

COMBINATORIAL PROPERTIES OF PERMUTATIONS AND PERMUTATION STATISTICS

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ABSTRACT

Some of the most interesting and fundamental objects in the study of combinatorics are permutations. *Permutations* are typically defined to be an arrangement of the numbers $\{1, 2, \dots, n\}$, and they appear in countless problems and applications throughout mathematics. Sometimes we are particularly interested in observing or counting some key property of a given permutation. A *permutation statistic* is a function that takes a particular permutation and returns specific information about it, such as how many numbers are only adjacent to smaller numbers. Each permutation statistic gives rise to many counting questions, such as finding all possible permutations which have that particular statistic, or perhaps finding a subset of those permutations where that statistic is maximized. These numbers can sometimes be described by generating functions or polynomials which give light to the beauty and structure of mathematics, sometimes even in fields outside combinatorics. Some examples of this will be given in Chapter 1. In the rest of this dissertation, we will explore several of these statistics including new ones and variations on some of the more well known statistics.

In the Chapter 2, we focus on shuffle compatibility for cyclic permutations. The *shuffle* of two permutations is the set of all permutations that have both original permutations as sub-sequences. A statistic is said to be *shuffle compatible* if its values over all possible shuffles of two permutations is completely determined by the statistic on the two original permutations, together with their lengths. Shuffle compatibility is implicit in Stanley's work on P -partitions and was also studied by Gessel and Zhuang. Shuffle compatibility is also useful in studying mathematical objects outside of combinatorics, such as quasisymmetric functions. More recently, an analogous definition of shuffle compatibility has been defined for *cyclic* permutations, which are permutations arranged in a circle so that the last element is considered adjacent to the first. In Chapter 2, we study a Lifting Lemma that can prove shuffle compatibility for some statistics on circular permutations based on known results for those statistics on linear permutations.

One well-studied permutation statistic is the *peak set*, which is the set of all indices of a permutation where an element is adjacent to two smaller elements. Primarily spearheaded by Davis, Nelson, Petersen, and Tenner, there has been recent interest in studying an analogue of this

statistic known as the *pinnacle set*, which are the values of the elements at the indices of the peak set. Davis et al. proposed a number of unanswered questions about this statistic, which later led to a series of papers on the topic. In Chapter 3 we will present multiple results that attempt to answer some of these questions, including some original formulas and also some alternate combinatorial proofs of known results. These will include a bijection for counting the number of sets that could be the pinnacle set of some permutation, formulas and recursions for counting the number of permutations with a given pinnacle set, along with a new proof for a weighted sum of those numbers, and a recursion for counting the number of distinct orderings in which the elements of a pinnacle set can appear within a permutation.

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CHAPTER 1

INTRODUCTION

1.1 Definitions, notation, and motivation

Throughout, we will let $\mathbb{N}$ and $\mathbb{P}$ be the nonnegative and positive integers, respectively. Given $m, n \in \mathbb{P}$ we use the notation $[m, n] = \{m, m+1, \dots, n\}$ for the interval they define, and abbreviate $[1, n]$ to $[n]$. Let $\mathfrak{S}_n$ be the symmetric group of all permutations $\pi = \pi_1\pi_2 \dots \pi_n$ of $[n]$ written in one-line notation. If $\pi = \pi_1\pi_2 \dots \pi_n$ then n is called the *length* of π , written $\# \pi = |\pi| = n$. The hash symbol and absolute value sign will also be used for the cardinality of a set. In fact, we will often treat permutations as sets if no problem will result and write things such as $x \in \pi$ in place of the more cumbersome $\pi_i = x$ for some i .

Given a finite $A \subset \mathbb{P}$ then a *linear permutation* of A is a linear arrangement $\pi = \pi_1\pi_2 \dots \pi_n$ of the elements of A . This is a generalization of permutations in $\mathfrak{S}_n$ which are just linear arrangements of $A = [n]$. We let

$$L(A) = \{\pi \mid \pi \text{ is a linear permutation of } A\}.$$

We often drop “linear” if it is understood from context. For example,

$$L(\{1, 3, 6\}) = \{136, 163, 316, 361, 613, 631\}.$$

A *(linear) permutation statistic* is a function St whose domain is all linear permutations. Several common examples are as follows. A permutation $\pi = \pi_1\pi_2 \dots \pi_n$ has *descent set*

$$\text{Des } \pi = \{i \mid \pi_i > \pi_{i+1}\},$$

descent number

$$\text{des } \pi = \# \text{Des } \pi,$$

peak set

$$\text{Pk } \pi = \{i \mid \pi_{i-1} < \pi_i > \pi_{i+1}\},$$

peak number

$$\text{pk } \pi = \# \text{Pk } \pi,$$

major index

$$\text{maj } \pi = \sum_{i \in \text{Des } \pi} i,$$

inversion set

$$\text{Inv } \pi = \{(i, j) \mid i < j \text{ and } \pi_i > \pi_j\},$$

and inversion number

$$\text{inv } \pi = \# \text{Inv } \pi.$$

To illustrate, if $\pi = 4218596$ then $\text{Des } \pi = \{1, 2, 4, 6\}$, $\text{des } \pi = 4$, $\text{Pk } \pi = \{4, 6\}$, $\text{pk } \pi = 2$, $\text{maj } \pi = 1 + 2 + 4 + 6 = 13$, $\text{Inv } \pi = \{(1, 2), (1, 3), (2, 3), (4, 5), (4, 7), (6, 7)\}$, and $\text{inv } \pi = 6$. We will be especially concerned with a particular type of statistic which is determined by the descent set in the following sense. Statistic St is a *descent statistic* if $\text{Des } \pi = \text{Des } \sigma$ and $\# \pi = \# \sigma$ implies $\text{St } \pi = \text{St } \sigma$. Note that Des itself, des , Pk , pk , and maj are all descent statistics. A statistic which is not a descent statistic would be $\text{inv } \pi$ since 312 and 213 both have the same Des but different inv .

We will often need to evaluate a statistic St on a set of permutations Π . So we define the *distribution* of St over Π to be

$$\text{St } \Pi = \{\{\text{St } \pi \mid \pi \in \Pi\}\}.$$

Note that this is a set with multiplicity or *multiset* since St may evaluate to the same thing on various members of Π . We will sometimes use exponents to denote multiplicities where, as usual, no exponent means multiplicity 1. For example

$$\text{des}\{123, 132, 312, 321\} = \{\{0, 1, 1, 2\}\} = \{\{0, 1^2, 2\}\}.$$

Permutation statistics are interesting objects of study not only because of their relation to other areas of mathematics such as quasisymmetric functions, but also for their innate beauty and structure. For example, while maj and inv typically evaluate to different numbers on the same permutation, their distributions over $\mathfrak{S}_n$ are always the same. In general, statistics with the same distribution as maj and inv are known as *Mahonian statistics*, and all of them have the following beautiful generating function:

$$\sum_{\pi \in \mathfrak{S}_n} q^{\text{maj } \pi} = [n]_q!,$$

where $[n]_q! = (1)(1+q)(1+q+q^2) \cdots (1+q+q^2 \cdots q^{n-1})$. This is known as the q -factorial, which derives its name from the property that if we substitute $q = 1$, then we get $[n]_q! = n!$. Notice that in our generating function, letting $q = 1$ will result in a sum over all permutations in $\mathfrak{S}_n$ on the left-hand side, which is then equal to $n!$ as expected. In addition to the factorial, q -analogues exist for many other numbers as well including binomial coefficients, where each factorial is replaced with the corresponding q -factorial. The study of these different q -analogues is a subject in its own right, and Mahonian statistics is only one of many places where they naturally appear.

Another class of statistics with interesting generating functions is known as the *Eularian statistics*, which consist of all those statistics whose distribution over $\mathfrak{S}_n$ is the same as des . Their generating function is as follows:

$$\sum_{\pi \in \mathfrak{S}_n} q^{\text{des } \pi} = (1-q)^{n+1} \sum_{m \geq 0} (m+1)^n q^m.$$

This distribution is so important that it is denoted by $A_n(q)$ and called the n^{th} *Eularian polynomial*. Additionally, these polynomials themselves have an exponential generating function given by

$$\sum_{n \geq 0} A_n(q) \frac{x^n}{n!} = \frac{q-1}{q - e^{(q-1)x}}.$$

Even some of the non-Eularian descent statistics turn out to be closely related to $A_n(q)$. For example, if we define the generating function for pk as follows:

$$P_n(q) = \sum_{\pi \in \mathfrak{S}_n} q^{\text{pk } \pi},$$

then we get the relation,

$$A_n(q) = \frac{(1+q)^{n-1}}{2^{n-1}} P_n\left(\frac{4q}{(1+q)^2}\right).$$

Our goal will be to examine analogues of a few of these permutation statistics.

1.2 Chapter 2 introduction

If $\pi \in L(A)$ and $\sigma \in L(B)$ where $A \cap B = \emptyset$ then these permutations have *shuffle set*

$$\pi \sqcup \sigma = \{\tau \in L(A \uplus B) \mid \pi, \sigma \text{ are subwords of } \tau\}$$

where a subword of a permutation τ is a subsequence of (not necessarily consecutive) elements of τ . For example,

$$25 \sqcup 73 = \{2573, 2753, 2735, 7253, 7235, 7325\}. \quad (1.1)$$

Whenever we write the shuffle of two permutations, we will tacitly assume that they are from disjoint sets.

An important use of the shuffle set is in computing the product of two fundamental quasisymmetric functions, which form a basis for the algebra QSym of quasisymmetric functions. These fundamental functions are indexed by the descent sets of permutations, and when two are multiplied together the result is a sum of fundamental quasisymmetric functions over the descent sets of all shuffles of the original permutations. In order to show that multiplication is well defined (and hence that QSym really is an algebra), one needs shuffle compatibility as defined below. More information about QSym can be found in the texts of Luoto, Mykytiuk, and van Willigenburg [18], Sagan [23], or Stanley [25]. Roughly speaking, a statistic St is shuffle compatible if the distribution of St over $\pi \sqcup \sigma$ depends only on $\text{St } \pi$, $\text{St } \sigma$, and the lengths $\#\pi$ and $\#\sigma$. To be precise, St is *shuffle compatible* if for all quadruples $\pi, \pi', \sigma, \sigma'$ with $\text{St } \pi = \text{St } \pi'$, $\text{St } \sigma = \text{St } \sigma'$, $\#\pi = \#\pi'$, and $\#\sigma = \#\sigma'$ we have

$$\text{St}(\pi \sqcup \sigma) = \text{St}(\pi' \sqcup \sigma').$$

For example, from equation (1.1) it is easy to see that

$$\text{des}(25 \sqcup 73) = \{\{1^3, 2^3\}\}$$

The reader can check that this is also the distribution $\text{des}(12 \sqcup 43)$ which is because des is shuffle compatible. Shuffle compatibility is implicit in Stanley's theory of P -partitions [24] and he proved a result implying the shuffle compatibility of des . Gessel and Zhuang [12] were the first to define the concept explicitly and prove many shuffle compatibility results, including those for some of the other statistics defined previously.

Theorem 1.2.1 ([12]). *The statistics*

$$\text{Des}, \text{des}, \text{Pk}, \text{pk}$$

are all shuffle compatible. □

Other work on shuffle compatibility has been done by Baker-Jarvis and Sagan [2], by Oğuz [20], and by Grinberg [15].

Recently Adin, Gessel, Reiner, and Roichman [1] introduced a cyclic version of quasisymmetric functions with a corresponding cyclic shuffle operation. A linear permutation $\pi = \pi_1\pi_2 \dots \pi_n$ has a corresponding *cyclic permutation* $[\pi]$ which is the set of all its rotations, namely

$$[\pi] = \{\pi_1\pi_2 \dots \pi_n, \pi_2 \dots \pi_n\pi_1, \dots, \pi_n\pi_1 \dots \pi_{n-1}\}.$$

For example

$$[3725] = \{3725, 7253, 2537, 5372\}$$

so that

$$[3725] = [7253] = [2537] = [5372].$$

The reader should bear in mind the difference between the notation $[n]$ for an interval of integers and $[\pi]$ for a cyclic permutation. We let

$$C(A) = \{[\pi] \mid [\pi] \text{ is a cyclic permutation of } A\}.$$

To illustrate

$$C(\{2, 3, 5, 7\}) = \{[2357], [2375], [2537], [2735], [2573], [2753]\}.$$

We can lift the linear permutation statistics we have introduced to the cyclic realm as follows. Define the *cyclic descent set* and *cyclic descent number* for a linear permutation $\pi = \pi_1\pi_2 \dots \pi_n$ to be

$$\text{cDes } \pi = \{i \in [n] \mid \pi_i > \pi_{i+1} \text{ where } i \text{ is taken modulo } n\}$$

and

$$\text{cdes } \pi = \# \text{cDes } \pi,$$

respectively. Returning to $\pi = 4218596$ we have $\text{cDes } \pi = \{1, 2, 4, 6, 7\}$ and $\text{cdes } \pi = 5$. Now a cyclic permutation $[\pi]$ will have *cyclic descent set*

$$\text{cDes}[\pi] = \{\{\text{cDes } \sigma \mid \sigma \in [\pi]\}\}$$

and *cyclic descent number*

$$\text{cdes}[\pi] = \text{cdes } \pi.$$

Note that $\text{cdes}[\pi]$ is well defined since all elements of $[\pi]$ have the same number of cyclic descents. To illustrate

$$\text{cDes}[3725] = \{\{\text{cDes } 3725, \text{cDes } 7253, \text{cDes } 2537, \text{cDes } 5372\}\} = \{\{\{1, 3\}^2, \{2, 4\}^2\}\}$$

and $\text{cdes}[3725] = 2$. An important point later will be that if S is one of the sets in $\text{cDes}[\pi]$ then all of the other member sets can be obtained by adding i to the elements of S modulo $n = \# \pi$ as i runs over $[n - 1]$.

We deal with peaks in a similar manner, defining the *cyclic peak set* and *cyclic peak number* in the linear case by

$$\text{cPk } \pi = \{i \in [n] \mid \pi_{i-1} < \pi_i > \pi_{i+1} \text{ where } i \text{ is taken modulo } n\}$$

and

$$\text{cpk } \pi = \# \text{cPk } \pi,$$

respectively. For $\pi = 4218593$ we have $\text{cPk } \pi = \{1, 4, 6\}$ and $\text{cpk } \pi = 3$. The extension to cyclic permutations is as expected

$$\text{cPk}[\pi] = \{\{\text{cPk } \sigma \mid \sigma \in [\pi]\}\}$$

and

$$\text{cpk}[\pi] = \text{cpk } \pi.$$

In general, a *cyclic permutation statistic* is any function cSt whose domain is cyclic permutations. And cSt is a *cyclic descent statistic* if $\text{cDes}[\pi] = \text{cDes}[\sigma]$ and $\#\pi = \#\sigma$ implies $\text{cSt}[\pi] = \text{cSt}[\sigma]$. All four of our cyclic statistics are cyclic descent statistics.

The definitions for shuffles follow the same pattern already established. Given $[\pi] \in C(A)$ and $[\sigma] \in C(B)$ where $A \cap B = \emptyset$ we define their *cyclic shuffle set* to be

$$[\pi] \sqcup [\sigma] = \{[\tau] \in C(A \uplus B) \mid [\pi], [\sigma] \text{ are circular subwords of } [\tau]\}.$$

Alternatively, there are rotations π' and σ' of π and σ , respectively, which are both linear subwords of τ . To illustrate,

$$[13] \sqcup [24] = \{[1324], [1342], [1234], [1432], [1243], [1423]\}. \quad (1.2)$$

Call a cyclic permutation statistic cSt *cyclic shuffle compatible* if for all quadruples $[\pi], [\pi'], [\sigma], [\sigma']$ with $\text{cSt}[\pi] = \text{cSt}[\pi']$, $\text{cSt}[\sigma] = \text{cSt}[\sigma']$, $\#\pi = \#\pi'$, and $\#\sigma = \#\sigma'$ we have

$$\text{cSt}([\pi] \sqcup [\sigma]) = \text{cSt}([\pi'] \sqcup [\sigma']).$$

The aim of the Chapter 2 will be to study this concept. In particular, we will prove the following theorem, which was my contribution to the paper I worked on with Domagalski, Liang, Sagan, Schmidt, and Sietsema [7].

Theorem 1.2.2. *The statistics*

$$\text{cDes}, \text{cDes}, \text{cPk}, \text{cPk}$$

are all cyclic shuffle compatible.

□

Our primary tool for doing this will be a Lifting Lemma (2.1.3) which can prove shuffle compatibility for certain cyclic statistic from the shuffle compatibility of their linear counterparts. The theorem also requires a certain property pertaining to how the largest element of a circular permutation contributes to the given statistic, but thankfully the property holds for the above statistics.

One interesting nuance about many linear permutation statistics, including those mentioned above, is that they depend heavily on the indices of certain elements within a permutation. Because of this, it can sometimes be difficult to define the cyclic version of these statistics, and when we do, it is not always obvious which properties or formulas from the linear case carry through. One way of side-stepping this difficulty however is to consider statistics which depend on the values within a permutation as opposed to the indices. In the next chapter, we focus on a particular statistic with this property known as the pinnacle set.

1.3 Chapter 3 introduction

Define the *pinnacle set* of a permutation $\pi \in \mathfrak{S}_n$ to be

$$\text{Pin } \pi = \{\pi_i \mid \pi_{i-1} < \pi_i > \pi_{i+1}\} \subseteq [3, n].$$

Pinnacle sets have generated a lot of recent interest in part because of their close relation to the peak set, since the pinnacle set is just the values at the indices of the peak set. Peak sets have been widely studied with many interesting results. For instance, it is easy to see that $S \subseteq [2, n-1]$ is the peak set of some $\pi \in \mathfrak{S}_n$ if and only if no two elements of S are consecutive, and so the number of possible peak sets is a Fibonacci number. One could also ask how many permutations have a given peak set. This question was answered by Billey, Burdzy and Sagan.

Theorem 1.3.1 ([3]). *If $n \in \mathbb{P}$ and $S \subseteq [2, n]$ then*

$$\#\{\pi \in \mathfrak{S}_n \mid \text{Pk } \pi = S\} = p(S; n)2^{n-\#S-1}$$

where $p(S; n)$ is a polynomial in n depending on S . □

It is natural to study the values at the peak indices. Some of the initial work related pinnacle sets (although under a different name) was done by Strehl in [26] where he studied permutations in which pinnacles and non-pinnacles alternated. The concept was rediscovered independently by Davis et al. in [5] where they proved several initial results, including that the number of sets $P \subseteq [n]$ that could be a pinnacle set for some permutation is counted by a central binomial coefficient. Not long afterwards Domagalski et al. gave a formula in [9] for the number of permutations in $\mathfrak{S}_n$

having a given pinnacle set P using the principle of inclusion and exclusion. This was a paper that I coauthored, and those results in it that were originally mine have been included as the first two main results in Chapter 3. Around the same time, Diaz-Lopez et al. gave a different formula for the same number by keeping track of *vale sets*, which are those elements of a permutation π smaller than those on either side. Both of these formulas however ran in time exponential in the number of elements of P .

Shortly afterwards, Falque et al. published some results in [10] which included a recursive formula for the number of permutations having a fixed pinnacle set that ran in time $O(nk^2)$ where k is the number of elements in P and n is the length of the permutations being counted. Finally, both Fang [11] and myself [19], Theorem 3.5.3 below, independently proved results which computed the number in $O(k^2 \log n + k^4)$ time.

In addition to results that pertained to the number of permutations with a given pinnacle set, Rusu in [21] studied transforming a permutation with a given pinnacle set into another permutation with the same pinnacle set such that the pinnacle set was preserved in all intermediate steps of the transformation. Additional work was also done by Rusu and Tenner in [22] characterizing the orderings in which the elements of some pinnacle set P could appear within a permutation with pinnacle set P . Later, both Fang [11] and Minnich [19], Theorem 3.4.5 below, independently found means of calculating the number of such orderings of some pinnacle set P with k elements that ran in $O(k^2)$ time.

Following Davis et al. we call a set P an *admissible pinnacle set* if there is some permutation π with $\text{Pin } \pi = P$. Note that if $P = \text{Pin } \pi$ for some $\pi \in \mathfrak{S}_n$ then P is also a pinnacle set of some $\pi' \in \mathfrak{S}_{n'}$ for all $n' \geq n$ since one can just add values larger than n to the beginning of π in decreasing order. Davis et al. found a criterion for P to be admissible which will be useful in the sequel. This result was stated in recursive fashion, but it is clearly equivalent to the following non-recursive version.

