CUBIC FOURFOLDS AND THE KUGA-SATAKE CONSTRUCTION

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ABSTRACT

Given a $K3$ surface S, the Kuga-Satake construction associates to S an abelian variety $KS(S)$ known as the Kuga-Satake variety. Many similarities between cubic fourfolds X and $K3$ surfaces S have been studied, particularly via Hodge theory by Hassett and derived categories by Kuznetsov. We study how the Kuga-Satake construction fits into this theory by studying the Kuga-Satake varieties of cubic fourfolds and their associated $K3$ surfaces, endormorphism algebras of cubic fourfolds, and the derived category $D^b(KS(S))$.

To my parents.

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TABLE OF CONTENTS

LIST OF SYMBOLS

A_Λ The discriminant group of the lattice Λ

$\mathrm{Br}(X)$ The Brauer group of X

$\mathbb{C}$ The complex numbers

C The moduli space of smooth cubic fourfolds

C_d Cubic fourfolds admitting a labeling of discriminant d

$\mathrm{c}_i(\cdot)$ the i-th Chern class

$Cl(V, q)$ The Clifford algebra associated to (V, q)

$Cl^+(V, q)$ The even Clifford algebra associated to (V, q)

$Cores$ Corestriction of algebras

$\mathrm{CSpin}(V)$ The Clifford group on (V, q)

$D^b(X)$ The bounded derived category of coherent sheaves on a variety X

$\mathrm{End}(V)$ The Hodge endomorphism algebra of the Hodge structure V

$\bigwedge^n$ The n-th exterior algebra of V

Φ_K The Fourier-Mukai transform with kernel K

$\mathrm{Gr}(k, V)$ The Grasmannian of $k-$dimensional subspaces of V

$H^n(X, V)$ Singular cohomology of a variety X with coefficients in V

$H^n(X, V)_0$ Primitive cohomology of X with respect to some ample divisor

$h^{p,q}(V)$ The (p, q) Hodge number of V

$HH_i(X)$ The $i-$th Hochschild homology of $D^b(X)$

$\sim$ Isogeny of abelian varieties

κ_S The Kuga-Satake class of S

$KS(V)$ The Kuga-Satake variety of a Hodge structure V

(Λ, b) A lattice Λ with bilinear form b

$l(\Lambda)$ The number of generators of A_Λ

$\mathcal{N}_d$ The moduli space of polarized $K3$ surfaces of degree d

$NS(S)$ The Néron-Severi group of S

$O(V)$ The orthogonal group on (V, q)

$\mathcal{O}_X$ The structure sheaf of a variety X

PGL The projective general linear group

$\rho(S)$ The Picard rank of S

ρ_V The algebraic representation defining the Hodge structure V

$\mathbb{Q}$ the rational numbers

$\mathbb{R}$ the real numbers

S A $K3$ surface

(S, L) A polarized $K3$ surface S with polarization L

$T(S)$ The transcendental lattice of S

$T(V)$ The tensor algebra associated to V

$\mathcal{T}$ A triangulated category

V A free $\mathbb{Z}-$module of finite rank or a finite-dimensional $\mathbb{Q}-$vector space

$V^\vee$ The dual Hodge structure to V

(V, Q) A polarized Hodge structure V with polarization Q

$V(n)$ The $n-$th Tate Twist of the Hodge structure V

X A variety, typically a cubic fourfold unless otherwise noted

$\chi(X, \mathcal{O}_X)$ the arithmetic genus

$\mathbb{Z}$ the integers

CHAPTER 1

INTRODUCTION

Associating abelian varieties to complex projective varieties has often been an effective strategy in algebraic geometry. Since many tools have been developed for studying abelian varieties, it is advantageous to be able to transfer geometric questions about projective varieties to questions about abelian varieties. A classic example of such an abelian variety is the Jacobian variety $J(C)$ associated to a complex projective curve C. In fact, the Torelli Theorem states that a nonsingular complex projective curve C is completely determined by its Jacobian $J(C)$ and the theta divisor. There have been several approaches to associating abelian varieties to higher dimensional varieties including Albanese varieties, intermediate Jacobian varieties, and Kuga-Satake varieties. Our study will focus on the Kuga-Satake construction, first developed in [KS67].

Kuga-Satake varieties have been traditionally used to study $K3$ surfaces. We say that a Hodge structure is of $K3$-type if it a weight two, polarized Hodge structure V such that $V^{2,0} = 1$. Of course, given a $K3$ surface S, its middle cohomology $H^2(S, \mathbb{Z})$ is an example of such a Hodge structure. We will review the Kuga-Satake construction in Chapter 3. The construction takes a weight two Hodge structure V of $K3$-type and associates to it a Hodge structure of weight one. We will see that this corresponds geometrically to inputting a $K3$ surface S into the construction and getting an abelian variety $KS(S)$ out of it. It is known that the construction gives an inclusion of Hodge structures

$$V \hookrightarrow H^1(KS(V), \mathbb{Q}) \otimes H^1(KS(V), \mathbb{Q}),$$

so, in particular, it can be shown that the Hodge structure V can be recovered from the Hodge structure on KS and so the Kuga-Satake construction is injective! This motivates the potential for transferring geometric questions on S to questions on the abelian variety $KS(S)$ instead. There is a Torelli theorem for $K3$ surfaces as well. That is, two $K3$ surfaces S_1 and S_2 are isomorphic if and only if there is a Hodge isometry $H^2(S_1, \mathbb{Z}) \cong H^2(S_2, \mathbb{Z})$ [PSS71]. In particular, the Kuga-Satake variety $KS(S)$ determines the $K3$ surface S.

Kuga-Satake varieties have proven useful in answering geometric questions about $K3$ surfaces. For example, it is an important ingredient in Deligne's proof of the Weil conjectures for $K3$ surfaces [Del71]. More recently, the construction has been used in proving cases of the Hodge conjecture for examples of self-products of $K3$ surfaces. We will discuss such an example of [Sch10] in Chapter 4.

The Kuga-Satake construction has been generalized to varieties other than $K3$ surfaces. In [Mor85], Morrison studied the Kuga-Satake varieties associated to abelian surfaces. We detail this example in Example 3.2.9. Further, work has been done by Voisin in [Voi05] to provide an alternate viewpoint to the construction by putting a weight two Hodge structure on the exterior algebra $\bigwedge^* V$. In our study, we aim to investigate the Kuga-Satake variety $KS(X)$ associated to a cubic fourfold X.

It has been noted by many that the Hodge structure on the middle cohomology $H^4(X,\mathbb{Z})$ of a cubic fourfold X looks similar to $H^2(S,\mathbb{Z})$ of a $K3$ surface S. In fact, its Tate Twist $H^4(X,\mathbb{Z})(1)$, which we will define in Chapter 2, is actually a Hodge structure of $K3$-type. This observation was first put on solid footing by Hassett in [Has00] where the notion of an associated $K3$ surface was formalized. In particular, we say that a $K3$ surface S is associated to a cubic fourfold X if there is a Hodge isometry

$$H^4(X,\mathbb{Z})(1) \supset K^\perp \to f^\perp \subset H^2(S,\mathbb{Z}) \tag{1}$$

for appropriate sub-Hodge structures $K^\perp$ and $f^\perp$. Associated $K3$ surfaces have been studied extensively by Hassett and others since then, and they play a role in rationality conjectures for cubic fourfolds. In particular, it is conjectured that a cubic fourfold is rational if and only if it possesses an associated $K3$ surface.

Our goal is to study the Kuga-Satake construction for cubic fourfolds within the framework of associated $K3$ surfaces. In Chapter 3 we study the Kuga-Satake construction for cubic fourfolds. This leads to our first main result relating the Kuga-Satake construction to the theory of associated $K3$ surfaces.

Theorem 1.0.1. *Suppose* (X, K) *is a special cubic fourfold with associated* $K3$ *surface* (S, f) *as in* (1)*. Then* $KS(X) \sim KS(S)^2$*, where* $\sim$ *denotes isogeny of abelian varieties.*

We also show in Section 3.4 that these methods apply to other types of varieties. In particular there is a notion of associated $K3$ surfaces for Gushel-Mukai fourfolds. As a corollary to Theorem 1.0.1, we show:

Corollary 1.0.2. Suppose X is a Gushel-Mukai fourfold and S is a $K3$ surface associated to X. Then $KS(X) \sim KS(S)^4$.

Next, we turn our attention to endomorphism algebras in Chapter 4. Given a $K3$ surface S, one can consider the set of Hodge endomorphisms $T(S)_{\mathbb{Q}} \to T(S)_{\mathbb{Q}}$ on its transcendental lattice, which we will denote by $\mathrm{End}(T(S))$. In his foundational paper in [Zar83], Zarhin studied this endomorphism algebra and was able to show that $\mathrm{End}(T(S))$ is in fact a field. Further, Zarhin shows that it is a number field that is either of totally real type or a CM-field. Our goal is to study $\mathrm{End}(T(X))$ for a cubic fourfold X.

In doing so, we must take a couple of important detours. First, we study Nikulin's lattice theory in [Nik80]. Most of this lattice theory is developed for even, unimodular lattices. However, $H^4(X, \mathbb{Z})$ is an odd, unimodular lattice, so we spend time developing the appropriate lattice theory for odd unimodular lattices. This results in:

Proposition 1.0.3. *Suppose* T *is an even lattice of signature* $(21 - \rho, 2)$*. If* $13 \leq \rho \leq 21$*, then there exists a primitive embedding* $T \hookrightarrow \Lambda_{C4}$ *such that* $T^{\perp} \cong N$ *is odd.*

We also study the image of the period map for cubic fourfolds, which was developed by Laza and Looijenga in [Laz10] and [Loo09]. We prove the existence of a cubic fourfold X with $\mathrm{End}(T(X))$ of totally real type. Further, we study how endomorphism algebras $\mathrm{End}(T(X))$ of cubic fourfolds X are related to endomorphism algebras $\mathrm{End}(T(S))$ of their associated $K3$ surfaces S.

In the final chapter, we take a derived category approach to associated $K3$ surfaces that was inspired by Kuznetsov in [Kuz16]. Kuznetsov studied the derived category $D^b(X)$ of a cubic fourfold X and noted that it contains a subcategory

$$\mathcal{A}_X := \langle \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle^{\perp}$$

that looks quite similar to the derived category $D^b(S)$ of a $K3$ surface S. Kuznetsov conjectures that a cubic fourfold X is rational if and only if $\mathcal{A}_X \cong D^b(S)$ for a $K3$ surface S.

Addington and Thomas in [AT14] and others in [BLM+21] show that Kuznetsov's conjecture for the rationality of cubic fourfolds is actually equivalent to the conjecture inspired by Hassett's Hodge theoretic approach. This inspires us to study how the Kuga-Satake construction fits into the derived picture by investigating $D^b(KS(S))$ and $D^b(KS(X))$. In doing so, we study the Kuga-Satake Hodge conjecture, which is a special case of the Hodge conjecture itself. We show the following:

Proposition 1.0.4. *There exist examples of associated $K3$ surfaces that satisfy the Kuga-Satake Hodge conjecture.*

Assuming the Kuga-Satake Hodge conjecture, which by the previous proposition is a fact in some cases, we are able to construct several functors involving the derived categories of Kuga-Satake varieties. We construct various Fourier-Mukai functors that result in the following commutative diagram in Chapter 5:

$$\begin{array}{ccccc} D^b(S) & & \xrightarrow{\Phi_{\mathcal{O}_{Z_S}}} & & D^b(KS(S)^2) \\ \Big\downarrow{\scriptstyle \Phi} & & & & \Big\downarrow{\scriptstyle \Phi_{\mathcal{O}_{\Gamma_f}}} \\ D^b(X) & \xleftarrow[\Phi_{\mathcal{O}_{Z_X}}]{} & D^b(KS(X)^2) & \xleftarrow[\Phi_{\mathcal{O}_{\Gamma_\Delta}}]{} & D^b(KS(X)) \end{array}$$

and end by outlining possible next directions for research.

CHAPTER 2

BACKGROUND

In this chapter we provide the main background theory and definitions required for our study. This chapter is outlined as follows:

§2.1 We introduce basic Hodge theory and standard definitions, including the Hodge decomposition, Hodge diamond, and polarizations of Hodge structures. Important examples of Hodge-theoretical constructions are given, morphisms of Hodge structures are discussed, and we catalogue some basic facts that will be used later.

§2.2 We briefly recall basic definitions and lattice theory that will be sufficient to use throughout. Discriminants of lattices and discriminant groups are discussed and the most important examples of lattices that we use are constructed. More advanced lattice theory is developed in Chapter 4 where it is primarily used in the context of endomorphism algebras.

§2.3 We discuss facts about abelian varieties which are crucial for us in the context of the Kuga-Satake construction. In particular, we mainly approach complex tori and abelian varieties with their Hodge theory kept in mind. The Hodge diamond of an abelian variety is discussed in our context. The rest of the section is dedicated to showing a one-to-one correspondence between abelian varieties and polarized, integral Hodge structures of weight one.

§2.4 In this section we define $K3$ surfaces and give basic examples. The Hodge diamond of a $K3$ surface is discussed.

§2.5 We discuss cubic fourfolds, the main objects of our consideration throughout the rest of the dissertation. In particular, we point out important similarities between $K3$ surfaces and cubic fourfolds, and introduce the notion of associated $K3$ surfaces in the sense of Hassett. Motivating rationality questions are also discussed.

2.1 Hodge Theory

In the following, suppose that V is either a free $\mathbb{Z}-$module of finite rank or a finite-dimensional $\mathbb{Q}-$vector space. A reference for additional background in this section is [Huy16].

Definition 2.1.1. For $n \in \mathbb{Z}$, a weight-n Hodge structure on V is a direct sum decomposition of the complexification $V_{\mathbb{C}} := V \otimes_{\mathbb{Z}} \mathbb{C}$ as

$$V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}$$

subject to the condition that $\overline{V^{p,q}} = V^{q,p}$ where $\overline{v \otimes z} := v \otimes \overline{z}$ for $v \in V$ and $z \in \mathbb{C}$. We call the Hodge structure integral if V is a free $\mathbb{Z}-$module and we call it rational if V is a $\mathbb{Q}-$vector space. We say that a Hodge structure is of type (k, k) if $V^{p,q} \neq 0$ only for $(p, q) = (k, k)$.

Example 2.1.2. The most important examples of Hodge structures that we consider are provided by cohomology of smooth projective varieties. Let X be a smooth complex projective variety and let $V = H^n(X, \mathbb{Q})$ denote singular cohomology. Then $V_{\mathbb{C}} = H^n(X, \mathbb{Q}) \otimes \mathbb{C} = H^n(X, \mathbb{C})$ by the universal coefficient theorem. We can decompose $V_{\mathbb{C}}$ as

$$H^n(X, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X)$$

where $H^{p,q} = H^q(X, \Omega^p)$ and Ω^p is the sheaf of differential $p-$forms on X. This defines a weight-n Hodge structure on V. Note that the same construction works when applied to the integral Hodge structure with $V_{\mathbb{Z}} = H^n(X, \mathbb{Z})/\text{torsion}$.

Definition 2.1.3. Let X be a smooth complex projective variety. The Hodge numbers of X are defined to be $h^{p,q}(X) := \dim(H^{p,q}(X))$.

The Hodge numbers of a variety are often kept track of in the Hodge diamond. For example, in dimension 2, the Hodge diamond of a surface looks like:

$$\begin{array}{ccccc} & & h^{2,2} & & \\ & h^{2,1} & & h^{1,2} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{1,0} & & h^{0,1} & \\ & & h^{0,0} & & \end{array}$$

The Hodge diamond has several symmetries. In particular, $h^{p,q} = h^{q,p}$ since $\overline{H^{p,q}} = H^{q,p}$ and $h^{p,q} = h^{n-p,n-q}$ due to Serre duality. These facts are often referred to as Hodge symmetry.

There are several standard Hodge-theoretical constructions that we will use throughout. The following are all defined similarly for integral and rational Hodge structures:

Example 2.1.4.

1. Suppose V and W are both Hodge structures of weight n. Then we can define the *direct sum* Hodge structure, which is again of weight n, by
$$(V \oplus W)^{p,q} := V^{p,q} \oplus W^{p,q}.$$
2. Suppose V is a Hodge structure of weight n and W is a Hodge structure of weight m. Then we can define the *tensor product* Hodge structure of weight $n + m$ by
$$(V \otimes W)^{p,q} := \bigoplus_{p+q=n+m} (V^{p_1,q_1} \otimes W^{p_2,q_2}),$$
where $p_1 + p_2 = p$ and $q_1 + q_2 = q$.
3. Suppose V is a Hodge structure of weight n. Then we define the *dual* Hodge Structure $V^\vee$ of weight-$(-n)$ by
$$(V^\vee)^{-p,-q} := \mathrm{Hom}_{\mathbb{C}}(V^{p,q}, \mathbb{C}) \quad (p + q = n)$$
4. Define the *Tate Hodge structure* $\mathbb{Z}(n)$ as the weight $-2n$ Hodge structure on $\mathbb{Z}$ such that $\mathbb{Z}(n)^{-n,-n}$ is one-dimensional and $\mathbb{Z}(n)^{p,q} = 0$ if $(p, q) \neq (-n, -n)$. This is a Hodge structure of type $(-n, -n)$. If we just say *the* Tate structure, we are referring to $\mathbb{Z}(1)$. The rational Tate Hodge structures $\mathbb{Q}(n)$ are defined similarly by replacing $\mathbb{Z}$ with $\mathbb{Q}$ everywhere

5. Given an integral Hodge structure V of weight n, we can increase or decrease its weight by defining the *Tate twist* of V by

$$V(k) := V \otimes \mathbb{Z}(k).$$

This is a Hodge structure of weight $n - 2k$. The Tate twist is defined analogously for rational Hodge structures.

Now that we have defined Hodge structures, we will need to be able to consider morphisms between them.

Definition 2.1.5. Suppose V and W are Hodge structures of the same weight n. Then we define a morphism of Hodge structures as a linear map $f : V \to W$ such that $f_{\mathbb{C}}(V^{p,q}) \subset W^{p,q}$. If the weights of V and W are different, the only morphism of Hodge structures is the zero map, called the trivial morphism.

If the two Hodge structures do not have the same weight, then it is possible to define a morphism of Hodge structures between certain Tate twists. Suppose V has weight n and W has weight $n + 2k$ and we have a linear map $f : V \to W$ satisfying $f_{\mathbb{C}}(V^{p,q}) \subset W^{p+k,q+k}$ then we have a morphism of Hodge structures $f : V \to W(k)$.

We can also understand Hodge structures on V through algebraic representations of $\mathbb{C}^*$ on V, i.e. morphisms of real algebraic groups $\rho : \mathbb{C}^* \to GL(V_{\mathbb{R}})$.

Proposition 2.1.6. *Let V be a finite-dimension $\mathbb{Q}-$vector space. Then there is a bijection*

$$\{\textit{Hodge structures of weight } n \textit{ on } V\} \leftrightarrow \{\textit{real algebraic representations } \rho : \mathbb{C}^* \to GL(V_{\mathbb{R}})\}$$

such that $\rho(r) = r^n$ for all $r \in \mathbb{R}^*$.

Proof. For a full proof see [VG00, Proposition 1.4]. We will sketch the idea of the proof here. Given a rational Hodge structure of weight n, we can associate an algebraic representation by

$$\rho : \mathbb{C}^* \to GL(V_{\mathbb{R}}), z \mapsto (\rho(z) : v \mapsto (z^p \bar{z}^q)v)$$

for $v \in V^{p,q}$. In the other direction, an algebraic representation ρ defines a Hodge structure that is given by the decomposition $V^{p,q} = \{v \in V_{\mathbb{C}} \mid \rho(z)v = (z^p \bar{z}^q)v\}$. □

We now interpret our previous constructions of Hodge structures in terms of this correspondence.

Example 2.1.7. 1. The Tate Hodge structure $\mathbb{Z}(k)$ corresponds to the algebraic representation given by $\rho(z)(v) = (z\bar{z})^{-k}v$ for each $v \in V(k)$.

2. If V and W are two Hodge structures corresponding to algebraic representations ρ_V and ρ_W then $V \otimes W$ corresponds to the algebraic representation $\rho_V \otimes \rho_W$. In particular, taking $W = \mathbb{Z}(1)$ to be the Tate Hodge structure, the Tate twist $V(1)$ corresponds to the algebraic representation $\rho_{V(1)} : z \mapsto (z\bar{z})^{-1}\rho_V(z)$.

3. A morphism of Hodge structures can be interpreted as a linear map $f : V \to W$ that satisfies $f(\rho_V(z)v) = \rho_W(z)(f(v))$ for $v \in V$.

4. If V is a Hodge structure, then $\mathrm{End}(V)$ also has a canonical Hodge structure. If ρ is the representation corresponding to the Hodge structure V, then the representation

$$\rho_{\mathrm{End}}(z)(A) = \rho(z)(A)(\rho(z))^{-1}.$$

corresponds to the Hodge structure on $\mathrm{End}(V)$.

For most of the following, we consider polarized Hodge structures. This is motivated by the fact that the cohomology of varieties we are working with also come equipped with a lattice structure given by the standard intersection form.

Definition 2.1.8. Let V be a rational Hodge structure of weight n and suppose ρ_V is its corresponding algebraic representation. Then a polarization of a Hodge structure is a morphism of Hodge structures

$$Q : V \otimes V \to \mathbb{Q}(-n)$$

such that $Q(v, \rho_V(i)w)$ defines a positive definite symmetric form on $V_{\mathbb{R}}$. We call the pair (V, Q) a polarized Hodge structure.

Example 2.1.9. Let X be a complex projective variety of dimension n. Let $\omega \in H^2(X, \mathbb{Q})$ correspond to an ample divisor. Then for $k \leq n$ define a pairing by

$$Q(v, w) := (-1)^{\frac{k(k-1)}{2}} \int_X v \wedge w \wedge \omega^{n-k}$$

where $\wedge$ denotes the standard wedge product on $H^*(X, \mathbb{Q})$. Then Q defines a polarization on the primitive cohomology $H^k(X, \mathbb{Q})_0$, which is defined as $\langle \omega \rangle^\perp \subset H^k(X, \mathbb{Q})$. This polarization Q is sometimes called the Hodge-Riemann pairing.

In particular, we are interested in the case where $n = k = 2$, i.e. middle cohomology of a surface.

Example 2.1.10. If S is a smooth projective surface then middle cohomology can be decomposed as $H^2(S, \mathbb{Q}) = H^2(S, \mathbb{Q})_0 \oplus \mathbb{Q} \cdot \omega$, where ω corresponds to $\mathcal{O}_S(1)$. In this case, the Hodge-Riemann pairing gives

$$Q(v, w) := -\int_X v \wedge w.$$

Of course, this is just the standard intersection form up to a sign change. This provides a polarization on $H^2(S, \mathbb{Q})_0$. To provide a polarization on the full $H^2(S, \mathbb{Q})$, modify Q to be positive on $\mathbb{Q} \cdot \omega$ (to satisfy the positive-definite requirement of the definition).

We will often need to consider sub-Hodge structures or irreducible Hodge structures.

Definition 2.1.11. Let V be a Hodge structure of weight-n.

- A sub-Hodge structure W of a Hodge structure V is a sub-module or sub-vector space $W \subset V$ such that the Hodge structure on W is completely induced by the Hodge structure on V. In other words,

$$W^{p,q} = W_\mathbb{C} \cap V^{p,q}$$

  defines the Hodge structure on W.
- In case of integral Hodge structures, a sub-Hodge structure $W \subset V$ is called primitive if V/W is torsion-free.
- A Hodge structure is called irreducible if it contains no non-trivial, proper, primitive sub-Hodge structures.

We compile some facts concerning polarizations and sub-Hodge structures.

Remark 2.1.12. Let (V, Q) be a polarized Hodge structure of weight-n.

- If the Hodge structure has even dimension $n = 2k$, then Q defines a $(-1)^{k-p}$–definite form on the subspace $V_\mathbb{R} \cap (V^{p,q} \oplus V^{q,p})$. If ρ is the algebraic representation defining the Hodge structure on V, then $\rho(i)$ acts as i^{p-q} on $V^{p,q}$. Now, $i^{p-q} = i^{2p-2k} = (-1)^{k-p}$. Now since Q is a polarization, we have that $(-1)^{k-p}Q(v,v) = Q(v,\rho(i)v) > 0$ by definition.

