RENORMALIZATION WITH THE GRADIENT FLOW

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ABSTRACT

For all of its successes, the Standard Model (SM) of particle physics cannot explain the observed asymmetry of the matter and antimatter contents of the universe. Toward a solution for this problem, Andrei Sakharov proposed in 1967 three necessary and sufficient conditions for any extension of the accepted model to be able to produce such an imbalance. In particular, the combined parity (P) and charge conjugation (C) symmetry must be significantly violated by fundamental interactions. While there is some CP violation in the electroweak sector of the Standard Model, it is grossly insufficient to account for the observed difference. A historically attractive probe into sources of CP violation beyond the Standard Model (BSM) has been the neutron electric dipole moment (nEDM). The experimental upper bound on its value lies several order of magnitude above the lower bound imposed by the Standard Model, providing a large window to search for CP-violating BSM phenomena. There are many potential sources. At hadronic scales, these interactions may be encoded by effective local operators of SM fields. In order to disentangle their contributions, their hadronic matrix elements must be precisely precisely determined, which is currently only possible within the framework of lattice quantum chromodynamics (LQCD). The primary difficulty in the computation of these matrix elements is their renormalization, which mixes the effective operators. Since the only available scale to parametrize the mixing is the lattice spacing, these computations are prone to potential power divergences related to lower-dimensional operators in the continuum limit. In this thesis, we propose to use the gradient flow to temper these divergences. The gradient flow is essentially a gauge-covariant smearing of the quantum fields. It introduces a fifth dimension, the flow time, that controls the extent of the smearing. Critically, the flow time also provides an alternative scale to the lattice spacing. This allows us to define the effective operators through a short-flow-

time expansion, which enjoys a smooth continuum limit for fixed, nonzero flow times. the expansion coefficients can be determined on the lattice, so long as their ultraviolet behavior is constrained in some manner. The natural way to do this is through perturbation theory, though the calculations are made much more difficult by the introduction of Gaussian damping factors. In this thesis, we comprehensively construct the perturbation theory and renormalization of the gradient flow from the ground up, introducing along the way a new method for calculating dimensionally-regularized loop integrals with difficult angular dependence. This method relies heavily on the Schwinger proper time representation of propagators and handles the angular pieces through a combinatorial tensor decomposition. Using this novel technique, we calculate the renormalization constants and short-flow-time coefficients of a handful of physically interesting operators, including the topological charge density and chromoelectric dipole moments. We further use the gradient flow to define a pure-lattice renormalization scheme along with an induced renormalization group flow, which we connect to more phenomenologically amenable renormalization schemes using our new perturbative techniques.

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Introduction

In the late 1920s, it became apparent that the quantum mechanics of Erwin Schrödinger and Werner Heisenberg could not fully treat the quantization of the electromagnetic field. Due to the manifest Lorentz covariance of Maxwell's equations, it was evident from the beginning that a properly quantized theory of electrodynamics should also exhibit this covariance. Unfortunately, the propagators of Schrödinger's theory were nonvanishing over the whole of spacetime, signaling a violation of causality. Moreover, Louis de Broglie's wavelike interpretation of the electron implied a wavelike nature for both the matter and forces in the quantum theory. To that end, Max Born, Pascual Jordan, and Heisenberg constructed a free field theory in 1925 by treating the degrees of freedom as an infinite set of quantized harmonic oscillators [1]. Paul Dirac further showed in 1927 that this structure could replicate the Einstein coefficients [2]. The pivotal step was, however, his introduction of the Dirac equation [3], the first successful relativistic wave equation. Schrödinger himself had first attempted to use the relativistic dispersion relation to construct his Hamiltonian, giving what would later be called Klein-Gordon equation for scalar fields [4, 5]. Lacking the full consideration of spin, this formalism could not reproduce the Bohr levels in hydrogen, so it was scrapped for the familiar Schrödinger equation.

As it turns out, it is not the Schrödinger equation, but the Hamiltonian operator that fails, perceived as acting on a single-particle Hilbert space. Indeed, this is partially why the original Klein-Gordon equation failed. Dirac, too, originally held a single-particle interpretation of his equation, implying for each state of energy E an accompanying state of energy $-E$. While immaterial in a free theory, the energy spectrum of the interaction Hamiltonian was unbounded below when including electrodynamics. Dirac proposed a sea of negative-energy eigenstates, all filled with negative-energy electrons save for a number of effectively positively-charged "holes," presumed to be protons. It was hoped that this would indirectly bound the

Hamiltonian from below through the Pauli exclusion principle, but, notwithstanding a grave misinterpretation of the Fock space, the stability of atoms and the vast discrepancy between the masses of the proton and electron were enough to condemn this picture [6]. Though the proton was out of the question, Dirac maintained that there was a fundamental importance to this symmetry under charge conjugation.

Carl Anderson's 1932 discovery of the positron rectified the situation [7]. The "antielectron" field, formally identical to the electron but for its positive electric charge, replaced the Dirac sea; the positron not only fit the bill for the negative energy eigenstates of the Dirac equation, restoring the positive-definiteness of the Hamiltonian, but its oppositely-signed currents flowed backwards relative to those of the electron, fixing also the problem of propagation over spacelike intervals.

These developments form the basis of second quantization, wherein fields are promoted to local operators acting on a multiparticle Fock space of excitations of the vacuum. The field operators, themselves subjected to the canonical quantization conditions, compose a complete set of quantum harmonic oscillators with a particle corresponding to each excitation. The excitations are generated by the coefficients of the Fourier decomposition, having now been promoted to ladder operators acting on the multiparticle states. The coefficients of the positive-frequency terms locally produce particles, while their negative-frequency counterparts produce antiparticles. Much of this work on configuration-space is due to Wolfgang Pauli and Jordan, who proved the commutation relations were Lorentz invariant [8], and to Vladimir Fock, who constructed the Hilbert space and worked out — along with Eugene Wigner and Jordan — canonical (anti)commutation relations for bosons (fermions) consistent with spin and statistics [9, 10]. All of these advances allowed for the construction of the S-matrix, which produces all observables.

In 1949, Freeman Dyson introduced the Dyson series [11], which gave a perturbative construction of the S-matrix. Specifically, his introduction of time-ordered correlation functions guaranteed causality by forcing amplitudes outside the lightcone to vanish. Gian Carlo Wick then proposed in 1950 a combinatorial decomposition of the matrix elements produced by the Dyson series. He related Dyson's time-ordered products to field contractions and normal-ordered products, the latter of which gave vanishing contributions to scattering amplitudes. This reduction expressed the matrix elements in terms of simple two-point functions and interaction vertices, forming from the local-field perspective a basis for Richard Feynman's diagrammatic approach.

Feynman himself preferred a particle theory, motivated by his and John Wheeler's earlier development of absorbers. They had built a generalized classical electrodynamics from the Lagrangian point of view, which was possible due to their unorthodox usage of both advanced and retarded waves and their dismissal of electrical self-interactions [12,13]. The interactions were confined to the lightcone with a delta distribution, which Feynman realized could be relaxed in a small neighborhood to generalize the behavior of electrodynamics at large energies. To explain the universality of the electron's mass and charge, Wheeler suggested that all electrons are one singular entity, traveling on a complicated, looping world line. Then on any time slice, those sections traveling toward the plane may be considered positrons, while those pointing away were electrons. Their action was particularly clean, and the absence of fields simplified both the mathematics and visualization. Feynman successfully incorporated fields into their theory, though he ultimately dismissed them as bikeshedding, leaving him suspicious of the Hamiltonian formalism. Indeed, when he moved on to developing a quantized absorber theory, he found his theory to be incompatible with the typical Hamiltonian methods of the time. He was later introduced to an idea of Dirac, that between two points in time, the path-dependence of a particle's trajectory could be related to an overall complex phase on the wavefunction, where the argument was proportional to the action along the path. Infinitesimally iterating along a finite interval, he found that the propagation of a particle could be described by a sum over all possible paths weighed by a phase equal to the associated action [14]. This new path integral was a natural setting for his quantum absorber theory.

The measurement of the Lamb shift eventually forced Feynman to reconsider the self-energy of the electron. Following the suggestion of Hans Kramers, he worked with Hans Bethe to calculate the self-energy in his path integral formalism, finding that the infinite result could be tamed by smearing the delta function in the Lagrangian for the absorber theory, amounting to a physical cutoff on the spacing of the points of self-interaction [15]. This was an early example of regularization and renormalization, where a measurable parameter is redefined to be the finite difference of two formally infinite quantities. Around this time, Feynman developed simpler methods for path integral calculations, culminating in his 1949 introduction of Feynman diagrams [16]. He, employing his heuristic "spacetime" method, proceeded to calculate the leading radiative correction to the electron's anomalous magnetic moment. Julian Schwinger and Shin'ichirō Tomonaga concomitantly arrived at the same result as Feynman through the local-field picture, to which Dyson subsequently proved the spacetime formulation was equivalent. The triplicate determination of the anomalous magnetic moment and Bethe's calculation of the Lamb shift were at this point the most accurate calculations in physics, bringing renormalization and the new quantum electrodynamics (QED) to the theoretical fore.

The 1950s and '60s were largely spent building models of the weak and strong interactions, catalyzed by Feynman's new, efficient methods and the continuing success of QED. There was a shift in attention to Noetherian symmetries and the role of Lie groups. Chen-Ning Yang and Robert Mills expanded the earlier work of Herman Weyl to describe the relationship between the allowable interactions and the symmetry group of the theory [17]. With the further work of Murray Gell-Mann providing physical consequences of group-theoretic considerations, quantum field theory matured into the study of gauge theories, the paradigm now being Yang-Mills theory [18]. Additionally, the inclusion of fermions into the path integral was finally treated in full with the implementation of Grassmann calculus, owed chiefly to David Candlin [19] (who is disgracefully absent from the modern literature). After Chien-Shiung Wu demonstrated a violation of parity in the electroweak interaction [20], there was a renewed interest in discrete symmetries as well, akin to the Dirac's earlier notion of charge symmetry. This was not the only symmetry broken by weak interactions. In order to incorporate both parity-conserving and parity-violating interactions into a gauge theory of the weak nuclear force, Sheldon Glashow, Adbus Salam, and John Ward developed a semisimple gauge theories with massive vector bosons [21–23]. Unfortunately, these proposals broke gauge symmetry and could not be renormalized. As a consequence of the broken gauge symmetry, it was expected that the theory would contain massless Nambu-Goldstone bosons. In 1964, three collaborations – one consisting of Peter Higgs, one of Robert Brout and François Englert, the third of Gerald Guralnik, C. Richard Hagen, and Tom Kibble – determined that the Nambu-Goldstone bosons could in the presence of a spontaneously broken symmetry combine with massless gauge bosons to generate particle masses [24–26]. Dubbed the Higgs mechanism, this explained the absence of Goldstone bosons while producing massive gauge bosons in a manner consistent with gauge symmetry. Steven Weinberg incorporated these arguments into Glashow's and Salam's earlier theories, unifying the electroweak and electromagnetic forces as one gauge theory, now commonly called the Glashow-Weinberg-

Salam (GWS) theory.

These successes inspired a Yang-Mills theory for Gell-Mann's quark model, quantum chromodynamics, but the nature of quarks would not receive a satisfactory treatment until the demonstration by H. David Politzer, David Gross, and Frank Wilczek that asymptotic freedom could be dynamically realized from the self-interaction of the gauge field [27, 28]. All of this was enabled by two major advancements. The first was the general gauge-fixing procedure of Ludvig Fadeev and Victor Popov, which removed the overcounting of gauge configurations in the path integral [29]. The second was a deeper understanding of renormalization granted by both the proof by Gerardus 't Hooft and Martinus Veltman that Yang-Mills theories are renormalizable [30, 31] and the identification of the renormalization group by Kenneth Wilson [32, 33].

The '70s saw the grand synthesis of quantum and statistical field theories, led by Wilson's systematization of the scaling principles of Curtis Callan and Kurt Symanzik [34–36]. Wilson viewed the cutoffs of conventional renormalization as threshold scales beyond which the laws of physics were unknown. He explained how to encode irresolvable high-energy phenomena at low energies by successively "integrating out" highly energetic degrees of freedom, thus recasting current theories as effective theories for an as-yet-unknown ultraviolet (UV) theory. In this way, divergences induced by highly local and energetic interactions were seen to be artifacts of sending Wilson's thresholds to infinity. Renormalization was then just a demand that the physics must be insensitive to the mathematical choices made in imposing a cutoff. In an attempt to probe the nonperturbative confinement of quarks in hadrons, Wilson proposed defining the theory on a discrete spacetime lattice from which the physical theory could be recovered in the continuum and infinite volume limits [37]. Path integrals constituted an especially natural setting for this lattice field theory, and the path integral formula for scattering amplitudes could be easily translated to the discrete language. While perturbation theory had been extremely successful for weak interaction strengths, one could now generate numerical predictions for strongly coupled theories. A critical component of lattice field theory is the Euclideanization of the action. Interestingly, this stipulation led to the only mathematically rigorous definition of the path integral.

Anticipating the discovery of the Higgs boson in 2012 and the confirmation of the Glashow-Weinberg-Salam model for electroweak unification, the superstructure of the Standard Model (SM) of particle physics was reasonably complete, standing as the most precise and predictive theory of Nature ever constructed. A major blemish on its record, however, has been its inability to account for the obvious asymmetry of matter and antimatter in the Universe. Aside from an exceedingly small contribution from the electroweak sector, the SM predicts a largely democratic universe, producing matter and antimatter at roughly equal rates. Much of the work following Wilson was dedicated to unification of the forces and beyond-the-Standard-Model (BSM) extensions to fill in the proliferating gaps between theory and experiment. A standard feature of these theories is a measurable violation of the discrete charge-parity (CP) symmetry, in concordance with the Sakharov conditions for baryogenesis. The mechanisms for CP-violation are typically mediated by heavy particles, detectable only at very large energies. It is conceivable, however, that signatures of this broken symmetry are visible in very accessible systems, the archetypical example being the hypothetical neutron electric dipole moment (nEDM). Due to confinement at low energies, baryons are in general poorly defined in perturbation theory. On the other hand, Wilson's lattice field theory is perfectly suited for these low-energy systems. In this regime, the high-energy BSM interactions are irresolvable. Instead, one may consider effective local interactions built from only the low-energy modes of the theory. In the Wilsonian picture, the low-energy Lagrangian is supplemented with an infinite tower of effective operators, corresponding to the potential UV completions. The potential contribution of each such interaction to the nEDM may be computed on the lattice by inserting the operators into hadronic matrix elements with electromagnetic currents. After a suitable renormalization, these results can be compared with several experimental measurements to isolate the physical contributions and identify appropriate BSM extensions. As we will discuss in Ch. 4, the renormalization of these operators is highly nontrivial on the lattice, forming the motivation for current manuscript.

In what follows, we develop a method for circumventing the impediments to lattice renormalization. The chief difficulty is the treatment of low-dimensional operators. Under renormalization, the whole ensemble of operators within a theory is mixed; that is to say, each operator is renormalized with an infinite series of virtual corrections from every other operator. It follows by dimensional analysis that the series coefficients of lower-dimensional operators must have positive engineering dimension. This becomes problematic under lattice regularization. Since the only internal scale is the lattice spacing, which has units of length, the coefficients must be parametrized by inverse powers of the spacing. If one hopes to make predictions, the spacing must be taken to zero, so that the model truly simulates a physical, continuous universe. This is the continuum limit. Any matrix elements containing the lower-dimensional operators are thus seen to diverge, and numerical uncertainty dominates any predictions.

Recently, the gradient flow formalism has become an attractive tool for regulating divergences on the lattice. It characterizes a gauge-covariant parabolic smearing of quantum fields into a new dimension, the flow time t. We have proposed to use the gradient flow formalism to define matrix elements on the lattice. The primary benefit of the flow is that it introduces a second scale, t, which may be used to parametrize the operator mixing. The flowed operators may be expanded as a linear combinations of physical, unflowed operators with coefficients depending on the flow time. Then at fixed, positive t, we may be allowed to perform the continuum limit (and infinite volume limit) free of power divergences, since the power divergences are now represented by inverse powers of the flow time. BBecause the flow is a continuum theory, the Wilson coefficients can be calculated in perturbation theory, and the divergent parts may be systematically removed for small enough values of the coupling on the lattice. Conveniently, the perturbative renormalization of the flowed fields is strictly multiplicative, involving only a single new field renormalization. Nevertheless, flowed perturbation theory is generally very difficult.

The typical regularization of perturbation theory involves the analytic continuation of the spacetime dimension d to the complex plane [31]. One then must be able to find a coordinate basis in which some $d - n$-dimensional subspace can be integrated directly by symmetry, leaving some integer n integrals to be solved by standard means. Spherical coordinates are typically used, since the problem is reduced to a single radial integral. This is only useful, however, when the integrand itself has no angular dependence or when a simple enough parametrization of the integrand exists for which there is no explicit angular dependence. The gradient flow introduces Gaussian factors for each field propagator that defy all standard methods. The key to our proposed method is this author's extension of Schwinger parametrization [38]. The procedure involves a combinatorial analysis of the Lorentz structure of tensor integrals. So far, this has only been rigorously justified to two-loop order, but the path to generalization is clear. All perturbative results obtained herein will be derived with this new method.

For demonstration, we first renormalize QCD in Minkowski space, obtaining the standard one-loop results. All of the subsequent calculations will utilize the Euclidean version of this scheme. We calculate the renormalization constant for flowed fermions the one-loop self-energy, which is so far absent from the literature. More importantly, we calculate the flowed mixing coefficients for CP-violating operators up to dimension six, excluding four-fermion operators (a subject of future work). These are genuinely new results. In Part 4 we apply this procedure to the quark chromoelectric dipole moment operator. In so doing, we are able to compute a smooth extrapolation of the dominant mixing coefficient over a wide range of energies. The results concerning perturbative renormalization have been published in five papers [39–43]; two related papers are in preparation. A third paper being prepared is a standalone treatment of the new combinatorial method for tensor decomposition, and a fourth presents a renormalization group flow induced by the gradient flow and an algorithm for nonperturbative renormalization with perturbative matching.

Part I

Exposition

This first part serves to provide a theoretical background of the forthcoming material. The first chapter lays the technical foundation for quantum field theory upon which we construct quantum chromodynamics (QCD), the standard theory of the strong interaction governing the dynamics of quarks and gluons at high energies. QCD further gives rise to hadronic matter at intermediate scales and – residually, at low energies – to nuclear matter. Once the Lagrangian is constructed, we explore the perturbative expansion, gauge fixing, and renormalization. We discuss the impediments to performing calculations connecting disparate energy regimes, focusing in particular on the need for nonpeturbative methods and the concept of effective field theory (EFT), wherein some degrees of freedom that are unresolvable at some reference scale are systematically removed, leaving a series of effective contributions characterizing the virtual presence of the full theory. This leads naturally into operator mixing and the operator-product expansion (OPE). These concepts are finally translated into the language of lattice field theory, giving us lattice quantum chromodynamics (LQCD), a numerical framework for studying QCD nonperturbatively on a discretized spacetime lattice. Since they are sensitive to the whole of the theory, lattice methods provide a natural setting for the study of bound states, such as the neutron, which are perturbatively inaccessible.

The second half of this part introduces baryon asymmetry and the Sakharov conditions for breaking such a symmetry, focusing on the violation of the combined charge-parity ($\mathcal{CP}$) symmetry. We discuss potential beyond-the-Standard-Model (BSM) sources of $\mathcal{CP}$ violation and their signatures in the EFT, leading to the concept of the neutron electric dipole moment (nEDM), a promising experimental probe of $\mathcal{CP}$ violation. This involves the insertion of effective operators into nucleonic correlation functions at hadronic energy scales accessible only nonperturbatively. We thus recast the problem into a lattice representation, where the QCD corrections to the effective sources may be determined numerically. Critically, this involves defining a renormalization scheme that is amenable to both the lattice and perturbation theory, while being sensitive to all operator mixing. For this we apply the gradient flow, the details of which are deferred to Part II.

The treatment here is moderately long, but the intention is that all techniques and formulae employed in the following parts be given a firm foundation and that this thesis should be reasonably self-contained. There are several nonstandard conventions which are chosen to avoid a proliferation of unnecessary numerical constants or normalizations. For example, we choose a basis of *skew*-hermitian generators for the $\mathfrak{su}(N)$ Lie algebra. The author's opinion is that this is more naturally derived from the definition of the algebra. A fortunate consequence is that the imaginary unit does not appear in any calculations involving color algebra. This does, however, alter the definitions of the group invariants, so all results in this thesis differ commensurately from much of the literature. We also introduce a new general method to calculate dimensionally-regularized Feynman integrals in both Minkowski and Euclidean spaces; this is treated fully in App. .

Chapter 1

Quantum Field Theory

The principal difference between quantum field theory and quantum mechanics is Lorentz covariance. Since the action is relativistically invariant, the Lagrangian formulation of classical field theory provides a natural foundation for a relativistic theory. The specific approach we take in constructing a QFT relies on the Feynman path integral, which expresses a quantum field theory with the manifest symmetries of the Lagrangian perspective. More importantly, when the strength of interaction for some phenomenon is too large, typical perturbative approximation techniques become invalid. The only systematic nonperturbative treatment of a quantum field theory is lattice field theory, which relies wholly on the discretization of the path integral. When the theory is QCD, the discrete analogue is lattice quantum chromodynamics (LQCD, Ch. 4), which is central to the following chapters.

1.1 Relativistic Field Theory

The fundamental object containing the entire dynamics of a field theory is the Lagrangian density functional:

$$\mathcal{L} = \mathcal{L}[\{\phi_i\}, \{\dot{\phi}_i\}](x), \tag{1.1}$$

where ϕ_i and $\dot{\phi}_i$, $i \in [n]$ represent some n fields and their conjugate momenta. These degrees of freedom assume the traditional role of generalized coordinates, and since they are themselves functions of both spatial and temporal coordinates, the action is defined over the whole of spacetime:

$$S[\{\phi_i\}, \{\dot{\phi}_i\}] = \int d^4x \; \mathcal{L}[\{\phi_i\}, \{\dot{\phi}_i\}](x). \tag{1.2}$$

For each species of dynamical field, the Lagrangian contains a Lorentz invariant free-field contribution determined by its spin, for example

spin-0, real scalar ϕ	Klein-Gordon Lagrangian	$\mathcal{L}_{KG} = -\frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}\mu^2\phi^2,$
spin-1/2, spinors $\bar{\psi}, \psi$	Dirac Lagrangian	$\mathcal{L}_D = \bar{\psi}(\not{\partial} + m)\psi,$
spin-1, vector A_μ	Proca Lagrangian	$\mathcal{L}_P = \frac{1}{g^2}\,\mathrm{Tr}\left[\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + M^2 A_\mu A^\mu\right],$

and so on, following the Bargmann-Wigner construction [44]. The interactions are then determined by the imposition of a local gauge symmetry, following the Yang-Mills construction, which is generated by the unitary action a compact (semi)simple Lie group G at each spacetime coordinate. In order that the theory remain invariant under the action of such a group, the ordinary derivative must be promoted to a gauge covariant derivative,

$$\partial \to D = \partial + A. \tag{1.3}$$

The second term is the gauge connection, which is associated with the gauge boson generated by the symmetry group and takes values in the Lie algebra. Because derivatives are formally defined at two infinitesimally separated points, the local transformation acts on the field at each point separately. A tracks the change in the gauge transformation between these points. In particular, its transformation under the gauge group exactly cancels the change in the transformation between the two points under the derivative, ensuring local invariance of the action. To illustrate, consider a local gauge transformation $g \in G$ parametrized by the exponential map,

$$g(x) = e^{-\omega(x)}, \qquad \omega(x) = \omega^a(x) t^a, \tag{1.4}$$

where t^a are the generators of the Lie algebra $\mathfrak{g}$, that acts on fermions as

$$\psi \xrightarrow{g} \psi' = e^{-\omega}\psi, \qquad \bar{\psi} \xrightarrow{g} \bar{\psi}' = \bar{\psi}e^{-\omega^\dagger} = \bar{\psi}e^{\omega}. \tag{1.5}$$

Under the action of g, the kinetic term in the Dirac Lagrangian differentiates both the gauge transformation and the fermion field, so that

$$\mathcal{L}_D \xrightarrow{g} \mathcal{L}'_D = \bar{\psi}g^\dagger[i(\partial\!\!\!/ g) + ig\partial\!\!\!/ - mg]\psi = \mathcal{L}_D - i\bar{\psi}(\partial\!\!\!/\omega)\psi. \tag{1.6}$$

If we consider the covariant derivative, the connection transforms as

$$A \xrightarrow{g} A' = gAg^\dagger + g\partial g^\dagger, \tag{1.7}$$

since it is in the adjoint representation. Hence

$$i\bar{\psi}A\!\!\!/\psi \xrightarrow{g} i\bar{\psi}'A\!\!\!/'\psi' = i\bar{\psi}g^\dagger g(A\!\!\!/ - \partial\!\!\!/\omega^\dagger)g^\dagger g\psi = i\bar{\psi}A\!\!\!/\psi + i\bar{\psi}(\partial\!\!\!/\omega)\psi, \tag{1.8}$$

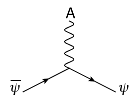
which precisely compensates for the new term in Eq. 1.6.

In general, there will be a new term in D associated to each simple factor of a semisimple gauge group, since the Lie algebra is then a direct sum of simple algebras. In this way, each factor of the gauge group generates a gauge boson. Additionally, since the symmetry is continuous by construction, it corresponds to a conserved charge by Noether's theorem. More specifically, there is a conserved charge for each generator of the Lie algebra. We will treat this construction specifically in the case of quantum chromodynamics, Ch. 3. It should be noted that the definition in Eq. 1.3 implicitly includes the coupling g in the gauge field

A. The standard convention is to write $D = \partial + gA$, but our choice to rescale $gA \to A$ will prove notationally cleaner, particularly in Part II.

When inserted into the Dirac Lagrangian, the connection term in the covariant derivative produces an interaction of the form

$$\bar{\psi} \not{A} \psi = \qquad (1.9)$$



which characterizes the radiation of an A boson by an initial fermion $\bar{\psi}$ that exits in the state ψ. This specific interaction, analogous to the canonical momentum of classical electrodynamics, is called minimal. Of course, there may be other gauge-invariant interactions, but our later discussion of effective field theory (Sec. 1.8) will clarify their absence from typical applications.

1.2 The Generating Functional

Once we have developed the action, we may encode the theory into a sum over histories, the generating functional;

$$Z = \int \mathcal{D}\phi \; e^{iS[\phi]}, \qquad (1.10)$$

where the integral is meant to be taken over all configurations of the fields, ϕ. The generating functional acts as a partition function for the field configurations, where states are distributed according to the "Boltzmann" factor e^{iS}. The addition of static source fields to the Lagrangian permits the use of the Schwinger-Dyson equations to generate all Green functions, or expectation values. Specifically, for some dynamical field ϕ and static source

J, the Lagrangian is augmented by

$$\mathcal{L}_{source} = J\phi. \tag{1.11}$$

Successive functional differentiation of the path integral with respect to the source at some coordinates x_i brings down as many powers of $i\phi(x_i)$. Shutting off the sources and normalizing by the source-free generating functional $Z_0 = Z[J = 0]$ to remove an infinite background of vacuum fluctuations, we are able to compute all physical observables. Given some operator

$$\mathcal{O}(x_1, \ldots, x_N) = \Gamma\phi(x_1)\cdots\phi(x_N), \tag{1.12}$$

where the differential, spacetime, and gauge structures are generically encoded in the quantity Γ, we have

$$\langle\mathcal{O}\rangle = \frac{1}{Z_0}\int \mathcal{D}\phi\ \mathcal{O}e^{iS[\phi]} = \prod_{i=1}^{N}\frac{-i\delta}{\delta J(x_i)} \cdot \frac{1}{Z_0}Z[J]\bigg|_{J=0}. \tag{1.13}$$

This formula elucidates the probabilistic nature of the path integral; it gives the expectation value for a function $\mathcal{O}$ of random variable ϕ distributed by e^{iS}. With the knowledge of all correlation functions, the theory is effectively solved, although this is generally easier stated than practiced. Eq. 1.13 indeed generates all interactions, but we must carefully unpack it before defining both its perturbative and nonperturbative treatments.

1.3 Grassmann Numbers

First we discuss the implementation of fermions. In order to uphold Fermi-Dirac statistics, any two spinor fields must anticommute. This is accomplished by treating fermions as Grassmann numbers, which are most easily characterized algebraically. Indeed, there needn't be a a rigorous justification for the entire calculus [45], since ultimately we will only be interested in integration over entire factors of the Grassmann algebra, defined as follows. Let $\{\theta_i\}$ be a basis for an n-dimensional vector space V. The Grassmann algebra $\Lambda(V)$ is a 2^n-dimensional unital algebra defined by equipping V with the product

$$\theta_i\theta_j + \theta_j\theta_i = \{\theta_i, \theta_j\} = 0, \quad \forall i, j \in [n]. \tag{1.14}$$

We may construct functions of the Grassmann numbers θ_i as elements of the ring of formal power series in the variables θ_i over the complex numbers, $\mathbb{C}[[\theta_1, \dots, \theta_n]]$. A direct consequence of the multiplicative law is that every generator is a zero divisor, in particular a square root of zero:

$$\theta_i^2 = \frac{1}{2}\{\theta_i, \theta_i\} = 0. \tag{1.15}$$

It follows that Taylor series truncate quickly; all functions are at most affine with respect to each variable:

$$f(\theta_i) = b_0 + b_1\theta_i, \tag{1.16}$$

for some complex b_i. We can thus define the Berezin integral of a function of a single Grassmann number [46]:

$$\int d\theta f(\theta) = a \int d\theta + b \int d\theta\ \theta, \tag{1.17}$$

where we have assumed linearity over the complex numbers. With the additional requirement that, since we are integrating over all θ, the integral must be translationally invariant, we have

$$\int d\theta\ f(\theta) = a \int d\theta + b \int d\theta\ (\theta + \eta) = (a - b\eta) \int d\theta + b \int d\theta\ \theta, \tag{1.18}$$

for another Grassmann number η. The first integral must vanish, since

$$a \int d\theta = (a - b\eta) \int d\theta, \tag{1.19}$$

and a, b are generic. The second integral above is simply an arbitrary normalization factor, with the conventional choice of

$$\int d\theta \ \theta = 1. \tag{1.20}$$

Multiple integration is easily found by extension, with the convention that for n variables θ_i,

$$\int d\theta_n \cdots \int d\theta_1 \ \theta_1 \cdots \theta_n = 1. \tag{1.21}$$

We are now able to calculate multivariate Gaussian integrals,

$$\int d\bar{\theta}_1 d\theta_1 \cdots \int d\bar{\theta}_n d\theta_n \ e^{-\bar{\theta}_i A_{ij} \theta_j},$$

for some $2n$ generators $\theta_i, \bar{\theta}_i$ and an n-dimensional Hermitian matrix A. Since the only terms that survive the Taylor expansion are linear in each variable, and each of these is totally antisymmetric, a unitary rotation U of the variables contributes an overall factor of $\det U$. Thus, by diagonalizing A we find the Matthews-Salam formula [47];

$$\int d\bar{\theta}_1 d\theta_1 \cdots \int d\bar{\theta}_n d\theta_n \ e^{-\bar{\theta}_i A_{ij} \theta_j} = \det A, \tag{1.22}$$

contrasting the standard Gaussian integral, which goes as $1/\sqrt{\det A}$. We may now express the Dirac field as a linear combination of Grassmann numbers ψ_i with smooth coefficients

$u_i(x)$ forming a basis for Dirac spinors:

$$\psi(x) = u_i(x)\psi_i, \tag{1.23}$$

where the Einstein summation convention is implied; we will adopt this notation for the rest of this work, unless otherwise specified.

1.4 Perturbative Expansion of the Generating Functional

The full generating functional is rarely exactly soluble. The free field theory, on the other hand, consists of strictly quadratic actions and can be transformed into a product of manifestly integrable Gaussians as we will see in Ch. 2. Denoting by g a generic coupling generated by a gauge interaction, the Lagrangian of any theory may be decomposed into free (0) and interacting (I) pieces (and perhaps a source term (S)),

$$\mathcal{L} = \mathcal{L}_0 + g\mathcal{L}_I \qquad (+\mathcal{L}_S)\,, \tag{1.24}$$

where exclusively the interaction Lagrangian may contain higher powers in the coupling. For some fields collectively referred to as ϕ, we see immediately that

$$\langle\mathcal{O}\rangle = Z^{-1}[0]\int\mathcal{D}\phi\; e^{i\int\mathcal{L}}\mathcal{O} = Z^{-1}[0]\int\mathcal{D}\phi\; e^{i\int\mathcal{L}_0}e^{ig\int\mathcal{L}_I}\mathcal{O} = \frac{Z_0[0]}{Z[0]}\langle e^{ig\int\mathcal{L}_I}\mathcal{O}\rangle_0, \tag{1.25}$$

where we have absorbed the source term into $\mathcal{L}_0$ with a suitable shift of variables and defined the free partition function:

$$\langle\mathcal{O}\rangle_0 = Z_0^{-1}[0]\int\mathcal{D}\phi\; e^{i\int\mathcal{L}_0}\mathcal{O}, \tag{1.26}$$

with the obvious normalization:

$$Z_0[0] = \int \mathcal{D}\phi \; e^{i\int \mathcal{L}_0}, \tag{1.27}$$

which gives us

$$Z[0] = \int \mathcal{D}\phi \; e^{i\int \mathcal{L}} = Z_0[0] \left\langle e^{ig\int \mathcal{L}_I} \right\rangle. \tag{1.28}$$

Thus,

$$\langle \mathcal{O} \rangle = \frac{\langle e^{ig\int \mathcal{L}_I} \mathcal{O} \rangle_0}{\langle e^{ig\int \mathcal{L}_I} \rangle_0}, \tag{1.29}$$

is now a partition function over the distribution defined by $\mathcal{L}_0$.

There is a subtlety here. The Boltzmann factor e^{iS} is purely oscillatory, so the path integral does not converge as written. We may for now regulate this integral by adding an infinitesimal imaginary shift to the energy of each field $\tilde{\phi}(p)$, $p_0 \to p_0(1 - i\varepsilon)$, or equivalently by adding to the Lagrangian a term

$$\mathcal{L}_\varepsilon = i\varepsilon\phi^2 \tag{1.30}$$

for each field (or pair of adjoint fields) ϕ. Expectation values are then defined in the limit as $\varepsilon \to 0$:

$$\langle \mathcal{O} \rangle = \lim_{\varepsilon \to 0} Z^{-1}[0] \int \mathcal{D}\phi \; e^{i\int \mathcal{L} + i\int \mathcal{L}_\varepsilon} \mathcal{O}. \tag{1.31}$$

We will often ignore this entirely, keeping the factor of $i\varepsilon$ implicit. Indeed, when we pass to Euclidean space in Sec. 1.11 the limit will commute with the integral, and we may then remove it explicitly.

The form 1.29 begs a formal series expansion in g:

$$\langle e^{ig\int \mathcal{L}_I}\mathcal{O}\rangle_0 = \sum_{i=0}^{\infty}\frac{i^n}{n!}\cdot g^n\left\langle\left(\int \mathcal{L}_I\right)^n \mathcal{O}\right\rangle_0 . \tag{1.32}$$

Since the distribution here is a (regulated) multivariate Gaussian, we are free to invoke Isserlis' theorem [48]:

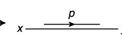
$$\langle \phi_1 \cdots \phi_{2n}\rangle_0 = \frac{1}{2^n n!}\sum_{\pi\in S_{2n}} s_\pi \langle \phi_{\pi(1)}\phi_{\pi(2)}\rangle_0 \cdots \langle \phi_{\pi(2n-1)}\phi_{\pi(2n)}\rangle_0, \tag{1.33}$$

where by S_{2n} we denote the symmetric group on $2n$ letters, and $s_\pi = \pm 1$ represents a symmetrization factor for potentially anticommuting fields. Each of these two-point pairings is called a Wick contraction after Gian Carlo Wick who introduced a similar construction[1] to physics within the canonical (operator) formalism. Note that we chose an even number of fields so that the correlator does not trivially vanish. The denominator of Eq. 1.25 is clearly the vacuum expectation value, which contributes only vacuum fluctuations without external states. Typically then, we will simply ignore this normalization and define the generating functional to produce only amplitudes with external fields. At each order in Eq. 1.32, the integrand is a sum over all allowed contractions of the fields. Each set of contractions is decomposed into a number of free propagators and some vertex factors coming from $\mathcal{L}_I$ with an integral for each vertex that preserves the locality of the interaction. The propagators for each species of particle are simply the Green functions for its equations of motion. Vertex factors are determined by taking the n-point functions at leading order and

[1] In physics, Isserlis' product-moment theorem is generally called Wick's probability theorem. The author of this thesis holds the opinion that this is a misnomer, since Wick's formulation relates time- and normal-ordered products of operator-valued fields, making no explicit reference to any Gaussian distribution. Of course, while the two approaches are equivalent (for quadratic free Lagrangians), the attribution to Isserlis is more appropriate in the functional formalism due to the probabilistic nature of the path integral.

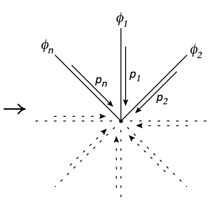
removing the external propagators. Together, the sets of values assigned to the propagators and fundamental vertices form the Feynman rules for the theory, which allow amplitudes to be built pictorially and calculated expediently. It is typically simplest to define them in momentum space, where the mathematical expressions are fairly uncomplicated. Roughly, propagators of momentum p are represented by oriented lines,

$$\langle\phi(x_1)\phi(x_2)\rangle \xrightarrow{\mathcal{F}} (2\pi)^4\delta^{(4)}(p_1+p_2)\langle\tilde{\phi}(p_2)\tilde{\phi}(p_1)\rangle \to \tag{1.34}$$



while vertices are given by an intersection of a number rays equal to the number of interacting fields:

$$\langle\phi(x_1)\cdots\phi(x_n)\rangle \xrightarrow{\mathcal{F}} (2\pi)^4\delta^{(4)}(p_1+\cdots+p_n)\langle\tilde{\phi}(p_1)\cdots\tilde{\phi}(p_n)\rangle \to \tag{1.35}$$



With some more rules ensuring the proper weight of each contribution, each term in Eq.1.33 may be written as a Feynman diagram, which is labeled according to the rules above. Feynman diagrams will be the primary tool for calculations in this thesis. In Chs. 2 and 3, we calculate the Feynman rules for QCD explicitly and detail methods of calculating expectation values through the use of Feynman diagrams.

In a perturbative series, closed loops appear when there are fewer external fields than interacting fields at any order in the coupling. Each increasing order has an extra factor of the interaction Lagrangian, which increases the number of loops. We then speak of these interchangeably; the first nonvanishing order is defined to be the zeroth order or the "tree level," while subsequent orders n are called n-loop corrections. Explicitly, if we describe some function f as a series in the coupling g beginning at order g^m, then we have

$$f = g^m \sum_{n=0}^{\infty} f^{(n)} g^n. \tag{1.36}$$

In this case $f^{(0)}$ is the tree-level contribution, and the other $f^{(n)}$ are n-loop corrections. This should clarify our later discussion of vacuum diagrams, which have no external states and therefore contain loops in their tree-level contributions.

