COMPUTATIONAL TOOLS FOR REAL TOPOLOGICAL HOCHSCHILD HOMOLOGY

By

Chloe Lewis

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ABSTRACT

Algebraic K-theory is an invariant of rings that relates to interesting questions in many mathematical subfields including geometric topology, algebraic geometry, and number theory. Although these connections have generated great interest in the study of algebraic K-theory, computations are quite difficult. This prompted the development of trace methods for algebraic K-theory in which one studies more computable invariants of rings (and their topological analogues) that receive maps from K-theory. One such approximation called topological Hochschild homology (THH) has proven foundational to progress in computations of K-theory via trace methods. The Bökstedt spectral sequence is one of the main tools for computing THH.

Hesselholt and Madsen developed a C_2-equivariant analogue of algebraic K-theory for rings with anti-involution called Real algebraic K-theory. A Real version of the trace methods story unfolds in this context by studying an approximation of Real K-theory called Real topological Hochschild homology (THR). The main result of this thesis is the construction of a Real Bökstedt spectral sequence which computes the equivariant homology of THR. We then extend our techniques to the case of another equivariant Hochschild theory called G-twisted topological Hochschild homology and construct a spectral sequence which computes the G-equivariant homology of H-twisted THH when $H \leq G$ are finite subgroups of S^1.

Finally, this thesis explores the algebraic structures present in Real topological Hochschild homology. When the input is commutative, THH has the structure of a Hopf algebra in the homotopy category. Work of Angeltveit-Rognes further shows that this structure lifts to the Bökstedt spectral sequence. We show in this thesis that when the input is commutative, THR has a Hopf algebroid structure in the C_2-equivariant stable homotopy category.

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CHAPTER 1

INTRODUCTION

The motivation for the work undertaken in this thesis is rooted in the study of an important invariant of rings called algebraic K-theory. To a ring R, we may associate a sequence of abelian groups, denoted by $K_n(R)$, called the algebraic K-theory groups. Building off of work of Grothendieck for the case $n = 0$, Quillen [Qui73] defined the nth algebraic K-theory group of R for $n > 0$ to be the homotopy group

$$K_n(R) := \pi_n(\mathrm{B}GL(R)^+).$$

Here, $\mathrm{B}GL(R)^+$ is the +-construction of the classifying space of the infinite general linear group $GL(R)$. Although K-theory is an invariant of algebraic objects, its definition utilizes topological notions from homotopy theory, and algebraic K-theory demonstrates deep connections between topology and algebra. Further details about historical developments in K-theory, including other constructions, may be found in [Wei13].

The study of algebraic K-theory has uncovered deep, and often unexpected, connections to many areas of mathematics including algebraic geometry, geometric topology, and number theory. These connections relate to foundational theorems in these fields, such as the s-cobordism theorem, which is relevant to the classification of manifolds [Bar64], and the Kummer-Vandiver conjecture, a conjecture in algebraic number theory dating back to the 1800s [Kur92]. Although such connections prompted an interest in the study of algebraic K-theory in the second half of the 20th century, progress was hindered by the incredible difficulty of K-theory computations. Perhaps the open question that best illustrates just how difficult K-theory computations are is that of $K_n(\mathbb{Z})$; more than 50 years after Quillen defined algebraic K-theory for rings, we still do not know all of the K-groups of the integers.

Seeking to gain a computational foothold, algebraic topologists developed a research program known as trace methods in which other, more computationally accessible invariants of rings (and their topological analogues) are studied as approximations of algebraic K-theory.

One such approximation is an invariant of rings from classical algebra called Hochschild homology. Hochschild homology receives a map from K-theory called the Dennis trace. Although Hochschild homology, denoted by HH, is significantly more computable than K-theory, the trace map is not an especially good approximation. One can understand this as a failure of Hochschild homology to capture some of the topological information used to define K-theory, given that HH is an entirely algebraic construction.

The development of so-called "brave new algebra" gave us another try at approximating K-theory. In this setting, also referred to as higher algebra, one works with topological objects that mimic the properties of objects from classical algebra. One example is that of ring spectra, the topological analogue of rings. The construction of Hochschild homology can be translated using this language of higher algebra to define a theory of topological Hochschild homology, denoted by THH, which is an invariant of ring spectra. An examination of the trace map between K-theory and THH provides a better approximation than our first attempt and has proven enormously useful in the trace methods approach to understanding K-theory.

Of particular relevance to our story is work of Bökstedt [BHM93], who constructed a spectral sequence which takes input data from classical Hochschild homology groups and produces information about the homology of THH:

$$E^2_{*,*} = \mathrm{HH}_*(H_*(R); k) \Rightarrow H_*(\mathrm{THH}(R); k).$$

The Bökstedt spectral sequence is quite useful in THH calculations, leveraging the computational accessibility of Hochschild homology to understand the better, higher algebra approximation we have in THH.

A generalized retelling of this story where we consider inputs with group actions has been a focus of homotopy theory in recent years. This led to definitions of equivariant algebraic K-theories. One example is Real algebraic K-theory, denoted by KR, for rings with the C_2-action of involution. Real K-theory was originally defined by Hesselholt and Madsen in [HM15], along with an equivariant version of topological Hochschild homology incorporating the involution action called Real topological Hochschild homology; we denote this invariant by THR. A definition of THR via a simplicial bar construction which lends itself well to computational work was given recently by Dotto, Moi, Patchkoria, and Reeh in [Dot+20].

Recently, work of Angelini-Knoll, Gerhardt, and Hill further filled in details of the Real trace methods story. In [AGH21], the authors develop a theory of Real Hochschild homology, denoted by $\underline{\mathrm{HR}}$, which takes inputs from equivariant algebra with an appropriate notion of an involution.

	Algebra	Topology
Non-equivariant inputs	HH	THH
Inputs with involution	$\underline{\mathrm{HR}}$	THR

We may now ask a question in this Real setting which motivated the use of the Bökstedt spectral sequence in classical THH computations: can we use information about Real Hochschild homology to better understand THR? The main result of this thesis answers this question in the affirmative with the construction of a Real Bökstedt spectral sequence.

Classically, THH inherits an action of the circle group S^1. Real topological Hochschild homology is constructed similarly, but with additional structure encoding the C_2-action of the involution. This construction yields an $S^1 \rtimes C_2 \cong O(2)$-action, and THR is an $O(2)$-equivariant spectrum. As all dihedral groups D_{2m} are subgroups of $O(2)$, we may restrict THR to a D_{2m}-equivariant spectrum. Thus, our Real Bökstedt spectral sequence computes the D_{2m}-equivariant homology of THR using input data from Real D_{2m}-Hochschild homology. This result is restated as Theorem 5.2.4 in the text.

Theorem. *Let A be a ring spectrum with anti-involution and let E be a commutative D_{2m}-ring spectrum. If $\underline{E}_{\circledast}(N_{D_2}^{D_{2m}} A)$ and $\underline{E}_{\circledast}(N_e^{D_{2m}} \iota_e^* A)$ are both flat as modules over $\underline{E}_{\circledast}$ and if A has free $(\iota_{D_2}^* E)$- and $\iota_e^* E$-homology then there is a Real Bökstedt spectral sequence of the form*

$$E^2_{*,\circledast} = \underline{\mathrm{HR}}_*^{D_{2m}}(\underline{(\iota_{D_2}^* E)}_{\circledast}(A)) \Rightarrow \underline{E}_{\circledast}(\iota_{D_{2m}}^* \mathrm{THR}(A)).$$

Here, $E_{\circledast}$ denotes an $\underline{RO}(D_{2m})$-graded homology theory, which we review in Section 4.5.

The techniques used to construct the Real Bökstedt spectral sequence may be extended to a different flavor of equivariant topological Hochschild homology. For G, a finite subgroup of S^1, the G-twisted topological Hochschild homology is an S^1-spectrum which incorporates an action of the cyclic group into the invariant constructions. In this twisted setting there is an analogous algebraic theory called Hochschild homology for Green functors.

	Algebra	Topology
Non-equivariant inputs	HH	THH
Inputs with involution	$\underline{\mathrm{HR}}$	THR
Inputs with G-action	$\underline{\mathrm{HH}}^G$	THH_G

Work of Adamyk, Gerhardt, Hess, Klang, and Kong [Ada+22] constructs a twisted Bökstedt spectral sequence which computes the G-equivariant homology of THH_G.

$$E^2_{s,\star} = \underline{\mathrm{HH}}_s^{\underline{E}_\star, G}(\underline{E}_\star(R)) \Rightarrow \underline{E}_{s+\star}(\iota_G^* \mathrm{THH}_G(R)).$$

In this thesis, we construct a spectral sequence which computes the G-equivariant homology of H-twisted THH, for a subgroup H of G. This result is restated as Theorem 5.3.3 in Chapter 5.

Theorem. *Let $H \leq G$ be finite subgroups of S^1 and let $g = e^{2\pi i/|G|}$ be a generator of G. Let R be an H-ring spectrum and E a commutative G-ring spectrum. Assume that g acts trivially on E and that $\underline{E}_{\circledast}(N_H^G R)$ is flat as a module over $\underline{E}_{\circledast}$. If R has $(\iota_H^* E)$-free homology, then there is a relative twisted Bökstedt spectral sequence*

$$E^2_{s,\circledast} = \underline{\mathrm{HH}}_H^G(\underline{(\iota_H^* E)}_{\circledast}(R))_s \Rightarrow \underline{E}_{s+\circledast}(\iota_G^* \mathrm{THH}_H(R)).$$

Taking $G = H$ in this theorem recovers the spectral sequence of [Ada+22]. In the case of $H = e$, this result also gives a new spectral sequence converging to the G-equivariant homology of ordinary THH.

One way to gain computational traction in calculations involving the classical Bökstedt spectral sequence is to utilize the algebraic structures present in THH and in the spectral sequence itself. Angeltveit and Rognes show in [AR05] how to use simplicial constructions to induce a Hopf algebra structure in the homotopy category on $\mathrm{THH}(R)$ when the ring spectrum R is commutative. The authors further show that, under appropriate flatness conditions, this Hopf algebra structure lifts to the Bökstedt spectral sequence. We take on a similar exploration of algebraic structures for Real topological Hochschild homology with the goal of lifting this structure to the Real Bökstedt spectral sequence in the future. We find that when the input is commutative, THR has the structure of a Hopf algebroid, a generalized notion of Hopf algebras, in the C_2-equivariant stable homotopy category. This result is restated as Theorem 6.2.10 in Chapter 6.

Theorem. *Let A be a commutative C_2-ring spectrum. The Real topological Hochschild homology of A is a Hopf algebroid in the C_2-equivariant stable homotopy category.*

1.1 Organization

We begin by recalling the classical constructions of Hochschild homology, topological Hochschild homology, and the Bökstedt spectral sequence in Chapter 2. In Chapter 3, we provide some necessary definitions from equivariant algebra including those of Mackey functors, Green functors, and equivariant norms. Chapter 4 describes equivariant analogues of the classical Hochschild invariants; here we recall definitions of Real topological Hochschild homology, Real Hochschild homology, twisted topological Hochschild homology, and Hochschild homology for Green functors.

In Chapter 5, we construct a Bökstedt spectral sequence for Real topological Hochschild homology and an equivariant Bökstedt spectral sequence for twisted topological Hochschild homology. Finally, Chapter 6 recalls the algebraic structures present in THH and proves the existence of a Hopf algebroid structure in the Real equivariant setting.

CHAPTER 2

TOPOLOGICAL HOCHSCHILD HOMOLOGY

In the introduction we discussed the extraordinary difficulty but deep interest in computing algebraic K-theory. The trace methods program arising from homotopy theory offers an approach to K-theory computations by way of approximation. Rather than studying the algebraic K-theory groups themselves, we instead investigate other, more computable invariants and the maps they receive from K-theory. In this chapter we describe the construction of some of these ring and ring spectra invariants. We begin by recalling a classical invariant of rings (or, more generally, of unital, associative algebras) from algebra called Hochschild homology. Following this, we describe the construction of the analogous topological version of this theory. This topological version of Hochschild homology, denoted by THH, plays a crucial role in the trace methods story; THH and a closely related invariant called topological cyclic homology have proven to be good approximations of K-theory. Much of the progress in K-theory computations relies on being able to compute THH. One tool which assists in these computations is the Bökstedt spectral sequence, which bridges the algebraic and topological Hochschild theories.

2.1 Hochschild homology

A first approximation of algebraic K-theory one might consider is Hochschild homology, an invariant of rings and algebras from the world of classical algebra. Hochschild homology is constructed as a simplicial abelian group so we begin by reviewing the definition of a simplicial object. These simplicial objects generalize the notion of a simplicial set.

Definition 2.1.1. A *simplicial object* $K_\bullet$ in a category $\mathcal{C}$ is a sequence of objects K_n in $\mathcal{C}$ for $n \geq 0$ with *face maps* $d_n : K_n \to K_{n-1}$ and *degeneracy maps* $s_n : K_n \to K_{n+1}$ obeying the following relations:

$$d_i d_j = d_{j-1} d_i \qquad \text{if } i < j$$

$$d_i s_j = \begin{cases} s_{j-1} d_i & i < j \\ id & i = j \text{ or } i = j+1 \\ s_j d_{i-1} & i > j+1 \end{cases}$$

$$s_i s_j = s_{j+1} s_i \qquad \text{if } i \leq j.$$

We now construct the simplicial object used to define Hochschild homology.

Definition 2.1.2. Let k be a commutative ring, A an associative, unital k-algebra and M an A-bimodule. The *cyclic bar construction on A with coefficients in M* is a simplicial abelian group, denoted by $B^{cy}_\bullet(A;M)$, and is defined as follows:

$$\begin{array}{c} \vdots \\ \downarrow\ \uparrow\ \downarrow\ \uparrow\ \downarrow\ \uparrow\ \downarrow\ \uparrow\ \downarrow \\ M \otimes A \otimes A \otimes A \\ d_0 \downarrow\ s_0 \uparrow\ d_1 \downarrow\ s_1 \uparrow\ d_2 \downarrow\ s_2 \uparrow\ \ \downarrow d_3 \\ M \otimes A \otimes A \\ d_0 \downarrow\ s_0 \uparrow\ d_1 \downarrow\ s_1 \uparrow\ \ \downarrow d_2 \\ M \otimes A \\ d_0 \downarrow\ s_0 \uparrow\ \ \downarrow d_1 \\ M \end{array}$$

The nth level of this simplicial object is the $(n+1)$-fold tensor product $B_n(A;M) = M \otimes A^{\otimes n}$. All of the tensor products are taken over k but we omit this from the notation. The face maps $d_i : M \otimes A^{\otimes n} \to M \otimes A^{\otimes (n-1)}$ are defined by

$$d_i = \begin{cases} \psi_R \otimes id^{\otimes (n-1)} & i = 0 \\ id_M \otimes id^{\otimes (i-1)} \otimes \mu \otimes id^{\ \otimes (n-i-1)} & 0 < i < n \\ (\psi_L \otimes id^{\otimes (n-1)}) \circ \tau & i = n \end{cases}$$

where ψ_L and ψ_R are the left and right A-module actions of M, respectively. The map $\mu : A \otimes A \to A$ is the multiplication on the algebra A and τ is the "twist map" which permutes the last tensor copy of A to the left of M.

The degeneracy maps $s_i : B_n(A;M) \to B_{n+1}(A;M)$ are defined to be

$$s_i = id_M \otimes id^{\otimes i} \otimes \eta \otimes id^{\otimes(n-i)}.$$

In the above notion, η denotes the unit map of the algebra, $\eta : k \to A$.

We define the *Hochschild homology of A with coefficients in M* to be the homotopy groups of the geometric realization of the cyclic bar construction,

$$\mathrm{HH}_n(A;M) = \pi_n(|B^{cy}_\bullet(A;M)|).$$

Alternatively, one may define Hochschild homology from a totally homological algebra perspective by taking the homology of the chain complex constructed from tensor copies of A.

Definition 2.1.3. Let k be a commutative ring, A an associative, unital k algebra, and M an A-bimodule. The *Hochschild complex* is the chain complex $(C_\bullet(A,M), b)$ with

$$C_n(A,M) = M \otimes A^{\otimes n}.$$

We define maps d_i on $C_n(A,M)$ for $0 \le i \le n$ as in Definition 2.1.2. The sum

$$b = \sum_{i=0}^{n} (-1)^i d_i$$

defines a boundary; one may check that $b^2 = 0$. The *Hochschild homology of A with coefficients in M* is the homology of this chain complex,

$$\mathrm{HH}_n(A;M) := H_n(C_\bullet(A,M), b).$$

The Dold-Kan correspondence between the category of simplicial abelian groups and the category of non-negatively graded chain complexes shows that these two definitions of Hochschild homology are equivalent.

Convention. In the case of $M = A$ where we take coefficients in the algebra as a bimodule over itself, we denote the Hochschild homology by

$$\mathrm{HH}_n(A;A) = \mathrm{HH}_n(A),$$

omitting the coefficients from the notation.

We now outline some results which are useful for computing Hochschild homology groups.

Definition 2.1.4. Let A be an algebra. The *opposite algebra*, denoted A^{op}, is an algebra consisting of the same elements and additive structure as A but with a reversed multiplicative structure so that the product is defined by

$$\mu^{op} : A^{op} \otimes A^{op} \to A^{op}$$
$$a \otimes b \mapsto ba.$$

Definition 2.1.5. For a k-algebra A, the *enveloping algebra* A^e is the tensor product $A \otimes_k A^{op}$. Multiplication is defined component-wise and we can consider A as a left A^e-module via the multiplication $(a \otimes b)c = acb$.

Because the Hochschild complex is a resolution of A as an A^e-module ([Lod13], 1.1.12) we can understand Hochschild homology as a Tor group.

Proposition 2.1.6 (see, for instance, [Lod13] 1.1.13)**.** *If A is projective as a module over k, then for any A-bimodule M there is an isomorphism*

$$\mathrm{HH}_n(A, M) \cong \mathrm{Tor}_n^{A^e}(M, A).$$

This relationship between Hochschild homology and Tor has analogous statements in equivariant Hochschild settings and is an important tool for computing Hochschild groups in both the equivariant and non-equivariant settings.

Returning to our trace methods motivation, we recall that there is a map called the Dennis trace from the algebraic K-theory groups of a ring A to its Hochschild homology. By this map, we consider $\mathrm{HH}(A)$ as a K-theory approximation,

$$K_n(A) \xrightarrow{\text{Dennis trace}} \mathrm{HH}_n(A).$$

A natural follow up when presented with such an approximation is to ask how close we come to capturing K-theory. This is akin to asking how close the Dennis trace comes to being an isomorphism. Unfortunately, what we gained in computational traction with Hochschild homology, we traded for accuracy; the Dennis trace is not a very good K-theory approximation. In part, we recognize this as the failure of Hochschild homology to incorporate the homotopy theoretic information used to construct the algebraic K-theory groups. We now consider another Hochschild invariant, this time one coming from topology, in an effort to better approximate algebraic K-theory.

2.2 Topological Hochschild homology

The preceding construction of Hochschild homology can be translated to a topological setting in order to define a theory of topological Hochschild homology, denoted by THH. To do so, we appeal to the world of so-called "brave new algebra," also known as higher algebra, to find topological analogues which mimic important properties of classical algebraic objects. This translation utilizes the following dictionary:

Classical Algebra	Higher Algebra
Rings	Ring spectra
Tensor product, $\otimes$	Smash product, $\wedge$
Integers, $\mathbb{Z}$	Sphere spectrum, $\mathbb{S}$
HH	THH

We begin this section by recalling some definitions in higher algebra one needs to understand this dictionary. Following that, we describe the construction of THH and recount how it fits into the story of trace methods for algebraic K-theory.

Definition 2.2.1. A *ring spectrum* is a monoid in a symmetric monoidal category of spectra. In other words, a ring spectrum E admits a unit map $\eta : \mathbb{S} \to E$ and a product map $\mu : E \wedge E \to E$ subject to the unitality relation

$$\begin{array}{ccccc} \mathbb{S} \wedge E & \xrightarrow{\eta \wedge id} & E \wedge E & \xleftarrow{id \wedge \eta} & E \wedge \mathbb{S} \\ & \underset{\cong}{\searrow} & \downarrow{\scriptstyle \mu} & \underset{\cong}{\swarrow} & \\ & & E & & \end{array}$$

and the associativity relation

$$\begin{CD} E\wedge E\wedge E @>{\mu\wedge id}>> E\wedge E \\ @V{id\wedge\mu}VV @VV{\mu}V \\ E\wedge E @>>{\mu}> E. \end{CD}$$

Example 2.2.2 (Eilenberg-Mac Lane spectrum). Any ring A gives rise to a ring spectrum HA called the Eilenberg-Mac Lane spectrum of A. These Eilenberg-Mac Lane spectra have the property that

$$\pi_n(HA) = \begin{cases} A & \text{if } n = 0 \\ 0 & n > 0. \end{cases}$$

Since ring spectra are analogous to rings in higher algebra, we also wish to have an analogous notion of a module.

Definition 2.2.3. Let A be a ring spectrum. We say that a spectrum M is a *left A-module spectrum* if there is a map of spectra $\psi_L : A \wedge M \to M$ such that the following diagrams commute:

$$\begin{array}{ccc} \mathbb{S}\wedge M & \xrightarrow{\eta\wedge id} & A\wedge M \\ & {\scriptstyle\cong}\searrow \quad \swarrow{\scriptstyle\psi_L} & \\ & M & \end{array}$$

$$\begin{CD} A\wedge M\wedge A @>{\mu\wedge id}>> A\wedge M \\ @V{id\wedge\psi_L}VV @VV{\psi_L}V \\ A\wedge M @>>{\psi_L}> M. \end{CD}$$

The definition of a right A-module is analogous. If M is a left A-module via an action ψ_L and a right A-module via ψ_R such that the following diagram commutes,

$$\begin{CD} M\wedge A\wedge M @>{id\wedge\psi_L}>> M\wedge A \\ @V{\psi_R\wedge id}VV @VV{\psi_R}V \\ A\wedge M @>>{\psi_L}> M, \end{CD}$$

we say that M is an A-*bimodule*.

These constructions in higher algebra mimic the familiar ones of classical algebra with the symmetric monoidal smash product playing the role of the tensor product. Akin to the notion of a relative tensor, we also have relative smash product in spectra.

Definition 2.2.4. Let A be a ring spectrum. If M is a right A-module spectrum with action ψ and N is a left A-module spectrum with action ϕ, the *relative smash product* $M \wedge_A N$ is the coequalizer in spectra:

$$M \wedge A \wedge N \underset{id \wedge \phi}{\overset{\psi \wedge id}{\rightrightarrows}} M \wedge N \longrightarrow M \wedge_A N.$$

Remark 2.2.5. Following this definition, we may translate a classical isomorphism to the setting of higher algebra. If A is a ring spectrum and M is a left A-module we can regard A as a right module over itself. Then the relative smash product $A \wedge_A M$ is the coequalizer

$$A \wedge A \wedge M \underset{id \wedge \psi_L}{\overset{\mu \wedge id}{\rightrightarrows}} A \wedge M \longrightarrow A \wedge_A M.$$

But the associativity of a left module action ensures that $\psi_L(\mu \wedge id) = \psi_L(id \wedge \psi_L)$ so we have that

$$A \wedge_A M \cong M.$$

Topological Hochschild homology (THH) was first defined in the 1980s by Bökstedt [Bök85b]. A more modern description of THH (including constructions utilizing the notion of an associative smash product in a category of spectra that was not yet developed at the time of Bökstedt's original publication) is given in Chapter 9 of [Elm+97].

Definition 2.2.6. Let A be a ring spectrum and M an (A, A)-bimodule. The *cyclic bar construction on A with coefficients in M*, denoted $B^{cy}_\bullet(A; M)$, is a simplicial spectrum whose n-simplices are $M \wedge A^{\wedge n}$ and which has the following face and degeneracy maps:

$$d_i = \begin{cases} \psi_R \wedge id^{\wedge(n-1)} & i = 0 \\ id_M \wedge id^{\wedge(i-1)} \wedge \mu \wedge id^{\wedge(n-i-1)} & 0 < i < n \\ (\psi_L \wedge id^{\wedge(n-1)}) \circ \tau & i = n \end{cases}$$

$$s_i = id_M \wedge id^{\wedge i} \wedge \eta \wedge id^{\wedge(n-i)}.$$

As before, τ is the twist map that brings the right-most factor of A to the left position. We can also encode this data into a simplicial diagram,

$$\begin{array}{c}
\vdots \\
\downarrow \; \uparrow \; \downarrow \; \uparrow \; \downarrow \; \uparrow \; \downarrow \; \uparrow \; \downarrow \\
M \wedge A \wedge A \wedge A \\
{\scriptstyle d_0}\big\downarrow \quad {\scriptstyle d_1}\big\downarrow \quad {\scriptstyle d_2}\big\downarrow \quad \big\downarrow{\scriptstyle d_3} \\
M \wedge A \wedge A \wedge A \\
{\scriptstyle d_0}\big\downarrow{\scriptstyle s_0}\big\uparrow{\scriptstyle d_1}\big\downarrow{\scriptstyle s_1}\big\uparrow{\scriptstyle d_2}\big\downarrow{\scriptstyle s_2}\big\uparrow \;\; \big\downarrow{\scriptstyle d_3} \\
M \wedge A \wedge A \\
{\scriptstyle d_0}\big\downarrow{\scriptstyle s_0}\big\uparrow{\scriptstyle d_1}\big\downarrow{\scriptstyle s_1}\big\uparrow \;\; \big\downarrow{\scriptstyle d_2} \\
M \wedge A \\
{\scriptstyle d_0}\big\downarrow{\scriptstyle s_0}\big\uparrow \;\; \big\downarrow{\scriptstyle d_1} \\
M.
\end{array}$$

The *topological Hochschild homology of a ring spectrum A with coefficients in a bimodule M* is then defined to be the spectrum given by the geometric realization

$$\mathrm{THH}(A; M) \coloneqq |B^{cy}_{\bullet}(A; M)|.$$

Convention. If we take coefficients in the ring spectrum A as a bimodule over itself, we call this simply the topological Hochschild homology of A and denote it by

$$\mathrm{THH}(A) \coloneqq |B^{cy}_{\bullet}(A; A)|.$$

At each level of the cyclic bar construction which defines $\mathrm{THH}(A)$, the twist map τ acts as a cyclic operator, inducing an action of the cyclic group C_{n+1} at simplicial level n. This operator allows us to classify the cyclic bar construction as a special type of simplicial object.