- The restriction of the polarization Q to any sub-Hodge structure $W \subset V$ defines a polarization. So, any sub-Hodge structure of a polarizable Hodge structure is also polarizable. We will use this fact when defining polarizations on transcendental and primitive sub-lattices of cohomology rather than full cohomology.
- If V is a polarized rational Hodge structure, then any sub-Hodge structure $W \subset V$ defines a direct sum decomposition $V = W \oplus W^{\perp}$, where the orthogonal complement is taken with respect to the polarization Q. This is because the polarization defines a symmetric, positive definite bilinear form on a finite dimensional vector space in the rational case. If V is an integral Hodge structure, then $W \oplus W^{\perp}$ is in general only a finite index Hodge sub-structure.

This distinction will be important for us to keep in mind in Chapter 3.

2.2 Lattice Theory

We will make use of lattices throughout. The basic notions are defined here, with more advanced theory explored in Chapter 4.

Definition 2.2.1.

1. A lattice is a pair (Λ, b) where Λ is a free $\mathbb{Z}-$module of finite rank and
$$b : \Lambda \times \Lambda \to \mathbb{Z}$$
is a non-degenerate symmetric bilinear form.
2. We call the lattice even if $b(x,x) \in 2\mathbb{Z}$ for all $x \in \Lambda$ and odd otherwise. In particular, we need only one element x with $b(x,x)$ odd to be called an odd lattice.
3. The discriminant of a lattice is the determinant of the Gram matrix of b with respect to any arbitrary basis of Λ, simply denoted by $\mathrm{disc}(\Lambda)$.

Given any lattice, there is a natural injection into its dual lattice $\Lambda^* := \mathrm{Hom}(\Lambda, \mathbb{Z})$ given by

$$\begin{aligned} i_\Lambda \colon \Lambda &\hookrightarrow \Lambda^* \\ x &\mapsto b(x,-) \end{aligned}$$

The injectivity follows since the lattice is non-degenerate by definition.

Definition 2.2.2. The discriminant group of a lattice is defined to be $A_\Lambda := \Lambda^*/\Lambda$. This is a finite group of order $|\mathrm{disc}(\Lambda)|$ by [Huy16]. The lattice Λ is called unimodular if A_Λ is trivial, which happens if and only if $\mathrm{disc}(\Lambda) = \pm 1$. The minimal number of generators of A_Λ is denoted by $l(\Lambda)$.

Definition 2.2.3. Given two lattices Λ_1 and Λ_2, we can define the direct sum lattice $\Lambda_1 \oplus \Lambda_2$ naturally by $(x_1 + x_2, y_1 + y_2)_{\Lambda_1 \oplus \Lambda_2} := (x_1, y_1)_{\Lambda_1} + (x_2, y_2)_{\Lambda_2}$ where $x_1, y_1 \in \Lambda_1$ and $x_2, y_2 \in \Lambda_2$.

Note that the discriminant is multiplicative over direct sums, a fact that will be useful in later sections. The following are the most common examples of lattices of interest to us.

Example 2.2.4.

1. I denotes the lattice of rank one with Gram matrix simply (1). Its direct sum of n-copies will be denoted by I_n. This is clearly a unimodular lattice of rank n.
2. The hyperbolic lattice U is the rank two lattice with Gram matrix
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$
This is a unimodular lattice of discriminant $\mathrm{disc}(U) = -1$.
3. The E_8-lattice is the unique positive definite, even, unimodular lattice of rank 8. It can be described by the set of points
$$E_8 := \left\{(x_i) \in \mathbb{Z}^8 \cup (\mathbb{Z} + \tfrac{1}{2})^8 \mid \Sigma x_i \in 2\mathbb{Z}\right\}.$$
4. The lattice A_2 is the rank 2 lattice with Gram matrix
$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$
Note that this lattice is not unimodular. It has $\mathrm{disc}(A_2) = 3$ and discriminant group $A_{A_2} \cong \mathbb{Z}/3\mathbb{Z}$. One can more generally define lattices of type A_n coming from root systems, but we will only make use of A_2.

We will often consider sub-lattices. If $L \hookrightarrow \Lambda$ is an injective morphism of free $\mathbb{Z}$-modules, then one can use the natural inclusions
$$L \hookrightarrow \Lambda \hookrightarrow \Lambda^* \hookrightarrow L^*$$

to show the relation $\mathrm{disc}(L) = \mathrm{disc}(\Lambda) \cdot (\Lambda : L)^2$ [Huy16, 14.0.2], where $(\Lambda : L)$ denotes the index of the subgroup L in Λ.

Definition 2.2.5. We call an embedding of lattices $L \hookrightarrow \Lambda$ primitive if its cokernel is torsion free.

As a word of caution, note that if $L_1 \hookrightarrow \Lambda$ is a primitive embedding and $L_2 := L_1^{\perp}$, where the orthogonal complement is taken inside of Λ, then the inclusion $L_1 \oplus L_2 \hookrightarrow \Lambda$ is not necessarily primitive, though it is of finite index.

We also consider the twists of lattices in the following sections.

Definition 2.2.6. Given a lattice (Λ, b), its twist $\Lambda(m)$ is defined by changing the form on Λ by multiplication by m. In other words, $b_{\Lambda(m)} := m \cdot b_{\Lambda}$.

The discriminant of a twisted lattice is related to the discriminant of the original lattice via

$$\mathrm{disc}(\Lambda(m)) = \mathrm{disc}(\Lambda) \cdot m^{rk(\Lambda)}.$$

where $\mathrm{rk}(\Lambda)$ denotes the rank of Λ.

2.3 Abelian Varieties

Abelian varieties will be important to us in the following, since the Kuga-Satake construction results in an abelian variety $KS(V)$. In this section, we collect basic facts about abelian varieties that we will use later. A reference for additional background is [Mil86] or [BL04]. A similar exposition of background is given in [Mac16].

Definition 2.3.1. Let V be a complex vector space of dimension n and let $\Lambda \subset V$ be a full lattice, i.e. Λ has rank $2n$ and $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} = V$. Then a complex torus is defined to be

$$X := V/\Lambda.$$

It is well known that if $X = V/\Lambda$ is a complex torus, then $\pi_1(X) \cong H_1(X, \mathbb{Z}) \cong \Lambda$. Additionally, it is a fact from algebraic topology that for complex tori, $H^1(X, \mathbb{Z}) = \mathrm{Hom}(H_1(X), \mathbb{Z})$. So for the case of a complex torus, $H^1(X, \mathbb{Z}) = \mathrm{Hom}(\Lambda, \mathbb{Z})$.

To compute cohomology of a complex torus, it is enough to make use of the Künneth formula and note that as a real manifold, $X \cong (\mathbb{R}/\mathbb{Z})^{2n} \cong (S^1)^{2n}$ where S^1 denotes the unit circle [Mil86, Section 15]. This gives us that

$$\dim H^k(X, \mathbb{Z}) = \binom{2n}{k}$$

and if the complex torus is algebraic, it has Hodge numbers given by

$$h^{p,q}(X) = \binom{n}{p} \cdot \binom{n}{q}.$$

So, if we have an algebraic complex torus of dimension 2 (which we will call an abelian surface shortly), its Hodge diamond is as follows:

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & & 2 & \\ 1 & & 4 & & 1 \\ & 2 & & 2 & \\ & & 1 & & \end{array}$$

Now, we are working towards defining abelian varieties from complex tori. To do this, we first need to discuss Chern classes and polarizations of complex tori.

Let X be a complex torus (or more generally, any complex manifold). Recall the exponential sheaf sequence:

$$0 \longrightarrow 2\pi i\mathbb{Z} \longrightarrow \mathcal{O}_X \xrightarrow{e^{2\pi i}} \mathcal{O}_X^* \longrightarrow 0$$

This induces the usual long exact sequence in cohomology:

$$\cdots \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \to \cdots.$$

Definition 2.3.2. Identify $H^1(X, \mathcal{O}_X^*) \cong \mathrm{Pic}(X)$ and let a line bundle $L \in \mathrm{Pic}(X)$. Define the first Chern class of the line bundle L to be the image of L under c_1 as defined above, i.e. $c_1(L) \in H^2(X, \mathbb{Z})$.

When $X := V/\Lambda$ is a complex torus, by Künneth formula, $H^2(X,\mathbb{Z}) \cong H^1(X,\mathbb{Z}) \bigwedge H^1(X,\mathbb{Z}) \cong \mathrm{Hom}(\Lambda,\mathbb{Z}) \bigwedge \mathrm{Hom}(\Lambda,\mathbb{Z}) \cong \mathrm{Hom}(\Lambda \wedge \Lambda,\mathbb{Z})$, so the first Chern class $c_1(L)$ can be identified with a $\mathbb{Z}-$valued alternating form on the lattice Λ. Further, we can identify exactly when an alternating form on the lattice Λ is induced from the first Chern class of a line bundle.

Proposition 2.3.3. *Let $E : V \times V \to \mathbb{R}$ be an alternating form. Then the following are equivalent:*

- *E represents the first Chern class of a line bundle L.*
- *$E(\Lambda,\Lambda) \subset \mathbb{Z}$ and $E(iv,iw) = E(v,w)$ for all $v,w \in V$.*

Proof. See [BL04, Proposition 2.1.6]. □

This motivates the definition of a polarization of a complex torus. We say that a line bundle L is positive definite if $E(v,iv) > 0$ for all $v \in V$ where E is the alternating form associated to the line bundle L as in Proposition 2.3.3.

Definition 2.3.4. A polarization of a complex torus X is defined to be the first Chern class of a positive definite line bundle L. This means that a polarization is given by an alternating form $E : \Lambda \times \Lambda \to \mathbb{Z}$ under the above correspondence such that the $\mathbb{R}-$linear extension of E satisfies $E(iv,iw) = E(v,w)$ for all $v,w \in V$ and $E(v,iv) > 0$ for all $v \in V$ (positive definite). Compare this definition to the definition of a polarization of a Hodge structure given by 2.1.8.

The following is well known, due to Lefschetz.

Proposition 2.3.5. *[Lef21] If L is a positive definite line bundle on a complex torus X, then X is an algebraic variety with L^3 very ample. So, the map $\varphi_{L^3} : X \to \mathbb{P}^N$ is an embedding.*

We are finally ready to define an abelian variety.

Definition 2.3.6. An abelian variety is defined to be a complex torus equipped with a positive definite line bundle. In particular, the above proposition implies that an abelian variety is a complex, projective variety. In fact, it is also an algebraic group.

An important notion for us in Chapter 3 will be that of an isogeny of abelian varieties, as we will often consider Kuga-Satake varieties up to isogeny.

Definition 2.3.7. A morphism of abelian varieties $f : A \to B$ is called an isogeny if it is surjective and has finite (zero-dimensional) kernel. The degree of the isogeny is the degree of the field extension $[k(A) : f^*k(B)]$. Isogeny is typically denoted by $A \sim B$.

There are several equivalent notions of isogeny as seen in [Mil86].

Proposition 2.3.8. *The following are equivalent:*

- $f : A \to B$ *is an isogeny.*
- $\dim(A) = dim(B)$ *and* f *is surjective.*
- $dim(A) = dim(B)$ *and* $ker(f)$ *is finite.*
- f *is finite, flat, and surjective.*

It is often more appropriate to consider abelian varieties up to isogeny. For example, doing so makes direct sums work well. For example, given an abelian subvariety $A_1 \subset A$, there always exists another abelian subvariety $A_2 \subset A$ such that $A \sim A_1 \times A_2$. Therefore, every abelian variety is isogenous to a direct sum of simple abelian varieties. Such statements do not hold when considering abelian varieties up to isomorphism instead of abelian varieties up to isogeny.

Now that we have defined the necessary notions, we relate complex tori and abelian varieties back to Hodge structures. The following proposition will be our key consideration.

Proposition 2.3.9. *There is a one-to-one correspondence between the set of complex tori and the set of integral Hodge structures of weight one. Additionally, abelian varieties correspond to polarized, integral Hodge structures of weight one.*

Proof. Let $X = V/\Lambda$ be a complex torus. Then by definition we have $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} = V$ is a complex vector space, so it comes equipped with a complex structure J, i.e. $J^2 = -1$. Set $H^{1,0}$ to be the $i-$eigenspace under J. In other words, $H^{1,0} = \{v \in V : Jv = iv\}$. Similarly, set $H^{0,1}$ to be the $(-i)-$eigenspace under J. Then clearly $H = H^{1,0} \oplus H^{0,1}$ defines a weight one Hodge structure.

On the other hand, given a weight one integral Hodge structure on $H_{\mathbb{Z}}$, write $H_{\mathbb{C}} = H^{1,0} \oplus H^{0,1}$. Then $H_{\mathbb{Z}} \hookrightarrow H^{1,0}$ and $H^{1,0}/H_{\mathbb{Z}}$ is a complex torus.

Now, the definition of a polarization on a complex torus was carefully constructed to exactly mirror that of a polarization of a Hodge structure, so abelian varieties correspond to polarized

Hodge structures. ◻

Corollary 2.3.10. Further, there is a one-to-one correspondence between the set of abelian varieties up to isogeny and the set of polarized, *rational* Hodge structures of weight one.

This consideration will be especially crucial for us when working with Kuga-Satake varieties that are constructed from rational Hodge structures.

2.4 K3 Surfaces

In this section we recall relevant facts about $K3$ surfaces.

Definition 2.4.1. A $K3$ surface S is a smooth projective variety of dimension 2 that has trivial canonical bundle, i.e. $\omega_S \cong \mathcal{O}_S$, and zero irregularity, i.e. $H^1(S, \mathcal{O}_S) = 0$.

Example 2.4.2. We list a few basic examples of $K3$ surfaces.

- A smooth quartic $S \subset \mathbb{P}^3_{\mathbb{C}}$ is a $K3$ surface. One of the more notable examples is that of the Fermat quartic given by

$$Z(x^4 + y^4 + z^4 + w^4) \subset \mathbb{P}^3_{\mathbb{C}}.$$

  It is easy to see that such a smooth quartic is indeed $K3$. From the standard short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-4) \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{O}_S \longrightarrow 0$$

  we have that $H^1(S, \mathcal{O}_S) = 0$ since $H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) = 0 = H^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-4))$. The canonical bundle is calculated directly from adjunction formula [Har13], giving $\omega_S \cong \mathcal{O}_S(-3 - 1 + 4) \cong \mathcal{O}_S$.

- A smooth double cover of $\mathbb{P}^2_{\mathbb{C}}$ branched along a smooth sextic curve is a $K3$ surface S of degree 2. For example, one can use adjunction to see that ω_S is trivial.

- Let A be an abelian surface over $k = \mathbb{C}$ (any algebraically closed field k where $\mathrm{char}(k) \neq 2$ will also work). Then A always comes equipped with a natural involution $\iota(a) = -a$ for all $a \in A$. This involution has 16 fixed points and so A/ι has 16 double point singularities. Its minimal resolution $S \to A/\iota$ is a $K3$ surface that is called a Kummer surface.

Proposition 2.4.3. *The Hodge diamond of any $K3$ surface has the following form:*

$$
\begin{array}{ccccc}
 & & 1 & & \\
 & 0 & & 0 & \\
1 & & 20 & & 1 \\
 & 0 & & 0 & \\
 & & 1 & &
\end{array}
$$

Proof. We have that $H^{0,0} = H^0(S, \mathcal{O}_S) = 1$, so by Serre duality $h^{0,0} = h^{2,2} = 1$. By definition, $h^{0,1} = 0$ since $H^1(S, \mathcal{O}_S) = 0$. By symmetry this gives $h^{1,0} = h^{0,1} = h^{2,1} = h^{1,2} = 0$, so all of the interesting Hodge numbers appear in the middle of the diamond. Of course, we also have that $h^{2,0} = 1$ by definition since $H^0(S, \Omega^2) \cong H^0(S, \mathcal{O}_S) = 1$. So, we only need to determine $h^{1,1}$.

The above gives us that the Euler characteristic $\chi(S, \mathcal{O}_S) = 2$. So the Noether formula

$$\chi(S, \mathcal{O}_S) = \frac{c_1^2 + c_2}{12}$$

tells us that $c_2 = 24$, where c_i is the $i-$th Chern class of the tangent bundle. Note that $c_1 = 0$ since ω_S is trivial. Now, c_2 is equal to the topological Euler characteristic, and the above shows us that $b_1 = b_4 = 1$ and $b_2 = b_3 = 0$. So, we have $24 = 1 - 0 + b_3 - 0 + 1$. This implies $b_3 = 22$, so we must have $h^{1,1} = 20$. This completes the Hodge diamond. □

Remark 2.4.4. Equipped with its intersection form, $H^2(S, \mathbb{Z})$ has the structure of an even unimodular lattice. This lattice is denoted by Λ_{K3}, and it can be proved that for any $K3$ surface S it is isomorphic to the lattice $E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$, called the $K3$ lattice.

Sub-lattices of $H^2(S, \mathbb{Z})$ will also be considered throughout. In particular, we will need the Néron-Severi group and the transcendental lattice of S.

Definition 2.4.5. Given a $K3$ surface S, the Néron-Severi group is defined to be

$$NS(S) := H^2(S, \mathbb{Z}) \cap H^{1,1}(S).$$

$NS(S)_{\mathbb{Q}}$ can be defined similarly as $H^2(S,\mathbb{Q}) \cap H^{1,1}(S)$. The transcendental lattice of S is defined to be the orthogonal complement to $NS(S)$ with respect to the intersection form as

$$T(S) \coloneqq NS(S)^{\perp} \subset H^2(S,\mathbb{Z}).$$

Note that when considered as rational Hodge structures, we have a direct sum decomposition $H^2(S,\mathbb{Q}) = NS(S)_{\mathbb{Q}} \oplus T(S)_{\mathbb{Q}}$, a fact that will be useful later.

The rank of $NS(S)$ is called the Picard rank of S, denoted $\rho(S)$. For a projective $K3$ surface $S \subset \mathbb{P}^n_{\mathbb{C}}$, the Picard rank ranges between $1 \leq \rho(S) \leq 20$. A $K3$ surface of maximal Picard rank 20 is sometimes called a *singular* $K3$ surface.

We will often have to work with polarized $K3$ surfaces in the following.

Definition 2.4.6. A polarized $K3$ surface (of degree $2d$) is a pair (S, L) where S is a projective $K3$ surface and $L \in \text{Pic}(S)$ is a primitive, ample line bundle with $L^2 = 2d$.

It is a well known fact that polarized $K3$ surfaces of degree $2d$ exist for arbitrary $d > 0$, see for example [Bea11]. We will denote the moduli space of polarized $K3$ surfaces of degree d by $\mathcal{N}_d$.

2.5 Cubic Fourfolds

Throughout this section we work over $k = \mathbb{C}$.

Definition 2.5.1. A cubic fourfold is a smooth hypersurface $X \subset \mathbb{P}^5$ of degree 3.

Remark 2.5.2. Cubic hypersurfaces (not necessarily smooth) in $\mathbb{P}^5$ are parametrized by $\mathbb{P}(\mathbb{C}[x,y,z,u,v,w]_3)$, since they are given by a choice of a degree 3 homogenous polynomial in 6 variables. This space has dimension

$$\binom{n+d}{d} - 1 = \binom{8}{3} - 1 = 55$$

so $\mathbb{P}(\mathbb{C}[x,y,z,u,v,w]_3) \cong \mathbb{P}^{55}$.

The smooth cubic fourfolds correspond to a dense open set $U \subset \mathbb{P}^{55}$. Therefore, the moduli space of cubic fourfolds is

$$\mathcal{C} = [U/\text{PGL}_6(\mathbb{C})].$$

Note that U has dimension 55 since it is dense in $\mathbb{P}^{55}$ and $\mathrm{PGL}_6(\mathbb{C})$ has dimension $6^2 - 1 = 35$ so that $\dim(C) = 55 - 35 = 20$.

Cubic fourfolds are of great interest for rationality problems. That is, given a cubic fourfold X, does there exist a birational map $\varphi : \mathbb{P}^n \dashrightarrow X$? Rationality of cubic surfaces is long known by classical methods. On the other hand, cubic threefolds are known to be irrational by methods of [CG72] using the intermediate Jacobian. Through work of Hassett, cubic fourfolds are expected to show more mixed behavior, though most cubic fourfolds are expected to be irrational. We will explore this idea through Hodge theory in this section. Before seeing more general theory, we briefly give a few examples of cubic fourfolds that are known to be rational. See [Has16, Section 1] for additional details.

Example 2.5.3.

1. A cubic fourfold X containing two disjoint planes P_1 and P_2 can be shown to be rational. The map is constructed in the expected way, by taking a points $p_1 \in P_1$ and $p_2 \in P_2$ and joining them with a line to generally get a third point on X. In this way one can construct a birational map $\varphi : P_1 \times P_2 \dashrightarrow X$.
2. Let X be a cubic fourfold containing a plane P and another projective surface W such that

$$\deg(W) - \langle P, W \rangle$$

is odd, where $\langle\, , \,\rangle$ denotes the intersection pairing. Then X can be shown to be rational by [Has99, Corollary 2.2]. We will revisit this example in Chapter 5.
3. If M is a skew-symmetric $2n \times 2n$ matrix then the determinant of M can be written as

$$\det(M) = \mathrm{Pf}(M)^2$$

where $\mathrm{Pf}(M)$ is a homogenous form of degree n known as the Pfaffian of M. A Pfaffian cubic fourfold is the zero locus of this form:

$$X = Z(\mathrm{Pf}(M)) \subset \mathbb{P}^5$$

It was shown in [Tre84] that Pfaffian cubic fourfolds are rational.

Now we explore the Hodge theory of cubic fourfolds. Suppose $X \subset \mathbb{P}^5$ is a cubic fourfold. The Hodge diamond of X is of the form:

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 1 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
0 & & 1 & & 21 & & 1 & & 0 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 1 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}$$

Remark 2.5.4. Let X be a cubic fourfold.

1. Equipped with its intersection form, $H^4(X, \mathbb{Z}) \cong I_{21} \oplus I_2(-1)$ by [Has00, Proposition 2.1.2]. Therefore, we can see that $H^4(X, \mathbb{Z})$ is a unimodular lattice of signature $(21, 2)$. It is an odd lattice since $(h^2)^2 = 3$ for a hyperplane class h.
2. We also consider primitive cohomology $H^4(X, \mathbb{Z})_0 := \{h^2\}^\perp$. The signature of this lattice is $(20, 2)$ but it is no longer unimodular since $H^4(X, \mathbb{Z})_0 \cong A_2 \oplus U^2 \oplus E_8^2$ where each component is defined in 2.2.4. Since the discriminant is multiplicative, we have $\mathrm{disc}(H^4(X, \mathbb{Z})_0) = 3$.
3. As was the case with $K3$ surfaces, we can define the Néron-Severi group and transcendental lattice analogously. Since the Hodge conjecture holds for a cubic fourfold X by [Zuc77], the Néron-Severi group is given by $NS(X) \cong H^4(X, \mathbb{Z}) \cap H^{2,2}(X)$, and $TS(X) := NS(X)^\perp \subset H^4(X, \mathbb{Z})$. The Picard rank $\rho(X)$ is the rank of $NS(X)$ and $1 \leq \rho(X) \leq 21$.

This lattice structure looks very similar to that of a $K3$ surface, though the dimensions and signatures are not quite the same. That leads us to our next set of definitions.

Definition 2.5.5. 1. A cubic fourfold X is called special if there exists an algebraic surface $T \subset X$ that is not homologous to a complete intersection.

2. A labelling of a cubic fourfold is a choice of a rank two saturated sublattice $K \subset H^{2,2}(X, \mathbb{Z})$ such that $h^2 \in K$. We call a sublattice $K \subset L$ saturated if for every $l \in L$, if $nl \in K$ for some $n \in \mathbb{Z}$, then $l \in K$. We denote a labelled cubic fourfold by (X, K). Note that a cubic fourfold may have more than one labeling.

3. The discriminant d of a labelled cubic fourfold is the determinant of the restriction of the intersection form on $H^4(X, \mathbb{Z})$ to the sublattice K. We denote a cubic fourfold with a labeling K of discriminant d by (X, K_d).

Remark 2.5.6. Note that a very general cubic fourfold is not special. Indeed by [Voi86], for a very general cubic fourfold, $H^{2,2}(X, \mathbb{Z}) \cong \mathbb{Z}h^2$. So for a very general cubic fourfold, any algebraic surface $T \subset X$ is homologous to a complete intersection. Therefore, the notion of a special cubic fourfold really is special.

Note that $K_d^\perp$ now has signature $(19, 2)$ just as the primitive cohomology $f^\perp := H^2(S, \mathbb{Z})_0$ of a polarized $K3$ surface (S, f) does. This motivates the following important definition.

