STOCHASTIC AND DETERMINISTIC FINITE-TIME SYSTEM IDENTIFICATION

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ABSTRACT

Identifying a high-fidelity model of nonlinear dynamic systems is a prerequisite for achieving desired specifications in any model-based control design technique. This is because, most control design methods rely on the availability of an accurate model of the system dynamics and coarse dynamics models without generalization guarantees typically induce controllers that are either overly conservative with poor performance or violate spatiotemporal constraints imposed on the system when applied to the true system.

This dissertation investigates the finite-time identification of deterministic and stochastic systems. First in Chapter 2, a novel finite-time distributed identification method is introduced for nonlinear interconnected systems. A distributed concurrent learning (CL) based discontinuous gradient descent (GD) update law is presented to learn uncertain interconnected subsystems' dynamics by minimizing the identification error for a batch of previously recorded data collected from each subsystem as well as its neighboring subsystems. The state information of neighboring interconnected subsystems is acquired through direct communication. Finite-time Lyapunov stability analysis is performed and easy-to-check rank conditions on the distributed memories data of subsystems are obtained, under which finite-time stability of the distributed identifier is guaranteed. These rank conditions replace the restrictive persistence of excitation (PE) conditions which are hard and even impossible to achieve and verify.

Next, Chapter 3 presents a fixed-time system identifier for continuous-time nonlinear systems. A novel adaptive update law with discontinuous gradient flows of the identification errors is presented that leverages CL to guarantee the learning of uncertain dynamics in a fixed time. The CL approach retrieves a batch of samples stored in a memory and the update law simultaneously minimizes the identification error for current stream of samples as well as past memory samples. Fixed-time Lyapunov stability analysis certifies fixed-time convergence to the stable equilibria of the GD flow of the system identification error under easy-to-verify rank conditions.

In Chapter 4, an online data-regularized CL-based stochastic GD is presented for function approximation with noisy data. A fixed-size memory of past experiences is repeatedly used in the

update law along with the current streaming data to provide probabilistic convergence guarantees with much-improved convergence rates (i.e, linear instead of sublinear) and less restrictive data-richness requirements. This approach allows us to leverage the Lyapunov theory to provide probabilistic guarantees that assure convergence of the parameters to a probabilistic ultimate bound exponentially fast, provided that a rank condition on the stored data is satisfied. This analysis shows how the quality of the memory data affects the ultimate bound and can reduce the effects of the noise variance on the error bounds.

In Chapter 5, deterministic and stochastic fixed-time stability of autonomous nonlinear discrete-time (DT) systems are studied. Lyapunov conditions are first presented under which the fixed-time stability of deterministic DT systems is certified. Extensions to systems under deterministic perturbations as well as stochastic noise are then considered. For the former, the sensitivity to perturbations for fixed-time stable DT systems is analyzed, and it is shown that fixed-time attractiveness is resulted from the presented Lyapunov conditions. For the latter, sufficient Lyapunov conditions for fixed-time stability in probability of nonlinear stochastic DT systems are presented. The fixed upper bound of the settling-time function is derived for both fixed-time stable and fixed-time attractive systems, and the stochastic settling-time function fixed upper bound is derived for stochastic DT systems.

Finally, using the results of Chapter 5, in Chapter 6, a fixed-time identifier for modeling unknown DT nonlinear systems without requiring the PE condition is developed. A data-driven update law based on a modified GD update law is presented to learn the system parameters, which relies on CL. Fixed-time convergence guarantees are provided for the modified GD update law under a rank condition on the recorded data. To guarantee fixed-time convergence, fixed-time Lyapunov analysis is leveraged.

To my deceased father who was the best inspiration in my whole life,
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CHAPTER 1

INTRODUCTION AND LITERATURE REVIEW

System identification approaches are typically categorized into batch (offline) or incremental (online) identification methods. Batch identification relies on the availability of a rich set of samples that are collected offline. In offline learning settings, rich data are assumed to be available and collected a priori for learning. This includes system identification using the subspace method and its variants [1–5] and the RL value function learning using the least-square temporal difference (LSTD) leaning [6–9]. In both cases, the least-squares (LS) method [10,11] is most commonly used to estimate unknown parameters by minimizing the estimation error. In offline learning settings, rich data are assumed to be available and collected a priori for learning. This includes system identification using the subspace method and its variants [1–5] and the RL value function learning using the least-square temporal difference (LSTD) leaning [6–9]. In both cases, the least-squares (LS) method [10, 11] is most commonly used to estimate unknown parameters by minimizing the estimation error.

While classical offline learning methods provide asymptotic-sample guarantees (i.e, convergence to the actual parameters under infinite number of samples), finite-sample guarantees have been widely considered recently [1–3, 12–26]. These methods provide error bounds for every finite number of samples, only after a sufficiently large number of samples that satisfy the PE condition. Existing finite-sample results for offline system identification are typically limited to linear systems, as they are inspired by the subspace method, in which the linear dynamics structure is used to constructs a Hankel matrix from the input–output pairs. Finally, the offline setting does not comply with adaptive control settings for which the data samples are streaming and rich PE data are not available a priori. To adapt to a new situation in adaptive control settings, offline or least-square methods are not satisfactory, since one has to compute the new estimate from the scratch after a new sample becomes available. To circumvent this issue, recursive methods leverage the new data and modify online the immediately past estimate accordingly. The LS estimate for linearly parameterized approximators (e.g. linear in parameter system identification [27] and linearized parametrization of value functions in RL using a single-layer network [28]) can be derived in a recursive manner. However, in general, the recursive implementation of LS estimate for nonlinear systems is a daunting challenge.

However, samples must satisfy restrictive independent and identically distributed (i.i.d) conditions which are hard or even impossible to verify and obtain in closed-loop control systems. Moreover, offline learning cannot account for changes in system dynamics. On the other hand, online learning provides a framework to learn a system model on the fly and using the stream of data collected from the system dynamics in real-time. Nevertheless, restrictive persistence of excitation (PE) conditions [10, 29] must be satisfied to guarantee parameter convergence (which turns to assure generalization guarantee). Satisfying and verifying PE conditions in real-time pose limitations on certifying parameter convergence of online system identifiers. Moreover, parameter convergence guarantees are mainly achieved asymptotically or exponentially.

Concurrent learning (CL) technique has been leveraged to relax the PE condition [30–43]. Chowdhary et al. [35, 36] presented a CL update law for adaptive control systems in which the identification error is minimized for not only current samples but also a batch of recorded samples. An easy-to-verify condition on the richness of data is then derived to guarantee the exponential parameter error convergence, which replaces the restrictive PE condition with a rank condition on the recorded samples. In CL methods, past recorded data are replayed along with the current stream of data in the update law to not only minimize the identification error for the current data but also for the batch of recorded data. CL has been recently extended to adaptive control [33, 44, 45], optimal and robust control [39, 46] networked control [32, 38, 47], continuous and discrete-time system identification [37, 40–42, 48]. In the most of previously mentioned studies, the asymptotic convergence of the estimated parameters is guaranteed under an easy-to-verify rank condition rather than PE condition. Recently, a few CL-based methods [40–43] provided the finite-time convergence for the estimated parameters. However, all the aforementioned identification approaches are dealing with identification of a single dynamic system.

A class of nonlinear multi-agent systems is interconnected systems which are composed of several (possibly heterogeneous) physically connected subsystems influencing each other's behavior. Numerous engineering systems with practical relevance belong to this class of systems, including intelligent buildings, power systems, transportation infrastructure and urban traffic systems. Typically, distributed control and monitoring methods for interconnected systems rely on high-fidelity models of the subsystems. Designing controllers based on coarse dynamic models and without generalization guarantees may induce closed-loop systems with poor performance or may even result in instability. Moreover, failure in accurate and timely identification of the dynamics of a single subsystem may snowball into an entire network instability due to the physical interconnections among subsystems.

However, identifying the dynamics of interconnected systems is challenging due to the physical interconnections among the subsystems. This makes the existing system identification methods for single-agent systems not directly applicable to interconnected systems. Developing system identifiers with finite-time guarantees for interconnected systems is of utmost importance in practice, since it allows the designer to preview and quantify the identification errors. The preview and qualification of the error bounds can in turn be leveraged by the control and/or monitoring systems to avoid conservationism. Otherwise, the conservationism introduced due to slow or asymptotic convergence can degrade the interconnected system performance.

Different types of multi-agent systems' learning approaches, classified as centralized, decentralized, and distributed identification methods, typically employed in control of multi-agent systems [49–53], can be adopted to identify interconnected system dynamics. Centralized identification methods rely on the existence of a learning center that receives data from all subsystems and identifies the dynamics of the entire network. The centralized approach, however, comes at a high computation and communication cost and requires access to the global knowledge of the subsystems' interconnection network. By contrast, in the decentralized learning, an independent identifier is allocated for every subsystem which only relies on the subsystem's own information to identify its dynamics. Since there is no exchange of state information among the subsystems, decentralized identifiers are unable to identify the interconnection terms in the dynamics of subsystems. On the other hand, distributed learning methods can accurately identify the interconnected system dynamics by employing a local identifier for every subsystem while allowing it to communicate its state information with its neighboring interconnected identifiers. In contrast to the centralized method, in the distributed identification approach, no access to the global knowledge of the interconnection network is required.

The distributed identification of interconnected systems can be performed either online or offline. Generally, in offline (batch) distributed identification [54], a rich batch of data must be collected from each subsystem and its neighboring subsystems to provide high confidence generalization guarantees across the entire operating regimes of subsystems [12–16]. In the batch learning, where finite-time or non-asymptotic convergence refers to generalized guarantees provided by finite number of samples, satisfying the condition of independent and identically distributed (i.i.d) is difficult to obtain and hard to verify in closed-loop interconnected systems. On the other hand, online distributed identification, which is the problem of interest in this dissertation, uses online data from each subsystem and its neighboring subsystems to learn the dynamics of interconnected system in real time. Nevertheless, standard approaches for online identification require the restrictive persistence of excitation (PE) condition [10, 29] to ensure generalization and exact parameter convergence. This includes online identification of interconnected systems using both decentralized [55, 56] and distributed [57, 58] learning approaches. The PE condition, however, is hard to achieve and to verify online and its satisfaction is much more challenging for interconnected subsystems compared to single-agent systems. This is because the regressor's PE condition for a subsystem not only depends on the richness of its own data but also the interactive data collected from its interaction with its neighboring subsystems.

To satisfy the regressor's PE condition in interconnected systems, all subsystems must synchronously inject probing noises into their control systems to excite their dynamics and consequently to produce rich data for the entire network of subsystems. Designing such a probing noise for every subsystem to collectively satisfy the regressors' PE conditions for all subsystems while not jeopardizing the overall system stability is a daunting challenge due to the subsystems' interconnectivity: the probing noise can snowball in the entire network and lead to the system instability. Therefore, designing an identification method for interconnected systems without requiring restrictive PE conditions, for which their satisfaction can deteriorate the system's stability and performance, is of vital importance.

For interconnected systems, the identification error dynamics are only guaranteed to be locally uniformly ultimately bounded [55–58].

Based on the concept of finite-time stability [59], several finite-time control methods have been developed for output feedback control [60] and multi-agent system consensus [61, 62]. In finite-time control design methods, the controllers are designed to guarantee finite-time stability of the system dynamics or tracking error dynamics where either no learning is accomplished or some observers are used along with identifiers whose identification precision is not taken into account; therefore, there is no requirement on the data richness.

Moreover, several distributed asymptotic-convergent estimators have been designed in [63, 64] to estimate the system states or a specific parameter for multi-agent systems with known dynamics for which the learning objectives and therefore the rich data recording do not exist. In contrast, in the multi-agent system identification, a precise model of the system is not available and the richness of the employed data affects the identification results. For the multi-agent system identification, specifically interconnected systems, finite-time approaches are essential to assure collecting rich data to identify the system dynamics in finite time. However, finite-time identification of interconnected systems is unsettled.

Therefore, in the second chapter of this dissertation, we aim to identify the interconnected system dynamics in finite time by proposing a novel distributed discontinuous CL-based estimation law without requiring the standard regressors' PE condition.

In finite-time CL-based system identifiers [40,41], the convergence settling-time is a function of initial parameters' estimation error and varies with initial parameter estimation variations. In finite-time convergence, the amplitude of the initial parameter estimation error is of great importance because if the initial error is not bounded, then it is hard to guarantee convergence of the parameters to their true values in a limited time. Moreover, the settling time of their convergence depends on the initial parameters' estimation error and thus cannot be computed a priori since the true values of the system parameters are unknown.

In practice, having an accurate fixed-time identification method, for which the convergence time is independent of the initial errors is of utmost importance and allows to preview and quantify the identification errors, which can be leveraged by the control system to avoid conservatism.

Based on the notion of fixed-time stability [65], various fixed-time control methods are extensively developed in neural control [66], event/self-triggered consensus [67], team-triggered consensus [68], and prescribed performance control [69, 70]. It is worth noting that in fixed-time control design methods, the controller is designed to assure fixed-time tracking error or stability for known system dynamics: No learning is taken placed and conditions on the richness of data are therefore not required. In sharp contrast, system identification requires learning unknown dynamics while the controller is usually not designed for the sake of identifying dynamics. Therefore, designing data-efficient system identifier that requires limited access to samples collected from system dynamics is of vital importance.

Fixed-time observer-based controllers [66] and observers [71–76] are investigated to estimate the system states [66, 71–73], disturbance [74, 75] and uncertainty [76] where the settling time usually depends on the observer gains satisfying a Hurwitz condition on the observer gains matrix. Moreover, a high-fidelity model of the system is assumed to be known in existing fixed-time observers and/or the controller is designed to achieve fixed-time convergence. Therefore, the problem of learning and rich data collection does not appear in these approaches. However, in sharp contrast, in system identification, neither a high-fidelity model of the system is available, nor the controller is usually designed for the sake of learning the dynamics. Therefore, novel approaches are required to assure collecting rich data for identifying system dynamics in fixed time, which is surprisingly unsettled.

Although [77] and [78] presented fixed-time identification methods, they rely on the PE condition which is hard to verify and certify in real-time. The authors in [79] and [80] introduced two short fixed-time stable parameter estimation algorithms by relaxing the PE condition to an interval excitation condition; however, short fixed-time stability which ensures stability in a finite interval of time, is a weak form of fixed-time stability. A fixed-time convergent method for time-varying parameter identification is given in [81] that requires an analogous condition to the PE condition called injectivity which requires the minimum singular value of the regressor to be always strictly positive. Furthermore, in [81], the learning rate must satisfy some constraints and to check these constraints the minimum and maximum singular values of the regressor and the upper bound of the unknown parameters are needed to be known which are hard to compute and check online.

Despite, the emergent need for an accurate fixed-time identification method in practical cases, to the best of our knowledge, no fixed-time identification algorithm without requiring the PE condition is presented in the literature. Therefore, in chapter 3 of this dissertation, we present a fixed-time identification algorithm independent of the initial estimation errors that eliminates the PE condition by integrating the capabilities of CL and fixed-time learning. The novel idea of employing the recorded data in a discontinuous gradient flow along with the current data in the update law has overcome the challenge of proposing a fixed-time CL method without requiring a PE condition.

In control theory, function mapping in the presence of noisy data has been a long-standing challenge and amounts to a joint optimization over data-richness satisfaction and function error reduction. This is because, learning a set of parameters by minimizing a loss function does not necessarily minimizes the expected parameter estimation error, unless a set of rich data is used for learning. For instance, in system identification, for which the aim is to learn the unknown dynamics of a system from collected data, to ensure the system parameters' convergence to their actual values, data samples must be persistently exciting (PE) [10]. Otherwise, a set of system parameters is learned without any convergence guarantee, even though the estimation error for the set of collected data is minimized. The parameters convergence cannot be guaranteed because the set of collected data used for leaning the system parameters is not a good representative of the entire state space. As another example, the PE condition over collected data must be satisfied in reinforcement learning (RL) to assure convergence of the RL agent to an optimal policy that minimizes the cumulative cost of control actions [82]. In online settings, simultaneous satisfaction of the PE condition and learning of the function parameters requires solving a joint optimization over the data and the function error to assure rich data collection and learning an optimal set of parameters, respectively.

A popular approach for online learning for streaming settings is the stochastic gradient descent (SGD) method [83–88]. However, when the data have temporal dependencies, as shown in [84], naive implementation of SGD does not show a satisfactory performance. The data-drop technique drops a large number of samples from the stream to obtain nearly independent samples [84]. However, there is no systematic approach to check whether the stream data are independent and can provide convergence guarantees and which samples to drop or collect to satisfy the data richness conditions, which can result in wasting a lot of samples.

Recently, several studies have focused on accelerating SGD methods for linear regressions corrupted by noise. In [89], a projected SGD based algorithm with weighted iterate-averaging is presented. The convergence rate, however, is sublinear and the function under optimization is assumed to be strongly convex. A high-order tuner is presented in [90] for time-varying regressors that guarantees exponential convergence of parameter estimates to a bound depending on the noise statistics. Nevertheless, a regularization term is added to penalize the deviation of the parameters from their initial values, which can lead to a bias from the optimal value. To overcome the problem of highly correlated steaming data, a SGD with reverse experience replay is developed in [84] that divides data into small buffers and runs SGD backwards on the data stored in the individual buffers. This method guarantees a sublinear convergence rate for linear regressors. A non-asymptotic convergence analysis of a variant of SGD is presented in [91], in which the learning rate is selected according to the expected data streams to improve the convergence rate. In [92] a stochastic average gradient method is presented for optimizing strongly convex functions to achieve a linear convergence rate. The work of [93] leverages the importance sampling approach to improve the convergence rate of the SGD. Most of these existing results are presented for finite training sets for which the loss function is sum of a finite set of strongly convex functions. However, as shown later, for online time-varying regression, under which the data samples are streaming, the strong convexity is satisfied under the PE condition on the streaming data. Besides, in existing mentioned results, the bounds on the smoothness of the function and Lipschitzness of its gradient are assumed fixed. In sharp contrast, we aim to pave the way to change these bounds and thus to improve the convergence rate and reduce the ultimate bound of the parameters' estimation error while reducing the PE condition to a rank condition on the stored data. Therefore, in chapter 4 of this dissertation, an online data-regularized concurrent learning-based stochastic gradient descent (CL-based SGD) update law is presented for function approximation with noisy measurements.

The Lyapunov stability theory has a longstanding history as a powerful tool in control theory to obtain many important results in the design of a variety of controllers and adaptation laws. The basic framework of the Lyapunov stability theory provides conditions under which their satisfaction guarantees the stability of the system in some sense. While finding a function satisfying these conditions, called Lyapunov function, is generally challenging, controllers and update laws can be developed to make a candidate Lyapunov function enforce the stability conditions.

The Lyapunov theory generally provides conditions to assure the states of a system convergence to an equilibrium state. The qualitative guarantees that are provided for the convergence time determine the stability type, ranging from asymptotic stability, exponential stability, finite-time stability to fixed-time stability. While asymptotic stability and exponential stability provide assurance that the system's states eventually converge to an equilibrium, many real-world practical systems demand intense time response constraints, which makes these types of stabilities insufficient. Therefore, a surge of interest has emerged in the control community in studying finite-time stability to design control systems and adaptation laws that exhibit finite-time convergence to an equilibrium point.

Finite-time stability [59] has been studied for continuous-time (CT) and discrete-time (DT) deterministic and stochastic systems [94–97]. Moreover, finite-time stability concept has been extensively applied for the finite-time control of DT [98–100] and CT [101–103] systems, as well as finite-time identification [40–42, 104–110]. In the finite-time stability, however, the settling (i.e., convergence) time, depends on the system's initial condition, and, thus, cannot be specified a priori. Moreover, when the magnitude of the initial condition is large, it can lead to an unacceptable convergence time guarantee. Fixed-time stability, on the other hand, imposes a stronger requirement on the settling time, because it requires convergence guarantees with a pre-specified bound on the settling-time function, independent of the initial condition. Fixed-time stability of deterministic and stochastic CT systems, respectively, studied in [65] and [111], have been widely studied within the frameworks of fixed-time control design [66–70, 112–116], fixed-time observer design [71–76] and fixed-time identification [43, 77–81].

While most real-world systems are CT in nature, DT systems are of great importance since systems are typically discretized and controlled with digital computers and micro-controllers in real-world applications. DT Lyapunov analysis is different from its CT counterpart and the analysis applied for CT systems' fixed-time stability can not be employed for DT systems. Moreover, development of Lyapunov conditions that guarantee fixed-time stability of DT deterministic and stochastic systems is challenging due to the requirement of having a fixed upper bound for the convergence time. This fixed-time bound represents a priori computable time of convergence independent of the initial conditions. Even though finite-time stability of DT deterministic [42, 97, 117] and stochastic [118, 119] systems are recently studied, fixed-time stability of DT deterministic and stochastic systems is surprisingly unsettled, despite its practical importance. This gap motivates us to present fixed-time Lyapunov stability conditions that pave the way for the realization of fixed-time control and identification of DT systems through designing appropriate controllers and adaptation laws, respectively.

Lyapunov theory can also be leveraged to study the behavior of uncertain systems. There are typically two types of uncertainties in control systems: randomness which is caused by a noise in a stochastic system, and deterministic unknown perturbations with known bounds (here, we call the deterministic systems affected by deterministic perturbations as perturbed deterministic systems). The stability results are typically presented in terms of stability in probability for stochastic systems' stability [94,95,111,119,120], which guarantees convergence in probability to an equilibrium point, and in terms of attractiveness to a bounded set for perturbed systems. Thus, in Chapter 5 of this dissertation, we develop fixed-time stability conditions for both deterministic and stochastic DT autonomous nonlinear systems.

To relax the PE condition, concurrent learning (CL) has been widely leveraged [36] [32, 35, 37–39]. In this approach, the identification error is minimized for not only current samples but also a set of recorded samples. using recorded past data during learning allows us to replace a verifiable rank condition on the memory data with the strong PE condition. The convergence guarantees, however, are limited to the exponential or asymptotic convergence of parameters' errors. Besides, most of these results are presented for the identification of continuous-time systems. Nevertheless, in practice, due to the employment of digital computers for controlling the systems, a discrete-time model is typically needed.

Despite its importance, few results are available on the identification of discrete-time systems [42, 48, 108, 109, 121, 122]. To improve the convergence, the work of [108] presented a finite-time identifier for discrete-time systems. However, it requires online invertibility of a regressor matrix and its inverse computation, which makes it inapplicable for online learning of a large number of unknown parameters' identification. The work of [121] presented an estimation method using dynamic regressor extension and mixing for both continuous-time and discrete-time systems; however, their results on finite-time convergence are limited to continuous-time systems. The work in [48] presented a concurrent learning-based function approximator for discrete-time systems without the PE condition requirement and ensured the asymptotic convergence of the estimated parameters. The authors in [122] presented a framework for processing gradient algorithms where finite-time algorithms are given using nabla fractional-order calculus. In [122], time-varying learning rates that reach zero along with converging to the optimal solution are employed to converge to the optimal solution regardless of the initial conditions. Although, no fixed time of convergence and fixed-time Lyapunov analysis are given in [122]. The works of [42] and [109] proposed finite-time CL identifiers for discrete-time systems' dynamics identification where rigorous finite-time Lyapunov analysis guaranteed finite-time convergence.

While finite-time identifiers have significantly improved the convergence time of classical system identifiers that rely on standard gradient descent, the settling-time upper bound in these methods is a function of the initial parameters' estimation error. Therefore, the settling time of convergence becomes unbounded as the initial condition's norm approaches infinity. Moreover, in finite-time convergence, a bound for the settling time cannot be computed because it would depend on the unknown true values of the system parameters. Therefore, it is of vital importance to develop a fixed-time identification method in which the settling-time function upper bound is independent of the initial errors. This will allow us to quantify the identification errors over time, which leads to less conservative control design methods that rely on the fixed-time identified system model. This motivates us to propose an online identifier for discrete-time systems with guaranteed fixed-time convergence properties using CL with a rank condition on the memory data which eliminates the requirement of restrictive PE condition.

Although fixed-time controls, identifiers, and observers have been extensively employed for continuous-time systems [43,66,68–70,72,73,123–127], fixed-time methods for discrete-time systems are generally unsettled due to the lack of fixed-time stability analysis of discrete-time systems. The extension of fixed-time stability analysis from continuous-time systems to discrete-time systems is far from trivial. Recently, we presented fixed-time stability for stochastic and deterministic discrete-time systems in [128], which opens the door to developing fixed-time learning algorithms. Therefore, it is desirable to present a fixed-time learning method that can eliminate the restrictive PE condition for discrete-time systems' identification due to the need for an accurate fixed-time identification method in real-world applications. Therefore, in Chapter 6 of this dissertation, a fixed-time concurrent learning (FxTCL) algorithm for discrete-time systems is presented to 1) ensure fixed-time parameter convergence independent of the initial estimation errors and 2) relax the PE condition to a rank condition on the recorded data using CL.

1.1 Organization of the dissertation

Based on the above-elaborated problems, the brief contribution and organization of this dissertation are as follows.

Chapter 2 presents a novel distributed discontinuous CL-based estimation law without requiring the standard regressors' PE condition, to identify the interconnected system dynamics in finite time. To this end, a distributed finite-time identifier is allocated to every subsystem that leverages local communication to not only learn the subsystem's own dynamics but also the interconnected dynamics based on its own state and input data, and its neighbors' state information. Moreover, in order to relax the regressors' PE condition and guarantee finite-time convergence, a discontinuous distributed CL-based gradient descent update law is presented. Using the presented update law, every local identifier minimizes the identification error at the current time based on the current stream of data from its own state and that of its neighbors as well as the identification error for data collected in a rich distributed memory. The dynamics of the gradient flows are analyzed using finite-time stability and it is shown that for every subsystem an easy-to-verify rank condition on the matrix containing the recorded filtered regressor data (that is used to avoid state derivative measurements) is sufficient to ensure finite-time convergence. Two different cases are considered in this chapter: 1) Realizable system identification for which there is a set of model parameters that can make the identification error zero. That is, the minimum functional approximation error (MFAE) is zero and is realized by an optimal set of unknown system parameters; and 2) non-realizable system identification for which there are no model parameters that result in zero identification error. For case 2, the subsystems have mismatch identification errors and their MFAEs are nonzero. In both cases, linearly parameterized universal approximators such as radial basis function neural networks are used to model the uncertain system functions. It is shown that under a verifiable rank condition, the proposed approach results in finite-time zero identification error for case 1 (which is a special form of case 2) and finite-time attractiveness to a bound near zero for case 2.

Chapter 3 presents a a fixed-time identification algorithm independent of the initial estimation errors that eliminates the PE condition by integrating the capabilities of CL and fixed-time learning. The novel idea of employing the recorded data in a discontinuous gradient flow along with the current data in the update law has overcome the challenge of proposing a fixed-time CL method without requiring a PE condition. Here, by leveraging the CL technique, unlike [81], no persistence of excitation or injectivity condition on the regressor and no upper bound knowledge of the unknown parameters is required for fixed time convergence. In the proposed fixed-time concurrent learning (FxTCL), the settling time is independent of the initial parameter estimation error. Therefore, given the recorded data, the settling time of convergence can be computed a priori regardless of the initial parameter estimation error. Consequently, the presented FxTCL update law guarantees learning a high-fidelity model of the system with a priori computable and fixed time of convergence. A fixed settling time of convergence for the identifier provides a priori computable convergence bound. This, in turn, allows quantifying system uncertainty for the control design and provides mechanisms to avoid designing overly conservative controllers caused by long-lasting large model estimation errors.

Chapter 4 presents an online data-regularized concurrent learning-based stochastic gradient descent (CL-based SGD) update law is presented for function approximation with noisy measurements. Inspired by the concurrent learning for deterministic settings, a novel parameter estimation update law is presented that replaces the typical gradient estimation methods with a memory-augmented gradient update law. That is, the gradient update law not only minimizes the current estimated estimation error but also the estimation error for past historic data stored in a fixed-size memory. This is in sharp contrast with mini-batch SGD in which a mini-batch of data are randomly selected to estimate the noisy gradient. Using the Lyapunov theory, probabilistic guarantees are provided for the parameters estimation errors, provided that a rank condition on the stored data is satisfied. It is also shown that the parameters estimation errors converge exponentially to a probabilistic ultimate bound. The ultimate bound depends on the noise variance of the function approximation as well as approximation error and richness of the recorded memory data.

Chapter 5 presents fixed-time stability conditions for both deterministic and stochastic DT autonomous nonlinear systems. First, fixed-time stability for equilibria of deterministic DT autonomous systems is defined. That is, a settling-time function is defined with a fixed upper bound independent of the initial condition. We then present Lyapunov theorems for fixed-time stability of both unperturbed and perturbed deterministic DT systems. Moreover, the sensitivity of fixed-time stability properties to perturbations of systems is investigated under the assumption of the existence of a locally Lipschitz discrete Lyapunov function. It is ensured that fixed-time stability is preserved under perturbations in the form of fixed-time attractiveness. Furthermore, sufficient Lyapunov conditions for fixed-time stability in probability of stochastic DT systems and their stochastic settling-time function are presented.

Chapter 6 of this dissertation presents a fixed-time concurrent learning (FxTCL) algorithm for discrete-time systems to 1) ensure fixed-time parameter convergence independent of the initial estimation errors and 2) relax the PE condition to a rank condition on the recorded data using CL. In the presented FxTCL, the settling-time upper bound is independent of the initial parameter estimation error. To achieve this goal, a modified gradient-descent update law is presented for learning the unknown system parameters. This update law reuses past collected data at every time instance and leverages discontinuous and non-integer powers of the identification errors. The Lyapunov analysis presented in Chapter 5 is then leveraged to guarantee fixed-time convergence of the system parameters to their true values.

Chapter 7 summarizes and concludes this dissertation and provides future research directions.

The contributions of this dissertation are published or submitted in the following journal papers.

Farzaneh Tatari, Hamidreza Modares, Christos Panayiotou, Marios Polycarpou, “Finite-time Distributed Identification for Nonlinear Interconnected Systems”, IEEE/CAA Journal of Automatica Sinica, vol. 9, no. 7, pp. 1–12, Jul. 2022.

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of Discrete-time Autonomous Systems," in IEEE/CAA Journal of Automatica Sinica, vol. 10, no. 4, pp. 945-956, April 2023, doi: 10.1109/JAS.2023.123405.

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CHAPTER 2

FINITE-TIME DISTRIBUTED IDENTIFICATION FOR NONLINEAR INTERCONNECTED SYSTEMS

2.1 Introduction

In this chapter, first, a novel finite-time distributed CL identification method is presented for nonlinear interconnected systems. The proposed discontinuous distributed CL estimation law ensures the finite-time convergence of the approximated parameters without requiring the regressors' PE condition. In the proposed distributed CL, every distributed identifier leverages a local state communication with its neighboring subsystems to collect and employ a rich distributed memory to relax the regressor's PE condition and identify its own interconnected subsystem dynamics in finite time. Then, based on finite-time Lyapunov analysis, when there is zero MFAE, the finite-time convergence of interconnected system parameters is ensured through rigorous proofs. For the case with non-zero MFAE, finite-time attractiveness of the interconnected system parameters' estimation error is guaranteed. Finally, the upper bounds of the settling-time functions for the finite convergence time are provided as a function of distributed memory data richness.

Notation The network of subsystems in an interconnected system is shown by a bidirectional graph $G(\mathcal{V}, \Sigma)$, where $\mathcal{V} = \{1, 2, \dots, N\}$ is the set of vertices representing N subsystems and $\Sigma \subset \mathcal{V} \times \mathcal{V}$ is the set of graph edges. $(i, j) \in \Sigma$ indicates that there exists an edge from node i to node j which indicates the interconnection between subsystems i and j. The set of neighbors of node i is shown by $N_i = \left\{j : (j, i) \in \Sigma\right\}$ and $|N_i|$ is the cardinality measure of the set N_i, $i = 1, ..., N$. Throughout this chapter, I is the identity matrix of appropriate dimension. $stack(x, y)$ is an operator which stacks the columns of x and y vectors on top of one another. $\|x\|$ denotes the vector norm for $x \in \mathbb{R}^n$, $\|A\|$ shows the induced 2-norm of the matrix A. $\lambda_{min}(A)$ and $\lambda_{max}(A)$ denote the minimum and maximum eigenvalues of the matrix A, respectively.

2.2 Preliminaries and Problem Formulation

Preliminaries Consider the following nonlinear system with the equilibrium point in the origin,

$$\dot{y}(t) = F(t, y), \quad y(0) = y_0, \tag{2.1}$$

where $y \in \mathcal{D}_y$, $F : \mathbb{R}_+ \times \mathcal{D}_y \mapsto \mathcal{D}_y$ and $\mathcal{D}_y \subset \mathbb{R}^n$ is an open neighborhood of the origin.

Definition 1 (Persistence of excitation [29]) A signal $y(t)$ is persistently exciting if there are positive scalars η_1, η_2 and $\mathcal{T} \in \mathbb{R}_+$, such that the following condition on $y(t)$ (PE condition) is satisfied for $\forall t \in \mathbb{R}_+$,

$$\eta_1 I \leq \int_t^{t+\mathcal{T}} y(\tau) y^T(\tau) d\tau \leq \eta_2 I.$$

Definition 2 (Finite-time stability [59]) The system (2.1) is said to be

1) finite-time stable, if it is asymptotically stable and any solution $y(t, y_0)$ of (2.1) reaches the equilibrium point in finite time, i.e., $y(t, y_0) = 0, \forall t \geq T(y_0)$, where $T : \mathcal{D}_y \mapsto \mathbb{R}_+ \cup \{0\}$ is the settling-time function.

2) finite-time attractive to an ultimate bounded set Y around origin, if any solution $y(t, y_0)$ of (2.1) reaches Y in finite-time and stays there $\forall t \geq T(y_0)$ where $T : \mathcal{D}_y \mapsto \mathbb{R}_+ \cup \{0\}$ is the settling-time function.

Lemma 1 [59] Suppose that there exists a positive definite continuous function $V : \mathcal{D}_y \mapsto \mathbb{R}_+ \cup \{0\}$ in an open neighborhood of the origin and there exist real numbers $\alpha > 0$ and $0 < r_1 < 1$ such that $V(y)$ is positive definite and

$$\dot{V}(y) \leq -\alpha V^{r_1}(y).$$

Then, the system (2.1) is finite-time stable with a finite settling-time

$$T(y_0) \leq \frac{1}{\alpha(1 - r_1)} V^{1-r_1}(y_0),$$

for all $y_0 \in \mathcal{D}_y$.

Fact 1: In general, for a vector $x = [x_1, x_2, ..., x_n]^T \in \mathbb{R}^n$, the p-norm is defined as $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$. Moreover, for positive constants r and s, if $0 < r < s$, then based on Hölder inequality [129], one obtains

$$\|x\|_s \leq \|x\|_r \leq n^{\frac{1}{r}-\frac{1}{s}} \|x\|_s.$$

Problem Formulation Consider the following nonlinear interconnected system composed of N uncertain subsystems described by

$$\dot{x}_i(t) = f_i(x_i(t)) + g_i(x_i(t))u_i(t) + \Delta_i(x_i(t), x_j(t)|_{j\in N_i}), \quad i = 1, ..., N, \tag{2.2}$$

where $x_i = [x_{i1}, x_{i2}, ..., x_{in}] \in \mathcal{D}_i \subset \mathbb{R}^n$ is the state and $u_i \in \mathcal{D}_u \subset \mathbb{R}^m$ is the control input of subsystem $i, i = 1, ..., N$; $\mathcal{D}_i$ and $\mathcal{D}_u$ are compact sets. $f_i : \mathcal{D}_i \mapsto \mathbb{R}^n$, $g_i : \mathcal{D}_i \mapsto \mathbb{R}^{n\times m}$ and $\Delta_i : \mathcal{D}_{N_i} \mapsto \mathbb{R}^n$ are the unknown nonlinear drift, input and interconnection terms with $\mathcal{D}_{N_i} \subset \mathbb{R}^{n(|N_i|+1)}$, respectively.

This chapter aims to present an identification method to learn the unknown dynamics of the nonlinear interconnected system (2.2) in finite time and in a distributed fashion.

Assumption 1 $f_i(x_i(t))$ and $g_i(x_i(t))$ are both locally Lipschitz in $x_i(t)$ and $\Delta_i(x_i(t), x_j(t)|_{j\in N_i})$ is locally Lipschitz in $x_{N_i}(t)$ where $x_{N_i}(t) = stack\,(x_i(t), x_j(t))\big|_{j\in N_i}$.

In order to learn the subsystems' uncertain dynamics in a distributed fashion, first, every subsystem dynamics (2.2) is formulated into a distributed filtered regressor form. The distributed filtered-regressor form presents the subsystems' states with a time-varying regressor for which the dynamic flow of regressors are known and depend on the subsystem's states and inputs, as well as its neighbors' states. This will allow to present update laws without requiring to measure the state derivatives of the subsystems and their neighbors [10].

To develop filtered regressors, linearly parameterized adaptive approximation models are first used to respectively represent $f_i(x_i)$, $g_i(x_i)$ and $\Delta_i(x_i, x_j(t)|_{j\in N_i})$ for every subsystem $i, i = 1, ..., N$ as follows,

$$f_i(x_i(t)) = \hat{f}_i(x_i(t), \Theta_i^*) + e_{f_i}(x_i(t)), \tag{2.3}$$

$$g_i(x_i(t)) = \hat{g}_i(x_i(t), \Phi_i^*) + e_{g_i}(x_i(t)), \tag{2.4}$$

$$\Delta_i(x_i(t), x_j(t)|_{j\in N_i}) = \hat{\Delta}_i(x_i(t), x_j(t)|_{j\in N_i}, \Psi_i^*) + e_{\Delta_i}(x_i(t), x_j(t)|_{j\in N_i}), \tag{2.5}$$

where

$$\hat{f}_i(x_i(t), \Theta_i^*) = \Theta_i^{*T} \varphi_i(x_i(t)), \tag{2.6}$$

$$\hat{g}_i(x_i(t), \Phi_i^*) = \Phi_i^{*T} \chi_i(x_i(t)), \tag{2.7}$$

$$\hat{\Delta}_i(x_i(t), x_j(t)|_{j\in N_i}, \Psi_i^*) = \Psi_i^{*T} \upsilon_i(x_i(t), x_j(t)|_{j\in N_i}), \tag{2.8}$$

$$e_{f_i}(x_i) = \sup_{x_i\in\mathcal{D}_i} \left\| f_i(x_i) - \hat{f}_i(x_i, \Theta_i) \right\|, \tag{2.9}$$

$$e_{g_i}(x_i) = \sup_{x_i\in\mathcal{D}_i} \left\| g_i(x_i) - \hat{g}_i(x_i, \Phi_i) \right\|, \tag{2.10}$$

$$e_{\Delta_i}(x_i, x_j|_{j\in N_i}) = \sup_{x_i\in\mathcal{D}_i, x_j\in\mathcal{D}_j} \left\| \Delta_i(x_i, x_j|_{j\in N_i}) - \hat{\Delta}_i(x_i, x_j|_{j\in N_i}, \Psi_i) \right\|. \tag{2.11}$$

The matrices $\Theta_i^* \in \mathcal{D}_f \subset \mathbb{R}^{p_i\times n}$, $\Phi_i^* \in \mathcal{D}_g \subset \mathbb{R}^{q_i\times n}$, $\Psi_i^* \in \mathcal{D}_\Delta \subset \mathbb{R}^{r_i\times n}$ represent the unknown optimal adaptive parameters for the approximators given as follows:

$$\Theta_i^* = \arg\min_{\Theta_i\in\mathcal{D}_f} \{e_{f_i}(x_i)\}, \tag{2.12}$$

$$\Phi_i^* = \arg\min_{\Phi_i\in\mathcal{D}_g} \{e_{g_i}(x_i)\}, \tag{2.13}$$

$$\Psi_i^* = \arg\min_{\Psi_i\in\mathcal{D}_\Delta} \{e_{\Delta_i}(x_i, x_j|_{j\in N_i})\}, \tag{2.14}$$

and $\varphi_i : \mathcal{D}_i \mapsto \mathbb{R}^{p_i}$, $\chi_i : \mathcal{D}_i \mapsto \mathbb{R}^{q_i}$, $\upsilon_i : \mathcal{D}_{N_i} \mapsto \mathbb{R}^{r_i}$ are the basis functions, such that p_i, q_i and r_i are the number of linearly independent basis functions to approximate $f_i(x_i)$, $g_i(x_i)$ and $\Delta(x_i, x_j|_{j\in N_i})$, respectively. The quantities $e_{f_i}(x_i)$, $e_{g_i}(x_i)$ and $e_{\Delta_i}(x_i, x_j|_{j\in N_i})$, defined in (2.9)-(2.11) are, respectively, the MFAEs for $f_i(x_i)$, $g_i(x_i)$ and $\Delta_i(x_i, x_j|_{j\in N_i})$, denoting the residual approximation errors for the case of optimal parameters. As a special case, if the adaptive approximation models $\hat{f}_i(x_i,\Theta_i)$, $\hat{g}_i(x_i,\Phi_i)$ and $\hat{\Delta}_i(x_i,x_j|_{j\in N_i},\Psi_i)$ can exactly approximate the unknown functions $f_i(x_i)$, $g_i(x_i)$ and $\Delta_i(x_i,x_j|_{j\in N_i})$, respectively, then, $e_{f_i} = e_{g_i} = e_{\Delta_i} = 0$.

Remark 1 Generally, adaptive approximators can be classified into linearly parameterized and nonlinearly parameterized [10]. Linearly parameterized approximators are more common in the literature of adaptive control because they provide mechanism to derive stronger analytical results for stability and convergence. Linearly parameterized approximators are different and more general than linear models. In linear models, the entire structure of the system is assumed to be linear. In linearly parameterized approximators, the unknown nonlinearities are estimated by nonlinear approximators, where the weights (parameter estimates) appear linearly with respect to nonlinear basis functions.

Remark 2 The linearly parameterized approximation models as given in (2.6), (2.7) and (2.8), are linear in parameters Θ_i^*, Φ_i^*, and Ψ_i^*, respectively, and their corresponding basis functions $\varphi_i(x_i(t))$, $\chi_i(x_i(t))$ and $\upsilon_i(x_i(t),x_j(t)|_{j\in N_i})$, respectively, contain some nonlinear functions. We consider two different cases [130]: 1) in the first case, the hypothesis class is assumed to be realizable. That is, the identification is realizable as there is a perfect hypothesis within the hypothesis class (i.e, basis functions and their corresponding optimum weights) that generates no error. 2) in the second case, the hypothesis class is assumed to be not realizable (all system parameters make some identification error). For the first case, the nonlinear basis functions completely capture the subsystem dynamics (i.e., $e_{f_i}(x_i(t))$, $e_{g_i}(x_i(t))$ $e_{\Delta_i}(x_i(t),x_j(t)|_{j\in N_i}$, given in (2.9)-(2.11), are zero) and only parametric uncertainty exists and therefore, the MFAE is zero. For the second case, the basis functions cannot fully capture the dynamics of the subsystems and mismatch error exists and therefore, the MFAE is nonzero for all hypotheses.

By using (2.3)-(2.8), each subsystem dynamics (2.2) can be rewritten as

$$\dot{x}_i(t) = W_i^{*T} z_i(x_i(t),u_i(t),x_j(t)|_{j\in N_i}) + \varepsilon_i(x_i(t),u_i(t),x_j(t)|_{j\in N_i}), \tag{2.15}$$

where $W_i^* \in \mathbb{R}^{(p_i+q_i+r_i)\times n}$, $z_i(x_i, u_i, x_j|_{j\in N_i}) \in \mathbb{R}^{(p_i+q_i+r_i)}$,

$$W_i^* = [\Theta_i^{*T}, \Phi_i^{*T}, \Psi_i^{*T}]^T,$$

$$z_i(x_i, u_i, x_j|_{j\in N_i}) =[\varphi_i^T(x_i), u_i^T\chi_i^T(x_i), \upsilon_i^T(x_i, x_j|_{j\in N_i})]^T,$$

$$\varepsilon_i(x_i(t), u_i(t), x_j(t)|_{j\in N_i}) =e_{f_i}(x_i(t)) + e_{g_i}(x_i(t))u_i + e_{\Delta_i}(x_i(t), x_j(t)|_{j\in N_i}).$$

Assumption 2 The approximation error ε_i are bounded inside compact sets $\mathcal{D}_i$, $\mathcal{D}_u$ and $\mathcal{D}_{N_i}$. That is, $\sup\limits_{x_i\in\mathcal{D}_i, x_j\in\mathcal{D}_j, u_i\in\mathcal{D}_u} \|\varepsilon_i(x_i, u_i, x_j|_{j\in N_i})\| \leq b_\varepsilon$ with $b_\varepsilon \geq 0$. The approximators' basis functions are also bounded in the mentioned compact sets.