Definition 2.2.7 ([Lod13], 6.1.2)**.** A *cyclic object* $C_\bullet$ in a category $\mathcal{C}$ is a simplicial object which is further endowed with a map

$$t_n : C_n \to C_n,$$

such that $(t_n)^{n+1} = id$, and the map t interacts with the face and degeneracy maps in the following ways:

$$d_i t_n = \begin{cases} d_n & i = 0 \\ t_{n-1} d_{i-1} & 1 \leq i \leq n \end{cases}$$

$$s_i t_n = \begin{cases} (t_{n+1})^2 s_n & i = 0 \\ t_{n+1} s_{i-1} & 1 \leq i \leq n. \end{cases}$$

Proposition 2.2.8 ([Lod13], Theorem 7.1.4)**.** *The geometric realization of a cyclic object has an action of the circle group, S^1.*

Since $B^{cy}_\bullet(A)$ is a cyclic object, $\mathrm{THH}(A)$ has an S^1-action. Further, $\mathrm{THH}(A)$ can be constructed as a genuine S^1-spectrum.

In the preceding section we discussed the failure of classical Hochschild homology to serve as a good approximation of algebraic K-theory and sought to remedy this by considering an invariant constructed in the world of topology. For a ring A, it was shown that topological Hochschild homology also receives a Dennis trace map from K-theory:

$$K_n(A) \xrightarrow{\text{topological Dennis trace}} \pi_n(\mathrm{THH}(HA)),$$

where HA is the Eilenberg-Mac Lane spectrum of the ring A. In fact, the classical Dennis trace factors as a composition of the above map with a map

$$\pi_n(\mathrm{THH}(HA)) \to \mathrm{HH}_n(A).$$

This is a particular case of a map which relates THH and HH called linearization.

Proposition 2.2.9. *Let A be a (-1)-connected ring spectrum. There is a map*

$$\pi_n \mathrm{THH}(A) \to \mathrm{HH}_n(\pi_0 A)$$

which is an isomorphism for $n = 0$.

With this topological trace we do indeed get a better approximation of K-theory from THH; in fact, much of the recent progress made in K-theory computations has involved computing THH and a closely related invariant called topological cyclic homology.

The question now is how to compute topological Hochschild homology. Recall that the motivation for trace methods was to find a computationally approachable approximation of K-theory. We now describe a tool which bridges the algebraic and topological theories of Hochschild homology to help us compute THH.

2.3 The Bökstedt spectral sequence

The Bökstedt spectral sequence is a computational tool that uses the classical, algebraic Hochschild groups (which are computationally more approachable than their topological counterparts) to determine the homology of THH. This work of Bökstedt originally appeared in [Bök85b] at a time before many of the tools of higher algebra (including a symmetric monoidal category of spectra with an associative smash product) were developed. Here we recall a more modern construction of this spectral sequence as it appears in [Elm+97].

We begin with a result about the existence of spectral sequences for simplicial spectra.

Proposition 2.3.1 ([Elm+97], X.2.9). *Let $X_\bullet$ be a proper simplicial spectrum and let E be any spectrum. There is a natural homological spectral sequence $\{E^r_{p,q}X_\bullet\}$ such that*

$$E^2_{p,q} = H_p(E_q(X_\bullet)) \Rightarrow E_{p+q}(|X_\bullet|) \tag{2.1}$$

which converges strongly.

Recalling Definition 2.2.6, we see that topological Hochschild homology is a simplicial spectrum, so by Proposition 2.3.1 the skeletal filtration gives rise to a spectral sequence converging to the homology of THH. Here, for a commutative ring spectrum E, a general E_*-homology theory is defined by setting $E_*(A) = \pi_*(A \wedge E)$. Setting $E = Hk$ (the Eilenberg-Mac Lane spectrum of k) recovers ordinary homology with coefficients in a field k. Thus, to compute $H_*(\mathrm{THH}(A);k)$ the E^2-page has the form:

$$E^2_{p,*} = H_p(H_*(B^{cy}_\bullet(A);k)).$$

We would like to identify this E_2-term as something familiar and computable; indeed we shall see that this is Hochschild homology.

The notation $H_*(X_\bullet; k)$ denotes applying the homology functor to the simplicial object $X_\bullet$ level-wise. For example, taking homology of the cyclic bar construction at level p we have

$$H_*(\overbrace{A \wedge A \wedge \ldots \wedge A}^{p+1}; k) = \pi_*(\overbrace{A \wedge A \wedge \ldots \wedge A}^{p+1} \wedge Hk).$$

The isomorphism described in Remark 2.2.5 allows us to write this as

$$\pi_*(A \wedge A \wedge \ldots \wedge A \wedge Hk) \cong \pi_*(A \wedge Hk \wedge_{Hk} A \wedge Hk \wedge_{Hk} \ldots \wedge_{Hk} A \wedge Hk).$$

Here, because $\pi_*(Hk) = k$ is a field, all modules over it are flat including the module $\pi_*(A \wedge Hk)$. Thus the Künneth spectral sequence collapses and the homotopy of this product can be written

$$\begin{aligned} \pi_*(A \wedge Hk \wedge_{Hk} A \wedge Hk \wedge_{Hk} \ldots \wedge_{Hk} A \wedge Hk) &\cong \\ \pi_*(A \wedge Hk) \otimes_{\pi_*(Hk)} \pi_*(A \wedge Hk) \otimes_{\pi_*(Hk)} \ldots \otimes_{\pi_*(Hk)} \pi_*(A \wedge Hk) &= \\ H_*(A; k) \otimes_k H_*(A; k) \otimes_k \ldots \otimes_k H_*(A; k) & \end{aligned}$$

A simplicial complex which is built from $(n+1)$-tensor copies of a ring at the nth level is precisely how we constructed the Hochschild complex. One can verify that the boundary maps in the Hochschild complex coincide with the d_1-differential in the spectral sequence, thus we have the following result of Bökstedt:

Proposition 2.3.2. *Let A be a ring spectrum and k a field. There is a spectral sequence called the Bökstedt spectral sequence which has the form:*

$$E^2_{*,*} = \mathrm{HH}_*(H_*(A); k) \Rightarrow H_*(\mathrm{THH}(A); k).$$

This result has been foundational in many THH computations. For example, in [Bök85a] Bökstedt uses this spectral sequence to compute $\mathrm{THH}(\mathbb{F}_p)$ and $\mathrm{THH}(\mathbb{Z})$. These computations have been essential to many calculations in algebraic K-theory.

CHAPTER 3

TOOLS FROM EQUIVARIANT ALGEBRA

The goal of the following chapters is to reconstruct much of the previously discussed machinery in an equivariant setting where the invariants we consider encode a group action on the input. Before constructing these equivariant Hochschild theories, we take this chapter to introduce some key objects of study in equivariant algebra and related notions in equivariant topology.

Convention. In this thesis we use D_2 to denote the group of two elements $1, \omega$ such that $\omega^2 = 1$. We elect to use this naming rather than the more common notation of C_2 for cyclic groups or $\mathbb{Z}/2$ in anticipation of generalized results for dihedral D_{2m}-equivariant objects.

3.1 Mackey and Green functors

We now recall the definition of Mackey functors which one may think of as the analogue of abelian groups in the equivariant setting. We then give several examples of common Mackey functors to illustrate what these objects look like and how one might use them.

Convention. In this section we always take G to be a finite group.

Definition 3.1.1. A *G-Mackey functor* $\underline{M}$ is a pair (M_*, M^*) of functors, one covariant and one contravariant, from the category of finite G-sets to the category of abelian groups

$$M_*, M^* : \mathcal{S}et_G^{fin} \to \mathcal{A}b$$

with the following properties:

(a) $M_*(X) = M^*(X)$ for all finite G-sets X. This group is denoted by $\underline{M}(X)$.

(b) M_* and M^* take disjoint unions of finite G-sets to direct sums of groups.

(c) A pullback diagram in $\mathcal{S}et_G^{fin}$ of the form

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} \\ Z & \xrightarrow[g]{} & W \end{array}$$

is taken to a commutative diagram of abelian groups.

$$\begin{array}{ccc} \underline{M}(X) & \xrightarrow{M_*(f)} & \underline{M}(Y) \\ {\scriptstyle M^*(h)}\uparrow & & \uparrow{\scriptstyle M^*(k)} \\ \underline{M}(Z) & \xrightarrow[M_*(g)]{} & \underline{M}(W). \end{array}$$

The fact that every finite G-set can be written as a disjoint union of orbits of the form G/H in combination with property (b) above means that to completely describe a G-Mackey functor $\underline{M}$, it is sufficient to determine the structure on just the orbits. This yields a more concrete description of $\underline{M}$ which we take to be our definition of a Mackey functor going forward.

Definition 3.1.2. A *G-Mackey functor* $\underline{M}$ consists of the following data:

1. An abelian group $\underline{M}(G/H)$ associated to each subgroup H of G.
2. A transfer map $tr_K^H : \underline{M}(G/K) \to \underline{M}(G/H)$ for each subgroup $K < H \leq G$.
3. A restriction map $res_K^H : \underline{M}(G/H) \to \underline{M}(G/K)$ for each subgroup $K < H \leq G$.
4. An action of the Weyl group $W_G(H) = N_G(H)/H$ on $\underline{M}(G/H)$ for all subgroups $H \leq G$.

This data is subject to the following relations:

(a) If $J < K < H$, then we have $tr_J^H = tr_K^H tr_J^K$ and $res_J^H = res_J^K res_K^H$.

(b) If $K < H \leq G$, then $res_K^H(x) = \gamma \cdot res_K^H(x)$ for all $x \in \underline{M}(G/H)$ and $\gamma \in W_H(K)$.

(c) If $K < H \leq G$, then $tr_K^H(\gamma \cdot x) = tr_K^H(x)$ for all $x \in \underline{M}(G/K)$ and $\gamma \in W_H(K)$.

(d) For all subgroups K, J in H and $x \in \underline{M}(G/(J \cap K))$,

$$res_J^H tr_K^H(x) = \sum_{\gamma \in W_H(J)} \gamma \cdot tr_{J \cap K}^J(x)$$

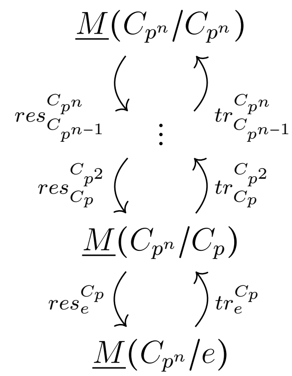


Figure 3.1 The Lewis diagram for a C_{p^n}-Mackey functor.

Convention. For the remainder of this thesis, the underline notion will be used to denote an object in equivariant algebra.

Remark 3.1.3. From this definition, we see that a trivial e-Mackey functor is just an abelian group. We may thus think of Mackey functors as a generalization of abelian groups in the world of equivariant algebra.

The collection of data which defines a Mackey functor may be organized visually into what is referred to as a Lewis diagram, first introduced by Gaunce Lewis in [Lew88]. We elect to omit the Weyl action in all Lewis diagrams presented in this thesis. In Figures 3.1 and 3.2 we provide two different examples of Lewis diagrams for Mackey functors to illustrate this definition.

Example 3.1.4 (Lewis diagram for a C_{p^n}-Mackey functor)*.* We can describe a C_{p^n}-Mackey functor by the collection of abelian groups associated to the orbits $C_{p^n}/e, C_{p^n}/C_p, ..., C_{p^n}/C_{p^n}$, the transfer and restriction maps between them, and the Weyl actions at each orbit. The Lewis diagram has a ladder-like structure, as seen in Figure 3.1.

Example 3.1.5 (Lewis diagram for a dihedral Mackey functor)*.* For a dihedral group D_{2p}, where p is prime, we obtain a branched diagram from the cyclic rotation subgroup C_p and the reflection subgroup D_2. We depict this in Figure 3.2 for the case of D_6.

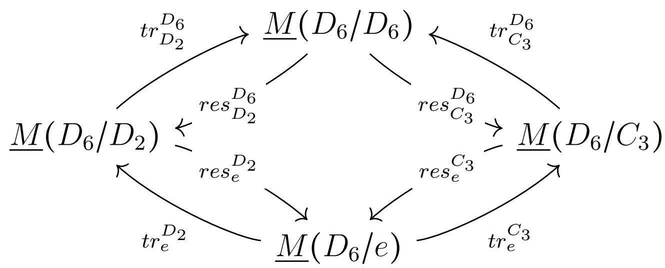


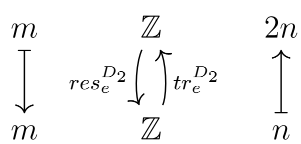
Figure 3.2 The Lewis diagram for a D_6-Mackey functor.

Definition 3.1.6. Let $\underline{M}$ and $\underline{N}$ be G-Mackey functors. A *map of Mackey functors* $\phi : \underline{M} \to \underline{N}$ is a homomorphism $\phi_H : \underline{M}(G/H) \to \underline{N}(G/H)$ for each subgroup $H \leq G$, such that the maps ϕ_H are $W_G(H)$-equivariant and commute with the restriction and transfer.

Equipped with the definitions of objects and morphisms in the category of G-Mackey functors (which we denote by $Mack_G$), we now turn to some important examples of Mackey functors.

Definition 3.1.7. Let B be an abelian group with a trivial G-action. There is a G-Mackey functor called the *constant Mackey functor*, which is denoted by $\underline{B}$, consisting of the group $B = \underline{B}(G/H)$ at each orbit. The restriction maps are the identity and the transfer maps tr_K^H are multiplication by the index $|H/K|$ for all subgroups $K < H \leq G$. All the Weyl actions are trivial.

Example 3.1.8. The case of a constant Mackey functor with $B = \mathbb{Z}$ arises often in equivariant algebra. A diagram of this constant D_2-Mackey functor is depicted below.



Example 3.1.9. Later in this section we explain how one takes the homotopy and homology of a G-equivariant spectrum. In the case of $G = D_2$, taking homology with coefficients in the D_2-Mackey functor $\underline{\mathbb{F}}_2$ yields a widely studied equivariant homology theory. The constant

D_2-Mackey functor $\underline{\mathbb{F}}_2$ is given by:

$$\begin{array}{c} \mathbb{F}_2 \\ {\scriptstyle id}\downarrow \uparrow {\scriptstyle 0} \\ \mathbb{F}_2. \end{array}$$

We now recall a special object in the category of G-Mackey functors which will play the role of a unit in this setting. We begin first with a definition needed to understand this Mackey functor.

Definition 3.1.10. For a group G, the *Burnside ring* $A(G)$ of G is the group completion of the monoid of isomorphism classes of finite G-sets with disjoint union.

Example 3.1.11. The G-Burnside Mackey functor, denoted by $\underline{A}$ is the Burnside ring $A(H)$ at each orbit G/H with the trivial Weyl action. For $K < H$, the transfer map sends $[X] \in A(K)$ to the induction $[H \times_K X]$. The restriction map sends an H-set $[X]$ to its underlying K-set $[\iota_K^* X]$.

The category $Mack_G$ has a symmetric monoidal product on it; for two G-Mackey functors $\underline{M}$ and $\underline{N}$, the box product is a G-Mackey functor denoted by $\underline{M} \;\square\; \underline{N}$. This product has a categorical definition as a left Kan extension. More concrete formulas in the case of $G = C_{p^n}$ may be found in Section 1 of [Lew88]. We will not reproduce these definitions here, although we remark that these formulas underscore the computational complexity of equivariant algebra. For instance, although the box product plays the role of a tensor product in the world of Mackey functors, the box product of two Mackey functors is not simply their level-wise tensor.

The Burnside Mackey functor given in Example 3.1.11 is the unit for this box product. Thus, for any Mackey functor $\underline{M}$ we have that $\underline{A} \;\square\; \underline{M} \cong \underline{M}$.

Definition 3.1.12. Let M be a group with G-action. The *fixed point Mackey functor* $\underline{M}$ has at each orbit (G/H) the H-fixed points of M,

$$\underline{M}(G/H) = M^H = \{m \in M \mid hm = m, \forall h \in H\}.$$

Restriction maps are inclusions of fixed points and the transfer map tr_K^H is given by a sum over the Weyl actions

$$tr_K^H(m) = \sum_{\gamma \in W_H(K)} \gamma \cdot m.$$

We have already seen an example of a fixed point Mackey functor; if we consider $\mathbb{Z}$ as a group with a trivial G-action, we recover the constant Mackey functor $\underline{\mathbb{Z}}$. At each orbit $\underline{\mathbb{Z}}(G/H)$ we have $\mathbb{Z}^H = \mathbb{Z}$ since $\mathbb{Z}$ has a trivial G-action. All restriction maps are thus the identity. The transfer maps

$$tr_K^H(m) = \sum_{\gamma \in W_H(K)} \gamma \cdot m = \sum_{\gamma \in W_H(K)} m$$

send an integer m to $|H/K| \cdot m$ since all the Weyl group actions are trivial. This is precisely the structure of the constant Mackey functor described in 3.1.7 and depicted in the D_2-case in Example 3.1.8. More generally, for an abelian group B, one could endow B with the trivial G action. The fixed point Mackey functor coincides with the constant Mackey functor.

A key example of Mackey functors that will be used extensively in the following chapters of this thesis is the equivariant homotopy Mackey functor. We want to recognize the full equivariance present in a G-spectrum when we take its homotopy and the classical notion of a group is insufficient in this regard. Instead, we need the additional equivariant structure of a Mackey functor to capture the homotopy of a G-spectrum. Before recounting the definition of equivariant homotopy, we first recall two notions of a graded Mackey functor, noting that in the G-equivariant setting, we often take gradings in the real representation ring $RO(G)$.

Definition 3.1.13. A *$\mathbb{Z}$-graded G-Mackey functor* $\underline{M}_*$ is a set $\{\underline{M}_n \mid n \in \mathbb{Z}\}$ of G-Mackey functors. A map of $\mathbb{Z}$-graded G-Mackey functors is a set $\{\underline{M}_k \to \underline{N}_k\}$ of maps of G-Mackey functors.

Definition 3.1.14. An *$RO(G)$-graded Mackey functor* $\underline{M}_\star$ is a set $\{\underline{M}_\alpha \mid \alpha \in RO(G)\}$ of G-Mackey functors. A map between two $RO(G)$-graded Mackey functors $\underline{M}_\star$ and $\underline{N}_\star$ is a set of maps of Mackey functors $\{\underline{M}_\alpha \to \underline{N}_\alpha\}$.

In anticipation of the work conducted with graded Mackey functors in later chapters, we also introduce the definition of a graded box product.

Definition 3.1.15 ([LM06], Definition 2.4)**.** Let $\underline{M}_*$ and $\underline{N}_*$ be $\mathbb{Z}$-graded Mackey functors as defined in 3.1.13. The graded box product $\underline{M}_* \Box \underline{N}_*$ is given by

$$(\underline{M}_* \Box \underline{N}_*)_n := \bigoplus_{i+j=n} \underline{M}_i \Box \underline{N}_j.$$

We can also define a box product for the $RO(G)$-graded Mackey functors of Definition 3.1.14. Given two $RO(G)$-graded Mackey functors $\underline{M}_\star$ and $\underline{N}_\star$ we have

$$(\underline{M}_\star \Box \underline{N}_\star)_\alpha := \bigoplus_{\beta+\gamma=\alpha} \underline{M}_\beta \Box \underline{N}_\gamma.$$

Using these conventions, we recall that one may consider the $\mathbb{Z}$-graded or $RO(G)$-graded homotopy of a G-spectrum.

Definition 3.1.16. Let X be a G-spectrum. The *$\mathbb{Z}$-graded homotopy of X* is a graded Mackey functor given by

$$\underline{\pi}_n^G(X)(G/H) := \pi_n(X^H).$$

To define the $RO(G)$-graded homotopy, let $\alpha = [\beta] - [\gamma] \in RO(G)$ where β and γ are finite dimensional real representations of G. Then for $H \leq G$,

$$\underline{\pi}_\alpha^G(X)(G/H) := [S^\beta \wedge G/H_+, S^\gamma \wedge X]_G$$

is defined to be the *$RO(G)$-graded homotopy Mackey functor.*

Convention. In this thesis we will use the notation $*$ to denote an integer grading. The five-point star $\star$ will denote an $RO(G)$-grading.

Analogous to the case of abelian groups, we have an Eilenberg-Mac Lane functor which takes a G-Mackey functor $\underline{M}$ to a G-spectrum $H\underline{M}$. The spectrum $H\underline{M}$ has the property that

$$\underline{\pi}_n^G(H\underline{M}) = \begin{cases} \underline{M} & n = 0 \\ 0 & n \neq 0. \end{cases}$$

As described above in the case of homotopy, we also wish to preserve the full equivariant structure when considering an appropriate homology theory for a G-spectrum X. Again, the structure that incorporates the equivariant structure is not a group, but a Mackey functor. We can thus study the G-equivariant homology of a G-spectrum with the following definition. This equivariant notion of homology will play an important role in our construction of the Real Bökstedt spectral sequence in Chapter 5.

Definition 3.1.17. Let E be a commutative G-ring spectrum and let X be a G-spectrum. We define the *$RO(G)$-graded E-homology of X* to be

$$\underline{E}_\star(X) := \underline{\pi}^G_\star(X \wedge E).$$

In the case of $E = H\underline{\mathbb{F}}_p$ we this is homology with coefficients in the constant Mackey functor $\underline{\mathbb{F}}_p$, $\underline{H}^G_\star(X;\underline{\mathbb{F}}_p)$.

Having presented Mackey functors as an equivariant analogue to abelian groups and noted the symmetric monoidal product on this category, we now turn to the question of what a ring looks like in the world of equivariant algebra.

Definition 3.1.18. A *G-Green functor* $\underline{R}$ is an associative monoid in the category $Mack_G$. Explicitly, $\underline{R}$ is a G-Mackey functor with a product map $\mu : \underline{R} \,\square\, \underline{R} \to \underline{R}$ and a unit map $\eta : \underline{A} \to \underline{R}$ such that the follow associativity and unitality diagrams commute:

$$\begin{array}{ccccc} \underline{A} \,\square\, \underline{R} & \xrightarrow{\eta \,\square\, id} & \underline{R} \,\square\, \underline{R} & \xleftarrow{id \,\square\, \eta} & \underline{A} \,\square\, \underline{R} \\ & \searrow^{\cong} & \downarrow{\scriptstyle \mu} & \swarrow^{\cong} & \\ & & \underline{R} & & \end{array}$$

$$\begin{array}{ccc} \underline{R} \,\square\, \underline{R} \,\square\, \underline{R} & \xrightarrow{id \,\square\, \mu} & \underline{R} \,\square\, \underline{R} \\ {\scriptstyle \mu \,\square\, id}\downarrow & & \downarrow{\scriptstyle \mu} \\ \underline{R} \,\square\, \underline{R} & \xrightarrow{\mu} & \underline{R}. \end{array}$$

If we further require a commutative diagram

$$\begin{array}{ccc} \underline{R} \,\square\, \underline{R} & \xrightarrow{\tau} & \underline{R} \,\square\, \underline{R} \\ & {\scriptstyle \mu}\searrow \quad \swarrow{\scriptstyle \mu} & \\ & \underline{R} & \end{array}$$

where τ is the switch map that permutes the two copies of $\underline{R}$, we say that $\underline{R}$ is a *commutative G-Green functor*.

Green functors serve as the analogue to rings in the world of equivariant homotopy theory; we see this connection in numerous ways. For example, a G-Green functor $\underline{R}$ yields a G-ring spectrum as its Eilenberg-Mac Lane spectrum $H\underline{R}$. If we take the equivariant homotopy of a G-ring spectrum X, $\underline{\pi}_n(X)$ has the structure of a Green functor. Green functors will thus serve as the input to the equivariant algebraic Hochschild theory considered in the following chapters.