Definition 2.5.7. Let (X, K) be a special cubic fourfold and (S, f) a polarized $K3$ surface. We say that S is associated to X if there is a Hodge isometry

$$H^4(X, \mathbb{Z})(1) \supset K^\perp(1) \xrightarrow{\varphi} f^\perp \subset H^2(S, \mathbb{Z})$$

i.e. φ is an isomorphism of weight 2 Hodge structures that respects lattice structures.

We explore conditions for when a cubic fourfold to have an associated $K3$ surface. The following is given in [Has16, Proposition 20]. First, a lemma is needed.

Lemma 2.5.8. *Let $d > 0$ be a positive integer such that $d \equiv 0$ or $2 \pmod 6$. Then there exists an isomorphism of lattices*

$$K_d^\perp \longrightarrow f^\perp$$

if and only if d is not divisible by $4, 9$ or any odd prime $p \equiv 2 \pmod 3$.

This motivates the following definition.

Definition 2.5.9. We call an even, positive integer d admissible if d is not divisible by $4, 9$ or any odd prime $p \equiv 2 \pmod 3$.

The first few admissible values are $d = 14, 26, 38, 42$. Now, we are ready to state Hassett's theorem as the following.

Theorem 2.5.10. *A labelled special cubic fourfold (X, K_d) admits an associated $K3$ surface if and only if d is admissible.*

Proof. See [Has16, Proposition 20]. □

We would like to understand how certain moduli spaces of cubic fourfolds relate to moduli spaces of $K3$ surfaces. First we fix some notation.

As before, let $\mathcal{C}_d \subset \mathcal{C}$ denote special cubic fourfolds admitting a labeling of discriminant d. It can be shown that $\mathcal{C}_d \subset \mathcal{C}$ is an irreducible divisor, and it is non-empty if and only if $d \geq 8$ and $d \equiv 0, 2 \pmod 8$ [Has16, Theorem 13]. These divisors are often referred to as "Hassett divisors". Further, let $\mathcal{C}_d'$ denote cubic fourfolds X together with a choice of saturated embedding of K_d into $H^4(X, \mathbb{Z})$. Let $\mathcal{N}_d$ denote polarized $K3$ surfaces of degree d. Then Hassett proves the following results.

Proposition 2.5.11. *Let d be an admissible value. Then $\mathcal{C}_d'$ is irreducible and there exists an open immersion of $\mathcal{C}_d'$ into $\mathcal{N}_d$.*

Further, we know when cubic fourfolds admit multiple associated $K3$ surfaces. We will use this fact later.

Corollary 2.5.12. Let d be an admissible value. If $d \equiv 2 \pmod 6$ then $\mathcal{C}_d$ is birational to $\mathcal{N}_d$. Otherwise $\mathcal{C}_d$ is birational to a quotient of $\mathcal{N}_d$.

This possible ambiguity in having multiple associated $K3$ surfaces is addressed by Orlov's Theorem. We will discuss derived categories in much more detail in Chapter 5, but it is relevant to mention this result here.

Theorem 2.5.13. *[Orl97, Theorem 3.3] Let S_1 and S_2 be smooth projective $K3$ surfaces over $\mathbb{C}$.*

Then S_1 and S_2 are derived equivalent, i.e. $D^b(S_1) \cong D^b(S_2)$ if and only if there exists a Hodge isometry $T(S_1) \cong T(S_2)$ between transcendental lattices.

Now, if S_1 and S_2 are both associated to a labelled cubic fourfold (X, K), then we have Hodge isometries

$$H^2(S_1, \mathbb{Z})_0 \cong K^\perp \cong H^2(S_2, \mathbb{Z})_0.$$

Hence $T(S_1) \cong T(S_2)$ since the above are Hodge isometries. Therefore, by Orlov's Theorem, we conclude that if a cubic fourfold has multiple associated $K3$ surfaces then they are all derived equivalent.

We end this section by giving a brief census of what is currently known about rationality of cubic fourfolds in terms of Hassett divisors C_d:

1. Cubic fourfolds in C_{14} are known to be rational. Indeed, it can be shown that C_{14} is the closure of the Pfaffian locus that we saw earlier in this section.
2. In [RS18] it is shown by Russo and Staglianò that all cubic fourfolds in C_{26} and C_{38} are rational.
3. Further, Russo and Staglianò show in [RS19] that all cubic fourfolds in C_{42} are rational.

Note that $14, 25, 38, 42$ are in fact the first few admissible values of discriminants d. Since this is precisely when a cubic fourfold possesses an associated $K3$ surface S by Theorem 2.5.10, this gives some evidence towards the following conjecture.

Conjecture 2.5.14. A cubic fourfold X is rational if and only if it possesses an associated $K3$ surface S.

Note, however, that there are no examples of irrational cubic fourfolds that are known to date, even though the conjecture implies that most cubic fourfolds should be irrational. We take this rationality conjecture as motivation to further exploring cubic fourfolds and their associated $K3$ surfaces in the following chapters.

CHAPTER 3

KUGA-SATAKE VARIETIES

In this chapter we prove our first main result. Specifically, we aim to study the Kuga-Satake varieties of cubic fourfolds and their associated $K3$ surfaces in the interest of finding a relationship between them. The chapter is outlined as follows:

§3.1 We review Clifford algebras of quadratic forms. A Clifford algebra is a unital associative algebra that is associated to a vector space or a $\mathbb{Z}-$module equipped with a quadratic form q. These special algebras are one of the main ingredients in the Kuga-Satake construction, so we recall basic definitions and facts concerning them here, as well as briefly mentioning the Clifford group.

§3.2 In this section, we recall the Kuga-Satake construction and show that it produces an abelian variety. We show that the construction works for cubic fourfolds and provide an example of a situation where the Kuga-Satake variety can be explicitly described in the case of an abelian surface.

§3.3 We begin this section by proving an important lemma that is used to describe Kuga-Satake varieties associated to direct sums of Hodge structures. Lemma 3.3.1 allows us to describe additional types of Kuga-Satake varieties up to isogeny. We use this to prove one of our main results: If a cubic fourfold X has an associated $K3$ surface S, then $KS(X) \sim KS(S)^2$, where $\sim$ denotes isogeny of abelian varieties. We also prove a lemma that allows us to show a partial converse in the situation where we know more about the discriminants involved. This relates the Kuga-Satake construction to the theory of associated $K3$ surfaces.

§3.4 In this section, we take an interesting interlude to show that our results can be applied to a different class of varieties that are known as Gushel-Mukai fourfolds. We quickly define these varieties and show that their Hodge theory is related to $K3$ surfaces in a similar way to cubic fourfolds. In particular, concepts such as "special", "admissible discriminant", and "associated $K3$ surface" are all generalized. We end with a couple of corollaries using our main theorem in Section 3.3 to show similar results for Gushel-Mukai fourfolds.

3.1 Clifford Algebras

One of the main ingredients in the Kuga-Satake construction is the Clifford algebra. This is a type of algebra that can be associated to any free $\mathbb{Z}-$module of finite rank equipped with a quadratic form or finite dimensional vector space that is equipped with a quadratic form.

Suppose V is a free $\mathbb{Z}-$module of finite rank attached with a quadratic form q. Recall the tensor algebra associated to V is given by

$$T(V) \coloneqq \bigoplus_{i\geq 0} V^{\otimes i}.$$

There is a natural $\mathbb{Z}/2\mathbb{Z}$-grading on $T(V)$ given by decomposing it into even and odd components as

$$T(V) = T^+(V) \oplus T^-(V)$$

where $T^+(V) \coloneqq \bigoplus_{i\geq 0} V^{\otimes 2i}$ and $T^-(V) \coloneqq \bigoplus_{i\geq 0} V^{\otimes (2i+1)}$.

Definition 3.1.1. The Clifford algebra associated to (V, q) is defined to be

$$C(V, q) \coloneqq T(V)/\langle v \otimes v - q(v) \mid v \in V\rangle .$$

Also define $C_K(V, q) \coloneqq C(V, q) \otimes_{\mathbb{Z}} K$ where we allow $K \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$.

Note that the ideal defining $C(V, q)$ is generated by even elements so the natural $\mathbb{Z}/2\mathbb{Z}$ grading on $T(V)$ descends to a grading on $C(V, q)$ giving

$$C(V, q) = C^+(V, q) \oplus C^-(V, q).$$

The even Clifford algebra $C^+(V, q)$ will be of particular interest to us. Note that the even Clifford algebra is again a sub-algebra of the Clifford algebra.

The tensor algebra $T(V)$ comes with a natural anti-involution $\iota : T(V) \to T(V), v_1 \otimes \ldots \otimes v_n \mapsto v_n \otimes \ldots \otimes v_1$. This descends to an anti-automorphism on $C(V)$ since the ideal defining $C(V)$ is invariant under ι, which we also denote by ι.

Proposition 3.1.2. *The dimension of the Clifford algebra is given by* $\dim_K(C_K(V,q)) = 2^n$ *where* $n = \dim(V)$.

Proof. By [Huy16, Section 4.1.1], one can define an isomorphism $C_K(V,q) \cong \bigwedge^* V_K$ between the Clifford algebra and exterior algebra of V. Under this isomorphism, a choice of an orthogonal basis $v_1, \ldots, v_n$ of V gives a basis of $C_K(V,q)$ of the form $v_1^{a_1} \cdot \ldots \cdot v_n^{a_n}$ for $a_i \in \{0,1\}$. There are 2^n such basis elements. □

Definition 3.1.3. The Clifford group associated to (V,q) is defined to be

$$\mathrm{CSpin}(V) := \{g \in C(V,q)^* \,|\, gVg^{-1} \subset V\}.$$

To study the Clifford group in the next section, we use the fact that there is a natural orthogonal representation

$$\tau : \mathrm{CSpin}(V) \to O(V)$$

given by $g \mapsto (v \mapsto g \cdot v \cdot g^{-1})$. By orthogonal representation we mean $\tau(g)$ preserves q for any g. Since $q(w) = w^2$ for all $w \in V$, we see that $q(g\cdot v\cdot g^{-1}) = (g\cdot v\cdot g^{-1})(g\cdot v\cdot g^{-1}) = q(v)(g\cdot g^{-1}) = q(v)$, so the representation is orthogonal.

3.2 The Kuga-Satake Construction

The Kuga-Satake construction was first defined in [KS67]. The construction allows us to pass from certain Hodge structures of weight two to Hodge structures of weight one. Geometrically, this was first used in associating an abelian variety to a $K3$ surface. Since the original construction was developed, the Kuga-Satake construction has been generalized further. In particular, [Voi05] provides an alternative viewpoint to the construction, and [Mor85] applies the construction to abelian surfaces.

We adapt the Kuga-Satake construction for cubic fourfolds X and explore the relationship of $KS(X)$ with $KS(S)$ for associated $K3$ surfaces S.

Definition 3.2.1. A Hodge structure V is said to be of $K3-$type if it is a Hodge structure of weight two and $v^{2,0} = 1$, where $v^{p,q} := \dim(V^{p,q})$.

Example 3.2.2. $K3$ surfaces and cubic fourfolds provide the most important examples of Hodge structures of $K3-$type for our purposes.

1. Let S be a $K3$ surface. Then $H^2(S,\mathbb{Z})$ is of $K3-$type. We can also see that primitive cohomology $H^2(S,\mathbb{Z})_0$ and the transcendental lattice $T(S)$ are sub-Hodge structures of $K3-$type.
2. Let X be a cubic fourfold. Then $H^4(X,\mathbb{Z})$ is of weight 4. Consider its Tate twist $V = H^4(X,\mathbb{Z})(1)$. This is now a Hodge structure of weight 2 with $v^{2,0} = 1$, since $h^{3,1}(X,\mathbb{Z}) = 1$.

Now, let V be a Hodge structure of $K3$-type and suppose that it has polarization $-q$. The Kuga-Satake construction takes this weight 2 Hodge structure and gives a Hodge structure of weight one. The first step will be to endow $C_{\mathbb{R}}(V,-q)$ with a complex structure.

Let V be one of the polarized Hodge structures in the example above. By the Hodge index theorem, if V has rank n then $-q$ has signature $(n-2,2)$. So, there is a choice of orthonormal basis of $V_{\mathbb{R}}$ such that $-q$ can be written as $-q = -E_1^2 - E_2^2 + E_3^2 + \cdots + E_n^2$. Let e_i correspond to E_i in the Clifford algebra and set $J := e_1 \cdot e_2 \in Cl_{\mathbb{R}}(V,-q)$. Since $e_1e_2 = -e_2e_1$ we have that $J^2 = (e_1e_2)(e_1e_2) = -(e_1)^2(e_2)^2 = -1$. Therefore multiplication by J defines a complex structure on $Cl_{\mathbb{R}}(V,-q)$. Note that J is independent of the choice of orthonormal basis (up to a sign).

Now, with the complex structure defined by J in hand, we define the weight one Hodge structure on $Cl_{\mathbb{R}}(V,-q)$ by the algebraic representation:

$$\begin{aligned} \rho : \mathbb{C}^* &\longrightarrow \mathrm{GL}(Cl_{\mathbb{R}}(V,-q)) \\ x+iy &\longmapsto (v \mapsto (x+Jy)(v)) \end{aligned}$$

for any $v \in V_{\mathbb{R}}$. We perform the same construction to get a weight one Hodge structure on $Cl^+_{\mathbb{R}}(V,-q)$ and $Cl^-_{\mathbb{R}}(V,-q)$ as well.

Now, suppose V is an integral Hodge structure. Then $Cl^+(V,-q) \subset Cl^+_{\mathbb{R}}(V,-q)$ is a full lattice so the quotient $Cl^+_{\mathbb{R}}(V,-q)/Cl^+(V,-q)$ defines a complex torus. We show below that this is indeed an abelian variety.

Definition 3.2.3. The Kuga-Satake variety associated to the integral Hodge structure V of $K3$-type is defined as

$$KS(V) := Cl^+_{\mathbb{R}}(V, -q)/Cl^+(V, -q).$$

In order to show that this complex torus defines an abelian variety, it is sufficient to construct a polarization by our discussion in Proposition 2.3.9. We first choose two orthogonal vectors $v_1, v_2 \in V$ such that $q(v_i) > 0$. This is possible since the signature of q is $(2, n-2)$. Now define a polarization by

$$\begin{array}{rcl} Q : Cl^+(V,q) \times Cl^+(V,q) & \longrightarrow & Q(-1) \\ (x,y) & \longmapsto & \mathrm{tr}(v_1 \cdot v_2 \cdot \iota(x) \cdot y) \end{array} \tag{2}$$

where $\mathrm{tr}(L)$ denotes the trace of an endormorphism L on $Cl^+(V,q)$ and ι denotes the anti-automorphism of the Clifford algebra defined previously.

Lemma 3.2.4. *Q defines a polarization for the weight one Hodge structure on $Cl^+(V,q)$. Therefore, the Kuga-Satake variety $KS(V)$ is an abelian variety.*

Proof. [Huy16, Proposition 4.2.5] □

Now that we have defined the Kuga-Satake variety $KS(V)$ associated to a Hodge structure V of $K3-$type, and realized it as an abelian variety, we discuss basic properties of Kuga-Satake varieties. In the following, we may drop the polarization q in $Cl(V,q)$ to simplify notation.

Lemma 3.2.5. *If* $\dim_{\mathbb{C}}(V) = n$*, then* $\dim_{\mathbb{C}}(KS(V)) = 2^{n-2}$.

Proof. We saw previously in Proposition 3.1.2 that $\dim_{\mathbb{C}}(Cl(V)) = 2^n$. Therefore $\dim_{\mathbb{R}}(Cl^+_{\mathbb{R}}(V)) = 2^{n-1}$. It follows that $\dim_{\mathbb{C}}(KS(V)) = 2^{n-2}$ □

So in general the dimension of Kuga-Satake varieties will be quite high. This creates a difficulty in giving an explicit geometric description of $KS(V)$ in many cases.

Definition 3.2.6. Let X be a variety of dimension n such that $H^n(X,\mathbb{Z})(k)$ is a Hodge structure of $K3-$type for some Tate twist of degree k. Then we define the Kuga-Satake variety associated to X to be the Kuga-Satake variety associated to the particular Hodge structure $V = H^n(X,\mathbb{Z})(k)$, i.e. $KS(X) := KS(H^n(X,\mathbb{Z})(k))$ with the polarization as defined in (2).

One of the more important facts that comes from the construction is that we can recover the Hodge structure on V from the Hodge structure on $KS(V)$. We provide a proof of this fact in the case of a cubic fourfold.

Proposition 3.2.7. *Suppose X is a smooth cubic fourfold and $V = H^4(X, \mathbb{Z})$. Then there is an inclusion of Hodge structures of weight four*

$$V \hookrightarrow Cl^+(V(1)) \otimes Cl^+(V(1)) \otimes \mathbb{Q}(-1).$$

Furthermore, the Hodge structure of weight four on V can be recovered from the Hodge structure of weight one on $KS(X)$.

Proof. The proof is similar to that of [Huy16, Proposition 4.2.6], adapted to our situation. Choose an element $v_0 \in V(1)$ with $q(v_0) \neq 0$. Consider the map

$$\begin{aligned} \varphi : V(2) \quad &\hookrightarrow \quad \mathrm{End}(Cl^+(V(1))) \\ v \quad &\longmapsto \quad (\varphi(v) := f_v : w \mapsto v \cdot w \cdot v_0). \end{aligned}$$

This is an embedding since $f_v(v_1 \cdot v_0) = q(v_0) v \cdot v_1$ for all $v_1 \in V$. We must now show that φ is a morphism of Hodge structures of weight zero.

Recall that a Hodge structure on V corresponds to a representation ρ_V of $\mathbb{C}^*$ on V. Let the induced representations $\rho_V, \rho_{V(1)}, \rho_{V(2)}, \rho_{Cl}$, and ρ_{End} correspond to the Hodge structures on $V, V(1), V(2), Cl^+(V(1))$, and $\mathrm{End}(Cl^+(V(1)))$ respectively. Then, to show that φ is a morphism of Hodge structures we must show

$$f_{\rho_{V(2)}(z)(v)} = \rho_{End}(z) \cdot f_v \tag{3}$$

for all $z \in \mathbb{C}^*$ and $v \in V(2)$ by Example 2.1.7(3). By definition, the evaluation of the left hand side for any $w \in Cl^+(V(1))$ is given by

$$f_{\rho_{V(2)}(z)(v)}(w) = (\rho_{V(2)}(z)(v)) \cdot w \cdot v_0.$$

The evaluation of the right hand side of (3) for any $w \in Cl^+(V(1))$ is given by

$$(\rho_{End}(z) \cdot f_v)(w) = \rho_{Cl}(z) f_v(\rho_{Cl}(z)^{-1}(w)) = \rho_{Cl}(z)(v \cdot \rho_{Cl}(z)^{-1} w \cdot v_0)$$

by Example 2.1.7(4). So, it suffices to show the following identity holds:

$$\rho_{V(2)}(z)(v) = \rho_{Cl}(z) \cdot v \cdot \rho_{Cl}(z)^{-1}.$$

This computation is checked in [Huy16, Proposition 2.6]. Note that by definition, $\rho_{Cl}(z)$ acts via left multiplication by an element in $Cl^+(V_{\mathbb{R}})$. This follows since for $z = x + iy \in \mathbb{C}$, $x, y, J \in Cl^+(V_{\mathbb{R}})$, where $J = e_1 \cdot e_2$ as before. This identity shows that this element, which we identify with $\rho_{Cl}(z)$, is contained in the Clifford group $\mathrm{CSpin}(V(2))$.

Now, use the fact that $Cl^+(V(1))^* \cong Cl^+(V(1)) \otimes \mathbb{Q}(1)$ to get

$$\mathrm{End}(Cl^+(V(1))) \cong Cl^+(V(1)) \otimes Cl^+(V(1))^* \cong Cl^+(V(1)) \otimes Cl^+(V(1)) \otimes \mathbb{Q}(1).$$

Combined with the above result, this gives

$$V(2) \hookrightarrow Cl^+(V(1)) \otimes Cl^+(V(1)) \otimes \mathbb{Q}(1)$$

and tensoring with $\mathbb{Q}(-2)$ gives the desired embedding of Hodge structures

$$V \hookrightarrow Cl^+(V(1)) \otimes Cl^+(V(1)) \otimes \mathbb{Q}(-1).$$

Next suppose we have a Hodge structure of weight four on V as above. Then we get a weight one Hodge structure on $Cl^+(V(1))$. We claim that this Hodge structure can be used to recover the Hodge structure on V. Recall that we have constructed the map φ above of Hodge structures of weight zero. Since $\rho_{Cl}(z) \in \mathrm{CSpin}(V(2))$, so we apply the orthogonal representation

$$\tau : \mathrm{CSpin}(V(2)) \to O(V(2))$$

to get $\tau(\rho_{Cl}(z)) = \rho_{V(2)}(z)$. In this way we recover the Hodge structure of weight zero on $V(2)$. Now, by definition, $\rho_{V(2)}(z) = (z\bar{z})^{-2}\rho_V(z)$. This allows us recover the Hodge structure on V by $\rho_V(z) = (z\bar{z})^2\rho_{V(2)}(z)$. □

Corollary 3.2.8. The Kuga-Satake Construction is injective:

$$KS : \left\{ \begin{array}{l} \text{polarized Hodge structures of} \\ K3-\text{type} \end{array} \right\} \hookrightarrow \left\{ \begin{array}{l} \text{polarized Hodge structures of} \\ \text{weight one} \end{array} \right\}.$$

Proof. By the previous proposition, we can recover the Hodge structure on V directly from the Hodge structure on $KS(V)$ via the orthogonal representation $\tau : \mathrm{CSpin}(V(k)) \to O(V(k))$ for the appropriate Tate-Twist of degree k. $\square$

We now study an example of a Kuga-Satake variety that we can explicitly describe. The following is due to Morrison in [Mor85], who first studied Kuga-Satake varieties associated to abelian surfaces.

Example 3.2.9. Let A be an abelian surface. Then $KS(A) \sim A^8$. We will outline a sketch of the main ideas of the proof.

We saw previously that the Hodge diamond of an abelian surface looks like:

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & & 2 & \\ 1 & & 4 & & 1 \\ & 2 & & 2 & \\ & & 1 & & \end{array}$$

This implies that the Kuga-Satake variety $KS(A)$ has dimension $2^{6-2} = 16$, so A^8 is at least a reasonable candidate. We also know from our discussion in Chapter 2 that an abelian variety is completely determined by its weight one Hodge structure. Therefore, this problem reduces to comparing $H^1(A, \mathbb{Z})$ and $H^1(KS(A), \mathbb{Z})$.

Recall from Chapter 2 that for an abelian surface, $H^2(A, \mathbb{Z}) \cong \bigwedge^2 H^1(A, \mathbb{Z})$ as Hodge structures. With this in mind, identify $H^2(A, \mathbb{Z})$ as the subspace of $\mathrm{Hom}(H^1(A, \mathbb{Z})^*, H^1(A, \mathbb{Z}))$ given by alternating morphisms. Similarly, identify $H^2(A, \mathbb{Z})^*$ with a subspace of $\mathrm{Hom}(H^1(A, \mathbb{Z}), H^1(A, \mathbb{Z})^*)$.

With these identifications, one can define a map:

$$\begin{array}{ccc} \wedge^2 H^1(A,\mathbb{Z}) & \longrightarrow & \mathrm{End}(H^1(A,\mathbb{Z}) \oplus H^1(A,\mathbb{Z})^*) \\ v & \longmapsto & \begin{pmatrix} 0 & v \\ -v^* & 0 \end{pmatrix}. \end{array}$$

Now, this morphism induces a bijective map on the Clifford algebra as

$$Cl^+(\wedge^2 H^1(A,\mathbb{Z})) \to \mathrm{End}(H^1(A,\mathbb{Z})) \oplus \mathrm{End}(H^1(A,\mathbb{Z})^*).$$

It is possible to show by direct computation that $\mathrm{End}(H^1(A,\mathbb{Z})) \oplus \mathrm{End}(H^1(A,\mathbb{Z})^*)$ is isomorphic as a Hodge structure to $(H^1(A,\mathbb{Z}) \oplus H^1(A,\mathbb{Z})^*)^4$, for example [Huy16, Proposition 4.3.1].

Since an abelian variety is always isogenous to its dual [Mil86, Chapter 7], it follows that for abelian surfaces we have

$$KS(A) \sim (A \times \hat{A})^4 \sim A^8.$$

In the next section, we develop more theory and look at additional examples of Kuga-Satake varieties.