A distributed filtered regressor is now formulated to circumvent the requirement of measuring $\dot{x}_i(t)$, and is leveraged by the update law later. For regressor filtering, the dynamics (2.15) is rewritten as

$$\dot{x}_i = -Cx_i + W_i^{*T} z_i(x_i, u_i, x_j|_{j\in N_i}) + Cx_i + \varepsilon_i, \tag{2.16}$$

where $C = cI, c > 0$ and $i = 1, ..., N$. The state space solution to state model (2.16) can be expressed as

$$x_i =e^{-Ct}x_i(0) + \int_0^t e^{-C(t-\tau)}[W_i^{*T} z_i(x_i(\tau), u_i(\tau), x_j(\tau)|_{j\in N_i}) + Cx_i(\tau) + \varepsilon_i(\tau)]d\tau. \tag{2.17}$$

Now, one can rewrite (2.17) as follows,

$$x_i =W_i^{*T} d_i(t) + cl_i(x_i) + e^{-Ct}x_i(0) + \varepsilon_{x_i}(t), \tag{2.18}$$

$$\begin{aligned} \dot{d}_i(t) &= -cd_i(t) + z_i(x_i, u_i, x_j|_{j\in N_i}),\ d_i(0) = 0,\\ \dot{l}_i(t) &= -Cl_i(x_i) + x_i(t),\ l_i(0) = 0, \quad i = 1, ..., N, \end{aligned} \tag{2.19}$$

where $l_i(t) = \int_0^t e^{-C(t-\tau)}x_i(\tau)d\tau$ is the filtered regressor version of $x_i(t)$, $\varepsilon_{x_i} = \int_0^t e^{-C(t-\tau)}\varepsilon_i(\tau)d\tau$ and $x_i(0)$ is the initial state of (2.16), $d_i(t) = \int_0^t e^{-c(t-\tau)} z_i(x_i(\tau), u_i(\tau), x_j(\tau)|_{j\in N_i})d\tau$ is the distributed filtered regressor of $z_i(x_i(t), u_i(t), x_j(t)|_{j\in N_i})$.

Dividing (2.18) by $n_i = 1 + d_i^T(t)d_i(t) + l_i^T(t)l_i(t)$ as a normalizing signal, one has,

$$\bar{x}_i(t) = W_i^{*T}\bar{d}_i(t) + c\bar{l}_i(t) + e^{-Ct}\bar{x}_i(0) + \bar{\varepsilon}_i(t), \tag{2.20}$$

where $\bar{d}_i = \frac{d_i}{n_i}$, $\bar{l}_i = \frac{l_i}{n_i}$, $\bar{x}_i = \frac{x_i}{n_i}$ and $\bar{\varepsilon}_i = \frac{\varepsilon x_i}{n_i}$. It is implied by Assumption 2 that $\bar{\varepsilon}_i(t)$ is also bounded, i.e.,

$$\sup_{x_i \in \mathcal{D}_i, x_j \in \mathcal{D}_j, u_i \in \mathcal{D}_u} \|\bar{\varepsilon}_i(x_i, u_i, x_j|_{j\in N_i})\| \leq b_{\bar{\varepsilon}}$$

for some $b_{\bar{\varepsilon}} \geq 0$ and $\|\bar{d}_i(t)\| < 1$.

To approximate the uncertainties $f_i(x_i)$, $g_i(x_i)$ and $\Delta_i(x_i, x_j|_{j\in N_i})$ in a distributed finite-time fashion without the need for satisfaction of the PE condition on the regressor, the chapter objective is to propose a finite-time distributed CL approach that guarantees every interconnected subsystem i parameter estimation error, $\tilde{W}_i(t) := \hat{W}_i(t) - W_i^*$, is:

1) finite-time stable for distributed adaptive approximators with zero MFAE;

2) finite-time attractive to a bounded set around zero for distributed adaptive approximators with non-zero MFAE; where $\hat{W}_i(t) = [\hat{\Theta}_i^T(t), \hat{\Phi}_i^T(t), \hat{\Psi}_i^T(t)]^T \in \mathbb{R}^{(p_i+q_i+r_i)\times n}$, $\hat{\Theta}_i(t)$, $\hat{\Phi}_i(t)$ and $\hat{\Psi}_i(t)$ are, respectively, the estimated parameter matrices of W_i^*, Θ_i^*, Φ_i^* and Ψ_i^* at time t for the subsystem i and $\tilde{W}_i(t) := \hat{W}_i(t) - W_i^* := [\tilde{\Theta}_i^T(t), \tilde{\Phi}_i^T(t), \tilde{\Psi}_i^T(t)]^T$ such that $\tilde{\Theta}_i(t) := \hat{\Theta}_i(t) - \Theta_i^*$, $\tilde{\Phi}_i(t) := \hat{\Phi}_i(t) - \Phi_i^*$, $\tilde{\Psi}_i(t) := \hat{\Psi}_i(t) - \Psi_i^*$, $i = 1, ..., N$.

2.3 Finite-time Distributed Concurrent Learning

In this section, a finite-time distributed parameter estimation law for approximating the uncertainties of the nonlinear interconnected system (2.2) is presented. The convergence analysis of the proposed method is presented based on the Lyapunov approach.

Consider the distributed approximator for subsystem i to be of the form

$$\hat{\bar{x}}_i(t) = \hat{W}_i^T(t)\bar{d}_i(t) + c\bar{l}_i(t) + e^{-Ct}\bar{x}_i(0). \tag{2.21}$$

The state estimation error of the subsystem i is obtained as

$$e_i(t) = \hat{\bar{x}}_i(t) - \bar{x}_i(t). \tag{2.22}$$

The state estimation error $e_i(t)$, which is later employed in the proposed parameter update law, is accessible online, because $\hat{\bar{x}}_i(t)$ is computed online by the approximator (2.21) and $\bar{x}_i(t)$ is the normalized measurable state of the system. However, for the sake of parameter convergence analysis, using (2.20) and (2.21), $e_i(t)$ in (2.22) is rewritten as

$$e_i(t) = \tilde{W}_i^T(t)\bar{d}_i(t) - \bar{\varepsilon}_i(t). \tag{2.23}$$

To use CL, that employs experienced data along with current data in the update law of the distributed identifier parameters, the memory data is recorded in the memory stacks $M_i \in \mathbb{R}^{(p_i+q_i+r_i)\times P_i}$, $L_i \in \mathbb{R}^{n\times P_i}$ and $X_i \in \mathbb{R}^{n\times P_i}$ for each interconnected subsystem i, $i = 1, ..., N$ at times $\tau_1, ..., \tau_{P_i}$ as

$$\begin{aligned} &M_i = [\bar{d}_i(\tau_1), \bar{d}_i(\tau_2), ..., \bar{d}_i(\tau_{P_i})], \quad L_i = [\bar{l}_i(\tau_1), \bar{l}_i(\tau_2), ..., \bar{l}_i(\tau_{P_i})], \\ &X_i = [\bar{x}_i(\tau_1), \bar{x}_i(\tau_2), ..., \bar{x}_i(\tau_{P_i})], \end{aligned} \tag{2.24}$$

where P_i denotes the number of data points recorded in each stack of subsystem i. The memory stack M_i captures the interactive data samples for which their richness depends on collective richness of the subsystem's state itself as well as its neighbors. The number of data points P_i, for $i = 1, ..., N$, is chosen such that M_i is full-row rank and contains as many linearly independent elements as the dimension of the distributed filtered regressor $d_i(t)$ (i.e., the total number of linearly independent basis functions for $f_i(x_i)$, $g_i(x_i)$ and $\Delta_i(x_i, x_j|_{j\in N_i})$), given in (2.18), that is called as rank condition on M_i and requires $P_i \geq p_i + q_i + r_i$, for $i = 1, ..., N$.

In order for the matrix M_i to be full-row rank, one needs to collect at least $p_i + q_i + r_i$ number of data samples. Therefore, one can check the full-row rank condition on the data matrix M_i online after recording $p_i + q_i + r_i$ number of data points in the memory stacks of the subsystem i. Whenever the full-row rank condition on M_i is satisfied, i.e.,

$$rank(M_i) = p_i + q_i + r_i,$$

one can stop recording data samples in the corresponding subsystem's memory stacks.

The error $e_i^h(t)$ for the h^{th} recorded data is defined as follows

$$e_i^h(t) = \hat{\bar{x}}_i^h(t) - \bar{x}_i(\tau_h), \tag{2.25}$$

where

$$\hat{\bar{x}}_i^h(t) = \hat{W}_i^T(t)\bar{d}_i(\tau_h) + c\bar{l}_i(\tau_h) + e^{-Ct}\bar{x}_i(0), \tag{2.26}$$

is the state estimation at time $0 \leq \tau_h < t$, $h = 1, ..., P_i$ employing the current estimated parameters in $\hat{W}_i(t)$ and the recorded $\bar{d}_i(\tau_h)$ and $\bar{l}_i(\tau_h)$.

The error $e_i^h(t)$, which is later employed in the proposed parameter update law, is accessible online, since, $\hat{\bar{x}}_i^h(t)$ is computed online by (2.26) using the online estimated $\hat{W}_i(t)$ and the memory stacks' elements M_i and L_i, and $\bar{x}_i(\tau_h)$ is accessible from the memory stack X_i of the corresponding subsystem i. For analysis purposes, using (2.20) and (2.26), one can rewrite (2.25) as follows

$$e_i^h(t) = \tilde{W}_i^T(t)\bar{d}_i(\tau_h) - \bar{\varepsilon}_i(\tau_h). \tag{2.27}$$

Remark 3 In the distributed approximator (2.21), the received neighboring states appear in the distributed filtered regressor $\bar{d}_i(t)$, as given in (2.19). Therefore, the richness of the local neighboring data affects the richness and rank condition satisfaction of the distributed data stored in memory M_i.

Finite-time Distributed Concurrent Learning Estimation Law

Now, the finite-time distributed estimation law for the unknown parameters in the interconnected subsystem i approximator (2.21) is proposed as

$$\dot{\hat{W}}_i(t) = -\Gamma_i(\Xi_G\bar{d}_i(t)\lfloor e_i^T(t)\rceil^{\gamma_i} + \Xi_C\sum_{h=1}^{P_i}\bar{d}_i(\tau_h)\lfloor e_i^{hT}(t)\rceil^{\gamma_i}), \tag{2.28}$$

where $\lfloor . \rceil^{\gamma_i} := |.|^{\gamma_i} sign(.)$ with $|.|$ and $sign(.)$ understood in component-wise sense and $0 \leq \gamma_i < 1$. The matrices $\Gamma_i, \Xi_G, \Xi_C \in \mathbb{R}^{(p_i+q_i+r_i)\times(p_i+q_i+r_i)}$ are positive definite, $\Gamma_i > 0$ is the learning rate matrix, $\Xi_C = \xi_C I$ and $\Xi_G = \xi_G I$ with scalars $\xi_C > 0$ and $\xi_G > 0$. The proposed estimation law is distributed since the current $\bar{d}_i(t)$ and recorded $\bar{d}_i(\tau_h)$ are distributed filtered regressors depending not only on the subsystem i states but also on its neighboring states. In (2.28), the first term $\Xi_G \bar{d}_i(t) \lfloor e_i^T(t) \rceil^{\gamma_i}$ is a gradient descent term, containing the current state approximation error for the subsystem i, and the second term $\Xi_C \sum_{h=1}^{P_i} \bar{d}_i(\tau_h) \lfloor e_i^{hT}(t) \rceil^{\gamma_i}$, containing the experienced data of subsystem i, is the distributed CL term.

Remark 4 In (2.28), the weights Ξ_C and Ξ_G are not necessarily equal and one of the two estimation terms can be prioritized over the other by choosing appropriate ξ_C and ξ_G, respectively. Generally, in (2.28) choosing high learning rates Γ_i or weights ξ_C can increase the convergence rate. However, it may also lead to chattering in the estimated parameters. Once combined with the control design, this chattering can result in poor control performance or even instability.

Remark 5 For every distributed identifier that uses (2.28), the shared neighboring states on the learning time length, not only affect the current value of the distributed regressor, $\bar{d}_i(t)$, but are also influential on the richness of the distributed memory employed in the second term of (2.28). This entirely discriminates the current work from single system's finite-time CL-based identification methods [40–43].

In the following, the convergence properties for distributed adaptive approximators with zero and nonzero MFAEs are investigated.

Finite-time Convergence Properties for Distributed Adaptive Approximators with Zero MFAEs ($\bar{\varepsilon}_i(t) = 0$)

The theorem below shows that using the proposed finite-time distributed concurrent learning method (2.28), for distributed adaptive approximators with zero MFAEs, i.e. $\bar{\varepsilon}_i(t) = 0$, the estimated parameters $\hat{W}_i(t)$ converge to their optimal values in finite time.

Theorem 1 Let the distributed approximator for every nonlinear interconnected subsystem i in (2.2), be given by (2.21), whose parameters are adjusted by the update law of (2.28) with $0 \leq \gamma_i < 1$ and a distributed filtered regressor given by (2.19), for $i = 1, ..., N$. Let Assumptions 1-2 hold. Once the full-row rank condition on M_i, $i = 1, .., N$ is satisfied, then for every i^{th} adaptive distributed approximator with zero MFAE, i.e., $\bar{\varepsilon}_i(t) = 0$, the distributed parameter estimation law (2.28) ensures finite-time convergence of $\tilde{W}_i(t)$ to zero for all interconnected subsystems within the following settling-time function

$$T \leq \max_{i=1,\dots,N} \frac{2\|\tilde{W}_i(0)\|^{1-\gamma_i}}{\zeta_i \beta_i (1-\gamma_i)}, \tag{2.29}$$

where $\zeta_i = \xi_C \lambda_{min}^{\frac{\gamma_i+1}{2}}(S_i)$, $\beta_i = 2\lambda_{min}(\Gamma_i)$ and $S_i = \sum_{h=1}^{P_i} \bar{d}_i(\tau_h)\bar{d}_i^T(\tau_h)$.

Proof 1 Choosing the following Lyapunov function candidate

$$V(t) = \sum_{i=1}^{N} V_i(t) = \frac{1}{2}\sum_{i=1}^{N} tr\{\tilde{W}_i^T(t)\Gamma_i^{-1}\tilde{W}_i(t)\}, \tag{2.30}$$

one has

$$\alpha_i^{-1}\|\tilde{W}_i(t)\|^2 \leq V_i(t) \leq \beta_i^{-1}\|\tilde{W}_i(t)\|^2, \tag{2.31}$$

where $\alpha_i = 2\lambda_{max}(\Gamma_i), \beta_i = 2\lambda_{min}(\Gamma_i)$.

The time derivative $\dot{V}_i$ for $i = 1, ..., N$, using (2.23), (2.27) and (2.28), yields,

$$\begin{aligned} \dot{V}_i(t) &= tr\{\tilde{W}_i^T(t)\Gamma_i^{-1}\dot{\tilde{W}}_i(t)\} \\ &= tr\{-\Xi_G \tilde{W}_i^T(t)\bar{d}_i(t)\lfloor \bar{d}_i^T(t)\tilde{W}_i(t)\rceil^{\gamma_i} - \Xi_C \tilde{W}_i^T(t)\sum_{h=1}^{P_i}\bar{d}_i(\tau_h)\lfloor \bar{d}_i^T(\tau_h)\tilde{W}_i(t)\rceil^{\gamma_i}\}. \end{aligned} \tag{2.32}$$

One knows that

$$\begin{aligned} \tilde{W}_i^T(t)\bar{d}_i(t)\lfloor \bar{d}_i^T(t)\tilde{W}_i(t)\rceil^{\gamma_i} &= \sum_{i=1}^{n} |(\tilde{W}_i^T(t)\bar{d}_i(t))_i|^{\gamma_i+1} \\ &= \|\tilde{W}_i^T(t)\bar{d}_i(t)\|_{\gamma_i+1}^{\gamma_i+1}, \end{aligned} \tag{2.33}$$

and based on Fact 1

$$\|\tilde{W}_i^T(t)\bar{d}_i(t)\| \leq \|\tilde{W}_i^T(t)\bar{d}_i(t)\|_{\gamma_i+1}, \tag{2.34}$$

holds for $0 < \gamma_i + 1 < 2$. Therefore, using (2.32)-(2.34), one obtains,

$$\begin{aligned} \dot{V}_i(t) &\leq -\xi_G\|\tilde{W}_i^T(t)\bar{d}_i(t)\|^{\gamma_i+1} - \xi_C\sum_{h=1}^{P_i}\|\tilde{W}_i^T(t)\bar{d}_i(\tau_h)\|^{\gamma_i+1} \\ &\leq -\xi_C\sum_{h=1}^{P_i}(\tilde{W}_i^T(t)\bar{d}_i(\tau_h)\bar{d}_i^T(\tau_h)\tilde{W}_i(t))^{\frac{\gamma_i+1}{2}}. \end{aligned} \tag{2.35}$$

Therefore,

$$\dot{V}_i(t) \leq -\xi_C \lambda_{min}^{\frac{\gamma_i+1}{2}}(S_i)\|\tilde{W}_i(t)\|^{\gamma_i+1}, \tag{2.36}$$

where $S_i = \sum_{h=1}^{P_i} \bar{d}_i(\tau_h)\bar{d}_i^T(\tau_h)$. Using (2.31), (2.36) gives

$$\dot{V}_i(t) \leq -\zeta_i \beta_i^{\frac{\gamma_i+1}{2}} V_i^{\frac{\gamma_i+1}{2}}(t),$$

where $\zeta_i = \xi_C \lambda_{min}^{\frac{\gamma_i+1}{2}}(S_i)$ and based on Lemma 1, it is proved that for every subsystem i, $i = 1, ..., N$, $\tilde{W}_i(t)$ is finite-time stable with the following settling-time function

$$T_i(\tilde{W}_i(0)) \leq \frac{2\|\tilde{W}_i(0)\|^{1-\gamma_i}}{\zeta_i \beta_i (1-\gamma_i)}.$$

Therefore, the whole interconnected system dynamics can be identified in finite time within the following settling time,

$$T \leq \max_{i=1,...,N} T_i(\tilde{W}_i(0)) = \max_{i=1,...,N} \frac{2\|\tilde{W}_i(0)\|^{1-\gamma_i}}{\zeta_i \beta_i (1-\gamma_i)}.$$

This completes the proof.

Corollary 1 Let the assumptions and statements of Theorem 1 hold. Then, for adaptive distributed approximators with zero MFAEs, i.e., $\bar{\varepsilon}_i(t) = 0$, the state estimation error $e_i(t)$ for every subsystem i, $i = 1, ..., N$, is finite-time stable.

Proof 2 The proof is a direct consequence of Theorem 1.

Remark 6 As shown in (2.29), the settling time function of the identifier depends on the minimum eigenvalue of the distributed memory matrix, $\lambda_{min}(S_i)$. Therefore, to improve the convergence speed, an optimization over recorded data can be performed to replace old data with new ones as more data becomes available to maximize the minimum eigenvalue of the distributed memory matrix to reduce the convergence time.

Finite-time Convergence Properties for Distributed Adaptive Approximators with Non-zero MFAEs ($\bar{\varepsilon}_i(t) \neq 0$)

The following theorem gives the finite-time convergence properties for the distributed parameter estimation law (2.28) of distributed adaptive approximators with non-zero MFAEs, $\bar{\varepsilon}_i(t) \neq 0$, in interconnected systems' identification.

Theorem 2 Let the distributed approximator for nonlinear interconnected subsystem (2.2) given by (2.21), whose parameters are adjusted by the update law of (2.28) with $0 < \gamma_i < 1$ and a regressor given in (2.19). Consider that Assumptions 1-2 hold. Once the full-row rank condition on M_i for $i = 1, ..., N$ is met, then for adaptive distributed approximators with non-zero MFAEs, i.e., $\bar{\varepsilon}_i(t) \neq 0$, the proposed parameter estimation law (2.28) guarantees that for every subsystem i, $i = 1, ..., N$, the $\tilde{W}_i(t)$ is finite-time attractive by the following bounded set,

$$S^i_{\tilde{W}} = \{\tilde{W}_i(t) : \|\tilde{W}_i(t)\| \leq \sqrt{\frac{\lambda_{max}(\Gamma_i)}{\lambda_{min}(\Gamma_i)}}\bar{\mu}_i\}, \ \forall t \geq T, \tag{2.37}$$

where

$$\bar{\mu}_i = \begin{cases} max\{\dfrac{b_{\bar{\varepsilon}}}{\min\{\lambda^{\frac{1}{2}}_{min}(D_i(t)), \bar{\lambda}_i\}}, (\frac{\omega_i}{\zeta_i\delta})^{\frac{1}{\gamma_i}}\}, & \bar{d}_i(t) \neq 0, \\ max\{\frac{b_{\bar{\varepsilon}}}{\bar{\lambda}_i}, (\frac{\omega_i}{\zeta_i\delta})^{\frac{1}{\gamma_i}}\}, & \bar{d}_i(t) = 0, \end{cases} \tag{2.38}$$

$$T \leq \max_{i=1,...,N} \frac{2\|\tilde{W}_i(0)\|^{1-\gamma_i}}{\zeta_i\beta_i(1-\delta)(1-\gamma_i)}, \tag{2.39}$$

$$\bar{\lambda}_i = \min_{h=1,...,P_i}(\lambda^{\frac{1}{2}}_{min}(D_i(\tau_h))), D_i(t) = \bar{d}_i(t)\bar{d}^T_i(t), \omega_i = n^{\frac{1-\gamma_i}{2}} b^{\gamma_i}_{\bar{\varepsilon}}(\xi_G + P_i\xi_C),$$

and $0 < \delta < 1$.

Proof 3 Choose the same Lyapunov function (2.30) that satisfies (2.31). The time derivation of V_i employing (2.23), (2.27) and (2.28) gives,

$$\begin{aligned} \dot{V}_i(t) = tr\{ & -\Xi_G\tilde{W}^T_i(t)\bar{d}_i(t)\lfloor\bar{d}^T_i(t)\tilde{W}_i(t) - \bar{\varepsilon}^T_i(t)\rceil^{\gamma_i} \\ & -\Xi_C\tilde{W}^T_i(t)\sum_{h=1}^{P_i}\bar{d}_i(\tau_h)\lfloor\bar{d}^T_i(\tau_h)\tilde{W}_i(t) - \bar{\varepsilon}^T_i(\tau_h)\rceil^{\gamma_i}\}. \end{aligned} \tag{2.40}$$

Consider in the component-wise sense that $|(\bar{d}^T_i(t)\tilde{W}_i(t))_k| \geq |(\bar{\varepsilon}_i(t))_k|$, for $k = 1, ..., n$. Note that the previous inequality is required for $\bar{d}_i(t) \neq 0$. If $\bar{d}_i(t) = 0$ then the first term in (2.28) is zero and in the second term of (2.28), the data collection assures that $\bar{d}_i(\tau_h) \neq 0, h = 1, ..., P_i$.

Therefore, $sign(\bar{d}^T_i(t)\tilde{W}_i(t) - \bar{\varepsilon}^T_i(t)) = sign(\bar{d}^T_i(t)\tilde{W}_i(t))$ is obtained. Then, for any $y, \bar{y} \in \mathbb{R}$ and $0 < \gamma_i < 1$, one has [129],

$$|y + \bar{y}|^{\gamma_i} < |y|^{\gamma_i} + |\bar{y}|^{\gamma_i}.$$

Therefore, defining $y = (\vec{d}_i^T(t)\tilde{W}_i(t))_k - (\bar{\varepsilon}_i(t))_k$ and $\bar{y} = (\bar{\varepsilon}_i(t))_k$, one obtains that for all $k = 1, ..., n$,

$$
\begin{aligned}
&|(\vec{d}_i^T(t)\tilde{W}_i(t))_k|^{\gamma_i} = |(\vec{d}_i^T(t)\tilde{W}_i(t))_k - (\bar{\varepsilon}_i(t))_k + (\bar{\varepsilon}_i(t))_k|^{\gamma_i} \\
&\leq |(\vec{d}_i^T(t)\tilde{W}_i(t))_k - (\bar{\varepsilon}_i(t))_k|^{\gamma_i} + |(\bar{\varepsilon}_i(t))_k|^{\gamma_i} \Rightarrow \\
&|(\vec{d}_i^T(t)\tilde{W}_i(t))_k|^{\gamma_i} - |(\bar{\varepsilon}_i(t))_k|^{\gamma_i} \leq |(\vec{d}_i^T(t)\tilde{W}_i(t))_k - (\bar{\varepsilon}_i(t))_k|^{\gamma_i},
\end{aligned}
$$

and then in the component-wise sense,

$$
-|\vec{d}_i^T(t)\tilde{W}_i(t) - \bar{\varepsilon}_i(t)|^{\gamma_i} \leq -|\vec{d}_i^T(t)\tilde{W}_i(t)|^{\gamma_i} + |\bar{\varepsilon}_i(t)|^{\gamma_i}. \tag{2.41}
$$

Now, using (2.41), (2.40) is upper bounded by

$$
\begin{aligned}
\dot{V}_i(t) \leq tr\{ &-\Xi_G \tilde{W}_i^T(t)\bar{d}_i(t)(\lfloor \vec{d}_i^T(t)\tilde{W}_i(t)\rceil^{\gamma_i} - |\bar{\varepsilon}_i(t)|^{\gamma_i} sign(\vec{d}_i^T(t)\tilde{W}_i(t))) \\
&- \Xi_C \tilde{W}_i^T(t) \sum_{h=1}^{P_i} \bar{d}_i(\tau_h)(\lfloor \vec{d}_i^T(\tau_h)\tilde{W}_i(t)\rceil^{\gamma_i} - |\bar{\varepsilon}_i(\tau_h)|^{\gamma_i} sign(\vec{d}_i^T(\tau_h)\tilde{W}_i(t)))\}.
\end{aligned} \tag{2.42}
$$

Recall that in $\lfloor . \rceil^{\gamma_i}$, $|.|$ and $sign(.)$ are employed in the component-wise sense, i.e.

$$
\begin{aligned}
\lfloor \vec{d}_i^T(t)\tilde{W}_i(t)\rceil^{\gamma_i} = [&|(\vec{d}_i^T(t)\tilde{W}_i(t))_1|^{\gamma_i} sign((\vec{d}_i^T(t)\tilde{W}_i(t))_1), \\
&|(\vec{d}_i^T(t)\tilde{W}_i(t))_2|^{\gamma_i} sign((\vec{d}_i^T(t)\tilde{W}_i(t))_2), \\
&\vdots \\
&|(\vec{d}_i^T(t)\tilde{W}_i(t))_n|^{\gamma_i} sign((\vec{d}_i^T(t)\tilde{W}_i(t))_n)],
\end{aligned}
$$

$$
\begin{aligned}
|\bar{\varepsilon}_i(t)|^{\gamma_i} sign(\vec{d}_i^T(t)\tilde{W}_i(t)) = [&|(\bar{\varepsilon}_i(t))_1|^{\gamma_i} sign((\vec{d}_i^T(t)\tilde{W}_i(t))_1), |(\bar{\varepsilon}_i(t))_2|^{\gamma_i} sign((\vec{d}_i^T(t)\tilde{W}_i(t))_2), \\
&\ldots, |(\bar{\varepsilon}_i(t))_n|^{\gamma_i} sign((\vec{d}_i^T(t)\tilde{W}_i(t))_n)].
\end{aligned}
$$

Therefore, using (2.33), (2.34), $\|\bar{d}_i(t)\| \leq 1$ and (2.42), one obtains,

$$
\begin{aligned}
\dot{V}_i(t) \leq &-\xi_G \|\tilde{W}_i^T(t)\bar{d}_i(t)\|^{\gamma_i+1} + \xi_G \|\tilde{W}_i(t)\| \||\bar{\varepsilon}_i(t)|^{\gamma_i}\| \\
&- \xi_C \sum_{h=1}^{P_i} \|\tilde{W}_i^T(t)\bar{d}_i(\tau_h)\|^{\gamma_i+1} + \xi_C P_i \||\bar{\varepsilon}_i(\tau_h)|^{\gamma_i}\| \|\tilde{W}_i(t)\|.
\end{aligned} \tag{2.43}
$$

Since $\| |\bar{\varepsilon}_i(t)|^{\gamma_i} \| = \sqrt{\sum_{k=1}^{n} |(\bar{\varepsilon}_i(t))_k|^{2\gamma_i}} = \|\bar{\varepsilon}_i(t)\|_{2\gamma_i}^{\gamma_i}$ and by Hölder's inequality

$$\|\bar{\varepsilon}_i(t)\|_{2\gamma_i} \leq n^{\frac{1-\gamma_i}{2\gamma_i}} \|\bar{\varepsilon}_i(t)\|, \tag{2.44}$$

holds for all $0 < 2\gamma_i < 2$, it is given that

$$\begin{aligned}\dot{V}_i(t) \leq &-\xi_C \sum_{h=1}^{P_i} (\tilde{W}_i^T(t)\bar{d}_i(\tau_h)\bar{d}_i^T(\tau_h)\tilde{W}_i(t))^{\frac{\gamma_i+1}{2}} + \xi_G n^{\frac{1-\gamma_i}{2}} \|\bar{\varepsilon}_i(t)\|^{\gamma_i} \|\tilde{W}_i(t)\| \\ &+ \xi_C P_i n^{\frac{1-\gamma_i}{2}} \|\bar{\varepsilon}_i(\tau_h)\|^{\gamma_i} \|\tilde{W}_i(t)\|.\end{aligned}$$

Therefore,

$$\dot{V}_i(t) \leq -\zeta_i \|\tilde{W}_i(t)\|^{\gamma_i+1} + \omega_i \|\tilde{W}_i(t)\|, \tag{2.45}$$

where

$$\omega_i = n^{\frac{1-\gamma_i}{2}} b_{\bar{\varepsilon}}^{\gamma_i} (\xi_G + P_i \xi_C)$$

.

In the following, (2.45) is rewritten as

$$\begin{aligned}\dot{V}_i(t) \leq &-\zeta_i \|\tilde{W}_i(t)\|^{\gamma_i+1} + \omega_i \|\tilde{W}_i(t)\| \\ \leq &-\zeta_i (1-\delta) \|\tilde{W}_i(t)\|^{\gamma_i+1} - \zeta_i \delta \|\tilde{W}_i(t)\|^{\gamma_i+1} + \omega_i \|\tilde{W}_i(t)\|,\end{aligned}$$

where $0 < \delta < 1$. Hence,

$$\dot{V}_i(t) \leq -\zeta_i (1-\delta) \|\tilde{W}_i(t)\|^{\gamma_i+1}, \;\; \bar{\mu}_i \leq \|\tilde{W}_i(t)\|, \tag{2.46}$$

where

$$\bar{\mu}_i = \begin{cases} max\{\frac{b_{\bar{\varepsilon}}}{\min\{\lambda_{min}^{\frac{1}{2}}(D_i(t)), \bar{\lambda}_i\}}, (\frac{\omega_i}{\zeta_i \delta})^{\frac{1}{\gamma_i}}\}, & \bar{d}_i(t) \neq 0, \\ max\{\frac{b_{\bar{\varepsilon}}}{\bar{\lambda}_i}, (\frac{\omega_i}{\zeta_i \delta})^{\frac{1}{\gamma_i}}\}, & \bar{d}_i(t) = 0. \end{cases}$$

From (2.31) and (2.46), it follows that

$$\dot{V}_i(t) \leq -\zeta_i (1-\delta) \beta_i^{\frac{\gamma_i+1}{2}} V_i^{\frac{\gamma_i+1}{2}}(t), \tag{2.47}$$

and by comparison principle one obtains

$$V_i(t) \leq (V_i^{\frac{1-\gamma_i}{2}}(0) - \frac{\zeta_i(1-\delta)(1-\gamma_i)\beta_i^{\frac{\gamma_i+1}{2}}}{2} t)^{\frac{2}{1-\gamma_i}},$$

then using (2.31), the above inequality ensures that $\tilde{W}_i(t)$ satisfies

$$\|\tilde{W}_i(t)\| \leq \sqrt{\alpha_i}(\beta_i^{\frac{\gamma_i-1}{2}} \|\tilde{W}_i(0)\|^{1-\gamma_i} - \frac{\zeta_i(1-\delta)(1-\gamma_i)\beta_i^{\frac{\gamma_i+1}{2}}}{2} t)^{\frac{1}{1-\gamma_i}},$$

for all $t < T_i(\tilde{W}_i(0))$. Then, for all $t > T_i(\tilde{W}_i(0))$, from (2.31), one obtains that $\tilde{W}_i(t)$ is bounded as

$$\|\tilde{W}_i(t)\| \leq \sqrt{\frac{\lambda_{max}(\Gamma_i)}{\lambda_{min}(\Gamma_i)}} \bar{\mu}_i, \quad \forall t \geq T_i(\tilde{W}_i(0)). \tag{2.48}$$

Therefore, for every subsystem i, $i = 1, ..., N$, the solutions of $\tilde{W}_i(t)$ are finite-time attractive to the bound in (2.48) where

$$T_i(\tilde{W}_i(0)) \leq \frac{2\|\tilde{W}_i(0)\|^{1-\gamma_i}}{\zeta_i\beta_i(1-\delta)(1-\gamma_i)}.$$

Therefore, all the solutions of $\tilde{W}_i$, $i = 1, ..., N$ for the interconnected system are finite-time attractive to the bound given in (2.37) in the following settling time,

$$T \leq \max_{i=1,\ldots,N} T_i(\tilde{W}_i(0)) = \max_{i=1,\ldots,N} \frac{2\|\tilde{W}_i(0)\|^{1-\gamma_i}}{\zeta_i\beta_i(1-\delta)(1-\gamma_i)}.$$

This completes the proof.

Corollary 2 Let the assumptions and statements of Theorem 2 hold. Then, for adaptive distributed approximators with non-zero MFAEs, i.e., $\bar{\varepsilon}_i(t) \neq 0$, the state estimation error $e_i(t)$ for every subsystem i, $i = 1, ..., N$, is finite-time attractive.

Proof 4 The proof is a direct consequence of Theorem 2.

Remark 7 In Theorem 2, for $\gamma_i = 0$, using (2.31) and (2.45) it can be shown that for every interconnected subsystem $i, i = 1, ..., N$, if $\zeta_i > \omega_i$, $\tilde{W}_i(t)$ is finite-time stable and the interconnected system can be exactly identified with zero MFAE.

Remark 8 As discussed in [79], the concurrent learning approach is based on the combination of a gradient descent algorithm with an auxiliary static feedback update law, which can be viewed as a type of σ-modification [10] and allows the requirement on persistence of excitation to be relaxed by keeping enough measurements in memory. Here, the same extension is applied to the proposed distributed finite-time concurrent learning in (2.28). Theoretical support of this claim is provided in Theorem 2 to show the finite-time attractiveness of the proposed parameter update law (2.28) in case of nonzero MFAEs.

Remark 9 For distributed adaptive approximators with non-zero MFAEs, the richness of the distributed data stored in M_i influences the finite settling time as well as the error bound. Accordingly, $\tilde{W}_i(t)$ converges to a narrower bound in faster time by maximizing $\lambda_{min}(S_i)$ that minimizes the error bound and the settling time respectively given in (2.37) and (2.39). Therefore, after the rank condition satisfaction, optimization over recorded data can improve the convergence results for every subsystem where one can replace new distributed samples with old ones in M_i, $i = 1, ..., N$, if $\lambda_{min}(S_i)$ increases to result in a faster convergence to a lower error bound.

Remark 10 Similar to the concurrent learning literature [30, 33–37, 39, 41] and most system identifiers for nonlinear systems, in this chapter it is assumed that the subsystem states are measurable. Even though the subsystems' states are measurable, the finite-time identification of interconnected systems without the persistency of excitation requirement is challenging. Online finite-time identification of the interconnected system dynamics under output measurements assumption is a direction for future research. This requires coupled distributed identifier and observer design for every subsystem to be able to identify the subsystem dynamics and observe its states interactively in finite-time.

Remark 11 In the proposed finite-time distributed concurrent learning estimation law (2.28), if the concurrent learning term regarding the past historical data is eliminated, the following finite-time distributed gradient descent estimation law that only depends on the current distributed data is obtained as

$$\dot{\hat{W}}_i(t) = -K_i \bar{d}_i(t) \lfloor e_i^T(t) \rceil^{\gamma_i}, \tag{2.49}$$

with $K_i > 0$. According to the analysis provided in the previous theorems, similar results are obtained for the estimation law (2.49) provided that $\bar{d}_i(t)$ is persistently excited for every subsystem

i. The finite-time distributed gradient descent law (2.49) is similar to the estimation law for a single system in Algorithm 1 of [106] where the short finite-time input to state stability of the mentioned learning law (ensuring stability in a finite and limited time interval) has been proven, provided that the regressor $\bar{d_i}(t)$ is nullifying in finite time.

2.4 Simulation Results

Now, the proposed finite-time distributed CL method performance for a nonlinear interconnected system identification is examined in comparison with the finite-time distributed gradient descent estimation method given in (2.49). The considered nonlinear interconnected system contains 3 inverted interconnected pendulums as depicted in Fig. 2.1. Every inverted pendulum i [131], subject to control input u_i is described by

$$\begin{cases} \dot{x}_{i1} = f_{i1}(x_i) + g_{i1}(x_i) + \Delta_{i1}(x_i(t), x_j(t)|_{j\in N_i}), \\ \quad = x_{i2} \\ \dot{x}_{i2} = f_{i2}(x_i) + g_{i2}(x_i) + \Delta_{i2}(x_i(t), x_j(t)|_{j\in N_i}) \\ \quad = \frac{g}{l} sinx_{i1} + \frac{u_i}{m_i l^2} + \sum_{j\in N_j} \frac{k_{i,j}a^2}{m_i l^2}(sinx_{j1}cosx_{j1} - sinx_{i1}cosx_{i1}), \end{cases} \tag{2.50}$$

where $x_{i1} = \theta_i\,(rad)$ is the angular position and $x_{i2} = \dot{\theta}_i\,(rad/s)$ is the angular velocity, for the inverted pendulum i, $i = 1,2,3$. The gravity acceleration g is $g \approx 10\,\frac{m}{s^2}$, m_i is the mass of the i^{th} rod ($m_i = 0.25\,kg$, $i = 1,2,3$), l is the length of each rod ($l = 2\,m$), a is the distance from the pivot to the center of gravity of the rod ($a = 1\,m$), $k_{i,j}$ $(\frac{kg}{s^2})$ is the spring constant which interconnects subsystem i to subsystem j, $j \in N_i$, with $k_{i,j} = k_{j,i}$ and $k_{1,2} = k_{1,3} = 1.5$, $k_{2,3} = 2$. In this system, due to the physical limitations, x_i domain is defined by $D_i = [D_{i1}, D_{i2}]^T$ where $D_{i1} = [-6,6]$ and $D_{i2} = [-4,4]$ for $i = 1,2,3$. The initial states and parameters are chosen from the interval $[-2,2]$ and stabilizing controllers $u_i = -0.06x_i$ for $i = 1,2,3$ are employed. Every interconnected subsystem i dynamics in (2.50) is unknown. For every subsystem i, $i = 1,2,3$, the proposed finite-time distributed concurrent learning identifier employs the following basis functions,

$$z_i(x_i, u_i, x_j|_{j\in N_i}) = [x_{i2}, \sin x_{i1}, u_i, \sin x_{i1}\cos x_{i1}, \sin x_{j1}\cos x_{j1}]^T_{j\in N_i}. \tag{2.51}$$

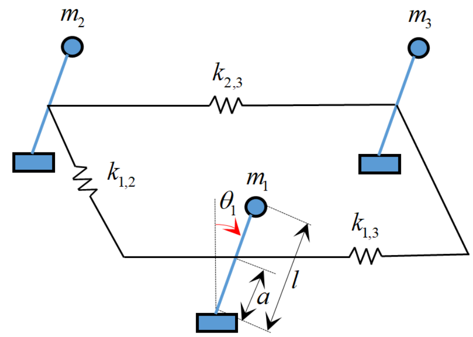

Figure 2.1: Interconnection network of the physically interconnected inverted pendulums.

While the regressor vector $z_i(x_i, u_i, x_j|_{j\in N_i})$ is exciting over some time period, it is not persistently exciting. The relaxed excitation condition without the PE requirement is achieved without injecting any exciting probing noise to the subsystems' controllers.

Therefore, the approximation of (2.50) for every subsystem i is as follows,

$$\begin{cases} \dot{x}_{i1} = p_1^i x_{i2}, \\ \dot{x}_{i2} = p_2^i sinx_{i1} + p_3^i u_i + p_4^i sinx_{i1}cosx_{i1} + \sum\limits_{j\in N_j, r=5,\ldots,4+|N_j|} p_r^i sinx_{j1}cosx_{j1}, \end{cases} \tag{2.52}$$

where based on the system descriptions, the true parameters for the three interconnected subsystems are as follows,

$$[p_1^1, p_2^1, p_3^1, p_4^1, p_5^1, p_6^1] = [1, 5, 1, -3, 1.5, 1.5],$$
$$[p_1^2, p_2^2, p_3^2, p_4^2, p_5^2, p_6^2] = [1, 5, 1, -3.5, 1.5, 2],$$
$$[p_1^3, p_2^3, p_3^3, p_4^3, p_5^3, p_6^3] = [1, 5, 1, -3.5, 2, 1.5].$$

We set $\Gamma_i = 3I$, $\xi_G = 1$, $\xi_C = 0.1$, $\gamma_i = 0.5$ for $i = 1, 2, 3$. We chose $\xi_G > \xi_C$, to prioritize current data to the recorded data in the proposed learning method (2.28) and $P_i = 10$, $i = 1, 2, 3$, which are set to be greater than 6, the number of independent basis functions for every subsystem. To have a fair speed and the precision comparison of the mentioned methods for approximating $\hat{f}_i(x_i)$ and $\hat{g}_i(x_i)$ on the domain D_i, and $\hat{\Delta}_i(x_i, x_j|_{j\in N_i})$ on the domain of D_{N_i}, the following online learning errors are computed for every subsystem i, $i = 1, 2, 3$,

$$E_{f_i}(t) = \int_{\mathcal{D}_i} \|e_{f_i}(x_i(t))\| d^n x_i, \quad E_{g_i}(t) = \int_{\mathcal{D}_i} \|e_{g_i}(x_i(t))\| d^n x_i,$$
$$E_{\Delta_i}(t) = \int_{\mathcal{D}_{N_i}} \|e_{\Delta_i}(x_{N_i}(t))\| d^{n(N_i+1)} x_{N_i}, \tag{2.53}$$

where the notation $\int_{\mathcal{D}_i} \|e_{f_i}(x_i(t))\| d^n x_i$ indicates that the integral of $\|e_{f_i}(x_i(t))\|$ is calculated over an n-dimensional region $\mathcal{D}_i$. The simulations are done in MATLAB with Euler integration and the sample time is equal to 0.05 seconds. In the simulation results, the proposed finite-time distributed concurrent learning method and finite-time distributed gradient descent approach, given in (2.49), are respectively labeled by FTDCL and FTDGD. Fig. 2.2 shows the approximated parameters using the proposed finite-time distributed concurrent learning approach and finite-time distributed gradient descent method (given in (2.49)) for three interconnected subsystems. Fig. 2.2 clearly shows that the approximated parameters using the proposed finite-time distributed concurrent learning method have converged to the true parameters, while because of the lack of persistence of excitation, the estimated parameters for the finite-time distributed gradient descent failed to converge to the true parameters. Figs. 2.3 to 2.5 depict the online learning errors $E_{f_i}(t)$, $E_{g_i}(t)$, and $E_{\Delta_i}(t)$ for, respectively, the three interconnected subsystems, $i = 1, 2, 3$, where the results of the proposed finite-time distributed concurrent learning show the finite-time convergence of all errors to zero while the learning errors for finite-time distributed gradient descent method did not converge to the origin due to the lack of regressor's PE condition.