Theorem 1.3.2 ([5]). *Let $P = \{p_1 < p_2 < \dots < p_d\} \subset \mathbb{P}$. The set P is an admissible pinnacle set*

if and only if we have

$$p_i > 2i$$

for all $i \in [d]$.

□

In Chapter 3 we will present many more results on pinnacle sets, including a bijection for counting the number of admissible pinnacle sets (Theorem 3.1.4), multiple methods for counting the number of permutations with a given pinnacle set (Theorem 3.2.1 and Theorem 3.5.3), and a method for counting the number of orderings in which the elements of some pinnacle set P could appear within a permutation with pinnacle set P (Theorem 3.4.5).

CHAPTER 2

CYCLIC SHUFFLING

This chapter will be structured as follows. Rather than proving each of the cases of Theorem 1.2.2 in an ad hoc manner, we will develop a method for lifting linear shuffle compatibility results to cyclic ones. This will be done in the next section. Then in Section 2.2 we will apply our Lifting Lemma to each of the four statistics in turn. Finally, we will end with a section of comments and a future direction for research.

We will first set some notation. In the notations like $C(A)$ and $L(A)$ we will often drop the parentheses if they would cause double delimiters. For example $L[n]$ is all the linear permutations of $[n]$.

If $A \subset \mathbb{P}$ and $n \in \mathbb{N}$ then let

$$A + n = \{a + n \mid a \in A\}.$$

In particular,

$$[m] + n = \{n + 1, n + 2, \dots, n + m\}.$$

We will also need to add integers to sets modulo some $m \in \mathbb{P}$. So if $A \subseteq [m]$ then define

$$A + n \pmod{m} = \{a + n \pmod{m} \mid a \in A\}$$

where the representatives are chosen to be in $[m]$. For example if $A = \{2, 4, 5\}$ and $m = 6$ then

$$A + 3 = \{5, 7, 8\}$$

and

$$A + 3 \pmod{6} = \{1, 2, 5\}.$$

We will also need the notion of standardization. Suppose $A, B \subset \mathbb{P}$ with $\#A = \#B = n$ and $\pi = \pi_1\pi_2 \dots \pi_n \in L(A)$. The *standardization of π to B* is

$$\text{std}_B \pi = f(\pi_1)f(\pi_2) \dots f(\pi_n) \in L(B)$$

where $f : A \rightarrow B$ is the unique order-preserving bijection between A and B . To illustrate, if $A = \{2, 4, 6, 7\}$ and $B = \{1, 3, 8, 9\}$ then $\text{std}_B(4762) = 3981$. If $B = [n]$ then we write just $\text{std } \pi$ for $\text{std}_{[n]} \pi$ and call this the *standardization* of π . Standardization for cyclic permutations is defined in the analogous manner.

Finally, if Π and Π' are sets of permutations and St is a statistic, then a function $f : \Pi \rightarrow \Pi'$ is *St-preserving* if

$$\text{St } f(\pi) = \text{St } \pi$$

for all $\pi \in \Pi$. Functions preserving cyclic permutation statistics are defined in the obvious way.

2.1 The lifting lemma

In this section we will provide a general method for proving cyclic shuffle compatibility results as corollaries of linear ones. We start with a result about cyclic descent statistics which is a cyclic analogue of one in the linear case [2].

Lemma 2.1.1. *Let cSt be a cyclic descent statistic. For any four cyclic permutations $[\pi]$, $[\pi']$, $[\sigma]$, $[\sigma']$ such that*

$$\text{std}[\pi] = \text{std}[\pi'] \quad \text{and} \quad \text{std}[\sigma] = \text{std}[\sigma']$$

we have

$$\text{cSt}([\pi] \sqcup [\sigma]) = \text{cSt}([\pi'] \sqcup [\sigma']).$$

Proof. Since cSt is a cyclic descent statistic, its values only depends on the relative order of adjacent elements. So it suffices to prove the case when

$$[\pi] \uplus [\sigma] = [\pi'] \uplus [\sigma'] = [m + n]$$

where $m = \#\pi = \#\pi'$ and $n = \#\sigma = \#\sigma'$. For simplicity, let $A = [m]$ and $B = [n] + m$.

We claim that it suffices to find, for any π and σ as in the previous paragraph, a cSt -preserving bijection

$$[\pi] \sqcup [\sigma] \rightarrow \text{std}_A[\pi] \sqcup \text{std}_B[\sigma].$$

For from this map and the hypothesis of the lemma we have

$$\text{cSt}([\pi] \sqcup [\sigma]) = \text{cSt}(\text{std}_A[\pi] \sqcup \text{std}_B[\sigma]) = \text{cSt}(\text{std}_A[\pi'] \sqcup \text{std}_B[\sigma']) = \text{cSt}([\pi'] \sqcup [\sigma']).$$

We will show the existence of this bijection by induction on the size of the set of what we will call out-of-order pairs

$$O = \{(i, j) \in \pi \times \sigma \mid i > j\}.$$

If $\#O = 0$ then $[\pi] \in C(A)$ and $[\sigma] \in C(B)$. It follows that $[\pi] = \text{std}_A[\pi]$ and $[\sigma] = \text{std}_B[\sigma]$ so the identity map will do.

For the induction step, let $\#O > 0$. Then there must be a pair $(i, i-1) \in O$. Let $\pi'' = (i-1, i)\pi$ and $\sigma'' = (i-1, i)\sigma$ where $(i-1, i)$ is the transposition which exchanges $i-1$ and i . We will be done if we can construct a cSt preserving bijection

$$T_i : [\pi] \sqcup [\sigma] \rightarrow [\pi''] \sqcup [\sigma''].$$

This is because π'', σ'' have fewer out-of-order pairs and so, by induction, there is a cSt-preserving bijection $[\pi''] \sqcup [\sigma''] \rightarrow \text{std}_A[\pi''] \sqcup \text{std}_B[\sigma'']$ which, when composed with T_i , will finish the construction.

Define T_i by

$$T_i[\tau] = \begin{cases} [(i-1, i)\tau] & \text{if } i-1, i \text{ are not cyclically adjacent in } [\tau], \\ [\tau] & \text{else.} \end{cases}$$

We must first check that T_i is well defined in that $T_i[\tau] \in [\pi''] \sqcup [\sigma'']$. This is true if $i-1$ and i are not cyclically adjacent since $i-1$ and i have been swapped in all three cyclic permutations involved. If they are adjacent then the relative order of the elements of $[\tau]$ corresponding to $[\pi]$ and $[\pi'']$ are the same, and similarly for $[\sigma]$ and $[\sigma'']$. So leaving $[\tau]$ fixed again gives a shuffle in the range.

Finally, we need to verify that T_i is cSt preserving. Since cSt is a descent statistic, it suffices to show that T_i is cDes preserving. Certainly this is true of $[\tau]$ is fixed. And if it is not, then

$i - 1, i$ are not cyclically adjacent in $[\tau]$. But switching $i - 1$ and i could only change a cyclic descent into a cyclic ascent or vice-versa if these two elements were adjacent. So in this case $\text{cDes}[(i - 1, i)\tau] = \text{cDes}[\tau]$ and we are done. $\square$

As an example of the map T_i , consider the shuffle set in (1.2). Here we can take $i = 3$ since $3 \in 13$ and $2 \in 24$. For $[\tau] = [1324]$ we have 2 and 3 cyclically adjacent so

$$T_3[1324] = [1324] \in [12] \sqcup [34]$$

as desired. On the other hand, in $[1342]$ the 2 and 3 are not cyclically adjacent so

$$T_3[1342] = [1243].$$

Note that $[1243] \in [12] \sqcup [34]$ and

$$\text{cDes}[1243] = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 4\}\} = \text{cDes}[1342].$$

We can use the previous lemma to drastically cut down on the number of cases which need to be checked to obtain cyclic shuffle compatibility. In particular, the component permutations in the shuffles to be considered can be on consecutive intervals of integers. And one can keep one component of the shuffle constant while the other varies.

Corollary 2.1.2. *Suppose that cSt is cyclic descent statistic. The following are equivalent.*

(a) *The statistic cSt is cyclic shuffle compatible.*

(b) *If $\text{cSt}([\pi]) = \text{cSt}([\pi'])$ where $[\pi], [\pi'] \in C[m]$ and $[\sigma] \in C([n] + m)$ for some $m, n \in \mathbb{N}$ then*

$$\text{cSt}([\pi] \sqcup [\sigma]) = \text{cSt}([\pi'] \sqcup [\sigma]).$$

(c) *If $\text{cSt}([\sigma]) = \text{cSt}([\sigma'])$ where $[\sigma], [\sigma'] \in C([n] + m)$ and $[\pi] \in C[m]$ for some $m, n \in \mathbb{N}$ then*

$$\text{cSt}([\pi] \sqcup [\sigma]) = \text{cSt}([\pi] \sqcup [\sigma']).$$

Proof. We will prove the equivalence of (a) and (b) since the equivalence of (a) and (c) is similar. Clearly (a) implies (b). For the converse, let $\pi, \pi', \sigma, \sigma'$ be any four permutations satisfying the hypothesis of the cyclic shuffle compatible definition and let $m = \#\pi = \#\pi', n = \#\sigma = \#\sigma'$. Also let $A = [m], B = [n] + m, A' = [m] + n$, and $B' = [n]$. Then, using Lemma 2.1.1 and (b) alternately

$$\begin{aligned}
\text{cSt}([\pi] \sqcup [\sigma]) &= \text{cSt}(\text{std}_A[\pi] \sqcup \text{std}_B[\sigma]) \\
&= \text{cSt}(\text{std}_A[\pi'] \sqcup \text{std}_B[\sigma]) \\
&= \text{cSt}(\text{std}_{A'}[\pi'] \sqcup \text{std}_{B'}[\sigma]) \\
&= \text{cSt}(\text{std}_{A'}[\pi'] \sqcup \text{std}_{B'}[\sigma']) \\
&= \text{cSt}([\pi'] \sqcup [\sigma'])
\end{aligned}$$

which is what we wished to prove. $\square$

In order to prove the Lifting Lemma, we will need to define two functions. For $i \in [n]$ we construct the *splitting map* $S_i : C[n] \rightarrow L[n]$ as follows. If $[\pi] \in C[n]$ then let $S_i[\pi]$ be the unique linear permutation in $[\pi]$ which starts with i . For example,

$$S_3[45132] = 32451.$$

Also define the *maximum removal map* $M : C[n] \rightarrow L[n-1]$ by first applying S_n to $[\pi]$ and then removing the initial n . To illustrate

$$M[45132] = 1324.$$

Note that M is a bijection. We similarly define $M : C([n] + m) \rightarrow L([n-1] + m)$ using $m+n$ in place of n , as in

$$M[67354] = 3546.$$

Finally, we say that y is *between* x and z in $[\pi]$ if this is true of any linear permutation in $[\pi]$ where x occurs to the left of z . If $[\pi] \in C[m], i \in [m]$, and $[\sigma] \in C([n] + m)$ define

$$[\pi] \sqcup_i [\sigma] = \{[\tau] \in [\pi] \sqcup [\sigma] \mid \text{only elements of } \sigma \text{ are between } m+n \text{ and } i \text{ in } [\tau]\}.$$

For example,

$$[12] \sqcup [34] = \{[1234], [1243], [1324], [1423], [1342], [1432]\}$$

with

$$[12] \sqcup_1 [34] = \{[1234], [1243], [1324]\}$$

and

$$[12] \sqcup_2 [34] = \{[1423], [1342], [1432]\}.$$

Clearly $[\pi] \sqcup [\sigma]$ is the disjoint union of the $[\pi] \sqcup_i [\sigma]$ for $i \in [m]$.

Lemma 2.1.3 (Lifting Lemma). *Let cSt be a cyclic descent statistic and St be a shuffle compatible linear descent statistic such that the following conditions hold.*

(a) *For any $[\tau], [\tau']$ of the same length*

$$\text{St}(M[\tau]) = \text{St}(M[\tau']) \quad \text{implies} \quad \text{cSt}[\tau] = \text{cSt}[\tau'].$$

(b) *Given any $[\pi], [\pi'] \in C[m]$ such that $\text{cSt}[\pi] = \text{cSt}[\pi']$, there exists a bijection $f : [m] \rightarrow [m]$ such that for all i and $j = f(i)$*

$$\text{St}(S_i[\pi]) = \text{St}(S_j[\pi']).$$

Then cSt is cyclic shuffle compatible.

Proof. By Corollary 2.1.2, it suffices to show that if $[\pi], [\pi']$ are as given in (b) and $\sigma \in C([n] + m)$ then $\text{cSt}([\pi] \sqcup [\sigma]) = \text{cSt}([\pi'] \sqcup [\sigma])$. The remarks preceding the lemma show that this reduces to proving

$$\text{cSt}([\pi] \sqcup_i [\sigma]) = \text{cSt}([\pi'] \sqcup_j [\sigma]) \tag{2.1}$$

for all $i \in [m]$ and $j = f(i)$.

It follows that $\text{St}(S_i[\pi]) = \text{St}(S_j[\pi'])$ by (b). So, since St is shuffle compatible, we have $\text{St}(S_i[\pi] \sqcup \sigma') = \text{St}(S_j[\pi'] \sqcup \sigma')$ where $\sigma' = M[\sigma]$. Thus there is an St -preserving bijection

$$\theta : S_i[\pi] \sqcup \sigma' \rightarrow S_j[\pi'] \sqcup \sigma'.$$

This gives rise to a map

$$\theta' : [\pi] \sqcup_i [\sigma] \xrightarrow{M} S_i[\pi] \sqcup \sigma' \xrightarrow{\theta} S_j[\pi'] \sqcup \sigma' \xrightarrow{M^{-1}} [\pi'] \sqcup_j [\sigma]$$

Since this is a bijection, to show (2.1) it suffices to prove that θ' is cSt-preserving. So take $[\tau] \in [\pi] \sqcup_i [\sigma]$ and $[\tau'] = \theta'[\tau]$. Then $M[\tau'] = \theta \circ M[\tau]$. Since θ is St preserving, we have $\text{St } M[\tau'] = \text{St } M[\tau]$. But then hypothesis (a) implies that $\text{cSt}[\tau] = \text{cSt}[\tau']$ which is what we wished to prove. $\square$

2.2 Applications

The hypotheses in the Lifting Lemma may seem strange at first glance. But they can be quite easy to verify, making it a useful tool. We will see four instances of this by proving Theorem 1.2.2 using its aid.

Theorem 2.2.1. *The statistic cDes is cyclic shuffle compatible.*

Proof. We will verify the hypotheses of Lemma 2.1.3 using $\text{St} = \text{Des}$. For (a), suppose that $\#\tau = \#\tau' = n$ and

$$\text{Des}(M[\tau]) = \text{Des}(M[\tau']). \quad (2.2)$$

Let us assume that τ, τ' were chosen from their cyclic equivalence class so that $\tau_1 = \max \tau$ and similarly for τ' . Then $M[\tau] = \tau_2 \tau_3 \dots \tau_n$. And by the choice of τ_1 we see that

$$\text{Des } \tau = \{1\} \uplus (\text{Des } M[\tau] + 1). \quad (2.3)$$

Since the same statement holds with τ' in place of τ and (2.2) holds, we have $\text{Des } \tau = \text{Des } \tau'$. But $\text{Des } \tau$ is one set in the multiset $\text{cDes}[\tau]$, and the others are gotten by adding each $i \in [n]$ to all elements of $\text{Des } \tau$ modulo n . The same being true of $\text{cDes}[\tau']$ shows that $\text{cDes}[\tau] = \text{cDes}[\tau']$ as desired.

For (b), we are given $[\pi], [\pi'] \in C[m]$ with $\text{cDes}[\pi] = \text{cDes}[\pi']$ and must construct the necessary bijection. From this assumption, we can choose π and π' such that

$$\text{cDes } \pi = \text{cDes } \pi' = A. \quad (2.4)$$

for some set A . Define $f : [m] \rightarrow [m]$ by $f(\pi_k) = \pi'_k$ for all $k \in [m]$. Let $i = \pi_k$ and $j = \pi'_k$. To show that $\text{Des}(S_i[\pi]) = \text{Des}(S_j[\pi'])$ there are two cases depending on whether $k - 1 \in A$ or not, where $k - 1$ is taken modulo m .

If $k - 1 \notin A$ then π_k, π'_k are not the second elements in cyclic descents of their respective permutations. From this and equation (2.4) it follows that

$$\text{Des}(S_i[\pi]) = A - k + 1 \pmod{m} = \text{Des}(S_j[\pi']).$$

If $k - 1 \in A$ then the same equalities hold with A replaced by A' which is A with $k - 1$ removed. This completes the proof of (b) and of the theorem. $\square$

Theorem 2.2.2. *The statistic cdes is cyclic shuffle compatible.*

Proof. We proceed as in the previous proof with $\text{St} = \text{des}$. For (a) we assume $\text{des}(M[\tau]) = \text{des}(M[\tau'])$. But from equation (2.3) we see that $\text{des } \tau = \text{des}(M[\tau]) - 1$ and the same is true for τ' . Combining this with our initial assumption gives $\text{des } \tau = \text{des } \tau'$. But τ, τ' begin with their largest element so that

$$\text{cDes}[\tau] = \text{des } \tau = \text{des } \tau' = \text{cDes}[\tau']$$

which is what we wished to show.

To prove (b), we are given $\text{cdes}[\pi] = \text{cdes}[\pi']$. As in the preceding paragraph, choose π, π' to begin with their largest elements so that $\text{cdes}[\pi] = \text{des } \pi$ and similarly for π' . Since $\text{des } \pi = \text{des } \pi'$ there is a bijection $\theta : \text{Des } \pi \rightarrow \text{Des } \pi'$. Extend this map to $\theta : [m] \rightarrow [m]$ by using any bijection between the complements of $\text{Des } \pi$ and $\text{Des } \pi'$. The proof that θ has the desired property is now similar to that for Des and so is left to the reader. $\square$

Theorem 2.2.3. *The statistic cPk is cyclic shuffle compatible.*

Proof. This proof parallels the one for cDes , so we will only mention the highlights. For (a) one sees that $\text{Pk } \tau = \text{Pk } M[\tau] + 1$ since $\tau_1 = \max \tau$ is not a peak in τ and is removed in $M[\tau]$. But this still implies that $\text{Pk } \tau = \text{Pk } \tau'$ and the rest of this part of the demonstration goes through.

For (b), the map θ is constructed in exactly the same way using $A = \text{cPk } \pi = \text{cPk } \pi'$. The only difference with the remaining part of the proof is that there are two subcases when $k - 1 \notin A$ depending on whether $k - 2 \in A$ or not. If $k - 2 \in A$ then one loses the peak which was at π_{k-2} in $S_i[\pi]$. On the other hand, the peak set stays the same modulo rotation if $k - 2 \notin A$. But since the analogous statements hold for π' , the manipulations in these subcases are as before. $\square$

The proof of the next result is based on the proof of Theorem 2.2.3 in much the same way that the demonstrations of Theorems 2.2.1 and 2.2.2 are related. So the details are left to the reader.

Theorem 2.2.4. *The statistic cpk is cyclic shuffle compatible.* $\square$

Lest it appear that cyclic shuffle compatibility follows exactly the same lines as the linear case, let us point out a place where they differ. A *birun* of π is a maximal monotone factor (subsequence of consecutive elements). Let $\text{bru } \pi$ be the number of biruns of π . For example, $\text{bru } 125346 = 3$ because of the biruns 125, 53, and 346. It is easy to see [12] that the birun statistic is not linearly shuffle compatible. A *cyclic birun* of $[\pi]$ is defined in the obvious manner and denoted $\text{cbru}[\pi]$. Returning to our example, $\text{cbru}[125346] = 4$ because of the biruns above and 61.

