DIFFUSION FOR A DISCRETE LINDBLAD MASTER EQUATION WITH PERIODIC HAMILTONIAN

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ABSTRACT

A quantum particle restricted to a lattice of points has been well studied in many different contexts. In the absence of considering the interaction with its environment, the particle simply undergoes ballistic transport for many suitable Hamiltonian operators. The evolution becomes much more complicated when considering environmental interaction, which leads to the so-called Lindblad master equation. When considering this master equation, the Lindbladian term dominates the dynamics of the particle, leading to diffusive propagation. In this document, we prove diffusion is indeed present in the context of a periodic Hamiltonian. Additionally, we show that the diffusion constant is inversely proportional to the particles' coupling strength with its environment.

This dissertation is dedicated to my wife, Lilianne
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GLOSSARY OF TERMS

- $\mathcal{B}(\mathcal{H}) = \{A : \mathcal{H} \to \mathcal{H} : ||A||_{\mathcal{H}} < \infty\}$, the set of bounded operators in a Hilbert space $\mathcal{H}$
- $\mathcal{B}_1(\mathcal{H}) = \{A : \mathcal{H} \to \mathcal{H} : \mathrm{tr}A < \infty\}$, the set of operators in $\mathcal{H}$ with finite trace
- $M_d(\mathcal{A})$, the set of $d \times d$ matrices with entries in $\mathcal{A}$
- $SA(\mathcal{H})$, the set of self-adjoint operators on a Hilbert space $\mathcal{H}$
- $C_0(X)$, the set of continuous functions on a metric space X which vanish at infinity
- $(\tau_y f)(x) = f(x + y)$, the translation operator by y
- $(X_j f)(x) = x_j f(x)$, the position operators

CHAPTER 1: INTRODUCTION

Consider a single quantum particle living in a closed quantum system given by the Hilbert space $\mathcal{H}$. The pure states of this particle are given by wave functions $|\psi_t\rangle \in \mathcal{H}$ which evolve in time via the Schrödinger Equation

$$\partial_t|\psi_t\rangle = -iH|\psi_t\rangle$$

for some self-adjoint Hamiltonian operator $H \in SA(\mathcal{H})$ which represents the energy of the system. When analyzing entangled quantum systems, the associated density matrices $\rho_t := |\psi_t\rangle\langle\psi_t| \in \mathcal{B}_1(\mathcal{H})$ are typically used. A *density matrix* is any bounded positive operator ρ satisfying $\mathrm{tr}\rho = 1$. For convenience, we will denote $\rho_t(x,y) := \langle x|\psi_t\rangle\langle\psi_t|y\rangle$ as the kernel of ρ_t. The equation governing the evolution of density matrices in a closed quantum system is simply given by the related von Neumann equation

$$\partial_t\rho_t = -i[H,\rho_t], \tag{1}$$

where here $[A,B] = AB - BA$ represents the usual commutator. In this document, we consider a quantum particle restricted to a discrete lattice of points $\mathbb{Z}^d$, which amounts to letting our Hilbert space be $\mathcal{H} = \ell^2(\mathbb{Z}^d)$. This could simulate the particle being in a rigid crystalline structure and such models are widely used in modern day literature (See for example [1], [2], [3]). On its own, the solution ρ_t of equation (1) will behave ballistically in the limit $t \to \infty$ for many standard Hamiltonian operators, i.e. translation invariant or periodic operators. That is, for the 2nd position moments given by

$$\langle X_t^2\rangle := \sum_{x\in\mathbb{Z}^d} |x|^2\rho_t(x,x),$$

we have the following asymptotic relation:

$$\langle X_t^2\rangle \sim t^2$$

for large times t. Many recent works have shown that the particle's dynamics will drastically change to exhibit diffusion, i.e.

$$\langle X_t^2 \rangle \sim t$$

when subject to some form of random disorder. For instance, [3], [4] showed diffusive propagation for the tight binding Markov-Schrödinger model consisting of a random potential which fluctuates stochastically in time. In [5] and [6], diffusion was proven for a quantum particle coupled to a field of bosons having random thermal state in dimensions $d \geqslant 3$, as well as a quantum particle coupled to an array of heat baths, respectively. Additionally, it is conjectured that the Anderson model consisting of a random static potential will similarly produce diffusion for dimension $d \geqslant 3$ provided the disorder strength is sufficiently small. Heuristically, the random potential produces this diffusive effect due to the wave scattering off of the random background and producing random phases. These phases eventually build up over time and lead to an overall decoherence of the wave.

Most of the recent works in this area do not consider how the particle couples with its environment. For example, in the consideration of the particle being trapped in a crystal, the wave function could interact with free boson gasses in the crystal caused by quantized vibrations [1]. This situation is a lot more nuanced, though the equations of motion for such a particle have been well-established in both open quantum theory [7] as well as quantum information theory [8]. In the thermodynamic limit, the evolution of a one-particle density matrix taking into account environmental coupling may be approximated by the Lindblad equation

$$\partial_t \rho_t = -i[H, \rho_t] + g\left(\Psi(\rho_t) - \frac{1}{2}\{\Psi^*(I), \rho_t\}\right), \tag{2}$$

where Ψ is some completely positive operator, $g > 0$, and $\{A, B\} = AB + BA$ is the anti-commutator (See Section 4 for a derivation of this equation). The new term

$$\mathcal{L}(\rho_t) := \Psi(\rho_t) - \frac{1}{2}\{\Psi^*(I), \rho_t\} \tag{3}$$

in this expression is called the *Lindbladian* operator, which describes the coupling of the particle with its environment. A parameter g is introduced in (2) to allow us to control the strength of this coupling. In [2] and [9], it was shown that Lindbladian interaction induces diffusive behavior for quantum particles with translation-invariant Hamiltonian operators similar to the effect of adding a disordered potential. In [1], diffusion was shown for a model involving the Anderson Hamiltonian and an environmental interaction term similar to but distinct from a Lindbladian. It is thus natural to wonder whether the Lindbladian will be sufficient to contribute to diffusive behavior in other contexts as well. The present document continues this work by proving diffusion for a single quantum particle in an open quantum system coupled with an environment in the case of a periodic Hamiltonian.

CHAPTER 2: STATEMENTS OF MAIN RESULTS

For the remainder of this document, we will assume $H : \mathcal{B}_1(\ell^2(\mathbb{Z}^d)) \to \mathcal{B}_1(\ell^2(\mathbb{Z}^d))$ is Q-periodic. That is, let $Q \in M_d(\mathbb{Z})$ be an invertible matrix so that $\{Qx : x \in \mathbb{Z}^d\}$ defines a sublattice of points in $\mathbb{Z}^d$. Then, assume the Hamiltonian operator H satisfies $[H, \tau_{Qx}] = 0$ for all $x \in \mathbb{Z}^d$ where τ_x denotes the translation operator by x. The various assumptions required for the Lindbladian are given in Section 5.3. However, I will outline in general what we need here.

First, after diagonalizing the Lindbladian via a Fourier transform, we may fiber along the momentum variable k, which yields

$$\widehat{\mathcal{L}}_k = T_k - \mathcal{D}_k$$

for T_k an integral operator and $\mathcal{D}_k$ a multiplication operator. This is described by some authors as the gain-loss framework (See [10], [1]) where T_k is the gain term and $\mathcal{D}_k$ is the loss term. At the zero fiber, we must guarantee that the kernel of the Lindbladian is nondegenerate, which will help in calculations involving the spectrum. For the gain term, we must assume a local uniform lower bound, and for the loss term, we must assume a uniform upper and lower bound. Finally, the Lindbladian must abide by reflection invariance, in order to reflect certain symmetries present in the environment. Some authors [1], [9] may utilize certain physically realizable assumptions such as detailed balance or a gapping in the spectrum of the Lindbladian at the zero fiber. However, as we will see later, our assumptions are sufficient to prove a gapping in the spectrum of the Lindbladian and Hamiltonian together at the zero fiber.

The main result proved in this document is the following central limit theorem:

Theorem 1. *Let H be a Q-periodic Hamiltonian and $\mathcal{L}$ a Lindbladian satisfying Assumptions 1-4 in Section 5.3. Then, there exists a drift constant $v \in \mathbb{R}^d$ and a positive definite* *diffusion matrix* $\boldsymbol{D} = (D_{i,j})_{i,j} \in M_d(\mathbb{C})$ *such that for all initial conditions* $\rho_0 \in \mathcal{B}_1(\ell^2(\mathbb{Z}^d))$,

$$\lim_{\tau\to\infty} \sum_{x\in\mathbb{Z}^d} e^{i\frac{1}{\sqrt{\tau}}(x-\tau t v)\cdot Q^{-1}k} \rho_{\tau t}(x,x) = [\mathrm{tr}\rho_0] e^{-t\sum_{i,j} D_{i,j}k_i k_j}, \tag{4}$$

where ρ_t *is a solution of (2). In addition, if the initial condition* ρ_0 *satisfies the regularity assumption*

$$\sum_{x\in\mathbb{Z}^d} |x|^2 \rho_0(x,x) < \infty,$$

then the drift and diffusion constants are equivalent to:

$$v = \lim_{t\to\infty} \frac{1}{[\mathrm{tr}\rho_0]t} \sum_{x\in\mathbb{Z}^d} x\rho_t(x,x), \tag{5}$$

$$D_{i,j} = \lim_{t\to\infty} \frac{1}{2[\mathrm{tr}\rho_0]t} \sum_{x\in\mathbb{Z}^d} ((Q^T)^{-1}(x-tv))_i((Q^T)^{-1}(x-tv))_j \rho_t(x,x). \tag{6}$$

For $Q = I$, this yields the known result of diffusion for translation-invariant Hamiltonian operators given in [2], [9]. Some authors proving similar central limit theorems assume the initial condition $\rho_0 = \delta_0$ or a zero-drift condition $v = 0$ to simplify the calculations. Our method of proof allows for a generalization of this, as given above.

In order to derive the Lindblad equation of motion (2), one must assume that the coupling strength with the environment is small. Therefore, of particular interest is the case $0 < g \ll 1$. The method of proof used in this document allows for the diffusion matrix to be expressed as a function of this parameter, which is an improvement over even the translation-invariant case in [2], [9]. Assuming a uniform upper bound on the gain term, as well as an ergodicity assumption, we are able to prove the following result concerning the asymptotics of the diffusion for small g:

Theorem 2. *Let* $\boldsymbol{D}(g)$ *be the diffusion matrix in Theorem 1 and suppose* $\mathcal{L}$ *additionally satisfies Assumptions 5 and 6 in Section 5.3. Then*

$$0 < \lim_{g\to 0^+} g\boldsymbol{D}(g) < \infty. \tag{7}$$

That is, $\mathbf{D}(g) \sim \frac{1}{g}$ for small g. This is consistent with previous results since turning off the coupling with the environment will simply lead to ballistic motion.

The rest of this document is organized as follows. In Chapter 4, we provide some background into the Lindblad equation (2), including a derivation found in various quantum information theory sources, i.e. [8]. We also discuss Markov jump processes, which are necessary to fully describe the structure of the Lindbladian. In this chapter, we also provide an assumption from [11], and show how it is utilized to express the Lindbladian in a simpler form as is done in [9]. In Chapter 5, we introduce some structure and properties of the Lindbladian. First, we partially diagonalize the Hamiltonian and Lindbladian operators using a generalized Fourier transform. This allows us to state the various assumptions necessary to prove Theorems 1 and 2. We also compute the spectrum of our operators using results from [12]. Theorems 1 and 2 are proved in Chapter 6, and we discuss in Chapter 7 some additional research questions related to this work.

CHAPTER 3: BACKGROUND

In this chapter, we provide some history into the Lindbladian as well as continuous-time Markov processes, which will be necessary to analyze certain properties of the Lindbladian. We also state the *translation-covariance* assumption, and show how this is used to decompose the Lindbladian in a nicer way.

3.1 The Open Quantum System

In an arbitrary open quantum system, the total state space $\mathcal{H}_T$ is given by a composite system comprised of a system of interest $\mathcal{H}_S$ and the system corresponding to the environment $\mathcal{H}_E$. That is, $\mathcal{H}_T = \mathcal{H}_S \otimes \mathcal{H}_E$. The goal is to derive the equations of motion for the system of interest (2). There are two main approaches for deriving equation (2), one of which comes from open quantum theory, and can be found in various sources such as [13], [7]. The central idea is to assume the total system $\mathcal{H}_T$ is a closed quantum system, and thus its density matrices ρ_T satisfy (1). In order to determine the evolution equation for the density matrices ρ in the system of interest, we must trace out the extraneous degrees of freedom in the environment, i.e. $\rho = \mathrm{tr}_E \rho_T$. After using various physical assumptions such as the system of interest and the environment are noncorrelated for all times, the state of the environment is thermal for all times, and the so-called *rotating wave* approximation, we arrive at the master equation (2). What I present in Section 4.2 is a different approach from quantum information theory using generators of dynamical semigroups.

3.2 Derivation of the Master Equation

3.2.1 The Schrödinger Picture

In quantum information theory, the evolution equation for density matrices ρ_t in the system of interest $\mathcal{H}_S$ should be of the form

$$\partial_t \rho_t = \mathcal{G}\rho_t$$

for some time-independent operator $\mathcal{G}$. For our purposes, we shall assume that $\mathcal{G}$ is bounded; however, this assumption was relaxed in [14]. This differential equation is easily solved as

$$\rho_t = e^{t\mathcal{G}}\rho_0, \tag{8}$$

and we assume that $\Phi_t := e^{t\mathcal{G}}$ is a *dynamical semigroup* with generator given by $\mathcal{G}$. That is, $\Phi_t : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ is a bounded one-parameter family of operators satisfying $\Phi_{t+s} = \Phi_t \circ \Phi_s$ for all $t, s \geqslant 0$ and $\lim\limits_{t\to 0^+} \operatorname{tr}|\Phi_t\rho - \rho| = 0$ for all $\rho \in \mathcal{B}_1(\mathcal{H})$. As evidenced by equation (8) and the fact that density matrices have unit trace, it is reasonable to assume $\operatorname{tr}(\Phi_t\rho) = \operatorname{tr}\rho$ for all $\rho \in \mathcal{B}_1(\mathcal{H})$, i.e. Φ_t is *trace-preserving*. Furthermore, due to the composite nature of the problem, we must guarantee positivity not only for the semigroups acting on the system of interest $\mathcal{H}_S$, but for semigroups acting on larger systems containing $\mathcal{H}_S$ as a subsystem. In mathematical terms, we must guarantee positivity for the extended operators $\Phi_t \otimes \mathbb{1}_n : M_n(\mathcal{B}_1(\mathcal{H})) \to M_n(\mathcal{B}_1(\mathcal{H}))$ given by

$$\Phi_t \otimes \mathbb{1}_n(\rho \otimes E_{ij}) = \Phi_t\rho \otimes E_{ij}$$

for all $n \in \mathbb{N}$ where E_{ij}, $1 \leqslant i, j \leqslant n$ are matrix units spanning $M_n(\mathbb{C})$. That is, we require Φ_t to be *completely positive*. Thus, this approach amounts to finding an explicit form for the generator of a completely positive trace-preserving (CPTP) dynamical semigroup known as a *quantum Markov semigroup*.

3.2.2 The Heisenberg Picture

Thus far, we have considered everything in the Schrödinger picture; that is, the states depend on time whereas operators/observables are time-independent. However, most authors (See for instance [8]) categorize the generators of quantum Markov semigroups in the Heisenberg picture instead. In this picture, states are time-independent whereas observables $X_t \in \mathcal{B}(\mathcal{H})$ are dependent on time. Operators and density matrices are related due to the following. For any map $\mathcal{A} : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ in the Schrödinger picture, there is a corresponding map $\mathcal{A}^T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ uniquely defined in the Heisenberg picture by the relation

$$\operatorname{tr}(X\mathcal{A}(\rho)) = \operatorname{tr}(\mathcal{A}^T(X)\rho)$$

and vice-versa. In the Heisenberg picture, we wish to categorize the generators of the semigroups $\Phi_t^T := e^{t\mathcal{G}^T}$. In this space, the condition $\lim\limits_{t\to 0^+} \operatorname{tr}|\Phi_t\rho - \rho|$ is replaced with *norm-continuity*; that is, $\lim\limits_{t\to 0^+} ||\Phi_t^T - \mathbb{1}|| = 0$. The trace-preserving condition is replaced by $\Phi_t^T(I) = I$, i.e. Φ_t^T must be *unital*. So in the Heisenberg picture, we wish to completely categorize the generators of completely positive unital dynamical semigroups $\Phi_t^T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$. Related to this categorization arises the concept of a *completely dissipative* operator.