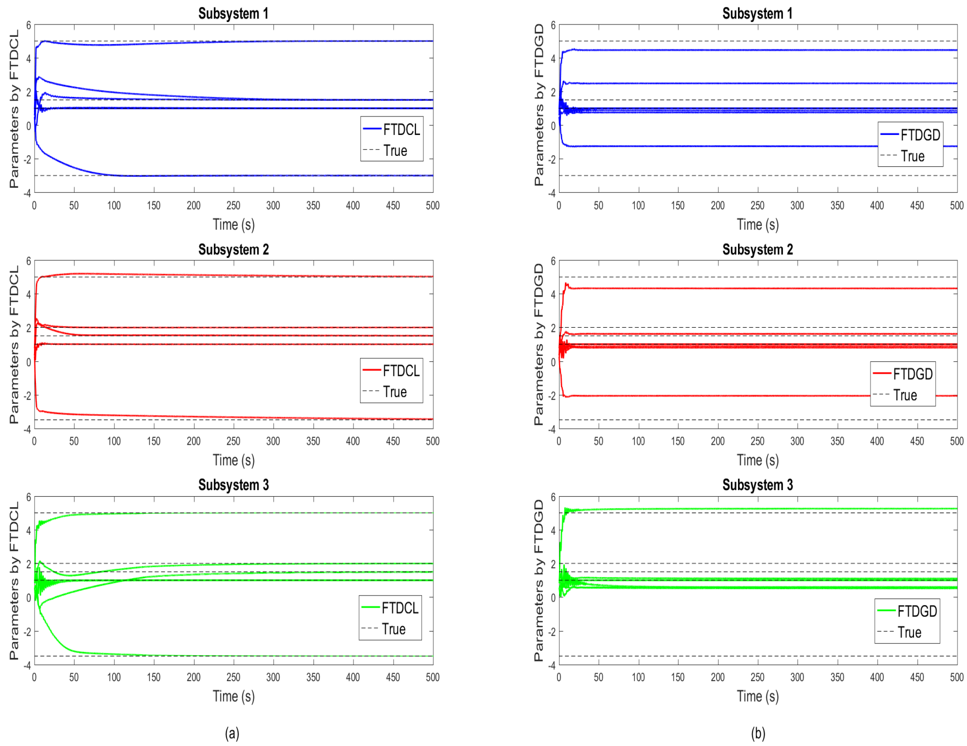

Figure 2.2: (a). Parameters of finite-time distributed concurrent learning (FTDCL) identifiers for subsystems 1, 2 and 3. (b). Parameters of finite-time distributed gradient descent (FTDGD) identifiers for subsystems 1, 2 and 3.

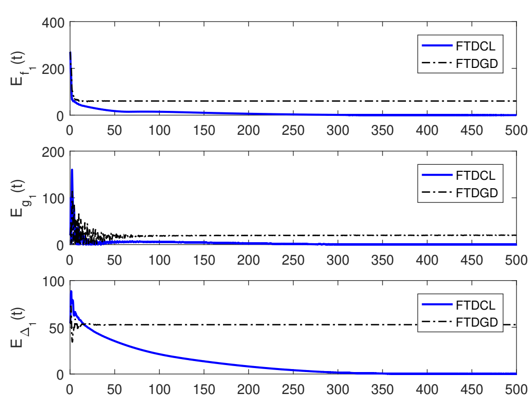

Figure 2.3: Online learning errors $E_{f_1}(t)$, $E_{g_1}(t)$, and $E_{\Delta_1}(t)$ for interconnected subsystem 1.

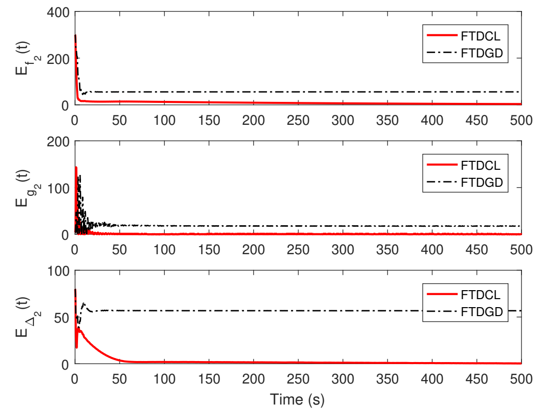

Figure 2.4: Online learning errors $E_{f_2}(t)$, $E_{g_2}(t)$, and $E_{\Delta_2}(t)$ for interconnected subsystem 2.

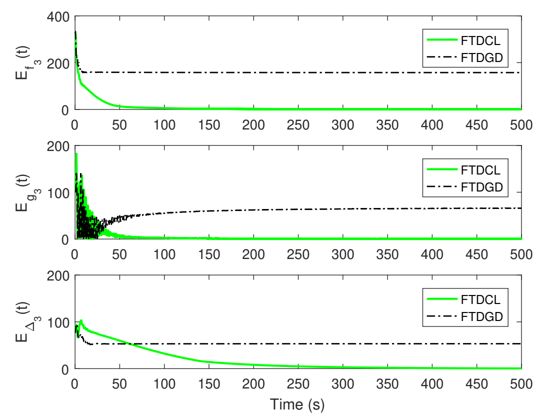

Figure 2.5: Online learning errors $E_{f_3}(t)$, $E_{g_3}(t)$, and $E_{\Delta_3}(t)$ for interconnected subsystem 3.

2.5 Conclusion

In this chapter, a finite-time distributed concurrent learning method for interconnected systems' identification in finite time is introduced. Leveraging local state communication among interconnected subsystems' identifiers enabled them to identify every subsystem's own dynamics as well as its interconnections' dynamics. In this method, distributed concurrent learning relaxed the regressors' persistence of excitation (PE) conditions to rank conditions on the recorded distributed data in the memory stack of the subsystems. It is shown that the precision and convergence speed of the proposed finite-time distributed learning method depends on the spectral properties of the distributed recorded data. Simulation results show that the proposed finite-time distributed concurrent learning has outperformed the finite-time distributed gradient descent in both terms of precision and convergence speed.

CHAPTER 3

FIXED-TIME SYSTEM IDENTIFICATION USING CONCURRENT LEARNING

3.1 Introduction

In this chapter, first, a novel discontinuous update law is presented that employs CL to identify system uncertainties in a fixed time that can be computed a priori. Fixed-time convergence guarantee is certified under a rank condition on recorded experienced data rather than PE condition. Second, the rigorous analysis based on fixed-time Lyapunov stability certifies the convergence of the discontinuous gradient flow equipped with CL to zero for the case where minimum functional approximation error (MFAE) is zero and under a rank condition on stored data. Moreover, for adaptive approximators with non-zero MFAE, it is ensured that by employing the proposed algorithm, the parameters estimation errors are fixed time attractive to an ultimate bound. Third, the fixed-time upper bounds of the estimated parameters' settling-times, independent of the initial parameter estimation error, are derived for adaptive approximators with zero and non-zero MFAEs.

Notation Throughout this chapter, the following notation is adopted. $\mathbb{R}$ and $\mathbb{R}_+$ denote the set of real and positive real numbers, respectively. $\|.\|$ is used to denote the Euclidean norm for a vector and induced 2-norm for a matrix. $tr(.)$ indicates trace of a matrix. $\lambda_{min}(A)$ and $\lambda_{max}(A)$ denote the minimum and maximum eigenvalues of matrix A respectively. I is the identity matrix of appropriate dimension.

3.2 Preliminaries and Problem Formulation

Preliminaries Consider

$$\dot{y}(t) = F(t, y), \quad y(0) = y_0, \tag{3.1}$$

where $y \in \mathcal{D}_y$, $F : \mathbb{R}_+ \times \mathcal{D}_y \mapsto \mathcal{D}_y$, is a nonlinear function on the open neighborhood $\mathcal{D}_y$ of the origin. Assume the origin is an equilibrium point of (3.1).

Definition 3 [29] The bounded signal $y(t)$ is said to be persistently exciting if there exist positive scalars μ_1, μ_2 and $\mathcal{T} \in \mathbb{R}_+$ such that $\forall t \in \mathbb{R}_+$, $\mu_1 I \leq \int_t^{t+\mathcal{T}} y(\tau)y^T(\tau)d\tau \leq \mu_2 I$.

Definition 4 [65] The system (3.1) is said to be

1) fixed-time stable, if it is asymptotically stable and $\forall y_0 \in \mathcal{D}_y$ any solution $y(t)$ of (3.1) reaches the equilibrium point at some finite-time moment, i.e., $y(t) = 0, \forall t \geq T(y_0)$, where $T : \mathcal{D}_y \mapsto \mathbb{R}_+ \cup \{0\}$ is the settling-time function and the settling-time function $T(y_0)$ is bounded, i.e., $\exists T_{max} > 0 : T(y_0) \leq T_{max}, \forall y_0 \in \mathcal{D}_y$.

2) fixed-time attractive by a bounded set $\mathcal{Y}$ around zero, if $\forall y_0 \in \mathcal{D}_y$ any solution $y(t)$ of (1) reaches $\mathcal{Y}$ in some finite-time moment $t = T(y_0)$ and remains there, $\forall t \geq T(y_0), T : \mathcal{D}_y \mapsto \mathbb{R}_+ \cup \{0\}$ is the settling-time function and the settling-time function $T(y_0)$ is bounded by some $T_{max} > 0$.

Lemma 2 [65] Let there exist a continuous positive definite function $V : \mathcal{D}_y \mapsto \mathbb{R}_+ \cup \{0\}$ in an open neighborhood of the origin and real positive numbers $\alpha, \beta, r_1, r_2 > 0$ such that $0 < r_1 < 1$ and $1 < r_2$. Let also any solution $y(t)$ of (3.1) satisfy the inequality,

$$\dot{V}(y(t)) \leq -\alpha V^{r_1}(y(t)) - \beta V^{r_2}(y(t)). \tag{3.2}$$

Then, the system (3.1) is fixed-time stable and

$$T(y_0) \leq \frac{1}{\alpha(1 - r_1)} + \frac{1}{\beta(r_2 - 1)}. \tag{3.3}$$

Fact 1. In general, $\forall x \in \mathbb{R}^n$ with $0 < r < s$, one has

$$\|x\|_s \leq \|x\|_r \leq n^{\frac{1}{r} - \frac{1}{s}} \|x\|_s.$$

This is a consequence of Hölder inequality [129].

Problem Formulation Consider the following nonlinear system,

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t), \tag{3.4}$$

where $x \in \mathcal{D}_x \subset \mathbb{R}^n$ and $u \in \mathcal{D}_u \subset \mathbb{R}^m$ are the state and input vectors, respectively; $\mathcal{D}_x$ and $\mathcal{D}_u$ are compact sets. Let $f : \mathcal{D}_x \mapsto \mathbb{R}^n$ and $g : \mathcal{D}_x \mapsto \mathbb{R}^{n \times m}$ be the unknown nonlinear system and input dynamics, respectively. The overarching objective of this chapter is to present novel model learning approaches to learn these uncertain dynamics in a fixed time.

Assumption 3 $x(t)$ is a measurable state vector, and $f(x(t))$ and $g(x(t))$ are both locally Lipschitz in $x(t)$.

Here, linearly parameterized adaptive approximation models [10] are used to, respectively, represent $f(x(t))$ and $g(x(t))$ as follows,

$$f(x(t)) = \hat{f}(x(t), \Theta_f^*) + e_f(x(t)), \tag{3.5}$$

$$g(x(t)) = \hat{g}(x(t), \Theta_g^*) + e_g(x(t)), \tag{3.6}$$

where

$$\hat{f}(x(t), \Theta_f^*) = \Theta_f^{*T} \varphi(x(t)), \tag{3.7}$$

$$\hat{g}(x(t), \Theta_g^*) = \Theta_g^{*T} \chi(x(t)). \tag{3.8}$$

The matrices $\Theta_f{}^* \in \mathcal{D}_f \subset \mathbb{R}^{p\times n}$ and $\Theta_g{}^* \in \mathcal{D}_g \subset \mathbb{R}^{q\times n}$ denote the unknown optimal parameters of the adaptive approximation models, defined as follows

$$\Theta_f^* = \arg \min_{\Theta_f \in \mathcal{D}_f} \{ \sup_{x(t)\in\mathcal{D}_x} \left\| f(x(t)) - \hat{f}(x(t), \Theta_f) \right\| \}, \tag{3.9}$$

$$\Theta_g^* = \arg \min_{\Theta_g \in \mathcal{D}_g} \{ \sup_{x(t)\in\mathcal{D}_x} \left\| g(x(t)) - \hat{g}(x(t), \Theta_g) \right\| \}, \tag{3.10}$$

where $\mathcal{D}_f$ and $\mathcal{D}_g$ are compact sets. The vectors $\varphi : \mathcal{D}_x \mapsto \mathbb{R}^p$ and $\chi : \mathcal{D}_x \mapsto \mathbb{R}^q$, denote the basis functions, while p and q are the number of linearly independent basis functions for approximating $f(x(t))$ and $g(x(t))$, respectively. The quantities $e_f(x(t)) \in \mathbb{R}^n$ and $e_g(x(t)) \in \mathbb{R}^{n\times m}$ are the MFAEs for $f(x(t))$ and $g(x(t))$, respectively, representing the residual approximation error in the case of optimal parameters. In the special case that the unknown functions $f(x(t))$ and $g(x(t))$ can be approximated exactly by the adaptive approximation models $\hat{f}(x(t), \Theta_f)$ and $\hat{g}(x(t), \Theta_g)$, respectively, then $e_f(x(t)) = e_g(x(t)) = 0$.

Using (3.5)-(3.8), the system dynamics (3.4) can be written as

$$\dot{x}(t) = \Theta^{*T} z(x(t), u(t)) + \varepsilon(x(t), u(t)), \tag{3.11}$$

where $\Theta^* = [\Theta_f^{*T}, \Theta_g^{*T}]^T \in \mathbb{R}^{(p+q)\times n}$, $z(x(t),u(t)) = [\varphi^T(x(t)), u^T(t)\chi^T(x(t))]^T \in \mathbb{R}^{(p+q)}$, and $\varepsilon(x(t),u(t)) = e_f(x(t)) + e_g(x(t))u(t)$.

Assumption 4 For the given compact sets $\mathcal{D}_x$ and $\mathcal{D}_u$, the approximators' basis functions are bounded. Moreover, the approximation error $\varepsilon(x(t),u(t))$ is bounded by an upper bound $b_\varepsilon \geq 0$, i.e.,

$$\sup_{x\in\mathcal{D}_x, u\in\mathcal{D}_u} \|\varepsilon(x,u)\| \leq b_\varepsilon$$

.

Remark 12 Assumption 3 ensures the existence and uniqueness of the solution of system (3.4) and Assumption 4 is standard in the literature based on universal approximator characteristics [29].

In this chapter, since, $\dot{x}(t)$ is not available for measurement, a regressor filtering method [39] is used to obviate its requirement in the presented update law. Therefore, to proceed with regressor filtering, dynamics (3.11) is written as

$$\dot{x}(t) = -Cx(t) + \Theta^{*T} z(x(t),u(t)) + Cx(t) + \varepsilon(t), \tag{3.12}$$

where $C = cI, c > 0$. The solution of (3.11) can be expressed as

$$x(t) = \Theta^{*T} d(t) + Cl(t) + e^{-Ct}x(0) + \varepsilon_f(t), \tag{3.13}$$

$$\dot{d}(t) = -cd(t) + z(x(t),u(t)),\ d(0) = 0,$$
$$\dot{l}(t) = -Cl(t) + x(t),\ l(0) = 0, \tag{3.14}$$

where $l(t) = \int_0^t e^{-C(t-\tau)}x(\tau)d\tau$ is the filtered regressor of $x(t)$, $d(t) = \int_0^t e^{-c(t-\tau)}z(x(\tau),u(\tau))d\tau$ is the filtered regressor of $z(x(t),u(t))$, $\varepsilon_f(t) = \int_0^t e^{-C(t-\tau)}\varepsilon(\tau)d\tau$, and $x(0)$ is the initial state of (3.12).

Dividing (3.13) by the normalizing signal $n_s = 1 + d^T(t)d(t) + l^T(t)l(t)$, one has,

$$\bar{x}(t) = \Theta^{*T}\bar{d}(t) + c\bar{l}(t) + e^{-Ct}\bar{x}(0) + \bar{\varepsilon}(t), \tag{3.15}$$

where $\bar{d} = \frac{d}{n_s}$, $\bar{l} = \frac{l}{n_s}$, $\bar{x} = \frac{x}{n_s}$ and $\bar{\varepsilon} = \frac{\varepsilon_f}{n_s}$. Note that Assumption 4 implies that $\bar{\varepsilon}(t)$ is also bounded, i.e.,

$$\sup_{x\in\mathcal{D}_x, u\in\mathcal{D}_u} \|\bar{\varepsilon}(x,u)\| \leq b_{\bar{\varepsilon}}$$

and $\|\bar{d}(t)\| < 1$.

Now, let the approximator of (3.15) for the system (3.4) be of the form

$$\hat{\bar{x}}(t) = \hat{\Theta}^T(t)\bar{d}(t) + c\bar{l}(t) + e^{-Ct}\bar{x}(0), \tag{3.16}$$

where $\hat{\Theta}(t) = [\hat{\Theta}_f^T(t), \hat{\Theta}_g^T(t)]^T \in \mathbb{R}^{(p+q)\times n}$, $\hat{\Theta}_f(t)$ and $\hat{\Theta}_g(t)$ are, respectively, the estimation of parameters matrices Θ^*, Θ_f^* and Θ_g^* at time t. The state estimation error, $e(t)$, for system (3.4) is defined as

$$e(t) = \hat{\bar{x}}(t) - \bar{x}(t) = \tilde{\Theta}^T(t)\bar{d}(t) - \bar{\varepsilon}(t), \tag{3.17}$$

where $\tilde{\Theta}(t) := \hat{\Theta}(t) - \Theta^* := [\tilde{\Theta}_f^T(t), \tilde{\Theta}_g^T(t)]^T$ is the parameter estimation error with

$$\tilde{\Theta}_f(t) := \hat{\Theta}_f(t) - \Theta_f^*, \tilde{\Theta}_g(t) := \hat{\Theta}_g(t) - \Theta_g^*$$

.

To fulfill the fixed-time learning of the uncertainties $f(x)$ and $g(x)$ in the system (3.4) without the requirement of the PE condition on the stream of data, a fixed-time CL method is presented next to guarantee that the parameter estimation error $\tilde{\Theta}(t)$ dynamics are:
1) fixed-time stable for adaptive approximators with zero MFAE.
2) fixed-time attractive within a bounded set around zero for adaptive approximators with non-zero MFAE.

3.3 Fixed-time Concurrent Learning Identifier

In this section, a novel fixed-time update law is presented to approximate the uncertainties of the system (3.4) that leverages the CL in its adaptive update law to eliminate the requirement of the PE condition. In this section, the introduced update law employs discontinuous gradient flows of the estimation errors to optimize the estimation error for current samples as well as samples collected in a recorded data stack, and the convergence analysis of the dynamics of the gradient update law is presented based on fixed-time Lyapunov stability.

To employ the CL technique, which uses recorded experienced data along with current data in the update law, the past data is collected and stored in the memory stacks $M \in \mathbb{R}^{(p+q)\times P}$,

$L \in \mathbb{R}^{n \times P}$ and $X \in \mathbb{R}^{n \times P}$, at times $\tau_1, ..., \tau_P$ as,

$$M = [\bar{d}(\tau_1), \bar{d}(\tau_2), ..., \bar{d}(\tau_P)], \quad L = [\bar{l}(\tau_1), \bar{l}(\tau_2), ..., \bar{l}(\tau_P)],$$
$$X = [\bar{x}(\tau_1), \bar{x}(\tau_2), ..., \bar{x}(\tau_P)], \tag{3.18}$$

where P is the number of data points stored in every stack. The number of data points P is chosen so that M contains as many linearly independent elements as the dimension of $d(t)$ (i.e., the total number of linearly independent basis functions for $f(x(t))$ and $g(x(t))$), given in (3.13). That is the rank of M must be $p + q$ which requires $P \geq p + q$.

Define the error $e_h(t)$ for the h^{th} recorded sample as

$$e_h(t) = \hat{\bar{x}}_h(t) - \bar{x}(\tau_h), \tag{3.19}$$

where

$$\hat{\bar{x}}_h(t) = \hat{\Theta}^T(t)\bar{d}(\tau_h) + c\bar{l}(\tau_h) + e^{-Ct}\bar{x}(0), \tag{3.20}$$

is the state estimation at time $0 \leq \tau_h < t$, $h = 1, ..., P$, using the current estimated parameters matrix $\hat{\Theta}(t)$ and the recorded $\bar{d}(\tau_h)$ and $\bar{l}(\tau_h)$. Substituting $\bar{x}(\tau_h)$, from (3.15), in (3.19) leads to

$$e_h(t) = \tilde{\Theta}^T(t)\bar{d}(\tau_h) - \bar{\varepsilon}(\tau_h). \tag{3.21}$$

In the proposed FxTCL method that is presented next, the stored data in M is selected based on data recording algorithm in [34, 36] to maximize

$$\frac{\lambda_{min}(S)}{\lambda_{max}(S)}$$

where $S = \sum_{h=1}^{P} \bar{d}(\tau_h)\bar{d}^T(\tau_h)$.

Fixed-time concurrent learning update law The proposed fixed-time CL update law for the parameters in the system approximator (3.16) is given as

$$\dot{\hat{\Theta}}(t) = -\Gamma[\Xi_G \bar{d}(t)(\lfloor e^T(t)\rceil^{\gamma_1} + \lfloor e^T(t)\rceil^{\gamma_2}) + \Xi_C \sum_{h=1}^{P} \bar{d}(\tau_h)(\lfloor e_h^T(t)\rceil^{\gamma_1} + \lfloor e_h^T(t)\rceil^{\gamma_2})], \tag{3.22}$$

where $\lfloor . \rceil^{\gamma} := |.|^{\gamma} sign(.)$ with $|.|$ and $sign(.)$ understood in component-wise sense and $0 \leq \gamma_1 < 1$, $\gamma_2 > 1$. The matrices $\Gamma, \Xi_G, \Xi_C \in \mathbb{R}^{(p+q)\times(p+q)}$, $\Gamma > 0$ is the positive definite learning rate matrix, $\Xi_C = \xi_C I$ and $\Xi_G = \xi_G I$ with positive constants $\xi_C > 0$ and $\xi_G > 0$. The above update law has two learning terms, the first term $\Xi_G \bar{d}(t)(\lfloor e^T(t) \rceil^{\gamma_1} + \lfloor e^T(t) \rceil^{\gamma_2})$ containing the current state approximation error that is a nonlinear gradient descent term, and the second term, $\Xi_C \sum_{h=1}^{P} \bar{d}(\tau_h)(\lfloor e_h^T(t) \rceil^{\gamma_1} + \lfloor e_h^T(t) \rceil^{\gamma_2})$, contains the experienced data, is the CL term. The weights Ξ_C and Ξ_G do not need to be equal and by setting appropriate ξ_C and ξ_G, respectively, one of the two learning terms can be prioritized over the other.

Remark 13 In the update law (3.22) by leveraging CL technique, discontinuous gradient flows of the current and stored identification errors are concurrently employed to, respectively, minimize the estimation error for the current stream of data and recorded memory samples. Discontinuous gradient-based adaptation of past data enables the update law (3.22) to converge to the optimal parameters in a fixed time regardless of the initial parameters' estimation error. Therefore, given the recorded data, the fixed time of convergence can be computed a priori in this method.

The convergence properties of the proposed method are investigated for adaptive approximators with zero and non-zero MFAEs in the following.

Fixed-time Convergence Properties for Adaptive Approximators with Zero MFAEs ($\bar{\varepsilon}(t) = 0$)

The following theorem demonstrates the fixed-time convergence of the estimated parameters to their optimal values for the proposed FxTCL method (3.22), in adaptive approximators with zero MFAEs, i.e., $\bar{\varepsilon}(t) = 0$.

Theorem 3 Consider the approximator for nonlinear system (3.4) given in (3.16), whose parameters are adjusted according to the update law of (3.22) with $0 \leq \gamma_1 < 1$, $\gamma_2 > 1$ and a regressor given in (3.14). Let Assumptions 3-4 hold. Once the rank condition on M is met, then for adaptive approximators with zero MFAEs, $\bar{\varepsilon}(t) = 0$, the proposed update law (3.22) guarantees the fixed-time convergence of $\tilde{\Theta}(t)$ to zero for $t > T$ where the settling-time is bounded by $T \leq T_{max}$

and

$$T_{max} = \frac{2}{\alpha_1 c_2^{\frac{\gamma_1+1}{2}}(1-\gamma_1)} + \frac{2}{\alpha_2 c_2^{\frac{\gamma_2+1}{2}}(\gamma_2-1)}, \tag{3.23}$$

such that $c_2 = 2\lambda_{min}(\Gamma)$, $\alpha_1 = \xi_C \lambda_{min}^{\frac{\gamma_1+1}{2}}(S)$, $\alpha_2 = \xi_C n^{\frac{1-\gamma_2}{2}} \lambda_{min}^{\frac{\gamma_2+1}{2}}(S)$.

Proof 5 Consider the Lyapunov function candidate

$$V(t) = \frac{1}{2} tr\{\tilde{\Theta}^T(t)\Gamma^{-1}\tilde{\Theta}(t)\}. \tag{3.24}$$

We know that

$$c_1^{-1}\|\tilde{\Theta}(t)\|^2 \leq V(t) \leq c_2^{-1}\|\tilde{\Theta}(t)\|^2, \tag{3.25}$$

where $c_1 = 2\lambda_{max}(\Gamma), c_2 = 2\lambda_{min}(\Gamma)$.

The time derivative of V using (3.17), (3.21) and (3.22) yields,

$$\begin{aligned}\dot{V}(t) =& tr\{\tilde{\Theta}^T(t)\Gamma^{-1}\dot{\tilde{\Theta}}(t)\} \\ =& tr\{-\Xi_G\tilde{\Theta}^T(t)\bar{d}(t)(\lfloor \bar{d}^T(t)\tilde{\Theta}(t)\rceil^{\gamma_1} + \lfloor \bar{d}^T(t)\tilde{\Theta}(t)\rceil^{\gamma_2}) - \Xi_C\tilde{\Theta}^T(t)\sum_{h=1}^{P}\bar{d}(\tau_h)(\lfloor \bar{d}^T(\tau_h)\tilde{\Theta}(t)\rceil^{\gamma_1} \\ &+ \lfloor \bar{d}^T(\tau_h)\tilde{\Theta}(t)\rceil^{\gamma_2})\}. \end{aligned} \tag{3.26}$$

One knows that

$$\begin{aligned}\tilde{\Theta}^T(t)\bar{d}(t)\lfloor \bar{d}^T(t)\tilde{\Theta}(t)\rceil^{\gamma_1} &= \sum_{i=1}^{n}|(\tilde{\Theta}^T(t)\bar{d}(t))_i|^{\gamma_1+1} \\ &= \|\tilde{\Theta}^T(t)\bar{d}(t)\|_{\gamma_1+1}^{\gamma_1+1}, \end{aligned} \tag{3.27}$$

and using Fact 1

$$\|\tilde{\Theta}^T(t)\bar{d}(t)\| \leq \|\tilde{\Theta}^T(t)\bar{d}(t)\|_{\gamma_1+1}, \tag{3.28}$$

holds for $0 < \gamma_1 + 1 < 2$. By using (3.27), (3.28) and

$$\|\tilde{\Theta}^T(t)\bar{d}(t)\| \leq n^{\frac{\gamma_2-1}{2(\gamma_2+1)}}\|\tilde{\Theta}^T(t)\bar{d}(t)\|_{\gamma_2+1}, \tag{3.29}$$

that holds based on Fact 1 for $0 < \gamma_1 + 1 < 2 < \gamma_2 + 1$, one obtains,

$$\dot{V}(t) \leq -\xi_G(\|\tilde{\Theta}^T(t)\bar{d}(t)\|^{\gamma_1+1}+n^{\frac{1-\gamma_2}{2}}\|\tilde{\Theta}^T(t)\bar{d}(t)\|^{\gamma_2+1}) - \xi_C(\sum_{h=1}^{P}\|\tilde{\Theta}^T(t)\bar{d}(\tau_h)\|^{\gamma_1+1} + n^{\frac{1-\gamma_2}{2}}\sum_{h=1}^{P}\|\tilde{\Theta}^T(t)\bar{d}^T(\tau_h)\|^{\gamma_2+1}).$$

Thus,

$$\dot{V}(t) \leq -\xi_C\sum_{h=1}^{P}(\tilde{\Theta}^T(t)\bar{d}(\tau_h)\bar{d}^T(\tau_h)\tilde{\Theta}(t))^{\frac{\gamma_1+1}{2}} - \xi_C n^{\frac{1-\gamma_2}{2}}\sum_{h=1}^{P}(\tilde{\Theta}^T(t)\bar{d}(\tau_h)\bar{d}^T(\tau_h)\tilde{\Theta}(t))^{\frac{\gamma_2+1}{2}}. \tag{3.30}$$

One can rewrite (3.30) as follows,

$$\dot{V}(t) \leq -\alpha_1\|\tilde{\Theta}(t)\|^{\gamma_1+1}-\alpha_2\|\tilde{\Theta}(t)\|^{\gamma_2+1}, \tag{3.31}$$

where $\alpha_1 = \xi_C\lambda_{min}^{\frac{\gamma_1+1}{2}}(S)$, $\alpha_2 = \xi_C n^{\frac{1-\gamma_2}{2}}\lambda_{min}^{\frac{\gamma_2+1}{2}}(S)$ and since, $S = \sum_{h=1}^{P}\bar{d}(\tau_h)\bar{d}^T(\tau_h) > 0$, we have $\alpha_1 > 0$ and $\alpha_2 > 0$.

Employing (3.25), (3.31) gives

$$\dot{V}(t) \leq -\alpha_1 c_2^{\frac{\gamma_1+1}{2}}V^{\frac{\gamma_1+1}{2}}(t) - \alpha_2 c_2^{\frac{\gamma_2+1}{2}}V^{\frac{\gamma_2+1}{2}}(t). \tag{3.32}$$

Let us introduce the following inequalities

$$V^{\gamma_2+1}(t) \leq V^{\gamma_1+1}(t) \leq V(t) \ \ ,\forall V(t) \leq 1, \tag{3.33}$$

$$V^{\gamma_2+1}(t) > V^{\gamma_1+1}(t) > V(t) \ \ ,\forall V(t) > 1. \tag{3.34}$$

Hence, from (3.25), (3.32), and (3.34), when $V(t) > 1$, one obtains

$$\dot{V}(t) \leq -\alpha_2 c_2^{\frac{\gamma_2+1}{2}}V^{\frac{\gamma_2+1}{2}}(t). \tag{3.35}$$

Thus, for any $\tilde{\Theta}(t)$ such that $V(\tilde{\Theta}(0)) > 1$, (3.35) ensures $V(\tilde{\Theta}(t)) \leq 1$ for all $t \geq T_1 = \frac{2}{\alpha_2 c_2^{\frac{\gamma_2+1}{2}}(\gamma_2-1)}$.

Then when $V(t) \leq 1$, using inequality (3.33) for every $\gamma_2 + 1 > \gamma_1 + 1 > 1$, it follows from (3.32) that,

$$\dot{V}(t) \leq -\alpha_1 c_2^{\frac{\gamma_1+1}{2}} V^{\frac{\gamma_1+1}{2}}(t). \tag{3.36}$$

and we derive $V(\tilde{\Theta}(t)) = 0$ for $t \geq T_2$ where $T_2 = \frac{2}{\alpha_1 c_2^{\frac{\gamma_1+1}{2}}(1-\gamma_1)}$. Therefore, $V(\tilde{\Theta}(t)) = 0$ for $\forall t \geq T_{max}$ where

$$T_{max} = T_1 + T_2 = \frac{2}{\alpha_1 c_2^{\frac{\gamma_1+1}{2}}(1-\gamma_1)} + \frac{2}{\alpha_2 c_2^{\frac{\gamma_2+1}{2}}(\gamma_2 - 1)},$$

and it implies that $\tilde{\Theta}(t) = 0$ for $\forall t \geq T_{max}$.

Fixed-time Convergence Properties for Adaptive Approximators with Non-zero MFAEs ($\bar{\varepsilon}(t) \neq 0$)

The fixed-time convergence properties of the proposed FxTCL update law for adaptive approximators with non-zero MFAEs, $\bar{\varepsilon}(t) \neq 0$, are given in the next theorem.

Theorem 4 Consider the approximator for nonlinear system (3.4), given in (3.16), with parameters adjusted by the update law of (3.22) with $0 \leq \gamma_1 < 1, \gamma_2 > 1$ and a regressor given in (3.14). Let Assumptions 3-4 and the rank condition on M hold. Then, for adaptive approximators with non-zero MFAEs, the proposed update law (3.22) guarantees that

1) for $\gamma_1 = 0$, if

$$\max\{\sqrt{\frac{2\lambda_{max}(\Gamma)}{(2\lambda_{min}(\Gamma))^{\gamma_2+1}}}, \sqrt{\frac{\lambda_{max}(\Gamma)}{\lambda_{min}(\Gamma)}}\} < \frac{\min\{\alpha_4, \alpha_3\}}{\omega}$$

, then $\tilde{\Theta}(t)$ is fixed-time convergent to zero for $t > T$ and $T \leq T_{max}$ with

$$T_{max} = \frac{2}{\alpha(\gamma_2 - 1)} + \frac{2}{\alpha_3\sqrt{c_2} - \omega\sqrt{c_1}}, \tag{3.37}$$

2) for $0 < \gamma_1 < 1$, if $\sqrt{\frac{2\lambda_{max}(\Gamma)}{(2\lambda_{min}(\Gamma))^{\gamma_2+1}}} < \frac{\alpha_4}{\omega}$, then $\tilde{\Theta}(t)$ is fixed-time attractive with the following bound

$$\|\tilde{\Theta}(t)\| \leq \sqrt{\frac{\lambda_{max}(\Gamma)}{\lambda_{min}(\Gamma)}} min\{\sqrt{2\lambda_{max}(\Gamma)}, \bar{\mu}\}, \quad \forall t \geq T, \tag{3.38}$$

such that $T \leq T_{max}$,

$$T_{max} = \frac{2}{\alpha(\gamma_2 - 1)} + \frac{2(1-(c_2^{-0.5}\min(\bar{\mu}, \sqrt{c_1}))^{1-\gamma_1})}{\alpha_3(1-\delta)(1-\gamma_1)c_2^{\frac{\gamma_1+1}{2}}}, \tag{3.39}$$

$$\bar{\mu} = \begin{cases} max\{\frac{b_{\bar{\varepsilon}}}{\min\{\lambda_{min}^{\frac{1}{2}}(D(t)), \bar{\lambda}_h\}}, (\frac{\omega}{\alpha_3\delta})^{\frac{1}{\gamma_1}}\}, & \bar{d}(t) \neq 0, \\ max\{\frac{b_{\bar{\varepsilon}}}{\bar{\lambda}_h}, (\frac{\omega}{\alpha_3\delta})^{\frac{1}{\gamma_1}}\}, & \bar{d}(t) = 0, \end{cases} \tag{3.40}$$

where

$$\bar{\lambda}_h = \min_{h=1,...,P} \lambda_{min}^{\frac{1}{2}}(D(\tau_h))\}, \ D(t) = \bar{d}(t)\bar{d}^T(t), \alpha = \alpha_3 c_2^{\frac{\gamma_2+1}{2}} - \omega\sqrt{c_1},$$

$$\alpha_3 = \xi_G \lambda_{min}^{\frac{\gamma_1+1}{2}}(D(t)) + \xi_C \lambda_{min}^{\frac{\gamma_1+1}{2}}(S), \alpha_4 = 2^{1-\gamma_2} n^{\frac{1-\gamma_2}{2}}(\xi_G \lambda_{min}^{\frac{\gamma_2+1}{2}}(D(t)) + \xi_C \lambda_{min}^{\frac{\gamma_2+1}{2}}(S)),$$

$$\omega = (\xi_G + P\xi_C)[n^{\frac{2-\gamma_1}{4}} b_{\bar{\varepsilon}}^{\gamma_1} + b_{\bar{\varepsilon}}^{\gamma_2}].$$

Proof 6 Consider the Lyapunov function candidate (3.24) that satisfies (3.25). The time derivative of V using (3.17), (3.21) and (3.22) yields,

$$\begin{aligned} \dot{V}(t) =& tr\{-\Xi_G \tilde{\Theta}^T(t)\bar{d}(t)(\lfloor \bar{d}^T(t)\tilde{\Theta}(t) - \bar{\varepsilon}^T(t)\rceil^{\gamma_1} + \lfloor \bar{d}^T(t)\tilde{\Theta}(t) - \bar{\varepsilon}^T(t)\rceil^{\gamma_2}) \\ & - \Xi_C \tilde{\Theta}^T(t) \sum_{h=1}^{P} \bar{d}(\tau_h)(\lfloor \bar{d}^T(\tau_h)\tilde{\Theta}(t) - \bar{\varepsilon}^T(\tau_h)\rceil^{\gamma_1} + \lfloor \bar{d}^T(\tau_h)\tilde{\Theta}(t) - \bar{\varepsilon}^T(\tau_h)\rceil^{\gamma_2})\}. \end{aligned} \tag{3.41}$$

Consider in the component-wise sense that $|(\bar{d}^T(t)\tilde{\Theta}(t))_i| \geq |\bar{\varepsilon}_i(t)|$, for $i = 1, ..., n$. It is worth mentioning that the last inequality is required when $\bar{d}(t) \neq 0$. Because, if $\bar{d}(t) = 0$ then the first term in (3.22) will be zero and in the second term of (3.22), called the CL term, the data collection algorithm ensures that $\bar{d}(\tau_h) \neq 0, h = 1, ..., P$.

Therefore, $sign(\bar{d}^T(t)\tilde{\Theta}(t) - \bar{\varepsilon}^T(t)) = sign(\bar{d}^T(t)\tilde{\Theta}(t))$ is implied. For any $y_1, y_2 \in \mathbb{R}$ and $0 \leq \gamma_1 < 1$, the following inequality holds [129],

$$|y_1 + y_2|^{\gamma_1} \leq |y_1|^{\gamma_1} + |y_2|^{\gamma_1}.$$

Therefore, defining $y_1 = (\bar{d}^T(t)\tilde{\Theta}(t))_i - \bar{\varepsilon}_i(t)$ and $y_2 = \bar{\varepsilon}_i(t)$, one obtains that for all $i = 1, ..., n$

$$|(\bar{d}^T(t)\tilde{\Theta}(t))_i|^{\gamma_1} = |(\bar{d}^T(t)\tilde{\Theta}(t))_i - \bar{\varepsilon}_i(t) + \bar{\varepsilon}_i(t)|^{\gamma_1} \leq |(\bar{d}^T(t)\tilde{\Theta}(t))_i - \bar{\varepsilon}_i(t)|^{\gamma_1} + |\bar{\varepsilon}_i(t)|^{\gamma_1} \Rightarrow$$

$$|(\bar{d}^T(t)\tilde{\Theta}(t))_i|^{\gamma_1} - |\bar{\varepsilon}_i(t)|^{\gamma_1} \leq |(\bar{d}^T(t)\tilde{\Theta}(t))_i - \bar{\varepsilon}_i(t)|^{\gamma_1},$$

and then in the component-wise sense,

$$-|\bar{d}^T(t)\tilde{\Theta}(t)-\bar{\varepsilon}(t)|^{\gamma_1}\leq -|\bar{d}^T(t)\tilde{\Theta}(t)|^{\gamma_1}+|\bar{\varepsilon}(t)|^{\gamma_1}. \tag{3.42}$$

For $\gamma_2>1$, the following inequality [129] holds,

$$|y_1+y_2|^{\gamma_2}\leq 2^{\gamma_2-1}(|y_1|^{\gamma_2}+|y_2|^{\gamma_2}).$$

Thus, for $y_1=(\bar{d}^T(t)\tilde{\Theta}(t))_i-\bar{\varepsilon}_i(t)$ and $y_2=\bar{\varepsilon}_i(t)$, one has

$$|(\bar{d}^T(t)\tilde{\Theta}(t))_i|^{\gamma_2}=|(\bar{d}^T(t)\tilde{\Theta}(t))_i-\bar{\varepsilon}_i(t)+\bar{\varepsilon}_i(t)|^{\gamma_2}\leq 2^{\gamma_2-1}(|(\bar{d}^T(t)\tilde{\Theta}(t))_i-\bar{\varepsilon}_i(t)|^{\gamma_2}+|\bar{\varepsilon}_i(t)|^{\gamma_2})\Rightarrow$$
$$|(\bar{d}^T(t)\tilde{\Theta}(t))_i|^{\gamma_2}-2^{\gamma_2-1}|\bar{\varepsilon}_i(t)|^{\gamma_2}\leq 2^{\gamma_2-1}|(\bar{d}^T(t)\tilde{\Theta}(t))_i-\bar{\varepsilon}_i(t)|^{\gamma_2},\;\; i=1,\ldots,n,$$

and then component-wisely, for $\gamma_2>1$, it follows that

$$-|\bar{d}^T(t)\tilde{\Theta}(t)-\bar{\varepsilon}(t)|^{\gamma_2}\leq -2^{1-\gamma_2}|\bar{d}^T(t)\tilde{\Theta}(t)|^{\gamma_2}+|\bar{\varepsilon}(t)|^{\gamma_2}. \tag{3.43}$$

Now, using (3.42)-(3.43), $\dot{V}(t)$ in (3.41) is upper bounded by

$$\begin{aligned}\dot{V}(t)\leq tr\{&-\Xi_G\tilde{\Theta}^T(t)\bar{d}(t)(\lfloor\bar{d}^T(t)\tilde{\Theta}(t)\rceil^{\gamma_1}+2^{1-\gamma_2}\lfloor\bar{d}^T(t)\tilde{\Theta}(t)\rceil^{\gamma_2})\\
&+\Xi_G\tilde{\Theta}^T(t)\bar{d}(t)(|\bar{\varepsilon}(t)|^{\gamma_1}sign(\bar{d}^T(t)\tilde{\Theta}(t))+|\bar{\varepsilon}(t)|^{\gamma_2}sign(\bar{d}^T(t)\tilde{\Theta}(t)))\\
&-\Xi_C\tilde{\Theta}^T(t)\sum_{h=1}^{P}\bar{d}(\tau_h)(\lfloor\bar{d}^T(\tau_h)\tilde{\Theta}(t)\rceil^{\gamma_1}+2^{1-\gamma_2}\lfloor\bar{d}^T(\tau_h)\tilde{\Theta}(t)\rceil^{\gamma_2})\\
&+\Xi_C\tilde{\Theta}^T(t)\sum_{h=1}^{P}\bar{d}(\tau_h)(|\bar{\varepsilon}(\tau_h)|^{\gamma_1}sign(\bar{d}^T(\tau_h)\tilde{\Theta}(t))+|\bar{\varepsilon}(\tau_h)|^{\gamma_2}sign(\bar{d}^T(\tau_h)\tilde{\Theta}(t)))\}.\end{aligned}$$

Therefore, using (3.27)-(3.29) and Fact 1, one obtains,

$$\begin{aligned}\dot{V}(t)\leq&-\xi_G\|\tilde{\Theta}^T(t)\bar{d}(t)\|^{\gamma_1+1}-\xi_C\sum_{h=1}^{P}\|\tilde{\Theta}^T(t)\bar{d}(\tau_h)\|^{\gamma_1+1}+\xi_G\|\tilde{\Theta}^T(t)\|(\||\bar{\varepsilon}(t)|^{\gamma_1}\|+\||\bar{\varepsilon}(t)|^{\gamma_2}\|)\\
&+\xi_CP\|\tilde{\Theta}(t)\|(\||\bar{\varepsilon}(\tau_h)|^{\gamma_1}\|+\||\bar{\varepsilon}(\tau_h)|^{\gamma_2}\|)-2^{1-\gamma_2}n^{\frac{1-\gamma_2}{2}}(\xi_G\|\tilde{\Theta}^T(t)\bar{d}(t)\|^{\gamma_2+1}\\
&+\xi_C\sum_{h=1}^{P}\|\tilde{\Theta}^T(t)\bar{d}^T(\tau_h)\|^{\gamma_2+1}).\end{aligned} \tag{3.44}$$

Since $\||\bar{\varepsilon}(t)|^{\gamma_1}\|=\sqrt{\sum_{i=1}^{n}|\bar{\varepsilon}_i(t)|^{2\gamma_1}}=\|\bar{\varepsilon}(t)\|_{2\gamma_1}^{\gamma_1}$, and by using Fact 1,

$$\|\bar{\varepsilon}(t)\|_{2\gamma_2}\leq\|\bar{\varepsilon}(t)\|,\qquad \|\bar{\varepsilon}(t)\|_{2\gamma_1}\leq n^{\frac{1-\gamma_1}{2\gamma_1}}\|\bar{\varepsilon}(t)\|, \tag{3.45}$$

holds for all $0 < 2\gamma_1 < 2 < 2\gamma_2$. Using (3.45), (3.44) leads to

$$\dot{V}(t) \le -\xi_G(\tilde{\Theta}^T(t)\bar{d}(t)\bar{d}^T(t)\tilde{\Theta}(t))^{\frac{\gamma_1+1}{2}} - \xi_C \sum_{h=1}^{P}(\tilde{\Theta}^T(t)\bar{d}(\tau_h)\bar{d}^T(\tau_h)\tilde{\Theta}(t))^{\frac{\gamma_1+1}{2}}$$
$$+ \xi_G\|\tilde{\Theta}(t)\|(n^{\frac{1-\gamma_1}{2}}\|\bar{\varepsilon}(t)\|^{\gamma_1}+\|\bar{\varepsilon}(t)\|^{\gamma_2}) + \xi_C P\|\tilde{\Theta}(t)\|(n^{\frac{1-\gamma_1}{2}}\|\bar{\varepsilon}(\tau_h)\|^{\gamma_1}+\|\bar{\varepsilon}(\tau_h)\|^{\gamma_2})$$
$$- 2^{1-\gamma_2}n^{\frac{1-\gamma_2}{2}}(\xi_G(\tilde{\Theta}^T(t)\bar{d}(t)\bar{d}^T(t)\tilde{\Theta}(t))^{\frac{\gamma_2+1}{2}} + \xi_C\sum_{h=1}^{P}(\tilde{\Theta}^T(t)\bar{d}(\tau_h)\bar{d}^T(\tau_h)\tilde{\Theta}(t))^{\frac{\gamma_2+1}{2}}).$$

Therefore,

$$\dot{V}(t) \le -\alpha_3\|\tilde{\Theta}(t)\|^{\gamma_1+1} - \alpha_4\|\tilde{\Theta}(t)\|^{\gamma_2+1} + \omega\|\tilde{\Theta}(t)\|, \tag{3.46}$$

where

$$\alpha_3 = \xi_G\lambda_{min}^{\frac{\gamma_1+1}{2}}(D(t)) + \xi_C\lambda_{min}^{\frac{\gamma_1+1}{2}}(S), \alpha_4 = 2^{1-\gamma_2}n^{\frac{1-\gamma_2}{2}}(\xi_G\lambda_{min}^{\frac{\gamma_2+1}{2}}(D(t)) + \xi_C\lambda_{min}^{\frac{\gamma_2+1}{2}}(S)),$$
$$\omega = (\xi_G + P\xi_C)[n^{\frac{1-\gamma_1}{2}}b_{\bar{\varepsilon}}^{\gamma_1} + b_{\bar{\varepsilon}}^{\gamma_2}].$$

By using (3.25), (3.46) is written as,

$$\dot{V}(t) \le -\alpha_3 c_2^{\frac{\gamma_1+1}{2}}V^{\frac{\gamma_1+1}{2}}(t) - \alpha_4 c_2^{\frac{\gamma_2+1}{2}}V^{\frac{\gamma_2+1}{2}}(t) + \omega\sqrt{c_1}V^{\frac{1}{2}}(t), \tag{3.47}$$

and employing inequality (3.34) when $V(t) > 1$, one obtains

$$\begin{aligned}\dot{V}(t) &\le -\alpha_4 c_2^{\frac{\gamma_2+1}{2}}V^{\frac{\gamma_2+1}{2}}(t) + \omega\sqrt{c_1}V^{\frac{1}{2}}(t)\\ &\le -\alpha_4 c_2^{\frac{\gamma_2+1}{2}}V^{\frac{\gamma_2+1}{2}}(t) + \omega\sqrt{c_1}V^{\frac{\gamma_2+1}{2}}(t)\\ &\le -\alpha V^{\frac{\gamma_2+1}{2}}(t),\end{aligned} \tag{3.48}$$

where $\alpha = \alpha_4 c_2^{\frac{\gamma_2+1}{2}} - \omega\sqrt{c_1}$ is positive if

$$\alpha_4 c_2^{\frac{\gamma_2+1}{2}} > \omega\sqrt{c_1} \;\; \Rightarrow \;\; \frac{\alpha_4}{\omega} > \sqrt{\frac{2\lambda_{max}(\Gamma)}{(2\lambda_{min}(\Gamma))^{\gamma_2+1}}}. \tag{3.49}$$

Thus, if (3.49) is met, for any $\tilde{\Theta}(t)$ such that $V(\tilde{\Theta}(0)) > 1$, (3.46) ensures $V(\tilde{\Theta}(t)) \le 1$ for all $t \ge T_3 = \frac{2}{\alpha(\gamma_2-1)}$.