Theorem 2.2.5. *The statistic cbru is cyclic shuffle compatible.*

Proof. Clearly cyclic biruns always begin at a cyclic peak and end at a cyclic valley, or vice-versa. So $\text{cbru}[\pi] = 2 \text{cpk}[\pi]$ and the result follows from Theorem 2.2.4. $\square$

2.3 Remarks and an open question

2.3.0.1 Cyclic pattern avoidance and statistics

We say that $[\sigma] \in C[n]$ *contains the pattern* $[\pi] \in C[m]$ if there is some subsequence $[\sigma']$ of $[\sigma]$ such that $\text{std}[\sigma'] = [\pi]$. Otherwise $[\sigma]$ *avoids* $[\pi]$. For any set $[\Pi]$ of cyclic permutations we let

$$\text{Av}_n[\Pi] = \{[\sigma] \in C[n] \mid [\sigma] \text{ avoids } [\pi] \text{ for all } [\pi] \in [\Pi]\}.$$

Callan in [4] calculated $\# \text{Av}_n[\Pi]$ when $[\Pi]$ consists of a single element of $C[4]$. Gray, Lanning, and Wang then contributed work on cyclic packing of patterns [13] and patterns in colored cyclic

permutations [14]. Most recently, Domagalski, Liang, Minnich, Sagan, Schmidt, and Sietsema determined $\# \text{Av}_n[\Pi]$ for all $[\Pi] \subseteq C[4]$ [8]. They also calculated the generating function

$$D_n([\Pi]; q) = \sum_{[\sigma] \in \text{Av}_n[\Pi]} q^{\text{cDes}[\sigma]}.$$

for the same sets of patterns.

2.3.1 Cyclic maj

Recall that the *major index* of π is

$$\text{maj } \pi = \sum_{i \in \text{Des } \pi} i.$$

Stanley's theory of P -partitions [24] shows that if $\#\pi = m$ and $\#\sigma = n$ then

$$\sum_{\tau \in \pi \sqcup \sigma} q^{\text{maj } \tau} = q^{\text{maj } \pi + \text{maj } \sigma} \begin{bmatrix} m+n \\ m \end{bmatrix}_q$$

where the last factor is a q binomial coefficient. Note that this is a q -analogue of the fact that $\#(\pi \sqcup \sigma) = \binom{m+n}{m}$.

It is easy to see that

$$\#([\pi] \sqcup [\sigma]) = (m+n-1) \binom{m+n-2}{m-1}. \quad (2.5)$$

Indeed, write each $[\tau]$ in the shuffle so that it starts with π_1 . Then there are $m+n-1$ more slots to be filled and $m-1$ of them must be for the elements $\pi_2\pi_3 \dots \pi_m$ in that order. The remaining slots can be filled with any of the n rotations of σ for a total of $n \binom{m+n-1}{m-1}$ choices. This is equivalent to the formula above. In [1] the following question was raised.

Question 2.3.1. *Is there a cyclic analogue of maj which has nice properties such as giving a q -analogue of equation (2.5)?*

One possible analogue was given by Ji and Zhang in [16] where they defined cyclic maj of $[\pi]$ to be $\text{maj}(\pi')$ where π' is the representative of $[\pi]$ starting with the largest element in the cyclic permutation. However, Liang, Sagan, and Zhuang noted in [17] that this analogue was not shuffle compatible. They proposed a second possibility, which was a multiset containing the sums of cDes for each rotation of $[\pi]$, but this also failed to be shuffle compatible and so the search for a cyclic maj remains an open pursuit in combinatorics.

CHAPTER 3

PINNACLE SETS

This chapter will be structured as follows. First, we will state a result originally proven by Davis et al. for which we were able to find a simpler proof. Next, defining $p_n(P)$ to be the number of permutations in $\mathfrak{S}_n$ with pinnacle set P , we give a formula to directly count $p_n(P)$ using the principle of inclusion and exclusion which runs in exponential time in the size of n . Next, we will give a combinatorial proof for a weighted sum of the numbers $p_n(P)$ originally proven by Fang in [11] using more algebraic methods. Following this, we give a recursion for counting the number of *admissible orderings*, which are orderings of the elements of some set P that can appear in a permutation with pinnacle set P . Finally, we finish by giving a recursive way of finding $p_n(P)$ that runs in time $O(k^2 \log n + k^4)$ where n is the length of the permutations being counted and $k = \#P$.

3.1 Counting the number of admissible pinnacle sets

In their paper, Davis et al. used lattice paths to give a bijective proof for the following theorem where n is fixed and $\lfloor m \rfloor$ is the floor function.

Theorem 3.1.1 ([5]). *If*

$$\mathcal{A}_n = \{P \mid P = \text{Pin } \pi \text{ for some } \pi \in \mathfrak{S}_n\}$$

then

$$\#\mathcal{A}_n = \binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}.$$

We were able to come up with a second bijective proof of the result that is simpler in the sense that it does not use lattice paths.

Our strategy will be as follows. First, we will introduce the set of interleaved permutations which are obviously counted by the desired binomial coefficient. Next, we will associate with each admissible pinnacle set P a particular permutation π such that $\text{Pin } \pi = P$. This permutation will be called right canonical because its pinnacles will be as far right as possible. Finally, we will show that the set of interleaved permutations and the set of right canonical permutations are, in fact, the same. This will complete the proof of the theorem.

An *interleaved permutation* $\pi \in \mathfrak{S}_n$ is one constructed in the following manner. Pick any $A \subseteq [2, n]$ with $\#A = \lfloor \frac{n-1}{2} \rfloor$.

I1 Fill the first $\lfloor \frac{n-1}{2} \rfloor$ even positions of π with the elements of A in increasing order.

I2 Fill the remaining positions of π with the elements of $\bar{A} = [n] - A$ in increasing order.

As an example, suppose $n = 9$ and $A = \{2, 3, 7, 9\}$. After step I1 we have

$$\pi = _ 2 _ 3 _ 7 _ 9 _.$$

Since $\bar{A} = \{1, 4, 5, 6, 8\}$, after I2 we have the full interleaved permutation

$$\pi = 1 \ 2 \ 4 \ 3 \ 5 \ 7 \ 6 \ 9 \ 8. \quad (3.1)$$

Let

$$\mathcal{I}_n = \{\pi \in \mathfrak{S}_n \mid \pi \text{ is interleaved}\}.$$

Clearly $\pi \in \mathcal{I}_n$ is completely determined by the choice of A . It follows immediately that

$$\#\mathcal{I}_n = \binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}. \quad (3.2)$$

Now given an admissible pinnacle set $P = \{p_1 < p_2 < \dots < p_d\} \subset [n]$ we wish to construct a permutation $\pi \in \mathfrak{S}_n$ with $\text{Pin } \pi = P$. We use the following algorithm to construct the *right canonical permutation* π from P . We first deal with the case where n is odd. Let $\bar{P} = [n] - P$.

C1 Place elements of $\bar{P}$ in π moving right to left, starting with the largest unused element of $\bar{P}$ and then decreasing until an element less than the largest unused element of P is placed.

C2 Place the largest unused element of P in the rightmost unused position.

C3 Iterate C1 and C2 until all elements of P and $\bar{P}$ are placed.

If n is even, the only change to this procedure is that we fill both π_n and π_{n-1} with elements of $\bar{P}$ before considering whether to place an element of P . To illustrate, consider $n = 9$ and $P = \{4, 7, 9\}$.

So $\bar{P} = \{1, 2, 3, 5, 6, 8\}$. Here is the construction of π where, at each stage, we note whether C1 or C2 is being used.

step	C1	C2	C1	C2	C1	C1	C2	C1	C1
π	8	98	698	7698	57698	357698	4357698	24357698	124357698

So the right canonical permutation for $P = \{4, 7, 9\}$ is $\pi = 124357698$. Note that $\text{Pin } \pi = P$. Furthermore, this is the same permutation as obtained in (3.1). However, neither the sets A nor $\bar{A}$ equals P . Let

$$C_n = \{\pi \in \mathfrak{S}_n \mid \pi \text{ is right canonical}\}.$$

We first need to show that C1–C3 is well defined in that every position of π gets filled and that we always have $\text{Pin } \pi = P$.

Lemma 3.1.2. *If $P \subset [n]$ is an admissible set then C1–C3 produces a permutation π with $\text{Pin } \pi = P$.*

Thus

$$\#C_n = \#\mathcal{A}_n.$$

Proof. Clearly the second sentence follows from the first. For the first sentence, we will present details for the case when n is odd. If n is even, then one can just place the largest element of $\bar{P}$ in position n and proceed as in the odd case.

The following notation will be useful. Let

$$P = \{p_1 < p_2 < \dots < p_d\},$$

$$\bar{P} = \{\bar{p}_1 < \bar{p}_2 < \dots < \bar{p}_{n-d}\}.$$

We will also let P_p and $\bar{P}_p$ denote the elements of P and of $\bar{P}$, respectively, which have not been used during the placement of $\pi_n, \pi_{n-1}, \dots, \pi_p$.

We will use reverse induction on the position p being filled in π . When $p = n$, we have $\bar{P} \neq \emptyset$ since 1, which is always a non-pinnacle, must be in $\bar{P}$. So there is an element $\bar{p}_{n-d}$ to place in position n . Furthermore this element can not be a pinnacle since it is the last element of the permutation, which agrees with the fact that it is in $\bar{P}$.

Suppose that $\pi_n, \pi_{n-1}, \dots, \pi_{p+1}$ have been constructed. Suppose first that $\pi_{p+1} \in \bar{P}$. One subcase is if either $P_p = \emptyset$, or $P_p \neq \emptyset$ and $\pi_{p+1} > \max P_p$. We must show that $\bar{P}_p \neq \emptyset$ so that we can let $\pi_p = \max \bar{P}_p$. This is true when $P_p = \emptyset$ since $|P_p \uplus \bar{P}_p| = p$. If the second option holds then we have $\pi_{p+1} > \max P_p$. But there must be at least two elements of $\bar{P}$ smaller than $\max P_p$ since P is admissible and so there is some permutation making $\max P_p$ a pinnacle. Also, these elements must still be in $\bar{P}_p$ since elements of this set are placed in decreasing order right to left. Thus this set is nonempty as desired. Furthermore, π_p is not a pinnacle since it is smaller than π_{p+1} .

Now consider the subcase when $\pi_{p+1} < \max P_p$. Then we let $\pi_p = \max P_p$ which is well defined. But we must show that π_p is a pinnacle. We know $\pi_p > \pi_{p+1}$. So there remains to check whether one can construct π_{p-1} with $\pi_{p-1} < \pi_p$. For this, it suffices to show that $\bar{P}_{p-1} \neq \emptyset$ since then we will have $\pi_{p-1} = \max \bar{P}_{p-1} < \pi_{p+1} < \pi_p$. Note that this will also finish the induction step.

We claim that if $\pi_p = p_i$ and $\pi_{p+1} = \bar{p}_j$ then $j > i$. It will then follow that $\bar{p}_{j-1}$ exists and can be used for π_{p-1} . But by Theorem 1.3.2 we have $p_i > \bar{p}_{i+1}$. Indeed, if $p_i < \bar{p}_{i+1}$ then at most the elements $p_1, \dots, p_{i-1}, \bar{p}_1, \dots, \bar{p}_i$ are less than p_i so that $p_i \leq 2i$, a contradiction. Also, elements of $\bar{P}$ are placed in decreasing order with p_i being placed as early as possible with a smaller element to its right. The desired bound on j follows. $\square$

We are now ready to give our proof of Theorem 3.1.1.

Theorem 3.1.3. *We have $C_n = \mathcal{I}_n$. Thus*

$$\#\mathcal{A}_n = \binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}.$$

Proof. The second statement follows directly from the first, Lemma 3.1.2, and equation (3.2). So we only need to prove that the two sets are the same. We will consider the case when n is odd, as the even case is similar.

We begin by showing that any right canonical permutation π is interleaved. That is to say, the subword consisting of all even indices is an increasing sequence, and the subword consisting of all odd indices is an increasing sequence starting with 1.

In terms of the placement of 1, note that π_1 is not a pinnacle. And since non-pinnacles are placed in decreasing order from right to left we must have $\pi_1 = 1$.

To finish this direction, it is enough to show that for any elements π_i and π_{i+2} , we have that $\pi_{i+2} > \pi_i$. Note that we are done immediately if π_i and π_{i+2} are either both pinnacles or both non-pinnacles since the construction places them in decreasing order from right to left. If π_{i+2} is a pinnacle and π_i is not, then by the pinnacle assumption $\pi_{i+2} > \pi_{i+1}$. And since non-pinnacles are placed in decreasing order right to left $\pi_{i+1} > \pi_i$. Combining the two inequalities gives the desired result. Finally, suppose π_i is a pinnacle and π_{i+2} is not. Then π_{i+1} is not a pinnacle, being adjacent to π_i . And, by construction, π_{i+1} must be the first available non-pinnacle right to left which is smaller than π_i . It follows that $\pi_{i+2} > \pi_i$.

For set containment the other way, let π be an interleaved permutation. It suffices to show that if the elements of π are placed right to left then they follow C1–C3. Consider π_i placed after π_{i+1} with $1 < i < n$. The boundary cases when $i = 1$ or n are similar. If $\pi_i < \pi_{i+1}$ then π_i is a non-pinnacle and π_{i+1} is either a non-pinnacle or a pinnacle. In the first case, the non-pinnacles are being placed in decreasing order as desired. In the second, the previously placed non-pinnacle is π_{i+2} . So the same conclusion holds by the interleaving condition. Now consider the possibility $\pi_i > \pi_{i+1}$. By the interleaving condition, $\pi_{i-1} < \pi_{i+1}$ so π_i is a pinnacle. Either π_{i+2} is a pinnacle or not, the latter possibility including the case that π_{i+2} does not exist. If it is, then the interleaving condition shows that pinnacles are being placed in decreasing order. If π_{i+2} is not a pinnacle, then this fact and the interleaving condition again imply $\pi_i < \pi_{i+2} < \pi_{i+3}$. It follows that π_i was placed after the first smaller non-pinnacle and, by the interleaving condition one last time, that any pinnacles to its right are larger. This completes the proof of the other containment. $\square$

The above construct gives us a bijection.

Theorem 3.1.4. *Given a set A and $k \in \mathbb{N}$ we let $\binom{A}{k}$ be the set of all k -element subsets of A , we have the bijection*

$$\psi : \left(\begin{array}{c} [2, n] \\ \lfloor \frac{n-1}{2} \rfloor \end{array} \right) \rightarrow \mathcal{A}_n$$

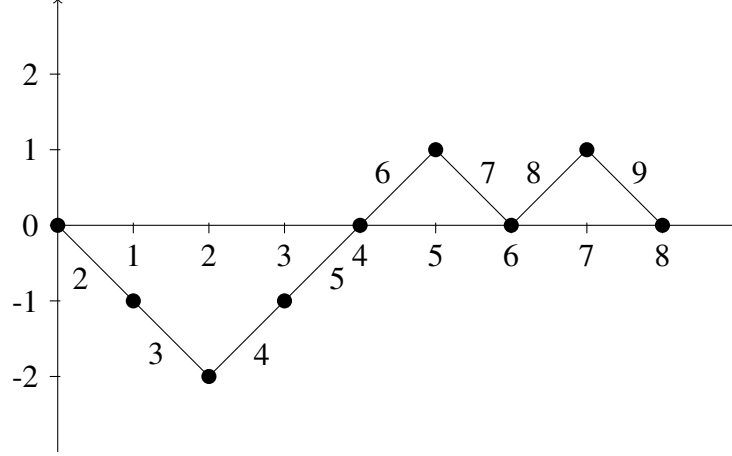


Figure 3.1 The lattice path L for $A = \{2, 3, 7, 9\}$

given by

$$\psi(A) = \text{Pin } \pi$$

where π is the interleaving permutation corresponding to A .

In [5], the authors proved Theorem 3.1.1 using a bijection

$$\phi : \left(\begin{matrix} [2, n] \\ \lfloor \frac{n-1}{2} \rfloor \end{matrix} \right) \rightarrow \mathcal{A}_n$$

defined as follows. An *up-down lattice path* L starts at the origin and uses steps which are either up (U) or down (D) parallel to the vectors $[1, 1]$ and $[1, -1]$, respectively. For more information about lattice paths, see the text of Sagan [23]. It will be convenient to index the steps of L with $[2, n]$ and write $L = p_2 p_3 \dots p_n$. Associate with $A \in \left(\begin{matrix} [2, n] \\ \lfloor \frac{n-1}{2} \rfloor \end{matrix} \right)$ the lattice path L such that

$$p_i = \begin{cases} D & \text{if } i \in A, \\ U & \text{if } i \notin A. \end{cases}$$

To illustrate, if $n = 9$ and $A = \{2, 3, 7, 9\}$ as in the example beginning this section then

$$L = DDUUUDUD$$

as depicted in Figure 3.1 where each step is labeled by its index. We now define

$$\phi(A) = \{i \mid \text{in } L \text{ either } p_i = U \text{ strictly below the } x\text{-axis, or } p_i = D \text{ weakly above the } x\text{-axis}\}.$$

Continuing our example, $\phi(\{2, 3, 7, 9\}) = \{4, 7, 9\} = \psi(\{2, 3, 7, 9\})$. This is not an accident.

Proposition 3.1.5. *We have*

$$\phi = \psi.$$

Proof. We will give the proof for n odd as the even case is similar. Let $l = (n - 1)/2$. We need to show that $\phi(A) = \psi(A)$ for all $A \in \binom{[2,n]}{l}$. Suppose $A = \{a_1 < a_2 < \dots < a_l\}$ and $\bar{A} = [n] - A = \{\bar{a}_1 < \bar{a}_2 < \dots < \bar{a}_{n-l}\}$. Let L and π be the lattice path and interleaved permutation, respectively, associated with A . So $\psi(A) = \text{Pin } \pi$ and there will be two cases depending on whether a pinnacle of π comes from A or $\bar{A}$

In the first case, suppose $a_i \in \text{Pin } \pi$. Since π is interleaved, this is equivalent to $a_i = \pi_{2i} > \pi_{2i+1} = \bar{a}_{i+1}$. Recall that a_i indexes the i th D step of L , and similarly for $\bar{a}_{i+1}$ and U steps. So the previous inequality is equivalent to step $p_{a_i} = D$ being preceded by more up steps than down steps. And this is precisely the condition for a_i to be the index of a down step weakly above the x -axis, which means it is in $\phi(A)$. Thus this case is complete.

In a similar manner, one proves that $\bar{a}_i \in \text{Pin } \pi$ if and only if $\bar{a}_i$ is the index of an up step strictly below the x -axis. This completes the second case and the proof. $\square$

3.2 Permutations with a given pinnacle set

Given an admissible set P of cardinality d , we may also want to count the number of permutations $\pi \in \mathfrak{S}_n$ such that $\text{Pin } \pi = P$; we denote this number by $p_n(P)$. There does not seem to be an expression for $p_n(P)$ analogous to the one in Theorem 1.3.1 for peak sets. In [5], Davis et al. found expressions for $p_n(P)$ when $\#P \leq 2$ as well as bounds for general P , and asked whether an exact formula could be given in the general case. Such an expression was given in [6] as a summation. In this section we will give another sum which is asymptotically more efficient.

Since our sum will involve a significant amount of new notation, we will collect it here and then explain its relevance afterwards. Fix $n \in \mathbb{P}$. Suppose we have an admissible pinnacle set $P = \{p_1 < p_2 < \dots < p_d\}$ for permutations in $\mathfrak{S}_n$. We use the convention $p_0 = 0$ and $p_{d+1} = n + 1$ and let

$$n_i = p_{i+1} - p_i - 1$$

for $0 \leq i \leq d$. Let

$$D = \{1_l, 1_r, 2_l, 2_r, \dots, d_l, d_r\}$$

and give the following total order to D 's elements

$$1_l < 1_r < 2_l < 2_r < \dots < d_l < d_r.$$

We call i_l and i_r the elements of *rank* i in D . If $B \subseteq D$ then we will let

$$b = \#B$$

and

$$r_j = \text{the rank of the } j\text{th smallest element of } B$$

for $1 \leq j \leq b$. We also define

$$b_i = \text{the number of elements in } B \text{ with rank at least } i.$$

Note that we always have $b_1 = b$ and $b_{d+1} = 0$ since d is the largest rank. For example, if $d = 4$, then $D = \{1_l, 1_r, 2_l, 2_r, 3_l, 3_r, 4_l, 4_r\}$ and one possible B might be $B = \{1_l, 3_l, 3_r, 4_r\}$ which has $r_1 = 1, r_2 = 3, r_3 = 3, r_4 = 4$ and $b_1 = 4, b_2 = 3, b_3 = 3, b_4 = 1, b_5 = 0$. We can now state the first main result of this section.

