, some of which were written in foreign languages, and some were embedded in a foreign country's traditional game contexts. Then, we shared how perplexed we felt when trying to figure out what the problems were asking. This experience of feeling lost allowed us to discuss how to be mindful of multilingual, multicultural students' emotional struggles in American classrooms. Third, we used the curriculum spaces analysis table (Drake et al., 2015a) as a tool to understand the teachers' curriculum resources and discuss their planning and teaching. The analysis table had question prompts that teachers could ask themselves as they read curriculum resources. Going through each question, the teachers wrote ideas of how to plan or enact the curriculum problem to ensure students' learning opportunities. Drawing on concrete

examples and reflections, we explicitly discussed the teachers' ongoing challenges and hopeful changes in their curriculum use and their teaching for equitable learning. The teachers hoped to learn about diverse cultures from their students and incorporate them into mathematics lessons while considering their time constraints.

While we looked into examples from the literature, their interpretations and views on their official curriculum were naturally brought up. The discussions often went on to brainstorm ideas for making changes when the teachers noticed curriculum spaces within the textbook problems. I did not explicitly ask them to plan lessons based on our discussion during the first four meetings. It was because the pedagogical imagination phase was to explore possibilities before taking practical, particular class and school contexts into consideration as suggested in Skovsmose and Borba's (2004) model. By the end of the fourth meeting, the teachers interpreted their curriculum resources through the lens of curriculum spaces a few times, and they expressed that they felt comfortable moving on to the practical organization phase in order to make a lesson plan and enact it in the classroom.

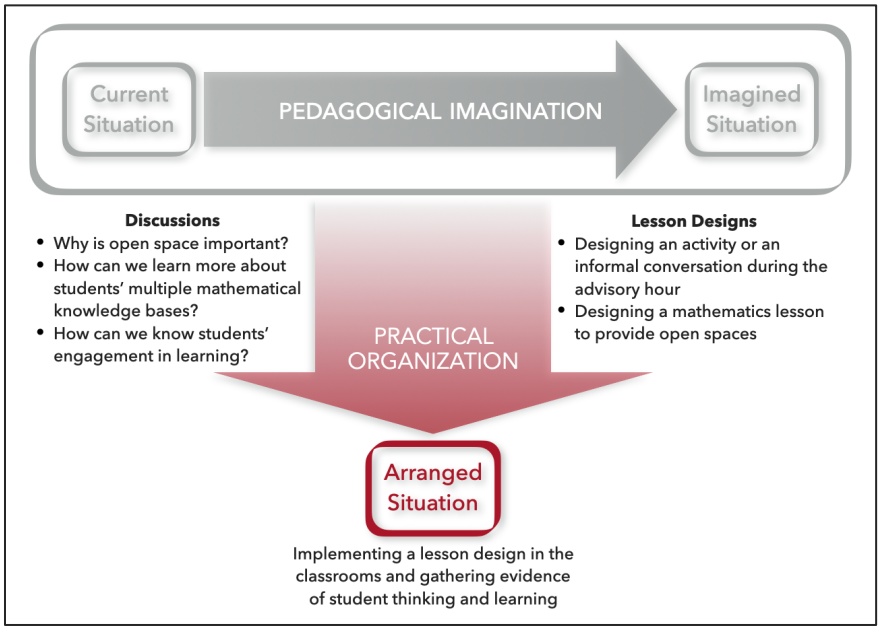
Throughout the four meetings, we talked about their challenges in the current situation in terms of their overall curriculum use to value students' cultural experiences. As we iterated our conversation around what we had been doing, what we could have done, and what we wanted to change, the teachers were able to identify the current situation using the literature-provided terminologies such as students' multiple mathematical knowledge bases and open spaces. They stated that they hoped to change their planning and teaching to incorporate students' multiple mathematical knowledge bases while balancing between conceptual understanding and procedural fluency and while keeping the work doable given the limited time available at work.

The Practical Organization Phase (Figure 4)

We began our fifth meeting with a discussion on the value and importance of open spaces. The guiding discussion questions included (a) Why is it important to open curriculum spaces? For whom, is it important? (b) Reflecting on your current teaching practice and the readings, what can we do to ensure to provide students with open spaces? (c) Have you noticed curriculum spaces in your curriculum resources? This discussion was to help us establish a shared goal to change teaching practices in ways that consider students' multiple mathematical knowledge bases before thinking about what we could have done differently. Based on the teachers' desires to learn more about their students' diverse backgrounds, we figured that their advisory hours would serve as a place in which teachers could interact with students more closely and freely under no pressure. In our fifth meeting, we brainstormed together to plan for the upcoming advisory hour.

Figure 4.

The Data Collection Phase 2: The Practical Organization



Since the teachers were already having informal conversations with students during the

advisory hours, we narrowed our discussion down to ways in which we could connect between students' multiple mathematical knowledge bases and mathematics lessons. It was the biggest challenge for the teachers when they thought about making lessons more relevant to their students. Although the teachers wanted to consider students' multiple mathematical knowledge bases in their planning and teaching, they felt overwhelmed to utilize the current resources (the textbooks and the school/district professional learning opportunities) to make changes. As one way to address this challenge, during the fifth meeting, the teachers planned an activity that could give them information about both students' cultural experiences and their mathematical thinking.

Ms. Jay planned to facilitate an activity that could engage students in thinking about using the percentage concept to express their ethnicities and personalities because she knew many students were mixed in terms of race and ethnicity. Also, she thought that through this activity, she could formatively assess students' understanding of percentages which was necessary to develop an understanding of growth rates in the upcoming lessons. Ms. Mer planned to facilitate a conversation that could engage students in sharing special foods and recipes in their family or culture. This conversation would not directly ask students to connect with mathematical thinking; rather, she was seeking to gather some information about students' cultural experiences so that she could plan a lesson about percentages. The teachers also planned to record students' responses for our discussion and future use for lesson planning. Ms. Jay collected student work because she used a handout for the activity, and Ms. Mer took a note while students shared out their favorite foods and recipes special to their families and cultures. After implementing their plans during their advisory hours and at the beginning of mathematics lessons, the teachers shared how the activity and conversation went, what they learned about

students, and how they could use this interaction with students for lesson designs in our sixth meeting.

Our seventh and eighth meetings were devoted to individually and collaboratively designing a lesson that could draw on students' multiple mathematical knowledge bases. Before the seventh meeting, the teachers read Drake et al.'s (2015b) article about strategies to open curriculum spaces, which provided the teachers with some practical insights into analyzing their official curriculum and considering potential changes they could make. In addition, we discussed how teachers could know whether students were engaged in mathematics problems and what teachers could do if they noticed students' struggles with participating in lessons. Then, they selected the dates that they would like me to visit their classrooms for the observation, which gave them a particular problem in the textbook to plan. The teachers individually thought about how to provide students with open spaces before we met to discuss and provide each other feedback and support during the eighth meeting. During these planning meetings, we focused on investigating curriculum spaces, drawing on students' mathematical knowledge bases, and planning to collect evidence of student participation and mathematical thinking.

The ninth meeting was replaced with classroom observations and short conversations between and after classes. During the enactment of lessons, the teachers paid attention to student engagement in the lessons and collected evidence of student thinking for the purpose of reflection after the class. I as a researcher observed both students' participation in the lessons and teachers' pedagogical moves, while videotaping lessons with the camera focusing on the teachers and the classroom whiteboard. The teachers were asked to write a reflection journal right after their enactment of planned lessons based on the evidence of student learning they observed. Reflection journal prompts included (a) How did your students participate in the open spaces?

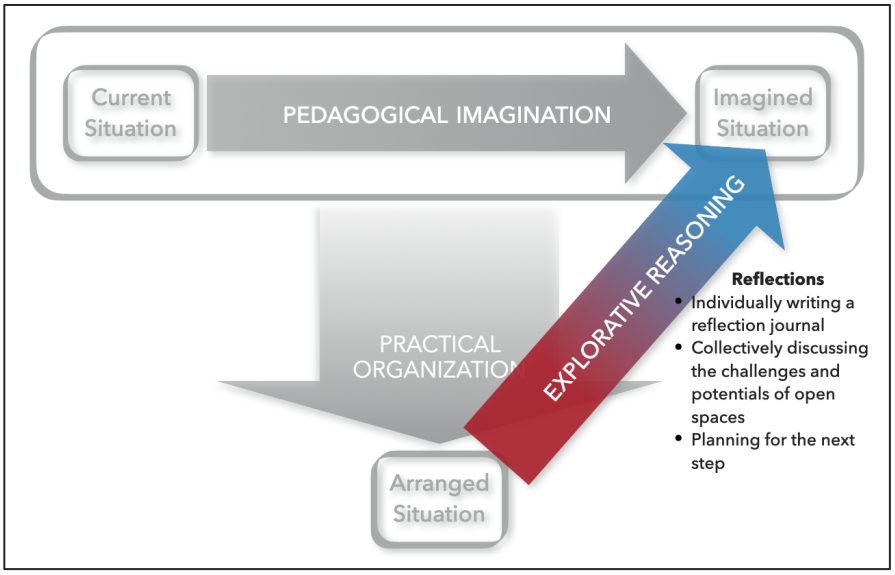
How do you know? (b) What did you notice and wonder about teaching mathematics through open spaces? (c) What would you do differently if you teach it again? Their reflection notes and my observation notes were shared for us to have a collective reflection meeting for the explorative reasoning phase.

The Explorative Reasoning Phase (Figure 5)

In our tenth meeting, the teachers shared their reflections on the lesson that included their observations of student learning and wonders about how to sustain student engagement in mathematics problems. We looked across our reflection and observation notes and discussed common concerns and wonders. Also, the teachers shared their perceptions of how their participation in this study influenced their planning for daily lessons. Although we shared a couple of ideas about possible directions for the next iteration, we agreed to take a break from weekly meetings until we could plan a school or district workshop focusing on students' multiple mathematical knowledge bases and lesson designs for open spaces.

Figure 5.

The Data Collection Phase 3: The Explorative Reasoning



Data Analysis

The analysis of these data was guided by an interpretive qualitative research method (Saldaña, 2016) to answer the research questions; (a) In what ways do mathematics teachers use curriculum resources in their daily lessons drawing on students' multiple mathematical knowledge bases? (b) How does teacher agency, when supported in a conducive context, allow for teachers' capacity to open curriculum spaces? Data sources for the analysis include video and audio recordings of weekly meetings and transcripts, classroom observation recorded videos, Google slides used for individual and collective notes during meetings, student work examples, teachers' individual reflection journal entries, and the researcher's field notes and memos. Concerns about validity were addressed in two ways. First, multiple sources of data were collected to cross-reference. Second, the triangulation through checking in with the participants allowed me to be careful about the possibility of misinterpretation of participants' intentions (Maxwell, 2012). Any ambiguity in the interview responses was addressed by follow-up questions during and after the meetings.

In what ways do mathematics teachers use curriculum resources in their daily lessons drawing on students' multiple mathematical knowledge bases?

To answer the first research question, I focused on the processes of how the teachers planned a lesson based on their interpretations of the curriculum and their perceptions of students' multiple mathematical knowledge bases. Before applying specific codes to the data sources, I repeatedly watched the videos of meetings, read the transcripts, and took memos about the main points, flows, or my wonders in order to gauge the overall process. Even though the seventh and eighth meetings focused on the teachers' plan for a specific lesson in order for the researcher to observe the classroom enactment, we also discussed the use of curriculum

resources with regard to opening curriculum spaces in other meetings. The teachers' lesson design process was not captured in a single interview but discussed throughout ten meetings, and hence it was important to understand how the process developed by looking across multiple meeting transcripts. Then, I applied a structural coding approach (Saldaña, 2016) in order to identify both teachers' design processes and their interpretation of curriculum spaces. The unit of analysis was an episode that includes a set of teachers' utterances (from one sentence to multiple sentences) that involves structural codes—lesson design process phases and curriculum spaces.

First, I identified teachers' design activities based on Brown's (2009) lesson design process structure: selecting, interpreting, reconciling, accommodating, and modifying resources. The *selecting* code was applied to a set of a teacher's utterances when a teacher mentioned a resource that could be potentially used in mathematics lessons. Since both teachers used the official curriculum as the main resource for their daily lesson planning, they did not explicitly mention that they selected the curriculum as a resource. Instead, the teachers brought up problems in the official curriculum during the discussions and explained their interpretations of the problems. In these cases, a selecting code was applied to a set of teacher utterances indicating that the teacher was attending to a problem in the curriculum. Since the goal of this structural coding was to understand how teachers made decisions on the curriculum resources, I focused on identifying teachers' comments on a resource that led to their interpretations. In other words, if a resource was brought up as an isolated word without interpretations, it did not get a selecting code. One example of selecting codes represents how Ms. Jay attended to the introduction to the curriculum problem. She did not mention that she selected the curriculum as a resource, but she opened the student edition of the curriculum and read out loud the first question:

This does bring up … So, it starts off saying, suppose you give a friend two $1

> bills and your friend gives you eight quarters. Would you consider this a fair trade? So, that's the very first question.

The *interpreting* code was applied to a set of a teacher's utterances when the teachers described the meaning, the instructional goals, or their anticipation of student responses. This code followed after teachers indicated their attention to a particular problem or resource, and it ended when the teacher's utterance switched to the next phase (reconciling) or a different topic. For example, after she attended to the first question in the curriculum, Ms. Jay continued to explain:

> So, I can start asking their opinions? Like, would you think that's fair or not? You get answers like, 'Yeah, it's the same amount,' or 'No because no one uses quarters.' Well, yeah, you can connect it with them. Because, you know, some students do use quarters or changes, if they need to go do laundry, or if that's how they get candy from machines. Or if some kids don't use cash at all, and their credit cards are what they're used to.

This quote shows Ms. Jay's interpretation of the introduction page of a problem in the textbook drawing on her anticipation of her students' responses to the question. Because the teachers focused on incorporating open spaces in their lessons, the majority of their interpretations of the curriculum resources were made through a lens of curriculum spaces. I will present structural codes for their interpretations of curriculum spaces later in this section.

The *reconciling* code was applied to a set of a teacher's utterances when the teachers described the alignments or discrepancies between the goals of the curriculum and their own goals, particularly regarding open spaces. This code was followed by teachers' interpretation of resources, and it ended when the teacher's utterance switched to the next phase (accommodating)

or a different topic. In some episodes, however, teachers' interpretations ended without leading to reconciliation. It was when the teachers implicitly agreed with the problem's approach or when the discussion did not specifically focus on how to use the problem. One example of reconciling code is followed by Ms. Jay's interpretation of the curriculum's introduction page when she continued to explain:

> Well, yeah, that would be a really cool discussion. But then, I'm like, how do I reconnect it back to the math if we just start talking about money and preferred ways of payment? Like, that kind of [conversation] doesn't seem like has anything to do with coming up with the equations like the second part.

This comment shows her recognition that the question in the introduction would not serve as a connection to the main problem of the lesson.

Not all the reconciling codes were followed by an accommodating code because when the teachers expressed that a selected resource could serve to meet their goals for open spaces, they did not need to make adaptations to accommodate students' needs and interests. Also, the teachers' comments on their reconciliation were sometimes intertwined with their interpretations. When the teachers focused on how to utilize the resources rather than making explicit comments on their analysis, their interpretation was implicitly embedded in their reconciliation. For example, when Ms. Jay read the second part after the introduction, she said:

> Then, okay, let's look at another trade. Do you think this one is fair? For this trade, I never think about this. But do you think it's relatable? I mean, students who would not trade land pieces? I don't know.

She asked questions herself because the problem did not seem aligned with her goal for open spaces, which also implied that she thought the problem would not be relatable for students. For

such cases, I coded a set of utterances as one code—*interpreting/reconciling*—rather than exclusively separating the two phases because the goal of this structural coding for the design process was less about identifying the process itself and more about how the teachers came to a decision about the curriculum use.

The *accommodating* code was applied to a set of a teacher's utterances when the teachers mentioned their consideration of students' interests and needs. This code followed after teachers' explicit or implicit reconciliation of goals, especially when the teachers noticed the dissonance between goals. Some reconciling and interpreting/reconciling codes ended without leading to an accommodating code because the teachers did not always come up with ideas to consider students' interests. One example of the accommodating code is when Ms. Jay shared an idea to help students understand what fair trade means by drawing on her own story that students might find more relatable:

> So, okay, I grew up with siblings, a sister and a brother, I think most of my students have. So, when they think of fair trade, usually the older siblings trick the younger ones into thinking they're trading something fair, but they're really not great things. Like, "Oh, look, you're getting a lot bigger things than I am." Like, whatever I'm getting is far more valuable.

Since providing students with open spaces to connect with their life experiences calls for teachers' attention to students' out-of-school experiences, the accommodating code also captured teachers' perceptions of students' multiple mathematical knowledge bases and their ideas to learn about students. Ms. Mer's suggestion to utilize advisory hours as a place to learn about students is one example of this code:

> Having some conversations and even just conversations about what they're going

> to do over break, what activities their families do, and how they spend their time [would be helpful]. It's because, think about *Comparing and Scaling*, [which] has all those proportions, […] If I can get some of that info from them, I could start to build that into some of the [problem contexts]. And so, it may even just be like every day an advisory for the next week.

The *modifying* code was applied to a set of a teacher's utterances when the teachers described the ways in which they make changes to the curriculum. For example, after sharing her own childhood story about unfair trade between siblings, Ms. Jay mentioned her plan:

> I think that's even better than this coin stuff. Yeah, you can just ask them to share their tricks. How to trick their younger sisters and brothers. […] And then I think they will understand what fair trade means and they can jump into this problem [about trading land pieces].

Even though each teacher created a single lesson plan (Step 8) for the purpose of the researcher's classroom observation (Step 9) as a result of the *practical organization* phase, I applied this coding process to all transcripts of the meetings. It was because the teachers constantly interacted with the curriculum and shared their perceptions and potential use of the curriculum during the weekly meetings as we discussed open spaces and students' multiple mathematical knowledge bases. Although the design processes did not always surface in a linear fashion, these structural codes allowed the analysis to focus on what they noticed in the curriculum resources (interpreting/reconciling), what they considered when opening spaces (accommodating), and what they decided on their use of curriculum resources (modifying).

After identifying teachers' design activities, I focused on their interpreting and reconciling phases. I identified teachers' perceptions of curriculum spaces following Land et al.'s

(2019) scheme for curriculum analysis. Sub-codes within this category include open spaces for multiple strategies, open spaces for explanations, open spaces for real-life connections, closed spaces for multiple strategies, closed spaces for explanations, and closed spaces for real-life connections. Land et al.'s (2019) analysis identified *conflicting spaces* when the curriculum engages students in an open space, then immediately followed by a closed space. In my analysis, however, the teachers did not explicitly mention conflicting spaces within the curriculum. Figure 6 presents a part of the transcript of Ms. Mer's interpretation of open/closed spaces within the curriculum. Within a problem in the *Comparing and Scaling* unit, (1) she noticed open spaces for multiple strategies and (2) explained her reasoning for such interpretation. Then, (3) she highlighted this open space once again, and (4) she described a closed space for real-life connections.

Figure 6.

An Example of Teachers' Interpretation of Curriculum Spaces

So, this problem is kind of like, combining some of their strategies from the whole unit and using this idea of percents, like it says, that doesn't involve money. If you click over to the Initial Challenge, it's, it's talking about this chimp food and percentages. And then a couple of questions for them to kind of explore. And (1) I think that it's really open in the sense that it has space for multiple strategies like this whole unit. (2) It's because, it just kind of says they need 225 scoops, how many [scoops of each do they need]? So, kids can use percent, they can change it to a ratio, [or] they can use a proportion. [Earlier problems] teach them these percent tape strips, and so [student could use them, too]. The second question as well: they have this much left, it needs to be this percent, how many scoops do they need? (3) So, I think [it offers open spaces] in the sense of kind of kids bringing their own strategies, multiple representations. That's gonna be, it's going to be really open. Like I said, (4) I don't know about this whole chimp food recipe contexts. Like, I just don't know if [students] will connect to [it]. You know, thinking about if you're trying to do something with percent that's not money and this kind of mixture thing, there's probably a lot more you could do than these chimp foods, although I mean, for some kids, it would be open and interesting for them, but that's where I think the contexts, like most kids don't have much connection to this context. [It] may be interesting, but it's not gonna be relevant to them.

After identifying codes for both design processes and curriculum spaces, I revisited the data set in order to open-code teachers' perceptions and explanations in association with their decisions on how to use the curriculum resources. This open coding approach allowed me to

capture the teachers' reasoning behind their interaction with the curriculum and illustrate how they opened curriculum spaces in detail. These codes were also utilized in the analysis for the second research question.