3.2 Equivariant norms

In the preceding section of this chapter we held a finite group G constant and described what it means to be a G-equivariant version of an abelian group or a ring. We also considered the equivariant topological notion of a G-spectrum. We now recall two change-of-group functors that play an important role in equivariant homotopy theory. The first one allows us to take a G-spectrum and create an H-spectrum when $H \leq G$ by remembering only the H-equivariance.

$$\iota_H^* : Sp^G \to Sp^H.$$

We now recall a functor in the opposite direction which creates a G-spectrum from an H-spectrum for a finite group G and subgroup H. The original construction of this functor, called an equivariant norm is due to Hill, Hopkins, and Ravenel [HHR16]. For an H-spectrum X, we denote its norm to G by $N_H^G X$. The norm is symmetric monoidal meaning it enjoys the property that

$$N_H^G(X \wedge Y) \cong N_H^G X \wedge N_H^G Y.$$

In the commutative case, the norm is also left adjoint to the restriction functor we described above. In particular, Corollary 2.28 of [HHR16] gives us that for a commutative G-ring spectrum X,

$$N_H^G(\iota_H^* X) \cong X \otimes (G/H).$$

Remark 3.2.1. A particular application of this isomorphism will be utilized in the following chapters; if A is a D_2-spectrum, then $N_e^{D_2}\iota_e^* A$ is the D_2-spectrum $A \wedge A$ with the action given by τ, the permutation of the two smash copies of A. We make use of this fact in the Real Hochschild constructions in Chapter 4.

Hill and Hopkins show that this symmetric monoidal norm of spectra can be used to define a norm functor for finite $H \leq G$,

$$N_H^G : Mack_H \to Mack_G.$$

We now recall this definition of the norm in Mackey functors.

Definition 3.2.2 ([HH16], Definition 5.9)**.** Let $H \leq G$ be finite groups and let $\underline{M}$ be an H-Mackey functor. The *norm from* H *to* G *of* $\underline{M}$ is defined

$$N_H^G \underline{M} := \underline{\pi}_0^G(N_H^G H\underline{M})$$

where $H\underline{M}$ is the Eilenberg-Mac Lane spectrum of $\underline{M}$.

Work of Mazur, Hill-Mazur, and Hoyer ([Maz13], [HM19], [Hoy14]) develop explicit formulas for norms of Mackey functors without passing to spectra though we do not reproduce these formulas here. This notion of a change-of-group functor in equivariant algebra also appeared under a different guise in earlier work of Bouc [Bou00].

A restriction functor in the opposite direction may also be defined for Mackey functors.

Definition 3.2.3. [see [Hoy14], Section 2.3] Let $\underline{M}$ be a G-Mackey functor and H a subgroup of G. There is a functor called the *restriction*,

$$\iota_H^* : Mack_G \to Mack_H.$$

The restriction takes $\underline{M}$ to its underlying H-Mackey functor $\iota_H^* \underline{M}$ is given by

$$\iota_H^* \underline{M}(H/K) := \underline{M}(G \times_H (H/K)).$$

In the cases relevant to us (when G is a cyclic group or dihedral group) we have that $\iota_H^* \underline{M}(H/K) = \underline{M}(G/K)$ and the transfer maps, restriction maps, and Weyl actions are simply the H-restriction of those maps in $\underline{M}$.

Remark 3.2.4. Although we focused on norms for finite groups in this section, the more complex question of norms for compact Lie groups has been considered in [Ang+18] and [BDS22] with a particular goal of constructing a norm to S^1. That one can make sense of such norms to groups which are not finite plays an important role in interpreting Hochschild constructions as a norm. See [Ang+18] for further discussion in the case of THH and [AGH21] in the case of THR.

In this chapter we developed the language of equivariant algebra, as summarized in the dictionary below:

Classical Algebra	Equivariant Algebra
Abelian groups	Mackey functors
Tensor product, $\otimes$	Box product, $\square$
Integers, $\mathbb{Z}$	Burnside Mackey functor, $\underline{A}$
Rings	Green functors.

In the following chapters we will use this dictionary to translate trace methods for algebraic K-theory, and in particular the construction of the Bökstedt spectral sequence, to an equivariant setting where the inputs and their invariants have an action of involution.

CHAPTER 4

EQUIVARIANT THEORIES OF TOPOLOGICAL HOCHSCHILD HOMOLOGY

Equipped with the foundations of equivariant algebra, we now return to the story of trace methods for algebraic K-theory that began in Chapter 2. This time we take an equivariant perspective, studying an algebraic K-theory (and related approximations) that recognizes a group action on the input. Hesselholt and Madsen defined such a D_2-equivariant refinement of K-theory called Real algebraic K-theory (KR) for rings with the action of involution [HM15]. Real algebraic K-theory is a generalization of Hermitian K-theory. For a discrete ring with anti-involution A in which 2 is invertible, taking the D_2-fixed points of $KR(A)$ recovers Karoubi's connective Hermitian K-theory [Kar73].

As we observed in the classical case, despite the interest in Real algebraic K-theory computations, they are quite difficult. We again employ a trace methods approach to approximate KR with more computable equivariant invariants. In this chapter, we review the constructions of these invariants, called Real Hochschild homology and Real topological Hochschild homology.

A different equivariant perspective on Hochschild invariants one may wish to consider is that of a C_n-action on the input. This produces a different theory called Hochschild homology for C_n-Green functors (in the algebraic setting) and twisted topological Hochschild homology (in the topological setting). In Chapter 5 we will extend some of the techniques used to develop computational tools for Real topological Hochschild homology to this twisted theory so we take the opportunity in this chapter to include relevant background on twisted Hochschild constructions as well.

4.1 Equivariant simplicial objects

Recall that in Chapter 2 we defined Hochschild homology as a simplicial object in the category of abelian groups. Analogously, THH was a simplicial spectrum. By defining a cyclic operator at each simplicial level, we showed that the bar constructions which define

HH and THH were in fact cyclic objects, giving their geometric realization an action of S^1. We now review some additional equivariant definitions of simplicial objects.

To set notation, let D_{2m} denote the dihedral group generated by two elements

$$D_{2m} = < \omega, t \mid \omega^2 = t^m = 1, \omega t \omega = t^{-1} > .$$

Note that when $m = 1$, this is the cyclic group on two elements typically denoted C_2. In anticipation of our discussion about dihedral-equivariant objects in algebra and topology, we elect to use the notation D_2 for this group instead.

Definition 4.1.1. A *dihedral object* $L_\bullet$ in a category $\mathcal{C}$ is a simplicial object in $\mathcal{C}$ together with a $D_{2(n+1)}$-action on L_n specified by the action of the generators:

$$t_n : L_n \to L_n \quad \text{and} \quad \omega_n : L_n \to L_n$$

such that:

1. $\omega_n t_n = t_n^{-1} \omega_n$
2. $d_0 t_{n+1} = d_n$
3. $d_i t_n = t_{n-1} d_{i-1}$ if $1 \le i \le n$
4. $s_0 t_n = t_{n+1}^2 s_n$
5. $s_i t_n = t_{n+1} s_{i-1}$ if $1 \le i \le n$
6. $d_i \omega_n = \omega_{n-1} d_{n-i}$ if $0 \le i \le n$
7. $s_i \omega_n = \omega_{n+1} s_{n-i}$ if $0 \le i \le n$

Definition 4.1.2. A *Real simplicial object* $M_\bullet$ is a simplicial object together with maps $\omega_n : M_n \to M_n$ for each $n \ge 0$ which square to the identity ($\omega_n^2 = id_{M_n}$) and obey relations 6 and 7 in Definition 4.1.1.

Remark 4.1.3. By Theorem 5.3 of [FL91], the geometric realization of a dihedral object has an action of the orthogonal group $O(2)$ and the geometric realization of a Real simplicial object has a D_2-action.

At times, it is useful to work with a subdivided simplicial object which supports a D_2-action. The appropriate subdivision in this case is attributed to Segal [Seg73] and Quillen.

Definition 4.1.4. The *Segal-Quillen subdivision* of a simplicial object $X_\bullet$, denoted $\mathrm{sq}X_\bullet$, is the simplicial object with k-simplices

$$\mathrm{sq}X_k = X_{2k+1}.$$

Let d_i and s_i denote the face and degeneracy maps of $X_\bullet$. The face and degeneracy maps $\tilde{d}_i$ and $\tilde{s}_i$ of the subdivision $\mathrm{sq}X_\bullet$ are given by

$$\tilde{d}_i = d_i d_{2k+1-i}$$

$$\tilde{s}_i = s_{2k-i} s_i.$$

The geometric realizations of a simplicial object and its Segal-Quillen subdivision are homeomorphic [Spa00]. Further, if $X_\bullet$ is a dihedral or Real simplicial set then the homeomorphism $|X_\bullet| \cong |\mathrm{sq}X_\bullet|$ is D_2-equivariant (see [AGH21], Section 2.1).

4.2 Real topological Hochschild homology

In Section 2.2, we discussed topological Hochschild homology, an invariant of ring spectra. We now wish to consider a Real notion of ring spectra. These are a particular kind of D_2-equivariant ring spectra called ring spectra with anti-involution. A genuinely equivariant topological Hochschild homology theory for these ring spectra with anti-involution can be constructed to encode this D_2-action of involution. This invariant, called Real topological Hochschild homology (THR), was first introduced by Hesselholt and Madsen in their work on Real algebraic K-theory [HM15], given in the style of Bökstedt's original construction of THH [Bök85b]. Dotto, Moi, Patchkoria, and Reeh [Dot+20] subsequently gave a construction of THR using a dihedral bar construction analogous to the definition of THH via the cyclic bar construction which we recalled in Section 2.2. In this section, we recall the definition of THR via the dihedral bar construction, beginning with a formal description of its input.

Definition 4.2.1. A *ring spectrum with anti-involution* is a pair (A, ω) consisting of a ring spectrum A and a map $\omega : A \to A$ such that $\omega^2 = id$ and the following diagram commutes

$$\begin{array}{ccccccc} A \wedge A & \xrightarrow{\mu} & A & & \xrightarrow{\omega} & & A \\ \| & & & & & & \| \\ A \wedge A & \xrightarrow{\omega \wedge \omega} & A \wedge A & \xrightarrow{\tau} & A \wedge A & \xrightarrow{\mu} & A. \end{array}$$

Here, τ is the switch map that permutes the two copies of A and μ is the product on the ring spectrum. Equivalently, one may define the anti-involution to be a map $\omega : A^{op} \to A$ such that $\omega^2 = id$. We use these descriptions of the anti-involution interchangeably.

Example 4.2.2. Let A be a commutative D_2-ring spectrum. Since A is commutative, $A^{op} = A$ and the D_2-action on A defines an anti-involution.

Definition 4.2.3. A *map of ring spectra with anti-involution* $f : (A, \omega) \to (B, \tau)$ is a morphism of ring spectra $f : A \to B$ that commutes strictly with the involutions ω and τ.

Let (A, ω) be a ring spectrum with anti-involution and M an A-bimodule with left action map ψ_L and right action map ϕ_R. We may define an A-bimodule M^{op} via

$$A \wedge M \xrightarrow{\tau} M \wedge A \xrightarrow{id \wedge \omega} M \wedge A \xrightarrow{\psi_R} M$$

$$M \wedge A \xrightarrow{\tau} A \wedge M \xrightarrow{\omega \wedge id} A \wedge M \xrightarrow{\psi_L} M.$$

Definition 4.2.4. Let (A, ω) be a ring spectrum with anti-involution. An (A, ω)-*bimodule* is a pair (M, σ) which consists of an A-bimodule M and a map of A-bimodules $\sigma : M^{op} \to M$ such that $\sigma^2 = id$.

With this description of the inputs to Real topological Hochschild homology, we now proceed to recall the dihedral bar construction.

Definition 4.2.5 ([Dot+20], Section 2.2)**.** Let (A, ω) be a ring spectrum with anti-involution and (M, σ) an (A, ω)-bimodule. The *dihedral bar construction of* (A, ω) *with coefficients in* (M, σ) is a Real simplicial spectrum (in the sense of Definition 4.1.2) and is denoted by $B^{di}_\bullet(A; M)$. This spectrum has k-simplices

$$B^{di}_k(A; M) = M \wedge A^{\wedge \mathbf{k}}.$$

The simplicial structure maps in this spectrum are the same as those of the cyclic bar construction of Definition 2.2.6.

Furthermore, the dihedral bar construction has a level-wise involution W. At level k, let $\mathbf{k}$ be the D_2-set of integers $k = \{1, ..., k\}$ with a D_2-action of $\gamma(i) = k+1-i$ for γ a generator of D_2. The involution is given by

$$W : M \wedge A^{\wedge \mathbf{k}} \xrightarrow{id \wedge A_{\gamma(1)} \wedge \ldots \wedge A_{\gamma(k)}} M \wedge A^{\wedge \mathbf{k}} \xrightarrow{\sigma \wedge \omega^{\wedge \mathbf{k}}} M \wedge A^{\wedge \mathbf{k}}.$$

For example, at $k = 3$ the involution W is defined by

$$M \wedge A_1 \wedge A_2 \wedge A_3 \longrightarrow M \wedge A_3 \wedge A_2 \wedge A_1 \xrightarrow{\sigma \wedge \omega \wedge \omega \wedge \omega} M \wedge A_3 \wedge A_2 \wedge A_1.$$

Definition 4.2.6. The *Real topological Hochschild homology* of a ring spectrum with anti-involution (A, ω) with coefficients in the bimodule (M, σ) is the D_2-spectrum given by the geometric realization of the dihedral bar construction,

$$\mathrm{THR}(A; M) := |B^{di}_{\bullet}(A; M)|.$$

Convention. Taking coefficients in A, we simplify notation and write

$$\mathrm{THR}(A) := \mathrm{THR}(A, A).$$

Remark 4.2.7. Recall from Section 2.2 that the cyclic operators in the cyclic bar construction realized to give $\mathrm{THH}(A)$ an S^1-action. We note that the dihedral bar construction inherits much of the same structure as the cyclic bar construction, including the cyclic operators. The additional data present in the dihedral bar construction comes from the D_2-action of the anti-involution on each level. Thus, we see that each level $B^{di}_n(A)$ has a $C_n \rtimes D_2 = D_{2n}$-action on it. These actions assemble to an action of $O(2)$ on the geometric realization, thus $\mathrm{THR}(A)$ is an $O(2)$-spectrum.

4.3 Real Hochschild homology

Having recalled a Real-equivariant theory of topological Hochschild homology, we now seek a Real algebraic Hochschild theory to complete the translation of the trace methods approach to the Real equivariant setting. A Hochschild homology theory for rings and algebras equipped with an anti-involution called dihedral homology is described in Section 5.2 of [Lod13]. However, this theory is does not fully capture the Real equivariant structure present. In particular, one sees this failure when attempting to construct a Real linearization map. In Proposition 2.2.9, we recalled that a connective link between THH and HH was the linearization map $\pi_n(\mathrm{THH}(R)) \to \mathrm{HH}_n(\pi_0(R))$ from the homotopy groups of THH to the Hochschild homology of the ring $\pi_0(R)$.

Given a ring spectrum with anti-involution A, a Real linearization map should take $\mathrm{THR}(A)$ to the Hochschild homology of $\pi_0(A)$. This is not, however, simply a ring with involution; since A is a D_2-spectrum, its homotopy forms a graded D_2-Mackey functor $\underline{\pi}_n^{D_2}(A)$. Thus, a true algebraic analogue of THR should be constructed using the language of equivariant algebra we developed in Chapter 3 and take inputs in a D_2-Mackey functor that encodes the action of involution.

In this section, we recall Angelini-Knoll, Gerhardt, and Hill's [AGH21] construction of the algebraic analogue of THR, Real Hochschild homology. We begin with a definition of the appropriate input for this Real Hochschild theory, a particular kind of D_2-Mackey functor called a discrete E_σ-ring.

Definition 4.3.1. A *discrete E_σ-ring* consists of the following:

1. A D_2-Mackey functor $\underline{M}$ such that there is an associative product on $\underline{M}(D_2/e)$ for which the Weyl action is an anti-homomorphism.

2. An $N_e^{D_2}\iota_e^*\underline{M}$-bimodule structure on $\underline{M}$ with right action $\psi_L : N_e^{D_2}\iota_e^*\underline{M} \,\square\, \underline{M} \to \underline{M}$ and left action $\psi_R : \underline{M} \,\square\, N_e^{D_2}\iota_e^*\underline{M} \to \underline{M}$. We further require that ψ restricts to the usual module action over the enveloping algebra (see 2.1.5) on $\underline{M}(D_2/e)$.

3. A unit element $1 \in \underline{M}(D_2/D_2)$ such that $res(1) = 1 \in \underline{M}(D_2/e)$.

Remark 4.3.2. The data of a discrete E_σ-ring is essentially that of a Hermitian Mackey functor plus the unit condition in (3) above. The definition of Hermitian Mackey functors is due to Dotto and Ogle; for further details see [DO19].

Example 4.3.3. Let A be a ring spectrum with anti-involution. Then $\underline{\pi}_0^{D_2}(A)$ has the structure of a discrete E_σ-ring.

Remark 4.3.4. A discrete E_σ-ring can also be described as an algebra in D_2-Mackey functors over an E_σ-operad (see [AGH21], 6.3).

Before introducing the definition of Real Hochschild homology, we recall the notion of a two-sided bar construction in Mackey functors.

Definition 4.3.5. Let $\underline{R}$ be an associative G-Green functor with right module $\underline{M}$ and left module $\underline{N}$. The *two-sided bar construction* $B_\bullet(\underline{M}, \underline{R}, \underline{N})$ is a simplicial Mackey functor with k-simplices

$$B_k(\underline{M}, \underline{R}, \underline{N}) = \underline{M} \ \Box \ \underline{R}^{\Box k} \ \Box \ \underline{N}.$$

The face maps are defined by

$$d_i = \begin{cases} \psi_R \ \Box \ id_{\underline{R}}^{\Box(k-1)} \ \Box \ id_{\underline{N}} & i = 0 \\ id_{\underline{M}} \ \Box \ id_{\underline{R}}^{\Box(i-1)} \ \Box \ \mu \ \Box \ id_{\underline{R}}^{\Box(k-i-1)} \ \Box \ id_{\underline{N}} & 0 < i < k \\ id_{\underline{M}} \ \Box \ id_{\underline{R}}^{\Box(k-1)} \ \Box \ \psi_L & i = k \end{cases}$$

where ψ represents the left and right module actions of $\underline{N}$ and $\underline{M}$ respectively and μ is the Green functor multiplication. The degeneracy maps are given by

$$s_i = id_{\underline{M}} \ \Box \ id_{\underline{R}}^{\Box i} \ \Box \ \eta \ \Box \ id_{\underline{R}}^{\Box(k-i)} \ \Box \ id_{\underline{N}},$$

where η is the unit map from the Burnside Mackey functor $\underline{A}^G \to \underline{R}$.

We now define the left and right module action maps needed to describe the coefficients of the two-sided bar construction which defines Real Hochschild homology.

Definition 4.3.6. Recall from Definition 4.3.1 that the data of a discrete E_σ-ring $\underline{M}$ includes a right module structure $\psi_R : \underline{M} \,\square\, N_e^{D_2}\iota_e^*\underline{M} \to \underline{M}$. This action can be lifted to view $N_{D_2}^{D_{2m}}\underline{M}$ as a right $N_e^{D_{2m}}\iota_e^*\underline{M}$-module. We define this module structure via the induced action map

$$N_{D_2}^{D_{2m}}\underline{M} \,\square\, N_e^{D_{2m}}\iota_e^*\underline{M} \xrightarrow{\cong} N_{D_2}^{D_{2m}}(\underline{M} \,\square\, N_e^{D_2}\iota_e^*\underline{M}) \xrightarrow{N_{D_2}^{D_{2m}}(\psi_R)} N_{D_2}^{D_{2m}}\underline{M},$$

where the isomorphism on the left arises from the fact that the norm is symmetric monoidal.

We have another way to consider D_2 sitting inside D_{2m} as a subgroup. Let $\zeta = e^{2\pi i/2m}$. Then $\zeta D_2 \zeta^{-1}$ is a distinct order 2 subgroup of D_{2m}. However, since these groups are distinct only up to a change of generator, we have an equivalence of categories,

$$c_\zeta : Mack^{D_2} \to Mack^{\zeta D_2 \zeta^{-1}}.$$

This equivalence may similarly be defined in spectra and we will use the same notation to denote it. In our bar construction we will wish to take right coefficients in the norm $N_{\zeta D_2\zeta^{-1}}^{D_{2m}} c_\zeta \underline{M}$. The left module structure $\psi_L : N_e^{D_2}\iota_e^*\underline{M} \,\square\, \underline{M} \to \underline{M}$ defines a left $N_e^{D_2}\iota_e^*\underline{M}$-module structure on $N_{\zeta D_2\zeta^{-1}}^{D_{2m}} c_\zeta \underline{M}$ by composing the isomorphism given by the symmetric monoidal property of the norm,

$$N_e^{D_2}\iota_e^*\underline{M} \,\square\, N_{\zeta D_2\zeta^{-1}}^{D_{2m}} c_\zeta\underline{M} \cong N_{\zeta D_2\zeta^{-1}}^{D_{2m}}(N_e^{\zeta D_2\zeta^{-1}}\iota_e^*\underline{M} \,\square\, c_\zeta\underline{M})$$

with the induced left action,

$$N_{\zeta D_2\zeta^{-1}}^{D_{2m}}(N_e^{\zeta D_2\zeta^{-1}}\iota_e^*\underline{M} \,\square\, c_\zeta\underline{M}) \xrightarrow{N_{\zeta D_2\zeta^{-1}}^{D_{2m}}(c_\zeta(\psi_L))} N_{\zeta D_2\zeta^{-1}}^{D_{2m}} c_\zeta\underline{M}.$$

Equipped with the necessary information to define left and right module action maps for our coefficients, we now recall Angelini-Knoll, Gerhardt, and Hill's definition of Real Hochschild homology as a two-sided bar construction.

Definition 4.3.7 ([AGH21], Definition 6.15)**.** Let $\underline{M}$ be a discrete E_σ-ring. Let $\underline{\mathrm{HR}}_\bullet^{D_{2m}}(\underline{M})$ denote the two-sided bar construction

$$B_\bullet(N_{D_2}^{D_{2m}}\underline{M}, N_e^{D_{2m}}\iota_e^*\underline{M}, N_{\zeta D_2\zeta^{-1}}^{D_{2m}} c_\zeta\underline{M})$$

and define the *Real D_{2m}-Hochschild homology of $\underline{M}$* to be the $\mathbb{Z}$-graded D_{2m}-Mackey functor

$$\underline{\mathrm{HR}}_*^{D_{2m}}(\underline{M}) \coloneqq H_*(\underline{\mathrm{HR}}_\bullet^{D_{2m}}(\underline{M})),$$

where H_* denotes taking the homology of the dg Mackey functor associated to $\underline{\mathrm{HR}}_\bullet^{D_{2m}}(\underline{M})$.

Remark 4.3.8. Theorem 6.20 of [AGH21] gives a Real linearization map

$$\underline{\pi}_n^{D_{2m}} \mathrm{THR}(A) \to \underline{\mathrm{HR}}_n^{D_{2m}}(\underline{\pi}_0^{D_2} A),$$

which is one piece of evidence justifying that $\underline{\mathrm{HR}}$ is the algebraic analogue of THR. The construction of the Real Bökstedt spectral sequence in the following chapter relates $\underline{\mathrm{HR}}$ to the equivariant homology of THR, further justifying that this is the correct definition of an algebraic analogue for THR.

A useful relationship in Hochschild computations is that between Hochschild homology and the Tor functor - we noted this relationship in the classical setting in Lemma 2.1.6. In their work defining $\underline{\mathrm{HR}}$, the authors also prove an analogous result for Real Hochschild homology.

Lemma 4.3.9 ([AGH21], Proposition 6.19)**.** *Let $\underline{M}$ be a discrete E_σ-ring. If $\underline{M}$ is flat as a module over the D_{2m}-Burnside Mackey functor there is an isomorphism of D_{2m}-Mackey functors*

$$\underline{\mathrm{HR}}_*^{D_{2m}}(\underline{M}) \cong \underline{\mathrm{Tor}}_*^{N_e^{D_{2m}} \iota_e^* \underline{M}}(N_{D_2}^{D_{2m}} \underline{M}, N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} c_\zeta \underline{M}).$$

4.4 Twisted topological Hochschild homology

Rather than considering equivariant inputs with a involution action, we could also consider a set of Hochschild invariants which capture the equivariance of a cyclic group C_n-action. We begin by taking inputs in C_n-equivariant ring spectra and recall here the theory of C_n-twisted topological Hochschild homology, denoted by THH_{C_n}, first defined by the authors in [Ang+18].

Definition 4.4.1 ([Ang+18], Definition 8.1)**.** Let R be an associative orthogonal C_n-ring spectrum indexed on the trivial universe $\mathbb{R}^\infty$. The *C_n-twisted cyclic bar construction* on R, denoted by $B_\bullet^{cy,C_n}(R)$, is a simplicial spectrum which has k-simplices

$$B_k^{cy,C_n}(R) = R^{\wedge(k+1)}.$$

For g a generator of C_n, let α_k denote the composition that wraps the last factor of R around to the front then acts,

$$\alpha_k : R \wedge \overbrace{R \wedge \ldots \wedge R}^{k} \to R \wedge R \wedge \overbrace{R \wedge \ldots \wedge R}^{k-1} \to {}^gR \wedge R \wedge \overbrace{R \wedge \ldots \wedge R}^{k-1},$$

where ${}^gR$ denotes the action of g on R. We define the face maps of $B_k^{cy,C_n}(R)$ by

$$d_i = \begin{cases} id^{\wedge i} \wedge \mu \wedge id^{\wedge(k-i-1)} & 0 \leq i < k \\ (\mu \wedge id^{\wedge(k-1)}) \circ \alpha_k & i = k. \end{cases}$$

Let η denote the unit map of the ring spectrum R. The degeneracy maps in this simplicial object are

$$s_i = id^{\wedge(i+1)} \wedge \eta \wedge id^{\wedge(k-i)}.$$

We then define the C_n-twisted topological Hochschild homology of R to be the geometric realization of this simplicial construction.