3.3 Kuga-Satake Varieties & Associated K3 Surfaces

Before proving our main result, we need a lemma that allows us to analyze the Kuga-Satake variety of a direct sum of Hodge structures.

Lemma 3.3.1. *Suppose V is a Hodge structure of $K3$ type and V can be written as a direct sum of Hodge structures $V = V_1 \oplus V_2$. Without loss of generality, suppose V_2 is of type $(1,1)$ and V_1 is of $K3-$type. Then $Cl^+(V)$ is isomorphic to $2^{\dim V_2 - 1}$ copies of $Cl^+(V_1) \times Cl^-(V_1)$. Thus $KS(V) \cong KS(V_1 \oplus V_2) \sim KS(V_1)^{2^{\dim V_2}}$.*

Proof. The even Clifford algebra decomposes as

$$Cl^+(V) = Cl^+(V_1 \oplus V_2) \cong (Cl^+(V_1) \otimes Cl^+(V_2)) \oplus (Cl^-(V_1) \otimes Cl^-(V_2)).$$

Since V is of $K3-$type, we must have that either V_1 or V_2 are also of $K3-$type and so the other must be pure of type $(1,1)$. Since V_2 is of type $(1,1)$, this means that the representation defined by V_2 is trivial, so by [Poz13, Remark 3.2.6] we have that $Cl^+(V_1) \otimes Cl^+(V_2) \cong Cl^+(V_1)^{2^{\dim(V_2)-1}}$ as Hodge structures and similarly $Cl^-(V_1) \otimes Cl^-(V_2) \cong Cl^-(V_1)^{2^{\dim(V_2)-1}}$ as Hodge structures. So we have an isomorphism of Hodge structures:

$$Cl^+(V) \cong (Cl^+(V_1) \oplus Cl^-(V_1))^{2^{\dim(V_2)-1}}.$$

Now, since the Kuga-Satake variety defined by $Cl^+(V)$ and the Kuga-Satake variety defined by $Cl^-(V)$ are isogenous by [Huy16, Remark 4.2.3], we have that $KS(V) \sim KS(V_1)^{2^{\dim(V_2)}}$. □

Remark 3.3.2. The above lemma is very useful for switching between different Hodge structures coming from the cohomology of a variety used to define Kuga-Satake varieties.

- Let X be a variety of dimension n and let $H^n(X, \mathbb{Z})(k)$ be of $K3-$type for some Tate twist. Then the transcendental lattice $T(X)$ is also of $K3-$type so we can define the Kuga-Satake variety associated to $T(X)$ as $KS(T(X))$ (up to appropriate Tate Twist if necessary). This is now an abelian variety of dimension $2^{\dim(T(X))-2} = 2^{\dim(H^n(X,\mathbb{Z}))-\rho(X)-2}$ where $\rho(X)$ is the Picard rank of X. For a $K3$ surface S, for example, the dimension of $KS(T(S))$ is $2^{20-\rho(S)}$.

  It is natural to ask if there is a relationship between $KS(X) := KS(H^n(X, \mathbb{Z})(k))$ and $KS(T(X))$. Lemma 3.3.1 implies that $KS(X) \sim KS(T(X))^{2^{\rho(X)}}$, a rather useful fact to keep on hand.

  An interesting example occurs when the Picard rank is maximal. For example, if a $K3$ surface has Picard rank $\rho(S) = 20$, then $\dim(KS(T(S))) = 1$, i.e. it is an elliptic curve. This shows that in the case of maximal Picard rank, $KS(S)$ is isogenous to a product of elliptic curves.

- If X is a projective variety and h is an ample class, then one can consider primitive cohomology $h^\perp := H^n(X, \mathbb{Z})_0(k)$ with respect to the ample class. Then $H^n(X, \mathbb{Z})_0(k)$ is again a Hodge structure of $K3-$type, so we can define a Kuga-Satake variety by $KS(V, h) := KS(H^n(X, \mathbb{Z})_0(k))$. By again using Lemma 3.3.1 we have that $KS(X) \sim KS(X, h)^2$.

- Putting the above together, we have the following relations between Kuga-Satake varieties up to isogeny:

$$KS(X) \sim KS(X,h)^2 \sim KS(T(X))^{2^\rho}.$$

So we can easily translate between different Kuga-Satake varieties that are natural in our context.

With these new facts we can explore some new important examples of Kuga-Satake varieties. We will use the example of a Kummer surface again in Chapter 5.

Example 3.3.3. Recall the construction of a Kummer surface outlined in Example 2.4.2. Given an abelian surface A, one can associate to it a Kummer surface S which is always a $K3$ surface. Since we know $KS(A) \sim A^8$ for an abelian surface by Example 3.2.9, a Kummer surface is a natural next choice to understand the Kuga-Satake construction.

The Hodge structure on $H^2(S,\mathbb{Q})$ of a Kummer surface S is isomorphic to $H^2(A,\mathbb{Q}) \oplus \mathbb{Q}(-1)^{16}$ by [Huy16, Example 3.2.5]. Essentially, the cohomology of the Kummer surface is given by the cohomology of the abelian surface plus cohomology coming from the 16 exceptional divisors coming from the blowup. In particular, this also implies that $\rho(S) = \rho(A) + 16$.

By the same argument, it is also known that $T(S)_{\mathbb{Q}} \cong T(A)_{\mathbb{Q}}$ is an isomorphism of Hodge structures. Therefore, we have that

$$KS(T(S)) \sim KS(T(A)).$$

Now, using the remarks following Lemma 3.3.1, we have a sequence of isogenies

$$KS(S) \sim KS(T(X))^{2^{\rho(S)}} \sim KS(T(A))^{2^{\rho(S)}} \sim KS(A)^{2^{16}}$$

since $\rho(S) = \rho(A) + 16$. Since $KS(A) \sim A^8$, we can conclude

$$KS(S) \sim A^{2^{19}}.$$

We also mention the following example of Paranjape in [Par88, Main Theorem] since we will refer to it in the following chapters. The proof is quite involved, so we only state the result here.

Example 3.3.4. Let $S \to \mathbb{P}^2$ be a double cover that is ramified over 6 lines in general position. We saw in Example 2.4.2 that such an S must be a $K3$ surface. Paranjape computes that

$$KS(T(S)) \sim P^{2^{18}}$$

where P is a certain abelian fourfold that is known as a Prym variety. It is constructed geometrically as a cover of a certain genus 5 curve. See [Par88, Section 2] for more background and explicit details.

Combined with Lemma 3.3.1, we have that for $K3$ surfaces S which are realized as double covers of $\mathbb{P}^2$ ramified over six lines

$$KS(S) \sim KS(T(S))^2 \sim (P^{2^{18}})^2 \sim P^{2^{19}},$$

so the Kuga-Satake variety in this case can be explicitly described as a product of specially constructed abelian fourfolds up to isogeny.

We now come to our first main result in which we relate the Kuga-Satake variety of a cubic fourfold X and its associated $K3$ surface S.

Theorem 3.3.5. *Suppose (X, K) is a special cubic fourfold with associated $K3$ surface (S, f). Then $KS(X) \sim KS(S)^2$.*

Proof. Since S is associated to X, $K^\perp \cong f^\perp$. Let $V = H^4(X, \mathbb{Z})$. Then $V_\mathbb{Q} \cong K^\perp_\mathbb{Q} \oplus K_\mathbb{Q}$ and $V(1)_\mathbb{Q}$ is a weight two rational Hodge structure of $K3$ type. Additionally, $V(1)_\mathbb{Q} \cong f^\perp_\mathbb{Q} \oplus K(1)_\mathbb{Q}$ where $f^\perp_\mathbb{Q}$ is of $K3$ type and $K(1)_\mathbb{Q}$ is of type $(1, 1)$. Apply Lemma 3.3.1 with $V_1 = f^\perp_\mathbb{Q}$ and $V_2 = K(1)_\mathbb{Q}$. Then $\dim V_2 = 2$ and $KS(X) \sim KS(f^\perp)^{2^{\dim V_2}} = KS(f^\perp)^4$. Finally, by Remark 3.3.2, there is an isogeny $KS(S) \sim KS(f^\perp)^2$. Combined with the above, we arrive at the desired result $KS(X) \sim KS(f^\perp)^4 \sim KS(S)^2$. □

We can also prove a partial converse to the above theorem. To do that, we need the following lemma.

Lemma 3.3.6. *Suppose A, B are Hodge structures of $K3-$type of the same rank and C is a Hodge structure that is pure of type $(1,1)$. Suppose further that $A \oplus C \cong B \oplus C$. Then $A \cong B$ as Hodge structures.*

Proof. First observe that $A^{1,1} \cong B^{1,1}$ since both of these are trivial Hodge structures of the same rank. Further, we have that $(A \oplus C)^{2,0} = A^{2,0}$ and $(B \oplus C)^{2,0} = B^{2,0}$ since C is pure of type $(1,1)$. Let $\varphi : A \oplus C \to B \oplus C$ be the Hodge isomorphism in the statement. Then we must have

$$\varphi_{\mathbb{C}}((A^{2,0}) = \varphi_{\mathbb{C}}((A \oplus C)^{2,0}) = (B \oplus C)^{2,0} = B^{2,0}.$$

Since φ is a Hodge morphism, $\varphi_{\mathbb{C}}(A^{0,2}) = B^{0,2}$ as well. It follows that $A \cong B$. □

Theorem 3.3.7. *Suppose d is an admissible value, (X, K_d) is a labelled cubic fourfold of discriminant d and (S, f) is a polarized $K3$ surface of degree d. Suppose $KS(X) \sim KS(S)^2$. Then S is associated to X.*

Proof. Since d is an admissible value, we have that $f^\perp \cong K_d^\perp$ as lattices by Lemma 2.5.8. To show that S is associated to X, it suffices to show that $f^\perp \cong K_d^\perp$ as Hodge structures.

Note that $KS(K_d^\perp \oplus K_d) \sim KS(X)$ and $KS(f^\perp \oplus K_d) \sim KS(S)^2$ since K_d is pure of type $(1,1)$ and $KS(S)^2 \sim KS(f^\perp)^4$. Since $KS(X) \sim KS(S)^2$, injectivity of the Kuga-Satake construction implies $K_d^\perp \oplus K_d \cong f^\perp \oplus K_d$ as Hodge structures. Lemma 3.3.6 implies that $K_d^\perp \cong f^\perp$ as Hodge structures. □

The additional assumption that we begin with an admissible value of d is necessary. Consider the following where $d = 2$ is not admissible.

Example 3.3.8. Consider Example 19 of [Has16]. A cubic fourfold X containing a plane P gives a $K3$ surface (S, f) of degree 2. However, if K is the rank two labelling of X then [VG05, 9.7] shows that $K^\perp \subset f^\perp$ as an index 2 sublattice. In particular, there is no lattice isomorphism $K^\perp \cong f^\perp$ and S is not associated to X.

However, we still have an embedding $K^\perp \hookrightarrow f^\perp$ of lattices of rank 21. Therefore $\mathrm{rk}(f^\perp/K^\perp) = 0$, so by [Huy16, Remark 2.5] there is an isogeny $KS(K^\perp) \sim KS(f^\perp)$. So, $KS(X) \sim KS(K^\perp)^4 \sim$

$KS(f^{\perp})^4 \sim KS(S)^2$.

In particular, $KS(X) \sim KS(S)^2$, but S is not associated to X under this choice of labeling and polarization. However, it is unclear if X may be associated to S under a different choice of polarization (of admissible degree).

3.4 Gushel-Mukai Varieties

Our result can be applied to other types of varieties. We provide an aside on one such example here, that of Gushel-Mukai varieties. These varieties are studied in [DIM15] and [Deb20] where basic definitions and properties are given. We will focus on Gushel-Mukai fourfolds as our previous results are applicable to them.

Definition 3.4.1. A Gushel-Mukai fourfold X is a smooth complex Fano variety of dimension 4 and Picard rank 1, so that $\mathrm{Pic}(X) = \mathbb{Z}H$, such that X is of index 2 and degree 10. That is, $-K_X \equiv 2H$ and $H^4 = 10$.

The following proposition gives further geometric attributes of Gushel-Mukai fourfolds, these are sometimes taken as a definition.

Proposition 3.4.2. *[Deb20, Theorem 1.1] A smooth Gushel-Mukai fourfold X as given by the above definition can be described as*

$$X = \mathrm{Gr}(2, V_5) \cap \mathbb{P}(W_9) \cap Q \subset \mathbb{P}(\textstyle\bigwedge^2 V_5)$$

where Q is a quadric and $W_9 \subset \bigwedge^2 V_5$ is a vector subspace of dimension 9 *and V_5 is a vector space of dimension* 5.

These varieties are studied because they have interesting properties similar to those of cubic fourfolds. In particular, questions about their rationality are open and they have interesting period maps and derived categories (as studied in [KP18] in particular).

The Hodge diamond of a Gushel-Mukai fourfold looks similar to that of a cubic fourfold. A full computation is provided by [IM11, Lemma 4.1] and the Hodge diamond looks like the following:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 1 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
0 & & 1 & & 22 & & 1 & & 0 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 1 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}
$$

As was the case with cubic fourfolds, all of the interesting structure comes from the middle cohomology. We will refer to middle cohomology $\Lambda_X := H^4(X, \mathbb{Z})$ as the Gushel-Mukai lattice. Equipped with the standard intersection form, it is a unimodular lattice of signature $(22, 2)$ [DIM15, Proposition 5.1], i.e. $\Lambda_X \cong (1)^{22} \oplus (-1)^2$. By the classification of unimodular lattices, $22 - 2 = 20$ is not divisible by 8. So, in particular, Λ_X is an odd, unimodular lattice as in the case of the cubic fourfold lattice.

We would like to establish the notion of a "special" Gushel-Mukai fourfold as was done with cubic fourfolds. Using the description in 3.4.2, set $G := \mathrm{Gr}(2, V_5)$. Then $H^4(G, \mathbb{Z})$ embeds into $H^4(X, \mathbb{Z})$ as a rank 2 sub-lattice. Its orthogonal complement is sometimes referred to as the "vanishing cohomology" of X, i.e. $H^4(X, \mathbb{Z})_{\mathrm{van}} := H^4(G, \mathbb{Z})^{\perp} \subset H^4(X, \mathbb{Z})$. The vanishing cohomology is an even lattice of signature $(20, 2)$ and it is isomorphic to $H^4(X, \mathbb{Z})_{\mathrm{van}} \cong 2E_8 \oplus 2U \oplus 2A_2$, where all of the components are as defined in Chapter 2. $H^4(X, \mathbb{Z})_{\mathrm{van}}$ always contains a special rank 2 sublattice Λ_2 as defined in [DIM15, Proposition 5.1].

Definition 3.4.3. We call a Gushel-Mukai fourfold X special if it contains a surface T that does not come from $\mathrm{Gr}(2, V_5)$. This is equivalent to the condition that the rank of $H^{2,2}(X) \cap H^4(X, \mathbb{Z})$ is at least 3.

Note that a very general Gushel-Mukai fourfold is not special by [DIM15, Corollary 4.6]. Recall that for special cubic fourfolds, we had to choose a rank 2 sublattice K. Since the dimension of $H^4(X,\mathbb{Z})$ for Gushel-Mukai fourfolds is one higher, we must choose a special rank 3 sublattice. A labeling of a Gushel-Mukai fourfold is a choice of a primitive, positive-definite rank 3 sublattice $K \subset H^4(X,\mathbb{Z})$ that contains Λ_2. We define the discriminant of (X,K) to be the determinant of the intersection form on K.

Proposition 3.4.4. *[Deb20, Proposition 4.11] Let (X,K) be a special, labelled Gushel-Mukai fourfold of discriminant d. Then $K^\perp$ is isomorphic to $H^2(S,\mathbb{Z})_0$ for a polarized $K3$ surface (S,f) of degree d if and only if $d \equiv 2,4 \pmod 8$ and the only primes that divide d are $p \equiv 1 \pmod 4$.*

As with cubic fourfolds, the proposition motivates the following definitions.

Definition 3.4.5.

1. An integer satisfying the conditions of Proposition 3.4.4 is called an admissible discriminant.
2. We say that the $K3$ surface (S,f) satisfying the condition of Proposition 3.4.4 is associated to the Gushel-Mukai fourfold X.

The first admissible values of d are $10,20,26$. There is a conjecture relating the existence of associated $K3$ surfaces to rationality similar to the conjecture for cubic fourfolds.

Conjecture 3.4.6. A Gushel-Mukai fourfold X is rational if and only if there is an associated $K3$ surface.

Now that we have set up the theory of Gushel-Mukai fourfolds similarly to that of cubic fourfolds, we can apply the proof of Theorem 3.3.5 to get:

Corollary 3.4.7. Suppose X is a Gushel-Mukai fourfold and S is a $K3$ surface associated to X. Then $KS(X) \sim KS(S)^4$.

It is natural to ask if these concepts for Gushel-Mukai fourfolds and cubic fourfolds are intertwined. There is a notion of an "associated cubic fourfold" for Gushel-Mukai fourfolds. This is addressed in the following proposition.

Proposition 3.4.8. *[DIM15, Proposition 6.6] Let* (X, K_X) *be a labelled Gushel-Mukai fourfold of discriminant d. Then* $K_X^{\perp}$ *is isomorphic to* $K_Y^{\perp}$ *for a special, labelled cubic fourfold* (Y, K_Y) *(also of discriminant d) if and only if either:*

1. $d \equiv 2, 20 \pmod{24}$ *and the only odd primes that divide d are* $p \equiv \pm 1 \pmod{12}$ *or*
2. $d \equiv 12, 66 \pmod{72}$ *and the only primes* $p \geq 5$ *that divide d are* $p \equiv \pm 1 \pmod{12}$.

In particular, the first values of such a d are $2, 12, 26, 44$. Again, we can apply 3.3.5 to get:

Corollary 3.4.9. Let Y be a cubic fourfold associated to a Gushel-Mukai fourfold X. Then $KS(X) \sim KS(Y)^2$.

In particular, note that the values of d for which Gushel-Mukai fourfolds and cubic fourfolds possess an associated $K3$ surface S are not disjoint. Consider the following intriguing example.

Example 3.4.10. For $d = 26$, a labelled cubic fourfold (Y, K_Y) of discriminant 26 possesses a unique associated $K3$ surface S by Corollary 2.5.12 since $26 \equiv 2 \pmod{4}$. Combining our above results, we get that (Y, K_Y) is associated to a Gushel-Mukai fourfold (X, K_X) also of discriminant 26 and so we have $KS(X) \sim KS(Y)^2 \sim KS(S)^4$.

CHAPTER 4

ENDOMORPHISM ALGEBRAS

In this chapter, we study endomorphism algebras of cubic fourfolds. Endomorphism algebras of $K3$ surfaces have been extensively studied and used to answer Hodge conjecture type questions. This chapter is outlined as follows:

§4.1 Given a rational Hodge structure V, one can study its endormorphism algebra $\mathrm{End}_{\mathrm{Hod}}(V)$ given by all homomorphisms of Hodge structures $V \to V$. This was studied in detail by Zarhin in [Zar83] which we outline. In particular, Zarhin showed that for an irreducible Hodge structure V of $K3$-type, $\mathrm{End}_{\mathrm{Hod}}(V)$ is a field and furthermore, it is a number field that is either totally real or a CM field. We explore how these results are useful for answering Hodge conjecture type questions. Next, we discuss known results for the transcendental lattice $T(S)$ of $K3$ surfaces in both cases, and finish by outlining Van Geemen's proof in [VG08] that there exists $K3$ surfaces S with transcendental lattice $T(S)$ such that $\mathrm{End}_{\mathrm{Hod}}(T(S)) = F$ for any given totally real number field F.

§4.2 In this section, we outline Nikulin's lattice theory for even, unimodular lattices in [Nik80]. We modify this theory for odd, unimodular lattices with applications to Λ_{C4} for a cubic fourfold X.

§4.3 We study the period map for cubic fourfolds through work of Laza and Looijenga. In particular, we work towards understanding the image of the period map.

§4.4 In this section we prove our result for the existence of cubic fourfolds with endomorphism algebras of totally real type.

§4.5 We explore some applications of endomorphism algebras in the case of $K3$ surfaces associated to cubic fourfolds in the sense of Section 2.5. We prove a few results as corollaries to the previous sections, and in the vein of [Sch10], we explore connections between Kuga-Satake varieties and endomorphism algebras of totally real type in the associated $K3$ framework.

4.1 Endomorphism Algebras of K3 Surfaces

We begin this chapter by examining known results about endomorphism algebras for $K3$ surfaces. These results inspire our approach for studying endomorphism algebras for cubic fourfolds. Most of the following results are found in [Zar83] and [VG08].

Let S denote a $K3$ surface. In this section, we consider the Hodge structure for the transcendental lattice $T(S)$ instead of the full middle cohomology $H^2(S,\mathbb{Z})$. Recall the definition as follows.

Definition 4.1.1. Let V be a Hodge structure of $K3-$type. The transcendental lattice T of V is the minimal primitive sub-Hodge structure $T \subset V$ with $V^{2,0} = T^{2,0} \subset T_{\mathbb{C}}$.

Note that the transcendental lattice is clearly still of $K3-$type since $h^{2,0}(T) = h^{2,0}(V) = 1$. In the case of a $K3$ surface, $T(S)$ will denote the transcendental lattice of $H^2(S,\mathbb{Z})$.

It is important to note that the transcendental lattice is an irreducible Hodge structure. The following result is well-known, we include a proof for the sake of completeness.

Proposition 4.1.2. *The transcendental lattice $T(V)$ of a Hodge structure V of $K3-$type is an irreducible Hodge structure of $K3-$type.*

Proof. Let $S \subset T$ be a nontrivial sub-Hodge structure of the transcendental lattice. If S is pure of type $(1,1)$ then $(S^{\perp})^{(2,0)} \subset V^{2,0} \subset S^{\perp}_{\mathbb{C}}$ where the $\perp$ is taken inside T. So $S^{\perp} \subset T$ is a nontrivial Hodge structure of $K3-$type. This contradicts minimality of T. If S is not pure of type $(1,1)$, then $V^{2,0} \subset S_{\mathbb{C}}$ so minimality and primitivity of T again implies that $S = T$. So, T contains no nontrivial sub-Hodge structures and by the discussion above, T is a Hodge structure of $K3-$type. Therefore the transcendental lattice is an irreducible Hodge structure of $K3-$type. □

Recall the definition of a homomorphism of Hodge structures from Definition 2.1.5. We use the following notation for the main object of study in this chapter.

Remark 4.1.3. Let V, W be (rational) Hodge structures. Let $\mathrm{Hom}(V,W)$ denote the $\mathbb{Q}-$vector space of homomorphisms of Hodge structures from V to W. Then the endomorphism algebra of V is defined to be $\mathrm{End}(V) := \mathrm{Hom}(V,V)$. Note that $\mathrm{End}(V)$ naturally has the structure of a $\mathbb{Q}-$algebra with the product given by composition of endomorphisms.

Lemma 4.1.4. *If V is an irreducible Hodge structure then* $\mathrm{End}(V)$ *is a division algebra.*

Proof. Note that kernels and images of homomorphisms of Hodge structures are sub-Hodge structures. Therefore, if V is irreducible, then for any nonzero $f \in \mathrm{End}(V)$, its kernel must be zero and the image must be V by Proposition 4.1.2. This means it must be an isomorphism. In particular, f has an inverse and so $\mathrm{End}(V)$ is a division algebra. □

In particular, the above implies that $\mathrm{End}(T(S))$ is a division algebra for the transcendental lattice of a $K3$ surface S. Zarhin shows that much more holds in this case:

Theorem 4.1.5. *[Zar83, Theorem 1.6] Let S be a $K3$ surface. Then* $\mathrm{End}(T(S))$ *is a commutative field.*

Proof. There is a natural homomorphism $i : \mathrm{End}(T(S)) \to Hom_{\mathbb{C}} H^{2,0}(S) = \mathbb{C}$. Since $\mathrm{End}(T(S))$ is a division algebra by Lemma 4.1.4, $i : \mathrm{End}(T(S)) \hookrightarrow \mathbb{C}$ must be an embedding and so it is a commutative field. □

The same proof works for $\mathrm{End}(T(X))$ of a cubic fourfold. Note that since $T(S)$ is finite dimensional, $\mathrm{End}(T(S))$ is necessarily a number field. We will consider two main types of number fields in the following theory, totally real number fields and CM fields.

Definition 4.1.6. Let K be a number field.