As I iterated coding cycles, a few themes emerged. First, I grouped open codes for teachers' perceptions of students' multiple mathematical knowledge bases. These codes captured the school context serving an increasing number of multicultural, multilingual students, the teachers' ongoing endeavors to attend to individual students' learning opportunities considering their diverse backgrounds, and the teachers' perceived challenges in such work. Second, I grouped open codes for how teachers perceived potential and challenges in designing lessons for open spaces. These codes captured the teachers' desires and struggle to balance between following the school's official curriculum and incorporating culturally relevant problem contexts. Third, I grouped open codes for teachers' perceptions of teaching practices and teacher learning opportunities. These codes captured the teachers' instructional decisions when encountering dilemmas or challenges as well as their experiences and suggestions about professional learning opportunities. Particularly, I applied In Vivo coding in order to record the teachers' probes as much as possible to "preserve participants' meanings of their views and actions in the coding itself" (Charmaz, 2006, p. 55). These open codes were used to identify relationships among structural codes during the second cycle of data analysis.

Then, I applied a holistic coding approach (Saldaña, 2016). I looked for relationships among structural codes, patterns in codes, and similarities and differences between the two teachers. As an interpretive qualitative analysis, I began by describing both teachers' lesson design processes, following *interpreting/reconciling*, *accommodating*, and *modifying*. Then I focused on illustrating how each process was supported by their perceptions of students' multiple

mathematical knowledge bases and curriculum spaces.

How does teacher agency, when supported in a conducive context, allow for teachers' capacity to open curriculum spaces?

To answer the second research question, I focused on identifying dimensions and aspects of teacher agency in the teachers' explanations of their lesson designs. Priestley et al.'s (2015) teacher agency framework provided the structure for the three codes (*iterational*, *projective*, and *practical-evaluative* dimensions) and three sub-codes for the *practical-evaluative* dimension. The unit of analysis was an episode where a teacher's utterances involved the aspects and dimensions of teacher agency, which typically expanded more than one sentence because teachers tended to elaborate on their perceptions, decisions, and reasons in length. First, the *iterational* dimension code was applied when the teachers elaborated on their life histories and past experiences including both personal and professional. In addition to identifying this *iterational* dimension code, I also applied an open coding approach in order to capture the main idea of each episode, and then I categorized these open codes into sub-codes within the *iterational* dimension of teacher agency. Table 2 summarizes sub-codes of the *iterational* dimension code and examples.

Table 2.

The Iterational Dimension Code and Its Sub-codes

Code	Sub-Code	Example
Iterational Dimension	Past Learning Experience	"I always try to emphasize it because when I learned math, I always thought there was only one way to do something."
	Past Teaching Experience	"It was the old school of teaching math."

Table 2 (cont'd)

Code	Sub-Code	Example
Iterational Dimension	Perception of Diversity	"If they're unfamiliar, how do we know if they're unfamiliar? And some kids are familiar, and some kids aren't. And that's the hardest thing when you're meeting so many different backgrounds."
	Perception of Curriculum Adaptation	"I mean, it was tough though. I've taught this lesson seven years and so it's even hard thinking about it with that [goal in mind]." "I had a hard time just because I wasn't familiar with the material."

Second, the *projective* dimension code was applied when the teachers reasoned their interpretations of resources or explanations for their instructional decisions based on their short-term goals or long-term goals. Also, sub-codes have emerged through open coding approach. Because in this study, the teachers and I established a shared goal to design a lesson for open spaces, the teachers expressed their goals to make mathematics lessons more relevant to their students the most frequently. Table 3 summarizes sub-codes of the *projective* dimension code and examples.

Table 3.

The Projective Dimension Code and Its Sub-codes

Code	Sub-Code	Example
Projective Dimension	Making Math Relevant	"How to make that relevant to them when they do these deep problems"
	Understanding/Explaining why	"I want to make sure that they're showing what they're doing as well because I want it to be visible for me." "How did you get that number? What was your thinking? Because I also feel like kids really learn a lot from each other."

Table 3 (cont'd)

Code	Sub-Code	Example
	Seeing/Using Math in Life	"You know that age-old question of 'When will I ever use this?' […] At least, [opening spaces] allows them to see that there is a place for math, always like in their life. And so, that's the ultimate goal for me."
	Goals for Lesson-specific Math Topics	"This problem is like combining some of their strategies from the whole unit and using this idea of percentage."

Third, the *practical-evaluative* dimension code was applied when the teachers elaborated on their considerations of the current situations and possible changes. Priestley et al. (2015) also suggested that within this dimension, three particularly influential aspects could surface: *cultural* (ideas, values, etc.), *material* (resources), and *structural* (social structure such as relationships). After open-coding influences in the teachers' explanations of lesson designs, I categorized them into sub-codes, applying the three aspects. Table 4 summarizes sub-codes and examples of the *practical-evaluative* dimensions.

Table 4.

The Practical-Evaluative Dimension Code and Its Sub-codes

Code	Sub-Code	Example

Practical-Evaluative Dimension	Material	• Resource from the literature
		  ○ "It would be an easy add-on to kind of help with bringing them in."
		• The official curriculum
		  ○ "It was … more like deeper problems rather than just the problems here and there."

	Cultural	• Rapport with students
		  ○ "I guess just even asking the most basic question, 'Which country were you born in?' because we were in a very diverse world. A lot of students are not just born in the US. So that would be an excellent conversation starter."
		  ○ "What was something fun you did over our snow days?"

Table 4 (cont'd)

Code	Sub-Code	Example

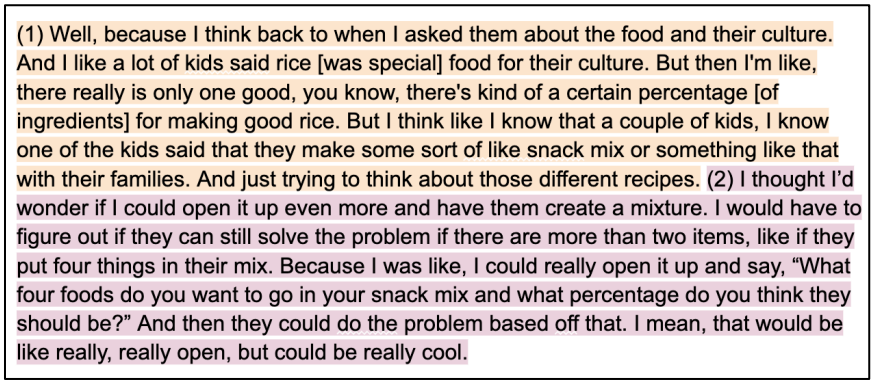
	Structural	• Advisory hours
		  ○ "I'd say, at the beginning and end of math class, I try to do those kinds of things. But we also do have an advisory class now. […] Kind of unstructured, so I'm able to learn a lot about that group through that class."
		• Support from each other
		  ○ "I'm glad we talked about this."
		  ○ "See, like us bouncing ideas off each other. I think we got it!"
		  ○ "So, I thought it was nice to be able to use this time to talk about changes we could make to learn about it."

After identifying codes and sub-codes of teacher agency dimensions, I looked for the relationships among the three dimensions in order to understand how the teachers drew from the *iterational* dimension (the past), orienting toward the *projective* dimension (the future), and achieved the agentic engagement in the curriculum adaptations from the *practical-evaluative* dimension (the present). Then, I looked for evidence of how these codes surfaced within the lesson design processes that were identified for the first research question. This allowed me to illustrate how teachers made decisions on the curriculum use supported by teacher agency.

For example, in one episode of Ms. Mer's modification, she explained her decision on her curriculum adaptations by drawing from her advisory hour conversation (Figure 7, (1)), which is coded as a *structural aspect—Advisory hours* in the *practical-evaluative* dimension code. She continued explaining that she considered open spaces for both mathematical thinking and life experiences (Figure 7, (2)), which is coded as the *projective* dimension. This example shows that Ms. Mer utilized her informal conversation with students (the structural aspect in the practical-evaluative dimension) and oriented toward the goal of opening curriculum spaces (the projective dimension) in the process of making modifications to a curriculum problem.

Figure 7.

An Example of the Relationship Between Codes



CHAPTER 4: FINDINGS ON OPENING CURRICULUM SPACES

The first research question of this study attends to ways in which mathematics teachers open curriculum spaces to draw on students' multiple mathematical knowledge bases. Since the participating teachers used the school's official curriculum set as the main resource for planning daily lessons, their lesson design for open spaces began with them attending to the curriculum problems (*selecting* codes) that led to the interpretation through a lens of curriculum spaces. This chapter reports on what the participating teachers noticed in the curriculum (*interpreting* and *reconciling*), how they learned about students' multiple mathematical knowledge bases (*accommodating*), and how they used the curriculum to design lessons for open spaces (*adding, modifying,* or *omitting*). Then, this chapter illustrates how they enacted problems in the classroom using their lesson design.

Designing Open Spaces

Interpreting the Curriculum and Reconciling Goals

During the collaborative pedagogical imagination phase of the study, where the teachers and I investigated culturally relevant mathematics task examples (Chval et al., 2021; Mathews et al., 2022) and strategies for opening spaces (Drake et al., 2015b), the teachers looked into problem sequences in multiple units of their curriculum through a lens of curriculum spaces. Overall, the teachers reasoned that open spaces exist within their curriculum that help students to connect with their prior understanding of mathematics because the problems directly ask students to utilize and explain multiple solution strategies. On the other hand, the teachers found fewer open spaces within the curriculum that engage students in making connections between mathematics and their life experiences. As they shared their interpretations of the curriculum in terms of open spaces, the teachers concurrently attended to how they wanted to enact the

curriculum to make sure to provide open spaces. In other words, the teachers' interpretations of the curriculum in terms of open spaces were entangled with their reconciliation between the curriculum's approach and their plans for enactment.

Curriculum Spaces for Mathematical Thinking

Across multiple units and problems, the teachers constantly pointed out that the problem sequences were well attuned to encouraging students to use multiple strategies and explain their thinking. This observation was casually brought up when the teachers were sharing their reflections on teaching during that week. (We typically started our weekly meetings with a brief share-out on how lessons and their interactions with students went.) For example, after Ms. Jay taught one problem in the unit about exponential growth, *Growing Growing Growing*, she said, "When I looked at the questions, these are really good for students to come up with some argumentation among them." She excitedly ("That was fun!") explained that the sequence of problems prompted students to come up with two different equations, which led them to have a spontaneous class discussion to decide which equation was "correct" to describe the problem situation mathematically. Even though Ms. Jay had not planned to make it a whole class discussion, she improvised the conversation because she noticed that the curriculum problems engaged students in such argumentation among various solution paths and "learn from each other" by explaining their thinking processes.

After reading Drake et al.'s (2015b) article about the strategies for opening spaces, Ms. Mer said, "[The CMP does] encourage multiple strategies a lot. A lot of problems are really open-ended." Recall that Drake et al. (2015b) suggested practical ways to make small changes in daily lessons by a) rearranging lesson components to frontload problem-solving, b) adjusting numbers or offering choices, c) encouraging multiple representations and strategies, and d)

making authentic connections to real-world experiences. Based on her previous experience of teaching CMP, Ms. Mer highlighted that the curriculum, in general, frontloads problem-solving, which means that teachers would not have to do the extra work for rearranging lesson components. Also, she noticed that CMP problem sequences progressed in a way that would make it easy to offer choices or provide low-floor questions.

For most of the problems in the curriculum explicitly asking students to explain their strategies, the teachers not only acknowledged such open spaces within the curriculum, but they also interpreted it as something that they should make sure to happen during the lesson. Ms. Jay wrote in her curriculum space analysis table that she would put students in groups to make posters and "share their poster and thinking with the class." Similarly, Ms. Mer claimed that "[Spaces for reasoning] would just be kind of how [teachers] decide to share out the different things [during the whole class summary]." As such, the teachers' interpretation of curriculum spaces to draw on students' mathematical thinking was inseparably tied to their decision on the enactment.

Although the teachers found that many problems could allow students to explore various ways to solve a problem, they also pointed out the existence of closed problems in the curriculum. For example, Ms. Mer observed this flow in the unit about the similarity of figures, *Stretching and Shrinking*: Problem 3.1 engages students in exploring how to create and compare reptiles in their own ways, Problem 3.2 provides a set of practice problems, and Problem 3.3 engages students in utilizing various strategies to solve real-world related problems. She explained that Problem 3.1 has "a lot of open spaces" because it engages students in "different sense-making and strategies in how they figure it." Then, she interpreted Problem 3.2 as closed spaces because "these are a lot of those, just like naked number problems." However, she did not

think it was necessary to make a change to Problem 3.2 for its closedness. She claimed that "You have to have some of those kinds of naked number problems at some point for some kind of practice." Like this example, the teachers interpreted curriculum spaces across sequences of problems rather than isolating one problem or question before making decisions on whether they needed to make changes.

In short, the teachers noticed the existence of open and closed spaces within the curriculum for students to utilize their own strategies and explain the reasoning behind their thinking. Particularly, the teachers perceived that the curriculum lends itself to coherently emphasizing the importance of multiple representations, strategies, and reasoning, which reinforced teachers to attend to such opportunities during their enactment. In other words, the teachers found that the curriculum was well aligned with the goal to provide students with learning opportunities to build on their mathematical thinking as they developed an understanding of a new mathematical concept.

When it comes to opportunities for connecting mathematics with students' life experiences, the teachers' interpretations of the curriculum revealed the dissonance between the curriculum's problem contexts and their goal for open spaces. For example, when Ms. Mer talked about Problem 3.3, whose problem situation involves building a bridge over a river and figuring out the distance, she wondered, "Do I think any of the kids have ever thought about doing this, like using old school map and measuring? I don't know if that's really a real-life connection for them." In the following section, I describe what teachers noticed in the curriculum in terms of open spaces for students' connections with their life experiences.

Curriculum Spaces for Real-Life Experiences

When teachers discussed open spaces focusing on how the curriculum encouraged

students to make connections with their life experiences, they distinguished real-world related problem contexts from imaginary-world contexts. While they perceived that imaginary-world problem contexts would not necessarily be relevant to their students' cultural experiences, the teachers expressed their belief that such contexts could be effectively used for student understanding. For example, Ms. Mer wrote in her curriculum space analysis table, that Problem 2.1 in the *Stretching and Shrinking* unit was "a heavy workload, but has a good context that students enjoy and remember." She highlighted that even if the characters do not exist in students' real life or the video games they play, the story about the characters that continue throughout the sequence of problems helps students engage in problem-solving. The main inquiry of this problem set is to figure out who are imposters that are not similar figures to the Wump family by graphing other characters with given rules. Although she mentioned that the introduction of the problem brought up computer games probably to make the problem context relevant to students' interests, she emphasized that her students liked talking about the Wump family characters and finding imposters rather than computer games:

> [It has a good context] because […] it's like this little character that they graph, and it's just like a trapezoid with legs and [a] little face. But the kids think it's so cute, and they will remember it. I bet if you ask the 8th graders if they remember Mug Wump, [they will say yes]. What I first was taught about CMP was that there were these stories or contexts that keep coming up or that they try to make overall stories. And so, the kids just think it's a cute character. Because it's kind of not difficult to graph, but it takes time and energy to do it. [Students] feel very accomplished when they do it, and so I know through the years that the kids have

really remembered it. And then, they love how it changes the goofy characters [by the different rules to graph].

Because she had observed students' engagement in the problems, she wanted to preserve the problem context without making changes. Even if the characters and stories were imaginary and not directly relevant to students' real life, she mentioned that she could also encourage students to connect the story with their experiences. She imagined asking students to think about a situation of being an imposter in a family or "their family trees and how genetics have changed over time, [such as] who has been stretched."

Similarly, when Ms. Jay taught one lesson about exponential growth in the unit, *Growing Growing Growing*, she noticed that students were very engaged in the storyline. It was a story about a peasant in an ancient kingdom requesting the king and queen to grant her the chess champion reward by an exponential rule. She particularly appreciated that the peasant story developed through multiple questions because once students understood the problem context, they could focus on constructing mathematical representations and interpretations. Both teachers commented that this story would be a familiar "fairy tale" as students would have been exposed to some sort of story about an ancient kingdom featuring the king and the queen regardless of their cultural backgrounds. Ms. Jay did not want to change this imaginary-world context to make it more directly relevant to students' real life. Instead, she suggested asking students to reflect on the story with the question, "Have you ever tried to make a deal with someone or a bargain, and then they want to counteroffer you because [the offer] is not so great for [them]?" She thought that it might help students relate to the peasant problem's storyline because the problem sequences presented different offers, and the peasant could decide the best option for her.

As demonstrated in these examples, teachers noticed that some problem contexts were

not necessarily relevant to their students' life experiences and rather situated in an imaginary world, and they interpreted that those problem contexts could be powerful to engage students in developing mathematical understanding. Thus, they did not find it critical to change imaginary-world situations to be more relevant to students' real life. Instead, they considered adding extra questions that could help students make personal connections to the problem story, such as having them think about 'imposters' among their family members or 'make a deal' with someone.

Contrasting to these "fairy tale story" problems, the teachers wondered whether students would be able to relate to the real-world related problem situations. For example, one problem in the 7th-grade unit, *Comparing and Scaling*, deals with a situation of making orange juice using the concentrate and explores the percentages to make the strong or light orange flavor. Ms. Mer noticed that although the problem was well designed to engage students in exploring the meaning of percentages, students might wonder why someone would make orange juice with the concentrate because they would not be accustomed to making juice with the concentrate. The teachers agreed that students might not find some problem contexts relatable even if they were using real-life objects. For another example, Ms. Mer shared that her students pointed out that one problem context in the unit, *Stretching and Shrinking,* which used magazines as a unit length to figure out the mystery teacher's height, could be updated to use objects other than magazines. While students found the story interesting, they thought it seemed outdated because their school did not have magazine racks in the hallways anymore.

In addition, the teachers shared their concerns about how to implement "traditional" real-world examples. Ms. Mer commented on one such problem context, "Even though they are real-world [related], […], they feel very routine and very kill-and-drill type thing." She further argued

that she might be able to create an open space by changing the enactment without necessarily changing the problem context. She shared her plan for having students experience "traditional" real-world examples:

> I feel like, I've actually never done the experiment, but I think I'm going to this year just because I'm like, "Let's give it a try." [I haven't done the shadow experiment] mostly because it's in the middle of winter and we don't have much sun, or sunshine here [in the Midwest]. But we're going to do the mirror. And I'm interested to see how that helps students with that concept and lesson because sometimes they do have a really hard time understanding that you can see in the mirror and that it makes those similar triangles. So, I'll be interested to see how it goes because that's [going to be] a super open curriculum problem where we talk about this real thing and then they're going to go experience it and measure it themselves and kind of [solve] problems and figure it out and how that makes similar figures.

With the exponential growth concept, Ms. Jay also planned to let students experience traditional real-life examples. The problem suggests an activity to repeatedly fold and cut a paper so as to observe how the thickness of a pile of ballots would change. In addition to this hands-on activity, Ms. Jay considered asking students to come up with their own examples that could represent a particular mathematical concept. She shared her plan for the enactment, "I was trying to think how students can come up with their own examples. So, I was thinking of giving the students an opportunity to come up with different examples of where they could see exponential growth."

In sum, teachers noticed fewer opportunities for students' real-life connection as written in the curriculum compared to opportunities for mathematical thinking. While they viewed that

the problem contexts do not always have to be relevant to student life experience as imaginary characters can also help students develop mathematical understanding, they wanted to add questions to make sure students have a chance to think about the problem context and story based on their experiences. Moreover, when teachers noticed that some of the real-world contexts in the curriculum would be less relatable for their students, they spontaneously extended the discussion to brainstorm how to enact the curriculum. For example, they wanted to add questions and activities that could engage students to leverage their experiences to develop mathematical understanding. One way of achieving this goal can be letting students conduct a classroom experiment for themselves, such as a mirror experiment or cutting ballots. Another way would be asking students to come up with their own examples like Ms. Jay's plan for the exponential growth lesson. Particularly, when they interpreted that some of the real-life problem contexts could be more relevant to students' real-life experiences, they considered possibilities for them to bring their own students' experiences into their lesson plans. In the following sections, I will report on how the teachers attended to understanding their students' multiple mathematical knowledge bases and planned one of their lessons including open spaces.

Accommodating Students' Interests and Needs

As the teachers noticed that the curriculum offered fewer opportunities for students to connect problem contexts with their life experiences, teachers navigated ways in which they could learn about their students' cultural experiences and use them as resources for teaching mathematics. Even if teachers could encounter a good example of students' cultural referents in their own life, it might not be readily applicable to mathematics topics. If they did not leave a note to remind themselves of the potential usage of the example for a particular mathematics topic or lesson, it would be likely to be forgotten. Thus, the teachers wanted to learn about their

students' multiple mathematical knowledge bases that could specifically be incorporated into their upcoming mathematics lessons. As one way to learn about students' multiple mathematical knowledge bases, particularly students' cultural experiences, the teachers facilitated informal conversations with students during advisory hours and at the beginning of mathematics lessons. Because teachers viewed advisory hours as a space for both teachers and students to be able to relax and "get to know each other," they thought that they could learn about students' cultural experiences through this informal conversation.