For a bounded operator L, Lindblad [8] defines the dissipation function $D(L) : \mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ by

$$D(L; X, Y) := L(X^\dagger Y) - L(X^\dagger)Y - X^\dagger L(Y).$$

A bounded operator L is then said to be *completely dissipative* if the following conditions hold:

(i) $L(\mathbb{1}) = 0$,

(ii) $L(X^\dagger) = L(X)^\dagger$ for all $X \in \mathcal{B}(\mathcal{H})$, and

(iii) $D(L \otimes \mathbb{1}_n, X, X) \geqslant 0$ for all $X \in M_n(\mathcal{B}(\mathcal{H}))$ and all $n \in \mathbb{N}$.

3.2.3 Lindblad's Representation Theorem

In [8], Lindblad produces the following representation theorem for the generators of completely positive, unital, and norm-continuous dynamical semigroups in the Heisenberg picture:

Proposition 1 (Lindblad, 1976)**.** *Let $L : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a bounded map and define $\Phi_t := e^{tL}$. The following are equivalent:*

(i) *Φ_t is completely positive, unital, and norm-continuous*

(ii) *L is completely dissipative*

(iii) *There exists a completely positive map Ψ and a self-adjoint operator H such that for all $X \in \mathcal{B}(\mathcal{H})$,*

$$L(X) = i[H, X] + \Psi(X) - \frac{1}{2}\{\Psi(I), X\}. \tag{9}$$

Transforming (9) back into the Schrödinger picture will yield the arbitrary form for a Lindbladian acting on density matrices. Assume $\mathcal{G}^T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is the bounded generator given in (9) with corresponding completely positive map Ψ^*. Then

$$\begin{aligned} \mathrm{tr}(X\mathcal{G}(\rho)) &= \mathrm{tr}(\mathcal{G}^T(X)\rho) \\ &= \mathrm{tr}(i[H, X]\rho + \Psi^*(X)\rho - \frac{1}{2}\{\Psi^*(I), X\}\rho) \\ &= \mathrm{tr}X(-i[H, \rho] + \Psi(\rho) - \frac{1}{2}\{\Psi^*(I), \rho\}). \end{aligned}$$

Thus, $\mathcal{G}(\rho) = -i[H, \rho] + \Psi(\rho) - \frac{1}{2}\{\Psi^*(I), \rho\}$ and the evolution equation for density matrices ρ_t in an open quantum system is given by the master equation (2), after introducing the scaling parameter g.

3.3 The Translation-Covariance Assumption

As the evolution equation (2) may be decomposed into a purely Hamiltonian part and a purely Lindbladian part, let us begin by considering these terms separately. For the Lindbladian operator (3), we may yield a further decomposition using the Choi-Kraus Theorem given in [15], [16]:

Proposition 2 (Choi/Kraus, 1975)**.** *A linear map $\mathcal{A} : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ is completely positive if and only if*

$$\mathcal{A}\rho = \sum_j A_j \rho A_j^\dagger$$

for $A_j \in \mathcal{B}_1(\mathcal{H})$.

Applying this to the completely positive maps Ψ, we may express (3) as

$$\mathcal{L}\rho = \sum_j (V_j \rho V_j^\dagger - \frac{1}{2}\{V_j^\dagger V_j, \rho\})$$

for $V_j \in \mathcal{B}_1(\mathcal{H})$. Due to Proposition 1, there exists a corresponding quantum Markov semigroup $\Phi_t : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ given by $\Phi_t = e^{t\mathcal{L}}$. Let us assume that this Markov semigroup is *translation-covariant.* That is, for τ_x the translation operator by x, assume

$$\Phi_t(\tau_x^* \rho \tau_x) = \tau_x^* \Phi_t(\rho) \tau_x \tag{10}$$

for every density matrix $\rho \in \mathcal{B}_1(\mathcal{H})$ and every $x \in \mathbb{Z}^d$. This assumption reflects certain symmetries present in the environment and is thus a very physically reasonable assumption to make. Assuming translation-covariance of the semigroup allows us to categorize a certain class of Lindbladians by utilizing a very helpful theorem from Holevo [11], which I state here in the Schrödinger picture:

Proposition 3 (Holevo, 1993)**.** *Let L be the generator of a translation-covariant trace-*

preserving dynamical semigroup in $\mathcal{B}_1(\ell^2(\mathbb{Z}^d))$. Then

$$L\rho = -i[H,\rho] + \int_{\mathbb{T}^d} \sum_j (V_\theta L_j(\theta)\rho L_j(\theta)^* V_\theta^* - \{L_j(\theta)^* L_j(\theta), \rho\}) d\theta,$$

where $H \in SA(\ell^2(\mathbb{Z}^d))$ satisfies $[H, \tau_x] = 0$ for all $x \in \mathbb{Z}^d$, V_θ is a unitary representation of $\mathbb{T}^d$, $L_j(\theta)$ are weak- measureable functions satisfying $[L_j(\theta), \tau_x] = 0$ for all $x \in \mathbb{Z}^d$, and the integral*

$$\int_{\mathbb{T}^d} \sum_j L_j(\theta) L_j(\theta)^*$$

weak- converges.*

Utilizing Proposition 3, our Lindbladian operator may further be decomposed as

$$\mathcal{L}(\rho) = \int_{\mathbb{T}^d} \sum_j (e^{i\theta X} L_j(\theta)\rho L_j(\theta)^* e^{-i\theta X} - \{L_j(\theta)^* L_j(\theta), \rho\}) d\theta$$

where X represents the position operator. That is, we may express the completely positive map Ψ in (3) as

$$\Psi(\rho) = \int_{\mathbb{T}^d} d\theta e^{i\theta X} M_\theta(\rho) e^{-i\theta X} \tag{11}$$

where the operators

$$M_\theta(\rho) := \sum_j L_j(\theta)\rho L_j(\theta)^*$$

commute with translations. An important result of this decomposition is that the operators Ψ are also translation-covariant. This yields a very nice structure for the Lindbladian.

3.4 Generators of Markov Jump Processes

In this section, we give some background into continuous-time Markov processes, which may be found in various sources such as [17], [18]. In particular, we will focus on the specific Markov process known as a *jump process*. Jump processes are related to the structure of the Lindbladian, as will be shown in more detail in Section 5.2. In Chapter 6, we will utilize this structure to prove the small g asymptotic result given by Theorem 2.

Let X_t be a continuous-time Markov process on a locally compact metric space with homogeneous transition functions given by T_t. That is,

$$\mathbb{E}[f(X_t)|\mathcal{F}_s] = T_{t-s}f(X_s),$$

where $\mathcal{F}_t$ is the natural filtration $(\mathcal{F}_t) = (\sigma(X_u, u \leqslant t))$. According to the Chapman-Kolmogorov equation, the family of transition functions $\{T_t : t \geqslant 0\}$ form a dynamical semigroup (Also see Hille-Yosida theory for more details). Hence we may define the infinitesimal generator $\mathcal{A}$ of the Markov process in the usual way by

$$(\mathcal{A}f)(x) := \lim_{t \to 0^+} \frac{1}{t}(T_t f - f).$$

The generator of a Markov process is a way to describe how the process moves from point to point in infinitesimally small increments, and thus it is important to be able to categorize the process in a meaningful way. In general, generators of Markov processes may be very complex. However, we may restrict to a specific class of Markov process to be able to categorize them quite nicely. For instance, if we assume the transition functions act on the space $C_0(X)$ and are contractive $(||T_t|| \leqslant 1 \quad \forall t)$ and norm-continuous $(\lim_{t \to 0^+} ||T_t f - f|| = 0 \quad \forall f \in C_0(X))$, the Markov process is called a *Feller process*. For a Feller process, the infinitesimal generator may be categorized via the following Proposition by Revuz and Yor [17]:

Proposition 4. *Let X_t be a real-valued Feller process on a locally compact smooth manifold X. Then the infinitesimal generator $\mathcal{A}$ is given by*

$$\begin{aligned}(\mathcal{A}f)(x) = c(x)f(x) + b(x) \cdot \nabla f(x) + \frac{1}{2} div\; a(x) \nabla f(x) \\ + \int_{X \setminus \{x\}} \left[f(y) - f(x) - \frac{y - x}{1 + |y - x|^2} \cdot \nabla f(x) \right] R(dy|x),\end{aligned}$$

where $R(dy|x)$ is a positive conditional Radon measure on $X \setminus \{x\}$, $a(x)$ is symmetric and nonnegative, and $c(x) \leqslant 0$.

Heuristically, this generator describes a Markov process which moves from a position x via translation by $b(x)$, diffuses via a gaussian with covariance $a(x)$, and jumps via $R(\cdot|x)$. The term $c(x)$ describes the killing probability, allowing for the process to be terminated at some future time. While this is the most general form for the generator of a Feller process, we are only interested in the special case where a particle's movement is governed solely by jumps, i.e. a pure jump process. In this process, the particle waits an exponential time at a position x, jumps to a position y instantaneously, then repeats this process, jumping to a new position. For a pure jump process, the infinitesimal generator will simply be given by

$$(\mathcal{A}f)(x) = \int_{X\setminus\{x\}} [f(y) - f(x)]R(dy|x), \tag{12}$$

where the rate at which the particle jumps from y to x is given by $R(dy|x)$.

CHAPTER 4: LINDBLADIAN STRUCTURE AND ASSUMPTIONS

4.1 Quasi-Momentum Space

Let us now take a Fourier transform and consider our operators in the momentum representation. We define our Fourier transform on the square-integrable kernel $\rho_t(x,x')$ in the following way. First, we may split $\mathbb{Z}^d$ into a finite set of equivalence classes $\Sigma = \mathbb{Z}^d/\sim$ such that for $x,y \in \mathbb{Z}^d$, $x \sim y$ if and only if $x - y \in \{Qn : n \in \mathbb{Z}^d\}$. Then for $\sigma, \sigma' \in \Sigma$,

$$\widehat{\rho}_t(p,p')_{\sigma,\sigma'} = \sum_{x\in\sigma, x'\in\sigma'} e^{-i\left(x\cdot Q^{-1}p - x'\cdot Q^{-1}p'\right)}\rho_t(x,x') \tag{13}$$

where $\widehat{\rho}_t(p,p') : \mathbb{T}^{2d} \to \mathcal{B}_1(\mathbb{C}^{|\Sigma|})$ is now matrix-valued. Applying this transform to the maps (11) yields

$$\begin{aligned}
\Psi(\widehat{\rho}_t)(p,p')_{\sigma,\sigma'} &= \int_{\mathbb{T}^d} d\theta \sum_{x\in\sigma, x'\in\sigma'} e^{-i(x\cdot Q^{-1}(p-Q\theta) - x'\cdot Q^{-1}(p'-Q\theta))} M_\theta(\rho_t)(x,x') \\
&= \int_{\mathbb{T}^d} d\theta \widehat{M_\theta(\rho_t)}(p - Q\theta, p' - Q\theta)_{\sigma,\sigma'}.
\end{aligned}$$

Using the specific form for M_θ, we decompose this operator further. We note that since $L_j(\theta)$ are translation-invariant, we may define

$$L_j(\theta; x-y) := \langle x | L_j(\theta) | y \rangle$$

and using this notation,

$$\begin{aligned}
&\widehat{M_\theta(\rho_t)}(p,p')_{\sigma,\sigma'} \\
&\quad = \sum_{x\in\sigma, x'\in\sigma'} e^{-i(x\cdot Q^{-1}p - x'\cdot Q^{-1}p')} \sum_{n,m=1}^{|\Sigma|} \sum_{y\in\sigma_n, y'\in\sigma'_m} L_j(\theta; x-y)\rho_t(y,y')L_j(\theta; x'-y')^* \\
&\quad = \sum_{n,m=1}^{|\Sigma|} \sum_{x\in\sigma-\sigma_n} e^{-ix\cdot Q^{-1}p} L_j(\theta; x)\widehat{\rho}_t(p,p')_{\sigma_n,\sigma'_m} \sum_{x'\in\sigma'-\sigma'_m} (e^{-ix'\cdot Q^{-1}p'} L_j(\theta; x'))^* \\
&\quad = \sum_j (\widehat{L}_j(\theta; p)\widehat{\rho}_t(p,p')\widehat{L}_j(\theta; p')^\dagger)_{\sigma,\sigma'},
\end{aligned}$$

where

$$\widehat{L}_j(\theta;p)_{\sigma,\sigma'} := \sum_{x\in\sigma-\sigma'} e^{-ix\cdot Q^{-1}p} L_j(\theta;x).$$

Therefore,

$$\Psi(\widehat{\rho}_t)(p,p') = \int_{\mathbb{T}^d} d\theta \widehat{M}_\theta(p-Q\theta, p'-Q\theta)[\widehat{\rho}_t(p-Q\theta, p'-Q\theta)]$$

for the operators $\widehat{M}_\theta$ given by

$$\widehat{M}_\theta(p,p')[A] := \sum_j \widehat{L}_j(\theta;p) A \widehat{L}_j(\theta;p')^\dagger.$$

We then define

$$\widehat{\rho}_{t;k}(p) := \widehat{\rho}_t\left(p-\frac{k}{2}, p+\frac{k}{2}\right) \tag{14}$$

where $p, k \in \mathbb{T}^d$ and we think about $\widehat{\rho}_{t;k}$ as fibers over $\widehat{\rho}_t$, indexed by $k \in \mathbb{T}^d$. We note that for the density matrix $\rho_t := |\psi_t\rangle\langle\psi_t|$, we have

$$\begin{aligned}
(\widehat{|\psi_t\rangle\langle\psi_t|})_k(p)_{\sigma,\sigma'} &= \sum_{x\in\sigma, x'\in\sigma'} e^{-i\left(x\cdot Q^{-1}\left(p-\frac{k}{2}\right) - x'\cdot Q^{-1}\left(p+\frac{k}{2}\right)\right)} \psi_t^*(x)\psi_t(x') \\
&= \widehat{\psi}_t^*\left(p-\frac{k}{2}\right)_\sigma \widehat{\psi}_t\left(p+\frac{k}{2}\right)_{\sigma'}
\end{aligned}$$

where

$$\widehat{\psi}_t(p)_\sigma := \sum_{x\in\sigma} e^{ix\cdot Q^{-1}p}\psi_t(x).$$

Since $|\psi_t\rangle \in \ell^2(\mathbb{Z}^d)$, $|\widehat{\psi}_t\rangle \in L^2(\mathbb{T}^d; \mathbb{C}^{|\Sigma|})$. Cauchy-Schwarz then yields $(\widehat{|\psi_t\rangle\langle\psi_t|})_k \in L^1(\mathbb{T}^d; \mathcal{B}_1(\mathbb{C}^{|\Sigma|}))$. By extension, we have $\widehat{\rho}_{t;k} \in L^1(\mathbb{T}^d; \mathcal{B}_1(\mathbb{C}^{|\Sigma|}))$ given any density matrix $\rho_t \in \mathcal{B}_1(\ell^2(\mathbb{Z}^d))$. It will be useful later to define the pairing

$$\langle A, B\rangle := \mathrm{tr} \int_{\mathbb{T}^d} dp A(p)B(p) \tag{15}$$

whenever $A(p)B(p) \in L^1(\mathbb{T}^d; \mathcal{B}_1(\mathbb{C}^{|\Sigma|}))$.

Using this fibering, we may write equation (11) as

$$\Psi(\widehat{\rho}_{t;k})(p) = \int_{\mathbb{T}^d} d\theta \widehat{M}_\theta \left(p - \frac{k}{2} - Q\theta, p + \frac{k}{2} - Q\theta\right) [\widehat{\rho}_{t;k}(p - Q\theta)] =: (T_k \widehat{\rho}_{t;k})(p). \tag{16}$$

Remark: At the $k = 0$ fiber, this integral takes on the simple form

$$(T_0 A)(p) = \int_{\mathbb{T}^d} dp' \widehat{M}_{Q^{-1}(p-p')}(p', p')[A(p')].$$

To compute $\Psi^*(I)$, we observe:

$$\begin{aligned}
&\mathrm{tr}(\Psi^*(I)\widehat{\rho}_t(p,p')) \\
&\qquad = \mathrm{tr}(\Psi(\widehat{\rho}_t)(p,p')) \\
&\qquad = \int_{\mathbb{T}^d} dp \int_{\mathbb{T}^d} d\theta \sum_j \mathrm{tr}(\widehat{L}_j(\theta; p - Q\theta)^\dagger \widehat{L}_j(\theta; p - Q\theta)\widehat{\rho}_t(p - Q\theta, p - Q\theta)) \\
&\qquad = \mathrm{tr} \int_{\mathbb{T}^d} d\theta \sum_j |\widehat{L}_j(\theta; p)|^2 \widehat{\rho}_t(p, p'),
\end{aligned}$$

and so $(\Psi^*(I)\widehat{\rho}_t)(p, p') = D(p)\widehat{\rho}_t(p, p')$ where

$$D(p) := \int_{\mathbb{T}^d} d\theta \sum_j |\widehat{L}_j(\theta; p)|^2. \tag{17}$$

Similarly, $(\widehat{\rho}_t \Psi^*(I))(p, p') = \widehat{\rho}_t(p, p')D(p')$. This allows us to write the simplified form of our Lindblad operator (3) in the momentum representation as

$$\widehat{\mathcal{L}}_k = T_k - \mathcal{D}_k,$$

where

$$(\mathcal{D}_k A)(p) = \frac{1}{2}\left(D\left(p - \frac{k}{2}\right) A(p) + A(p) D\left(p + \frac{k}{2}\right)\right)$$

and T_k is given in (16).