Theorem 3.2.1. *Given $n \in \mathbb{P}$ and admissible $P = \{p_1 < p_2 < \dots < p_d\}$ we have*

$$p_n(P) = 2^{n-2d-1} \sum_{B \subseteq D: |B| \leq d} (-1)^b (d-b)! \left(\prod_{i=0}^{b-1} (d+1-i-r_{b-i}) \right) \left(\prod_{i=0}^d (d+1-i-b_{i+1})^{n_i} \right).$$

To prove this, it will be convenient to convert the linear permutations we have been studying into cyclic ones in order to avoid considering boundary cases. Let

$$[\mathfrak{S}_n] = \{[\pi] \mid \pi \in \mathfrak{S}_n\}.$$

We define the *pinnacle set* of $[\pi] = [\pi_1 \pi_2 \dots \pi_n]$ to be

$$\text{Pin}[\pi] = \{\pi_i \mid \pi_{i-1} < \pi_i > \pi_{i+1} \text{ where subscripts are taken modulo } n\}.$$

Continuing our example from the last paragraph

$$\text{Pin}[1324] = \{3, 4\}.$$

Note in particular that $\text{Pin}[12] = \{2\}$ and, more generally, $n \in \text{Pin}[\pi]$ for any $[\pi] \in [\mathfrak{S}_n]$ where $n \geq 2$.

Lemma 3.2.2. *For $n \in \mathbb{P}$, there is a bijection between linear permutations in $\mathfrak{S}_n$ with pinnacle set P and cyclic permutations in $[\mathfrak{S}_{n+1}]$ with pinnacle set $P' = P \cup \{n+1\}$.*

Proof. Given a linear π , append the element $n+1$ to the end of π and take the corresponding equivalence class in $\mathfrak{S}_{n+1}$ to form an element of $[\mathfrak{S}_{n+1}]$. The map is clearly invertible and does not destroy or create any pinnacles for elements in $[n]$. Since $n+1 \geq 2$, we know that $n+1$ will become a pinnacle. Therefore the map has the desired properties concerning the pinnacle set. $\square$

Consider some admissible pinnacle set $P = \{p_1 < p_2 < \dots < p_d\}$. Given the above lemma, we may count the number of permutations in $\mathfrak{S}_n$ with pinnacle set P by counting the number of cyclic permutations $[\pi] \in [\mathfrak{S}_{n+1}]$ with pinnacle set $P' = P \cup \{n+1\}$ where we let $p_{d+1} = n+1$. Therefore, much of what follows will be in regards to cyclic permutations with pinnacle set P' .

A *factor* of a (cyclic) permutation is a subsequence of consecutive elements. We may attempt to construct a $[\pi]$ with pinnacle set P' by first putting the elements of P' in some cyclic order, and then placing all elements in $\overline{P'} = [n+1] - P'$ into either decreasing factors starting with some p_i , or into increasing factors ending with some p_i . Such a $[\pi]$ will then be completely determined by the increasing/decreasing factors that each element of $\overline{P'}$ falls into, and we will call every such assignment a *placement*. Note that it is possible for multiple placements to result in the same permutation since each vale (an element of $[\pi]$ smaller than the elements on either side) can be part of the factor on either side. For example, start with a desired pinnacle set $\{4, 5\}$ and place non-pinnacles between these elements to form the cyclic permutation $[\pi] = [14325]$. Then $[\pi]$ would be associated with a placement where the decreasing factor starting with 4 is 43 and the increasing factor ending with 5 is 25. But it would also be associated with a placement having these factors be 432 and 5, respectively.

It is also possible, depending on the placement, that $[\pi]$ will not have pinnacle set P' if no sufficiently small elements are placed between two pinnacles. In our example above, this could have happened if we had placed 1, 2 and 3 all in the increasing factor ending in 5, resulting in the cyclic permutation $[41235]$ in which only 5 is a pinnacle. It is true, however, that any $[\pi]$ so constructed will have a pinnacle set that is a subset of P' since every non-pinnacle was placed so that its factor contains an p_i which is the largest element. For our arguments, we will focus on counting placements and then convert them into permutations later.

Fix a cyclic ordering of the pinnacle indices and write it as $[\tau] = [\tau_1 \cdots \tau_{d+1}] \in [\mathfrak{S}_{d+1}]$. An example is shown in Figure 3.2 where $\tau = [7612354]$. Now given a placement consistent with this ordering, for every space between two adjacent elements in $[\tau]$ define the *dale set* of this placement to consist of all elements between the two corresponding pinnacles that are also smaller than both pinnacles. So in Figure 3.2 the dales are outlined by triangles with solid lines as sides. If p_i is the smaller of the two pinnacles, then we say that the dale has *rank* i . Note that the rank is from the index of p_i and not its actual value. We will further denote the rank as either i_l or i_r depending on whether the dale is to the left, or right of the pinnacle p_i . In Figure 3.2 the dale ranks are given along the x -axis. Define the *dale rank set* $D_{[\tau]}$ to be the set of the dale ranks of $[\tau]$. And define the *master dale rank set* to be

$$D = \{1_l, 1_r, 2_l, 2_r, \dots, d_l, d_r\}$$

so that $D \supseteq D_{[\tau]}$ for all $[\tau]$. In Figure 3.2, we have that $D_{[\tau]} = \{1_l, 1_r, 2_r, 3_r, 4_l, 4_r, 6_l\}$ while $D = \{1_l, 1_r, 2_l, 2_r, \dots, 6_l, 6_r\}$. Note that, by our definitions, there will be no dales in the case where $d = 0$.

Clearly $D_{[\tau]}$ will be a subset of D consisting of exactly $d + 1$ elements if $d > 0$, and empty otherwise. We can derive further information about $D_{[\tau]}$ if we want, such as how it will always contain both 1_l and 1_r if $d > 0$, how it will never contain both d_l and d_r if $d > 1$, and how $D_{[\tau]}$ will never be able to have certain combinations of the higher ranked dales. These facts are not necessary for proving our formula, although further analysis of them might help to improve its efficiency.

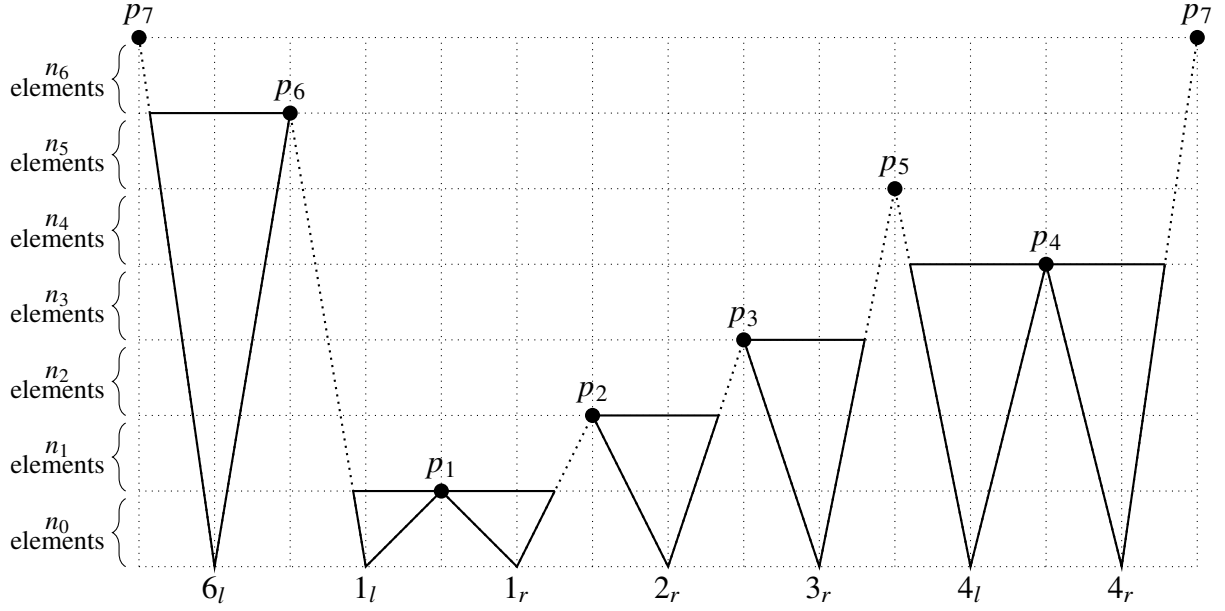


Figure 3.2 Example of a pinnacle set ordering $[\tau] = [7612354]$ with corresponding dales

Lemma 3.2.3. *For $n \in \mathbb{P}$, a given placement will correspond to a permutation $[\pi] \in [\mathfrak{S}_{n+1}]$ with pinnacle set P' if and only if every dale is non-empty.*

Proof. First, suppose $d = 0$. In this case, the theorem is trivial since there are no dales. And every placement will automatically result in only one pinnacle, namely $n + 1$, as long as $n > 0$.

Now suppose $d > 0$. Clearly if any dale of rank i (whether left or right) is empty, then the pinnacle p_i will have no smaller elements between itself and the higher pinnacle next to it, which will force p_i to not be a pinnacle. On the other hand, if all dales have at least one element, then the space between any two pinnacles will always contain an element smaller than both, and all elements of P' will in fact be pinnacles. $\square$

We can now enumerate all placements corresponding to a given cyclic ordering of the indices of the pinnacle set P' .

Lemma 3.2.4. *Given an admissible pinnacle set P' , fix an order $[\tau]$ of the pinnacle indices. The total number of placements with order $[\tau]$ that will result in a permutation with pinnacle set P' is*

given by

$$\sum_{B \subseteq D_{[\tau]}} (-1)^b \prod_{i=0}^d 2^{n_i} (d+1-i-b_{i+1})^{n_i}$$

where b , d , the b_i , and the n_i are defined above.

Proof. We will use the Principle of Inclusion and Exclusion or PIE. We let our universal set be all possible placements with no restrictions. We then wish to exclude any placement where at least one dale is empty. Therefore, if B is some subset of the dales, we must be able to count the number of placements where all dales in B (and possibly others) are empty.

First consider the case when $B = \emptyset$. There are $2(d+1)$ factors of which $2i$ only exist below p_i . So each of the n_i non-pinnacles between p_i and p_{i+1} may be placed in any of the $2(d+1-i)$ factors that are long enough to extend above p_i . As an example, in Figure 3.2.3 if we look between the horizontal boundary lines for the elements counted by n_2 we see there are $10 = 2(6+1-2)$ such factors represented by the diagonal lines (solid or dotted) which intersect the region.

For non-empty B , each dale of rank at least $i+1$ that we require to be empty will result in a loss of two additional factors, and so there are only $2(d+1-i-b_{i+1})$ choices. Therefore, for a given B , the total number of placements guaranteeing the dales in B are empty is

$$\prod_{i=0}^d 2^{n_i} (d+1-i-b_{i+1})^{n_i}.$$

To use the PIE, we must also attach the sign $(-1)^{|B|} = (-1)^b$ to this term before summing. Therefore, given a fixed order $[\tau]$ of the pinnacle indices of P' , we have that the total number of placements that will result in a permutation with pinnacle set P' is

$$\sum_{B \subseteq D_{[\tau]}} (-1)^b \prod_{i=0}^d 2^{n_i} (d+1-i-b_{i+1})^{n_i}.$$

Finally, when $B = D_{[\tau]}$ then $b_1 = \#B = d+1$. So we can ignore this term because the product has a factor of $d+1-b_1 = 0$. □

The above formula must be summed over all possible $[\tau]$ to give a final count for the number of $[\pi]$ with $\text{Pin}[\pi] = P'$. This results in computationally expensive double sum. Also, note that

in the above formula there may be multiple B resulting in the same term. For example, $\{1_l, 2_r, 5_l\}$ is not the same as $\{1_r, 2_r, 5_l\}$ even though both produce the same b_i . We will take care of this redundancy when we optimize our formula below.

To fix the double sum problem, note that each B in Lemma 3.2.4 is a subset of the master dale rank set D . We will fix some subset $B \subseteq D$ and count the number of orderings $[\tau]$ that will produce a $D_{[\tau]}$ which can have B as a subset. This will allow us to just sum over all subsets $B \subseteq D$ without having to keep track of $[\tau]$. Furthermore, we only have to sum over the subsets B of cardinality at most d since requiring more than d dales to be empty is impossible for an admissible pinnacle set.

Lemma 3.2.5. *Fix some $B \subseteq D$ with $|B| \leq d$. The number of orderings $[\tau]$ that will produce a $D_{[\tau]}$ such that $B \subseteq D_{[\tau]}$ is given by*

$$(d - b)! \prod_{i=0}^{b-1} (d + 1 - i - r_{b-i})$$

where b , d , and the r_i are defined as above.

Proof. We will start by viewing all $d + 1$ pinnacles as separate and then adjoin them in pairs in such a way so that the desired dales are formed. Here, “adjoining a pair of pinnacles” means requiring that they be adjacent in $[\tau]$.

We start with the dale of rank r_b the largest rank in B . In that case, the only way to generate such a dale is to order p_{r_b} so that one of the $d + 1 - r_b$ higher pinnacles is directly to its left or right depending on whether the corresponding element of B is a left or right rank, respectively. So select one such pinnacle and adjoin it to the appropriate side of p_{r_b} .

Next we will examine the dale in B with the next highest rank, r_{b-1} . If r_{b-1} is a smaller rank than r_b , we may once again select a taller pinnacle to place next to $p_{r_{b-1}}$, on either the left or right as necessary, in order to produce the desired dale. This time however, although there are $d + 1 - r_{b-1}$ pinnacles higher than $p_{r_{b-1}}$, one of them is unavailable since we have already adjoined two of the higher-ranked pinnacles together. More specifically, because of adjoining a higher pinnacle with p_{r_b} , we know that one taller pinnacle cannot be joined to its left and another cannot be joined to its right. So no matter whether r_{b-1} corresponded to a left or right dale, there is one less option.

Therefore, the number of ways to append a larger pinnacle is $d + 1 - r_{b-1} - 1$. On the other hand, if $r_{b-1} = r_b$ then we need to adjoin a second pinnacle to p_{r_b} on the side opposite the one used when considering r_b . Again, the pinnacle already adjoined to p_{r_b} removes one option so the number of choices is $d + 1 - r_{b-1} - 1$ as before. So in either case we have the same number of possibilities. Similar consideration show that, in general, each r_{b-i} results in $d + 1 - i - r_{b-i}$ choices for adjoining pinnacles. Note that for this argument we are using the fact that $b \leq d$ since if $b = d + 1$ then the string of pinnacles would wrap into a circle before creating the final dale.

Once all dales have been created by the above process, we only need to count the number of ways to join the resulting strings of pinnacles together. Since we have adjoined pinnacles together b times, we have $d + 1 - b$ strings which we then must arrange in a circle. This can be done in $(d + 1 - b - 1)! = (d - b)!$ ways. Therefore,

$$(d - b)! \prod_{i=0}^{b-1} (d + 1 - i - r_{b-i})$$

is the number of orderings $[\tau]$ that will allow for a given B to be a subset of $D_{[\tau]}$. $\square$

We are now in a position to prove Theorem 3.2.1 which we restate here for ease of reference.

Theorem 3.2.6. *Given $n \in \mathbb{P}$ and admissible $P = \{p_1 < p_2 < \dots < p_d\}$ we have*

$$p_n(P) = 2^{n-2d-1} \sum_{B \subseteq D: |B| \leq d} (-1)^b (d - b)! \left(\prod_{i=0}^{b-1} (d + 1 - i - r_{b-i}) \right) \left(\prod_{i=0}^d (d + 1 - i - b_{i+1})^{n_i} \right).$$

Proof. It is easy to verify the formula if $d = 0$, so we assume $d > 0$. From Lemma 3.2.3, the number of permutations $\pi \in \mathfrak{S}_{n+1}$ with pinnacle set P equals the number of cyclic permutations $[\pi] \in [\mathfrak{S}_{n+1}]$ with pinnacle set $P' = P \cup \{n + 1\}$. So we will count the latter. From Lemma 3.2.4, the number of placements which correspond to a cyclic permutation with pinnacle set P' is given by

$$\sum_{[\tau]} \sum_{B \subseteq D_{[\tau]}} (-1)^b \prod_{i=0}^d 2^{n_i} (d + 1 - i - b_{i+1})^{n_i}$$

where the outer sum is over all possible cyclic orderings $[\tau]$ of the index set of P' . We now wish to swap the summations so that the outer sum is over all $B \subseteq D$ with $|B| \leq d$. We may restrict to

size at most d since any larger B will either consist of a combination of dales that cannot exist, or will require all $d + 1$ dales to be empty which is impossible because of the assumption that $d > 0$. In order to interchange the summations we must multiply the term corresponding to each B by the number of distinct permutations $[\tau]$ that could have generated it. This was counted in Lemma 3.2.5, and so we get the formula

$$\sum_{B \subseteq D: |B| \leq d} (-1)^b (d - b)! \left(\prod_{i=0}^{b-1} (d + 1 - i - r_{b-i}) \right) \left(\prod_{i=0}^d 2^{n_i} (d + 1 - i - b_{i+1})^{n_i} \right)$$

for the number of placements.

Now we seek to turn the placements into permutations. Since all dales are guaranteed to be non-empty, we have that every permutation corresponding to one of these placements will have $d + 1$ non-pinnacle elements that are part of both a decreasing factor and an increasing factor. This means that every such corresponding $[\pi]$ has been counted by 2^{d+1} placements. Dividing by this, and also pulling some common factors of two out from the second product, we have

$$\begin{aligned} p_n(P) &= 2^{-d-1} \prod_{i=0}^d 2^{n_i} \sum_{B \subseteq D: |B| \leq d} (-1)^b (d - b)! \left(\prod_{i=0}^{b-1} (d + 1 - i - r_{b-i}) \right) \left(\prod_{i=0}^d (d + 1 - i - b_{i+1})^{n_i} \right) \\ &= 2^{n-2d-1} \sum_{B \subseteq D: |B| \leq d} (-1)^b (d - b)! \left(\prod_{i=0}^{b-1} (d + 1 - i - r_{b-i}) \right) \left(\prod_{i=0}^d (d + 1 - i - b_{i+1})^{n_i} \right) \end{aligned}$$

where this is the formula we set out to prove. $\square$

In [5], explicit formulas were given for $p_n(P)$ when $|P| \leq 3$. These expressions follow easily from the previous result.

Corollary 3.2.7. *We have the following values for $p_n(P)$.*

(1) *If $P = \emptyset$ then*

$$p_n(P) = 2^{n-1}.$$

(2) *If $P = \{l\}$ where $3 \leq l \leq n$ then*

$$p_n(P) = 2^{n-2}(2^{l-2} - 1).$$

(3) If $S = \{l, m\}$ where $l \geq 3$, $m \geq 5$, and $l < m \leq n$, then

$$p_n(P) = 2^{n+m-l-5}(3^{l-1} - 2^l + 1) - 2^{n-3}(2^{l-2} - 1).$$

Proof. In each of the results we apply Theorem 3.2.6.

(1) When $d = 0$, the first product in Theorem 3.2.6 is always empty and the second always equals one. Therefore, everything reduces immediately to $p_n(P) = 2^{n-1}$, as desired.

(2) When $d = 1$ we have $n_0 = l - 1$, $n_1 = n - l$. Therefore, we have the following possibilities for B , and the corresponding terms in the summation

- $B = \emptyset : 2^{l-1}$
- $B = \{1_l\}$ or $\{1_r\} : -1$

which when substituted into the formula gives

$$p_n(P) = 2^{n-3}(2^{l-1} - 2) = 2^{n-2}(2^{l-2} - 1).$$

(3) When $d = 2$ we have $n_0 = l - 1$, $n_1 = m - l - 1$, and $n_2 = n - m$. Additionally, the first inner product will always zero out if $2_r, 2_l$ are both in B . Therefore, we have the following possibilities for B , and the corresponding terms in the summation:

- $B = \emptyset : (2)3^{l-1}2^{m-l-1}$
- $B = \{1_l\}$ or $\{1_r\} : (-2)2^{l-1}2^{m-l-1}$
- $B = \{2_l\}$ or $\{2_r\} : (-1)2^{l-1}$
- $B = \{1_l, 1_r\} : 2$
- $B = \{1_l, 2_r\}$ or $\{1_r, 2_r\}$ or $\{1_l, 2_l\}$ or $\{1_r, 2_l\} : 1$.