Ms. Jay's Learning About Her Students

At the time of our discussion for advisory hour plans (our fifth meeting), she was about to teach problems in *Growing Growing Growing*, so Ms. Jay wanted to gather some information about students' multiple mathematical knowledge bases related to the concept of exponential growth. However, she did not feel comfortable asking students to generate exponential examples in their daily life, in part because she found it difficult to come up with real-life examples for herself. Alternatively, she planned for a conversation about the concept of percentages because it would still be the fundamental concept to figure out the exponential growth rates. She said that her students' backgrounds could be expressed with percentages in many ways, such as comparing student populations in terms of different ethnicities or explaining how individuals are mixed races. She facilitated a mathematics activity that could allow students to express themselves and their prior mathematical understanding.

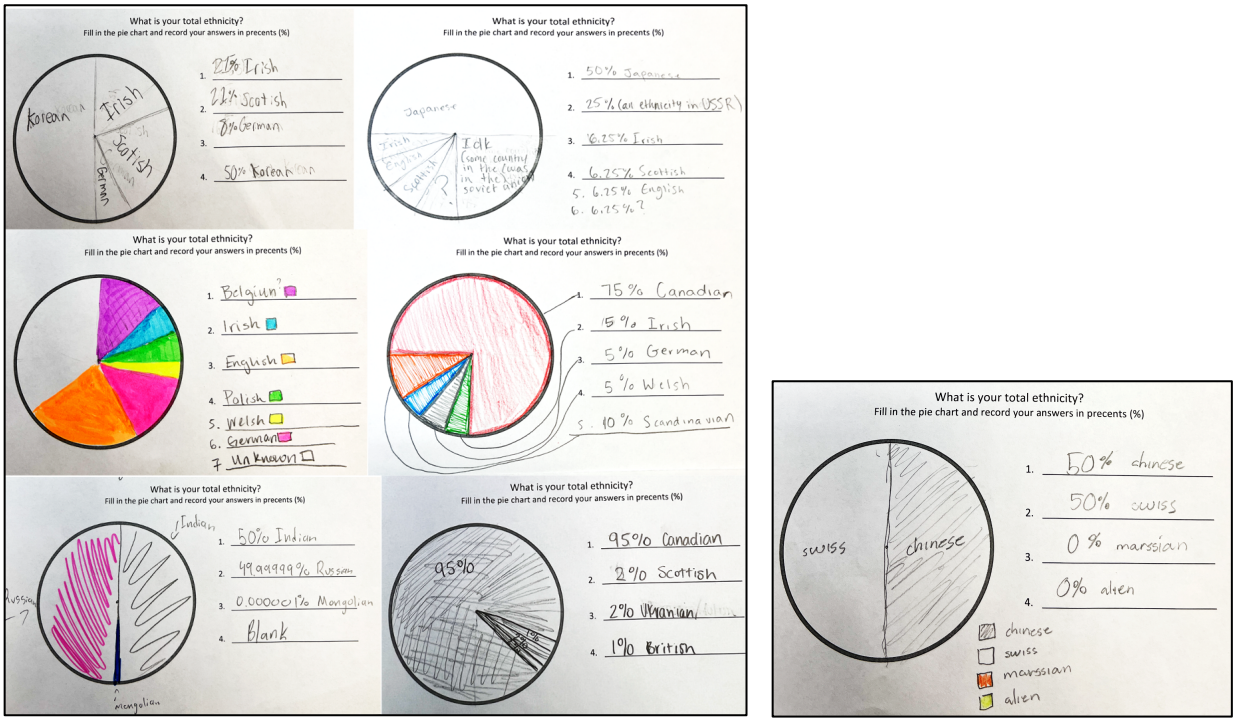
During our reflection meeting after the activity, she reported that she initiated the activity by asking students to think about percentages of themselves. She posed a question, "What is your total ethnicity?" and had students draw a pie chart of themselves on a worksheet before freely sharing their charts with her and other students. To be mindful of those students who would not

want to talk about their backgrounds, she mentioned that they could use personality traits or characteristics (e.g., friendly, kind, funny, etc.) for their pie chart as well. Moreover, she said that she had this activity with all of her mathematics classes beyond the advisory hour students because she thought it would be cool to learn about students' perceptions of their heritages, especially about those students who she would otherwise have had no chance to recognize their ethnicities.

Through this pie-chart activity, Ms. Jay provided students a chance to recognize that they are from all over the world. In their written responses, students indicated their ethnicity as 100% (e.g., Indian, Black, White, Egyptian, etc.), 50% to 50% (e.g., White to Asian, Iranian to Turkish, African American to Indian, Lebanese to Palestinian, etc.), or multiple components more than two. Figure 8 (Left) captures examples of three or more ethnicities students put in their pie chart. In our reflection meeting, we agreed that students' responses provided a glimpse of their unique stories about their heritage, their conversations with family members, and their perceptions of themselves. Ms. Jay described that her students were very engaged in expressing themselves and interacted with one another, "Wow, you are that too? That's cool, we are both so and so!" It is worth noting that students came up with the ethnicity components not based on factual investigations but based on their beliefs, perceptions, or creativity. For example, some students indicated, "x% unknown" or "50% mom and 50% dad," or "0% [Martian] and 0% alien" (Figure 8, Right). Ms. Jay said that she was conscious of making non-judgmental comments and validated all of these responses while students were sharing their charts with her and other students.

Figure 8.

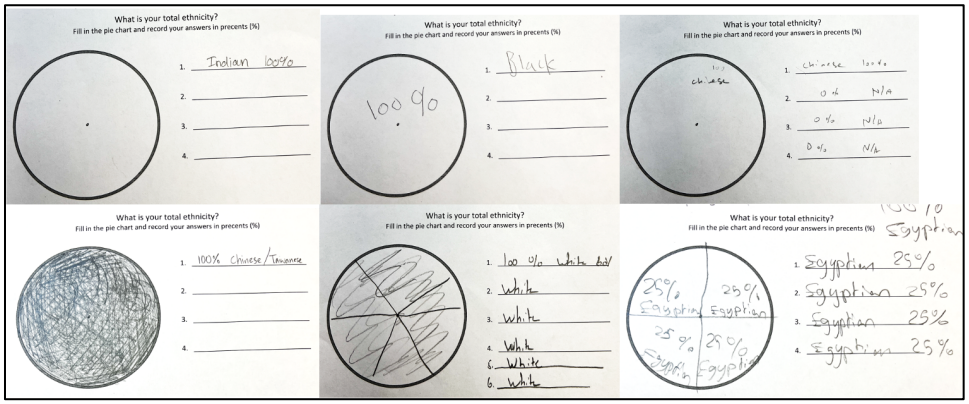
Student Work Examples: Ethnicity Pie Charts



Moreover, these examples provide some information about students' understanding of percentages. As shown in Figure 8, some students visualized relatively very small areas in a chart to represent small percentages such as "1% British" or "0.000001% Mongolian." Some students, though, did not write their responses in percentages as instructed on the worksheet. Also, various representations for the same percentage were employed by different students. For example, Figure 9 shows various ways that students drew a pie chart to represent 100%. Some students viewed 100% as a whole (i.e., an empty circle with a center point or a shaded circle), whereas some other students divided a circle into four or six pieces and labeled these small pieces with the same ethnicity (e.g., Egyptian or White). One student labeled "Chinese/Taiwanese" as a whole, 100%.

Figure 9.

Student Work Examples: Expressing 100%



While Ms. Jay assessed that many students seemed comfortable converting percentages into a pie chart, as prior knowledge of growth rates, she also mentioned that if she noticed students struggling with computing growth rates, she would bring up this pie-chart activity to help students solidify their understanding of percentages and growth rates. With a big smile, she stated that she was so glad that this activity "worked out well" in that she could learn more about students' perceptions of themselves and she could use this information for future interaction with students both in advisory hour communications and during the mathematics lessons.

Ms. Mer's Learning About Her Students

As she was planning for the unit, *Comparing and Scaling*, which includes "so many mixture problems" like the orange juice problem, Ms. Mer assumed that she might be able to get some cultural references if she asked students to share their favorite food and recipes. At first, she was thinking about asking a question, "What are you going to do over break?" Then, she changed it to "What is a food that's special to your culture or family?" because she thought it could specifically yield conversations about ratios among ingredients. In fact, she always displays a class screen with the goals of the day's lesson and one ice-breaker question so students can talk to each other as they walk into her classroom and settle in their seats. These daily

random questions could possibly be anything. When I visited her classroom, the day's question was "Would you rather eat only breakfast food all day or never eat breakfast food?", for example. So, she thought that her students would not find it strange to talk about special food and recipes if she threw the question about special food at the beginning of mathematics lessons. In this case, her purpose would be not only to let students chat about out-of-school experiences but also to gather some information from students so she could use it for her lesson planning.

One thing to highlight in her plan for this conversation was that she wanted to take notes about students' responses. She thought that she would be able to quickly take notes during the advisory hour whereas she gets easily distracted in regular mathematics classrooms. She implied the complexity of teachers' role in the classroom:

> In my math classes…I'm listening [to what students are sharing], but I need to be a little bit more mindful about listening to them. Which I'm sure we're all guilty of. Like, when kids are talking to us, we're not always fully listening because we're doing twenty other things. In advisory when I am not doing all those other things, I can sit and be really mindful about what they're talking to me about.

In our reflection meeting, she reported that she *really* listened to and took notes of students' responses because she did not make it as an activity with a worksheet as Ms. Jay did. Ms. Mer said that she received a wide range of responses from her students, including some she found familiar (e.g., cinnamon rolls or latkes) and some she had never heard of (e.g., French cashews, Polish croissants, or rice variations). She also noticed that many students talked about something made with rice, which she thought reflected the large Asian population in the area. She self-assessed the conversations to be successful in that she could be able to gather "quite a variety of responses" that might be useful for her problem adaptation. When asked about her perception of

student participation in the conversation, she explained that with her daily random questions, she kept explicitly encouraging students to respect others' responses even if they could not relate to them. After having these conversations, she considered changing one orange juice mixture problem context to a situation where an Asian student wants to make rice because "[making rice] has a pretty specific ratio and scaling the recipe up or down [to have] the right consistency, stickiness."

Similar to Ms. Jay's facilitation for her class activity, Ms. Mer also strived to create a space for students to freely share out (or not share out) so students could recognize the variety of special foods from all different cultures. Furthermore, she considered utilizing the information she learned from this conversation in adapting curriculum problem contexts. She said she could also use the different special food ideas and recipes for check-up quizzes. It was because she thought that changing (or creating) problem contexts would be easier for her formative assessment purposes than with the curriculum set whose problem contexts often span across multiple problems. She revisited her idea about using students' special food to change the problem context when we discussed lesson plans for open spaces in the following section.

Summary: Learning About Students' Multiple Mathematical Knowledge Bases

The teachers strived to learn about students' multiple mathematical knowledge bases in the pursuit of planning mathematics lessons that could provide students with open spaces for making connections between mathematics problems and their life experiences. By facilitating conversations about students' diverse cultural stories, the teachers provided students with an opportunity to reflect on their family cultures and share their stories. In addition, students might also have a chance to view the cultural, ethnic, and linguistic diversity in the classroom as something *cool* and *normal* rather than through a sense of superiority or inferiority because the

teachers seemed conscious to facilitate class conversations in a way that respects differences.

One interesting point about these activities to learn about students' mathematical knowledge bases was that both teachers facilitated the conversation with all of their mathematics classes, not solely with advisory hour students. They explained that it would not take up too much time during lessons, and yet it could provide them with resourceful information for planning lessons. For these activities, the teachers found it worth spending class time on in that they could consider making connections between students' cultural, out-of-school experiences and mathematics problems.

Modifying the Curriculum Resources to Include Open Spaces

Based on the weekly reflections about their teaching practices with regard to students' multiple mathematical knowledge bases and curriculum resources, the teachers planned one of the upcoming lessons to include open spaces for students' multiple mathematical knowledge bases and invited me to observe their classroom enactment. This section illustrates the teachers' lesson plans to include open spaces. Partly because they were in different grade levels using different problems, the teachers' approaches to *open* curriculum spaces were different from each other. Ms. Jay added one question that asks students to interpret the problem context based on their experience, and Ms. Mer changed the problem context to use students' own experiences. By making these modifications, the teachers hoped to provide students with open spaces for connecting with their experiences. Also, both teachers ensured to incorporate open spaces for connecting with students' prior mathematical understanding afforded by the curriculum.

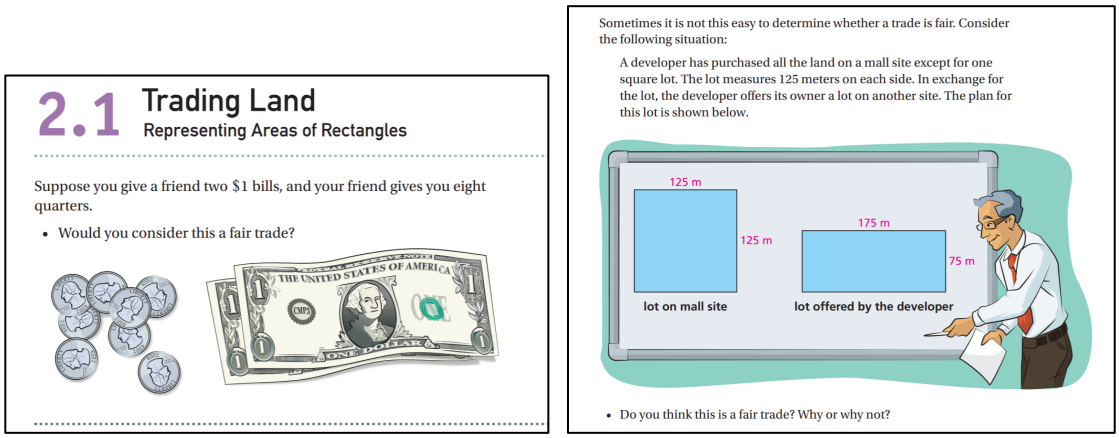
Ms. Jay's Lesson Plan for Open Spaces

Ms. Jay was planning a lesson for Problem 2.1 in the unit, *Frogs, Fleas, and Painted Cubes*. Prior to this problem, students had learned about expressing relationships among side

lengths, perimeters, and areas of rectangles. Using the same area context, the current problem asks students to compare two rectangles in terms of their areas. It is contextualized in a situation where a land developer offers the owner of a piece of land 'a trade' to build a mall, and it begins with the question, "Do you think this is a fair trade? Why or why not?" (Figure 10, Right). This question can potentially provide an open space for students in that students can explain their thoughts about fair trade using their own words because the problem has not suggested a particular way that students need to follow to answer the question. Before this open question, there is an introduction page (Figure 10, Left) that shows graphics of money. It reads, "Suppose you give a friend two $1 bills, and your friend gives you eight quarters. Would you consider this a fair trade?"

Figure 10.

Problem 2.1 in Frogs, Fleas, and Painted Cubes (pp. 24 ~ 25)



Her first reaction to this introduction page was that she would jump to let students share their opinions on this exchange between bills and coins. Then she brainstormed what her students would say and realized that the class conversation about this exchange might go away from describing 'fair trade' and more toward sharing experiences of using coins or cashless payments (e.g., "Who would like to carry coins these days?"). She was unsure of how much time

she would end up spending if she threw this question to students because she could not see the purpose of this money context. She went on reading the next page to recognize that such conversation about coins or cash would have nothing to do with exchanging two different rectangular land pieces:

> I can talk about the money but then, what does that have to do with the land, the size of the land? Talking about like, how small the coins are and the size of the dollars, but they're worth the same, but the land doesn't work that way. The perimeters are the same. But their areas are not, and you want more area.

She interpreted that the trading land pieces situation connected back to Problem 1.3 where students compared two rectangles with the same perimeters, and she realized that the coins vs. bills situation on the first page was designed to open a space for talking about what 'fair trade' means. She decided not to "go in-depth with the money" and focus on "talk[ing] about how that's a fair trade." While she was sharing it, she was reminded of her childhood growing up with siblings when she and her siblings used to trick each other to make an unfair trade look fair. She said:

> Usually, the older sibling tricks the younger ones into thinking they're trading something fair, but they're really not great things. With my brother, all the time, … you break [one cookie] in half and say, "You're getting two cookies, give me your big bowl of ice cream. I'll give you *two* cookies."

This example from her own story inspired her to formulate a question, "Have you ever traded something on someone that was fair? You made them think it was fair? Or you didn't get the fair amount?" She planned to facilitate the class conversation so that students would reflect on what aspects they need to compare between two sides in order to decide the fairness of trade. For the

money exchange situation, students would compare the value of money; for the land exchange situation, students would compare the area of rectangles. By adding one more question, "Have you ever traded something on someone that was fair (or unfair)?", she anticipated for students to connect the meaning of fair trade with their personal experiences. She believed that with this connection, students could easily jump into the main problem of fair trade between two rectangles that have the same perimeter and different areas.

In short, Ms. Jay included one extra question that could invite students to make sense of the original problem context based on their personal experiences. She did not change the problem context because she found the context about the areas of rectangles connected with the previous problem situations. For the purpose of providing open spaces for students to connect with their life experiences, she planned to let students share personal stories about fair/unfair exchanges with family members or friends before mathematically investigating fair/unfair trades between two land pieces described in the problem.

Ms. Mer's Lesson Plan for Open Spaces

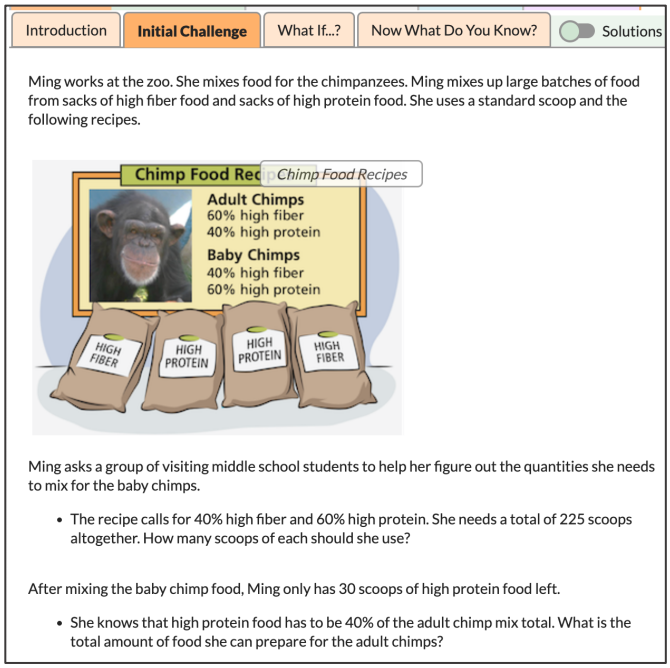
Ms. Mer was planning a lesson for Problem 3.3 in the *Comparing and Scaling* unit that encourages students to "combine some of their strategies from the whole unit and using this idea of percent [in a real-life situation], like [the introduction] says, that doesn't involve money." The original problem asks students to figure out the number of scoops using the given percentages of ingredients, whose context involved making food mix for chimpanzees in the zoo (Figure 11). Ms. Mer noticed the open space in terms of students' opportunities to come up with their own ideas that can center the whole class discussion around multiple different strategies to answer the questions:

It's really open in the sense that it has space for multiple strategies like [in] this

whole unit. […] They can change it to a ratio, they can use a proportion, [and previous problems in the unit] teach them these percent tape strips [so students could use them]. […] I think that's in the sense of kids bringing their own strategies, and multiple representations. It's going to be really open.

Figure 11.

Problem 3.3 in Comparing and Scaling (Screenshot from the digital curriculum)



In contrast, she was not convinced if students would relate to the problem situation of making chimp food by saying, "Most kids don't have much connection to this context. Maybe [it is] interesting, but it's not going to be relevant to them." Based on this interpretation, she wanted to replace the context with something else rather than adding more information or activity that could help students understand this chimps' food recipe context. It was because she thought that she could easily find alternative situations, such as snack mix, paint mixture for making different colors, or color mixes in a bag of M&M chocolate, all of which do not involve money as the curriculum suggested. She identified the essential aspects entailed in the problem and considered

them while selecting an alternative context. As long as a new context contains the same structure as the chimps' recipe problems, such as using two different percentages (40% and 60%) and using two different recipes (adults and babies), she stated that "there could be so much more openness in the context than this chimp problem."

Among many other possibilities for a new context, Ms. Mer thought about connecting back to her class conversation about special food items in students' families or cultures. At first, she wanted to utilize a recipe using rice because she learned that a lot of her students' special foods contained rice. As she mentioned during the reflection meeting, she thought that making rice would use the concepts of percentage because it needs the ratios of water and rice. However, when we met for planning this lesson, she said she would not use the rice example. It was because she recognized that with rice situations, it would be hard to keep the same structure of the chimp's recipe problem because there would be "a certain percentage for good rice," namely one with the right stickiness. Thus, she decided not to opt for the rice recipe and looked for another possible context that could provide different percentage combinations. Her thought process about making rice context briefly illustrates how she selected an alternative context, interpreted it for potential adaptation, and decided to keep searching for other contexts. In other words, she interacted with the information she gathered from her students in a similar way that teachers use a curriculum (Sherin & Drake, 2009). This might mean that she treated students' examples of life experiences shared through her informal conversations as potential curriculum resources.