Let us now focus on the Hamiltonian term. In the momentum representation, we note that due to the periodicity of the Hamiltonian, $(H\rho_t)(x,x')$ becomes

$$(\widehat{H\rho_t})(p,p')_{\sigma,\sigma'} = \sum_{x\in\sigma, x'\in\sigma'} e^{-i\left(x\cdot Q^{-1}p - x'\cdot Q^{-1}p'\right)} \sum_{n=1}^{|\Sigma|}\sum_{y\in\sigma_n} H(x,y)\rho_t(y,x')$$
$$= \sum_{n=1}^{|\Sigma|} \widehat{H}(p)_{\sigma,\sigma_n}\widehat{\rho}_t(p,p')_{\sigma_n,\sigma'},$$

where

$$\widehat{H}(p)_{\sigma,\gamma} := \sum_{x\in\sigma-\gamma} e^{-ix\cdot Q^{-1}p} H(x+\gamma,\gamma). \tag{18}$$

Similarly,

$$(\widehat{\rho_t H})(p,p')_{\sigma,\sigma'} = \sum_{n=1}^{|\Sigma|} \widehat{\rho}_t(p,p')_{\sigma,\sigma_n}\widehat{H}(p')_{\sigma_n,\sigma'}$$

and therefore applying the Fourier transform to $[H,\rho_t](x,x')$ yields

$$(\mathcal{J}_k\widehat{\rho}_{t;k})(p) := \widehat{H}\left(p-\frac{k}{2}\right)\widehat{\rho}_{t;k}(p) - \widehat{\rho}_{t;k}(p)\widehat{H}\left(p+\frac{k}{2}\right).$$

Combining this with the Lindblad operator, we may write the evolution equation (2) as

$$\partial_t\widehat{\rho}_{t;k} = -\mathcal{G}_k\widehat{\rho}_{t;k} \tag{19}$$

where

$$\mathcal{G}_k = i\mathcal{J}_k - g(T_k - \mathcal{D}_k). \tag{20}$$

4.2 The Jump Process

As our method of proof involves a perturbation argument similar to approaches taken in [1], [3], and [2], it is natural to consider the $k=0$ fiber. At $k=0$, we have some additional structure for the Lindbladian that will be very useful in the proof of positivity for the density matrix $\mathbf{D}$. First, for the translation-invariant case $Q=I$, the operators $\widehat{L}_j$

and $\widehat{\rho}_t$ are no longer matrix-valued, and hence commute. This yields

$$(\widehat{\mathcal{L}}_0\widehat{\rho}_t)(p) = \int_{\mathbb{T}^d} d\theta \sum_j |\widehat{L}_j(\theta; p-\theta)|^2 \widehat{\rho}_t(p-\theta) - \int_{\mathbb{T}^d} d\theta \sum_j |\widehat{L}_j(\theta; p)|^2 \widehat{\rho}_t(p).$$

This is of the form (12) where the rate of jumping from θ to p is given by $\sum_j |\widehat{L}_j(p-\theta;\theta)|^2 d\theta$ and thus $\widehat{\mathcal{L}}_0$ is the generator for a pure jump process in the translation-invariant case. However, this is not true for an arbitrary Q. In order to yield this structure for the Lindbladian, we first must project onto the subspace given by $\ker\mathcal{J}_0$.

Lemma 1. *Let* Π *be the projection onto* $\ker\mathcal{J}_0$. *Then* $\Pi\widehat{\mathcal{L}}_0\Pi$ *is the generator for a jump process on* $\mathbb{C}^{|\Sigma|} \times \mathbb{T}^d$.

Proof. Since $\widehat{H}(p)$ is a $|\Sigma| \times |\Sigma|$ matrix, we may list the eigenvalues as $\lambda_1(p), \cdots, \lambda_{|\Sigma|}(p)$ with corresponding eigenvectors $\psi_1(p), \cdots, \psi_{|\Sigma|}(p)$. Denote $E_{ij}(p) := |\psi_i(p)\rangle\langle\psi_j(p)|$ as the corresponding matrix element in this basis. As the kernel of $\mathcal{J}_0$ is the commutant of $\widehat{H}$, in the above framework, we may write that Π is the projection onto diagonal matrices in the basis $\{\psi_i(p)\}_{i=1}^{|\Sigma|}$, i.e.

$$(\Pi A)(p) = \sum_{i=1}^{|\Sigma|} A_{ii}(p) E_{ii}(p).$$

Furthermore, for any matrix $A(p)$,

$$\begin{aligned}
(\Pi T_0 \Pi A)(p) &= \sum_j E_{jj}(p) \int_{\mathbb{T}^d} dp' \widehat{M}_{Q^{-1}(p-p')}(p', p')[\Pi A(p')] E_{jj}(p) \\
&= \sum_{i,j} \int_{\mathbb{T}^d} dp' A_{ii}(p') \langle\psi_j(p)| \widehat{M}_{Q^{-1}(p-p')}(p', p')[E_{ii}(p')] |\psi_j(p)\rangle E_{jj}(p)
\end{aligned}$$

and similarly,

$$\begin{aligned}
(\Pi\mathcal{D}_0\Pi A)(p) &= \sum_j E_{jj}(p) \sum_i A_{ii}(p)(D(p)E_{ii}(p) + E_{ii}(p)D(p)) E_{jj}(p) \\
&= \sum_j A_{jj}(p) \langle\psi_j(p)| D(p) |\psi_j(p)\rangle E_{jj}(p).
\end{aligned}$$

The operator $\Pi\widehat{\mathcal{L}}_0\Pi$ will thus be given by

$$(\Pi\widehat{\mathcal{L}}_0\Pi A)(p) = \sum_{i,j}\left\{\int_{\mathbb{T}^d} dp' r((j,p),(i,p'))A(i,p') - \int_{\mathbb{T}^d} dp' r((i,p'),(j,p))A(j,p)\right\} E_{jj}(p) \tag{21}$$

where

$$\begin{aligned} r((j,p),(i,p')) &:= \langle\psi_j(p)|\widehat{M}_{Q^{-1}(p-p')}(p',p')[E_{ii}(p')]|\psi_j(p)\rangle \\ &= \sum_k |\langle\psi_j(p)|\widehat{L}_k(Q^{-1}(p-p');p')|\psi_i(p')\rangle|^2 \geqslant 0 \end{aligned}$$

and $A(j,p) = A_{jj}(p)$. This is again of the form (12) and hence $\Pi\widehat{\mathcal{L}}_0\Pi$ is the generator for a jump process on $\mathbb{C}^{|\Sigma|}\times\mathbb{T}^d$ with rate of jumping from (i,p') to (j,p) given by $r((j,p),(i,p'))dp'$. □

4.3 Assumptions

At this point, we make some additional assumptions, which are slightly stronger conditions than are often taken for Lindblad operators of this form (See for instance [1]). We assume the following:

Assumptions:

1. (Nondegeneracy of the kernel) $\ker\widehat{\mathcal{L}}_0^T = \langle I\rangle$,

2. (Uniform Dissipation at all Momenta) $\frac{1}{C} \leqslant D(p) \leqslant C$ for some $C > 0$,

3. (Reflection invariance) $[\mathcal{L}, R] = 0$ for the reflection operator $(R\psi)(x) = \psi(-x)$,

4. (Local Uniform Positivity of the Integral Kernel) There exist constants $\delta > 0$ and $\chi > 0$ such that

   $$\widehat{M}_{Q^{-1}(p-p')}(p',p')[A(p')] \geqslant \frac{1}{\chi} I$$

   for all operators $A \in L^1(\mathbb{T}^d;\mathcal{B}_1(\mathbb{C}^{|\Sigma|}))$ satisfying $A(p') \geqslant 0$ and $\mathrm{tr}A(p') = 1$ whenever $|p-p'| < \delta$,

5. (Uniform Boundedness of the Integral Kernel) There exists a constant $\chi > 0$ such that

$$\widehat{M}_{Q^{-1}(p-p')}(p',p')[A(p')] \leqslant \chi I$$

for all operators $A \in L^1(\mathbb{T}^d; \mathcal{B}_1(\mathbb{C}^{|\Sigma|}))$ satisfying $A(p') \geqslant 0$ and $\mathrm{tr}A(p') = 1$ and all $p, p' \in \mathbb{T}^d$,

6. (Ergodicity of $\Pi\widehat{\mathcal{L}}_0\Pi$) For a.e. $p \in \mathbb{T}^d$ and every function $\phi \geqslant 0$ with $\int_{\mathbb{T}^d} dp\phi(p) = 1$, there exists $n \in \mathbb{N}$ such that $\left(\frac{1}{\Pi\mathcal{D}_0\Pi}\Pi T_0\Pi\right)^n \phi(p) > 0$.

Assumptions 1-4 are needed to prove Theorem 1, and Theorem 2 additionally requires Assumptions 5 and 6. I now describe these assumptions in detail, as well as some useful implications of each.

We note that clearly, $\widehat{\mathcal{L}}_0^T I = 0$. The first assumption guarantees that I is in fact the only equilibrium eigenvector for $\widehat{\mathcal{L}}_0^T$. The second assumption is utilized in Lemma 2 to guarantee a gapping in the spectrum of $\mathcal{G}_0$. Looking closer at the third assumption, we see that the reflection operator R is actually the operator $(\widehat{R}A_k)(p)_{\sigma,\sigma'} = A_{-k}(-p)_{-\sigma,-\sigma'}$ in momentum space. Hence the condition $[\widehat{R}, \widehat{\mathcal{L}}_k] = 0$ yields

$$\begin{aligned} -(\nabla_k \widehat{\mathcal{L}}_k|_{k=0} A)(p)_{\sigma,\sigma'} &= (\widehat{R}\nabla_k \widehat{\mathcal{L}}_k|_{k=0} A)(-p)_{-\sigma,-\sigma'} \\ &= (\nabla_k \widehat{\mathcal{L}}_k|_{k=0} \widehat{R}A)(-p)_{-\sigma,-\sigma'} \\ &= (\nabla_k \widehat{\mathcal{L}}_k|_{k=0} A)(p)_{\sigma,\sigma'}. \end{aligned}$$

So the third assumption guarantees

$$\nabla_k \widehat{\mathcal{L}}_k|_{k=0} = 0. \tag{22}$$

That is, we have zero-drift for a particle governed solely by the Lindbladian.

Let us define the operator

$$\mathcal{O}_n := e^{-r_n(i\mathcal{J}_0 + g\mathcal{D}_0)} T_0 e^{-r_{n-1}(i\mathcal{J}_0 + g\mathcal{D}_0)} T_0 \cdots T_0 e^{-r_0(i\mathcal{J}_0 + g\mathcal{D}_0)}$$

for some real numbers $r_0, \cdots, r_n > 0$. We remark that

$$((i\mathcal{J}_0 + g\mathcal{D}_0)A)(p) = K(p)A(p) + A(p)K^\dagger(p) \tag{23}$$

for

$$K(p) := i\widehat{H}(p) + \frac{g}{2}D(p)$$

is simply a sum of multiplication operators. This implies

$$(e^{-t(i\mathcal{J}_0+g\mathcal{D}_0)}A)(p) = e^{-tK(p)}A(p)e^{-tK^\dagger(p)}.$$

Let $A \in L^1(\mathbb{T}^d; \mathcal{B}_1(\mathbb{C}^{|\Sigma|}))$ satisfy $A(p) \geqslant 0$ for all $p \in \mathbb{T}^d$ and $\langle I, A\rangle = 1$. Assumption 4 then yields the following:

$$\begin{aligned}
(\mathcal{O}_n A)(p) &= (e^{-r_n(i\mathcal{J}_0+g\mathcal{D}_0)}T_0\mathcal{O}_{n-1}A)(p) \\
&= e^{-r_nK(p)}(T_0\mathcal{O}_{n-1}A)(p)e^{-r_nK^\dagger(p)} \\
&= e^{-r_nK(p)}\left\{\int_{\mathbb{T}^d} dp_1 \widehat{M}_{Q^{-1}(p-p_1)}(p_1,p_1)[(\mathcal{O}_{n-1}A)(p_1)]\right\}e^{-r_nK^\dagger(p)} \\
&\geqslant \frac{1}{\chi}|e^{-r_nK(p)}|^2 \int_{|p-p_1|<\delta} dp_1 \mathrm{tr}((\mathcal{O}_{n-1}A)(p_1)).
\end{aligned}$$

Due to Gronwall's Inequality and Assumption 2,

$$|e^{-r_nK(p)}|^2 \geqslant e^{-Cgr_n}$$

and hence,

$$(\mathcal{O}_n A)(p) \geqslant \frac{1}{\chi}e^{-Cgr_n}\int_{|p-p_1|<\delta} dp_1 \mathrm{tr}((\mathcal{O}_{n-1}A)(p_1)).$$

Repeating this argument n times, we have

$$(\mathcal{O}_n A)(p) \geqslant \frac{|\Sigma|^{n-1}}{\chi^n}e^{-Cg(r_0+\cdots+r_n)}\int_{|p-p_1|<\delta} dp_1 \cdots \int_{|p_{n-1}-p_n|<\delta} dp_n \mathrm{tr}(A(p_n)).$$

If n is sufficiently large ($n\delta > 2\pi$), we will have

$$\int_{|p-p_1|<\delta} dp_1 \cdots \int_{|p_{n-1}-p_n|<\delta} dp_n \mathrm{tr}(A(p_n)) \geqslant C_{n,\delta}\int_{\mathbb{T}^d} \mathrm{tr}(A(p)) = C_{n,\delta}$$

for some constant $C_{n,\delta} > 0$ and so $(\mathcal{O}_n A)(p) > 0$ for all $p \in \mathbb{T}^d$.

Let us now define the family of operators $x(t) := e^{t(i\mathcal{J}_0 + g\mathcal{D}_0)} e^{-t\mathcal{G}_0}$ for $t > 0$. For this family, we have

$$\frac{d}{dt} x(t) = g e^{t(i\mathcal{J}_0 + g\mathcal{D}_0)} T_0 e^{-t(i\mathcal{J}_0 + g\mathcal{D}_0)} x(t).$$

Hence,

$$x(t) = \sum_{n=0}^{\infty} g^n \int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-1}} ds_n e^{s_1(i\mathcal{J}_0+g\mathcal{D}_0)} T_0 e^{(s_2-s_1)(i\mathcal{J}_0+g\mathcal{D}_0)} T_0 \cdots T_0 e^{-s_n(i\mathcal{J}_0+g\mathcal{D}_0)}$$

for some $n \in \mathbb{N}$ and some constants $s_1, \cdots, s_n > 0$. This finally implies

$$(e^{-t\mathcal{G}_0} A)(p) = \sum_{n=0}^{\infty} g^n \int_0^t ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{n-1}} ds_n (\mathcal{O}_n A)(p)$$

where the real numbers $r_0, \cdots, r_n$ in $\mathcal{O}_n$ are defined by $r_0 = s_n$, $r_1 = s_{n-1} - s_n, \cdots$, $r_{n-1} = s_1 - s_2$, $r_n = t - s_1$. So Assumption 4 guarantees that

$$\begin{aligned} &(e^{-t\mathcal{G}_0} A)(p) > 0 \; \forall \; A \in L^1(\mathbb{T}^d; \mathcal{B}_1(\mathbb{C}^{|\Sigma|})) \text{ satisfying} \\ &\qquad\qquad A(p) \geqslant 0 \; \forall \; p \in \mathbb{T}^d \text{ and } \langle I, A \rangle = 1. \end{aligned} \tag{24}$$

The fifth assumption will be utilized in Section 6.2 to bound the invariant state of $\mathcal{G}$. Finally, the sixth assumption is utilized to guarantee ergodicity of the underlying jump for the jump process $\Pi \widehat{\mathcal{L}}_0 \Pi$.