1) If $\gamma_1 = 0$, for the case that $V(t) \leq 1$, using (3.33) and (3.47), one obtains

$$\dot{V}(t) \leq -\alpha_3 \sqrt{c_2} V^{\frac{1}{2}}(t) + \omega \sqrt{c_1} V^{\frac{1}{2}}(t). \tag{3.50}$$

Therefore, satisfying

$$\frac{\alpha_3}{\omega} > \sqrt{\frac{\lambda_{max}(\Gamma)}{\lambda_{min}(\Gamma)}}, \tag{3.51}$$

we have $V(\tilde{\Theta}(t)) = 0$ for $t \geq T_4$ where

$$T_4 = \frac{2}{\alpha_3 \sqrt{c_2} - \omega \sqrt{c_1}}.$$

Therefore, for $\gamma_1 = 0$, once M rank condition, (3.49) and (3.51) are satisfied then $V(\tilde{\Theta}(t)) = 0$ for $t \geq T_{max}$ where

$$T_{max} = T_3 + T_4 = \frac{2}{\alpha(\gamma_2 - 1)} + \frac{2}{\alpha_3 \sqrt{c_2} - \omega \sqrt{c_1}}.$$

This completes the proof of part 1.

2) If $0 < \gamma_1 < 1$, for the case when $V(t) \leq 1$, using (46), one obtains

$$\begin{aligned}\dot{V}(t) \leq & -\alpha_3 \|\tilde{\Theta}(t)\|^{\gamma_1+1} + \omega \|\tilde{\Theta}(t)\| \\ \leq & -\alpha_3(1-\delta)\|\tilde{\Theta}(t)\|^{\gamma_1+1} - \alpha_3 \delta \|\tilde{\Theta}(t)\|^{\gamma_1+1} + \omega \|\tilde{\Theta}(t)\|,\end{aligned}$$

where $0 < \delta < 1$. Hence,

$$\dot{V}(t) \leq -\alpha_3(1-\delta)\|\tilde{\Theta}(t)\|^{\gamma_1+1}, \ \ \bar{\mu} \leq \|\tilde{\Theta}(t)\| \leq \sqrt{c_1}, \tag{3.52}$$

where

$$\bar{\mu} = \begin{cases} max\{\frac{b_{\bar{\varepsilon}}}{\min\{\lambda_{min}^{\frac{1}{2}}(D(t)), \bar{\lambda}_h\}}, (\frac{\omega}{\alpha_3 \delta})^{\frac{1}{\gamma_1}}\}, & \bar{d}(t) \neq 0, \\ max\{\frac{b_{\bar{\varepsilon}}}{\bar{\lambda}_h}, (\frac{\omega}{\alpha_3 \delta})^{\frac{1}{\gamma_1}}\}, & \bar{d}(t) = 0. \end{cases}$$

Using (3.25), (3.52) implies that

$$\dot{V}(t) \leq -\alpha_3(1-\delta) c_2^{\frac{\gamma_1+1}{2}} V^{\frac{\gamma_1+1}{2}}(t),$$

and using comparison principle and (3.25), one obtains

$$V(t) \le (V^{\frac{1-\gamma_1}{2}}(T_3) - \frac{\alpha_3(1-\delta)(1-\gamma_1)c_2^{\frac{\gamma_1+1}{2}}}{2}t)^{\frac{2}{1-\gamma_1}}$$
$$\le c_2^{-1}(min(\bar{\mu}, \sqrt{c_1}))^2,$$

then, the above inequality shows that $\tilde{\Theta}(t)$ satisfies (3.38), for all $t > T_{max}$ where $T_{max} = T_3 + T_5$ and

$$T_5 = \frac{2(1-(c_2^{-0.5}\min(\bar{\mu}, \sqrt{c_1}))^{1-\gamma_1})}{\alpha_3(1-\delta)(1-\gamma_1)c_2^{\frac{\gamma_1+1}{2}}}.$$

Therefore, for $0 < \gamma_1 < 1$, it is concluded that once M rank condition and (3.49) are met then the solutions $\tilde{\Theta}(t)$ are finite-time attractive by the bound given in (3.38) and

$$T \le \frac{2}{\alpha(\gamma_2 - 1)} + \frac{2(1-(c_2^{-0.5}\min(\bar{\mu}, \sqrt{c_1}))^{1-\gamma_1})}{\alpha_3(1-\delta)(1-\gamma_1)c_2^{\frac{\gamma_1+1}{2}}}.$$

This completes the proof.

Remark 14 The convergence set (3.38) can be kept small by maximizing $\lambda_{min}(S)$ that maximizes α_3 and helps to minimize $\bar{\mu}$ in (3.40). Furthermore, choosing $P = p + q$ (satisfying $P \ge p + q$) helps to maximize $\frac{\lambda_{min}(S)}{\lambda_{max}(S)}$ and minimize ω. Moreover, maximizing $\lambda_{min}(S)$ results in a faster convergence time for $\tilde{\Theta}(t)$ as can be found in (3.23), (3.37), and (3.39). These results completely coincide with the obtained results in [34, 36]. Therefore, while applying the proposed FxTCL, the data recording algorithm in [34, 36] is used, where appropriate data is selected to maximize $\frac{\lambda_{min}(S)}{\lambda_{max}(S)}$.

Remark 15 In [81], for zero MFAE, it is shown that once the regressor satisfies the injectivity condition (analogous to the PE condition) for $\gamma_1 = 0$ and $\gamma_2 > 1$, the estimated parameters are fixed-time stable and for $\gamma_1 \in (0, 1)$ and $\gamma_2 > 1$, the estimated parameters are just ultimately bounded. Moreover, the learning rate must satisfy some constraints and to check these constraints the online knowledge of the minimum and maximum singular values of the regressor and the upper bound of the unknown parameters are required which are hard to compute and check online.

However, by employing CL technique, in the proposed method for approximators with zero MFAE, it is shown that for $\gamma_1 \in [0,1)$ and $\gamma_2 > 1$, the estimated parameters are fixed-time stable and no constraint is imposed on the learning rates. Moreover, for both approximators with zero and nonzero MFAEs, no upper bound for the unknown parameters is required anymore. Above all, in sharp contrast to [81], employing the CL technique has eliminated the PE or injectivity requirement in the proposed FxTCL method for both approximators with zero and nonzero MFAEs.

Comparison with other methods Three approaches will be considered in the next section for comparison.

1) *Concurrent learning [36]:* The asymptotically converging CL has the following update law,

$$\dot{\hat{\Theta}}(t) = -\Gamma_C(\Sigma_G \bar{d}(t)e^T(t) + \Sigma_C \sum_{h=1}^{P} \bar{d}(\tau_h)e_h^T(t)), \tag{3.53}$$

where $\Gamma_C > 0$, $\Sigma_G = \sigma_G I$, $\Sigma_C = \sigma_C I$ with positive constants $\sigma_G > 0, \sigma_C > 0$. In contrary to the proposed method (3.22), the update law (3.53) can just guarantee asymptotic convergence of the estimated parameters rather than the fixed-time convergence.

2) *Fixed-time parameter estimation [77, 78, 81]:* The fixed-time parameter estimation law [77, 78, 81] is as follows

$$\dot{\hat{\Theta}}(t) = -K\bar{d}(t)(\lfloor e^T(t)\rceil^{\gamma_1} + \lfloor e^T(t)\rceil^{\gamma_2}), \tag{3.54}$$

for some $K > 0$. In contrast to the proposed method (3.22), (3.54) requires the PE [77, 78] or injectivity [81] condition on the regressor to guarantee fixed-time convergence. However, the proposed FxTCL method (3.22) employs past recorded experienced data to obviate the PE requirement while update law (3.55) only employs current data and require PE or injectivity condition.

3) *Finite-time concurrent learning [41]: The finite-time CL introduced in [41] uses the following update law,*

$$\dot{\hat{\Theta}}(t) = -\Gamma'(K_1\bar{d}(t)e^T(t) + K_2 \sum_{h=1}^{P} \bar{d}(\tau_h)e_h^T(t) + K_3 \frac{\Sigma_{h=1}^{P} \bar{d}(\tau_h)e_h^T(t)}{\|\Sigma_{h=1}^{P} \bar{d}(\tau_h)e_h^T(t)\|}), \tag{3.55}$$

where $\Gamma' > 0$, $K_j = k_j I > 0$ *with constant* $k_j > 0$ *for* $j = 1, 2, 3$. *In contrast to the proposed method* (3.22)*, the settling time of* (3.55) *depends on the initial parameter estimation error* $\tilde{\Theta}(0)$ *as follows*

$$T \leq T_{max}(\tilde{\Theta}(0)) = \frac{2}{\alpha'} ln \frac{\alpha' \|\tilde{\Theta}(0)\| + \beta' \sqrt{2\lambda_{min}(\Gamma')}}{\beta' \sqrt{2\lambda_{min}(\Gamma')}},$$

$$\alpha' = 2\lambda_{min}(\Gamma')\lambda_{min}(k_1 D(t) + k_2 S), \quad \beta' = k_3 \sqrt{2\lambda_{min}(\Gamma')} \frac{\lambda_{min}(S)}{\lambda_{max}(S)}. \tag{3.56}$$

Remark 16 It is notable that to guarantee fixed-time convergence for update law (3.54), not only injectivity or PE condition is required but also its learning rate, K, must satisfy some constraints on the entire time of learning where the minimum and maximum singular values of the regressor and the upper bound of the unknown parameters are needed to be known. Moreover, CL (3.53) only guarantees the asymptotic convergence of the estimated parameters and finite-time CL (3.55) can only ensure that there is a finite-time of convergence that cannot be computed due to the dependence of the settling time on initial parameter estimation error. Therefore, the proposed fixed-time CL update law (3.22) that guarantees fixed-time convergence regardless of the initial parameter estimation error and does not require PE condition under the rank condition of recorded data, intuitively outperforms the previously mentioned methods.

3.4 Simulation Results

In this section, the performance of the proposed fixed time CL is numerically examined in comparison with asymptotically converging CL, fixed-time parameter estimation, and finite-time CL, given by (3.53), (3.54) and (3.55), respectively.

In the following examples, the x domain is defined by $\mathcal{D}_x = [x_L, x_H]$ where $x_L = -2$ and $x_H = 2$, initial values and the controllers are all set to zero and a small exponential sum of sinusoidal input is injected into the system controller to ensure the rank condition on the collected data where data selection procedure in [34, 36] for maximizing $\frac{\lambda_{min}(S)}{\lambda_{max}(S)}$ is employed for CL, finite-time CL and the proposed fixed-time CL methods. To fairly compare the speed and precision of the mentioned online learning methods for approximating $\hat{f}(x)$ and $\hat{g}(x)$ on the whole domain of

x as time evolves, the following learning errors are computed online

$$E_f(t) = \int_{\mathcal{D}} \|e_f(x(t))\| d^n x, \quad E_g(t) = \int_{\mathcal{D}} \|e_g(x(t))\| d^n x.$$

The simulations are done in MATLAB with Euler integration with the sample time equal to 0.001 seconds. In the simulations, the results of the proposed fixed-time CL, asymptotically converging CL, fixed-time parameter estimation, and finite-time CL methods respectively given in (3.22), (3.53)-(3.55), are labeled by FxTCL, CL, FxT, and FTCL, respectively.

Example 1: Adaptive approximators with zero MFAEs

Consider the following system

$$\dot{x}(t) = p_1 x(t) + p_2 x(t)\cos(x(t)) + p_3 e^{-x(t)} u(t), \tag{3.57}$$

where the regressors are fully known as $z(x(t), u(t)) = [x(t), x(t)\cos(x(t)), e^{-x(t)}u(t)]$ with $p+q = 3$. The unknown parameters are $[p_1, p_2, p_3] = [-0.5, 0.5, 0.5]$. We set $P = 3$ for CL, finite-time CL and FxTCL methods. Let $\xi_G = \sigma_G = k_1 = 3, \xi_C = \sigma_C = k_2 = 1, k_3 = 0.1$. Set $\Gamma' = \Gamma = \Gamma_C = K = I$, $\gamma_1 = 0.5$ and $\gamma_2 = 2$.

Fig. 3.1 depicts the true parameters and the approximated parameters for CL, fixed-time parameter estimation, finite-time CL, and the proposed fixed-time CL methods. In Fig. 3.1, CL, finite-time CL and the proposed fixed-time CL succeeded in convergence to true parameters where FxTCL resulted in faster convergence in comparison with CL and finite-time CL. As shown in Fig. 3.1, fixed-time method (3.54), did not succeed in convergence to the true parameters due to the lack of the PE condition. The online learning errors $E_f(t)$ and $E_g(t)$ are plotted in Fig. 3.2 where FxTCL shows faster converging to the origin in comparison with the other methods. The integral absolute errors (IAEs) of $E_f(t)$ and $E_g(t)$ for all methods are computed in Table 3.1 where FxTCL with IAEs 14.81 and 2.58, respectively, for $E_f(t)$ and $E_g(t)$ has resulted in the best precision of online learning in comparison with other mentioned methods. Using data selection algorithm in [34, 36] to maximize $\frac{\lambda_{min}(S)}{\lambda_{max}(S)}$, the maximum of $\lambda_{min}(S)$ is obtained as 0.07. Therefore, using (3.23), the upper bound of the settling-time for the proposed FxTCL is obtained as $T_{max} = 57$ seconds. Figs. 3.1 and 3.2 show that the settling-time for the FxTCL method satisfies the expected

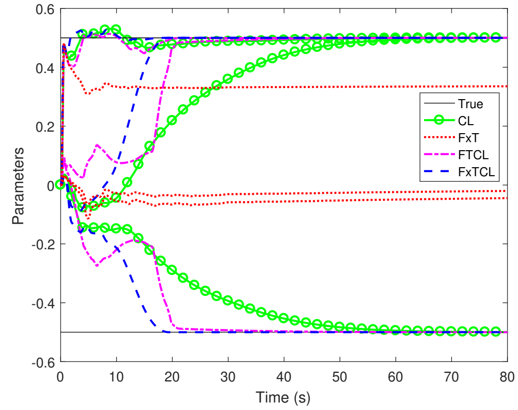

Figure 3.1: Estimated parameters for approximators with zero MFAE.

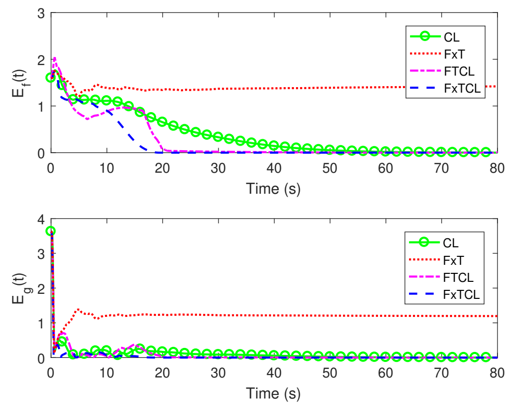

Figure 3.2: Online learning errors for approximators with zero MFAE.

Table 3.1: Learning errors comparison

	Example 1		Example 2	
	IAE $E_f(t)$	IAE $E_g(t)$	IAE $E_f(t)$	IAE $E_g(t)$
CL	30.02	7.37	1448	6079
FxT	111.37	96.33	3252	10587
FTCL	19.11	5.40	1319	4022
FxTCL	14.81	2.58	907	2281

settling-time bound, however, the experienced stacks are not prerecorded and the experienced data is collected online during the learning. It is worth noting that the obtained bound for the FxTCL is valid for any initial condition in $\mathcal{D}_x$.

Example 2: Adaptive approximators with non-zero MFAEs Now, consider the following system

$$\dot{x}(t) = x(t)\sin(0.5x(t)) + (3+\cos(x(t)))u(t), \tag{3.58}$$

where the associated $f(x)$ and $g(x)$ are fully unknown uncertainties. In this example, since there is no knowledge about the exact regressors, it is expected that the learning errors are fixed-time attractive to a bound near zero. Therefore, radial basis function neural networks that are linearly parameterized universal approximators are used. Here, we consider 5 radial basis functions defined as $\exp(-\frac{\left\|x-c_j\right\|^2}{2\sigma_j{}^2})$, $j = 1, 2, ..., 5$, where the centroids c_j are uniformly picked on the interval $[x_L, x_H] = [-2, 2]$ and the spreads are all fixed to $\sigma_j = 1.2$. Therefore, the employed regressor is

$$z(x(t), u(t)) = [e^{-\frac{\|x(t)-(-2)\|^2}{2(1.2)^2}}, ..., e^{-\frac{\|x(t)-(2)\|^2}{2(1.2)^2}}, e^{-\frac{\|x(t)-(-2)\|^2}{2(1.2)^2}}u(t), ..., e^{-\frac{\|x(t)-(2)\|^2}{2(1.2)^2}}u(t)]^T,$$

with 10 independent basis functions that leads to setting $P = 10$. Thus, the approximation of (3.58) is given as

$$\dot{x}(t) = \hat{\Theta}^T(t)z(x(t), u(t)) = [p_1, p_2, ..., p_{10}]z(x(t), u(t)).$$

Let $\Gamma = \Gamma' = \Gamma_C = K = I$, and $\xi_G = \sigma_G = k_1 = 1$, $\xi_C = \sigma_C = k_2 = 0.2$, $k_3 = 0.02$ and $\gamma_1 = 0.5$ and $\gamma_2 = 2$. It should be noted that in finite-time CL, increasing $k_3 = 0.02$ causes chattering in the approximation error. The mentioned learning methods led to the approximated parameters depicted in Fig. 3.3. In Fig. 3.3(a), it is shown that the fixed-time parameter estimation method cannot guarantee convergence of parameters to their true values, due to the lack of PE. Fig. 3.3(d) shows that the proposed FxTCL satisfying the rank condition, succeeded in convergence to the suitable parameters while CL and FTCL methods need more time for convergence as respectively shown in parts (b) and (c) of Fig. 3.3. The steady state approximations for the uncertainties $f(x)$ and $g(x)$, on the x-domain $\mathcal{D}_x$, are depicted in Fig. 3.4 where the steady-state approximations of FxTCL could better match the true values of $f(x)$ and $g(x)$ in comparison with other methods. As the comparison of the learning errors $E_f(t)$ and $E_g(t)$ in Fig. 3.5 shows, the fixed-time parameter estimation, did not perform well in learning the uncertainty due to the lack of the PE condition.

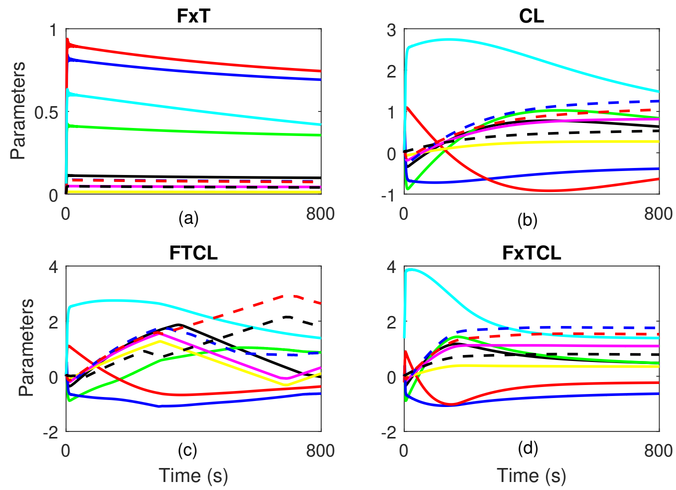

Figure 3.3: Estimated parameters for approximators with non-zero MFAE: (a). Estimated parameters by FxT method, (b). Estimated parameters by CL method, (c). Estimated parameters by FTCL method, (d). Estimated parameters by FxTCL.

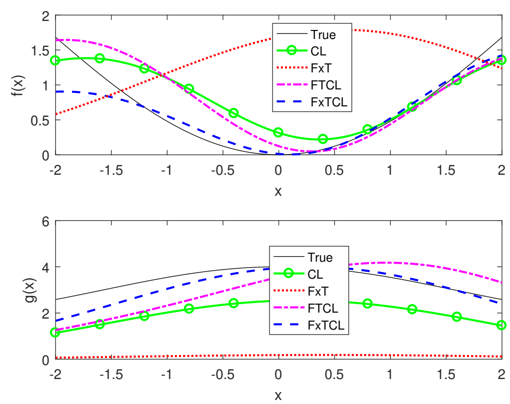

Figure 3.4: Steady state uncertainty approximations.

However, the proposed FxTCL, FTCL and CL errors showed bounded convergence near zero in Fig. 3.5, where FxTCL method is faster in converging to a smaller bound near zero in comparison with others. Furthermore, based on the results of IAEs for $E_f(t)$ and $E_g(t)$ in Table 1, FxTCL resulted in the lowest learning errors during the whole time of online learning in comparison with other mentioned methods.

In the presented numerical results, it is shown that the proposed FxTCL has outperformed other mentioned methods both in terms of precision and convergence speed.

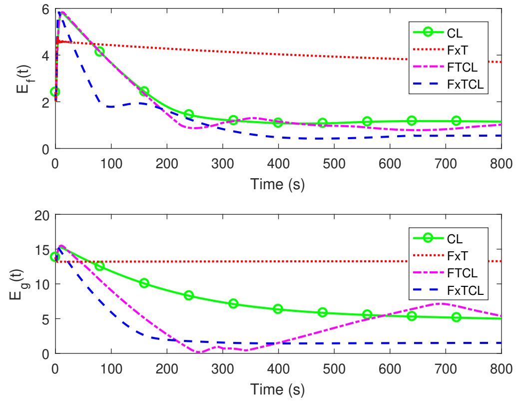

Figure 3.5: Online learning errors for approximators with non-zero MFAE.

3.5 Conclusion

In this chapter, a fixed-time concurrent learning system identification method is introduced without the persistence of excitation (PE) requirement. In this method, the concurrent learning relaxes the requirement of the PE condition to a rank condition on the memory stack of recorded data. It is shown that the richness of the recorded experienced data depends on the minimum eigenvalue properties of the stack of regressor's data which influences the speed and precision of the proposed fixed-time concurrent learning method. Simulation results are given where it is shown that the proposed fixed-time concurrent learning has outperformed other mentioned methods in both terms of precision and convergence speed.

CHAPTER 4

ONLINE IDENTIFICATION OF NOISY FUNCTIONS VIA A DATA-REGULARIZED LEARNING APPROACH

4.1 Introduction

This chapter presents online learning rule that paves the way to designing an active learning approach to collect informative data that improves the convergence rate as well as reduce the effect of the noise variance on the estimation. This is in sharp contrast to the existing online approaches that require independent and identically distributed (i.i.d) data samples for which there is no systematic approach to verify them. More specifically, it is shown that as the data is streaming, the fixed-size memory data can be updated to improve the strong convexity properties of the data-regularized loss function to reduce the ultimate bound and improve the convergence speed. The exponential convergence rate is also guaranteed under a rank condition on the matrix of the memory data. The rate of convergence to the ultimate bound also depends on the quality of the stored data. More specifically, the employed data-regularized loss function is strongly convex as long as a rank condition on the fixed-size memory data is satisfied, and does not impose any bias on the estimated parameters. The strong convexity parameter of the loss function depends on the maximum and minimum eigenvalues of the memory data matrix, which can be improved by replacing new samples with old ones. The presented online data-regularized CL-based SGD ensures a finite-sample performance guarantee by providing a bound of the estimated parameters for every time step. Moreover, it is shown how the function approximation with noisy measurements can be leveraged in system identification and RL applications. Simulation examples are also provided to verify the effectiveness of the proposed approach and the results are compared with the standard SGD.

Notation $\mathbb{R}$, $\mathbb{N}$, and $\mathbb{Z}_+$ respectively show the set of real, natural and all nonnegative integers. $\|.\|$ denotes the Euclidean norm for vectors and induced 2-norm for matrices. $tr(.)$ indicates trace of a matrix. The minimum and maximum eigenvalues of matrix A are, respectively, denoted by $\lambda_{min}(A)$ and $\lambda_{max}(A)$. The matrix I denotes the identity matrix of appropriate dimensions. We use $\gamma_1 \circ \gamma_2$ to denote the composition of two functions γ_1 and γ_2 where $\gamma_i : \mathbb{R} \mapsto \mathbb{R}$, for $i = 1, 2$. A function $\alpha : \mathbb{R} \geqslant 0 \mapsto \mathbb{R} \geqslant 0$ is a $\mathcal{K}$-function if it is continuous, strictly increasing and $\alpha(0) = 0$; it is a $\mathcal{K}_\infty$-function if it is a $\mathcal{K}$-function and also $\alpha(s) \to \infty$ as $s \to \infty$; and it is a positive definite function if $\alpha(s) > 0$ for all $s > 0$, and $\alpha(0) = 0$. A function $\beta : \mathbb{R} \geqslant 0 \times \mathbb{R} \geqslant 0 \to \mathbb{R} \geqslant 0$ is a $\mathcal{KL}$-function if, for each fixed $k \geqslant 0$, the function $\beta(\cdot, k)$ is a $\mathcal{K}$-function, and for each fixed $s \geqslant 0$, the function $\beta(s, \cdot)$ is decreasing and $\beta(s, k) \to 0$ as $k \to \infty$.

All random variables are assumed to be defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with Ω as the sample space, $\mathcal{F}$ as its associated Borel σ-algebra and $\mathbb{P}$ as the probability measure. For a random variable $w : \Omega \longrightarrow \mathbb{R}^n$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with some abuse of notation, the statement $w \in \mathbb{R}^n$ is used to state the dimension of the random variable. Finally, $\mathbb{E}[X]$ denotes the expected value of the random variable X on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

4.2 Preliminaries

Consider the following stochastic discrete-time (DT) dynamics system

$$x(k+1) = F(x(k), v(k)), \tag{4.1}$$

where $x(k) \in \mathcal{D} \subset \mathbb{R}^n$ is the measurable state vector and $\mathcal{D}$ is a compact set; $v(k) \in \mathbb{R}^n$ is a zero-mean independent white noise and $F : \mathcal{D} \times \mathbb{R}^n \mapsto \mathbb{R}^n$.

The following definition and lemmas are introduced for stability analysis and convergence of this stochastic system.

Definition 5 [132] Consider the stochastic system (4.1) and fix $\epsilon \in (0, 1)$. The system is said to be practical stable in probability (PS-P) if there exist a positive constant γ and a class $\mathcal{KL}$-function $\beta(\cdot, \cdot)$ such that

$$\mathbb{P}\left\{\left\|x(k)\right\| \leq \beta\left(\left\|x_0\right\|, k\right) + \gamma\right\} \geq 1 - \epsilon.$$

The following lemma gives the criterion on practical stability in probability (PS-P) for the system (4.1).

Lemma 3 [132] The system (4.1) is PS-P if there exist a positive definite function $V(x(k))$ and real scalar $d \geq 0$ and $\mathcal{K}_\infty$-functions $\alpha_1, \alpha_2, \alpha_3$ such that

$$\alpha_1(\|x(k)\|) \leq V(x(k)) \leq \alpha_2(\|x(k)\|),$$

and

$$\mathbb{E}\left[V\left(x(k+1)\right)\right] - V(x(k)) \leq -\alpha_3(\|x(k)\|) + d, \tag{4.2}$$

where $\alpha_3 \circ \alpha_2^{-1}$ is a convex function.

Definition 6 [133] The origin of the system (4.1) is said to be exponentially bounded in mean square with exponent a if there exist constants $0 < a < 1$, $c_1 \geq 0$ and $c_2 > 0$ such that

$$E\|x(k)\|^2 \leq c_1 + c_2(1-a)^k. \tag{4.3}$$

Remark 17 Definition 6 does not necessarily imply that $E\|x(k)\|^2$ decreases monotonically for all k. It only implies that the bound on $E\|x(k)\|^2$ decreases exponentially and as $k \to \infty$ the mean square of the process is bounded by $E\|x(\infty)\|^2 \leq c_1$ where c_1 depends on the noise disturbing the system.

Lemma 4 [134] For the system (4.1), if there exists a function $V(x(k))$ with $V(0) = 0$ such that, $\forall k \geq 0$, 1) $\mathbb{E}[V(x(k))] \geq c\mathbb{E}[\rho(\|x(k)\|)]$, and 2) $\mathbb{E}[V(x(k+1))] - \mathbb{E}[V(x(k))] \leq M - a\mathbb{E}[V(x(k))]$, for some $\rho(.) \in \mathcal{K}$, and constants $c > 0$, $M \geq 0$, and $0 < a < 1$, then

$$c\mathbb{E}[\rho(\|x(k)\|)] \leq \mathbb{E}[V(x(k))] \leq (1-a)^k \mathbb{E}[V(x(0))] + M \sum_{i=0}^{k-1} (1-a)^i, \tag{4.4}$$

and $\lim_{k \to \infty} \mathbb{E}[\rho(\|x(k)\|)] \leq \frac{M}{ca}$.

The following definitions are also used throughout the chapter.

Definition 7 [135] The sequence $\{x(k)\}_{k=1}^{\infty}$ converges to x^* with a linear (exponentially fast) rate if $\|x(k+1) - x^*\| \leq \gamma\|x(k) - x^*\|$, or equivalently if $\|x(k+1) - x^*\| \leq \gamma^k \|x(0) - x^*\|$ for some $\gamma \in (0, 1)$. The convergence is sublinear if $\gamma = 1$. For the linear (sublinear) convergence, the error rate is $O(\gamma^k)$ (O($1\frac{}{k)}$) .

Definition 8 (Markov's Inequality [136]). Let X be a non-negative random variable. Then for all real positive constant $a > 0$,

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}[\mathbb{X}]}{a}. \tag{4.5}$$

Definition 9 (Jason's inequality [137]): If X is a random variable and φ is a convex function, then $\varphi(\mathbb{E}[X]) \leq \mathbb{E}[\varphi(X)]$.

Definition 10 (Persistently exciting (PE) [29]) The bounded vector signal $\varphi(x(k)) \in \mathbb{R}^n$ is PE if there exist a natural number N and $\alpha > 0$ such that

$$\sum_{k=\tau}^{\tau+N} \varphi(x(k))\varphi^T(x(k)) \geq \alpha I, \quad \forall \tau \in \mathbb{Z}_+. \tag{4.6}$$

Definition 11 (Strongly Convex and Smooth Functions [138]) A convex function f is said to be α-strongly convex if

$$f(y) \geq f(x) + f(x)^T(y-x) + \frac{\alpha}{2}||y-x||^2 \tag{4.7}$$

Moreover, a continuously differentiable function f is β-smooth if its gradient is β-Lipschitz. That is, if

$$||f(x) - f(y)|| \leq \beta||x-y|| \tag{4.8}$$

A twice differentiable function f is α-strongly convex if for all x, one has $\nabla^2 f(x) \geq \alpha\, I$ and is β-smooth if for all x, one has $\nabla^2 f(x) \leq \beta\, I$.

4.3 Problem Formulation and Motivation

Stochastic Function Identification: Problem Formulation Consider the following DT function

$$y(k) = f(x(k)) + v(k), \tag{4.9}$$

where $x \in \mathcal{D} \subset \mathbb{R}^n$ is the measurable state vector and $\mathcal{D}$ is a compact set; $f : \mathcal{D} \mapsto \mathbb{R}^n$, $v \in \mathbb{R}^n$ is an additive zero-mean independent white noise.

Assumption 5 The function $f(.)$ is unknown; however, its noisy measurements $y(k)$ as well as $x(k)$ are available for measurement.

The noise $v(k)$ in system (4.9) satisfies the following assumption.

Assumption 6 For $\forall s, t \in \mathbb{Z}_+, \mathbb{E}\left\{v(s)\right\} = \mathbb{E}\left\{v(t)\right\} = 0$, and

$$\mathbb{E}\left\{v(s)^T v(t)\right\} = \begin{cases} \sigma^2, s = t, \\ 0, s \neq t. \end{cases}$$

In this chapter, we consider the problem of online DT function identification, $f(x(k))$, from streaming noisy measurements and recorded experienced data.

Here, linearly parameterized adaptive approximation models [10] are employed to represent $f(x(k))$ as follows,

$$f(x(k)) = \Theta^{*T} \varphi(x(k)) + \varepsilon(x(k)), \tag{4.10}$$

where the matrix $\Theta^* \in \mathcal{D}_\Theta \subset \mathbb{R}^{q \times n}$ denotes the unknown optimal parameters of the approximater, given by

$$\Theta^* = \arg \min_{\Theta \in \mathcal{D}_\Theta} \{ \underset{x(k) \in \mathcal{D}}{sup} \left\|\varepsilon(x(k))\right\| \}, \tag{4.11}$$

where $\mathcal{D}_\Theta$ is a compact set. The measurable vector $\varphi : \mathcal{D} \mapsto \mathbb{R}^q$ denotes the basis functions, while q is the number of linearly independent basis functions for approximating $f(x(k))$. The quantity $\varepsilon(x(k)) \in \mathbb{R}^n$ is the minimum functional approximation error (MFAE) for $f(x(k))$. If the unknown functions $f(x(k))$ can be approximated exactly by the model, one has $\varepsilon(x(k)) = 0$.

Assumption 7 For the compact set $\mathcal{D}$, the approximators' basis functions are bounded i.e., $b_1 \leq \|\varphi(x(k))\| \leq b_2, \forall x \in \mathcal{D}$, where $b_1 \geq 0$ and $b_2 > 0$. Moreover, the approximation error $\varepsilon(x(k))$ is bounded by $b_\varepsilon \geq 0$, i.e.,

$$\sup_{x \in \mathcal{D}} \|\varepsilon(x)\| \leq b_\varepsilon.$$

Now using (4.10), (4.9) can be written as

$$y(k) = \Theta^{*T} \varphi(x(k)) + \varepsilon(x(k)) + v(k). \tag{4.12}$$

Let the approximator of (4.12) be of the form

$$\hat{y}(k) = \hat{\Theta}^T(k) \varphi(x(k)), \tag{4.13}$$

where $\hat{\Theta}(k) \in \mathbb{R}^{q \times n}$ is the estimation of parameter matrix Θ^* at time k. The approximation error $e(k)$ is defined as

$$e(k) = \hat{y}(k) - y(k) = \tilde{\Theta}^T(k)\varphi(k) - \varepsilon(k) - v(k), \tag{4.14}$$

where $\tilde{\Theta}(k) := \hat{\Theta}(k) - \Theta^*$ is the parameter estimation error.

The goal of function approximation is to learn the unknown parameters vector Θ^* by $\hat{\Theta}$ by fitting (4.13) to data samples. This function approximation problem has many applications in machine learning. For example, identification of dynamic systems and learning the value function in reinforcement learning can be transformed into this regressor, as shown later. The following definition is needed to formalize the goal of this chapter, stated in Problem 1.

Definition 12 [139] Let $0 < \epsilon < 1$. Let a learning algorithm A is designed to iteratively learn the unknown parameters Θ^* and let its output at iteration or time k be $\hat{\Theta}(k)$. We say that the set $S_\Theta(k)$ is a probabilistic bound of the learning algorithm at time k, if $\mathbb{P}[\Theta^* - \hat{\Theta}(k) \in S_\Theta(k)] \geq (1 - \epsilon)$ for time k and after. We say S_Θ is the probabilistic ultimate bound of the learning if $\mathbb{P}[\lim_{k\to\infty}(\Theta^* - \hat{\Theta}(k)) \in S_\Theta] \geq (1 - \epsilon)$.

Problem 1 Consider the function (4.12) and let its approxixmator be (4.13). Let Assumptions 5-7 be satisfied. Design an iterative learning algorithm A such that $\Theta^* - \hat{\Theta}(k)$ converges exponentially fast to a probabilistic ultimate bound with minimum size. Moreover, the algorithm A provides finite-sample guarantees for the estimation error.

Remark 18 The probabilistic ultimate bound in Definition 12 can be achieved by assuring that the error dynamics of the learning, i.e, $\tilde{\Theta}$, is PS-P (which is defined in Definition 5). In this case, the exponential convergence to the ultimate bound is characterized by the $\beta(\cdot, \cdot)$ function, and the ultimate bound is characterized by γ. Moreover, the sets $S_\Theta(k)$ and S_Θ are balls of radius with $\beta(\Theta(0), k) + \gamma$ and γ, respectively.

Remark 19 Note that due to the noise in (4.12), no point estimate $\hat{\Theta}(k)$ of the unknown parameters Θ^* can entirely predict the outcome of the stochastic function (4.12). Therefore, a high-confidence set for the estimation is typically found to provide guarantees on how far the estimated parameters can be from the optimal parameters. In our proposed approach, we will show that this set has a transient part that goes to zero as k goes to infinity and the steady state part depends on the noise variance and the MFAE. An efficient algorithm is then the one that makes the transient response faster (i.e, faster convergence to the ultimate bound) and also makes the size of the ultimate bound smaller (more confidence and robust learning).

Standard SGD algorithms are typically presented to reduce the computational complexity of the optimization for the case where the noise and MFAE are both identically zero in (4.12). To clarify this, consider the case where the noise and MFAE are both identically zero and N samples $\{\phi(x(k), y(k))\}_{k=1}^{N}$ are collected that span the entire space of the function. Then, optimizing the following finite sum of the function approximation error using either least squares or gradient decent guarantees convergence to the optimal parameter,

$$\ell_{\tilde{\Theta}}(k) = \sum_{k=1}^{N} e(k)^2. \tag{4.15}$$

However, when N is large, then, to reduce the computational complexity, the SGD randomly samples from the set of N data samples and perform the following gradient descent only on the randomly selected sample,

$$\hat{\Theta}(k+1) = \hat{\Theta}(k) - \xi_k \, \varphi(k) e^T(k), \tag{4.16}$$

where ξ_k is the learning rate. That is, the SGD optimizes the instantaneous loss function

$$\ell_{\tilde{\Theta}}(k) = \frac{1}{2} e^T(k) e(k). \tag{4.17}$$

Convergence of the SGD is shown to include a transient part and a steady-state part (depending on the variance of the estimation of the entire gradient with the gradient of one sample). However, in our setting, the ultimate bound of the estimation error depends on the inherent measurement noise and not on randomly selecting from available set of samples. Besides, for the time-varying regressor for which the data samples are streaming and not available at once, the signals $\varphi(k)$ interacting with the parameter estimates must remain PE during the estimation procedure. If PE condition is not satisfied, the parameters converge to either a wrong value or do not converge at all. The following Lemma shows that if the PE condition is satisfied, then the optimization solution becomes unique due to strong convexity of the empirical average of the loss function.

Lemma 5 Consider the function approximation problem with noisy measurements (4.12) and the approximation error (4.14). Let the SGD algorithm (4.16) be used to learn the unknown parameters Θ^*, where ξ_k is a time-varying learning rate satisfying $\xi_k > 0, \forall k \in \mathbb{N}$ and the sequence $\{\xi_k\}_{k=1}^{\infty}$ converges to 0 and that $\sum_{k=1}^{\infty} \xi_k = \infty$. Then, at any time $t \geq 1$, the SGD optimizes the empirical average of the loss unction given by

$$\ell_{\tilde{\Theta}}(t) = \frac{1}{2t} \sum_{k=1}^{t} e^T(k)e(k). \tag{4.18}$$

Moreover, it converges to a unique solution if the signal $\varphi(k)$ is PE.

Proof 7 While the SGD (4.16) optimizes the instantaneous loss function defined in (4.17) at every time step, it is shown in [140] that using a learning rate that satisfies the conditions provided in the statement of the lemma, the update law (4.16), any any time $t \geq 1$, optimizes the empirical average of the observations in (4.18). The gradient and hessian of the instantaneous error $\ell_{\tilde{\Theta}}(k)$ at $\tilde{\Theta}(k)$ are, respectively, given as

$$\begin{aligned} \nabla \ell_{\tilde{\Theta}}(k) &= \varphi(k)e^T(k) \\ &= \varphi(k)\varphi^T(k)\tilde{\Theta}(k) - \varphi(k)\varepsilon^T(k) - \varphi(k)v^T(k), \end{aligned} \tag{4.19}$$

and

$$\nabla^2 \ell_{\tilde{\Theta}}(k) = \varphi(k)\varphi^T(k). \tag{4.20}$$

Therefore, for the empirical average function $\ell_{\tilde{\Theta}}(t)$, this becomes

$$\nabla^2 \ell_{\tilde{\Theta}}(t) = \frac{1}{t} \sum_{k=1}^{t} \varphi(x(k))\varphi^T(x(k)). \tag{4.21}$$

If the signal $\varphi(k)$ is PE (i.e., if it satisfies (4.6)), then $\nabla^2 \ell_{\tilde{\Theta}}(t) \geq \alpha I$ and thus $\ell_{\tilde{\Theta}}(t)$ is strongly convex after some time $t \geq N$. Therefore, due ot the strong convexity, a unique solution to the optimization problem is found. This completes the proof.