1. A number field K is a totally real number field if every embedding $i : K \hookrightarrow \mathbb{C}$ satisfies $i(K) \subset \mathbb{R}$.
2. A number field K is a totally imaginary field if there is no embedding $i : K \hookrightarrow \mathbb{R}$.
3. K is a CM-field if it is totally imaginary and if it is a quadratic extension K/F of a totally real number field F.

The most basic examples are provided by quadratic fields $K = \mathbb{Q}(\sqrt{d})$ for $d \in \mathbb{Z}$ square-free. If $d > 0$, the field is totally real. If $d < 0$, the field is totally imaginary and it is a CM-field with $F = \mathbb{Q}$.

This leads us to the following important result of Zarhin. We will use it frequently in the remainder of the chapter.

Theorem 4.1.7. *Let V be a Hodge structure of $K3-$type and suppose* $\mathrm{End}(V)$ *is a commutative field (for example, consider* $\mathrm{End}(T(S))$ *for a $K3$ surface S). Then either*

1. $\mathrm{End}(V)$ *is a totally real number field or*
2. $\mathrm{End}(V)$ *is a CM-field.*

Proof. See [Zar83, Theorem 1.5.1]. □

Endomorphism algebras have proven to be useful tools for studying the Hodge conjecture for products. Given a smooth projective variety X, recall that the Hodge conjecture states that the space of Hodge classes $\mathrm{Hod}^k(X)$ of degree $2k$ given by

$$\mathrm{Hod}^k(X) := H^{2k}(X, \mathbb{Q}) \cap H^{k,k}(X)$$

is spanned by algebraic k-cycles. The conjecture is of course still open for $k \neq 0, 1, \dim(X) - 1, \dim(X)$.

Now, we explore how endomorphism algebras naturally appear in the Hodge conjecture for self-products. For smooth projective varieties X, Y, the Künneth formula and linear algebra give

$$\begin{aligned} H^s(X \times Y, \mathbb{Q}) &\cong \bigoplus_{l+m=s} H^l(X, \mathbb{Q}) \otimes H^m(Y, \mathbb{Q}) \\ &\cong \bigoplus_{l+m=s} \mathrm{Hom}(H^{2\dim(X)-l}(X, \mathbb{Q}), H^m(Y, \mathbb{Q})). \end{aligned}$$

Note that each summand is a Hodge substructure of the left side. It is easy to see that under this correspondence, Hodge cycles of degree $2k$ correspond to homomorphisms of Hodge structures.

Now, apply the above to $X = Y = S$ for a $K3$ surface S. Note that the Hodge conjecture is not known in general for the self-product $S \times S$. The only nontrivial case in the Hodge conjecture in this case is that of $H^4(S \times S, \mathbb{Q})$.

We know that $H^1(S, \mathbb{Q}) = 0$ and $H^3(S, \mathbb{Z}) = 0$ by Proposition 2.4.3. We can also see that

$$H^4(X, \mathbb{Q}) \otimes H^0(Y, \mathbb{Q}) \cong \mathrm{Hom}(H^0(S, \mathbb{Q}), H^0(S, \mathbb{Q}))$$

and

$$H^0(X, \mathbb{Q}) \otimes H^4(Y, \mathbb{Q}) \cong \mathrm{Hom}(H^4(S, \mathbb{Q}), H^4(S, \mathbb{Q}))$$

are spanned by the classes $\{p\} \times S$ and $S \times \{p\}$ respectively for a point p, so they are algebraic. The only remaining case is $\mathrm{Hom}_{\mathrm{Hod}}(H^2(S, \mathbb{Q}), H^2(S, \mathbb{Q}))$.

$H^2(S, \mathbb{Q})$ splits as a direct sum of Hodge structures as

$$H^2(S, \mathbb{Q}) = NS(S)_{\mathbb{Q}} \oplus T(S)_{\mathbb{Q}}.$$

Now, since $NS(S)_{\mathbb{Q}}$ is pure of type $(1, 1)$ and $T(S)_{\mathbb{Q}}$ is irreducible by Proposition 4.1.2, we have that $\mathrm{Hom}_{\mathrm{Hod}}(NS(S)_{\mathbb{Q}}, T(S)_{\mathbb{Q}}) = 0$ and $\mathrm{Hom}_{\mathrm{Hod}}(T(S)_{\mathbb{Q}}, NS(S)_{\mathbb{Q}}) = 0$. In the first case, observe that the image of such a Hodge morphism would define a Hodge sub-structure of $T(S)_{\mathbb{Q}}$. In the second case, the fact that $NS(S)_{\mathbb{Q}}^{2,0} = 0$ implies that the kernel of such a Hodge morphism would define a Hodge sub-structure of $T(S)_{\mathbb{Q}}$. The Neron-Severi part $NS(S)_{\mathbb{Q}}$ is algebraic by definition and $\mathrm{Hom}_{\mathrm{Hod}}(NS(S)_{\mathbb{Q}}, NS(S)_{\mathbb{Q}})$ is spanned by classes of curves $C_1 \times C_2 \subset S \times S$.

Therefore, we have reduced the problem of the Hodge conjecture for self-products of $K3$ surfaces $S \times S$ to considering

$$\mathrm{Hom}_{\mathrm{Hod}}(T(S)_{\mathbb{Q}}, T(S)_{\mathbb{Q}})$$

and this is of course just $\mathrm{End}(T(S)_{\mathbb{Q}})$ by definition. So, the Hodge conjecture for $S \times S$ is equivalent to the condition that $\mathrm{End}(T(S)_{\mathbb{Q}})$ be generated by algebraic classes. This is the main motivation for studying endomorphism algebras of Hodge structures.

By Theorem 4.1.7, studying $\mathrm{End}(T(S))$ reduces to two cases. The CM case is well understood due to a result of Mukai.

Theorem 4.1.8. *If $f \in \mathrm{End}(T(S))$ is a Hodge isometry, then f corresponds to the class of an algebraic cycle in $H^4(S \times S, \mathbb{Q})$.*

Proof. See [Muk03, Theorem 2]. □

However, in [RM08, Theorem 5.4], the author uses Mukai's theorem and shows the following:

Theorem 4.1.9. *If $\mathrm{End}(T(S))$ is a CM field, then it is spanned by Hodge isometries. In particular, the Hodge conjecture holds for self-products of $K3$ surfaces $S \times S$ if $\mathrm{End}(T(S))$ is of CM type.*

On the other hand, if $\text{End}(T(S))$ is totally real, then the problem of the Hodge conjecture for self products is much more open. In the totally real case, it is easy to show that the only Hodge isometries in $\text{End}(T(S))$ are $\pm\text{id}$. So Mukai's result no longer helps. We will note some examples and results about the Hodge conjecture in the totally real case in Section 4.5.

For the remainder of this section, we will outline Van Geemen's proof for the existence of $K3$ surfaces S with $\text{End}(T(S))$ totally real. We will point out where details of Van Geemen's proof will no longer work in the case of cubic fourfolds, which will lead to our focus in the next two sections. First, a Hodge-theoretic lemma is needed.

Lemma 4.1.10. *Let F be a totally real field and let $m \in \mathbb{Z}, m \geq 3$. Then there exists an irreducible, polarized Hodge structure (V, ρ, ψ) of $K3-$type with $\text{End}(V) = F$ and $\dim_F(V) = m$, where ρ denotes the algebraic representation defining the Hodge structure and ψ denotes the polarization.*

Proof. See [VG08, Lemma 3.2]. □

Now, Van Geemen shows that such a Hodge structure can be realized as the transcendental lattice of a $K3$ surface. We reproduce the proof here directly to illustrate the changes needed in our study of cubic fourfolds.

Proposition 4.1.11. *[VG08, Proposition 3.3] Given a totally real number field F and integer $m \geq 3$ such that $m[F : \mathbb{Q}] \leq 10$, there exists an $(m-2)-$dimensional family of $K3$ surfaces such that $\text{End}(T(S)) = F$ for a general member of that family.*

Proof. Let (V, ρ, ψ) be a polarized, irreducible Hodge structure of $K3-$type with $\text{End}(V) = F$ and period ω by Lemma 4.1.10. Choose a free $\mathbb{Z}-$module $T \subset V$ of rank $d = \dim_{\mathbb{Q}}(V)$ such that ψ is integer valued on $T \times T$. Then by [Nik80, Theorem 1.10.1], there is a primitive embedding of T into the $K3$ lattice Λ_{K3}, $T \hookrightarrow \Lambda_{K3}$. The surjectivity of the period map for $K3$ surfaces implies the existence of a polarized $K3$ surface S with $\text{End}(T(S)) \cong T$ corresponding to the period ω. The proof of [VG08, Lemma 3.2] shows that there are $(m-2)$ moduli. □

So, if we want to work towards a similar result for cubic fourfolds, there are two main difficulties in following a similar process.

First, the lattice theory used by Van Geemen was developed by Nikulin in [Nik80] for even, unimodular lattices. While the $K3$ lattice Λ_{K3} is an even, unimodular lattice, the cubic fourfold lattice Λ_{C4} is odd and unimodular, and the lattice $\Lambda_{C4,0}$ is even and non-unimodular. Odd, unimodular lattices are easier to understand than even, non-unimodular lattices. We study odd, unimodular lattices in Section 4.2 and apply the results to Λ_{C4} instead of $\Lambda_{C4,0}$.

Second, the proof uses surjectivity of the period map for $K3$ surfaces. It is well known that the period map for cubic fourfolds is not surjective. However, by work of Laza and Looijenga, we understand the image of the period map for cubic fourfolds explicitly. We dedicate Section 4.3 to the study of this period map.

4.2 Odd Unimodular Lattices

In [Nik80, Theorem 1.12.2], Nikulin proves an important result about the existence of embeddings of sub-lattices into even, unimodular lattices. The goal of this section is to work through Nikulin's lattice theory to prove a similar result for odd unimodular lattices.

Recall that a lattice is a pair (L, b) where L is a free $\mathbb{Z}-$module of finite rank and $b : L \times L \to \mathbb{Z}$ is a non-degenerate, integral, symmetric, bilinear form on L. We saw the following examples in Chapter 2, but we compile them here as they will be the most important examples for this chapter.

Example 4.2.1. Given an algebraic variety V of dimension n, the intersection form on the variety endows its middle cohomology $H^n(V, \mathbb{Z})$ with the structure of a lattice.

1. For a $K3$ surface S, $H^2(S, \mathbb{Z})$ is isomorphic to the lattice $\Lambda_{K3} = E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$. This lattice is even and unimodular.
2. For a cubic fourfold X, $H^4(X, \mathbb{Z})$ is isomorphic to the lattice $\Lambda_{C4} = \langle 1 \rangle^{21} \oplus \langle -1 \rangle^2$ [Has16]. This is an odd, unimodular lattice since $b(h^2, h^2)_{C4} = 3 \notin 2\mathbb{Z}$ for a hyperplane section h, and $\mathrm{disc}(\Lambda_{C4}) = 1$.
3. Denote the orthogonal complement of h^2 in Λ_{C4} by $\Lambda_{C4,0} = \{h^2\}^\perp$. This is an even lattice and $\Lambda_{C4,0} \subset \Lambda_{C4}$. However, $\Lambda_{C4,0} \subset \Lambda_{C4}$ is not unimodular. $\Lambda_{C4,0} \cong A_2 \oplus U^{\oplus 2} \oplus E_8^{\oplus 2}$ by

[Has16] where

$$A_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Hence $\mathrm{disc}(A_2) = 3$. Since the discriminant is multiplicative across direct sums, $\Lambda_{C4,0}$ is not unimodular.

We need several numerical facts in this section. Most of the following are found in [Huy16, Chapter 14] or [Zar83]. We provide proofs when it was not available in literature.

For a lattice L, we denote its dual free lattice $\mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$ by L^*. For any integral lattice L, there is a canonical finite index inclusion $i_L : L \hookrightarrow L^*$ of order $|\mathrm{disc}(L)|$, given by $l \mapsto b(l, -)$. This is well-defined since L is an integral lattice.

Lemma 4.2.2. *If an embedding of lattices $S \hookrightarrow L$ has finite index, then*

$$disc(S) = disc(L) \cdot (L : S)^2.$$

Proof. We have the chain of inclusions

$$S \hookrightarrow L \hookrightarrow L^* \hookrightarrow S^*$$

so that

$$|S^*/S| = |L/S| \cdot |L^*/L| \cdot |S^*/L^*|.$$

This gives

$$\mathrm{disc}(S) = \mathrm{disc}(L) \cdot (L : S) \cdot |S^*/L^*|.$$

Now, it is easy to show that S^*/L^* and L/S are isomorphic, so $|S^*/L^*| = |L/S| = (L : S)$, giving the result. □

Recall the following definition.

Definition 4.2.3. Define the discriminant group of S to be $A_S = S^*/S$, which is a finite abelian group of order $|\mathrm{disc}(S)|$.

We extend the bilinear form on S naturally to a bilinear form on S^* as follows: Since S^*/S is a finite group, given $s_1, s_2 \in S^*$ we have $n \cdot s_1, m \cdot s_2 \in S$ for some $n, m \in \mathbb{Z}$. Define

$$b(s_1, s_2)_{S^*} = \frac{1}{nm} b(n \cdot s_1, m \cdot s_2)_S.$$

It is easy to see that this is well-defined. Note that the form on S^* restricts to the original form on S, and the form on S^* is now $\mathbb{Q}$-valued. The following description of $b(\, , \,)_{S^*}$ will be useful.

Lemma 4.2.4. *If $s^* \in S^*$ and $l \in S$ then $b(s^*, l)_{S^*} = s^*(l)$.*

Proof. Since S has finite index in S^*, $ds^* \in S$ for some d. That is, some multiple of s^* can be described as $ds^* = b(x, \,)_S$ for some $x \in S, d \in \mathbb{Z}$. Therefore

$$b(s^*, l)_{S^*} = \frac{1}{d} b(x, l)_S = \frac{1}{d} b(x, -)_S(l) = \frac{1}{d} \cdot ds^*(l) = s^*(l).$$

□

Definition 4.2.5. Given an abelian group A, we will use $l(A)$ throughout to denote the minimal number of generators of A. This may be referred to as the length of A.

Example 4.2.6. By [Huy16, Example 14.0.3], for $\Lambda = \Lambda_{C4,0}$, $A_\Lambda = \mathbb{Z}/3\mathbb{Z}$. Since A_Λ is a cyclic group, $l(A) = 1$.

Remark 4.2.7. If A is a finite abelian group then we can decompose A into its $p-$components A_p. If b is a bilinear form on A then its restriction to A_p will be denoted b_p. Note that $b = \bigoplus_p b_p$.

Definition 4.2.8. Define the discriminant bilinear form $b_S : A_S \times A_S \to \mathbb{Q}/\mathbb{Z}$ to be

$$b_S(s_1 + S, s_2 + S) = \langle s_1, s_2 \rangle_{S^*} + \mathbb{Z},$$

where $s_1, s_2 \in S^*$. We will be primarily concerned with odd lattices. However, if a lattice L is even then one can further define the discriminant quadratic form similarly by $q_L : A_L \to \mathbb{Q}/2\mathbb{Z}$.

It will be useful to decompose the discriminant form as follows:

Lemma 4.2.9. *If S is an integral lattice then its discriminant bilinear form decomposes as $(b_S)_p = b_{S \otimes \mathbb{Z}_p}$.*

Proof. See [Nik80, Proposition 1.7.1]. □

The discriminant form is an important tool in studying the classification of lattices.

Definition 4.2.10. We say two lattices S and L have the same genus if $L \otimes \mathbb{Z}_p \cong S \otimes \mathbb{Z}_p$ for all primes p and $L \otimes \mathbb{R} \cong S \otimes \mathbb{R}$.

Note that all equivalent forms have the same genus, but not all forms of the same genus are equivalent. Consider the following result of Nikulin.

Theorem 4.2.11. *The genus of an even lattice L is determined by its signature (l_+, l_-) and discriminant form q_L.*

Proof. [Nik80, Corollary 1.9.4] □

Now we will begin to explore embeddings of lattices in more detail.

Definition 4.2.12. Fix a lattice S. A lattice L such that $S \hookrightarrow L$ is an embedding and L/S is a finite abelian group will be called an overlattice of S.

Given an overlattice L of S, we have a series of embeddings

$$S \hookrightarrow L \hookrightarrow L^* \hookrightarrow S^*$$

where each embedding is of finite index.

Define $H_L := L/S$. Taking quotients by S, we have a chain

$$H_L := L/S \subset L^*/S \subset S^*/S = A_S$$

where A_S is the discriminant group. We would like to understand subgroups of the discriminant group A_S. To do that, we need a few lemmas.

Definition 4.2.13. Let b_S be the discriminant bilinear form on A_S. We say a subgroup $H \subset A_S$ is b_S-isotropic if $b_S|_{H\times H} = 0$.

The following lemma was presented in [Nik80, Proposition 1.4.1] for even overlattices. We adapt the lemma for all overlattices. In doing so, we consider b_S-isotropic subgroups instead of q_S-isotropic subgroups as Nikulin does, since discriminant quadratic forms are only defined for even lattices.

Lemma 4.2.14. *Fix a lattice S. There is a bijective correspondence between finite index overlattices $S \subset L$ and b_S–isotropic subgroups $H \subset A_S$.*

Proof. Let L be a finite index overlattice of S. Then the corresponding b_S–isotropic subgroup of A_S is given by $H_L = L/S$. This is clearly a subgroup of A_S. To see why it is isotropic, choose $l_1, l_2 \in L$. Then $b_S(\bar{l_1}, \bar{l_2}) = \langle l_1, l_2 \rangle_L + \mathbb{Z} = \bar{0} + \mathbb{Z}$, where $\langle l_1, l_2 \rangle_L \in \mathbb{Z}$ since L is an integral lattice.

In the other direction, let $H \subset A_S = S^*/S$ be a b_S–isotropic subgroup. Consider the quotient map $q : S^* \to A_S$. and the preimage $L := q^{-1}(H)$ of H. L contains S since $H \subset S^*/S$ and $S \hookrightarrow L$ is of finite index since $S \subset L \subset S^*$ and $S \hookrightarrow S^*$ is of finite index. L is integral since H being b_S–isotropic implies $\langle\, , \rangle_{S^*}$ restricts to an integral form on L. These constructions are clearly inverses of each other which gives the bijection in the statement. □

The following technical lemma is necessary for studying subgroups of A_S.

Lemma 4.2.15. *Suppose L is a finite index overlattice of S. Then the following all hold.*

1. *$L^*/S = H_L^{\perp}$ in A_S*
2. *$H_L^{\perp}/H_L \cong L^*/L = A_L$*
3. *$(b_S|H_L^{\perp})/H_L = b_L$*

Proof.

1. Let $l^* \in L^*$ and $m \in L$. Then $b_S(\bar{l^*}, \bar{m}) = \langle l^*, m \rangle_S + \mathbb{Z} = l^*(m) + \mathbb{Z} = \bar{0} + \mathbb{Z}$ since $l^* \in L^* = Hom(L, \mathbb{Z})$. So $L^*/S \subset H_L^{\perp}$.

   Let $\bar{s^*} \in H_L^{\perp}$. Then $b_S(\bar{s}, \bar{l}) = 0$ for all $l \in L$.

   $\implies \langle s^*, l \rangle_{S^*} \in \mathbb{Z}$, for all $l \in L$

   $\implies s^*(l) \in Z$ for all $l \in L$ by Lemma 4.2.4

   $\implies s^* \in L^*$

   $\implies \bar{s^*} \in L^*/S$.

   Therefore $L^*/S = H_L^{\perp}$.
2. Since $L^*/S = H_L^{\perp}$ and $H_L = L/S$, we can see that $A_L := L^*/L \cong (L^*/S)/(L/S) = H_L^{\perp}/H_L$ as claimed.

3. Note that H_L is b_S−isotropic. Also $\langle\, ,\, \rangle_{S^*}|_{L^*} = \langle\, ,\, \rangle_{L^*}$ since $L^* \hookrightarrow S^*$. The discriminant-bilinear form $b_L : A_L \times A_L \to \mathbb{Q}/\mathbb{Z}$ is given by $\langle\, ,\, \rangle_{L^*} + \mathbb{Z}$. By the previous part, $A_L = H_L^{\perp}/H_L$. So, b_S induces a bilinear form $H_L^{\perp}/H_L \times H_L^{\perp}/H_L \to \mathbb{Q}/\mathbb{Z}$ by $\langle\, ,\, \rangle_{S^*} + \mathbb{Z}$. In other words, $(b_S|H_L^{\perp})/H_L = b_L$.

This completes the proof. □

Definition 4.2.16. We say two lattices S and K are orthogonal, denoted $S \perp K$, if there is a primitive embedding of S into a unimodular lattice L such that $(S)_L^{\perp} \cong K$.

The following proposition relates the discriminant forms of orthogonal lattices. Our argument is similar to [Nik80, Proposition 1.5.1], but we do not require L to be an even lattice and we consider discriminant-bilinear forms instead of discriminant-quadratic forms. Note that $S \oplus K \hookrightarrow L$ is of finite index. We will apply the lemmas above for this embedding.

Proposition 4.2.17. *Two lattices S and K are orthogonal if and only if $(A_S, b_S) \cong (A_K, -b_K)$.*

Proof. We first prove the forward direction. So, let S and K be two lattices that are primitively embedded in a unimodular lattice L. In particular, we have a primitive embedding $S \hookrightarrow L$ where L is unimodular with $S^{\perp} = K$ and $S \oplus K \hookrightarrow L$ is of finite index. So, we have the following chain of inclusions

$$S \oplus K \hookrightarrow L \hookrightarrow L^* \hookrightarrow (S \oplus K)^*.$$

Taking quotients, this gives a map ϕ as a composition:

$$\phi : H_L := L/(S \oplus K) \to L^*/(S \oplus K) \to (S \oplus K)^*/(S \oplus K) \cong A_S \oplus A_K.$$

Composing with either projection to A_S or A_K, this composition gives two maps $\phi_S : H_L \to A_S$ and $\phi_K : H_L \to A_K$. We claim that both maps are isomorphisms which would show that $A_S \cong A_K$ as groups.

First, note that since S is primitive in L, we have that $L \cong S \oplus (L/S)$. Now $L^* \cong S^* \oplus (L/S)^*$ and so projection gives us a surjective map $f^* : L^* \to S^*$. Now, the key point to note is that L is unimodular, so $L \cong L^*$. This gives us a surjection $f^* : L \cong L^* \to S^*$. Given $s^* \in S^*$, let $l_s \in L$ be an element such that $f^*(l_s) = s^*$. Since K is also primitive in L, we can apply the same argument to K^*. So we have elements $l_s \in L, l_K \in K$ such that $\phi_S(l_s) = \bar{s^*}$ and $\phi_K(l_k) = \bar{k^*}$. This shows that each map is surjective.

Next, we see that that ϕ_S and ϕ_K are embeddings since S and K are primitive in L: Suppose $\phi_S(\bar{l}) = 0$ for some $l \in L$. Write $l = s_l^* + k_l^*$ where $s_l^* \in S^*$ and $k_l^* \in K^*$ since $L \subset S^* \oplus K^*$. Then $s_l^* \in S$ since $\phi_S(\bar{l}) = \bar{s_l^*} = 0$, so $l = s_l + k_l^*$ where $s_l \in S$. Now, $S \oplus K \subset L$ is of finite index, so $n \cdot l = n \cdot s_l + n \cdot k_l^* \in S \oplus K$ for some $n \in \mathbb{Z}$. This implies that $n \cdot k_l^* \in K$. But L/K is torsion-free since K is primitive in L. Therefore $k_l^* \in K$. So $l \in S \oplus K$, which means $\bar{l} = 0$. Injectivity of ϕ_K can be seen similarly.

We now have isomorphisms $A_S \cong A_K$. To see why $b_S \cong -b_K$ under this isomorphism, apply Lemma 4.2.15 to $b_{S \oplus K} = b_S \oplus b_K$ and b_L. Setting $L/(S \oplus K) = H_L$, by proof of Lemma 4.2.14, we see that H_L is $b_{S\oplus K}-$isotropic, $H_L^{\perp} = L^*/(S \oplus K)$, and $b_L = ((b_S \oplus b_K)|H_L^{\perp}/H_L$. But, $L \cong L^*$, so $H_L^{\perp} = H_L$. This means $H_L^{\perp}$ is $b_{S\oplus K}-$isotropic and so $b_L = ((b_S \oplus b_K)|H_L^{\perp}/H_L = 0$, giving $b_S = -b_K$. So, we have $(A_S, b_S) \cong (A_K, -b_K)$. This completes the proof of the forward direction.