4.4 The Spectrum

In order to compute the spectrum of $\mathcal{G}_0$, we must utilize a result from Deimling [12] and Schaefer [19] on the eigenvalues on the boundary of the spectral radius. For a Banach space X, we define a *total cone* $K \subset X$ to be a closed convex set such that $\lambda K \subset K$ for all $\lambda \geqslant 0$, $K \cap (-K) = \{0\}$, and $\overline{K - K} = X$. The *dual cone* of K is then defined as

$$K^* := \{x^* \in X^* : \mathrm{Re} x^*(x) \geqslant 0 \text{ on } K\}.$$

An operator $T \in \mathcal{B}(X)$ is considered to be *quasicompact* if $T^n = T_1 + T_2$ for some $n \in \mathbb{N}$, T_1 is bounded with $r(T_1) < (r(T))^n$ and T_2 is compact. Here, $r(T) = \sup_{\lambda \in \sigma(T)} |\lambda|$ denotes the spectral radius of T.

Proposition 5 (Theorem 19.5 in Deimling, 1985 and Proposition 5.1 in Schaefer, 1974)**.** *Let X be a Banach space, $K \subset X$ a total cone, and $T \in \mathcal{B}(X)$ a positive quasicompact operator satisfying that for each $x \in K\backslash\{0\}$, there exists $n \in \mathbb{N}$ such that $x^*(T^n x) > 0$ for all $x^* \in K^*\backslash\{0\}$. Then for $r(T)$ the spectral radius of T, we have the following:*

(a) $r(T) > 0$ and $r(T)$ is a simple eigenvalue with a positive eigenvector v such that $x^*(v) > 0$ for all $x^* \in K^*\backslash\{0\}$.

(b) $|\lambda| < r(T)$ for all $\lambda \in \sigma(T)\backslash\{r(T)\}$.

We can then utilize Proposition 5 to prove the following Lemma regarding the spectrum of our operator at the zero fiber:

Lemma 2. *The operator $\mathcal{G}_0$ defined in (20) has spectrum given by*

$$\sigma(\mathcal{G}_0) = \{0\} \cup \Sigma_0,$$

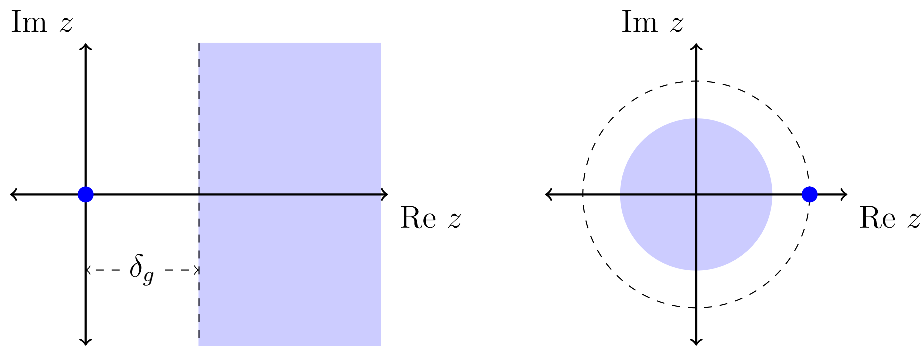
where $\Sigma_0 \subseteq \{\mathrm{Re} z \geqslant \delta_g\}$ for some $\delta_g > 0$. Furthermore, 0 is a nondegenerate eigenvalue of $\mathcal{G}_0$ for which the corresponding eigenvector F_{eq} satisfies $F_{eq}(p) > 0$ for a.e. $p \in \mathbb{T}^d$.

Proof. Due to (20) we have $\sigma_{ess}(\mathcal{G}_0) = \sigma_{ess}(i\mathcal{J}_0 + g\mathcal{D}_0)$ since T_0 is compact and essential spectrum is invariant under compact perturbations. Due to (23) and Lemma 4, we have

$$\begin{aligned}\sigma_{ess}(i\mathcal{J}_0 + g\mathcal{D}_0) &= \bigcup_p \sigma(K(p)\cdot + \cdot K^\dagger(p)) \\ &\subseteq \bigcup_p \left(\sigma(K(p)\cdot) + \sigma(\cdot K^\dagger(p))\right).\end{aligned}$$

Assumption 2 yields

$$\bigcup_p \sigma(K(p)\cdot), \bigcup_p \sigma(\cdot K^\dagger(p)) \subseteq \left\{z \in \mathbb{C} : \mathrm{Re} z \geqslant \frac{g}{2C}\right\}$$



(a) $\sigma(\mathcal{G}_0)$ (b) $\sigma(e^{-\mathcal{G}_0})$

Figure 1: A visualization of the spectrum of $\mathcal{G}_0$. In (a), $\sigma(\mathcal{G}_0)$ is shown with isolated eigenvalue at 0 and the remaining spectrum included in the right half plane after a gap δ_g. In (b), $\sigma(e^{-\mathcal{G}_0})$ is shown with isolated eigenvalue at 1 and the remaining spectrum included in the unit circle.

and so $\sigma_{ess}(\mathcal{G}_0) \subseteq \left\{z \in \mathbb{C} : \text{Re} z \geqslant \frac{g}{C}\right\}$. The remaining spectrum will be discrete spectrum.

As suggested in Figure 1, we will compute the discrete spectrum of $\mathcal{G}_0$ by considering the related operator $e^{-\mathcal{G}_0}$ and utilizing Proposition 5. This operator will act on elements of $L^1(\mathbb{T}^d; SA(\mathbb{C}^{|\Sigma|}))$. Therefore, we will consider the total cone given by $K = \{A \in L^1(\mathbb{T}^d; SA(\mathbb{C}^{|\Sigma|})) : A(p) \geqslant 0 \text{ for a.e. } p \in \mathbb{T}^d\}$. We note that $e^{-\mathcal{G}_0}$ is certainly a positive operator. To see that it is quasicompact, we first realize that $\sigma(\mathcal{G}_0) \subseteq \{z \in \mathbb{C} : \text{Re} z \geqslant 0\}$ implies $\sigma(e^{-\mathcal{G}_0}) \subseteq \{z \in \mathbb{C} : |z| \leqslant 1\}$. Then due to our above observations about the essential spectrum, we additionally have $\sigma_{ess}(e^{-\mathcal{G}_0}) \subseteq \{z \in \mathbb{C} : |z| \leqslant c\}$ for some $c < 1$. Define the counter-clockwise contour Γ such that $|z| < 1$ for all $z \in \Gamma$, $\sigma_{ess}(e^{-\mathcal{G}_0}) \subset \text{int } \Gamma$, and Γ does not intersect any eigenvalues of $e^{-\mathcal{G}_0}$. Then for

$$T_1 := \tfrac{1}{2\pi i} \int_\Gamma e^{-z} \tfrac{1}{z - \mathcal{G}_0} dz \quad \text{and} \quad T_2 := e^{-\mathcal{G}_0} - T_1,$$

T_1 will be bounded and T_2 will be compact since it is finite rank. Additionally, $r(T_1) < 1 = r(e^{-\mathcal{G}_0})$ due to the fact that $\mathcal{G}_0^\dagger I = 0$. Thus, $e^{-\mathcal{G}_0}$ is quasicompact. Due to (24), we may apply Proposition 5 to the operator $e^{-\mathcal{G}_0}$. This yields 1 is a simple eigenvalue, $e^{-\mathcal{G}_0} F_{eq} = F_{eq}$ for some strictly positive equilibrium eigenvector F_{eq}, and $|\lambda| < 1$ for all eigenvalues $\lambda \neq 1$. This is equivalent to $\mathcal{G}_0^\dagger$ having a one-dimensional kernel given by $\langle I \rangle$. This then implies that $\mathcal{G}_0$ has a one-dimensional kernel as well. Furthermore, the rest of the eigenvalues of $e^{-\mathcal{G}_0^\dagger}$ will lie strictly inside $B_1(0)$. Hence the rest of the discrete spectrum of $\mathcal{G}_0$ will lie in $\{\mathrm{Re} z \geqslant c_g\}$ for some constant $c_g > 0$. So Lemma 2 holds with $\delta_g := \min\left(\frac{g}{C}, c_g\right)$. □

This Lemma states that there is a unique density matrix in the kernel of $\mathcal{G}_0$. We shall label this equilibrium eigenvector F_{eq}. Since 0 is an isolated point of the spectrum, we may define the Riesz projection onto this eigenvector in the normal way as

$$P_0 = \frac{1}{2\pi i}\int_\Gamma \frac{1}{z - \mathcal{G}_0} dz = F_{eq}\langle I, \cdot\rangle, \tag{25}$$

where Γ is a counterclockwise contour in $\rho(\mathcal{G}_0)$ whose interior contains the eigenvalue 0 and no other point of $\sigma(\mathcal{G}_0)$, and the pairing $\langle\cdot,\cdot\rangle$ is given in (15).

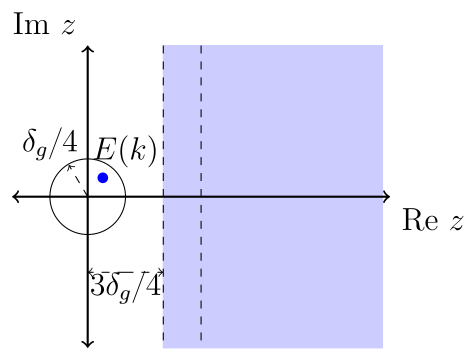


Figure 2: A visualization of the spectrum of $\mathcal{G}_k$ for sufficiently small k, which consists of a simple isolated eigenvalue $E(k) \in \{z \in \mathbb{C} : |z| \leqslant \delta_g/4\}$ and the remaining spectrum in some set $\Sigma_k \subseteq \{\mathrm{Re} z \geqslant 3\delta_g/4\}$.

We also note that the spectrum of $\mathcal{G}_k$ moves continuously with k and hence for k sufficiently small,

$$\sigma(\mathcal{G}_k) = \{E(k)\} \cup \Sigma_k$$

where $E(k)$ is some isolated nondegenerate eigenvalue in $\{z \in \mathbb{C} : |z| \leqslant \delta_g/4\}$ and $\Sigma_k \subseteq \{\mathrm{Re} z \geqslant 3\delta_g/4\}$ (See Figure 2). Since $E(k)$ is isolated and nondegenerate, we may consequently define the one-dimensional Riesz projection onto the corresponding eigenvector as

$$P_k = \frac{1}{2\pi i}\int_{\Gamma} \frac{1}{z - \mathcal{G}_k} dz, \tag{26}$$

where Γ is a counterclockwise contour in $\rho(\mathcal{G}_k)$ whose interior contains the eigenvalue $E(k)$ and no other point of $\sigma(\mathcal{G}_k)$.

4.5 A Generalized Dissipation Condition

Let us now return to the operator $\mathcal{G}_0^T$ acting on observables $X \in \mathcal{B}(L^2(\mathbb{T}^d))$ in the Heisenberg picture. Due to Proposition 1, $\mathcal{G}_0^T$ will be completely dissipative. In this section, we prove a generalized version of dissipation using the results from Lemma 2.

First, note that $\mathcal{G}_0^T = i\mathcal{J}_0 - g\widehat{\mathcal{L}}_0^T$ where

$$\widehat{\mathcal{L}}_0^T X = \Psi(X) - \frac{1}{2}\{\Psi(I), X\}$$

for every $X \in L^2(\mathbb{T}^d; \mathcal{B}(\mathbb{C}^{|\Sigma|}))$ and

$$\Psi(X)(p) = \int_{\mathbb{T}^d} d\theta \widehat{M}_\theta^T(p,p)[X(p + Q\theta)] = \int_{\mathbb{T}^d} d\theta \sum_j \widehat{L}_j(\theta; p)^\dagger X(p + Q\theta)\widehat{L}_j(\theta; p).$$

Due to Proposition 2 in Lindblad [8], we may further decompose this operator as

$$\Psi(X)(p) = \int_{\mathbb{T}^d} d\theta W_\theta^\dagger(p)(X(p + Q\theta) \otimes 1)W_\theta(p)$$

for some functions $W : \mathbb{T}^d \times \mathbb{T}^d \to \mathcal{B}(\mathbb{C}^{|\Sigma|} \otimes \mathcal{K})$ and some auxiliary Hilbert space $\mathcal{K}$. Recall the dissipation function $D(\mathcal{G}_0^T) : L^2(\mathbb{T}^d; \mathcal{B}(\mathbb{C}^{|\Sigma|})) \times L^2(\mathbb{T}^d; \mathcal{B}(\mathbb{C}^{|\Sigma|})) \to L^2(\mathbb{T}^d; \mathcal{B}(\mathbb{C}^{|\Sigma|}))$ given by

$$D(\mathcal{G}_0^T; X, Y) := \mathcal{G}_0^T(X^\dagger Y) - \mathcal{G}_0^T(X^\dagger)Y - X^\dagger \mathcal{G}_0^T(Y).$$

Using our decomposition of Ψ, we observe that

$$D(\mathcal{G}_0^T; X(p), X(p)) = -\int_{\mathbb{T}^d} d\theta |(X(p + Q\theta) \otimes 1)W_\theta(p) - W_\theta(p)X(p)|^2 \leqslant 0.$$

For a function $F \in L^2(\mathbb{T}^d; \mathcal{B}(\mathbb{C}^{|\Sigma|}))$, define the following inner product on $L^2(\mathbb{T}^d; \mathcal{B}(\mathbb{C}^{|\Sigma|}))$:

$$\langle X, Y \rangle_F := \frac{1}{2}\mathrm{tr}\int_{\mathbb{T}^d} F(p)\{X^\dagger(p), Y(p)\}.$$

We then have for F_{eq} the equilibrium eigenfunction of $\mathcal{G}_0$,

$$\begin{aligned}\mathrm{Re}\langle X, \mathcal{G}_0^T X\rangle_{F_{eq}} &= -\frac{1}{4}\mathrm{tr}\int_{\mathbb{T}^d} dp F_{eq}(p)[D(\mathcal{G}_0^T; X(p), X(p)) + D(\mathcal{G}_0^T; X^\dagger(p), X^\dagger(p))] \\ &= \frac{1}{4}\mathrm{tr}\int_{\mathbb{T}^d} dp F_{eq}(p) \int_{\mathbb{T}^d} d\theta \left[|(X(p+Q\theta)\otimes 1)W_\theta(p) - W_\theta(p)X(p)|^2 \right. \\ &\qquad \left. + |(X^\dagger(p+Q\theta)\otimes 1)W_\theta(p) - W_\theta(p)X^\dagger(p)|^2 \right].\end{aligned}$$

Hence, since $F_{eq}(p) > 0$ for a.e. $p \in \mathbb{T}^d$ by Lemma 2, $\mathrm{Re}\langle X, \mathcal{G}_0^T X\rangle_{F_{eq}} = 0$ if and only if $(X(p+Q\theta)\otimes 1)W_\theta(p) = W_\theta(p)X(p)$ and $(X^\dagger(p+Q\theta)\otimes 1)W_\theta(p) = W_\theta(p)X^\dagger(p)$ for a.e. $p, \theta \in \mathbb{T}^d$. Multiplying by $W_\theta(p)^\dagger$ and integrating, these imply $\Psi(X)(p) = \Psi(I)(p)X(p)$ and $\Psi(X)(p) = X(p)\Psi(I)(p)$ for a.e. $p \in \mathbb{T}^d$. Hence $(\widehat{\mathcal{L}_0^T} X)(p) = 0$ and so by Assumption 1, $X = I$ and $(\mathcal{G}_0^T X)(p) = 0$. This string of implications yields the observation that

$$(\mathcal{G}_0^T X)(p) \neq 0 \text{ for a.e. } p \in \mathbb{T}^d \Rightarrow \mathrm{Re}\langle X, \mathcal{G}_0^T X\rangle_{F_{eq}} > 0. \tag{27}$$

CHAPTER 5: DIFFUSIVE PROPAGATION FOR MASTER EQUATION WITH PERIODIC HAMILTONIAN

5.1 Proof of Main Result

We now have everything we need to prove Theorem 1.