When we substitute all these into the formula, we get

$$\begin{aligned} p_n(P) &= 2^{n-5}[(2)3^{l-1}2^{m-l-1} - (4)2^{l-1}2^{m-l-1} - (2)2^{l-1} + (2)2^{m-l-1} + 4] \\ &= 2^{n-5}[(2)3^{l-1}2^{m-l-1} - (4)2^{l-1}2^{m-l-1} + (2)2^{m-l-1}] - 2^{n-5}[(2)2^{l-1} - 4] \\ &= 2^{n+m-l-5}(3^{l-1} - 2^l + 1) - 2^{n-3}(2^{l-2} - 1) \end{aligned}$$

as desired. □

We can make Theorem 3.2.6 more efficient by summing over certain weak compositions rather than subsets. A *weak composition* of $n \in \mathbb{N}$ is a sequence $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_k]$ of nonnegative integers called *parts* such that $\sum_i \alpha_i = n$. In this case we write $\alpha \models n$ or $|\alpha| = n$ where $|\alpha| = \sum_i \alpha_i$. To $B \subseteq D$ we associate the composition $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_d]$ where α_i is the number of dales in B of rank i . To illustrate, for the example in Figure 3.2 the corresponding composition is $\alpha = [2, 1, 1, 2, 0, 1]$. Note that all the necessary parameters for D can be read off of α . In particular

$$r_j = \min\{i \mid \alpha_1 + \alpha_2 + \dots + \alpha_i \geq j\},$$

and

$$b_i = \alpha_i + \alpha_{i+1} + \dots + \alpha_d.$$

Note that

$$b = b_1 = |\alpha|.$$

Thus we will be able to sum over the following set

$$C(d) = \{\alpha = [\alpha_1, \alpha_2, \dots, \alpha_d] \mid \alpha_i \in [0, 2] \text{ for all } i \text{ and } |\alpha| \leq d\}.$$

We must find how many B correspond to a given α . If $\alpha_i = 0$ then B contains no dales of rank i . If $\alpha_i = 2$ then B contains both dales of rank i . So the only choice comes if $\alpha_i = 1$ in which case B could contain either i_l or i_r . Letting

$$o = \text{the number of } \alpha_i = 1$$

we see that the number of B represented by α is 2^o . Thus we have proved the following result.

Corollary 3.2.8. *Given $n \in \mathbb{P}$ and admissible $P = \{p_1 < p_2 < \dots < p_d\}$ we have*

$$p_n(P) = 2^{n-2d-1} \sum_{\alpha \in C(d)} (-1)^b 2^o (d-b)! \left(\prod_{i=0}^{b-1} (d+1-i-r_{b-i}) \right) \left(\prod_{i=0}^d (d+1-i-b_{i+1})^{n_i} \right). \quad \square$$

P	DLHHIN	DLMSSS
$\{3, 5, 7, 9, 11, 13, 15, 17, 19, 21\}$	9.2×10^{-5}	0.72
$\{3, 6, 9, 12, 15, 18, 21, 24, 27, 30\}$	0.11	0.73
$\{3, 7, 11, 15, 19, 23, 27, 31, 35, 39\}$	9.5	0.73
$\{3, 8, 13, 18, 23, 28, 33, 38, 43, 48\}$	210	0.78

Table 3.1 Run times in seconds compared when most n_i are equal

In order to compare this formula to the one in [6], we need to introduce some notation. The *vale set* of a permutation π is

$$\text{Val } \pi = \{\pi_i \mid \pi_{i-1} > \pi_i < \pi_{i+1}\}.$$

Call a pair (P, T) *n-admissible* if there is a permutation $\pi \in \mathfrak{S}_n$ with $\text{Pin } \pi = P$ and $\text{Val } \pi = T$. Define

$$\mathcal{V}_n(P) = \{T \mid (P, T) \text{ is } n\text{-admissible}\}.$$

Theorem 3.2.9 ([6]). *Given $n \in \mathbb{P}$ and admissible P with $\#P = d$ we have*

$$p_n(P) = 2^{n-2d-1} \sum_{T \in \mathcal{V}_n(P)} \prod_{p \in P} \binom{N_{PT}(p)}{2} \prod_{t \in [n] - (P \oplus T)} N_{PT}(t)$$

where $P_i = \{p \in P \mid p < i\}$, $T_i = \{t \in T \mid t < i\}$, and $N_{PT}(i) = \#T_i - \#P_i$. □

In order to estimate the number of terms in this sum, we need a formula for $\#\mathcal{V}_n(P)$. Let

$$K(d) = \{\alpha = [\alpha_1, \alpha_2, \dots, \alpha_d] \models d \mid \alpha_1 + \alpha_2 + \dots + \alpha_k \geq k \text{ for all } k \in [d]\}.$$

Theorem 3.2.10 ([6]). *Given $n \in \mathbb{P}$ and admissible $P = \{p_1 < p_2 < \dots < p_d\}$ we have*

$$\#\mathcal{V}_n(P) = \sum_{\alpha \in K(d)} \binom{n_0 - 1}{\alpha_1} \prod_{i=2}^d \binom{n_{i-1}}{\alpha_i}. \quad \square$$

We can now compare the number of terms in the sums of Corollary 3.2.8 and Theorem 3.2.9. In the former we have $c(d) := \#C(d) \leq 3^d$ terms, where the inequality comes from the fact that every $\alpha_i \in \{0, 1, 2\}$. In the latter, we have $v_n(P) := \#\mathcal{V}_n(P)$ terms which depends on n and P , and

not just d as seen in Theorem 3.2.10. If $n_1 \leq 4$ and $n_i \leq 3$ for $i \geq 2$ then each of the binomial coefficients in the sum is at most 3 and so $v_n(P)$ could be significantly smaller than $c(d)$. But if even one of the n_i is large, then the inequality will be reversed. For example, suppose $n_1 \geq 2d + 1$ and take $\alpha = [d, 0, 0, \dots, 0] \in K(d)$. Then, by Stirling's approximation,

$$v_n(P) \geq \binom{2d}{d} \sim \frac{4^d}{\sqrt{\pi d}}$$

which will eventually be greater than 3^d . So, for fixed d , there are only finitely many n such that $v_n(P) \leq c(d)$. Thus, in most cases, Corollary 3.2.8 will be more efficient. We should mention that Diaz-Lopez, Insko, and Nilsen [6] have come up with a refinement of the ideas in [6] which permits the product of binomial coefficients in Theorem 3.2.10 to be replaced by 2^d .

The observations of the previous paragraph are borne out by actual computer computations. In Tables 3.1 and 3.2 we show the results of computing $p_{1000}(P)$ for various sets P (first column) with constant d by the algorithm in [6] (second column) and our algorithm (third column). The run times are in seconds and are the average over 10 trials for each set using a 15-inch 2017 MacBook Pro with a 3.1 GHz Quad-Core Intel Core i7 processor. In Table 3.1 the n_i for $0 < i < d$ are constant in each set, but allowed to increase as one goes down the table. As expected, the algorithm using vales starts out orders of magnitude faster than the one using dales but quickly becomes orders of magnitude slower, with the latter's times being virtually constant. Similar behaviour is shown in the two parts of Table 3.2 which keep all of the n_i for $0 \leq i < d$ constant except for one which is allowed to grow. Note the difference in growth rate of the vale algorithm between increasing n_4 (upper chart) and n_0 (lower chart).

Another advantage to this approach is that it can be modified to count $\#O(P)$, the number of admissible orderings of an admissible pinnacle set P . First, if we fix $n > 0$ we have that Lemma 3.2.2 will again allow us to reduce to the case of cyclic orderings of the pinnacle set P' for permutations in $\mathfrak{S}_{n+1}$. We now prove the following intermediate result.

Increase n_4 with other n_i constant

P	DLHHIN	DLMSSS
$\{3, 5, 7, 9, 11\}$	2.9×10^{-5}	0.0014
$\{3, 5, 7, 9, 21\}$	7.1×10^{-5}	0.0014
$\{3, 5, 7, 9, 31\}$	0.00012	0.0015
$\{3, 5, 7, 9, 41\}$	0.00017	0.0015

Increase n_0 with other n_i constant

P	DLHHIN	DLMSSS
$\{3, 5, 7, 9, 11\}$	2.9×10^{-5}	0.0014
$\{13, 15, 17, 19, 21\}$	0.012	0.0015
$\{23, 25, 27, 29, 31\}$	0.26	0.0015
$\{33, 35, 37, 39, 41\}$	1.8	0.0015

Table 3.2 Run times in seconds compared when most n_i are constant

Lemma 3.2.11. *Consider a cyclic ordering $[\tau]$ with dale set $D_{[\tau]}$ and corresponding r_j . The ordering is admissible if and only if*

$$j \leq n_0 + n_1 + \cdots + n_{r_j-1}$$

for all $j \in [d + 1]$.

Proof. Note that, by definition of the n_i and r_i , the right hand side of the inequality is simply the number of non-pinnacles small enough to be placed in any of the dales having rank at least r_j . So if for any j we have $j > n_0 + n_1 + \cdots + n_{r_j-1}$, then there will be at least $j + 1$ dales having rank at most r_j . This means there would not be enough small non-pinnacle elements to fill them all. Therefore, any such ordering is not admissible. On the other hand, if we have that $j \leq n_0 + n_1 + \cdots + n_{r_j-1}$ for all j , then we may always fill all the dales by placing the smallest non-pinnacle in the lowest ranked dale, and proceeding upwards. The inequalities guarantee that we will always have enough non-pinnacles to do this at every step, and so we are done. $\square$

Since the problem is trivial if $d = 0$, so we may also assume that $d > 0$. We also define for the master dale rank set D

$$D' = D - \{1_l, 1_r\}$$

and for any subset B

$$\delta_B = \begin{cases} 1 & \text{if } j \leq n_0 + n_1 + \cdots + n_{r_j-1} \text{ for all } j \in [b], \\ 0 & \text{otherwise.} \end{cases}$$

With this notation, we can count admissible orderings.

Theorem 3.2.12. *If $d \in \mathbb{P}$ and P is admissible then*

$$\#O(P) = \sum_{B \subseteq D': |B|=d-1} \delta_{B \cup \{1_l, 1_r\}} \prod_{i=0}^{d-2} (d+1-i-r_{d-1-i}).$$

Proof. We first wish to sum over all possible orderings, partitioned by their dales. Since every dale set for $d > 0$ is guaranteed to have $d+1$ elements and contain $\{1_l, 1_r\}$, we may index the dales by taking $B \subseteq D'$ where $|B| = d-1$. We then consider the following summation

$$\sum_{B \subseteq D': |B|=d-1} \prod_{i=0}^{d-2} (d+1-i-r_{d-1-i}).$$

Clearly this sums over every possible dale set once, and the expression inside comes from Lemma 3.2.5, which counts the number of cyclic orderings of P' which have dales containing those in B . However, due to the restrictions placed on the size of B and the comments above, this expression will count those cyclic orderings of P' which have dales equal to $B \cup \{1_l, 1_r\}$ instead of just a subset. Therefore, no ordering can be counted twice by two different B 's and so every ordering is accounted for exactly once in the above summation, making the total $d!$.

Finally, using Lemma 3.2.11, we may exclude from this sum precisely those orderings which are not admissible by writing it as

$$\sum_{B \subseteq D': |B|=d-1} \delta_{B \cup \{1_l, 1_r\}} \prod_{i=0}^{d-2} (d+1-i-r_{d-1-i}).$$

This completes the proof. □

We may also rewrite our result in terms of compositions for a faster summation. Lemma 3.2.11 still holds as the r_j are the same whether or not the dales sets are represented as compositions, but now we will need make some new definitions. Let

$$C'(d) = \{\alpha = [\alpha_1, \alpha_2, \dots, \alpha_{d-1}] \models [d-1] \mid \alpha_i \in [0, 2] \text{ for all } i\},$$

and if $\alpha \models b$

$$\delta_\alpha = \begin{cases} 1 & \text{if } j \leq n_0 + n_1 + \dots + n_{r_j-1} \text{ for all } j \in [b], \\ 0 & \text{otherwise.} \end{cases}$$

Also, if $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_{d-1}]$ then define

$$2 \oplus \alpha = [2, \alpha_1, \alpha_2, \dots, \alpha_{d-1}].$$

The following result follows from Theorem 3.2.12 in much the same way that Corollary 3.2.8 followed from Theorem 3.2.6. So the proof is omitted.

Corollary 3.2.13. *If $d \in \mathbb{P}$ and P is admissible then*

$$\#O(P) = \sum_{\alpha \in C'(d)} \delta_{2 \oplus \alpha} 2^o \prod_{i=0}^{d-2} (d+1-i-r_{d-1-i}).$$

□

3.3 A formula for the weighted sum of $p_n(P)$

We do not know of a simpler closed formula for $p_n(P)$ than the one given above, but we can get a simplification if we look at a weighted sum.

Define a *Dyck path* with k steps to be a path starting at $(0, 0)$, ending at $(k, 0)$, that uses only down steps $[1, -1]$ and up steps $[1, 1]$, and that never drops below the x -axis. A *Dyck walk* of length k is a generalised version of a Dyck path where we allow the path to end at any point (k, y) for $y \geq 0$. We then define a *modified Dyck walk* to be a Dyck walk where the last step may possibly drop below the x -axis. In other words, a modified Dyck walk is a walk starting at $(0, 0)$ consisting of only up steps and down steps that may end at any height $h \geq -1$, and that does not drop below the x -axis except possibly at the final step. Define R_k be the set of y -coordinate sequences of modified Dyck walks of length k . A walk $r \in R_k$ takes the form $(r_0, r_1, \dots, r_k)$ where r_i is the height of the

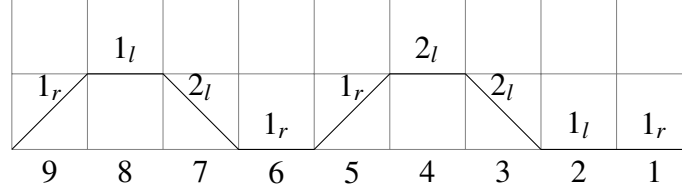


Figure 3.3 An example of a walk in $M_9(P)$ for $p = \{9, 7, 5, 3\}$

end of the i -th step and $r_0 = 0$. From this, we have that $r_i \geq 0$ for all $0 \leq i < k$, $r_k \geq -1$, and $r_i = r_{i-1} \pm 1$ for all $1 \leq i \leq k$.

We will give a generalization of Theorem 1.2 found in [11] with a simpler, combinatorial proof.

Theorem 3.3.1. *For $n \geq 1$ and $P = \{p_1 > p_2 > \dots > p_k\} \subseteq [n]$, we take the convention that $p_0 = n + 1$ and $p_{k+1} = 1$. Then if we take the convention that $0^0 = 1$, we have the following.*

$$\sum_{Q \subseteq P} 2^{|Q|+1} |p_n(Q)| = 2^{n-k} \sum_{r \in R_k} \prod_{i=0}^k (r_i + 1)^{p_i - p_{i+1}}.$$

This will be proven later, but first we need to set up some combinatorial objects.

Define the set of *decorated Motzkin walks*, $M_n(P)$, to be the set of all lattice walks with n steps, starting at $(0, 0)$, consisting of steps $[1, -1]$, $[1, 0]$, and $[1, 1]$, such that every step starts weakly above the x -axis, and where the steps are numbered $1, 2, \dots, n$ from right to left. Note that this will allow walks that can go below the x -axis only on their final step. Furthermore, we require that in each walk of $M_n(P)$ the step numbered k be either the up step $[1, 1]$ or the down step $[1, -1]$ if $k \in P$, and the horizontal step $[1, 0]$ if $k \notin P$. Finally, we give every step of the walk two labels. The first is an integer in $[h + 1]$ where h is the height of the starting point of the step. The second label is either a right or left designation subject to the condition that down steps are always labeled as left and up steps always labeled as right.

An example of an element in $M_n(P)$ is given in Figure 3.3 where the number of each step is at the bottom of the figure and the label of each step is directly above the step. Note that for simplicity, we have combined the two labels by giving l and r subscripts to the height labels. For instance, the label given to the third step in Figure 3.3 is 2_l meaning it has a height label of 2 and a left designation.

The count for the number of walks in $M_n(P)$ is given by the following result.

Theorem 3.3.2. *Suppose $n \geq 1$ and $P \subseteq [n]$. Then the formula*

$$|M_n(P)| = 2^{n-k} \sum_{r \in R_k} \prod_{i=0}^k (r_i + 1)^{p_i - p_{i+1}}$$

gives the count for the number of distinct decorated Motzkin walks.

Proof. If we consider any walk in $M_n(P)$ we note that by removing all horizontal steps we reduce the walk to one in R_k . We may also reverse this process by starting with a walk $r \in R_k$ and adding horizontal steps as specified by P . Therefore, to find $|M_n(P)|$ we need only sum over all sequences $r \in R_k$ and for each resulting walk, determine how many different labelings are possible.

Consider all walks in $M_n(P)$ corresponding to a fixed $r \in R_k$. Note that the height of the step corresponding to p_{i+1} and the heights of the steps between p_i and p_{i+1} will have starting point at height r_i since the height can only change on a step corresponding to an element in P . Therefore, there are $r_i + 1$ choices for each of the labels of these $p_i - p_{i+1}$ steps. Additionally, since only horizontal steps have a choice for subscripts, we must multiply by a factor of 2 for each element not in P . Therefore, the total count is as stated. $\square$

Given $\pi \in \mathfrak{S}_n$ and arbitrary $P \subseteq [n]$, let $[\pi']$ be the cyclic permutation obtained by adding the element $n + 1$ to the end of π and wrapping it around into a circle, and let $P' = P \cup \{n + 1\}$. We then have that the cyclic permutations with $\text{cPin}[\pi'] = P'$ are in bijection to those in $p_n(P)$, since this operation is invertible, and preserves all pinnacles except $n + 1$. Define an *intermediate set* to be the set of all elements between (but not including) two consecutive elements of P' in $[\pi']$. Intermediate sets may be empty. Given π a *cyclic vale* is any vale of $[\pi']$. Because $n + 1$ will always be a pinnacle, all cyclic vales will be in $[n]$ and will alternate with the pinnacles, which means there will be exactly $k + 1$ of them in $[\pi']$ where k is the number of pinnacles of π .

In order to prove Theorem 3.3.1, we will interpret the sum

$$\sum_{Q \subseteq P} 2^{|Q|+1} |p_n(Q)|$$

as the number of cyclic permutations such that $[\pi'] \in [\mathfrak{S}_{n+1}]$, $\text{cPin}[\pi'] = Q' = Q \cup \{n+1\} \subseteq P'$, and where every element of Q is marked as either left or right subjected to certain restrictions. If we note that every element of a cyclic permutation is either a pinnacle, a vale, or in the middle of an increasing or decreasing sequence of three elements, we may impose the following labeling.

1. Any element that is a cyclic vale may be marked as either right or left.
2. Any element of P that is not a cyclic vale is always marked as right.
3. Any element not in P that is in the middle of a decreasing sequence of 3 elements is always marked as right.
4. Any element not in P that is in the middle of an increasing sequence of 3 elements is always marked as left.

Let us call the set of all such permutations $V_n(P)$. Note that by this convention, all markings are forced except those at the cyclic vales, which are all free, and therefore the cardinality of $V_n(P)$ is

$$\sum_{Q \subseteq P} 2^{|Q|+1} |p_n(Q)|$$

since the number of cyclic vales is $|Q| + 1$ and since the number of cyclic permutations with $\text{cPin}[\pi'] = Q'$ is $|p_n(Q)|$.

Next, we define a map f from $M_n(P)$ to $V_n(P)$. To do this, we read the steps of some $m \in M_n(P)$ one at a time and use it to build up a cyclic permutation. The process is described as follows.

1. Start with the cyclic permutation consisting of only $p_0 = n+1$. In all future steps, an intermediate set will be said to be *available* if neither of the elements in P' that bound it have a left label. To this end, p_0 is assumed to have a right label.
2. Take the current leftmost step of walk m and insert its step number, i , into the cyclic permutation in the following manner. Let the label of the step be h_x . Counting clockwise from p_0 , insert i into the h -th available intermediate set and give it the left/right label

corresponding to the subscript x . If the intermediate set was empty, there is only one way to insert i_x . If the intermediate set is non-empty, look at the current smallest element of that set, v , and place i adjacent to v on the side corresponding to v 's label. Note that if k was in P' , the available intermediate sets for future steps will be renumbered. Finally, delete the step numbered i from the walk.

3. Repeat until all elements have been placed and the walk is empty.

As an example of this algorithm, suppose that $P = \{9, 7, 5, 3\}$ and consider the walk in Figure 3.3. To compute the output of the map in $V_9(P)$ we successively generate the following permutations as we read the steps of P from left to right, inserting the next element as indicated by the number and label of the step in the walk. Elements in P are in bold to make it easier to see the intermediate sets. Commentary is provided for the first few insertions to aid the reader.

$[10_r]$

$[10_r, 9_r]$: 9_r is inserted in the only intermediate set available, splitting it in two.