Definition 4.4.2. Let U be a complete S^1-universe and let $\tilde{U} = \iota_{C_n}^* U$ be the pullback of U to C_n. For R an associative orthogonal C_n-ring spectrum indexed on $\tilde{U}$, the *C_n-twisted topological Hochschild homology* of R is defined

$$\mathrm{THH}_{C_n}(R) = N_{C_n}^{S^1}(R) = \mathcal{I}_{\mathbb{R}^\infty}^{U} |B_\bullet^{cy,C_n}(\mathcal{I}_{\tilde{U}}^{\mathbb{R}^\infty} R)|.$$

Here, $\mathcal{I}$ denotes a change of universe functor.

There is an algebraic theory associated to C_n-twisted THH, which takes inputs in C_n-Green functors. The theory of Hochschild homology for Green functors was first defined by Blumberg, Gerhardt, Hill, and Lawson. We first recall the definition of G-twisted Hochschild homology for a G-Green functor $\underline{R}$, denoted by $\underline{\mathrm{HH}}_*^G(\underline{R})$ (see [Blu+19] Section 2.3).

Definition 4.4.3. Let $G < S^1$ be a finite subgroup and let $g \in G$. For a G-Green functor $\underline{R}$ and a left module $\psi : \underline{R} \square \underline{M} \to \underline{M}$, we can define a g-twisted module structure on $\underline{M}$, denoted ${}^g\underline{M}$, where the action map is the composition

$$\begin{array}{ccc} \underline{R} \square \underline{M} & & \\ {\scriptstyle g\square 1}\downarrow & \searrow^{{}^g\psi} & \\ \underline{R} \square \underline{M} & \xrightarrow{\psi} & \underline{M}. \end{array}$$

Definition 4.4.4. Let $G < S^1$ be a finite subgroup and let $g = e^{2\pi i/|G|}$ be a generator of G. For a G-Green functor $\underline{R}$ and an $\underline{R}$-bimodule $\underline{M}$, we denote the *G-twisted cyclic nerve* by $\underline{B}_\bullet^{cy,G}(\underline{R}; {}^g\underline{M})$. This is a simplicial Mackey functor which has k simplices

$$\underline{B}_k^{cy,G}(\underline{R}; {}^g\underline{M}) = {}^g\underline{M} \square \underline{R}^{\square k},$$

and face maps

$$d_i = \begin{cases} \psi_R \wedge id^{\wedge(k-1)} & i = 0 \\ id_{\underline{M}} \wedge id^{\wedge i-1} \wedge \mu \wedge id^{\wedge(k-i-1)} & 0 < i < k \\ ({}^g\psi_L \wedge id^{\wedge(k-1)}) \circ \tau & i = k. \end{cases}$$

Here, τ is the map which wraps the last factor $\underline{R}$ around to the front. The degeneracy maps are given by

$$s_i = id_{\underline{M}} \wedge id^{\wedge i} \wedge \eta \wedge id^{\wedge(k-i-1)}.$$

We now recall a relative version of G-twisted Hochschild homology where the input is an H-spectrum for $H \leq G$.

Definition 4.4.5 ([Blu+19], 3.2.6)**.** Let $H < G$ be a finite subgroup of S^1 and let $g = e^{2\pi i/|G|}$ be a generator of G as above. For an associative H-Green functor $\underline{R}$ we define the *G-twisted Hochschild homology* of $\underline{R}$ as the homology of the simplicial Mackey functor

$$\underline{\mathrm{HH}}_H^G(\underline{R})_* = H_*(\underline{B}_\bullet^{cy,G}(N_H^G\underline{R}; {}^gN_H^G\underline{R})).$$

We will utilize these twisted Hochschild homology definitions in the construction of a spectral sequence computing the equivariant homology of THH_{C_n} in Section 5.3.

4.5 Equivariant Hochschild homology for graded inputs

In the following chapter, our construction of the Real Bökstedt spectral sequence requires us to make sense of Real Hochschild homology for a graded discrete E_σ-ring; recall from our discussion of the classical Bökstedt spectral sequence for a ring A in Section 2.3 that the E^2-page was the Hochschild homology of the graded ring $H_*(A;k)$. In the classical case we only consider $\mathbb{Z}$-graded inputs. In the construction of equivariant Bökstedt spectral sequences, the necessary flatness conditions that arise when one does not restrict to field coefficients are more likely to hold if one uses an equivariant grading. For example, the equivariant Bökstedt spectral sequence for C_n-twisted THH constructed in [Ada+22] takes inputs in $RO(C_n)$-graded Green functors.

As we saw in Definition 4.3.7, however, $\underline{\text{HR}}$ is defined with a two-sided bar construction that involves taking equivariant norms and restrictions of the input. This complicates the question of gradings; although discrete E_σ-rings are D_2-Mackey functors, if one only considers the $RO(D_2)$-graded homology, it is unclear how to restrict to the trivial group and then norm back to D_2 while preserving the $RO(D_2)$-grading. We therefore need a grading which contains representations of other groups; the appropriate grading in this equivariant setting with norms is in $\underline{RO}(G)$, as considered in [HHR17], [AB18], and [Hil22].

Let G be a finite group. An element in $\underline{RO}(G)$ is a pair (H, α) where H is a subgroup of G and α is a virtual H-representation. Note the contrast with an element of $RO(G)$ which is only a virtual G-representation. We see that in our case, an $\underline{RO}(D_2)$-grading will allow us to grade on representations of e and D_2, as desired.

We have the following definition of the $\underline{RO}(G)$-graded homotopy of a G-spectrum due to Hill, Hopkins, and Ravenel.

Definition 4.5.1 ([HHR17], Definition 2.7)**.** Let X be a G-spectrum. For each pair (H, α) consisting of a subgroup $H \leq G$ and a virtual orthogonal representation α of H, define the $\underline{RO}(G)$*-graded homotopy Mackey functor* to be the G-Mackey functor $\underline{\pi}_{\circledast}(X)$ where, for

$\circledast = (H, \alpha)$, we set

$$\underline{\pi}_{H,\alpha}(X)(T) \coloneqq [(G_+ \wedge_H S^\alpha) \wedge T_+, X]_G \cong [S^\alpha \wedge \iota_H^* T_+, \iota_H^* X]_H = \underline{\pi}_\alpha^H(\iota_H^* X)(\iota_H^* T)$$

for each finite G-set T.

Convention. We will use $\circledast$ to denote an $\underline{RO}(G)$-grading. We continue to denote an $RO(G)$-grading by $\star$.

Definition 4.5.2. For a G-spectrum X and a commutative G-ring spectrum E, the $\underline{RO}(G)$-graded E-homology of X is defined to be

$$\underline{E}_\circledast(X) \coloneqq \underline{\pi}_\circledast(X \wedge E).$$

Remark 4.5.3. The $\underline{RO}(G)$-graded homotopy and homology Mackey functors are both examples of a more general construction called an $\underline{RO}(G)$-graded Mackey functor due to Angeltveit and Bohmann [AB18].

In Chapter 3 we noted that the category of G-Mackey functors is equipped with a symmetric monoidal product. Similarly, this category of $\underline{RO}(G)$-graded Mackey functors has a symmetric monoidal product (see, for instance [Hil22]). For $\underline{RO}(G)$-graded Mackey functors $\underline{M}_\circledast$ and $\underline{N}_\circledast$ we denote this product, which we refer to as the $\underline{RO}(G)$-graded box product, by $\underline{M}_\circledast \,\square\, \underline{N}_\circledast$. The notion of a product allows to consider monoids in this category.

Definition 4.5.4. An $\underline{RO}(G)$*-graded Green functor* is an associative monoid in the category of $\underline{RO}(G)$-graded Mackey functors with respect to the graded box product.

Definition 4.5.5. Let $\underline{R}_\circledast$ be an $\underline{RO}(G)$-graded G-Green functor. A *left* $\underline{R}_\circledast$*-module* is an $\underline{RO}(G)$-graded Mackey functor $\underline{M}_\circledast$ with an action map

$$\psi_L : \underline{R}_\circledast \,\square\, \underline{M}_\circledast \to \underline{M}_\circledast,$$

such that diagrams analogous to those in Definition 2.2.3 commute. A right $\underline{R}_\circledast$-module is defined similarly.

Definition 4.5.6. Let $\underline{R}_{\circledast}$ be an $\underline{RO}(G)$-graded G-Green functor. We say that an $\underline{RO}(G)$-graded $\underline{R}_{\circledast}$-module $\underline{M}_{\circledast}$ is *flat* if the functor $(-) \square_{\underline{R}_{\circledast}} \underline{M}_{\circledast}$ is exact.

Our motivation for the use of these $\underline{RO}(G)$-graded objects was to define an appropriately graded input to Real Hochschild homology, the construction of which includes equivariant norms. A grading in $\underline{RO}(G)$ is the appropriate setting for constructions which involve an equivariant norm as demonstrated by work of Angeltveit and Bohmann on $\underline{RO}(G)$-graded Tambara functors which takes a more categorical perspective [AB18]. The interplay of norms with $\underline{RO}(G)$-graded objects in equivariant homotopy is also considered by Hill in [Hil22].

In the case of an ungraded input, discussed in Section 4.3, $\underline{\mathrm{HR}}$ took inputs in discrete E_σ-rings. We now define a notion of $\underline{RO}(D_2)$-graded discrete E_σ-rings.

Definition 4.5.7. An *$\underline{RO}(D_2)$-graded discrete E_σ-ring* $\underline{M}_{\circledast}$ is the following data:

1. An D_2-Mackey functor $\underline{M}_{(H,\alpha)}$ for each subgroup $H \leq D_2$ and virtual H-representation α such that $\underline{M}_{\circledast}(D_2/e)$ forms a graded ring with anti-involution. That is, we have an associative product,

$$\underline{M}_{\circledast}(D_2/e) \otimes \underline{M}_{\circledast}(D_2/e) \to \underline{M}_{\circledast}(D_2/e)$$

   where the domain has the action of swapping the two copies of $\underline{M}_{\circledast}(D_2/e)$.

2. An $N_e^{D_2}\iota_e^*\underline{M}_{\circledast}$-bimodule structure on $\underline{M}_{\circledast}$. We further require that the action restricts to the standard action of $\underline{M}_{\circledast}(D_2/e) \otimes \underline{M}_{\circledast}(D_2/e)^{op}$ on $\underline{M}_{\circledast}(D_2/e)$.

3. A designated unit $1 \in \underline{M}_{\circledast}(D_2/D_2)$ which restricts to $1 \in \underline{M}_{\circledast}(D_2/e)$.

We claim that the $\underline{RO}(D_2)$-graded homotopy Mackey functor of a ring spectrum with anti-involution forms an $\underline{RO}(D_2)$-graded discrete E_σ-ring. A useful perspective in proving this statement is the view of ring spectra with anti-involution as algebras in D_2-spectra over an E_σ-operad. It is this interpretation which motivates the name and definition of a discrete E_σ-ring in equivariant algebra. Formally, this gives the following definition.

Definition 4.5.8 ([AGH21], Corollary 3.10)**.** An *E_σ-ring* A is a D_2-spectrum such that

1. the spectrum $\iota_e^* A$ is an E_1-ring with anti-involution, denoted $\tau : \iota_e^* A^{op} \to \iota_e^* A$ and given by the action of the generator of the Weyl group.

2. the spectrum A is an $E_0 - N_e^{D_2}\iota_e^* A$-algebra and applying the restriction functor ι_e^* to the $N_e^{D_2}\iota_e^* A$-module structure map gives $\iota_e^* A$ the standard $\iota_e^* A \wedge \iota_e^* A^{op}$-module structure.

Proposition 4.5.9. *The $\underline{RO}(D_2)$-graded homotopy of a ring spectrum with anti-involution (equivalently, an E_σ-ring) A forms an $\underline{RO}(D_2)$-graded discrete E_σ-ring.*

Proof. By Definition 4.5.1 we see that each $\underline{\pi}_{(H,\alpha)}(A)$ is a D_2-Mackey functor. Further, we see by this definition that the restriction to the orbit (D_2/e) recovers the non-equivariant homotopy of $\iota_e^* A$. This is clear in the case of $H = e$. When $H = D_2$ we have that

$$\underline{\pi}_{(D_2,\alpha)}(A)(D_2/e) = [S^\alpha, A]_e = \pi_{|\alpha|}(\iota_e^* A).$$

Thus, restricting to the orbit (D_2/e) in both cases recovers the non-equivariant homotopy of the underlying spectrum. From Definition 4.5.8, we know that $\iota_e^* A$ is an E^1-ring with anti-involution. Thus we have a map $\iota_e^* A \wedge \iota_e^* A \to A$ with a swap action on the domain. This induces the desired map on homotopy.

To define the module structure specified in condition 2 of Definition 4.5.7, we recall that since A is an E_σ-ring, it has an $N_e^{D_2}\iota_e^* A$-bimodule structure. We denote these module action maps by ψ'_R and ψ'_L. We use these maps to induce module action maps on $\underline{RO}(D_2)$-graded homotopy,

$$\psi'_R : \underline{\pi}_{\circledast}(A \wedge N_e^{D_2}\iota_e^* A) \to \underline{\pi}_{\circledast}(A),$$

and similarly for ψ'_L. Since the $\underline{RO}(D_2)$-graded homotopy functor is D_2-lax monoidal there is a map

$$\underline{\pi}_{\circledast}(A) \,\square\, N_e^{D_2}\iota_e^* \underline{\pi}_{\circledast}(A) \to \underline{\pi}_{\circledast}(A \wedge N_e^{D_2}\iota_e^* A).$$

Precomposing this map with ψ'_R yields the desired right-module structure,

$$\psi_R : \underline{\pi}_{\circledast}(A) \,\square\, N_e^{D_2}\iota_e^* \underline{\pi}_{\circledast}(A) \to \underline{\pi}_{\circledast}(A).$$

The left module action ψ_L is defined analogously. Thus $\underline{\pi}_{\circledast}(A)$ is an $N_e^{D_2}\iota_e^*\underline{\pi}_{\circledast}(A)$-bimodule. Above, we argued that restriction to the orbit (D_2/e) recovers the non-equivariant homotopy of ι_e^*A. By Definition 4.5.8, ι_e^*-restriction of the module structure map recovers the standard action. Thus the induced map on homotopy is also the standard action of $\underline{\pi}_{\circledast}(A)(D_2/e) \otimes \underline{\pi}_{\circledast}(A)(D_2/e)^{op}$. □

The $\underline{RO}(D_2)$-graded equivariant homology of a ring spectrum with anti-involution will be the input of Real Hochschild homology in the Real Bökstedt spectral sequence constructed in Chapter 5. We now define $\underline{\text{HR}}$ for $\underline{RO}(D_2)$-graded discrete E_σ-rings. To make sense of the Real Hochschild homology two-sided bar construction in the graded setting, we use the $N_e^{D_2}\underline{\pi}_{\circledast}(A)$-module structure of $\underline{\pi}_{\circledast}(A)$ to give the necessary module structures which define the coefficients in the two-sided bar construction.

Proposition 4.5.10. *Let $\underline{M}_{\circledast}$ be an $\underline{RO}(D_2)$-graded discrete E_σ-ring. Then $N_{D_2}^{D_{2m}}\underline{M}_{\circledast}$ is a right $N_e^{D_{2m}}\iota_e^*\underline{M}_{\circledast}$-module and $N_{\zeta D_2\zeta^{-1}}^{D_{2m}}c_\zeta\underline{M}_{\circledast}$ is a left $N_e^{D_{2m}}\iota_e^*\underline{M}_{\circledast}$-module.*

Proof. Since $\underline{M}_{\circledast}$ is an $\underline{RO}(D_2)$-graded discrete E_σ-ring there is an $N_e^{D_2}\iota_e^*\underline{M}_{\circledast}$-bimodule structure on $\underline{M}_{\circledast}$,

$$\psi_R' : N_e^{D_2}\iota_e^*\underline{M}_{\circledast} \,\square\, \underline{M}_{\circledast} \to \underline{M}_{\circledast}$$

$$\psi_L' : \underline{M}_{\circledast} \,\square\, N_e^{D_2}\iota_e^*\underline{M}_{\circledast} \to \underline{M}_{\circledast}.$$

We then define the desired module structures over the $\underline{RO}(D_2)$-graded box product to be the same composites as specified in the ungraded case by Definition 4.3.6. □

With these module actions in hand, we define a two-sided bar construction for an $\underline{RO}(D_2)$-graded discrete E_σ-ring $\underline{M}_{\circledast}$,

$$\underline{\text{HR}}_\bullet^{D_{2m}}(\underline{M}_{\circledast}) := B_\bullet(N_{D_2}^{D_{2m}}\underline{M}_{\circledast}, N_e^{D_{2m}}\iota_e^*\underline{M}_{\circledast}, N_{\zeta D_2\zeta^{-1}}^{D_{2m}}c_\zeta\underline{M}_{\circledast})$$

with the same face and degeneracy maps as in Definition 4.3.7, taken here over the $\underline{RO}(D_2)$-graded box product. The Real Hochschild homology of the $\underline{RO}(D_2)$-graded discrete E_σ-ring $\underline{M}_{\circledast}$ is the homology of this two-sided bar construction.

In Section 5.3, we also wish to consider a notion of Hochschild homology for $\underline{RO}(G)$-graded Green functors. Recall from Definition 4.4.4 that Hochschild homology for Green functors is defined using a cyclic bar construction, rather than the kind of two-sided bar construction we have been discussing for Real Hochschild homology. The final face map of the cyclic bar construction picks up an additional sign in the graded case - see Section 4.1 of [Ada+22] for a discussion of this point in the $\mathbb{Z}$- and $RO(G)$-graded settings. Before defining a notion of $\underline{RO}(G)$-graded Hochschild homology for G-Green functors, we address the question of signs in the final face map of the twisted cyclic bar construction in the $\underline{RO}(G)$-graded case.

Let G be a finite subgroup of S^1 and let β and γ be two finite dimensional real representations of G. The switch map on the representation spheres, $S^\beta \wedge S^\gamma \to S^\gamma \wedge S^\beta$, specifies an element in the Burnside ring $A(G)$, which we denote by $\sigma(\beta, \gamma)$. The rotating isomorphism of $\underline{RO}(G)$-graded Mackey functors is a map

$$\tau : \underline{M}_{\circledast} \,\square\, \underline{N}_{\circledast} \to \underline{N}_{\circledast} \,\square\, \underline{M}_{\circledast}.$$

We restrict to working one subgroup at a time in the $\underline{RO}(G)$-graded box product, so at level (H, α), the rotating isomorphism is as defined in [Ada+22], Definition 4.1.4 with the sign in the switch map coming from the Burnside ring $A(H)$.

Definition 4.5.11. We say an $\underline{RO}(G)$-graded Green functor $\underline{R}_{\circledast}$ is *commutative* if $\mu\tau = \mu$ where μ is the multiplication on $\underline{R}_{\circledast}$ and τ is the rotating isomorphism $\underline{R}_{\circledast} \,\square\, \underline{R}_{\circledast} \to \underline{R}_{\circledast}$.

With this description of how $\underline{RO}(G)$-graded Green functors commute past each other, we are ready to define an $\underline{RO}(G)$-graded notion of the twisted cyclic nerve.

Definition 4.5.12. Let G be a finite subgroup of S^1 and let $g = e^{2\pi i/|G|} \in G$ be a generator. For $\underline{R}_{\circledast}$, an $\underline{RO}(G)$-graded G-Green functor and $\underline{M}_{\circledast}$ an $\underline{R}_{\circledast}$-module, we define the *$G$-twisted cyclic nerve* of $\underline{R}_{\circledast}$ with coefficients in ${}^g\underline{M}_{\circledast}$ to be the simplicial $\underline{RO}(G)$-graded Mackey functor which has k-simplices

$$B_k^{cy,G}(\underline{R}_{\circledast}, {}^g\underline{M}_{\circledast}) = {}^g\underline{M}_{\circledast} \,\square\, \underline{R}_{\circledast}^{\square k}.$$

The face map d_0 applies the right module action of $\underline{R}_\circledast$ to ${}^g\underline{M}_\circledast$. For $1 \le i \le k$, the face map d_i multiplies the ith and $(i+1)$st copies of $\underline{R}_\circledast$. The final face map d_k incorporates the rotating isomorphism by rotating the last factor around to the front and then applying the left module action of $\underline{R}_\circledast$ to ${}^g\underline{M}_\circledast$. Explicitly, this is given by

$$ {}^g\underline{M}_\circledast \,\square\, \underline{R}_\circledast^{\square k} \xrightarrow{\tau_k} \underline{R}_\circledast \,\square\, {}^g\underline{M}_\circledast \,\square\, \underline{R}_\circledast^{\square(k-1)} \xrightarrow{{}^g\psi \square id} {}^g\underline{M}_\circledast \,\square\, \underline{R}_\circledast^{\square(k-1)}. $$

where τ_k denotes iterating the rotating isomorphism k times in order to bring the last factor to the front. Recall from Definition 4.4.3 that ${}^g\psi$ denotes the g-twisted module action on ${}^g\underline{M}_\circledast$. This twisted action is defined analogously in the $\underline{RO}(G)$-graded setting. The degeneracy maps in this simplicial object are the usual maps induced by inclusion via the unit.

In Section 5.3, we consider the case of relative Hochschild homology for $\underline{RO}(G)$-graded Green functors. With our definition of the graded twisted cyclic nerve, we can now define a relative equivariant Hochschild homology for these graded inputs.

Definition 4.5.13. Let $H \le G$ be finite subgroups of S^1 and let $\underline{R}_\circledast$ be an $\underline{RO}(G)$-graded associative Green functor for H. The G-twisted Hochschild homology of $\underline{R}_\circledast$ is

$$ \underline{\mathrm{HH}}_H^G(\underline{R}_\circledast)_* := H_*(B_\bullet^{cy,G}(N_H^G \underline{R}_\circledast)). $$

CHAPTER 5

SPECTRAL SEQUENCE CONSTRUCTIONS

In this chapter we describe the construction of a Bökstedt spectral sequence relating the theories of Real Hochschild homology and Real topological Hochschild homology. This construction parallels the one we described for the classical Bökstedt spectral sequence in Section 2.3, however the presence of equivariant norms in the Real equivariant case is a notable difference. These norms require us to place additional hypotheses on the ring spectra inputs to ensure that we may recognize the E^2-page of the spectral sequence is Real Hochschild homology. The existence of norms on this E^2-page also necessitates the use of a more complicated grading convention, as discussed in Section 4.5. We begin this chapter by recalling Hill's notion of free homology in an equivariant setting, particularly as it relates to this question of how to treat equivariant norms in the Real Bökstedt spectral sequence construction. Following this, in Section 5.2 we construct the Real Bökstedt spectral sequence converging to the $\underline{RO}(D_{2m})$-graded equivariant homology of THR. Finally, we extend our techniques to the setting of twisted THH, generalizing the results of [Ada+22].

5.1 Free homology

In contrast with the classical Bökstedt spectral sequence, the equivariant norms present in the two-sided bar construction defining Real Hochschild homology necessitate additional freeness conditions on the input in order to construct a Bökstedt spectral sequence. Before the construction, we take this opportunity to recall Hill's notion of free homology in an equivariant setting and a related lemma describing the interaction of the norm functor with free homology. Finally, we prove a corollary of the lemma which will be used in the construction of the spectral sequence.

Convention. For a G spectrum E, we let $\underline{E}_{\circledast}$ denote the $\underline{RO}(G)$-graded equivariant homotopy Mackey functor of E, $\underline{\pi}^G_{\circledast}(E)$.

Definition 5.1.1 ([Hil22])**.** Let A be a G-spectrum and let E be a commutative G-ring spectrum. We say A has *free E-homology* if $E \wedge A$ splits as a wedge of E-modules of the form

$$E \wedge (G_+ \wedge_H S^\alpha)$$

where α is a virtual representation of H, a subgroup of G.

Hill shows that this class of spectra with E-free homology is closed under operations such taking coproducts, restriction, and the norm. Of particular relevance to our work constructing the Real Bökstedt spectral sequence is a description of how the homology functor interacts with the equivariant norm under freeness hypotheses.

Lemma 5.1.2 ([Hil22], Corollary 3.30)**.** *Let A be an H-spectrum for $H \leq G$ and let E be a commutative G-ring spectrum. If A has free $(\iota_H^* E)$-homology, then there is a natural isomorphism*

$$\underline{E}_{\circledast}(N_H^G A) \cong N_H^G(\underline{(\iota_H^* E)}_{\circledast}(A)).$$

This result, which allows us to permute the norm functor and the homology functor, will be important in our construction of the Real Bökstedt spectral sequence. In particular, we make use of the following consequence of this lemma.

Corollary 5.1.3. *Let A be a ring spectrum with anti-involution and E be a commutative D_{2m}-ring spectrum such that $\iota_e^* A$ has free $\iota_e^* E$-homology. Then there is an isomorphism*

$$\underline{E}_{\circledast}(N_e^{D_{2m}} \iota_e^* A) \cong N_e^{D_{2m}} \iota_e^* (\underline{(\iota_{D_2}^* E)}_{\circledast}(A))$$

Proof. The assumption that $\iota_e^* A$ has free $\iota_e^* E$-homology allows us to apply the result in 5.1.2. We have

$$\underline{E}_{\circledast}(N_e^{D_{2m}} \iota_e^* A) \cong N_e^{D_{2m}}(\underline{\iota_e^* E}_{\circledast}(\iota_e^* A)).$$

The restriction functor commutes with homology so we on the right hand side we can write

$$N_e^{D_{2m}}(\underline{\iota_e^* E}_{\circledast}(\iota_e^* A)) \cong N_e^{D_{2m}} \iota_e^* (\underline{(\iota_{D_2}^* E)}_{\circledast}(A)).$$

which gives us the desired isomorphism. □

Remark 5.1.4. This freeness condition is always satisfied for equivariant-$H\underline{\mathbb{F}}_p$ homology since $\iota_e^* H\underline{\mathbb{F}}_p$ is considered by [Rav03] Theorem 3.1.2.g.