Proof. To prove the central limit theorem (4), we note that

$$\mathrm{tr}\int_{\mathbb{T}^d} dp\widehat{\rho}_{t;k}(p) = \int_{\mathbb{T}^d} dp \sum_{\sigma\in\Sigma}\sum_{x,x'\in\sigma} e^{-i\left(x\cdot Q^{-1}\left(p-\frac{k}{2}\right)-x'\cdot Q^{-1}\left(p+\frac{k}{2}\right)\right)}\rho_t(x,x') = \sum_x e^{ix\cdot Q^{-1}k}\rho_t(x,x)$$

and hence,

$$\sum_x e^{i\frac{1}{\sqrt{\tau}}(x-\tau t v)\cdot Q^{-1}k}\rho_{\tau t}(x,x) = e^{-i\frac{1}{\sqrt{\tau}}\tau t v\cdot Q^{-1}k}\mathrm{tr}\int_{\mathbb{T}^d} dp\widehat{\rho}_{\tau t;k/\sqrt{\tau}}(p).$$

where the drift constant v is to be chosen later. Since $\widehat{\rho}_{t;k} = e^{-t\mathcal{G}_k}\widehat{\rho}_{0;k}$, this then gives that this term is equivalent to

$$e^{-i\frac{1}{\sqrt{\tau}}\tau t v\cdot Q^{-1}k}\mathrm{tr}\int_{\mathbb{T}^d} dp\left(e^{-\tau t\mathcal{G}_{k/\sqrt{\tau}}}\widehat{\rho}_{0;k/\sqrt{\tau}}\right)(p).$$

Consider the Riesz projection $P_{k/\sqrt{\tau}}$ as defined in (26). Introducing the projections $P_{k/\sqrt{\tau}}$ and $1-P_{k/\sqrt{\tau}}$ after the semigroup yields

$$\begin{aligned}&\sum_x e^{i\frac{1}{\sqrt{\tau}}(x-\tau t v)\cdot Q^{-1}k}\rho_{\tau t}(x,x)\\ &\quad= e^{-i\frac{1}{\sqrt{\tau}}\tau t v\cdot Q^{-1}k}\left[\langle I,\widehat{\rho}_{0;k/\sqrt{\tau}}\rangle e^{-\tau t E(k/\sqrt{\tau})} + \mathrm{tr}\int_{\mathbb{T}^d} dp\left(e^{-\tau t\mathcal{G}_{k/\sqrt{\tau}}}(1-P_{k/\sqrt{\tau}})\widehat{\rho}_{0;k/\sqrt{\tau}}\right)(p)\right].\end{aligned}$$

Let us first deal with the second term in this expression. Define a contour Γ surrounding $\sigma(\mathcal{G}_{k/\sqrt{\tau}})\backslash\{E(k/\sqrt{\tau})\}$ such that $\mathrm{Re}\Gamma \geqslant \delta_g/2$ (which is possible for sufficiently large τ, say $\tau \geqslant \frac{1}{\varepsilon}$ for some $\varepsilon > 0$). As $\mathcal{G}_{k/\sqrt{\tau}}$ is bounded, we may additionally choose Γ to be bounded, and

$$\sup_{\tau\geqslant\frac{1}{\varepsilon}, z\in\Gamma}\left\|\frac{1}{z-\mathcal{G}_{k/\sqrt{\tau}}}\right\| =: M < \infty.$$

We then have

$$\begin{aligned}
&\left\|e^{-i\frac{1}{\sqrt{\tau}}\tau tv\cdot Q^{-1}k}\mathrm{tr}\int_{\mathbb{T}^d} dp\left(e^{-\tau t\mathcal{G}_{k/\sqrt{\tau}}}(1-P_{k/\sqrt{\tau}})\widehat{\rho}_{0;k/\sqrt{\tau}}\right)(p)\right\| \\
&\qquad\leqslant \left\|\mathrm{tr}\int_{\mathbb{T}^d} dp\int_{\Gamma} dz e^{-\tau tz}\frac{1}{z-\mathcal{G}_{k/\sqrt{\tau}}}\widehat{\rho}_{0;k/\sqrt{\tau}}(p)\right\| \\
&\qquad\leqslant \sup_{\tau\geqslant\frac{1}{\varepsilon},z\in\Gamma}\left\|\frac{1}{z-\mathcal{G}_{k/\sqrt{\tau}}}\right\| ||\widehat{\rho}_{0;k/\sqrt{\tau}}||\ |\Gamma| e^{-\tau t\inf_{z\in\Gamma}\mathrm{Re}z} \\
&\qquad\leqslant M|\Gamma|\langle I,\widehat{\rho}_{0;k/\sqrt{\tau}}\rangle e^{-\tau t\delta_g/2}.
\end{aligned}$$

We note that

$$\langle I,\widehat{\rho}_{0;k/\sqrt{\tau}}\rangle=\sum_x e^{i\frac{1}{\sqrt{\tau}}x\cdot Q^{-1}k}\rho_0(x,x)\to\mathrm{tr}\rho_0$$

as $\tau\to\infty$ and hence this term will vanish in the large τ limit. For the first term, we use the Taylor expansion

$$E(k/\sqrt{\tau})=E(0)+\frac{1}{\sqrt{\tau}}\sum_i\partial_i E(0)k_i+\frac{1}{2\tau}\sum_{i,j}\partial_i\partial_j E(0)k_ik_j+o\left(\frac{1}{\tau}\right).$$

Since $E(0)$ is the isolated eigenvalue of $\mathcal{G}_0$, Lemma 2 gives $E(0)=0$. Using Feynman-Hellman,

$$\partial_i E(0)=\langle I,\partial_i\mathcal{G}_k|_{k=0}F_{eq}\rangle.$$

Hence, if we choose the drift constant to be

$$v=iQ^T\langle I,\nabla_k\mathcal{G}_k|_{k=0}F_{eq}\rangle, \tag{28}$$

then $e^{-i\frac{1}{\sqrt{\tau}}\tau tv\cdot Q^{-1}k}$ will cancel with $e^{-\tau t\frac{1}{\tau}\sum_i\partial_iE(0)k_i}$. In this case,

$$\sum_x e^{i\frac{1}{\sqrt{\tau}}(x-\tau tv)\cdot Q^{-1}k}\rho_{\tau t}(x,x)=\left(\sum_x e^{i\frac{1}{\sqrt{\tau}}x\cdot Q^{-1}k}\rho_0(x,x)\right)e^{-\frac{t}{2}\sum\limits_{i,j}\partial_i\partial_jE(0)k_ik_j}+o(1).$$

Again using the fact that $\langle I,\widehat{\rho}_{0;k/\sqrt{\tau}}\rangle\to\mathrm{tr}\rho_0$ and choosing the diffusion matrix $\mathbf{D}$ to be defined as

$$D_{i,j}=\frac{1}{2}\partial_i\partial_jE(0), \tag{29}$$

this yields

$$\lim_{\tau\to\infty}\sum_x e^{i\frac{1}{\sqrt{\tau}}(x-tv)\cdot Q^{-1}k}\rho_{\tau t}(x,x) = [\mathrm{tr}\rho_0]e^{-t\sum_{i,j}D_{i,j}k_ik_j}$$

as desired.

Now let us assume the initial condition ρ_0 satisfies $\sum\limits_{x\in\mathbb{Z}^d}|x|^2\rho_0(x,x) < \infty$. The solution to the evolution equation (19) is given by $\widehat{\rho}_{t;k} = e^{-t\mathcal{G}_k}\widehat{\rho}_{0;k}$ and so we have the following for the right-hand side of (5):

$$\begin{aligned}\frac{1}{t}\sum_{x\in\mathbb{Z}^d}x\rho_t(x,x) &= -\frac{i}{t}Q^T\mathrm{tr}\int_{\mathbb{T}^d}dp\nabla_k\widehat{\rho}_{t;k}(p)|_{k=0}\\ &= -\frac{i}{t}Q^T\mathrm{tr}\int_{\mathbb{T}^d}dp\nabla_k(e^{-t\mathcal{G}_k}\widehat{\rho}_{0;k})(p)|_{k=0}.\end{aligned}$$

After using the formula for the derivative of a semigroup and using the fact that $e^{-t\mathcal{G}_k^\dagger}I = I$ due to conservation of quantum probabilities, we have

$$\frac{1}{t}\sum_{x\in\mathbb{Z}^d}x\rho_t(x,x) = \frac{i}{t}Q^T\mathrm{tr}\int_{\mathbb{T}^d}dp\int_0^t ds\nabla_k\mathcal{G}_k|_{k=0}e^{-s\mathcal{G}_0}\widehat{\rho}_{0;0}(p) - \frac{i}{t}Q^T\mathrm{tr}\int_{\mathbb{T}^d}(\nabla_k\widehat{\rho}_{0;0})(p).$$

The second term in this expression trivially vanishes in the large t limit. For the first term, we insert the Riesz projections P_0 and $1-P_0$ after the semigroup to yield the two terms

$$\frac{i}{t}Q^T\mathrm{tr}\int_{\mathbb{T}^d}dp\int_0^t ds\nabla_k\mathcal{G}_k|_{k=0}F_{eq}(p)\langle I,\widehat{\rho}_{0;0}\rangle + \frac{i}{t}Q^T\mathrm{tr}\int_{\mathbb{T}^d}dp\int_0^t ds\nabla_k\mathcal{G}_k|_{k=0}e^{-s\mathcal{G}_0}(1-P_0)\widehat{\rho}_{0;0}(p).$$

For the projection off of P_0, we may draw a bounded contour Γ around $\sigma(\mathcal{G}_0)\backslash\{0\}$ such that $\mathrm{Re}\Gamma \geqslant \frac{\delta_g}{2}$. Hence

$$\begin{aligned}&\frac{i}{t}Q^T\mathrm{tr}\int_{\mathbb{T}^d}dp\int_0^t ds\nabla_k\mathcal{G}_k|_{k=0}e^{-s\mathcal{G}_0}(1-P_0)\widehat{\rho}_{0;0}(p)\\ &\qquad= \frac{i}{t}Q^T\mathrm{tr}\int_{\mathbb{T}^d}dp\int_0^t ds\nabla_k\mathcal{G}_k|_{k=0}\frac{1}{2\pi i}\int_\Gamma dze^{-sz}\frac{1}{z-\mathcal{G}_0}\widehat{\rho}_{0;0}(p)\\ &\qquad= iQ^T\mathrm{tr}\int_{\mathbb{T}^d}dp\nabla_k\mathcal{G}_k|_{k=0}\frac{1}{2\pi i}\int_\Gamma\frac{1}{tz}(1-e^{-tz})\frac{1}{z-\mathcal{G}_0}\widehat{\rho}_{0;0}(p).\end{aligned}$$

This will vanish in the large t limit since $\frac{1}{tz}(1-e^{-tz}) \to 0$ as $t\to\infty$ for $\mathrm{Re}z > 0$. Therefore,

$$\lim_{t\to\infty}\frac{1}{t}\sum_{x\in\mathbb{Z}^d}x\rho_t(x,x) = iQ^T\mathrm{tr}\int_{\mathbb{T}^d}dp\nabla_k\mathcal{G}_k|_{k=0}F_{eq}(p)\langle I,\widehat{\rho}_{0;0}\rangle = iQ^T[\mathrm{tr}\rho_0]\langle I,\nabla_k\mathcal{G}_k|_{k=0}F_{eq}\rangle.$$

So $v = \lim_{t\to\infty} \frac{1}{[\mathrm{tr}\rho_0]t} \sum_{x\in\mathbb{Z}^d} x\rho_t(x,x)$ due to our definition of the drift constant (28).

We perform a similar analysis in calculating the expression (6) for the diffusion matrix **D**. We observe:

$$
\begin{aligned}
&\frac{1}{2t}\sum_{x\in\mathbb{Z}^d}((Q^T)^{-1}(x-tv))_i((Q^T)^{-1}(x-tv))_j\rho_t(x,x)\\
&\qquad = -\frac{1}{2t}\mathrm{tr}\int_{\mathbb{T}^d} dp(\partial_i - it((Q^T)^{-1}v)_i)(\partial_j - it((Q^T)^{-1}v)_j)\widehat{\rho}_{t;k}(p)|_{k=0}\\
&\qquad = -\frac{1}{2t}\mathrm{tr}\int_{\mathbb{T}^d} dp[\partial_i\partial_j - it((Q^T)^{-1}v)_i\partial_j - it((Q^T)^{-1}v)_j\partial_i\\
&\qquad\qquad - t^2((Q^T)^{-1}v)_i((Q^T)^{-1}v)_j](e^{-t\mathcal{G}_k}\widehat{\rho}_{0;k})(p)|_{k=0}.
\end{aligned}
$$

After using the formula for the derivative of a semigroup and using the fact that $e^{-t\mathcal{G}_k^\dagger}I = I$ due to conservation of quantum probabilities, we have

$$
\frac{1}{2t}\sum_{x\in\mathbb{Z}^d}((Q^T)^{-1}x)_i((Q^T)^{-1}x)_j\rho_t(x,x) = \sum_{n=1}^{7} N_n(t)
$$

where

$$
\begin{aligned}
N_1(t) &= -\frac{1}{2t}\mathrm{tr}\int_{\mathbb{T}^d} dp(\partial_i\partial_j\widehat{\rho}_{0;0})(p),\\
N_2(t) &= -\frac{1}{2t}\mathrm{tr}\int_{\mathbb{T}^d} dp\int_0^t ds\left\{\partial_i\mathcal{G}_k|_{k=0}e^{-s\mathcal{G}_0}(\partial_j\widehat{\rho}_{0;0})(p) + \partial_j\mathcal{G}_k|_{k=0}e^{-s\mathcal{G}_0}(\partial_i\widehat{\rho}_{0;0})(p)\right\},\\
N_3(t) &= -\frac{1}{2t}\mathrm{tr}\int_{\mathbb{T}_d} dp\int_0^t ds\int_0^s dr\left\{\partial_i\mathcal{G}_k|_{k=0}e^{-(s-r)\mathcal{G}_0}\partial_j\mathcal{G}_k|_{k=0}e^{-r\mathcal{G}_0}\widehat{\rho}_{0;0}(p)\right.\\
&\qquad\qquad \left. + \partial_j\mathcal{G}_k|_{k=0}e^{-(s-r)\mathcal{G}_0}\partial_i\mathcal{G}_k|_{k=0}e^{-r\mathcal{G}_0}\widehat{\rho}_{0;0}(p)\right\},\\
N_4(t) &= \frac{1}{2t}\mathrm{tr}\int_{\mathbb{T}^d} dp\int_0^t ds\partial_i\partial_j\mathcal{G}_k|_{k=0}e^{-s\mathcal{G}_0}\widehat{\rho}_{0;0}(p),\\
N_5(t) &= \frac{1}{2}\mathrm{tr}\int_{\mathbb{T}^d} dp\left\{i((Q^T)^{-1}v)_i(\partial_j\widehat{\rho}_{0;0})(p) + i((Q^T)^{-1}v)_j(\partial_i\widehat{\rho}_{0;0})(p)\right\},\\
N_6(t) &= \frac{1}{2}\mathrm{tr}\int_{\mathbb{T}^d} dp\int_0^t ds\left\{i((Q^T)^{-1}v)_i\partial_j\mathcal{G}_k|_{k=0}e^{-s\mathcal{G}_0}\widehat{\rho}_{0;0}(p)\right.\\
&\qquad\qquad \left. + i((Q^T)^{-1}v)_j\partial_i\mathcal{G}_k|_{k=0}e^{-s\mathcal{G}_0}\widehat{\rho}_{0;0}(p)\right\},\ \text{and}\\
N_7(t) &= \frac{t}{2}((Q^T)^{-1}v)_i((Q^T)^{-1}v)_j\langle I, \widehat{\rho}_{0;0}\rangle.
\end{aligned}
$$

We first note that $\lim_{t\to\infty} N_1(t) = 0$ so this first term is negligible in the large time limit. Let us now consider projecting onto and off of the eigenspace $\langle F_{eq}\rangle$ using the Riesz projection P_0 given in (25). Introducing $P_0 + (1 - P_0)$ after every semigroup in the above expression yields the following. For the second term, we have

$$\begin{aligned} N_2(t) = &-\frac{1}{2t}\mathrm{tr}\int_{\mathbb{T}^d} dp \int_0^t ds \left\{\partial_i \mathcal{G}_k|_{k=0} e^{-s\mathcal{G}_0}(1-P_0)(\partial_j \widehat{\rho}_{0;0})(p)\right. \\ &\qquad\qquad\qquad\qquad \left. + \partial_j \mathcal{G}_k|_{k=0} e^{-s\mathcal{G}_0}(1-P_0)(\partial_i \widehat{\rho}_{0;0})(p)\right\} \\ &-\frac{1}{2}\left\{i((Q^T)^{-1}v)_i \langle I, \partial_j \widehat{\rho}_{0;0}\rangle + i((Q^T)^{-1}v)_j \langle I, \partial_i \widehat{\rho}_{0;0}\rangle\right\}, \end{aligned}$$

the second part of which simply cancels with $N_5(t)$. For the first part of this term, we use the fact that off of the equilibrium eigenvalue, we may draw a contour Γ enclosing the rest of the spectrum of $\mathcal{G}_0$ such that $\mathrm{Re}\Gamma \geqslant C > 0$ as per Lemma 2. This yields

$$\begin{aligned} N_2(t) + N_5(t) &= -\frac{1}{2t}\mathrm{tr}\int_{\mathbb{T}^d} dp \int_0^t ds \left\{\partial_i \mathcal{G}_k|_{k=0} \frac{1}{2\pi i}\int_\Gamma e^{-sz}\frac{1}{z-\mathcal{G}_0}(\partial_j \widehat{\rho}_{0;0})(p)\right. \\ &\qquad\qquad \left. + \partial_j \mathcal{G}_k|_{k=0} \frac{1}{2\pi i}\int_\Gamma e^{-sz}\frac{1}{z-\mathcal{G}_0}(\partial_i \widehat{\rho}_{0;0})(p)\right\} \\ &= -\frac{1}{2t}\mathrm{tr}\int_{\mathbb{T}^d} dp \left\{\partial_i \mathcal{G}_k|_{k=0} \frac{1}{2\pi i}\int_\Gamma \frac{1}{z}(1-e^{-tz})\frac{1}{z-\mathcal{G}_0}(\partial_j \widehat{\rho}_{0;0})(p)\right. \\ &\qquad\qquad \left. + \partial_j \mathcal{G}_k|_{k=0} \frac{1}{2\pi i}\int_\Gamma \frac{1}{z}(1-e^{-tz})\frac{1}{z-\mathcal{G}_0}(\partial_i \widehat{\rho}_{0;0})(p)\right\}. \end{aligned}$$