1.5 Gauge Fixing

Suppose we want to calculate the propagation amplitude for a free vector boson A between two points x and y, that is, the Green function for the Yang-Mills equations of motion with zero coupling:

$$\langle A_\nu(y) A_\mu(x) \rangle = \int \mathcal{D}A \; A_\nu(y) A_\mu(x) e^{iS[A]}. \tag{1.37}$$

Since we are working in the free theory, where all interactions are quadratic, this integral should be exactly solvable by Gaussian integration (*v.i.*, Sec. 2.1.1). Transforming to momentum space with four-momentum q, the Lagrangian goes as[2]

$$\tilde{\mathcal{L}} \sim \int_q \tilde{A}^\nu (q^2 g_{\mu\nu} - q_\mu q_\nu) \tilde{A}^\mu, \tag{1.39}$$

[2]We use the shorthand notations

$$\int_x := \int d^d x \qquad \text{and} \qquad \int_p := \int \frac{d^d p}{(2\pi)^d} \tag{1.38}$$

for position and momentum integrals respectively, where the range of integration is the entire real line for each component of x or p in d-dimensions ($\mathbb{R}^d$). The context will always make clear whether we are integrating over position or momentum, and therefore when we should include the $(2\pi)^d$ normalization in the integral measure.

where the expression in parentheses has a null eigenvector q_μ, corresponding to the unphysical longitudinal polarization of A, making it singular. In order to extract a Green function, we must somehow remove this unphysical degree of freedom and fix the gauge. Unfortunately, removing removing the unphysical degrees of freedom destroys the gauge invariance and the unitarity of the path integral.

Faddeev and Popov solved this problem for generally nonabelian gauge theories by removing redundant gauge configurations in the functional integral [29]. Notice that the gauge invariance of the action partitions the symmetry group G into equivalence classes consisting of the (infinite) orbit of each configuration A. Each orbit contributes an infinite volume factor to the functional integral which represents the infinite number of physically indistinct configurations within it. Consequently, the integral measure overcounts each gauge orbit and is not normalizable by any finite volume. By restricting to a surface which intersects each gauge orbit once, we may restrict the domain of integration to the set of representatives of each orbit; in other words, we fix the gauge by imposing a constraint on the action in the form of a functional F such that

$$F[A] = 0. \tag{1.40}$$

Guaranteeing that the induced surface intersects each gauge orbit once is, however, impossible for the entire space of configurations, since a global section (global basis of coordinates) cannot be defined for nonabelian theories in general. This characterizes the Gribov ambiguity in choosing a representative for each orbit with a global choice of gauge [49]. We can circumvent this problem in perturbation theory, since the Dyson series is defined in the neighborhood of a specific classical vacuum and is thus strictly local. We will encounter the nonperturbative breakdown of this loophole in Ch. 4. Given $F[A]$ unambiguously, we may impose the gauge condition by integrating over the space of gauge transformations $g \in G$ of A:

$$\int \mathcal{D}g \; \delta(F[gAg^{-1}]) \det \delta_g F = 1, \tag{1.41}$$

which is inserted into the path integral:

$$\int \mathcal{D}A \; e^{iS} = \int \mathcal{D}[A,g] \; e^{iS} \delta(F) \det \delta_g F, \tag{1.42}$$

where gauge invariance allows us to ignore the gauge transformation within the delta function. The measure $\mathcal{D}[A,g]$ represents separate integrations over the equivalence classes of A and the orbits generated by g. The Jacobian determinant may be represented as a Gaussian path integral over some Grassmann-valued scalar fields $c = c^a t^a$ and $\bar{c} = \bar{c}^a t^a$ in the adjoint representation,

$$\det \delta_g F = \int \mathcal{D}[c, \bar{c}] \; \exp\left\{ T_F^{-1} \int \mathrm{Tr}\, \bar{c}(\delta_g F) c \right\}. \tag{1.43}$$

This can be treated as a phase if we rescale the exponent by a factor of i, producing a new contribution to the action:

$$\int \mathcal{D}A \; e^{iS} = \int \mathcal{D}[A, g, c, \bar{c}] \; e^{i(S + S_{FP})} \delta(F), \tag{1.44}$$

where

$$S_{FP} = \int \mathcal{L}_{FP}, \qquad \mathcal{L}_{FP} = T_F^{-1} \, \mathrm{Tr}\, \bar{c}(\delta_g F) c \tag{1.45}$$

is the Faddeev-Popov action. The rescaling by i generates an immaterial factor of $i^{\dim A}$; this may be absorbed into the functional measure, though it will in any case be cancelled by the normalization factor Z_0. The Faddeev-Popov action defines two virtual, anticommuting scalar fields $c, \bar{c}$, called ghosts and antighosts, which exactly cancel the unphysical polarizations. The typical generalization of the Lorenz gauge condition is expressed by the gauge-fixing function

$$F[A] = \partial_\mu A^\mu - \omega, \tag{1.46}$$

for a smooth function ω. In this case, the Jacobian assumes the form

$$\delta_g F = \partial_\mu D^\mu, \tag{1.47}$$

where the covariant derivative acts on the adjoint representation of G:

$$Dc = \partial c + [A, c]. \tag{1.48}$$

The arbitrary function ω may be removed from the path integral by integrating in ω the entire generating functional with a Gaussian weight, which scales the entire integral by some volume $1/N$:

$$\begin{aligned} \int \mathcal{D}A\ e^{iS} &= N \int \mathcal{D}\omega\ \exp\left\{\frac{i}{g^2\xi}\int \operatorname{Tr}\omega^2\right\}\int \mathcal{D}[A,g,c,\bar{c}]\ e^{i(S+S_{FP})}\delta(\partial_\mu A^\mu - \omega) \\ &= \int \mathcal{D}[A,g,c,\bar{c}]\ e^{i(S+S_{FP})}\exp\left\{\frac{i}{g^2\xi}\int \operatorname{Tr}\left(\partial_\mu A^\mu\right)^2\right\}. \end{aligned} \tag{1.49}$$

The resulting exponential contains another action defining the gauge-fixing Lagrangian,

$$\mathcal{L}_{gf} = \frac{1}{g^2\xi}\operatorname{Tr}\left(\partial_\mu A^\mu\right)^2. \tag{1.50}$$

Now, the gauge is determined by the choice of some positive real scalar ξ. Gauges with this choice of the function F are known as renormalizable-ξ (R_ξ) gauges. We can see that as ξ

tends to zero, finiteness of the action requires that $\partial_\mu A^\mu = 0$. This choice, called the Landau gauge condition, is equivalent to the classical Lorenz condition. The most common gauge — and the choice for all major results in this thesis — is the Feynman gauge, $\xi = 1$. The gauge-fixing Lagrangian contributes to the quadratic term in the gauge fields, so that now

$$\tilde{\mathcal{L}} \sim \int \tilde{A}^\nu \left[q^2 g_{\mu\nu} - \left(1 - \frac{1}{\xi}\right) q_\mu q_\nu \right] \tilde{A}^\mu, \tag{1.51}$$

which has an invertible kernel:

$$\langle \tilde{A}_\nu(-q) \tilde{A}_\mu(q) \rangle \sim \frac{1}{q^2} \left[g_{\mu\nu} - (1 - \xi) \frac{q_\mu q_\nu}{q^2} \right]. \tag{1.52}$$

We will generally work in the Feynman gauge, $\xi = 1$, since the gauge field propagator is particularly simple with this choice. Somehow, removing the redundant degrees of freedom also decouples the unphysical polarizations.

This may be explained through the notion of BRST (after Becchi, Rouet, Stora, and Tyutin) symmetry, which remains unbroken even after gauge fixing [50]. Notice that the gauge-fixing condition forces $\partial^2 \omega = 0$, which is precisely the form of the equations of motion for free ghosts.We might then imagine a gauge-like transformation generated by ghosts with some other constant Grassmann variable θ ensuring overall commutativity; this is the BRST transformation:

$$\phi \to \phi + \theta s \phi, \tag{1.53}$$

where s is called the Slavnov differential. To avoid using the equations of motion, we rearrange the Lagrangian in terms of an auxiliary Nakanishi-Lautrup field $B = B^a t^a$ [51, 52],

$$\mathcal{L}_{gf} = \frac{1}{g^2 T_F} \operatorname{Tr}\left\{B\ \partial_\mu A^\mu - \frac{\xi}{4T_F} B^2\right\}, \tag{1.54}$$

which may be integrated out of the functional integral since it does not propagate:

$$\frac{\xi}{2T_F} B - \partial_\mu A^\mu = 0 \Rightarrow \mathcal{L}_{gf} = \frac{1}{g^2 \xi} \operatorname{Tr}\left(\partial_\mu A^\mu\right)^2. \tag{1.55}$$

The Slavnov operator is defined so that the action on fermion and gauge fields goes as a gauge transformation (Eq. 1.5 and Eq. 1.7) generated by c with $\theta s\phi = \delta\phi$,

$$\delta\psi = -\theta c\psi, \qquad \delta\bar{\psi} = -\theta\bar{\psi}c, \qquad \delta A_\mu = \theta D_\mu c, \tag{1.56}$$

while ghosts transforms in such a way to maintain invariance of $D_\mu c^a$,

$$\delta c = -\theta c^2 = -\frac{1}{2}\theta f^{abc} t^a c^b c^c. \tag{1.57}$$

The antighosts cancel the variation of the gauge-fixing term,

$$\delta\bar{c} = \theta B, \tag{1.58}$$

and the auxiliary field is unchanged,

$$\delta B = 0. \tag{1.59}$$

With the help of the Jacobi identity, the full gauge-fixed gauge field Lagrangian is found to be invariant under the action of s. Moreover, the BRST transformation is nilpotent,

$$s^2\phi = 0, \tag{1.60}$$

for any field ϕ, so it determines a cohomology on the (pseudo-inner product) space of states. Since s generates a continuous symmetry, there is a conserved charge called the ghost number. Schematically, for the space Ω_n of states of ghost number n, the n^{th} BRST transformation maps

$$s : \Omega_n \to \Omega_{n+1}, \qquad |a, n\rangle \mapsto |b, n+1\rangle \tag{1.61}$$

Since it is nilpotent, we have $\mathcal{B}_n \subset \mathcal{Z}_n$, where

$$\mathcal{Z}_n = \text{Ker}\left\{s : \Omega_n \to \Omega_{n+1}\right\}, \qquad \mathcal{B}_n = \text{Im}\left\{s : \Omega_{n-1} \to \Omega_n\right\}, \tag{1.62}$$

and the space of states annihilated by s (closed states) is divided into equivalence classes determined by the n^{th} BRST cohomology group,

$$H^n_{BRST} = \frac{\mathcal{Z}_n}{\mathcal{B}_n}, \tag{1.63}$$

the space of BRST-invariant states (BRST-closed) which are not themselves BRST variations (BRST-exact). In the zero coupling limit, the variations above show that s converts antighosts to auxiliary fields (equivalent to gauge bosons polarized longitudinally, opposite the direction of propagation) and converts gauge bosons to ghosts, so that the longitudinal bosons and ghosts are exact states. There is an analogous anti-BRST cohomology generated by $s^\dagger$ that determines an equivalent condition on the antighost states. The space of states then has the Hodge decomposition

$$\Omega = \bigoplus_n \Omega_n = \mathcal{H}_\Delta \oplus \mathrm{Im}\, s \oplus \mathrm{Im}\, s^\dagger \tag{1.64}$$

where we denote the space of harmonic states by $\mathcal{H}_\Delta = \{\phi \in \Omega | \Delta\phi = 0\}$ with $\Delta = s^\dagger s + s s^\dagger$. By $\mathrm{Im}\, s$ we represent the direct sum of all $\mathcal{B}_n$ and analogously for $\mathrm{Im}\, s^\dagger$.

Kugo and Ojima showed that all physical states must reside in H^0_{BRST} [53], which is isomorphic to the space of harmonic forms by Hodge's theorem. Indeed, these must be closed states of positive-semidefinite norm. The states in the images of s and $s^\dagger$ have zero norm due to nilpotency, so that all nonzero-norm states must be harmonic. To ensure that the images of s and $s^\dagger$ exhaust all zero-norm states, it is sufficient to demand that the inner product is nondegenerate on the space of physical states. Then any closed state ϕ is uniquely determined by requiring that it is also co-closed, $s^\dagger \phi = 0$. That is, since $\phi = s^\dagger \psi$ for some state ψ, and $s s^\dagger \psi = 0$, the nondegeneracy of the inner product requires that ϕ itself must vanish, so that there are no nontrivial co-closed (zero-norm) states in $\mathcal{Z}_n$. In other words, the anti-BRST operator $s^\dagger$ fixes the gauge in the space of states in the cohomology of s so long as the norm is nondegenerate. It follows further that all harmonic states are either exclusively positive- or negative-definite. Assuming to the contrary that there are two harmonic states ϕ and ψ of respectively positive and negative norm, we may construct a homotopy $\chi_t = t\phi + (1-t)\psi$ so that for some t_0, the state χ_{t_0} has zero norm, and so must be an element of $\mathrm{Im}\, s$. Then the norm is definite on $\mathcal{H}_\Delta$, and we may simply impose the positivity of the norm with a suitable field redefinition. The harmonic space is then naturally associated with the unique physical states. We see that the addition of any exact or co-exact state to an element of the cohomology group has a vanishing contribution to physical matrix elements, which reflects a residual symmetry of these states under gauge transformations. Indeed, if we rescale the ghosts as $gc \to c$, the new terms in the gauge-fixed action may be rewritten as

$$S_{FV} + S_{gf} = \frac{1}{g^2 T_F} s \,\mathrm{Tr} \int \bar{c} \left(\partial_\mu A^\mu - \frac{\xi}{4T_F} B \right), \tag{1.65}$$

confirming that the entire Faddeev-Popov construction is BRST-exact, having no bearing on physical predictions.

The BRST construction demonstrates the relation between the redundant degrees of freedom in the path integral and the unphysical polarizations. All physically equivalent gauge configurations are related up to a (co-)exact form, which is associated with the unphysical states of the theory. Since the (co-)exact forms have norm zero, they vanish in inner products and do not affect correlation functions. The nondegeneracy of the inner product requires negative-norm states in the total Hilbert space, corresponding to the unphysical polarizations of the gauge fields. These are precisely compensated by the ghost fields, which carry the opposite sign due to their fermionic nature. Intuitively, since the Faddeev-Popov ghosts are Grassmann-valued, their contribution to the path integral in d-dimensions goes as $\det\left(\partial^2\right)^d$. On the other hand, free gauge bosons are complex fields following the same wave equation, so they contribute $\det\left(\partial^2\right)^{-d/2}$. In four dimensions, the ghosts cancel exactly two degrees of freedom, the two longitudinal polarizations. We will see this cancellation explicitly when we calculate the first radiative correction to the gluon propagator in Sec. 3.2.1.

1.6 Renormalization

In a free field theory, once the two-point correlation function is known, the theory is solved. One may determine the propagation amplitude for a single particle over any spacetime interval, which corresponds to an inner product in the Fock space of free eigenstates. As interactions are introduced, however, there are excitations for any number of particles, and the propagator loses its intuitive interpretation. The eigenspace of the interacting Hamiltonian is, per Haag's theorem [54], unitarily inequivalent to the free case, so that no isomorphism may be found between the free Fock space and interacting Hilbert spaces; in fact, the interacting space need not even be a Fock space. This section introduces renormalization, which circumvents the assumptions of Haag's theorem, and allows us to loosely construct a space of interacting multiparticle states. We may analyze the spectrum of the propagator in the interacting theory by inserting the completeness relation on the new Hilbert space. Since momentum is conserved, Hamiltonian eigenstates are simultaneously momentum eigenstates, and we may express the sum over states as a sum over zero-momentum eigenstates $|\lambda, 0\rangle$ integrated over all boosts $|\lambda, \mathbf{p}\rangle$ of the resting states:

$$\mathbb{1} = \sum_{\lambda} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega(\lambda, \mathbf{p})} |\lambda, \mathbf{p}\rangle\langle\lambda, \mathbf{p}|, \tag{1.66}$$

where $\omega(\lambda, \mathbf{p}) = \sqrt{\mathbf{p}^2 + m_\lambda^2}$ is the energy of the state $|\lambda, \mathbf{p}\rangle$ with physical mass m_λ. For simplicity, we specialize to ϕ^4 theory with the Lagrangian[3]

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \phi_0)(\partial^\mu \phi_0) - \frac{1}{2} m_0^2 \phi_0^2 - g_0 \phi_0^4. \tag{1.67}$$

[3] The subscript zeroes are written here for "bare" quantities in anticipation of later results.

This process will be treated with more care in the following chapter. For now, we simply quote the relevant results. The free propagator is given by[4]

$$\tilde{S}_0^{(0)}(p, m_0) = \frac{-i}{p^2 + m_0^2 - i\varepsilon}. \tag{1.68}$$

We now insert Eq. 1.66 into the time-ordered (causal) two-point function:

$$\begin{aligned} \langle \phi(y)\phi(x) \rangle_0 &= \langle 0 | \mathcal{T} \phi_0(y) \phi_0(x) | 0 \rangle \\ &= \sum_\lambda \int \frac{d^3 p}{(2\pi)^3} \frac{\theta(x_0 - y_0) e^{ip(x-y)} + \theta(y_0 - x_0) e^{ip(y-x)}}{2\omega(\lambda, \mathbf{p})} |\langle 0 | \phi_0(0) | \lambda, 0 \rangle|^2, \end{aligned} \tag{1.69}$$

giving us the exact propagator for mass m_λ,

$$\langle 0 | \mathcal{T} \phi_0(y) \phi_0(x) | 0 \rangle = \sum_\lambda S(x - y, m_\lambda) |\langle 0 | \phi_0(0) | \lambda, 0 \rangle|^2 = \int_0^\infty \frac{dM}{2\pi} \rho(M) S(x - y, M), \tag{1.70}$$

where

$$\begin{aligned} S(x - y, M) &= \int \frac{d^3 p}{(2\pi)^3} \frac{\theta(x_0 - y_0) e^{ip(x-y)} + \theta(y_0 - x_0) e^{ip(y-x)}}{2\omega(\lambda, \mathbf{p})} \\ &= \int \frac{d^4 p}{(2\pi)^4} \frac{-i}{p^2 + M^2 - i\varepsilon} e^{ipx} \end{aligned} \tag{1.71}$$

is the free propagator for a state of mass M. This is the Källén-Lehmann spectral representation [55, 56], with the spectral density

$$\rho(M) = \sum_\lambda 2\pi \delta(M - m_\lambda) |\langle 0 | \phi_0(0) | \lambda, 0 \rangle|^2 = 2\pi \delta(M - m_\phi) \cdot Z_\phi + \cdots, \tag{1.72}$$

[4] We have included the $i\varepsilon$ term discussed in Sec. 1.4 in order that integrals may be solved by Wick rotation. (See Sec. 1.11.)

where $Z_\phi = |\langle 0|\phi_0(0)|1,0\rangle|^2$ ($\lambda = 1$ is a shorthand for a one-particle state), m_ϕ is the mass of the single-particle state, and the truncated terms represent bound and multiparticle states[5]. Passing to momentum space, we may finally write

$$\langle 0|\mathcal{T}\tilde{\phi}_0(-p)\tilde{\phi}_0(p)|0\rangle = \frac{-iZ_\phi}{p^2 + m_\phi^2 - i\varepsilon} + \cdots. \tag{1.73}$$

Z_ϕ is the vacuum expectation value of a single particle state including all self-interactions. We may absorb it into the normalization of the fields by defining $\phi_0 = Z_\phi^{1/2}\phi$, so that

$$\langle 0|\mathcal{T}\tilde{\phi}(-p)\tilde{\phi}(p)|0\rangle = \frac{-i}{p^2 + m_\phi^2 - i\varepsilon} + \cdots, \tag{1.74}$$

Here, the renormalized field ϕ now accounts for all of the quantum fluctuations induced by interactions. The mass m_ϕ in Eq. 1.74 is the physical mass of a single-particle state in the interacting theory, since it is the eigenvalue of the squared momentum operator. This is to be contrasted with the parameter m_0 in the Lagrangian, which corresponds to the mass of the free theory. Evidently, in the presence of interactions, the pole of the propagator is shifted from $-m_0^2$ to $-m_\phi^2$. This is the essence of renormalization; the inclusion of interactions in a field theory perturbs physical, measurable quantities away from the bare parameters of the Lagrangian.

We may similarly define the renormalized mass and coupling through $Z_m m^2 = Z_\phi m_0^2$ and

[5]Bound states may contribute poles at masses larger than m_ϕ, while multiparticle states, which may have a continuum of masses greater than $2m_\phi$, form a branch cut from $2m_\phi$ to positive infinity. We have also quietly discarded the constant – typically vanishing – contribution from the ground state. In ϕ^4 theory, this term vanishes by Lorentz invariance

$Z_g g = Z_\phi^2 g_0$, so that the Lagrangian may be cleanly rewritten as

$$\mathcal{L} = -\frac{1}{2} Z_\phi (\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} Z_m m^2 \phi^2 - Z_g g \phi^4. \tag{1.75}$$

Since the free theory undergoes no renormalization due to quantum fluctuations, $Z = 1$ for all parameters in the bare Lagrangian (1.67). Then for each renormalization constant Z_i we have the decomposition

$$Z_i = 1 + \delta_i, \tag{1.76}$$

where $\delta_i = \mathcal{O}(g_0)$ vanishes as the coupling goes to zero. This allows us to write the renormalized Lagrangian:

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 - g\phi^4 - \frac{1}{2}\delta_\phi (\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}\delta_m m^2 \phi^2 - \delta_g g \phi^4. \tag{1.77}$$

The last three terms above are called counterterms. They produce vertices analogous to those of the bare Lagrangian, and they will prove useful in the next section.

At this point, there is a proliferation of unknowns in the Lagrangian. We must relate the bare, renormalized, and physical parameters. In general, in order to calculate the renormalized parameters, $\{a_i\}$, we need as many equations, called renormalization conditions, relating physical observables to calculable[6] functions $\{f_i\}$ of the bare parameters $\{a_i^0\}$:

$$a_i = f_i(a_1^0, a_2^0, ...). \tag{1.78}$$

The renormalization conditions reflect the shift in physical parameters due to the intro-

[6] These will often be implicit functions of the renormalized parameter; see below, Eqs 1.92 and 1.94.

duction of interactions to the Lagrangian. Since new interactions change the equations of motion, the solutions change correspondingly. Critically, renormalization depends on the scale at which we define the renormalization conditions. As we will discuss in the following section, loop integrals in quantum field theory typically require a regularization scheme to be made well-defined. This introduces a momentum scale related to the size of the regulator which must be fixed by the choice of renormalization conditions. Since the coupling in particular depends on this renormalization scale, the speed of (asypmtotic) convergence of the perturbation series is largely determined by the choice of scale. The best choice is usually that which minimizes the standard logarithmic corrections encountered in perturbation theory.

The spectral representation gives us a natural choice of conditions for the mass and field-strength renormalizations. In the example above, since the physical mass m_ϕ of the scalar field ϕ is defined as the location of the kinematical pole of the exact propagator, we can unambiguously define the renormalized mass m to be the physical mass, a convention known as on-shell renormalization. This fixes also the field-strength renormalization, which may be defined as i times the residue at the pole mass. Further renormalization conditions are required for any other dynamical fields and all couplings. In the case of an interaction of n fields with coupling g_n, one may fix the renormalized couplings by imposing

$$\langle \phi(x_1)...\phi(x_n) \rangle_{\text{scale}} = \langle \phi(x_1)...\phi(x_n) \rangle^{(0)}, \tag{1.79}$$

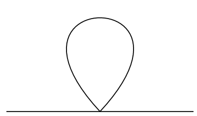
where the subscript "scale" represents some choice fixing the renormalization scale, often chosen in terms of the mass and n external momenta. This forces any loop corrections to cancel δ_{g_n} at each order, thereby defining Z_{g_n}. If we are able to choose a finite number of renormalization conditions which fix each normalization to all orders, the theory is said to be renormalizable.

Let us briefly return to the idea of BRST symmetry. Since it is unbroken after gauge fixing, we may expect that all loop corrections in perturbations theory remain BRST-invariant and that there are no symmetry-breaking counterterms. Then, given the most general BRST-invariant Lagrangian for some gauge theory, all possible counterterms will be encoded in the definitions of the renormalized fields, masses, and couplings. In the case that this Lagrangian contains only a finite number of terms, we have also a finite number of counterterms, and the theory is renormalizable.

1.7 Regularization

In the perturbative expansion, as the interaction Lagrangian is successively inserted into a correlation function, the increasing number of virtual contractions leads to closed loops of contracted vertices. In the momentum space representation, the spacetime integral associated with each insertion generates a delta distribution over the sum of all ingoing momenta, corresponding to a conservation of momentum at that vertex. In a loop with some n vertices, the momenta of any $n-1$ internal propagators fix the value of the n^{th} momentum, so that there is an overall integral over the total loop momenta. For example, in ϕ^4 theory the first loop correction appears at $\mathcal{O}(g_0)$:

$$g_0\tilde{S}_0^{(1)}(p) =$$



$$= ig_0\frac{-i}{p^2+m_0^2-i\varepsilon}\left[-\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\frac{-i}{k^2+m_0^2-i\varepsilon}\right]\frac{-i}{p^2+m_0^2-i\varepsilon}, \tag{1.80}$$

where $\tilde{S}_0^{(1)}$ is the next-to-leading order (NLO) contribution to the perturbation expansion of the bare propagator:

$$\tilde{S}_0 = \sum_{n=0}^{\infty} g_0^n \tilde{S}_0^{(n)}. \tag{1.81}$$

The integral above diverges quadratically as the loop momenta becomes infinite, as may be seen by transforming to Euclidean (*v.i.*, Sec. 1.11) spherical coordinates so that the magnitude of the loop momentum is explicit:

$$\int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2 + m_0^2 - i\varepsilon} = \lim_{\Lambda\to\infty} \frac{1}{8\pi^2} \int_0^{\Lambda} dk \frac{k^3}{k^2 + m_0^2} = \lim_{\Lambda\to\infty} \frac{m_0^2}{(4\pi)^2} \left[\frac{\Lambda^2}{m_0^2} - \log\left(1 + \frac{\Lambda^2}{m_0^2} \right) \right]. \tag{1.82}$$

This form of the integral, though formally infinite, demonstrates the effect of a hard cutoff on the loop momentum. If we drop the limit above, we may interpret our theory to be only well-defined in the infrared region $0 \le k^2 \le \Lambda^2$ up to some threshold scale Λ. This is an elementary example of regularization, where a divergent integral is recast as the limit of some divergent sequence of finite integrals. Bare quantities are thus defined with a particular regularization which is reflected in the explicit form of the Z-factors. On the other hand, the renormalization conditions relate the renormalized parameters to measurable quantities, which may not depend on the choice of regulator. Renormalized quantities are thus defined in the limit which removes the regulator, and the renormalization constants are defined such that the limit exists.

The cutoff regularization above breaks gauge symmetry, so it is of limited practical use. Instead, divergent Feynman integrals are typically treated in dimensional regularization, wherein the spacetime dimension is analytically continued away from four dimensions to a generically complex value d. In order to calculate integrals in d dimensions, we must transform to a coordinate system where some number of dimensions may be integrated directly by symmetry. The easiest and most common practice is to find some parametrization such that the integrand depends solely on the magnitude of the loop momentum. In this case, the integrand is made spherically-symmetric, and the $(d-1)$-dimensional solid angle integral may be read off, leaving only the radial integral:

$$\int \frac{d^dk}{(2\pi)^d} f(k^2) = \frac{1}{(2\pi)^d}\int d\Omega_{d-1}\int_0^\infty dk\ k^{d-1} f(k^2) = 2\frac{(4\pi)^{2-d/2}}{\Gamma(d/2)}\int_0^\infty dk\ k^{d-1} f(k^2). \tag{1.83}$$

Clearly, now the degree of divergence of the integrand is dependent on the dimension, which is customarily written as $d = 4 - 2\epsilon$, where 2ϵ is the radius of some neighborhood of the physical value of $d = 4$. After the momentum integral is calculated, the result is expanded in a Laurent series about $\epsilon = 0$. Terms which go as inverse powers of ϵ represent the same divergences as in cutoff regularization. In the example above, the dimensionally-regularized integral is

$$-\frac{1}{2} i g_0 \int \frac{d^dk}{(2\pi)^d}\frac{-i}{k^2 + m_0^2 - i\varepsilon} = \frac{1}{2} i g_0 \frac{m_0^2}{(4\pi)^2}\left[\frac{1}{\epsilon} + \log\left(\frac{4\pi}{m_0^2}\right) - \gamma_E + 1 + \mathcal{O}(\epsilon)\right], \tag{1.84}$$

where $\gamma_E = 0.57721...$ is the Euler-Mascheroni constant. Since the integration measure is now d dimensional, we account for the deviation by changing the canonical dimension of the bare coupling to $[g_0] = 4 - d$ through definition of the renormalized coupling:

$$g_0 = \mu^{2\epsilon} Z_g g, \tag{1.85}$$

where μ is some momentum scale. After renormalization, this compensates for the awkward dimensionful logarithm in Eq. 1.84, since the $\mu^{2\epsilon} = 1 + \epsilon \log \mu^2 + \mathcal{O}(\epsilon^2)$, and each logarithm is paired with a $1/\epsilon$ pole, giving an overall contribution of $\log\left(4\pi\mu^2/m_0^2\right)$. One drawback of dimensional regularization is that the spacetime algebra generated must too be generalized to d-dimensions, which is typically a nontrivial procedure, especially in the case of parity-violating interactions.

Regardless of the chosen regulator, we may now write the NLO propagator:

$$\begin{aligned}\tilde{S}_0(p) &= \tilde{S}_0^{(0)}(p) + g_0\tilde{S}_0^{(1)}(p) + \mathcal{O}(g_0^2) \\ &= \frac{-i}{p^2+m_0^2-i\epsilon} + g_0\frac{-i}{p^2+m_0^2-i\epsilon}i\Sigma^{(1)}\frac{-i}{p^2+m_0^2-i\epsilon} + O(g_0^2),\end{aligned} \tag{1.86}$$

where $\Sigma^{(1)}$ represents an appropriately regularized version of the bracketed expression in Eq. 1.80. Continuing to higher orders, this sum easily seen to be a geometric series, where $i\Sigma^{(n)}$ represents the one-particle-irreducible (1PI) correlation function of order n; *viz.*, the sum of all Feynman diagrams at $\mathcal{O}(g_0^n)$ whose graphs contain no bridges. Chaining these together by tree-level bridges gives us the series

$$\begin{aligned}\tilde{S}_0(p) &= \frac{-i}{p^2+m_0^2-i\varepsilon}\sum_{n=0}^{\infty}\left(i\Sigma\frac{-i}{p^2+m_0^2-i\varepsilon}\right)^n \\ &= \frac{-i}{p^2+m_0^2-\Sigma-i\varepsilon} \\ &= \frac{-i}{p^2+m_0^2-g_0\Sigma^{(1)}-i\varepsilon+\mathcal{O}(g_0^2)}.\end{aligned} \tag{1.87}$$

In both regularization schemes above, $\Sigma^{(1)} \propto m_0^2$, so that it can be absorbed into some shift in the mass given by $\Sigma = -m_0^2\Sigma_m + \mathcal{O}(g_0^2)$:

$$\tilde{S}_0(p) = \frac{-i}{p^2+m_0^2\left(1+\Sigma_m^{(1)}\right)-i\varepsilon+\mathcal{O}(g_0^2)}. \tag{1.88}$$

In dimensional regularization, we find

$$\Sigma_m = -\frac{1}{2}\frac{g_0}{(4\pi)^2}\left[\frac{1}{\epsilon} + \log\left(\frac{4\pi}{m_0^2}\right) - \gamma_E + 1 + \mathcal{O}(\epsilon)\right] + \mathcal{O}(g_0). \tag{1.89}$$

This suggests that the mass m_0 in the Lagrangian and the mass satisfying the Klein-Gordon equation of motion differ in the interacting theory. The infinite shift from m_0 in the physical mass represents the energy of the particle interacting with its own field, the self-energy of ϕ_0, which receives contributions from infinitely many loop corrections. Since the measured value is obviously finite, the "bare" mass m_0 must compensate for the infinite shift. We reinterpret the bare mass as the (potentially infinite) rest mass of a single-particle excitation of ϕ_0, dressed with all virtual self-energy interactions.

At the next order in the coupling, there are momentum-dependent terms that diverge as the regulator is removed, which we include by writing $\Sigma = -m_0^2\Sigma_m - p^2\Sigma_p$. Resumming these contributions and replacing the bare parameters with their renormalized counterparts, the exact renormalized two-point function may be written as

$$\tilde{S}(p) = \frac{-i}{p^2 Z_\phi(1+\Sigma_p) + m^2 Z_m(1+\Sigma_m) - i\varepsilon} = \frac{-i}{p^2 + m_\phi^2 - i\varepsilon}, \tag{1.90}$$

where the second equivalence enforces the Källén-Lehmann representation. We immediately see that both $Z_\phi(1+\Sigma_p)$ and $Z_m(1+\Sigma_m)$ must be finite as we remove the regulator and that the pole mass is given by

$$m_\phi^2 = \lim_{\epsilon\to 0} m^2 \frac{Z_m}{Z_\psi}\frac{1+\Sigma_m}{1+\Sigma_p}. \tag{1.91}$$

In the on-shell scheme, the condition $m = m_\phi$ fixes both the mass and field renormalization constants. In terms of the bare propagator, this means that we encounter a pole at $p^2 = -m^2$, which translates to

$$\Sigma\big|_{p^2=-m^2} = 0. \tag{1.92}$$

Further, the residue must be $-iZ_\phi$. This means that

$$-iZ_\phi = \lim_{p^2\to -m^2}(p^2+m^2)\frac{-iZ_\phi}{p^2+m^2-\Sigma-i\varepsilon} = -iZ_\phi \cdot \lim_{p^2\to -m^2}\left(1-\frac{d\Sigma}{dp^2}\right)^{-1}, \tag{1.93}$$

so that the field renormalization is fixed by

$$\left.\frac{d\Sigma}{dp^2}\right|_{p^2=-m^2} = 0. \tag{1.94}$$

We may then simply read off $Z_m^{-1} = 1+\Sigma_m$ and $Z_\phi^{-1} = 1+\Sigma_p$.

The physical and renormalized masses, however, do not coincide in general, and the renormalization condition may not be so easily expressed. An important example of this type is the minimal subtraction (MS) scheme in dimensional regularization. Since we require only that renormalized correlation functions remain finite as the regulator is removed, we are free to choose how much of the finite part of any bare quantity is left in the renormalized correlator and how much enters the renormalization constants. In dimensional regularization, each loop can contribute a pole of the form $1/\epsilon$. At some loop-order n, there are potentially as many overlapping divergences which occur when all vertices coincide at some spacetime point, contributing a factor of $1/\epsilon^n$. Any subset of these vertices may also overlap, so that at order n, each renormalization constant is of the form

$$Z^{(n)} = \sum_{k=0}^{n}\frac{a_{n,k}}{\epsilon^k}. \tag{1.95}$$

For $k > 0$, the coefficients $a_{n,k}$ are fixed, but we are free to fix all $a_{n,0}$ with the renormalization conditions. The simplest choice is of course that $a_{n,0} = 0$ for some scale μ, which is the MS prescription. We can slightly improve the convergence of the perturbation expansion by shifting the renormalization scale. In dimensional regularization, each pole carries with it the finite bit $\log(4\pi) - \gamma_E$, just as in Eqs. 1.84 and 1.89. The term $\log(4\pi)$ comes from the momentum integral measure and the angular integral, which contribute a factor of $(4\pi)^{2-d/2}$. The term γ_E comes from the radial integral, which may be expressed as a gamma function with a pole at $d = 4$. After renormalizing the coupling, we can absorb these terms into the renormalization scale by shifting

$$\mu^2 \to \bar{\mu}^2 = 4\pi e^{-\gamma_E} \mu^2. \tag{1.96}$$

This is the modified minimal subtraction ($\overline{\text{MS}}$) scheme. In the case of the scalar field, we may completely define the renormalized mass and field by requiring that $a_{n,0} = 0$ at the scale $\bar{\mu}$. From Eq. 1.89, we see that

$$Z_m = 1 + \frac{g}{(4\pi)^2} \frac{1}{2\epsilon} + \mathcal{O}(g^2), \tag{1.97}$$

which defines the physical mass:

$$m_\phi^2 = \lim_{\epsilon \to 0} m^2 \frac{Z_m}{Z_\psi} \frac{1 + \Sigma_m}{1 + \Sigma_p} = m^2 \left\{ 1 - \frac{g}{(4\pi)^2} \cdot \frac{1}{2} \left[\log\left(\frac{\bar{\mu}^2}{m^2} \right) + 1 \right] + \mathcal{O}(g) \right\}, \tag{1.98}$$

since the denominator is $1 + \mathcal{O}(g^2)$. This of course depends on the value of $\bar{\mu}$, but we may choose this on a case-by-case basis in order that the leading logarithms vanish. In the present case, this means choosing $\bar{\mu} = m$.

The divergences encountered in this section may be understood as artifacts of some very high-energy phenomena which are irresolvable near the energy at which the action was defined. This is the focus of Sec. 1.8.