5.2 Construction of the spectral sequence

In Definition 4.2.6, we presented THR as a dihderal bar construction but Corollary 2.12 of [Dot+20] shows that we could also think of the D_2-spectrum THR as a two-sided bar construction using a multiplicative double coset formula. This result was extended by Angelini-Knoll, Gerhardt, and Hill in [AGH21] to a multiplicative double coset formula for THR as a D_{2m}-spectrum. In this section, we use these results to construct a Real Bökstedt spectral sequence converging to the D_{2m}-equivariant homology of THR as a D_{2m}-spectrum.

Definition 5.2.1. Let R be a unital ring spectrum with a right module M and left module N. The *two-sided bar construction* $B_\bullet(M, R, N)$ is a simplicial spectrum with k-simplices

$$B_k(M, R, N) = M \wedge R^{\wedge k} \wedge N.$$

The face and degeneracy maps are given by

$$d_i = \begin{cases} \phi \wedge id_R^{\wedge(k-1)} \wedge id_N & i = 0 \\ id_M \wedge id_R^{\wedge(i-1)} \wedge \mu \wedge id_R^{\wedge(k-i-1)} \wedge id_N & 0 < i < k \\ id_M \wedge id_R^{\wedge(k-1)} \wedge \psi & i = k. \end{cases}$$

$$s_i = id_M \wedge id_R^{\wedge i} \wedge \eta \wedge id_R^{\wedge(k-i)} \wedge id_N.$$

Here, ϕ and ψ denotes the right and left module actions of M and N respectively. The map μ is the ring spectrum product and η is the unit map $\mathbb{S} \to R$.

Recall that a ring spectrum with anti-involution is an E_σ-ring as in Definition 4.5.8. This interpretation yields a right $N_e^{D_{2m}} \iota_e^* A$-module structure on $N_{D_2}^{D_{2m}} A$ and a left module structure on $N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} c_\zeta A$ in spectra via the same compositions of maps which we saw for Mackey functors in Definition 4.3.6. The following lemma gives us a characterization of the D_{2m}-spectrum THR via a two-sided bar construction.

Lemma 5.2.2 ([AGH21], Theorem 5.9)**.** *Let A be a flat ring spectrum with anti-involution. There is a stable equivalence of D_{2m}-spectra*

$$\iota^*_{D_{2m}} N^{O(2)}_{D_2} A \simeq N^{D_{2m}}_{D_2} A \wedge^{\mathbb{L}}_{N^{D_{2m}}_{e} \iota^*_e A} N^{D_{2m}}_{\zeta D_2 \zeta^{-1}} c_\zeta A.$$

In particular, this gives an equivalence

$$\iota^*_{D_{2m}} \operatorname{THR}(A) \simeq |B_\bullet(N^{D_{2m}}_{D_2} A, N^{D_{2m}}_{e} \iota^*_e A, N^{D_{2m}}_{\zeta D_2 \zeta^{-1}} c_\zeta A)|.$$

This result generalizes Corollary 2.12 of Dotto, Moi, Patchkoria, and Reeh in [Dot+20], which proves the stable equivalence of spectra in the case of $m = 1$.

Remark 5.2.3. In Proposition 4.5.9 we showed that the $\underline{RO}(D_2)$-graded homotopy of a ring spectrum with anti-involution A forms an $\underline{RO}(D_2)$-graded discrete E_σ-ring. Note that the equivariant $E_\circledast$-homology of a ring spectrum with anti-involution is also an an $\underline{RO}(D_2)$-graded discrete E_σ-ring. This is because the $E_\circledast$-homology is defined by taking the $\underline{RO}(D_2)$-graded homotopy of the spectrum $A \wedge E$ and this product is also a ring spectrum with anti-involution.

The main result of this chapter is the following theorem.

Theorem 5.2.4. *Let A be a ring spectrum with anti-involution and let E be a commutative D_{2m}-ring spectrum. If $\underline{E}_\circledast(N^{D_{2m}}_{D_2} A)$ and $\underline{E}_\circledast(N^{D_{2m}}_{e} \iota^*_e A)$ are both flat as modules over $\underline{E}_\circledast$ and if A has free $(\iota^*_{D_2} E)$- and $\iota^*_e E$-homology then there is a Real Bökstedt spectral sequence of the form*

$$E^2_{*,\circledast} = \underline{\operatorname{HR}}^{D_{2m}}_*(\underline{(\iota^*_{D_2} E)}_\circledast(A)) \Rightarrow \underline{E}_\circledast(\iota^*_{D_{2m}} \operatorname{THR}(A)).$$

In the above, we let $\underline{E}_\circledast$ denote the $\underline{RO}(D_{2m})$-graded homotopy Mackey functor $\underline{\pi}_\circledast(E)$. Before presenting a proof, we recall Theorem 2.3.1 which gave a general construction of a spectral sequence associated to a simplicial spectrum, $X_\bullet$. We note that proof of this result also goes through equivariantly.

Proposition 5.2.5. *Let E be a commutative G-spectrum and $X_\bullet$ be a proper simplicial G-spectrum. There exists a strongly convergent spectral sequence*

$$E^2_{*,\circledast} = H_*(\underline{E}_\circledast(X_\bullet)) \Rightarrow E_{*+\circledast}(|X_\bullet|).$$

As in the non-equivariant case, the simplicial filtration

$$...F_{p-1} \subset F_p \subset F_{p+1} \subset ... \subset X_\bullet$$

gives rise to such a spectral sequence. Here, the E^1-page is given by

$$E^1_{p,\circledast} = \underline{E}_\circledast(F_p/F_{p-1}).$$

Since the multiplicative double coset formula for THR is a simplicial spectrum, we can apply this result to construct the Real Bökstedt spectral sequence.

Proof of Theorem 5.2.4. We begin by taking E-homology of the double bar construction from the multiplicative double coset formula for $\iota^*_{D_{2m}}\,\mathrm{THR}(A)$. At the pth level we see that $E^1_{p,\circledast}$ takes the form

$$\begin{aligned}\underline{E}_\circledast(N_{D_2}^{D_{2m}}A \wedge \overbrace{N_e^{D_{2m}}\iota_e^*A \wedge ... \wedge N_e^{D_{2m}}\iota_e^*A}^{p} \wedge N_{\zeta D_2\zeta^{-1}}^{D_{2m}}c_\zeta A) &= \\ \underline{\pi}_\circledast(N_{D_2}^{D_{2m}}A \wedge N_e^{D_{2m}}\iota_e^*A \wedge ... \wedge N_e^{D_{2m}}\iota_e^*A \wedge N_{\zeta D_2\zeta^{-1}}^{D_{2m}}c_\zeta A \wedge E) &\cong \\ \underline{\pi}_\circledast(N_{D_2}^{D_{2m}}A \wedge E \wedge_E N_e^{D_{2m}}\iota_e^*A \wedge E \wedge_E ... \wedge_E N_e^{D_{2m}}\iota_e^*A \wedge E \wedge_E N_{\zeta D_2\zeta^{-1}}^{D_{2m}}c_\zeta A \wedge E)&\end{aligned}$$

where the final isomorphism is that of Remark 2.2.5.

By Corollary 3.20 of [Hil22] and the flatness of $\underline{E}_\circledast(N_{D_2}^{D_{2m}}A)$ and $\underline{E}_\circledast(N_e^{D_{2m}}\iota_e^*A)$ there is a Künneth isomorphism which allows us to split this as a product of homotopy Mackey functors,

$$\begin{aligned}\underline{\pi}_\circledast(N_{D_2}^{D_{2m}}A \wedge E \wedge_E N_e^{D_{2m}}\iota_e^*A \wedge E \wedge_E ... \wedge_E N_e^{D_{2m}}\iota_e^*A \wedge E \wedge_E N_{\zeta D_2\zeta^{-1}}^{D_{2m}}c_\zeta A \wedge E) &\cong \\ \underline{\pi}_\circledast(N_{D_2}^{D_{2m}}A \wedge E) \mathop{\square}_{\underline{E}_\circledast} \underline{\pi}_\circledast(N_e^{D_{2m}}\iota_e^*A \wedge E) \mathop{\square}_{\underline{E}_\circledast} ... \mathop{\square}_{\underline{E}_\circledast} \underline{\pi}_\circledast(N_e^{D_{2m}}\iota_e^*A \wedge E) \mathop{\square}_{\underline{E}_\circledast} \underline{\pi}_\circledast(N_{\zeta D_2\zeta^{-1}}^{D_{2m}}c_\zeta A \wedge E) &= \\ \underline{E}_\circledast(N_{D_2}^{D_{2m}}A) \mathop{\square}_{\underline{E}_\circledast} \underline{E}_\circledast(N_e^{D_{2m}}\iota_e^*A) \mathop{\square}_{\underline{E}_\circledast} ... \mathop{\square}_{\underline{E}_\circledast} \underline{E}_\circledast(N_e^{D_{2m}}\iota_e^*A) \mathop{\square}_{\underline{E}_\circledast} \underline{E}_\circledast(N_{\zeta D_2\zeta^{-1}}^{D_{2m}}c_\zeta A).&\end{aligned}$$

Recall from Corollary 5.1.3 that when A has free $\iota_e^* E$-homology as we assumed, there is an isomorphism

$$\underline{E}_{\circledast}(N_e^{D_{2m}} \iota_e^* A) \cong N_e^{D_{2m}} \iota_e^* (\underline{(\iota_{D_2}^* E)}_{\circledast}(A)) \tag{5.1}$$

Furthermore, the hypothesis that A has free $(\iota_{D_2}^* E)$-homology in conjunction with Lemma 5.1.2, yields an isomorphism

$$\underline{E}_{\circledast}(N_{D_2}^{D_{2m}} A) \cong N_{D_2}^{D_{2m}} (\underline{(\iota_{D_2}^* E)}_{\circledast}(A)). \tag{5.2}$$

Finally, since $c_\zeta A$ is just viewing A as a D_2-spectrum with a different generator for D_2, if A has free $(\iota_{D_2}^* E)$-homology, then $c_\zeta A$ has free $(\iota_{\zeta D_2 \zeta^{-1}}^* E)$-homology under this equivalence of categories in spectra. Thus we have an isomorphism,

$$\underline{E}_{\circledast}(N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} c_\zeta A) \cong N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} c_\zeta (\underline{(\iota_{D_2}^* E)}_{\circledast}(A)) \tag{5.3}$$

The isomorphisms in 5.1, 5.2, and 5.3 allow us to conclude that at level p, the E^2-page of the spectral sequence is isomorphic to

$$N_{D_2}^{D_{2m}} (\underline{(\iota_{D_2}^* E)}_{\circledast}(A)) \underset{\underline{E}_{\circledast}}{\Box} N_e^{D_{2m}} \iota_e^* (\underline{(\iota_{D_2}^* E)}_{\circledast}(A))^{\Box_{\underline{E}_{\circledast}} p} \underset{\underline{E}_{\circledast}}{\Box} N_{\zeta D_2 \zeta^{-1}}^{D_{2m}} c_\zeta (\underline{(\iota_{D_2}^* E)}_{\circledast}(A)).$$

This is precisely the pth simplicial level of the two-sided bar construction which defines the Real D_{2m}-Hochschild homology of the $\underline{RO}(D_2)$-graded equivariant E-homology of A:

$$\underline{\mathrm{HR}}_{\bullet}^{D_{2m}} (\underline{(\iota_{D_2}^* E)}_{\circledast}(A)).$$

At each simplicial level, we have identified the E_1-page of the spectral sequence with the complex that computed Real Hochschild homology. A diagram chase in the style of the one on p. 111 of [May06] shows that the d_1 differential of the spectral sequence agrees with the differential in $\underline{\mathrm{HR}}$ and hence on the E_2-page we get Real Hochschild homology as in the statement. □

A particular case of interest occurs when $E = H\underline{\mathbb{F}}_2$ and $m = 1$; the Real Bökstedt spectral sequence allows us to compute the D_2-equivariant homology of $\mathrm{THR}(A)$ as a D_2-spectrum.

Note that in this case we do not need an additional flatness assumption about $\underline{E}_{\circledast}(N_{D_2}^{D_2} A)$ since this norm is trivial. Similarly, we do not require freeness assumptions since we are in the case described by Remark 5.1.4. Thus we have the following corollary:

Corollary 5.2.6. *Let A be a ring spectrum with anti-involution and such that $\underline{H}_{\circledast}^{D_2}(A;\underline{\mathbb{F}}_2)$ and $\underline{H}_{\circledast}^{D_2}(N_e^{D_2}\iota_e^* A;\underline{\mathbb{F}}_2)$ are flat as modules over $H\underline{\mathbb{F}}_{2\circledast}$. Then there is a Real Bökstedt spectral sequence*

$$E^2_{*,\circledast} = \underline{\mathrm{HR}}_*^{D_2}(\underline{H}_{\circledast}^{D_2}(A;\underline{\mathbb{F}}_2)) \Rightarrow \underline{H}_{\circledast}^{D_2}(\iota_{D_2}^* \mathrm{THR}(A);\underline{\mathbb{F}}_2).$$

Finally, we remark that there are interesting equivariant spectra for which these flatness conditions will hold. One example is the Real bordism spectrum, $MU_{\mathbb{R}}$. This is a D_2-equivariant ring spectrum first studied by Landweber [Lan68] and Fuji [Fuj76] which played an important role in work of Hill, Hopkins, and Ravenel on the Kervaire invariant one problem [HHR16]. In particular, we have that the $H\underline{\mathbb{F}}_2$-homology of $MU_{\mathbb{R}}$ is polynomial over $H\underline{\mathbb{F}}_2$ and hence satisfies the flatness condition required to use the Real Bökstedt spectral sequence.

5.3 Extensions to twisted topological Hochschild homology

In [Ada+22], the authors construct a Bökstedt spectral sequence for a different flavor of equivariant topological Hochschild homology which takes inputs in G-ring spectra for G a finite subgroup of S^1. Adamyk, Gerhardt, Hess, Klang, and Kong proved the existence of the following Bökstedt spectral sequence which converges to the $RO(G)$-graded G-equivariant homology of G-twisted THH:

Lemma 5.3.1 ([Ada+22], Theorem 4.2.7)**.** *Let $G < S^1$ be a finite subgroup and $g = e^{2\pi i/|G|}$ a generator of G. Let R be a G-ring spectrum and E a commutative G-ring spectrum such that g acts trivially on E. If $\underline{E}_\star(R)$ is flat over $\underline{E}_\star$ then there is a twisted Bökstedt spectral sequence*

$$E^2_{s,\star} = \underline{\mathrm{HH}}_s^{\underline{E}_\star,G}(\underline{E}_\star(R)) \Rightarrow \underline{E}_{s+\star}(\iota_G^* \mathrm{THH}_G(R)).$$

Remark 5.3.2. If g acts trivially on E, Lemma 4.2.5 of [Ada+22] shows that there is an isomorphism of left $\underline{E}_\star(R)$-modules, ${}^g\underline{E}_\star(R) \cong \underline{E}_\star({}^gR)$. The proof of this lemma shows there is an isomorphism at the level of spectra, $E \wedge {}^g R \cong {}^g(E \wedge R)$, hence the result also goes through when passing to $\underline{RO}(G)$-graded homotopy.

We note that G-twisted THH is an S^1-spectrum. In the preceding result, the authors considered the equivariant homology of the G-restriction of THH_G. However we could also consider the spectrum given by taking the G-restriction of H-twisted THH when H is a subgroup of G. In doing so, we find there is an equivariant Bökstedt spectral sequence which computes the G-equivariant homology of $\iota_G^* \mathrm{THH}_H(R)$ and it has on its E^2-page the relative theory of Hochschild homology for Green functors given in Definition 4.4.5. Recall that the relative construction of the twisted cyclic nerve involved taking equivariant norms N_H^G. Thus once again, we require the use of an $\underline{RO}(G)$-grading to ensure that our grading scheme respects the norm.

Theorem 5.3.3. *Let $H \leq G$ be finite subgroups of S^1 and let $g = e^{2\pi i/|G|}$ be a generator of G. Let R be an H-ring spectrum and E a commutative G-ring spectrum. Assume that g acts trivially on E and that $\underline{E}_\circledast(N_H^G R)$ is flat as a module over $\underline{E}_\circledast$. If R has $(\iota_H^* E)$-free homology, then there is a relative twisted Bökstedt spectral sequence*

$$E^2_{s,\circledast} = \underline{\mathrm{HH}}_H^G((\underline{\iota_H^* E})_\circledast(R))_s \Rightarrow \underline{E}_{s+\circledast}(\iota_G^* \mathrm{THH}_H(R)).$$

Proof. By Proposition 5.2.5 we have a spectral sequence

$$E^2_{*,\circledast} = H_*(\underline{E}_\circledast(B^{cy}_\bullet(N_H^G R; {}^g N_H^G R))) \Rightarrow \underline{E}_{s+\circledast}(|B^{cy}_\bullet(N_H^G R; {}^g N_H^G R)|).$$

On the right hand side, this is the twisted cyclic bar construction which defines $\iota_G^* \mathrm{THH}_H(R)$. We wish to identify the E^2-page with $\underline{RO}(G)$-graded relative Hochschild homology for Green functors.

We apply the homology functor $\underline{E}_\circledast(-)$ level-wise to the relative twisted cyclic bar construction. At the nth level this gives

$$\begin{aligned}\underline{E}_\circledast({}^gN_H^GR\wedge \overbrace{N_H^GR\wedge\ldots\wedge N_H^GR}^{n}) &=\\ \underline{\pi}_\circledast^G({}^gN_H^GR\wedge N_H^GR\wedge\ldots\wedge N_H^GR\wedge E) &\cong\\ \underline{\pi}_\circledast^G({}^gN_H^GR\wedge E\wedge_E N_H^GR\wedge E\wedge_E\ldots\wedge E\wedge_E N_H^GR\wedge E) &\cong\\ \underline{\pi}_\circledast^G({}^gN_H^GR\wedge E)\mathbin{\square_{\underline{E}_\circledast}}\underline{\pi}_\circledast^G(N_H^GR\wedge E)\mathbin{\square_{\underline{E}_\circledast}}\ldots\mathbin{\square_{\underline{E}_\circledast}}\underline{\pi}_\circledast^G(N_H^GR\wedge E) &=\\ \underline{E}_\circledast({}^gN_H^GR)\mathbin{\square_{\underline{E}_\circledast}}\underline{E}_\circledast(N_H^GR)\mathbin{\square_{\underline{E}_\circledast}}\ldots\mathbin{\square_{\underline{E}_\circledast}}\underline{E}_\circledast(N_H^GR).&\end{aligned}$$

Here the first isomorphism is that given by 2.2.5. The flatness of $\underline{E}_\circledast(N_H^GR)$ as a module over $\underline{E}_\circledast$ and an application of the Künneth theorem yields the second isomorphism in the above.

By the freeness assumption in the statement of the theorem and by Lemma 5.1.2, we get an isomorphism

$$\underline{E}_\circledast(N_H^GR)\cong N_H^G(\underline{(\iota_H^*E)}_\circledast R). \tag{5.4}$$

Combining this isomorphism with the equivalence in Remark 5.3.2 further yields

$$\underline{E}_\circledast({}^gN_H^GR)\cong {}^g\underline{E}_\circledast(N_H^GR)\cong {}^gN_H^G((\underline{\iota_H^*E})_\circledast R). \tag{5.5}$$

The isomorphisms in 5.4 and 5.5 allow us to conclude that $E^1_{n,\circledast}$ is isomorphic to the following:

$${}^gN_H^G((\underline{\iota_H^*E})_\circledast R)\mathbin{\square_{\underline{E}_\circledast}}\overbrace{N_H^G((\underline{\iota_H^*E})_\circledast R)\mathbin{\square_{\underline{E}_\circledast}}\ldots\mathbin{\square_{\underline{E}_\circledast}}N_H^G((\underline{\iota_H^*E})_\circledast R)}^{n}.$$

Term-wise, this is precisely the nth level of the twisted cyclic bar construction which defines $\underline{\mathrm{HH}}_H^G(\underline{(\iota_H^*E)}_\circledast(R))$ as given in Definition 4.5.12. A diagram chase in the style of the one on p. 111 of [May06] shows that d_1 differential of the spectral sequence agrees with the differential in $\underline{\mathrm{HH}}_H^G$.

We thus conclude that the relative twisted Bökstedt spectral sequence takes the form

$$\underline{E}^2_{s,\circledast}=\underline{\mathrm{HH}}_H^G(\underline{(\iota_H^*E)}_\circledast(R))_s\Rightarrow\underline{E}_{s+\circledast}(\iota_G^*\,\mathrm{THH}_H(R)).$$

□

CHAPTER 6

REAL ALGEBRAIC STRUCTURES

Bökstedt spectral sequence calculations are often quite complex, therefore identifying any additional algebraic structures present in the spectral sequence can provide a computational foothold. An exploration of the algebraic structure in topological Hochschild homology was undertaken by McClure, Swanzel, and Vogt (see [MSV97]) and by Angeltveit and Rognes, who showed in [AR05] that for A a commutative ring spectrum, $\mathrm{THH}(A)$ is an A-Hopf algebra in the stable homotopy category by constructing simplicial algebraic structure maps. Since the spectral sequence construction we recalled in Theorem 2.3.2 utilized the simplicial filtration on THH, they showed that this Hopf algebra structure lifts to the Bökstedt spectral sequence. In this chapter, we utilize techniques analogous to those of Angeltveit and Rognes to show that $\mathrm{THR}(A)$ is a Hopf algebroid in the D_2-equivariant stable homotopy category when A is a commutative ring spectrum with anti-involution. Section 6.1 recalls the argument that there is a Hopf algebra structure on THH, beginning with a characterization of THH as the tensor of a ring spectrum with a simplicial circle in the commutative case. In Section 6.2, we translate this argument to the Real equivariant setting to show that $\mathrm{THR}(A)$ has a Hopf algebroid structure.

Since our aim is to recall the existence of a Hopf algebra structure in THH and identify a similar structure in THR, we begin this chapter by recalling the definition of a Hopf algebra in spectra.

Definition 6.0.1. Let R be a commutative ring spectrum. A unital, associative *R-algebra* is an R-module spectrum A which has a unit map $\eta : R \to A$ and a product map $\mu : A \wedge_R A \to A$, subject to the condition that the following unitality and associativity diagrams commute. All smash products are taken over R, though we omit this from the notation.

$$\begin{array}{ccccc} R\wedge A & \xrightarrow{\eta\wedge id} & A\wedge A & \xleftarrow{id\wedge\eta} & A\wedge R \\ & {\scriptstyle\cong}\searrow & \downarrow{\scriptstyle\mu} & \swarrow{\scriptstyle\cong} & \\ & & A & & \end{array} \tag{6.1}$$

$$\begin{array}{ccc} A \wedge A \wedge A & \xrightarrow{\mu \wedge id} & A \wedge A \\ {\scriptstyle id \wedge \mu}\big\downarrow & & \big\downarrow{\scriptstyle \mu} \\ A \wedge A & \xrightarrow[\mu]{} & A \end{array} \tag{6.2}$$

If A is further endowed with a counit map $\varepsilon : A \to R$ and a coassociative, counital coproduct $\delta : A \to A \wedge A$ subject to the compatibility relations depicted in the diagrams below, we say that A is an *R-bialgebra*.

$$\begin{array}{ccccc} A \wedge A & \xleftarrow{\delta} & A & \xrightarrow{\delta} & A \wedge A \\ & {\scriptstyle \varepsilon \wedge id}\searrow & {\scriptstyle id}\big\downarrow & \swarrow{\scriptstyle id \wedge \varepsilon} & \\ & & R \wedge A \cong A \cong A \wedge R & & \end{array} \tag{6.3}$$

$$\begin{array}{ccc} A & \xrightarrow{\delta} & A \wedge A \\ {\scriptstyle \delta}\big\downarrow & & \big\downarrow{\scriptstyle id \wedge \delta} \\ A \wedge A & \xrightarrow[\delta \wedge id]{} & A \wedge A \wedge A \end{array} \tag{6.4}$$

$$\begin{array}{ccc} R & \xrightarrow{id} & R \\ & {\scriptstyle \eta}\searrow \quad \nearrow{\scriptstyle \varepsilon} & \\ & A & \end{array} \tag{6.5}$$

$$\begin{array}{ccc} R & \xrightarrow{\cong} & R \wedge R \\ {\scriptstyle \eta}\big\downarrow & & \big\downarrow{\scriptstyle \eta \wedge \eta} \\ A & \xrightarrow[\delta]{} & A \wedge A \end{array} \tag{6.6}$$

$$\begin{array}{ccc} A \wedge A & \xrightarrow{\mu} & A \\ {\scriptstyle \varepsilon \wedge \varepsilon}\big\downarrow & & \big\downarrow{\scriptstyle \varepsilon} \\ R \wedge R & \xrightarrow[\cong]{} & R \end{array} \tag{6.7}$$

$$\begin{array}{ccccc} A \wedge A & \xrightarrow{\mu} & A & \xrightarrow{\delta} & A \wedge A \\ {\scriptstyle \delta \wedge \delta}\big\downarrow & & & & \big\uparrow{\scriptstyle \mu \wedge \mu} \\ A \wedge A \wedge A \wedge A & & \xrightarrow[id \wedge \tau \wedge id]{} & & A \wedge A \wedge A \wedge A \end{array} \tag{6.8}$$

Here, τ denotes the flip map which permutes two factors.