Since $\mathrm{Re}\Gamma \geqslant C > 0$, $\lim_{t\to\infty} \frac{1}{tz}(1-e^{-tz}) = 0$ for $z \in \Gamma$. Hence this term vanishes as well for $t \to \infty$. For the fourth term, we note that

$$\begin{aligned} N_4(t) = &\frac{1}{2t}\mathrm{tr}\int_{\mathbb{T}^d} dp \int_0^t ds \partial_i \partial_j \mathcal{G}_k|_{k=0} e^{-s\mathcal{G}_0}(1-P_0)\widehat{\rho}_{0;0}(p) \\ &+ \frac{1}{2}\mathrm{tr}\int_{\mathbb{T}^d} dp \partial_i \partial_j \mathcal{G}_k|_{k=0} F_{eq}(p)\langle I, \widehat{\rho}_{0;0}\rangle. \end{aligned}$$

In a very similar manner to the previous calculation, the first part of this term vanishes as

$t \to \infty$. For $N_6(t)$, another similar calculation yields

$$N_6(t) = \frac{1}{2}\mathrm{tr}\int_{\mathbb{T}^d} dp \int_0^t ds\, \big\{ i((Q^T)^{-1}v)_i \partial_j \mathcal{G}_k|_{k=0} e^{-s\mathcal{G}_0}(1-P_0)\widehat{\rho}_{0;0}(p)$$
$$+ i((Q^T)^{-1}v)_j \partial_i \mathcal{G}_k|_{k=0} e^{-s\mathcal{G}_0}(1-P_0)\widehat{\rho}_{0;0}(p) \big\}$$
$$- t((Q^T)^{-1}v)_i((Q^T)^{-1}v)_j \langle I, \widehat{\rho}_{0;0}\rangle.$$

Due to the lack of a t in the denominator, the first part of this term will not vanish as $t \to \infty$. In fact, the first part of this term will tend to

$$\frac{1}{2}\mathrm{tr}\int_{\mathbb{T}^d} dp \{ i((Q^T)^{-1}v)_i \partial_j \mathcal{G}_k|_{k=0} \mathcal{G}_0^{-1}(1-P_0)\widehat{\rho}_{0;0}(p)$$
$$+ i((Q^T)^{-1}v)_j \partial_i \mathcal{G}_k|_{k=0} \mathcal{G}_0^{-1}(1-P_0)\widehat{\rho}_{0;0}(p) \}.$$

Finally, for $N_3(t)$, applying P_0 to both semigroups yields

$$-\frac{1}{2t}\mathrm{tr}\int_{\mathbb{T}^d} dp \int_0^t ds \int_0^s dr\, \big\{ \partial_i \mathcal{G}_k|_{k=0} e^{-(s-r)\mathcal{G}_0} P_0 \partial_j \mathcal{G}_k|_{k=0} e^{-r\mathcal{G}_0} P_0 \widehat{\rho}_{0;0}(p)$$
$$+ \partial_j \mathcal{G}_k|_{k=0} e^{-(s-r)\mathcal{G}_0} P_0 \partial_i \mathcal{G}_k|_{k=0} e^{-r\mathcal{G}_0} P_0 \widehat{\rho}_{0;0}(p) \big\}$$
$$= -\frac{1}{2t}\mathrm{tr}\int_{\mathbb{T}^d} dp \int_0^t ds \int_0^s dr\, \{ \partial_i \mathcal{G}_k|_{k=0} F_{eq}(p) \langle I, \partial_j \mathcal{G}_k|_{k=0} F_{eq}\rangle$$
$$+ \partial_j \mathcal{G}_k|_{k=0} F_{eq}(p) \langle I, \partial_i \mathcal{G}_k|_{k=0} F_{eq}\rangle \} \langle I, \widehat{\rho}_{0;0}\rangle$$
$$= -\frac{1}{2t}\big\{ i((Q^T)^{-1}v)_i \cdot i((Q^T)^{-1}v)_j + i((Q^T)^{-1}v)_j \cdot i((Q^T)^{-1}v)_i \big\} \frac{1}{2}t^2 \langle I, \widehat{\rho}_{0;0}\rangle$$
$$= \frac{t}{2}((Q^T)^{-1}v)_i((Q^T)^{-1}v)_j \langle I, \widehat{\rho}_{0;0}\rangle.$$

This will fully cancel with $N_7(t)$ and the second part of $N_6(t)$. For the remainder of the terms in $N_3(t)$, we again use the contour Γ. We observe that

$$
\begin{aligned}
&-\frac{1}{2t}\mathrm{tr}\int_{\mathbb{T}^d} dp\int_0^t ds\int_0^s dr\,\big\{\partial_i\mathcal{G}_k|_{k=0}e^{-(s-r)\mathcal{G}_0}P_0\partial_j\mathcal{G}_k|_{k=0}e^{-r\mathcal{G}_0}(1-P_0)\widehat{\rho}_{0;0}(p)\\
&\qquad\qquad\qquad\qquad\qquad\qquad +\partial_j\mathcal{G}_k|_{k=0}e^{-(s-r)\mathcal{G}_0}P_0\partial_i\mathcal{G}_k|_{k=0}e^{-r\mathcal{G}_0}(1-P_0)\widehat{\rho}_{0;0}(p)\big\}\\
&\quad = -\frac{1}{2t}\mathrm{tr}\int_{\mathbb{T}^d} dp\int_0^t ds\int_0^s dr\,\big\{\partial_i\mathcal{G}_k|_{k=0}F_{eq}(p)\langle I,\partial_j\mathcal{G}_k|_{k=0}e^{-r\mathcal{G}_0}(1-P_0)\widehat{\rho}_{0;0}\rangle\\
&\qquad\qquad\qquad\qquad\qquad\qquad +\partial_j\mathcal{G}_k|_{k=0}F_{eq}(p)\langle I,\partial_i\mathcal{G}_k|_{k=0}e^{-r\mathcal{G}_0}(1-P_0)\widehat{\rho}_{0;0}\rangle\big\}\\
&\quad = -\frac{1}{2t}\mathrm{tr}\int_{\mathbb{T}^d} dp\int_0^t ds\int_0^s dr\,\big\{i((Q^T)^{-1}v)_i\partial_j\mathcal{G}_k|_{k=0}e^{-r\mathcal{G}_0}(1-P_0)\widehat{\rho}_{0;0}(p)\\
&\qquad\qquad\qquad\qquad\qquad\qquad +i((Q^T)^{-1}v)_j\partial_i\mathcal{G}_k|_{k=0}e^{-r\mathcal{G}_0}(1-P_0)\widehat{\rho}_{0;0}(p)\big\}\\
&\quad \to -\frac{1}{2}\mathrm{tr}\int_{\mathbb{T}^d} dp\,\big\{i((Q^T)^{-1}v)_i\partial_j\mathcal{G}_k|_{k=0}\mathcal{G}_0^{-1}(1-P_0)\widehat{\rho}_{0;0}(p)\\
&\qquad\qquad\qquad\qquad\qquad\qquad +i((Q^T)^{-1}v)_j\partial_i\mathcal{G}_k|_{k=0}\mathcal{G}_0^{-1}(1-P_0)\widehat{\rho}_{0;0}(p)\big\}\,.
\end{aligned}
$$

This precisely cancels with the first part of $N_6(t)$ as shown above. Similarly, placing the projections in the reverse order for $N_3(t)$ gives

$$
\begin{aligned}
&-\frac{1}{2t}\mathrm{tr}\int_{\mathbb{T}^d} dp\int_0^t ds\int_0^s dr\,\big\{\partial_i\mathcal{G}_k|_{k=0}e^{-(s-r)\mathcal{G}_0}(1-P_0)\partial_j\mathcal{G}_k|_{k=0}e^{-r\mathcal{G}_0}P_0\widehat{\rho}_{0;0}(p)\\
&\qquad\qquad\qquad\qquad\qquad\qquad +\partial_j\mathcal{G}_k|_{k=0}e^{-(s-r)\mathcal{G}_0}(1-P_0)\partial_i\mathcal{G}_k|_{k=0}e^{-r\mathcal{G}_0}P_0\widehat{\rho}_{0;0}(p)\big\}\\
&\quad \to -\frac{1}{2}\mathrm{tr}\int_{\mathbb{T}^d} dp\,\big\{\partial_i\mathcal{G}_k|_{k=0}\mathcal{G}_0^{-1}(1-P_0)\partial_j\mathcal{G}_k|_{k=0}F_{eq}(p)\\
&\qquad\qquad\qquad\qquad +\partial_j\mathcal{G}_k|_{k=0}\mathcal{G}_0^{-1}(1-P_0)\partial_i\mathcal{G}_k|_{k=0}F_{eq}(p)\big\}\,\langle I,\widehat{\rho}_{0;0}\rangle.
\end{aligned}
$$

Finally, performing the projection $1-P_0$ on both semigroups in $N_3(t)$ requires two contours

Γ and Γ', each with strictly positive real part and such that $\Gamma \cap \Gamma' \neq \varnothing$. We then have

$$
\begin{aligned}
-\frac{1}{2t}\mathrm{tr}\int_{\mathbb{T}^d} dp \int_0^t ds \int_0^s dr \,\{&\partial_i \mathcal{G}_k|_{k=0} e^{-(s-r)\mathcal{G}_0}(1-P_0)\partial_j \mathcal{G}_k|_{k=0} e^{-r\mathcal{G}_0}(1-P_0)\widehat{\rho}_{0;0}(p) \\
&+\partial_j \mathcal{G}_k|_{k=0} e^{-(s-r)\mathcal{G}_0}(1-P_0)\partial_i \mathcal{G}_k|_{k=0} e^{-r\mathcal{G}_0}(1-P_0)\widehat{\rho}_{0;0}(p)\} \\
= -\frac{1}{2t}\mathrm{tr}\int_{\mathbb{T}^d} dp \Big\{ &\partial_i \mathcal{G}_k|_{k=0} \frac{1}{2\pi i}\int_\Gamma dz \frac{1}{2\pi i}\int_{\Gamma'} dz' \\
&\frac{1}{z'-z}\left(\frac{1}{tz}(1-e^{-tz}) - \frac{1}{tz'}(1-e^{-tz'})\right)\frac{1}{z-\mathcal{G}_0}\partial_j \mathcal{G}_k|_{k=0}\frac{1}{z'-\mathcal{G}_0}\widehat{\rho}_{0;0}(p) \\
&+ \partial_j \mathcal{G}_k|_{k=0} \frac{1}{2\pi i}\int_\Gamma dz \frac{1}{2\pi i}\int_{\Gamma'} dz' \\
&\frac{1}{z'-z}\left(\frac{1}{tz}(1-e^{-tz}) - \frac{1}{tz'}(1-e^{-tz'})\right)\frac{1}{z-\mathcal{G}_0}\partial_i \mathcal{G}_k|_{k=0}\frac{1}{z'-\mathcal{G}_0}\widehat{\rho}_{0;0}(p)\Big\} \\
\to 0.&
\end{aligned}
$$

Putting all terms $N_n(t)$ together and taking the limit as $t \to \infty$ then gives the following expression for the right-hand side of (6):

$$
\begin{aligned}
\lim_{t\to\infty} \frac{1}{2[\mathrm{tr}\rho_0]t} \sum_{x\in\mathbb{Z}^d} &((Q^T)^{-1}(x-tv))_i((Q^T)^{-1}(x-tv))_j \rho_t(x,x) \\
&= -\frac{1}{2}\mathrm{tr}\int_{\mathbb{T}^d} dp \partial_i \mathcal{G}_k|_{k=0}\mathcal{G}_0^{-1}(1-P_0)\partial_j \mathcal{G}_k|_{k=0} F_{eq}(p) \\
&\quad - \frac{1}{2}\mathrm{tr}\int_{\mathbb{T}^d} dp \partial_j \mathcal{G}_k|_{k=0}\mathcal{G}_0^{-1}(1-P_0)\partial_i \mathcal{G}_k|_{k=0} F_{eq}(p) \\
&\quad + \frac{1}{2}\mathrm{tr}\int_{\mathbb{T}^d} dp \partial_i\partial_j \mathcal{G}_k|_{k=0} F_{eq}(p).
\end{aligned}
\tag{30}
$$

Using second-order perturbation theory, we have

$$
\begin{aligned}
\partial_i\partial_j E(0) = \langle I, \partial_i\partial_j \mathcal{G}_k|_{k=0} F_{eq}\rangle &+ \langle I, \partial_i \mathcal{G}_k|_{k=0}\mathcal{G}_0^{-1}(1-P_0)\partial_j \mathcal{G}_k|_{k=0} F_{eq}\rangle \\
&+ \langle I, \partial_j \mathcal{G}_k|_{k=0}\mathcal{G}_0^{-1}(1-P_0)\partial_i \mathcal{G}_k|_{k=0} F_{eq}\rangle
\end{aligned}
$$

and hence due to our definition of the diffusion matrix (29),

$$
D_{i,j} = \frac{1}{2}\partial_i\partial_j E(0) = \lim_{t\to\infty} \frac{1}{2[\mathrm{tr}\rho_0]t} \sum_{x\in\mathbb{Z}^d} ((Q^T)^{-1}(x-tv))_i((Q^T)^{-1}(x-tv))_j \rho_t(x,x).
$$

It is clear that the diffusion matrix $\mathbf{D}$ is symmetric. We wish to further show that this matrix is positive definite. To show positivity for the final term in (30), let us consider a solution $\widehat{\rho}_{t;k}$ of the evolution equation (19) with initial condition $\widehat{\rho}_{0;k} = F_{eq}$. That is, $\widehat{\rho}_{t;k} = e^{-t\mathcal{G}_k}F_{eq}$. Similar to the above analysis, we have for $z \in \mathbb{C}^d$,

$$
\begin{aligned}
0 &\leqslant \frac{1}{2t}\sum_{x\in\mathbb{Z}^d}\rho_t(x,x)\left|\sum_i((Q^T)^{-1}x)_i z_i\right|^2 \\
&= \frac{1}{2t}\sum_{i,j}\sum_{x\in\mathbb{Z}^d}((Q^T)^{-1}x)_i((Q^T)^{-1}x)_j\rho_t(x,x)z_i^* z_j \\
&= -\frac{1}{2t}\sum_{i,j}\mathrm{tr}\int_{\mathbb{T}^d}dp\partial_i\partial_j(e^{-t\mathcal{G}_k}F_{eq}-F_{eq})(p)|_{k=0}z_i^* z_j.
\end{aligned}
$$

As $e^{-t\mathcal{G}_k}$ is a dynamical semigroup, $\lim_{t\to 0}\frac{e^{-t\mathcal{G}_k}F_{eq}-F_{eq}}{t} = -\mathcal{G}_k$. Hence taking a limit as $t \to 0$ of the above expression yields

$$
\frac{1}{2}\sum_{i,j}\mathrm{tr}\int_{\mathbb{T}^d}dp\partial_i\partial_j\mathcal{G}_k|_{k=0}F_{eq}(p)z_i^* z_j \geqslant 0.
$$

To show positivity for the first two terms in (30), we note that due to (22), the expression may be simplified using the modified inner product introduced in Section 5.5. We observe:

$$
\begin{aligned}
&\frac{1}{2}\mathrm{tr}\int_{\mathbb{T}^d}dp\partial_i\mathcal{J}_k|_{k=0}\mathcal{G}_0^{-1}(1-P_0)\partial_j\mathcal{J}_k|_{k=0}F_{eq}(p) \\
&\qquad + \frac{1}{2}\mathrm{tr}\int_{\mathbb{T}^d}dp\partial_j\mathcal{J}_k|_{k=0}\mathcal{G}_0^{-1}(1-P_0)\partial_i\mathcal{J}_k|_{k=0}F_{eq}(p) \\
&= \frac{1}{2}\mathrm{tr}\int_{\mathbb{T}^d}dp(\partial_i\widehat{H}(p)\mathcal{G}_0^{-1}(1-P_0)(\partial_j\widehat{H}F_{eq})(p) + \partial_i\widehat{H}(p)\mathcal{G}_0^{-1}(1-P_0)(F_{eq}\partial_j\widehat{H})(p)) \\
&\qquad + \frac{1}{2}\mathrm{tr}\int_{\mathbb{T}^d}dp(\partial_j\widehat{H}(p)\mathcal{G}_0^{-1}(1-P_0)(\partial_i\widehat{H}F_{eq})(p) + \partial_j\widehat{H}(p)\mathcal{G}_0^{-1}(1-P_0)(F_{eq}\partial_i\widehat{H})(p)) \\
&= \mathrm{Re}\langle\partial_j\widehat{H},(\mathcal{G}_0^T)^{-1}(1-P_0)\partial_i\widehat{H}\rangle_{F_{eq}} \\
&\qquad + \mathrm{Re}\langle\partial_i\widehat{H},(\mathcal{G}_0^T)^{-1}(1-P_0)\partial_j\widehat{H}\rangle_{F_{eq}}.
\end{aligned}
$$

Therefore, for $z \in \mathbb{C}^d\backslash\{0\}$,

$$
\begin{aligned}
\mathrm{Re}\langle z,\mathbf{D}z\rangle &\geqslant 2\mathrm{Re}\langle\sum_i z_i\partial_i\widehat{H},(\mathcal{G}_0^T)^{-1}(1-P_0)\sum_i z_i\partial_i\widehat{H}\rangle_{F_{eq}} \\
&= 2\mathrm{Re}\langle\Phi,\mathcal{G}_0^T(1-P_0)\Phi\rangle_{F_{eq}}
\end{aligned}
$$

where $\Phi := \sum_i z_i(\mathcal{G}_0^T)^{-1}\partial_i \widehat{H}$. Due to (27), this expression will be strictly positive and so $\mathbf{D}$ is positive definite. $\square$

5.2 The Small g Limit

We wish to show the diffusion is $O\left(\frac{1}{g}\right)$ in the small g limit as per Theorem 2. This requires us to first analyze the limit of the equilibrium eigenvector F_{eq}.