$[10_r, 8_l, 9_r, 7_l]$: 7_l is inserted in the second intermediate set because its label is 2_l ; from here to the end of the algorithm, no other elements will be inserted between 9 and 7, or between 7 and 10 because $7 \in P$ and step 7 has a left label.

$[10_r, 6_r, 8_l, 9_r, 7_l]$: 6_r is inserted to the left of the smallest element, 8, of the first intermediate set because 8 has a left label.

$[10_r, 6_r, 5_r, 8_l, 9_r, 7_l]$: Inserting 5_r splits the only available intermediate set into two (one from 10 to 5 and one from 5 to 9) since $5 \in P$ and both are available since 5 has a right label.

$[10_r, 6_r, 5_r, 4_l, 8_l, 9_r, 7_l]$

$[10_r, 6_r, 5_r, 3_l, 4_l, 8_l, 9_r, 7_l]$

$[10_r, 6_r, 2_l, 5_r, 3_l, 4_l, 8_l, 9_r, 7_l]$

$[10_r, 6_r, 1_r, 2_l, 5_r, 3_l, 4_l, 8_l, 9_r, 7_l]$

The last line is the final output.

We will now prove some properties about this map so that we may show it is well-defined.

Lemma 3.3.3. *In the final result, all elements in P' that are not cyclic vales are guaranteed to have right labels.*

Proof. This follows because when an element of P' with a left label is initially inserted, it will become a cyclic vale. Indeed, it is a cyclic vale upon insertion since it is the smallest element inserted so far. And in all future steps no elements are ever inserted next to it on account of the intermediate sets on either side being unavailable. So it will remain a cyclic vale until the end. $\square$

Lemma 3.3.4. *The output of f will always have pinnacles a subset of P'*

Proof. Consider some element $i \notin P'$. Since all elements become vales when initially placed, it is enough to show that i remains next to one of its larger adjacent elements through the end of the

algorithm. But this is immediate because if i is labeled as left, then no future element will ever be inserted directly to the right of i , and similarly for a right label. Since no element is ever removed, we have that i will remain next to a larger element and therefore fail to be a pinnacle. $\square$

Lemma 3.3.5. *In the final result, all elements $i \notin P'$ that are also not vales will be in the middle of a decreasing sequence of three elements if i has a right label, and in the middle of an increasing sequence of three elements if i has a left label.*

Proof. First, since i cannot be a vale by assumption and since it cannot be a pinnacle by Lemma 3.3.4, the only remaining possibilities are that it is in the middle of a decreasing or increasing sequence. If such an element i has a left label, then the algorithm forces all elements which are less than i and placed adjacent to i to be to its left. Therefore, no element smaller than i will be placed to its right, which makes it impossible for i to end up in the middle of a decreasing sequence. Therefore, it must be in the middle of an increasing one. A similar proof holds for when i has a right label. $\square$

Lemma 3.3.6. *At any step of the algorithm, if p_i was the last element of P' to be placed, then there will be $r_i + 1$ available intermediate sets for the next step.*

Proof. Suppose that element i was just placed in $[\pi']$. If $i \in P'$ corresponds to an up step, then i will have a right label and the net effect on the permutation will be to replace the intermediate set it landed in with two new ones on either side, resulting in an increase of one available intermediate set. If $i \in P'$ corresponds to a down step however, it will have a left label and therefore make the intermediate set it lands in unavailable, for a net loss of one intermediate set. Finally, if $i \notin P'$, then i cannot change the number of intermediate sets since they are completely determined by the positions of the elements in P' . This, combined with the fact that at the start of the process there is always one available intermediate set and that $r_0 = 0$, is enough to show the result by induction. $\square$

Proposition 3.3.7. *The algorithm defined by f is well defined. In particular it always terminates, and outputs a permutation in $V_n(P)$.*

Proof. To prove that f terminates, note that there will always be at least one available intermediate set in which to place elements because every $r_i \geq 0$, for $0 \leq i < k$ and so by Lemma 3.3.6 there will always be at least one intermediate set at each step. Lemma 3.3.4 shows that the pinnacles will be a subset of P' , and Lemmas 3.3.3 and 3.3.5 together show that those elements which are not cyclic vales will be marked as specified by the set $V_n(P)$. Finally, since the algorithm marks all cyclic vales, we have that f is well-defined. $\square$

We now construct the inverse $g = f^{-1}$ by reversing each step of the algorithm for f .

1. Start with a permutation $[\pi'] \in V_n(P)$. We will build a walk $m \in M_n(P)$ from right to left with steps numbered $1, 2, \dots, n$. To build step number i , we remove the element i from $[\pi']$ in the following way.
2. Locate the smallest element, i , of $[\pi']$, and remove it to form $[\sigma]$. Now in $[\sigma]$ take note of the intermediate set, S , from which i was removed. Count how many available intermediate sets S is clockwise from p_0 and call this number h . Then add a step to the walk m with label h_x where x corresponds to whether the element was labeled left or right.

The step is placed so that it begins at height $h - 1$ where h is the number of available intermediate sets in $[\pi']$ after i is removed. The slope of the new step is determined as follows.

- a) If $i \notin P$, the step will be a horizontal step.
 - b) If $i \in P$ and i has a left label, the step will be a down step.
 - c) If $i \in P$ and i has a right label, the step will be an up step.
3. Label $[\sigma]$ as $[\pi']$ and continue with the above process until $[\pi']$ consists of only p_0 . The resulting walk is the final output of the map.

We must prove that this map is well-defined.

Lemma 3.3.8. *At every step of the process, the pinnacles of $[\pi']$ will always be in P' , regardless of how many elements have been removed.*

Proof. To see this, note that $[\pi']$ starts with all pinnacles in P' by assumption, and no non-pinnacle can ever become a pinnacle since we always remove the smallest element. $\square$

Lemma 3.3.9. *At every step where i is removed from $[\pi']$ to form $[\sigma]$, the intermediate set in $[\sigma]$ from which i was removed will be available.*

Proof. Suppose the intermediate set in $[\sigma]$ from which i was removed was unavailable in $[\sigma]$, which would force one of the two bounding elements in P' to have a left label. Now any element $p \in P'$ with a left label is a cyclic vale at the start of the algorithm, and since there are no other elements in P' between p and i to be pinnacles, it must be that $i > p$. But this contradicts the assumption that the smallest element is being removed. Therefore, i must have been removed from an available intermediate set. $\square$

Lemma 3.3.10. *The algorithm g will produce a connected walk.*

Proof. Consider any step m_i formed by removing element i from $[\pi']$ and let m_{i-1} be the step placed just before that (when $i-1$ was removed from $[\pi']$). Then by the definition of g , if r_i and r_{i-1} are the starting heights of steps m_i and m_{i-1} respectively, we have that $r_i + 1$ must be the number of available intermediate sets in $[\pi']$ right after i was removed while $r_{i-1} + 1$ was the number of available intermediate sets in $[\pi']$ right before i was removed. We now have three cases.

1. Case 1: Suppose that i was not in P' . Then the number of available intermediate sets will be unchanged once i is removed, and since g will cause m_i to be horizontal in this case, we have that m_i will end at height r_{i-1} , thereby connecting it to the previous step.
2. Case 2: Suppose that $i \in P'$ and i had a left label. Then the intermediate set it occupied will become available by Lemma 3.3.9 and there will be a net increase of one available intermediate set. Since in this case g causes m_i to have downward slope, we have that m_i will end at height r_{i-1} , thereby connecting it to the previous step.

3. Case 3: Suppose that $i \in P'$ and i had a right label. Then the intermediate sets on each side of i must be available or else Lemma 3.3.9 will be contradicted when i is removed. Removing i will then join these available intermediate sets into one available set, and there will be a loss of one available intermediate set. Since in this case g causes m_i to have upward slope, we have that m_i will end at height r_{i-1} , thereby connecting it to the previous step.

Therefore, since in every case we have that the newly placed step connects to the previous, we will have that the final walk will be connected. $\square$

Proposition 3.3.11. *The walk produced by the algorithm is in $M_n(P)$.*

Proof. By definition, g causes every step i to have its label in $[r_i + 1]$ because the numeric label for i is chosen from the available intermediate sets after it is removed, which is $r_i + 1$. Also, g forces all slanted steps to have the required left/right label by definition. Additionally, the walk will be connected by Lemma 3.3.10 and also begin on the x -axis because there will be only one available intermediate set when the process is finished (because at the end, all that is left of the permutation is $[p_0]$). Finally, every step of the walk must start weakly above the x -axis because the only time there can be no available intermediate sets in $[\pi']$ is at the start of the algorithm, since removing an element always leaves behind an available intermediate set by Lemma 3.3.9. This means that r_i must be at least zero for all i except possibly the last one, which forces all steps to begin weakly above the x -axis.

Therefore, the result $m \in M_n(P)$ and we have that the map is well-defined. $\square$

Proposition 3.3.12. *The maps f and g defined above are inverses of each other.*

Proof. First, given a walk $m \in M_n(P)$, we let m_i be the step numbered i , that is, the i -th step from the right in the original m (note that if segments are deleted from m , we do not re-index or renumber the remaining segments). Let f_i be the step in f that deletes walk segment m_i and adds element i to $[\pi']$, and let g_i be the step in g that adds walk segment m_i and removes element i from $[\pi]$. Then we have that $f = f_1 \circ f_2 \circ \cdots \circ f_n$ and $g = g_n \circ g_{n-1} \circ \cdots \circ g_1$ by definition.

For a given walk $m \in M_n(P)$ we define S_i to be the state of the walk and the permutation $[\pi']$ after applying $f_{i+1} \cdots \circ f_{n-1} \circ f_n$. Under this notation, we have that S_n is the initial state consisting of the original walk m and $[\pi'] = [n + 1]$ and that S_0 is the final state consisting of the empty walk and the final permutation. We can then say that when applying f to a walk, each f_i will convert state S_i into state S_{i-1} . Therefore, if we can show that g_i applied to state S_{i-1} gives state S_i , we will have that $g \circ f$ is the identity because the g_i will move back through all the states step by step.

Consider the walk and permutation at some state S_i . If we apply f_i to this state, we will delete step m_i with label j_x and add element i to $[\pi']$. Then j will be the number of the intermediate set into which i is inserted and the left/right label of i in π will be x . This will give us state S_{i-1} . If we then apply g_i , we will remove i and add step m_i to walk m with label j_x .

Now we must show that the height and slope of m_i are as they were in S_i . As was proven already in Lemma 3.3.6, the height of the start of step m_i in S_i was $h - 1$ where h is the number of available intermediate sets in $[\pi']$ in state S_i . But by definition of g acting on state S_{i-1} , the height of the start of the step that g places must be $h - 1$ again since the number of intermediate sets after i is removed from $[\pi']$ is the same as it was in S_i . Therefore, m_i will be placed at the same height. To show that slope is preserved, we consider each of the three cases. If m_i started as horizontal in S_i , then $i \notin P$ because of the restrictions imposed on $M_n(P)$ and when we apply g_i to state S_{i-1} we will get a horizontal step back. If m_i was an up step, we have by the restrictions imposed on $M_n(P)$ that $m_i \in P$ and that m_i has a right label. Since the left/right label for n in $[\pi']$ is the same as that for m_i , we have that g_i will make m_i an up step again. A similar proof shows that if m_i was a down step before it was removed, it will be a down step again after it is returned. Therefore, g_i undoes f_i .

Now for a given permutation $[\pi'] \in V_n(P)$ we define S'_i to be the state of the walk and permutation $[\pi']$ after applying $g_i \cdots \circ g_2 \circ g_1$. Under this notation, we have that S'_n is the final state consisting of the final walk and permutation $[\pi'] = [n + 1]$, and that S'_0 is the initial state consisting of the empty walk and the initial permutation. We can then say that when applying g to a permutation, each g_i will convert state S'_{i-1} into state S'_i . Therefore, if we can show that f_i applied to state S'_i gives state S'_{i-1} , we will have that $f \circ g$ is the identity because the f_i will move back

through all the states step by step.

Consider the walk and permutation at some state S_{i-1} and suppose we apply g_i where we remove element i with label x from $[\pi']$ and add step m_i to the walk. Then we will be in state S'_i where the numeric label of m_i will indicate the intermediate set, I , from which i was removed and the left/right label will be x . If we then apply f_i , we will add i back to the intermediate set corresponding to the label of m_i with left/right label x , and so we have that i will get back to $[\pi']$ in I and have the same label as in state S'_{i-1} .

We must now show that the position of i in $[\pi']$ is as it was in S'_{i-1} . If i was the only element in I in state S'_{i-1} , it will arrive back in the same position in $[\pi']$ and we are done. If i was not the only element, we note that since it was a vale and since all non-pinnacles in I must form a decreasing sequence to the left of i and an increasing sequence to the right of i (or become pinnacles themselves), we have that i was initially positioned next to the second smallest element in I . By the labeling conventions imposed on $[\pi']$, any element in I to the left of i will have a right label and any elements in S to its right will have a left label. Therefore, no matter which of these two elements becomes the new vale after i is removed, we are guaranteed that f_i will put i adjacent to this element again, and that it will be on the left if the element is labeled left and the right if it is labeled right. Therefore, f_i undoes g_i .

Since f and g undo each other step by step, we have that they are inverses of each other. $\square$

Since the maps are inverses, we have that the sets are in bijection which finishes the proof of Theorem 3.3.1. We finish with a few remarks.

Corollary 3.3.13. *If we define W_k to be the set of y coordinate sequences of all walks using steps $[1, 1]$ and $[1, -1]$ and ending at any height, then*

$$\sum_{Q \subseteq P} 2^{|Q|+1} |p_n(Q)| = 2^{n-k} \sum_{w \in W_k} \prod_{m=0}^k (w_m + 1)^{p_m - p_{m+1}}$$

where we sum over W_k instead of R_k .

Proof. To see this, we note that if any walk $w \in W_k$ has a step that starts below the x -axis, then for some i there will exist a y -coordinate of the walk, $w_i = -1$, for which $i \neq k$. This will cause

$w_i + 1 = 0$ and $p_i > p_{i+1}$ so that the weight of walk w zeros out. Therefore, the sum reduces to summing over R_k . $\square$

As a final comment, we note that if we require P to not contain the element 1 then Theorem 3.3.1 reduces to Theorem 1.2 in [11]. This is because the rightmost step of every walk in $M_n(P)$ will then be horizontal, which prevents all walks from dipping below the x -axis, and so we can safely restrict our definition of R_k in Theorem 3.3.1 to require that all r_k are non-negative, which is the same set that Fang used. Dividing both sides of Theorem 3.3.1 by two then gives Fang's formula.

3.4 Recursion for admissible orderings

Let ω be any ordering of the elements in some set $S \subseteq [n]$. Then for any permutation π of a set of positive integers containing S , we say that $\text{ord}(\pi) = \omega$ if the elements in S appear in the same relative order in π as they do in ω .

Now consider some $P = \{p_1 < p_2 < \dots < p_k\} \subseteq [n]$ which need not be a pinnacle set (note the different indexing from the previous section). For the following theorems, we define an *admissible ordering* of P to be any ordering ω of the elements of P such that there exists a $\pi \in \mathfrak{S}_n$ with $\text{Pin}(\pi) = P$ and $\text{ord}(\pi) = \omega$. If such a π exists, we call it a *witness* to the ordering. Under this definition, it makes sense to talk about the admissible orderings even when P is not an admissible pinnacle set, since there will just be zero orderings. We will also define $N = \{n_1 < n_2 < \dots < n_{k+1}\}$ to be the set of the first $k + 1$ positive integers that are not in P .

Theorem 3.4.1. *Suppose $P = \{p_1 < p_2 < \dots < p_k\} \subseteq [n]$ is an arbitrary subset of $[n]$. Let $N = \{n_1 < n_2 < \dots < n_{k+1}\}$ be the set of the first $k + 1$ positive integers that are not in P . Then given a fixed ordering ω of P , there exists a $\pi \in \mathfrak{S}_n$ with $\text{Pin}(\pi) = P$ and $\text{ord}(\pi) = \omega$ if and only if there exists a permutation τ of the elements $P \cup N$ with $\text{Pin}(\tau) = P$ and $\text{ord}(\tau) = \omega$.*

Proof. Let $\pi \in \mathfrak{S}_n$ be a witness to some admissible ordering of P . We will build up a permutation τ on the elements of $P \cup N$ which has $\text{Pin}(\tau) = P$. An example of this process follows the proof.

To start, let $\tau = \pi$ and identify each of the $k + 1$ blocks of consecutive non-pinnacles in τ . For each block, delete all but the smallest non-pinnacle. Finally, take the set of all $k + 1$ remaining non-pinnacles in the result and standardize them to the set N to get permutation τ . Then τ will be a permutation of the elements of $P \cup N$ by definition with elements of P in order ω . We must also have that $\text{Pin}(\tau) = P$ since every $p \in P$ started as a pinnacle in π , and the elements on either side of p could only decrease throughout the steps of this process. Since all other elements end up adjacent to a pinnacle, there can be no added pinnacles, and so we have τ as specified.

To see the reverse direction, simply take any τ with a given ordering of the elements of P as the pinnacle set and add the elements $[n] \setminus (P \cup N)$ in increasing order to the end of τ . The result will be in $\mathfrak{S}_n$ and have pinnacle set P in the same order as in τ . $\square$

As an example, suppose that $n = 8$, $P = \{3, 7\}$ and $\pi = 57642318$. Then $N = \{1, 2, 4\}$, $\omega = 73$, and we build up τ in the following steps:

$$\tau = 57642318$$

$$\tau = 57231$$

$$\tau = 47231$$

We may now state a corollary which will always allow us to reduce to the case where we only need to look for witnesses in $\mathfrak{S}_{2k+1}$

Corollary 3.4.2. *Given $P = \{p_1 < p_2 < \cdots < p_k\}$ and N as above, standardize the set $P \cup N$ and let P' and N' be the sets that P and N standardize to, respectively. Then the number of admissible orderings of set P for $\pi \in \mathfrak{S}_n$ is the same as the number of admissible orderings of set P' for $\pi \in \mathfrak{S}_{2k+1}$.*

Proof. Simply standardize the permutation τ given in the proof of the previous theorem. $\square$

For example, if we standardize the $\tau = 47231$ we had at the end of the previous example, we would get $\tau = 45231 \in \mathfrak{S}_5$ where now $P' = \{5, 3\}$.

In all that follows we will assume that

$$P = \{p_1 < p_2 < \dots < p_k\} \text{ and } n = 2k + 1 \quad (3.3)$$

since counting admissible orders when n is larger can be reduced to this case. Note that because of this, specifying P is enough to determine N since $N = [2k + 1] \setminus P$.

Let $O(P)$ be the set of admissible orderings of set P , and $o(P) = |O(P)|$. More generally, if we let $N_i = \{n_1 < n_2 < \dots < n_i\}$ be the i smallest elements of N , we define

$$O_i(P) = \{\omega \in S_{P \cup N_i} \mid \text{there is } \pi \in \mathfrak{S}_{2k+1} \text{ with } \text{ord}(\pi) = \omega \text{ and } \text{Pin}(\pi) = P\}$$

where S_A is the set of all permutations of the set A . We further define $o_i(P) = |O_i(P)|$ and note that $O_0(P) = O(P)$.

Essentially, $O_i(P)$ “keeps track” of the positions of not only the elements of P , but of the positions of the i smallest elements of N , too. As an example, consider the set $P = \{3, 5\}$. Then $o(P) = 2$ since all orderings are admissible. However, $o_1(P) = 4$ since

$$O_1(P) = \{135, 315, 513, 531\}$$

with corresponding witnesses

$$\{13254, 23154, 45132, 45231\}.$$

Note that the orderings 351 and 153 are not included however because there is no way to insert the elements 2 and 4 to get a permutation in $\mathfrak{S}_5$ having pinnacle set P .

In the following results, we will need to talk about witnesses to specific orderings in $O_i(P)$. The following lemma will allow us to make certain assumptions about the witness we wish to pick.