Finally, if a bialgebra A is equipped with an antipode $\chi : A \to A$ such that the following diagram commutes, A is called an *R-Hopf algebra*.

$$\begin{array}{ccccc} A \wedge A & & \xrightarrow{\chi \wedge id} & & A \wedge A \\ \delta \uparrow & & & & \downarrow \mu \\ A & \xrightarrow{\varepsilon} & R & \xrightarrow{\eta} & A \\ \delta \downarrow & & & & \uparrow \mu \\ A \wedge A & & \xrightarrow[id \wedge \chi]{} & & A \wedge A \end{array} \tag{6.9}$$

These are the types of structures we wish to identify in THR and in the Real Bökstedt spectral sequence. We begin in Section 6.1 by recalling how one endows classical topological Hochschild homology with a Hopf algebra structure.

6.1 Topological Hochschild homology is a Hopf algebra

In Definition 2.2.6, we presented topological Hochschild homology as the geometric realization of a cyclic bar construction. To define algebraic structure maps on THH, a different perspective proves useful in which THH(A) may be thought of as the tensor of A with the simplicial circle when A is a commutative ring spectrum. The strategy for endowing THH(A) with a Hopf algebra structure is to first define structure maps on simplicial circles and then lift those to THH. We take a very similar approach to produce the algebraic structure maps on THR in Section 6.2, so we take the opportunity in this section to reproduce the arguments for classical THH with detail in order to illustrate the proof technique.

To begin, recall that the standard n-simplex Δ^n consists of the convex hull of points $(0, ..., 0, 1, 0, ..., 0) \in \mathbb{R}^{n+1}$.

Definition 6.1.1 ([Lod13])**.** The standard model for the simplicial circle $S^1_\bullet$ is a quotient of the 1-simplex $\Delta^1/\partial\Delta^1$. Here, Δ^1 is the simplicial object $\Delta^1_n = \{x_0, ..., x_{n+1}\}$, with face maps defined by

$$d_j(x_i) = \begin{cases} x_i & \text{if } i \leq j \\ x_{i-1} & \text{if } i > j \end{cases}$$

and degeneracy maps defined by

$$s_j(x_i) = \begin{cases} x_i & \text{if } i \le j \\ x_{i+1} & \text{if } i > j. \end{cases}$$

The quotient $\Delta^1/\partial\Delta^1$ identifies x_0 with x_{n+1}.

Remark 6.1.2. By examining the images of the degeneracy maps s_i, we see that the only non-degenerate simplices are $x_0 \in \Delta^1_0$ and $x_1 \in \Delta^1_1$. Thus, upon geometric realization we recognize the familiar model of a circle with a single 0-cell and a single 1-cell. This model is depicted in Figure 6.1.

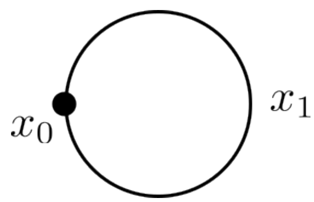


Figure 6.1 The simplicial circle $S^1_\bullet$.

For a commutative ring spectrum A, a direct, level-wise association of the cyclic bar construction on A with the tensor product of A with $S^1_\bullet$ yields the following isomorphism. For an explanation of what it means to tensor a spectrum with a simplicial object we direct the reader to Section 3 of [AR05].

Proposition 6.1.3 ([AR05])**.** *Let A be commutative ring spectrum. There is an isomorphism of simplicial ring spectra*

$$\mathrm{THH}(A) \cong A \otimes S^1_\bullet.$$

Remark 6.1.4. The above isomorphism was also proven in the topological (non-simplicial) case in Theorem B of [MSV97]. In this chapter we follow the proof techniques of Angeltveit and Rognes [AR05] so that in future work we may lift the algebraic structure of $\mathrm{THR}(A)$ to the Real Bökstedt spectral sequence.

To endow THH(A) with an A-Hopf algebra structure one starts by constructing maps on simplicial circles and then tensoring those simplicial maps with A to obtain maps on THH(A). We present this argument contained in [AR05] in three steps, beginning with the existence of an A-algebra structure on THH(A).

Lemma 6.1.5. *Let A be a commutative ring spectrum. The topological Hochschild homology of A,* THH(A)*, is an associative, unital A-algebra.*

Proof. The algebraic structure maps arise from maps of simplicial circles. We have a unit map

$$\eta : A \to \mathrm{THH}(A)$$

induced by the map to the basepoint, $\eta : * \to S^1_\bullet$. Applying the functor $A \otimes (-)$ to the map depicted in Figure 6.2 yields the desired map η.

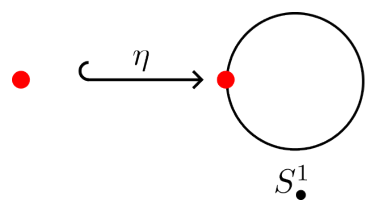


Figure 6.2 The simplicial map inducing a unit on THH.

In a similar fashion we obtain a product map

$$\mu : \mathrm{THH}(A) \wedge_A \mathrm{THH}(A) \to \mathrm{THH}(A)$$

from the simplicial map which folds one copy of $S^1_\bullet$ onto the other, as shown in Figure 6.3.

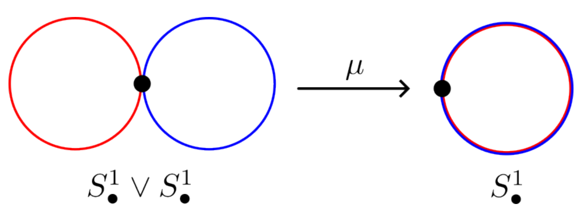


Figure 6.3 The simplicial map inducing a product on THH.

To verify unitality and associativity we check that diagrams of simplicial circles in the style of Diagrams 6.1 and 6.2 in Definition 6.0.1 commute. The diagram demonstrating associativity is given in Figure 6.4.

In this chapter we will not include all simplicial commutative diagrams but we provide some as illustrative examples. In Figure 6.4, we denote the association of the blue and green copies of $S^1_\bullet$ via the fold along the top by a thickened circle colored with green and blue. We use a similar color coding in the remaining figures of this chapter to keep track of which simplicial objects are associated in the diagrams.

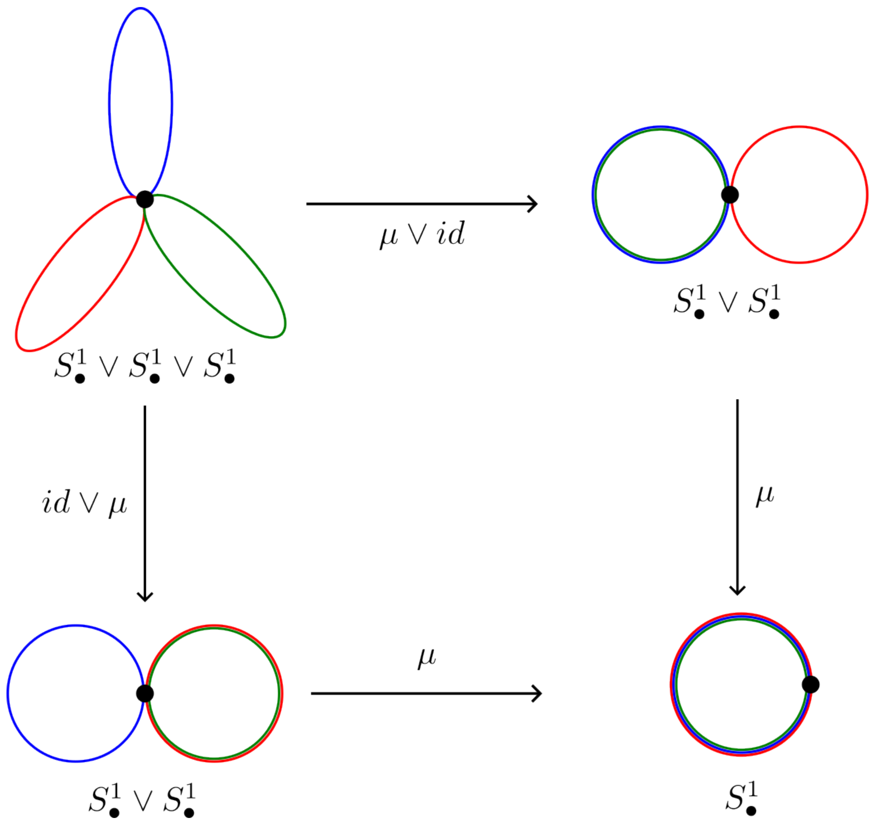


Figure 6.4 Simplicial commutative associativity diagram for THH.

If we tensor Figure 6.4 with A, we obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{THH}(A) \wedge_A \mathrm{THH}(A) \wedge_A \mathrm{THH}(A) & \xrightarrow{\mu \wedge id} & \mathrm{THH}(A) \wedge_A \mathrm{THH}(A) \\ {\scriptstyle id\wedge\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ \mathrm{THH}(A) \wedge_A \mathrm{THH}(A) & \xrightarrow[\mu]{} & \mathrm{THH}(A) \end{array}$$

demonstrating that $\mathrm{THH}(A)$ is an associative algebra. A similar check shows that η satisfies the unitality diagram hence $\mathrm{THH}(A)$ is a unital, associative A-algebra. □

Topological Hochschild homology can also be endowed with a coalgebra structure, however the definition of a simplicial coproduct map is not as straightforward as the one used to define the product. A coproduct on $\mathrm{THH}(A)$ is a map $\mathrm{THH}(A) \to \mathrm{THH}(A) \wedge_A \mathrm{THH}(A)$. The natural topological map which creates two copies of S^1 from one is a pinch map; however, this map is not simplicial when we use our standard simplicial model of the circle. To remedy this, we instead define a coproduct on a different model of the simplicial circle, which, upon tensoring with A, yields a model of THH which is homotopy equivalent to the model presented above.

Definition 6.1.6 ([AR05], Section 3)**.** The *double model* of $S^1_\bullet$, denoted by $dS^1_\bullet$ is the simplicial set

$$dS^1_\bullet = (\Delta^1 \sqcup \Delta^1) \sqcup_{\partial\Delta^1 \sqcup \partial\Delta^1} \partial\Delta^1$$

and is depicted in Figure 6.5. We denote the tensor product with this model, $A \otimes dS^1_\bullet$ by $d\,\mathrm{THH}$.

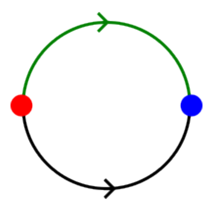


Figure 6.5 The double circle $dS^1_\bullet$.

In order to make use of this double model in the construction of a coproduct on THH, we must demonstrate that there is a homotopy equivalence $d\,\mathrm{THH}(A) \simeq \mathrm{THH}(A)$. We now recall the proof of this fact by Angeltveit-Rognes.

Lemma 6.1.7 ([AR05], Lemma 3.8)**.** *Let A be a commutative ring spectrum which is cofibrant as an $\mathbb{S}$-module. The double model $d\,\mathrm{THH}(A)$ is weakly equivalent to $\mathrm{THH}(A)$ via the simplicial collapse map $\pi : dS^1_\bullet \to S^1_\bullet$ which crushes the second copy of Δ^1 in the double circle.*

Proof. Consider the pushout diagram of simplicial sets given in Figure 6.6. In this diagram, the top map associates the two points and the left map includes them as the boundary of the 1-simplex.

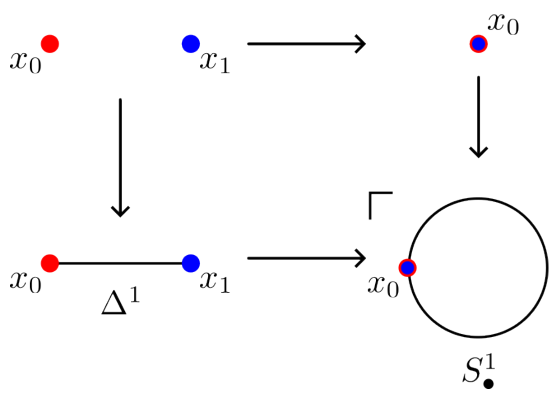


Figure 6.6 A pushout diagram which gives $S^1_\bullet$.

Since the tensor product preserves pushouts, tensoring this diagram with A gives

$$\begin{array}{ccc} A \wedge A & \longrightarrow & A \\ \downarrow & \ulcorner & \downarrow \\ B(A) & \longrightarrow & \mathrm{THH}(A) \end{array}$$

where $B(A)$ is the two-sided bar construction $B(A, A, A) = A \otimes \Delta^1$.

We could similarly consider the diagram in Figure 6.7 which shows $dS^1_\bullet$ as a pushout.

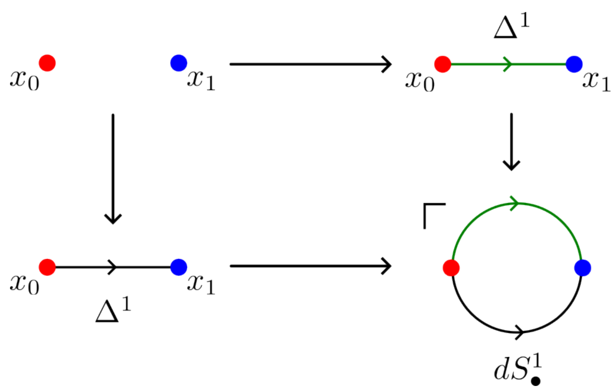


Figure 6.7 A pushout diagram which gives $dS^1_\bullet$.

The tensor of the diagram in Figure 6.7 with A gives a commutative diagram

$$\begin{array}{ccc} A \wedge A & \longrightarrow & B(A) \\ \downarrow & \ulcorner & \downarrow \\ B(A) & \longrightarrow & d\mathrm{THH}(A). \end{array}$$

There is a map of pushout diagrams

$$\begin{array}{ccccc} B(A) & \longleftarrow & A \wedge A & \longrightarrow & A \\ \| & & \| & & \uparrow \simeq \\ B(A) & \longleftarrow & A \wedge A & \longrightarrow & B(A) \end{array}$$

where the weak equivalence on the right is the augmentation from the two-sided bar construction to its right hand coefficients as in [Elm+97], IV.7.2. Since pushouts preserve weak equivalences, we have $d\,\mathrm{THH}(A) \simeq \mathrm{THH}(A)$ as desired. We note that geometrically, this weak equivalence is induced by the map $\pi : dS^1_\bullet \to S^1_\bullet$ which collapses the second copy of Δ^1 in the double circle to a point. We will refer to the homotopy between $d\,\mathrm{THH}(A)$ and $\mathrm{THH}(A)$ by π also. □

We can now use the double model to define a coalgebra structure on THH. In contrast to the algebra structure, we only have a coalgebra structure on $\mathrm{THH}(A)$ in the stable homotopy category since the coproduct map must factor through the weak equivalence described above.

Lemma 6.1.8 ([AR05])**.** *Let A be a commutative ring spectrum. Then* $\mathrm{THH}(A)$ *is a counital A-coalgebra in the stable homotopy category. Further, this coalgebra structure is compatible with the algebra structure so that* $\mathrm{THH}(A)$ *is in fact an A-bialgebra in the homotopy category.*

Proof. The counit

$$\epsilon : \mathrm{THH}(A) \to A$$

is induced by the simplicial collapse map from $S^1_\bullet$ to a point, as shown in Figure 6.8.

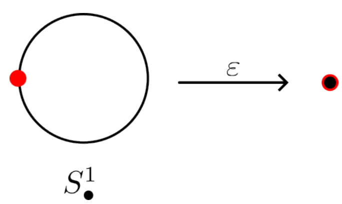


Figure 6.8 The simplicial map inducing a counit on THH.

To define the simplicial pinch map which induces the coproduct

$$\delta : \mathrm{THH}(A) \to \mathrm{THH}(A) \wedge_A \mathrm{THH}(A),$$

we use the double model of the circle. This map is shown in Figure 6.9. The map δ is thus induced by a composition of a simplicial coproduct $\delta' : dS^1_\bullet \to S^1_\bullet \vee S^1_\bullet$ and the homotopy equivalence π^{-1},

$$\mathrm{THH}(A) \xrightarrow{\pi^{-1}} d\,\mathrm{THH}(A) \xrightarrow{\delta'} \mathrm{THH}(A) \wedge_A \mathrm{THH}(A).$$

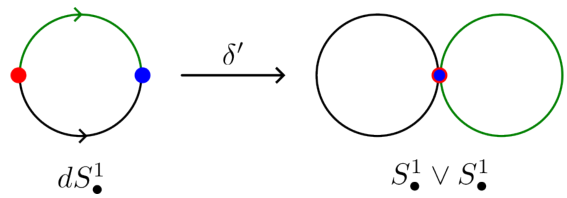


Figure 6.9 The simplicial map inducing a coproduct on THH.

A check that these maps of simplicial circles satisfy the bialgebra compatibility diagrams given in Definition 6.0.1 completes the proof. We omit this. □

Lemma 6.1.9 ([AR05], Theorem 3.9)**.** *If A is a commutative ring spectrum, then* $\mathrm{THH}(A)$ *is an A-Hopf algebra in the stable homotopy category.*

Proof. The double circle has an antipode map χ', depicted in Figure 6.10, which induces an antipodal map on THH via the composition

$$\chi : \mathrm{THH}(A) \xrightarrow{\pi^{-1}} d\,\mathrm{THH}(A) \xrightarrow{\chi'} d\,\mathrm{THH}(A) \xrightarrow{\pi} \mathrm{THH}(A).$$

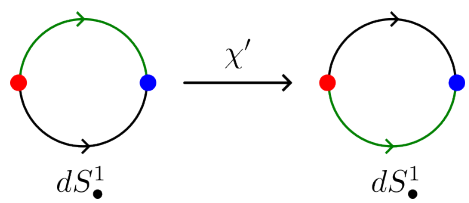


Figure 6.10 The simplicial map inducing an antipode on THH.

Equipped with this map and the results of Lemmas 6.1.5 and 6.1.8, all that remains to check is that the Hopf compatibility diagram in 6.9 commutes. We omit this part of the proof but direct the reader to the diagram in 3.10 of [AR05] for further details. □

6.2 Real topological Hochschild homology is a Hopf algebroid

In this section, we follow the technique of the proofs presented earlier in this chapter to determine the algebraic structure of $\mathrm{THR}(A)$ when A is a commutative ring spectrum with anti-involution. In particular, we define all algebraic structure maps as ones induced by maps of simplicial objects in anticipation of lifting the structure to the Real Bökstedt spectral sequence in future work.

The Hopf algebra structure on THH was induced by maps on simplicial circles since THH is a tensor product with S^1. Recall from Remark 4.2.7 that $\mathrm{THR}(A)$ is an $O(2)$-spectrum. For a nice class of ring spectra with anti-involution, we can recognize THR as a tensor with $O(2)$.

Definition 6.2.1. An orthogonal D_2-spectrum A indexed on a complete universe $\mathcal{U}$ is *well-pointed* if $A(V)$ is well pointed in Top^{D_2} for all finite dimensional orthogonal D_2-representations V. Further, we say a D_2-spectrum A is *very well-pointed* if it is well-pointed and the unit map $S^0 \to A(\mathbb{R}^0)$ is a Hurewicz cofibration in Top^{D_2}.

Proposition 6.2.2 ([AGH21], Proposition 4.9)**.** *Let A be a commutative D_2-ring spectrum which is very well pointed. Then there is a weak equivalence of D_2-spectra*

$$N_{D_2}^{O(2)} A \simeq A \otimes_{D_2} O(2).$$

Note that in the Real equivariant setting, our tensor product defining THR occurs over D_2. In our case, we utilize the fact that the category of commutative monoids in the category of orthogonal D_2-spectra is tensored over the category of D_2-sets (see Section 4.1 of [AGH21]). More generally, we have that G-spectra are tensored in G-sets which allows us to define the tensor product of a G-spectrum over G as a coequalizer.

Definition 6.2.3. Let A be a commutative G-ring spectrum and $X_\bullet$ a simplicial G-set. The *tensor product over G* of A with X is the coequalizer

$$A \otimes G \otimes X_\bullet \underset{id\otimes\gamma_2}{\overset{\gamma_1\otimes id}{\rightrightarrows}} A \otimes X_\bullet \longrightarrow A \otimes_G X_\bullet$$

where γ_1 is the G-action applied to A and γ_2 is the G-action on $X_\bullet$.

The standard simplicial model on $O(2)$ (see 6.3 of [Lod13], for instance) is the geometric realization of a simplicial complex of dihedral groups:

$$\cdots D_8 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; D_6 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; D_4 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; D_2.$$

Let the group D_{2m} be generated by an element ω of order 2 and by t, an element of order m with the usual dihedral relations. The face maps at simplicial level n, $d_i : D_{2(n+1)} \to D_{2n}$ are defined on the generator t by

$$d_i(t^j) = \begin{cases} t^j & j \leq i \\ t^{j-1} & j > i \end{cases}$$

$$d_n(t^j) = \begin{cases} t^j & j < n \\ 1 & j = n \end{cases}$$

The degeneracy maps on t, $s_i : D_{2(n+1)} \to D_{2(n+2)}$, are given by

$$s_i(t^j) = \begin{cases} t^j & j \leq i \\ t^{j+1} & j > i. \end{cases}$$

We further specify that these are D_2-equivariant maps so $d_i(\omega t^j) = \omega d_i(t^j)$ and $s_i(\omega t^j) = \omega s_i(t^j)$ which defines the face and degeneracy maps on the entire simplicial object.

To view $\mathrm{THR}(A)$ as the D_2-tensor product of A with $O(2)$, we actually require a different simplicial model of $O(2)$. We take the standard simplicial model and apply a Segal-Quillen subdivision as in Definition 4.1.4. In keeping with the conventions of [AGH21], we denote this subdivided $O(2)$ by $O(2)_\bullet$, regarded as a simplicial D_2-set.

One can check that most of the cells in this simplicial object are degenerate, thus upon geometric realization our model of $O(2)_\bullet$ can be depicted as two subdivided circles. Explicitly, the simplicial structure of $O(2)_\bullet$ at levels 0 and 1 is as follows. At simplicial level 0 it is the group

$$D_4 = \langle t_0, \omega \mid \; t_0^2 = 1 = \omega^2, \; t_0\omega = \omega t_0 \rangle = \{1, t_0, \omega, \omega t_0\}.$$

To emphasize that t is the generator of the group at the 0th level, we denote it by a subscript 0. At simplicial level 1 we have

$$D_8 = \langle t_1, \omega \mid \; t_1^4 = 1 = \omega^2, \; t_1\omega = \omega t_1^3 \rangle = \{1, t_1, t_1^2, t_1^3, \omega, \omega t_1, \omega t_1^2, \omega t_1^3\}.$$

The elements 1, t_1^2, ω, and ωt_1^2 in simplicial level 1 are in the image of the degeneracy maps. Thus, upon geometric realization to $O(2)_\bullet$, we only retain 1-cells indexed by $t_1, t_1^3, \omega t_1$, and ωt_1^3. This simplicial model $O(2)_\bullet$ is depicted in Figure 6.11. For ease of notion, we will cease

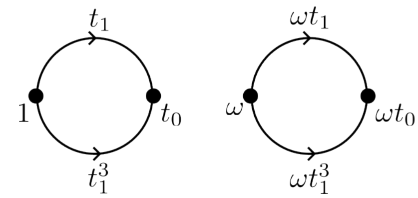


Figure 6.11 The simplicial model $O(2)_\bullet$.

to label every cell in the remaining figures of this chapter.

We obtain a D_4-action on $O(2)_\bullet$ where ω swaps the two circles (as seen in Figure 6.12) and t reflects within each circle (see Figure 6.13).

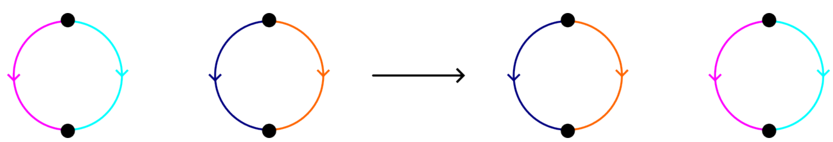


Figure 6.12 The action of ω on $O(2)_\bullet$.

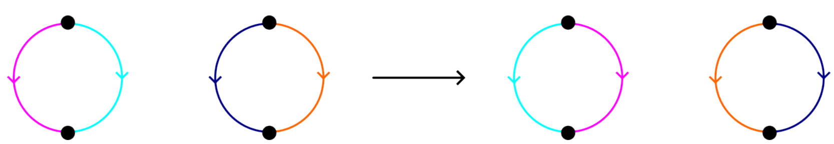


Figure 6.13 The action of t on $O(2)_\bullet$.

We now use this simplicial structure to define algebraic structure maps on THR.

Lemma 6.2.4. *Let A be a commutative D_2-ring spectrum. The Real topological Hochschild homology of A is a commutative A-algebra in D_2-spectra.*

Proof. As in the proof of Lemma 6.1.5, we define algebraic structure maps on $\mathrm{THR}(A)$ by defining maps on simplicial $O(2)_\bullet$. Note that in addition to constructing all maps so that they are simplicial, we must also ensure that the maps are D_2-equivariant.