Now we suppose S and K are two lattices, and there exists an isomorphism $(A_S, b_S) \cong (A_K, -b_K)$. Since $A_{S\oplus K} \cong A_S \oplus A_K$, we consider the graph of this isomorphism Γ in $A_{S\oplus K}$. Observe that Γ is $b_{S\oplus K}$-isotropic since $b_S = -b_K$. So, by Lemma 4.2.14, Γ corresponds to an overlattice L. The unimodularity follows from the last part of Lemma 4.2.15. This proves the backward direction. □

The following lemma is adapted from [Nik80, Proposition 1.12.1], again by replacing even lattices with arbitrary lattices, and q_S with b_S.

Lemma 4.2.18. *A lattice with signature (s_+, s_-) and bilinear-discriminant form b_S exists if and only if a lattice with signature (s_-, s_+) and bilinear-discriminant form $-b_S$ exists.*

Proof. If a lattice S has signature (s_+, s_-) and bilinear form $\langle\, , \rangle_S$, then by multiplying the form by -1 we get a lattice T with signature (s_-, s_+) and bilinear form $\langle\, , \rangle_T = -\langle\, , \rangle_S$. Extending $\langle\, , \rangle_T$ to T^* we have $\langle\, , \rangle_{T^*} = -\langle\, , \rangle_{S^*}$. Therefore, $b_T = \langle\, , \rangle_{T^*} + \mathbb{Z} = -\langle\, , \rangle_{S^*} + \mathbb{Z} = -b_S$, so T has signature

(s_-, s_+) and bilinear-discriminant form $-b_S$ as required. The other direction is implied as well. □

Before discussing the embedding theorem for odd unimodular lattices, we need a result of Nikulin. In particular, we need the corollary to [Nik80, Theorem 1.16.5]. In the following, by an invariant (t_+, t_-, b_S) of a lattice L, we mean a lattice L with signature (t_+, t_-) and discriminant bilinear form b_S.

Theorem 4.2.19. *An odd lattice S with invariants (t_+, t_-, b_S) exists if the following conditions are satisfied simultaneously:*

1. $t_+ \geq 0$
2. $t_- \geq 0$
3. $t_+ + t_- > l(A_S)$
4. $t_+ + t_- > l(A_{S_2}) + 2$.

Proof. See [Nik80, Corollary 1.16.6]. □

Now, we can put everything together to state the embedding theorem for odd unimodular lattices. Applying Theorem 4.2.19 we have the following useful statement:

Corollary 4.2.20. There exists a primitive embedding of an even lattice S with invariants (t_+, t_-, b_S) into an odd unimodular lattice L of signature (l_+, l_-) with odd orthogonal complement K if the following conditions are satisfied:

1. $l_+ - t_+ \geq 0$
2. $l_- - t_- \geq 0$
3. $l_+ + l_- - t_+ - t_- \geq l(A_S)$
4. $l_+ + l_- - t_+ - t_- > l(A_S) + 2$.

Proof. Assuming the conditions above, Theorem 4.2.19 implies the existence of an odd lattice K with invariants $(l_+ - t_+, l_- - t_-, -b_S)$. In particular, note that S and K are orthogonal by Proposition 4.2.17. In particular, it implies that there exists a primitive embedding of an even lattice S with invariants (t_+, t_-, b_S) into some odd unimodular lattice L of signature (l_+, l_-). □

We will explore how the above embedding results apply in the case of a cubic fourfold. Recall that for a cubic fourfold X, $H^4(X,\mathbb{Z}) \cong \Lambda_{C4}$ is an odd unimodular lattice of signature $(21,2)$.

Proposition 4.2.21. *Suppose T is an even lattice of signature $(21-\rho,2)$. If $13 \leq \rho \leq 21$, then there exists a primitive embedding $T \hookrightarrow \Lambda_{C4}$ such that $T^{\perp} \cong N$ is odd.*

Proof. Conditions 1 and 2 of Corollary 4.2.20 are automatically satisfied since $\rho \geq 0$. Now, denoting $(l_+, l_-) = (21,2)$ and $(t_+, t_-) = (21-\rho,2)$ we have

$$l_+ + l_- - t_+ - t_- = 21 + 2 - (21-\rho) - 2 = \rho.$$

Next the conditions 3 and 4 of Corollary 4.2.20 are satisfied if $\rho \geq l(A_T)$ and $\rho > l(A_{T_2}) + 2$.

Note that T is a free $\mathbb{Z}-$module so $T \cong \mathbb{Z}^{23-\rho}$. So

$$T^* = \mathrm{Hom}(T,\mathbb{Z}) \cong \mathrm{Hom}(\mathbb{Z}^{23-\rho},\mathbb{Z}) \cong \bigoplus_{23-\rho} \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \cong \mathbb{Z}^{23-\rho}.$$

So $l(T^*) = 23-\rho$. Hence the quotient $A_L = L^*/L$ satisfies $l(A_L) \leq 23-\rho$ since the image of the generators of L^* under the quotient map form a (not necessarily minimal) generating set of A_L.

Now, $\rho \geq 23-\rho \geq l(A_T)$ for $\rho > 12$, so condition 3 is satisfied for $\rho > 12$. For condition 4, note that $l(A_T) = \max_p l(A_{T_p})$. So, $\rho > 25-\rho \geq l(A_T)+2 \geq l(A_{T_2})+2$ for $\rho \geq 13$. Therefore, all conditions of 4.2.20 are satisfied for $13 \leq \rho \leq 21$ and the required embedding $T \hookrightarrow \Lambda_{C4}$ exists. □

We use this result in Section 4.4.

4.3 Period Maps

In this section, we will first recall the period map for $K3$ surfaces. In particular, it is well known that the period map is surjective. A similar result for cubic fourfolds is false. In particular the works of Voisin, Laza, and Looijenga shows that, unlike the situation for $K3$ surfaces, the period map for cubic fourfolds is not surjective. However, we know the exact image of the map.

Definition 4.3.1. Define the moduli space of marked $K3$ surfaces to be

$$N := \{(S,\varphi)\}/\sim$$

where S is a $K3$ surface and φ is a marking of S, i.e. an isomorphism of lattices

$$\varphi : H^2(S,\mathbb{Z}) \to \Lambda_{K3}.$$

Two marked $K3$ surfaces (S_1, φ_1) and (S_2, φ_2) are considered to be equivalent under $\sim$ if there is an isomorphism $f : S_1 \to S_2$ preserving the marking.

Definition 4.3.2. Consider the $\mathbb{C}$−linear extension of the bilinear form $\langle\ ,\ \rangle$ on the complex vector space $\Lambda_{K3,\mathbb{C}} := \Lambda_{K3} \otimes_{\mathbb{Z}} \mathbb{C}$. The period domain associated with Λ_{K3} is defined to be

$$D := \{x \in \mathbb{P}(\Lambda_{K3,\mathbb{C}}) \mid \langle x, x\rangle = 0, \langle x, \bar{x}\rangle > 0\}$$

The period domain D is naturally related to Hodge structures of $K3$−type by the following observation.

Proposition 4.3.3. *There exists a bijection between D and the set of Hodge structures of $K3$−type on Λ_{K3} such that for all non-zero $(2,0)$−classes σ:*

1. $\langle \sigma, \sigma\rangle = 0$
2. $\langle \sigma, \bar{\sigma}\rangle > 0$
3. *σ is perpendicular to $\Lambda_{K3}^{1,1}$.*

Proof. See [Huy16, Proposition 6.1.2]. □

Definition 4.3.4. The period map for $K3$ surfaces is the map $N \to D$ defined by sending a marked $K3$ surface (S, φ) to the complex line $H^0(S, \Omega^2) \subset H^2(S, \mathbb{C}) \overset{\varphi}{\cong} \Lambda_{K3,\mathbb{C}}$. By the above proposition, this line determines a point in the period domain D.

Now, we state the Torelli theorem for $K3$ surfaces, which implies that a $K3$ surface is determined by its Hodge structure.

Theorem 4.3.5. *The period map for $K3$ surfaces $N \to D$ is surjective.*

Proof. See [Tod80, Theorem 1]. □

With the period map for $K3$ surfaces as motivation, we also consider the period map for cubic fourfolds. We have seen previously that the Hodge structure of cohomology of cubic fourfolds is similar to that of $K3$ surfaces, so it is reasonable to expect that some similar results may hold.

Definition 4.3.6. Define the period domain for cubic fourfolds as

$$\mathcal{D} = \{\omega \in \mathbb{P}(\Lambda_{C4,0} \otimes_{\mathbb{Z}} \mathbb{C}) \mid \langle \omega, \omega\rangle = 0, \langle \omega, \bar{\omega}\rangle < 0\}_0$$

where the final subscript denotes a choice of a connected component.

Let $\mathcal{M}_0$ denote the space of marked, smooth cubic fourfolds where a marking means an isomorphism $H^4(X,\mathbb{Z}) \cong \Lambda_{C4}$. Then the period map is given by $\mathcal{M}_0 \to \mathcal{D}/\Gamma$ by sending X to $H^1(X,\Omega^3)$. Note that here Γ represents the monodromy group.

Proposition 4.3.7. *The period map for cubic fourfolds is injective.*

Proof. See [Voi86]. □

The map is not surjective, we now discuss the image of the period map for cubic fourfolds. We have the following definitions from [Laz10].

Definition 4.3.8.

1. Let Λ_{C4} denote the cubic fourfold lattice and fix $h \in \Lambda_{C4}$ such that $h^2 = 3$. For any rank two sublattice $M \hookrightarrow \Lambda_{C4}$ primitively embedded in Λ_{C4} with $h \in M$, define a hyperplane $\mathcal{D}_M$ to be

$$\mathcal{D}_M = \{\omega \in \mathcal{D} \mid \omega \perp M\}.$$

   Note that $\mathcal{D}_M$ is the restriction of a hyperplane $H_M \subset \mathbb{P}(\Lambda_{C4,0} \otimes_{\mathbb{Z}} \mathbb{C})$ to $\mathcal{D}$.

2. We say $\mathcal{D}_M$ is a hyperplane of determinant $d = \det(M)$.
3. The hyperplanes of a given determinant form an arrangement of hyperplanes in the period domain $\mathcal{D}$ and the arrangements of determinant 2 and 6 are denoted by $\mathcal{H}_2$ and $\mathcal{H}_6$.

Remark 4.3.9. The motivation for considering $\mathcal{H}_2$ and $\mathcal{H}_6$ comes from [Has00], where Hassett notes that the period map for cubic fourfolds misses this hyperplane arrangement. Laza and Looijenga showed that in fact, the period map is surjective onto the complement of this arrangement. The image of this arrangement in $\mathcal{D}/\Gamma$ coincides with the image of the union of two hypersurfaces $\mathcal{H}_2 \cup \mathcal{H}_6$ by [Has00, Proposition 3.2.2, Proposition 3.2.4].

Theorem 4.3.10. *The image of the period map for cubic fourfolds is the complement of the hyperplane arrangements $\mathcal{H}_2 \cup \mathcal{H}_6$.*

Proof. See [Laz10, Theorem 1.1]. □

We use this result in the next section. We now discuss endomorphism algebras of Hodge structures of cubic fourfolds.

4.4 Endomorphism Algebras of Cubic Fourfolds

In this section, we prove the result for the existence of cubic fourfolds X with $\mathrm{End}(T(X))$ of totally real type.

Lemma 4.4.1. *Let (V, h, ψ) be a polarized weight 2 Hodge structure with $v^{2,0} = 1$ and denote $V^{2,0} = \mathbb{C}\omega$ for an element $\omega \in V_{\mathbb{C}}$. Then (V, h, ψ) is an irreducible Hodge structure of $K3$-type if and only if $\psi_{\mathbb{C}}(\omega, v) = 0$ for $v \in V$ implies $v = 0$.*

Proof. See [VG08, Lemma 1.7]. □

In Proposition 4.2.21, we showed that there exists a primitive embedding of an even lattice T into Λ_{C4} under certain conditions. We interpret this in geometric situations for cubic fourfolds.

In the following statement, let $\mathcal{D}_T$ denote the period domain associated to $T_{\mathbb{C}}$, i.e.

$$\mathcal{D}_T = \{\omega \in \mathbb{P}(T \otimes_{\mathbb{Z}} \mathbb{C}) \mid \langle \omega, \omega \rangle = 0, \langle \omega, \bar{\omega} \rangle < 0\} \subset \mathbb{P}(T_{\mathbb{C}}).$$

Additionally, let $\mathcal{H}_2, \mathcal{H}_6 \subset \mathcal{D}_{\Lambda_{C4,0}} \subset \mathbb{P}(\Lambda_{C4,0,\mathbb{C}})$ be the (open subsets of) hyperplanes as in Definition 4.3.8. Note that if $T_{\mathbb{C}} \subset \Lambda_{C4,0,\mathbb{C}}$, then $\mathcal{D}_T = \mathcal{D}_{\Lambda_{C4,0}} \cap \mathbb{P}(T_{\mathbb{C}})$.

Theorem 4.4.2. *Let T be an even lattice of signature $(21 - \rho, 2)$ with $13 \leq \rho \leq 21$. Then there exists a primitive embedding $i : T \hookrightarrow \Lambda_{C4}$. Further, assume that $\mathcal{D}_T$ is not contained in $(\mathcal{H}_2 \cap \mathbb{P}(T_{\mathbb{C}})) \cup (\mathcal{H}_6 \cap \mathbb{P}(T_{\mathbb{C}}))$. Then there exists cubic fourfold X such that under this embedding $T(X) \cong T$.*

Proof. By Proposition 4.2.21, there exists a primitive embedding $T \hookrightarrow \Lambda_{C4}$. Let $N = T^{\perp}$. To define a Hodge structure on T, we need to choose a period $\omega \in T_{\mathbb{C}}$ so that the Hodge structure on T is of $K3$-type and such that ω is in the image of the period map for cubic fourfolds. This will then show that T can be realized as the transcendental lattice of a cubic fourfold X.

Since T is a lattice, it comes equipped with a bilinear form ψ. Our goal is to a find period in $\mathcal{D}_T$ so that (T, ω, ψ) is a polarized Hodge structure of weight 2. Moreover, by Lemma 4.4.1, the Hodge structure corresponding to a period $\omega \in \mathcal{D}_T$ is an irreducible Hodge structure if $\psi_{\mathbb{C}}(\omega, t) \neq 0$ for all nonzero $t \in T$. For any nonzero $t \in T$, define a hyperplane

$$H_t := \{[\omega] \mid \psi_{\mathbb{C}}(\omega, t) = 0\} \subset \mathbb{P}(T_{\mathbb{C}}).$$

Now since the period domain $\mathcal{D}_T$ associated to $T_{\mathbb{C}}$ is a nonempty open subset in a quadric in $\mathbb{P}(T_{\mathbb{C}})$, we have that $H_t \cap \mathcal{D}_T$ is of codimension ≥ 1 in $\mathcal{D}_T$ by [VG08, Lemma 3.2]. This implies that $\mathcal{D}_T \neq \bigcup_t (H_t \cap \mathcal{D}_T)$. This together with our hypothesis on $\mathcal{D}_T$ implies that $\mathcal{D}_T \neq \bigcup_t (H_t \cap \mathcal{D}_T) \cup (\mathcal{H}_2 \cap \mathcal{D}_T) \cup (\mathcal{H}_6 \cap \mathcal{D}_T)$. Choosing ω corresponding to a point in the complement gives $\omega \in T_{\mathbb{C}} \subset \Lambda_{C4,0,\mathbb{C}}$. Since ω is in the image of the period map, this defines a cubic fourfold X with $H^4(X, \mathbb{Z}) \cong \Lambda_{C4}$. Now the transcendental lattice $T(X)$ and the lattice T are both irreducible Hodge substructures of $K3$ type in $H^4(X, \mathbb{Z})(1)$ with nontrivial intersection. This follows by Proposition 4.1.2 and by our hypothesis on T. By definition of irreducibility of Hodge structures, this implies that $T = T(X)$. This finishes the proof. □

Remark 4.4.3. A similar embedding result to the above is shown in [May11, Corollary 5.2] in terms of roots of lattices.

We use the following lemma in what comes next:

Lemma 4.4.4. *Let F be a totally real field. Then for any integer $m \geq 3$ there exist rational Hodge structures of $K3$-type (V, ω, ψ) with $\mathrm{End}_{Hod}(V) = F$ and $dim_F V = m$.*

Proof. See [VG08, Lemma 3.2]. □

Remark 4.4.5. The signature of ψ that is constructed in the proof of [VG08, Lemma 3.2] is given by $(m[F : \mathbb{Q}] - 2, 2)$.

We have the following proposition towards proving the existence of cubic fourfolds X with $\mathrm{End}(T(X))$ a totally real field:

Proposition 4.4.6. *Let F be a totally real field, $m \geq 3$ an integer, and (V, ω, ψ) a rational $K3$-type Hodge structure of signature $(m[F : \mathbb{Q}] - 2, 2)$ with $\mathrm{End}_{Hod}(V) = F$ and $dim_F V = m$. If $3 \leq m[F : \mathbb{Q}] \leq 10$, then $V_{\mathbb{C}} \subset \Lambda_{C4,0,\mathbb{C}}$.*

Further, if $\omega \notin (\mathcal{H}_2 \cap \mathbb{P}(V_{\mathbb{C}})) \cup (\mathcal{H}_6 \cap \mathbb{P}(V_{\mathbb{C}}))$, then there exists a cubic fourfold X such that

$\mathrm{End}(T(X)) = F$ *and* $\dim_F(T(X)) = m$.

Proof. Choose a free $\mathbb{Z}$-module $T \subset V$ of rank $\dim_{\mathbb{Q}} V$ such that ψ is integer valued on $T \times T$. This is possible: Otherwise, we can choose a new T by multiplying the basis elements by appropriate scalars to clear denominators. Now, apply Proposition 4.2.21 to get a primitive embedding $T \hookrightarrow \Lambda_{C4,0}$. This gives $T_{\mathbb{C}} = V_{\mathbb{C}} \hookrightarrow \Lambda_{C4,0,\mathbb{C}}$. The hypothesis of Proposition 4.2.21 are satisfied since the signature of ψ is given by $(m[F : \mathbb{Q}] - 2, 2)$. Since the corresponding period $\omega \in T_{\mathbb{C}}$ is in the image of the period map, apply Theorem 4.4.2 to find a cubic fourfold X such that $T(X) \cong T$. Such an X gives the result, i.e. $\mathrm{End}(T(X)) = F$. □

The above proposition illustrates the difficulty in proving an analogue of Van Geemen's Proposition 4.1.11 for cubic fourfolds. In particular, ω may be contained in $(\mathcal{H}_2 \cap \mathbb{P}(V_{\mathbb{C}})) \cup (\mathcal{H}_6 \cap \mathbb{P}(V_{\mathbb{C}}))$. If this happens, then ω is not in the image of the period map. However, we show below that there exist examples of cubic fourfolds X with $\mathrm{End}(T(X))$ of totally real type, though we do not have any control on F.

Lemma 4.4.7. *Let T be an irreducible, rational Hodge structure of $K3-$type. If* $\dim_{\mathbb{Q}}(T)$ *is odd then* $F = \mathrm{End}(T)$ *is totally real.*

Proof. The proof is taken from [Huy16, Remark 3.3.14]. We have that

$$\dim_Q(T) = \dim_F(T) \cdot [F : \mathbb{Q}].$$

So, $[F : \mathbb{Q}]$ divides $\dim_Q(T)$. But $[F : \mathbb{Q}]$ is even for any CM-field. Therefore, if $\dim_{\mathbb{Q}}(T)$ is odd, then F is totally real. □

Proposition 4.4.8. *For a cubic fourfold X with even Picard rank $\rho(X)$,* $\mathrm{End}(T(X))$ *is of totally real type.*

Proof. The proof is immediate: Since $\dim_{\mathbb{Q}}(T(X)) = 23 - \rho(X)$, we can see that $\mathrm{End}(T(X))$ is totally real by Lemma 4.4.7. □

Corollary 4.4.9. There exist cubic fourfolds for any even Picard rank $2 \leq \rho \leq 20$ with $\mathrm{End}(T(X))$ a totally real field.

Proof. By [Laz21, Section 1], there exists cubic fourfolds of any Picard rank $1 \leq \rho(X) \leq 21$. Apply Proposition 4.4.8 to complete the proof. □

4.5 Endomorphism Algebras and Associated K3 Surfaces

Now that we have shown that cubic fourfolds of real type exist, we relate endomorphism algebras of cubic fourfolds to those of associated $K3$ surfaces when this makes sense and to the Kuga-Satake construction.

We now know that there exist $K3$ surfaces and cubic fourfolds that are of real type. However, it is a priori unclear if there exists associated $K3$ surfaces such that the $K3$ surface and the cubic fourfold are both of real type. We prove that such examples do exist.

Lemma 4.5.1. *If a $K3$ surface S is associated to a cubic fourfold X, then either both* $\mathrm{End}(T(S))$ *and* $\mathrm{End}(T(X))$ *are CM fields or both are totally real fields. Additionally, the Picard ranks are related by* $\rho(X) = \rho(S) + 1$.

Proof. By definition, S being associated to X means that there is a Hodge isometry $g : K^{\perp}(1) \to H^2(S, \mathbb{Z})_0$ for some labeling $K(1) \subset H^4(X, Z)(1)$. In particular, note that $T(X) \subset K^{\perp}$ and $T(S) \subset H^2(S, \mathbb{Z})_0$. Since the map g is a Hodge isometry, it must respect both the Hodge structures and the lattice structures. So, we have that $T(X)^{2,0} = K^{\perp}(1)^{2,0} \cong H^2(S, \mathbb{Z})_0^{2,0} = T(S)^{2,0}$. So, $T(X) \subset K^{\perp}(1) \cong H^2(S, \mathbb{Z})_0$ with $T(X)^{2,0} = T(S)^{2,0}$. By irreducibility of $T(S)$ and $T(X)$ from Proposition 4.1.2, we have that $T(X) \cong T(S)$. This gives the first part of the statement. This also implies that $\rho(X) = \rho(S) + 1$ □

We need an additional statement in order to prove the main result in the subsection.

Theorem 4.5.2. *There exists associated $K3$ surfaces of any Picard rank* $1 \leq \rho(S) \leq 20$.

Proof. See [ABP20, Theorem 1.1]. The authors use intersections of Hassett divisors to prove their result. □

Now, we prove the following proposition.

Proposition 4.5.3. *There exists $K3$ surfaces S associated to cubic fourfolds X such that* $\mathrm{End}(T(S))$ *and* $\mathrm{End}(T(X))$ *are both totally real.*

Proof. Choose a cubic fourfold X and associated $K3$ surface S such that the Picard rank $\rho(X)$ of X is even. This is possible by Theorem 4.5.2. Now, by Corollary 4.4.9 we have that $\text{End}(T(X))$ is totally real. By Lemma 4.5.1, $\text{End}(T(S))$ must be totally real as well. This concludes the proof. □

The remainder of this section is motivated by [Sch10]. In that paper, the author proves the following result for $K3$ surfaces.

Theorem 4.5.4. *Let S be a $K3$ surface and let $\text{End}(T(S))$ be totally real of degree d over $\mathbb{Q}$. Let $KS(S)$ denote the Kuga-Satake variety of S. Then there exists another abelian variety B such that $KS(T(S)) \sim B^{2^{d-1}}$ and the endomorphism algebra of B can be described by*

$$\text{End}_{\mathbb{Q}}(B) = Cores_{E/\mathbb{Q}} C^+(\Phi)$$

where $\Phi : T(S) \times T(S) \to \text{End}(T(S))$ is the symmetric bilinear form from [Zar83], C^+ is the associated even Clifford algebra, and $Cores$ is the corestriction of algebras.

Proof. See [Sch10, Theorem 1]. □

Schlickewei uses this theorem to prove an interesting result, namely that the Hodge conjecture holds for $S \times S$ where S is a $K3$ surface that is realized as a double cover of $\mathbb{P}^2$ ramified over six lines.

Since the proof of Theorem 4.5.4 only requires that T is an irreducible Hodge structure of $K3-$type and $\text{End}(T(S))$ is of totally real type (see Section 3.3 of [Sch10]), the same result holds for the Kuga-Satake varieties of cubic fourfolds. Using our result in Theorem 3.3.5, we have the following corollary.