On one hand, when the excitation of the signal $\phi(x)$ decays quickly, the online SGD cannot receive necessary amount of information about Θ^* and fails to estimate it correctly. If the PE condition is not satisfied, the error terms (4.18) is only convex over time and not strongly convex, and thus can become zero even if Θ^* converges to a wrong value. On the other hand, even though gradient descent achieves linear convergence rate for a strongly convex function, SGD does not enjoy the linear convergence rate of gradient descent under strong convexity and only achieve sublinear convergence rate [141, 142]. Sublinear convergence rate, however, is not strong because it has the property that the longer you run the algorithm, the less progress it makes. A fundamental question is how to develop new online learning algorithms that achieve linear convergence rates and under relaxed PE condition that can be easily verified and improved.

To fulfill the learning of the uncertainties $f(x)$ in (4.9) without requiring the PE condition on the data stream, a data-regularized CL-based SGD based learning is proposed next to ensure that the parameter estimation error $\tilde{\Theta}(k)$ dynamics is PS-P, and thus solve Problem 1. Before presenting our algorithm, the following subsection shows two applications of the function identification with noisy measurements.

Motivation for Function Identification: Value Learning in Reinforcement Learning and System identification as Function Identifiers In this subsection, we show that the value learning in RL and the model learning in system identification can be formalized as instances of the problem of stochastic DT function identification. **Value Function Learning** Consider the system described by the following stochastic nonlinear difference equation

$$x(k+1) = G(x(k), u(k)) + v(k), \tag{4.22}$$

where $x \in \mathcal{D} \subset \mathbb{R}^n$ and $u \in \mathcal{D}_u \subset \mathbb{R}^m$ are the system's states and inputs, respectively; $\mathcal{D}$ and $\mathcal{D}_u$ are compact sets, $G : \mathcal{D} \times \mathcal{D}_u \mapsto \mathcal{D}$ is the dynamics function, and $v(k)$ is the zero mean white noise at time k with covariance Σ.

A stage cost or reward function for the state x and action u at time k is considered as $r(x(k), u(k))$. For a fixed control policy $\pi : \mathcal{D} \mapsto \mathcal{D}_u$, the cost-to-go for a single realization and the initial condition $x(0)$ is defined by

$$J(x(0), \pi) = \mathbb{E}\left[\sum_{k=0}^{N} r(x(k), \pi(x(k))) \right]. \tag{4.23}$$

Defining the value function for the policy π as $V^{\pi}(x) := J(x, \pi)$, the following Bellman equation is obtained

$$V^{\pi}(x(k)) = r(x(k), u(k)) + \mathbb{E}[V^{\pi}(x(k+1))], \tag{4.24}$$

where $u(k) = \pi(x(k))$ and $x(k+1)$ is the system's next state under the control action $u(k)$. Based on (4.24), consider the Bellman operator introduced below,

$$TV^{\pi}(x(k)) := r(x(k), u(k)) + \mathbb{E}[V^{\pi}(x(k+1))]. \tag{4.25}$$

Then, the Bellman equation (4.24) becomes

$$V^{\pi}(x) = TV^{\pi}(x). \tag{4.26}$$

To solve (4.26) for value function, the value function is typically parametrized in the form of $V^{\pi}_{\Theta}(x(k)) = \Theta^{*T} \phi(x(k))$ where Θ^* are the unknown optimal parameters of the approximation model. The goal is to learn the unknown parameter Θ^* using data.

The fact that the Bellman equation (4.26) is a contraction map [143] is leveraged in many studies by the stochastic approximation to learn the parameters of the parametrized value function. The exact value of $TV^{\pi}(x)$ is not available due to the expectation operator and only its noisy estimates are provided typically using the temporal difference approach. Defining $L(\hat{\Theta}(k), x(k)) = [V^{\pi}_{\Theta}(x(k)) - TV^{\pi}_{\Theta}(x(k))]^2$, then only its noisy measurements are available due to the expectation operator in $TV^{\pi}_{\Theta}(x(k))$. The goal is to learn the probabilistic ultimate bound S_{Θ}, such that $\mathbb{P}[\lim_{k\to\infty}(\Theta^* - \hat{\Theta}(k)) \in S_{\Theta}] \geq (1-\epsilon)$ for $\epsilon \in (0, 1)$, while only noisy measurements are available for the loss function $L(\hat{\Theta}(k), x(k))$. Iteratively solving the Bellman's equation by solving this optimization problem using noisy samples is at the heart of reinforcement learning algorithms such as policy iteration and value iteration.

System Dynamics Identification

Consider the following DT system,

$$x(k+1) = f(x(k)) + g(x(k))u(k) + v(k), \tag{4.27}$$

where $x \in \mathcal{D} \subset \mathbb{R}^n$ and $u \in \mathcal{D}_u \subset \mathbb{R}^m$ are the system's states and inputs, respectively; $\mathcal{D}$ and $\mathcal{D}_u$ are compact sets; $f : \mathcal{D} \mapsto \mathbb{R}^n$, and $g : \mathcal{D} \mapsto \mathbb{R}^{n\times m}$ are the unknown nonlinear drift and input terms,respectively, and $v(k) \in \mathbb{R}^n$ is a zero-mean independent white noise with covariance Σ. The system identification aim is to learn the unknown dynamics in (4.27), namely to approximate $f(x(k))$ and $g(x(k))$.

Linearly parameterized adaptive approximation models are employed to, respectively, represent $f(x(k))$ and $g(x(k))$ as follows,

$$f(x(k)) = \Theta_f^{*T}\psi(x(k)) + e_f(x(k)), \tag{4.28}$$

$$g(x(k)) = \Theta_g^{*T}\chi(x(k)) + e_g(x(k)), \tag{4.29}$$

where the matrices $\Theta_f^* \in \mathcal{D}_{\Theta_f} \subset \mathbb{R}^{r\times n}$ and $\Theta_g^* \in \mathcal{D}_{\Theta_g} \subset \mathbb{R}^{s\times n}$ denote the unknown optimal parameters of the adaptive approximation models, where $\mathcal{D}_{\Theta_f}$ and $\mathcal{D}_{\Theta_g}$ are compact sets. The vectors $\psi : \mathcal{D} \mapsto \mathbb{R}^r$ and $\chi : \mathcal{D} \mapsto \mathbb{R}^s$, are the computable basis functions; r and s are the number of linearly independent basis functions to, respectively, approximate $f(x(k))$ and $g(x(k))$. In (4.28) and (4.29), $e_f(x(k)) \in \mathbb{R}^n$ and $e_g(x(k)) \in \mathbb{R}^{n\times m}$ are, respectively, the MFAEs for $f(x(k))$ and $g(x(k))$. If $f(x)$ and $g(x)$ can be approximated exactly by the models $\Theta_f^T\psi(x)$ and $\Theta_g^T\chi(x)$, respectively, one has $e_f(x) = e_g(x) = 0$.

Using (4.28)-(4.29), the system dynamics (4.27) is rewritten as

$$x(k+1) = \Theta^{*T}z(x(k), u(k)) + \varepsilon(x(k), u(k)) + v(k), \tag{4.30}$$

where $\Theta^* = [\Theta_f^{*T}, \Theta_g^{*T}]^T \in \mathbb{R}^{(r+s)\times n}$ and $z(x(k), u(k)) = [\psi^T(x(k)), u^T(k)\chi^T(x(k))]^T \in \mathbb{R}^{(r+s)}$ and $\varepsilon(x, u) = e_f(x) + e_g(x)u$.

Now, consider the approximator be

$$\hat{x}(k+1) =\hat{\Theta}^T(k)z(x(k), u(k)), \tag{4.31}$$

where $\hat{\Theta}(k) = [\hat{\Theta}_f^T(k), \hat{\Theta}_g^T(k)]^T \in \mathbb{R}^{(r+s)\times n}$, $\hat{\Theta}_f(k)$ and $\hat{\Theta}_g(k)$ are respectively the estimation for parameter matrices Θ^*, Θ_f^* and Θ_g^* at time k.

Let $\hbar_\Theta(x(k))$ be defined as

$$\hbar_\Theta(x(k)) = \hat{x}(k+1) - x(k+1)$$
$$= \tilde{\Theta}^T z(x(k), u(k)) - \varepsilon(k) - v(k), \tag{4.32}$$

where $\tilde{\Theta}(k) := \hat{\Theta}(k) - \Theta^* := [\tilde{\Theta}_f^T(k), \tilde{\Theta}_g^T(k)]^T$ is the parameter estimation error with $\tilde{\Theta}_f(k) := \hat{\Theta}_f(k) - \Theta_f^*$, $\tilde{\Theta}_g(k) := \hat{\Theta}_g(k) - \Theta_g^*$.

The goal is to learn the probabilistic ultimate bound S_Θ, such that $\mathbb{P}[\lim_{k\to\infty}(\Theta^* - \hat{\Theta}(k)) \in S_\Theta] \geq (1-\epsilon)$ for $\epsilon \in (0,1)$, while only noisy measurements are available for the loss function $L(\hat{\Theta}(k), x(k)) = [\hbar_\Theta(x(k))]^2$. The availability of $x(k+1)$ at time k, which is required based on (4.32), can be relaxed either by employing regressor filtering [42] or using estimators [37].

4.4 Data-regularized Concurrent Learning-based SGD for Function Identifier with noisy measurements

In this section, a data-regularized CL-based SGD update law is presented to approximate the function given in (4.9). In sharp contrast to SGD and mini-batch SGD version, rather than estimating the gradient of the error using a single (current) sample or mini-batch of random samples, the presented approach approximates the gradient flows of the estimation errors using the current data as well as fixed samples collected in recorded data stacks. Leveraging a fixed-size memory of data, for which the data are selected based on an easy-to-verify data-richness condition, rather than random, allows us to not only eliminate the PE condition requirement on the stream of data, but also to improve the convergence rate and providing guarantees using the Lyapunov theory. The convergence analysis of the dynamics of the data-regularized CL-based SGD estimation law is given based on practical stability in probability.

To employ the data-regularized CL-based SGD, which uses recorded experienced data along with current data, the previous data are collected and stored in the memory stacks $\mathcal{M} \in \mathbb{R}^{q\times P}$ and $\mathcal{Y} \in \mathbb{R}^{n\times P}$, at times $\tau_1, ..., \tau_P$ as,

$$\mathcal{M} = [\varphi(\tau_1), \varphi(\tau_2), ..., \varphi(\tau_P)], \;\; \mathcal{Y} = [y(\tau_1), y(\tau_2), ..., y(\tau_P)], \tag{4.33}$$

where P is the number of data points stored in the history stacks. P is determined such that $\mathcal{M}$ contains as many linearly independent elements as the dimension of $\varphi(k)$ (i.e., the number of linearly independent basis functions for $f(x(k))$). That is, $P \geq q$.

The error $e_h(k)$ for the h^{th} recorded sample, but using the current estimation of the function parameters at time k, is

$$e_h(k) = \hat{y}_h(k) - y(\tau_h), \tag{4.34}$$

where

$$\hat{y}_h(k) = \hat{\Theta}^T(k)\varphi(\tau_h), \tag{4.35}$$

is the estimation at time $0 \leq \tau_h < k$, $h = 1, ..., P$, using the current estimated parameters matrix $\hat{\Theta}(k)$ and the recorded $\varphi(\tau_h)$. Substituting $y(\tau_h)$, from (4.12), in (4.34) leads to

$$e_h(k) = \tilde{\Theta}^T(k)\varphi(\tau_h) - \varepsilon(\tau_h) - v(\tau_h). \tag{4.36}$$

Now, the following data-regularized loss function is considered

$$\ell_{\tilde{\Theta}}(k) = \frac{1}{2}e^T(k)e(k) + \frac{1}{2}\sum_{h=1}^{P} e_h^T(k)e_h(k). \tag{4.37}$$

The following lemma guarantees that this data-regularized objective function is strongly convex in the absence of PE condition when the rank condition on $\mathcal{M}$ is satisfied.

Lemma 6 The loss function (4.37) is strongly convex if the matrix $\mathcal{M}$ in (4.33) is full-row rank.

Proof 8 The gradient and hessian of $\ell_{\tilde{\Theta}}(k)$ in (4.37) at $\tilde{\Theta}(k)$ are respectively defined as

$$\begin{aligned}\nabla\ell_{\tilde{\Theta}}(k) &= \varphi(k)e^T(k) + \sum_{h=1}^{P}\varphi(k)e_h^T(k) \\ &= \varphi(k)\varphi^T(k)\tilde{\Theta}(k) - \varphi(k)\varepsilon^T(k) - \varphi(k)v^T(k) \\ &+ \sum_{h=1}^{P}\{\varphi(\tau_h)\varphi^T(\tau_h)\tilde{\Theta}(k) - \varphi(\tau_h)\varepsilon^T(\tau_h) - \varphi(\tau_h)v^T(\tau_h)\},\end{aligned} \tag{4.38}$$

and

$$\nabla^2\ell_{\tilde{\Theta}}(k) = \varphi(k)\varphi^T(k) + \sum_{h=1}^{P}\varphi(\tau_h)\varphi^T(\tau_h). \tag{4.39}$$

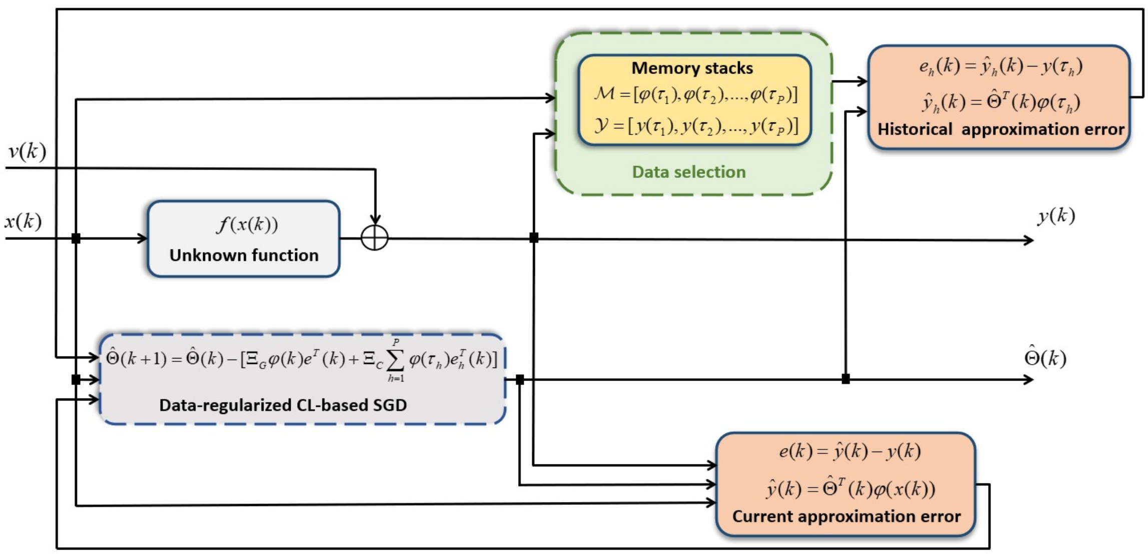

Figure 4.1: Data-regularized CL-based SGD for noisy function identification

In (4.39), the satisfaction of $\mathcal{M}$ rank condition keeps $S = \sum_{h=1}^{P} \varphi(\tau_h)\varphi^T(\tau_h) > 0$ which ensures the strong convexity of $\ell_{\tilde{\Theta}}(k)$, i.e., $\nabla^2 \ell_{\tilde{\Theta}}(k) > 0$. The data-regularized CL-based SGD parameter estimation law presented in the next subsection, which is obtained by minimizing (4.37), results in the linear convergence of the approximated parameters' error to a probabilistic ultimate bound and thus solves Problem 1. Moreover, it is shown how the selection of stored data in $\mathcal{M}$ which maximizes the condition number $\frac{\lambda_{min}(S)}{\lambda_{max}(S)}$ with $S = \sum_{h=1}^{P} \varphi(\tau_h)\varphi^T(\tau_h)$, improves the convergence rate and reduces the parameters' estimation error bound. This is in contrast to standard SGD for which there is no systematic approach for data selection to reduce the convergence bound and improve the convergence rate. The proposed data-regularized CL-based SGD update law for the approximator (4.13) is given as

$$\hat{\Theta}(k+1) = \hat{\Theta}(k) - [\Xi_G \varphi(k) e^T(k) + \Xi_C \sum_{h=1}^{P} \varphi(\tau_h) e_h^T(k)]. \tag{4.40}$$

The matrices $\Xi_G, \Xi_C \in \mathbb{R}^{q \times q}$, are the positive definite learning rate matrices for, respectively, the gradient descent term $(\varphi(k)e^T(k))$ and concurrent learning term $(\sum_{h=1}^{P} \varphi(\tau_h)e_h^T(k))$ where $\Xi_C = \xi_C I$ and $\Xi_G = \xi_G I$ with constants $\xi_C > 0$ and $\xi_G > 0$. Fig. 4.1 shows the introduced data-regularized CL-based SGD for noisy function identification.

The stochastic convergence properties of the proposed data-regularized CL-based SGD are investigated next.

Stochastic Convergence Properties for Data-regularized Concurrent Learning-based SGD Update Law

In this section, first PS-P of the estimated parameters in convergence to their optimal values is ensured using the proposed data-regularized CL-based SGD method (4.40). Then, it is discussed that the proposed data-regularized CL-based SGD method (4.40) guarantees the finite-sample boundedness in probability of the estimated parameters' error in every step k.

Exponential Probabilistic Ultimate Boundedness of Parameters' Estimation Error

Theorem 5 Consider the approximator of nonlinear function in (4.9) given in (4.13), whose parameters are adjusted according to the update law of (4.40). Let Assumptions 5-7 hold. Once the rank condition on $\mathcal{M}$ is satisfied and $\xi_C > 0$ and $\xi_G > 0$ are chosen such that

$$\frac{1}{2}(\xi_G b_2^2 + \xi_C \lambda_{max}(S))^2 < \xi_C \lambda_{min}(S) + \xi_G b_1^2 < 1, \tag{4.41}$$

then, the update law (4.40) guarantees PS-P and exponential probabilistic ultimate boundedness of $\tilde{\Theta}(k) = \hat{\Theta}(k) - \Theta^*$. That is, for $\epsilon \in (0, 1)$,

$$\mathbb{P}\{\|\tilde{\Theta}(k)\| \leq \sqrt{\frac{b}{\epsilon}}\} \geq 1 - \epsilon, \tag{4.42}$$

where $b = \frac{2d}{\lambda_{min}(Q)}$,

$$d = \frac{2B^2 b_\varepsilon^2}{\lambda_{min}(Q)} + C(\sigma), \tag{4.43}$$

$$\lambda_{min}(Q) = 2\xi_C \lambda_{min}(S) + 2\xi_G b_1^2 - (\xi_G b_2^2 + \xi_C \lambda_{max}(S))^2, \tag{4.44}$$

$$B = b_2[\xi_G + \xi_C P + \xi_G^2 b_2^2 + \xi_G \xi_C b_2^2 P + \xi_G \xi_C \lambda_{max}(S) + \xi_C^2 \lambda_{max}(S) P], \tag{4.45}$$

$$C(\sigma) = b_\varepsilon^2 b_2^2 (\xi_G + P\xi_C)^2 + b_2^2 \sigma^2 (\xi_G^2 + \xi_C^2 P^2). \tag{4.46}$$

Proof 9 Consider the Lyapunov function candidate

$$V(k) = tr\{\tilde{\Theta}^T(k)\tilde{\Theta}(k)\}. \tag{4.47}$$

One knows that

$$V(\tilde{\Theta}(k)) \leq \|\tilde{\Theta}(k)\|^2. \tag{4.48}$$

For the given $V(k)$ one has

$$\begin{aligned}\mathbb{E}[V(k+1)] - V(k) &= \mathbb{E}[tr\{\tilde{\Theta}^T(k+1)\tilde{\Theta}(k+1)\}] - tr\{\tilde{\Theta}^T(k)\tilde{\Theta}(k)\} \\ &= tr\{\mathbb{E}[\tilde{\Theta}^T(k+1))\tilde{\Theta}(k+1)] - \tilde{\Theta}^T(k)\tilde{\Theta}(k)\}.\end{aligned} \tag{4.49}$$

Using $\tilde{\Theta}(k) = \hat{\Theta}(k) - \Theta^*$ in (4.40) gives

$$\tilde{\Theta}(k+1) = \tilde{\Theta}(k) - [\Xi_G \varphi(k) e^T(k) + \Xi_C \sum_{h=1}^{P} \varphi(\tau_h) e_h^T(k)]. \tag{4.50}$$

Substituting (4.50) in (4.49) gives

$$\begin{aligned}\mathbb{E}[V(k+1)] - V(k) = tr\{\mathbb{E}[(\tilde{\Theta}^T(k) - [\Xi_G e(k)\varphi^T(k) + \Xi_C \sum_{h=1}^{P} e_h(k)\varphi^T(\tau_h)])(\tilde{\Theta}(k) \\ - [\Xi_G \varphi(k) e^T(k) + \Xi_C \sum_{h=1}^{P} \varphi(\tau_h) e_h^T(k)])] - \tilde{\Theta}^T(k)\tilde{\Theta}(k)\}.\end{aligned} \tag{4.51}$$

Using (4.14) and (4.36), one has

$$\begin{aligned}\mathbb{E}[V(k+1)] - V(k) = tr\{\mathbb{E}[(\tilde{\Theta}^T(k) - [\Xi_G \tilde{\Theta}^T(k)\varphi(k)\varphi^T(k) - \Xi_G \varepsilon(k)\varphi^T(k) - \Xi_G v(k)\varphi^T(k) \\ + \Xi_C \sum_{h=1}^{P} \tilde{\Theta}^T(k)\varphi(\tau_h)\varphi^T(\tau_h) - \Xi_C \sum_{h=1}^{P} \varepsilon(\tau_h)\varphi^T(\tau_h) - \Xi_C \sum_{h=1}^{P} v(\tau_h)\varphi^T(\tau_h)])(\tilde{\Theta}(k) \\ - [\Xi_G \varphi(k)\varphi^T(k)\tilde{\Theta}(k) - \Xi_G \varphi(k)\varepsilon^T(k) - \Xi_G \varphi(k) v^T(k) + \Xi_C \sum_{h=1}^{P} \varphi(\tau_h)\varphi^T(\tau_h)\tilde{\Theta}(k) \\ - \Xi_C \sum_{h=1}^{P} \varphi(\tau_h)\varepsilon^T(\tau_h) - \Xi_C \sum_{h=1}^{P} \varphi(\tau_h) v^T(\tau_h)])] - \tilde{\Theta}^T(k)\tilde{\Theta}(k)\}.\end{aligned} \tag{4.52}$$

By employing $S = \sum_{h=1}^{P} \varphi(\tau_h)\varphi^T(\tau_h)$ and $\phi(k) = \varphi(k)\varphi^T(k)$, (4.52) is written as,

$$\begin{aligned}\mathbb{E}[V(k+1)] - V(k) = tr\{\mathbb{E}[(\tilde{\Theta}^T(k) - [\Xi_G \tilde{\Theta}^T(k)\phi(k) - \Xi_G \varepsilon(k)\varphi^T(k) - \Xi_G v(k)\varphi^T(k) \\ + \Xi_C \tilde{\Theta}^T(k) S - \Xi_C \sum_{h=1}^{P} \varepsilon(\tau_h)\varphi^T(\tau_h) - \Xi_C \sum_{h=1}^{P} v(\tau_h)\varphi^T(\tau_h)])(\tilde{\Theta}(k) - [\Xi_G \phi(k)\tilde{\Theta}(k) \\ - \Xi_G \varphi(k)\varepsilon^T(k) - \Xi_G \varphi(k) v^T(k) + \Xi_C S\tilde{\Theta}(k) - \Xi_C \sum_{h=1}^{P} \varphi(\tau_h)\varepsilon^T(\tau_h) \\ - \Xi_C \sum_{h=1}^{P} \varphi(\tau_h) v^T(\tau_h)])] - \tilde{\Theta}^T(k)\tilde{\Theta}(k)\}.\end{aligned} \tag{4.53}$$

Based on the independency of noise $v(k)$ and Assumption 6, the expectation of cross terms multiplication of $v(k)$ with $\tilde{\Theta}(k)$ and $\varepsilon(k)$, for every $k \in \mathbb{Z}_+$, is equal to zero and hence omitted. Therefore, (4.53) is rewritten as follows

$$\begin{aligned}
&\mathbb{E}[V(k+1)] - V(k) = tr\{\mathbb{E}[\tilde{\Theta}^T(k)\tilde{\Theta}(k) - 2\Xi_G\tilde{\Theta}^T(k)\phi(k)\tilde{\Theta}(k) - 2\Xi_C\tilde{\Theta}^T(k)S\tilde{\Theta}(k) \\
&+ \Xi_G^2\tilde{\Theta}^T(k)\phi^T(k)\phi(k)\tilde{\Theta}(k) + 2\Xi_G\Xi_C\tilde{\Theta}^T(k)S\phi(k)\tilde{\Theta}^T(k) + \Xi_C^2\tilde{\Theta}^T(k)S^TS\tilde{\Theta}^T(k) \\
&+ 2\Xi_G\varepsilon(k)\varphi^T(k)\tilde{\Theta}(k) + 2\Xi_C\sum_{h=1}^{P}\varepsilon(\tau_h)\varphi^T(\tau_h)\tilde{\Theta}(k) - 2\Xi_G^2\varepsilon(k)\varphi^T(k)\phi^T(k)\tilde{\Theta}(k) \\
&+ \Xi_G^2\varepsilon(k)\varphi^T(k)\varphi(k)\varepsilon^T(k) - 2\Xi_G\Xi_C\sum_{h=1}^{P}\varphi(\tau_h)\varepsilon^T(\tau_h)\phi^T(k)\tilde{\Theta}(k) - 2\Xi_G\Xi_C\varepsilon(k)\varphi^T(k)S\tilde{\Theta}(k) \\
&- 2\Xi_C^2\tilde{\Theta}^T(k)S^T\sum_{h=1}^{P}\varphi(\tau_h)\varepsilon^T(\tau_h) + 2\Xi_G\Xi_C\sum_{h=1}^{P}\varphi(\tau_h)\varepsilon^T(\tau_h)\varphi(k)\varepsilon^T(k) \\
&+ \Xi_G^2 v(k)\varphi^T(k)\varphi(k)v^T(k) + \Xi_C^2\sum_{h=1}^{P}\varepsilon(\tau_h)\varphi^T(\tau_h)\sum_{h=1}^{P}\varphi(\tau_h)\varepsilon^T(\tau_h) \\
&+ \Xi_C^T\Xi_C\sum_{h=1}^{P}v(\tau_h)\varphi^T(\tau_h)\sum_{h=1}^{P}\varphi(\tau_h)v^T(\tau_h)] - \tilde{\Theta}^T(k)\tilde{\Theta}(k)\}. \qquad (4.54)
\end{aligned}$$

Now using

$$Q = 2\Xi_C S + 2\Xi_G\phi(k) - \Xi_G^2\phi^T(k)\phi(k) - 2\Xi_G\Xi_C\phi(k)S - \Xi_C^2 S^T S, \qquad (4.55)$$

one obtains the upper bound of (4.54) as

$$\mathbb{E}[V(k+1)] - V(k) \leq -\tilde{\Theta}^T(k)\lambda_{min}(Q)\tilde{\Theta}(k) + 2\|\tilde{\Theta}(k)\|Bb_\varepsilon + C(\sigma), \qquad (4.56)$$

where $\lambda_{min}(Q)$, B and $C(\sigma)$ are given in (4.44)-(4.46).

Knowing

$$2\|\tilde{\Theta}(k)\|Bb_\varepsilon - \frac{1}{2}\lambda_{min}(Q)\|\tilde{\Theta}(k)\|^2 \leq \frac{2B^2b_\varepsilon^2}{\lambda_{min}(Q)},$$

one rewrites (4.56) as

$$\begin{aligned}
\mathbb{E}[V(k+1)] - V(k) &\leq -\frac{1}{2}\lambda_{min}(Q)\|\tilde{\Theta}(k)\|^2 + \frac{2B^2b_\varepsilon^2}{\lambda_{min}(Q)} + C(\sigma) \\
&= -\alpha_3(\|\tilde{\Theta}(k)\|) + d, \qquad (4.57)
\end{aligned}$$

where d is given in (4.43) and

$$\alpha_3(\|\tilde{\Theta}(k)\|) = \frac{1}{2}\lambda_{min}(Q)\|\tilde{\Theta}(k)\|^2.$$

Using (4.48), one can rewrite (4.57) as follows

$$\begin{aligned} \mathbb{E}[V(\tilde{\Theta}(k+1))] - V(\tilde{\Theta}(k)) &\leq -\frac{1}{2}\lambda_{min}(Q)\|\tilde{\Theta}(k)\|^2 + d \\ &\leq -\frac{1}{2}\lambda_{min}(Q)V(\tilde{\Theta}(k)) + d. \end{aligned} \tag{4.58}$$

Taking expectation on both sides of the above equation, and using $\mathbb{E}[\frac{1}{2}\lambda_{min}(Q)V(\tilde{\Theta}(k))] \geq \frac{1}{2}\lambda_{min}(Q)\mathbb{E}[V(\tilde{\Theta}(k))]$ derived from Jason's inequality, one has

$$\mathbb{E}[V(\tilde{\Theta}(k+1))] - \mathbb{E}[V(\tilde{\Theta}(k))] \leq -\frac{1}{2}\lambda_{min}(Q)\mathbb{E}[V(\tilde{\Theta}(k))] + d. \tag{4.59}$$

Now using Lemma 4 and (4.59), in order to show that the mentioned bound in (4.42) is exponential bounded in probability, one needs

$$0 < \frac{1}{2}\lambda_{min}(Q) < 1. \tag{4.60}$$

In order to satisfy $0 < \frac{1}{2}\lambda_{min}(Q)$, using (4.44), one needs to choose $\xi_C > 0$ and $\xi_G > 0$ such that

$$\frac{1}{2}(\xi_G b_2^2 + \xi_C\lambda_{max}(S))^2 < \xi_C\lambda_{min}(S) + \xi_G b_1^2, \tag{4.61}$$

and to meet $\frac{1}{2}\lambda_{min}(Q) < 1$ or $\lambda_{min}(Q) < 2$, using (4.44), one obtains

$$\begin{aligned} &2\xi_C\lambda_{min}(S) + 2\xi_G b_1^2 - (\xi_G b_2^2 + \xi_C\lambda_{max}(S))^2 < 2 \Rightarrow \\ &(\xi_G b_2^2 + \xi_C\lambda_{max}(S))^2 + 2(1 - \xi_C\lambda_{min}(S) - \xi_G b_1^2) > 0 \Rightarrow \\ &0 < \xi_C\lambda_{min}(S) + \xi_G b_1^2 < 1. \end{aligned} \tag{4.62}$$

Since $\xi_C > 0$ and $\xi_G > 0$ are chosen such that (4.41) is met, (4.61) and (4.62) are also satisfied and this leads to (4.60).

Thus, if $\mathbb{E}[V(\tilde{\Theta}(k))] > \frac{2d}{\lambda_{min}(Q)}$, then $\mathbb{E}[V(\tilde{\Theta}_{k+1})] - \mathbb{E}[V(\tilde{\Theta}(k))] < 0$, whereas, after $\mathbb{E}[V(\tilde{\Theta}(k))]$ enters the set

$$\mathcal{D}_{\tilde{\Theta}} = \{\tilde{\Theta}(k) : \mathbb{E}[V(\tilde{\Theta}(k))] \leq b\}, \tag{4.63}$$

it is possible to have $\mathbb{E}[V(\tilde{\Theta}_{k+1})] - \mathbb{E}[V(\tilde{\Theta}(k))] \geq 0$ where $b = \frac{2d}{\lambda_{min}(Q)}$. However, $\mathbb{E}[V(\tilde{\Theta}(k))]$ stays within the positive invariant set $\mathcal{D}_{\tilde{\Theta}}$. Thus, for $\mathbb{E}[V(\tilde{\Theta}(0))] > b$, ultimately one has $\mathbb{E}[V(\tilde{\Theta}(k))] \leq b$.

Based on the Markov's inequality, for any $\epsilon \in (0, 1)$, one has

$$\mathbb{P}\{V(\tilde{\Theta}(k)) > \frac{b}{\epsilon}\} \leq \frac{\epsilon \mathbb{E}[V(\tilde{\Theta}(k))]}{b} \leq \epsilon. \tag{4.64}$$

Thus, using (4.48), it yields that

$$\mathbb{P}\{\|\tilde{\Theta}(k)\|^2 > \frac{b}{\epsilon}\} \leq \frac{\epsilon \mathbb{E}[V(\tilde{\Theta}(k))]}{b} \leq \epsilon, \tag{4.65}$$

which leads to

$$\mathbb{P}\{\|\tilde{\Theta}(k)\| \leq \sqrt{\frac{b}{\epsilon}}\} \geq 1 - \epsilon. \tag{4.66}$$

Therefore, based on Lemma 3 and Definition, $\tilde{\Theta}$ is PS-P and exponential probabilistic ultimate bounded to the bound given in (4.42). This completes the proof.

Finite-sample Boundedness of the Parameters' Estimation Error in probability The proposed data-regularized CL-based SGD update rule (4.40) guarantees the exponential probabilistic ultimate boundedness of the parameter estimation error as $k \to \infty$. Moreover, the following lemma ensures that the proposed method can also guarantee finite-sample boundedness in probability of the parameters' estimation error $\tilde{\Theta}(k)$ at any time k where $k > P$. Therefore, it solves Problem 1.

Lemma 7 Consider the approximator of nonlinear function in (4.9) given in (4.13), whose parameters are adjusted according to the update law of (4.40). Let Assumptions 5-7 hold. Once the rank condition on $\mathcal{M}$ is satisfied and $\xi_C > 0$ and $\xi_G > 0$ are chosen such that (4.41) is met, then the proposed update law (4.40) guarantees finite-sample boundedness in probability at every time $k > P$ for parameter estimation error $\tilde{\Theta}(k) = \hat{\Theta}(k) - \Theta^*$. That is,

$$\mathbb{P}\{\|\tilde{\Theta}(k)\| \leq \sqrt{\frac{b_k}{\epsilon}}\} \geq 1 - \epsilon, \tag{4.67}$$

where

$$b_k = (1 - \frac{1}{2}\lambda_{min}(Q))^k \|\tilde{\Theta}(0)\|^2 + \frac{d}{\frac{1}{2}\lambda_{min}(Q)}, \tag{4.68}$$

is a constant for every time $k > P$, $\lambda_{min}(Q)$ and d are respectively given in (4.44) and (4.43).

Proof 10 Using (4.59) and (4.4) in Lemma 4, one has

$$\begin{aligned}\mathbb{E}[V(\tilde{\Theta}(k))] \leq &(1-\frac{1}{2}\lambda_{min}(Q))^k\mathbb{E}[V(\tilde{\Theta}(0))] + d\sum_{i=0}^{k-1}(1-\frac{1}{2}\lambda_{min}(Q))^i \Rightarrow \\ \mathbb{E}[\|\tilde{\Theta}(k)\|^2] \leq &(1-\frac{1}{2}\lambda_{min}(Q))^k\|\tilde{\Theta}(0)\|^2 + \frac{d}{\frac{1}{2}\lambda_{min}(Q)}.\end{aligned} \tag{4.69}$$

Now, using Markov's inequality, for every $0 < \epsilon < 1$, one has

$$\mathbb{P}\{V(\tilde{\Theta}(k)) > \frac{b_k}{\epsilon}\} \leq \frac{\epsilon\mathbb{E}[V(\tilde{\Theta}(k))]}{b_k} \leq \epsilon, \tag{4.70}$$

which implies (4.67) with b_k is given in (4.68). Therefore, (4.67) represents the finite-sample bound in probability of $\sqrt{\frac{b_k}{\epsilon}}$ for $\|\tilde{\Theta}(k)\|$ at every finite-time $k > P$. This completes the proof.

Remark 20 As discussed in [124], the concurrent learning approach is based on the combination of a gradient descent algorithm with an auxiliary static feedback update law, which can be viewed as a type of σ-modification [10] and ensures bounded exponential convergence without the PE requirement by keeping enough measurements in memory. Here, the same extension is applied to the proposed data-regularized CL-based SGD in (4.40).

Remark 21 The parameter estimation law (4.40) converges exponentially fast to a bound which depends on the noise variance. Employing the memory data selection algorithm [144] which maximizes $\frac{\lambda_{min}(S)}{\lambda_{max}(S)}$ helps to shrink the convergence bound of parameters' estimation error in (4.42). Moreover, leveraging rich memory data in terms of maximizing $\frac{\lambda_{min}(S)}{\lambda_{max}(S)}$ leads to a narrower finite-sample bound in (4.67) for the parameters' estimation error. Intuitively, maximizing $\frac{\lambda_{min}(S)}{\lambda_{max}(S)}$ provides a higher convexity parameter for the introduced data-regularized loss function (4.37).

Remark 22 The analysis of this section shows that Based on (4.69), the error $\mathbb{E}[\|\tilde{\Theta}(k)\|^2]$ rate at any time k is $O((1-\frac{1}{2}\lambda_{min}(Q))^k) + O(\frac{d}{\lambda_{min}(Q)})$. Therefore, the parameter estimation error with a linear rate of $O((1-\frac{1}{2}\lambda_{min}(Q))^k)$ converges to the bound $\frac{d}{\frac{1}{2}\lambda_{min}(Q)}$ shrinking by rich memory data selection through maximizing $\frac{\lambda_{min}(S)}{\lambda_{max}(S)}$ that maximizes $\lambda_{min}(Q)$. Note that $\lambda_{min}(S)$ amounts to the α-strong convexity and $\lambda_{max}(S)$ amounts to the β-smoothness. Therefore, through selecting data to reuse, the condition number of the function under optimization is improved and consequently the learning rate is improved.

4.5 Simulations

In this section, the performance of the presented data-regularized CL-based SGD for online approximators with zero and non-zero MFAEs is compared with SGD [145] whose estimation law is given as follows,

$$\hat{\Theta}(k+1) = \hat{\Theta}(k) - \Gamma_G[\varphi(k)e^T(k)], \tag{4.71}$$

where $\Gamma_G = \gamma_G I$, with positive constants $\gamma_G > 0$.

In the examples, the simulation time span is $[k_0, k_f]$ with $k_0 = 0$ and $k_f = 10000$, and the x domain is defined by $\mathcal{D} = [x_L, x_H]$ with $x_L < x_H$, $x_L, x_H \in \mathbb{R}$ and $\mathcal{D}$ is quantized by $[x_L : \frac{x_H - x_L}{k_f - k_0} : x_H]$. In the proposed data-regularized CL-based SGD method, ξ_C and ξ_G are chosen such that (4.41) is satisfied and setting $\xi_G > \xi_C$, the current data is prioritized over recorded data. For SGD (4.71), let $\gamma_G = 0.1$. In all cases, the initial parameters' values are all set to zero. The additive measurement noise $v(k)$ is a zero-mean independent white noise with uniform distribution in the interval $[-\bar{v}, \bar{v}]$, i.e., $v(k) = -\bar{v} + 2\bar{v}(rand)$, where $rand$ is used to generate pseudorandom scalar with uniform distribution on the interval $(0,1)$. Two different values of $\bar{v} = 0.01$ and $\bar{v} = 0.1$ which respectively lead to the variances $\sigma^2 = 3\times10^{-5}$ and $\sigma^2 = 3\times10^{-3}$ for the defined noise are respectively employed for Examples 1 and 2.

To have a fair comparison between the mentioned methods for approximating $f(x)$ on the whole domain of x, the learning error, given below, is calculated.

$$E(k) = \mathbb{E}[\int_{\mathcal{D}} \|e(x(k))\| d^n x],$$

where the expected value is estimated by averaging over several realizations of the learning algorithms, starting from the same initial condition. In the simulations, the results of the proposed data-regularized CL-based SGD and SGD methods are respectively labeled by CL-SGD and SGD.

Example 1: Approximator with zero MFAE ($\varepsilon(k) = 0$)

Consider the following function

$$y(k) = p_1 e^{-x(k)} + p_2 e^{-x(k)} \sin(x(k)) + v(k),$$

where the parameters $[p_1, p_2]$ are unknown and the regressors are known as

$$z(x(k)) = [e^{-x(k)}, e^{-x(k)}\sin(x(k))],$$

with $q = 2$. The unknown parameters are $[p_1, p_2] = [-0.5, 0.5]$ and $\mathcal{D}$ is given with $x_L = 0$ and $x_H = 2$. We set $P = 2$ for data-regularized CL-based SGD methods. Let $\xi_G = 0.1$, $\xi_C = 0.01$ for data-regularized CL-based SGD method. Based on the obtained results, the rank condition on $\mathcal{M}$ matrix is satisfied in the first $q = 2$ steps. Therefore, P is chosen as $P = 2$ which satisfies $P \geq q$. After P steps, the data selection algorithm [144] is employed to improve the richness of the recorded memory data.

Fig. 4.2 depicts the true parameters and the approximated parameters for data-regularized CL-based SGD and SGD methods for two different noise variances $\sigma^2 = 3\times10^{-5}$ and $\sigma^2 = 3\times10^{-3}$. For both noise variances, Fig. 4.2 shows that while SGD could not converge to the vicinity of true parameters, but data-regularized CL-based SGD succeeded in convergence to the vicinity of true parameters. The online learning error $E(k)$ of data-regularized CL-based SGD and SGD for different noise variances $\sigma^2 = 3 \times 10^{-5}$ and $\sigma^2 = 3 \times 10^{-3}$ are plotted in Fig. 4.3 where data-regularized CL-based SGD shows converging to the vicinity of the origin while SGD could not approach zero. However, in Figs. 4.2 and 4.3, the converged values for the case with higher noise variance $\sigma^2 = 3 \times 10^{-3}$ show larger variations in comparison with lower noise variance $\sigma^2 = 3 \times 10^{-5}$, as expected from (4.42). The integral absolute errors (IAEs) of $E(k)$ for data-regularized CL-based SGD and SGD methods are computed in Table 4.1 where data-regularized CL-based SGD with $E(k)$ IAEs, 348.41 and 480.97, respectively, for noise variances $\sigma^2 = 3\times10^{-5}$ and $\sigma^2 = 3 \times 10^{-3}$, has resulted in better precision of online learning compared with SGD.

In this example, using data selection algorithm [144], one obtains $\lambda_{min}(S) = 0.24$, $\lambda_{max}(S) = 1.01$, $b_1 = 0.2$ and $b_2 = 1$. Since in this example $b_\varepsilon = 0$, for $\epsilon = 0.2$, the probabilistic bounds in (4.42) for the defined noise with variances $\sigma^2 = 3 \times 10^{-5}$ and $\sigma^2 = 3 \times 10^{-3}$ is, respectively, obtained as 0.06 and 0.6. Fig. 4.4 shows that for 20 different implementations of the data-regularized CL-based SGD method for noise variances $\sigma^2 = 3\times10^{-5}$ and $\sigma^2 = 3\times10^{-3}$, the parameter estimation error $\tilde{\Theta}(k)$ stays within the specified bounds. **Example 2: Approximators**

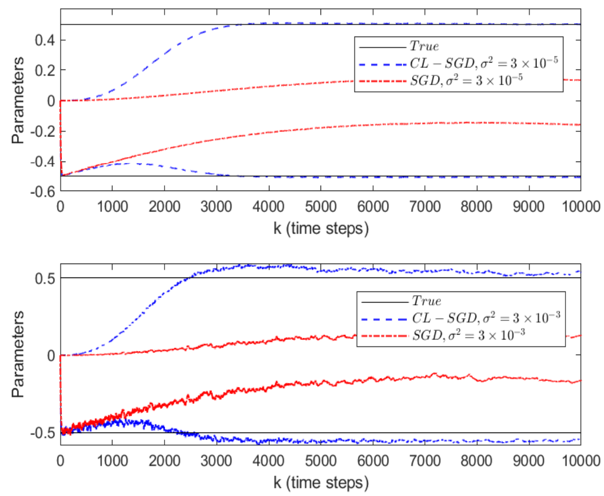

Figure 4.2: Parameters' estimation for approximators with zero MFAE.