Then, she described that she moved on to examine a snack mix context, remembering that when they had a conversation about special food for families and cultures, some students said they made snack mixes for family hiking. Using a snack mix idea, she thought that she could

make the mathematics problem even more open and challenging by having students create a mixture of two or more ingredients. She explained, "I could really open it up and say, 'What four foods do you want to go in your snack mix? What percentage do you think they should be?' And then, they could do the problem based on that." Because the original problem uses two ingredients with percentages, she wanted to make sure that with more than two ingredients, the problem structure would stay the same as the original problem, and students would still be able to engage in the same mathematical thinking. This consideration might have influenced her to include the 'should total to 100%' condition in her modified problem (Figure 12).

Figure 12.

Ms. Mer's Modified Problem (Part 1)

You are making the ultimate snack mix for our camp's movie night!

- Name your Mix.
- Include at least 2 food items (up to 5).
    - Describe how much percent each item will be.
    - Should total to 100%
- Make Version 2
    - Include the same items.
    - Change the percent amounts.

In the original problem, two percentages for each recipe add up to 100%, and yet it does not need to be explicitly stated in the problem because the two percentages are given numbers. Thus, it is an important point to consider when students are to select two or more ingredients. In addition, regarding the point that students would have different percentages of various ingredients to make their own snack mix, she replaced the original problem's adult vs. baby recipes by letting students create two different snack mix recipes. With this change, she thought that "[it] could be fun too, because, in their group, it kind of connects them personally to it. And then I think I could still pose the same questions, the same open questions to the whole class no matter what they choose."

In short, Ms. Mer selected the snack mix context based on her consideration of students' interests, which was influenced by her informal conversations about favorite/special food items. Also, she made sure to preserve the structure of questions provided in the curriculum because she trusted that those questions could help students examine multiple strategies and explain their solutions. Thus, Ms. Mer modified the curriculum context to incorporate students' multiple knowledge bases by letting students use their selection of food items to solve mathematics problems. For this snack mix context, she anticipated that students would share out various ingredients to their own liking because she had already posed so many random questions related to foods.

Enacting Open Spaces

As illustrated in the previous sections, the teachers designed lessons with the instructional goal of providing open spaces for students to draw on their multiple mathematical knowledge bases—i.e., teachers viewed students' prior understanding and experiences as resources for their learning. Applying Brown's (2009) terms, on the one hand, teachers decided to *offload* the curriculum to adhere to questions and situations in the curriculum because it generally encourages students to construct mathematical understanding in their own ways. On the other hand, they decided to *adapt* the curriculum to include direct questions that could encourage students to reflect on their life experiences because some problems do not explicitly support students in making connections with their own experiences.

This section reports on how the teachers' lesson plans for open spaces were enacted in the classrooms, based on the researcher's class observations, the teachers' reflection journal after the enactment, and the collective reflection meeting among the teachers and the researcher. Although this study focused on teachers' lesson design activities, understanding how the plans

are enacted can enrich explanations for the teachers' pedagogical design capacities (Amador, 2016). During the planning meetings, the teachers did not explicitly share their plans for the entire class time but rather focused on how to open a space to connect with life experience. There were a few similarities between the two teachers' classrooms regarding how they enacted open spaces. First, during the introduction of the lesson, they reviewed mathematics concepts and problem-solving strategies from the previous lessons that were related to the current lesson. Second, during the *Launch* phase, where students are introduced to a new problem, they opened a space for students to connect with their life experiences. Third, during the *Explore* (students' small group work on problems) and *Summarize* (a whole class discussion) phases, the teachers encouraged students to use multiple strategies and explain their strategies. Lastly, both teachers attended to creating an inclusive classroom environment that seemed to manifest students' engagement in open spaces.

Introduction: Connecting to Students' Mathematical Thinking

At the beginning of the lessons I observed, both teachers presented the topic and learning goals of the day through the projector screen and reviewed the mathematics concepts from the previous lessons before introducing the day's problem. For Ms. Jay, it was mainly because, in the exit ticket quizzes of the previous lesson, she noticed a lot of confusion in students' understanding of equations involving perimeters and areas of rectangles. She "decided not to grade" students' answers to the quizzes, and she instead planned to revisit the previous problem together as a warm-up before jumping into a new problem. Even though she said she would "go over the problem" with students and she took the lead in the conversation, she was not demonstrating the procedure but facilitating a class conversation about how to make sense of equations. For this review, she projected her notebook on the whiteboard and drew a rectangle on

her notebook. She explained the relationship between its sides, the perimeter, and the area, building off students' responses. For example, one student expressed that the perimeter of Ms. Jay's rectangle would be "Width plus Width plus Length plus Length," and Ms. Jay wrote down the student's answer next to her rectangle and asked students if they could see where this student's expression came from.

Similarly, Ms. Mer facilitated a class conversation about expressing percentages as the review. She thought that for her lesson plan, students would need to utilize various ways of representing and figuring out percentages and quantities. Particularly, since it was right after two snow days, she wanted to give students a chance to review what they did in the last mathematics lessons. So, she focused the class conversation on sharing out different ways to interpret an equation as well as different ways to figure out missing quantities in given equations. For instance, when she asked, "Who thinks they have a way of figuring out this answer?" students shared out the strategies they took: "I can find 10% and multiply it by 8" or "I'm trying to find the whole." Then, Ms. Mer used each student's strategy to further ask the whole class if it made sense and if there could be another way of approaching the question.

As such, both teachers' lessons began by providing students with an opportunity for reflecting on their previous lessons before developing mathematical understanding within a new problem context. More importantly, during their reviews, both teachers called for students to explain concepts or strategies in their own words rather than converging to one standardized way of thinking. This opportunity could potentially allow students to make sense of mathematical ideas based on their prior understanding. In other words, the teachers provided students with an open space to draw on their mathematical thinking.

Launch: Connecting to Students' Real-Life Experiences

After taking about 7 to 10 minutes to review, the teachers introduced a new problem for the day. Both teachers brought up the questions planned to provide students with an open space for real-life connections. As the outcome of the teachers' design process, Ms. Mer modified the curriculum's problem context to have students select snack items while keeping the structure of mathematics questions. Ms. Jay added one question to ask about students' experiences of fair/unfair trade with families or friends. Albeit these differences in their approaches to modifying the curriculum, both teachers' introduction to the new problem provided students with an opportunity to connect the problem context with their out-of-school experiences.

Following Ms. Mer's brief introduction to the first part of her modified problem (Figure 12), each group of 3 or 4 students selected food items with percentages to make a snack mix and created a name for the mix. As freely talking about their favorite snacks and foods with groupmates, some groups decided to allow each person to nominate one item with an equal percentage, while some other groups discussed different percentages for different items based on the group's agreement on particular favorite ones. Only a few items did seem popular across the three classes I observed, and in general, there was a wide range of different selections of snack mix ingredients. Relatively common items chosen by students include Cheese-its and M&M, and relatively unique items include Canadian smarties, Sushi, Ice, and Halal marshmallow. While student groups were selecting ingredients, Ms. Mer circulated the room and checked in with each group. Having selected items, some groups allocated percentages of each ingredient evenly (e.g., 20% for each of five ingredients or 25% for each of four ingredients), whereas some other groups used different percentages to make up 100% in total (e.g., 60%: 20%: 20%, 30%: 15%: 15%: 30%: 10%; or 30%: 2%: 25%: 21.5%: 21.5%). Interestingly, no group selected a single

ingredient with 100% or two ingredients with any percentage combinations.

By allowing students to freely select snack mix ingredients and name their mix (e.g., "Csnack mix" when all items begin with the letter C, "Nex mix" for version 1 and "Next mix" for version 2, or "CHWEET"), Ms. Mer let students make their experience be the lesson's problem context. In addition, students were asked to choose the number of items between 2 and 5 at their preferences, which could serve as a low-floor entry that invites students to begin their learning from where they would want. As Ms. Mer observed during this activity and reported during our post-reflection meeting, it seemed obvious that students were excited to share their favorite snacks and talk about food items, which perhaps influenced their continued engagement in solving the subsequent mathematics questions.

During Ms. Jay's *Launch*, she read the introduction page of the textbook that was about exchanging two $1 bills with eight quarters (Figure 10 Left), and she posed the question, "Do you think this is fair trade? Or do you think it is not really a fair trade? Why?" Some students voted for 'not fair' and explained that bills are better than coins to carry. Some other students voted for 'fair' and claimed that two $1 bills and eight quarters are the same amounts of money, and occasionally coins can be useful to pay the exact amount of money. Then, students started to argue against the opposite voters by bringing up specific examples, such as "coins are useful when using a vending machine." That is, students had all different reasons for deciding which is better between coins and bills, and they had not yet built a shared understanding of what 'fair trade' would mean. Ms. Jay paused the students' sharing out their claims and reasons to tell them her own childhood story about how she tricked her younger brother (taking one $10 bill from him and giving him two $1 bills). Then she asked students if they also had similar experiences by posing, "Has anyone had a thought that they made a deal that was fair, but it was not fair? As an

older sibling or as a younger sibling?" Students started to share out their stories to highlight how unfair it sounds in retrospect. Every time someone shared their unfair trade story, students laughed out loud. Examples of students' unfair trade stories include one cup of orange juice to take over the family computer, a smaller room with a "nice" view for a larger room, and exchanging foreign currency with US dollars, among others.

Thinking about unfair trade examples might have helped students understand what values they should compare in order to not make one side take advantage of the trade. When Ms. Jay presented the problem situation (Figure 10, Right), students immediately evaluated that the deal was in favor of the land developer. Students pointed out that people would be tricked by a bigger number on the rectangle than on the square in the diagram. Also, students mentioned that some people might see it as a fair trade because the perimeters were equal, which led them to claim that people should compare the areas instead of perimeters. Instead of telling students that fair trade means exchanging items of equal value, Ms. Jay shared her childhood story and let students share their stories of making (mostly) unfair deals with family members. As Ms. Jay mentioned during the planning and reflection meetings, her students would feel comfortable sharing their experiences when she first shares her own story.

In summary, Ms. Mer's students had a chance to create a mathematics problem using their selection of items and percentages, and Ms. Jay's students had an opportunity to construct a shared meaning of 'fair trade' based on their own personal stories. One important similarity between the two teachers' classes was the teachers' reactions to students' responses. Both teachers showed that they were listening to students, encouraged students to listen to each other, and validated students' responses. In other words, while students shared their life experiences during the Launch, both of the teachers implicitly paid attention to accommodate a classroom

environment to be inclusive. This point will be further discussed in the later section.

Explore and Summarize: Connecting to Students' Mathematical Thinking

After being introduced to the problem situation, students started to work on the problem individually, with a partner, or in a group. During this *Explore* phase of the lesson, the teachers explicitly encouraged students to use whichever strategies that made sense to them. Also, when the teacher facilitated class discussions during the *Summarize* phase, they focused on sharing students' different approaches. As the teachers repeatedly expressed throughout our weekly meetings, the curriculum, by its design, provides students with opportunities for taking various approaches and explaining their solutions, the teachers explicitly encouraged students to use their own strategies and called for an explanation about their thinking. Thus, the teachers accommodated an open space during the *Explore* and *Summarize* phases of lessons where students could perceive that there could be more than one way to get the correct answer and that it was more important to explain how they got it than what they got.

In Ms. Mer's classroom, student desks were organized into seven groups, and the *Explore* phase of lessons was devoted to group work. Once the student groups came up with their snack mix recipes and the name of their mix, Ms. Mer presented the questions that students should use their group's recipe (Figure 13). Circulating the classroom, she checked in with each group and answered questions from the students. It was notable that in many cases, students asked her questions to clarify if they understood the problem correctly or clarify their confusion in their strategies rather than seek her direct explanation for the right solution. Thus, the conversations among groupmates and between students and the teacher foregrounded students' current understanding of percentages and their development of them. Later in the reflection meeting, she explained that students had learned over time (it was their second semester with her) that their

mathematics teacher would not give one simple answer and instead, she would ask them back what they thought and why they felt challenged. As students moved out of "fun snack mix chat" and into talking about mathematics, Ms. Mer noticed that some students began to struggle. It might be partly because students were using the percentages of their own choices and found the numbers complicated to manipulate as some groups had percentages with decimals. When students seemed lost, Ms. Mer pointed them to the class's earlier review about various ways to interpret percentages and ratios. This way, she maintained the open space so that students could continue drawing on their previous understanding.

Figure 13.

Ms. Mer's Modified Problem (Part 2 and Part 3)

Use your snack mix Version 1: • Each camper gets 1 cup of Snack Mix. • There are 210 campers. How many cups of each item do you need to make enough for the whole camp?	Use your snack mix Version 2: • After making mix #1 you only have 30 cups of ingredient #1 left. • How much of a mix Version 2 can you make?

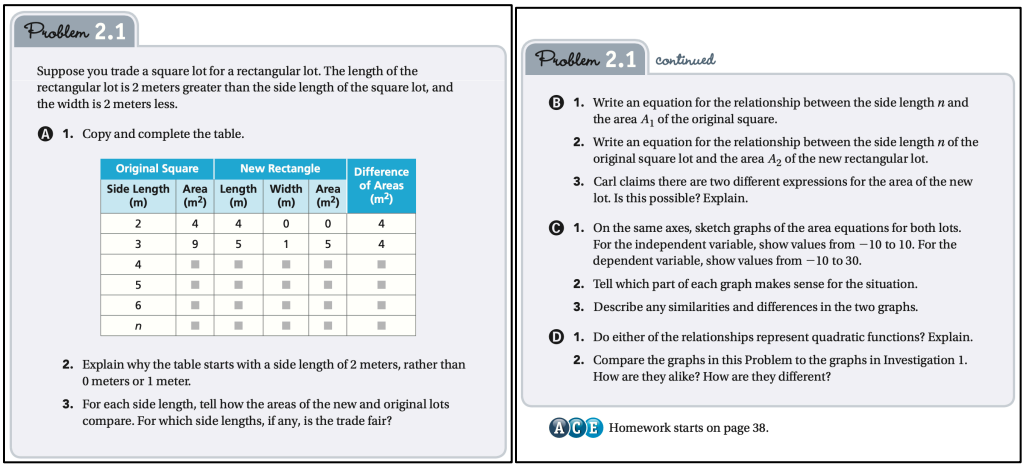
During the whole class summary discussion, Ms. Mer posed the question, "What did you do to figure out those total amounts? What were some things, some strategies that we used to figure out the cups?" Because all groups were using different numbers of ingredients and different percentages to answer the question, she focused the class discussion on similarities and differences in students' approaches. For instance, one student explained their group's strategy by saying, "We took our percentages and changed it to decimals, then we multiplied by 210." Ms. Mer revoiced the student's explanation and asked, "Did anybody do a different strategy?" Like this example, she constantly implied that there could be other ways to get correct answers and that the class discussion could serve as an opportunity to examine if others' strategies make sense to them. For the questions using Version 2, some students said they were "lucky" because

their numbers were easy to manipulate. One group's first ingredient accounted for 10%, and they simply needed to multiply by 10 to get the total number of cups. For another example, one group shared that their first ingredient was 15% and the second one was 30%. So, they thought they only needed to double the cups from the first to the second ingredient. Ms. Mer drew the whole class's attention to these "lucky" cases and highlighted that it could be an easy proportional relationship depending on the numbers given. In addition to facilitating the whole class discussion for students to pay attention to various ways to solve the problem, she kept asking, "Why did you do that?" and "Where is [this number] coming from?" as students shared their ideas. This shows her emphasis on students elaborating on their thinking as well.

In Ms. Jay's classroom, student desks formed three rows facing the whiteboard, and the *Explore* phase of lessons was devoted to working with a partner. After the class conversation about the land developer's trade proposal, she handed out a printed table and let students work in pairs to answer the questions on the next page of the textbook (Figure 14). She circulated the classroom and kept encouraging students to draw rectangles and think about where the number came from as they filled in the table.

Figure 14.

Problem 2.1 in Frogs, Fleas, and Painted Cubes (pp. 26 ~ 27)



Problem 2.1

Suppose you trade a square lot for a rectangular lot. The length of the rectangular lot is 2 meters greater than the side length of the square lot, and the width is 2 meters less.

A 1. Copy and complete the table.

Original Square		New Rectangle			Difference of Areas (m^2)
Side Length (m)	Area (m^2)	Length (m)	Width (m)	Area (m^2)	
2	4	4	0	0	4
3	9	5	1	5	4
4					
5					
6					
n					

2. Explain why the table starts with a side length of 2 meters, rather than 0 meters or 1 meter.
3. For each side length, tell how the areas of the new and original lots compare. For which side lengths, if any, is the trade fair?

Problem 2.1 continued

B 1. Write an equation for the relationship between the side length n and the area A_1 of the original square.
2. Write an equation for the relationship between the side length n of the original square lot and the area A_2 of the new rectangular lot.
3. Carl claims there are two different expressions for the area of the new lot. Is this possible? Explain.

C 1. On the same axes, sketch graphs of the area equations for both lots. For the independent variable, show values from −10 to 10. For the dependent variable, show values from −10 to 30.
2. Tell which part of each graph makes sense for the situation.
3. Describe any similarities and differences in the two graphs.

D 1. Do either of the relationships represent quadratic functions? Explain.
2. Compare the graphs in this Problem to the graphs in Investigation 1. How are they alike? How are they different?

ACE Homework starts on page 38.

Since questions 2 and 3 ask students to provide explanations of why and how they make sense of the table, the whole class discussion focused on sharing their explanations in their own words. Also, when students shared their equations for the problem part B, she asked, “Why?”, “Why not?”, or “Where did you get that?”

On the next day, when Ms. Jay moved to Problem 2.2, she facilitated the summary discussion by having different students walk up to the whiteboard and share out how they solved the problem. To write equations for different rectangles areas in association with the diagrams, students used different color markers and explained their approaches. Ms. Jay kept checking in with the whole class and asked, “Did someone do it differently?” for some of which cases, students volunteered to come up to the board and shared different drawing ideas.

In summary, during the *Explore* and *Summarize* phases of a lesson, teachers encouraged students to consider different ways to solve problems and explain how and why their solutions worked. The teachers provided students with open space so that students could explore various ways to develop mathematical understanding. Even though such open space was brought up drawing from the curriculum, both teachers consistently kept asking students to attend to differences in various strategies. Across different classroom configurations (i.e., group work setting vs. individual/pair work setting), the teachers enacted open spaces in a way that could allow students to reflect on their mathematical thinking rather than acquiring one particular strategy throughout the Explore and Summarize phases.

Inclusive Classroom Environments

Throughout lesson phases (*Launch, Explore*, and *Summarize*), the teachers worked to engage students in drawing from their multiple mathematical knowledge bases. In other words, they enacted open spaces in a way that did not shift toward a closed space and did keep

encouraging students to connect with their prior knowledge and experiences. Given that enacting lesson plans can be influenced by the teachers' in-the-moment decisions depending on the interaction with the students (Amador, 2016), their classroom environments can also play a role in their enactment of open spaces. Both teachers' classrooms similarly showcased an inclusive learning environment in that the teachers paid special attention to accommodating students' sense of belonging to the classroom regardless of their cultural and linguistic backgrounds.

Entering Ms. Jay's classroom, it felt cozy and inviting. Perhaps, it was because she was standing in front of the classroom door, with warm smiles, to greet students during a five-minute break between classes, and students coming in and out of the classroom casually chatted with her (about their birthday plans for instance) or had a conversation about their homework or quizzes. Three walls of her classroom displayed group posters about the exponential growth from Problem 2.2 in *Growing Growing Growing*, along with some random drawing pieces created by students. On the back wall of the classroom hung a copy of the classroom norms on which many students signed their names. The norms included "Be respectful of others," "Be inclusive, open-minded, supportive, and understanding," and "Be an active listener, and value others' thinking" among others.

Ms. Mer also welcomed students in front of her classroom. As they entered the room, students looked at the whiteboard to find the lesson goals, resources needed, and today's topic question. Every day, students get a random question that they can chat about with groupmates while coming into the room and waiting for the class to begin. When I visited her classroom, the question was "What was something fun you did over our snow days?" By the moment the bell rang, and Ms. Mer called for students' attention to announce some school events and weekly plans, the classroom was full of laughs. Almost every student raised their hand to share their

experiences during snow days, and she let all the volunteers have a chance to speak. Later in the reflection meeting, she explained that since it was right after two consecutive snow days, she wanted to spend a little more time than usual talking about their experiences. She also shared her one-year-old daughter's first day out playing with snow. By posing one question that could invite students to share their out-of-school experiences with no pressure, she said she had focused on creating a classroom culture where students could be aware and respectful of classmates' different stories and opinions.