Lemma 3. *The equilibrium eigenvector F_{eq} for $\mathcal{G}_0$ converges weakly as $g \to 0$ to the equilibrium eigenvector for $\Pi\widehat{\mathcal{L}}_0\Pi$, where Π is the projection onto ker $\mathcal{J}_0$. Furthermore, this eigenvector is strictly positive.*

Proof. First, we must guarantee the existence of $\operatorname*{w-lim}_{g\to 0} F_{eq}$. To do so requires a uniform bound on $||F_{eq}||_2$, which guarantees we may pass to a weakly convergent subsequence. Since $F_{eq} \in \ker \mathcal{G}_0$, (20) yields

$$F_{eq} = \left(\mathcal{D}_0 + \frac{i}{g}\mathcal{J}_0\right)^{-1} T_0 F_{eq}. \tag{31}$$

Since

$$\left(\mathcal{D}_0 + \frac{i}{g}\mathcal{J}_0\right) A(p) = \left(\frac{1}{2}D(p) + \frac{i}{g}\widehat{H}(p)\right) A(p) + A(p)\left(\frac{1}{2}D(p) - \frac{i}{g}\widehat{H}(p)\right)$$

is a sum of multiplication operators, (31) becomes

$$F_{eq} = \int_0^\infty dt e^{-t\left(\frac{1}{2}D(p)+\frac{i}{g}\widehat{H}(p)\right)}(T_0F_{eq})(p)e^{-t\left(\frac{1}{2}D(p)-\frac{i}{g}\widehat{H}(p)\right)}.$$

Due to Gronwall's inequality and Assumption 2, we have $||e^{-t\left(\frac{1}{2}D(p)\pm\frac{i}{g}\widehat{H}(p)\right)}|| \leqslant e^{-\frac{1}{2C}t}$, and hence

$$\begin{aligned}
||F_{eq}||_2 &\leqslant \int_0^\infty dt||e^{-t\left(\frac{1}{2}D(p)+\frac{i}{g}\widehat{H}(p)\right)}||\; ||(T_0F_{eq})(p)||_2||e^{-t\left(\frac{1}{2}D(p)-\frac{i}{g}\widehat{H}(p)\right)}|| \\
&\leqslant \int_0^\infty dt e^{-\frac{1}{C}t}||(T_0F_{eq})(p)||_2 \\
&\leqslant C\int_{\mathbb{T}^d} dp'||\widehat{M}_{Q^{-1}(p-p')}(p',p')[F_{eq}(p')]||_2.
\end{aligned}$$

Due to Lemma 2, we may apply Assumption 5 to bound this kernel to yield

$$||F_{eq}||_2 \leqslant C\chi \int_{\mathbb{T}^d} dp' ||I||_2 \mathrm{tr} F_{eq}(p') = C\chi |\Sigma|^{1/2}.$$

Hence $||F_{eq}||_2$ is uniformly bounded and a certain subsequence of F_{eq} converges weakly as $g \to 0$ to some matrix F_{eq}^0. This implies F_{eq} converges weakly as well and it must also converge to F_{eq}^0.

Due to Lemma 6, taking the weak limit of (31) as $g \to 0$ yields

$$F_{eq}^0 = \Pi(\Pi \mathcal{D}_0 \Pi)^{-1} \Pi T_0 F_{eq}^0 \tag{32}$$

where Π is the projection onto ker $\mathcal{J}_0$. Hence F_{eq}^0 is an equilibrium eigenvector for $\Pi \widehat{\mathcal{L}}_0 \Pi$. Due to Lemma 1, F_{eq}^0 will be nonnegative, as it is the equilibrium eigenvector for the generator of a jump process. In addition, $F_{eq}^0 \neq 0$ since $\langle I, F_{eq} \rangle = 1$ for all g implies $\langle I, F_{eq}^0 \rangle = 1$ as well. Therefore, due to Assumption 4 (using a suitable normalization of F_{eq}^0), F_{eq}^0 will in fact be strictly positive. □

Lemma 3 shows that F_{eq} converges weakly to F_{eq}^0 satisfying $F_{eq}^0(p) = \sum_{i=1}^{|\Sigma|} w_i(p) E_{ii}(p)$ for some $w_i(p)$ and the matrix elements $E_{ij}(p) = |\psi_i(p)\rangle\langle\psi_j(p)|$ for the basis $\{\psi_i(p)\}_{i=1}^{|\Sigma|}$ of $\widehat{H}(p)$ given in the proof of Lemma 1. Utilizing Corollary 1 in Appendix C, we can now prove Theorem 2.

Proof. To begin, consider expression (30). To leading order in g, the diffusion will be

$$\begin{aligned} D_{i,j} = &\frac{1}{2} \mathrm{tr} \int_{\mathbb{T}^d} dp \partial_i \mathcal{J}_k|_{k=0} (i\mathcal{J}_0 - g\widehat{\mathcal{L}}_0)^{-1} \partial_j \mathcal{J}_k|_{k=0} F_{eq}(p) \\ &+ \frac{1}{2} \mathrm{tr} \int_{\mathbb{T}^d} dp \partial_j \mathcal{J}_k|_{k=0} (i\mathcal{J}_0 - g\widehat{\mathcal{L}}_0)^{-1} \partial_i \mathcal{J}_k|_{k=0} F_{eq}(p) + O(1). \end{aligned}$$

Multiplying by g and taking the limit $g \to 0^+$ using Corollary 1 yields

$$\begin{aligned} gD_{i,j} \to &-\frac{1}{2} \mathrm{tr} \int_{\mathbb{T}^d} dp \Pi \partial_i \mathcal{J}_0|_{k=0} (\Pi \widehat{\mathcal{L}}_0 \Pi)^{-1} \Pi \partial_j \mathcal{J}_k|_{k=0} F_{eq}^0 \\ &- \frac{1}{2} \mathrm{tr} \int_{\mathbb{T}^d} dp \Pi \partial_j \mathcal{J}_0|_{k=0} (\Pi \widehat{\mathcal{L}}_0 \Pi)^{-1} \Pi \partial_i \mathcal{J}_k|_{k=0} F_{eq}^0, \end{aligned}$$

where Π is the projection onto $\ker \mathcal{J}_0$ and $F_{eq}^0 = \lim_{g \to 0} F_{eq}$ as in the proof of Lemma 3. For an arbitrary function $F(p)$, we calculate

$$(\Pi \partial_j \mathcal{J}_k|_{k=0} F)(p) = -\frac{1}{2} \sum_{i=1}^{|\Sigma|} E_{ii}(p)(\partial_j \widehat{H}(p) F(p) + F(p) \partial_j \widehat{H}(p)) E_{ii}(p).$$

In particular, this yields

$$(\Pi \partial_j \mathcal{J}_k|_{k=0} F_{eq}^0)(p) = -\sum_{i=1}^{|\Sigma|} w_i(p) \langle \psi_i(p) | \partial_j \widehat{H}(p) | \psi_i(p) \rangle E_{ii}(p).$$

Let $z \in \mathbb{C}^d \backslash \{0\}$. If we denote $\Phi(p) := \sum_{i=1}^{|\Sigma|} \langle \psi_i(p) | \sum_j \partial_j \widehat{H}(p) z_j | \psi_i(p) \rangle E_{ii}(p)$, we will have

$$\begin{aligned}
\lim_{g \to 0^+} \mathrm{Re}\langle z, g\mathbf{D} z \rangle &= -\mathrm{Re}\langle \Phi, (\Pi \widehat{\mathcal{L}}_0 \Pi)^{-1} \Phi F_{eq}^0 \rangle \\
&= -\mathrm{Re}\langle \Phi F_{eq}^0, (\Pi \widehat{\mathcal{L}}_0 \Pi F_{eq}^0)^{-1} \Phi F_{eq}^0 \rangle \\
&= -\langle \Phi F_{eq}^0, ((\Pi \widehat{\mathcal{L}}_0 \Pi F_{eq}^0)^{\dagger})^{-1} \mathrm{Re}(\Pi \widehat{\mathcal{L}}_0 \Pi F_{eq}^0)(\Pi \widehat{\mathcal{L}}_0 \Pi F_{eq}^0)^{-1} \Phi F_{eq}^0 \rangle \\
&= -\mathrm{Re}\langle (\Pi \widehat{\mathcal{L}}_0 \Pi F_{eq}^0)^{-1} \Phi F_{eq}^0, \Pi \widehat{\mathcal{L}}_0 \Pi F_{eq}^0 (\Pi \widehat{\mathcal{L}}_0 \Pi F_{eq}^0)^{-1} \Phi F_{eq}^0 \rangle.
\end{aligned}$$

Due to the proof of Lemma 3, $\Pi \widehat{\mathcal{L}}_0 \Pi$ is the generator for a jump process with a unique positive invariant state given by F_{eq}^0. Hence Lemma 5 applies and this term will be strictly positive. Hence $\mathbf{D}(g) = O\left(\frac{1}{g}\right)$. $\square$

CHAPTER 6: CONCLUSIONS AND FUTURE WORK

A quantum particle's dynamics are seemingly governed solely by its interaction with the environment in the case of a Lindblad master equation. Indeed, [9] and [2] showed diffusion was present in this context for a translation-invariant Hamiltonian. This document showed that in the more general Q-periodic Hamiltonian context, diffusive propagation also occurred. In [1], diffusion was shown for an Anderson Hamiltonian (though the Lindbladian used in that paper was not the generator of a completely positive semigroup). In each case, the presence of a Lindbladian caused the dynamics of the particle to exhibit diffusion. It is then natural to wonder whether this behavior occurs for other Hamiltonians as well.

For instance, consider the Anderson model (2) whose Hamiltonian operator is given by

$$H_\omega = -\Delta + \lambda V_\omega \tag{33}$$

where the potentials V_ω are diagonal operators with ω given by i.i.d. random variables and the parameter λ measures the strength of the disorder. We should also suspect diffusion to be present for this Hamiltonian in the context of Lindbladian environmental interaction. However, one would expect the disorder to affect the asymptotics of the diffusion for small g, as the $g = 0$ case should yield localization for large enough disorder. Thus we make the following conjecture.

Conjecture 1. *Let ρ_t be a solution of (2) with initial condition $\rho_0 \in \mathcal{B}_1(\ell^2(\mathbb{Z}^d))$ and $g > 0$. If H_ω satisfies (33) with λ sufficiently large, then the quantum particle whose density matrix is given by ρ_t exhibits diffusive propagation with diffusion matrix $\boldsymbol{D}(g)$ satisfying $\boldsymbol{D}(g) = O(g)$ for small g.*

A particularly interesting subcase of the one-dimensional Anderson model is the random dimer model. In this model, the random variables $\omega(x)$ for $x \in \mathbb{Z}$ are chosen from the set $\{-1, 1\}$ with the additional requirement that $\omega(2x) = \omega(2x+1)$ for every $x \in \mathbb{Z}$. That is, the random variables are chosen in dimer pairs. This particular case of the Anderson model is interesting, since it was shown that (without Lindbladian interaction) the dynamics of the quantum particle change depending on the value of λ. For instance, the particle's dynamics will be localized as with the usual Anderson model for $\lambda > 1$, yet diffusive for $\lambda = 1$ and superdiffusive for $0 < \lambda < 1$ [20], [21]. This leads us to the following conjecture.

Conjecture 2. *Let ρ_t be a solution of (2) with initial condition $\rho_0 \in \mathcal{B}_1(\ell^2(\mathbb{Z}^d))$ and $g > 0$. If H_ω is the random dimer Hamiltonian, then the quantum particle whose density matrix is given by ρ_t exhibits diffusive propagation with diffusion matrix $\boldsymbol{D}(g)$. For small g, we have the following asymptotics:*

- *If $0 \leqslant \lambda \leqslant 1$, then $\boldsymbol{D}(g) = O(g^{-1+2\lambda})$*
- *If $\lambda > 1$, then $\boldsymbol{D}(g) = O(g)$.*

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APPENDIX A: A SPECTRUM RESULT FOR COMMUTING OPERATORS

Lemma 4. *Let A and B be bounded operators on a Banach space with $[A,B]=0$. Then*

$$\sigma(A+B)\subseteq\sigma(A)+\sigma(B).$$

Proof. We first note that $[A,B]=0$ implies $[(A-z)^{-1},(B-w)^{-1}]=0$ for all $z\in\rho(A), w\in\rho(B)$ since

$$[(A-z)^{-1},(B-w)^{-1}]=(A-z)^{-1}(B-w)^{-1}[A-z,B-w](B-w)^{-1}(A-z)^{-1}=0.$$

Let $z\notin\sigma(A)+\sigma(B)$ so that $\sigma(A)$ and $z-\sigma(B)$ are two disjoint compact sets. We may thus define a bounded counterclockwise contour Γ enclosing $\sigma(A)$ such that $\mathrm{int}\Gamma$ contains no part of $z-\sigma(B)$. Using this contour, we define an operator

$$\Phi:=\frac{1}{2\pi i}\int_\Gamma dw(w+B-z)^{-1}(w-A)^{-1}$$

which will be bounded since $\mathrm{dist}(\Gamma,\sigma(A)),\mathrm{dist}(\Gamma,z-\sigma(B))>0$. Since the resolvents of A and B commute,

$$\begin{aligned}(A+B-z)\Phi &= \frac{1}{2\pi i}\int_\Gamma dw(A-w)(w+B-z)^{-1}(w-A)^{-1}\\ &\quad+\frac{1}{2\pi i}\int_\Gamma dw(B+w-z)(w+B-z)^{-1}(w-A)^{-1}\\ &= -\frac{1}{2\pi i}\int_\Gamma dw(w+B-z)^{-1}+\frac{1}{2\pi i}\int_\Gamma dw(w-A)^{-1}\\ &= 0+I.\end{aligned}$$

So $\Phi=(A+B-z)^{-1}$ and since Φ is bounded, $z\in\rho(A+B)$. $\square$

APPENDIX B: NEGATIVITY OF A JUMP PROCESS

Lemma 5. *Suppose $\mathcal{L}$ is the generator for a jump process on a compact metric space X with nonnegative jump rate $r(x,y)dy$, i.e.*

$$(\mathcal{L}f)(x) = \int_X dy\, r(x,y)f(y) - \int_X dy\, r(y,x)f(x) =: (Kf)(x) - D(x)f(x).$$