1.8 Effective Field Theory

Much of the formalism regarding renormalization and regularization was largely considered unphysical and *ad hoc* until the introduction of the Wilsonian or effective picture of quantum field theory [57–63]. From this perspective, a theory is defined only up to some scale Λ, above which the precise physics is unknown. Specifically, the cutoff is placed on the range of integration in the generating functional, where the measure is decomposed into a product measures for each Fourier mode:

$$\mathcal{D}\tilde{\phi} = \prod_k d\tilde{\phi}(k), \tag{1.99}$$

with $k^2 \leq \Lambda^2$. However, since the Minkowski metric is indefinite, we must rotate to Euclidean space in order that k^2 be bounded below, deferring once again the details of the Wick rotation to Sec. 1.11. Returning to the prototypical case of ϕ^4 theory, we define the Euclidean generating functional,

$$Z = Z_0 \int \mathcal{D}\tilde{\phi}\ e^{-\int \tilde{\mathcal{L}}[\tilde{\phi}]}, \tag{1.100}$$

with the bare Lagrangian in Euclidean momentum space,

$$\tilde{\mathcal{L}} = \frac{1}{2}\tilde{\phi}(k^2 + m^2)\tilde{\phi} + g\tilde{\phi}^4, \tag{1.101}$$

where we have dropped the subscripts denoting bare quantities for simplicity. The measure is decomposed as above with the restriction that $0 \leq k^2 \leq \Lambda^2$. This form allows us to partition the range of integration to isolate arbitrary bands of energy within the functional integral. Fixing some scaling parameter $x \in (0,1)$, we restrict our attention to the momentum shell $x\Lambda^2 \le k^2 \le \Lambda^2$ and split the momentum-space fields into UV (+) and IR (−) components:

$$\tilde{\phi}(k) = \theta(x\Lambda^2 - k^2)\tilde{\phi}_-(k) + \theta(k^2 - x\Lambda^2)\tilde{\phi}_+(k), \tag{1.102}$$

where $\theta(x)$ is the Heaviside step function (the discontinuity needn't be defined; any convention may be absorbed into the inequalities defining the range of k without loss of generality). Dropping the step functions for cleanliness of notation and plugging this into the Lagrangian, we find that

$$\tilde{\mathcal{L}}[\tilde{\phi}] = \tilde{\mathcal{L}}[\tilde{\phi}_-] + \tilde{\mathcal{L}}[\tilde{\phi}_+] + 4g\tilde{\phi}_-^3\tilde{\phi}_+ + 6g\tilde{\phi}_-^2\tilde{\phi}_+^2 + 4g\tilde{\phi}_-\tilde{\phi}_+^3, \tag{1.103}$$

and the generating functional may be rewritten

$$Z = Z_0 \int \mathcal{D}\tilde{\phi}_- \; e^{-\int \tilde{\mathcal{L}}[\tilde{\phi}_-]} \int \mathcal{D}\tilde{\phi}_+ \; e^{-\int \tilde{\mathcal{L}}[\tilde{\phi}_+]} \exp\left\{-\int \left[4g\tilde{\phi}_-^3\tilde{\phi}_+ + 6g\tilde{\phi}_-^2\tilde{\phi}_+^2 + 4g\tilde{\phi}_-\tilde{\phi}_+^3\right]\right\}. \tag{1.104}$$

The first two exponentials define free field theories for the ultraviolet and infrared modes separately, so there is a propagator for each set of fields (±) that vanishes outside the associated range of momenta. We can thus proceed as in standard perturbation theory with restrictions on the ranges of loop momenta. Notice that, since the integral of any smooth function of k is well-defined on the upper band $x\Lambda^2 \le k^2 \le \Lambda^2$, all loops generated by the UV modes are strictly finite. For example, the $\tilde{\phi}_-^2\tilde{\phi}_+^2$ interaction simply represents a finite shift in the mass of the IR modes, since the $\tilde{\phi}_+$ fields may be contracted to form a finite loop, leaving a smooth momentum-dependent function times $\tilde{\phi}_-^2$. Combining the other vertices through similar contractions, we may construct diagrams with any number n of external $\tilde{\phi}_-$

fields[7]. In each case, the $\tilde{\phi}_+$ integrals may again be reduced to smooth, finite functions f_n, so that we have an infinite series of terms of the form $f_n(k)\phi_-^n$. The functions f_n may be expanded in a Taylor series, which allows us to rearrange each of the new n-point interactions as another infinite series of operators, containing n fields and arbitrarily many spacetime derivatives and with constant coefficients. Continuing this process to all orders in the coupling, we effectively solve the path integral in the range $x\Lambda^2 \leq k^2 \leq \Lambda^2$, a process known as integrating-out the UV modes. We have added in turn an infinite tower of operators to the Lagrangian, consisting of all possible local products of $\tilde{\phi}_-$, its derivatives, and its mass:

$$\tilde{\mathcal{L}} \to \tilde{\mathcal{L}}_{\text{eff}} = \tilde{\mathcal{L}}[\tilde{\phi}_-] + \sum_{n_1,n_2,n_3} C_{n_1 n_2 n_3} m^{n_1} \partial^{2n_2} \tilde{\phi}_-^{n_3}. \tag{1.105}$$

Though we have removed the explicit dependence on $\tilde{\phi}_+$, the low-energy effective Lagrangian $\mathcal{L}_{\text{eff}}$ retains some artifacts of the ultraviolet degrees of freedom, the Wilson coefficients $C_{n_1 n_2 n_3}$ and the effective operators. The resulting generating functional defines a low-energy effective field theory (EFT) which approximates the original theory below some scale $\Lambda_0 = x\Lambda$.

Notice that, for generic dimension d, the fields ϕ have canonical dimension $d/2 - 1$, which restricts the dimension of the coefficients to

$$[C_{n_1 n_2 n_3}] = d - n_3\left(\frac{d}{2} - 1\right) - 2n_2 - n_1. \tag{1.106}$$

The only intrinsic scale in the defining loop integrals of each Wilson coefficient is the cutoff Λ, which means $C_{n_1 n_2 n_3} \sim \Lambda^{d - n_3\left(\frac{d}{2} - 1\right) - 2n_2 - n_1}$. Then all operators of dimension greater

[7]Barring a few exceptions, such as vertices containing one or three powers of $\tilde{\phi}_-$, since they cannot be constructed by contracting only $\tilde{\phi}_+$ fields.

than d have coefficients which go as inverse powers of the cutoff, and may thereby be ignored for a sufficiently large cutoff. These are called irrelevant operators. Near $d = 4$, the only combinations of n_i for which the Wilson coefficients have nonnegative mass dimension are

$$\begin{aligned} (n_1, n_2, n_3) = (0,0,1):&\quad \tilde{\phi}_- & (1,0,2):&\quad m\tilde{\phi}_-^2 \\ (0,0,2):&\quad \tilde{\phi}_-^2 & (1,0,3):&\quad m\tilde{\phi}_-^3 \\ (0,0,3):&\quad \tilde{\phi}_-^3 & (1,1,1):&\quad m\partial^2\tilde{\phi}_- \\ (0,0,4):&\quad \tilde{\phi}_-^4 & (2,0,1):&\quad m^2\tilde{\phi}_- \\ (0,1,1):&\quad \partial^2\tilde{\phi}_- & (2,0,2):&\quad m^2\tilde{\phi}_-^2 \\ (0,1,2):&\quad \partial^2\tilde{\phi}_-^2 & (3,0,1):&\quad m^3\tilde{\phi}_-. \\ (1,0,1):&\quad m\tilde{\phi}_- \end{aligned} \tag{1.107}$$

All of the above operators with odd powers of $\tilde{\phi}_-$ are impossible to construct with the available vertices. The operator $m\tilde{\phi}_-^2$ is in principle allowed, but it can be absorbed into the definitions of m and Λ. This leaves only four operators:

$$\mathcal{L}_{\text{eff}} = \mathcal{L}[\tilde{\phi}_-] + c_1\partial^2\tilde{\phi}_-^2 + c_2 m^2\tilde{\phi}_-^2 + c_3\tilde{\phi}_-^4 + c_4\tilde{\phi}_-^2 + \cdots, \tag{1.108}$$

where the ellipsis represents all higher-dimensional operators. Aside from the last term, all of the new operators are already present in the Lagrangian, so their reappearance signifies a shift in the corresponding coupling of each term due to fluctuations in the UV band. The final term is similarly innocuous, representing another shift in the mass. In fact, we could have anticipated its presence from the quadratic pole in Eq. 1.82, since the Wilson coefficient c_4 goes as Λ^2 by virtue of Eq. 1.106. We then observe that all of the new operators are either irrelevant or may be absorbed into the bare parameters of the original Lagrangian; that is to say, the ultraviolet modes renormalize the bare parameters in the original Lagrangian.

Of course, we are still left with an infinite number of higher-dimensional operators with Wilson coefficients suppressed by the inverse powers of the cutoff. Since these couplings have negative mass dimension, they are rendered nonrenormalizable by simple power-counting arguments. The only choice to restore renormalizability is to send the cutoff to infinity. While this removes the nonrenormalizable interactions, it also forces the relevant Wilson coefficients — those which renormalize the bare Lagrangian — to diverge. These are exactly the infinite shifts we found in our naïve approach to renormalization. It is then natural that we encounter divergences in loop integrals; we must assume that there is some intrinsic granularity to our observations, some scale at which our experiments cannot resolve the ultralocal, high-energy dynamics. Ignoring this scale is tantamount to introducing local products of quantum fields that generate local divergences. Fortunately, as we have previously seen, it is often possible to absorb these divergences into the definitions of our bare and physical Lagrangians with a suitable scheme for regularization and renormalization.

The arguments presented here for the case of scalar field theory are simple to generalize to more robust theories. Indeed, for any Lagrangian of arbitrarily many species of interacting fields, integrating out the ultraviolet modes generates both a shift in the bare parameters of the base theory and an infinite tower of local products of fields, masses, and derivatives. In the general case, the only restriction on the higher-dimensional operators is that they reflect the symmetries of the original Lagrangian. Aside from that, the inclusion of all other operators is perfectly allowed — and in fact compulsory.

1.9 The Renormalization Group

The Wilsonian picture of quantum field theory presents renormalization as a shift in the low-energy parameters of a theory due to ultraviolet fluctuations within some momentum shell $x\Lambda^2 \le k^2 \le \Lambda^2$. The parameter x describes a smooth dependence on the size of this range. Treating $1 - x$ as an infinitesimal deviation from the sharp cutoff Λ, we define a continuous transformation from a theory defined below the cutoff Λ to a theory defined below $x\Lambda$. We may iterate this procedure for infinitely many of these transformations, which defines a composition law. This gives us an algebraically-closed set of smooth transformations with an identity element ($x = 1$), known as the renormalization group[8] (RG). We may view the process of successively removing UV modes as following a trajectory in the space of Lagrangian field theories: the initial conditions are defined by the cutoff and the base Lagrangian, and the trajectory is described by the Wilson coefficients. The parameter x determines the distance traveled along this path.

From this perspective, the space of all quantum field theories is simply the ring of local polynomials in the field variables and their derivatives, the fundamental degrees of freedom in QFT. Any theory is thus defined given a cutoff scale and a set of initial couplings. In the case that all couplings vanish[9], we recover a free field theory. The only constructible interactions are thus quadratic, so there is no contact between the IR and UV fields. Integrating out the UV modes then contributes an overall constant to the path integral, which may be absorbed into the overall normalization. This means that all couplings are unaffected under RG flows; a free theory remains free and requires no renormalization.

[8] It is in fact not a group but a semigroup, since integrating out a momentum shell is not invertible.

[9] Except the for kinetic terms: for any Lagrangian with n terms, there are $n-1$ degrees of freedom involved in its normalization. We retain the kinetic term to preserve dynamics and avoid a trivial theory.

The nescience of the free theory to Wilsonian renormalization characterizes a fixed point in the space of all theories, a point invariant under RG flows. Since the action in strictly quadratic, this is known as the Gaussian fixed point. Perturbation theory is then simply a study of the theories within an infinitesimal neighborhood of the Gaussian fixed point. If the process of integrating out UV modes drives us toward the fixed point — that is, if the Wilson coefficients negate the initial couplings at some energy scale — then the theory is said to be asymptotically free. This is the case in quantum chromodynamics, where the strong coupling constant vanishes at a sufficiently large energy.

The applicability of perturbation theory rests on size of the coupling, which is determined by the Wilson coefficients at some fixed x. This is equivalent to renormalizing the bare theory at the scale $x\Lambda$. On the other hand, like the renormalization scale, the scaling parameter is completely unphysical, so it cannot influence physical predictions. We may impose this consistency condition by mandating that for any renormalized function $\Gamma = Z^{-1}\Gamma_0$,

$$\frac{d\Gamma}{dx} = 0. \tag{1.109}$$

In perturbation theory with renormalization scale μ, this is equivalent to

$$\mu\frac{d\Gamma}{d\mu} = 0. \tag{1.110}$$

While the renormalized correlation function is itself scale-invariant, it is a function of renormalization constants and couplings, both of which depend on the renormalization scale. We can expand Eq. 1.110 to account for this dependence explicitly:

$$\begin{aligned}\mu\frac{d\Gamma}{d\mu} &= \mu\frac{\partial\Gamma}{\partial\mu} + \mu\frac{\partial g}{\partial\mu}\frac{\partial\Gamma}{\partial g} + \mu\frac{\partial m}{\partial\mu}\frac{\partial\Gamma}{\partial m} + \mu\frac{\partial Z^{-1}}{\partial\mu}\frac{\partial\Gamma}{\partial Z^{-1}} \\ &= \mu\frac{\partial\Gamma}{\partial\mu} + \beta\cdot\frac{\partial\Gamma}{\partial g} + \gamma_m\cdot m\frac{\partial\Gamma}{\partial m} - \gamma_Z\cdot Z^{-1}\frac{\partial\Gamma}{\partial Z^{-1}} \\ &= 0.\end{aligned} \tag{1.111}$$

In the second line, we introduced the beta function

$$\beta(g) = \mu\frac{\partial g}{\partial\mu} \tag{1.112}$$

and the anomalous dimensions

$$\gamma_m(g) = \mu\frac{\partial\log m}{\partial\mu}, \qquad \gamma_Z(g) = \mu\frac{\partial\log Z}{\partial\mu}. \tag{1.113}$$

The terms $Z^{-1}\frac{\partial\Gamma}{\partial Z^{-1}}$ and $m\frac{\partial\Gamma}{\partial m}$ simply count the number of fields and masses present in Γ. Then for n_Z fields with n_m mass insertions, we can write the Callan-Symanzik equation for Γ:

$$\left[\mu\frac{\partial}{\partial\mu} + \beta\cdot\frac{\partial}{\partial g} + n_m\gamma_m - \frac{1}{2}n_Z\gamma_Z\right]\Gamma = 0, \tag{1.114}$$

which explicitly encodes the scaling behavior of each parameter in a renormalized quantity. In general, there will be additional terms containing a beta function for each coupling and anomalous dimensions for each field and each mass. The beta function is particularly important, because it tells us how far away from the Gaussian fixed point we are allowed to flow before the perturbation series is invalidated by large values of the coupling.

The anomalous dimension and beta function allow us to connect predictions made at disparate energy scales. Given some quantity renormalized at a scale μ_0, we can flow to any other scale μ by direct integration of the related RG functions. The beta function gives us a separable differential equation which implicitly defines the coupling constants:

$$\frac{\mu^2}{\mu_0^2} = \exp\left\{2\int_{g(\mu_0)}^{g(\mu)} \frac{dg}{\beta(g)}\right\}. \tag{1.115}$$

The anomalous dimensions give us similar equations for the Z-factors. In the case of the mass, for example, we find

$$\frac{m^2(\mu)}{m^2(\mu_0)} = \exp\left\{2\int_{g(\mu_0)}^{g(\mu)} \frac{\gamma_m(g)}{\beta(g)}dg\right\}. \tag{1.116}$$

Using the Callan-Symanzik equation to extract the RG functions perturbatively, we can solve these integrals to arbitrary order in the coupling. The typical expansions in the coupling are[10]

$$\beta(g) = -g^3 \sum_{n=0}^{\infty} b_n g^{2n}, \tag{1.117}$$

and

$$\gamma_i(g) = -g^2 \sum_{n=0}^{\infty} \gamma_i^{(n)} g^{2n}. \tag{1.118}$$

In Sec. 3.3, we will calculate b_0 and $\gamma_m^{(0)}$ in quantum chromodynamics.

1.10 Operator Mixing

As we saw in Sec. 1.8, the removal of some high energy modes of a theory introduces an infinite series of effective interactions to the Lagrangian. It is often the case that the precise nature of physics is unknown at high energies, so that the best we can do is consider the

[10] The coupling constant typically only appears in even powers in physical theories.

most general effective Lagrangian consistent with the desired symmetries. Typically, then, we are left with some low-energy, renormalizable Lagrangian $\mathcal{L}$ with couplings g_i and a tower of higher-dimensional effective operators encoding the effects of the high energy modes we removed. Analogously to Eq. 1.105, we may fix some UV scale $\Lambda_0 = x\Lambda$ above which all degrees of freedom shall be integrated out, so that the effective Lagrangian may be written

$$\mathcal{L}_{eff} = \mathcal{L} + \sum_i C_i(\Lambda_0)\mathcal{O}_i \tag{1.119}$$

with effective interactions $\mathcal{O}_i$ and couplings (Wilson coefficients) C_i. In the sum above, the index i runs over a basis for the ring of polynomials in the field variables. This can be arranged so that i represents all operators of a given engineering dimension, and the sum runs over all dimension-n operators and all dimensions n. Then for any n, the associated operators and Wilson coefficients have consistent dimensions. Just as the case of the scalar field theory, where we observed a shift in the mass due to the presence of a ϕ^4 interaction, the bare correlators of the effective theory are sensitive to contributions from all high-energy operators, and the Wilson coefficients will shift according to

$$C_i \to C_i + \delta C_i, \tag{1.120}$$

where the variation has the general dependence

$$\delta C_i = f(g_1, g_2, ..., C_1, C_2, ...). \tag{1.121}$$

for some polynomial f. Since there is an infinite number of Wilson coefficients, there is also an infinite number of possible counterterms, and the theory is formally nonrenormalizable.

At any order in the perturbative expansion, however, the number of counterterms is finite, so we can absorb all corrections into the bare parameters. We are thus able to define an effective renormalization scheme valid to any finite order in the coupling. The interdependence of the Wilson coefficients characterizes the mixing under renormalization of operators in an EFT.

As the cutoff removed, the low-energy correlators diverge, but they may be calculated within a suitable regularization scheme, and — critically — they depend the scaling parameter x. After renormalization at some scale μ, this dependence is dimensionally transmuted to the variable μ^2/Λ_0^2. Further, the effective operators vanish from the total Lagrangian, since they couple with inverse powers of the cutoff. Of course, we may still like to consider the effects of these operators from a low-energy perspective by inserting effective interactions into correlation functions of an interacting theory defined solely by the low-energy Lagrangian, $\mathcal{L}$. Since the cutoff scale is removed, we must choose a scheme to regulate the Feynman integrals, conventionally chosen to be dimensional-regularization. In principle, the operators $\mathcal{O}_i$ remain sensitive to the other effective interactions, so that the corrections to any bare operator may be expressed in a series over renormalized operators:

$$\mathcal{O}_0 = \sum_i Z_i \mathcal{O}_i, \tag{1.122}$$

with some divergent renormalization constants Z_i that capture the corrections to $\mathcal{O}_0$ due to the renormalized operator O_i.

The divergent mixing of the effective operator basis is the result of many fields localizing at a single point. Formally, in order to maintain unitarity of the generating functional, the inner product on the space of states must be normalizable. This can be ensured by treating each state as a distribution over all momentum states. Unfortunately, this also introduces products of distributions centered at the same point, leading to overlap divergences. These arguments are the motivation for the operator-product expansion (OPE), which allows us to define renormalized products of many fields [64–66].

Hidden in Eq. 1.122 is the fact that there is a regulator built into both the bare operators and the renormalization constants that suppresses the divergent UV modes. In the case of cutoff regularization, the scale Λ places an upper bound on the ranges of virtual momenta. Equivalently, predictions are only viable for length scales greater than $1/\Lambda$, so the momentum cutoff places a lower limit on the separation of field distributions. As long as the regulator is present, there will be an arbitrarily small displacement between the central points of any pair of fields. Then, defining the renormalized operators order-by-order in the couplings g_i so that there is a finite number of counterterms to be absorbed, we may rewrite Eq. 1.122 as an asymptotic expansion, the operator-product expansion for $\mathcal{O}_0$:

$$\mathcal{O}_0(x_1, ..., x_n; \Lambda) \xrightarrow{x_i \to x} \sum_i Z_i(\Lambda)\mathcal{O}_i(x), \tag{1.123}$$

where there is a coordinate x_i for each field in the bare operator $\mathcal{O}_0$. In general, the renormalization of bare operators $(\mathcal{O}_i)_0$ follows a matrix equation

$$(\mathcal{O}_i)_0 = Z_{ij}\mathcal{O}_j, \tag{1.124}$$

where the renormalization constants Z_{ij} quantify the fluctuations in $(\mathcal{O}_i)_0$ due to the operators $\mathcal{O}_j$.

The OPE is particularly useful because it holds at the operator level. This allows us to insert the full expansion into correlation functions containing the bare operator. For some initial and final states $|i\rangle$ and $\langle f|$ renormalized by some product of low-energy renormalization constants Z_{fi}, we have

$$\langle f|\mathcal{O}_i|i\rangle_0 = Z_{fi}^{-1} Z_{ij} \langle f|\mathcal{O}_j|i\rangle. \tag{1.125}$$

The renormalized correlation functions and the mixing matrix may be expanded in the renormalizable couplings g_i. For simplicity, we define $\Gamma_i = \langle f|\mathcal{O}_i|i\rangle$. In the case of a single coupling g the perturbative expansions are

$$Z_{ij} = \sum_{n=0}^{\infty} g^{2n} Z_{ij}^{(n)} \tag{1.126}$$

and

$$\Gamma_i = \sum_{n=0}^{\infty} g^{2n} \Gamma_i^{(n)}. \tag{1.127}$$

We can use these expressions to iteratively construct the OPE, starting at the leading order. At NLO, Eq. 1.125 takes the form

$$(\Gamma_i)_0^{(0)} + g_0^2 (\Gamma_i)_0^{(1)} + \mathcal{O}(g_0^4) = Z_{fi}^{-1} \left(Z_{ij}^{(0)} + g^2 Z_{ij}^{(1)} + \mathcal{O}(g^4) \right) \left(\Gamma_j^{(0)} + g^2 \Gamma_j^{(1)} + \mathcal{O}(g^4) \right). \tag{1.128}$$

Since the bare operator corresponds to $g_i = 0$, we see immediately that $Z_{ij}^{(0)} = \delta_{ij}$. Inserting the renormalized coupling[11] $g_0 = Z_g g$ and collecting like powers of g, we have

$$(\Gamma_i)_0^{(0)} = Z_{fi}^{-1} \cdot Z_{ij}^{(0)} \Gamma_j^{(0)} = Z_{fi}^{-1} \Gamma_i^{(0)}, \tag{1.129}$$

and

$$(\Gamma_i)_0^{(1)} = Z_{fi}^{-1} \cdot \left(\Gamma_i^{(1)} + Z_{ij}^{(1)} \Gamma_j^{(0)} \right). \tag{1.130}$$

[11] with a scale μ^ϵ if working in dimensional regularization

These equations will be indispensable to our analysis of the mixing of $\mathcal{CP}$ -violating operators in Part III.

By dimensional analysis it is clear that for operators $\mathcal{O}_i$ with $[\mathcal{O}_i] = d_i$, the elements of the renormalization matrix have dimension

$$[Z_{ij}] = d_i - d_j. \tag{1.131}$$

Thus the mixing structure is determined by the nature of the chosen regulator. In cutoff schemes, the mixing goes as $\Lambda^{d_i - d_j}$, so any operators for which $d_j > d_i$ are suppressed by the cutoff scale and may be neglected in the OPE. For $d_j \leq d_i$, the all operators must be taken into account. We saw this phenomenon explicitly in the one-loop correction of the scalar propagator, Eq. 1.82: the ϕ^2 operator contributed a correction of order Λ^2, while ϕ^4 term contributed a logarithm. Dimensional-regularization, on the other hand, mixes operators only of the same dimension, as evident in Eq. 1.84, since there is no scale to compensate for the difference in dimension. In any case, there are only finitely many operators of a given dimension, so there are also only finitely many counterterms at any order in the coupling, and the OPE may be truncated. In particular, there are finitely many contributions to Eqs. 1.129 and 1.130.

1.11 Euclidean Field Theory

In Sec. 1.4, we mentioned that the path integral must be regulated with an imaginary shift in the Lagrangian, $\mathcal{L} \to \mathcal{L} + \mathcal{L}_\varepsilon$, with a term of the form

$$\mathcal{L}_\varepsilon = i\varepsilon\phi^2 \tag{1.132}$$

for every species of dynamical field ϕ. This addition makes the modulus of the Boltzmann factor decay exponentially,

$$\left|e^{iS}\right|^2 = e^{i\int(\mathcal{L}+\mathcal{L}_\varepsilon)}e^{-i\int(\mathcal{L}+\mathcal{L}_\varepsilon^*)} = e^{-2\int \Im\mathcal{L}_\varepsilon}, \tag{1.133}$$

so that the path integral converges in principle. As far as perturbative calculations are concerned, the extra term in the action serves only to define free correlation functions unambiguously. Indeed, in our expression for the scalar two-point correlator, Eq. 1.71, the $i\varepsilon$ prescription avoids any real poles in the energy (p_0) integral. The poles are shifted to the second and fourth quadrants,

$$p_0 = \pm\sqrt{\omega^2(\lambda,\mathbf{p}) - i\varepsilon} \approx \pm\omega(\lambda,\mathbf{p}) \mp i\varepsilon, \tag{1.134}$$

so that the integral is soluble by integration over a semicircular contour in the upper or lower half-plane. The specific contour of integration is fixed by the sign of $x_0 - y_0$: the lower contour vanishes at infinity for $x_0 > y_0$, while the upper contour vanishes for $x_0 < y_0$, and the p_0 integral is determined by the residue at the enclosed pole. With this prescription, we find that the two-point function describes the propagation of a particle from y to x for $x_0 > y_0$ or x to y for $y_0 > x_0$, consistent with causality. The choice

$$\mathcal{L}_\varepsilon = -i\varepsilon\phi^2 \tag{1.135}$$

would lead to backward propagation.

There is, however, a much deeper problem. In defining the functional integral, Eq. 1.13, we have assumed the existence of a translationally-invariant measure $\mathcal{D}\phi$ on the infinite-

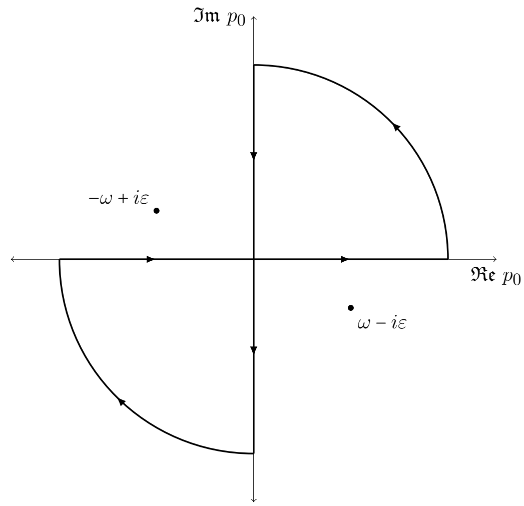


Figure 1.1: The contour of integration defining the Wick rotation

dimensional Hilbert-space of functions ϕ. Following Riesz' Lemma [67, 68], it is clear that this cannot exist in any nontrivial way. Instead, it is possible that the state space may be densely embedded in a larger Banach space [69, 70], allowing us to define the path integral via the Wiener measure [71]. The only known construction of this sort is valid only for Riemannian manifolds [72–74], so one must work with a Euclidean signature. In order to have a well-defined (free) theory, we must somehow relate Minkowskian QFT to Euclidean QFT. In fact, the $i\varepsilon$ prescription allows us to do exactly that[12]. Since the poles have been shifted into the second and fourth quadrants, the integration contour may be defined as in Fig. 1.1. There are no poles enclosed by the contour, so the entire integral vanishes. Further, the curved paths have a vanishing contribution by Jordan's lemma, so the integrals over the remaining contours must precisely cancel. Then we see that

$$\int_{-\infty}^{\infty} dp_0 \frac{-i}{-p_0^2 + \mathbf{p}^2 + m^2 - i\varepsilon} = \int_{-i\infty}^{i\infty} dp_0 \ \frac{-i}{-p_0^2 + \mathbf{p}^2 + m^2 - i\varepsilon} = \int_{-\infty}^{\infty} dp_4^E \frac{1}{{p_4^E}^2 + \mathbf{p}^2 + m^2}, \tag{1.136}$$

[12] while working in a free theory. As we saw in the spectral representation, the presence of interactions can introduce additional poles corresponding to bound states and a branch cut for multiparticle states.

where we have defined the Euclidean (E) time coordinate $p_0 = ip_4^E$[13]. In effect, this change of variables transforms the $(-,+,+,+)$ Minkowski metric into the $(+,+,+,+)$ Euclidean metric. The process of analytically continuing to complex p_0 and transitioning to Euclidean space is called a Wick rotation. The change of variables above extends to all 4-vectors, including derivatives, as follows:

$$\begin{aligned}
&\text{coordinates:} && x_0 = -ix_4^E, && x_i = x_i^E, \\
&\text{momenta:} && p_0 = ip_4^E, && p_i = p_i^E, \\
&\text{derivatives:} && \partial_0 = i\partial_4^E, && \partial_i = \partial_i^E, \\
&\text{vector fields:} && A_0 = iA_4^E, && A_i = A_i^E, \\
&\text{gamma matrices:} && \gamma_0 = \gamma_4^E, && \gamma_i = -i\gamma_4^E,
\end{aligned} \tag{1.137}$$

where the subscript i indicates a spatial component. We can then define the Euclidean action with the above replacements. In scalar field theory, the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}(\partial^2 - m^2)\phi - g\phi^4 = \frac{1}{2}(\partial^{E^2} - m^2)\phi - g\phi^4 =: -\mathcal{L}^E. \tag{1.138}$$

We have defined the Lagrangian with a negative sign so that the Euclidean action is positive-definite, and the Boltzmann factor is decaying:

$$iS = \int d^4x\mathcal{L} = -\int d^4x^E\mathcal{L}^E = -S^E. \tag{1.139}$$

[13]More generally, $p_0 = ip_d$

The path integral can now be written in Euclidean space:

$$Z^E = \left(Z_0^E\right)^{-1} \int \mathcal{D}\phi e^{-S^E}, \tag{1.140}$$

where the measure $\mathcal{D}\phi$ is the Wiener measure on the space of continuous paths ϕ on the underlying d-dimensional Euclidean space. From this perspective, it makes sense to define the physical Minkowskian path integral as the Wick rotation of a well-defined Euclidean theory. The precise conditions for this to be possible are given by the Osterwalder–Schrader axioms [75–77]. One of the conditions, reflection positivity, requires that any Euclidean correlation function is nonnegative when its spacetime arguments are symmetric about the $x_4 = 0$ hyperplane. When this holds, the Minkowski functions obtained by Wick rotation are time-ordered. Under a rotation $x_4 = e^{i\theta}x_0$ with $\theta = \pi/2 - \varepsilon$, the ordering of the real part of the time coordinates remains the same, so the ordering of the fields holds as well. Expanding about small ε, we see that $x_4 = i(1 - i\varepsilon)x_0$ induces the shift $p^2 \to p^2 - i\varepsilon p_0^2$ in momentum space. We can absorb the positive factor of p_0^2 into ε, so that the Minkowskian propagators are

$$\frac{1}{p^2 + m^2} \to \frac{1}{p^2 + m^2 - i\varepsilon}, \tag{1.141}$$

reproducing the earlier $i\varepsilon$ prescription. So long as no poles are encountered in the interval $\theta \in [0, \pi/2]$, we may take the limit $\varepsilon \to 0$.

In the interacting theory, it is possible that many more poles appear, potentially even in the first and third quadrants [78]. Under these circumstances, calculations require more careful treatment [79]. There is, at the time of this writing, no known construction of an interacting theory as above. The full path integral is then treated heuristically and defined only in terms of the algebraic manipulations that produce intuitively correct correlation functions.

Chapter 2

Perturbation Theory

2.1 Feynman Rules

In Section 1.4, we presented a perturbative treatment of the generating functional. This depended critically on the assumption that we could absorb the free-field portion of the Boltzmann factor into the path integral measure. This conveniently allowed us to define expectation values with respect to the Gaussian density of the free theory and therefore to express the n^{th} moments — equivalently the n-point correlation functions — in terms of the two-point functions for each pair of fields. In this section, we derive the rules for the propagators and vertices needed to translate the perturbation series into a graphical representation in terms of Feynman diagrams. As with the perturbation series, much of the material in the present chapter has been discussed or introduced in previous sections. Whereas it was then only cursorily treated, this chapter serves to flesh out those methods that will be integral to the rest of this work. In particular, we introduce in Sec. 2.5 a new method for treating the notoriously cumbersome angular-dependent quantities one encounters within dimensional regularization. Toward this construction, we illustrate the loop expansion in detail, deriving Feynman rules along the way, and briefly describe the common parametrization schemes for loop integrals upon which the new method is based.

2.1.1 Propagators

The two-point functions, or propagators, of a field ϕ are the most fundamental building block of perturbative calculations. As a first example, we consider a scalar field ϕ which evolves according to the Klein-Gordon Lagrangian (see Sec. 1.1). The propagator may be constructed as in Eq. 1.32, taking $\mathcal{O} = \phi(y)\phi(x)$:

$$\langle\phi(y)\phi(x)\rangle = \frac{Z_0[0]}{Z[0]}\sum_{i=0}^{\infty}\frac{i^n}{n!}\cdot g^n\left\langle\left(\int\mathcal{L}_I\right)^n\phi(y)\phi(x)\right\rangle_0 = \langle\phi(y)\phi(x)\rangle_0 + \mathcal{O}(g). \tag{2.1}$$

If we reintroduce the source field $J(x)$, we can recast the free propagator $\langle\phi(y)\phi(x)\rangle_0$ above as a derivative of the free generating functional as in Sec. 1.2,

$$\langle\phi(y)\phi(x)\rangle_0 = -\frac{\delta^2}{\delta J(y)\delta J(x)}\frac{1}{Z_0[0]}\int\mathcal{D}\phi\; e^{i\int(\mathcal{L}_0+J\phi)}\bigg|_{J=0}, \tag{2.2}$$

Recall that the free Lagrangian is quadratic in each species of field, so we may write

$$\mathcal{L}_0(x) = \frac{1}{2}\int_y\phi(y)K(y,x)\phi(x), \tag{2.3}$$

where K represents the integral kernel of the free action[1]. In this case, the kernel is given by $\delta(y-x)(\partial_x^2 - m^2)$.The above integrand is a shifted Gaussian in the field ϕ. Since the path integral measure $\mathcal{D}\phi$ is translation-invariant, we are free to "complete the square" in the exponent by redefining ϕ up to a shift

$$\phi(x) = \phi'(x) - \int_y K^{-1}(x,y)J(y), \tag{2.4}$$

[1]We will often drop the spacetime arguments of the fields and associated kernels along with delta functionals for ease of notation. This practice is perfectly analogous to suppressing matrix indices and writing Kronecker deltas as the identity matrix 1 or I.

and the path integral is greatly simplified:

$$\frac{1}{Z_0[0]}\int \mathcal{D}\phi\, e^{\frac{1}{2}i\int \phi K\phi} e^{i\int J\phi} = \frac{1}{Z_0[0]}\int \mathcal{D}\phi'\, e^{\frac{1}{2}i\int \phi' K\phi'} e^{-\frac{1}{2}i\int JK^{-1}J} = e^{-\frac{1}{2}i\int JK^{-1}J}. \quad (2.5)$$

In the last equivalence above, we took the ϕ-independent source term out of the integrand and cancelled the remaining integral with the normalization constant $Z_0[0]$. In this form, it is trivial to take derivatives with respect to the source field J, and for the free propagator, we find

$$\langle\phi(y)\phi(x)\rangle_0 = -\frac{\delta^2}{\delta J(y)\delta J(x)} e^{-\frac{1}{2}i\int JK^{-1}J}\bigg|_{J=0} = iK^{-1}(y,x). \quad (2.6)$$

The inverse kernel is easily calculated by Fourier transformation. For scalar fields we have

$$\delta(x-y) = \int_z K(x,z)K^{-1}(z,y) = (\partial_x^2 - m^2)\int_p e^{ip(x-y)}\tilde{K}^{-1}(p) = -\int_p e^{ip(x-y)}(p^2+m^2)\tilde{K}^{-1}(p), \quad (2.7)$$

from which it follows that

$$-(p^2+m^2)\tilde{K}^{-1}(p) = 1 \quad (2.8)$$

and

$$\langle\phi(y)\phi(x)\rangle_0 = \int_p e^{ip(y-x)}\frac{-i}{p^2+m^2}. \quad (2.9)$$

We derive the propagators for other fields in an identical manner. For fermions of mass m, we have the free Lagrangian $\mathcal{L}_0(x) = \int_y \overline{\psi}(y)K(y,x)\psi(x)$ with the kernel $K(y,x) = \delta(y-x)(\not{\partial}+m)$. Introducing Grassmann-valued spinor source fields $\bar{\eta}$ and η for the ψ and its adjoint $\overline{\psi}$ respectively, we may write the free fermion propagator as

$$\langle\psi(y)\overline{\psi}(x)\rangle_0 = \frac{\delta^2}{\delta\bar{\eta}(y)\delta\eta(x)}\frac{1}{Z_0[0]}\int \mathcal{D}\psi\mathcal{D}\overline{\psi}\; e^{i\int\left(\overline{\psi}K\psi+\bar{\eta}\psi+\eta\overline{\psi}\right)}\bigg|_{\bar{\eta},\eta=0}. \quad (2.10)$$

Note the difference in sign as compared to the scalar case due to the anticommutation of the field $\overline{\psi}$ and the derivative with respect to η. Shifting again the integration variables,

$$\overline{\psi}(x) = \overline{\psi}'(x) - \int_y \bar{\eta}(y) K^{-1}(y,x) \qquad \text{and} \qquad \psi(x) = \psi'(x) - \int_y K^{-1}(x,y)\eta(y), \tag{2.11}$$

we obtain

$$\langle \psi(y)\overline{\psi}(x)\rangle_0 = -\frac{\delta^2}{\delta\bar{\eta}(y)\delta\eta(x)} e^{-i\int \bar{\eta}K^{-1}\eta}\bigg|_{\bar{\eta},\eta=0} = iK^{-1}(y,x). \tag{2.12}$$

As before, we can extract the inverse with a Fourier transform, so that

$$\langle \psi(y)\overline{\psi}(x)\rangle_0 = i\int_p e^{ip(y-x)} \frac{-i\not{p} + m}{p^2 + m^2}. \tag{2.13}$$

Finally, for gauge bosons we have the free Lagrangian constructed in Sec. 1.5:

$$\mathcal{L}_0 = \frac{1}{2g^2} \operatorname{Tr}\left\{\partial_{[\mu} A_{\nu]} \partial^{[\mu} A^{\nu]}\right\} + \frac{1}{\xi g^2} \operatorname{Tr}\left\{\partial^\mu A_\mu \partial^\nu A_\nu\right\}. \tag{2.14}$$

Integrating by parts and taking the trace, we find the kernel

$$K^{ab}_{\mu\nu}(y,x) = -\frac{T_F}{g^2}\delta^{ab}\delta(y-x)\left[\delta_{\mu\nu}\partial_x^2 - \left(1 - \frac{1}{\xi}\right)\partial_{x,\mu\nu}\right], \tag{2.15}$$

which allows us to express the generating functional as before:

$$Z_0[J] = \frac{1}{Z_0[0]} \int \mathcal{D}A \exp\left\{i \int \left[A^{\mu a} K^{ab}_{\mu\nu} A^{\nu b} + J^a_\mu A^{\mu a}\right]\right\}. \tag{2.16}$$

Shifting the integration variable once again as

$$A^a_\mu(x) = A'^a_\mu(x) - \frac{1}{2}\int_y K^{-1}{}^{ab}_{\mu\nu}(x,y)J^{\nu b}(y), \tag{2.17}$$

we have

$$Z_0[J] = \exp\left\{-\frac{1}{4}i\int_{xy} J^{\mu a}(x)K^{-1}{}^{ab}_{\mu\nu}(x,y)J^{\nu b}(y)\right\}. \tag{2.18}$$

From this form, it is clear that

$$\langle A^b_\nu(y)A^a_\mu(x)\rangle_0 = \frac{1}{4}i\left\{K^{-1}{}^{ab}_{\mu\nu}(x,y) + K^{-1}{}^{ba}_{\nu\mu}(y,x)\right\} = \frac{1}{2}iK^{-1}{}^{ba}_{\nu\mu}(y,x), \tag{2.19}$$

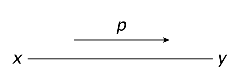
where the inverse kernel is given by

$$K^{-1}{}^{ab}_{\mu\nu}(x,y) = \frac{g^2}{T_F}\int_p e^{ip(x-y)}\frac{\delta^{ab}}{p^2}\left\{\delta_{\mu\nu} - (1-\xi)\frac{p_{\mu\nu}}{p^2}\right\}. \tag{2.20}$$

In Minkowski space, there is one more step required to define the propagators. As we mentioned before, the path integral measure is oscillatory, so it must be regularized by introducing a small imaginary part to the Lagrangian. The appropriate prescription for exponential decay was written in Eqs. 1.30 and 1.141, and with this final ingredient we can write the Feynman rules for each propagator.