Their classroom environments, both physically and socially, showed that the teachers cared for students' sense of belonging to the class as a learning community. In addition to establishing classroom cultures, the teachers were mindful of potential struggles, even if temporary, of those students who recently moved to the USA from other countries. Both teachers explicitly allowed students to use a translator if needed as I could observe a few students switching between two languages on their computers. Also, they considered pairing students who speak the same language. For example, when a new student came to the school in the middle of the semester, Ms. Jay learned that the student felt more comfortable speaking in Chinese than in English. So, she had the student sit next to another student who could speak Chinese because "the translation is never perfect. So, I think by having them work together, talk it out, they're doing really well compared to the first semester." Even though the initial purpose of pairing them up was to help the new student adjust and feel connected with others by using their home language, she said that she was surprised to observe how much more the other student, who was already fluent in both Chinese and English, was engaged in the lessons than before.

Since Ms. Jay speaks Arabic, she was able to listen to which students talked to each other in Arabic, and occasionally she participated in their conversation. Acknowledging that she could

communicate in English and Arabic only, she mindfully let students use any language they found helpful with each other, especially for those who were in their early transition to the USA. For example, when I was in her classroom for the observation, one Korean student seemed confused by an equation in the textbook while working on problems. The student asked another Korean student sitting two seats away a clarifying question in Korean. Then, the other student explained the meaning of the equation using the two languages mixed.

Similarly, Ms. Mer recalled that she paired up two Japanese students to sit together when one recently moved to the USA. She said that she usually let those pairs work together for about a month and then put them in different groups. It was because she had observed that in many cases, after a month or two, new students became quite accustomed to American classrooms and gained a willingness to mingle with others. This is not to imply that students who move to the USA from other countries will have no issues with speaking in English after a couple of months. Rather, she attended to whether new students felt comfortable and confident enough to participate in the class work. In other words, these teachers believed that allowing students to use translators alone would not be sufficient; and more importantly, they focused on helping students feel connected with the teacher and other students. Moreover, Ms. Mer shared that she was still practicing the proper Chinese pronunciation of one student's name. She admitted that it was so hard to make the right sound and tone in Chinese, and every time she tried to call his name, he burst into laughter because she sounded awkward in Chinese. This example illustrates how Ms. Mer diffuses the power relationship between native and non-native English speakers. Students in her classroom could imagine when the roles are reversed anybody can sound funny when speaking unfamiliar languages.

These episodes provide a glimpse of how the teachers had already paid attention to

establishing relationships with their students when they participated in this study. Ms. Jay's reflection on her conversation with one Chinese student shows that her student was openly sharing his struggle with languages, which also explains why she could not simply disregard multilingual students' struggles. She said, "It was heartbreaking" because the student told her, "I know the answer, but I don't know how to say [in English]." Sometimes they communicated with each other using a smartphone translator so that the student could speak in Chinese, and the teacher could read its translation into English. And she was relieved when the student got along well with the other Chinese students.

In summary, the teachers were implicitly and explicitly establishing their classroom environments to be accepting of multicultural and multilingual aspects of various student demographics. While I cannot make a generalized claim about multilingual students' participation in the teachers' lessons, at least for some students in their classrooms, English being their second language did not stop them from participating in classroom activities. During the whole class summary discussions I observed, students were not shy about raising their hands to share their thoughts even if at times they needed to pause to find words to articulate or even if they had a clear foreign accent.

Explorative Reasoning for Opening Curriculum Spaces

In their individual reflection journals and in our last meeting for a group reflection, the teachers shared their perspectives and challenges of designing and enacting open spaces. Both teachers were not sure how to self-evaluate their lessons when students went from "a high participation" in talking about their life experiences to "a lower participation" in solving challenging mathematics problems. On the one hand, the teachers found it meaningful in that students were "all able to at least kind of enter the problem." On the other hand, they seemed a

little bit discouraged that students did not seem as engaged in mathematics as they were in 'non-math' conversations.

As they wondered how to balance students' engagement in the two aspects of multiple mathematical knowledge bases, their focus seemed to be more on supporting mathematical thinking than students' connections with culture, home, or community experiences. When asked what they would do differently if they had a chance to teach the same lesson again, both teachers discussed teaching practices and teacher moves that were related to enhancing students' understanding through multiple strategies and reasoning, and they made no comment on their reflection regarding their students' cultural identities. Even though they understood the importance of open spaces for real-life connections in culturally nondominant students' identity development, they might view such connections as interesting launch activities and not consider them as continuous influences on students' mathematical investigations. In this sense, while their lessons attended to both aspects of students' multiple mathematical knowledge bases, the connections with students' life experiences received less emphasis than those with students' mathematical thinking.

When one of Ms. Jay's classes discussed what fair trade means based on students' life experiences, one African American student raised his hand and said, "When people come to America from other places for a job opportunity, that is not a fair game because Americans are not kind to immigrants, and they do not want to keep them at all." It might or might not reflect his own perspective on immigrants; it might or might not be related to his perceptions of the reality of underrepresented people's lives in the USA. Ms. Jay responded, "That's a good point," then she told the whole class that the student was being "creative" and moved on to share her own childhood story on fair trade. During the break right after the class, she talked to me about

how the student always tried to make “fun” comments during lessons. From my observation during the break time before and after the class as well as at lunchtime, Ms. Jay and the student seemed to have friendly individual conversations a lot, suggesting that the student valued Ms. Jay’s attention. While I found his response about “not a fair game” interesting and wanted to hear more, Ms. Jay did not talk about it with him or with me after the class, so I did not bring it up again in our group reflection meeting. However, I recognized the need to reflect on my own positionality as one of those people who immigrated to America for a job opportunity, which also influenced my role as a researcher.

I sensed that Ms. Jay did not perceive his answer as an opportunity to attend to the issues and tensions in our society and instead kindly redirected the class conversation back to ‘less creative’ life examples. However, upon reflection, I felt uncomfortable approaching this issue with her mainly because I was worried that I would be applying unintended pressures on her to develop the same level of critical awareness as I had as an international researcher. Working with white American teachers, I was conscious of my own influence as a researcher with an international background on their work. Specifically, I was afraid to imply that she inadvertently missed out on an open space while it was somewhat apparent to my eyes. I wondered if we shared a cultural and linguistic background (i.e., if I were also a white American researcher), I could have felt confident enough to open up a discussion with her. This case study created an environment where I am one of the underrepresented populations in the group, but ironically, I have the power to relate to their underrepresented students’ frustration in the classrooms, yet I was not comfortable with such a position.

At the same time, I wondered how teachers (especially those who are from the dominant culture) can develop critical awareness without having a conversation with people like me. Since

I was too conscious of my positionality, I did not share my personal confusion with Ms. Mer when she posed a daily question, "Would you rather eat only breakfast food all day or never eat breakfast food?" during one of my classroom observations. I looked at the question on the board but could not quite get what it was asking because of the cultural differences between America and my home country. Once students started chatting about it, and I could hear 'waffles' and 'pancakes,' I recognized that there exists this "breakfast food" category in American culture, whereas people in my home culture usually eat similar food for breakfast, lunch, and dinner without necessarily specifying "breakfast food." Although I found it an interesting cultural difference, I wondered if students who are from a culture similar to mine would share the same misunderstanding as I had. If I shared this with the teachers, it would have served as a realization moment for them that otherwise they would not have considered. However, I kept it to myself because I did not want to make her feel hesitant to pose those questions while feeling overwhelmed to be aware of all possible cultural considerations. Perhaps, I need more time to build a strongly trustworthy relationship with the teachers so as to openly discuss and encourage us to share critical perspectives in our collaborative work. And perhaps, I need to have other people with different cultural backgrounds—teachers, researchers, and/or students—so as to share and learn about the cultural diversity from each other.

Summary of Findings on Opening Curriculum Spaces

When two secondary mathematics teachers set an instructional goal of providing students with opportunities to draw on their multiple mathematical knowledge bases, the teachers designed open spaces by interpreting the curriculum, reconciling the goals, and adapting the curriculum. First, the teachers used the school's official curriculum, CMP (*Connected Mathematics Project*), as the main resource for the lesson design. They interpreted the

curriculum in terms of open spaces that afford opportunities to draw on mathematical thinking and real-life experiences. Teachers noticed that the problems and questions in the curriculum were very well aligned with the goal to attend to students' mathematical thinking. Thus, they did not find it necessary to make changes with regard to open spaces for mathematical thinking, and they decided to enact the curriculum as suggested (i.e., curriculum use: offloading). On the other hand, the teachers noticed that there were fewer opportunities for students to connect the problem contexts with their own experiences. When they interpreted the problem contexts, they distinguished real-life situations from "fairy tale stories." They perceived that students could genuinely enjoy working on problems contextualized in an imaginary world (e.g., the Wump family or a peasant's reward), which led the teachers to decide not to make changes. When it came to problem contexts that involved the real world (e.g., making orange juice), the teachers hoped to make it more relevant to their students' life experiences.

In order to accommodate students' interests and needs, the teachers leveraged their interaction with students through small activities during advisory hours and at the beginning of mathematics lessons. They collected information about students' cultural experiences (e.g., ethnicity percentages and special food for the family and culture) for the purpose of utilizing them to design mathematics lessons. Then, they decided to let students share their experiences instead of them selecting and creating a problem context using a specific cultural referent. This was in part because they recognized that they would not be able to understand other cultures enough to create a problem when they are so accustomed to their own culture, and also because their students represent so many different cultural backgrounds. Thus, they adapted the curriculum resources to ask students to share their experiences related to the problem context or to create problem contexts on their own using student choices.

In their lesson designs, teachers aimed to create open spaces in a way that could invite students to freely share their thinking and experience. The teachers began lessons by providing students with an opportunity to review their problem-solving strategies from the previous lesson. Then they encouraged students to make sense of the problem contexts based on their own life experiences. Ms. Jay shared her own childhood story first and asked students if they had a similar experience, and Ms. Mer asked students to select the kind, number, and percentages of different items and use them in solving problems. While students worked on the problems and shared their solutions with the whole class, the teachers kept encouraging students to come up with their own strategies and share different solutions, emphasizing to explain what they were thinking. Throughout the lesson phases, the teachers made sure to keep the spaces open, which was also supported by the inclusive classroom culture. They observed that their students were more engaged than usual when they could use their experiences as learning resources.

CHAPTER 5: FINDINGS ON INFLUENCES OF TEACHER AGENCY

The second research question of this study asked to illustrate how teacher agency allowed for teachers' pedagogical design capacity to *open* curriculum spaces. Considering that pedagogical design capacity cannot be enacted without teacher agency (Priestley et al., 2015), this study sought to understand ways in which teacher agency influenced teachers' decision-making during their design processes. Drawing on the relational, ecological nature of teacher agency (Emibayer & Mische, 1998; Priestley et al., 2015), this chapter reports on the contextual aspects that influenced teachers' agentic engagement in changing their use of curriculum resources.

Since the teachers utilized the official curriculum as the main resource, they began their design activities by reading and interpreting the curriculum. Through a lens of curriculum spaces, both teachers noticed that the curriculum coherently encouraged students to use their own strategies and explain their mathematical thinking whereas there were fewer opportunities to connect with students' cultural or life experiences. As the teachers read the curriculum, their interpretations of the problems often concurred with their reconciliation of goals. It could be because their intention of reading the curriculum was to decide how to use the curriculum—the extent to which they want to adhere to the curriculum or make changes—in the process of planning. Based on their interpretation and reconciliation, the teachers' decisions on curriculum use were different. For multiple strategies and explanations, they decided to use the curriculum without making changes; for real-life connections, they decided to make modifications to the curriculum. The first section of this chapter describes findings on how teacher agency influenced the teachers' decision to adhere to the curriculum as provided. The second section follows to illustrate how teacher agency influenced the teachers' decision to adapt the curriculum.

Offloading the Curriculum: Multiple Strategies and Explanations

Influence of the Iterational Dimension

The analysis revealed that the *iterational* dimension of teacher agency supported the ways the teachers interpreted the curriculum in terms of students' learning opportunities for drawing on their own strategies and explaining their thinking, both when talking about the curriculum in general and focusing on one problem for planning a specific lesson. In these episodes, the data show that the teachers elaborated on how the curriculum's approach to teaching and learning mathematics was aligned with teacher beliefs. In their reasoning, they drew on their personal and professional experiences.

Ms. Mer explained that she believes that it is important to foster student understanding of the 'why' behind mathematical concepts, drawing on her own learning experience as a student:

> I always try to emphasize it because when I learned math, I always thought there was only one way to do something, the way that I was told to do it. When I got to college and learned, "Oh, that's not always the case." And I learned truly why we did certain methods like FOIL or cross multiply and divide, all those tricks that we were taught. When I learned why we do those things, I was like, "Oh, wow, this is so much more exciting and makes more sense." So, I always want to try to let kids see that [...] math isn't just one way to do things.

Her learning experiences in the two very different environments—one focusing on acquiring one way of solving problems and the other focusing on understanding why—had influenced her teaching to emphasize "math isn't just one way to do things." Having taught the CMP curriculum for more than ten years, she developed trust in the curriculum over time as she found it aligned with her teaching philosophy, and it engaged her students in developing conceptual

understanding. In this case, her past experience as a learner provided a foundation for her belief in a teaching approach focusing on students' mathematical understanding.

In Ms. Jay's case, she contrasted her teaching experiences to highlight that she values student-centered teaching approach. She explained that it is important to allow students to utilize their own strategies and share different solutions with classmates because "kids really learn a lot from each other, maybe even more so than from us." When students shared their solution strategies, Ms. Jay said she always called for students' explanations of how they came up with the strategy and why they thought it was mathematically correct:

> I feel like when a student explains something to other students, sometimes it clicks better, and so I like every time, for my students, to emphasize "How did you get that?", "What were you thinking?", "Where did that number come from?" And then, I think that helps other kids rather than me just telling them exactly where to look. [It is] because I can tell them all day long, but if they hear from another student, they'll be like, "Oh! I see that too!"

As they interpreted the curriculum in terms of open spaces for mathematical thinking, both teachers "built upon past achievements, understandings and patterns of action" (Priestley et al., 2015, p. 24), and recognized that their goals for open spaces were aligned with the curriculum's vision. Both teachers expressed that they believe it is important to allow students to make sense of mathematics in their own ways and ask them to explain it. This belief, drawing from their past experiences as a learner (Ms. Mer) and a teacher (Ms. Jay), can explain why they agreed with the curriculum's approach during their design process and further took an active role in facilitating student-centered learning during the enactment. Although this *iterational* dimension of teacher agency cannot exclusively determine teachers' agentic actions, it can

suggest that when teachers decide to adhere to an official curriculum, it does not mean that they simply follow the curriculum because it is given to them. Instead, it can be because they interpreted the curriculum to figure that they agree with the curriculum's approach based on their past experiences.

For Ms. Mer, her teaching experiences using the same curriculum reinforced her to develop trust in the curriculum over time. She did not necessarily need to change her teaching approach based on her interpretations because she was already familiar with enacting the curriculum as the curriculum developers suggested. In other words, even if the data represent her *iterational* dimension of teacher agency, it is hard to parse out its influence on her curriculum use. In contrast, Ms. Jay's case can illustrate how she changed her role to be an active agent in the classroom supporting students in developing multiple strategies, based on her past teaching experiences. That is, she utilized the official curriculum in a way that aligned with her belief instead of deploying her agency to reject the curriculum. In the following, I report on how and why Ms. Jay shifted her teaching approach.

Influence of the Practical-Evaluative Dimension

The data about Ms. Jay's explanations for her curriculum use suggest that the official curriculum, CMP, had a strong influence on her shift from a teacher-centered toward a student-centered approach. That is, the official curriculum served her as a resource (the *material* aspect of the *practical-evaluative* dimension) that helped her take an agentic action to change her teaching approach. Although it was her first year of using the CMP curriculum, Ms. Jay said she could immediately tell that her students were engaged in such a student-centered learning environment. She described that in the first week of using the CMP curriculum, she had to drastically change her planning and teaching approach because she recognized her planning

routines from the previous school did not work the same way. She summarized her previous planning and teaching approach:

> It was the old school [style] of teaching math, where you have problems on the board, [students] take the notes, they do the homework, that was it. [For planning,] I just write out my lesson on a line piece of paper, which would literally just be the problems that I'm going to show them. And so that whole first year was just like a ton of planning. After that, it was just so easy. I knew the whole curriculum, and I had everything done. So, I just pulled out the lesson of the day, [wrote] on the board, and kids copied. Pull it back in the next class, next lesson, so just very easy.

Then, she compared this "old school" style with her new approach:

> Starting out this year, it was tough because it was not like the other curriculum. It was more group-based, more word problems, more like, I guess, *deeper problems* rather than just the problems here and there. So, [in the first week,] I tried writing out my lesson plans, kind of like how I did last year. And I was like, "This isn't working. I don't like it. It's not the same. I don't know how to write things out." For this, I changed it up. As I [solve problems] with my classes, I'm writing notes, and that worked out a whole lot better than pre-plan. Because at the moment, I see what's actually happening with the responses kids are getting, and what they don't get, so I know how to plan (after seeing students' responses).

She changed her teaching approach from demonstrating how to get one right answer to brainstorming together with students to solve problems. That is, she positioned herself not as the authority of mathematical knowledge but as a collaborator in her classroom. Based on her past

experiences with the two distinct teaching approaches (the *iterational* dimension), she figured that it is important to pay attention to what her students would think and say about the problems. Since she taught the same content standards in the 8th-grade mathematics in her previous school, she owned her lesson plans using a different curriculum, which she might have referenced this year. She mentioned that she used to look at the other curriculum from her previous school in addition to the CMP curriculum at the beginning of this year, and then she shifted to solely using the CMP curriculum. As she believed that "understand[ing] where the math is coming from" is better than rote memorization of procedures, she appreciated that the curriculum was designed to emphasize student understanding. Even if it was challenging to shift away from her typical teaching approach to a new one, she accepted the challenge because "better [for student learning] is probably going to be more difficult [for planning and teaching]." As such, the school's official curriculum (the *material* aspect in the *practical-evaluative* dimension) strongly influenced Ms. Jay's agentic capacity.

Adapting the Curriculum: Real-Life Connections

Since the teachers noticed that the curriculum offered fewer opportunities for students to draw on their life experiences, they decided to adapt the curriculum to include open spaces for real-life connections. As the first step to making such adaptations, the teachers hoped to learn more about their students' diverse cultural experiences. The data show that it was because the teachers recognized the challenges of attending to diverse cultural backgrounds (the *iterational* dimension). Also, the data suggest that the teachers leveraged their informal conversations during advisory hours and at the beginning of mathematics lessons (the *structural* aspect of the *practical-evaluative* dimension) to learn more about their students with the hope of better accommodating students' interests and needs (the *accommodating* phase in the design process).

Drawing from what they learned from the accommodating phase, the teachers adapted the curriculum in order to incorporate open spaces that allow students to connect with their life experiences. Within this *modifying* phase of the design process, the data suggest that although the teachers found curriculum adaptations challenging in general (the *iterational* dimension), they leveraged the resources available to them (the *material* aspect of the *practical-evaluative* dimension), the rapport built between teachers and students (the *cultural* aspect of the *practical-evaluative* dimension), and the collaborative space where we shared the instructional goals to make 'small adjustments' in their lessons and supported each other (the *structural* aspect of the *practical-evaluative* dimension).

Influence of the Iterational Dimension

In their reasoning for designing open spaces, the teachers described their perceptions of students' cultural experiences. First, the teachers thought it was a natural phenomenon that their students represented many different cultural backgrounds because it simply reflected a wide range of demographics of the area where the school is located. In fact, the teachers listed at least nine different languages that could be spoken in their classrooms. In addition to the diversity in students' home languages, the teachers acknowledged a number of different ethnicities and mixed races in student demographics. Even though such diversity in one classroom could have overwhelmed teachers, the teachers expressed it was cool to learn about different cultures from their students. They mentioned that they had informal conversations with students when they had a chance, which help them understand their students' out-of-school experiences. Also, they observed that their students talked about different perspectives on things from different cultures. For example, earlier on the day we had this discussion, Ms. Mer heard two students talking about the temperature on different scales in the hallway. She recalled that one student said it was zero

degrees outside and the other student got confused. This confusion arose because one student meant zero degrees in Celsius, and the other student misunderstood it in Fahrenheit. Then, she heard the students talking about the conversion between Fahrenheit and Celsius. As such, the teachers perceived that diversity was a big part of the school culture.

For the teachers' perceptions of cultural diversity as normal and cool to learn about, one possible explanation would be their own life experiences associated with cultural diversity. Even though both teachers are from the USA, they have family members who are from outside of the States. Ms. Jay's parents are from Middle Eastern countries, so she can speak Arabic and cook Middle Eastern food using her family's recipes. Ms. Mer has a brother whose fiancée is a Chinese American "patisserie," who shares various kinds of Asian foods and local restaurants to try out. The teachers' multicultural family environment might have helped them develop an open mind toward trying new things themselves and perceiving such diversity as part of their lives. Moreover, both teachers did not limit their perceptions of the diversity in their students only to cultural and linguistic backgrounds. Besides the diversities in race, ethnicity, and language use, the teachers included each student's uniqueness, personality, neighborhood, gender identity, and needs for special education in their list of classroom diversity.