Lemma 3.4.3. *Consider any $\omega \in O_i(P)$ where the conditions in (3.3) hold and where $i < p_1$. Then if ω' is obtained by permuting the elements in $[i]$ in ω among their indices, then $\omega' \in O_i(P)$ also.*

Proof. Since ω is an admissible ordering, there must exist at least one witness to it which we will call π . Because $n = 2k + 1$ we know that in π the pinnacles and non-pinnacles, which are

the elements in P and not in P respectively, alternate beginning and ending with a non-pinnacle. Therefore each element of $[i]$ will be adjacent to two pinnacles larger than itself, and since all values in $[i]$ are smaller than all pinnacles, this will still be the case after permuting the elements in $[i]$. Therefore, no elements in $[i]$ can become pinnacles. Additionally, no pinnacles can be lost because the only elements to move are those in $[i]$ and if an element of $[i]$ adjacent to a pinnacle gets swapped, it will just be replaced by a different value in $[i]$ that is still smaller than the pinnacle. Finally, we have that any element j that is not in P or $[i]$ cannot become a pinnacle because neither j nor the two elements of P initially adjacent to it will move during the swap. Therefore, no pinnacles are gained or lost, and so the resulting permutation will be a witness to the corresponding ordering ω' , which shows it must be in $O_i(P)$. $\square$

Given $P \neq \emptyset$ and corresponding N , consider the set $[2k + 1] \setminus \{p_1, n_1\}$ and standardize so that the elements are in $[2k - 1]$. Let P' be the set that the elements in $P \setminus \{p_1\}$ get mapped to which will always be $\{p_2 - 2, p_3 - 2, \dots, p_k - 2\}$ if $i < p_1$. Then define the *reduction operator* $r(P) = P'$.

We now wish to form a bijection between a subset of the elements in $O_i(P)$ and those in some $O_{i'}(P')$. To this end, for $j \in \{0, 1, 2\}$ we define

$$O_i^j(P) = \{\omega \in O_i(P) \mid p_1 \text{ is adjacent to } j \text{ elements in } [i]\}$$

and let $o_i^j(P) = |O_i^j(P)|$.

We then have the following result.

Lemma 3.4.4. *Suppose $i \geq 0$, that the conditions in (3.3) hold, and that P is non-empty. Then if for some $j \in \{0, 1, 2\}$ we have that $i < p_1$ and $O_i^j(P) \neq \emptyset$ and we let $P' = r(P)$, then we have the following*

$$o_i^j(P) = \begin{cases} i(i-1)o_{i-1}(P') & \text{if } j = 2, \\ 2io_i(P') & \text{if } j = 1, \\ o_{i+1}(P') & \text{if } j = 0. \end{cases}$$

Proof. Note that since we have that $O_i^j(P) \neq \emptyset$, there must exist at least j elements in $[i]$ and $2 - j$ elements not in $[i]$ but less than p_1 to surround p_1 . Therefore, we have the inequalities $i \geq j$ and $i + 2 - j < p_1$.

We will consider the case where $j = 2$ in detail and note the changes that must occur for when $j = 0, 1$. Note first that by Lemma 3.4.3 we have that the existence of an ordering $\omega \in O_i^2(P)$ in which ω contains the factor $x p_1 y$ implies the existence of one containing the factor $1 p_1 2$ and visa versa. If we let

$$A = \{\omega \in O_i^2(P) \mid 1 p_1 2 \text{ is a factor of } \omega\}$$

then we will have that

$$o_i^2(P) = i(i - 1) \cdot |A|.$$

where we have the factor of $i(i - 1)$ because there are i ways to choose an element in $[i]$ to the left of p_1 and $i - 1$ ways to choose a different element in $[i]$ to the right. We may now define a map between set A and $O_{i-1}(P')$ in which we remove the factor $1 p_1$ from ω and standardize to the set $P' \cup [i - 1]$. To show this ordering has a witness, first consider any witness, π , to ω and note that π must also contain the factor $1 p_1 2$ since elements in P must alternate with those that do not, and so no new elements can be inserted between $1, p_1$, and 2 . If we then remove $1 p_1$ from π and standardize to get π' , we are left with a witness to the ordering ω' . Note that the element to the left of 1 in π , if it exists, must have originally been a pinnacle, and since in π' it will end up being adjacent to 1 again, it must remain a pinnacle. Furthermore, since 2 cannot become a pinnacle, and since the relative order of all other elements was preserved, we have that the pinnacle set of π' is P' which shows that ω' is admissible.

To reverse this map, consider some $\omega' \in O_{i-1}(P')$ and standardize to the set $(P \cup [i]) \setminus \{p_1, 1\}$. Note that since $i \geq j = 2$, we have that ω' will contain the element 1 , which will standardize to 2 . Replace 2 with $1 p_1 2$ to get the ordering ω which will be in A if we can show it has a witness. To show this, consider any witness π' to ω' , standardize to $[2k + 1] \setminus \{p_1, 1\}$, and replace 2 with $1 p_1 2$ to get π , which will be a witness to ω . An argument similar to the one above shows that the pinnacle set will be P , and so we have that ω is admissible, completing the proof in this case.

Since these maps are clearly inverses, we have that $|A| = o_{i-1}(P')$ which completes the case when $j = 2$.

For the case where $j = 1$, we can use a similar proof where we use Lemma 3.4.3 to show that

$$o_i^2(P) = 2i \cdot |\{\omega \in O_i^2(P) \mid 1p_1 \text{ is a factor of } \omega\}|.$$

and then define a bijection to simply delete p_1 from ω and standardize to $P' \cup [i]$ to get ω' . Care must be taken to prove this results in an ordering in $O_i(P')$. To prove it has a witness, one can use the fact that $i + 2 - j < p_1$ to argue for the existence of a witness $\pi \in \mathfrak{S}_{2k+1}$ to ω which contains the factor $1p_1(p_1 - 1)$ where $p_1 - 1 \notin [i]$. Replacing this factor with 1 and standardizing will result in a witness for ω' with pinnacle set P' . The fact that $i + 2 - j < p_1$, together with the fact that $i \geq j$, must be used again when showing that the reverse of this map is well-defined.

For the case where $j = 0$, we may define our bijection between $O_i^j(P)$ and $O_{i+1}(P')$ by taking some $\omega \in O_i^j(P)$, replacing p_1 with 0, and standardizing to $P' \cup [i + 1]$ to get ω' . To prove that ω has a witness, one can use the fact that $i + 2 - j < p_1$ to argue for the existence of a witness $\pi \in \mathfrak{S}_{2k+1}$ to ω which contains the factor $(p_1 - 2)p_1(p_1 - 1)$ where $p_1 - 1, p_1 - 2 \notin [i]$. Replacing this factor with 0 and standardizing will then result in a witness for ω' , which can be shown to be in $O_{i+1}(P')$. The fact that $i + 2 - j < p_1$ must be used again when showing that the reverse of this map is well-defined. $\square$

Given some set P , we make the following definition:

$$\delta_j = \begin{cases} 1 & \text{if } p_1 > j, \\ 0 & \text{otherwise.} \end{cases}$$

Then we can state our main result.

Theorem 3.4.5. *Let $|P| \geq 1$, $P' = r(P)$, $0 \leq i < p_1$, and suppose the conditions in (3.3) hold. Then we have the following recursion*

$$o_i(P) = i(i - 1)o_{i-1}(P') + (2i)\delta_{i+1}o_i(P') + \delta_{i+2}o_{i+1}(P').$$

Proof. Note that $O_i(P)$ can be partitioned into three (possibly empty) subsets depending on whether p_1 is directly adjacent to either two, one, or no elements of N_i . It follows that

$$o_i(P) = o_i^2(P) + o_i^1(P) + o_i^0(P).$$

Therefore, we will have our result if we can show that

$$o_i^j(P) = \begin{cases} i(i-1)o_{i-1}(P') & \text{if } j = 2, \\ 2i\delta_{i+1}o_i(P') & \text{if } j = 1, \\ \delta_{i+2}o_{i+1}(P') & \text{if } j = 0. \end{cases}$$

If $O_i^j(P)$ is non-empty, then we have our result by Lemma 3.4.4 since we will have that $i+2-j < p_1$ and the corresponding delta is 1. Therefore, we only need to concern ourselves with the case where $o_i^j(P) = 0$, in which we must prove that the corresponding expression for $o_i^j(P)$ is also zero.

First, we claim that $o_i^j(P) = 0$ if and only if either P is not admissible, $i+2-j \geq p_1$, or $j > i$. That each of these three conditions independently forces $o_i^j(P) = 0$ is not hard to see since we will either have that $o_i(P) = 0$ or that there are not enough of the necessary elements to surround p_1 and have it still be a pinnacle. For the reverse direction, we prove the contrapositive. Suppose that P is admissible, that $i+2-j < p_1$, and that $j \leq i$. In this case, chose any witness to any ordering of P and call it π . Now by our two inequalities, we know there must exist two elements, x and y , in π such that j of them are in $[i]$ and $2-j$ of them are between i and p_1 . If we then swap x and y with the pair of elements to either side of p_1 in π , we will have a witness to an ordering in $O_i^j(P)$. The fact that the pinnacle set of this witness is indeed P can be proven similarly to Lemma 3.4.3 since π alternated pinnacles and non-pinnacles, and since all elements swapped were smaller than all pinnacles. Therefore, we have an ordering in $O_i^j(P)$, which proves it is non-empty. This finishes the proof of the claim.

Therefore, in the case where $o_i^j(P) = 0$ we may suppose that at least one of these three conditions is true. If $j > i$, we immediately have that all three expressions for $o_i^j(P)$ are zero and so we still have equality even in the case when $j > i$. If we assume that $i+2-j \geq p_1$ for some $j \in \{0, 1, 2\}$,

we must have that $j < 2$ since otherwise we contradict our assumption that $i < p_1$. But if $j = 0, 1$, we immediately have that δ_{i+2-j} is zero and so again we have equality.

To finish the proof, all we have left to show is that all three expressions are zero if $j \leq i$, $i + 2 - j < p_1$, but P is not admissible. Clearly if P' fails to be admissible too, all three terms will immediately be zero, so suppose that P' is admissible. This is equivalent to saying that for every element $p \in P'$, the number of elements in P' less than p is exceeded by the number of elements not in P' less than p . This will still be the case for all corresponding pinnacles in P since both elements added are smaller than all other pinnacles, and so if P fails to be admissible, it must be because the added element of P , p_1 , is less than 3. But supposing $j \leq i$ and $i + 2 - j < p_1$ together force $p_1 > 2$, and so we have a contradiction.

Therefore, since in all cases where $o_i^j(P) = 0$ we have that the corresponding expression is also zero, we have our result. $\square$

Corollary 3.4.6. *This recursion can be used to compute $o(P) = o_0(P)$. Furthermore, the number of terms that need to be computed will be asymptotically equal to $k^2/4$.*

Proof. First, note that every step reduces the size of P by one, and so if we induct on k , we have the base case $P = \emptyset$, and $|N| = 1$ which always has only one ordering. In order to show that we may repeatedly use this recursion however, we must show that we never need to compute some $o_i(P)$ where $i \geq p_1$.

First, note that on the first step when $i = 0$, we must have that $i < p_1$ since $P \neq \emptyset$. Now consider any application of the recursion on $o_i(P)$ where $i < p_1$. Since $p'_1 = p_2 - 2$ forces $p'_1 \geq p_1 - 1$ by definition of $r(P)$, we know that $p'_1 > i - 1$ and so the recursion may be applied on $o_{i-1}(P')$. Now we consider $o_i(P')$. If $i < p'_1$ we may apply the recursion; if not, then we will have that $i \geq p_1 - 1$ which means $p_1 \leq i + 1$. But then we will have that $\delta_{i+1} = 0$, and so we will not have to calculate $o_i(P')$. Finally, we consider $o_{i+1}(P')$. If $i + 1 < p'_1$ we may apply the recursion; if not, then we will have that $i + 1 \geq p_1 - 1$ and so $p_1 \leq i + 2$. But then we will have that $\delta_{i+2} = 0$, and so we will not have to calculate $o_{i+1}(P')$. Therefore, the recursion never generates a term that cannot then be further broken down.

As to the number of terms that need to be calculated, let $P^j = r^j(P)$ applying the r operator j times where $0 \leq j \leq k$. Note that since i starts out as 0 and i can only increase by at most 1 each step, we must have that $i \leq j$ for any term $o_i(P^j)$. Additionally, since all permutations have elements in $[2(k-j)+1]$, we know that $i \leq k-j+1$ since the remaining $k-j$ elements are in P^j . Therefore when calculating terms of the form $o_i(P^j)$, if $j \leq k/2$ we know that $0 \leq i \leq j$ while if $k/2 \leq j \leq k$ we know that $0 \leq i \leq k-j+1$. Therefore, if we sum up all the different values that i can take over all values of j , we have that the total number of terms is at most

$$\sum_{j=0}^{\lfloor k/2 \rfloor} (j+1) + \sum_{j=\lceil (k+1)/2 \rceil}^k (k-j+2) = \sum_{j=0}^{\lfloor k/2 \rfloor} (j+1) + \sum_{j=0}^{\lfloor k/2 \rfloor} (j+2) = \binom{\lfloor k/2 \rfloor + 2}{2} + \binom{\lfloor k/2 \rfloor + 2}{2} + \lfloor k/2 \rfloor + 1$$

which is asymptotically $k^2/4$. □

3.5 Block representation of pinnacle sets

Given a (not necessarily admissible) set $P = \{p_1 < p_2 < \dots < p_k\} \subseteq [n]$, we may choose to represent P as an array of some length $n \geq p_k$ consisting of 0s and 1s such that a given index i contains a 1 if and only if $i \in P$. For example, if we had

$$P = \{3, 6, 7\}$$

then the corresponding array for $n = 7$ would be

$$[0, 0, 1, 0, 0, 1, 1].$$

For some fixed n , we can clearly go back and forth between these representations without loss of information, and the array will correspond to an admissible pinnacle set if and only if it is a ballot sequence, that is, at every index the number of 0s exceeds the number of 1s occurring before that index (we will not use this fact, but it is interesting). Now for the purposes of our proofs, it will be useful to think of P using the array notation because it allows use to visually divide P up into “blocks,” which is to say, we may break the array up into sub-arrays. For instance, in our previous example if we let

$$B_1 = [0, 0, 1], B_2 = [0, 0, 1, 1]$$

then we may say that P can be written as $B_1 B_2$ where we concatenate the arrays to recover P . We may then apply algorithms recursively to each block, and combine the results to say something about P . For our purposes, it won't matter how we break up P provided that all blocks are non-empty, and so we will not (at this point) structure our notation to indicate how much of P is contained in B_1 .

As an aside, but something that will not be used in the proof, we may choose our blocks carefully so that they correspond to trees. This is done by simply looking at the binary tree representation of P (which is given in greater detail below), and letting B_1 consist of those indices appearing in the first tree, B_2 of those indices appearing in the second tree, and so on. Our approach is a little more general however as it allows us use blocks that contain “part” of a tree, which will be useful in our applications. Using this particular division into blocks however will allow us to say some things about the tree representation of P later.

We now must introduce some new ideas and notation.

Given a block B , which is to say, an array of some length m containing only 0s and 1s, we may define a set, P_B , to contain all those indices in B that are 1s. We then define $P(B)$ to be the set of permutations in $\mathfrak{S}_m$ with pinnacle set P_B , and further define $p(B) = |P(B)|$. For instance, suppose that in our running example we took $B_1 = [0, 0, 1]$. Then $P_{B_1} = \{3\}$ and $p(B_1) = 2$ since there are two permutations in $\mathfrak{S}_3$ that have pinnacle set $\{3\}$. On the other hand, if we took $B_2 = [0, 0, 1, 1]$ we would have $P_{B_2} = \{3, 4\}$ and $p(B_2) = 0$ since P_{B_2} is not an admissible pinnacle set.

We now take this a step further. Given a block B of length n with corresponding pinnacle set P_B , it will be desirable to modify P_B by adding additional elements larger than n (that is to say, we wish to force additional pinnacles). Furthermore, we will want to require that these added elements are indistinguishable from one another, which is a deviation from normal pinnacle sets where all pinnacles are different. Pinnacle sets are still defined however even when considering permutations with repeated numbers, and so the notion still makes sense.

More formally, define $P(B)^i$ to be all permutations of the multiset $\{1, 2, \dots, m\} \cup \{(n+1)^i\}$ that have pinnacle multiset $P_B \cup \{(n+1)^i\}$, and define $p(B)^i = |P(B)^i|$. For instance, if $B =$

$[0, 0, 1, 0, 0, 0]$, then $P_B = \{3\}$ and $m = 6$. To compute $P(B)^2$, we must ask which permutations of the set $\{1, 2, 3, 4, 5, 6, 7, 7\}$ have pinnacle set $\{3, 7, 7\}$, of which there are 36 (there are 3 ways to decide the order of 3, 7, 7 within the permutation, 2 ways to decide the order of 1 and 2 on either side of 3, and 6 ways to arrange the elements 4, 5, 6 to fill the remaining gaps between the pinnacles). On the other hand, $p(B)^5 = 0$ because there would be too many pinnacles forced.

There is a similar idea for forcing cyclic vales. Suppose once again that we have a block B of length n with corresponding pinnacle set P_B . It will be useful to talk about the set of permutations having not only pinnacle set P_B , but also a cyclic vale set that contains a certain set of desired elements. In fact, similar to how above we added new, large elements and forced them to be pinnacles, we will want to add new, small elements and force them to be cyclic vales.

More formally, define $P(B)_j$ to be all permutations of the multiset $\{1, 2, \dots, n\} \cup \{0^j\}$ that have pinnacle set P_B and cyclic vale set containing the multiset $\{0^j\}$, and define $p(B)_j = |P(B)_j|$. For instance, if $B = [0, 1, 1]$, then $P_B = \{2, 3\}$ and $m = 3$. To compute $P(B)_2$, we must ask how many permutations of the set $\{0, 0, 1, 2, 3\}$ have pinnacle set $\{2, 3\}$ and also cyclic vale set containing $\{0^2\}$, of which there are 6: there are 6 ways to order the elements 1, 2, 3 and then we may always insert the two 0s in a unique way in the remaining positions around 2 and 3 to force them to be pinnacles. On the other hand, $P(B)_4 = 0$ because the only way to force all four zeros to be cyclic vales is if they alternate with the numbers 1, 2, 3, but then this will force 1 to become a pinnacle.

While the above ideas may seem strange, hopefully they will feel more natural in a minute. The gist is that in addition to requiring that a particular set becomes a pinnacle set, we may now throw in additional large elements and require them to be pinnacles, or additional small elements and require them to be cyclic vales. In fact, we will often do both at once, and to this end, define the notation $P(B)_j^i$ to be all permutations of the multiset $\{1, 2, \dots, n\} \cup \{0^j, (n+1)^i\}$ that have pinnacle multiset $P_B \cup \{(n+1)^i\}$ and vales set containing $\{0^j\}$. In the event that no such permutations exist, we simply say that $P(B)_j^i = \emptyset$. Note that if any such permutations do exist, it will always be the case that no two elements of $\{(n+1)^i\}$ are adjacent, and that no two elements of $\{0^j\}$ are adjacent, as that would not allow them to all be pinnacles or cyclic vales respectively. This fact will be helpful

in our first proof.

Finally, we need to remind the reader of cyclic permutations. In general, any linear permutation in $\{1, 2, \dots, n\} \cup \{0^j, (n+1)^i\}$ included in $P(B)_j^i$, that is, having pinnacle multiset $P_B \cup \{(n+1)^i\}$ and cyclic vale set containing $\{0^j\}$, corresponds to a cyclic permutation in $\{1, 2, \dots, n, \infty\} \cup \{0^j, (n+1)^i\}$ having cyclic pinnacle multiset $P_B \cup \{(n+1)^i, \infty\}$ and cyclic vale set containing $\{0^j\}$. This correspondence is achieved simply by appending ∞ onto the beginning of the linear permutation, and wrapping it around into a cyclic permutation. The reverse operation is also well defined. Therefore, we define $cP(B)_j^i$ to be the set of cyclic permutations corresponding to the linear ones in $P(B)_j^i$.

Theorem 3.5.1. *Suppose that some block B (as described above) of length n is decomposed into two, non-empty parts B_1 and B_2 such that the concatenation B_1B_2 gives back B . Suppose further that the number of ones in B_2 is b . Then*

$$p(B)_j^i = \sum_{a=1}^{b+i+1} p(B_1)_j^{a-1} p(B_2)_a^i$$

Proof. First, let L be the set of all elements corresponding to those in block B_1 together with the set $\{0^j\}$. Similarly let H be the set of all remaining elements in $[n]$ not in L together with the set $\{(n+1)^i, \infty\}$. Then L will be the “low” set and H the “high” set in the sense that all elements in L will be smaller than all elements in H , and given any $[\pi] \in cP(B)_j^i$, we have that all elements in $[\pi]$ will be in either L or H . Since $p(B)_j^i = |cP(B)_j^i|$ it suffices to consider only cyclic permutations to prove the result.