Corollary 4.5.5. Let S be a $K3$ surface associated to a cubic fourfold X such that $\text{End}(T(S))$ and $\text{End}(T(X))$ are totally real of degree d over $\mathbb{Q}$. Then there exists another abelian variety B such that $KS(X) \sim B^{2^{\rho(X)+d-1}}$ and the endomorphism algebra of B can be described by

$$\text{End}_{\mathbb{Q}}(B) = Cores_{E/\mathbb{Q}} C^+(\Phi)$$

as in Theorem 4.5.4.

Proof. By Theorem 3.3.5, we have $KS(X) \sim KS(S)^2$. We also saw in Chapter 3 that $KS(S) \sim KS(T(S))^{2^{\rho(S)}}$. In Lemma 4.5.1, we showed that $\rho(X) = \rho(S) + 1$. Putting this together, we have a string of isogenies:

$$KS(X) \sim KS(S)^2 \sim (KS(T(S)))^{2^{\rho(S)+1}} \sim (B^{2^{d-1}})^{2^{\rho(X)}} \sim B^{2^{\rho(X)+d-1}}.$$

The description of the endomorphism algebra of B is direct from Theorem 4.5.4. □

CHAPTER 5

DERIVED CATEGORIES AND FUTURE WORK

In this chapter, we explore Kuznetsov's derived categorical approach to studying cubic fourfolds. In particular, we work towards trying to understand the relationship between $D^b(KS(X))$ and $D^b(KS(S))$ when S is a $K3$ surface associated to X. The goal is to examine how the derived category of Kuga-Satake varieties may fit into the general framework. The chapter is outlined as follows:

§5.1 We discuss the background material on derived categories with a particular focus on exceptional objects, semiorthogonal decompositions, Fourier-Mukai transforms, and Hochschild homology. The section ends with an explicit calculation for $D^b(S)$ of a $K3$ surface.

§5.2 In this section, we outline Kuznetsov's approach to studying the derived categories of cubic fourfolds. In particular we define the Kuznetsov component $\mathcal{A}_X$, prove facts about the derived category $D^b(X)$ of a cubic fourfold X such as its Hochschild homology and semiorthogonal decompositions, and note similarities between Kuznetsov's approach using derived categories and Hassett's approach using Hodge theory. We end the section with a few important examples, including Pfaffian cubic fourfolds and cubic fourfolds containing a plane.

§5.3 We discuss recent works which show that the derived categorical and Hodge theoretic approaches to studying cubic fourfolds and associated $K3$ surfaces are the same. In particular, we mention the important results of [AT14] and [BLM+21]. These results, along with our result in Theorem 3.3.5 motivate our attempt to relate derived categories of Kuga-Satake varieties into the general framework.

§5.4 In this section, we study the Kuga-Satake Hodge Conjecture, which is a special case of the standard Hodge conjecture. We use this conjecture to obtain an algebraic cycle used in the construction of Fourier-Mukai transforms. We prove that the Kuga-Satake Conjecture holds for at least some associated $K3$ surfaces.

§5.5 We use the Kuga-Satake Hodge conjecture result of the previous section to construct Fourier-

Mukai transforms involving the derived category of the Kuga-Satake variety.

§5.6 In our final section, we sketch potential future directions of research using the Fourier-Mukai functors.

5.1 Background: Derived Categories

Throughout this section, $\mathcal{T}$ will denote a triangulated category. We assume basic familiarity with abelian and triangulated categories. For a reference, see [Huy06, Chapter 1].

Definition 5.1.1. Suppose $\mathcal{T}_1, \ldots, \mathcal{T}_n$ is a sequence of full triangulated subcategories of $\mathcal{T}$ such that for all $i < j, \mathrm{Hom}_{\mathcal{T}}(\mathcal{T}_j, \mathcal{T}_i) = 0$ and $\mathcal{T}_1, \ldots, \mathcal{T}_n$ generate $\mathcal{T}$. (That is, $\mathcal{T}$ is the smallest full triangulated subcategory containing all of $\mathcal{T}_1, \ldots, \mathcal{T}_n$.) Then we say $\mathcal{T}_1, \ldots, \mathcal{T}_n$ is a semiorthogonal decomposition of $\mathcal{T}$ and we write $\mathcal{T} = \langle \mathcal{T}_1, \ldots, \mathcal{T}_n \rangle$.

Much of our focus will be on semiorthogonal decompositions. A standard method for constructing semiorthogonal decompositions is using exceptional collections.

Definition 5.1.2. An object $E \in \mathcal{T}$ is called exceptional if $\mathrm{Hom}(E, E) = k$ and $\mathrm{Ext}^m(E, E) = 0$ for all $m \neq 0$. In addition, we call a collection of exceptional objects $\{E_1, \ldots, E_n\}$ an exceptional collection if $\mathrm{Ext}^m(E_i, E_j) = 0$ for all $i > j$.

Recall that $D^b(X) := D^b(\mathrm{Coh}(X))$ denotes the bounded derived category of coherent sheaves on a variety X. The next proposition allows us to use exceptional collections to construct semiorthogonal decompositions of $D^b(X)$.

Proposition 5.1.3. *[BO95, Theorem 3.5] Let X be a smooth projective variety and suppose $\{E_1, \ldots, E_n\}$ is an exceptional collection in $D^b(X)$. Define its orthogonal complement to be*

$$\mathcal{A} := \langle E_1, \ldots, E_n \rangle^{\perp} := \left\{ F \in D^b(X) : Ext^{\bullet}(E_i, F) = 0 \; \forall i = 1, \ldots, n \right\}.$$

Then there is a semiorthogonal decomposition:

$$D^b(X) = \langle \mathcal{A}, E_1, \ldots, E_n \rangle.$$

We also consider maps between derived categories. The most natural way to do so is using Fourier-Mukai transforms, defined as follows.

Definition 5.1.4. Let X, Y be smooth projective varieties and let $K \in D^b(X \times Y)$. Denote the two natural projections by $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$. Then the Fourier-Mukai transform with kernel K is defined to be

$$\begin{aligned} \Phi_K : D^b(X) &\longrightarrow D^b(Y) \\ \mathcal{F} &\longmapsto Rp_{2*}(Lp_1^*\mathcal{F} \otimes^L K) \end{aligned}$$

where $R\text{–}_*$ denotes the derived direct image, $L\text{–}^*$ denotes the derived pullback, and $\otimes^L$ denotes the derived tensor product.

In the following, we only consider Fourier-Mukai functors between derived categories.

Example 5.1.5. We list a few examples of Fourier-Mukai kernels:

- Consider the structure sheaf of the diagonal $\mathcal{O}_\Delta \in D^b(X \times X)$. Then the Fourier-Mukai transform with kernel $K = \mathcal{O}_\Delta$ gives the identity on $D^b(X)$, since $Rp_{2*}(p_1^*\mathcal{F} \otimes^L \mathcal{O}_\Delta) = \mathcal{F}$. So $\Phi_{\mathcal{O}_\Delta} \cong \mathrm{id}$.
- If $f : X \to Y$ is a morphism of varieties, consider the structure sheaf of the graph $\mathcal{O}_{\Gamma_f}$ of f. Then the Fourier-Mukai transform with kernel $K = \mathcal{O}_{\Gamma_f}$ is isomorphic to the derived pushforward $\Phi_{\mathcal{O}_{\Gamma_f}} : D^b(X) \to D^b(Y)$.
- If A is an abelian variety and $\hat{A}$ is its dual, then $A \times \hat{A}$ comes equipped with a special line bundle called the Poincaré bundle $\mathcal{P}$, see [Mil86] for reference. The Fourier-Mukai functor $\Phi_{\mathcal{P}} : D^b(A) \to D^b(\hat{A})$ gives an equivalence of categories. This is an important example since A and $\hat{A}$ are in general not isomorphic varieties, so it illustrates that non-isomorphic varieties may be derived equivalent.

An important invariant of a triangulated category $\mathcal{T}$ is its Hochschild homology $HH_\bullet(\mathcal{T})$. The following facts and additional background can be found in [Kuz09]. We are primarily interested in Hochschild homology of derived categories and their admissible subcategories. In this context, we use $HH_\bullet(X)$ to denote $HH_\bullet(D^b(\mathrm{Coh}(X))$. It is an invariant in the sense that a derived equivalence $D^b(X) \cong D^b(Y)$ induces an isomorphism $HH_\bullet(X) \cong HH_\bullet(Y)$.

Hochschild homology is directly related to Hodge theory through the Hochschild-Kostant-

Rosenberg Theorem, which is useful for explicitly describing $HH_\bullet(X)$:

Theorem 5.1.6 (HKR Isomorphism Theorem)**.** *Let X be a smooth projective variety and let $H^{p,q}(X)$ denote its standard Hodge decomposition as in Chapter 2. Then there is an isomorphism of graded $k-$vector spaces*

$$HH_i(X) \cong \bigoplus_{q-p=i} H^{p,q}(X).$$

With this theorem we can compute Hochschild homology of many varieties of interest since we know their Hodge diamonds. For example, this applies to $K3$ surfaces, cubic fourfolds, abelian varieties, and Gushel-Mukai varieties.

Hochschild homology also works nicely with semiorthogonal decompositions.

Proposition 5.1.7. *[Kuz09, Theorem 7.3] Let X be a smooth projective variety and suppose there is a semiorthogonal decomposition of its derived category as $D^b(X) = \langle \mathcal{A}_1, \ldots, \mathcal{A}_n \rangle$. Then its Hochschild homology decomposes as a direct sum:*

$$HH_\bullet(X) \cong \bigoplus_{i=1}^{n} HH_\bullet(\mathcal{A}_i).$$

The following is also useful in combination with the above proposition:

Lemma 5.1.8. *[Kuz10, Lemma 2.5] If E is an exceptional object in $D^b(X)$, then the sub-triangulated category $\langle E \rangle$ is equivalent to the derived category of a vector spaces $D^b(k)$. In particular, its Hochschild homology is $k[0]$.*

Proof. The functor $D^b(k) \to D^b(X), V \mapsto V \otimes E$ is fully faithful since E is an exceptional object, so $D^b(k) \cong \langle E \rangle$. □

We briefly mention yet another useful tool, called the Serre functor.

Definition 5.1.9. Let k be a field and $\mathcal{T}$ be a k-linear triangulated category. A Serre functor on $\mathcal{T}$ is an exact functor $\mathcal{S}_\mathcal{T} : T \to T$ such that

$$\mathrm{Hom}(\mathcal{F}, \mathcal{S}_\mathcal{T}(\mathcal{G})) \cong \mathrm{Hom}(\mathcal{G}.\mathcal{F})^\wedge$$

for all $\mathcal{F}, \mathcal{G} \in \mathcal{T}$ and the dual is taken in the category of k-vector spaces.

For a smooth projective variety X, there exists a Serre Functor given by $\mathcal{S}_{D^b(X)}(\mathcal{F}) = \mathcal{F} \otimes \omega_X[dim(X)]$, where ω_X is the canonical sheaf of X. By *the Serre Functor of* X, we mean this functor. We apply these tools to a $K3$ surface now.

Example 5.1.10. Let S be a $K3$ surface. Recall that its Hodge diamond has the form:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

We use the HKR Theorem to compute the Hochschild homology of $D^b(S)$ by reading off the columns of the Hodge diamond as

$$HH_\bullet(S) \cong k[2] \oplus k^{22}[0] \oplus k[-2].$$

The canonical sheaf ω_S of a $K3$ surface is trivial, so its Serre functor is the shift by 2 functor, $\mathcal{S}_{D^b(S)} \cong [2]$.

5.2 The Kuznetsov Viewpoint

In Chapter 2, we saw a conjecture motivated by the framework of associated $K3$ surfaces via a Hodge theoretical approach: A cubic fourfold is rational if and only if it possesses an associated $K3$ surface. In [Kuz10], Kuznetsov provides an alternative approach to establishing a criterion for rationality via derived categories. We summarize the main ideas here.

In the following, let X denote a cubic fourfold and $D^b(X)$ its derived category of coherent sheaves. Consider the line bundles $\mathcal{O}_X, \mathcal{O}_X(-1), \mathcal{O}_X(-2)$ on X. Computing their cohomology is a direct application of the Kodaira Vanishing Theorem. Recall that the Kodaira Vanishing Theorem says that if $\mathcal{L}$ is an ample invertible sheaf on X, then $H^i(X, \mathcal{L}^{-1}) = 0$ for $i \leq 4$. We have that

$\dim(H^0(X, \mathcal{O}_X)) = 1$ and

$$\dim(H^n(X, \mathcal{O}_X(k))) = 0 \text{ for } -2 \leq k \leq 0, (n, k) \neq (0, 0).$$

Now, the basic properties of Ext imply that $\{\mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2)\}$ form an exceptional collection in $D^b(X)$.

Definition 5.2.1. Let X be a cubic fourfold and consider the exceptional collection

$$\{\mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2)\}$$

in $D^b(X)$ as shown above. Then the Kuznetsov component $\mathcal{A}_X$ of X is the orthogonal complement of this exceptional collection:

$$\mathcal{A}_X := \langle \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle^{\perp}.$$

The Kuznetsov component appears in the semiorthogonal decomposition:

Lemma 5.2.2. *Let X be a cubic fourfold and let $\mathcal{A}_X$ be the Kuznetsov component as defined above. Then there is a semiorthogonal decomposition of $D^b(X)$ as*

$$D^b(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle .$$

Proof. Apply Proposition 5.1.3. □

The sub-category $\mathcal{A}_X$ has interesting properties. We compute its Hochschild homology as follows.

Example 5.2.3. We use the HKR Theorem and the Hodge diamond of the cubic fourfold X, reproduced here for reference:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 1 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
0 & & 1 & & 21 & & 1 & & 0 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 1 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}
$$

Recall by Lemma 5.1.7 that

$$HH_\bullet(X) \cong \left(\bigoplus_{i=0}^{2} HH_\bullet(\mathcal{O}_X(i))\right) \bigoplus HH_\bullet(\mathcal{A}_X).$$

By Lemma 5.1.8, $\{\mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2)\}$ form an exceptional collection, so each of them has Hochschild homology $\cong k[0]$. Therefore, using the above decomposition and tallying the middle column of the Hodge diamond via the HKR Theorem, we have that $HH_0(X) \cong k^{25}$, and therefore $HH_0(\mathcal{A}_X) \cong k^{22}$. The exceptional objects do not contribute to Hochschild homology outside of the degree 0 part, so we have that

$$HH_\bullet(\mathcal{A}_X) \cong k[2] \oplus k^{22}[0] \oplus k[-2].$$

It is also possible to prove that the Serre functor on $\mathcal{A}_X$ is shifting by 2, i.e. $\mathcal{S}_{\mathcal{A}_X} \cong [2]$. Comparing with Example 5.1.10, the resemblance between the triangulated category $\mathcal{A}_X$ and the derived category $D^b(S)$ for a $K3$ surface S is striking! This observation leads to Kuznetsov's conjecture in [Kuz10, Conjecture 1.1].

Conjecture 5.2.4. A cubic fourfold X is rational if and only if the triangulated subcategory $\mathcal{A}_X \subset D^b(X)$ is equivalent to the derived category of a $K3$ surface, i.e. $\mathcal{A}_X \cong D^b(S)$ for some $K3$ surface S.

There are several known examples of rational cubic fourfolds X such that $\mathcal{A}_X \cong D^b(S)$ for a $K3$ surface S. We mention two such well known examples here (and compare to the examples seen in Chapter 2).

Pfaffian cubics provide the most explicit examples where the Kuznetsov component of X is realized as the derived category of a $K3$ surface. The following is proved in [Kuz06]:

Theorem 5.2.5. *Let X be a Pfaffian cubic fourfold. Then there exists a smooth $K3$ surface S such that $\mathcal{A}_X \cong D^b(S)$.*

Let W be a $6-$dimensional vector space and consider $\mathbb{P}(\wedge^2 W^*)$, the space of skew forms on W. Define the Pfaffian space as

$$\mathrm{Pf}(W) := \left\{\omega \in \mathbb{P}(\wedge^2 W^*) \mid \omega \text{ is degenerate}\right\}.$$

Recall that a Pfaffian cubic fourfold is given by $X = \mathbb{P}(V) \cap \mathrm{Pf}(W^*)$ for some $6-$dimensional vector space W and $6-$dimensional vector subspace $V \subset \wedge^2 W^*$. In [Kuz10, Section 3], Kuznetsov constructs the $K3$ surface explicitly as the zero locus of a global section on $V^* \otimes \mathcal{O}_{\mathrm{Gr}(2,W)}(1)$ on the Grassmanian $\mathrm{Gr}(2, W)$. Here $\mathcal{O}_{\mathrm{Gr}(2,W)}(1)$ is the very ample class corresponding to the Plücker embedding. In other words, $S = Z(s) \subset \mathrm{Gr}(2, W)$ for a global section s.

Before discussing cubic fourfolds that contain a plane, we take a brief detour into twisted $K3$ surfaces. We recall the basic terminology here and refer the reader to [Huy16, Chapter 18] for more detailed background.

Definition 5.2.6. Let X be any scheme.

1. An Azumaya algebra $\mathcal{A}$ over X is defined to be an $\mathcal{O}_X-$algebra that is coherent as an $\mathcal{O}_X-$module and étale locally isomorphic to $M_n(\mathcal{O}_X)$.
2. An Azumaya algebra $\mathcal{A}$ is called trivial if $\mathcal{A} \cong \mathrm{End}(E)$ for a locally free sheaf E.

3. Two Azumaya algebras are defined to be equivalent, i.e. $\mathcal{A}_1 \sim \mathcal{A}_2$ if there exists locally free sheaves $\mathcal{E}_1, \mathcal{E}_2$ such that $\mathcal{A}_1 \otimes \operatorname{End}(\mathcal{E}_1) \cong \mathcal{A}_2 \otimes \operatorname{End}(\mathcal{E}_2)$.
4. The Brauer group $\operatorname{Br}(X)$ is defined to be the set of equivalence classes of Azumaya algebras on X where the equivalence relation is given above and the group operation is tensor product $\otimes$:

$$\operatorname{Br}(X) := \{\text{Azumaya algebras } \mathcal{A}\}/\sim .$$

With the definition of a Brauer class, we can define a twisted $K3$ surface as follows.

Definition 5.2.7. A twisted $K3$ surface is a pair (S, α) where $\alpha \in \operatorname{Br}(S)$.

Now, let X be a cubic fourfold containing a plane P. Then Kuznetsov relates the theory of such cubic fourfolds to twisted $K3$ surfaces.

Theorem 5.2.8. *[Kuz10, Theorem 4.3] If X is a cubic fourfold containing a plane, then its Kuznetsov component can be realized as the derived category of a twisted $K3$ surface. That is, there exists a $K3$ surface S and a Brauer class $\beta \in \operatorname{Br}(S)$ such that $\mathcal{A}_X \cong D^b(S, \beta)$.*

In particular, to prove that the Kuznetsov component of a cubic fourfold containing a plane is equivalent to the derived category of a $K3$ surface, it is sufficient to prove that the Brauer class β that appears in the above theorem is trivial.

Example 5.2.9. Let X be a cubic fourfold containing a plane P and a projective surface T not homologous to P. Consider the intersection index given by

$$\delta(T) := \deg(T) - \langle P, T \rangle.$$

We saw in Chapter 2 that such a cubic fourfold is rational if $\delta(T)$ is odd by a result of Hassett. Furthermore, such examples exist by [Has99]. Kuznetsov shows in [Kuz10, Proposition 4.7] that such a cubic fourfold with odd intersection index has trivial Azumaya class β. Therefore, this provides an example of a rational cubic fourfold such that $\mathcal{A}_X \cong D^b(S)$.

The above examples illustrate why one may expect there to be a direct relationship between Kuznetsov's conjecture for the rationality of cubic fourfolds and the conjecture using Hassett's

Hodge theoretical approach which we will explore in the next section.

It is worth mentioning that there is an analogous derived categorical approach for Gushel-Mukai fourfolds as well. As we explored in Section 3.4, there is a notion of associated $K3$ surface for Gushel-Mukai fourfolds Y. As in the case of cubic fourfolds, we can define a subcategory $\mathcal{A}_Y \subset D^b(Y)$ and it is conjectured that a Gushel-Mukai fourfold is rational if and only if $\mathcal{A}_Y \cong D^b(S)$ for a $K3$ surface S. See [KP18] for more details.

5.3 Connecting Hodge Theory and Derived Categories

In [AT14], Addington and Thomas showed that Kuznetsov's derived category criterion for rationality coincides with Hasset's associated $K3$ notion. This is summarized in [AT14, Theorem 1.1].

Theorem 5.3.1. *If $\mathcal{A}_X \cong D^b(S)$ for some $K3$ surface S, then $X \in C_d$ for some admissible value of d. Conversely, for each admissible value of d, the set of cubics $X \in C_d$ satisfying $\mathcal{A}_X \cong D^b(S)$ for some $K3$ surface S is a Zariski open dense subset of C_d.*

In light of the above theorem, Kuznetsov's conjecture would therefore imply that generic $X \in C_d$ for admissible d is rational. The proof is involved, but we mention some ingredients of their strategy here since the approach uses theory relevant to our study.

The authors begin by defining a weight two Hodge structure on the Kuznetsov component $\mathcal{A}_X$. They show that there is a Hodge isometry with the usual Mukai lattice $H^*(S, \mathbb{Z})$ when $\mathcal{A}_X \cong D^b(S)$, and interpret the relationship between this lattice and $H^4(X, \mathbb{Z})$. Next, they consider the intersection $C_d \cap C_8$ for admissible d, which they show to be a non-empty intersection. The motivation for considering C_8 comes from Theorem 5.2.8: Since every cubic fourfold in C_8 contains a plane by [Voi86], we already know that $\mathcal{A}_X \cong D^b(S, \alpha)$ for some twisted $K3$ surface for every $X \in C_8$. They then use lattice theory to show that there must be cubic fourfolds $X \in C_d \cap C_8$ where α is trivial, hence $\mathcal{A}_X \cong D^b(S)$ for a $K3$ surface S for these cubic fourfolds. Finally they use Deformation theory to show that $\mathcal{A}_X \cong D^b(S)$ for a Zariski open subset of C_d.

The authors mention in [AT14, Section 7.4] that it may be possible to use stability conditions of some kind to pass from a Zariski open subset of C_d to all of C_d. Indeed, using this approach the authors in [BLM+21, Part VI] show that "generic" can be replaced with "every". In other words, the two notions of associated $K3$ surfaces entirely coincide.

Theorem 5.3.2. *[BLM+21, Corollary 29.7] Let X be a cubic fourfold. Then X has an associated $K3$ surface in the sense of Hassett if and only if there exists a smooth projective $K3$ surface S such that $\mathcal{A}_X \cong D^b(S)$ in the sense of Kuznetsov.*

Addington and Thomas proved another interesting result about how the Hodge isometry that realizes a $K3$ surface as an associated $K3$ surface is induced by a certain algebraic cycle.

Theorem 5.3.3. *[AT14, Theorem 1.2] Let $X \in C_d$ be a special cubic fourfold of admissible discriminant. Then there exists a polarized $K3$ surface S of degree d and an algebraic cycle $Z \in A^3(X \times S)_{\mathbb{Q}}$ that realizes the Hodge isometry $K^{\perp}(1)_{\mathbb{Q}} \to H^2(S, \mathbb{Q})_0$. In other words, the associated condition can be realized by an algebraic cycle on $X \times S$.*

In the following, we proceed in a similar fashion for the Kuga-Satake construction. This motivates the next section.

5.4 Kuga-Satake Hodge Conjecture

Our initial goal is to construct a functor $D^b(S) \to D^b(K(S)^2)$. Of course, the natural approach is to construct a Fourier-Mukai transform using a kernel $K \in D^b(S \times KS(S)^2)$. Our idea is to use an algebraic cycle $Z \subset S \times KS(S)^2$ in the kernel K. The Fourier-Mukai transform is most meaningful when the class Z depends on the Kuga-Satake construction itself. This brings us to the Kuga-Satake Hodge Conjecture.

By the Kuga-Satake construction, there is an inclusion

$$H^2(S, \mathbb{Q}) \subset H^1(KS(S), \mathbb{Q}) \otimes H^1(KS(S), \mathbb{Q}).$$

By application of Künneth formula on $H^2(KS(S) \times KS(S), \mathbb{Q})$ we have that

$$H^1(KS(S), \mathbb{Q}) \otimes H^1(KS(S), \mathbb{Q}) \subset H^2(KS(S) \times KS(S), \mathbb{Q}).$$

Composing these inclusions gives us an injective map $i : H^2(S, \mathbb{Q}) \hookrightarrow H^2(KS(S)^2, \mathbb{Q}))$.