Also, suppose for a.e. $x \in X$, $\frac{1}{C} \leqslant D(x) \leqslant C$ for some $C > 0$ and let $\mathcal{L}$ have a unique nonnegative invariant state w. Assume that for a.e. $x \in X$ and every nonnegative function ϕ with $\int_X \phi(x)dx = 1$, there exists $n \in \mathbb{N}$ such that $(K^n\phi)(x) > 0$. Then for all $f \neq$ constant a.e.,

$$-Re\langle f, \mathcal{L}(wf)\rangle > 0.$$

Proof. We first observe that the adjoint operator will be given by

$$(\mathcal{L}^\dagger f)(x) = \int_X r(y,x)(f(y) - f(x))dy.$$

We then have since $\mathcal{L}w = 0$,

$$\begin{aligned}
-\mathrm{Re}\langle f, \mathcal{L}(wf)\rangle &= -\frac{1}{2}\mathrm{Re}\langle f, \mathcal{L}(wf)\rangle - \frac{1}{2}\mathrm{Re}\langle \mathcal{L}^\dagger f, (wf)\rangle \\
&= -\frac{1}{2}\mathrm{Re}\int_X\int_X dxdy\, f^*(x)[r(x,y)w(y)f(y) - r(y,x)w(x)f(x)] \\
&\quad - \frac{1}{2}\mathrm{Re}\int_X\int_X dxdy\, r(y,x)[f^*(y) - f^*(x)]w(x)f(x) \\
&= \frac{1}{2}\mathrm{Re}\int_X\int_X dxdy\, r(x,y)w(y)[|f(x)|^2 - 2f^*(x)f(y) + |f(y)|^2] \\
&= \frac{1}{2}\int_X\int_X dxdy\, r(x,y)w(y)|f(x) - f(y)|^2.
\end{aligned}$$

This integral is clearly nonngative. Since w is an invariant state for $\mathcal{L}$, we will have

$$w(x) = \left(\frac{1}{D}K\right)w(x) = \left(\frac{1}{D}K\right)^n w(x) \geqslant \frac{1}{C^n}(K^n w)(x)$$

for each $n \in \mathbb{N}$ and a.e. $x \in X$. Using a suitable normalization of w, this implies $w > 0$ a.e. By way of contradiction, suppose $-\mathrm{Re}\langle f, \mathcal{L}(wf)\rangle = 0$. Then for a.e. $x, y \in X$ with

$r(x,y) > 0$, we have $f(x) = f(y)$. However, our assumption yields

$$\int_X dy_1 \cdots \int_X dy_{n-1} \int_X dy r(x,y_1) r(y_1,y_2) \cdots r(y_{n-2},y_{n-1}) r(y_{n-1},y) \phi(y) > 0$$

for a.e. $x \in X$ and every function $\phi \geqslant 0$ with $\int_X \phi(x)dx = 1$. Hence there exists a positive measure set $E \subseteq X^{n-1}$ such that

$$r(x,y_1), r(y_1,y_2), \cdots, r(y_{n-2},y_{n-1}), r(y_{n-1},y) > 0$$

for a.e. $x, y \in X$ and a.e. $(y_1, \cdots, y_n) \in E$. Therefore, for a.e. $x, y \in X$ and a.e. $(y_1, \cdots, y_n) \in E$,

$$f(x) = f(y_1) = f(y_2) = \cdots = f(y_{n-2}) = f(y_{n-1}) = f(y)$$

and $f = \textit{constant}$ a.e., a contradiction. $\square$

APPENDIX C: A GENERALIZED LIMIT FOR RESOLVENTS

In [1], the following limit result was proved for resolvents of a particular form:

Proposition 6 (Fröhlich/Schenker, 2016)**.** *Let $\mathcal{H}$ be a Hilbert space. Suppose A is a normal operator on $\mathcal{H}$ with $ReA \geqslant 0$ and B is a bounded operator on $\mathcal{H}$ with $ReB \geqslant c > 0$. Then if Π denotes the projection onto the kernel of A,*

$$\lim_{\lambda\to\infty} \langle \phi, (\lambda A + B)^{-1}\psi\rangle_{\mathcal{H}} = \langle \Pi\phi, (\Pi B\Pi)^{-1}\Pi\psi\rangle_{ran\Pi}$$

for all $\phi, \psi \in \mathcal{H}$.

This Proposition, while elegant, has a couple of drawbacks. For one, it requires that A be normal. Also, it does not hold up to compact perturbations of B. What we prove below is a slightly stronger generalization of this Proposition using the concept of dilation spaces given in [22].

Lemma 6. *Let $\mathcal{H}$ be a Hilbert space. Let A and B be operators on $\mathcal{H}$ with $ReA \geqslant 0$, $ReB \geqslant c > 0$ and B bounded. Then if* $\text{w-lim}_{\lambda\to\infty} \psi_\lambda = \psi \in \mathcal{H}$,

$$\lim_{\lambda\to\infty} \langle \phi, (\lambda A + B)^{-1}\psi_\lambda\rangle_{\mathcal{H}} = \langle \Pi\phi, (\Pi B\Pi)^{-1}\Pi\psi\rangle_{ran\Pi}$$

for any $\phi \in \mathcal{H}$, where Π denotes the projection onto the kernel of A.

Proof. Let $z \in \mathbb{C}$ be such that $\mathrm{Re}z > 0$ and consider the operator $F(z) := (\lambda A + z)^{-1}$ for some fixed $\lambda > 0$. As this is an operator-valued Herglotz function with $\mathrm{Re}\langle\phi, F(z)\phi\rangle \geqslant 0$, there exists a positive operator-valued measure M such that

$$F(z) = \int \frac{1}{i\lambda t + z} dM(t).$$

Based on results in [22], there exists a dilation space $\mathcal{K} \supset \mathcal{H}$ and a minimal dilation N of M onto $\mathcal{K}$ where N is a projection-valued measure on the real line. Furthermore, if P is the projection of $\mathcal{K}$ onto $\mathcal{H}$, then since $\frac{1}{i\lambda t+z}$ is bounded,

$$\int \frac{1}{i\lambda t + z} dM(t) = P \int \frac{1}{i\lambda t + z} dN(t) P.$$

Since N is a projection-valued measure on $\mathbb{R}$, the spectral theorem dictates that $H := \int t dN(t)$ defines a self-adjoint operator. Hence by spectral mapping,

$$F(z) = P(i\lambda H + z)^{-1}P.$$

Now consider the operator $(\lambda A + B + z)^{-1}$. Since B is bounded, a Taylor expansion yields

$$\begin{aligned}(\lambda A + B + z)^{-1} &= F(z)\sum_{n=0}^{\infty}(-BF(z))^n \\ &= P(i\lambda H + z)^{-1}\sum_{n=0}^{\infty}(-PBP(i\lambda H + z)^{-1})^n P \\ &= P(i\lambda H + PBP + z)^{-1}P \\ &= P(i\lambda H + P(B-c)P + z + c)^{-1}P.\end{aligned}$$

Since $\mathrm{Re}A > 0$ and $\mathrm{Re}B \geqslant c > 0$, the limit $z \to 0$ may be taken on the left-hand side. Similarly, since $c > 0$, $\mathrm{Re}(iH) = 0$, and $\mathrm{Re}P(B-c)P \geqslant 0$, we may take a limit as $z \to 0$ on the right-hand side as well to give

$$(\lambda A + B)^{-1} = P(i\lambda H + P(B-c)P + c)^{-1}P.$$

Denote $h_\lambda := (i\lambda H + P(B-c)P + c)^{-1}P\psi_\lambda$. We note that $\mathrm{Re}(iH) = 0$ and $\mathrm{Re}P(B-c)P + c \geqslant c > 0$. Hence h_λ is bounded since

$$c||h_\lambda||^2 \leqslant \mathrm{Re}\,\langle h_\lambda, P\psi_\lambda\rangle \leqslant ||h_\lambda||\;||P\psi_\lambda||$$

implies $||h_\lambda|| \leqslant c^{-1}||P\psi_\lambda||$. Then since $P\psi_\lambda = i\lambda H h_\lambda + [P(B-c)P + c]h_\lambda$, this gives

$$|\lambda|\;||iHh_\lambda|| \leqslant (1 + c^{-1}||P(B-c)P + c||)||P\psi_\lambda||.$$

Now since $\psi_\lambda \rightharpoonup \psi$, Banach-Steinhaus yields $P\psi_\lambda$ is uniformly bounded. Therefore, $(I - \Pi_H)h_\lambda \rightharpoonup 0$ where Π_H is the projection onto the kernel of H. Since iH is normal, Π_H commutes with iH. Hence

$$\Pi_H[P(B-c)P + c]h_\lambda = \Pi_H P\psi_\lambda.$$

Furthermore, since $\Pi_H[P(B-c)P+c]\Pi_H$ is boundedly invertible, we have

$$(\Pi_H[P(B-c)P+c]\Pi_H)^{-1}\Pi_H P\psi_\lambda \rightharpoonup (\Pi_H[P(B-c)P+c]\Pi_H)^{-1}\Pi_H P\psi.$$

These together imply that $\Pi_H h_\lambda \rightharpoonup (\Pi_H[P(B-c)P+c]\Pi_H)^{-1}\Pi_H P\psi$. That is,

$$\lim_{\lambda\to\infty}\langle\phi, P(\lambda iH+P(B-c)P+c)^{-1}P\psi_\lambda\rangle_{\mathcal{H}} \\ = \langle\Pi_H P\phi, (\Pi_H[P(B-c)P+c]\Pi_H)^{-1}\Pi_H P\psi\rangle_{\mathrm{ran}\Pi_H P}$$

for all $\phi\in\mathcal{H}$. Now the kernel of A corresponds with the atoms of the operator-valued measure M. These clearly correspond to the atoms of the projection-valued measure N, which then further correspond with the kernel of H. Hence the kernel of H coincides with the kernel of A and we may replace Π_H by Π. Finally, since P is a projection from $\mathcal{K}$ onto $\mathcal{H}$, P may be removed from the final expression. □

Corollary 1. *Let $\mathcal{H}$ be a Hilbert space. Let A, $B\in\mathcal{B}(\mathcal{H})$ with $ReA\geqslant 0$ and $ReB\geqslant c>0$, and let K be compact in $\mathcal{H}$. Then if* w-$\lim_{\lambda\to\infty}\psi_\lambda=\psi\in\mathcal{H}$ *and $ker\Pi(B+K)\Pi = ker\Pi$ for Π the projection onto the kernel of A,*

$$\lim_{\lambda\to\infty}\langle\phi,(\lambda A+B+K)^{-1}\psi_\lambda\rangle_{\mathcal{H}} = \langle\Pi\phi,(\Pi(B+K)\Pi)^{-1}\Pi\psi\rangle_{ran\Pi}$$

for any $\phi\in\mathcal{H}$.

Proof. For an arbitrary compact K, we may write $K=K'+F$ where $\mathrm{Re}K'>-c$ and F has finite rank. As $\mathrm{Re}(B+K')\geqslant c'>0$ for some constant c', we may thus assume without loss of generality that $K=F$ has finite rank.

Let us write

$$(\lambda A+B+K)^{-1} = (\lambda A+B)^{-1} - (\lambda A+B)^{-1}K(\lambda A+B+K)^{-1}$$

or

$$S_\lambda(\lambda A+B+K)^{-1} = (\lambda A+B)^{-1}$$

where

$$S_\lambda := I + (\lambda A + B)^{-1}K.$$

Consider the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ where $\mathcal{H}_1 = \text{ran}P$ and $\mathcal{H}_2 = \text{ran}(1-P)$ for P the projection onto $(\ker K)^\perp$. This decomposition yields

$$S_\lambda = \begin{pmatrix} 1-P & (1-P)(\lambda A + B)^{-1}K \\ 0 & P + P(\lambda A + B)^{-1}K \end{pmatrix}.$$

Assuming for the moment that $P + P(\lambda A + B)^{-1}K$ is invertible, Schur complement gives

$$S_\lambda^{-1} = \begin{pmatrix} 1-P & -(1-P)(\lambda A + B)^{-1}K(P + P(\lambda A + B)^{-1}K)^{-1} \\ 0 & (P + P(\lambda A + B)^{-1}K)^{-1} \end{pmatrix}. \tag{34}$$

We observe that $P + P(\lambda A + B)^{-1}K$ will be invertible if and only if $P(\lambda A + B)^{-1}K$ does not have a nontrivial eigenvector corresponding to the eigenvalue -1.

By Lemma 6, $(\lambda A + B)^{-1}K$ converges weakly to $\Pi(\Pi B \Pi)^{-1}\Pi K$ as $\lambda \to \infty$. Then since K is compact and P is the projection onto ran$K$, $P(\lambda A + B)^{-1}K$ actually converges to $P\Pi(\Pi B \Pi)^{-1}\Pi K$ in norm. Hence taking $\lambda \to \infty$ in the eigenvalue equation $P(\lambda A + B)^{-1}K\psi = -\psi$ yields $\Pi(\Pi B \Pi)^{-1}\Pi K \psi = -\psi$ for $\psi \in \text{ran}P$. In fact, this equation reveals that $\psi \in \text{ran}\Pi$ as well and so $(\Pi B \Pi)^{-1}\Pi K \Pi \psi = -\psi$ for $\psi \in \text{ran}P \cap \text{ran}\Pi$. That is, $\Pi(B + K)\Pi\psi = 0$. Since we have assumed $\ker \Pi(B + K)\Pi = \ker\Pi$, $\psi = 0$ and therefore, $P + P\Pi(\Pi B \Pi)^{-1}\Pi K$ is invertible. Using perturbation theory and norm convergence, we may conclude that $P + P(\lambda A + B)^{-1}K$ is invertible for sufficiently large λ and so S_λ is also invertible with inverse given in block form by (34). Then since $(\lambda A + B + K)^{-1} =$

$S_\lambda^{-1}(\lambda A+B)^{-1}$, we have for any $\phi, \psi_\lambda \in \mathcal{H}$ with $\psi_\lambda \rightharpoonup \psi$,

$$\begin{aligned}
&\langle \phi, (\lambda A+B+K)^{-1}\psi_\lambda\rangle = \\
&\quad \langle (1-P)\phi, (1-P)(\lambda A+B)^{-1}\psi_\lambda\rangle \\
&\quad - \langle (1-P)\phi, (1-P)(\lambda A+B)^{-1}K(P+P(\lambda A+B)^{-1}K)^{-1}P(\lambda A+B)^{-1}\psi_\lambda\rangle \\
&\quad + \langle P\phi, (P+P(\lambda A+B)^{-1}K)^{-1}P(\lambda A+B)^{-1}\psi_\lambda\rangle.
\end{aligned}$$

Since P is the projection onto a finite rank operator, we may decompose each projection P in the above expression into a finite sum. Then utilizing Lemma 6, the limit of each term should exist as $\lambda \to \infty$. In particular, we should have

$$\begin{aligned}
&\lim_{\lambda\to\infty}\langle \phi, (\lambda A+B+K)^{-1}\psi_\lambda\rangle = \\
&\quad \langle (1-P)\phi, (1-P)\Pi(\Pi B\Pi)^{-1}\Pi\psi\rangle \\
&\quad - \langle (1-P)\phi, (1-P)\Pi(\Pi B\Pi)^{-1}\Pi K(P+P\Pi(\Pi B\Pi)^{-1}\Pi K)^{-1}P\Pi(\Pi B\Pi)^{-1}\Pi\psi\rangle \\
&\quad + \langle P\phi, (P+P\Pi(\Pi B\Pi)^{-1}\Pi K)^{-1}P\Pi(\Pi B\Pi)^{-1}\Pi\psi\rangle \\
&\quad = \langle \phi, S_\infty^{-1}\Pi(\Pi B\Pi)^{-1}\Pi\psi\rangle
\end{aligned}$$

where

$$S_\infty^{-1} := \begin{pmatrix} 1-P & (1-P)\Pi(\Pi B\Pi)^{-1}\Pi K \\ 0 & P+P\Pi(\Pi B\Pi)^{-1}\Pi K \end{pmatrix}.$$

Working backwards, this yields $S_\infty = I + \Pi(\Pi B\Pi)^{-1}\Pi K$ and therefore, we have

$$\lim_{\lambda\to\infty}\langle \phi, (\lambda A+B+K)^{-1}\psi_\lambda\rangle = \langle \phi, (I+\Pi(\Pi B\Pi)^{-1}\Pi K)^{-1}\Pi(\Pi B\Pi)^{-1}\Pi\psi\rangle.$$

By decomposing the space $\mathcal{H}$ into $\mathcal{H} = \operatorname{ran}(1-\Pi)\oplus \operatorname{ran}\Pi$, this may be simplified to

$$\lim_{\lambda\to\infty}\langle \phi, (\lambda A+B+K)^{-1}\psi_\lambda\rangle = \langle \Pi\phi, (\Pi(B+K)\Pi)^{-1}\Pi\psi\rangle$$

as desired. $\square$