We can define a unit map

$$\eta : A \to \mathrm{THR}(A)$$

which is induced by the inclusion of D_2 into $O(2)_\bullet$, as shown in Figure 6.14.

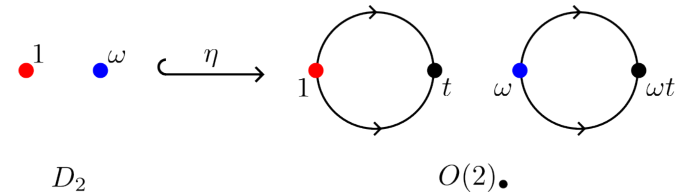
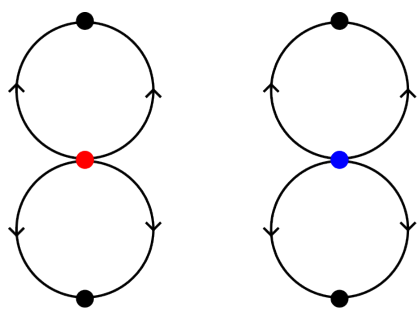


Figure 6.14 The simplicial map inducing a unit on THR.

Applying the functor $A \otimes_{D_2} (-)$ to the diagram yields our desired map.

We now wish to define a product map $\mu : \mathrm{THR}(A) \wedge_A \mathrm{THR}(A) \to \mathrm{THR}(A)$. Since we want this product to arise from a simplicial product, we must first recognize this smash product as a tensor of A with simplicial sets over D_2.



$O(2)_\bullet \vee_{D_2} O(2)_\bullet$

Figure 6.15 The simplicial wedge from which $\mathrm{THR}(A) \wedge_A \mathrm{THR}(A)$ arises.

We claim that $A \otimes_{D_2} (O(2)_\bullet \vee_{D_2} O(2)_\bullet)$ (depicted in Figure 6.15) is the smash product $\mathrm{THR}(A) \wedge_A \mathrm{THR}(A)$. To see this, note that the relative wedge $O(2)_\bullet \vee_{D_2} O(2)_\bullet$ is defined to be the pushout of the diagram

$$O(2)_\bullet \longleftarrow D_2 \longrightarrow O(2)_\bullet.$$

The tensor product in D_2-spectra preserves pushouts so upon applying the functor $A \otimes_{D_2} (-)$ we obtain a diagram

$$A \otimes_{D_2} O(2)_\bullet \longleftarrow A \otimes_{D_2} D_2 \longrightarrow A \otimes_{D_2} O(2)_\bullet$$

whose pushout is $A \otimes_{D_2} (O(2)_\bullet \vee_{D_2} O(2)_\bullet)$. By identifying $A \otimes_{D_2} O(2)_\bullet$ as $\mathrm{THR}(A)$ and $A \otimes_{D_2} D_2$ as A, we see this is the diagram

$$\mathrm{THR}(A) \longleftarrow A \longrightarrow \mathrm{THR}(A).$$

The pushout of this diagram defines the relative smash product $\mathrm{THR}(A) \wedge_A \mathrm{THR}(A)$ so we have identified $A \otimes_{D_2} (O(2)_\bullet \vee_{D_2} O(2)_\bullet)$.

Now we may define a product map

$$\mu : \mathrm{THR}(A) \wedge_A \mathrm{THR}(A) \to \mathrm{THR}(A)$$

which is induced by the map on simplicial copies of $O(2)$ that folds one copy of $O(2)$ onto the other, as depicted in Figure 6.16.

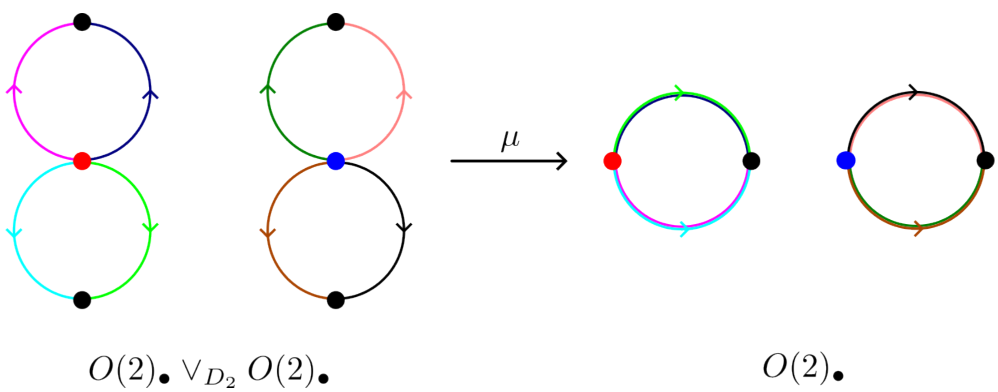


Figure 6.16 The simplicial map inducing a product on THR.

We note that both the product and unit are equivariant with respect to the swapping action on the circles and the action which reflects within each circle. We omit the pictures, but one can check that these maps satisfy commutative diagrams of simplicial sets for unitality and associativity analogous to those presented in Definition 6.0.1 in order to show that $\mathrm{THR}(A)$ is a unital, associative A-algebra. □

In the preceding proof we defined a unit map on $O(2)_\bullet$ by including D_2 as the points 1 and ω of $O(2)_\bullet$. However, our simplicial model of $O(2)$ also includes another pair of points that one could consider as the base points. The presence of two possible non-equivalent unit maps suggests that rather than inheriting a Hopf algebra structure like THH, Real topological Hochschild homology has the structure of a more general object called a Hopf algebroid. We now recall the definition of a Hopf algebroid from algebra.

Definition 6.2.5 ([Rav03], Definition A1.1.1)**.** A *Hopf algebroid* over a commutative ring k is a pair of commutative k-algebras (A, R) together with:

- a left unit map $\eta_L : A \to R$
- a right unit map $\eta_R : A \to R$
- a coproduct map $\delta : R \to R \otimes_A R$
- a counit map $\varepsilon : R \to A$
- an antipode map $\chi : R \to R$ which squares to the identity

such that all of the following diagrams commute,

$$\begin{array}{ccccc} R & \xleftarrow{\eta_R} & A & \xrightarrow{\eta_L} & R \\ & {\scriptstyle\varepsilon}\searrow & \downarrow{\scriptstyle id} & \swarrow{\scriptstyle\varepsilon} & \\ & & A & & \end{array} \tag{6.10}$$

$$\begin{array}{ccccc} R \otimes_A R & \xleftarrow{\delta} & R & \xrightarrow{\delta} & R \otimes_A R \\ & {\scriptstyle\varepsilon\otimes id}\searrow & \downarrow{\scriptstyle id} & \swarrow{\scriptstyle id\otimes\varepsilon} & \\ & & R & & \end{array} \tag{6.11}$$

$$\begin{array}{ccc} R & \xrightarrow{\delta} & R \otimes_A R \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle id\otimes\delta} \\ R \otimes_A R & \xrightarrow{\delta\otimes id} & R \otimes_A R \otimes_A R \end{array} \tag{6.12}$$

$$\begin{array}{ccc} A & \xrightarrow{\eta_L} & R \\ & {\scriptstyle\eta_R}\searrow & \downarrow{\scriptstyle\chi} \\ & & R \end{array} \tag{6.13}$$

$$\begin{array}{ccc} A & \xrightarrow{\eta_R} & R \\ & {\scriptstyle\eta_L}\searrow & \downarrow{\scriptstyle\chi} \\ & & R \end{array} \tag{6.14}$$

and such that there exist maps μ_R and μ_L such that the following diagram commutes.

$$\begin{array}{ccccc} R\otimes_k R & \xleftarrow{\chi\otimes id} & R\otimes_k R & \xrightarrow{id\otimes\chi} & R\otimes_k R \\ \downarrow{\scriptstyle\varphi} & & \downarrow & & \downarrow{\scriptstyle\varphi} \\ R & \overset{\mu_R}{\dashleftarrow} & R\otimes_A R & \overset{\mu_L}{\dashrightarrow} & R \\ \uparrow{\scriptstyle\eta_R} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\eta_L} \\ A & \xleftarrow{\varepsilon} & R & \xrightarrow{\varepsilon} & A \end{array} \tag{6.15}$$

Here, the map φ is the multiplication map on $R \otimes R$ as a k-algebra.

As Ravenel explains in Appendix A of [Rav03], Hopf algebroids were named suggestively since one is to think of them as as generalization of Hopf algebras in the way that a groupoid generalizes the notion of a group. When the left and right units coincide, $\eta_L = \eta_R$, R is simply an A-Hopf algebra.

We claim that $\mathrm{THR}(A)$ has a Hopf algebroid structure in the D_2-homotopy category. To show the existence of this structure requires us to define a coproduct map on $O(2)_\bullet$. Recall from the proof of Lemma 6.1.9, that in the case of THH, a double model of the simplicial circle was needed in order to define a simplicial coproduct. Similarly, we must use a double model of $O(2)_\bullet$,

$$dO(2)_\bullet := \mathrm{sq}(O(2)_\bullet) = \mathrm{sq}(\mathrm{sq}(D_{2(\bullet+1)})).$$

This model is depicted in Figure 6.17.

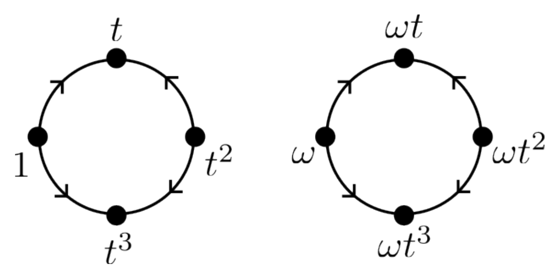


Figure 6.17 The double model $dO(2)_\bullet$.

We now demonstrate the D_2-equivalence between the model of THR given by $A\otimes_{D_2} O(2)_\bullet$ and the double model $d\,\mathrm{THR}(A) = A \otimes_{D_2} dO(2)_\bullet$, following the structure of Angeltveit and Rognes' argument presented in Lemma 6.1.7. To begin, we argue that THR and $d\,$THR can be understood as pushouts by taking the D_2-tensor of a diagram of simplicial objects.

Definition 6.2.6. For a ring spectrum R, we let $B(R)$ denote the double bar construction $B(R, R, R)$.

For clarity, we label copies of the ring spectrum R as R_i. A Segal-Quillen subdivision of the bar construction $B(R)$ has n-simplices given by the product $R_0 \wedge R_1 \wedge \ldots \wedge R_{2n+1} \wedge R_{2n+2}$. There is a level-wise D_2-action on $\mathrm{sq}B(R)$ given by swapping $R_i \leftrightarrow R_{2n+2-i}$. Thus the coefficients R_0 and R_{2n+2} are exchanged by the D_2-action on $\mathrm{sq}B(R)$.

Proposition 6.2.7. *Let A be a commutative D_2-ring spectrum. Then the Real topological Hochschild homology of A is represented as a pushout diagram in D_2-spectra given by,*

$$\begin{array}{ccc} A \wedge A & \longrightarrow & A \\ \downarrow & \ulcorner & \downarrow \\ sqB(A) & \longrightarrow & \mathrm{THR}(A) \end{array}$$

where $A \wedge A$ has a swap action and $sqB(A)$ has the D_2-action induced by Segal-Quillen subdivision. The map along the top is given by multiplication and the left hand map is the inclusion of $A \wedge A$ as the coefficients in the subdivided bar construction.

Proof. We wish to recognize this pushout diagram as one which arises from a diagram of simplicial objects whose pushout is $O(2)_\bullet$. We claim that the appropriate diagram is the following,

$$\begin{array}{ccc} D_4 & \longrightarrow & D_2 \\ \downarrow & & \downarrow \\ D_2 \otimes \mathrm{sq}\Delta^1 & \longrightarrow & O(2)_\bullet \end{array} \tag{6.16}$$

where the top map associates the points 1 and t in D_4 and the map on the left includes D_4 as the boundaries of the two subdivided 1-simplices. Figure 6.18 provides a geometric visualization of this diagram.

Applying the functor $A \otimes_{D_2} (-)$ to this entire diagram, it is clear that in the top right corner we have $A \otimes_{D_2} D_2 = A$. To see that $A \otimes_{D_2} D_4$ is the smash product $A \wedge A$ endowed with a swap action, consider the coequalizer

$$A \otimes D_2 \otimes D_4 \rightrightarrows A \otimes D_4 \longrightarrow A \otimes_{D_2} D_4.$$

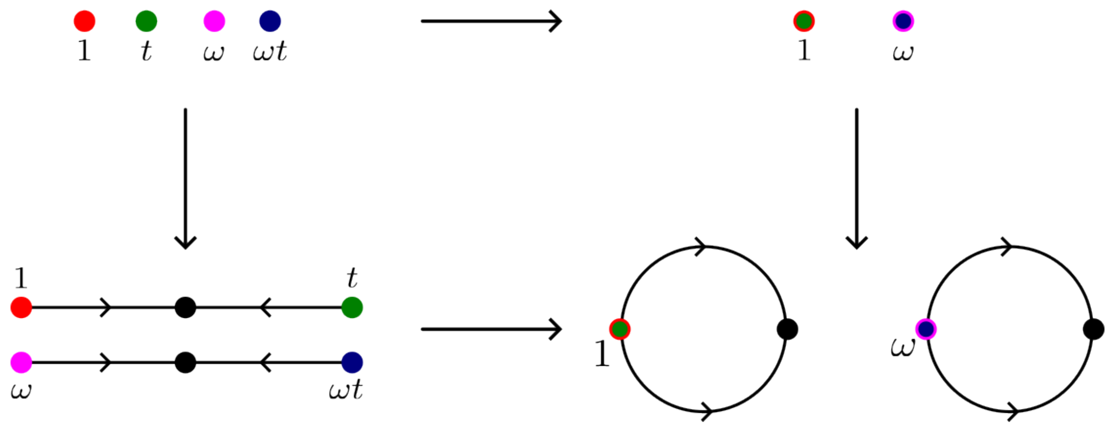


Figure 6.18 A pushout diagram which gives $O(2)_\bullet$.

The term on the left is a smash product of eight copies of A, indexed on pairs of elements $(\alpha, \beta) \in D_2 \times D_4$ and the term on the right is four copies of A, indexed by the elements of D_4. One map in the coequalizer associates the copies of A via the following association:

$$A_{1,1} \wedge A_{\omega,1} \to A_1 \qquad A_{1,\omega} \wedge A_{\omega,\omega} \to A_\omega$$

$$A_{1,t} \wedge A_{\omega,t} \to A_t \qquad A_{1,\omega t} \wedge A_{\omega,\omega t} \to A_{\omega t}.$$

The other map multiplies to associate these eight copies of A in a different way,

$$A_{1,1} \wedge A_{\omega,\omega} \to A_1 \qquad A_{\omega,1} \wedge A_{1,\omega} \to A_\omega$$

$$A_{1,t} \wedge A_{\omega,\omega t} \to A_t \qquad A_{\omega,t} \wedge A_{1,\omega t} \to A_{\omega t}.$$

In the coequalizer, we have $A_1 \wedge A_t$ which retains a D_2-action of t that swaps the copies. Thus we've identified the term in the top left corner of the pushout diagram that gives $\mathrm{THR}(A)$ as a tensor with a simplicial set.

Finally, we wish to identify $A \otimes_{D_2} (D_2 \otimes \mathrm{sq}\Delta^1)$ with $\mathrm{sq}B(A)$. To begin, we consider the Segal-Quillen subdivision of the 1-simplex Δ^1, whose simplicial structure was described in Definition 6.1.1. In the subdivision, $(\mathrm{sq}\Delta^1)_n = \Delta^1_{2n+1} = \{x_0, x_1, ..., x_{2(n+1)}\}$ and has the structure maps given by compositions of the fact and degeneracy maps in Δ^1 as described in Definition 4.1.4. Furthermore, the subdivided 1-simplex has a D_2-action given by $x_i \leftrightarrow x_{2(n+1)-i}$. Tensoring over the D_2 action of ω, we see that $A \otimes_{D_2} (D_2 \otimes \mathrm{sq}B(A))$ is $\mathrm{sq}B(A)$ which retains the D_2-action given by the subdivision. A level-wise comparison shows that $A \otimes (\mathrm{sq}\Delta^1)_n = (\mathrm{sq}B(A))_n$ and one may check that the face and degeneracy maps agree. Thus, we find that applying the functor $A \otimes_{D_2} (-)$ to the diagram in 6.16 yields the diagram in the statement of the proposition. The map $A \wedge A \to A$ is multiplication and the map $A \wedge A \to \mathrm{sq}B(A)$ is the inclusion of the two copies of A as the coefficients in the bar construction. These maps are D_2-equivariant and so, since the equivariant tensor product preserves pushouts, we have a pushout diagram in D_2-spectra which gives $\mathrm{THR}(A)$. □

We now employ a similar technique to recognize the double model of THR, which is defined as the tensor over D_2 with the double model $dO(2)_\bullet$ depicted in Figure 6.17, also arises from a simplicial pushout.

Proposition 6.2.8. *Let A be a commutative D_2-ring spectrum. Then $d\,\mathrm{THR}(A)$ is the pushout in D_2-spectra given by the diagram*

$$\begin{array}{ccc} A \wedge A & \longrightarrow & sqB(A) \\ \downarrow & \ulcorner & \downarrow \\ sqB(A) & \longrightarrow & d\,\mathrm{THR}(A) \end{array}$$

where $A \wedge A$ and $sqB(A)$ have the same D_2-actions as specified in the statement of Proposition 6.2.7.

Proof. Consider the pushout diagram of simplicial objects,

$$\begin{array}{ccc} D_4 & \longrightarrow & D_2 \otimes \mathrm{sq}\Delta^1 \\ \downarrow & \ulcorner & \downarrow \\ D_2 \otimes \mathrm{sq}\Delta^1 & \longrightarrow & dO(2)_\bullet . \end{array} \tag{6.17}$$

Here, both maps included D_4 as the boundary of $D_2 \otimes \mathrm{sq}\Delta^1$. This diagram is depicted geometrically in Figure 6.19.

We can make the same identifications of $A \otimes_{D_2} D_4$ and $A \otimes_{D_2} D_2 \otimes \mathrm{sq}\Delta^1$ as in the proof of Proposition 6.2.7. Since the equivariant tensor preserves pushouts, we see that applying the functor $A \otimes_{D_2} (-)$ to the diagram in 6.17 yields the pushout in the statement of the proposition. □

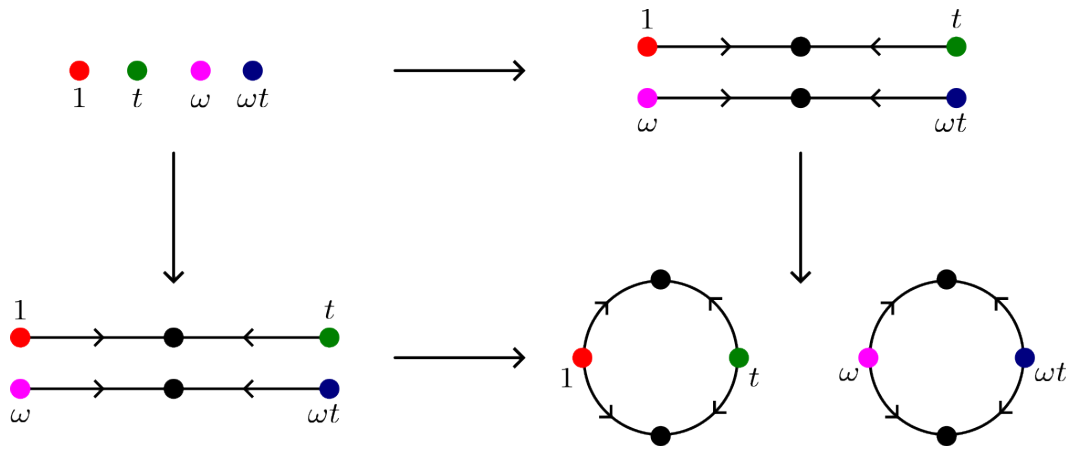


Figure 6.19 A pushout diagram which gives $dO(2)_\bullet$.

We now demonstrate that $\mathrm{THR}(A)$ and $d\,\mathrm{THR}(A)$ are D_2-weakly equivalent by giving a weak equivalence of pushout diagrams.

Lemma 6.2.9. *Let A be a commutative D_2-ring spectrum which is cofibrant as a D_2-spectrum. Then there is a D_2-weak equivalence*

$$\pi : d\,\mathrm{THR}(A) \xrightarrow{\simeq} \mathrm{THR}(A)$$

which is induced by the simplicial homotopy collapsing one half of each circle in $O(2)_\bullet$ to a point. This simplicial homotopy is depicted in Figure 6.20.

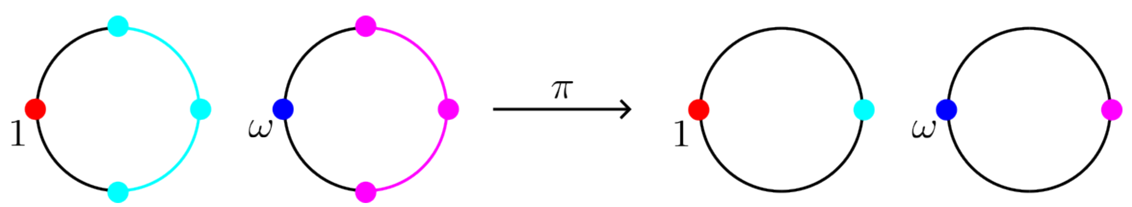


Figure 6.20 The simplicial homotopy inducing π on $d\,\mathrm{THR}$.

Proof. There is a commutative diagram of commutative D_2-spectra constructed from the pushout diagrams given in Proposition 6.2.7 and Proposition 6.2.8 of the form

$$\begin{array}{ccccc} \mathrm{sq}B(A) & \longleftarrow & A \wedge A & \longrightarrow & \mathrm{sq}B(A) \\ \| & & \| & & \downarrow \simeq \\ \mathrm{sq}B(A) & \longleftarrow & A \wedge A & \longrightarrow & A. \end{array} \tag{6.18}$$

Classically, there is a homotopy equivalence $B(A,A,A) \to A$ (see [Elm+97], IV. 7.3 and XII.1.2) defined level-wise on the bar construction by treating A as a constant simplicial object. Since the bar construction we are considering here is a subdivision of $B(A)$, the map inducing the equivalence $B(A,A,A) \to A$ at level $2n+1$ induces the equivalence $\mathrm{sq}B(A) \to A$ at level n. The homotopy is given by an iterated composite of unit and multiplication maps, which are all D_2-equivariant maps. Since pushouts preserve weak equivalences by [Elm+97] III.8.2 we obtain a weak equivalence between the pushout along the top row and the pushout along the bottom, $d\,\mathrm{THR}(A) \xrightarrow{\simeq} \mathrm{THR}(A)$.

Finally, we verify that this is a D_2-weak equivalence by checking that the map is a weak equivalence on the geometric fixed points, Φ^{D_2}. The geometric fixed points functor commutes with colimits and since $\mathrm{THR}(A)$ and $d\,\mathrm{THR}(A)$ are both colimits by Propositions 6.2.7 and 6.2.8, we have that

$$\Phi^{D_2}(d\,\mathrm{THR}(A)) \cong \mathrm{colim}(\Phi^{D_2}(\mathrm{sq}B(A)) \leftarrow \Phi^{D_2}(A \wedge A) \to \Phi^{D_2}(\mathrm{sq}B(A))$$
$$\Phi^{D_2}(\mathrm{THR}(A)) \cong \mathrm{colim}(\Phi^{D_2}(\mathrm{sq}B(A)) \leftarrow \Phi^{D_2}(A \wedge A) \to \Phi^{D_2}(A).$$

To compare the terms on the right, we recall that Φ^{D_2} commutes with the smash product and that this functor is applied level-wise to a simplicial object. Hence $\Phi^{D_2}(\mathrm{sq}B(A)) \cong \mathrm{sq}B(\Phi^{D_2}A)$ We then apply the same equivalence between a two-sided bar construction and its right coefficients described above to see that $\mathrm{sq}B(\Phi^{D_2}A) \simeq \Phi^{D_2}A$ and conclude that the equivalence $d\,\mathrm{THR}(A) \simeq A$ induced by the homotopy in 6.18 is a D_2-weak equivalence of spectra. □

Equipped with this homotopy equivalence between our two models of THR, we are now able to describe a coproduct structure and thus show that THR has the structure of a Hopf algebroid.

Theorem 6.2.10. *For a commutative D_2-ring spectrum A,* $\mathrm{THR}(A)$ *is a Hopf algebroid in the D_2-equivariant stable homotopy category.*

Proof. Here we consider the pair of D_2-spectra A and $\mathrm{THR}(A)$. Recall that the data of a Hopf algebroid includes both a left and right unit map. We will induce these maps $A \to \mathrm{THR}(A)$ by tensoring simplicial diagrams with A over D_2. The left and right units are given by the two possible inclusion maps depicted in Figures 6.21 and 6.22. Note that η_L is the map previously called η in Lemma 6.2.4.