Recall that by the Künneth formula, there is a chain of isomorphisms

$$\begin{aligned} H^4(S \times KS(S)^2, \mathbb{Q}) &\cong \bigoplus_{l+m=4} H^l(S, \mathbb{Q}) \otimes H^m(KS(S)^2, \mathbb{Q}) \\ &\cong \bigoplus_{l+m=4} \mathrm{Hom}(H^{4-l}(S, \mathbb{Q}), H^m(KS(S)^2, \mathbb{Q})). \end{aligned}$$

Therefore, the injection i corresponds to an element

$$\kappa_S \in H^4(S \times KS(S)^2, \mathbb{Q})$$

that is known as the Kuga-Satake class. This leads us to a special case of the standard Hodge Conjecture.

Conjecture 5.4.1. (Kuga-Satake Hodge Conjecture) The Kuga-Satake class κ_S constructed above is algebraic.

The Kuga-Satake Hodge Conjecture was developed for surfaces, but we can make a similar conjecture for cubic fourfolds with some quick alterations. If X is a cubic fourfold, then our formulation of Proposition 3.2.7 gives an inclusion

$$H^4(X, \mathbb{Q})(1) \subset H^1(KS(X), \mathbb{Q}) \otimes H^1(KS(X), \mathbb{Q}).$$

We can ignore the Tate twist if we are only concerned with cohomology. Now, by applying the Künneth formula in the same way we get an injective map $i : H^4(X, \mathbb{Q}) \hookrightarrow H^2(KS(X)^2, \mathbb{Q}))$. We now get a chain of isomorphisms

$$\begin{aligned} H^6(X \times KS(X)^2, \mathbb{Q}) &\cong \bigoplus_{l+m=6} H^l(X, \mathbb{Q}) \otimes H^m(KS(X)^2, \mathbb{Q}) \\ &\cong \bigoplus_{l+m=6} \mathrm{Hom}(H^{6-l}(X, \mathbb{Q}), H^m(KS(X)^2, \mathbb{Q})). \end{aligned}$$

So the inclusion corresponds to an element

$$\kappa_X \in H^6(X \times KS(X)^2, \mathbb{Q}).$$

Conjecture 5.4.2. (Kuga-Satake Hodge Conjecture for Cubic Fourfolds) The Kuga-Satake class κ_X constructed above is algebraic.

The conjecture is still open even for K3 surfaces and a few cases are known. In the next section, we assume that the Kuga-Satake Hodge conjecture holds in order to construct the Fourier-Mukai functors. Since this is a significant assumption, we provide an example of an associated $K3$ surface for which the conjecture holds.

We briefly survey currently known results for the Kuga-Satake Hodge Conjecture. As mentioned earlier, the conjecture is still open for all but a few special varieties.

Recall in Example 3.3.3 it was shown that for a Kummer surface S associated to an abelian surface S, $KS(S)$ can be explicitly described as $KS(S) \sim A^{2^{19}}$. It is known that the Kuga-Satake Hodge conjecture holds for Kummer surfaces. To show this, first consider the following result of [MZ99, Theorem 0.1] and [RM08, Theorem 3.15].

Theorem 5.4.3. *Let A be an abelian surface. Then the Hodge conjecture holds for products A^n for arbitrary $n \in \mathbb{Z}^+$.*

Now, one can show:

Proposition 5.4.4. *The Kuga-Satake Hodge conjecture holds for Kummer surfaces.*

Proof. See [Huy16, Example 4.3.4] or [VV22, Remark 2.15]. The following facts give the proof: 1) If S is a Kummer surface associated to an abelian surface A, then we have shown that $KS(S) \sim A^{2^{19}}$ in Example 3.3.3. 2) The Hodge conjecture holds for $A^{2^{19}}$ by Theorem 5.4.3. 3) The correspondence $T(S)_{\mathbb{Q}} \cong T(A)_{\mathbb{Q}}$ is algebraic (given by the graph of the rational map $A \dashrightarrow S$). □

Before proceeding with our main construction, we need one more result:

Lemma 5.4.5. *The set of complex projective Kummer surfaces are dense in the image of the period map for $K3$ surfaces. In other words, if (S, φ) denotes a marked $K3$ surface, then*

$$\mathcal{P} : \{(S, \varphi) \mid S \text{ is Kummer}\} \to \mathcal{D}$$

has dense image. Furthermore, this also holds for polarized $K3$ surfaces: For any $d > 0$, Kummer surfaces are dense in $\mathcal{N}_d$, the moduli space of polarized $K3$ surfaces with polarization of degree d.

Proof. See [Huy16, Remark 3.24]. □

Proposition 5.4.6. *There exist examples of associated $K3$ surfaces that satisfy the Kuga-Satake Hodge conjecture.*

Proof. By Lemma 5.4.5, Kummer surfaces are dense in $\mathcal{N}_d$ for all admissible discriminants d. Choose an admissible d satisfying $d \equiv 2 \pmod 6$, for example, $d = 14$. By Corollary 2.5.12, a dense open subset of $\mathcal{N}_d$ for a degree d are associated $K3$ surfaces. So, by density of Kummer surfaces in $\mathcal{N}_d$, there must exist Kummer surfaces that are associated $K3$ surfaces in $\mathcal{N}_d$ for such a d. Now, applying Proposition 5.4.4 we get examples of associated $K3$ surfaces for which the Kuga-Satake Hodge Conjecture holds. □

Corollary 5.4.7. Let X be a cubic fourfold with an associated $K3$ Kummer surface S. Then X satisfies the Kuga-Satake Hodge Conjecture for cubic fourfolds.

Proof. By Theorem 3.3.5, we have that $KS(X) \sim KS(S)^2 \sim (A^{2^{19}})^2 \sim A^{2^{20}}$. We also have that $T(X)_{\mathbb{Q}}(1) \cong T(S)_{\mathbb{Q}}$ is a Hodge isometry by the proof of Lemma 4.5.1. So, $T(X)_{\mathbb{Q}}(1) \cong T(S)_{\mathbb{Q}} \cong T(A)_{\mathbb{Q}}$ where the second isometry is algebraic. Applying Theorem 5.3.3, we see that $T(X)_{\mathbb{Q}}(1) \cong T(S)_{\mathbb{Q}}$ is induced by an algebraic cycle. Therefore, the correspondence $T(X)_{\mathbb{Q}}(1) \cong T(A)_{\mathbb{Q}}$ is algebraic as well. The result follows from by Theorem 5.4.3. □

The Kuga-Satake Hodge Conjecture is known in a few other cases as well. We mention these results and their applications:

Example 5.4.8.

1. The Kuga-Satake Hodge Conjecture holds for abelian surfaces by Theorem 5.4.3 together with Example 3.2.9.
2. In [Par88], Paranjape uses his description of the transcendental lattice of $K3$ surfaces that are realized as double covers of $\mathbb{P}^2$ ramified over 6 lines to show that the Kuga-Satake Hodge Conjecture holds for such $K3$ surfaces.
3. In [Sch10, Theorem 2], the author uses Paranjape's result and a study of the endomorphism algebra of $K3$ surfaces with real multiplication to show that the full Hodge conjecture holds for self-products $S \times S$ of $K3$ surfaces S that are double covers of $\mathbb{P}^2$ ramified over 6 lines.
4. Inspired by results of [Sch10], the author of [Var22] shows that if the Kuga-Satake Hodge

Conjecture holds for a family of $K3$ surfaces of generic Picard rank 16, then the Hodge conjecture holds for all powers of $K3$ surfaces in that family. In particular, this generalizes the results of [Sch10].

5.5 Derived Categories of Kuga-Satake Varieties

In this section, we assume that the Kuga-Satake Hodge conjecture for both $K3$ surfaces S and cubic fourfolds X holds.

We want to relate $D^b(KS(S))$ and $D^b(KS(X))$ when S is an associated $K3$ surface to X in an effort to examine how they fit into Kuznetsov's derived categorical viewpoint. We focus on $D^b(KS(S))$ since the approaches are similar. First, recall that $KS(S)$ is an abelian variety, so we have some basic facts that hold for $D^b(A)$ for any abelian variety A.

Example 5.5.1. Let A be an abelian variety and $\hat{A}$ denote its dual abelian variety.

1. Recall that the Poincaré bundle $\mathcal{P}$ on $A \times \hat{A}$ induces an equivalence of categories by the Fourier-Mukai transform with kernel $\mathcal{P}$, $D^b(A) \cong D^b(\hat{A})$.
2. We can use Theorem 5.1.6 to compute Hochschild homology for an abelian variety since

   $$HH_k(A) = \bigoplus_{p-q=k} H^{p,q}(A)$$

   and we know from Chapter 2 that the Hodge numbers of any abelian variety of dimension n are given by

   $$h^{p,q}(A) = \binom{n}{p} \cdot \binom{n}{q}.$$

   For example, if A is an abelian surface, then we have $HH_0 \cong k^6$.
3. If A is an abelian variety, then its canonical bundle ω_A is trivial. This is easily seen as the tangent bundle of any group variety is trivial, so the cotangent bundle is trivial and so the canonical bundle is trivial. It is a fact that for any variety X with trivial canonical bundle ω_X, its derived category $D^b(X)$ has no nontrivial semiorthogonal decompositions [KO15, Theorem 1.4].

We want to construct Fourier-Mukai functors $D^b(S) \to D^b(KS(S))$ (or to $D^b(KS(S)^2)$. Unfortunately, there is in general no geometric description of $KS(S)$. However, the Kuga-Satake Hodge Conjecture provides us with a potential algebraic cycle on $S \times KS(S)^2$.

Assuming the Kuga-Satake Hodge conjecture, let $\kappa_S \subset S \times KS(S)^2$ be the codimension 2 algebraic Kuga-Satake class. Taking Poincaré duals we get an algebraic surface $Z_S \overset{i}{\subset} S \times KS(S)^2$. In [VG00, 10.2], Van Geemen shows that the Kuga-Satake Hodge conjecture is equivalent to the following condition: Z_S induces an isomorphism $T(S) \cong T(S)$ via

$$T(S) \hookrightarrow H^2(S, \mathbb{Q}) \hookrightarrow H^2(KS(S)^2, \mathbb{Q}) \xrightarrow{\pi_* \phi^*} H^2(S, \mathbb{Q})$$

where π, ϕ are defined by the following diagram:

$$\begin{array}{ccc} Z_S & \xrightarrow{\phi} & KS(S)^2 \\ {\scriptstyle\pi}\downarrow & & \\ S & & \end{array}$$

Now, consider the sheaf $i_* \mathcal{O}_{Z_S}$, we refer to it as simply $\mathcal{O}_{Z_S}$ to simplify the notation. We construct the Fourier-Mukai transform with kernel $K = \mathcal{O}_{Z_S}$ to get a functor

$$\Phi_{\mathcal{O}_{Z_S}} : D^b(S) \to D^b(KS(S)^2).$$

We outline a possible approach to understanding properties of this functor in the next section.

Now, recall the basic examples of Fourier-Mukai functors given in Example 5.1.5. Let S be associated to a cubic fourfold X. Then by our result in Theorem 3.3.5, we have an isogeny $f : KS(S)^2 \to KS(X)$. The Fourier-Mukai functor with kernel given by the structure sheaf of the graph of f gives a pushforward functor:

$$\Phi_{\mathcal{O}_{\Gamma_f}} : D^b(KS(S)^2) \to D^b(KS(X)).$$

So this far we have constructed functors

$$\begin{array}{ccc} D^b(S) & \xrightarrow{\Phi_{\mathcal{O}_{Z_S}}} & D^b(KS(S)^2) \\ & & \downarrow{\scriptstyle \Phi_{\mathcal{O}_{\Gamma_f}}} \\ & & D^b(KS(X)) \end{array}$$

Now, we make similar constructions for $D^b(X)$. We construct a functor $D^b(KS(X)^2) \to D^b(X)$ in the same way as for the $K3$ surface. Let $Z_X \subset KS(X)^2 \times X$ be the Kuga-Satake Hodge cycle and consider $\mathcal{O}_{Z_X} \in D^b(KS(X)^2) \times X)$. This gives a functor

$$\Phi_{\mathcal{O}_{Z_X}} : D^b(KS(X)^2) \to D^b(X).$$

Finally we construct a functor $D^b(KS(X)) \to D^b(KS(X)^2)$ using the kernel given by the graph of the diagonal morphism $\Delta : KS(X) \to KS(X)^2$ Considering $\mathcal{O}_{\Gamma_\Delta} \in D^b(KS(X) \times KS(X)^2)$ we get a functor:

$$\Phi_{\mathcal{O}_{\Gamma_\Delta}} : D^b(KS(X)) \to D^b(KS(X)^2).$$

Putting this all together, we get the following diagram:

$$\begin{array}{ccccc} D^b(S) & & \xrightarrow{\Phi_{\mathcal{O}_{Z_S}}} & & D^b(KS(S)^2) \\ \Big\downarrow{\scriptstyle \Phi} & & & & \Big\downarrow{\scriptstyle \Phi_{\mathcal{O}_{\Gamma_f}}} \\ D^b(X) & \xleftarrow[\Phi_{\mathcal{O}_{Z_X}}]{} & D^b(KS(X)^2) & \xleftarrow[\Phi_{\mathcal{O}_{\Gamma_\Delta}}]{} & D^b(KS(X)) \end{array}$$

where $\Phi : D^b(S) \to D^b(X)$ is simply the composition. Note that this functor Φ is Fourier-Mukai since it is the composition of Fourier-Mukai functors. So, for a $K3$ surface S associated to a cubic fourfold X, we have constructed a functor $D^b(S) \to D^b(X)$ using the Kuga-Satake construction.

5.6 Future Work

The construction above raises some questions: What are the properties of the functor $\Phi : D^b(S) \to D^b(X)$? What is the image of Φ? What is the relationship between $D^b(S)$ and $\mathcal{A}_X$?

A key obstacle in answering these questions is a lack of understanding of the functors $\Phi_{\mathcal{O}_{Z_S}} : D^b(S) \to D^b(KS(S)^2)$. A possible approach to prove that $\Phi_{\mathcal{O}_{Z_S}}$ is fully faithful is to consider its right adjoint, which we will denote by Φ_R. It is well known that the functor $\Phi_{\mathcal{O}_{Z_S}}$ fully faithful if and only if $\Phi_R \circ \Phi_{\mathcal{O}_{Z_S}} \cong \mathrm{Id}$. The right adjoint has a description as follows:

Proposition 5.6.1. *Let X, Y be smooth projective varieties. Let $\Phi_K : D^b(X) \to D^b(Y)$ be a Fourier-Mukai functor with kernel $K \in D^b(X \times Y)$. Then Φ_K has a right adjoint given by Φ_{K_R}, where the kernel K_R is given by*

$$K_R := K^\vee \otimes^L Lq^*\omega_X[\dim(X)]$$

where $q : X \times Y \to X$ is projection.

Proof. See [Muk81, Theorem 2.2]. □

Note that this simplifies in our case since ω_S is trivial and $\dim(S) = 2$. So, we focus on the dual $(i_*\mathcal{O}_{Z_S})^\vee$. In the case that $Z_S \subset S \times KS(S)^2$ is smooth, we have the following:

Proposition 5.6.2. *If $i : X \hookrightarrow Y$ is a smooth closed subvariety of codimension c then*

$$(i_*\mathcal{O}_X)^\vee \cong Ri_*\omega_X \otimes^L \omega_Y^*[-c].$$

Proof. See [Huy06, Corollary 3.40]. □

If the algebraic cycle Z_S is smooth, the above proposition gives $(i_*\mathcal{O}_{Z_S})^\vee \cong i_*\omega_{Z_S}[-2]$. This reduces the Fourier-Mukai kernel of the adjoint to $K_R = i_*\omega_{Z_S}$. Using this to compute the composition $\Phi_R \circ \Phi_{\mathcal{O}_{Z_S}}$ is a possible approach to studying properties of $\Phi : D^b(S) \to D^b(X)$ in future work.

BIBLIOGRAPHY

[ABP20] Hanine Awada, Michele Bolognesi, and Claudio Pedrini. K3 surfaces and cubic fourfolds with abelian motive. *arXiv preprint arXiv:2012.05969*, 2020.

[AT14] Nicolas Addington and Richard Thomas. Hodge theory and derived categories of cubic fourfolds. *Duke Mathematical Journal*, 163(10):1885–1927, 2014.

[Bea11] Arnaud Beauville. *Surfaces algébriques complexes*. Springer, 2011.

[BL04] Christina Birkenhake and Herbert Lange. *Complex abelian varieties*, volume 302. Springer, 2004.

[BLM+21] Arend Bayer, Martí Lahoz, Emanuele Macrì, Howard Nuer, Alexander Perry, and Paolo Stellari. Stability conditions in families. *Publications mathématiques de l'IHÉS*, 133(1):157–325, 2021.

[BO95] Alexei Bondal and Dmitri Orlov. Semiorthogonal decomposition for algebraic varieties. *arXiv preprint alg-geom/9506012*, 1995.

[CG72] C Herbert Clemens and Phillip A Griffiths. The intermediate jacobian of the cubic threefold. *Annals of Mathematics*, 95(2):281–356, 1972.

[Deb20] Olivier Debarre. Gushel-mukai varieties. *arXiv preprint arXiv:2001.03485*, 2020.

[Del71] Pierre Deligne. La conjecture de weil pour les surfaces k 3. *Inventiones mathematicae*, 15:206–226, 1971.

[DIM15] Olivier Debarre, Atanas Iliev, and Laurent Manivel. Special prime fano fourfolds of degree 10 and index 2. *Recent advances in algebraic geometry*, 417:123, 2015.

[Har13] Robin Hartshorne. *Algebraic geometry*, volume 52. Springer Science & Business Media, 2013.

[Has99] Brendan Hassett. Some rational cubic fourfolds. *Journal of Algebraic Geometry*, 8(1):103–114, 1999.

[Has00] Brendan Hassett. Special cubic fourfolds. *Compositio Mathematica*, 120(1):1–23, 2000.

[Has16] Brendan Hassett. Cubic fourfolds, k3 surfaces, and rationality questions. In *Rationality problems in algebraic geometry*, pages 29–66. Springer, 2016.

[Huy06] Daniel Huybrechts. *Fourier-Mukai transforms in algebraic geometry*. Clarendon Press, 2006.

[Huy16] Daniel Huybrechts. *Lectures on K3 surfaces*, volume 158. Cambridge University Press, 2016.

[IM11] Atanas Iliev and Laurent Manivel. Fano manifolds of degree ten and epw sextics. In *Annales scientifiques de l'Ecole normale supérieure*, volume 44, pages 393–426, 2011.

[KO15] Kotaro Kawatani and Shinnosuke Okawa. Nonexistence of semiorthogonal decompositions and sections of the canonical bundle. *arXiv preprint arXiv:1508.00682*, 2015.

[KP18] Alexander Kuznetsov and Alexander Perry. Derived categories of gushel–mukai varieties. *Compositio Mathematica*, 154(7):1362–1406, 2018.

[KS67] Michio Kuga and Ichirô Satake. Abelian varieties attached to polarized k 3-surfaces. *Mathematische Annalen*, 169:239–242, 1967.

[Kuz06] Alexander Kuznetsov. Homological projective duality for grassmannians of lines. *arXiv preprint math/0610957*, 2006.

[Kuz09] Alexander Kuznetsov. Hochschild homology and semiorthogonal decompositions. *arXiv preprint arXiv:0904.4330*, 2009.

[Kuz10] Alexander Kuznetsov. Derived categories of cubic fourfolds. In *Cohomological and geometric approaches to rationality problems*, pages 219–243. Springer, 2010.

[Kuz16] Alexander Kuznetsov. Derived categories view on rationality problems. In *Rationality problems in algebraic geometry*, pages 67–104. Springer, 2016.

[Laz10] Radu Laza. The moduli space of cubic fourfolds via the period map. *Annals of mathematics*, pages 673–711, 2010.

[Laz21] Radu Laza. Maximally algebraic potentially irrational cubic fourfolds. *Proceedings of the American Mathematical Society*, 149(08):3209–3220, 2021.

[Lef21] Solomon Lefschetz. *On certain numerical invariants of algebraic varieties with application to abelian varieties*. American Mathematical Society, 1921.

[Loo09] Eduard Looijenga. The period map for cubic fourfolds. *Inventiones mathematicae*, 177(1):213–233, 2009.

[Mac16] Casey Machen. *Abelian varieties associated to Clifford algebras*. Michigan State University, 2016.

[May11] Evgeny Mayanskiy. Intersection lattices of cubic fourfolds. *arXiv preprint arXiv:1112.0806*, 2011.

[Mil86] James S Milne. Abelian varieties. *Arithmetic geometry*, pages 103–150, 1986.

[Mor85] David R Morrison. The kuga-satake variety of an abelian surface. *Journal of Algebra*, 92(2):454–476, 1985.

[Muk81] Shigeru Mukai. Duality between d (x) and d (^x) with its application to picard sheaves nagoya math. *J*, 81:153–175, 1981.

[Muk03] Shigeru Mukai. Vector bundles on a k3 surface. *arXiv preprint math/0304303*, 2003.

[MZ99] BJJ Moonen and Yu G Zarhin. Hodge classes on abelian varieties of low dimension. *arXiv preprint math/9901113*, 1999.

[Nik80] VV Nikulin. Integral symmetric bilinear forms and some of their applications. *Mathematics of the USSR*, 14, 1980.

[Orl97] Dmitri O Orlov. Equivalences of derived categories and k 3 surfaces. *Journal of Mathematical Sciences*, 84(5):1361–1381, 1997.

[Par88] Kapil Paranjape. Abelian varieties associated to certain k3 surfaces. *Compositio Mathematica*, 68(1):11–22, 1988.

[Poz13] Alice Pozzi. *The Kuga-Satake Construction: A Modular Interpretation*. PhD thesis, Concordia University, 2013.

[PSS71] Ilya Iosifovich Pyatetskii-Shapiro and Igor Rostislavovich Shafarevich. A torelli theorem for algebraic surfaces of type k3. *Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya*, 35(3):530–572, 1971.

[RM08] José J Ramón Marí. On the hodge conjecture for products of certain surfaces. *Collectanea mathematica*, 59:1–26, 2008.

[RS18] Francesco Russo and Giovanni Stagliano. Explicit rationality of some cubic fourfolds. *arXiv preprint arXiv:1811.03502*, 2018.

[RS19] Francesco Russo and Giovanni Staglianò. Trisecant flops, their associated k3 surfaces and the rationality of some fano fourfolds. *arXiv preprint arXiv:1909.01263*, 2019.

[Sch10] Ulrich Schlickewei. The hodge conjecture for self-products of certain k3 surfaces. *Journal of Algebra*, 324(3):507–529, 2010.

[Tod80] Andrei N Todorov. Applications of the kähler-einstein-calabi-yau metric to moduli of k3 surfaces. *Inventiones mathematicae*, 61:251–265, 1980.

[Tre84] Semion L Tregub. Three constructions of rationality of a cubic fourfold. *Vestnik Moskovskogo Universiteta. Seriya 1. Matematika. Mekhanika*, pages 8–14, 1984.

[Var22] Mauro Varesco. The hodge conjecture for powers of k3 surfaces of picard number 16. *arXiv preprint arXiv:2203.09778*, 2022.

[VG00] Bert Van Geemen. Kuga-satake varieties and the hodge conjecture. *The arithmetic and geometry of algebraic cycles*, pages 51–82, 2000.

[VG05] Bert Van Geemen. Some remarks on brauer groups of k3 surfaces. *Advances in Mathematics*, 197(1):222–247, 2005.

[VG08] Bert Van Geemen. Real multiplication on k3 surfaces and kuga satake varieties. *Michigan Mathematical Journal*, 56(2):375–399, 2008.

[Voi86] Claire Voisin. Théorème de torelli pour les cubiques de $\mathbb{P}^5$. *Inventiones mathematicae*, 86(3):577–601, 1986.

[Voi05] Claire Voisin. A generalization of the kuga-satake construction. *arXiv preprint math/0503122*, 2005.

[VV22] Mauro Varesco and Claire Voisin. Hyper–kähler manifolds of generalized kummer type and the kuga–satake correspondence. *Milan Journal of Mathematics*, 90(2):433–444, 2022.

[Zar83] Yu G Zarhin. Hodge groups of k3 surfaces. *Journal für die reine und angewandte Mathematik*, 1983.

[Zuc77] Steven Zucker. The hodge conjecture for cubic fourfolds. *Compositio Mathematica*, 34(2):199–209, 1977.