Nevertheless, the teachers perceived that it was important yet difficult for them to know about students' cultural experiences beyond superficial levels. On the one hand, teachers thought that they could bring a particular example of cultural referents that would interest some students who find it relevant to their own cultures. For example, Ms. Mer thought she could do an origami activity because her Japanese students would be able to reflect on their cultural experiences. She further argued that it could speak to not only Japanese students but also 'artistic students' who liked visualizing or drawing. On the other hand, the teachers thought that it was not

straightforward to foresee the effect of cultural references in mathematics problems when there were ten different cultural backgrounds in the classroom. As Ms. Mer put it:

> If they are unfamiliar, how do we know if they are unfamiliar? And some kids are familiar, and some kids aren't. And that's the hardest thing. When you are meeting so many different backgrounds and stuff, which is, I mean, part of what we're supposed to be able to do in those kinds of things, but it can be hard. And [they suggest], 'draw on their community and cultural knowledge,' and then you're feeling like, always step behind almost because you are waiting for that to come out and to add to your contexts. So, that can be a hard or overwhelming feeling, too.

The teachers contended that although having casual conversations allowed them to 'get to know' their students, it did not automatically lead them to design mathematics lessons using cultural references that were different from the teachers' own cultural experiences. For instance, when we discussed a problem context using Japanese costume in Matthews et al.'s (2022) book (Figure 15), they could not consider that Japanese students might not be familiar with a measurement unit, the yard.

Figure 15.

An Example of Culturally Relevant Mathematics Tasks

A naked fraction problem:
Add the fractions and show your work 5/8 + 1 2/3 =

Improved problem:
Yuki is making a traditional Japanese kimono for her daughter to wear for a special cultural ceremony. Her daughter's costume needs 5/8 yard of fabric and Yuki's costume needs 1 2/3 yards of fabric. How much fabric will Yuki need? What might Yuki's dream design look like?

(Matthews et al. 2022, p. 39)

It did not stand out to them even if teachers knew the fact that different countries use different measurement systems because they were so accustomed to the yard as a measurement. When the

researcher, an Asian person who is more comfortable with the metric system, pointed it out, Ms. Jay expressed the challenge of being cognizant of cultural differences by saying, "I wouldn't even think of that because I'm just like, *yard is yard*."

As in these examples, the teachers perceived their students' diverse cultural backgrounds as part of the school culture and considered bringing cultural references in teaching mathematics; at the same time, they expressed the challenges of attending to a wide range of different cultural backgrounds of their students and recognizing cultural differences when they were so accustomed to one culture themselves. This iterational dimension of teacher agency alone, drawing on their life and teaching experiences, might have hindered the teachers from taking agentic actions in association with accommodating students' interests and needs. However, as it got interplayed with the projective and practical-evaluative dimensions, the teachers were able to recognize that they could take advantage of the advisory hours in order to learn about students' backgrounds. The influences of those dimensions will be discussed later in separate sections.

Even if they leveraged their informal conversations with students to learn about cultural backgrounds for the purpose of the adaptation, they found it still challenging to create culturally relevant problem contexts. One main reason for such a challenge stemmed from their perceptions of curriculum adaptations. They perceived that making changes to the curriculum problems was hard work in and of itself for any teacher regardless of previous teaching experiences. Ms. Jay, who was using CMP for the first time, shared her frustration with reading the curriculum through a lens of curriculum spaces by saying, "I had a hard time just because I wasn't familiar with the material. Well, and I've never taught it through CMP before, so I had to kind of read through it, [thinking] what were they expecting me to [do?]" For her, making changes to the curriculum problems could be considered once she became familiar with the curriculum. She imagined that

it would have been much easier to navigate possibilities for making modifications if it was the unit that she just finished teaching (*Looking for Pythagoras*) because now she understood the lesson goals in the unit as a whole. One of the biggest challenges of curriculum adaptations for Ms. Jay was that she could not foresee student thinking or anticipate different strategies while she planned a lesson for a new curriculum. She thought that she would have been more confident in planning lessons and adapting problems if she had taught the unit before and collected student work examples.

Interestingly, Ms. Mer, who taught the same curriculum for multiple years, pointed out that making changes to the curriculum was still challenging even though she understood how problem sequences would develop and she had collected student work examples as Ms. Jay wished. She explained that her challenge involved her familiarity with the content by saying, "I've taught mine the same way for so many years, then it's hard to think about how I could do it differently." In addition, as both teachers noticed in the curriculum, CMP utilizes problem contexts whose stories can flow through a sequence of multiple problems. It may be this connectivity that influenced Ms. Mer to consider using a culturally relevant mathematics task for exit quizzes rather than making substantial changes to the core investigative problems in the curriculum.

As such, both teachers—one with no previous experience using the curriculum and the other with many years of experience—expressed that they perceived curriculum adaptations as hard work due in part to the nature of the curriculum as well as their previous experiences with the curriculum. This iterational dimension with regard to their perceptions of curriculum adaptations can influence them to decide to either withdraw themselves from modifying problems for open spaces or search for less overwhelming ways to open spaces. That is, the

iterational dimension alone cannot explain how the teachers achieve a sense of agency, but it can illustrate where individual teachers begin their decision-making processes. In the following sections, the influences of the other two dimensions will be explained.

Influence of the Projective Dimension

When teachers, as designers of mathematics lessons, go through a design process utilizing curriculum resources as design artifacts, they have instructional goals in their minds (Brown, 2009). Without specific goals, teachers would not be able to determine what resources to select or how to use the selected resources. In general, the participating teachers took short-term goals for each lesson, long-term goals for developing an understanding of big mathematical ideas, and long-term goals for enhancing student identity into consideration as they planned lessons. As the teachers and I discussed curriculum spaces and lesson plans throughout weekly meetings, our specific goal centered around finding ways to provide students with open spaces in daily lessons so that students could draw on their multiple mathematical knowledge bases.

To be clear, the teachers had not used the term "open spaces" to describe their instructional goal until I as a researcher introduced the concepts of curriculum spaces and students' multiple mathematical knowledge bases. They adopted the term as they recognized that it resonated with what they valued in their classrooms. As they were aware of the diversity in their classrooms in terms of race, ethnicity, and language, the teachers expressed their hopes for students to "feel like they can attempt, or have a place, or have a vision of how they fit into the [classroom]." Also, Ms. Jay highlighted the importance of opportunities for students to draw on their mathematical understanding by saying, "[Such opportunities] make it way more enjoyable and I think beneficial that they got to actually use what they practice and create something on their own." Furthermore, Ms. Mer emphasized the importance of opportunities for students to

connect mathematical ideas with their life experiences by stating that the ultimate instructional goal is "to make [math] not this subject that we study, but it's something that we're all a part of, and it can almost influence [our life]." In other words, the theory about open spaces for students' multiple mathematical knowledge bases provided the teachers with the language for what they aimed to achieve through this study's collaborative endeavor.

According to Priestley et al. (2015), this shared instructional goal relates to the projective dimension in that it could serve as a driving force toward desirable future directions as teachers reflect on their past experiences and navigate possible action items in the present. While this short-term object of designing lessons for open spaces had a continuous influence throughout this study as a guiding question, the analysis revealed that it particularly supported them in brainstorming how to mitigate the challenges of accommodating students' diverse interests and needs. For example, the teachers had an unstructured advisory hour every day for twenty minutes. While they found that during the advisory hour, they were "able to learn a lot about this group [of advisory hour students]" and "get to know them extra well," they did not necessarily view it as a potential place to gather information about students' multiple mathematical knowledge bases. They said they had not thought about connecting informal conversations with students with mathematics lesson topics before. As they established this shared instructional goal to provide students with open spaces, they recognized that they could leverage their interactions with students during the advisory hour to directly ask students about their experiences.

Influence of the Practical-Evaluative Dimension

With explicit instructional goals, the teachers achieved agency to learn about students' cultural backgrounds and adapt the problems to include open spaces for life experiences. In order to view themselves as agents to make such changes, teachers need to identify challenges and

navigate possible alternatives to alleviate the challenges (the *practical-evaluative* dimension). The teachers expressed that they were unsure how to incorporate students' cultural experiences into mathematics lessons when they had students from so many different backgrounds that they would have only a superficial level of understanding, and when they had an official curriculum whose problem contexts were connected throughout a unit. Despite this tension, they navigated practical approaches toward the instructional goals. They achieved agency to make changes in their lessons, evident in three aspects: the *material* (resources for opening spaces), the *cultural* (the rapport with students), and the *structural* (school's advisory hour and support from each other).

Material Aspect

In the process of adapting the curriculum, the teachers utilized resources provided by the researcher to mitigate their perceived challenges in curriculum adaptations. This material aspect in the practical-evaluative dimension, such as resource or physical environment, can influence the teachers' explorations of possible approaches and decisions on making changes (Priestley et al., 2015). During the discussion meetings, the teachers analyzed culturally relevant mathematics tasks (Matthews et al., 2022) and sought to gain some practical implications. Among various example tasks in the book, the teachers noticed that some tasks included extensive contextual information, which was superfluous to solving a problem mathematically. They were not convinced to facilitate class conversations about peripheral information, as Ms. Mer stated, "How much time do we spend on something like that? We can't get off topic." Ms. Jay wrote in her observation notes to point out that it was not clear how teachers could decide on this matter, "I noticed that the teachers' instructions were not specific […] So, I am wondering if students would be so off task or topic in deciding what is math."

With a tight pacing guide, the content standard to cover, and test schedules, the teachers felt like they had little flexibility in discussing problem situations. Their consciousness of limited class time was understandable because they were already dealing with the pressure to properly allocate class time between problems for developing conceptual understanding and problems for practicing procedures. As Ms. Mer candidly shared her struggles with "finding that balance [between deep problems and practice problems], and how to make that relevant to [students] when they do these deep problems. [I can't just say] 'Oh, here's a bunch of problems to practice on,' that is not very relevant." While teachers aimed for students to develop conceptual understanding and problem-solving skills, they also felt pressured to secure class time for practice problems. On top of this balance between deep and practice problems, teachers felt uncertain whether it is necessary to have class discussions about seemingly off-topic situations.

In contrast, the teachers were drawn to some other example tasks that added one simple question at the end, such as "How is this similar to or different from other [stores/stories/etc.] you know about?" (Matthews et al., 2022, p. 38). They perceived that this kind of additional question would not take up too much class time and could allow students to reflect on their personal or cultural experiences to better understand problem contexts. Thus, the teachers planned to directly ask students to share their experiences rather than creating a culturally relevant problem context with detailed background information. Particularly, Ms. Jay launched the problem by asking students to think about their experience of (un)fair trade to consolidate the meaning of 'equivalent' in equations. Because she noticed that the problem concerned area comparisons between rectangles, building on the previous problem, she did not want to create a whole new problem context. Instead, she decided to ask students to share their life experiences with regard to fair trade because it could help students make sense of equations based on

concrete examples of their own. Similarly, Ms. Mer appreciated such a simple additional question by saying:

> I'm thinking about, I just did a problem in *Accentuate the Negative*, and we were doing these real-life problems […] one that was about the game show. So, [I was thinking about] just little tweaks like, "Oh, how might this be like a game you played?" I mean, it would be an easy add-on to kind of help with bringing them in.

In addition to asking students a simple question, Ms. Mer utilized students' responses as problem contexts. When she was planning for a problem that was about food mixtures for chimpanzees, she considered replacing the context in the curriculum with her own students' special food recipes. However, she decided to let students create their own snack mix and use their ingredients and percentages to solve the problems. This way, she could encourage students to share their experiences and connect them with learning mathematics.

As such, the teachers found an 'easy add-on' as a feasible approach for their curriculum adaptations considering the realistic constraints, such as limited class time or planning hours. Their capacity for agency was influenced by the available resources which suggested that one simple add-on question could make a big change in student learning. While the teachers did not want to spend too much class time talking about peripheral information about problem contexts, they valued the opportunity for students to make sense of mathematics ideas based on their life experiences. They purposefully attended to the resource (the *material* aspect of the *practical-evaluative* dimension) and applied it to their lesson designs.

Cultural Aspect

In their processes of opening curriculum spaces, the teachers worked to build rapport

with students in learning about students' multiple mathematical knowledge bases. They did not frame the curriculum adaptations for open spaces as an isolated activity for a special lesson. Instead, they believed that learning about students' out-of-school experiences for the purpose of opening curriculum spaces would be similar to what teachers typically do to build rapport with students. This cultural aspect in the practical-evaluative dimension, such as ideas, beliefs, and values, can influence teachers' approaches to navigating possible changes they can make (Priestley et al., 2015).

For instance, when we discussed the hashtag activity as one way to learn about students' cultural backgrounds and utilize it as a problem context (Matthews et al., 2022), Ms. Mer argued that "it's important for us to try to remember […] that we do a lot of this stuff at the beginning of the year and that kind of fades away." Similar to this hashtag activity, Ms. Mer incorporated random questions into daily lessons through which she could build rapport with her students. Furthermore, she thought that those questions could prompt students to positively interact with each other. She gave us one example. When she posed a question, "What do you like better? Pizza or sushi?" It might also have been her way of showing her recognition of the Japanese population in the classroom. While students were chatting with their group, she heard one student saying, "Oh, what about pizza made out of sushi?" and another student agreed, "Oh, that's awesome!" Then, the first student said, "We finally agree upon something!" Like in this episode, Ms. Mer believes that opening a space for students' multiple mathematical knowledge bases would not be something completely different from what teachers would do in order to build close relationships with students. Ms. Mer claimed that building rapport can be a part of the process of opening spaces rather than a prerequisite:

I would also like to push back and say that even jumping in and opening up the

> spaces in your curriculum will give you [rapport]. You know that the question that I talked about with the snow day, I spend a lot of time on those questions in the first month or so…it may be related to my curriculum or not, but it's kind of the same idea of building that rapport. Sometimes before you have it, you just got to jump in and do it (opening spaces) so that you can get [rapport]. It's like a cycle. You got to ask the questions to get the answers to *open* the space and to ask the questions to get the trust. So, it's all connected.

She suggested the connection between building rapport and opening spaces for students' multiple mathematical knowledge bases. This perspective could allow teachers to feel less overwhelmed by curriculum adaptations because their effort to open curriculum spaces can be reinforced and interplayed with their close relationships with students. It means that the teachers were able to change their views on the curriculum adaptations from work that needs extra time and effort toward work similar to what they are already doing to build rapport. The teachers brought a sense of agency in exploring possible practical ways to adapt the curriculum for open spaces that leveraged their ideas of learning about students at the beginning of a new school year.

Structural Aspect

First, the teachers leveraged the school's advisory hours to learn about their students' different cultures that could potentially be utilized for the curriculum adaptations. The teachers viewed their free-formed advisory hours as a place for them to genuinely interact with students because they did not need to carry the burden related to pacing and testing as they would do during regular mathematics lessons. Then, they slightly repurposed their advisory hour to learn about students' cultural experiences that could potentially be utilized in their lesson planning. They facilitated either an informal conversation about special foods or a mathematics activity

about ethnicity pie charts during the school's advisory hour. Moreover, they ended up having a similar conversation/activity with all of their mathematics classes. For Ms. Jay, it was because she figured it was worth spending class time on in that she could learn about her students' prior understanding of percentages and cultural backgrounds. In addition, she thought it was meaningful to observe her students sharing their heritages with other students. For Ms. Mer, she did not feel like compromising her class time for the special food conversation because she anyways threw a random, icebreaker sort of question and spent a few minutes with students at the beginning of mathematics lessons. Her intention to pose non-mathematics questions at the beginning of each lesson was to help students feel at ease before jumping into mathematical thinking. For her question about special food, she had another purpose in her mind. She hoped to learn about students' cultural experiences so that she could use them in her lesson planning.

Even though the teachers typically perceived that class time would be too tight to add extra activities for 'off-topic' conversations, they decided to secure a small portion of lesson time for such informal conversations because they thought that otherwise, it would be hard for them to know diverse cultural references that their students would potentially bring to the classrooms. As explained in the previous section, the teachers were able to leverage this existing school hour (the *structural* aspect of the *practical-evaluative* dimension) because they had an explicit goal to open curriculum spaces (the *projective* dimension) even if they perceived that learning about diverse cultures would be challenging (the *iterational* dimension).

Second, the teachers acknowledged that they were able to adapt the curriculum to include open spaces because they supported each other in a collaboration group. This structural aspect in the practical-evaluative dimension, such as social relationships among members, can influence the teachers' agentic dispositions toward opening curriculum spaces (Priestley et al., 2015). The

teachers expressed that it would have been even more challenging if they did not have *this* space where they could explicitly and freely discuss planning, teaching, and reflection. For Ms. Jay, our weekly meetings served as a place for "bouncing ideas off each other," which could boost her confidence as a new teacher to the curriculum. When we brainstormed possible changes for the textbook problems, she stated that she would have a better sense next year when she would be able to collect her students' work examples. The biggest challenge for her to plan and teach "deep problems" involved her concern about not having concrete examples from students. While she could check some examples in the teacher's guide, she wanted to brainstorm her own solutions or examples. For instance, when talking about her lesson plan, we looked at the student textbook together and talked about the underlying mathematical ideas and how they unfolded through multiple questions. Toward the end of the meeting, she came up with the idea of sharing her own childhood story that could evoke students to interpret what 'fair trade' means based on their experiences. However, if it was by herself, it might not have been the case as she shared at the end of the meeting:

> I'm glad we talked about this…because looking at this, I probably just would have skipped the money part over. But now just talking about, what do you think is a fair trade? I even might start with my example of trading with my siblings. And then they'll like, "Oh! I know that." Yeah, sometimes they don't know what you're asking unless you give them an example.

In another meeting when we discussed how to make connections between mathematics lessons and informal conversations with students, she wanted to come up with real-life examples that could embed the concept of exponential growth. Since it was not straightforward to come up with an example, she got discouraged and said, "I don't think I have an idea," and later "I really

have no idea," once again. Since the two teachers were in different grade levels, Ms. Mer did not have specific examples for exponential growth, but she helped Ms. Jay think through the textbook problem contexts. Then Ms. Mer suggested posing a simple question, "When do you remember using percent?" because growth rates would use the percentage concept. Not long after, Ms. Jay suddenly had a light-bulb moment, "Oh! I just thought of one. So, you know how people are mixed races? So, what percent of like, Korean, are you?" Ms. Mer validated the idea by saying, "Yeah, genealogists. My husband did that ancestry DNA swab. And you could pull it up and [say] look at our percentages." Then, Ms. Jay's voice turned to sound more confident and excited saying:

> Yeah, yeah, that's actually cool. That way, they just know percents are out of 100, there's a whole versus parts, and yeah! It's not really exponential, but it's percent and then that also brings in culture. Like what percent are you of this versus this and this because a lot of the kids are mixed? … See, like us bouncing ideas off each other. I think we got it!

This shows that she could act as an agent of curriculum adaptation as she found the value of talking through the problems in the curriculum together with Ms. Mer and the researcher instead of planning by herself. It is important to mention that in many cases during the weekly meetings, Ms. Jay came up with ideas for the lesson plans even if she expressed her struggles with teaching CMP for the first time. Ms. Mer and I did not tell her what to do and just supported her to take her time to read the curriculum or talk about the problems. Specifically, Ms. Mer wholeheartedly showed her empathy in that she understood how vulnerable it would feel to teach mathematics using a new curriculum. Provided such validation and emotional support, Ms. Jay could bounce ideas off each other and achieve the agency to make adaptations for open spaces.

At the very first and last meetings with Ms. Mer, she shared that she viewed our weekly meetings as a place for her to continue learning and growing as a teacher. Even if she taught the same curriculum for multiple years, she believed that she should modify the problems to accommodate each year's students because she would have different students every year. Particularly, she wanted to improve her competence in making mathematics lessons more culturally relevant to her students. Through district-wide professional learning opportunities, Ms. Mer read books and articles about equitable teaching. She expressed her appreciation for such opportunities in that she could develop critical awareness, new perspectives, and knowledge relevant to issues around educational equity. Even after having discussions with other teachers in her school and district, she felt a bit overwhelmed by not having a concrete action item that she could integrate into her mathematics classroom. She hoped to have an opportunity to engage herself in making practical changes. Also, she pointed out that even with many years of teaching expertise, it would not make her feel confident to strive to self-inquiry:

> It's nice to do and it's hard to find the time to do it. Just kind of voluntarily, right? Even we go to things like our own staff meetings that we're required to do, and I'm just as bad as the kids, I'm like, "When can I get out of here?" If it's, especially if it's not useful, like today, we had a staff meeting. The first part was great. The second part was like, "Go work on your own." And the first thing I could think about was, "Get me out of here!" But it was partly because it had no guidance, it had no purpose, and we weren't committing to work on a certain thing. It was just like, "Do these, you have these choices of what to do." You know, and I often say like, especially because we have a new administration right now, I tell them, "I hate being told what to do, but I also want to be told what to

do." So, I thought it was nice to be able to use this time to talk about changes we could make to learn about it.