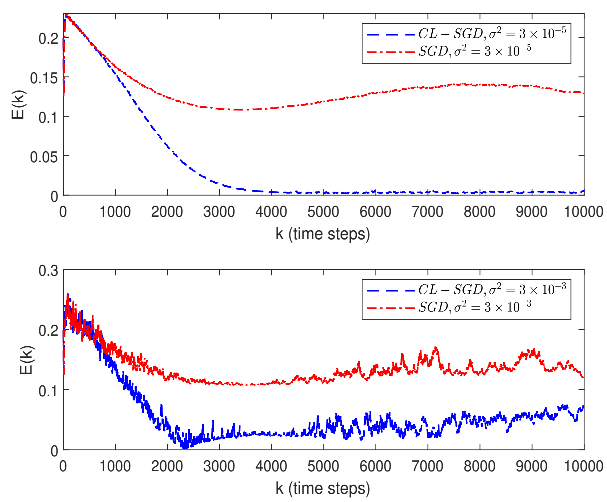

Figure 4.3: Online learning errors for approximators with zero MFAE.

Table 4.1: IAE $E(k)$ learning errors comparison

	Example 1		Example 2	
	$\sigma^2 = 3 \times 10^{-5}$	$\sigma^2 = 3 \times 10^{-3}$	$\sigma^2 = 3 \times 10^{-5}$	$\sigma^2 = 3 \times 10^{-3}$
CL-SGD	348.41	480.97	15052	15369
SGD	1342.9	1347.9	42210	42213

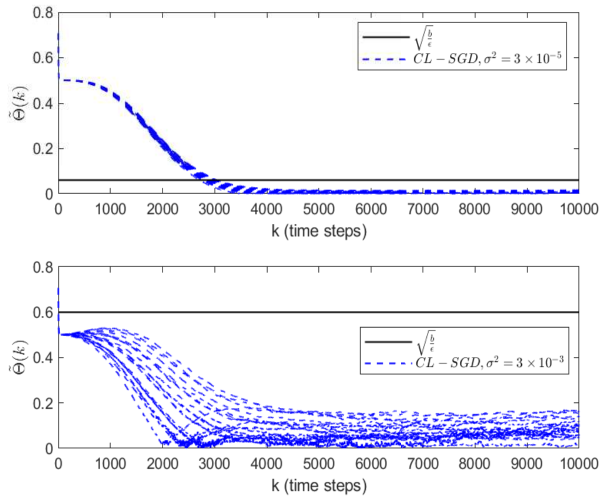

Figure 4.4: Online parameter estimation error of approximators with zero MFAE for 20 different implementations.

with non-zero MFAE ($\varepsilon(k) \neq 0$)

Now, consider the following function,

$$y(k) = 2 + \cos(x(k)) + v(k), \tag{4.72}$$

where the associated $f(x) = 2 + \cos(x(k))$ is fully unknown.

For this example, a radial basis function neural network is used with 5 radial basis functions

$$e^{-\frac{\|x(k)-c_i\|^2}{2\sigma_i{}^2}}, i = 1, 2, ..., 5,$$

where the centroids c_i are uniformly picked on $\mathcal{D} = [x_L, x_H] = [-2, 2]$ and the spreads $\sigma_i = 1.2$ for all basis functions. The rank condition on $\mathcal{M}$ matrix is satisfied in $q = 5$ steps; therefore, P is chosen as $P = q = 5$ satisfying $P \geq q$. The data selection algorithm in [144] is employed after the first 5 steps to improve the richness of the recorded data. The approximation of (4.72) is given as

$$\hat{y}(k) = \hat{\Theta}^T(k)\varphi(x(k)) = [p_1, p_2, ..., p_5][e^{-\frac{\|x(k)+2\|^2}{2(1.2)^2}}, ..., e^{-\frac{\|x(k)-2\|^2}{2(1.2)^2}}]^T.$$

Employing $\xi_G = 0.1, \xi_C = 0.05$ for data-regularized CL-based SGD method for noise variances $\sigma^2 = 3\times10^{-5}$ and $\sigma^2 = 3\times10^{-3}$ leads to the approximated parameters shown in Fig. 4.5. The SGD parameters in Fig. 4.5 did not converge to the suitable parameters, while data-regularized CL-based

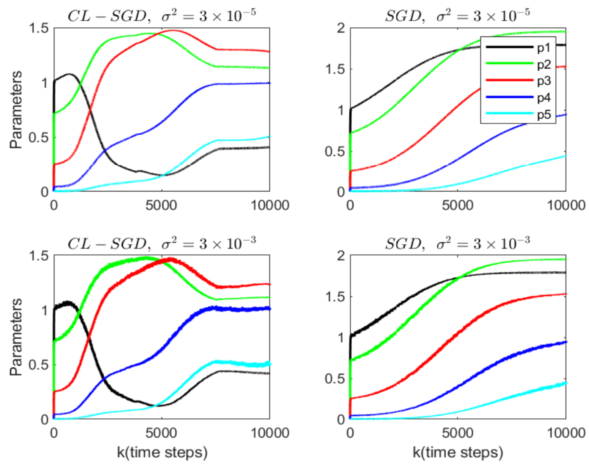

Figure 4.5: Parameters' estimation for approximators with non-zero MFAE.

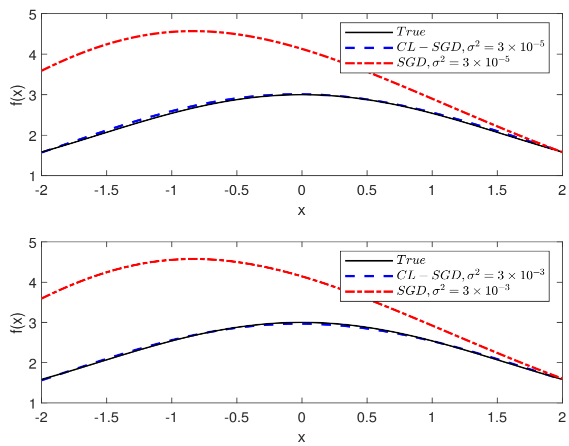

Figure 4.6: Steady-state uncertainty approximations.

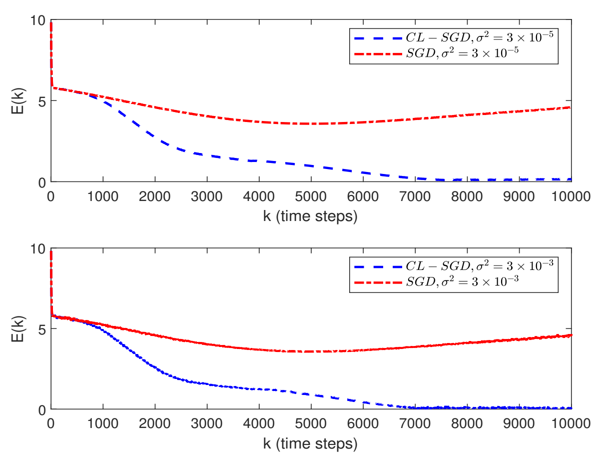

Figure 4.7: Identification errors for approximators with non-zero MFAE.

SGD succeeded in convergence to the appropriate parameters. The steady state approximations for the function $f(x)$ is given in Fig. 4.6 where for the two different values of the noise variances, it is depicted that the data-regularized CL-based SGD could better identify the unknown function compared with SGD. As the comparison of the learning error $E(k)$ in Fig. 4.7 shows, the SGD method could not perform well in the unknown function identification, however data-regularized CL-based SGD error showed ultimate bounded convergence near zero. Moreover, based on the IAE for $E(k)$ in Table 1, data-regularized CL-based SGD results in 15052 and 15369 for noise variances $\sigma^2 = 3 \times 10^{-5}$ and $\sigma^2 = 3 \times 10^{-3}$, respectively, which are lower in comparison with SGD.

4.6 Conclusion

This chapter presents a data-regularized concurrent learning-based stochastic gradient descent (CL-based SGD) method that leverages recorded data to guarantee linear (exponential) bounded convergence of the estimated parameters' error. It is shown that the richness of the memory data improves the speed of convergence and reduces the probabilistic bound of convergence. Lyapunov analysis guaranteed that the proposed data-regularized CL-based SGD method not only ensures the practical stability in probability of the estimated parameters' error but can ensure a finite-sample boundedness in probability of the estimated parameters' error. Simulation results verified that the employed data-regularized CL-based SGD could improve the speed and precision of convergence for the estimated parameters in comparison with SGD.

CHAPTER 5

DETERMINISTIC AND STOCHASTIC FIXED-TIME STABILITY OF DISCRETE-TIME AUTONOMOUS SYSTEMS

5.1 Introduction

In this chapter, we develop fixed-time stability conditions for both deterministic and stochastic DT autonomous nonlinear systems. First, fixed-time stability for equilibria of deterministic DT autonomous systems is defined. That is, a settling-time function is defined with a fixed upper bound independent of the initial condition. We then present Lyapunov theorems for fixed-time stability of both unperturbed and perturbed deterministic DT systems. Moreover, the sensitivity of fixed-time stability properties to perturbations of systems is investigated under the assumption of the existence of a locally Lipschitz discrete Lyapunov function. It is ensured that fixed-time stability is preserved under perturbations in the form of fixed-time attractiveness. Furthermore, sufficient Lyapunov conditions for fixed-time stability in probability of stochastic DT systems and their stochastic settling-time function are presented. The presented framework will pave the way for designing control laws with guaranteed satisfaction of a given performance measure in fixed time. Moreover, the presented stability results can be leveraged to develop fixed-time observers and identifiers for deterministic and stochastic DT systems, which are of great importance in control of safety-critical systems that highly rely on a system model and a state estimator to make less-conservative and feasible decisions. This is because fixed-time stability allows the system to preview and quantify probable errors in state estimators and identifiers considerably fast, which can be employed by the control system to avoid conservatism.

Notations: In this chapter, the following notations are employed. $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Z}$, $\mathbb{N}^+$, and $\mathbb{N}$ represent, respectively, the set of real numbers, non-negative real numbers, integer numbers, natural numbers except zero, and natural numbers. Moreover, $\mathbb{R}^n$ represents the set of $n \times 1$ real column vectors. $\|.\|$ is used to denote induced 2-norm for matrices and the Euclidean norm for vectors. The trace of a matrix A is indicated with $tr(A)$. $|.|$ denotes the absolute value of any scalar x. $\lfloor . \rfloor : \mathbb{R} \mapsto \mathbb{Z}$ is the floor function. $\Delta(.)$ is the DT difference operator for deterministic systems and is defined for a function $V(y(k)) : \mathbb{R}^n \mapsto \mathbb{R}^+$ as $\Delta V(y(k+1)) = V(y(k+1)) - V(y(k))$.

All random variables are assumed to be defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with Ω as the sample space, $\mathcal{F}$ as its associated Borel σ-algebra and $\mathbb{P}$ as the probability measure. For a random variable $v : \Omega \longrightarrow \mathbb{R}^n$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with some abuse of notation, the statement $v \in \mathbb{R}^n$ is used to state the dimension of the random variable. $\mathbb{E}[X]$ denotes the expected value of the random variable X on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. It is assumed that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ admits a sequence of mutually independent identically distributed random vectors $v(k), k \in \mathbb{N}$.

5.2 Fixed-time Stability for Deterministic Discrete-time Systems

In this section, the fixed-time stability of autonomous unperturbed deterministic DT systems is defined and the Lyapunov theorem specifying the sufficient conditions for their fixed-time stability is presented.

Consider the following nonlinear DT system,

$$y(k+1) = F(y(k)), \tag{5.1}$$

where $F : \mathcal{D}_y \mapsto \mathcal{D}_y, F(0) = 0$ is a nonlinear function on $\mathcal{D}_y$, and $\mathcal{D}_y$ is an open set with $0 \in \mathcal{D}_y$. Moreover, $y(k) \in \mathcal{D}_y \subseteq \mathbb{R}^n,\ k \in \mathbb{N}$ is the system state vector. For an initial condition $y(0)$, define the solution sequence $y(k),\ k \in \mathbb{N}_{y(0)} \subseteq \mathbb{N}$, where $\mathbb{N}_{y(0)}$ is the maximal interval of existence of $y(k)$ after which the solution may cease outside the domain of $F(.)$. Then, the solution sequence $y(k),\ k \in \mathbb{N}_{y(0)} \subseteq \mathbb{N}$ is uniquely defined in forward time for every initial condition $y(0) \in \mathcal{D}_y$ irrespective of whether or not the function $F(.)$ is a continuous function [97].

Before proceeding, the following definitions are needed.

Definition 13 *(Locally Lipschitz function)* A function $f(x)$ is locally Lipschitz on a domain $\Omega \subset \mathbb{R}^n$ if for each point in Ω there exist a neighborhood Ω_0 and a positive constant L such that

$$||f(x) - f(y)|| \leq L\ ||x - y||, \forall x \in \Omega_0, y \in \Omega_0. \tag{5.2}$$

Moreover, L is called the Lipschitz constant of $f(x)$.

The following definition extends the fixed-time stability definition presented in [65] for CT systems to DT systems.

Definition 14 *(Fixed-time stability)* Consider the DT nonlinear system (5.1). The zero solution of $y(k) = 0$ to the system (5.1) is said to be fixed-time stable, if there exist an open neighborhood $\mathcal{N}_y \subseteq \mathcal{D}_y$ of the origin and a settling time function $K : \mathcal{N}_y \backslash \{0\} \mapsto \mathbb{N}^+$, such that:

1) The system (5.1) is Lyapunov stable. That is, for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $||y(0)|| \leq \delta$, then $||y(k)|| \leq \epsilon$ for all $k \in \{0, ..., K(y(0)) - 1\}$.

2) For every initial condition $y(0) \in \mathcal{N}_y \backslash \{0\}$, the solution sequence $y(k)$ of (5.1) reaches the equilibrium point and remains there after $k > K(y(0))$ and $\forall y(0) \in \mathcal{N}_y$, where $K : \mathcal{N}_y \backslash \{0\} \mapsto \mathbb{N}^+$.

3) The settling-time function $K(y(0))$ is bounded, i.e., $\exists K_{max} \in \mathbb{N}^+ : K(y(0)) \leq K_{max}, \forall y(0) \in \mathcal{N}_y \backslash \{0\}$. DT nonlinear system (5.1) is globally fixed-time stable if it is fixed-time stable with $\mathcal{N}_y = \mathcal{D}_y = \mathbb{R}^n$.

Remark 23 If only conditions 1) and 2) of the above definitions are satisfied, the finite-time stability [59] is resulted. In contrast, the fixed-time stability imposes the additional condition 3). This requirement makes the upper bound of the settling time in the fixed-time stability independent of the initial condition, in contrast to the finite-time stability. Therefore, the fixed-time stability is a stronger type of stability than the finite-time stability.

The following theorem provides sufficient conditions under which the system (5.1) is fixed-time stable.

Theorem 6 Consider the nonlinear DT system (5.1). Suppose there is a Lyapunov function $V : \mathcal{D}_y \mapsto \mathbb{R}^+$ where $\mathcal{D}_y$ is an open neighborhood around the origin and there exist a neighborhood $\Omega_y \subset \mathcal{D}_y$ of the origin such that

$$V(y(0)) = 0, \tag{5.3}$$

$$V(y(k)) > 0, \quad y(k) \in \Omega_y \backslash \{0\}, \tag{5.4}$$

$$\Delta V(y(k+1)) \leq -\alpha \min\{\frac{V(y(k))}{\alpha}, \max\{V^{r_1}(y(k)), V^{r_2}(y(k))\}\}, \quad y(k) \in \Omega_y \backslash \{0\}, \tag{5.5}$$

for some positive constants $0 < \alpha < 1$, $0 < r_1 < 1$, and $r_2 > 1$. Then, the system (5.1) is fixed-time stable and has a settling-time function $K : \mathcal{N}_y \mapsto \mathbb{N}^+$ that satisfies

$$K(y(0)) \leq \lfloor \alpha^{\frac{1}{1-r_1}}(1-\alpha^{\frac{1}{1-r_1}}) \rfloor + \lfloor \alpha^{-1}(\alpha^{\frac{1}{1-r_2}}-1) \rfloor + 3, \tag{5.6}$$

for all $y(0) \in \mathcal{N}_y \backslash \{0\}$ where $\mathcal{N}_y$ is an open neighborhood of the origin. Moreover, if $\mathcal{D}_y = \mathbb{R}^n$, $V(.)$ is radially unbounded and (5.5) holds on $\mathbb{R}^n$, then system (5.1) is globally fixed-time stable.

Proof The Lyapunov stability of the system (5.1) can be concluded using similar arguments as of [97]. The proof of fixed-time stability consists of three parts. In the first part, we show that for $V(y(0)) \geq \alpha^{\frac{1}{1-r_2}}$, the settling time function is $K(y(0)) = 1$. In the second part, we show that if $\alpha^{\frac{1}{1-r_1}} < V(y(0)) < \alpha^{\frac{1}{1-r_2}}$, there exists a settling-time function with a fixed upper bound K^* (i.e., $K(y(0)) \leq K^*$) such that one has $V(y(k)) = 0,\ \forall k > K^*$. Finally, in the third part, for $V(y(0)) \leq \alpha^{\frac{1}{1-r_1}}$, the Lyapunov function reaches $V(k) = 0$ with the settling-time function $K(y(0)) = 1$.

Since $0 < r_1 < 1$ and $r_2 > 1$, one has

$$V^{r_2}(y(k)) \leq V^{r_1}(y(k)), \quad \forall V(y(k)) \leq 1, \tag{5.7}$$

$$V^{r_1}(y(k)) < V^{r_2}(y(k)), \quad \forall V(y(k)) > 1. \tag{5.8}$$

We first prove part 1 where $V(y(0)) \geq \alpha^{\frac{1}{1-r_2}}$. In this case, since $\alpha^{\frac{1}{1-r_2}} > 1$, using (5.8), (5.5) leads to

$$\Delta V(y(k+1)) \leq -\alpha \min\{\frac{V(y(k))}{\alpha}, V^{r_2}(y(k))\}. \tag{5.9}$$

Moreover, since $V(y(0)) \geq \alpha^{\frac{1}{1-r_2}}$, the above inequality for $k = 0$ yields

$$\Delta V(y(1)) \leq -V(y(0)). \tag{5.10}$$

Now, (5.10) implies that the settling time function is $K(y(0)) = 1$, for $V(y(0)) \geq \alpha^{\frac{1}{1-r_2}}$.

For part 2 where

$$\alpha^{\frac{1}{1-r_1}} < V(y(k)) < \alpha^{\frac{1}{1-r_2}},$$

based on (5.5), first we show that $V(k)$ reduces to $V(y(k)) \leq 1$ after some time where this time is upper bounded by a fixed constant K_1^*.

Note that for $1 < V(y(k)) < \alpha^{\frac{1}{1-r_2}}$, using (5.8), one has

$$\min\{\frac{V(y(k))}{\alpha}, \max\{V^{r_1}(y(k)), V^{r_2}(y(k))\}\} = V^{r_2}(y(k)), \tag{5.11}$$

Then, (5.11) and (5.5), lead to

$$V(y(k+1)) \leq V(y(k)) - \alpha V^{r_2}(y(k)). \tag{5.12}$$

The condition (5.12) holds for $k = 0, ..., K_1^* - 1$ where $1 < V(y(k)) < \alpha^{\frac{1}{1-r_2}}$. Therefore, using (5.12) for $k = 0, 1, ..., K_1^* - 1$, one has

$$\begin{aligned} V(y(1)) - V(y(0)) &\leq -\alpha V^{r_2}(y(0)), \\ V(y(2)) - V(y(1)) &\leq -\alpha V^{r_2}(y(1)), \\ &\vdots \\ V(y(K_1^* - 1)) - V(y(K_1^* - 2)) &\leq -\alpha V^{r_2}(y(K_1^* - 2)), \\ V(y(K_1^*)) - V(y(K_1^* - 1)) &\leq -\alpha V^{r_2}(y(K_1^* - 1)), \end{aligned} \tag{5.13}$$

where summing up the left and right-hand-side terms leads to

$$V(y(K_1^*)) - V(y(0)) \leq \sum_{k=0}^{K_1^*-1} -\alpha V^{r_2}(y(k)). \tag{5.14}$$

Using the fact that $V(y(k)) < V(y(k-1))$ and (5.14), one has

$$\sum_{k=0}^{K_1^*-1} V^{r_2}(y(k) \geq K_1^* V^{r_2}(y(K_1^* - 1)), \Rightarrow -\alpha \sum_{k=0}^{K_1^*-1} V^{r_2}(y(k) \leq -\alpha K_1^* V^{r_2}(y(K_1^* - 1)). \tag{5.15}$$

Employing (5.14) and (5.15) leads to

$$V(y(K_1^*)) - V(y(0)) \leq -\alpha K_1^* V^{r_2}(y(K_1^* - 1)), \tag{5.16}$$

which implies

$$K_1^* \leq \frac{V(y(0)) - V(y(K_1^*))}{\alpha V^{r_2}(y(K_1^* - 1))}. \tag{5.17}$$

Using $V(y(k)) < V(y(k-1))$, one can rewrite (5.17) as follows

$$K_1^* \leq \frac{V(y(0)) - V(y(K_1^*))}{\alpha V^{r_2}(y(K_1^*))}. \tag{5.18}$$

Having $1 < V(y(0)) < \alpha^{\frac{1}{1-r_2}}$ for $k < K_1^*$ and $V(y(K_1^*)) \leq 1$, (5.18) implies

$$K_1^* \leq \frac{\alpha^{\frac{1}{1-r_2}} - 1}{\alpha}, \tag{5.19}$$

which leads to the integer upper bound for K_1^* as follows

$$K_1^* \leq \lfloor \alpha^{-1}(\alpha^{\frac{1}{1-r_2}} - 1) \rfloor + 1. \tag{5.20}$$

Note that since for $k > K_1^*$ one has $V(y(k)) \leq 1$. Thus, for $\alpha^{\frac{1}{1-r_1}} < V(y(k)) \leq 1$, using (5.7) one has

$$\min\{\frac{V(y(k))}{\alpha}, \max\{V^{r_1}(y(k)), V^{r_2}(y(k))\}\} = V^{r_1}(y(k)),$$

which leads to rewriting (5.5) as follows

$$V(y(k+1)) \leq V(y(k)) - \alpha V^{r_1}(y(k)). \tag{5.21}$$

There exists a fixed positive integer K_2^* and time $k > K_2^*$ such that $V(k)$ reaches $V(y(k)) \leq \alpha^{\frac{1}{1-r_1}}$ and using (5.21) for $k = K_1^*, K_1^* + 1, ..., K_2^* - 1$ one obtains

$$\begin{aligned}
V(y(K_1^*+1)) - V(y(K_1^*)) &\leq -\alpha V^{r_1}(y(K_1^*)),\\
V(y(K_1^*+2)) - V(y(K_1^*+1)) &\leq -\alpha V^{r_1}(y(K_1^*+1)),\\
&\vdots\\
V(y(K_2^*-1)) - V(y(K_2^*-2)) &\leq -\alpha V^{r_1}(y(K_2^*-2)),\\
V(y(K_2^*)) - V(y(K_2^*-1)) &\leq -\alpha V^{r_1}(y(K_2^*-1)),
\end{aligned} \tag{5.22}$$

Summation of the left and right-half-side terms in (5.22) gives

$$V(y(K_2^*)) - V(y(K_1^*)) \leq -\alpha \sum_{i=0}^{K_2^*-K_1^*-1} V^{r_1}(y(K_1^*+i)). \tag{5.23}$$

Using the fact that $V(k) \le V(k-1)$ and (5.23), by employing a similar procedure as in (5.14)-(5.18), one obtains,

$$K_2^* - K_1^* \le \frac{V(K_1^*) - V(K_2^* - 1)}{\alpha V^{r_1}(y(K_2^* - 1))}. \tag{5.24}$$

Since $\alpha^{\frac{1}{1-r_1}} < V(y(k)) < 1$ for $k = K_1^*, K_1^* + 1, ..., K_2^* - 1$, (5.24) reduces to

$$K_2^* \le K_1^* + \lfloor \alpha^{\frac{1}{1-r_1}}(1 - \alpha^{\frac{1}{1-r_1}}) \rfloor + 1. \tag{5.25}$$

Using (5.20), (5.25) is rewritten as follows

$$K_2^* \le \lfloor \alpha^{-1}(\alpha^{\frac{1}{1-r_2}} - 1) \rfloor + \lfloor \alpha^{\frac{1}{1-r_1}}(1 - \alpha^{\frac{1}{1-r_1}}) \rfloor + 2. \tag{5.26}$$

At time $k > K_2^*$ for which $V(y(k)) \le \alpha^{\frac{1}{1-r_1}}$, (5.5) reduces to

$$\Delta V(y(k+1)) \le -V(y(k)), \tag{5.27}$$

which leads to $V(y(k+1)) = 0$ for $k \ge K_2^* + 1$. This completes the proof of part 2.

The proof of part 3 where $V(y(0)) \le \alpha^{\frac{1}{1-r_1}}$ is also derived based on (5.27) where $V(k)$ reaches zero with $K(y(0)) = 1$.

Hence, the Lyapunov function reaches $V(y(k)) = 0$ with the settling-time function $K(y(0))$ such that

$$K(y(0)) = 1, \quad V(y(0)) \ge \alpha^{\frac{1}{1-r_2}} \; and \; V(y(0)) \le \alpha^{\frac{1}{1-r_1}}, \tag{5.28}$$

and

$$K(y(0)) \le \lfloor \alpha^{-1}(\alpha^{\frac{1}{1-r_2}} - 1) \rfloor + \lfloor \alpha^{\frac{1}{1-r_1}}(1 - \alpha^{\frac{1}{1-r_1}}) \rfloor + 3, \quad \alpha^{\frac{1}{1-r_1}} < V(y(0)) < \alpha^{\frac{1}{1-r_2}}. \tag{5.29}$$

Therefore, the system is fixed-time stable, and the system trajectory converges to the origin with the settling-time function given in (5.6). This completes the proof.

Moreover, if $\mathcal{N}_y = \mathcal{D}_y = \mathbb{R}^n$ and $V(.)$ is radially unbounded, the global fixed-time stability follows using the same procedure. □

Remark 24 Based on the definitions of fixed-time [65] and finite-time [59] stabilities, an autonomous DT fixed-time stable system is also finite-time stable. Fixed-time stability, which requires stronger conditions in comparison with finite-time stability, needs to represent a fixed upper bound for the settling-time function. However, in finite-time stability, the settling-time is a function of the initial conditions and no fixed upper bound is provided. Therefore, fixed-time stability is a stronger type of stability than asymptotic, exponential and finite-time stabilities for DT systems.

5.3 Sensitivity to Deterministic Perturbation for Fixed-time Stable Discrete-time Systems

The system (5.1) usually describes a nominal model of the system that works under ideal conditions. Nevertheless, many real-world systems are under uncertainties and disturbances that affect the system's behavior. To account for these uncertainties, a more accurate representation of the system can be given by the following deterministic perturbed model

$$y(k+1) = F(y(k)) + g(k, y(k)), \tag{5.30}$$

where g represents a perturbation caused by disturbances, uncertainties, or modeling errors. This section investigates the solution behavior of the deterministic perturbed system (5.30) in a neighborhood of the fixed-time stable equilibrium of the nominal system (5.1).

Assumption 8 The perturbation term g is bounded, i.e.,

$$\sup_{\mathbb{N}^+ \times \mathcal{D}_y} \|g(k, y(k))\| < \delta_0, \tag{5.31}$$

for some $\delta_0 < \infty$.

The following definition extends the fixed-time attractiveness definition presented in [65] and [146] for CT systems to DT systems.

Definition 15 (Fixed-time attractiveness) The perturbed system (5.30) is said to be fixed-time attractive by a bounded set $\mathcal{Y}$ around the equilibrium point, if $\forall y(0) \in \mathcal{N}_y$ the solution sequence

$y(k)$ of (5.30) reaches $\mathcal{Y}$ in finite time $k > K(y(0))$ and remains there for all $k > K(y(0))$, where $K : \mathcal{N}_y \backslash \{0\} \mapsto \mathbb{N}^+$ is the settling-time function and the settling-time function $K(y(0))$ is bounded, i.e., $\exists K_{max} \in \mathbb{N}^+ : K(y(0)) \leq K_{max}, \forall y(0) \in \mathcal{N}_y$.

The following lemma is required in the proof of Lyapunov-based fixed-time attractiveness of perturbed deterministic systems.

Lemma 8 Let $V(y(k)) : \mathcal{D}_y \mapsto \mathbb{R}^+$ be a fixed-time Lyapunov function for the the nominal (unperturbed) system (5.1), i.e., $V(y(k))$ satisfies conditions (5.3)-(5.5) for the system (5.30) when $g = 0$. Let also $V(y(k))$ be locally Lipschitz continuous on $\mathcal{D}_y$ with Lipschitz constant L_V and Assumption 8 hold. Then, for the perturbed deterministic system (5.30), $V(k)$ satisfies

$$\Delta V(y(k+1)) \leq -\alpha \min\{\frac{V(y(k))}{\alpha}, \max\{V^{r_1}(y(k)), V^{r_2}(y(k))\}\} + L_V \|g(k, y(k))\|, \quad (5.32)$$

where $\Delta V(y(k+1))$ is computed along the solution of the unperturbed deterministic system.

Proof The proof is similar to [147], which is developed for exponential stability, and is thus omitted. □

The following theorem provides the behavior of deterministic fixed-time stable DT systems under bounded deterministic perturbations.

Theorem 7 Suppose there exists a Lyapunov function $V : \Omega_y \mapsto \mathbb{R}^+$ which is locally Lipschitz on an open neighborhood Ω_y of the origin with Lipschitz constant L_V and satisfies (5.3)-(5.5) for the nominal system (5.1) for some real positive numbers $\alpha, r_1, r_2 > 0$ such that $0 < \alpha < 1$, $0 < r_1 < 1$, and $r_2 > 1$. Let Assumption 8 hold. Then, around the origin, the system (5.30) is fixed-time attractive to the following bound

$$b_y = \{y \in \Omega_y : V(y) \leq \mathcal{B}\}, \quad (5.33)$$

where

$$\mathcal{B} = \begin{cases} (\frac{m_1 L_V \delta_0}{\alpha})^{\frac{1}{r_2}}, & 1 < V(y(0)) < \alpha^{\frac{1}{1-r_2}}, \\ (\frac{m_2 L_V \delta_0}{\alpha})^{\frac{1}{r_1}}, & \alpha^{\frac{1}{1-r_1}} < V(y(0)) \leq 1, \end{cases} \quad (5.34)$$

and its fixed-time bounded settling-time function is $K(y(0)) \le K^*$ where

$$K^* = \begin{cases} \lfloor \alpha_c^{-1}(\alpha^{\frac{1}{1-r_2}} - 1) \rfloor + 1, & 1 < V(y(0)) < \alpha^{\frac{1}{1-r_2}}, \\ \lfloor \alpha_d^{-1}(\alpha^{\frac{r_1}{r_1-1}} - \alpha) \rfloor + 1, & \alpha^{\frac{1}{1-r_1}} < V(y(0)) \le 1, \end{cases} \tag{5.35}$$

$\alpha_c = (1 - \frac{1}{m_1})\alpha$, $\alpha_d = (1 - \frac{1}{m_2})\alpha$. The constants $m_1 > 1$ and $m_2 > 1$ are selected such that

$$\begin{cases} \alpha \mathcal{B}^{r_2} - m_1 L_V \delta_0 > 0, & 1 < V(y(0)) < \alpha^{\frac{1}{1-r_2}}, \\ \alpha \mathcal{B}^{r_1} - m_2 L_V \delta_0 > 0, & \alpha^{\frac{1}{1-r_1}} < V(y(0)) \le 1 . \end{cases} \tag{5.36}$$

Proof According to Theorem 6, the origin is the fixed-time stable equilibrium for the unperturbed or nominal system (5.1).

Lemma 8 and (5.31) imply that

$$\Delta V(y(k+1)) \le -\alpha \min\{\frac{V(y(k))}{\alpha}, \max\{V^{r_1}(y(k)), V^{r_2}(y(k))\}\} + L_V \delta_0. \tag{5.37}$$

For $1 < V(y(0)) < \alpha^{\frac{1}{1-r_2}}$, (5.37) leads to

$$\Delta V(y(k+1)) \le -\alpha V^{r_2}(y(k)) + L_V \delta_0. \tag{5.38}$$

Having $1 < V(y(0)) < \alpha^{\frac{1}{1-r_2}}$ and $V(y(0)) > \mathcal{B}$, and using (5.36) and $m_1 > 1$, one has

$$\begin{aligned} \alpha \mathcal{B}^{r_2} - m_1 L_V \delta_0 > 0 &\Rightarrow -\alpha \mathcal{B}^{r_2} + m_1 L_V \delta_0 < 0, \\ &\Rightarrow -\alpha \mathcal{B}^{r_2} + L_V \delta_0 < 0, \end{aligned} \tag{5.39}$$

which results in

$$L_V \delta_0 < \frac{1}{m_1} \alpha \mathcal{B}^{r_2}. \tag{5.40}$$

For $y(0) \notin b_y$ ($V(y(0)) > \mathcal{B}$) and $1 < V(y(0)) < \alpha^{\frac{1}{1-r_2}}$, (5.38) and (5.40) imply that

$$\Delta V(y(k+1)) \le -\alpha V^{r_2}(k) + \frac{1}{m_1} \alpha \mathcal{B}^{r_2}. \tag{5.41}$$

Using $V(y(k)) > \mathcal{B}$, (5.41) is upper bounded as follows

$$\Delta V(y(k+1)) \le -\alpha_c V^{r_2}(y(k)), \tag{5.42}$$

such that $\alpha_c = (1 - \frac{1}{m_1})\alpha$ is positive. Using the results of part 2 in the proof of Theorem 6, (5.42) implies that for $y(0) \notin b_y$ and $1 < V(y(0)) < \alpha^{\frac{1}{1-r_2}}$ with $\alpha < m_1 L_V \delta_0$, $y(k)$ reaches the invariant set (5.33) within the fixed time steps $K^* = \lfloor \alpha_c^{-1}(\alpha^{\frac{1}{1-r_2}} - 1) \rfloor + 1$ and remains there after.

Using (5.37), for $\alpha^{\frac{1}{1-r_1}} < V(y(0)) \leq 1$, one has

$$\Delta V(y(k+1)) \leq -\alpha V^{r_1}(y(k)) + L_V \delta_0. \tag{5.43}$$

Having $\alpha^{\frac{1}{1-r_1}} < V(y(0)) \leq 1$ and $V(y(0)) > \mathcal{B}$, and using (5.36) and $m_2 > 1$, one has

$$\begin{aligned} \alpha \mathcal{B}^{r_1} - m_2 L_V \delta_0 > 0 &\Rightarrow -\alpha \mathcal{B}^{r_1} + m_2 L_V \delta_0 < 0, \\ &\Rightarrow -\alpha \mathcal{B}^{r_1} + L_V \delta_0 < 0. \end{aligned} \tag{5.44}$$

From (5.44), one obtains

$$L_V \delta_0 < \frac{1}{m_2} \alpha \mathcal{B}^{r_1}. \tag{5.45}$$

For $y(0) \notin b_y$ ($V(y(0)) > \mathcal{B}$) and $\alpha^{\frac{1}{1-r_1}} < V(y(0)) \leq 1$, then (5.43) and (5.45) imply that

$$\Delta V(y(k+1)) \leq -\alpha V^{r_1}(y(k)) + \frac{1}{m_2} \alpha \mathcal{B}^{r_1}. \tag{5.46}$$

Using $V(y(k)) > \mathcal{B}$, (5.46) is upper bounded as follows

$$\Delta V(y(k+1)) \leq -\alpha_d V^{r_1}(y(k)), \tag{5.47}$$

such that $\alpha_d = (1 - \frac{1}{m_2})\alpha$ is positive. Using the results of part 2 in Theorem 6 proof, (5.47) implies that for $y(0) \notin b_y$ and $\alpha^{\frac{1}{1-r_1}} < V(y(0)) < 1$ with $m_2 L_V \delta_0 < \alpha$, $y(k)$ reaches the invariant set (5.33) within the fixed time steps $K^* = \lfloor \alpha_d^{-1}(\alpha^{\frac{r_1}{r_1-1}} - \alpha) \rfloor + 1$ and remains in b_y ever after. This completes the proof. □

Remark 25 In (5.33), the bound $\mathcal{B}$ is either a function of m_1 or m_2, as given is (5.34). Notice that the fixed-time attractive bound (5.34) increases by choosing large values for m_1 or m_2 and accordingly the fixed-time of convergence given in (5.35) decreases. Therefore, the bigger we choose the bounded set $\mathcal{B}$, the shorter the fixed-time of convergence will be.

5.4 Fixed-time Stability in Probability for Stochastic Discrete-time Systems

Consider the DT nonlinear stochastic system given by

$$y(k+1) = f(y(k)) + g(y(k))\nu(k) \triangleq F(y(k),\nu(k)), \quad y(0) \overset{\text{a.s.}}{\equiv} y_0, \quad k \in \mathbb{N}, \tag{5.48}$$

where, for every $k \in \mathbb{N}, y(k) \in \mathcal{D} \subseteq \mathbb{R}^n$ is a $\mathcal{D}$-valued stochastic process with $y_0 \in \mathcal{D}$, and $\nu(k) \in \mathbb{R}^n, k \in \mathbb{N}$, is the independent and identically distributed zero-mean stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$. $f : \mathcal{D} \to \mathcal{D}$ and $g : \mathcal{D} \to \mathbb{R}^{n\times n}$ are continuous functions with $f(0) = 0$ and $g(0) = 0$ where $y_e = 0$ is the equilibrium of the system (5.48), if and only if $y(.)$ is $\mathbb{P}$-almost surely (a.s.) equal to zero (i.e., $y(.) \overset{\text{a.s.}}{\equiv} 0$) and is a solution of (5.48).

A stochastic process $y : [0, \kappa] \times \Omega \to \mathcal{D}$ is a solution sequence of (5.48) on the discrete-time interval $[0, \kappa]$ with initial condition $y(0) \overset{\text{a.s.}}{\equiv} y_0$ if $y(k)$ satisfies (5.48) almost surely.

The following definitions are given for stability in probability for the zero solution $y(k) \overset{\text{a.s.}}{\equiv} 0$ of the DT nonlinear stochastic system (5.48).

Definition 16 [119, 148]

1) The zero solution $y(k) \overset{\text{a.s.}}{\equiv} 0$ to (5.48) is Lyapunov stable in probability, if for every $\varepsilon > 0$ and $\rho \in (0, 1)$, there exist $\delta = \delta(\varepsilon, \rho) > 0$ such that, for all $||y_0||< \delta$,

$$\mathbb{P}\left(\sup_{k\in\mathbb{N}} \|y(k)\|> \varepsilon\right) \le \rho.$$

2) The zero solution $y(k) \overset{\text{a.s.}}{\equiv} 0$ to (5.48) is asymptotically stable in probability if it is Lyapunov stable in probability and, for every $\rho \in (0, 1)$, there exists $\delta = \delta(\rho) > 0$ such that if $||y_0||< \delta$, then

$$\mathbb{P}\left(\lim_{k\to\infty} \|y(k)\|= 0\right) \ge 1 - \rho.$$

3) The zero solution $y(k) \overset{\text{a.s.}}{\equiv} 0$ to (5.48) is globally asymptotically stable in probability if it is Lyapunov stable in probability and, for all $y_0 \in \mathbb{R}^n$,

$$\mathbb{P}\left(\lim_{k\to\infty} \|y(k)\|= 0\right) = 1.$$

4) The zero solution $y(k) \overset{\text{a.s.}}{\equiv} 0$ to (5.48) is exponentially stable in probability if for some $0 < \gamma < 1$ independent of ν, it is Lyapunov stable in probability and, for every $\rho \in (0, 1)$, there

exists $\delta = \delta(\rho) > 0$ such that if $||y_0||< \delta$, then

$$\mathbb{P}\left(\lim_{k\to\infty} \|\gamma^k y(k)\|= 0\right) \geq 1 - \rho.$$

5) The zero solution $y(k) \overset{\text{a.s.}}{\equiv} 0$ to (5.48) is globally exponentially stable in probability if for some $0 < \gamma < 1$ independent of ν, it is Lyapunov stable in probability and, for all $y_0 \in \mathbb{R}^n$,

$$\mathbb{P}\left(\lim_{k\to\infty} \|\gamma^k y(k)\|= 0\right) = 1.$$

Definition 17 [119] For the DT stochastic dynamical system (5.48) and $V : \mathcal{D} \to \mathbb{R}^+$, the difference operator $\mathbf{\Delta} V$ of y is given as follows,

$$\mathbf{\Delta} V(y) = \mathbb{E}[V(F(y, \nu))] - V(y), \quad y \in \mathcal{D}.$$

Note that the difference operator in Definition 5 is a deterministic function and does not involve the expectation of the system state trajectory and only involves the expectation over the random noise variable ν. Moreover, the random vectors $\nu(k)$, $k \in \mathbb{N}$, all have the same distribution.

In the following, sufficient conditions for Lyapunov, asymptotic and exponential stability in probability for the system (5.48) are given.

Lemma 9 [118, 148]: Consider the discrete-time nonlinear stochastic system (5.48) and assume that there exists a continuous function $V : \mathcal{D} \to \mathbb{R}^+$ such that

$$V(0) = 0,$$

$$V(y) > 0, \quad y \in \mathcal{D}, \quad y \neq 0,$$

$$\mathbf{\Delta} V(y) \leq 0, \quad y \in \mathcal{D}.$$

Then, the zero solution $y(k) \overset{\text{a.s.}}{\equiv} 0$ to (5.48) is Lyapunov stable in probability. Moreover, if

$$\mathbf{\Delta} V(y) < 0, \quad y \in \mathcal{D}, \quad y \neq 0,$$

then the zero solution $y(k) \overset{\text{a.s.}}{\equiv} 0$ to (5.48) is asymptotically stable in probability. Furthermore, if

$$\mathbf{\Delta} V(y) < -\gamma V(y), \quad 0 < \gamma < 1, \quad y \in \mathcal{D}, \quad y \neq 0,$$

then the zero solution $y(k) \overset{\text{a.s.}}{\equiv} 0$ to (5.48) is exponentially stable in probability. If $\mathcal{D} = \mathbb{R}^n$ and $V(\cdot)$ is radially unbounded, then the zero solution $y(k) \overset{\text{a.s.}}{\equiv} 0$ to (5.48) is globally asymptotically or exponentially stable in probability under the defined Lyapunov conditions.

The following definition provides the characteristics of stochastic DT systems under which they are fixed-time stable in probability.

Definition 18 *(Fixed-time stability in probability)* Consider the stochastic DT nonlinear system (5.48). The zero solution of $y(k) \overset{\text{a.s.}}{\equiv} 0$ to the system (5.48) is said to be fixed-time stable in probability, if there exist a stochastic process called stochastic settling time function $K(y, \cdot)$, such that:

1) The system (5.48) is Lyapunov stable in probability. That is, for every $\epsilon > 0$ and $\rho \in (0, 1)$, there exists a $\delta = \delta(\epsilon, \rho) > 0$ such that for all $y(0) \overset{\text{a.s.}}{\equiv} y_0 \in \mathcal{D}\backslash\{0\}$, if $||y(0)|| \leq \delta$, then

$$\mathbb{P}\left(\sup_{k \in [0, K(y_0, \nu))} \left\|y(k)\right\| > \varepsilon\right) \leq \rho.$$

2) For every initial condition $y(0) \overset{\text{a.s.}}{\equiv} y_0 \in \mathcal{D}\backslash\{0\}$, the solution sequence $y(k)$ is defined on $\left[0, K\left(y_0, \nu\right)\right)$, $\nu \in \Omega$, $y(k) \in \mathcal{D}\backslash\{0\}$, $k \in \left[0, K\left(y_0, \nu\right)\right)$, and

$$\mathbb{P}\left(\left\|y\left(K\left(y_0, \nu\right)\right)\right\| = 0\right) = 1.$$

3) The stochastic settling-time function $K(y, \cdot)$, for all $y \in \mathcal{D}$, is finite almost surely and there exist a fixed-time upper bound for the stochastic settling-time $K(y, \cdot)$, i.e., $\mathbb{E}[K(y_0, \nu)] \leq K_{max}$ where K_{max} is a positive integer.

The zero solution $y(k) \overset{\text{a.s.}}{\equiv} 0$ to (5.48) is globally fixed-time stable in probability if it is fixed time stable in probability with $\mathcal{D} = \mathbb{R}^n$.