In the diagrammatic representation, we assign to each scalar propagator an undirected solid line carrying some momentum p from a point x to another point y and one factor of

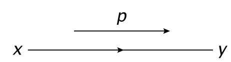
$$S_F(y-x) = \int_p e^{ip(y-x)}\frac{-i}{p^2+m^2-i\varepsilon} = \tag{2.21}$$



,

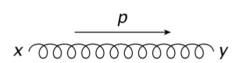
where the subscript "F" stands for Richard Feynman, to whom the $i\varepsilon$ prescription is due.

To fermion propagators we associate a directed solid line and a factor of

$$S_F(y-x) = \int_p e^{ip(y-x)} \frac{\not{p} + im}{p^2 + m^2 - i\varepsilon} = $$ 

$$. \qquad (2.22)$$

The notation S_F for scalars is customarily recycled for fermions, but context should make its usage clear. Gauge bosons are differently represented depending on their species. They are typically wavy (photons), zig-zag (electroweak bosons), or curly (gluons), but since this work is focused on QCD, we hereafter adopt the notation for gluons and write

$$D_{F\,\alpha\beta}^{\;ab}(y-x) = \frac{ig^2}{2T_F} \int_p e^{ip(y-x)} \frac{\delta^{ab}}{p^2} \left\{ \delta_{\alpha\beta} - (1-\xi) \frac{p_{\alpha\beta}}{p^2} \right\} = $$ 

$$. \qquad (2.23)$$

2.1.2 Vertices

In the last section, we considered only the leading-order terms in the perturbative series, Eq. 1.32. Each of these was generated by a strictly quadratic Lagrangian, since there was no insertion of the interaction part. At subleading orders, however, we encounter local products of fields with a higher degree. These correspond to interactions involving more than two fields, hence vertices in the diagrammatic representation, allowing for nontrivial n-point correlation functions for $n > 2$. The treatment of these structures is similar to the case of the two-point functions, the only difference being the presence of more complex differential or tensor structures and more propagators. As an introduction, we examine the quartic interaction of ϕ^4 theory, governed by the Lagrangian

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4, \qquad (2.24)$$

for a real scalar field ϕ. The first two terms above are immediately recognizable, constituting the Klein-Gordon Lagrangian. The last term is new; it describes an interaction of four scalar fields coupled to each other with strength $\lambda/4!$. Writing

$$\mathcal{L}_I = \frac{\lambda}{4!}\phi^4, \tag{2.25}$$

we can rewrite the scalar two-point function as

$$\langle\phi(y)\phi(x)\rangle = \frac{Z_0[0]}{Z[0]}\sum_{i=0}^{\infty}\frac{i^n}{n!}\left\langle\left(\frac{\lambda}{4!}\int_z\phi^4(z)\right)^n\phi(y)\phi(x)\right\rangle_0. \tag{2.26}$$

We have already considered the case $n = 0$. For $n = 1$, we have a single insertion of the interaction term, and we must calculate the function

$$\lambda\langle\phi(y)\phi(x)\rangle^{(1)} = \frac{i\lambda}{4!}\int_z\left\langle\phi^4(z)\phi(y)\phi(x)\right\rangle_0. \tag{2.27}$$

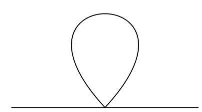
We again employ Isserlis' theorem to express the integrand as a product of propagators:

$$\begin{aligned}\left\langle\phi^4(z)\phi(y)\phi(x)\right\rangle_0 = {} & 4\cdot 3\cdot\langle\phi(y)\phi(z)\rangle_0\langle\phi(z)\phi(z)\rangle_0\langle\phi(z)\phi(x)\rangle_0 \\ & + 3\langle\phi(y)\phi(x)\rangle_0\langle\phi(z)\phi(z)\rangle_0\langle\phi(z)\phi(z)\rangle_0,\end{aligned} \tag{2.28}$$

where the numerical coefficients count the redundancies in pairing the fields; in the first term for example, there are four fields $\phi(z)$ which may be paired to $\phi(y)$ with three remaining to be paired with $\phi(x)$. In total there are $4\cdot 3 + 3 = 15$ terms, corresponding to the $(2\cdot 3 - 1)!!$ perfect pairings of the six fields involved. The second term above is called a disconnected diagram, since the integral over z does not involve the external states. The first factor is clearly just the free-field propagator, while the rest generates a vacuum fluctuation. This bit is identical to the $\mathcal{O}(\lambda)$ term in the denominator of Eq. 2.9, generated by the normalization Z_0/Z. As such, the two terms acquire opposite signs in the series expansion and can easily be seen to cancel. We thus ignore all disconnected diagrams, since they do not contribute to scattering processes.

The first term in Eq. 2.28 describes the first appreciable effect of the ϕ^4 interaction: an excitation of the scalar field propagates from a point x to a point z, where there is an interaction of four fields, after which it continues to y. At the midpoint, we see the propagation of another field from z back to the same point. This process was briefly treated in Sec. 1.7, represented graphically as

$$\langle\phi(y)\phi(z)\rangle_0\langle\phi(z)\phi(z)\rangle_0\langle\phi(z)\phi(x)\rangle_0 =$$



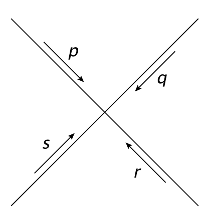
$$. \qquad (2.29)$$

The vertex of this graph symbolizes the quartic interaction, and aside from the attached propagators, it carries along with it a factor of $\frac{i\lambda}{4!}$. This additional structure is characteristic of any local product of fields. These will always correspond to vertices in the associated Feynman diagram and will carry information separate from the propagators. In general, we can isolate the structure of a vertex involving some n fields by calculating the n-point function of those fields at tree level, that is, with respect to a free background theory. In the case of the ϕ^4 interaction, we calculate the correlation function for four fields $\phi(x_1)$, $\phi(x_2)$,

$\phi(x_3)$, and $\phi(x_4)$:

$$\begin{aligned}
\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle^{(0)} &= \frac{i\lambda}{4!}\int_z \langle\phi^4(z)\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle_0 \\
&= i\lambda\int_z \langle\phi(z)\phi(x_1)\rangle_0\langle\phi(z)\phi(x_2)\rangle_0\langle\phi(z)\phi(x_3)\rangle_0\langle\phi(z)\phi(x_4)\rangle_0 \\
&= i\lambda\int_z S_F(z-x_1)\ S_F(z-x_2)\ S_F(z-x_3)\ S_F(z-x_4) \\
&= i\lambda\int_{p_1,p_2,p_3,p_4}(2\pi)^d\delta^{(d)}(p_1+p_2+p_3+p_4)\prod_{i=1}^{4}e^{-ip_ix_i}\tilde{S}_F(p_i).
\end{aligned} \tag{2.30}$$

In the second line, the factor of 4! was cancelled by the 24 equivalent contractions of the four fields in the interaction Lagrangian with the four external states. As in the final line, it is conventional to pass to momentum space, where the Feynman rules assume a relatively simple form. In momentum space, the spacetime integral over the interaction Lagrangian produces a delta function over the sum of the momenta, which ensures an overall conservation of momentum at each point of interaction. In our conventions for the vertex rules, all momenta are considered to be incoming. Since there will always be propagators attached to a vertex, we can treat them separately and simply drop the entire integral from the last line, so long as we remember to attach the appropriate delta function and integrate over all momenta. We can then simply read off the vertex rule from the remaining expression. In our example, the vertex rule is

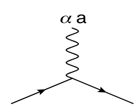


$$= i\lambda. \tag{2.31}$$

Other interactions are treated likewise. The vertex in Eq. 1.9, is derived from the three-point function of two fermions and a gauge field:

$$
\begin{aligned}
&\langle \psi(x_3) A^a_\alpha(x_2) \overline{\psi}(x_1) \rangle^{(0)} \\
&\qquad = i \int_z \langle \overline{\psi}(z) \not{A}(z) \psi(z) \psi(x_3) A^a_\alpha(x_2) \overline{\psi}(x_1) \rangle_0 \\
&\qquad = i \int_z \langle \psi(x_3) \overline{\psi}(z) \rangle_0 \langle A^b_\beta(z) A^a_\alpha(x_2) \rangle_0 \gamma^\beta t^b \langle \psi(z) \overline{\psi}(x_1) \rangle_0 \\
&\qquad = i \int_z S_F(x_3 - z) D_F{}^{ba}_{\beta\alpha}(z - x_2) \gamma^\beta t^b S_F(z - x_1) \rangle_0 \\
&\qquad = i \int_{p,q,r} (2\pi)^d \delta^{(d)}(p + q + r) e^{-irx_3} \tilde{S}_F(-r) \gamma^\beta t^b e^{-iqx_2} \tilde{D}_F{}^{ba}_{\beta\alpha}(q) e^{-ipx_1} \tilde{S}_F(p).
\end{aligned}
\tag{2.32}
$$

Here, when we amputate the gauge field propagator, we must leave behind a factor of $\delta_{\alpha\beta}\delta^{ab}$ for bookkeeping.[2] This leaves us with the rule



$$
= i\gamma_\alpha t^a. \tag{2.33}
$$

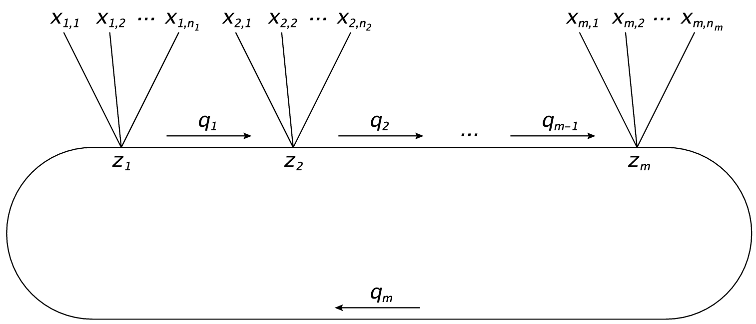
We will encounter more complicated vertices in the next chapter, but they are derived in the same manner.

2.2 Loop Integrals

The tools we have so far derived provide us with a dictionary for the transcription of any elementary scattering process in terms of spacetime integrals. The integrands are composed of propagators and local products of fields which are compactly represented in Feynman graphs by edges and vertices respectively. At the leading order, these graphs are simple forests[3], hence the term "tree-level". Eventually, however, the repeated insertion of inter-

[2]Since the propagator is the Green function for the free field kernel, the amputation of each external leg can be formally defined by acting on the external states by the appropriate kernel for that species and scaling accordingly to the identity matrix.

[3]Unless there are no external states, as in the case of vacuum expectation values or condensates



(a)

Figure 2.1: The most generic form of a loop within a Feynman diagram

action terms forces the number of interior fields to outgrow the number of external states. At this point, as we saw with the scalar propagator, the edges will form cycles or "loops" in the diagrammatic representation, where a sequence of propagators leads back to its own point of origin. We may treat the most general case as follows. Consider the loop in Fig. 2.1 connecting m vertices located at the points z_i for $i \in [m]$. Each vertex represents a product of $n_i + 2$ fields, so we attach as many propagators to each respective vertex. For every vertex i, two propagators connect to the neighboring vertices at points $z_{i\pm1}$ (with the indices taken mod m), leaving n_i propagators outside the loop leading either to external states or vertices at points $x_{i,j}$ for $j \in [n_i]$. The total contribution of this loop may be written

$$L(\mathbf{x}) = \int_{z_1,\ldots,z_m} \prod_{i=1}^{m} \left[V_i S(z_{i+1} - z_i) \prod_{j=1}^{n_i} S(z_i - x_{i,j}) \right] \tag{2.34}$$

where the boldface x is the set of all $x_{i,j}$, V_i is the Feynman rule for the i^{th} vertex, and S generically represents the propagator for any species of field[4]. Moving to momentum space,

[4]The caveat here is that the vertex rule and propagators must be written in the proper order when they represent noncommuting quantities, for example in the cases of fermions and ghosts. This ordering does not affect the integral, however, so we ignore it for the current discussion.

we write

$$L(\mathbf{x}) = \int_{z_1,\dots,z_m} \prod_{i=1}^{m} \left[\int_{q_i} e^{iq_i(z_{i+1}-z_i)} V_i \tilde{S}(q_i) \prod_{j=1}^{n_i} \int_{p_{i,j}} e^{ip_{i,j}(z_i - x_{i,j})} \tilde{S}(p_{i,j}) \right] \tag{2.35}$$

where q_i are the momenta transferred between points z_i and z_{i+1}, and $p_{i,j}$ are the momenta running to each z_i from the n_i external fields at the points $x_{i,j}$. Assuming that we can reorder the integrals with impunity, this may be simplified to

$$L(\mathbf{x}) = \int_{\mathbf{p},\mathbf{q},\mathbf{z}} \prod_{i=1}^{m} \left[e^{-iz_i(q_i - r_i - q_{i-1})} V_i \tilde{S}(q_i) \prod_{j=1}^{n_i} e^{-ip_{i,j}x_{i,j}} \tilde{S}(p_{i,j}) \right], \tag{2.36}$$

where we again use bold letters to represent entire sets of variables and introduce the shorthand

$$r_i = \sum_{j=1}^{n_i} p_{i,j}. \tag{2.37}$$

We may now perform the integrals over each z_i, leaving a string of delta functions as before:

$$L(\mathbf{x}) = \int_{\mathbf{p},\mathbf{q}} \prod_{i=1}^{m} \left[(2\pi)^d \delta^{(d)}(q_i - r_i - q_{i-1}) V_i \tilde{S}(q_i) \prod_{j=1}^{n_i} e^{-ip_{i,j}x_{i,j}} \tilde{S}(p_{i,j}) \right], \tag{2.38}$$

As we take the q-integrals sequentially from 1 to $m-1$, the delta functions enforce the relation

$$q_i = q_m + \sum_{j=1}^{i} r_j, \tag{2.39}$$

until only one remains, representing an overall conservation of momentum within the loop:

$$L(\mathbf{x}) = \int_{\mathbf{p},q_m} (2\pi)^d \delta^{(d)}(r_1 + \cdots + r_m) \prod_{i=1}^{m} \left[V_i \tilde{S}(q_i) \prod_{j=1}^{n_i} e^{-ip_{i,j}x_{i,j}} \tilde{S}(p_{i,j}) \right], \tag{2.40}$$

We can simplify this further by moving the external states to momentum space as well, defining

$$(2\pi)^d\delta^{(d)}(r_1+\cdots+r_m)\tilde{L}(\mathbf{p}) = \int_{\mathbf{x}} e^{i\mathbf{p}\cdot\mathbf{x}}L(\mathbf{x}), \tag{2.41}$$

from which it follows that

$$\tilde{L}(\mathbf{p}) = \prod_{i,j}\tilde{S}(p_{i,j})\cdot\int_q V_1\cdots V_m\tilde{S}(q+r_1)\cdots\tilde{S}(q+r_1+\cdots+r_{m-1})\tilde{S}(q). \tag{2.42}$$

Loops are typically treated independently, so we will amputate the external legs, and redefine

$$\tilde{L}(\mathbf{p}) := \tilde{L}_{\text{amp}}(\mathbf{p}) = \int_q V_1\cdots V_m\tilde{S}(q+r_1)\cdots\tilde{S}(q+r_1+\cdots+r_{m-1})\tilde{S}(q), \tag{2.43}$$

which is our master formula for loop integrals. A quantity of this form will be present for each independent cycle in a Feynman diagram. The number of loops or "loop order" then corresponds to the cyclomatic number (the first Betti number) of the corresponding undirected graph [80], given by

$$b_1 = e - v + b_0, \tag{2.44}$$

where e and v are respectively the numbers of edges and vertices, and b_0 is the zeroth Betti number or the number of connected components.

There is one more subtlety in writing down loop integrals. As mentioned in footnote 4, it becomes necessary to pay attention to the ordering of vertex and propagator rules for noncommuting objects like fermion and ghost fields. When Grassmann-odd fields form a loop, there must be at least two insertions of the corresponding interaction Lagrangian. These terms always introduce a pair of fields f and $\bar{f}$ with the basic field f on the right and the adjoint field $\bar{f}$ on the left, so that the loop is written as a string of products of the form $(\bar{f}\Gamma f)(\bar{f}\Gamma f)\cdots(\bar{f}\Gamma f)$ for some vertex structure Γ, where the parenthetical terms freely commute. The propagators are defined in the opposite order, and each field f can be contracted to the following $\bar{f}$ in the next insertion. This leaves the first and last fields in the string uncontracted. Though it can commute through any pair of fields, the first $\bar{f}$ must also pass through the final f to form a proper two-point function, so the loop acquires an overall negative sign. Further, since fermions carry representations of both gauge group and the Clifford algebra, the same argument results in a trace over each of these sets of indices.

2.3 Integral Parametrizations

As we discussed in Sec. 1.7, integrals encountered in perturbation theory often diverge for very large or very small values of the loop momenta, so they are rarely well-defined right out of the box. Instead we must choose a regularization scheme to ensure their convergence. So far, we have implicitly defined everything in an undetermined number of dimensions d, consistent with dimensional regularization. This was inconsequential in evaluating Feynman rules[5], but solving dimensionally regularized loop integrals will require more care. For any loop in a Feynman diagram that is not a self-loop, there will be associated a product of propagators in the form of Eq. 2.43, each with a quadratic dependence on its momentum in the denominator. This results in complicated angular contributions for each inner product $q \cdot r_i$. Though arduous even in four spacetime dimensions, their treatment is completely intractable — if not impossible — for generic d. Since we shall want to take smooth limits $d \to 4$ after renormalization, we must allow the dimension to assume non-integral values; indeed, d is generally considered complex. For this reason, the integral must be transformed

[5] so far, while there has been no parity violation

in such a way that some dimensions are removed by symmetry, leaving a small natural number of integrals over the remaining dimensions to be performed by hand. The most common practice is to find a parametrization of the integrand which depends only on the square of the loop momentum, after which everything can be recast in d-dimensional spherical coordinates. In this way, the entire $(d-1)$-dimensional spherical shell can be directly integrated, leaving a single radial integral to be solved by other techniques.

If the integrand is already spherically symmetric,

$$\int_q f(q^2) = \frac{1}{(2\pi)^d} \int_\Omega d\Omega \int_0^\infty r^{d-1} dr\ f(r^2), \tag{2.45}$$

where

$$d\Omega = \prod_{k=1}^{d-1} \sin^{d-k-1}(\phi_k) d\phi_k. \tag{2.46}$$

and where the angular domain Ω is defined by

$$\begin{aligned} &\phi_k \in [0, \pi); \quad\ \ k < d-1 \\ &\phi_k \in [0, 2\pi); \quad k = d-1, \end{aligned} \tag{2.47}$$

we may extract the solid angle as follows. We have assumed for now that d is an integer. Periodic symmetry allows us to write

$$\int_0^\pi d\phi_k\ \sin^{d-k-1}(\phi_k) = 2\int_0^{\pi/2} d\phi_k\ \sin^{d-k-1}(\phi_k); \quad k < d-1 \tag{2.48}$$

and

$$\int_0^{2\pi} d\phi_k\ \sin^{d-k-1}(\phi_k) = 4\int_0^{\pi/2} d\phi_k; \quad k = d-1, \tag{2.49}$$

so that

$$\begin{aligned}\int_\Omega d\Omega =&2\prod_{k=1}^{d-1}\left[2\int_0^{\pi/2} d\phi_k\ \sin^{d-k-1}(\phi_k)\right]\\ =&2\prod_{k=1}^{d-1} B\left(\frac{d-k}{2},\frac{1}{2}\right)\\ =&2\Gamma^{d-1}\left(\frac{1}{2}\right)\prod_{k=1}^{d-1}\Gamma\left(\frac{d-k}{2}\right)\bigg/\Gamma\left(\frac{d-k+1}{2}\right).\end{aligned}\tag{2.50}$$

The numerator of each factor cancels the denominator of the next, and we are left with

$$\int_\Omega d\Omega = 2\pi^{\frac{d-1}{2}}\frac{\Gamma(1/2)}{\Gamma(d/2)} = \frac{2\pi^{d/2}}{\Gamma(d/2)}.\tag{2.51}$$

Thus

$$\int_q f(q^2) = \frac{2(4\pi)^{-d/2}}{\Gamma(d/2)}\int_0^\infty\ r^{d-1}dr\ f(r^2).\tag{2.52}$$

2.3.1 Feynman Parametrization

If the integrand is not even, we must transform it to a spherical form, the standard for which is Feynman parameterization [16, 81]. A loop with N propagators has the general form

$$I^{n_I}_{\mu_J}(\mathbf{p};m_I) = \int_q \frac{q_{\mu_1}\cdots q_{\mu_n}}{\prod_{i=1}^N\left(s_i^2+m_i^2\right)^{n_i}},\tag{2.53}$$

where $I = \{1,\ldots,N\}$ and $J = \{1,\ldots,n\}$ are multi-indices, and where the product in the denominator runs over all propagators in the loop with their respective masses and momenta indexed by i. Each s_i in the denominator has the form $s_i = q + R_i$, where $R_i = r_1 + \cdots + r_i$, and r_i is as defined in Sec. 2.2. The identity (which we will prove in the next subsection)

$$\frac{1}{\prod_{i=1}^N\left(s_i^2+m_i^2\right)^{n_i}} = \frac{1}{B(n_1,\ldots,n_N)}\prod_{i=1}^N\int_0^1 dz_i\ z_i^{n_i-1}\cdot\frac{\delta\left(1-\sum_{i=1}^N z_i\right)}{\left[\sum_{i=1}^N z_i\left(s_i^2+m_i^2\right)\right]^{\sum_{i=1}^N n_i}}.\tag{2.54}$$

allows the denominator to be expressed as a sum, so that we can complete the square in the momentum of integration q:

$$I^{n_I}_{\mu_J}(p_I;m_I) = \frac{1}{B(n_1,\dots n_N)}\prod_{i=1}^{N}\int_0^1 dz_i\ z_i^{n_i-1}\cdot\int_q \frac{\delta\left(1-\sum_{i=1}^N z_i\right)}{\left[\sum_{i=1}^N z_i\left((q+R_i)^2+m_i^2\right)\right]^{\sum_{i=1}^N n_i}}q_{\mu_J}$$
$$= \frac{1}{B(n_1,\dots n_N)}\prod_{i=1}^{N}\int_0^1 dz_i\ z_i^{n_i-1}\cdot\int_q \frac{\delta\left(1-\sum_{i=1}^N z_i\right)}{\left[(q+Q)^2+\Delta\right]^{\sum_{i=1}^N n_i}}q_{\mu_J}, \tag{2.55}$$

where

$$\Delta = \sum_{i=1}^{N} z_i(R_i^2+m_i^2) - Q^2, \tag{2.56}$$

and

$$Q^\mu = \sum_{i=1}^{N} z_i R_i^\mu. \tag{2.57}$$

Under the change of variables $k = q + Q$, we have

$$I^{n_I}_{\mu_J}(p_I;m_I) = \frac{1}{B(n_1,\dots n_N)}\prod_{i=1}^{N}\int_0^1 dz_i\ z_i^{n_i-1}\cdot\int_k \frac{\delta\left(1-\sum_{i=1}^N z_i\right)}{\left[k^2+\Delta\right]^{\sum_{i=1}^N n_i}}(k-Q)_{\mu_J}, \tag{2.58}$$

and the parity of the integrand is more easily discernible. The product of vectors $(k-Q)_{\mu_J}$ is a polynomial in k, where the even-degree terms will survive integration, and the odd terms will vanish. The total momentum integral is therefore a sum over integrals of the form

$$\int_k f(k^2)k_{\mu_1}\cdots k_{\mu_{2n}}, \tag{2.59}$$

for some n. The $2n$-fold product ensures that the integral does not trivially vanish, and the tensor structure will be treated in the following section.

2.3.2 Schwinger Parametrization

There is an alternative parametrization introduced by Schwinger [38], which, though equivalent to Feynman parametrization, can be easier to use in practice. It relies on the identity

$$\frac{1}{A^n} = \frac{1}{\Gamma(n)} \int_0^\infty d\alpha \; e^{-A\alpha} \alpha^{n-1}, \tag{2.60}$$

which may be applied to the denominator of each propagator in Eq. 2.53:

$$I^{n_I}_{\mu_J}(\mathbf{p}; m_I) = \int_q \frac{q_{\mu_1} \cdots q_{\mu_n}}{\prod_{i=1}^N \left(s_i^2 + m_i^2\right)^{n_i}} = \int_q q_{\mu_J} \prod_{i=1}^N \int_0^\infty \frac{d\alpha_i}{\Gamma(n_i)} e^{-\alpha_i (s_i^2 + m_i^2)} \alpha_i^{n_i - 1} \tag{2.61}$$

If there are no factors of q_{μ_i}, the integral is a simple scalar quantity. Shifting the variable q to

$$k^\mu = q^\mu + \frac{1}{A} \sum_{i=1}^N \alpha_i R_i^\mu. \tag{2.62}$$

we have a d-fold product of identical Gaussian integrals in the components of k:

$$\begin{aligned} I^{n_I}_{\mu_J}(\mathbf{p}; m_I) &= \prod_{i=1}^N \int_0^\infty \frac{d\alpha_i \; \alpha_i^{n_i - 1}}{\Gamma(n_i)} \cdot \exp\left\{ \frac{1}{A} \left(\sum_{i=1}^N \alpha_i R_i \right)^2 - \sum_{i=1}^N \alpha_i (R_i^2 + m_i^2) \right\} \left(\int_{-\infty}^\infty \frac{dk}{2\pi} e^{-k^2 A} \right)^d \\ &= \prod_{i=1}^N \int_0^\infty \frac{d\alpha_i \; \alpha_i^{n_i - 1}}{\Gamma(n_i)} \cdot \exp\left\{ \frac{1}{A} \left(\sum_{i=1}^N \alpha_i R_i \right)^2 - \sum_{i=1}^N \alpha_i (R_i^2 + m_i^2) \right\} (4\pi A)^{-d/2}, \end{aligned} \tag{2.63}$$

where we have used the shorthand notation $A = \sum_{i=1}^N \alpha_i$.

This is as far as we will go in full generality. The bracketed term in Eq. 2.63 can most often be simplified in practical calculations, where a subset of masses may vanish, where the momenta R_i are interrelated, or where on-shell conditions may apply. As before, we have ignored potential tensor structures in the case of Schwinger parameters. These will be briefly treated in the next section, and some simple examples will be worked out in the Appendix.

The Schwinger parametrization can in fact be used to prove Eq. 2.54. Beginning from Eq. 2.61, we change variables to

$$\begin{aligned} z_i &= \alpha_i / A \qquad \text{for} \qquad 1 \le i < N \\ A &= \alpha_1 + \cdots + \alpha_N . \end{aligned} \tag{2.64}$$

The Jacobian is a simple arrowhead matrix,

$$J = \begin{bmatrix} \frac{\partial \alpha_1}{\partial z_1} & \dots & \frac{\partial \alpha_1}{\partial z_{N-1}} & \frac{\partial \alpha_1}{\partial A} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial \alpha_{N-1}}{\partial z_1} & \dots & \frac{\partial \alpha_{N-1}}{\partial z_{N-1}} & \frac{\partial \alpha_{N-1}}{\partial A} \\ \frac{\partial \alpha_N}{\partial z_1} & \dots & \frac{\partial \alpha_N}{\partial z_{N-1}} & \frac{\partial \alpha_N}{\partial A} \end{bmatrix} = \begin{bmatrix} A & \dots & 0 & z_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & A & z_{N-1} \\ -A & \dots & -A & 1 - \sum_{i=1}^{N-1} z_i \end{bmatrix}, \tag{2.65}$$

so the determinant can be quickly found by LU factorization:

$$|J| = A^{N-1}. \tag{2.66}$$

After changing variables, we have

$$\begin{aligned} I^{n_I}_{\mu_J}(\mathbf{p}; m_I) = &\int_q q_{\mu_J} \int_0^\infty dA \; A^{N-1} \prod_{i=1}^{N-1} \int_0^1 dz_i \; \frac{(A z_i)^{n_i - 1}}{\Gamma(n_i)} e^{-A z_i (s_i^2 + m_i^2)} \\ &\times \left(1 - \sum_{i=1}^{N-1} z_i\right)^{n_N - 1} \frac{A^{n_N - 1}}{\Gamma(n_N)} \exp\left\{-A \left(1 - \sum_{i=1}^{N-1} z_i\right)(s_N^2 + m_N^2)\right\}, \end{aligned} \tag{2.67}$$

where the integral over A is elementary:

$$I^{n_I}_{\mu_J}(\mathbf{p}; m_I) = \frac{1}{B(n_1, \ldots, n_N)} \int_q q_{\mu_J} \prod_{i=1}^{N-1} \int_0^1 dz_i \; z_i^{n_i - 1} \cdot \left(1 - \sum_{i=1}^{N-1} z_i\right)^{n_N - 1} \\ \times \left\{ \sum_{i=1}^{N-1} z_i (s_i^2 + m_i^2) + \left(1 - \sum_{i=1}^{N-1} z_i\right) (s_N^2 + m_N^2) \right\}^{-n_1 - \cdots - n_N} . \tag{2.68}$$

The final step is to trade the sums over z_i for a new variable z_N through the use of a delta function. It is clear from the definition of z_i that

$$0 \le z_N := 1 - \sum_{i=1}^{N-1} z_i \le 1, \tag{2.69}$$

so we can insert

$$1 = \int_0^1 dz_N \; \delta(1 - z_1 - \cdots - z_N) \tag{2.70}$$

into our current expression for $I^{n_I}_{\mu_J}$, replacing z_N where appropriate, and we reach the desired result:

$$I^{n_I}_{\mu_J}(\mathbf{p}; m_I) = \frac{1}{B(n_1, \ldots, n_N)} \int_q q_{\mu_J} \prod_{i=1}^{N} \int_0^1 dz_i \; z_i^{n_i - 1} \cdot \frac{\delta\left(1 - \sum_{i=1}^N z_i\right)}{\left[\sum_{i=1}^N z_i \left(s_i^2 + m_i^2\right)\right]^{\sum_{i=1}^N n_i}}. \tag{2.71}$$

2.4 Tensor Integral Decomposition

When we derived formulae for the Feynman and Schwinger parametric integrals, we largely ignored the factors of q_{μ_J}, since they play little part in the parametrizations themselves. On the other hand, they pose a significant obstacle to finding closed-form expressions for the momentum integrals[6]. Fortunately, we can always decompose tensor integrals into a linear

[6] The full parametric integrals rarely have a closed form, but we can at least solve the integrals over momentum variables, where the dimension d is most consequential

combination of Lorentz-covariant tensors whose coefficients scalar integrals.

The key observation is that any product $q_{\mu_J} = q_{\mu_1} \cdots q_{\mu_n}$ in the numerator of a momentum integral transforms as a n-tensor under Lorentz transformations. Then the solution must itself be expressible in terms of such n-tensors. The allowable structures must depend only on the external momenta within the integrand and any number of metric tensors, since these are the only available Lorentz-covariant quantities. Further, the tensors must be symmetric under any permutation of the spacetime indices μ_i. Then an n-tensor integral depending on some external momenta $p_1, \ldots, p_k$ is most generally expressible as

$$I^{n_I}_{\mu_J}(\mathbf{p}; m_I) = \sum_{\lambda} A_\lambda g^{\otimes \lambda_0/2}_{\{I_{\lambda_0}} p_1{}^{\otimes \lambda_1}_{I_{\lambda_1}} \cdots p_k{}^{\otimes \lambda_k}_{I_{\lambda_k}\}} \tag{2.72}$$

where the sum runs over all weak $(k+1)$-compositions λ of n with parts λ_i and with even λ_0. The braces around the multi-indices I_{n_i} represent symmetrization over the entire set μ_J, and the tensor product in the exponents indicates that each factor is a tensor product with λ_i factors. The coefficients A_λ are scalar integrals. This decomposition is akin to the Passarino-Veltman decomposition [82], which is ubiquitous in the literature on Feynman integral calculus.

We can in fact count the number p_λ of such compositions for each pair (n, k). It is well known that there are exactly $\binom{n+k-1}{k-1}$ weak k-compositions of an integer n. Then, given that the number of indices attached to metric tensors must be even, we seek the number of weak k-compositions of the remaining $n - \lambda_0$ indices, where $\lambda_0 = 2i$ with $0 \le i \le \lfloor n/2 \rfloor$. This is simply[7]

$$p_\lambda(n,k) = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-2i+k-1}{k-1}$$

$$= \begin{cases} 1 & k=0 \\ \lim_{m,\ell\to n,k} \binom{m+\ell-1}{\ell-1} {}_3F_2\left(1, \frac{1-m}{2}, -\frac{m}{2}; \frac{1-m-\ell}{2}, \frac{1-m-\ell}{2}; 1\right) & k>0 \end{cases}. \tag{2.73}$$

The function ${}_3F_2$ in the second line is a special case of the generalized hypergeometric function, defined by the series

$${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z) \coloneqq \sum k=0^\infty \frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{z^k}{k!}. \tag{2.74}$$

In order to determine the coefficients A_λ, we must reduce the tensor equation, Eq. 2.72, to a scalar form. For each composition λ, we can contract either side of the decomposition above with the tensor associated to λ. This saturates all indices in $I^{n_I}_{\mu_J}$, producing a system of as many linear equations for the scalar integrals A_λ.

The decomposition formula 2.72 is quite unwieldy, but it is often easy to write down for simple integrals. For example, an integral with two free indices and two external momenta can be written as

$$I^{n_I}_{\mu_1\mu_2}(p_1,p_2;m_I) = A_{(2,0,0)}g_{\mu_1\mu_2} + A_{(0,2,0)}p_{1\mu_1\mu_2} + A_{(0,1,1)}p_{1\{\mu_1}p_{2\mu_2\}} + A_{(0,0,2)}p_{2\mu_1\mu_2}, \tag{2.75}$$

[7]The limit is required in the final expression, because we have to recast the binomial coefficient in the first line in terms of Pochhammer symbols indexed by i in order to obtain a hypergeometric series. Since the arguments contain $-2i$, we must use the dimidiation and reflection formulae to obtain the index i at the cost of gamma functions of negative argument. We must then treat n and k as non-integral positive numbers for these manipulations, in order that the gamma functions are defined. After some simplification, we find that the limits back to integer values are well-behaved, and the second line follows.

where we indeed find $p_\lambda(2,2) = 4$ terms. Contracting the equation above with $g^{\mu_1\mu_2}$, we have

$$\int_q \frac{q^2}{\prod_{i=1}^N \left(s_i^2 + m_i^2\right)^{n_i}} = dA_{(2,0,0)} + A_{(0,2,0)} p_1^2 + 2A_{(0,1,1)} p_1 \cdot p_2 + A_{(0,0,2)} p_2^2. \tag{2.76}$$

Contracting likewise with the other three tensors in Eq. 2.75, we obtain four linear equations relating the coefficients A_λ to scalar integrals as above.

Eq. 2.72 is simplest when the integrand is already spherically symmetric, for example, in the context of Feynman parameters. Since the parametrized denominators depend only Lorentz scalars, there can be no tensors in the decomposition that depend on the external momenta, which covary with Lorentz transformations. Moreover, the denominators are even in the loop momentum q, so only the even tensor powers of q survive the parametrization. Then for a rank-$2n$ integral, the tensor decomposition contains only a symmetrized product of n metric tensors. This can be written explicitly as

$$S_{I_{2n}}^{(2n)} := g_{\{\mu_1\mu_2} \cdots g_{\mu_{2n-1}\mu_{2n}\}} = \frac{1}{2^n n!} \sum_{\pi \in S_{2n}} \prod_{i=1}^{n} g_{\mu_{\pi(2i-1)}\mu_{\pi(2i)}}, \tag{2.77}$$

which is simply a sum over all of the ways to distribute the $2n$ indices among the n metric tensors, that is, the perfect pairings of $[2n]$. There are of course $(2n)!$ permutations of the indices, but many of these are redundant. First, the metric tensors commute among themselves, so we overcount by $n!$ permutations of these factors. Second, each tensor is itself symmetric in its two indices, so we further overcount by 2^n. These two redundancies correspond to the S_n-action on n-tuples of pairs of indices and n $\mathbb{Z}_2$-actions on the pairs themselves. Overall then, each set of equivalent permutations is stabilized by the subgroup

$S_2 \wr_{[n]} S_n < S_{2n}$, which is simply the hyperoctahedral group B_n. The size of the redundancy is thus

$$|B_n| = \left|S_2 \wr_{[n]} S_n\right| = |S_2|^{|[n]|}\,|S_n| = 2^n n!, \tag{2.78}$$

and the total number of unique permutations is

$$\frac{(2n)!}{2^n n!} = (2n-1)!!, \tag{2.79}$$

the expected number of pairings over $2n$ objects. This explains the combinatorial prefactor of the sum in Eq. 2.103. The S tensors in Eq. 2.103 are a special case of the most general isotropic tensor of even rank:

$$\sum_{\pi \in \mathcal{M}_n} a_\pi \prod_{i=1}^{n} g_{\mu_{\pi(2i-1)}\mu_{\pi(2i)}} \tag{2.80}$$

where a_π are arbitrary coefficients (see for example [83]), and $\mathcal{M}_n := S_{2n}/B_n$ is the set of perfect matchings. Setting $a_\pi = 1$ for all permutations, we recover S. In particular, this allows us to sum over the group S_{2n} instead of the set M_n.

A spherically symmetric integral thus has the general decomposition

$$\int_q f(q^2) q_{\mu_1} \cdots q_{\mu_{2n}} = A S^{(2n)}_{\mu_1 \cdots \mu_{2n}}. \tag{2.81}$$

Since all permutations of the indices are contained within the tensor S, we can contract with any n-fold product of metric tensors containing these indices to solve for A. We may as well choose $g_{\mu_1\mu_2}\cdots g_{\mu_{2n-1}\mu_{2n}}$, so that

$$\int_q (q^2)^n f(q^2) = A g_{\mu_1\mu_2}\cdots g_{\mu_{2n-1}\mu_{2n}} S^{(2n)}_{\mu_1\cdots\mu_{2n}}. \tag{2.82}$$

As we will prove in App. , the contraction on the right simplifies to

$$g_{\mu_1\mu_2}\cdots g_{\mu_{2n-1}\mu_{2n}} S^{(2n)}_{\mu_1\cdots\mu_{2n}} = (d)_{n,2} =: 1/W_N \tag{2.83}$$

where

$$(x)_{n,k} = k^n (x/k)_n = k^n \frac{\Gamma(x/k+n)}{\Gamma(x/k)} \tag{2.84}$$

is the Pochhammer k-symbol. The general formula for this family of integrals is thus

$$\int_q f(q^2) q_{\mu_1}\cdots q_{\mu_{2n}} = \frac{(4\pi)^{-d/2}}{2^{n-1}\Gamma(d/2+n)} S^{(2n)}_{\mu_1\cdots\mu_{2n}} \int_0^\infty dq\; q^{d+2n-1} f(q^2) \tag{2.85}$$

after integrating out the spherical shell. This particular decomposition will be the most important in the following chapters.