For Ms. Mer, taking an agentic role in the curriculum adaptations was influenced by the collaborative planning and reflections with Ms. Jay and the researcher. By committing to meeting us once a week, she could secure time to push herself toward changing her lessons. In addition to her initial motivation for learning opportunities, she drew upon our collaboration toward the shared object (i.e., providing students with open spaces), where she could decide "what to do" collaboratively, to achieve her agency in the curriculum adaptations. As such, the teachers leveraged *this* space, where the teachers and the researcher together discussed open spaces for students' multiple mathematical knowledge bases, in order to make changes toward achieving the goals. As Ms. Mer wrote in her reflection journal, "Simply changing a context or making it more open-ended can really help students identify with the work more. It often seems difficult or hard to make these changes—but this study helped me show it doesn't have to be this huge switch."

Explorative Reasoning for Influences of Teacher Agency

Now, I reflect on what they meant by making "simple tweaks" in association with the influence of their teacher agency. First, both teachers as agents of making changes in their planning and teaching drew heavily from the school's official curriculum set to provide students with open spaces. As we discussed culturally relevant mathematics task examples through multiple meetings, the teachers constantly brought up their recently taught lessons or upcoming lessons and interpreted the curriculum through a lens of open spaces. Also, when they planned a lesson for open spaces, they began by reading the curriculum (both student textbooks and teacher guide materials) to figure out open or closed spaces in it. This may imply that teachers'

pedagogical design capacity would have been deployed with limited flexibility while the official curriculum played a central role in their designs.

For the teachers, one of the challenges in their work to open curriculum spaces stemmed from the nature of the curriculum—many problems in the curriculum are contextualized in real-world situations and engage students in exploring embedded mathematical ideas. As the teachers noticed open spaces for mathematical thinking within the curriculum, the teachers wanted to enact the problems without making changes. While maintaining these opportunities, the teachers aimed to include open spaces for real-life connections but found it challenging because the curriculum's problem contexts oftentimes went across multiple problems. That is, if the teachers would replace a problem context with a more culturally relevant one, they might have to make changes throughout the sequence of problems or the entire unit. Similarly, the teachers felt distant from the effort to teach mathematics for social justice. Although the teachers identified that "Fight for Social Justice" among culturally relevant practices (Matthews et al., 2022) was the least attended practice in their lessons, they were hesitant to focus on learning more about it because hardly could they imagine creating social justice mathematics problems on top of using the school's curriculum. In our meetings, the teachers repeatedly expressed their negotiation between the current and the imagined situations by making "small changes" instead of creating entirely new problems. One such simple strategy that they brought up multiple times was adding one question at the beginning or end of problems to explicitly ask students if they had a similar experience. This shows that the teachers' agency was achieved while having the curriculum as a given condition.

In order to see the impact of "small changes," the teachers planned to collect some sorts of data from students during the lesson and discussed them in the reflection meeting. When

asked how they would know if they provided students with open spaces, both teachers focused on observing the student's engagement in the lesson as evidence. While the teachers said that their lesson plans for open spaces worked out well in terms of students' overall engagement, they did not share any observations of how students drew on their multiple mathematical knowledge bases. Such specific observations could possibly have prompted a discussion about students' group dynamics during the Explore phase or ways teachers position students in selecting and sequencing student work for the Summarize phase. The teachers seemed to envision open spaces as the whole class rather than focusing on individual students' different participation. When the teachers made small changes to open spaces to "really help students identify with the work more," they certainly allowed all students to have the equal opportunity to enter open spaces. However, if students' multiple mathematical knowledge bases were not legitimized in the classroom learning community, the students might not view it as an open space for themselves.

In addition, both teachers had limited time and space at work that could be devoted to making curriculum changes. It also added pressure on my end in that I was responsible to help them use our collaboration time to meaningfully develop their teaching practices. Although we clearly communicated the expectations from each other and the logistics for the research process, I was being conscious of their time constraint and tried to not overwhelm the teachers during their participation. While I could share my sincere empathy with them, especially when I witnessed that the 5-minute break between classes was too short to reflect on each lesson or take notes about notice and wonder, my approach would have sent a message that they were supposed to make small changes using the current curriculum. Acknowledging that any teacher would relate to their time constraint, Ms. Mer wrote to suggest other mathematics teachers "try to do some small manageable changes" if they wanted to open curriculum spaces. She explained that

making small changes as the first step would be more meaningful than repeating the past pattern because of the overwhelmed feeling by the amount of time and effort needed to transform the curriculum. Furthermore, she expressed her hope to have regular meetings back with mathematics teachers in the district, which she found very helpful during the pandemic as teachers shared a lot of resources for remote/hybrid teaching. She said she could envision that such a regularly set time for teacher collaboration would help her and other mathematics teachers adapt the curriculum to integrate students' community knowledge. As the three of us experienced that having a regular meeting to reflect and plan for open spaces was helpful, collaboration in a bigger group of teachers using the same curriculum can have a stronger influence on the teachers' pedagogical design capacity. Despite her hope to have such meetings with district teachers, she was also respectful of other teachers' time and willingness to collaborate as a group. It would also call for the administrators' leadership to re-form the district teacher learning community.

Summary of Findings on Influences of Teacher Agency

The teachers in this study utilized an official curriculum which they interpreted as providing open spaces for students' mathematical thinking. Because they found that the curriculum's approach aligned with their beliefs about teaching and learning mathematics, the teachers planned to adhere to the curriculum. The analysis of this study revealed the influence of the official curriculum on the interplay between the teachers' beliefs (the *iterational* dimension) and their instructional goals (the *projective* dimension). Ms. Jay's case demonstrates that she was able to achieve agency in changing her teaching approach as she attended to the affordances of the official curriculum. Even if she held the same belief and goals in teaching and learning mathematics, her teaching and planning practices in the past did not seem to align with these.

This implies that she leveraged the official curriculum as a resource to achieve agency toward positioning herself in the classroom as a facilitator and co-explorer of multiple ways of thinking as opposed to guiding students to practicing one standard solution path.

When it comes to open spaces for connecting mathematics with students' life experiences, both teachers made changes in their lesson designs. The analysis revealed that these teachers' agentic engagement in such curriculum adaptations was supported by the interplay among their perceptions of both cultural diversity and curriculum adaptations (the *iterational* dimension), their explicit goals to open curriculum spaces (the *projective* dimension), and their persistence in navigating ways to leverage the existing resources (the *practical-evaluative* dimension). More specifically, the teachers drew from the contextual aspects within the practical-evaluative dimension of teacher agency. They mitigated their challenges in curriculum adaptations by attending to a suggestion from the literature—an 'easy add-on' can bring huge changes in student learning (the *material* aspect). Also, they connected open spaces for learning mathematics with open spaces for building rapport between teachers and students (the *cultural* aspect). This allowed them to relieve the pressure that they would need to put in extra time and effort in order to open curriculum spaces; instead, they viewed the work as an extension of building rapport with students. In addition, the teachers found the school's advisory hours and this study's weekly meeting hours helpful in their lesson designs (the *structural* aspect). They leveraged informal conversations during the school's advisory hours to learn about students' out-of-school experiences, and they viewed that having time and space with a colleague and a researcher regularly helped them pursue their goals to provide students with open spaces.

CHAPTER 6: DISCUSSION AND CONCLUSION

This study documented how mathematics teachers utilized curriculum resources to provide their students with open spaces that could encourage students to develop mathematical understanding by drawing on their mathematical thinking and life experiences. Also, by inviting participating teachers to collaborative group meetings in order to plan and reflect on their teaching through a lens of curriculum spaces, this study illustrated how the teachers' agency influenced them to enact pedagogical design capacity. The findings of this study suggest that teachers used the curriculum resources in different ways for the two aspects of multiple mathematical knowledge bases—students' mathematical thinking and life experiences. For students' mathematical thinking, teachers noticed that the curriculum engaged students in using multiple strategies and explaining their solutions, which led them to integrate the curriculum problems into their lessons without modification. In contrast, teachers noticed fewer opportunities in the curriculum for students to connect to real-life experiences relevant to problem situations, and they decided to adapt the problems to ensure students bring their life experiences to the class discussions. In the process of designing lessons to connect with students' life experiences, teachers leveraged their informal conversations with students to learn more about students' out-of-school experiences and cultural backgrounds. In this chapter, I discuss the significance, implications, and limitations of this study as well as directions for future research.

Discussion

The Harmony Between the Dominant and the Critical Axes

This study was motivated to mitigate the tension within the educational equity discourse that has stemmed from foregrounding different aspects of equity. For decades, educators have focused on developing students' conceptual understanding of mathematics using cognitively

demanding problems because it can help all students to achieve educational success (Boaler & Staples, 2008; Boaler, 2015). Although these *access* and *achievement* dimensions are necessary for educational equity in schools, they are not sufficient for *true* equity because the education systems and the official curricula tend to reflect one dominant culture while classrooms in the USA become greatly heterogeneous (Civil, 2012; Gay, 2018; Gutiérrez, 2007; Ladson-Billings, 2009; Martin, 2003; Tate, 1995). When scholars called for attention to students' cultural identity and mathematical power to analyze the world (Gutiérrez, 2007; Martin, 2015), it was not to dismiss the importance and potential of the access and achievement dimensions but to emphasize the need to take both *dominant* and *critical* axes into consideration when striving for educational equity. However, little attention was paid to how to harmonize the two axes in practice while the work was implicitly left to teachers' responsibility.

Drawing on the prior research literature that focused on prospective teachers' learning about open spaces for elementary school children's multiple mathematical knowledge bases (Aguirre & Zavala, 2013; Amador & Earnest, 2019; Drake et al., 2015a; Harper et al., 2018), the findings of this study contribute to expanding our understanding about how in-service secondary mathematics teachers can leverage the resources and contexts available to them in order to provide students with open spaces. The teachers in this study opened curriculum spaces by posing questions that could allow students to share their out-of-school experiences and use them as resources for their learning. This suggests that making as small changes as adding one question can serve as a strategy for teachers who find it more challenging to open spaces for students' funds of knowledge than multiple strategies and reasoning opportunities (Harper et al., 2018).

On the one hand, this result is promising in that in-service teachers can leverage their

school contexts and available resources to provide students with open spaces in their daily lessons. It can be a practical suggestion for in-service teachers given that prospective teachers in Aguirre et al.'s (2013) study spent a semester going into a school's neighborhood store to learn about ways of doing mathematics in the Mexican culture and creating mathematics tasks as a part of method course assignments. Along with the support from instructors, the time and effort these prospective teachers put into the adaptation work are not likely available for in-service teachers at their work. Moreover, not all but some of the prospective teachers in Aguirre et al.'s (2013) study created open spaces by successfully attending to both critical and dominant dimensions. Considering the time constraint in schools, making small changes will be more meaningful than making no changes at all.

On the other hand, the findings of this study support the claim that it is challenging for mathematics teachers to incorporate critical aspects into their daily lessons (Rubel, 2017), when the official curriculum does not proactively reflect the diversity of students' cultural backgrounds in the problem contexts or accept the plurality in doing mathematics from different cultures (Civil, 2016; Gay, 2018; Piatek-Jimenez et al., 2014). The teachers in this study expressed similar overwhelming feelings and struggles when considering the identity and power dimensions of equity. One of the main reasons for their challenges was that they thought it would be impossible to know about students' cultural experiences enough to modify the curriculum problems when their students represented so many different cultures.

As one way to combat this challenge, the teachers considered opening spaces in a way that their students could share their own examples instead of teachers themselves selecting a specific cultural reference. Specifically, Ms. Jay opened a space for students to share their stories of being tricked by unfair trades. She added this one question to open a conversation about what

'equal' means in equations, while she did not change the problem context. It was in part because she noticed it was built on the previous problem's context about comparing areas of rectangles. Ms. Mer changed the problem context from food mixtures for chimps to snack mixes of students' choices. Since she noticed that the chimp's recipe problem was one of the various examples of applying percentage concepts to real-world situations, which was not connected with other problems in the curriculum, she did not think that she needed to stick to this particular context. As she replaced the curriculum context with an alternative situation, she also opened a space for students to share their favorite snack items rather than she presents specific cultural food items.

The teachers' approach to incorporating students' multiple mathematical knowledge bases partially relied on creating a safe and respectful learning environment where students could feel comfortable with sharing their experiences that might or might not be related to their home and community cultures. Such an inclusive environment is essential, although insufficient, in the enactment of open spaces because it helps students understand that even if their experiences were different from their peers, their responses would be still valid. While all students were invited to bring their personal experiences, however, the teachers' approach did not explicitly consider whose knowledge was picked up by the teachers and other students in the classrooms. For example, when Ms. Jay asked students if they had experiences with unfair trades, all students might have recalled their own stories, and yet only a handful of students raised their hands to share with the class. Even though volunteering to share thoughts with the whole class should not be the sole measure of student engagement, we have no evidence for those students who did not verbally speak up whether they made meaningful connections between mathematics and their life experiences. Similarly, when Ms. Mer asked students to select their favorite snack items, we cannot assume that all students in a small group had an equal voice as they decided what items

should go in their snack mix with what percentages. While she circulated the room to converse with students and also shared her favorites, Ms. Mer did not seem to attend to students' group dynamics. Moreover, she did not ask students to share their snack mix names and items with the whole class, which contrasted with her summary discussion that focused on sharing different strategies to solve the problem. This contrast may imply that publicly validating students' multiple strategies are more important than validating students' life experiences. In order to make these learning opportunities more concrete, teachers could use exit quizzes as a channel to learn about each and every student's cultural experiences, as Ms. Mer suggested. Also, teachers could let students create a poster to display on the classroom walls, as Ms. Jay suggested, so other students could see different responses to learn from others or relate to others.

Paying attention to meaningful learning opportunities for students from non-dominant cultures calls for teachers' critical awareness beyond creating a learning environment that allows students to feel safe to share their experiences. According to Civil's (2012) identification of underrepresented students' learning opportunities, students in Ms. Mer's and Ms. Jay's classrooms were allowed to use their home *languages* with others who spoke the same language, but it was somewhat vague if students had opportunities to perceive that their funds of knowledge were *valorized,* or if they had opportunities to relate to the ways the *problem* was using mathematics. Although I should not expect to see evidence for all three aspects of learning opportunities (Civil, 2012) within a single lesson, it is important to discuss how the teachers (and the contexts they were in) could develop a critical awareness of students' learning opportunities as they design and enact open spaces. One explanation for the teachers not attending to the power relationships between different races, cultures, and languages might involve their definition of diversity which includes any quality that makes each student special. This view

might have influenced the teachers to put more emphasis on looking at the bright side of classroom diversity to provide equal learning opportunities and less on understanding inequitable or unjust issues inherent to such diversity. This perspective also resonated with their goals to open curriculum spaces. They strived to make mathematics more meaningful and relevant to their students as a whole group rather than accommodating a specific group of students with non-dominant cultural or linguistic backgrounds. Even if they showed genuine interest in learning about individual students' cultures when having informal conversations, such as Ms. Mer practicing Chinese pronunciation for a student's name, the marginalization of students' cultural identities was not taken into consideration in their designing or enacting open spaces. Given that it requires time and support for teachers to develop a critical awareness of educational equity and incorporate it into their practices (Aguirre et al., 2013; Bartell, 2013; Parker et al., 2017), the teachers in this study could also benefit from continuous teacher learning opportunities that bring critical perspectives in teachers' opening curriculum spaces.

From the teacher learning trajectory standpoint (Turner et al., 2012), the teachers in this study demonstrated the *Making Connections* phase of teaching practices to open curriculum spaces. During our earlier meetings, the teachers *attended* to and *elicited* both students' thinking by asking students to explain their strategies and having informal conversations with students to learn about their out-of-school experiences and cultural backgrounds, based on their *awareness* of open/closed spaces within the curriculum. Drawing on all of these *Initial Practices*, the teachers began to incorporate students' multiple mathematical knowledge bases in their instruction. Since they paid *significant attention* to students' mathematical thinking and *limited attention* to students' funds of knowledge, their adaptations to open spaces were in their *Emergent* stage (Aguirre et al., 2013). I view that it was "limited attention" because their

problem adaptations "connected superficially to the community context" (Aguirre et al., 2013, p. 182). Considering that this study was the teachers' first-time exploring possibilities to draw on students' multiple mathematical knowledge bases, they would be able to move toward the *Making Meaningful Connections* phase in their learning trajectory as they would learn more about students' cultural experiences over time.

In fact, the teachers in this study attempted to learn about their students' cultures by facilitating informal conversations and activities during advisory hours and at the beginning of mathematics lessons. It was purposeful to gather information about students' cultural experiences and utilize it to design upcoming lessons. When Ms. Mer facilitated a conversation about special foods and recipes for students' families and cultures, she expected to be able to replace the food-mixture context in the curriculum with one or some of the students' responses. Then she learned that students' special food items were so diverse that she could not select one. Furthermore, when she picked 'making rice' as a potential problem context that could resonate with her Asian students, she soon realized that making rice would not yield at least two different recipes as in the chimp's food context. Ms. Jay wanted to know students' prior understanding of percentages that would be rudimentary for developing a new concept in the upcoming problems. She facilitated an ethnicity pie-chart activity to provide students with opportunities for expressing themselves using mathematics. Although she gathered students' responses and said she could use them as a supplementary resource if students struggle with manipulating percentages, she did not directly use what she learned about students in her lesson design. These examples support the claim that it would not be straightforward for teachers to draw on students' funds of knowledge to adapt or create mathematics problems (Chval et al., 2021; Matthews et al., 2022; Rubel, 2017). That is, the *Making meaningful connections* phase will be achieved through multiple

iterations of practices including ‘try and error.’

Even if the teachers in this study did not directly utilize students’ particular cultural referents in their lesson plans, their informal conversations with students contributed to building rapport between the teacher and students. It is important to note that the teachers had already strived to connect with students and been genuinely interested in learning about different cultures from their students. This preexisting rapport might have influenced their students to feel okay with sharing their different cultural experiences as the teachers observed students’ enthusiastic engagement in these informal conversations. The teachers expressed their beliefs in the importance of getting to know their students throughout the year, and even further they claimed that opening curriculum spaces would enhance this rapport. This interplay between building rapport and opening curriculum spaces suggests that teachers can leverage their rapport with students in order to facilitate open spaces for students’ multiple mathematical knowledge bases because many teachers have various strategies to build rapport at the beginning of a year.

In addition, the stronger teachers can build rapport with students, the more likely the students would share their struggles in learning with teachers. For example, one of Ms. Jay’s Chinese students frankly expressed his challenges of using his second language, English, by stating that he knew the answer to a problem, but he could not say it in English. If the student did not trust that Ms. Jay would care about his challenges, he might not have shared it with her, and she would not have been aware of his struggle. Her connection with this Chinese student helped her to be conscious of allowing students to use their home languages in their learning (Civil, 2012). Pointing back to teachers’ critical awareness of students’ learning opportunities, however, we did not explicitly discuss how teachers could be cognizant of the influences that determine student-teacher interactions. Even if the teachers could utilize extra time during advisory hours to

talk with students, it would still be interactions between one teacher and multiple students with diverse backgrounds. It means that without teachers' purposeful orchestration, some students might have limited opportunities to develop rapport with the teacher.

While the teachers in this study attended to students' cultural and linguistic backgrounds to open spaces for multiple mathematical knowledge bases, their focus was more on students' cultural identity (the *identity* dimension in the *critical* axis) than developing mathematical eyes toward analyzing sociopolitical issues in the world around them (the *power* dimension in the *critical* axis). Drawing on their beliefs in the value of bringing students' multiple mathematical knowledge bases, the teachers successfully opened spaces for connecting mathematics with students' life experiences by leveraging informal conversations and rapport with students. However, these teachers did not necessarily commit to challenging the educational inequity and changing the mathematics tasks in a way that engages students in developing critical awareness of the world. Although the teachers were aware of the movement in the education field toward advocating for social justice based on their participation in school district professional development sessions, they did not feel confident to make such changes in their daily lessons. They explained that topics for social justice could be addressed in an interdisciplinary class where the learning goals would include both mathematics and social issues. Under the pressure of following the pacing guide and learning goals in day-to-day mathematics lessons, they argued that mathematics lessons for developing critical awareness would require extra class time and effort to significantly change the curriculum.

Their perspectives on the power dimension of educational equity support the resistance and tension in whether mathematics lessons should integrate social issues (Bartell, 2013; Felton & Koestler, 2015). In these teachers' cases, the main reason for such resistance was probably that

they did not have access to practical resources to shift their teaching practices. If they were provided a curriculum that put explicit emphasis on addressing community issues (e.g., Gutstein, 2003), they might have considered adapting the problem contexts by connecting with their students' community issues. Not because they did not care about the socio-political issues and the power of mathematics in analyzing and resolving the issues, but it might be because they could not create such teaching materials from scratch while using a school's official curriculum.