Lemma 10 Consider the nonlinear stochastic DT system (5.48) and the scalar system

$$V(x(k+1)) = \gamma(V(x(k))), \;\; x(k) \in \mathbb{R}^n, \tag{5.49}$$

where

$$\gamma(V(x(k))) = V(x(k)) - \alpha \min\{\frac{V(x(k))}{\alpha}, \max\{V^{r_1}(x(k)), V^{r_2}(x(k))\}\}, \tag{5.50}$$

such that $0 < \alpha < 1$, $0 < r_1 < 1$, and $r_2 > 1$. If there exists a continuous positive-definite function $V : \mathbb{R}^n \to \mathbb{R}^+$and the nondecreasing function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$such that

$$\mathbb{E}\left[V(F(y, \nu)\right] \leq \gamma(V(y)), \quad y \in \mathbb{R}^n,$$

then

$$V\left(y_0\right) \leq x_0, \quad x_0 \in \mathbb{R}^+$$

implies

$$\mathbb{E}[V(y(k))] \leq x(k), \quad k \in \mathbb{N},$$

where the sequence $x(k)$, $k \in \mathbb{N}$, satisfies (5.49).

Proof. This Lemma is an extension of finite-time stability conditions [119], which is provided for fixed-time stability conditions. The proof is similar and is omitted. □

The following theorem represents the sufficient Lyapunov conditions for fixed-time stability in probability for stochastic DT nonlinear systems.

Theorem 8 Consider the nonlinear stochastic system (5.48). If there exists a continuous and radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}^+$ such that

$$V(0) = 0, \tag{5.51}$$

$$V(y) > 0, \quad y \in \mathbb{R}^n \backslash \{0\}, \tag{5.52}$$

$$\mathbb{E}[V(F(y, \nu))] \leq \gamma(V(y)), \quad y \in \mathbb{R}^n \backslash \{0\}, \tag{5.53}$$

where $\gamma(.)$ is given in (5.50), then the zero solution $y(k) \overset{\text{a.s.}}{\equiv} 0$ to (5.48) is globally fixed-time stable in probability. Moreover, there exists a stochastic settling-time $K : \mathbb{R}^n \to \mathbb{N}$ such that

$$\mathbb{E}\left[K\left(y_0\right)\right] \leq \hat{K}\left(x_0\right) < K_{max}, \tag{5.54}$$

where $K(\cdot)$ is almost surely finite stochastic settling-time function and $\hat{K}\left(x_0\right)$ is the finite settling-time function of (5.49) and K_{max} is the fixed upper bound for $\hat{K}\left(x_0\right)$ and $\mathbb{E}\left[K\left(y_0\right)\right]$.

Proof Based on (5.50) and (5.53), one has

$$\mathbb{E}[V(F(y, \nu))] - V(y) \leq \gamma(V(y)) - V(y) < 0, \quad y \in \mathbb{R}^n \backslash \{0\}, \tag{5.55}$$

and hence, it follows from Lemma 9 that the zero solution $y(k) \overset{\text{a.s.}}{\equiv} 0$ to (5.48) is globally asymptotically stable in probability. Now, consider the nonlinear DT system (5.49) and note that, by Theorem 6, the zero solution $x(k) \equiv 0$ to (5.49) is globally fixed-time stable and there exists

$$\hat{K}(x_0) < \lfloor \alpha^{\frac{1}{1-r_1}}(1-\alpha^{\frac{1}{1-r_1}}) \rfloor + \lfloor \alpha^{-1}(\alpha^{\frac{1}{1-r_2}}-1) \rfloor + 3,$$

such that

$$x(k) = 0, \quad k \geq \hat{K}(x_0), \quad x_0 \in \mathbb{R}^+.$$

Now, let $V(y_0) < x_0, y(0) \overset{\text{a.s.}}{\equiv} y_0 \in \mathbb{R}^n$, and it follows from Lemma 10 that

$$\mathbb{E}[V(y(k))] = 0, \quad k \geq \hat{K}(x_0).$$

Since $V(y(k)), k \in \mathbb{N}$, is a nonnegative random variable, it follows that $V(y(k)) \overset{\text{a.s.}}{\equiv} 0$ for all $k \geq \hat{K}(x_0)$. Then, it follows from (5.51) and (5.52) that $y(k) \overset{\text{a.s.}}{\equiv} 0$ for all $k \geq \hat{K}(x_0)$. Therefore, there exists a stochastic settling-time $\mathbb{E}[K(y_0)] \leq \hat{K}(x_0)$ such that $y(k) = 0, k \geq K(y_0)$. Finally, since $\mathbb{E}[K(y_0)] \leq \hat{K}(x_0)$, it follows that

$$\mathbb{E}\left[K(y_0)\right] \leq \hat{K}(x_0) < \lfloor \alpha^{\frac{1}{1-r_1}}(1-\alpha^{\frac{1}{1-r_1}}) \rfloor + \lfloor \alpha^{-1}(\alpha^{\frac{1}{1-r_2}}-1) \rfloor + 3,$$

and hence, Definition 6 is satisfied. □

5.5 Example Illustration and Simulation

This sections provides examples to verify the correctness of the presented fixed-time stability results. Examples 1 and 2 are, respectively, presented for deterministic scalar and higher-order systems without uncertainties and perturbations. Examples 3 and 4 are counterexamples that show that if the Lyapunov conditions for a deterministic scalar or higher-order system guarantee its fixed-time stability, by adding noise to the system, the same Lyapunov candidate only guarantees exponential stability in probability, and not fixed-time stability in probability. These examples clearly show that moving from a fixed-time stable deterministic system to a stochastic system with the same dynamics, one might look for new Lyapunov function candidates than the one used for the deterministic system to show its fixed-time stability in probability, if there exists one.

Example 1. *(Fixed-time stable scalar deterministic discrete-time system)* Consider the scalar nonlinear DT system given as follows

$$y(k+1) = ay(k) - \alpha' sign(y(k)) \min\{|y(k)|/\alpha', \max\{|y(k)|^{r'_1}, |y(k)|^{r'_2}\}\}, \tag{5.56}$$

where $y(k) \in \mathbb{R}$, $k \in \mathbb{N}$, $\frac{1}{2} < a \leq 1$, $\alpha' \in (0,1)$, $r'_1 \in (0,1)$ and $r'_2 > 1$. Now, using Theorem 6, it is shown that the zero solution $y(k) = 0$ to (5.56) with $a = 1$ is globally fixed-time stable. Consider $V(y(k)) = y^2(k)$ and $y_L < y(0) < y_H$ where $y_L = \alpha'^{\frac{1}{1-r_1}}$ and $y_H = \alpha'^{\frac{1}{1-r_2}}$ (Note that if $y(0) > y_H$ or $y(0) < y_L$, then the zero solution $y(k) = 0$ for (5.56) with $a = 1$ is fixed-time stable with $K(y(0)) = 1$).

The difference of $V(y(k)) = y^2(k)$ is as follows,

$$\begin{aligned}\Delta V(y(k)) =& [ay(k) - \alpha' sign(y(k)) \min\{|y(k)|/\alpha', \max\{|y(k)|^{r'_1}, |y(k)|^{r'_2}\}\}]^2 - y^2(k)\\ =& (ay(k))^2 - 2a\alpha'|y(k)| \min\{|y(k)|/\alpha', \max\{|y(k)|^{r'_1}, |y(k)|^{r'_2}\}\}\\ &+ (\alpha' \min\{|y(k)|/\alpha', \max\{|y(k)|^{r'_1}, |y(k)|^{r'_2}\}\})^2 - y^2(k)\\ =& (a^2-1)y^2(k) + \alpha' \min\{|y(k)|/\alpha', \max\{|y(k)|^{r'_1}, |y(k)|^{r'_2}\}\}\times\\ &(-2a|y(k)| + \alpha' \min\{|y(k)|/\alpha', \max\{|y(k)|^{r'_1}, |y(k)|^{r'_2}\}\}).\end{aligned} \tag{5.57}$$

Using the fact that

$$|y(k)| > \alpha' \min\{|y(k)|/\alpha', \max\{|y(k)|^{r'_1}, |y(k)|^{r'_2}\}\}, \tag{5.58}$$

one has

$$\begin{aligned}&-2a|y(k)| + \alpha' \min\{|y(k)|/\alpha', \max\{|y(k)|^{r'_1}, |y(k)|^{r'_2}\}\} <\\ &(1-2a)\alpha' \min\{|y(k)|/\alpha', \max\{|y(k)|^{r'_1}, |y(k)|^{r'_2}\}\}.\end{aligned} \tag{5.59}$$

Therefore, using (5.59), (5.57) leads to,

$$\Delta V(y(k)) \leq (a^2-1)y^2(k) + (1-2a)\alpha'^2 \min\{y^2(k)/\alpha'^2, \max\{y^{2r'_1}(k), y^{2r'_2}(k)\}\}, \tag{5.60}$$

and using $V(y(k)) = y^2(k)$ one can rewrite (5.60) as follows,

$$\Delta V(y(k)) \leq (a^2-1)V(y(k)) + (1-2a)\alpha'^2 \min\{V(k)/\alpha'^2, \max\{V^{r'_1}(k), V^{r'_2}(k)\}\}. \tag{5.61}$$

Table 5.1: Parameters α', r'_1, r'_2, and fixed-time upper bound of settling-time function (K^*) for (5.56) with $a = 1$ and the initial condition $y(0) = 20$.

	α'	r'_1	r'_2	K^*	y_L	y_H
Case 1	0.4	0.2	1.2	59601	0.31	97.6
Case 2	0.7	0.9	1.1	2558	0.02	35.4
Case 3	0.3	0.6	1.3	34002	0.04	55.3
Case 4	0.7	0.9	10	3	0.02	1.04

Since $\frac{1}{2} < a \leq 1$, (5.61) is rewritten as

$$\Delta V(y(k)) \leq -\beta\alpha'^2 \min\{V(k)/\alpha'^2, \max\{V^{r'_1}(k), V^{r'_2}(k)\}\}, \tag{5.62}$$

where $\beta = (2a - 1)$ and for $\frac{1}{2} < a \leq 1$, $0 < \beta \leq 1$.

For $a = 1$, (5.62) leads to

$$\Delta V(y(k)) \leq -\alpha'^2 \min\{V(k)/\alpha'^2, \max\{V^{r'_1}(k), V^{r'_2}(k)\}\}, \tag{5.63}$$

which is analogous to (5.5) where $\alpha = \alpha'^2$, $r_1 = r'_1$ and $r_2 = r'_2$, and all the parameters conditions mentioned in Theorem 6 are satisfied. Therefore, it is shown that system (5.56) with $a = 1$ is globally fixed-time stable. Based on (5.6), the fixed upper bound for the settling-time function of system (5.56) with $a = 1$ is

$$K^* = \lfloor \alpha'^{\frac{2}{1-r'_1}}(1 - \alpha'^{\frac{2}{1-r'_1}}) \rfloor + \lfloor \alpha'^{-2}(\alpha'^{\frac{2}{1-r'_2}} - 1) \rfloor + 3. \tag{5.64}$$

The state trajectory of the system (5.56) with $a = 1$ is simulated in Fig. 5.1 for 4 different values of parameters α', r'_1 and r'_2 to verify the fixed-time convergence of the system (5.56) with $a = 1$, and $y(0) = 20$ such that $y_L < y(0) < y_H$ in Cases 1-3 and $y(0) > y_H$ for Case 4. As depicted in Fig. 5.1, the settling-time is less than K^* for Cases 1-3 where K^* is calculated using (5.64) and given in Table 5.5, and as mentioned in (5.28), for Case 4, $K(y(0)) = 1$. In Fig. 5.2, the state trajectory of system (5.56) with $a = 1$ and Case 1 parameters ($\alpha' = 0.4$, $r'_1 = 0.2$, $r'_2 = 1.2$) is simulated for 4 different initial conditions, $y(0) = 0.1$, $(y(0) < y_L)$, $y(0) = 8$, $(y_L < y(0) < y_H)$, $y(0) = 80$, $(y_L < y(0) < y_H)$ and $y(0) = 8000$, $(y_H < y(0))$ where as expected for $y(0) = 0.1$ and $y(0) = 8000$, the settling-time is $K(y(0)) = 1$, and for $y(0) = 8$ and $y(0) = 80$ the convergence to zero is achieved in few steps which ensures $K(y(0)) \leq K^*$.

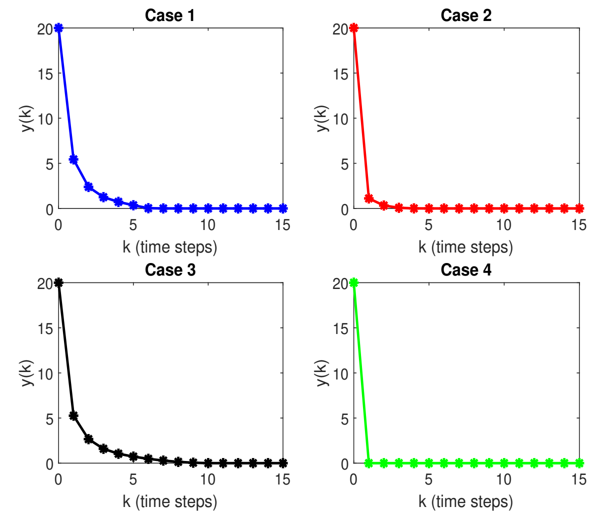

Figure 5.1: Different fixed times of convergence for system (5.56) with $a = 1$ and different values of α', r'_1 and r'_2.

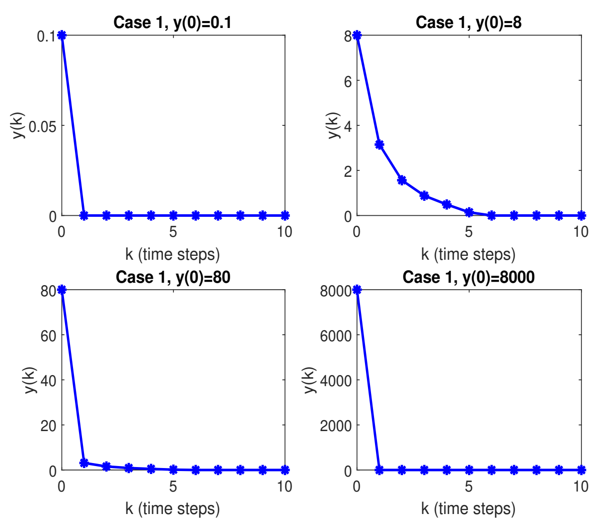

Figure 5.2: Fixed-time convergence for Case 1 of system (5.56) with $a = 1$ for different initial values.

However, for $\frac{1}{2} < a < 1$, based on (5.62), Lemma 9 and a similar procedure to Theorem 6 proof, one can show that the system (5.56) with $\frac{1}{2} < a < 1$ is exponentially stable.

It is worth to note that the autonomous system given in (5.56) with $a = 1$ can be considered as the following closed-loop system

$$y(k+1) = Ay(k) + Bu(k), \tag{5.65}$$

where $A = 1$, $B = 1$ and $u(k) = -\alpha' sign(y(k)) \min\{|y(k)|/\alpha', \max\{|y(k)|^{r'_1}, |y(k)|^{r'_2}\}\}$ with $y(k) \in \mathbb{R}$, $k \in \mathbb{N}$, $\alpha' \in (0,1)$, $r'_1 \in (0,1)$ and $r'_2 > 1$. In other words, in this example we have presented a fixed-time feedback controller

$$u(k) = -\alpha' sign(y(k)) \min\{|y(k)|/\alpha', \max\{|y(k)|^{r'_1}, |y(k)|^{r'_2}\}\},$$

which could stabilize the linear system (5.65) in a fixed amount of time.

Example 2. *(Fixed-time stable deterministic discrete-time higher-order system)* Consider the nonlinear DT system of order 3 given as follows

$$y_1(k+1) = y_1(k) - \bar{\alpha} sign(y_1(k)) \min\{|y_1(k)|/\bar{\alpha}, \frac{1}{3}\max\{[|y_1(k)|+|y_2(k)|+|y_3(k)|]^{\bar{r}_1}, [|y_1(k)|+|y_2(k)|+|y_3(k)|]^{\bar{r}_2}\}\}, \tag{5.66}$$

$$y_2(k+1) = y_2(k) - \bar{\alpha} sign(y_2(k)) \min\{|y_2(k)|/\bar{\alpha}, \frac{1}{3}\max\{[|y_1(k)|+|y_2(k)|+|y_3(k)|]^{\bar{r}_1}, [|y_1(k)|+|y_2(k)|+|y_3(k)|]^{\bar{r}_2}\}\}, \tag{5.67}$$

$$y_3(k+1) = y_3(k) - \bar{\alpha} sign(y_3(k)) \min\{|y_3(k)|/\bar{\alpha}, \frac{1}{3}\max\{[|y_1(k)|+|y_2(k)|+|y_3(k)|]^{\bar{r}_1}, [|y_1(k)|+|y_2(k)|+|y_3(k)|]^{\bar{r}_2}\}\}, \tag{5.68}$$

where $y(k) = [y_1(k), y_2(k), y_3(k)]^T \in \mathbb{R}^3$, $k \in \mathbb{N}$, $\bar{\alpha} \in (0,1)$, $\bar{r}_1 \in (0,1)$ and $\bar{r}_2 > 1$. Now, using Theorem 6, it is shown that the zero solution $y(k) = 0$ to the above higher-order system is globally fixed-time stable. Consider

$$V(y(k)) = |y_1(k)|+|y_2(k)|+|y_3(k)|, \tag{5.69}$$

and $V_L < V(0) < V_H$ where $V_L = \bar{\alpha}^{\frac{1}{1-\bar{r}_1}}$ and $V_H = \bar{\alpha}^{\frac{1}{1-\bar{r}_2}}$ (Note that if $V(0) > V_H$ or $V(0) < V_L$, then the zero solution $y(k) = 0$ of the above higher-order system is fixed-time stable with $K(y(0)) = 1$). The difference of (5.69) is as follows,

$$\Delta V(y(k)) = |y_1(k+1)| - |y_1(k)| + |y_2(k+1)| - |y_2(k)| + |y_3(k+1)| - |y_3(k)|, \tag{5.70}$$

where using (5.66)-(5.68) leads to

$$\begin{aligned}
\Delta V(y(k)) = & |y_1(k) - \bar{\alpha} sign(y_1(k)) \min\{|y_1(k)|/\bar{\alpha}, \frac{1}{3}\max\{[|y_1(k)| + |y_2(k)| + |y_3(k)|]^{\bar{r}_1}, \\
& [|y_1(k)| + |y_2(k)| + |y_3(k)|]^{\bar{r}_2}\}\}| - |y_1(k)| + |y_2(k) - \bar{\alpha} sign(y_2(k)) \min\{|y_2(k)|/\bar{\alpha}, \\
& \frac{1}{3}\max\{[|y_1(k)| + |y_2(k)| + |y_3(k)|]^{\bar{r}_1}, [|y_1(k)| + |y_2(k)| + |y_3(k)|]^{\bar{r}_2}\}\}| - |y_2(k)| \\
& + |y_3(k) - \bar{\alpha} sign(y_3(k)) \min\{|y_3(k)|/\bar{\alpha}, \frac{1}{3}\max\{[|y_1(k)| + |y_2(k)| + |y_3(k)|]^{\bar{r}_1}, \\
& [|y_1(k)| + |y_2(k)| + |y_3(k)|]^{\bar{r}_2}\}\}| - |y_3(k)|.
\end{aligned} \tag{5.71}$$

Consider

$$\begin{aligned}
a = & y_i(k), \\
b = & \bar{\alpha} sign(y_i(k)) \min\{|y_i(k)|/\bar{\alpha}, \frac{1}{3}\max\{[|y_1(k)| + |y_2(k)| + |y_3(k)|]^{\bar{r}_1}, \\
& [|y_1(k)| + |y_2(k)| + |y_3(k)|]^{\bar{r}_2}\}\},
\end{aligned} \tag{5.72}$$

where one knows that a and b have the same sign and for $i = 1, 2, 3$, $|a| \geq |b|$. Therefore one has

$$|a - b| = |a| - |b|. \tag{5.73}$$

Thus using (5.72)-(5.73), (5.71) is rewritten as follows

$$\begin{aligned}
\Delta V(y(k)) = & \\
& - \bar{\alpha}\min\{|y_1(k)|/\bar{\alpha}, \frac{1}{3}\max\{[|y_1(k)| + |y_2(k)| + |y_3(k)|]^{\bar{r}_1}, [|y_1(k)| + |y_2(k)| + |y_3(k)|]^{\bar{r}_2}\}\} \\
& - \bar{\alpha}\min\{|y_2(k)|/\bar{\alpha}, \frac{1}{3}\max\{[|y_1(k)| + |y_2(k)| + |y_3(k)|]^{\bar{r}_1}, [|y_1(k)| + |y_2(k)| + |y_3(k)|]^{\bar{r}_2}\}\} \\
& - \bar{\alpha}\min\{|y_3(k)|/\bar{\alpha}, \frac{1}{3}\max\{[|y_1(k)| + |y_2(k)| + |y_3(k)|]^{\bar{r}_1}, [|y_1(k)| + |y_2(k)| + |y_3(k)|]^{\bar{r}_2}\}\},
\end{aligned} \tag{5.74}$$

where (5.74) leads to

$$\begin{aligned}
\Delta V(y(k)) \leq & -\bar{\alpha}\min\{\frac{|y_1(k)| + |y_2(k)| + |y_3(k)|}{\bar{\alpha}}, \max\{[|y_1(k)| + |y_2(k)| + |y_3(k)|]^{\bar{r}_1}, \\
& [|y_1(k)| + |y_2(k)| + |y_3(k)|]^{\bar{r}_2}\}\}.
\end{aligned} \tag{5.75}$$

Using $V(y(k)) = |y_1(k)|+|y_2(k)|+|y_3(k)|$, (5.75) is rewritten as follows

$$\Delta V(y(k)) \leq -\bar{\alpha}\min\{\frac{V(y(k))}{\bar{\alpha}}, \max\{V(y(k))^{\bar{r}_1}, V(y(k))^{\bar{r}_2}\}\}, \tag{5.76}$$

which is analogous to (5.5) where $\alpha = \bar{\alpha}$, $r_1 = \bar{r}_1$ and $r_2 = \bar{r}_2$, and all the parameters conditions mentioned in Theorem 6 are satisfied. Therefore, it is shown that the higher-order system specified in (5.66)-(5.68) is globally fixed-time stable. Based on (5.6), the fixed upper bound for the settling-time function of this higher-order system is

$$K^* = \lfloor \bar{\alpha}^{\frac{1}{1-\bar{r}_1}}(1-\bar{\alpha}^{\frac{1}{1-\bar{r}_1}})\rfloor + \lfloor \bar{\alpha}^{-1}(\bar{\alpha}^{\frac{1}{1-\bar{r}_2}}-1)\rfloor + 3. \tag{5.77}$$

The state trajectories for the system (5.66)-(5.68) with $\bar{\alpha} = 0.47$, $\bar{r}_1 = 0.2$ and $\bar{r}_2 = 1.2$ are simulated in Figs. 5.3-5.5 for three different values of initial conditions. Based on the given parameters, one has $V_L = \bar{\alpha}^{\frac{1}{1-\bar{r}_1}} = 0.3892$ and $V_H = \bar{\alpha}^{\frac{1}{1-\bar{r}_2}} = 43.60$. Fig. 5.3 shows the system (5.66)-(5.68) trajectories reached the origin with $K(y(0)) = 1$ for initial conditions $y_1(0) = 0.1$, $y_2(0) = 0.01$, $y_3(0) = 0.001$ which imply that $V(0) < V_L$. Fig. 5.4 shows that for the 3 states' trajectories, it took several steps to reach the origin because the initial conditions are $y_1(0) = 9$, $y_2(0) = 9.5$, $y_3(0) = 10$ and $V_H < V(0) < V_L$. Moreover, in this case, using (6), the fixed upper bound for the convergence time is obtained as 93 steps where the convergence is achieved before this time. In Fig. 5.5 where the initial conditions are $y_1(0) = 90$, $y_2(0) = 900$, $y_3(0) = 9000$ and $V(0) < V_H$, the state trajectories reached the origin with $K(y(0)) = 1$.

Example 3. *(Lyapunov function candidate: from deterministic fixed-time stable scalar systems to their stochastic counterparts)*

In this counterexample we show that the deterministic global fixed-time stable system may not preserve its fixed-time stability under the same Lyapunov function candidate after it is exposed to stochastic noise.

Consider the scalar stochastic nonlinear DT system as follows

$$y(k+1) = ay(k) - \alpha' sign(y(k))\min\{|y(k)|/\alpha', \max\{|y(k)|^{r_1'}, |y(k)|^{r_2'}\}\} + by(k)\nu(k), \tag{5.78}$$

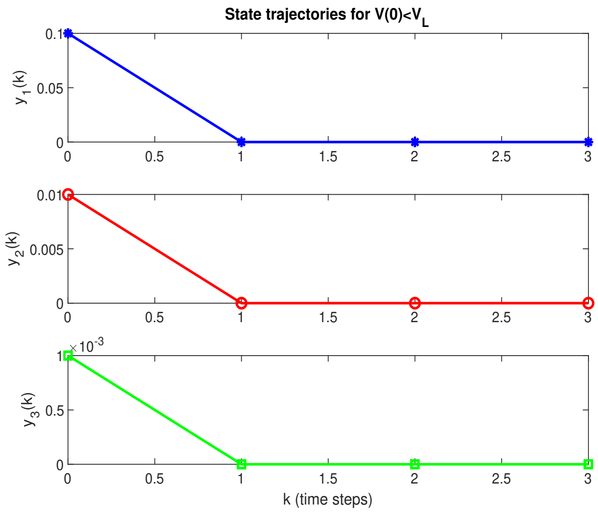

Figure 5.3: State trajectories for higher-order system (5.66)-(5.68) with $V(0) < V_L$.

where $y(k) \in \mathbb{R}$, $k \in \mathbb{N}$, $\alpha' \in (0,1)$, $r_1' \in (0,1)$ and $r_2' > 1$, $\nu(k) \in \mathbb{R}$ is a zero-mean stochastic noise with $\mathbb{E}[\nu(k)] = 0$ and $\mathbb{E}[\nu^2(k)] = \sigma^2$, $\frac{1}{2} < a \leq 1$ and $b < \sqrt{\frac{1-a^2}{\sigma^2}}$.

Now, using Theorem 8 and the results of Example 1, it is shown that the zero solution $y(k) \overset{\text{a.s.}}{\equiv} 0$ to (5.78) (the stochastic version of (5.56)) does not show global fixed-time stability in probability for $a = 1$ but preserves its exponential stability in probability for $\frac{1}{2} < a < 1$, using the same Lyapunov function as in Example 1.

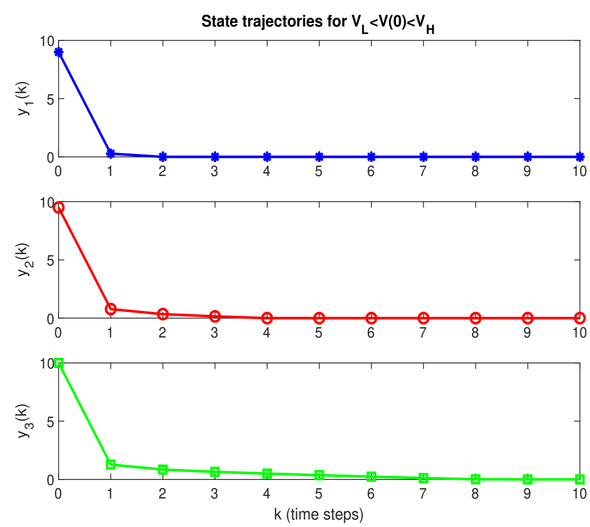

Figure 5.4: State trajectories for higher-order system (5.66)-(5.68) with $V_L < V(0) < V_H$.

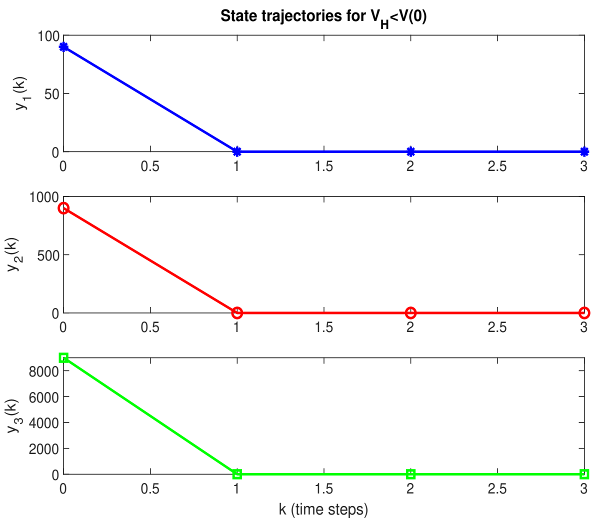

Figure 5.5: State trajectories for higher-order system (5.66)-(5.68) with $V_H < V(0)$.

Consider $V(y(k)) = y^2(k)$ such that for (5.78), one has

$$\mathbf{\Delta}V(y(k)) = \mathbb{E}[(ay(k) - \alpha' sign(y(k)) \min\{|y(k)|/\alpha', \max\{|y(k)|^{r'_1}, |y(k)|^{r'_2}\}\} + by(k)\nu(k))^2] - y^2(k), \tag{5.79}$$

where one can rewrite (5.79) as below,

$$\begin{aligned}\mathbf{\Delta}V(y(k)) = {}& \\ & \mathbb{E}[a^2y^2(k) + (\alpha' \min\{|y(k)|/\alpha', \max\{|y(k)|^{r'_1}, |y(k)|^{r'_2}\}\})^2 + b^2y^2(k)\nu^2(k) \\ & + 2aby^2(k)\nu(k) - 2a\alpha'|y(k)|\min\{|y(k)|/\alpha', \max\{|y(k)|^{r'_1}, |y(k)|^{r'_2}\}\} \\ & - 2b\alpha'\nu(k)|y(k)|\min\{|y(k)|/\alpha', \max\{|y(k)|^{r'_1}, |y(k)|^{r'_2}\}\}] \\ & - y^2(k).\end{aligned} \tag{5.80}$$

Applying expectation operator to the first term on the left-half-side of (5.80), leads to

$$\begin{aligned}\mathbf{\Delta}V(y(k)) = {}& \\ & (a^2 + b^2\sigma^2 - 1)y^2(k) + \alpha'^2 \min\{|y(k)|^2/\alpha'^2, \max\{|y(k)|^{2r'_1}, |y(k)|^{2r'_2}\}\} \\ & - 2a\alpha'|y(k)|\min\{|y(k)|/\alpha', \max\{|y(k)|^{r'_1}, |y(k)|^{r'_2}\}\}.\end{aligned} \tag{5.81}$$

Using (5.58), (5.81) leads to,

$$\Delta V(y(k)) \leq (a^2 + b^2\sigma^2 - 1)y^2(k) - (2a-1)\alpha'^2 \min\{y^2(k)/\alpha'^2, \max\{y^{2r'_1}(k), y^{2r'_2}(k)\}\}, \tag{5.82}$$

where using $V(y(k)) = y^2(k)$ one can rewrite (5.82) as follows,

$$\Delta V(y(k)) \leq (a^2 + b^2\sigma^2 - 1)V(k) - \beta\alpha \min\{V(k)/\alpha, \max\{V(k)^{r_1}(k), V(k)^{r_2}(k)\}\}, \tag{5.83}$$

where $\beta = 2a - 1$, $\alpha = \alpha'^2$, $r_1 = r'_1$ and $r_2 = r'_2$.

For $a = 1$, (5.83) reduces to

$$\Delta V(y(k)) \leq b^2\sigma^2 V(k) - \alpha \min\{V(k)/\alpha, \max\{V(k)^{r_1}(k), V(k)^{r_2}(k)\}\}. \tag{5.84}$$

However, (5.84) can not support the global fixed-time stability in probability of the system (5.78) with $a = 1$, due to the injected noise stochasticity, while in Example 1 it was shown that the same system without noise is fixed-time stable.

For $\frac{1}{2} < a < 1$ and $b < \sqrt{\frac{1-a^2}{\sigma^2}}$, one has $0 < \beta < 1$ and $a^2 + b^2\sigma^2 - 1 < 0$. Thus, using (5.83) one obtains

$$\begin{aligned} &\Delta V(y(k)) \leq \\ &-\beta\alpha \min\{V(k)/\alpha, \max\{V(k)^{r_1}(k), V(k)^{r_2}(k)\}\}. \end{aligned} \tag{5.85}$$

By using (5.85), Lemma 9 and a similar procedure to Theorem 8 proof, one can show that the system (5.78) with $\frac{1}{2} < a < 1$ and $b < \sqrt{\frac{1-a^2}{\sigma^2}}$ is exponentially stable in probability. Therefore, the stochastic system (5.78) preserves exponential stability in probability for $\frac{1}{2} < a < 1$ and

$$b < \sqrt{\frac{1-a^2}{\sigma^2}}.$$

Example 4. *(Lyapunov function candidate: from deterministic fixed-time stable higher-order systems to their stochastic counterparts)* In this counterexample we show that the deterministic global fixed-time stable higher-order system may not preserve its fixed-time stability under the same Lyapunov function candidate after it is exposed to stochastic noise. Consider the higher-order stochastic nonlinear DT system as follows

$$y_1(k+1) = y_1(k) - \bar{\alpha} sign(y_1(k)) \min\{|y_1(k)|/\bar{\alpha}, \frac{1}{3}\max\{[|y_1(k)|+|y_2(k)|+|y_3(k)|]^{\bar{r}_1},$$
$$[|y_1(k)|+|y_2(k)|+|y_3(k)|]^{\bar{r}_2}\}\} + y_1(k)\nu(k), \tag{5.86}$$
$$y_2(k+1) = y_2(k) - \bar{\alpha} sign(y_2(k)) \min\{|y_2(k)|/\bar{\alpha}, \frac{1}{3}\max\{[|y_1(k)|+|y_2(k)|+|y_3(k)|]^{\bar{r}_1},$$
$$[|y_1(k)|+|y_2(k)|+|y_3(k)|]^{\bar{r}_2}\}\} + y_2(k)\nu(k), \tag{5.87}$$
$$y_3(k+1) = y_3(k) - \bar{\alpha} sign(y_3(k)) \min\{|y_3(k)|/\bar{\alpha}, \frac{1}{3}\max\{[|y_1(k)|+|y_2(k)|+|y_3(k)|]^{\bar{r}_1},$$
$$[|y_1(k)|+|y_2(k)|+|y_3(k)|]^{\bar{r}_2}\}\} + y_3(k)\nu(k), \tag{5.88}$$

where $y(k) = [y_1(k), y_2(k), y_3(k)] \in \mathbb{R}^3$, $k \in \mathbb{N}$, $\bar{\alpha} \in (0,1)$, $\bar{r}_1 \in (0,1)$ and $\bar{r}_2 > 1$, $\nu(k) \in \mathbb{R}$ is a zero-mean stochastic noise with $\mathbb{E}[\nu(k)] = 0$ and $\mathbb{E}[|\nu(k)|] = c^2, 0 < c < \sqrt{\bar{\alpha}}$. Now, using Theorem 8 and the results of Example 2, it is shown that the zero solution $y(k) \overset{\text{a.s.}}{\equiv} 0$ to (5.86)-(5.88) (the stochastic version of (5.66)-(5.68)) does not show global fixed-time stability in probability, using the same Lyapunov function as in Example 2. Consider

$$V(y(k)) = |y_1(k)|+|y_2(k)|+|y_3(k)|, \tag{5.89}$$

such that for the system (5.86)-(5.88), one has

$$\begin{aligned}\mathbf{\Delta}V(y(k)) = {} & \mathbb{E}[|y_1(k) - \bar{\alpha} sign(y_1(k)) \min\{|y_1(k)|/\bar{\alpha},\\ & \frac{1}{3}\max\{[|y_1(k)|+|y_2(k)|+|y_3(k)|]^{\bar{r}_1},\\ & [|y_1(k)|+|y_2(k)|+|y_3(k)|]^{\bar{r}_2}\}\} + y_1(k)\nu(k)|]\\ & - |y_1(k)|+\mathbb{E}[|y_2(k) - \bar{\alpha} sign(y_2(k)) \min\{|y_2(k)|/\bar{\alpha},\\ & \frac{1}{3}\max\{[|y_1(k)|+|y_2(k)|+|y_3(k)|]^{\bar{r}_1},\\ & [|y_1(k)|+|y_2(k)|+|y_3(k)|]^{\bar{r}_2}\}\} + y_2(k)\nu(k)|]\\ & - |y_2(k)|+\mathbb{E}[|y_3(k) - \bar{\alpha} sign(y_3(k)) \min\{|y_3(k)|/\bar{\alpha},\\ & \frac{1}{3}\max\{[|y_1(k)|+|y_2(k)|+|y_3(k)|]^{\bar{r}_1}, [|y_1(k)|+|y_2(k)|\\ & + |y_3(k)|]^{\bar{r}_2}\}\} + y_3(k)\nu(k)|] - |y_3(k)|,\end{aligned} \tag{5.90}$$

Using triangle inequality for a, b (a and b are defined in (5.72)) and $c = y_i(k)v(k)$, for $i = 1, 2, 3$, as

$$|a - b + c| \leq |a - b| + |c|, \tag{5.91}$$

and employing (5.89), (5.72)-(5.73), and the noise properties, one can rewrite (5.90) as below,

$$\begin{aligned}\Delta V(y(k)) \leq & -\bar{\alpha}\min\{V(k)/\bar{\alpha}, \max\{V(k)^{\bar{r}1}, V(k)^{\bar{r}2}\}\} \\ & + c^2 V(k),\end{aligned} \tag{5.92}$$

One knows that (5.92) can not support the global fixed-time stability in probability of the system (5.86)-(5.88), due to the injected noise stochasticity, while in Example 2 it was shown that the same system without noise is fixed-time stable. By using Lemma 9, one can show that the system (5.86)-(5.88) preserved its exponential stability in probability for $0 < c < \sqrt{\bar{\alpha}}$.

5.6 Conclusion and Future work

This chapter addressed the fixed-time stability for deterministic and stochastic discrete-time (DT) autonomous systems based on fixed-time Lyapunov stability analysis. Novel Lyapunov conditions are derived under which the fixed-time stability of autonomous DT deterministic and stochastic systems is certified. The sensitivity to perturbations for fixed-time stable DT systems is analyzed and the analysis shows that fixed-time attractiveness can be resulted from the presented Lyapunov conditions. For both cases of fixed-time stable and fixed-time attractive systems, the fixed upper bounds of the settling-time functions are given. For the future work, we plan to employ the presented fixed-time stability analysis to develop fixed-time identifiers and controllers for DT systems.

CHAPTER 6

DISCRETE-TIME NONLINEAR SYSTEM IDENTIFICATION: A FIXED-TIME CONCURRENT LEARNING APPROACH

6.1 Introduction

The overarching objective of this chapter is to present a fixed-time concurrent learning (FxTCL) algorithm for discrete-time systems to 1) ensure fixed-time parameter convergence independent of the initial estimation errors and 2) relax the PE condition to a rank condition on the recorded data using CL. In the presented FxTCL, the settling-time upper bound is independent of the initial parameter estimation error. To achieve this goal, a modified gradient-descent update law is presented for learning the unknown system parameters. This update law reuses past collected data at every time instance and leverages discontinuous and non-integer powers of the identification errors. The Lyapunov analysis presented in our previous work in [128] is then leveraged to guarantee fixed-time convergence of the system parameters to their true values.

The main contributions of this chapter include the following. First, a novel discrete-time update law is presented using the CL technique that identifies system uncertainties in a fixed amount of time. Fixed-time convergence is guaranteed under a rank condition on recorded memory data, which is weaker than the standard PE condition. The rigorous analysis using fixed-time Lyapunov stability guarantees the convergence of estimated parameters' error to zero for adaptive approximators with parametric uncertainties under a condition on the learning rate. Third, a fixed-time upper bound for the parameters' estimation error settling-time function, independent of the initial parameter estimation error, is computed.

Notation $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{N}^+$, respectively, denote the sets of real, integer, and natural numbers without zero. $\|.\|$ denotes the Euclidean and induced 2 norms for vectors and matrices, respectively. $tr(.)$ shows the trace of a matrix. $\lambda_{min}(A)$ and $\lambda_{max}(A)$, respectively, show the minimum and maximum eigenvalues of matrix A. I is the identity matrix of appropriate dimensions. $\lfloor . \rfloor : \mathbb{R} \mapsto \mathbb{Z}$ denotes the floor function.

In general, for a vector $x = [x_1, x_2, ..., x_n]^T \in \mathbb{R}^n$, the p-norm is defined as $\|x\|_p = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}$.

Moreover, for positive constants r and s, if $0 < r < s$, based on Hölder inequality [129], one has $\|x\|_s \leq \|x\|_r \leq n^{\frac{1}{r}-\frac{1}{s}}\|x\|_s$.

Frobenius norm of matrix $A \in \mathbb{R}^{m\times n}$ defined as $\|A\|_F = \sqrt{tr(A^T A)}$, implies $\|A\| \leq \|A\|_F \leq \sqrt{\min(m,n)}\|A\|$.

6.2 Problem Formulation

Consider a nonlinear discrete-time system as below,

$$x(k+1) = f(x(k)) + g(x(k))u(k), \tag{6.1}$$

where $x \in \mathcal{D}_x \subset \mathbb{R}^n$ and $u \in \mathcal{D}_u \subset \mathbb{R}^m$ are, respectively, the system state and control input vectors, $\mathcal{D}_x$ and $\mathcal{D}_u$ are compact sets; the drift and input functions $f : \mathcal{D}_x \mapsto \mathbb{R}^n$ and $g : \mathcal{D}_x \mapsto \mathbb{R}^{n\times m}$ are functions with parametric uncertainty.

The functions $f(x)$ and $g(x)$ with parametric uncertainties are represented as

$$f(x(k)) = \Theta_f^{*T}\varphi(x(k)), \tag{6.2}$$

$$g(x(k)) = \Theta_g^{*T}\chi(x(k)), \tag{6.3}$$

where $\Theta_f^* \in \mathcal{D}_f \subset \mathbb{R}^{p\times n}$ and $\Theta_g^* \in \mathcal{D}_g \subset \mathbb{R}^{q\times n}$ are the optimal unknown parameters, and $\mathcal{D}_f$ and $\mathcal{D}_g$ are compact sets. $\varphi : \mathcal{D}_x \mapsto \mathbb{R}^p$ and $\chi : \mathcal{D}_x \mapsto \mathbb{R}^q$, are the basis functions, where p and q are, respectively, the number of linearly independent basis functions to approximate $f(x(k))$ and $g(x(k))$. Using (6.2)-(6.3), (6.1) is written as

$$x(k+1) = \Theta^{*T}z(x(k), u(k)), \tag{6.4}$$

where $\Theta^* = [\Theta_f^{*T}, \Theta_g^{*T}]^T \in \mathbb{R}^{(p+q)\times n}$, and $z(x(k), u(k)) = [\varphi^T(x(k)), u^T(k)\chi^T(x(k))]^T \in \mathbb{R}^{(p+q)}$.

The measurements of $x(k+1)$ are not accessible. Therefore, regressor filtering [10, 39, 42] of the system (6.4), gives

$$x(k) = \Theta^{*T}d(k) - l(k) + C^k x(0), \tag{6.5}$$

$$d(k+1) = cd(k) + z(x(k), u(k)),\ d(0) = 0,$$

$$l(k+1) = Cl(k) + Cx(k),\ l(0) = 0, \tag{6.6}$$

where $C = cI, -1 < c < 1$, $l(k) = \sum_{h=0}^{k-1} C^{k-h}x(h)$ is the filtered regressor of $x(k)$, and $d(k) = \sum_{h=0}^{k-1} c^{k-h-1}z(x(h), u(h))$ is the filtered regressor of $z(x(k), u(k))$. By dividing (6.5) to $n_s := 1 + d^T(k)d(k) + l^T(k)l(k)$, one has the normalized form of (6.5) given below,

$$\bar{x}(k) = \Theta^{*T}\bar{d}(k) - \bar{l}(k) + C^k\bar{x}(0), \tag{6.7}$$

where $\bar{d} = \frac{d}{n_s}$, $\bar{l} = \frac{l}{n_s}$, and $\bar{x} = \frac{x}{n_s}$.