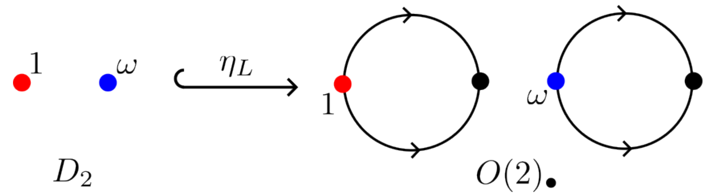


Figure 6.21 The simplicial map inducing the left unit on THR.

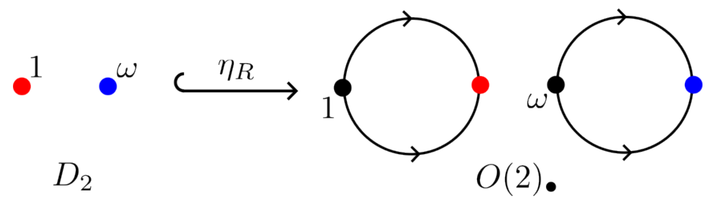


Figure 6.22 The simplicial map inducing the right unit on THR.

We also define a counit map

$$\varepsilon : \mathrm{THR}(A) \to A$$

by taking the D_2-tensor with A of the map which collapses each of the two circles in $O(2)_\bullet$ to a point. This is depicted in Figure 6.23.

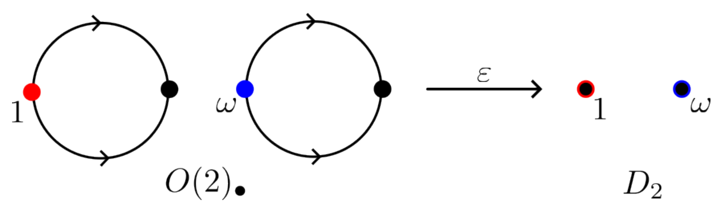


Figure 6.23 The simplicial map inducing a counit on THR.

Utilizing the double model of $O(2)_\bullet$, we define

$$\delta' : d\,\mathrm{THR}(A) \to \mathrm{THR}(A)$$

induced by the tensor over D_2 of the simplicial map in Figure 6.24 with A.

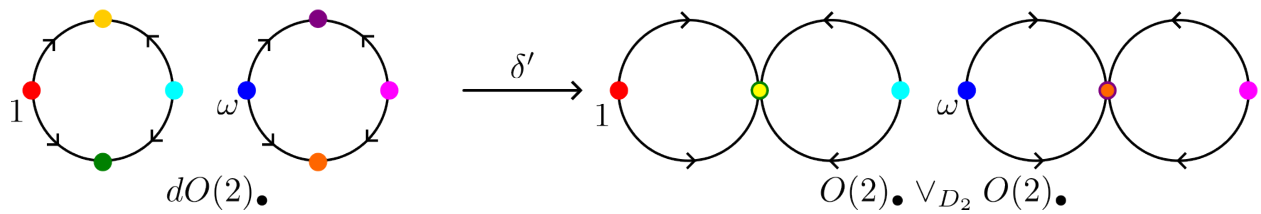


Figure 6.24 The simplicial map inducing a coproduct on THR.

A coproduct on THR is then given by the composition

$$\delta : \mathrm{THR}(A) \xrightarrow{\pi^{-1}} d\,\mathrm{THR}(A) \xrightarrow{\delta'} \mathrm{THR}(A) \wedge_A \mathrm{THR}(A),$$

where π is the equivalence from Lemma 6.2.9.

Finally, we define an antipodal map on THR induced by the map on $O(2)_\bullet$ which reflects each circle across an axis so that the base points swap. The action of this antipodal map, which we will call χ, is depicted in Figure 6.25.

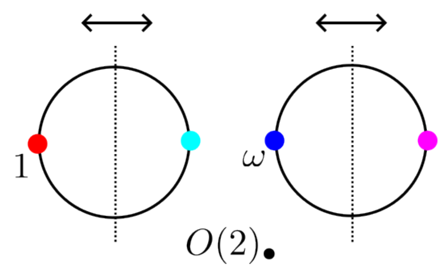


Figure 6.25 The action of the map χ which induces an antipode on THR.

To verify that these structure maps satisfy the commutativity relations of a Hopf algebroid as stated in Definition 6.2.5, we check that the relations hold for the simplicial maps of $O(2)_\bullet$. From a visual inspection it is clear that the antipodal map swaps the units. That the units and counit obey the relations depicted in Diagram 6.10 of Definition 6.2.5 is also clear.

To verify counitality, consider the diagram in Figure 6.26. For the sake of clarity in the diagram, we have not color-coded any of the cells in the second circle of $O(2)_\bullet$ but the identifications are precisely the same as shown in the first copy of the circle.
The simplicial homotopy from $\pi : dO(2)_\bullet \to O(2)_\bullet$ we described (and depicted in Figure 6.20) collapses the two copies of Δ^1 on the right hand side of each circle to a point. In Figure 6.26 above, this is represented by the diagonal arrow which collapses the yellow, orange,

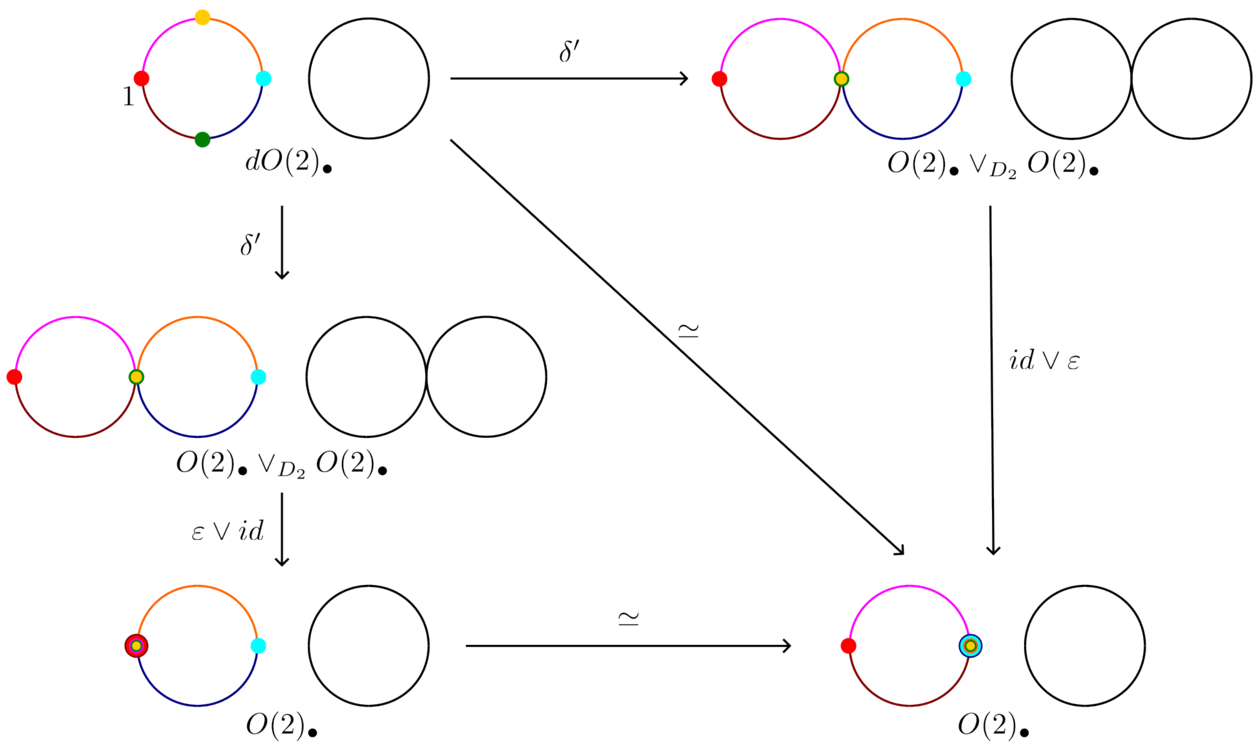


Figure 6.26 Simplicial counitality diagram for THR.

teal, blue, and green cells to a single point. This homotopy equivalence is the composition of $(id \vee \varepsilon) \circ \delta'$ along the right hand side of the diagram.

Equivalently, there is a simplicial homotopy from $\sigma : dO(2)_\bullet \to O(2)_\bullet$ which collapses the two copies of Δ^1 on the left hand side of each circle. In Figure 6.26, this homotopy collapses the green, pink, red, and maroon cells to one point. We see such a collapse occurring in the composite along the left hand side of the diagram. Thus we have the homotopy equivalence along the bottom of the diagram given by $\pi \circ \sigma^{-1}$ and we see that the counitality diagram commutes up to homotopy. We omit the diagram, but one may similarly check that the coassociativity relation holds.

Finally, we show the existence of two maps (denoted by μ_L and μ_R so as to suggest a left and right multiplication) which make Diagram 6.15 in Definition 6.2.5 commute. Once again, we will not label or color-code the second circle in $O(2)_\bullet$ in this diagram, but the associations are the same.

In Figure 6.27, the map μ_L is given by folding the orange-teal-navy circle onto the other

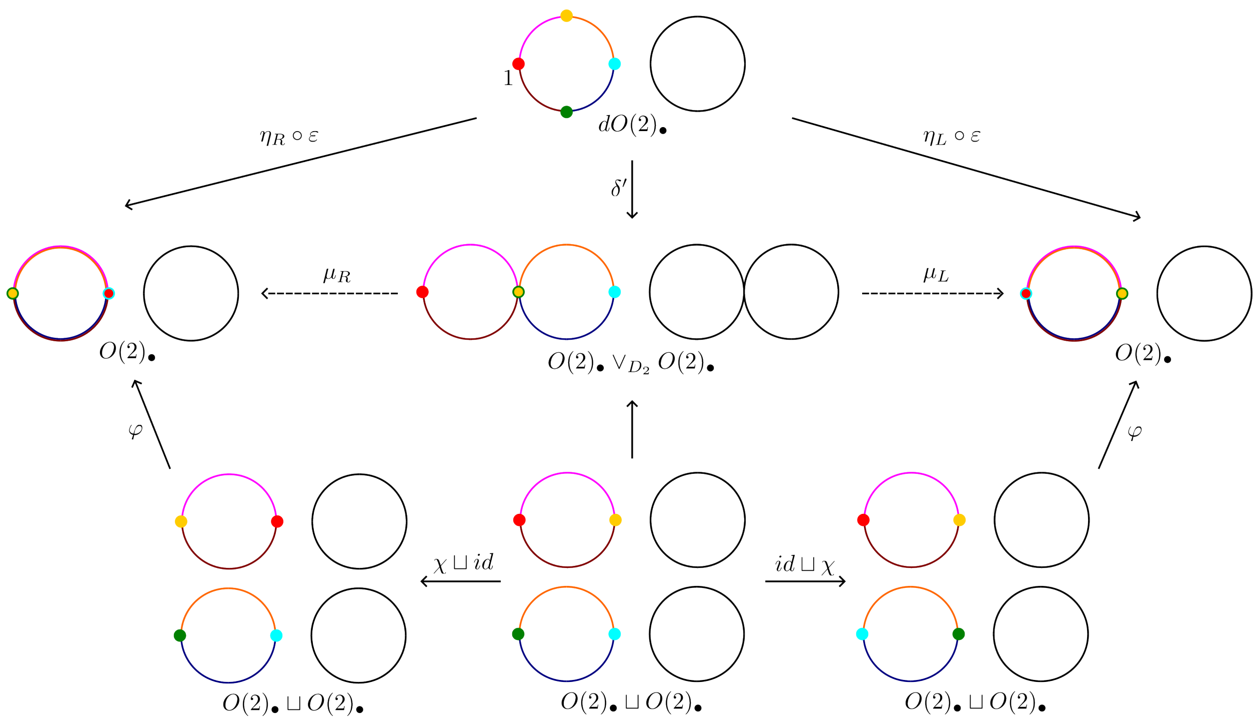


Figure 6.27 Simplicial Hopf algebroid compatibility diagram.

circle (a fold to the left). The map μ_R is a fold to the right, folding the pink-red-maroon circle on top of the other one. The map φ slides the copies of $O(2)_\bullet$ on top of each other. We claim composites $\mu_L \circ \delta'$ and $\mu_R \circ \delta'$ both factor through the map

$$\bar{\Delta}^2 \sqcup \bar{\Delta}^2 \hookrightarrow \partial\bar{\Delta}^2 \sqcup \partial\bar{\Delta}^2 \to \bar{\Delta}^1 \sqcup \bar{\Delta}^1 \to O(2)_\bullet,$$

which is shown below in Figure 6.28. Again, this figure depicts the contraction in one of the $O(2)_\bullet$ circles but the maps in the other disjoint copy of the circle are defined to be the same.

Here we take $\bar{\Delta}^2$ to be the subdivided Real 2-simplex which has a D_2-action that reflects across the vertical axis. We include the boundary of the subdivided Real 2-simplex into $\bar{\Delta}^2$ and then collapse through the 2-cells down to edge adc. The map f is given by folding the subdivided 1-simplex to the left and gluing a to c. This produces a copy of $O(2)_\bullet$ where the unit is included via η_L. If instead we fold the subdivided 1-simplex to the right along map g and glue c to a we produce $O(2)_\bullet$ where the unit has been included via η_R. Since the 1-simplex is contractible, both of these composites are null homotopic. We recover the wedge

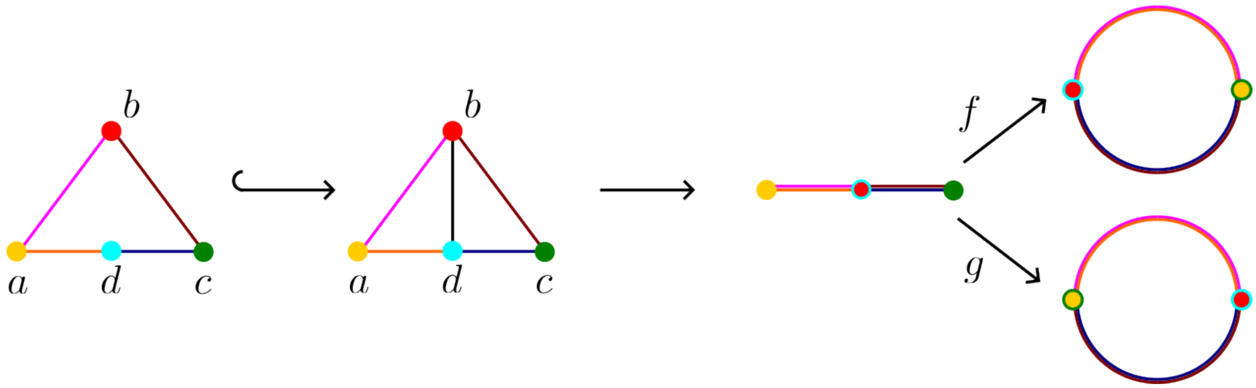


Figure 6.28 Simplicial contractibility factorization.

$O(2)_\bullet \wedge_{D_2} O(2)_\bullet$ at the center of the Hopf algebroid compatibility diagram in Figure 6.27 by gluing a to c in the boundary $\partial\bar{\Delta}^2$. Therefore we have that the maps $\mu_L \circ \delta'$ and $\mu_R \circ \delta'$ are null homotopic since they factor through the contractible 1-simplex and the verification that the diagram in Figure 6.27 is D_2-commutative up to homotopy is complete. We apply the functor $A \otimes_{D_2} (-)$ to the entire diagram and obtain the following diagram:

$$\begin{array}{ccccc}
A & \xleftarrow{\varepsilon \circ \pi} & d\,\mathrm{THR}(A) & \xrightarrow{\varepsilon \circ \pi} & A \\
\downarrow{\scriptstyle \eta_R} & & \downarrow{\scriptstyle \delta'} & & \downarrow{\scriptstyle \eta_L} \\
\mathrm{THR}(A) & \xleftarrow{\mu_R} & \mathrm{THR}(A) \wedge_A \mathrm{THR}(A) & \xrightarrow{\mu_L} & \mathrm{THR}(A) \\
\uparrow{\scriptstyle \varphi} & & \uparrow & & \uparrow{\scriptstyle \varphi} \\
\mathrm{THR}(A) \wedge_{\mathbb{S}} \mathrm{THR}(A) & \xleftarrow{\chi \wedge id} & \mathrm{THR}(A) \wedge_{\mathbb{S}} \mathrm{THR}(A) & \xrightarrow{id \wedge \chi} & \mathrm{THR}(A) \wedge_{\mathbb{S}} \mathrm{THR}(A)
\end{array}$$

We note that $A \otimes_{D_2} (O(2)_\bullet \sqcup O(2)_\bullet)$ is the smash product of two copies of $\mathrm{THR}(A)$ as algebras over the sphere spectrum. Because the simplicial diagram was D_2-commutative up to homotopy, so too is the diagram in spectra and the proof that $\mathrm{THR}(A)$ is a Hopf algebroid in the D_2-homotopy category when A is commutative is complete. □

Although $\mathrm{THR}(A)$ has an A-algebra structure (the one given in Lemma 6.2.4), it is not compatible with the coproduct. Specifically, the issue arises from the fact that the A-bimodule structure on THR is given by these two different unit maps that we defined. Hence we get a Hopf algebroid structure rather than a Hopf algebra structure.

In the classical case of topological Hochschild homology, the Hopf algebra structure on THH descends to a Hopf algebra structure on the spectral sequence. Specifically, the following theorem of Angeltveit and Rognes in the THH case motivated the work undertaken in this chapter for THR.

Theorem 6.2.11 ([AR05], Theorem 4.5)**.** *Let R be a commutative ring spectrum and consider the Bökstedt spectral sequence*

$$E^2_{*,*} = \mathrm{HH}_*(H_*(R;\mathbb{F}_p)) \Rightarrow H_*(\mathrm{THH}(R);\mathbb{F}_p).$$

If each term $E^r_{*,*}$ for $r \geq 2$ is flat over $H_*(R;\mathbb{F}_p)$ then the Bökstedt spectral sequence is a spectral sequence of $H_*(R;\mathbb{F}_p)$-Hopf algebras.

Though we do not recall the full proof of this theorem, we remark that the key step involves using the simplicial Hopf algebra structure maps on THH to define maps on the spectral sequence. Such an approach is made possible because the Bökstedt spectral sequence arises from a simplicial filtration of THH. For this reason, we were careful to construct all of the Hopf algebroid structure maps simplicially in this section. Although we do not consider whether this structure lifts to the Real Bökstedt spectral sequence in this thesis, we will return to this in future work.

BIBLIOGRAPHY

[AB18] Vigleik Angeltveit and Anna Marie Bohmann. "Graded Tambara functors". *Journal of Pure and Applied Algebra* 222.12 (2018), pp. 4126–4150.

[Ada+22] Katharine Adamyk, Teena Gerhardt, Kathryn Hess, Inbar Klang, and Hana Jia Kong. "Computational tools for twisted topological Hochschild homology of equivariant spectra". *Topology and its Applications* 316 (2022), p. 108102.

[AGH21] Gabriel Angelini-Knoll, Teena Gerhardt, and Michael A Hill. "Real topological Hochschild homology via the norm and Real Witt vectors". arXiv 2111.06970. pre-print. 2021.

[Ang+18] Vigleik Angeltveit, Andrew J Blumberg, Teena Gerhardt, Michael A. Hill, Tyler Lawson, and Michael A. Mandell. "Topological cyclic homology via the norm". *Documenta mathematica* 23 (2018), pp. 2101–2163.

[AR05] Vigleik Angeltveit and John Rognes. "Hopf algebra structure on topological Hochschild homology". *Algebraic & Geometric Topology* 5.3 (2005), pp. 1223–1290.

[Bar64] Dennis Barden. *On the structure and classification of differential manifolds.* Thesis (Ph.D.)–University of Cambridge. Apollo - University of Cambridge Repository, 1964.

[BDS22] Morten Brun, Bjørn Ian Dundas, and Martin Stolz. "Equivariant Structure on Smash Powers". arXiv:1604.05939. pre-print. 2022.

[BHM93] Marcel Bökstedt, Wu Chung Hsiang, and Ib Madsen. "The cyclotomic trace and algebraic K-theory of spaces". *Inventiones mathematicae* 111.1 (1993), pp. 465–539.

[Blu+19] Andrew J Blumberg, Teena Gerhardt, Michael A. Hill, and Tyler Lawson. "The Witt vectors for Green functors". *Journal of Algebra* 537 (2019), pp. 197–244.

[Bök85a] Marcel Bökstedt. *The topological Hochschild homology of Z and Z/p.* Universität Bielefeld, Fakultät für Mathematik, 1985.

[Bök85b] Marcel Bökstedt. "Topological Hochschild homology". *preprint, Bielefeld* 3 (1985).

[Bou00] Serge Bouc. *Non-additive exact functors and tensor induction for Mackey functors.* Vol. 683. American Mathematical Society, 2000.

[DO19] Emanuele Dotto and Crichton Ogle. "K-theory of Hermitian Mackey functors, real traces, and assembly". *Annals of K-Theory* 4.2 (2019), pp. 243–316.

[Dot+20] Emanuele Dotto, Kristian Moi, Irakli Patchkoria, and Sune Precht Reeh. "Real topological Hochschild homology". *Journal of the European Mathematical Society* 23.1 (2020), pp. 63–152.

[Elm+97] A.D. Elmendorf, I. Kriz, M.A. Mandell, and J.P. May. *Rings, modules, and algebras in stable homotopy theory.* 47. American Mathematical Soc., 1997.

[FL91] Zbigniew Fiedorowicz and Jean-Louis Loday. "Crossed simplicial groups and their associated homology". *Transactions of the American Mathematical Society* 326.1 (1991), pp. 57–87.

[Fuj76] Michikazu Fujii. "Cobordism theory with reality". *Mathematical Journal of Okayama University* 18.2 (1976), pp. 171–188.

[HH16] Michael A Hill and Michael J Hopkins. "Equivariant symmetric monoidal structures". arXiv 1610.03114. pre-print. 2016.

[HHR16] Michael A Hill, Michael J Hopkins, and Douglas C Ravenel. "On the nonexistence of elements of Kervaire invariant one". *Annals of Mathematics* (2016), pp. 1–262.

[HHR17] Michael A. Hill, Michael J. Hopkins, and Douglas C. Ravenel. "The slice spectral sequence for the C_4 analog of real K-theory". *Forum Mathematicum* 29.2 (2017), pp. 383–447.

[Hil22] Michael A. Hill. "Freeness and equivariant stable homotopy". *Journal of Topology* 15.2 (2022), pp. 359–397.

[HM15] Lars Hesselholt and Ib Madsen. "Real algebraic K-theory". to appear. 2015.

[HM19] Michael A Hill and Kristen Mazur. "An equivariant tensor product on Mackey functors". *Journal of Pure and Applied Algebra* 223.12 (2019), pp. 5310–5345.

[Hoy14] Rolf Hoyer. *Two topics in stable homotopy theory.* Thesis (Ph.D.)–The University of Chicago. ProQuest LLC, Ann Arbor, MI, 2014, p. 93. ISBN: 978-1321-03338-0.

[Kar73] Max Karoubi. "Périodicité de la K—Théorie hermitienne". *Algebraic K-Theory, III: Hermitian K-Theory and Geometric Applications (Proceedings of the Conf., Battelle Memorial Inst., Seattle, Wash., 1972).* Vol. 343. Springer. 1973, pp. 301–411.

[Kur92] Masato Kurihara. "Some remarks on conjectures about cyclotomic fields and K-groups of $\mathbb{Z}$". *Compositio Mathematica* 81.2 (1992), pp. 223–236.

[Lan68] Peter S Landweber. "Conjugations on complex manifolds and equivariant homotopy of MU". *Bulletin of the American Mathematical Society* 74.2 (1968), pp. 271–274.

[Lew88] L. Gaunce Lewis. "The RO (G)-graded equivariant ordinary cohomology of complex projective spaces with linear $\mathbb{Z}/\mathrm{p}$ actions". *Algebraic topology and transformation groups*. Springer, 1988, pp. 53–122.

[LM06] L Gaunce Lewis and Michael A Mandell. "Equivariant universal coefficient and Künneth spectral sequences". *Proceedings of the London Mathematical Society* 92.2 (2006), pp. 505–544.

[Lod13] Jean-Louis Loday. *Cyclic Homology*. Vol. 301. Springer Science & Business Media, 2013.

[May06] J. Peter May. *The geometry of iterated loop spaces*. Vol. 271. Springer, 2006.

[Maz13] Kristen Luise Mazur. *On the Structure of Mackey Functors and Tambara Functors*. Thesis (Ph.D.)–University of Virginia. ProQuest LLC, Ann Arbor, MI, 2013, p. 106. ISBN: 978-1303-45882-8.

[MSV97] James McClure, Roland Schwänzl, and Rainer Vogt. "$THH(R) \cong R \otimes S^1$ for E_∞ ring spectra". *Journal of Pure and Applied Algebra* 121.2 (1997), pp. 137–159.

[Qui73] Daniel Quillen. "Higher algebraic K-theory: I". *Higher K-Theories* (1973), pp. 85–147.

[Rav03] Douglas C. Ravenel. *Complex cobordism and stable homotopy groups of spheres*. Vol. 347. American Mathematical Society, 2003.

[Seg73] Graeme Segal. "Configuration-spaces and iterated loop-spaces". *Inventiones mathematicae* 21 (1973), pp. 213–221.

[Spa00] Jan Spaliński. "Homotopy theory of dihedral and quaternionic sets". *Topology* 39.3 (2000), pp. 557–572.

[Wei13] Charles A Weibel. *The K-book: An Introduction to Algebraic K-theory*. Vol. 145. American Mathematical Soc., 2013.