Especially when teachers utilized the school's official curriculum as the main teaching resource, prior research revealed that teachers found it challenging to attend to both critical and dominant axes (Cahnmann & Remillard, 2002; Clark et al., 2013; Rubel, 2017; Turner & Drake, 2016). On the one hand, the findings of this study provide evidence to support the claim that even if teachers intentionally attended to students' multiple mathematical knowledge bases, it can be challenging to design lessons drawing on students' diverse life experiences. On the other hand, the findings also provide a glimpse of the possibility to harmonize the two axes of Gutiérrez's (2007) equity framework. Ms. Mer's curriculum adaptation for the food mixture context describes how she carefully balanced the two. When she decided to change the context of food mixture recipes for chimps, it was to provide students with open spaces for real-life connections (the *critical* axis). Then, she continued to explain that she needed to keep the same structure of the questions in the curriculum in order to ensure open spaces for mathematical thinking (the *dominant* axis). This example demonstrates that it is possible for teachers to harmonize the critical and dominant aspects of educational equity and provide students from diverse backgrounds with open spaces.

The teachers' decision-making about the adaptation showcases Debarger et al.'s (2013) *productive adaptations*: they considered *students' needs* while following the curriculum's

approach and content, made *discursive connections* by bringing students' out-of-school experiences, and maintained *task complexity* by focusing on multiple strategies and explanations, especially during the enactment. Thus, the findings of this study are consistent with Choppin's (2008) case study in that the teachers made adaptations based on their observations of students. Taking one step further, the findings of this study revealed that teachers not only considered student thinking for adaptations but also took students' life experiences into account. Both teachers launched the problems by asking students to think about their own experiences and connect them with problems and then facilitated student inquiry by emphasizing multiple strategies and understanding how and why strategies work (Drake et al., 2015a).

The findings also highlight that teachers can adapt the curriculum without creating a completely new context to replace the problem context. The teachers viewed that adding one simple question to help students reflect on their own experiences could make a difference, which would be relatively easier to incorporate into their daily lessons compared to selecting a specific cultural example to create a problem. This is not to suggest that using specific cultural references would be less effective than asking students to share their own experiences. If a teacher brought up an Asian food recipe including rice, for example, it could be an opportunity for Asian students to view themselves in the problem contexts—as a *mirror* (Style, 1996)—and develop cultural competence. At the same time, other students could learn about Asian culture—as a *window* (Style, 1996)—and develop appreciation and respect toward different cultures. This can be another great way to support students from diverse backgrounds in developing cultural identity while learning mathematics (Gay, 2018). However, if the making rice context could not elicit students' mathematical thinking to utilize multiple strategies and representations, it would only afford superficial connections, which was why Ms. Mer decided to discard the rice context idea.

As demonstrated in her example, curriculum adaptations for open spaces would call for comprehending constraints and potentials of resources and negotiating to evaluate practical approaches. I will discuss more about this complexity in the following section.

Teacher Agency for Pedagogical Design Capacity

The findings of this study provide empirical evidence that supports the claim that teachers' curriculum use to achieve instructional goals can be influenced by contextual factors (Pietarinen et al., 2016; Priestley et al., 2015; Pyhältö et al., 2014). It means that teachers' pedagogical design capacity is enacted through the interaction between individual teachers and their situations, and such capacity cannot be solely explained as individual teachers' ability or passion. By illustrating how mathematics teachers incorporated students' multiple mathematical knowledge bases into their daily lessons, the findings of this study provide insights into understanding the material, structural, and cultural factors that influence teachers' agentic engagement in opening curriculum spaces. Based on their interpretations of the official curriculum, through a lens of open spaces, teachers decided to use the curriculum in different ways depending on the instructional goals. Their mixed approach to using the curriculum (offloading for mathematical thinking and adapting for life experiences) exemplifies the complexity of teachers' enactment of pedagogical design capacity (Leshota & Adler, 2018).

Given that the concept of students' multiple mathematical knowledge bases encompasses two aspects (i.e., students' prior understanding of mathematics and their life experiences), the teachers' decisions to offload and adapt the curriculum were different based on their interpretations regarding these two aspects. That is, having explicit instructional goals can influence teachers' enactment of pedagogical design capacity (Amador, 2016). When the goal was to encourage students to utilize various strategies and explain their thinking, the teachers

noticed that the official curriculum—problem-based middle school textbooks and the teacher's guide—coherently focused on such learning opportunities, and they implemented the curriculum without making changes. In contrast, when the goal involved providing students with opportunities to draw from their life experiences, they noticed that the official curriculum did not always offer such connections, which led them to adapt the curriculum.

In their design and enactment of open spaces for students' mathematical thinking, the official curriculum had an explicit influence. Ms. Jay's drastic shift in her planning and teaching toward centering on student thinking showcases the impact of curriculum resources on individual teachers' decisions on teaching practice. The curriculum is designed to enhance conceptual understanding of mathematical big ideas by engaging students in exploring relationships among multiple solution strategies, which is well aligned with open curriculum spaces for multiple strategies and explanations in Land et al.'s (2019) curriculum analysis. As she recognized that the curriculum's approach to teaching and learning mathematics was different from her previous teaching experience, she dropped her 'old school' style of planning and teaching and started to explore mathematical ideas together with her students. Considering that her previous school's official curriculum was another example of Standards-based curricula, the influence of this particular curriculum, CMP, could have stemmed from more specific features. One possible explanation would be that the CMP curriculum provides an overview of how mathematical big ideas develop across a unit (Edson et al., 2019). Each problem coherently makes connections with previous problems regarding not only mathematics concepts but also stories of problem contexts. As Ms. Jay consistently expressed during the weekly meetings, such connections seemed to help her facilitate class discussions building off of previous lessons.

Her decision to change her teaching approach is similar to one experienced teacher in

Collopy's (2003) study who developed a student-centered, inquiry-based teaching approach by interacting with the CMP curriculum. One may argue that their changes in teaching approaches are less to do with their agency and more to do with the requirement from the school policy and teachers might simply have implemented the official curriculum. However, even if it would have been mandated by the school, Ms. Jay could have minimized its impact on her usual teaching as Jenkins (2019) identified as the *reactive agency*. Instead of using her new curriculum in a reactive way or even a *passive* way, Ms. Jay actively changed her roles in the classroom as a collaborator and facilitator for student inquiry. This agentic engagement in her planning and enacting the curriculum can be further supported by her reflection on personal experiences. She compared her teaching experiences before and after using the CMP curriculum and expressed her appreciation that using CMP is more meaningful for student learning even if it is hard for her to plan as a new user. This reflection might also have influenced her to recognize the value of accepting such challenges in using a new curriculum (Felton & Koestler, 2015).

The influence of contextual factors in the teachers' design and enactment of open spaces for real-life experiences is more complicated. While the *iterational* dimension and the *projective* dimension can support teachers' reasoning for their instructional decisions, the teachers would not necessarily take agentic actions in their teaching practices without considering the practical possibilities and challenges in their situations (the *practical-evaluative* dimension). In other words, when the teachers noticed fewer opportunities for students to make connections with their life experiences within the curriculum, they could have decided not to adapt the curriculum to provide open spaces by considering the limited time and pressure from pacing and exam preparation. They were able to be engaged in agentic actions by leveraging their informal conversations during daily advisory hours to learn about students' cultural experiences and by

giving each other non-judgmental feedback and support throughout the weekly meetings. This result is consistent with Kneen et al.'s (2023) findings that accounted for the impact of the school management system and support among peer teachers on teachers' agentic engagement in curriculum development.

The findings in this study further suggest the potential of teachers' professional learning community as a place for teachers to achieve agency (Pietarinen et al., 2016). The teachers stated that their school district advocated for equitable teaching and learning in all subjects and called for teachers' special attention to integrating equitable teaching practices into daily lessons by engaging teachers in discussions about such practices. However, when the professional development sessions do not consider teacher agency (Rycroft-Smith & Macey, 2021), teachers often feel like being left alone not knowing where to start. As Ms. Mer expressed, "I hate being told what to do, but I also want to be told what to do," it is important to note that having autonomy would not suffice to take agentic actions for changes. Contrary to their perceptions of typical professional development sessions, they viewed our weekly meetings as a place for them to reflect on their teaching to aim at opening spaces for student learning and to make changes in their design of lessons. This may be due in part to the design of this study taking a participatory approach to proceed with the research process through joint decision-making among members (Skovsmose & Borba, 2004).

Drawing on the resources from the literature that helped them consider open spaces, the teachers collaboratively navigated practical ways to utilize students' cultural experiences. Even if their goals and motivations for open spaces were explicit, their curriculum adaptations were not immensely transformative but rather focused on *small adjustments* (Drake et al., 2015b). Their focus on an 'easy add-on' question might be related to their need to negotiate the time constraint

at work. Both teachers had five classes scheduled every day. There was a short break between classes, but they spent that time talking to students in front of the classroom door. It means they could not have sufficient time to reflect on each lesson and write reflection journals for their future reference. They said they were accustomed to relying on their memories instead of taking notes for reflections. They could not visit each other's classrooms to observe even though they were curious about how the other's lesson plans would work out in the classroom. In this context, teacher educators may not be able to expect teachers to spend extra time planning after-school hours or weekends. After dealing with administrative work, emails, and formative or summative assessments of student understanding, it would be hard to secure a solid time block devoted to making modifications to the curriculum problems. Without having a commitment by participating in this study, thus, it must be very challenging to keep pushing themselves to reflect on students' multiple mathematical knowledge bases to ensure students engage in open spaces. While accepting the current situation where they could not have sufficient time for planning and reflection, the teachers negotiated to make small changes to the problems that would not take up too much of their planning time.

Another reason for making small changes likely involves the CMP problems being so connected throughout a unit. Interestingly, while this official curriculum helped the teachers readily encourage students to focus on conceptual understanding, it also limited the teachers' flexible mobilization due to its connectivity. The teachers did not want to disregard such connections because they appreciated their affordances to support students' mathematical thinking. While the findings of this study expand our understanding of how teachers negotiated their daily interactions with the curriculum to achieve agency, they also highlight implicit influences of the curriculum on teacher agency in a way that constrains the teachers' curriculum

use. Given that the majority of published textbooks reflect the dominant culture with rare cultural relevance (Gay, 2018), if teachers use these textbooks as the main resource to design open spaces, it will be likely for teachers to have only limited room to achieve agency to incorporate students' multiple mathematical knowledge bases. This point suggests a few considerations for providing teacher learning opportunities. First, teachers can develop their practices to open curriculum spaces by interacting with educative features in the curriculum if it contains more open spaces for real-life connections. As Ms. Jay's case demonstrated, the curriculum can help teachers shift toward ambitious teaching practices. If the curriculum includes students' funds of knowledge with the same level of attention to mathematical thinking, teachers will be able to learn about accommodating students' interests and needs from the curriculum. Especially, teacher's guide materials can serve as resources for teachers learning about both students' funds of knowledge and opening strategies to make problems more locally relevant. Second, teachers can benefit from equipping critical awareness in their interactions with the curriculum. Even if textbooks and the school's official curriculum were written in a fashion that reflects only the dominant culture, teachers could critique such curriculum voice and consider opening spaces as they notice closed or conflicting spaces within the curriculum. Lastly, teachers can move toward the *Incorporating* phase (Turner et al., 2012) through collaborations with other colleagues using the same curriculum. Particularly, if teachers can discuss diverse cultural experiences with people from different cultural backgrounds than their own, it will not only enrich teachers' resources for adaptations but also help them develop critical awareness.

Limitations of The Study and Directions for Future Research

This study engaged two middle school mathematics teachers in a group collaboration that followed Skovsmose and Borba's (2004) critical research model of participatory action research.

This study was also guided by a case study approach in the data collection and analysis. The main objective of conducting a case study was to gain insights from a particular case that can inform to refine theories (Stake, 1994). This study zoomed in on a case where two mathematics teachers strived to open curriculum spaces in the context of a collaborative environment among the teachers and the researcher. It means that the findings of this study—highlighting the possibility of integrating open spaces for multiple mathematical knowledge bases in teachers' daily lesson design—should be understood by also taking account of the teachers' particular school situations, such as a wide range of cultural and linguistic diversity or the affordance of their school's advisory hour.

The assumption behind taking a participatory approach was that it would afford a place for teachers to exhibit agentic engagements in opening curriculum spaces. While the teachers successfully leveraged their participation in this study to change their teaching practices (Wright, 2021), the findings of this study were drawn from the data collected during the first cycle of Skovsmose and Borba's (2004) research model. That is, the findings of this study documented the teachers' design activities by capturing their temporal enactment of teacher agency. Given that teacher agency is relational (Pietarinen et al., 2016), the teachers' agency dimensions identified in this study cannot directly be generalized to other times and contexts.

This limitation suggests that if the teachers would be able to engage in the second cycle of the model—the *pedagogical imagination* phase, the *practical organization* phase, and the *explorative reasoning* phase—the findings would provide a deeper insight into understanding teachers' capacity to open curriculum spaces. One possible direction to extend the findings of this study will be conducting a case study that engages a group of teachers and researcher(s) to collaboratively work on opening curriculum spaces through multiple cycles of the critical

research model. Such a longitudinal study will be able to capture possible trajectories of teachers' curriculum adaptations for open spaces in accordance with mathematical big ideas embedded across multiple problems and units in the curriculum.

Particularly, the second cycle of such research can focus on facilitating teachers to conduct action research based on their reflection and inquiry from the first cycle. While this study benefitted from taking a participatory approach in that the teachers and the researcher collaboratively decided the activities and discussion topics for the following week, I as a researcher had an explicit research question going into this collaborative space. Thus, teachers' action research focusing on their own inquiry about curriculum use for equitable teaching can shed light on understanding teachers' perceptions of challenges in attending to students' multiple mathematical knowledge bases and their practical solutions to mitigate the challenges.

Moreover, future research can consider inviting students to participatory action research. The data sources in this study include student work examples and the classroom observation data that captured student participation in the lessons; and yet, no other data source was directly collected from the students such as interviews. It was mainly because the analysis of this study focused on the teachers' decision-making processes in their curriculum use. At the same time, the teachers were informed that this study was open to bringing student work examples or considering student participation. The teachers in this study gathered information about students' cultures and out-of-school experiences through informal conversations as well as collected student work examples for the purpose of reflections and group discussions. As the teachers in this study mentioned that one possible way to open curriculum spaces would be to ask students how they would want to change the problem contexts, student participation in collaborative action research can enrich our collection of possible adaptation examples. Especially, if students

from non-dominant cultural backgrounds bring their experiences into such collaborative work, it can allow teachers and students to come up with potential adaptations to reflect diverse cultural backgrounds in the classroom.

Another contextual influence specific to this study involves the researcher's participation in the discussions. Since it is important to develop trust for achieving shared goals in participatory research, I was committed to building a trustworthy relationship with the teachers while facilitating our weekly meetings. It required me to not only be transparent with my intentions to conduct this study but also present myself as who I am, which was inevitably intertwined with my own cultural identity. As our discussions involved students' cultural and linguistic diversities, I also shared my perspectives on the culturally relevant mathematics task examples and the problems in the CMP curriculum. My participation in these discussions seemed to influence the teachers' perceptions of difficulties in recognizing differences in different cultures. For example, if I did not share that the Japanese do not use the yard as a measurement for buying fabrics, the teachers would not have recognized it. So, they expressed their appreciation to learn from an Asian perspective. At the same time, such recognition might have influenced them to feel less confident in adapting problem contexts to use a specific cultural reference. In addition, my affiliation with the CMP curriculum as a graduate research assistant might have influenced our discussion. Even though I kept explicitly mentioning that it should not stop them from sharing their perceptions, opinions, or feedback about the curriculum, teachers might have felt uncomfortable sharing their critiques. If it was with a researcher who was not directly working for the curriculum development team, the teachers might have suggested making more changes to the curriculum.

Regarding the example with the yard measurement, it is not clear whether it matters to

consider different measurement systems in different cultures when bringing cultural experiences into problem contexts. A question to consider is: What are the potential benefits of changing the unit from yard to centimeters (e.g., honoring Japanese cultural practice), and do these benefits outweigh the potential costs (e.g., adding a step that is not central to the mathematical goal of the lesson)? This point leaves me wondering about potential future studies that focus on collaborations among people from diverse backgrounds. The discussions about different cultural experiences and different interpretations of curriculum spaces will provide insights into a deeper understanding of ways to accommodate students' diverse needs and interests.

Conclusion

As school classrooms become culturally and linguistically heterogeneous, increasing attention has been paid to equitable learning opportunities for all students from diverse backgrounds. Although mathematics educators have diligently scrutinized educational inequity that is compounded with larger sociopolitical issues, making actual changes in classrooms toward equitable learning opportunities is progressing slowly. Teacher educators cannot simply ascribe such slow changes in classrooms to individual teachers' responsibility and capacity. Even if individual teachers believe that teaching involves both consistent reflections on and learning about teaching, and even if they view themselves as a designer of lessons, it must be overwhelming and challenging work to incorporate specific cultural experiences into mathematics problems while attending to all the needs and requests from different stakeholders. That is, teachers' passion and commitment to promoting equitable learning opportunities, while important, should not be taken as the sole contributor to this post-modern educational reform.

Instead, sustainable collaboration among teachers, teacher educators, and curriculum developers should afford a place for teachers to both integrate evidence-based practices into their

lessons and provide the field with their repertoires to accommodate students' life experiences. Such collaborative space can influence teachers to recognize that attending to students' multiple mathematical knowledge bases is not a whole new approach but rather, it can be achieved by identifying what they have already been doing in order to learn about their students and connecting it with mathematics lessons. Having an explicit shared goal to provide students with meaningful learning opportunities, teachers and teacher educators can support one another in seeking ways to balance the focus of lessons between mathematical understanding and cultural identity. Even if the ways teachers open curriculum spaces would look different depending on their particular situations, their work can serve as a concrete example or an inspiration for other teachers. For this, curriculum developers can consider including teachers' adaptation examples in the teacher guide materials as suggestions for possible adaptations. Furthermore, curriculum developers can ultimately include teachers' (and their students') lived experiences within the curriculum in order to provide open spaces for students' multiple mathematical knowledge bases.

To close this chapter, I would like to bring back my story about the first potluck lunch and share my reflection from the perspective I have developed through this study. Around the time when I finished data collection for this study, I came across a short conversation with one member of my research team. It was just a brief chit-chat during a break between meetings, so I did not think there was something special to it at that moment. While I was analyzing the data for this study and listening to the recordings of teachers' stories about their interactions with students, I was reminded of this random conversation with my colleague. I have reconstructed the conversation below as accurately as I can possibly remember.

SP: By the way, I really enjoyed watching the TV show you recommended the other day. I like the main character. She is so cool and warm-hearted.

AE: Great! Did you finish all the episodes?

SP: No, not yet.

AE: Did you see the one with Joseph Gordon Levitt? Then you are close to the end.

SP: Umm. Who is that? I don't know.

AE: What? You don't know him? Come on! (Pulling up an internet search page on a computer and typing the actor's name)

SP: (Looking at the actor's photos on the computer screen) Oh, 조셉 고든 래빗! Of course, I know him from the movie, *500 일의 썸머*. I mean, *500 days of Summer*.

AE: Yeah? I remember he was in the movie *10 Things I Hate About You*.

SP: I am not sure if I watched it.

AE: (Typing the movie title on a search engine) It's a classic. It's referencing one of Shakespeare's plays, [The title]. Are you familiar with Shakespeare's work?

SP: Hmm, I might recognize the title if it was translated into Korean.

When I was at the first potluck lunch gathering, I was overwhelmed by unfamiliar cultural references and isolated myself to stay with my own thoughts. Well, I am still the same person who is shy to jump into a conversation about American culture. In this recent conversation, however, I can see how comfortably I expressed my confusion about different pronunciations of people and different titles with translations. Of course, the amount of time spent and experiences in a new culture matter. I could initiate the conversation in the first place because we had talked about TV shows before, and I was able to watch the one he recommended.

What I would like to draw attention to in this experience is the interplay of multiple

influences that eventually afforded me the confidence to express myself without hiding my cultural and linguistic backgrounds. The conversation might have ended when I said, "I don't know," if he had not shown me some photos of the actor. By bringing visual aids, he instead helped me (and perhaps himself as well) to recognize the different pronunciations of names. Also, I could express my confusion instead of withdrawing myself from the conversation because I trusted him that he would accept and respect the fact that my first language is not English. If he had never asked me about my experiences in Korea or my perspective on cultural differences between the two countries, I would not have felt comfortable enough to remind him that I am natively Korean. It was an open space for me to be able to express my cultural self and feel okay with pronouncing an American actor's name with a Korean accent.

While I have no idea if he was purposefully accommodating my cultural background by recommending American TV shows, asking questions about my home country, and pulling up visual aids of words and photos to help my understanding, nor what influenced him to do so, he nonetheless created an environment where I was comfortable enough to share. What I can say from this is that creating an open space (especially for people from nondominant cultures) can be as simple and casual as this conversation. Just like how simply and casually we could miss out on creating open spaces for the diversity among us.

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