2.5 A Novel Treatment of Schwinger Parametrization

We now describe a new general method, introduced in [40], for treating dimensionally-regularized momentum integrals. For this section, we will proceed in Euclidean space, where we are conveniently assured that the inner product of any two momenta is positive definite. This technique was in fact devised for the perturbative treatment of the gradient flow (see below, Part II), an innately Euclidean scheme [84]. In the next chapter, however, we will demonstrate an application in Minkowski space, which, due to the unfortunate necessity of

Wick rotations and contour integrals, is much less efficient.

Let us imagine a theory whose propagators generically assume the form[8]

$$\tilde{S}(q) = \frac{f(q)}{q^2 + m^2}, \tag{2.86}$$

where the numerator is some as yet undetermined smooth function f defined by the Lagrangian of that theory. In order to solve loop integrals in such a scenario, we would need a procedure for integrating products like

$$\tilde{S}(q)\tilde{S}(q+p_1)\cdots\tilde{S}(q+p_n) = \frac{f(q)}{q^2+m^2}\frac{f(q+p_1)}{(q+p_1)^2+m^2}\cdots\frac{f(q+p_n)}{(q+p_n)^2+m^2} \tag{2.87}$$

over loop momentum q. Again, the predominant difficulty is dealing with cross-terms $p_i \cdot q$ that introduce some angular dependence to the integrand. In familiar theories like the Standard Model, the function f is a simple polynomial. In QCD, for example, the most complicated numerator is in the fermion propagator, where $f(q) = -i\not{q} + m$. Feynman parameters are a perfectly good choice in this setting, because they at most introduce a shift in the integration variable, so the numerator remains a polynomial. For other functions we may not be so lucky, unless we can reduce the integrand to a (potentially tensor-valued) rational function of q. When f is smooth, we can do just that by expanding each propagator as a Maclaurin series in $p_i \cdot q$:

$$\tilde{S}(q+p_i) = \sum_{k_i=0}^{\infty} \frac{(p_i\cdot q)^n}{k_i!} \cdot \left.\frac{\partial_i^k \tilde{S}(q+p_i)}{\partial(p_i\cdot q)_i^k}\right|_{p_i\cdot q=0}. \tag{2.88}$$

[8] We have dropped Feynman prescription and the subscript F, since they are unnecessary in Euclidean space.

Each term in this series is now of the form $q_{I_{k_i+m_i}} f_{k_i}(q^2)$, where $I_{k_i+m_i}$ is a multi-index representing the some k_i indices donated by the factor $(p_i \cdot q)_i^k$ and another m_i indices which may be latent to the function f. We can repeat this expansion for each propagator with dependence on an external p_i, resulting in an integral with exactly the structure of Eq. 2.85 when $(k_1 + m_1) + \cdots + (k_n + m_n) = 2N$ for some integer N:

$$\begin{aligned} &\int_q q_{I_{k_1+m_1} \cdots I_{k_n+m_n}} f_{k_1}(q^2) \cdots f_{k_n}(q^2) \\ &\qquad = \frac{(4\pi)^{-d/2}}{2^{N-1}\Gamma(d/2+N)} S^{(2N)}_{\mu_1 \cdots \mu_{2N}} \int_0^\infty dq\ f_{k_1}(q^2) \cdots f_{k_n}(q^2)\ q^{2N+d-1}. \end{aligned} \tag{2.89}$$

This formula is, again, quite unmanageable in full generality, but we can find some nice formulae for simple cases when the exact form of the free propagator is known. In the present work, we are concerned with the cases $f(q) = e^{-q^2 t}$ and $f(q) = e^{-q^2 t}(-i\not{q} + m)$, which arise naturally in flowed perturbation theory, Sec. 5.3. The quantity t is a real, nonnegative parameter related to the “flow time,” which measures the delocalization or smearing of the fields. In this setting, Feynman parameters do more harm than good, since we have products of momentum in the denominators but sums in the arguments of the exponentials. Instead, the exponential factors suggest that we use Schwinger parameters, where everything is treated on the same footing:

$$\tilde{S}(q+p) = \frac{e^{-(q+p)^2 t}}{(q+p)^2 + m^2} = \int_0^\infty dz\ e^{-(q+p)^2(t+z)} e^{-m^2 z}. \tag{2.90}$$

This allows us to easily write the Taylor series for the cross term,

$$\tilde{S}(q+p) = \int_0^\infty dz\ e^{-m^2 z} e^{-p^2(t+z)} e^{-q^2(t+z)} \sum_{k=0}^{\infty} \frac{(-2(t+z))^k}{n!} p_{I_k} q_{I_k}, \tag{2.91}$$

and continue as in the general case.

In the following subsections, we should imagine that t is a small parameter which is only important up to logarithmic order. The reason for this is detailed in Ch. 7, which introduces the short-flow-time expansion. This assumption will make our life much easier, because it determines when we can truncate the Taylor series. The tensor decomposition will invariably result in a hypergeometric function with parameters related to the index of summation and argument related to the angle between external momenta. Because the series can be truncated, we require only a small number of hypergeometric functions at specific values of the parameters, a great many of which are known exactly in closed form. For the results in this thesis, we need to treat only a few simple cases, with at most three propagators per loop. We now derive the relevant formulae for these integrals, saving a brief discussion about their generalization until the end of this section.

2.5.1 Two Point Functions: One Loop, Two Propagators

When calculating two-point correlation functions at one-loop order, we regularly encounter integrals of the form

$$I = \int_q \frac{e^{-q^2 t}}{(q^2+m^2)^a} \frac{e^{-(q+p)^2 s}}{((q+p)^2+M^2)^b} q_{\mu_1} \cdots q_{\mu_\ell}, \tag{2.92}$$

for some positive numbers a, b, t, and s. The first factor is already quadratic in q, so we can leave it as is and use Eq. 2.91 to rewrite the second:

$$I = \int_q \frac{e^{-q^2 t}}{(q^2+m^2)^a} \int_z \frac{z^{b-1}}{\Gamma(b)} e^{-M^2 z} e^{-p^2(s+z)} e^{-q^2(s+z)} \sum_{n=0}^{\infty} \frac{(-2(s+z))^n}{n!} p_{I_n} q_{I_{n+\ell}}, \tag{2.93}$$

There are now two cases to consider: even ℓ and odd ℓ.

In the even case, we can relabel $\ell \to 2\ell$ and shift the summation index likewise from n to $2n$, since these will be the only terms that survive integration. Then

$$I = \sum\!\!\!\!\!\!\int_{n,z} \frac{z^{b-1}}{\Gamma(b)} \frac{(2(s+z))^{2n}}{(2n)!} e^{-M^2 z} e^{-p^2(s+z)} p_{I_{2n}} \int_q \frac{e^{-q^2(t+s+z)}}{(q^2+m^2)^a} q_{I_{2n+2\ell}}, \tag{2.94}$$

where we have introduced the shorthand notation $\sum\!\!\!\!\!\!\int_{n,z} = \sum_{n=0}^{\infty} \int_0^{\infty} dz$, the ranges of summation and integration being implied. We again employ Eq. 2.85 and transform to spherical coordinates, so that

$$\int_q \frac{e^{-q^2(t+s+z)}}{(q^2+m^2)^a} q_{I_{2n+2\ell}} = \frac{(4\pi)^{-d/2}}{2^{n+\ell-1}\Gamma(d/2+n+\ell)} S^{(2n+2\ell)}_{I_{2n+2\ell}} \int_0^{\infty} dq \frac{e^{-q^2(t+s+z)}}{(q^2+m^2)^a} q^{2n+2\ell+d-1} \tag{2.95}$$

Changing variables to $x = q^2/m^2$, the integral becomes

$$\int_0^{\infty} dq e^{-q^2(t+s+z)} q^{2n+2\ell+d-1} = \frac{1}{2} m^{d+2n+2\ell-2a} \int_0^{\infty} dx e^{-m^2(t+s+z)x} \frac{x^{d/2+n-1}}{(x+1)^a}, \tag{2.96}$$

which is just the integral representation of the Tricomi confluent hypergeometric function [85]:

$$U(a,b,z) = \frac{1}{\Gamma(a)} \int_0^{\infty} dt e^{-zt} t^{a-1} (1+t)^{b-a-1}. \tag{2.97}$$

The momentum integral is, finally,

$$\begin{aligned} &\int_q \frac{e^{-q^2(t+s+z)}}{(q^2+m^2)^a} q_{I_{2n+2\ell}} \\ &\qquad = \frac{(4\pi)^{-d/2}}{2^{n+\ell}} m^{d+2n+2\ell-2a} U\left(\frac{d}{2}+n+\ell, \frac{d}{2}+n+\ell-a+1, m^2(t+s+z)\right) S^{(2n+2\ell)}_{I_{2n+2\ell}}. \end{aligned} \tag{2.98}$$

When $a = 1$ this reduces to an incomplete gamma function, since

$$U(a,a,z) = e^z \Gamma(1-a,z). \tag{2.99}$$

An even simpler case is that of zero mass, where

$$\int_q \frac{e^{-q^2(t+s+z)}}{(q^2)^a} q_{I_{2n+2\ell}} = \frac{(4\pi)^{-d/2}}{2^{n+\ell}\Gamma(d/2+n+\ell)} \frac{\Gamma(d/2+n+\ell-a)}{(t+s+z)^{d/2+n+\ell-a}} S^{(2n+2\ell)}_{I_{2n+2\ell}}. \tag{2.100}$$

In the odd case, we relabel $\ell \to 2\ell+1$ and similarly reindex the sum from n to $2n+1$, so that

$$I = \sum\!\!\!\!\!\!\int_{n,z} \frac{z^{b-1}}{\Gamma(b)} \frac{(2(s+z))^{2n+1}}{(2n+1)!} e^{-M^2 z} e^{-p^2(s+z)} p_{I_{2n+1}} \int_q \frac{e^{-q^2(t+s+z)}}{(q^2+m^2)^a} q_{I_{2n+2\ell+2}}. \tag{2.101}$$

The momentum integral is identical the the even case with n shifted to $n+1$.

In any case, after performing the integral over q, we are left with a product of vectors p_{μ_i} that must be contracted with an isotropic tensor S. The simplest case is of course $\ell = 0$, where the tensor is completely saturated by the momenta:

$$p_{I_{2n}} S^{(2n+2\ell)}_{I_{2n+2\ell}} \xrightarrow{\ell \to 0} p_{I_{2n}} S^{(2n)}_{I_{2n}}. \tag{2.102}$$

Referring back to the definition of S, Eq. 2.103, it is clear that

$$p_{I_{2n}} S^{(2n)}_{I_{2n}} = \frac{1}{2^n n!} \sum_{\pi \in S_{2n}} p_{I_{2n}} \prod_{i=1}^{n} g_{\mu_{\pi(2i-1)}\mu_{\pi(2i)}} = \frac{1}{2^n n!} \sum_{\pi \in S_{2n}} (p^2)^n = (2n-1)!!(p^2)^n, \tag{2.103}$$

since $p_{I_{2n}}$ is invariant under permutations of the indices. All other cases may be found by repeated differentiation with respect to p. The chain rule on the product $p_{I_{2n}}$ gives us

$$\frac{\partial}{\partial p_\nu} p_{I_{2n}} = \sum_{i=1}^{2n} \delta_{\nu\mu_i} \prod_{j\neq i} p_{\mu_j} \tag{2.104}$$

Because the tensor S carries all of the indices on $p_{I_{2n}}$, every term is the sum above contracts to the same quantity, $p_{I_{2n-1}} S^{(2n)}_{I_{2n-1}\nu}$, and we find

$$\frac{\partial}{\partial p_\nu} p_{I_{2n}} = 2n p_{I_{2n-1}} S^{(2n)}_{I_{2n-1}\nu}. \tag{2.105}$$

On the other hand,

$$\frac{\partial}{\partial p_\nu} (p^2)^n = 2n(p^2)^{n-1} p_\nu, \tag{2.106}$$

so that

$$p_{I_{2n-1}} S^{(2n)}_{I_{2n-1}\nu} = (2n-1)!!(p^2)^{n-1} p_\nu \Rightarrow p_{I_{2n+1}} S^{(2n+2)}_{I_{2n+1}\nu} = (2n+1)!!(p^2)^n p_\nu. \tag{2.107}$$

Further useful cases are

$$p_{I_{2n}} S^{(2n+2)}_{I_{2n}\mu\nu} = (2n-1)!!(p^2)^n \left(\delta_{\mu\nu} + 2n\frac{p_{\mu\nu}}{p^2}\right), \tag{2.108a}$$

$$p_{I_{2n+1}} S^{(2n+4)}_{I_{2n+1}\mu\nu\rho} = (2n+1)!!(p^2)^n \left(\frac{1}{2}\delta_{\{\mu\nu}p_{\rho\}} + 2n\frac{p_{\mu\nu\rho}}{p^2}\right), \tag{2.108b}$$

$$p_{I_{2n}} S^{(2n+4)}_{I_{2n}\mu\nu\rho\sigma} = (2n-1)!!(p^2)^n \left(S^{(4)}_{\mu\nu\rho\sigma} + \frac{n}{2}\frac{p_{\{\mu\nu}\delta_{\rho\sigma\}}}{p^2} + \frac{(n-1)n}{6}\frac{p_{\mu\nu\rho\sigma}}{(p^2)^2}\right). \tag{2.108c}$$

These follow from the general identities[9]

$$\begin{aligned}\partial^{2m}_{J_{2m}} p_{I_{2n+2m}} S^{(2n+2m)}_{I_{2n+2m}} &= (2n+1)_{2m} p_{I_{2n}} S^{(2n+2m)}_{I_{2n}J_{2m}} \\ &= (2n+2m-1)!!\partial^{2m}_{J_{2m}} (p_2)^{n+m},\end{aligned} \tag{2.109}$$

and

$$\partial^{2m}_{J_{2m}} (p_2)^{n+m} = 2^m (n+1)_m \sum_{k=0}^{m} \frac{2^{k-m}}{(2k)!(m-k)!} \frac{n!}{(n-k)!} \frac{2^k}{(p^2)^k} p^{\otimes 2k}_{\{J_{2k}} \delta^{\otimes m-k}_{J_{2m-2k}\}}, \tag{2.110}$$

which are easily proven by induction on m. These combine to give us the general contraction rule for the even case.

$$p_{I_{2n}} S^{(2n+2m)}_{I_{2n}J_{2m}} = (2n-1)!!(p^2)^n \sum_{k=0}^{m} \frac{2^{k-m}}{(2k)!(m-k)!} \frac{n!}{(n-k)!} \frac{2^k}{(p^2)^k} p^{\otimes 2k}_{\{J_{2k}} \delta^{\otimes m-k}_{J_{2m-2k}\}}. \tag{2.111}$$

The odd case follows by differentiating once further.

2.5.2 Three Point Functions: One Loop, Three Propagators

In Sec. 8.2, we will calculate three-point functions at one-loop order, where we will encounter integrals of the form

$$I = \int_q \frac{e^{-q^2 t}}{(q^2+M^2)^a} \frac{e^{-(q+p_1)^2 s_1}}{((q+p_1)^2+m_1^2)^{b_1}} \frac{e^{-(q+p_2)^2 s_2}}{((q+p_2)^2+m_2^2)^{b_2}}. \tag{2.112}$$

[9] We use the shorthand $\partial^{2m}_{J_{2m}} = \frac{\partial}{\partial p_{\mu_1}} \cdots \frac{\partial}{\partial p_{\mu_{2m}}}$.

After absorbing the mixed denominators into the exponents with two Schwinger integrals, we will find a double sum over the cross-terms:

$$I = \sum\!\!\!\!\!\!\int_{n,x} \frac{x^{b_1-1}}{\Gamma(b_1)} \frac{(-2(s_1+x))^n}{n!} e^{-p_1^2(s_1+x)} e^{-m_1^2 x} \sum\!\!\!\!\!\!\int_{m,y} \frac{y^{b_2-1}}{\Gamma(b_2)} \frac{(-2(s_2+y))^m}{m!} e^{-p_2^2(s_2+y)} e^{-m_2^2 y} \\ \times \int_q \frac{e^{-q^2(t+s_1+x+s_2+y)}}{(q^2+M^2)^a} p_{1 I_n} p_{2 J_m} q_{I_n J_m}, \tag{2.113}$$

which is not as easily treated as the previous case. The momentum integral is easily decomposed: we seek even combinations of $n+m$, so that each index is either even ($m+n \to 2m+2n$), or each is odd ($m+n \to 2m+2n+2$). After this, the integration goes as before, and we find a Tricomi function.

In the former case, where both indices are even, the tensor decomposition on q gives us a factor of

$$P^E_{nm} := p_{1 I_{2n}} S^{(2n+2m)}_{I_{2n} J_{2m}} p_{2 J_{2m}}. \tag{2.114}$$

Fortunately, we have done most of the heavy lifting already in the last subsection. We can simply contract our master formula for two-point functions, Eq. 2.111, with $2m$ factors of a second momentum. With the current labels,

$$P^E_{nm} = (2n-1)!!(p_1^2)^n p_{2 J_{2m}} \sum_{k=0}^{m} \frac{2^{k-m}}{(2k)!(m-k)!} \frac{n!}{(n-k)!} \frac{2^k}{(p_1^2)^k} p_1{}^{\otimes 2k}_{\{J_{2k}} \delta^{\otimes m-k}_{J_{2m-2k}\}}. \tag{2.115}$$

Since $p_{2 J_{2m}}$ is totally symmetric in its $2m$ indices, the permutations within the summand are immaterial under contraction, so we find $(2m)!$ identical terms:

$$p_{2 J_{2m}} p_1{}^{\otimes 2k}_{\{J_{2k}} \delta^{\otimes m-k}_{J_{2m-2k}\}} = (2m)!(p_1 \cdot p_2)^{2k} (p_2^2)^{m-k}. \tag{2.116}$$

With some rearranging, we have

$$P^E_{nm} = (2n-1)!!(p_1^2)^n \frac{(2m)!}{2^m m!}(p_2^2)^m \sum_{k=0}^{m} \frac{2^{2k}}{(2k)!}\frac{m!}{(m-k)!}\frac{n!}{(n-k)!}\left(\frac{(p_1\cdot p_2)^2}{p_1^2 p_2^2}\right)^k. \tag{2.117}$$

The fraction in front of the sum is simply the definition of the double factorial on odd integers, Eq. 2.79. The summand can be further simplified using identities for the gamma function and Pochhammer symbols. We first recognize that

$$\frac{m!}{(m-k)!} = \frac{\Gamma(m-k+1+k)}{\Gamma(m-k+1)} = (m-k+1)_k \tag{2.118}$$

and likewise for the n-dependent factor. The reflection formula for Pochhammer symbols is

$$(m-k+1)_k = (-1)^k(-m)_k, \tag{2.119}$$

so

$$\frac{m!}{(m-k)!}\frac{n!}{(n-k)!} = (-m)_k(-n)_k. \tag{2.120}$$

The remaining bit can be rewritten as

$$\frac{2^{2k}}{(2k)!} = \frac{2^{2k}}{(2k)!}\frac{k\Gamma(k)}{k!} = \frac{2^{2k-1}}{\Gamma(2k)}\frac{\Gamma(k)}{k!}, \tag{2.121}$$

which is readily simplified further by means of the Legendre duplication formula:

$$\frac{\Gamma(2k)}{2^{2k-1}\Gamma(k)} = \frac{\Gamma(1/2)}{\Gamma(k+1/2)} = \frac{1}{(1/2)_k}. \tag{2.122}$$

We now have

$$P^E_{nm} = (2n-1)!!(2m-1)!!(p_1^2)^n(p_2^2)^m \sum_{k=0}^{m} \frac{(-n)_k(-m)_k}{(1/2)_k k!}\left(\frac{(p_1\cdot p_2)^2}{p_1^2p_2^2}\right)^k. \tag{2.123}$$

Since m is a nonnegative integer, the factor of $(-m)_k$ vanishes for $k > m$, and we can extend the upper limit of the sum to infinity without any repercussions. The result is exactly the Gaussian hypergeometric series:

$${}_2F_1(a,b;c;z) := \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}, \tag{2.124}$$

so

$$P^E_{nm} = (2n-1)!!(2m-1)!!(p_1^2)^n(p_2^2)^m {}_2F_1\left(-n,-m;\frac{1}{2};\Pi_{12}\right), \tag{2.125}$$

where

$$\Pi_i j := \cos^2\theta_{ij}, \tag{2.126}$$

and we have defined the angle θ_{ij} through the dot product

$$p_i\cdot p_j = |p_i||p_j|\cos\theta_{ij} \Rightarrow \frac{(p_1\cdot p_2)^2}{p_1^2p_2^2} = \cos^2\theta_{ij}. \tag{2.127}$$

The odd case,

$$P^O_{nm} := p_{1I_{2n+1}} S^{(2n+2m+2)}_{I_{2n+1}J_{2m+1}} p_{2J_{2m+1}}, \tag{2.128}$$

can again be found by differentiation, where on one hand,

$$p_{2\mu}\frac{\partial}{\partial p_{1\mu}} P^E_{n+1\ m} = \frac{\partial}{\partial p_{1\mu}} p_{1I_{2n+2}} S^{(2n+2m+2)}_{I_{2n+2}J_{2m}} p_{2J_{2m}\mu} = (2n+2)P^O_{nm}. \tag{2.129}$$

On the other hand,

$$p_{2\mu}\frac{\partial}{\partial p_{1\mu}}P^E_{n+1\ m} = p_{2\mu}\frac{\partial}{\partial p_{1\mu}}(2n+1)!!(2m-1)!!(p_1^2)^{n+1}(p_2^2)^m{}_2F_1\left(-n-1,-m;\frac{1}{2};\Pi_{12}\right)$$
$$= (2n+2)(2n+1)!!(2m-1)!!(p_1^2)^n(p_2^2)^m(p_1\cdot p_2)$$
$$\times\left\{{}_2F_1\left(-n-1,-m;\frac{1}{2};\Pi_{12}\right)+2(1-\Pi_{12}){}_2F_1\left(-n,-m-1;\frac{3}{2};\Pi_{12}\right)\right\}. \tag{2.130}$$

The braced term in the second line can be reduced using Gauss' contiguous relations,

$${}_2F_1\left(-n-1,-m;\frac{1}{2};\Pi_{12}\right)+2(1-\Pi_{12}){}_2F_1\left(-n,-m-1;\frac{3}{2};\Pi_{12}\right) = (2m+1){}_2F_1\left(-n,-m;\frac{3}{2};\Pi_{12}\right), \tag{2.131}$$

and the final result is

$$P^O_{nm} = (2n+1)!!(2m+1)!!(p_1^2)^n(p_2^2)^m(p_1\cdot p_2){}_2F_1\left(-n,-m;\frac{3}{2};\Pi_{12}\right). \tag{2.132}$$

Returning to Eq. 2.113 and remembering that the momentum integration itself introduces a confluent hypergeometric function, the general formula for three-point functions is once again totally unmanageable[10]. Although we cannot find a closed solution for any generic loop, the combinatorial contributions can be greatly simplified for certain configurations of the external momenta. If p_1 and p_2 are collinear, then $\Pi_{12} = 1$, and the hypergeometric function has unit argument. Since $1/2 > -n-m$, we can use Gauss' summation theorem:

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \qquad \mathfrak{R}(c) > \mathfrak{R}(a+b). \tag{2.133}$$

Alternatively, if the momenta are perpendicular, Π_{12} vanishes, and only the first term in the

[10] There might be a pattern here.

hypergeometric series survives, so

$$ {}_2F_1(a,b;c;0) = 1. \tag{2.134}$$

2.5.3 Vacuum Functions: Two Loops, Three Propagators

The final case we will consider is the two-loop vacuum diagram, which is generically expressible as

$$\int_{p,q} \frac{e^{-p^2 s_1}}{(p^2+m_1^2)^{n_1}} \frac{e^{-q^2 s_2}}{(q^2+m_2^2)^{n_2}} \frac{e^{-(p-q)^2 s_3}}{((p-q)^2+m_3^2)^{n_3}} p_{I_n} q_{J_m}, \tag{2.135}$$

where we once again handle the last factor using Schwinger parameters. After the Taylor expansion, we are left with a product of momenta $p_{I_n K_\ell} q_{J_m K_\ell}$, where ℓ is the index of summation. Since each momentum is integrated, each requires a tensor decomposition. Accordingly, the only scenarios in which the integral does not trivially vanish are those when n and m are both either odd or even (and therefore ℓ has the same parity). We consider the even case, where we must calculate

$$Q^E_{n\ell m} := p_{I_{2n}} S^{(2n+2\ell)}_{I_{2n}K_{2\ell}} S^{(2\ell+2m)}_{K_{2\ell}J_{2m}} q_{J_{2m}}. \tag{2.136}$$

We take a different, less direct approach to compute this contraction. Consider the integral

$$I_{nlm} = \int_r e^{-r^2 t} (r\cdot q)^{2m} p_{I_{2n}} S^{(2n+2\ell)}_{I_{2n}K_{2\ell}} r_{K_{2\ell}}. \tag{2.137}$$

The Gaussian in the integrand is just a weight function ensuring convergence. We can go about calculating this integral in two ways.

The first method begins by directly decomposing the tensor integral into a scalar integral and an isotropic tensor:

$$
\begin{aligned}
I_{nlm} &= p_{I_{2n}} q_{J_{2m}} S^{(2n+2\ell)}_{I_{2n}K_{2\ell}} \int_r e^{-r^2 t} r_{K_{2\ell}J_{2m}} \\
&= p_{I_{2n}} S^{(2n+2\ell)}_{I_{2n}K_{2\ell}} S^{(2\ell+2m)}_{K_{2\ell}J_{2m}} q_{J_{2m}} \frac{1}{(d)_{\ell+m,2}} \int_r e^{-r^2 t} (r^2)\ell + m \\
&= Q^E_{n\ell m} \frac{1}{(d)_{\ell+m,2}} \frac{(4\pi)^{-d/2}}{\Gamma(d/2)} \frac{\Gamma(d/2+\ell+m)}{t^{d/2+\ell+m}} \\
&= (4\pi t)^{-d/2} (2t)^{-\ell-m} Q^E_{n\ell m}.
\end{aligned}
\tag{2.138}
$$

where in the third line we used the definition of $Q^E_{n\ell m}$, Eq. 2.136, and the familiar transformation to spherical coordinates. We can invert this equation for $Q^E_{n\ell m}$, so that

$$
Q^E_{n\ell m} = (4\pi t)^{d/2} (2t)^{\ell+m} I_{nlm}. \tag{2.139}
$$

The second approach gives us an exact formula for the integral. We start by recognizing the definition of P^E_{nm} in the integrand, which we replace with the closed-form solution, Eq. 2.125:

$$
\begin{aligned}
I_{nlm} &= \int_r e^{-r^2 t} (r \cdot q)^{2m} p_{I_{2n}} S^{(2n+2\ell)}_{I_{2n}K_{2\ell}} r_{K_{2\ell}} \\
&= (2n-1)!!(2\ell-1)!!(p^2)^n q_{J_{2m}} \int_r e^{-r^2 t} (r^2)^\ell {}_2F_1\left(-n, -\ell; \frac{1}{2}; \Pi_{pr}\right) r_{J_{2m}}.
\end{aligned}
\tag{2.140}
$$

the hypergeometric function can be replaced by its series representation, whereafter the momentum integral is ready for another tensor decomposition:

$$
\begin{aligned}
I_{nlm} &= (2n-1)!!(2\ell-1)!!(p^2)^n q_{J_{2m}} \sum\!\!\!\!\!\!\int_{r,k} e^{-r^2 t}(r^2)^\ell \frac{(-n)_k(-\ell)_k}{(1/2)_k k!}\left(\frac{(p\cdot r)^2}{p^2 r^2}\right)^k r_{J_{2m}} \\
&= (2n-1)!!(2\ell-1)!!(p^2)^n q_{J_{2m}} \sum_{k=0}^{\infty} \frac{(-n)_k(-\ell)_k}{(1/2)_k k!}\frac{p_{I_{2k}}}{(p^2)^k}\int_r e^{-r^2 t}(r^2)^{\ell-k} r_{I_{2k}J_{2m}} \\
&= (2n-1)!!(2\ell-1)!!(p^2)^n \sum_{k=0}^{\infty} \frac{(-n)_k(-\ell)_k}{(1/2)_k k!}\frac{P^E_{km}}{(p^2)^k}\frac{1}{(d)_{k+m,2}}\int_r e^{-r^2 t}(r^2)^{\ell+m} \\
&= (2n-1)!!(2\ell-1)!!(p^2)^n \sum_{k=0}^{\infty} \frac{(-n)_k(-\ell)_k}{(1/2)_k k!}\frac{P^E_{km}}{(p^2)^k}\frac{(4\pi t)^{-d/2}}{2^{m+k} t^{m+\ell}}\frac{\Gamma(d/2+m+\ell)}{\Gamma(d/2+m+k)},
\end{aligned}
\tag{2.141}
$$

where in the third line we used

$$
p_{I_{2k}} S^{(2k+2m)}_{I_{2k}J_{2m}} q_{J_{2m}} = P^E_{km}. \tag{2.142}
$$

We again replace P^E_{km} with its closed form to write

$$
I_{nlm} = 2^\ell \alpha_{n\ell m}\frac{(4\pi t)^{-d/2}}{(2t)^{m+\ell}} \sum_{k=0}^{\infty} \frac{(-n)_k(-\ell)_k}{(1/2)_k}\frac{(2k-1)!!}{2^k k!}\frac{(d/2+m)_\ell}{(d/2+m)_k}\,{}_2F_1\left(-k,-m;\frac{1}{2};\Pi_{pq}\right), \tag{2.143}
$$

where we have defined

$$
\alpha_{n\ell m} := (2n-1)!!(2\ell-1)!!(2m-1)!!(p^2)^n(q^2)^m. \tag{2.144}
$$

The Legendre duplication formula allows us to cancel $(2k-1)!!$ in the numerator with $2^k(1/2)_k$ in the denominator. Then replacing the hypergeometric function with its series definition, we have

$$
I_{nlm} = 2^\ell \alpha_{n\ell m}\frac{(4\pi t)^{-d/2}}{(2t)^{m+\ell}} \sum_{k=0}^{\infty} \frac{(-n)_k(-\ell)_k}{k!}\frac{(d/2+m)_\ell}{(d/2+m)_k}\sum_{j=0}^{\infty}\frac{(-k)_j(-m)_j}{(1/2)_j}\frac{\Pi^j_{pq}}{j!}. \tag{2.145}
$$

Because $I_{n\ell m} \propto Q_{n\ell m}$, and because $Q_{n\ell m}$ is clearly finite by definition, we can reorder the sums, and use the fact that the factor of $(-n)_k(-\ell)_k$ forces the second sum to terminate:

$$I_{nlm} = 2^{\ell}\alpha_{n\ell m}\frac{(4\pi t)^{-d/2}}{(2t)^{m+\ell}}\sum_{j=0}^{\infty}\frac{(-m)_j}{(1/2)_j}\frac{\Pi_{pq}^{j}}{j!}\sum_{k=0}^{\min(n,\ell)}\frac{(d/2+m)_\ell}{(d/2+m)_k}\frac{(-k)_j(-n)_k(-\ell)_k}{k!}. \tag{2.146}$$

We now use the reflection formula to rewrite

$$(-k)_j = (-1)^j(k-j+1)_j. \tag{2.147}$$

We also use the ratio formulae,

$$\frac{(x)_m}{(x)_n} = \begin{cases} (x+m)_{n-m} & n \geq m \\ \frac{1}{(x+n)_{m-n}} & m \geq n, \end{cases} \tag{2.148}$$

to rewrite

$$(-n)_k(-\ell)_k = (-n)_j(-\ell)_j(j-n)_{k-j}(j-\ell)_{k-j} \tag{2.149}$$

and

$$(d/2+m)_k = (d/2+m)_j(d/2+m+j)_{k-j}, \tag{2.150}$$

so that

$$I_{nlm} = 2^{\ell}(d/2+m)_\ell\alpha_{n\ell m}\frac{(4\pi t)^{-d/2}}{(2t)^{m+\ell}}\sum_{j=0}^{\infty}\frac{(-n)_j(-\ell)_j(-m)_j}{(1/2)_j(d/2+m)_j}\frac{\Pi_{pq}^{j}}{j!} \times \sum_{k=0}^{\min(n,\ell)}\frac{(j-n)_{k-j}(j-\ell)_{k-j}}{(d/2+m+j)_{k-j}}\frac{(-1)^j(k-j+1)_j}{k!}. \tag{2.151}$$

The second sum can be reindexed, so that

$$\sum_{k=0}^{\min(n,\ell)} \frac{(j-n)_{k-j}(j-\ell)_{k-j}}{(d/2+m+j)_{k-j}} \frac{(k-j+1)_j}{k!} = \sum_{k=0}^{\infty} \frac{(j-n)_k(j-\ell)_k}{(d/2+m+j)_k k!} = {}_2F_1\left(n-j, \ell-j, \frac{d}{2}+m+j; 1\right), \tag{2.152}$$

which is further simplified by using Gauss' summation formula, Eq. 2.133:

$$\sum_{k=0}^{\min(n,\ell)} \frac{(j-n)_{k-j}(j-\ell)_{k-j}}{(d/2+m+j)_{k-j}} \frac{(k-j+1)_j}{k!} = \frac{\Gamma(d/2+m+j)\Gamma(d/2+n+\ell+m-j)}{\Gamma(d/2+m+n)\Gamma(d/2+m+\ell)}. \tag{2.153}$$

Rearranging the gamma functions into Pochhammer symbols, the final sum is

$$I_{nlm} = 2^\ell (d/2+m+n)_\ell \alpha_{n\ell m} \frac{(4\pi t)^{-d/2}}{(2t)^{m+\ell}} \sum_{j=0}^{\infty} \frac{(-n)_j(-\ell)_j(-m)_j}{(1/2)_j(1-d/2-n-\ell-m)_j} \frac{\Pi_{pq}^j}{j!}, \tag{2.154}$$

which is easily recognized as another generalized hypergeometric series:

$$I_{nlm} = \alpha_{n\ell m}(d+2m+2n)_{\ell,2} \frac{(4\pi t)^{-d/2}}{(2t)^{m+\ell}} {}_3F_2\left(-n, -\ell, -m; \frac{1}{2}, 1-\frac{d}{2}-n-\ell-m; \Pi_{pq}\right). \tag{2.155}$$

Combining this with Eq. 2.139, we reach the final result:

$$Q^E_{n\ell m} = \alpha_{n\ell m}(d+2m+2n)_{\ell,2}\ {}_3F_2\left(-n, -\ell, -m; \frac{1}{2}, 1-\frac{d}{2}-n-\ell-m; \Pi_{pq}\right). \tag{2.156}$$

2.5.4 Generalization

The calculations in the last three subsections have held to a pattern. In order to decompose the tensor integrals that appear in our Taylor series, we always need to find a way to contract strings of isotropic tensors S and some external momenta with some distribution of shared indices. The structure of the isotropic tensors may hint at a more general framework for computing quantities like $P_{nm}^{E,O}$ and $Q_{n\ell m}^{E,O}$. For each of these, we needed to attach some number of of external momenta p_1 to another momenta p_2 through all perfect matchings of their indices. It is tempting to try to embed this structure into a Brauer algebra $\mathfrak{B}_n(d)$ [86].

A Brauer algebra is an algebra over the ring $\mathbb{Z}[d]$ for some indeterminate number d with a basis of all perfect matchings of two sets $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$. This is most easily described in terms of Brauer diagrams, where the set $\{x_i\}$ is presented as a set of graph vertices in a horizontal line, and $\{y_i\}$ is a set of vertices placed in a parallel line below them. The matching defines a set of edges each joining any two of these $2n$ vertices. The composition law is concatenation of diagrams, where, for some diagrams A and B, $A \cdot B$ is formed by placing A above B and identifying the y vertices from A with the x vertices from B. When the concatenation of A and B forms some ℓ loops, the loop is deleted from the resulting diagram, product acquires a factor of d^ℓ, $A \cdot B = d^\ell AB$.

This is very similar to our construction. An S tensor is a sum of all perfect matchings of its indices, where the matchings are represented as pairs of indices distributed across products metric tensors. The indices can be identified with the vertex sets of a Brauer diagram, and the metric tensors form the edge set. Then contraction of any two tensors is equivalent to the sum over concatenations of the corresponding Brauer diagrams. In particular, a loop formed under concatenation of Brauer diagrams is the same as a trace formed by the contraction of isotropic tensors. Each contributes a factor of d and drops out of the resultant matching. In the context of the isotropic tensors, d is just the dimension of the underlying spacetime.

While the exact correspondence between the contraction formulae and Brauer algebras has not been fully worked out yet, there is a rich set of literature on the associated topics. Of particular interest are $\mathfrak{B}_n(d)$-modules, which are seemingly very closely related to the structure of the S tensors. This work is ongoing.

Chapter 3

Quantum Chromodynamics

3.1 Yang-Mills Theory

Quantum chromodynamics is the gauge theory for the strong interaction. It is easily constructed from only a few empirical principles. We begin with the fermionic Lagrangian, describing a set of uncoupled spin-1/2 quarks with $n_f = 6$ flavors:

$$\mathcal{L}_f = \sum_{i=1}^{n_f} \bar{\psi}_i(\not{\partial} + m_i)\psi_i \tag{3.1}$$

From now on the sum over flavors will be assumed, and we will drop the corresponding indices. The interacting theory is determined by imposing a local gauge symmetry with the condition of renormalizability. In order to choose a symmetry group, we observe that the branching fractions for mounic and hadronic decay channels in electron-positron strongly suggest that quarks compose a triplet representation of their gauge group [47]. Since $SO(3)$ and $SU(3)$ are the only compact, simple Lie groups up to isomorphism with three-dimensional irreps, it must be one of these two. However, quarks cannot be their own antiparticles, so we require a complex representation, which exclusively establishes $SU(3)$ as the gauge group. The fundamental triplet representation is defined over a vector space of three basis "color" charges held by the quarks. For full generality in choosing the number of colors N, we instead study $SU(N)$ for the remainder of this text.