Consider the approximator of (6.7) as follows,

$$\hat{\bar{x}}(k) = \hat{\Theta}^T(k)\bar{d}(k) - \bar{l}(k) + C^k\bar{x}(0), \tag{6.8}$$

where $\hat{\Theta}(k) = [\hat{\Theta}_f^T(k), \hat{\Theta}_g^T(k)]^T \in \mathbb{R}^{(p+q)\times n}$, $\hat{\Theta}_f(k)$ and $\hat{\Theta}_g(k)$ are, respectively, the estimated parameters' matrices for Θ^*, Θ_f^* and Θ_g^* at time k. The state estimation error is given as

$$e(k) = \hat{\bar{x}}(k) - \bar{x}(k) = \tilde{\Theta}^T(k)\bar{d}(k), \tag{6.9}$$

where $\tilde{\Theta}(k) := \hat{\Theta}(k) - \Theta^* := [\tilde{\Theta}_f^T(k), \tilde{\Theta}_g^T(k)]^T$ is the parameter estimation error such that $\tilde{\Theta}_f(k) := \hat{\Theta}_f(k) - \Theta_f^*$, $\tilde{\Theta}_g(k) := \hat{\Theta}_g(k) - \Theta_g^*$.

Problem 1: Consider the system (1), or equivalently (7). Let the system model (8) be used for identifying the unknown parameters of (7). Design a fixed-time update law to ensure that the parameter estimation error $\tilde{\Theta}(k)$ dynamics are fixed-time stable.

Remark 26 To our knowledge, fixed-time system identification for discrete-time systems has not been investigated in the literature. We present for the first time a solution to Problem 1 by developing a modified gradient descent-based update law that leverages the recorded past data to relax the PE condition.

6.3 Preliminaries

Definition 19 [29] The signal $d(k)$ is called persistently exciting if there are positive scalars υ_1, υ_2 and $T \in \mathbb{N}^+$ where $\forall\tau \in \mathbb{N}^+$, $\upsilon_1 I \leq \sum_{k=\tau}^{\tau+T} d(k)d^T(k) \leq \upsilon_2 I$.

Definition 20 *(Fixed-time stability [59])* Consider the system

$$z(k+1) = F(z(k)), \tag{6.10}$$

where $z \in \mathcal{D}_z$, $F : \mathcal{D}_z \mapsto \mathbb{R}^n$ and $\mathcal{D}_z$ is an open neighborhood of the origin which is the equilibrium point of (6.10). The nonlinear system (6.10) is fixed-time stable, if there is an open neighborhood $\mathcal{N}_z \subseteq \mathcal{D}_z$ of the origin and a settling time function $K : \mathcal{N}_z \backslash \{0\} \mapsto \mathbb{N}^+$, such that:

1) The system (6.10) is Lyapunov stable, i.e., for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $||z(0)|| \leq \delta$, then $||z(k)|| \leq \epsilon$ for all $k \in \{0, ..., K(z(0)) - 1\}$.

2) For every initial condition $z(0) \in \mathcal{N}_z \backslash \{0\}$, the solution sequence $z(k)$ of (6.10) reaches the equilibrium point and remains there after $k > K(z(0))$ and $\forall z(0) \in \mathcal{N}_z$, where $K : \mathcal{N}_z \backslash \{0\} \mapsto \mathbb{N}^+$.

3) The settling-time function $K(z(0))$ is bounded, i.e., $\exists K_{max} \in \mathbb{N}^+ : K(z(0)) \leq K_{max}, \forall z(0) \in \mathcal{N}_z \backslash \{0\}$.

Lemma 11 [128] Consider the nonlinear discrete-time system (6.10). Consider there is a continuous Lyapunov function $V : \mathcal{D}_z \mapsto \mathbb{R}$ where $\mathcal{D}_z$ is an open neighborhood around the origin and there exists a neighborhood $\Omega_z \subset \mathcal{D}_z$ of the origin such that $V(z(0)) = 0, V(z(k)) > 0, \ \ z(k) \in \Omega_z \backslash \{0\}$ and

$$\Delta V(z(k+1)) \leq -\alpha \min\{\frac{V(z(k))}{\alpha}, \max\{V^{r_1}(z(k)), V^{r_2}(z(k))\}\}, \ \ z(k) \in \Omega_z \backslash \{0\}, \tag{6.11}$$

for constants $0 < \alpha < 1$, $0 < r_1 < 1$, and $r_2 > 1$. Then, system (6.10) is fixed-time stable and has a settling time function $K : \mathcal{N}_z \mapsto \mathbb{N}^+$ that for all $z(0) \in \mathcal{N}_z \backslash \{0\}$ satisfies

$$K(z(0)) \leq \lfloor \alpha^{\frac{1}{1-r_1}} (1 - \alpha^{\frac{1}{1-r_1}}) \rfloor + \lfloor \alpha^{-1} (\alpha^{\frac{1}{1-r_2}} - 1) \rfloor + 3, \tag{6.12}$$

where $\mathcal{N}_z$ is an open neighborhood of the origin.

6.4 Fixed-time Concurrent Learning of the Unknown Discrete-time Dynamics

To employ CL for the approximation (6.8), the past data of (6.5)-(6.6) is recorded in the memory matrices $M \in \mathbb{R}^{(p+q) \times P}$, $L \in \mathbb{R}^{n \times P}$ and $X \in \mathbb{R}^{n \times P}$, at time steps $\tau_1, ..., \tau_P$,

$$M = [\bar{d}(\tau_1), \bar{d}(\tau_2), ..., \bar{d}(\tau_P)], \ \ L = [\bar{l}(\tau_1), \bar{l}(\tau_2), ..., \bar{l}(\tau_P)],$$
$$X = [\bar{x}(\tau_1), \bar{x}(\tau_2), ..., \bar{x}(\tau_P)], \tag{6.13}$$

where P (the number of stored data in each memory matrice) is chosen such that M is full-row rank, which is called M rank condition and requires $P \geq p + q$. Now, for the h^{th} stored data, the error $e_h(k)$ is defined as

$$e_h(k) = \hat{\bar{x}}_h(k) - \bar{x}(\tau_h), \tag{6.14}$$

where

$$\hat{\bar{x}}_h(k) = \hat{\Theta}^T(k)\bar{d}(\tau_h) - \bar{l}(\tau_h) + C^k\bar{x}(0), \tag{6.15}$$

is the state estimation at time step $0 \leq \tau_h < k$, $h = 1, ..., P$, using the recorded $\bar{d}(\tau_h)$ and $\bar{l}(\tau_h)$, and the current estimated parameters matrix $\hat{\Theta}(k)$. Substituting $\bar{x}(\tau_h)$ into (6.14), one obtains

$$e_h(k) = \tilde{\Theta}^T(k)\bar{d}(\tau_h). \tag{6.16}$$

The proposed FxTCL law for estimating the parameters of the system approximator is presented as follows

$$\hat{\Theta}(k+1) = \hat{\Theta}(k) - \Gamma[\Xi_G\bar{d}(k)e^T(k) + \Xi_C(\sum_{h=1}^{P}\bar{d}(\tau_h)(\lfloor e_h^T(k)\rceil^{\gamma_1} + \lfloor e_h^T(k)\rceil^{\gamma_2}))], \tag{6.17}$$

where $\lfloor . \rceil^{\gamma} := |.|^{\gamma} sign(.)$ with $|.|$ and $sign(.)$ understood in component-wise sense and $0 < \gamma_1 < 1$, $\gamma_2 > 1$. $\Gamma = \gamma I$ is the learning rate with constant $\gamma > 0$. $\Xi_C = \xi_C I$ and $\Xi_G = \xi_G I$ are weight matrices with constants $\xi_C > 0$ and $\xi_G > 0$, which can be set to prioritize one of the two learning terms (i.e. $\bar{d}(k)e^T(k)$ and $\sum_{h=1}^{P}\bar{d}(\tau_h)(\lfloor e_h^T(k)\rceil^{\gamma_1} + \lfloor e_h^T(k)\rceil^{\gamma_2})$) in (6.17) over the other. Moreover, before the few P steps of learning, required for filling the data stacks in (6.13) and satisfying the rank condition, we set $\Xi_C = 0$ such that (6.17) only employs current data to update the estimated parameters.

6.5 Fixed-time Convergent Analysis

In this section, the convergence analysis of the gradient update law dynamics is given based on fixed-time Lyapunov stability.

Theorem 9 Let the system (6.1) be approximated by (6.8), whose parameters are adjusted using (6.17) with $0 < \gamma_1 < 1$, $\gamma_2 > 1$ and a regressor given in (6.6). Let the rank condition on M is satisfied. If γ satisfies

$$\max\{\underline{a_u}, \underline{b_u}\} < \gamma < \min\{\frac{2}{n\xi_G}, \overline{b_u}, \overline{a_u}\}, \quad for \ \ 1 < \Upsilon, \tag{6.18}$$

$$\max\{\underline{a_l}, \underline{b_l}\} < \gamma < \min\{\frac{2}{n\xi_G}, \overline{a_l}, \overline{b_l}\}, \ \ for \ 0 < \Upsilon \leq 1, \tag{6.19}$$

$$\gamma = 0, \qquad\qquad for \quad \Upsilon = 0, \tag{6.20}$$

where $\Upsilon = \|\sum_{h=1}^{P} e_h^T(k-1)\|$. Then, the update law (6.17) ensures the fixed-time convergence of $\tilde{\Theta}(k)$ to zero for $k > K(\tilde{\Theta}_0)$ (i.e., it solves Problem 1). Besides, the settling time of convergence is given by

$$K(\tilde{\Theta}_0) \leq \max_{\alpha_i > 0}\{\lfloor \alpha_i^{\frac{2}{1-\gamma_1}}(1-\alpha_i^{\frac{2}{1-\gamma_1}})\rfloor + \lfloor \alpha_i^{-1}(\alpha_i^{\frac{2}{1-\gamma_2}} - 1)\rfloor\} + 3, \tag{6.21}$$

such that $\alpha_i = \min\{a_i, b_i\}, i = 1, 2$. For $1 < \Upsilon$, one has $i = 1$,

$$a_1 = a_u(\frac{\gamma}{n})^{\frac{\gamma_1+1}{2}}, b_1 = b_u(\frac{\gamma}{n})^{\frac{\gamma_2+1}{2}},$$

and $\eta_1 \geq \Upsilon^{\gamma_2 - 1}$; for $0 < \Upsilon \leq 1$, one has $i = 2$,

$$a_2 = a_l(\frac{\gamma}{n})^{\frac{\gamma_1+1}{2}}, b_2 = b_l(\frac{\gamma}{n})^{\frac{\gamma_2+1}{2}},$$

and $\eta_2 \geq \frac{1}{\Upsilon^{1-\gamma_1}}$; where

$$a_u = \xi_C[2\lambda_{min}^{\frac{\gamma_1+1}{2}}(S) - n\gamma\lambda_{max}^{\frac{\gamma_1+1}{2}}(S)(2\xi_G(n^{\frac{1-\gamma_1}{2}}) + \xi_C(n^{1-\gamma_1})], \tag{6.22}$$

$$b_u = 2\xi_C[(n^{\frac{1-\gamma_2}{2}})\lambda_{min}^{\frac{\gamma_2+1}{2}}(S) - n\gamma\lambda_{max}^{\frac{\gamma_2+1}{2}}(S)(\xi_G + \xi_C(n^{\frac{1-\gamma_1}{2}}) + 0.5\xi_C\eta_1)], \tag{6.23}$$

$$a_l = 2\xi_C[\lambda_{min}^{\frac{\gamma_1+1}{2}}(S) - n\gamma\lambda_{max}^{\frac{\gamma_1+1}{2}}(S)n^{\frac{1-\gamma_1}{2}}(\xi_G + \xi_C) + 0.5\xi_C(n^{\frac{1-\gamma_1}{2}})\eta_2], \tag{6.24}$$

$$b_l = \xi_C[2(n^{\frac{1-\gamma_2}{2}})\lambda_{min}^{\frac{\gamma_2+1}{2}}(S) - n\gamma\xi_G\lambda_{max}^{\frac{\gamma_2+1}{2}}(S)(2\xi_G + \xi_C)], \tag{6.25}$$

$$\underline{a_u} = \frac{a-1}{b}, \quad \overline{a_u} = \frac{a}{b}, \quad \underline{b_u} = \frac{(n^{\frac{1-\gamma_2}{2}})a - 1}{c}, \quad \overline{b_u} = \frac{(n^{\frac{1-\gamma_2}{2}})a}{c}, \tag{6.26}$$

$$\underline{a_l} = \frac{a-1}{d}, \quad \overline{a_l} = \frac{a}{d}, \quad \underline{b_l} = \frac{(n^{\frac{1-\gamma_2}{2}})a - 1}{e}, \quad \overline{b_l} = \frac{(n^{\frac{1-\gamma_2}{2}})a}{e}, \tag{6.27}$$

$$
\begin{aligned}
a =&2\xi_C\lambda_{min}^{\frac{\gamma_1+1}{2}}(S), \quad e = \lambda_{max}^{\frac{\gamma_2+1}{2}}(S)n[4\xi_C\xi_G + \xi_C^2],\\
b =&\lambda_{max}^{\frac{\gamma_1+1}{2}}(S)n[2\xi_C\xi_G(n^{\frac{1-\gamma_1}{2}}) + \xi_C^2(n^{1-\gamma_2})],\\
c =&\lambda_{max}^{\frac{\gamma_2+1}{2}}(S)n[4\xi_C\xi_G + 2\xi_C^2(n^{\frac{1-\gamma_1}{2}}) + \xi_C^2\eta_1(n^{1-\gamma_1})],\\
d =&\lambda_{max}^{\frac{\gamma_1+1}{2}}(S)n[(n^{\frac{1-\gamma_1}{2}})2\xi_C\xi_G + 2\xi_C^2(n^{\frac{1-\gamma_1}{2}}) + \xi_C^2\eta_2(n^{1-\gamma_1})],\\
S =& \sum_{h=1}^{P} \bar{d}(\tau_h)\bar{d}^T(\tau_h).
\end{aligned}
$$

Proof 11 Consider the Lyapunov function, $V(k)$ as follows

$$
V(k) = tr\{\tilde{\Theta}^T(k)\Gamma^{-1}\tilde{\Theta}(k)\}. \tag{6.28}
$$

where its change rate, $\Delta V(k) = V(k) - V(k-1)$, is given below,

$$
\begin{aligned}
\Delta V(k) &= tr\{\tilde{\Theta}^T(k)\Gamma^{-1}\tilde{\Theta}(k) - \tilde{\Theta}^T(k-1)\Gamma^{-1}\tilde{\Theta}(k-1)\}\\
&= tr\{(\tilde{\Theta}(k) - \tilde{\Theta}(k-1))^T\Gamma^{-1}(\tilde{\Theta}(k) + \tilde{\Theta}(k-1))\}.
\end{aligned} \tag{6.29}
$$

Using (6.17), (6.29) gives,

$$
\begin{aligned}
\Delta V(k) = tr\{&(-\Gamma[\Xi_G\bar{d}(k-1)e^T(k-1) + \Xi_C(\sum_{h=1}^{P}\bar{d}(\tau_h)(\lfloor e_h^T(k-1)\rceil^{\gamma_1} + \lfloor e_h^T(k-1)\rceil^{\gamma_2}))])^T\times\\
&\Gamma^{-1}(2\tilde{\Theta}(k-1) - \Gamma[\Xi_G\bar{d}(k-1)e^T(k-1) + \Xi_C(\sum_{h=1}^{P}\bar{d}(\tau_h)(\lfloor e_h^T(k-1)\rceil^{\gamma_1} + \lfloor e_h^T(k-1)\rceil^{\gamma_2}))])\},
\end{aligned} \tag{6.30}
$$

and using $\bar{D}(k) = \bar{d}^T(k)\bar{d}(k)$, (6.30) is rewritten as,

$$
\begin{aligned}
\Delta V(k) = tr\{ & -2\Xi_G e(k-1)e^T(k-1) - 2\Xi_C [\sum_{h=1}^{P} \lfloor e_h(k-1) \rceil^{\gamma_1} e_h^T(k-1) \\
& + \sum_{h=1}^{P} \lfloor e_h(k-1) \rceil^{\gamma_2} e_h^T(k-1)] + \Gamma \Xi_G^2 e(k-1)\bar{D}(k-1)e^T(k-1) \\
& + 2\Gamma \Xi_C \Xi_G \sum_{h=1}^{P} \lfloor e_h(k-1) \rceil^{\gamma_1} \bar{d}^T(\tau_h)\bar{d}(k-1)e^T(k-1) \\
& + 2\Gamma \Xi_C \Xi_G \sum_{h=1}^{P} \lfloor e_h(k-1) \rceil^{\gamma_2} \bar{d}^T(\tau_h)\bar{d}(k-1)e^T(k-1) \\
& + \Gamma \Xi_C^2 \sum_{h=1}^{P} \lfloor e_h(k-1) \rceil^{\gamma_1} \bar{d}^T(\tau_h) \sum_{h=1}^{P} \bar{d}(\tau_h) \lfloor e_h^T(k-1) \rceil^{\gamma_1} \\
& + 2\Gamma \Xi_C^2 \sum_{h=1}^{P} \lfloor e_h(k-1) \rceil^{\gamma_1} \bar{d}^T(\tau_h) (\sum_{h=1}^{P} \bar{d}(\tau_h) \lfloor e_h^T(k-1) \rceil^{\gamma_2}) \\
& + \Gamma \Xi_C^2 \sum_{h=1}^{P} \lfloor e_h(k-1) \rceil^{\gamma_2} \bar{d}^T(\tau_h) (\sum_{h=1}^{P} \bar{d}(\tau_h) \lfloor e_h^T(k-1) \rceil^{\gamma_2})\}. \qquad (6.31)
\end{aligned}
$$

Using $\| \lfloor \bar{d}^T(\tau_h)\tilde{\Theta}(k-1) \rceil^{\gamma_i} \| = \| \bar{d}^T(\tau_h)\tilde{\Theta}(k-1) \|_{2\gamma_i}^{\gamma_i}$ for $i = 1, 2$, and Facts 1-2, one rewrites (6.31) as follows

$$
\begin{aligned}
\Delta V(k) \leq & -2\xi_G \|e^T(k-1)\|^2 - 2\xi_C \sum_{h=1}^{P} \|e_h^T(k-1)\|_{\gamma_1+1}^{\gamma_1+1} - 2\xi_C \sum_{h=1}^{P} \|e_h^T(k-1)\|_{\gamma_2+1}^{\gamma_2+1} \\
& + n\gamma[\xi_G^2 \|e^T(k-1)\|^2 + 2\xi_C \xi_G \sum_{h=1}^{P} \|e_h^T(k-1)\|_{2\gamma_1}^{\gamma_1} \|e^T(k-1)\| \\
& + 2\xi_C \xi_G \sum_{h=1}^{P} \|e_h^T(k-1)\|_{2\gamma_2}^{\gamma_2} \| \|e^T(k-1)\| + \xi_C^2 (\sum_{h=1}^{P} \|e_h^T(k-1)\|_{2\gamma_1}^{\gamma_1})^2 \\
& + 2\xi_C^2 n^{\frac{1-\gamma_1}{2}} \sum_{h=1}^{P} \|e_h^T(k-1)\|^{\gamma_1} \sum_{h=1}^{P} \|e_h^T(k-1)\|^{\gamma_2} + \xi_C^2 (\sum_{h=1}^{P} \|e_h^T(k-1)\|_{2\gamma_2}^{\gamma_2})^2]. \qquad (6.32)
\end{aligned}
$$

Using $\sum_{h=1}^{P} \|e_h^T(k-1)\|^{\gamma_i} \leq (\sum_{h=1}^{P} \|e_h^T(k-1)\|)^{\gamma_i}$, $\|e_h^T(k-1)\| \leq \sum_{h=1}^{P} \|e_h^T(k-1)\|$, $i = 1, 2$, and

Fact 1, one has

$$\Delta V(k) \leq -(2\xi_G - n\gamma\xi_G^2)\|e^T(k-1)\|^2 - 2\xi_C \sum_{h=1}^{P} \|e_h^T(k-1)\|^{\gamma_1+1}$$
$$- 2\xi_C(n^{\frac{1-\gamma_2}{2}}) \sum_{h=1}^{P} \|e_h^T(k-1)\|^{\gamma_2+1} + n\gamma[2\xi_C\xi_G n^{\frac{1-\gamma_1}{2}}(\sum_{h=1}^{P} \|e_h^T(k-1)\|)^{1+\gamma_1}$$
$$+ 2\xi_C\xi_G(\sum_{h=1}^{P} \|e_h^T(k-1)\|)^{1+\gamma_2} + \xi_C^2(n^{1-\gamma_1})(\sum_{h=1}^{P} \|e_h^T(k-1)\|)^{2\gamma_1}$$
$$+ 2\xi_C^2 n^{\frac{1-\gamma_1}{2}}(\sum_{h=1}^{P} \|e_h^T(k-1)\|)^{\gamma_1+\gamma_2} + \xi_C^2(\sum_{h=1}^{P} \|e_h^T(k-1)\|)^{2\gamma_2}]. \tag{6.33}$$

For $\Upsilon > 1$ with $\Upsilon = \|\sum_{h=1}^{P} e_h^T(k-1)\|$, and knowing $0 < 2\gamma_1 < \gamma_1 + 1$, $\gamma_1 + 1 < \gamma_1 + \gamma_2 < \gamma_2 + 1 < 2\gamma_2$, one has

$$\Upsilon^{2\gamma_1} < \Upsilon^{\gamma_1+1}, \qquad \Upsilon^{\gamma_1+\gamma_2} < \Upsilon^{\gamma_2+1}. \tag{6.34}$$

Moreover, for $\Upsilon > 1$ one obtains

$$\Upsilon^{2\gamma_2} \leq \eta_1 \Upsilon^{\gamma_2+1} \quad for \quad \eta_1 \geq \Upsilon^{\gamma_2-1}. \tag{6.35}$$

Therefore, for $\Upsilon > 1$, using (6.34) and (6.35), (6.33) is rewritten as follows,

$$\Delta V(k) \leq -(2\xi_G - n\gamma\xi_G^2)\|e^T(k-1)\|^2 - 2\xi_C(\sum_{h=1}^{P} \|e_h^T(k-1)\|^{\gamma_1+1}$$
$$+ (n^{\frac{1-\gamma_2}{2}}) \sum_{h=1}^{P} \|e_h^T(k-1)\|^{\gamma_2+1}) + n\gamma[\eta_1\xi_C^2(\sum_{h=1}^{P} \|e_h^T(k-1)\|)^{\gamma_2+1}$$
$$+ 2(n^{\frac{1-\gamma_1}{2}})\xi_C\xi_G(\sum_{h=1}^{P} \|e_h^T(k-1)\|)^{1+\gamma_1} + 2\xi_C\xi_G(\sum_{h=1}^{P} \|e_h^T(k-1)\|)^{1+\gamma_2}$$
$$+ \xi_C^2(n^{1-\gamma_1})(\sum_{h=1}^{P} \|e_h^T(k-1)\|)^{\gamma_1+1} + 2\xi_C^2 n^{\frac{1-\gamma_1}{2}}(\sum_{h=1}^{P} \|e_h^T(k-1)\|)^{\gamma_2+1}], \tag{6.36}$$

For the first term of (6.36) to have $2\xi_G - n\gamma\xi_G^2 \geq 0$, one needs

$$\gamma \leq \frac{2}{n\xi_G}. \tag{6.37}$$

Hence, for $\Upsilon > 1$ and $\gamma \leq \frac{2}{n\xi_G}$, using (6.9), (6.16), and $S = \sum_{h=1}^{P} \bar{d}(\tau_h)\bar{d}^T(\tau_h)$, (6.36) is rewritten as

$$\Delta V(k) \leq -a_u\|\tilde{\Theta}(k-1)\|^{\gamma_1+1} - b_u\|\tilde{\Theta}(k-1)\|^{\gamma_2+1}, \tag{6.38}$$

where a_u and b_u are given in (6.22) and (6.23), respectively.

In order to have $0 < a_u < 1$ and $0 < b_u < 1$, γ should, respectively, satisfy the following inequalities

$$\underline{a_u} < \gamma < \overline{a_u}, \qquad \underline{b_u} < \gamma < \overline{b_u}, \tag{6.39}$$

where $\underline{a_u}$, $\overline{a_u}$, $\underline{b_u}$ and $\overline{b_u}$ are given in (6.26).

For the case where $0 < \Upsilon \leq 1$, and using $0 < 2\gamma_1 < \gamma_1 + 1$, $\gamma_1 + 1 < \gamma_1 + \gamma_2 < \gamma_2 + 1 < 2\gamma_2$, one obtains

$$\Upsilon^{\gamma_1+\gamma_2} < \Upsilon^{\gamma_1+1}, \qquad \Upsilon^{2\gamma_2} < \Upsilon^{\gamma_2+1}. \tag{6.40}$$

Furthermore, for $0 < \Upsilon \leq 1$, one has

$$\Upsilon^{2\gamma_1} \leq \eta_2 \Upsilon^{\gamma_1+1} \quad for \quad \eta_2 \geq \frac{1}{\Upsilon^{1-\gamma_1}}. \tag{6.41}$$

Therefore, for $0 < \Upsilon \leq 1$ and γ satisfying (6.37), using (6.40) and (6.41), (6.33) is rewritten as follows,

$$\begin{aligned}\Delta V(k) \leq & -2(\xi_C \sum_{h=1}^{P} \|e_h^T(k-1)\|^{\gamma_1+1} + (n^{\frac{1-\gamma_2}{2}}) \sum_{h=1}^{P} \|e_h^T(k-1)\|^{\gamma_2+1}) \\ & + n\gamma[\eta_2 \xi_C^2 n^{1-\gamma_1} (\sum_{h=1}^{P} \|e_h^T(k-1)\|)^{\gamma_1+1} + 2n^{\frac{1-\gamma_1}{2}} \xi_C \xi_G (\sum_{h=1}^{P} \|e_h^T(k-1)\|)^{1+\gamma_1} \\ & + 2\xi_C \xi_G (\sum_{h=1}^{P} \|e_h^T(k-1)\|)^{1+\gamma_2} + 2\xi_C^2 n^{\frac{1-\gamma_1}{2}} (\sum_{h=1}^{P} \|e_h^T(k-1)\|)^{\gamma_1+1} \\ & + \xi_C^2 (\sum_{h=1}^{P} \|e_h^T(k-1)\|)^{\gamma_2+1}],\end{aligned} \tag{6.42}$$

Thus, for $0 < \Upsilon \leq 1$ and $\gamma \leq \frac{2}{n\xi_G}$, (6.42) is rewritten as

$$\Delta V(k) \leq -a_l \|\tilde{\Theta}(k-1)\|^{\gamma_1+1} - b_l \|\tilde{\Theta}(k-1)\|^{\gamma_2+1}, \tag{6.43}$$

where a_l and b_l are, respectively, given in (6.24) and (6.25).

In order to have $0 < a_l < 1$ and $0 < b_l < 1$, γ should, respectively, satisfy the following inequalities

$$\underline{a_l} < \gamma < \overline{a_l}, \qquad \underline{b_l} < \gamma < \overline{b_l}, \tag{6.44}$$

where $\underline{a_l}$, $\overline{a_l}$, $\underline{b_l}$ and $\overline{b_l}$ are given in (6.27).

Therefore, in order to satisfy (6.37) and the inequalities (6.39) and (6.44) for $\Upsilon > 1$ and $0 < \Upsilon \leq 1$, respectively, γ needs to satisfy the following inequalities, respectively,

$$\max\{\underline{a_u}, \underline{b_u}\} < \gamma < \min\{\frac{2}{n\xi_G}, \overline{b_u}, \overline{a_u}\}, \qquad for \quad 1 < \Upsilon, \tag{6.45}$$

$$\max\{\underline{a_l}, \underline{b_l}\} < \gamma < \min\{\frac{2}{n\xi_G}, \overline{a_l}, \overline{b_l}\}, \quad for \quad 0 < \Upsilon \leq 1. \tag{6.46}$$

One obtains from (6.28) and Fact 2 that

$$V(\tilde{\Theta}(k)) \leq \frac{n}{\gamma}\|\tilde{\Theta}(k)\|^2 \;\Rightarrow\; \sqrt{\frac{\gamma}{n}}V^{\frac{1}{2}}(\tilde{\Theta}(k)) \leq \|\tilde{\Theta}(k)\|. \tag{6.47}$$

Using (6.47), one rewrites (6.38) and (6.43) as follows

$$\Delta V(k) \leq -a_i V^{\frac{\gamma_1+1}{2}}(k-1) - b_i V^{\frac{\gamma_2+1}{2}}(k-1),\; i = 1, 2, \tag{6.48}$$

where for $\Upsilon > 1$, $i = 1$,

$$a_1 = a_u(\frac{\gamma}{n})^{\frac{\gamma_1+1}{2}}, b_1 = b_u(\frac{\gamma}{n})^{\frac{\gamma_2+1}{2}},$$

for $0 < \Upsilon \leq 1$, $i = 2$,

$$a_2 = a_l(\frac{\gamma}{n})^{\frac{\gamma_1+1}{2}}, b_2 = b_l(\frac{\gamma}{n})^{\frac{\gamma_2+1}{2}}.$$

One can rewrite (6.48) as follows,

$$\Delta V(k) \leq -\alpha_i \max\{V^{\frac{\gamma_1+1}{2}}(k-1), V^{\frac{\gamma_2+1}{2}}(k-1)\}, \tag{6.49}$$

where $\alpha_i = \min\{a_i, b_i\}, i = 1, 2$. One knows that

$$\begin{aligned}&\min\{V(k-1), \alpha_i \max\{V^{\frac{\gamma_1+1}{2}}(k-1), V^{\frac{\gamma_2+1}{2}}(k-1)\}\}\\ &\leq \alpha_i \max\{V^{\frac{\gamma_1+1}{2}}(k-1), V^{\frac{\gamma_2+1}{2}}(k-1)\},\end{aligned} \tag{6.50}$$

implies

$$\begin{aligned}&-\alpha_i \max\{V^{\frac{\gamma_1+1}{2}}(k-1), V^{\frac{\gamma_2+1}{2}}(k-1)\} \leq\\ &-\alpha_i \min\{\frac{V(k-1)}{\alpha_i}, \max\{V^{\frac{\gamma_1+1}{2}}(k-1), V^{\frac{\gamma_2+1}{2}}(k-1)\}\}.\end{aligned} \tag{6.51}$$

Therefore, using (6.51), for $i = 1, 2$, (6.49) leads to

$$\Delta V(k) \le -\alpha_i \min\{\frac{V(k-1)}{\alpha_i}, \max\{V^{\frac{\gamma_1+1}{2}}(k-1), V^{\frac{\gamma_2+1}{2}}(k-1)\}\}. \tag{6.52}$$

Lemma 11 and (6.52) imply that $\tilde{\Theta}(k)$ converges to zero and a settling-time function is obtained as given in (6.21). By convergence of $\tilde{\Theta}(k)$ to zero which results in $\Upsilon = 0$, no further learning is required and one sets $\gamma = 0$ as given in (6.20). This completes the proof.

Remark 27 The settling-time fixed upper bound in (6.21) certifies that the richer the recorded data in terms of having a bigger ratio of $\frac{\lambda_{min}(S)}{\lambda_{max}(S)}$ (which leads to the bigger values of α_i), the smaller would be the upper bound for the settling-time in (6.21) and the identification converges in a faster fixed amount of time. Moreover, the fixed upper bound of the settling-time function can be computed along with the learning procedure and is obtained before $\tilde{\Theta}(k)$ convergence to zero.

Remark 28 Since the recorded regressors' data are normalized, the lower bounds of learning rate γ, $\max\{\underline{a_u}, \underline{b_u}\}$ and $\max\{\underline{a_l}, \underline{b_l}\}$, respectively, given in (6.18) and (6.19), are usually negative for $\xi_C < 1$. In order to ensure that a positive γ is chosen that satisfies (6.18)-(6.19) for $\Upsilon \neq 0$, γ can be chosen as follows,

$$\gamma = \max\{\min\{\frac{2}{n\xi_G}, \overline{b_u}, \overline{a_u}\} - \epsilon, \max\{\underline{a_u}, \underline{b_u}\} + \epsilon\}, \ \ 1 < \Upsilon,$$
$$\gamma = \max\{\min\{\frac{2}{n\xi_G}, \overline{a_l}, \overline{b_l}\} - \epsilon, \max\{\underline{a_l}, \underline{b_l}\} + \epsilon\}, \ \ 0 < \Upsilon \le 1,$$

where ϵ is a very small positive constant (such as $\epsilon = 0.01 \min\{\frac{2}{n\xi_G}, \overline{b_u}, \overline{a_u}\}$ for $1 < \Upsilon$ or $\epsilon = 0.01 \min\{\frac{2}{n\xi_G}, \overline{a_l}, \overline{b_l}\}$ for $0 < \Upsilon \le 1$). Moreover, to satisfy $\eta_1 \ge \Upsilon^{\gamma_2 - 1}$ and $\eta_2 \ge \frac{1}{\Upsilon^{1-\gamma_1}}$, one can choose $\eta_1 = \Upsilon^{\gamma_2 - 1}$ and $\eta_2 = \frac{1}{\Upsilon^{1-\gamma_1}}$, respectively.

Remark 29 The learning rate γ extracted from either (6.18) or (6.19), is not a fixed constant and it is an adaptive time-varying scalar due to employing time-varying adaptive constants $\eta_1 = \Upsilon^{\gamma_2 - 1}$ and $\eta_2 = \frac{1}{\Upsilon^{1-\gamma_1}}$, (respectively, satisfying (6.35) and (6.41)) where $\Upsilon = \|\Sigma_{h=1}^{P} e_h^T(k-1)\| = \|\Sigma_{h=1}^{P} \bar{d}^T(\tau_h)\tilde{\Theta}(k-1)\|$ depends on the parameter estimation error $\tilde{\Theta}(k)$ at every time k. Therefore, (6.17) is not the explicit Euler discretization of the continuous finite-time method given in [43]. Moreover, in this chapter, the adaptive time-varying γ is different from the time-varying discretization for continuous finite-time systems in [149], which preserves finite-time and fixed-time proprieties but guarantees convergence in infinite time. Furthermore, employing an adaptive time-varying learning rate matches with the concepts of other discrete and finite-time learning studies [109, 122].

6.6 Simulation Results and Discussion

In this section, the performance of the presented fixed-time concurrent learning is examined in comparison with traditional gradient descent (GD) [10], asymptotically converging concurrent learning (CL) [48] and finite-time concurrent learning (FTCL) [42, 109] with, respectively, the following estimation laws,

$$\begin{aligned}
&\hat{\Theta}(k+1) = \hat{\Theta}(k) - \Gamma_G \bar{d}(k) e^T(k), \\
&\hat{\Theta}(k+1) = \hat{\Theta}(k) - \Gamma_C[\Sigma_G \bar{d}(k) e^T(k) + \Sigma_C \sum_{h=1}^{P} \bar{d}(\tau_h) e_h^T(k)], \\
&\hat{\Theta}(k+1) = \hat{\Theta}(k) - \Gamma'[K_1 \bar{d}(k) e^T(k) + K_2(\sum_{h=1}^{P} \bar{d}(\tau_h) e_h^T(k) + \frac{\Sigma_{h=1}^{P} \bar{d}(\tau_h) e_h^T(k)}{\kappa + \|\Sigma_{h=1}^{P} \bar{d}(\tau_h) e_h^T(k)\|})],
\end{aligned} \tag{6.53}$$

where $\Gamma_G = \gamma_G I$, $\Gamma_C = \gamma_C I$, $\Gamma' = \gamma' I$, $\Sigma_G = \sigma_G I$, $\Sigma_C = \sigma_C I$, $K_1 = k_1 I$ and $K_2 = k_2 I$ with constants $\gamma_G > 0$, $\gamma_C > 0$, $\sigma_G > 0$, $\sigma_C > 0$, $\gamma' > 0$, $k_1 > 0$, $k_2 > 0$ and $\kappa > 0$.

The time interval for the simulation is given as $[k_0, k_f]$ with $k_0 = 0$ and $k_f = 1000$, and $\mathcal{D}_x = [x_L, x_H]$ where $x_L = 0, x_H = 2$ and $\mathcal{D}_x$ is discretized by $[x_L : \frac{x_H - x_L}{k_f - k_0} : x_H]$. In the presented FxTCL, γ is chosen to meet either (6.18) or (6.19) based on the value of Υ. We choose $\xi_G > \xi_C$ to prioritize current data over recorded data. For gradient descent, $\gamma_G = 0.6$, for CL according to [48], γ_C is chosen as $\gamma_C = \frac{1}{2\sigma_G + \sigma_C \lambda_{max}(S)}$ and for finite-time CL according to [109], γ' is chosen as

$$\gamma' = \frac{2k_1 \lambda_{min}(\bar{D}(k-1)) + 2k_2 \lambda_{min}(S)}{(k_1 \lambda_{max}(\bar{D}(k-1)) + k_2 \lambda_{max}(S)(1 + \frac{1}{\kappa}))^2}. \tag{6.54}$$

The controllers and initial values for all methods are zero. To ensure the rank condition on the recorded data, a very small exponential decaying sum of sinusoidal input is added to the controller for the data selection procedure [144] employed in FxTCL, FTCL, and CL methods. For the speed and precision comparison of the mentioned methods for approximating $f(x)$ and $g(x)$ on the entire domain of x, the following online learning errors are computed.

$$E_f(k) = \int_{\mathcal{D}_x} \|e_f(x(k))\| d^n x, \quad E_g(k) = \int_{\mathcal{D}_x} \|e_g(x(k))\| d^n x.$$

Consider the nonlinear system given below,

$$x(k+1) = p_1 e^{-x(k)} + p_2 e^{-x(k)} \cos(x(k)) + \frac{p_3}{1+x(k)} u(k),$$

where $[p_1, p_2, p_3]$ are the unknown parameters and the regressor is fully known as

$$z(x(k), u(k)) = [e^{-x(k)}, e^{-x(k)} \cos(x(k)), \frac{u(k)}{1+x(k)}],$$

where $p+q=3$. The values of unknown parameters are $[p_1, p_2, p_3] = [-1, 1.5, 1]$. We choose $P=3$ for FxTCL, FTCL and CL methods. Let $\sigma_G = 0.6$, $\sigma_C = 0.3$ for CL method, and $k_1 = 0.6$, $k_2 = 0.3$ and $\kappa = 0.4$ for FTCL method, and $\xi_G = 0.6$, $\xi_C = 0.3$, $\gamma_1 = 0.8$ and $\gamma_2 = 1.1$ for FxTCL method. Fig. 6.1 shows the true and the approximated parameters for FxTCL, FTCL, CL, and GD approaches. As depicted in Fig. 6.1, while GD did not succeed to converge to the true parameters, FxTCL, FTCL, and CL converged to true values. However, FxTCL converged faster to the true values compared with the other mentioned methods. The online learning errors $E_f(k)$ and $E_g(k)$ for the FxTCL, FTCL, CL, and GD are shown in Fig. 6.2 where FxTCL converged faster to zero in comparison with other mentioned methods.

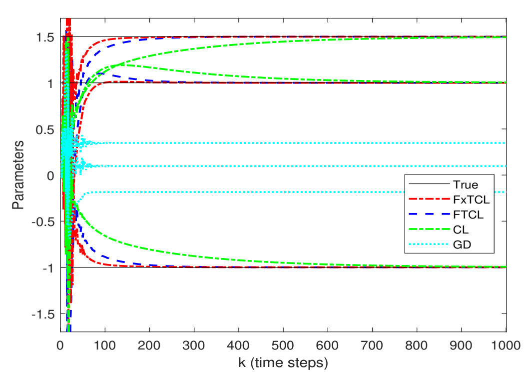

Figure 6.1: Estimated parameters.

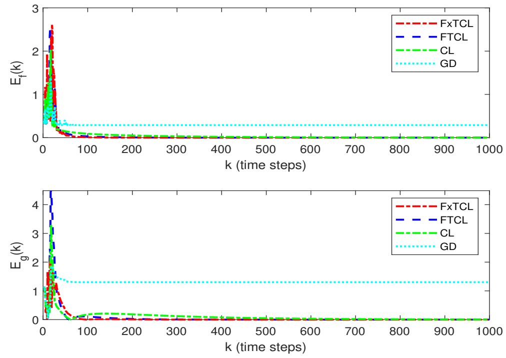

Figure 6.2: Online learning errors.

Table 6.1: Learning errors comparison

	IAE $E_f(k)$	IAE $E_g(k)$
FxTCL	33.84	50.93
FTCL	35.67	72.57
CL	45.01	101.85
GD	297.10	1295.6

The integral absolute errors (IAEs) of $E_f(k)$ and $E_g(k)$ for FxTCL, FTCL, CL, and GD methods are computed and given in Table 6.1 where FxTCL with IAEs 33.84 and 50.93 respectively for $E_f(k)$ and $E_g(k)$ has the lowest learning error compared with the other methods.

6.7 Conclusion

This chapter presented a fixed-time learning method for discrete-time system dynamics' identification where concurrent learning is used to relax the persistence of excitation requirement on the regressor to an easy-to-check rank condition of the recorded data. The learning rate conditions are achieved for fixed-time convergence based on discrete fixed-time analysis. The richness of the memory data in terms of the data spectral proprieties affects the speed of convergence for the presented fixed-time learning method. Simulations verify that the presented fixed-time concurrent learning convergence speed and precision have outperformed the other methods.

CHAPTER 7

CONCLUSION AND FUTURE WORK

In conclusion, first we introduced a finite-time distributed concurrent learning method for interconnected systems' identification in finite time. Leveraging local state communication among interconnected subsystems' identifiers enabled them to identify every subsystem's own dynamics as well as its interconnections' dynamics. In this method, distributed concurrent learning relaxed the regressors' persistence of excitation (PE) conditions to rank conditions on the recorded distributed data in the memory stack of the subsystems. It is shown that the precision and convergence speed of the proposed finite-time distributed learning method depends on the spectral properties of the distributed recorded data. Simulation results show that the proposed finite-time distributed concurrent learning has outperformed the finite-time distributed gradient descent in both terms of precision and convergence speed. For future work, we aim to develop finite-time distributed identifiers and observers to be employed in appropriate distributed controllers for interconnected systems.

Then we presented a fixed-time concurrent learning system identification method without the persistence of excitation (PE) requirement. In this method, the concurrent learning relaxes the requirement of the PE condition to a rank condition on the memory stack of recorded data. It is shown that the richness of the recorded experienced data depends on the minimum eigenvalue properties of the stack of regressor's data which influences the speed and precision of the proposed fixed-time concurrent learning method. Simulation results are given where it is shown that the proposed fixed-time concurrent learning has outperformed other mentioned methods in both terms of precision and convergence speed. For future work, it is intended to extend the existing results for discrete-time systems.

We also proposed a data-regularized concurrent learning-based stochastic gradient descent (CL-based SGD) method that leverages recorded data to guarantee linear (exponential) bounded convergence of the estimated parameters' error. It is shown that the richness of the memory data improves the speed of convergence and reduces the probabilistic bound of convergence. Lyapunov analysis guaranteed that the proposed data-regularized CL-based SGD method not only ensures the practical stability in probability of the estimated parameters' error but can ensure a finite-sample boundedness in probability of the estimated parameters' error. Simulation results verified that the employed data-regularized CL-based SGD could improve the speed and precision of convergence for the estimated parameters in comparison with SGD.

Furthermore, we presented the fixed-time stability for deterministic and stochastic discrete-time (DT) autonomous systems based on fixed-time Lyapunov stability analysis. Novel Lyapunov conditions are derived under which the fixed-time stability of autonomous DT deterministic and stochastic systems is certified. The sensitivity to perturbations for fixed-time stable DT systems is analyzed and the analysis shows that fixed-time attractiveness can result from the presented Lyapunov conditions. For both cases of fixed-time stable and fixed-time attractive systems, the fixed upper bounds of the settling-time functions are given.

Finally, we proposed a fixed-time learning method for discrete-time system dynamics' identification where concurrent learning is used to relax the persistence of excitation requirement on the regressor to an easy-to-check rank condition of the recorded data. The learning rate conditions are achieved for fixed-time convergence based on discrete fixed-time analysis. The richness of the memory data in terms of the data spectral proprieties affects the speed of convergence for the presented fixed-time learning method. Simulations verify that the presented fixed-time concurrent learning convergence speed and precision have outperformed the other methods.

For future work, we plan to employ the presented fixed-time stability analysis to develop fixed-time identifiers based on the dynamic regressor extension and mixing (DREM) technique. Moreover, the ideas of this dissertation can be easily extended to fixed-time controllers and identifiers for CT and DT systems.

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