Therefore given some $[\pi] \in cP(B)_j^i$ we may decompose it into maximal consecutive sequences of elements either entirely in H or entirely in L . Since such sequences from H must alternate with those in L , we have that there are the same number of each and we call this number the *alternations* of $[\pi]$ and denote it by a . Note that a must be at least 1 since we assumed both B_1 and B_2 were non-empty, and furthermore we know that a cannot exceed the number of cyclic pinnacles in H since every consecutive string of elements in H is adjacent to two values in L and so must contain

at least one cyclic pinnacle of $[\pi]$. Since every cyclic pinnacle element from H must either be one of the b pinnacles from B_2 , $n + 1$, or ∞ , we have an upper bound of $b + i + 1$ for a .

Therefore, if we fix some $1 \leq a \leq b + i + 1$, we may reduce to proving that the number of permutations in $cP(B)_j^i$ with a alternations is $|P(B_1)_j^{a-1}| |P(B_2)_a^i|$ since the values of a clearly partition all permutations in $cP(B)_j^i$.

We prove this by constructing a bijection. An example follows this proof. Let $l = |B_1|$, $h = |B_2|$, and suppose for the forward direction that $[\pi] \in cP(B)_j^i$ has a alternations. Then in $[\pi]$ we may replace the block of consecutive elements in H containing ∞ with the element ∞ and each of the other $a - 1$ blocks of consecutive elements in H with the element $l + 1$ to get $[\pi_L]$. Additionally, we may take the original $[\pi]$ and replace each of the a blocks of consecutive elements in L with the element 0 and subtract l from all remaining elements to get the cyclic permutation $[\pi_H]$. We must now show that $[\pi_L] \in cP(B_1)_j^{a-1}$ and $[\pi_H] \in cP(B_2)_a^i$.

From the construction given, it is easy to see that $[\pi_L]$ is a permutation of the elements $\{1, \dots, l\} \cup \{(l + 1)^{a-1}, \infty, 0^j\}$ and that $[\pi_H]$ is a permutation of the elements $\{1, \dots, h\} \cup \{(h + 1)^i, \infty, 0^a\}$. Therefore, we need only show that all the necessary cyclic pinnacles and vales are formed in both cases.

Consider $[\pi_L]$. Clearly we must have that every element in $\{(l + 1)^{a-1}\}$ is a cyclic pinnacle since they are larger than all elements in $\{0, 1, \dots, l\}$, and since none of them are adjacent to each other or to ∞ because everything in $\{(l + 1)^{a-1}, \infty\}$ originated from separate blocks of elements in $[\pi]$. Furthermore, given any element $x \in L$ the relative order between x and the elements on either side is the same in $[\pi_L]$ as it was in $[\pi]$, since the only change was that an element larger than x may have been replaced by another element larger than x . Therefore, since all elements in P_{B_1} and $\{0^j\}$ were cyclic pinnacles and vales respectively in $[\pi]$, they will continue to be so in $[\pi_L]$, and so we are done. The proof for showing $[\pi_H] \in cP(B_2)_a^i$ is almost identical, and therefore omitted.

Therefore, our map is well-defined. To invert it, suppose that $[\pi_L] \in cP(B_1)_j^{a-1}$ and $[\pi_H] \in cP(B_2)_a^i$. To combine them, first add l to all elements of $[\pi_H]$ to get $[\pi'_H]$. Then write $[\pi_L]$ in the form $[\infty L_1(l + 1) L_2(l + 1) \cdots (l + 1) L_a]$ where each L_i is a maximal consecutive sequence of

elements less than $l + 1$. Similarly, we may write π'_H in the form $[H_1 l H_2 l \cdots l H_a l]$ where each H_i is a maximal consecutive sequence of elements greater than l , and where H_1 contains ∞ . We then construct the cyclic permutation $[\pi] = [H_1 L_1 H_2 L_2 \cdots H_a L_a]$. It is easy to verify that $[\pi]$ contains j copies of the element 0, i copies of the element $n + 1$, and that all other elements are either ∞ or in B . Furthermore we note that given any element x in $[\pi]$, the relative sizes of x and its two adjacent elements will be the same as they were for x back in either $[\pi_L]$ or $[\pi_H]$. Therefore, all cyclic pinnacles and vales in $[\pi]$ are directly derived from those in $[\pi_L]$ or $[\pi_H]$. This means that all 0's will be vales, all elements of size $n + 1$ will be cyclic pinnacles, and all other cyclic pinnacles will correspond to a cyclic pinnacle in either B_L or B_H . Therefore, $[\pi] \in P(B)_j^i$.

Since these maps are clearly inverses step by step, we have our bijection, and therefore our result. $\square$

As an example of this process, consider $B = [00101001]$ where $B_1 = [001]$ and $B_2 = [01001]$. Then we have that $P_B = \{3, 5, 8\}$ while $P_{B_1} = \{3\}$ and $P_{B_2} = \{2, 5\}$ which correspond to the elements 5, 8 in P_B if you add $|B_1| = 3$. We may then consider the permutation $[\pi] = [\infty 0523081964097] \in {}_c P(B)_3^2$. If we write in boldface those elements of $[\pi]$ that are in set L defined in the proof, we have

$$[\pi] = [\infty \mathbf{0523081964097}]$$

and now it is easier to see that there are 4 alternations were we switch from an element of L to one in H and then back. Then we have

$$[\pi_L] = [\infty 042304140]$$

and

$$[\pi_H] = [\infty 02050631064]$$

which we get by standardizing

$$[\infty 05080964097]$$

This recursion is very useful, as it allows us to give an efficient method for calculating $p_n(P) = |\{\pi \in \mathfrak{S}_n \mid \text{Pin } \pi = P\}|$ using $O(k^2 \log n + k^4)$ arithmetic operations where $k = |P|$. It also allows us to derive results on the tree representation of pinnacles sets, which we do later.

For a pinnacle set represented by block B , we will call B a *segregated* block if it takes the form $B = [0^x, 1^y]$ in which we allow x and y to potentially be zero. That is to say, B must consist of some number of 0's followed by some number of 1's with no other alternations. Our method to calculate $p_n(P)$ has two parts. First, we will present an efficient formula for calculating $p(B)_j^i$ where B is a segregated block. We will then show that by using the recursion above, we may always write $P(A)$ for any block A in terms of these numbers, thereby giving a formula for the generic case.

Lemma 3.5.2. *Let $B = [0^x, 1^y]$ be a segregated block where $x + y = n$. Then if $S(n, m)$ is the Stirling number of the second kind, we have*

$$p(B)_j^i = 2^{x+j-c} \frac{c!(c-1)!}{i!} \sum_{m=0}^j \frac{1}{m!} \binom{c-m}{j-m} S(x, c-m)$$

where $c = i + y + 1$ is the number of cyclic pinnacles.

Proof. Given $\pi \in P(B)_j^i$, we may break π down into a sequence of pinnacles in the set $P = P_B \cup \{(n+1)^i\}$ and a sequence of intermediate permutations, where each intermediate permutation is defined to be a maximal set of consecutive elements in π consisting of no pinnacles. Furthermore, we know that the pinnacles must alternate with the intermediate permutations so that π takes the form $I_1 p'_1 I_2 p'_2 \cdots I_{y+i} p'_{y+i} I_c$ where I_m are the intermediate permutations, $P = \{p'_1, \dots, p'_{y+i}\}$, and $c = i + y + 1$ can be viewed as the number of cyclic pinnacles if we count ∞ . Therefore, since every pinnacle is larger than every non-pinnacle, it will be the case that given any ordering of the pinnacles P and any set of c nonempty intermediate permutations, every ordering of those intermediate permutations, when interleaved with the pinnacle elements, will form a permutation in $P(B)_j^i$. There are $(c-1)!/i!$ ways to order the pinnacle elements since i of them are indistinguishable, and so to get our result it is enough to show that the number of ways of generating and ordering c non-empty intermediate permutations is

$$2^{x+j-c} c! \sum_{m=0}^j \frac{1}{m!} \binom{c-m}{j-m} S(x, c-m)$$

Note that since intermediate permutations cannot contain pinnacles, they must contain exactly one cyclic vale, which will always be the smallest element. Additionally, we must have that each intermediate set can contain at most one 0, since containing more would force one of the 0's to not be a vale, which breaks the requirement of $P(B)_j^i$. Therefore, we may partition all possible sets of intermediate permutations by the number of permutations m that contain only the element 0. It then follows that each of the $c - m$ other intermediate permutations must contain at least one element of $[x]$, and will therefore be distinguishable. So for a given m the number of distinct orderings of a given set of intermediate permutations is $c!/m!$, as specified in the formula.

Now given an intermediate permutation, let its intermediate set be the corresponding set of elements in the permutation, which will have no repeats since we have already shown that the element 0 cannot appear twice. If an intermediate permutation has l elements, then we may assign $l - 1$ of them a left or right label, corresponding to which side of the smallest element they appear. Alternatively, if we start from an intermediate set and give all elements except the smallest a left/right label, then there will be exactly one way to construct an intermediate permutation from it. Namely, all left labeled elements will form a decreasing string to the left of the smallest element, and all the right labeled elements an increasing string to the right. Therefore, counting intermediate permutations can be reduced to counting intermediate sets and multiplying by a power of 2. Since every collection of intermediate sets will have $x + j$ elements total, and since all of them except the c cyclic vales will be given a label, this power of 2 will be as stated in the above relation.

Therefore, we have reduced to the case of counting the number of ways of partitioning the $x + i$ non-pinnacles into c non-empty intermediate sets such that no set contains more than one 0. This will force exactly j of the intermediate sets to contain a 0, and so all that remains is to place the final x elements in the set $[x]$. If we again fix m to be the number of intermediate sets containing only 0, there will be $\binom{c-m}{j-m} S(x, c-m)$ ways to create the remaining $c - m$ intermediate sets, namely, the number of ways of first partitioning the elements of $[x]$ into $c - m$ non-empty sets, and then

choosing $j - m$ of those sets to contain the remaining 0's. Since m can be at most j , and since m partitions all possible sets of intermediate sets, we may sum over m to get the final part of our formula, which completes the proof. $\square$

We can now prove our result.

Theorem 3.5.3. *The recursion in Theorem 3.5.1 together with the result in Lemma 3.5.2 is enough to compute $p(A)$ for a general block A of length n corresponding to a (not necessarily admissible) set P . Furthermore, if $|P| = k$ then this can be done using no more than $O(k^2 \log n + k^4)$ arithmetic operations.*

Proof. First we will show that the recursion and above lemma are enough to compute $p(A)_j^i$, which will then imply the result for $p(A) = p(A)_0^0$. Afterwards we will consider the run time.

Suppose that 0^x is the maximal sequence of 0's at the start of A , and that 1^y is the maximal sequence of 1's directly following it. Then we may write A as the block concatenation BA' where $B = [0^x, 1^y]$ is a segregated block and A' is the rest of block A . Note that A' must either be empty, or have fewer 1's in it than A since B will always take at least the first 1 that appears in A .

If A' is empty, then we have that A started segregated and we are done by Lemma 3.5.2. If not, then we may apply Theorem 3.5.1 to write $p(A)_j^i$ in terms of $p(B)_j^{i'}$ and $p(A')_j^{i'}$, the former of which can be computed directly from Lemma 3.5.2 and the later of which can be further broken down using Theorem 3.5.1 again. Since we know that A' must either be empty or have fewer 1's than A , we will only have to repeat this recursion a finite number of times before $p(A)_j^i$ is written entirely in terms of numbers that Lemma 3.5.2 can calculate, giving us our final answer.

To prove the bound for the run time, suppose that at each step the block break down was $A = A_0 = B_1 A_1$, $A_1 = B_2 A_2$, and so on until we have $A_{d-1} = B_d A_d$ where A_d is the first A_l to be empty. Since each B_l contains at least one 1, we have that d is bounded by k , the number of pinnacles.

It will be useful to find an upper bound on the different values for i and j that might appear in our calculations. Note that every time we apply the recursion to $p(A_l)_j^i$, we have that i must be

zero. This is because $i = 0$ initially when computing $p(A)_0^0$, and we only ever apply the recursion again on the second factor, which has the same i as at the start. However, the first factor will take the form $p(B_{l+1})_j^{i'}$ where i' ranges over the values 0 to b where b is the number of 1's in A_{l+1} . Therefore in the worst case, i' may reach a maximum of $k - 1$ when using Lemma 3.5.2 to compute $p(B_1)_j^{i'}$ and so in all cases will be bounded by k . Now suppose that we apply our recursion to $p(A_l)_j^i$ and let j' be the maximum value of j that can arise on any factor of the form A_{l+1} in the sum. We have that j' is at most $\max\{j, b + 1\}$ where $b + 1$ is maximized on the first step when b is the number of 1's in A_{l+1} , which is at most $k - 1$. Since j starts out at zero, and the value of b falls each step, we have that the maximum value that j can be is k .

We now wish to find the total run time of using Lemma 3.5.2 to calculate all the $p(B_l)_j^i$. First, let n_l be the number of 0's in block B_l and note that each n_l is clearly bounded by n , which is the original length of A . Then note that the quantity $c - m = y + i + 1 - m$ in Lemma 3.5.2 can be at most $k + 1$ since y is the number of 1's in the block B_l while i can be at most the number of ones in the block A_l , which totals to no more than k . Therefore, we only have to calculate Stirling numbers of the form $S(n_l, k')$ where l is bounded by k and k' is bounded by $k + 1$.

Using the formula

$$S(n_l, k') = \frac{1}{k'!} \sum_{d=0}^{k'} (-1)^{k'-d} \binom{k'}{d} d^{n_l}$$

we may first calculate all powers d^{n_l} where $d \in [k + 1]$ and $l \in [k]$ using fast exponentiation in at most $O(k^2 \log n)$ arithmetic operations. We may then calculate all binomial expressions of the form $\binom{k'}{d}$ with $k' \leq k + 1$ and $d \leq k'$ in $O(k^3)$ operations and store those too. From this information, we may calculate all Stirling numbers that will be needed in $O(k^3)$ additional time since the number of distinct n_l, k' values are bounded by $k + 1$. Furthermore, all other factorial and binomial expressions used in Lemma 3.5.2, along with the power of 2, can be calculated and stored at this point in no more than $O(k^2 \log n + k^3)$ time. Therefore, if we do this preparation first, we may calculate each $p(B_l)_j^i$ in $O(k)$ time from the stored data. Since i, j , and l are each bounded by k , there will be at most $O(k^3)$ distinct such terms, we have that a look up table for all the $p(B_l)_j^i$ can be generated in no more than $O(k^4)$ time.

Given this look up table, we may then compute and store the $p(A_l)_j^0$ in $O(k)$ algebraic operations each by starting with A_{m-1} and then accessing all the $p(B_l)_j^i$ and previously computed $p(A_l)_j^i$ to compute the next A_{l-1} . Since l and j are bounded by k , we have that this part of the algorithm takes at most $O(k^3)$ operations to get to $p(A_0)_0^0$, which is our final answer.

Therefore, the total run time is $O(k^2 \log n + k^3)$ to prepare for Lemma 3.5.2, plus $O(k^4)$ to create a look up table of the values in $p(B_k)_j^i$, plus a final $O(k^3)$ to use the recursion. Altogether, this results in a final run time of $O(k^2 \log n + k^4)$ as advertised. $\square$

Before concluding, we present one more application of our recursion where we prove a version of a formula conjectured by Falque, Novelli, and Thibon in [10]. In their paper, the authors note that there is a map between admissible pinnacle sets in $[n]$ and forests of complete binary trees on n vertices arranged as a sequence of trees from left to right, and such that the trees are labeled from left to right with nodes labeled in left suffix order. The map is given by the following algorithm.

1. Given some admissible pinnacle set $P = \{p_1 < p_2 < \dots < p_k\} \subset [n]$, start with the forest consisting of a single unlabeled node. Let $i = n$.
2. Label the rightmost unlabeled node i . If $i \in P$, give this node two unlabeled children. Otherwise, create a new tree with one unlabeled node to the left of all existing trees. Let $i = i - 1$.
3. Repeat the previous step until a node has been given the label 1, and then stop.

To reverse this map, we start with any complete binary forest with trees arranged in a sequence from left to right and with nodes labeled as specified above. We then simply let P be the set of labels of internal nodes. An example can be seen in Figure 3.4.

It is not hard to show that these maps are well defined and that they are inverses of each other. In what follows, we will use the letter F for forests, and T for single trees. We may then write an admissible pinnacle set as a tree sequence $(T_1, T_2, \dots, T_r)$. If a tree has only one vertex, denote it as

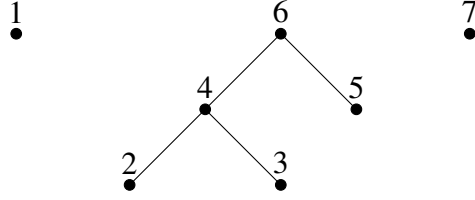


Figure 3.4 The complete binary forest corresponding to pinnacle set $P = \{4, 6\}$ for $n = 7$

O . We can also group consecutive trees together as an ordered forest; for instance, we may wish to say $(T_1, T_2, T_3, T_4) = (F_1, F_2)$ where $F_1 = (T_1)$ and $F_2 = (T_2, T_3, T_4)$.

Using this notion of trees, we may use the notation $p(F)$ to denote the number of permutations in $\mathfrak{S}_n$ having pinnacle set represented by (F) . It was conjectured in [10] that for pinnacle set $F = (O, T_2, T_3, \dots, T_r)$ we have the relation

$$p(F) = p(O, T_2)P(O, O, T_3, \dots, T_r).$$

Unfortunately, this is not true as can be seen by the counterexample when $n = 4$ and the pinnacles set is $\{4\}$. However, a very similar result can be proved.

Theorem 3.5.4. *For a sequence of trees $F = (O, T_2, T_3, \dots, T_r)$ encoding a pinnacle set, we have*

$$p(F) = \frac{1}{2}p(O, T_2)p(O, O, T_3, \dots, T_r).$$

Proof. Since the trees F are labeled from left to right, we have that all nodes in (O, T_2) will be smaller than all nodes in the rest of F . Therefore, we may consider a block composition B_1B_2 corresponding to this pinnacle set where B_1 corresponds only to the labels of the nodes in (O, T_2) while B_2 has the rest. Then by our notation, we will have that $p(B_1) = p(O, T_2)$ and $p(B_2) = p(T_3, \dots, T_r)$.

We then have by Theorem 3.5.1 that

$$p(F) = p(B_1B_2) = \sum_{m=0}^b p(B_1)^m p(B_2)_{m+1}.$$

Now note that if $m > 1$ we will have that $p(B_1)^m = 0$ since there will be too many pinnacles for the non-pinnacles to separate, and so this sum simplifies to

$$p(F) = p(B_1)p(B_2)_1 + p(B_1)^1 p(B_2)_2$$

First, note that $p(B_1) = p(B_1)^1$ since there is a simple bijection between the permutations they count. Namely, for any π counted by $p(B_1)$ there will always exist exactly one index where two non-pinnacles are adjacent and where we may insert a new largest element to get a permutation counted by $p(B_1)^1$. Similarly, we may reverse this bijection by deleting the largest element from some π' counted by $p(B_1)^1$. This will not create any new pinnacles because all non-pinnacles will still be adjacent to either the end of the permutation or to one of the other pinnacles.

We now consider a block B_3 corresponding to $(O, O, T_3, \dots, T_r)$, which will be B_2 with two zeros appended to the beginning. We claim that $p(B_3) = 2s$ where $s = p(B_2)_1 + p(B_2)_2$. To see this, note that every permutation counted by $p(B_3)$ will either have the 1 and 2 adjacent, or separate. If they are adjacent, replace both by a single 0 and subtract 2 from all other elements. This will not destroy any pinnacles since neither 1 nor 2 were originally pinnacles, and so the result will be a permutation counted by $p(B_2)_1$. Since this map is clearly reversible up to the ordering of 1 and 2, we also see that this is a two-to-one mapping.

On the other hand, suppose that 1 and 2 are not adjacent. Then it is easy to see, by reasons similar to above, that if we replace both 1 and 2 by separate 0's and then subtract 2 from all other elements we will end up with a permutation counted by $p(B_2)_2$. This will again be a two-to-one mapping and so we have that $p(B_3) = 2s$.

Therefore, we have

$$p(F) = p(B_1 B_2) = p(B_1)p(B_2)_1 + p(B_1)^1 p(B_2)_2 = p(B_1) (p(B_2)_1 + p(B_2)_2) = p(B_1) \frac{1}{2} p(B_3)$$

the last of which is equal to $\frac{1}{2} p(O, T_2) p(O, O, T_3, \dots, T_r)$ as desired. $\square$

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