Now, as in Sec. 1.1, in order that the Lagrangian maintain invariance under local gauge transformations, the derivative term must be made to transform covariantly. The true reason for our earlier construction is the locality of the transformation. Since this means ω is coordinate-dependent, the infinitesimally-separated fields under the derivative transform separately, and we need to consider their parallel transport on a path across the displacement:

$$\delta\psi = A_\mu \psi. \tag{3.2}$$

where A_μ is the gluon field, which assumes values in the Lie algebra $\mathfrak{su}(3)$. Adding this to the naïve derivative yields the gauge covariant derivative,

$$D_\mu \psi = (\partial_\mu + A_\mu)\psi, \tag{3.3}$$

which transforms as desired; *viz.*, for some $U \in SU(3)$,

$$D_\mu \psi \xrightarrow{U} D'_\mu \psi' = U D_\mu \psi. \tag{3.4}$$

Since $SU(N)$ is a matrix Lie group, the covariant derivative acts on fields in the fundamental representation – fermions in the present case – by multiplication as in Eq. 3.3. The gluons instead assume the adjoint representation, for which the connection is simply $ad_A(\cdot) = [A, \cdot]$, so the covariant derivative acts accordingly:

$$D_\mu A_\nu = \partial_\mu A_\nu + [A_\mu, A_\nu] = \left(\partial_\mu A^a_\nu + f^{abc} A^b_\mu A^c_\nu\right) t^a. \tag{3.5}$$

The replacement $\partial \to D$ adds an interaction piece to the fermionic Lagrangian,

$$\mathcal{L}_{int} = \bar{\psi} \not{A} \psi, \tag{3.6}$$

and may be identified with the minimal coupling prescription. We have now traded gauge variance for a new vector field, A, which further requires its own free Lagrangian in order to be dynamical. The Proca Lagrangian,

$$\mathcal{L}_P = \frac{1}{g^2}\,\mathrm{Tr}\left[\frac{1}{2}G_{\mu\nu}G^{\mu\nu} + M^2 A_\mu A^\mu\right], \tag{3.7}$$

where $G_{\mu\nu} = [D_\mu, D_\nu]$, describes free vector particles of mass M. The tensor $G_{\mu\nu} = G^a_{\mu\nu}t^a$ is known as the field-strength (curvature) tensor for the field A. The mass term in the above Lagrangian is not gauge invariant[1]; setting $M = 0$, we arrive at the Yang-Mills Lagrangian:

$$\mathcal{L}_{YM} = \frac{1}{2g^2}\,\mathrm{Tr}\,G_{\mu\nu}G^{\mu\nu}. \tag{3.8}$$

We now have the nonperturbative QCD Lagrangian, containing kinetic terms for both massive fermions and massless gauge bosons — respectively the quarks and gluons — and a minimal interaction term between them:

$$\mathcal{L}_{QCD} = \mathcal{L}_f + \mathcal{L}_{YM} + \mathcal{L}_{int} = \bar{\psi}(\not{D} + m)\psi + \frac{1}{2g^2}\,\mathrm{Tr}\,G_{\mu\nu}G^{\mu\nu}. \tag{3.9}$$

The gluon Lagrangian also hides two self-interactions generated by nonlinearities in the field-strength tensor

$$G^a_{\mu\nu} = \partial_{[\mu}A^a_{\nu]} + f^{abc}A^b_\mu A^c_\nu. \tag{3.10}$$

Because the gauge group in nonabelian, the commutator is generically nonzero, and the product $G_{\mu\nu}G^{\mu\nu}$ contains quadratic, cubic, and quartic interactions.

[1] This is not necessarily problematic, since the mass term may be acquired through a spontaneously broken symmetry, but there is no physical indication that the $SU(3)$ symmetry is broken.

From here, the generating functional produces all correlation functions:

$$\left\langle \prod_{i=1}^{n_G} G_{\mu_i}(z_i) \prod_{j=1}^{n_\psi} \psi(y_j) \prod_{k=1}^{n_\psi} \bar{\psi}(x_k) \right\rangle = \prod_{i=1}^{n_G} \frac{-i\delta}{\delta J^{\mu_i}(z_i)} \prod_{j=1}^{n_\psi} \frac{i\delta}{\delta \bar{\eta}(y_j)} \prod_{k=1}^{n_\psi} \frac{-i\delta}{\delta \eta(x_k)} \frac{1}{Z_0} \int \mathcal{D}[A, \psi, \bar{\psi}] e^{iS} \bigg|_{\bar{\eta},\eta,J=0}, \tag{3.11}$$

where we have inserted the appropriate numerical factors respecting fermionic statistics and modified the action to include the appropriate sources, as in Sec. 1.2:

$$S = \int d^4x \left[\mathcal{L}_{QCD} + J_\mu A^\mu + \bar{\psi}\eta + \bar{\eta}\psi\right]. \tag{3.12}$$

Here, in accordance with the fields they source, the J field is a Lorentz vector taking values in the adjoint representation of $SU(3)$, while the $\eta, \bar{\eta}$ fields are Grassmann-valued spinors.

In order to study QCD perturbatively, we must fix the gauge. Following the Faddeev-Popov procedure as before, we introduce two new terms to the action for the Faddeev-Popov and gauge-fixing Lagrangians (Eqs. 1.45 and 1.50) defined in an R_ξ gauge. The total Lagrangian is thus

$$\mathcal{L}_{QCD} = \mathcal{L}_D + \mathcal{L}_{YM} + \mathcal{L}_{gf} + \mathcal{L}_{FP} + \mathcal{L}_J, \tag{3.13}$$

where

$$\mathcal{L}_D = \mathcal{L}_f + \mathcal{L}_{int} = \bar{\psi}(\not{D} + m_i)\psi, \tag{3.14a}$$

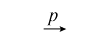
$$\mathcal{L}_{YM} = \frac{1}{2g^2} \operatorname{Tr} G_{\mu\nu} G^{\mu\nu}, \tag{3.14b}$$

$$\mathcal{L}_{gf} = \frac{1}{g^2\xi} \operatorname{Tr}\left(\partial_\mu A^\mu\right)^2, \tag{3.14c}$$

$$\mathcal{L}_{FP} = \frac{1}{g^2 T_F} \operatorname{Tr} \bar{c}(\partial_\mu D^\mu) c, \tag{3.14d}$$

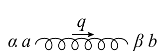
$$\mathcal{L}_J = J_\mu A^\mu + \bar{\psi}\eta + \bar{\eta}\psi + \bar{c}\kappa + \bar{\kappa}c, \tag{3.14e}$$

and we have introduced the Grassmann-odd, scalar source fields $\kappa, \bar{\kappa}$ for the ghosts. For now, all fields and parameters should be considered bare; we exclude the subscript zero for notational convenience. We may now construct the two-point Green functions, or propagators, for the fermions and gluons. Since we are calculating two-point functions, the only contributions at leading order in the coupling come from the kinetic part of the action, which is strictly quadratic in the fields. Following the Gaussian integration procedure in Sec. 2.1.1, we have three propagators. The fermion propagator is

$\langle\tilde{\psi}\tilde{\bar{\psi}}\rangle :$ 

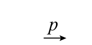
$$= \tilde{S}_F(p) = i\frac{\not{p} - m}{p^2 + m^2 - i\varepsilon}. \tag{3.15a}$$

We have already considered the gauge propagator in Secs. 1.5 and 2.1.1. In QCD, since the gauge group is $SU(N)$, we can replace $T_F = -1/2$ to avoid a proliferation of Dynkin indices. The gluon propagator is then

$\langle\tilde{A}\tilde{A}\rangle :$ 

$$= \tilde{D}_{F\,\alpha\beta}^{\;ab}(q) = -g^2\frac{i\delta^{ab}}{q^2 - i\varepsilon}\left[g_{\alpha\beta} - (1-\xi)\frac{q_\alpha q_\beta}{q^2 - i\varepsilon}\right]. \tag{3.15b}$$

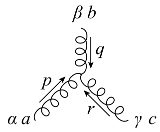
We did not consider the ghost propagator before, but its treatment is no different than that of any other field. Since free ghosts obey the Laplace (or Poisson for nonzero $i\varepsilon$) equation, their propagator is the well-known fundamental solution,

$\langle\tilde{c}\tilde{\bar{c}}\rangle :$ 

$$= \tilde{D}_F^{ab}(p) = -i\frac{g^2\delta^{ab}}{p^2 - i\varepsilon}. \tag{3.15c}$$

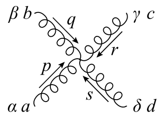
There are also four vertices involving higher powers of the fields. As we mentioned before, the Yang-Mills action contains vertices with three and four gluons:

$\langle \tilde{A}^3 \rangle$:



$$= -\frac{f^{abc}}{g^2}\left[(p-q)_\gamma g_{\alpha\beta} + (q-r)_\alpha g_{\beta\gamma} + (r-p)_\beta g_{\gamma\alpha}\right], \tag{3.16a}$$

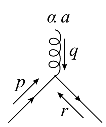
$\langle \tilde{A}^4 \rangle$:



$$\begin{aligned} = \frac{i}{g^2}\Big[& f^{abe} f^{cde} (g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\gamma\beta}) \\ & + f^{ace} f^{bde} (g_{\alpha\beta} g_{\gamma\delta} - g_{\alpha\delta} g_{\gamma\beta}) \\ & + f^{ade} f^{bce} (g_{\alpha\beta} g_{\gamma\delta} - g_{\alpha\gamma} g_{\beta\delta})\Big]. \end{aligned} \tag{3.16b}$$

The quark-quark-gluon vertex arises as a result of promoting the derivative to a covariant derivative, as in Sec. 1.1:

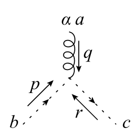
$\langle \tilde{\psi} \tilde{A} \tilde{\bar{\psi}} \rangle$:



$$= i\gamma_\alpha t^a. \tag{3.16c}$$

Likewise, at nonzero coupling the covariant derivative in the Faddeev-Popov action generates a ghost-ghost-gluon vertex:

$\langle \tilde{c} \tilde{A} \tilde{\bar{c}} \rangle$:



$$= -\frac{f^{abc}}{g^2} r_\alpha. \tag{3.16d}$$

3.2 Renormalization

Now that we have the Feynman rules for QCD, we can in principle calculate correlation functions to any order – at least in terms of momentum integrals. Nevertheless, one encounters difficulties extracting analytical results already at next-to-leading order.[2] After the tree-level, a new vertex appears for each factor of the interaction Lagrangian in Eq. 1.32,

[2] Except for vacuum correlation functions of nontrivial operators, whose leading order diagrams are already one-loop

and the extra legs contract to form loops. These fluctuations correspond to the mixing of all couplings in the Lagrangian. Hence, these diagrams represent the renormalizations of the bare fields and parameters. QCD contains six bare parameters: the normalizations of the fermion, gluon, and ghost wavefunctions; the strong coupling g_0; the fermion mass m_0, and the gauge-fixing parameter ξ_0. The renormalized fermions and masses are simply written

$$\psi_0 = Z_\psi \psi, \qquad \overline{\psi}_0 = \overline{\psi} Z_\psi, \qquad m_0 = Z_m m. \tag{3.17}$$

The gauge sector is not so straightforward. The parameter ξ_0 in particular requires renormalization in order to maintain gauge invariance. To see this, let us briefly consider the geometric series for the gluon propagator (see Eqs. 1.87 and 3.15b),

$$\tilde{D}^{ab}_{\alpha\beta}(q) = \tilde{D}_{F\alpha\beta}^{\;ab}(q) + \tilde{D}_{F\alpha\mu}^{\;ac}(q)\tilde{\Pi}^{(1)^{\mu\nu}\;cd}(q)\tilde{D}_{F\nu\beta}^{\;db}](q) + \cdots, \tag{3.18}$$

where $\tilde{\Pi}^{ab}_{\alpha\beta}(q)$ represents the sum over all 1PI diagrams in the propagator of a gauge boson. Lorentz invariance and the conservation of gluon charge restrict its form to $iq^2\delta^{ab}\left(\pi_1 g_{\alpha\beta} - \pi_2 \frac{q_\alpha q_\beta}{q^2}\right)$ for some functions $\pi_{1,2}(q)$. According to the transformation law for A, Eq. 1.7, a gauge transformation shifts any physical state by a total derivative that must identically vanish in order to uphold gauge invariance. In momentum space, this is the Slavnov-Taylor (ST) identity

$$q^\alpha \tilde{\Pi}^{ab}_{\alpha\beta}(q) = 0. \tag{3.19}$$

This additionally requires that $\pi_1 = \pi_2 =: \Pi$, so that only the transverse polarizations of the gluon propagator receive loop corrections. Since the gauge-fixing term is entirely longitudinal, we must ensure that it does not acquire an anomalous dimension. If we write

$$g_0 = \mu^\epsilon Z_g g, \qquad \xi_0 = Z_\xi \xi, \qquad A_0 = Z_A^{1/2} A, \tag{3.20}$$

the corresponding renormalized Lagrangian goes as $\frac{Z_A}{Z_\xi Z_g^2}$, which is restricted to one by the ST identity. This evidently allows us to discard one of these constants. To make later manipulations cleaner, we choose $Z_A = Z_\xi Z_g^2$.

Typically, the ghosts are renormalized symmetrically:

$$c_0 = Z_c^{1/2} c, \qquad \bar{c}_0 = \bar{c} Z_c^{1/2}. \tag{3.21}$$

We choose a different prescription in order to simplify the notation of Sec. 6.4. Like the gluon field, we have absorbed the coupling into the ghost fields,

$$gc \to c, \qquad g\bar{c} \to \bar{c} \qquad , \tag{3.22}$$

allowing us to write the ghost and gauge-fixing Lagrangians as a single, manifestly closed BRST variation, Eq. 1.65. With this rescaling, we choose

$$c_0 = Z_g Z_\xi^{1/2} Z_c c, \qquad \bar{c}_0 = \bar{c} Z_g Z_\xi^{-1/2}. \tag{3.23}$$

Modulo the rescaling by g, this choice is equivalent to Eq. 3.21, as the ghost renormalization is only relevant to $\bar{c}c$ pairs. Further, this asymmetric convention ensures that no renormalization constants enter the Slavnov-Taylor identities of flowed QCD, Part II.

We can now calculate all five renormalization constants with only four correlation func-

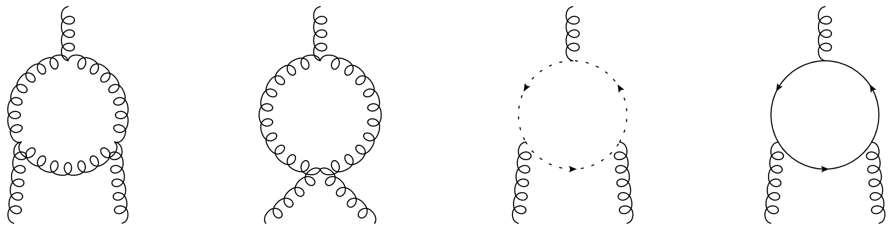


Figure 3.1: One-loop contributions to the three-gluon vertex

tions. Z_ξ may be extracted from the gluon propagator, Z_ψ and Z_m from the quark propagator, and Z_c from the ghost propagator. Z_g is computed with the gluon three-point function; this calculation is long and uses no new methods, so we simply quote the result for Z_g. It would typically be found to cancel the total divergence of four amputated diagrams, shown in Fig. 3.1. Since the tree-level is of order g_0^2, the renormalized three-point function will be proportional to Z_g^2. No factors of Z_A appear, since we amputate the legs. In dimensional regularization, the total pole may be canceled at leading order by setting

$$Z_g = 1 + \frac{g^2}{(4\pi)^2}\left(\frac{11}{6}T_A - \frac{2}{3}T_F n_f\right)\frac{1}{\epsilon} + \mathcal{O}(g^4). \tag{3.24}$$

For the remaining renormalization constants, we proceed in bare perturbation theory with a regulated dimension $d = 4 - 2\epsilon$. These are standard results, but for demonstration the following calculations use the robust method introduced in Sec 2.5 of this thesis to decompose difficult angular integrals in dimensional regularization.

3.2.1 Gluon Self-Energy

The perturbative expansion of the gluon propagator was given in Eq. 3.18. In this section we will calculate the one-loop correction $\tilde{\Pi}^{ab}_{\alpha\beta}(q)$. There are four Feynman diagrams at this order, pictured in Fig. 3.2. Two are totally gluonic, coming from the nonlinear terms in the gauge action. The third contains a ghost loop that exactly cancels the longitudinal terms in the pure-gauge diagrams. The last diagram has a quark loop representing the screening of the gluon field by virtual quark-antiquark pairs. We label the diagrams $[\Gamma_i]^{ab}_{\alpha\beta}(q)$ for $i \in \{a, b, c, d\}$ and set $\xi_0 = 1$ for the time being. The full results will be listed at the end. Using the Feynman rules, Eqs 3.15-3.16, diagram (a) may be written

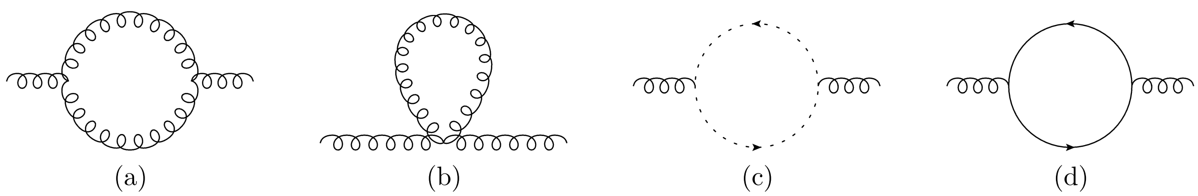


Figure 3.2: One-loop contributions to the vacuum polarization of the gluon field

$$\Gamma_a = -\int_k \frac{f^{acd}}{g_0^2}\left\{(q-k)_\delta g_{\alpha\gamma} + (q+2k)_\alpha g_{\gamma\delta} - (2q+k)_\gamma g_{\delta\alpha}\right\}\frac{g_0^2}{(q+k)^2 - i\varepsilon}$$
$$\times\frac{f^{bdc}}{g_0^2}\left\{-(2q+k)^\gamma g^\delta_\beta + (q+2k)_\beta g^{\delta\gamma} + (q-k)^\delta g^\gamma_\beta\right\}\frac{g_0^2}{k^2 - i\varepsilon}. \tag{3.25}$$

The structure constants simplify to $T_A\delta^{ab}$. To simplify the algebra, we define the three tensors,

$$N_{\alpha\beta} = (d-6)q_{\alpha\beta} + 5q^2 g_{\alpha\beta},$$
$$E_{\alpha\beta}{}^{\mu} = (2d-3)q_\alpha g^\mu_\beta + 2q^\mu g_{\alpha\beta}, \tag{3.26}$$
$$Z_{\alpha\beta}{}^{\mu\nu} = (2d-3)(g^\mu_\alpha g^\nu_\beta + g^\nu_\alpha g^\mu_\beta) + 2g_{\alpha\beta}g^{\mu\nu},$$

so that the numerator of the integrand may be written in powers of k:

$$\Gamma_a = -T_A\delta^{ab}\int_k \frac{N_{\alpha\beta} + E_{\alpha\beta}{}^{\mu}k_\mu + Z_{\alpha\beta}{}^{\mu\nu}k_{\mu\nu}}{(k^2 - i\varepsilon)((q+k)^2 - i\varepsilon)}. \tag{3.27}$$

Now, in order to perform the integral in d-dimensions, we must write the integrand as a spherically-symmetric function; that is, a function of only the magnitude k^2. The only impediment is the cross-term $2k{\cdot}q$ in the denominator. One could treat this with Feynman or Schwinger parametrization, which amount to completing the square in the denominator with some parametric integral and absorbing the shift into k by translation invariance. Instead, we propose to use a modified[3] Schwinger parametrization to write

$$\frac{1}{(q+k)^2 - i\varepsilon} = i\int_0^\infty dz\ e^{-i(q+k)^2 z - \varepsilon z} = i\int_0^\infty dz\ e^{-iq^2 z - ik^2 z - \varepsilon z}\sum_{n=0}^{\infty}\frac{(-2iz)^n}{n!}q^{I_n}k_{I_n}, \quad (3.28)$$

where the second equivalence is found by Taylor-expanding the cross-term $e^{-2izq\cdot k}$. Now the denominator is strictly symmetric, and the numerator is a polynomial in the components of k. All terms with an odd power of k will vanish due to the symmetry of the integration range, while the even terms may be decomposed to scalar integrals. We then split the sum into odd and even terms, reindexing $n \to 2n$ for the terms containing $N_{\alpha\beta}$ and $Z_{\alpha\beta}{}^{\mu\nu}$ and $n \to 2n+1$ for the term with $E_{\alpha\beta}{}^{\mu}$. After some simplification, we have

$$\begin{aligned}\Gamma_a = -iT_A\delta^{ab} \sum\!\!\!\!\!\!\int_{n,z} \frac{(2iz)^{2n}}{(2n)!} q^{I_{2n}} e^{-iq^2 z - \varepsilon z} \int_k \frac{e^{-ik^2 z}}{k^2 - i\varepsilon} k_{I_{2n}} \\ \times\left(N_{\alpha\beta} - \frac{2iz}{2n+1} q^{\mu_{2n+1}} E_{\alpha\beta}{}^{\mu} k_{\mu\mu_{2n+1}} + Z_{\alpha\beta}{}^{\mu\nu} k_{\mu\nu}\right),\end{aligned} \quad (3.29)$$

This form is ready for tensor decomposition. Using Eq. 2.83, we make the replacements

$$k_{I_{2n}} \to W_n S^{(2n)}_{I_{2n}} (k^2)^n \quad (3.30)$$

and utilize the contraction formulae for the isotropic tensors S, Eqs. 2.105-2.108. For the term with no accompanying powers of k, all indices are contracted, and we can write $q^{I_{2n}} S^{(2n)}_{I_{2n}} = (2n-1)!!(q^2)^n$. Similarly, for the terms of order one and two, we write $q^{I_{2n+1}} S^{(2n+2)}_{I_{2n+1}\mu} =$

[3] This form is useful in Minkowski space, where the square of any momentum needn't be positive.

$(2n{+}1)!!(q^2)^n q_\mu$ and $q^{I_{2n}} S^{(2n+2)}_{I_{2n}\mu\nu} = (2n{-}1)!!(q^2)^n \left(\delta_{\mu\nu} + 2n q_\mu q_\nu / q^2\right)$ respectively. Now, after performing all contractions, using $W_n = (d{+}2n)_{m,2} W_{n+m}$, and simplifying the combinatorial factors, we are able to take the momentum integrals to spherical coordinates. First, we transform to Euclidean spacetime with the Wick rotation $k_0 = ik_4$, so that

$$\int d^d k \ f(k^2) = i \int d^d x_E \ f(k_E^2). \tag{3.31}$$

In our case, the integrals are of the form

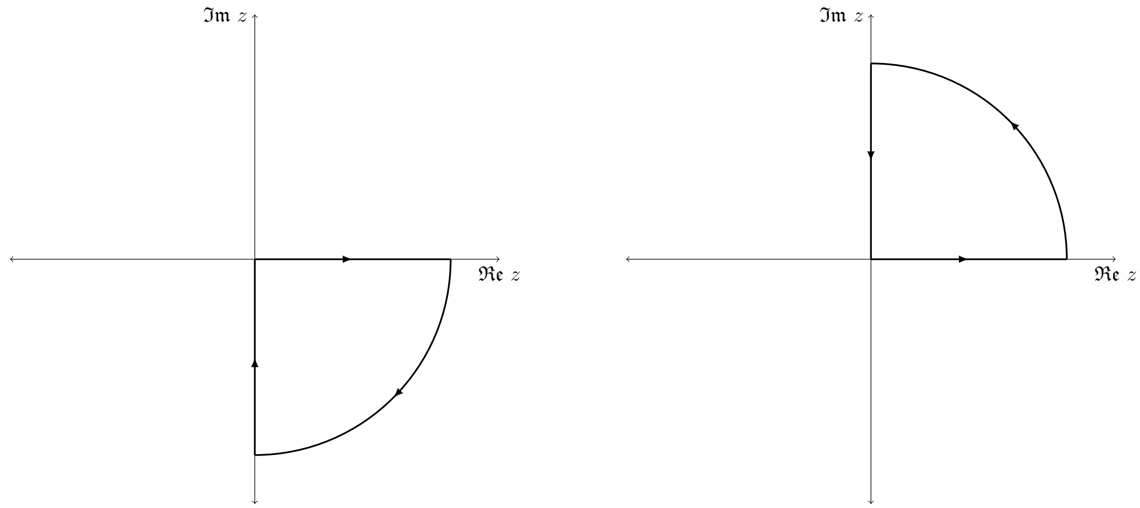
$$\int_0^\infty dz \ e^{-iq^2 z - \varepsilon z} \int_{k_M} e^{-ik_M^2 z} (k_M^2)^{n-1} z^m = i \int_0^\infty dz \ e^{-iq^2 z - \varepsilon z} \int_{k_E} e^{-ik_E^2 z} (k_E^2)^{n-1} z^m \tag{3.32}$$

for some nonnegative integers n and m. The momentum integral does not converge for real z, but we can circumvent this problem by writing the Schwinger integral as a contour integral (Fig. 3.3) about a quarter circle in the fourth (first) quadrant of the complex plane for positive (negative) $q^2 + k^2$, not unlike a Wick rotation. Since the Schwinger integral runs over an entire function, there are no poles enclosed, and the entire contour integral vanishes. We can further prove that the quarter-circular contours give a vanishing contribution in the limit of large R. For $Q \coloneqq q^2 + k^2 \gtrless 0$, the integrals are

$$\int_{\mathcal{C}_R^\pm} dz \ e^{\mp i|Q|z - \epsilon z} z^m = i \int_0^{\mp\pi/2} d\theta \ e^{-(\pm i|Q| + \epsilon) R e^{i\theta}} \left(Re^{i\theta}\right)^{m+1} = -\frac{\Gamma(m+1, (\pm i|Q| + \epsilon) R z)}{(\pm i|Q| + \epsilon)^{m+1}}\Bigg|_{z=1}^{\mp i}. \tag{3.33}$$

This quantity has the asymptotic behavior

$$-\frac{\Gamma(m+1, (\pm i|Q| + \epsilon) R z)}{(\pm i|Q| + \epsilon)^{m+1}} \overset{R\to\infty}{\sim} -\left((\pm i|Q| + \epsilon) R z\right)^m \exp\{-(\pm i|Q| + \epsilon) R z\} \tag{3.34}$$



(a) $q^2 + k^2 > 0$ (b) $q^2 + k^2 < 0$

Figure 3.3: Contours for rotating the Schwinger parameter z

for large R, so that the contributions from $\mathcal{C}_R^{\pm}$ diminish exponentially. The remaining integrals along the imaginary z-axis then allow us to write the relation

$$\int_0^R dz\; e^{\mp i|Q|z-\varepsilon z} z^m = (\mp i)^{m+1} \int_0^R dz\; e^{-|Q|z \pm i\varepsilon z} z^m, \tag{3.35}$$

and we can now discard ε. The full integral then becomes

$$\int_0^\infty dz\; e^{-iq^2 z} \int_{k_M} e^{-ik_M^2 z} (k_M^2)^{n-1} z^m = \pm \int_0^\infty dz\; e^{-q^2 z} \int_{k_E} e^{-k_E^2 z} (k^2)^{n-1} (\mp i z)^m \tag{3.36}$$

for $q^2 + k_E^2 \gtrless 0$. We now drop all Euclidean labels and write the result in spherical coordinates with Eq. 2.52:

$$\int_0^\infty dz\; e^{iq^2 z} \int_{k_M} e^{ik_M^2 z} (k_M^2)^{n-1} z^m = \pm \frac{(4\pi)^{-d/2}}{\Gamma(d/2)} \int_0^\infty dz\; e^{-q^2 z} (\mp i z)^m \frac{\Gamma(d/2+n-1)}{z^{d/2+n-1}}. \tag{3.37}$$

Putting all of this together and changing variables to $\zeta = q^2 z \geq 0$[4], the total is

$$\Gamma_a = -2iT_A \frac{\delta^{ab}}{q^2}\left(\frac{4\pi}{q^2}\right)^{-d/2} \sum\!\!\!\!\!\!\!\int_{n,\zeta} \frac{(\zeta)^{n-d/2}}{(d+2n-2)(d+2n)n!} e^{-\zeta}$$
$$\times \Big\{ \left[(d+2n-2)(3d+2n-3) + \zeta(3d+6n+4)\right] \delta_{\alpha\beta}$$
$$+ \left[2n(2d-3)(d+2n-2) - \zeta(d(3d+6n-8)+12)\right] \frac{q_{\alpha\beta}}{q^2} \Big\}. \tag{3.38}$$

Summing over n would produce an upper incomplete gamma function, so it is simpler to first integrate over ζ; the operations commute for $d < 4$, so this is allowed. After this, we may sum over n. The result is

$$\Gamma_a = 2iT_A \delta^{ab} \frac{q^2}{(4\pi)^2}\left(\frac{4\pi}{q^2}\right)^{2-d/2} \frac{B(d/2,d/2)\Gamma(2-d/2)}{2-d} \left\{(6d-5)\delta_{\alpha\beta} - (7d-6)\frac{q_{\alpha\beta}}{q^2}\right\}. \tag{3.39}$$

Replacing $d = 4 - 2\epsilon$ and expanding about ϵ to zeroth order, the final expression for diagram (a) is

$$\Gamma_a = -\frac{1}{18}\frac{T_A}{(4\pi)^2} \cdot i\delta^{ab} q^2 \left\{(57L_\epsilon + 116)T_{\alpha\beta} - 9(L_\epsilon + 2)\Lambda_{\alpha\beta} + \mathcal{O}(\epsilon)\right\}, \tag{3.40}$$

where the divergent bit is contained within

$$L_\epsilon = \frac{1}{\epsilon} + \log\left(\frac{4\pi}{q^2}\right) - \gamma_E, \tag{3.41}$$

and we have defined the transverse projector,

$$T_{\alpha\beta} = \delta_{\alpha\beta} - \frac{q_{\alpha\beta}}{q^2}, \tag{3.42}$$

[4]Without loss of generality, we can now assume $q^2 > 0$. We should in principle write the integral in terms of $\pm|q|$ again, but we would find the same result in both cases, since the $\pm$ sign of Eq. 3.37 is absorbed into the measure after changing variables.

and the longitudinal projector,

$$\Lambda_{\alpha\beta} = \frac{q_{\alpha\beta}}{q^2}. \tag{3.43}$$

The result with generic ξ_0 is

$$\Gamma_a = -\frac{1}{18}\frac{T_A}{(4\pi)^2}\cdot i\delta^{ab}q^2\left\{((75-18\xi_0)L_\epsilon + 9\xi_0^2 + 18\xi_0 + 89)T_{\alpha\beta} - 9(L_\epsilon + 2)\Lambda_{\alpha\beta} + \mathcal{O}(\epsilon)\right\}, \tag{3.44}$$

The second diagram produces a scaleless integral; that is, the integrand contains no dimensionful quantities other than the integration variable. In dimensional regularization these are defined to vanish, as we will later discuss in Sec. 7.3. For now, we may formally define the integral by inserting the identity

$$\frac{1}{q^2 + i\varepsilon} = \frac{1}{q^2 + \mu^2 + i\varepsilon} + \frac{1}{q^2 + i\varepsilon}\frac{\mu^2}{q^2 + \mu^2 + i\varepsilon} \tag{3.45}$$

where q is the integration variable, and μ is some fictitious positive mass parameter. In the present case, the first integral will only converge for $0 < d < 2$, while the second converges for $2 < d < 4$. If we calculate the integrals anyway, they give opposite contributions in the $d \to 4$ limit, and the total integral vanishes. Hence,

$$\Gamma_b = 0. \tag{3.46}$$

The third and fourth diagrams involve the quark and ghost fields, but are otherwise treated the same. One subtlety is that the fermion loop introduces a trace over the quark flavors. For the case of mass-degenerate flavor singlets, this simply introduces a factor of n_f, the number of fermion flavors. In any case, we disregard the mass, since it is not needed for the computation of Z_ξ. The results for diagrams (c) and (d) are

$$\Gamma_c = -\frac{1}{36}\frac{T_A}{(4\pi)^2} \cdot i\delta^{ab} q^2 \left\{(3L_\epsilon + 8)T_{\alpha\beta} + 9(L_\epsilon + 2)\Lambda_{\alpha\beta} + \mathcal{O}(\epsilon)\right\}, \tag{3.47}$$

and

$$\Gamma_d = \frac{4}{9}\frac{T_F n_f}{(4\pi)^2} \cdot i\delta^{ab} q^2 \left\{(3L_\epsilon + 5)T_{\alpha\beta} + \mathcal{O}(\epsilon)\right\}. \tag{3.48}$$

All diagrams may now be summed with proper weights accounting for redundant contractions. In diagram (a), there is a factor of 1/2! accounting for the interchange of vertices, a factor of 1/3! for each vertex to count the Wick contractions within the Feynman rule, and a factor of 3! for each vertex to count the ways to contract them. The symmetry factors are

$$s_a = \frac{1}{2}, \qquad s_b = \frac{1}{2}, \qquad s_c = 1, \qquad \text{and} \qquad s_d = 1. \tag{3.49}$$

Notice that the ghost loop now exactly cancels the longitudinal degrees of freedom for diagram (a) (and trivially (b)). Further, since the quark loop is completely transverse, the full one-loop correction is as well:

$$\tilde{\Pi}^{ab}_{\alpha\beta}(q) = \Pi(q^2) \cdot iq^2 \delta^{ab} T_{\alpha\beta} = (s_i \Gamma_i) + \mathcal{O}(g_0^2). \tag{3.50}$$

Adding everything up, we find that

$$\Pi(q^2) = -\frac{1}{(4\pi)^2}\left[\left(\frac{13 - 3\xi_0}{6}T_A - \frac{4}{3}T_F n_f\right)L_\epsilon + \frac{1}{4}\left(\xi_0^2 + 2\xi_0 + \frac{97}{9}\right)T_A - \frac{20}{9}T_F n_f + \mathcal{O}(\epsilon)\right] + \mathcal{O}(g_0^2). \tag{3.51}$$

We may resum the 1PI propagator as in Eq. 3.18 by noting that not only is T idempotent,

$$T_{\alpha\gamma}T^{\gamma}_{\beta} = T_{\alpha\beta}, \tag{3.52}$$

but it also projects the longitudinal modes to zero,

$$T_{\alpha\gamma}\Lambda^{\gamma}_{\beta} = 0, \tag{3.53}$$

so that $TD^{(0)} \propto T$. Then the bare geometric series simplifies to

$$[\tilde{D}_0]^{ab}_{\alpha\beta}(q) = g_0^2 \frac{i\delta^{ab}}{q^2} \left\{ \sum_{n=0}^{\infty} \left(g_0^2 \Pi(q^2) \right)^n \cdot T_{\alpha\beta} + \xi_0 \Lambda_{\alpha\beta} \right\} = g_0^2 \frac{i\delta^{ab}}{q^2} \left\{ \frac{1}{1 - g_0^2 \Pi(q^2)} T_{\alpha\beta} + \xi_0 \Lambda_{\alpha\beta} \right\}. \tag{3.54}$$

Inserting the appropriate Z-factors, the exact renormalized propagator simplifies to

$$\tilde{D}^{ab}_{\alpha\beta}(q) = \lim_{\epsilon\to 0} \mu^{2\epsilon} g^2 \frac{i\delta^{ab}}{q^2} \left\{ \frac{Z_\xi^{-1}}{1 - \mu^{2\epsilon} Z_g^2 g^2 \Pi(q^2)} T_{\alpha\beta} + \xi \Lambda_{\alpha\beta} \right\}. \tag{3.55}$$

Expanding both Z_ξ and Π to leading order, we see that the quantity

$$\frac{1 - g^2 Z_\xi^{(1)} + \mathcal{O}(g^4)}{1 - \mu^{2\epsilon} g^2 \Pi^{(1)} + \mathcal{O}(g^4)} = 1 - g^2 Z_\xi^{(1)} + \mu^{2\epsilon} g^2 \Pi^{(1)} + \mathcal{O}(g^4) \tag{3.56}$$

must be finite. At leading order, we may then write

$$Z_\xi^{(1)} = \frac{1}{\epsilon} \operatorname{Res}\left\{ \mu^{2\epsilon} \Pi^{(1)}, \epsilon \to 0 \right\} = -\frac{1}{(4\pi)^2} \left(\frac{13 - 3\xi_0}{6} T_A - \frac{4}{3} T_F n_f \right) \frac{1}{\epsilon}, \tag{3.57}$$

and

$$Z_\xi = 1 - \frac{g^2}{(4\pi)^2}\left(\frac{13-3\xi_0}{6}T_A - \frac{4}{3}T_F n_f\right)\frac{1}{\epsilon} + \mathcal{O}(g^4). \tag{3.58}$$

3.2.2 Fermion Self-Energy

Consider again the resummation of one-particle irreducible diagrams in the perturbative expansion of the fermion two-point function, $\tilde{S}(p)$. To proceed, it is algebraically convenient to write the leading order contribution to the fermion propagator as a formal inverse:

$$\tilde{S}^{(0)}(p) = \tilde{S}_F(p) = i\frac{-i\not{p}+m}{p^2+m^2-i\varepsilon} = -i(i\not{p}+m-i\varepsilon)^{-1} =: \frac{i}{i\not{p}+m-i\varepsilon} \tag{3.59}$$

Defining the sum of 1PI diagrams as $-i\Sigma(p)$, we may write

$$\tilde{S}_0(p) = \tilde{S}^{(0)}(p)\sum_{n=0}^{\infty}\left[-i\Sigma(p)\tilde{S}^{(0)}(p)\right]^n = \frac{i}{i\not{p}+m_0-\Sigma(p)-i\varepsilon}. \tag{3.60}$$

The function $\Sigma(p)$ is restricted by dimension to be of the form

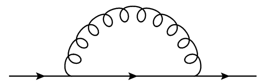
$$\Sigma(p) = \Sigma_p(p)i\not{p} + \Sigma_m(p)m_0. \tag{3.61}$$

Then, employing Eq. 3.17, the full renormalized propagator may be written

$$\tilde{S}(p) = \frac{iZ_\psi^{-1}}{(1-\Sigma_p)i\not{p}+(1-\Sigma_m)Z_m m - i\varepsilon}, \tag{3.62}$$

from which it is clear that Z_ψ exactly cancels the poles in Σ_p, and the product $Z_m Z_\psi$ exactly cancels those in Σ_m.

We now continue to the calculation of $\Sigma(p)$ to leading order. The one-loop contribution



(a)

Figure 3.4: The one-loop contribution to the quark self-energy

to the fermion self-energy is given by a single diagram, Fig. 3.4. Arranging the Feynman rules, we have

$$-i\Sigma^{(1)}(p) = -g_0^2 \int_k \gamma^\beta t^b \frac{i\not{k} + m_0}{k^2 + m_0^2 - i\varepsilon} \gamma^\alpha t^a \frac{\delta^{ab}}{(p+k)^2 - i\varepsilon} \left[\delta_{\alpha\beta} - (1-\xi_0) \frac{(p+k)_{\alpha\beta}}{(p+k)^2 - i\varepsilon} \right]. \quad (3.63)$$

At this point, it is useful to expand to leading order in the mass. Since the functions $\Sigma_{p,m}$ are dimensionless, the truncation error will be $\mathcal{O}(m_0^2/p^2)$. The perturbative regime is typically fairly high-energy, so the quark masses may be considered negligible, and these terms may be dropped:

$$i\Sigma^{(1)}(p) = g_0^2 \int_k \gamma^\beta t^b \frac{i\not{k} + m_0}{k^2 - i\varepsilon} \gamma^\alpha t^a \frac{\delta^{ab}}{(p+k)^2 - i\varepsilon} \left[\delta_{\alpha\beta} - (1-\xi_0) \frac{(p+k)_{\alpha\beta}}{(p+k)^2 - i\varepsilon} \right] + \mathcal{O}(m_0^2). \quad (3.64)$$

The momentum integrals are treated just as in the last section: rewrite the integrand with Schwinger parameters, Taylor-expand the cross-term, decompose the tensor integrals, rotate to Euclidean space, and integrate. In the end, we find that

$$\Sigma^{(1)}(p) = g_0^2 \frac{C_2(F)}{(4\pi)^2} \left[\xi_0 (L_\epsilon + 1) i\not{p} + ((\xi_0 + 3) L_\epsilon + 2\xi_0 + 4) m_0 + \mathcal{O}(m_0^2, \epsilon) \right], \quad (3.65)$$

leading to

$$\begin{aligned} \Sigma_p &= g_0^2 \frac{C_2(F)}{(4\pi)^2} \left[\xi_0 (L_\epsilon + 1) + \mathcal{O}(m_0^2, \epsilon) \right] + O(g_0^4), \\ \Sigma_m &= g_0^2 \frac{C_2(F)}{(4\pi)^2} \left[(\xi_0 + 3) L_\epsilon + 2\xi_0 + 4) + \mathcal{O}(m_0^2, \epsilon) \right] + O(g_0^4). \end{aligned} \quad (3.66)$$

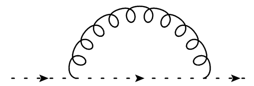


Figure 3.5: The one-loop contribution to the ghost self-energy

The renormalization constants can now be read off:

$$\begin{aligned} Z_\psi &= 1 + g^2 \frac{C_2(F)}{(4\pi)^2} \frac{\xi}{\epsilon} + O(g^4), \\ Z_m &= 1 + g^2 \frac{C_2(F)}{(4\pi)^2} \frac{3}{\epsilon} + O(g^4). \end{aligned} \tag{3.67}$$

3.2.3 Ghost Self-Energy

The final renormalization constant in QCD is the ghost field-strength renormalization, which is defined with the ghost two-point function. We may, as in the previous sections, write the exact bare ghost propagator as

$$\tilde{D}_0^{ab}(p) = \frac{-ig_0^2\delta^{ab}}{p^2 + i\varepsilon} \sum_{n=0}^{\infty} \left[i\Gamma(p) \frac{-ig_0^2}{p^2 + i\varepsilon} \right]^n = \frac{-ig_0^2\delta^{ab}}{p^2 - g_0^2\Gamma(p) + i\varepsilon}, \tag{3.68}$$

where $i\delta^{ab}\Gamma(p)$ represents the sum over 1PI diagrams. Inserting the proper Z-factors, the renormalized propagator is

$$\tilde{D}^{ab}(p) = \frac{-iZ_c^{-1}\mu^{2\epsilon}g^2\delta^{ab}}{p^2 - \mu^{2\epsilon}Z_g g^2\Gamma(p) + i\varepsilon}. \tag{3.69}$$

There is again only a single diagram at one-loop order, Fig. 3.5. The calculation is straightforward, and we find that

$$\Gamma(p) = \frac{1}{4}\frac{T_A}{(4\pi)^2} p^2 \left[(\xi_0 - 3)L_\epsilon - 4 + \mathcal{O}(\epsilon) \right] + \mathcal{O}(g_0^2). \tag{3.70}$$

At next-to-leading order, the effect of the coupling renormalization constant is shifted into $\mathcal{O}(g_0^4)$, so the only counterterm that appears is $Z_c^{(1)}$. Therefore

$$Z_c = 1 + g^2 \frac{T_A}{(4\pi)^2} \frac{3-\xi}{4} \frac{1}{\epsilon} + O(g^4). \tag{3.71}$$

3.3 The Running of QCD Parameters

Now that we have renormalized QCD to one-loop order, we may study the scaling of its parameters. The most straightforward RG function is the beta function:

$$\beta(g) = \mu \frac{\partial g}{\partial \mu}. \tag{3.72}$$

Writing the the renormalized coupling as $g = \mu^{-\epsilon} Z_g^{-1} g_0$, we find

$$\beta(g) = -\epsilon g - g \frac{\mu}{Z_g} \frac{\partial Z_g}{\partial \mu}. \tag{3.73}$$

In dimensional regularization, the renormalization constants are of the generic form

$$Z = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_{n,k} \frac{g^{2n}}{\epsilon^k}, \tag{3.74}$$

since the largest multiplicity of any pole is the maximum possible number of overlapping loop divergences, which is the loop order n. We now invert this series, writing

$$Z = \sum_{n=0}^{\infty} \frac{g^{2n}}{\epsilon^n} \sum_{k=0}^{n} a_{n,n-k} \epsilon^k. \tag{3.75}$$

Since the beta function is finite (in the perturbative region) for $d = 4$, the second term on the righthand side of Eq. 3.73, $\tilde{\beta} := \frac{\mu}{Z_g}\frac{\partial Z_g}{\partial \mu}$, should be finite as well. Then we may write

$$Z_g \tilde{\beta} = \mu \frac{\partial Z_g}{\partial \mu} = \beta \frac{\partial Z_g}{\partial g}. \tag{3.76}$$

Plugging in Eqs. 3.73 and 3.75 for the case of Z_g, this becomes

$$\tilde{\beta}\left\{a_{0,0} + \sum_{n=1}^{\infty}(2n+1)\frac{g^{2n}}{\epsilon^n}\sum_{k=0}^{n} a_{n,n-k}\epsilon^k\right\} = -\epsilon \sum_{n=1}^{\infty} 2n \frac{g^{2n}}{\epsilon^n}\sum_{k=0}^{n} a_{n,n-k}\epsilon^k. \tag{3.77}$$

At zero coupling $Z = 1$, so that $a_{0,0} = 1$ always. The finite portion of the left side is given by the all terms for which $k = n$. On the other hand, this is true for $k = n - 1$ on the right. Equating the finite parts, we have an expression for $\tilde{\beta}$:

$$\tilde{\beta} = -2K \sum_{n=1}^{\infty} n\ a_{n,1} g^{2n}, \qquad K^{-1} := 1 + \sum_{n=1}^{\infty}(2n+1)a_{n,0}g^{2n}. \tag{3.78}$$

All poles must exactly cancel on either side, so that this remains valid in the $d \to 4$ limit. Returning to Eq. 3.73, the general result for the beta function in dimensional regularization is

$$\beta(g) = -\epsilon g + 2K \sum_{n=1}^{\infty} n\ a_{n,1} g^{2n+1} \xrightarrow{\epsilon \to 0} 2K \sum_{n=1}^{\infty} n\ a_{n,1} g^{2n+1}. \tag{3.79}$$

In the cases of MS-like subtraction schemes, the renormalization constants contain no finite parts, so that $a_{n\geq 1,0} = 0$ and $K = 1$. The beta function is then simply a weighted sum over the simple poles at each order in the coupling:

$$\beta(g) = 2 \sum_{n=1}^{\infty} n\ a_{n,1} g^{2n+1}. \tag{3.80}$$

The standard perturbative expansion is reads

$$\beta(g) = -g^3 \sum_{n=0}^{\infty} b_n g^{2n}, \tag{3.81}$$

so that $b_n = -2(n+1)a_{n+1,1}$. Bearing in mind that

$$Z_g^{(n)} = \sum_{k=0}^{n} \frac{a_{n,k}}{\epsilon^k} \qquad \Rightarrow \qquad Z_g^{(1)} = \frac{a_{1,1}}{\epsilon}, \tag{3.82}$$

we quickly arrive at the one-loop QCD beta function [27, 28]:

$$\beta(g) = 2g^3 a_{1,1} + \mathcal{O}(g^5) = \frac{g^3}{(4\pi)^2}\left(\frac{11}{3}T_A - \frac{4}{3}T_F n_f\right) + \mathcal{O}(g^5) = -\frac{g^3}{(4\pi)^2}\left(\frac{11}{3}N - \frac{2}{3}n_f\right) + \mathcal{O}(g^5). \tag{3.83}$$

Evidently, the beta function decreases for $11N > 2n_f$. In the case of QCD, $N = 3, n_f = 6$, this is satisfied, and we see that the coupling decreases at high energy, corresponding to the asymptotic freedom of quarks. Conversely, at low energies the coupling grows, and perturbation theory becomes unreliable. At some scale Λ_{QCD}, the quarks become confined to hadrons, $SU(N)$ singlets composed of multiple quarks and/ or antiquarks. In perturbation theory,

$$\lim_{\mu \to \Lambda_{QCD}^+} g^2(\mu) = \infty. \tag{3.84}$$

Using the confinement scale as a reference point, we can integrate the perturbative beta function to find the one-loop running of the coupling:

$$\log \frac{\mu^2}{\Lambda_{QCD}^2} = -2\int_{g(\mu)}^{\infty} \beta^{-1}(g) dg = -\frac{1}{b_0 g^2} + \mathcal{O}(\log g^2). \tag{3.85}$$

Rearranging this, we find that

$$\frac{g^2(\mu)}{(4\pi)^2} = -\frac{3}{(11N - 2n_f)\log\frac{\mu^2}{\Lambda^2_{QCD}}} + \mathcal{O}(\log g^2). \tag{3.86}$$

For now, the only other coupling we need consider is the mass. The mass anomalous dimension is

$$\gamma_m = \frac{\partial \log m}{\partial \log \mu} = -\beta \frac{\partial \log Z_m}{\partial g}. \tag{3.87}$$

Of course, each power of g^2 within the derivative carries with it a series of poles, so we need to express the beta function as in Eq. 3.73 in order to find the simple pole. Instead of going through a finite-part analysis as before, we simply note that

$$\frac{\partial \log Z_m}{\partial g} = 2gZ_m^{(1)} + \mathcal{O}(g^3), \tag{3.88}$$

so

$$\gamma_m = g\left(\epsilon + \tilde{\beta}\right) \cdot 2gZ_m^{(1)} + \mathcal{O}(g^4) \xrightarrow{\epsilon \to 0} 6C_2(F)\frac{g^2}{(4\pi)^2} + \mathcal{O}(g^4). \tag{3.89}$$

Chapter 4

Lattice Quantum Chromodynamics

4.1 The Simplest Gauge Action

We define the $n \times m$ Wilson loop $W^{n\times m}_{\mu\nu}(x)$ as the path-ordered product of gauge links around a closed convex loop. This $SU_3(\mathbb{C})$ matrix encodes the holonomy of the gauge covariant derivative around discretized curves on a $4-D$ lattice. The simplest case, a so-called plaquette $W^{1\times 1}_{\mu\nu}(x)$, is given by

$$W^{1\times 1}_{\mu\nu}(x) \equiv U_\mu(x)U_\nu(x+a\hat{\mu})U^\dagger_\mu(x+a\hat{\nu})U^\dagger_\nu(x). \tag{4.1}$$

We may expand this product explicitly, keeping only terms up to order a^2, where a is the lattice spacing:

$$\begin{aligned} W^{1\times 1}_{\mu\nu}(x) &= e^{igaA_\mu(x)}e^{igaA_\nu(x+a\hat{\mu})}e^{-igaA_\mu(x+a\hat{\nu})}e^{-igaA_\nu(x)} \\ &= 1 + iga^2G_{\mu\nu}(x) - g^2a^4G_{\mu\nu}G_{\mu\nu}(x) + \mathcal{O}(a^6). \end{aligned} \tag{4.2}$$

The first line expresses the gauge links as exponentials. The second line expands pairwise the products of the exponentials in terms of the Baker-Campbell-Hausdorff (BCH) formula to second order in the lattice spacing. The third line expands the gauge potentials A about x to first order in a. This avoids more onerous commutators, as it allows us to immediately identify and ignore any higher order terms in the subsequent BCH expansion. The fourth line uses the BCH formula again to expand the final product, this time absorbing any cubic terms into the $\mathcal{O}(a^3)$ term, which cancel as we expand the exponential.

We may now express the lattice gauge action in terms of the plaquette $W^{1\times1}_{\mu\nu}(x)$:

$$S_G[U] \equiv \frac{2}{g^2}\sum_x\sum_{\mu<\nu}\mathfrak{Re}\mathrm{Tr}\left\{1 - W^{1\times1}_{\mu\nu}(x)\right\} = \frac{1}{2}a^4\sum_x\sum_{\mu,\nu}\Big[\mathrm{Tr}\left\{G_{\mu\nu}G_{\mu\nu}\right\} + \mathcal{O}(a^2)\Big]. \tag{4.3}$$

In the limit of zero lattice spacing, the sum over lattice points becomes an integral in $4-D$ Euclidean space with volume a^4; videlicet, $a^4\sum_x \to \int d^4x$ as $a \to 0$. Thus the lattice and continuum gauge actions are equal in the continuum limit:

$$S_G[U] \approx \frac{1}{2}a^4\sum_x\sum_{\mu,\nu}\mathrm{Tr}\left\{G_{\mu\nu}G_{\mu\nu}\right\} \xrightarrow{a\to0} \frac{1}{4}\int_{\mathbb{R}^4} d^4x\; G^a_{\mu\nu}G^a_{\mu\nu} \equiv S_G[A]. \tag{4.4}$$

4.2 The Naïve Dirac Action

The simplest discretization of the Dirac operator involves replacing the derivative with a symmetrized finite difference quotient and inserting gauge links to restore gauge invariance:

$$\begin{aligned} S^N_F[\psi,\bar\psi,U] &\equiv a^4\sum_x\left\{\frac{1}{2a}\sum_\mu\bar\psi(x)\gamma_\mu\big[U_\mu(x)\psi(x+a\hat\mu) - U_{-\mu}(x)\psi(x-a\hat\mu)\big] + m\bar\psi(x)\psi(x)\right\} \\ &\xrightarrow{a\to0} \int_{\mathbb{R}^4} d^4x\;\bar\psi\Big(\not{D} + m\Big)\psi \\ &= S_F[\psi,\bar\psi,U]. \end{aligned} \tag{4.5}$$

This may be recast more compactly by noting that x is quantized by the lattice spacing: $x = na$, which allows us to condense the gauge link dependence into a linear combination of Kronecker delta functions. Define

$$\mathcal{M}^N_{xy} \equiv \frac{1}{2a}\sum_\mu\left[\gamma_\mu(U_\mu)_x\delta_{x,y-a\hat\mu} - \gamma_\mu(U_\mu)_{x-a\hat\mu}\delta_{x,y+a\hat\mu}\right] + m\delta_{x,y}. \tag{4.6}$$

Then the above reduces to

$$S_F^N[\psi,\bar{\psi},U] = a^4 \sum_{x,y} \bar{\psi}_x \mathcal{M}_{xy}^N \psi_y. \tag{4.7}$$

This form is conducive to faster computation, especially when the fields have been previously generated at all lattice sites. The delta functions simply project out the values at the relevant points on the lattice.

A problem arises, however, when constructing the free fermion propagator from this action. Let us transform to momentum space to illustrate this, turning off the gauge potentials to restore the free particle action ($U \to 1$):

$$\begin{aligned} S_F^0[\psi,\bar{\psi}] &= a^4 \sum_{x=na} \bar{\psi}(na)\Big\{\frac{1}{2a}\sum_\mu \gamma_\mu\big[\psi(na+a\hat{\mu}) - \psi(na-a\hat{\mu})\big] + m\psi(na)\Big\} \\ &= a^4 \int_{-a/\pi}^{a/\pi} \frac{dk}{(2\pi)^4} \bar{\tilde{\psi}}(k)\Big\{\frac{i}{a}\sum_\mu \gamma_\mu sin\big(k_\mu a\big) + m\Big\}\tilde{\psi}(k). \end{aligned} \tag{4.8}$$

Thus, we have constructed the momentum-space propagator for naïve lattice fermions:

$$S_F(k,m) = \frac{i}{a}\sum_\mu \gamma_\mu sin\big(k_\mu a\big) + m. \tag{4.9}$$

In the chiral limit, the inverse propagator has roots at the origin and the boundaries of the first Brillouin zone ($x_p \in \{0, a/\pi\}$), so there are $2^{D=4} = 16$ poles which characterize the famous "doublers" problem. Various correction schemes and systematic limitations of discretization will be discussed later.

4.3 The Haar Measure

By construction, the QCD action is invariant under gauge transformations:

$$S[U] = S[U']. \tag{4.10}$$

Reasonably, we require also that any observables are gauge invariant, so the functional integral

$$Z = \int D[U]e^{-S[U]} \tag{4.11}$$

must be gauge invariant. This is tantamount to invariance under a change of variables. This restriction on the path integral translates to demanding invariance under the action of the gauge group G of the integral measure $D[U]$ for any measurable subset $U \subseteq G$. The Haar measure satisfies this requirement naturally. We will construct this explicitly. For any locally compact T_2 group G, let us define the left translate of a Borel set $U \in \sigma(G)$:

$$gU = \{gu \mid u \in U\}, \text{ for some } g \in G. \tag{4.12}$$

Intuitively, this object should be the same ”size” as the untranslated set. The goal is to find some measure $\mu(U)$, such that left translation by an element of the enclosing group does not affect this size; this is the Haar measure. We now state Haar's existence and uniqueness theorem for such a measure:

> *There exists a nontrivial, additive, regular measure on the Borel subsets of a locally compact Hausdorff group which is unique up to normalization, finite over compact sets, and invariant under left translation.*

Such a measure is called the left Haar measure:

$$\mu(U) = \mu(gU). \tag{4.13}$$

The details of the proof are beyond the scope of this thesis, but we may nonetheless apply the result to the gauge group in concern, $SU_3(\mathbb{C})$. Moreover, not only does there exist a left Haar measure in our case, but so also a right Haar measure, owing to the unimodularity of all compact Lie groups, $SU_N(\mathbb{C})$ included. We have, then, that

$$\mu(gU) = \mu(U) = \mu(Ug) \tag{4.14}$$

for any $g \in SU_N(\mathbb{C})$ with $U \in \sigma(G)$. Equipped with a measure, we may now consider integrals over locally compact groups. The invariance of the measure immediately implies the invariance of Lesbegue integrals when the integration variable is translated:

$$\int_U d\mu(u) f(u) = \int_U d\mu(gu) f(gu) = \int_U d\mu(u) f(gu) = \int_U d\mu(u) f(ug) \tag{4.15}$$

for some $u \in U$. This relationship is instrumental in finding exact solutions to many group integrals without requiring an explicit form of the Haar integral measure in terms of coordinates of the underlying manifold (odd sphere bundles in the case of $SU_N(\mathbb{C})$). We present a few useful results now:

$$\int_U d\mu(u) u_{ab} = 0 \tag{4.16}$$

$$\int_U d\mu(u) u_{ab} u_{cd} = 0 \tag{4.17}$$

$$\int_U d\mu(u) u_{ab} u^\dagger_{cd} = \delta_{ad}\delta_{bc} \tag{4.18}$$

$$\int_U d\mu(u) f(u) = \int_U d\mu(gu) f(gu) = \int_U d\mu(u) f(gu) = \int_U d\mu(u) f(ug) \tag{4.19}$$

Part II

The Gradient Flow

Chapter 5

The Flowed Formalism

The gradient flow belongs to a class of parabolic partial differential equations called geometric flows. In general, these equations describe the diffusion of some geometric quantity on a manifold. In particular, gradient flows in QFT are nonlinear heat equations on the space of configurations of a gauge field ϕ, which characterize its diffusion along some new dimension, the flow time t. Critically, this gives us the boundary condition that for $t = 0$ the flowed field Φ should coincide with the physical field ϕ. In order that the evolution is stable, the field should flow toward a local minimum of the action. This is accomplished by writing

$$\partial_t \Phi(x;t) = -\frac{\delta S[\Phi]}{\delta \Phi}, \qquad \Phi(x;0) = \phi(x), \tag{5.1}$$

where $S[\phi]$ is the action associated to the field ϕ at $t = 0$. Since the right side is proportional to the negative gradient of the action, we are assured that increasing the flow time drives the action toward a minimum as quickly as possible. We will find that this corresponds to a smearing of the gauge field in spacetime that suppresses ultraviolet modes.

5.1 The Yang-Mills Gradient Flow

In the case of the Yang-Mills action, Eq. 3.12, it is straightforward to verify that

$$\frac{\delta S_{YM}[A]}{\delta A_\mu^a} = -(D_\nu G_{\nu\mu})^a. \tag{5.2}$$

Plugging this into the schematic equation (5.1) above, we arrive at a prototypical Yang-Mills gradient flow equation:

$$\partial_t B_\mu = D_\nu G_{\nu\mu}[B], \qquad B_\mu\big|_{t=0} = A_\mu, \tag{5.3}$$

where B is the flowed counterpart of the gluon field A defined in the bulk of the $(d+1)$-dimensional half-space with coordinates $(x; t \geq 0)$. The boundary condition above enforces that Yang-Mills theory lives on the d-dimensional boundary at $t = 0$.

This form of the equation is not amenable to perturbation theory, however, due to the presence of nonrenormalizable longitudinal modes in the propagator

$$\tilde{D}^{ab}_{\alpha\beta}(q;t) = g_0^2 \frac{\delta^{ab}}{q^2}\left[\left(\delta_{\alpha\beta} - \frac{q_\alpha q_\beta}{q^2}\right)e^{-q^2 t} + \xi_0 \frac{q_\alpha q_\beta}{q^2}\right]. \tag{5.4}$$

The solution is to add a restoring force which fixes a plane normal to the gauge orbits [87]. For some gauge function F, the covariant derivative provides an appropriate tangent vector. Choosing a Lorenz-like gauge function $F = \partial \cdot B$ as in Sec. 1.5, we have [84]

$$\partial_t B_\mu = D_\nu G_{\nu\mu}[B] + \alpha_0 D_\mu \partial_\nu B_\nu, \qquad B_\mu\big|_{t=0} = A_\mu. \tag{5.5}$$

In Sec. 5.3, we will explore the perturbative solution to the flow equation. Considering only terms linear in B, the leading-order flow equation is simply

$$\partial_t B_\mu = \partial^2 B_\mu + (\alpha_0 - 1)\partial_{\mu\nu} B_\nu, \tag{5.6}$$

and the free bulk field is easily seen to diffuse according to a heat equation. This is most easily analyzed in momentum space with the standard Fourier analysis. Setting $\alpha_0 = 1$ for now, the linearized equation above simplifies to the standard multivariate heat equation, and the fundamental solution is the heat kernel:

$$\tilde{K}(x-y;t) \sim \int_p e^{ip(x-y)} e^{-p^2 t} = (4\pi t)^{-d/2} e^{\frac{-(x-y)^2}{4t}}. \tag{5.7}$$

This may be convoluted with the boundary condition, leading to the free-field solution

$$\tilde{B}_\mu(q;t) = (4\pi t)^{-d/2} \int d^d x \; e^{-iqx} \int d^d y \; e^{-\frac{(x-y)^2}{4t}} A(y) = e^{-q^2 t} \tilde{A}_\mu(q). \tag{5.8}$$

The Gaussian factor represents a delocalization of the gauge field over a d-dimensional sphere with root-mean-squared radius

$$\langle x \rangle_{rms}^2 = \int d^d x \; x^2 e^{-\frac{x^2}{4t}} = 2dt, \tag{5.9}$$

which sets a natural scale $\mu = (2dt)^{-1/2}$ for flowed computations. In practice, it is often simpler to choose $\mu = (2t)^{-1/2} e^{\gamma_E/2}$, corresponding to the $\overline{\text{MS}}$ subtraction point, so that all logarithms vanish in perturbation theory. The Gaussian smearing suppresses the high-energy modes of the boundary field, so that the gauge field is less singular at positive flow time.

The flow equation may be added as a constraint on the action through the use of Lagrange multiplier fields $L_\mu = L_\mu^a t^a$, giving us a new term in the action:

$$S = S_{YM} + S_{gf} + S_{FP} + S_B, \tag{5.10}$$

where

$$S_B = -2\int dt \int d^d x \ \mathrm{Tr}\left\{L_\mu(\partial_t B_\mu - D_\nu G_{\nu\mu} - \alpha_0 D_\mu \partial_\nu B_\nu)\right\}. \tag{5.11}$$

The coefficient of −2 accounts for the normalization $T_F = -1/2$ of the trace for the particular case of $SU(N)$. Now, the equation of motion for L_μ is exactly the gradient flow equation (5.5). The boundary condition $B_\mu(x;0) = A_\mu(x)$ is implemented by defining

$$B_\mu(x;t) = b_\mu(x,t) + \int d^d y \ K_{\mu\nu}(x-y;t)A_\nu(y), \tag{5.12}$$

where $b_\mu(x;0) = 0$, and $K_{\mu\nu}(x-y;t)$ is a heat kernel which solves linearized flow equation (to be discussed in Sec. 5.3). Since the latter term satisfies Eq. 5.6, the propagator between the L and A fields vanishes, leaving propagators of the form $\langle AA\rangle$, $\langle AB\rangle$, $\langle BB\rangle$, and $\langle LB\rangle$.

5.2 The Fermion Flow

The treatment of fermions in the flowed formalism differs slightly from that of the gauge fields [88]. Since the Dirac action breaks chiral symmetry and is only first order in the derivative, it is not suitable for a gradient flow. Instead, we may construct a covariant flow equation for fermions with the gauge-covariant Laplacian and its adjoint:

$$\Delta = D_\mu D_\mu, \qquad \overleftarrow{\Delta} = \overleftarrow{D}_\mu \overleftarrow{D}_\mu, \tag{5.13}$$

with

$$D_\mu = \partial_\mu + B_\mu, \qquad \overleftarrow{D}_\mu = \overleftarrow{\partial}_\mu - B_\mu, \tag{5.14}$$

Introducing flowed fermion fields χ and $\overline{\chi}$, we define the fermion flow equation

$$\partial_t \chi = \Delta \chi - \alpha_0 \partial_\mu B_\mu \chi, \qquad \chi\big|_{t=0} = \psi, \tag{5.15a}$$

and its adjoint

$$\overline{\chi} \overleftarrow{\partial}_t = \overline{\chi} \overleftarrow{\Delta} + \alpha_0 \overline{\chi} \partial_\mu B_\mu, \qquad \overline{\chi}\big|_{t=0} = \overline{\psi}. \tag{5.15b}$$

Since the covariant Laplacian coincides with the ordinary Laplacian at leading order, this prescription guarantees that the relaxation of the flowed fermions follows a heat equation as well.

Again, we constrain the action with some (Grassmann-odd) Lagrange multipliers λ and $\overline{\lambda}$, introducing another term to the action:

$$S_{\overline{\chi}\chi} = \int dt \int d^d x \, \left\{ \overline{\lambda}(\partial_t - \Delta + \alpha_0 \partial_\mu B_\mu)\chi + \overline{\chi}(\overleftarrow{\partial}_t - \overleftarrow{\Delta} - \alpha_0 \partial_\mu B_\mu)\lambda \right\} \tag{5.16}$$

with decompositions similar to Eq. 5.12 to enforce the boundary conditions.

5.3 Perturbation Theory

The flow equations, Eq. 5.5 and Eq. 5.15, constitute a system of coupled, nonlinear, parabolic PDEs for the flowed fields, so they are not soluble in any straightforward manner. On the other hand, at leading order in the coupling each reduces to a heat equation, which is readily integrated as in Sec. 5.1. The nonlinear terms may then be treated as perturbations. In the case of the gauge field, the flow equation may be written [84]

$$\partial_t B_\mu = \partial^2 B_\mu + (\alpha_0 - 1)\partial_{\mu\nu} B_\nu + R_\mu, \tag{5.17}$$

where

$$R_\mu = 2[B_\nu, \partial_\nu B_\mu] - [B_\nu, \partial_\mu B_\nu] + (\alpha_0 - 1)[B_\mu, \partial_\nu B_\nu] + [B_\nu, [B_\nu, B_\mu]] \tag{5.18}$$

is the nonlinear remainder that generates radiative corrections to the free solution. The kernel may be easily determined in momentum space:

$$\tilde{K}_{\mu\nu}(q;t) = \left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) e^{-q^2 t} + \frac{q_\mu q_\nu}{q^2} e^{-\alpha_0 q^2 t}, \tag{5.19}$$

leading to the solution

$$\tilde{B}_\mu(q;t) = \tilde{K}_{\mu\nu}(q;t)\tilde{A}_\nu(q) + \int_0^t ds\ \tilde{K}_{\mu\nu}(q;t-s)\tilde{R}_\nu(q;s). \tag{5.20}$$

We immediately find the flowed gauge field propagator:

$$\langle \tilde{B}^b_\nu(-q;s)\tilde{B}^a_\mu(q;t)\rangle^{(0)} = g_0^2 \frac{\delta^{ab}}{q^2}\left[\left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) e^{-q^2(t+s)} + \xi_0 \frac{q_\mu q_\nu}{q^2} e^{-\alpha_0 q^2 (t+s)}\right], \tag{5.21}$$

which includes the propagators $\langle AA\rangle$ and $\langle AB\rangle$ in the limits of vanishing t and s. The $\langle LB\rangle$ propagator is obtained by considering the Schwinger-Dyson equation for L:

$$\left\langle L^b_\nu(y;s)[\delta_{\mu\rho}\partial_t - \delta_{\mu\rho}\partial^2 - (\alpha_0 - 1)\partial_\mu\partial_\rho]B^a_\rho(x;t)\right\rangle = \delta^{ab}\delta_{\mu\nu}\delta^{(d)}(x-y)\delta(t-s), \tag{5.22}$$

with the condition that $\langle LB\rangle\big|_{s>t=0} = 0$. This has the unique solution

$$\langle L^b_\nu(y;s)B^a_\mu(x;t)\rangle^{(0)} = \int_q e^{iq(x-y)}\delta^{ab}\theta(t-s)\tilde{K}_{\mu\nu}(q;t-s), \tag{5.23}$$

called a (gauge boson) flow or kernel line.

The remainder contains terms quadratic and cubic in the bulk fields which correspond to new three- and four-point vertices. Writing

$$\begin{aligned}\tilde{R}^a_\mu(q;t) =& \frac{1}{2!}\int_{p_1,p_2}(2\pi)^d\delta^{(d)}(q+p_1+p_2)X^{(2,0)}(q,p_1,p_2)^{ab_1b_2}_{\mu\nu_1\nu_2}\tilde{B}^{b_1}_{\nu_1}(-p_1;s)\tilde{B}^{b_2}_{\nu_2}(-p_2;s)\\ &+\frac{1}{3!}\int_{p_1,p_2,p_3}(2\pi)^d\delta^{(d)}(q+p_1+p_2+p_3)\\ &\qquad\times X^{(3,0)}(q,p_1,p_2,p_3)^{ab_1b_2b_3}_{\mu\nu_1\nu_2\nu_3}\tilde{B}^{b_1}_{\nu_1}(-p_1;s)\tilde{B}^{b_2}_{\nu_2}(-p_2;s)\tilde{B}^{b_3}_{\nu_3}(-p_3;s),\end{aligned} \tag{5.24}$$

they are, respectively,

$$X^{(2,0)}(p,q,r)^{abc}_{\mu\nu\rho} = if^{abc}\left\{(r-q)_\mu\delta_{\nu\rho}+2q_\rho\delta_{\mu\nu}-2r_\nu\delta_{\rho\mu}+(\alpha_0-1)(q_\nu\delta\rho\mu-r_\rho\delta_{\mu\nu})\right\} \tag{5.25}$$

and

$$\begin{aligned}X^{(3,0)}(p,q,r,s)^{abcd}_{\mu\nu\rho\sigma} =& f^{abe}f^{cde}(\delta_{\mu\sigma}\delta_{\nu\rho}-\delta_{\mu\rho}\delta_{\sigma\nu})+f^{ade}f^{bce}(\delta_{\mu\rho}\delta_{\sigma\nu}-\delta_{\mu\nu}\delta_{\rho\sigma})\\ &+f^{ace}f^{dbe}(\delta_{\mu\nu}\delta_{\rho\sigma}-\delta_{\mu\sigma}\delta_{\nu\rho}).\end{aligned} \tag{5.26}$$

Inspecting the remainder term in S_B, it is obvious that these correspond to B^2L and B^3L vertices. As such, the kernel lines may only connect bulk gauge fields to the flow vertices $X^{(2,0)}$ and $X^{(3,0)}$. One subtlety of the flow lines is that they may not form closed loops.

Of course, these cannot appear in the perturbative expansion, but they are allowed when naïvely constructing all graphs, so one must take care to remove these diagrams manually in automated implementations [89].

The fermion flow is linearized analogously to the gauge fields:

$$\partial_t \chi = (\partial^2 + \Delta')\chi, \tag{5.27}$$

where $\Delta' = 2B_\mu \partial_\mu - (\alpha_0 - 1)\partial_\mu B_\mu + B_\mu B_\mu$. The leading-order equation is identical to the ordinary heat equation, so the fermion kernel is strictly Gaussian,

$$\tilde{J}(p;t) = e^{-p^2 t}, \tag{5.28}$$

and we have a general solution:

$$\tilde{\chi}(p;t) = \tilde{J}(p;t)\tilde{\psi}(p) + \int_0^t ds\ \tilde{J}(p;t-s)\tilde{\Delta}'\tilde{\chi}(p;s). \tag{5.29}$$

The adjoint flow is similar:

$$\tilde{\bar{\chi}}(p;t) = \tilde{\bar{\psi}}(p)\tilde{\bar{J}}(p;t) + \int_0^t ds\ \tilde{\bar{\chi}}(p;s)\tilde{\overleftarrow{\Delta}}'\tilde{\bar{J}}(p;t-s), \tag{5.30}$$

where $\overleftarrow{\Delta}' = -2B_\mu \partial_\mu + (\alpha_0 - 1)\partial_\mu B_\mu + B_\mu B_\mu$, and $\tilde{\bar{J}}(p;t) = e^{-p^2 t}$. The propagators are obtained exactly as before, leading to

$$\langle \tilde{\chi}(-p;s)\tilde{\bar{\chi}}(p;t)\rangle^{(0)} = \frac{-i\not{p} + m}{p^2 + m + 2} e^{-p^2(t+s)}, \tag{5.31}$$

$$\langle \tilde{\chi}(-p;t)\tilde{\bar{\lambda}}(p;s)\rangle^{(0)} = \theta(t-s)\tilde{J}(p;t-s) \tag{5.32}$$

$$\langle\tilde{\lambda}(-p;s)\tilde{\overline{\chi}}(p;t)\rangle^{(0)} = \theta(t-s)\tilde{\overline{J}}(p;t-s). \tag{5.33}$$

Rewriting the remainder as before,

$$\begin{aligned}\tilde{\Delta}'\tilde{\chi}(q;t) =& \frac{1}{1!}\int_{p_1,p_2}(2\pi)^d\delta^{(d)}(q+p_1+p_2)Y^{(1,1)}(q,p_1,p_2)^{a_1}_{\mu_1}\tilde{B}^{a_1}_{\mu_1}(-p_1;s)\tilde{\chi}(-p_2;s)\\ &+\frac{1}{2!}\int_{p_1,p_2,,p_3}(2\pi)^d\delta^{(d)}(q+p_1+p_2+p_3)\\ &\qquad\times Y^{(1,2)}(q,p_1,p_2,p_3)^{a_1a_2}_{\mu_1\mu_2}\tilde{B}^{a_1}_{\mu_1}(-p_1;s)\tilde{B}^{a_2}_{\mu_2}(-p_2;s)\tilde{\chi}(-p_3;s),\end{aligned} \tag{5.34}$$

we find two more vertices,

$$Y^{(1,1)}(p,q,r)^a_\mu = -it^a\left\{(1-\alpha_0)r_\mu + 2q_\mu\right\} \tag{5.35}$$

and

$$Y^{(1,2)}(p,q,r,s)^{ab}_{\mu\nu} = \delta_{\mu\nu}\left\{t^a,t^b\right\}, \tag{5.36}$$

corresponding to $\chi\overline{\lambda}B$ and $\chi\overline{\lambda}B^2$. The adjoint flow equation, too, generates two vertices;

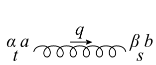
$$\overline{Y}^{(1,1)}(p,q,r)^a_\mu = it^a\left\{(1-\alpha_0)r_\mu + 2q_\mu\right\} \tag{5.37}$$

and

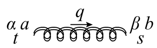
$$\overline{Y}^{(1,2)}(p,q,r,s)^{ab}_{\mu\nu} = \delta_{\mu\nu}\left\{t^a,t^b\right\}, \tag{5.38}$$

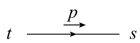
corresponding to $\lambda\overline{\chi}B$ and $\lambda\overline{\chi}B^2$.

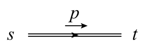
Altogether, we have the following Feynman rules. The flowed propagators are

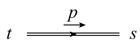
$\langle\tilde{B}\tilde{B}\rangle:$ 

$$= g_0^2\frac{\delta^{ab}}{q^2}\left[\left(\delta_{\alpha\beta}-\frac{q_\alpha q_\beta}{q^2}\right)e^{-q^2(t+s)} + \xi_0\frac{q_\alpha q_\beta}{q^2}e^{-\alpha_0 q^2(t+s)}\right], \tag{5.39a}$$

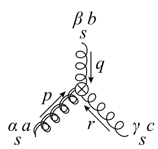
$\langle \tilde{L}\tilde{B}\rangle :$  $= \delta^{ab}\theta(t-s)\left[\left(\delta_{\alpha\beta} - \frac{q_\alpha q_\beta}{q^2}\right)e^{-q^2(t-s)} + \frac{q_\alpha q_\beta}{q^2}e^{-\alpha_0 q^2(t-s)}\right],$ (5.39b)

$\langle \tilde{\chi}\tilde{\bar{\chi}}\rangle :$  $= \dfrac{-i\not{p}+m}{p^2+m+2}e^{-p^2(t+s)},$ (5.39c)

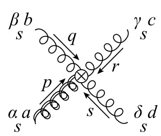
$\langle \tilde{\chi}\tilde{\bar{\lambda}}\rangle :$  $= \theta(t-s)e^{-p^2(t-s)},$ (5.39d)

$\langle \tilde{\lambda}\tilde{\bar{\chi}}\rangle :$  $= \theta(t-s)e^{-p^2(t-s)}.$ (5.39e)

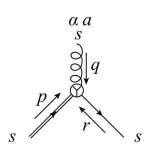
The vertices are

$\langle \tilde{B}^2\tilde{L}\rangle :$ 

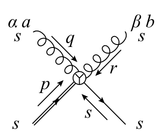
$$= if^{abc}\int_0^\infty ds\big[\delta_{\beta\gamma}(q-r)_\alpha + 2\delta_{\gamma\alpha}r_\beta - 2\delta_{\alpha\beta}q_\gamma + (\alpha_0-1)(\delta_{\alpha\beta}r_\gamma - \delta_{\gamma\alpha}q_\beta)\big], \quad (5.40a)$$

$\langle \tilde{B}^3\tilde{L}\rangle :$ 

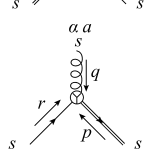
$$= -\int_0^\infty ds\big[f^{abe}f^{cde}(\delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\gamma\beta}) + f^{ace}f^{bde}(\delta_{\alpha\beta}\delta_{\gamma\delta} - \delta_{\alpha\delta}\delta_{\gamma\beta}) + f^{ade}f^{bce}(\delta_{\alpha\beta}\delta_{\gamma\delta} - \delta_{\alpha\gamma}\delta_{\beta\delta})\big], \quad (5.40b)$$

$\langle \tilde{\chi}\tilde{B}\tilde{\bar{\lambda}}\rangle :$ 

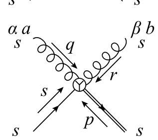
$$= -it^a\int_0^\infty ds\,[2r_\alpha + (1-\alpha_0)q_\alpha]\,, \quad (5.40c)$$

$\langle \tilde{\chi}\tilde{B}^2\tilde{\bar{\lambda}}\rangle :$ 

$$= \delta_{\alpha\beta}\{t^a, t^b\}\int_0^\infty ds, \quad (5.40d)$$

$\langle \tilde{\lambda}\tilde{B}\tilde{\bar{\chi}}\rangle :$ 

$$= it^a\int_0^\infty ds\,[2r_\alpha + (1-\alpha_0)q_\alpha]\,, \quad (5.40e)$$

$\langle \tilde{\lambda}\tilde{B}^2\tilde{\bar{\chi}}\rangle :$ 

$$= \delta_{\alpha\beta}\{t^a, t^b\}\int_0^\infty ds. \quad (5.40f)$$

Above, fermions are represented by oriented solid lines; gauge bosons by curly lines; fermionic flow lines by oriented, double solid lines; and gauge boson flow lines by double curly lines.

This notation differs notably from much of the literature, wherein all flow lines are single straight lines with an adjacent arrow indicating the direction of increasing flow time, determined by the attached Heaviside θ functions. In order to avoid a proliferation of arrows, we choose the double-line notation. The direction of flow time is unambiguous, since all subgraphs consisting of only flow lines and vertices are directed trees with each child vertex at a flow time less than or equal than that of its parent and the root at the maximum flow time (flow-line loops having been already excluded).

The integrals over s in the vertices are meant to be performed only after all attached legs are taken into the integrand. The flow vertices are inscribed by an X or Y to signify bosonic or fermionic vertices. Note that flow lines cannot be cut when constructing 1PI diagrams, since they represent genuine corrections to the flowed field.

Chapter 6

Renormalization and BRST Symmetry

A remarkable feature of the Yang-Mills gradient flow is that once the boundary theory is renormalized, the bulk gauge fields are finite to all orders. This is not the case for bulk fermions, though they may be rendered finite by a multiplicative field strength renormalization. in order to see this, we will evaluate the one-loop propagators of both the bulk gauge field and the bulk fermions. We fix $\alpha_0 = 1$ but leave ξ_0 free, since it requires renormalization at one-loop. To perform the momentum integrals, we use dimensional regularization with $d = 4 - 2\epsilon$ and employ the novel method introduced in App.

6.1 Gauge Field Self-Energy

In Chapter 3, we showed that the bare gluon propagator,

$$\begin{aligned}\langle \tilde{A}^b_\beta(-q)\tilde{A}^a_\alpha(q)\rangle_0 = \; & g_0^2 \frac{\delta^{ab}}{q^2}\left[\Pi_{\alpha\beta} + \xi_0\Lambda_{\alpha\beta}\right] \\ & - \frac{g_0^4}{(4\pi)^2}\frac{\delta^{ab}}{q^2}\left[\left(\frac{13-3\xi_0}{6}T_A + \frac{2}{3}n_f\right)L_0 + \frac{1}{4}\left(\xi_0^2 + 2\xi_0 + \frac{97}{9}\right)T_A + \frac{10}{9}n_f\right]\Pi_{\alpha\beta} \\ & + \mathcal{O}(g_0^6),\end{aligned} \tag{6.1}$$

may be renormalized by making the replacements

$$A_0 = Z_g Z_\xi A, \qquad g_0 = \mu^\epsilon Z_g g, \qquad \xi_0 = Z_\xi \xi. \tag{6.2}$$

At one-loop order, the propagator of the bulk gauge fields, $\langle \tilde{B}^b_\beta(-q)\tilde{B}^a_\alpha(q)\rangle$ is the sum of eight diagrams (Fig. 6.1). The first four of these diagrams ((a)-(d)) are identical to the

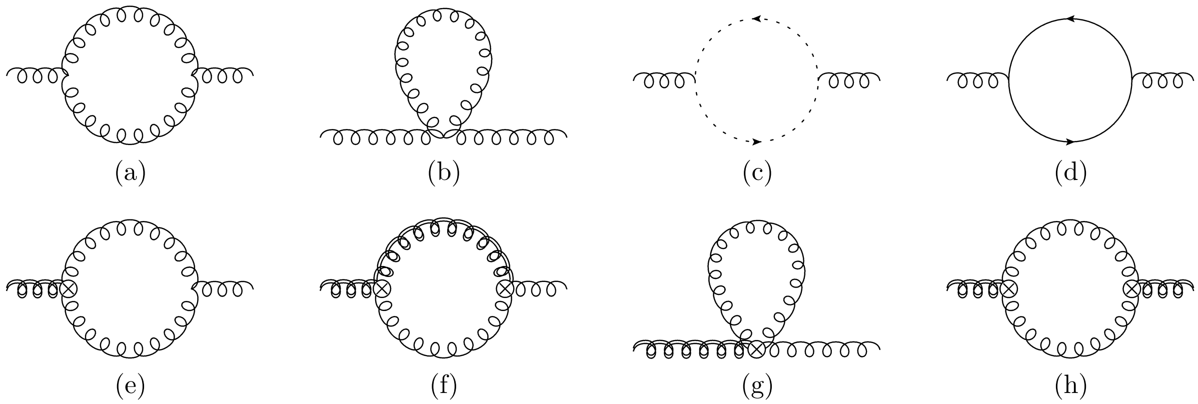


Figure 6.1: One-loop contributions to the propagator of the flowed gauge field

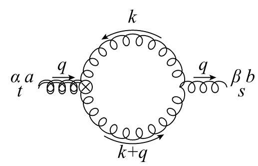


Figure 6.2: Diagram 6.1e with labels

unflowed diagrams up to the external fields. Since the difference is $\mathcal{O}(t)$, both the divergent and finite parts are unaffected at small t. The four additional diagrams ((e)-(h)) contain flow lines and vertices, representing the nonlinear terms in the flow equation. These are the first diagrams which exhibit the incomplete gamma integrals of App. , due to Gaussian factors within the loops. For demonstration, we will calculate diagram (d) explicitly. For brevity, we will set $\xi_0 = \alpha_0 = 1$, writing the full result only at the end.

Starting from the Feynman rules, Eqs. 5.39 and Eqs. 5.40, we have (dropping external indices and the outgoing AA leg)

$$\begin{aligned}\Gamma_e = \int_k \frac{if^{bcd}}{g_0^2}&\left[-(q+k)_\delta \delta_{\beta\gamma} + (2k-q)_\beta \delta_{\gamma\delta} + (2q-k)_\gamma \delta_{\delta\beta}\right] \\ &\times if^{adc}\left[\delta_{\delta\gamma}(2k-q)_\alpha - 2\delta_{\gamma\alpha}k_\delta + 2\delta_{\alpha\delta}(q-k)_\gamma\right] \\ &\times \int_0^t du\, \left(g_0^2 \frac{e^{-k^2 u}}{k^2}\right)\left(g_0^2 \frac{e^{-(q-k)^2 u}}{(q-k)^2}\right) e^{-q^2(t-u)}.\end{aligned} \tag{6.3}$$

Simplifying the numerical and color factors, collecting like terms in k, and writing the $(q-k)$ propagator in Schwinger parameters,

$$\begin{aligned}\Gamma_e = -g_0^2 T_A \int_0^t du \int_0^\infty dz\ e^{-q^2(z+t)} \int_k \frac{e^{-k^2(2u+z)}}{k^2} e^{2(k\cdot q)(u+z)} \\ \times\Big\{ \left[(d-5)q_\alpha q_\beta + 4q^2\delta_{\alpha\beta}\right] - 2\left[(d-2)(q_\alpha\delta_{\beta\mu} + q_\beta\delta_{\alpha\mu}) + 2q_\mu\delta_{\alpha\beta}\right] k_\mu \\ + 2\left[(d-2)(\delta_{\alpha\nu}\delta_{\beta\mu} + \delta_{\beta\nu}\delta_{\alpha\mu}) + 2\delta_{\mu\nu}\delta_{\alpha\beta}\right] k_\mu k_\nu \Big\},\end{aligned} \tag{6.4}$$

it is clear that the only term with angular dependence is $e^{2(k\cdot q)(u+z)}$. If we expand this as a MacLaurin series, we can again collect like powers of the loop momentum and discard all odd powers due to the symmetry of the integral. The first bracketed term above is constant with respect to k, so it multiplies only even powers of $k\cdot q$, and we can reindex $n \to 2n$. Likewise, the second and third terms are respectively odd and even in k, so they are reindexed according to $n \to 2n+1$ and $n \to 2n$:

$$\begin{aligned}\Gamma_e = -g_0^2 T_A \int_0^t du \int_0^\infty dz\ e^{-q^2(z+t)} \int_k \frac{e^{-k^2(2u+z)}}{k^2} \sum_{n=0}^{\infty} \frac{(2(u+z))^{2n}}{(2n)!} q_{I_{2n}} k_{I_{2n}} \\ \times\Big\{ \left[(d-5)q_\alpha q_\beta + 4q^2\delta_{\alpha\beta}\right] + 2\left[(d-2)(\delta_{\alpha\nu}\delta_{\beta\mu} + \delta_{\beta\nu}\delta_{\alpha\mu}) + 2\delta_{\mu\nu}\delta_{\alpha\beta}\right] k_\mu k_\nu \\ - \frac{4(u+z)}{2n+1}\left[(d-2)(q_\alpha\delta_{\beta\mu} + q_\beta\delta_{\alpha\mu}) + 2q_\mu\delta_{\alpha\beta}\right] q_{\mu_{2n+1}} k_{\mu_{2n+1}} k_\mu \Big\},\end{aligned} \tag{6.5}$$

All terms are now even in k, so they may be decomposed according to Eqs. 2.105-2.108 and replaced by:

$$q_{I_{2n}} k_{I_{2n}} \to \frac{1}{(d)_{n,2}} q_{I_{2n}} S^{(2n)}_{I_{2n}} = \frac{(2n-1)!!}{(d)_{n,2}} \left(q^2\right)^n \left(k^2\right)^n, \tag{6.6a}$$

$$q_{I_{2n+1}} k_{I_{2n+1}\mu} \to \frac{1}{(d)_{n+1,2}} q_{I_{2n+1}} S^{(2n+2)}_{I_{2n+1}\mu} = \frac{(2n+1)!!}{(d)_{n+1,2}} \left(q^2\right)^n \left(k^2\right)^{n+1} q_\mu, \tag{6.6b}$$

$$q_{I_{2n}} k_{I_{2n}\mu\nu} \to \frac{1}{(d)_{n+1,2}} q_{I_{2n}} S^{(2n+2)}_{I_{2n}\mu\nu} = \frac{(2n-1)!!}{(d)_{n+1,2}} \left(q^2\right)^n \left(k^2\right)^{n+1} \left(\delta_{\mu\nu} + 2n\frac{q_\mu q_\nu}{q^2}\right). \tag{6.6c}$$

The momentum integrals are now in the form of Eq. 2.52, and we can simplify the expression by making the substitutions $u = t\upsilon$, $z = t\zeta$, and $\tau = q^2 t$:

$$\Gamma_e = -g_0^2 \frac{T_A \delta^{ab}}{(4\pi)^2}(4\pi t)^{2-d/2} \int_0^1 d\upsilon \int_0^\infty d\zeta \sum_{n=0}^{\infty} \frac{\tau^n}{n!} \frac{(\zeta+\upsilon)^{2n}}{(\zeta+2\upsilon)^{d/2+n}} e^{-\tau(\zeta+1)} \times \Bigg\{ \frac{2\tau(\zeta+2\upsilon)}{d+2n-2}\left((d-5)\frac{q_\alpha q_\beta}{q^2} + 4\delta_{\alpha\beta}\right) - \frac{8\tau(\zeta+\upsilon)}{d+2n}\left((d-2)\frac{q_\alpha q_\beta}{q^2} + \delta_{\alpha\beta}\right) + \frac{8}{d+2n}\left((d+n-1)\delta_{\alpha\beta} + (d-2)n\frac{q_\alpha q_\beta}{q^2}\right)\Bigg\}. \tag{6.7}$$

There are a few ways to sum and integrate this expression. One can recast the factor $(\zeta+2\upsilon)^{-d/2-n}$ as a binomial series so that the integrals are simpler. Alternatively, integrating in υ first produces hypergeometric functions, and integrating in ζ or summing over n produces incomplete gamma functions. By replacing these special functions by their integral or series definitions, a complete solution can be obtained, but the intermediate expressions are fairly intractable and lead to the same result. Instead, it is far simpler to note that for the first two bracketed terms above are at least $\mathcal{O}(t)$ for all $n \geq 0$, as is the third term for $n \geq 1$, so they may be discarded. We are left with

$$\Gamma_e = -8g_0^2 \frac{T_A}{(4\pi)^2}\frac{d-1}{d}(4\pi t)^{2-d/2} \int_0^1 d\upsilon \int_0^\infty d\zeta \; \frac{e^{-\tau(\zeta+1)}}{(\zeta+2\upsilon)^{d/2}} \delta^{ab}\delta_{\alpha\beta} + \mathcal{O}(\tau). \tag{6.8}$$

Integrating over ζ, we have

$$\Gamma_e = -8g_0^2 \frac{T_A}{(4\pi)^2}\frac{d-1}{d}(4\pi t)^{2-d/2} e^{-\tau}\tau^{d/2-1} \int_0^1 d\upsilon \; e^{2\tau\upsilon}\Gamma\left(1-\frac{d}{2}, 2\tau\upsilon\right)\delta^{ab}\delta_{\alpha\beta} + \mathcal{O}(\tau). \tag{6.9}$$

Integrating over υ, we are left with

$$\Gamma_e = -8g_0^2 \frac{T_A}{(4\pi)^2} \frac{d-1}{d(d-2)} (4\pi t)^{2-d/2} \tau^{d/2-2} \times \left\{ e^{-\tau}\Gamma\left(2-\frac{d}{2}\right) - e^{\tau}\Gamma\left(2-\frac{d}{2}, 2\tau\right) \right\} \delta^{ab}\delta_{\alpha\beta} + \mathcal{O}(\tau), \tag{6.10}$$

which may be expanded in ϵ and t to zeroth order:

$$\Gamma_e = -3g_0^2 \frac{T_A}{(4\pi)^2} \left\{ \frac{1}{\epsilon} + \log(8\pi t) + \frac{5}{6} \right\} \delta^{ab}\delta_{\alpha\beta} + \mathcal{O}(\epsilon, t). \tag{6.11}$$

Proceeding as above for generic ξ_0, we find

$$\Gamma_e(q;t) = -\frac{1}{2} \cdot \frac{3}{2} g_0^2 \frac{T_A}{(4\pi)^2} (\xi_0 + 1) \left\{ \frac{1}{\epsilon} + \log(8\pi t) + \frac{5}{6} \right\} \delta^{ab} \left(\Lambda_{\alpha\beta} + \Pi_{\alpha\beta} \right) + \mathcal{O}(\epsilon, t), \tag{6.12a}$$

$$\Gamma_f(q;t) = \frac{1}{8} g_0^2 \frac{T_A}{(4\pi)^2} \left\{ (\xi_0 - 9) \left[\frac{1}{\epsilon} + \log(8\pi t) \right] + \frac{1}{2}(\xi_0 + 3) \right\} \delta^{ab} \left(\Lambda_{\alpha\beta} + \Pi_{\alpha\beta} \right) + \mathcal{O}(\epsilon, t), \tag{6.12b}$$

$$\Gamma_g(q;t) = \frac{1}{2} \cdot \frac{3}{4} g_0^2 \frac{T_A}{(4\pi)^2} \left\{ (\xi_0 + 3) \left[\frac{1}{\epsilon} + \log(8\pi t) \right] + \frac{1}{6}(5\xi_0 + 3) \right\} \delta^{ab} \left(\Lambda_{\alpha\beta} + \Pi_{\alpha\beta} \right) + \mathcal{O}(\epsilon, t), \tag{6.12c}$$

$$\Gamma_h(q;t) = \frac{1}{2} \cdot \mathcal{O}(t, s), \tag{6.12d}$$

where the symmetry factors have been written explicitly. There are three additional diagrams which are simply mirror images of (e)-(g), related by the interchange $t \leftrightarrow s$. Summing all contributions with external legs included, the bare propagator is

$$\begin{aligned}\langle \tilde{B}^b_\beta(-q)\tilde{B}^a_\alpha(q)\rangle_0 = {} & g_0^2\frac{\delta^{ab}}{q^2}\left[\Pi_{\alpha\beta}+\xi_0\Lambda_{\alpha\beta}\right] \\ & -\frac{g_0^4}{(4\pi)^2}\frac{\delta^{ab}}{q^2}\Bigg\{\Bigg[\left(\frac{13-3\xi_0}{6}T_A+\frac{2}{3}n_f\right)\left(\frac{1}{\epsilon}+\log\left(\frac{4\pi}{q^2}\right)-\gamma_E\right) \\ & \qquad +\frac{\xi_0+3}{4}T_A\left(\frac{2}{\epsilon}+\log(8\pi t)+\log(8\pi s)\right) \\ & \qquad +\frac{9\xi_0(\xi_0+4)+115}{36}T_A+\frac{10}{9}n_f\Bigg]\Pi_{\alpha\beta} \\ & \qquad +\Bigg[\frac{\xi_0(\xi_0+3)}{4}T_A\left(\frac{2}{\epsilon}+\log(8\pi t)+\log(8\pi s)\right) \\ & \qquad +\frac{\xi_0(\xi_0+1)}{2}T_A\Bigg]\Lambda_{\alpha\beta}+\mathcal{O}(\epsilon,t,s)\Bigg\} \\ & +\mathcal{O}(g_0^6).\end{aligned} \tag{6.13}$$

Replacing the bare coupling g_0 and gauge-fixing parameter ξ_0 by their renormalized counterparts as in Eq. 6.2, we find a finite result without any field renormalization:

$$\begin{aligned}\langle \tilde{B}^b_\beta(-q)\tilde{B}^a_\alpha(q)\rangle = {} & g^2\frac{\delta^{ab}}{q^2}\left[\Pi_{\alpha\beta}+\xi\Lambda_{\alpha\beta}\right] \\ & -\frac{g^4}{(4\pi)^2}\frac{\delta^{ab}}{q^2}\Bigg\{\Bigg[\left(\frac{13-3\xi}{6}T_A+\frac{2}{3}n_f\right)\left(\log\left(\frac{4\pi\mu^2}{q^2}\right)-\gamma_E\right) \\ & \qquad +\frac{\xi+3}{4}T_A\left(\log(8\pi\mu^2 t)+\log(8\pi\mu^2 s)\right) \\ & \qquad +\frac{9\xi(\xi+4)+115}{36}T_A+\frac{10}{9}n_f\Bigg]\Pi_{\alpha\beta} \\ & \qquad +\Bigg[\frac{\xi(\xi+3)}{4}T_A\left(\log(8\pi\mu^2 t)+\log(8\pi\mu^2 s)\right) \\ & \qquad +\frac{\xi(\xi+1)}{2}T_A\Bigg]\Lambda_{\alpha\beta}+\mathcal{O}(t,s)\Bigg\} \\ & +\mathcal{O}(g^6).\end{aligned} \tag{6.14}$$

Then, at least to one-loop order, the bulk gauge fields require no renormalization. In fact,

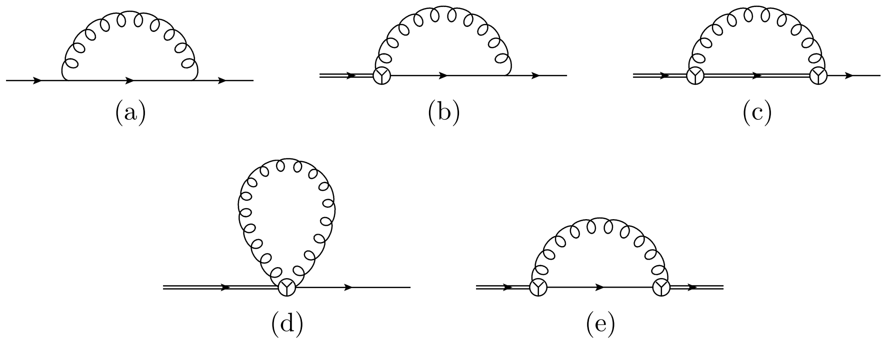


Figure 6.3: One-loop contributions to the propagator of the flowed fermion fields

we will show that there are no bulk counterterms for the gauge field at any order in Sec. 6.3.

6.2 Fermion Self-Energy

At one loop, the fermion self-energy receives contribution from eight diagrams with five unique topologies, Fig. 6.3. Since the relevant fermion masses, m_u, m_d, m_s, are all far less than than typical hadronic scales, $\Lambda \sim 1$ GeV, and since they are identically zero in the chiral limit, we consider them perturbations and keep only the leading order. Treating the integrals as we did in evaluating the gauge field propagator, we calculate:

$$\Gamma_a(p;t) = -g_0^2 \frac{C_2(F)}{(4\pi)^2} \frac{1}{p^2} \Bigg\{ \xi_0 \left[\frac{1}{\epsilon} + \log\left(\frac{4\pi}{p^2}\right) - \gamma_E + 1 \right] i\not{p} + \left[(3-\xi_0)\left(\frac{1}{\epsilon} + \log\left(\frac{4\pi}{p^2}\right) - \gamma_E \right) + 4 \right] m_0 \Bigg\} + \mathcal{O}(\epsilon, t, s, m_0^2), \tag{6.15a}$$

$$\Gamma_b(q;t) = -g_0^2 \frac{C_2(F)}{(4\pi)^2} \frac{\xi_0}{p^2} \left(\frac{1}{\epsilon} + \log(8\pi t) + 1 \right) \left(-i\not{p} + m_0 \right) + \mathcal{O}(\epsilon, t, s, m_0^2), \tag{6.15b}$$

$$\Gamma_c(q;t) = \mathcal{O}(t,s), \tag{6.15c}$$

$$\Gamma_d(q;t) = \frac{1}{2} \cdot g_0^2 \frac{C_2(F)}{(4\pi)^2} \frac{1}{p^2} \left[(\xi_0 + 3)\left(\frac{1}{\epsilon} + \log(8\pi t) \right) + \xi_0 + 1 \right] \left(-i\not{p} + m_0 \right) + \mathcal{O}(\epsilon, t, s, m_0^2), \tag{6.15d}$$

$$\Gamma_e(q;t) = \mathcal{O}(t,s). \tag{6.15e}$$

Summing these along with three additional diagrams related to (b)-(d) by the interchange $t \leftrightarrow s$, the total bare propagator is

$$\begin{aligned}
&\langle\tilde{\chi}(-p;s)\tilde{\bar{\chi}}(p;t)\rangle_0 \\
&\quad= \frac{-i\not{p}+m_0}{p^2} \\
&\quad\quad + g_0^2\frac{C_2(F)}{(4\pi)^2}\frac{1}{p^2}\Bigg\{-i\not{p}\left[\frac{3}{\epsilon}+\xi_0\log\left(\frac{4\pi}{\gamma' p^2}\right)+\frac{3-\xi_0}{2}\left(\log(8\pi t)+\log(8\pi s)\right)+1\right] \\
&\quad\quad\quad + m_0\left[(\xi_0-3)\left(\log\left(\frac{4\pi}{\gamma' p^2}\right)-\frac{1}{2}\log(8\pi t)-\frac{1}{2}\log(8\pi s)\right)-\xi_0-3\right] \\
&\quad\quad\quad + \mathcal{O}(\epsilon)\Bigg\} \\
&\quad\quad + \mathcal{O}(g_0^4, m_0^2, t, s).
\end{aligned} \tag{6.16}$$

Replacing the bare mass, coupling, and gauge-fixing parameter with renormalized parameters,

$$m_0 = Z_m m, \qquad g_0 = \mu^\epsilon Z_g g, \qquad \xi_0 = Z_\xi \xi, \tag{6.17}$$

the bare propagator becomes

$$\begin{aligned}
&\langle\tilde{\chi}(-p;s)\tilde{\bar{\chi}}(p;t)\rangle_0 \\
&\quad= \frac{-i\not{p}+m}{p^2} \\
&\quad\quad + g^2\frac{C_2(F)}{(4\pi)^2}\frac{1}{p^2}\Bigg\{-i\not{p}\left[\frac{3}{\epsilon}+\xi\log\left(\frac{4\pi\mu^2}{\gamma' p^2}\right)+\frac{3-\xi}{2}\left(\log\left(8\pi\mu^2 t\right)+\log\left(8\pi\mu^2 s\right)\right)+1\right] \\
&\quad\quad\quad + m\left[\frac{3}{\epsilon}+(\xi-3)\left(\log\left(\frac{4\pi\mu^2}{\gamma' p^2}\right)-\frac{1}{2}\log\left(8\pi\mu^2 t\right)-\frac{1}{2}\log\left(8\pi\mu^2 s\right)\right)-\xi-3\right] \\
&\quad\quad\quad + \mathcal{O}(\epsilon)\Bigg\} \\
&\quad\quad + \mathcal{O}(g^4, m^2, t, s),
\end{aligned} \tag{6.18}$$

and there is an overall pole of $3/\epsilon$ remaining in both the mass and kinetic terms. This may be canceled by defining a renormalized bulk fermion field:

$$\chi_0 = Z_\chi^{1/2}\chi, \qquad \overline{\chi}_0 = \overline{\chi} Z_\chi^{1/2}, \tag{6.19}$$

where

$$Z_\chi = 1 + g^2 \frac{C_2(F)}{(4\pi)^2}\frac{3}{\epsilon} + \mathcal{O}(g^4). \tag{6.20}$$

The fully renormalized one-loop propagator is thus

$$\begin{aligned}
&\langle\tilde{\chi}(-p;s)\tilde{\overline{\chi}}(p;t)\rangle \\
&\quad = \frac{-i\not{p} + m}{p^2} \\
&\quad + g^2 \frac{C_2(F)}{(4\pi)^2}\frac{1}{p^2}\Bigg\{ -i\not{p}\left[\xi \log\left(\frac{4\pi\mu^2}{\gamma' p^2}\right) + \frac{3-\xi}{2}\left(\log\left(8\pi\mu^2 t\right) + \log\left(8\pi\mu^2 s\right)\right) + 1\right] \\
&\qquad\qquad + m\left[(\xi - 3)\left(\log\left(\frac{4\pi\mu^2}{\gamma' p^2}\right) - \frac{1}{2}\log\left(8\pi\mu^2 t\right) - \frac{1}{2}\log\left(8\pi\mu^2 s\right)\right) - \xi - 3\right]\Bigg\} \\
&\quad + \mathcal{O}(g^4, m^2, t, s).
\end{aligned} \tag{6.21}$$

In the flowed action, the only fermionic counterterm allowed by gauge invariance, Grassmann parity, and the counting of engineering dimensions,

$$[\chi] = \frac{d-1}{2}, \qquad [\lambda] = \frac{d+1}{2}, \tag{6.22}$$

is proportional to

$$\int dt \int d^d x \ \left(\overline{\chi}\lambda + \overline{\lambda}\chi\right). \tag{6.23}$$

At positive flow time there is always at least one flowed propagator at every order in perturbation theory, so high-energy modes are Gaussian suppressed, and there are no local counterterms corresponding to the bulk fermions. This is not, however, the case on the boundary, where the corresponding term,

$$S_{\overline{\lambda},\lambda} = \int d^d x \, \left(\overline{\psi}\lambda|_{t=0} + \overline{\lambda}|_{t=0}\psi\right), \tag{6.24}$$

is required by BRST invariance. This forces us to reciprocally renormalize the fermionic Lagrange multipliers:

$$\lambda_0 = Z_\chi^{-1/2}\lambda, \qquad \overline{\lambda}_0 = \overline{\lambda} Z_\chi^{-1/2}. \tag{6.25}$$

We will return to this at the end of Sec. 6.3.

6.3 BRST Symmerty in $(d+1)$ Dimensions

The flow equation, Eq. 5.5, is invariant under a gauge transformation, Eq. 1.7, so long as the gauge function ω satisfies

$$\left(\partial_t - \alpha_0 D_\mu \partial_\mu\right)\omega = 0. \tag{6.26}$$

This condition may be fixed in a manner similar to the Faddeev-Popov construction, namely, by introducing a bulk ghost field d and a bulk antighost $\overline{d}$ with the action [84, 90]

$$S_{\overline{d}d} = -2\int dt \int d^d x \, \operatorname{Tr}\left\{\overline{d}(\partial_t - \alpha_0 D_\mu \partial_\mu)d\right\}. \tag{6.27}$$

The ghost field has the boundary condition $d_{t=0} = c$, while the antighost is left unfixed on the boundary, since it acts as a Lagrange multiplier generating a flow equation for d. The bulk ghost field then receives perturbative corrections just as the gauge and fermion fields do. In particular, the ghost field flow equation has the recursive solution

$$
\begin{aligned}
\tilde{d}(p;t) &= e^{-\alpha_0 p^2 t}\tilde{c}(p) \\
&\quad + \int_0^t ds\; e^{-\alpha_0 p^2 (t-s)} \int_{p_1,p_2} (2\pi)^d \delta^{(d)}(p+p_1+p_2) \\
&\qquad\qquad \times X^{(1,1)}(p,p_1,p_2)^{aa_1a_2}_{\mu_1} \tilde{B}^{a_1}_{\mu_1}(-p_1;s)\tilde{d}^{a_2}(-p_2;s),
\end{aligned}
\tag{6.28}
$$

which gives us the propagators

$$
\langle \tilde{d}^b(-p;s)\tilde{\bar{d}}^a(p;t)\rangle^{(0)} = \delta^{ab}\theta(t-s)e^{-\alpha_0 p^2 t} \tag{6.29}
$$

and, by virtue of the boundary condition,

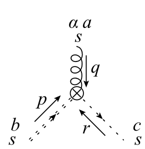
$$
\langle \tilde{d}^b(-p;s)\tilde{\bar{c}}^a(p;t)\rangle^{(0)} = g_0^2 \delta^{ab} \frac{e^{-\alpha_0 p^2 t}}{p^2}. \tag{6.30}
$$

There is also a vertex

$$
X^{(1,1)}(p,q,r)^{abc}_{\mu} = -i\alpha_0 f^{abc} r_\mu, \tag{6.31}
$$

giving us the Feynman rule

$\langle \tilde{d}\tilde{B}\tilde{\bar{d}}\rangle$:



$$
= i\alpha_0 f^{abc} r_\alpha, \tag{6.32}
$$

Where $\bar{d}$ is represented by a double dotted line, d by the standard dotted line. These fields generally do not enter perturbation theory, but they are necessary for a complete BRST-invariant action:

$$S = S_d + S_{d+1} \tag{6.33}$$

where the unflowed action is the sum

$$S_d = S_{YM} + S_D + S_{FP} + S_{gf}, \tag{6.34}$$

and the flowed part of the action is

$$S_{d+1} = S_B + S_{\overline{\chi}\chi} + S_{\overline{d}d}. \tag{6.35}$$

The boundary theory, S_d, was shown to be invariant under BRST transformations in Sec. 1.5. Extending that procedure to the flowed theory, the variations of the bulk gauge, fermion, and ghost fields are exactly like those at $t = 0$, Eqs. 1.56-1.57,

$$\delta\chi = -\theta d\chi, \quad \delta\overline{\chi} = -\theta\overline{\chi}d, \quad \delta B_\mu = \theta D_\mu d, \quad \delta d_\mu = -\theta d^2. \tag{6.36}$$

For each of these, the associated Lagrange multipliers transform similarly,

$$\delta\lambda = -\theta d\lambda, \quad \delta\overline{\lambda} = -\theta\overline{\chi}d, \quad \delta L_\mu = \theta[L_\mu, d], \tag{6.37}$$

with the exception of the bulk antighost field, whose variation has an unusual structure:

$$\delta\overline{d} = \theta\left\{D_\mu L_\mu - \{d, \overline{d}\} + \overline{\lambda}t^a\chi t^a - \overline{\chi}t^a\lambda t^a\right\}. \tag{6.38}$$

This last expression is derivable by extending the configuration space to include components of the gauge field in the t-direction, $B = (B_\mu, B_t)$, so that gauge transformations assume a $(d+1)$-dimensional form [91]. Under these variations, the total action is invariant:

$$\delta S = 0. \tag{6.39}$$

6.4 Perturbative Renormalizability

In order to prove the renormalizability, we follow Ref. [92], omitting many details. The Slavnov-Taylor (Ward) identities associated to the BRST symmetry of the flowed theory are generated by the Zinn-Justin (ZJ) equation [93], which requires a few definitions in advance. First we introduce a source J for each field:

$$S_J = \sum_\phi \int d^d x \; (\pm J_\phi \phi) + \sum_\Phi \int d^d x \; \int dt \; (\pm J_\Phi \Phi) \tag{6.40}$$

(where the sums are taken over all boundary fields ϕ and all bulk fields Φ with traces implied where necessary and signs accounting for canonical ordering of anticommuting fields), as well as a source K for each variation:

$$S_K = \sum_\phi \int d^d x \; (\pm K_\phi \delta\phi) + \sum_\Phi \int d^d x \; \int dt \; (\pm K_\Phi \delta\Phi). \tag{6.41}$$

We now define the effective action functional, which produces all 1PI correlation functions, as the Legendre transform of the energy with respect to the sources:

$$\Gamma[K, \phi, \Phi] = -\log Z[J, K] - S_J, \qquad Z[J, K] = \int \mathcal{D}[\phi, \Phi] e^{-S - S_J - S_K}. \tag{6.42}$$

Working in the off-shell scheme, one may simplify the following arguments by eliminating the Nakanishi-Lautrup field B^a and the antighost $\overline{c}^a$ through a shift in the effective action,

$$\tilde{\Gamma}[K,\phi,\Phi] = \Gamma[K,\phi,\Phi] - \frac{1}{T_F}\int d^dx \ \mathrm{Tr}\, B\frac{\delta\Gamma}{\delta B}, \tag{6.43}$$

which is absorbed by the source of the variation of the gauge field on the boundary, K_A. Now the Zinn-Justin equation assumes the form

$$\sum_{\phi}\int d^dx \ \left(\pm\frac{\delta\tilde{\Gamma}}{\delta\phi}\frac{\delta\tilde{\Gamma}}{\delta K_\phi}\right) + \sum_{\Phi}\int d^dx \ \int dt \ \left(\pm\frac{\delta\tilde{\Gamma}}{\delta\Phi}\frac{\delta\tilde{\Gamma}}{\delta K_\Phi}\right) = 0 \tag{6.44}$$

Renormalizability is then proven by induction on n, the order of the perturbative expansion of the effective action:

$$\tilde{\Gamma} = \sum_{n=0}^{\infty} g^{2n}\tilde{\Gamma}^{(n)} \tag{6.45}$$

where $\tilde{\Gamma}^{(0)} = S + S_K$ with all counterterms set to zero. Defining now the BRST operator in the form of a functional derivative,

$$\mathcal{D}_n = \sum_{\phi}\int d^dx \ \left(\frac{\delta\tilde{\Gamma}^{(n)}}{\delta\phi}\frac{\delta}{\delta K_\phi} + \frac{\delta\tilde{\Gamma}^{(n)}}{\delta K_\phi}\frac{\delta}{\delta\phi}\right) + \sum_{\Phi}\int d^dx \ \int dt \ \left(\frac{\delta\tilde{\Gamma}^{(n)}}{\delta\Phi}\frac{\delta}{\delta K_\Phi} + \frac{\delta\tilde{\Gamma}^{(n)}}{\delta K_\Phi}\frac{\delta}{\delta\Phi}\right), \tag{6.46}$$

with

$$\mathcal{D} = \sum_{n=0}^{\infty} g^{2n}\mathcal{D}_n, \tag{6.47}$$

we may rewrite the ZJ equation as

$$\mathcal{D}\tilde{\Gamma} = 0. \tag{6.48}$$

At tree-level, this is simply a statement of the BRST closure of the action:

$$\mathcal{D}_0\tilde{\Gamma}^{(0)} = 0, \tag{6.49}$$

which forms the base case of our induction. Since Eq. 6.48 holds at all orders in perturbation theory, we expand it and reindex,

$$\mathcal{D}\tilde{\Gamma} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} g^{2m+2n} D_n \tilde{\Gamma}^{(m)} = \sum_{n=0}^{\infty} g^{2n} \sum_{m=0}^{n} D_m \tilde{\Gamma}^{(n-m)} = 0, \tag{6.50}$$

so that for all n we find

$$\sum_{m=0}^{n} \mathcal{D}_m \tilde{\Gamma}^{(n-m)} = 0. \tag{6.51}$$

Now consider the divergent pieces, $\tilde{\Gamma}_\infty^{(n)}$, and suppose that our renormalization prescriptions, Eqs. ref, exactly cancel all divergences at n^{th} order. Then at the next order,

$$\mathcal{D}_0\tilde{\Gamma}^{(n+1)} = \mathcal{D}_0\tilde{\Gamma}^{(n+1)}_{\text{finite}} + \mathcal{D}_0\tilde{\Gamma}^{(n+1)}_{\infty} = -\sum_{m=1}^{n+1} \mathcal{D}_m \tilde{\Gamma}^{(n-m+1)}. \tag{6.52}$$

In our inductive hypothesis, however, we assumed that $\tilde{\Gamma}_\infty^{(m)} = 0$ for all $m \le n$. Thus all of the terms on the far right of Eq. 6.52 are finite, and we have

$$\mathcal{D}_0\tilde{\Gamma}^{(n+1)}_{\infty} = 0. \tag{6.53}$$

In principle, $\tilde{\Gamma}_\infty^{(n+1)}$ is allowed to have counterterms both in the bulk and on the boundary. The former may be immediately ruled out following the discussion preceding Eq. 6.25. Since external fields in the bulk will always induce at least one Gaussian damping factor, the relevant momentum integrals are convergent. The BPHZ theorem [94] then ensures that all boundary subdiagrams are finite as well, as the boundary theory is renormalized. Then there may be no divergent counterterms in the bulk.

This leaves only boundary counterterms, which must be proportional to local products of fields at $t = 0$ with mass dimension $d = 4$ and zero ghost number. The terms containing flow-time derivatives may be discarded, since no product satisfying these restrictions may contain Lagrange multiplier fields, which are required by the action. Since these terms do not appear, Eq. 6.44 in turn excludes the sources K of the variations of such fields. The remaining K_Φ, too, may be ruled out by mass dimension, leaving only the fields A, ψ, $\overline{\psi}$, c, K_A, K_ψ, $K_{\overline{\psi}}$, K_c, and the Lagrange multipliers on the boundaries, $L|_{t=0}$, $\overline{d}|_{t=0}$, $\lambda|_{t=0}$, and $\overline{\lambda}|_{t=0}$. The most general form of this counterterm (see Ref. [92], Eqs. 4.48 and 4.59), is determined up to seven formally divergent coefficients: one, w, for the gauge action; one for each the kinetic term, x_1, and the mass term, x_2, in the fermion action; three, y_1, y_2, and y_3, for the sources K_A, $K_{\psi(\overline{\psi})}$, and K_c; and one final constant, z, for the fermionic Lagrange multipliers, which are not ruled out by the ZJ equation. Choosing

$$\begin{aligned}
Z_\psi^{(n+1)} &= x_1 + 2y_2, \\
Z_c^{(n+1)} &= -y_1 - y_3, \\
Z_\chi^{(n+1)} &= x_1 + 4y_2 - 2z, \\
Z_g^{(n+1)} &= w, \\
Z_m^{(n+1)} &= x_2 - x_1, \\
Z_\xi^{(n+1)} &= 2y_1 - w,
\end{aligned} \tag{6.54}$$

exactly cancels all potential divergences in $\tilde{\Gamma}_\infty^{(n+1)}$, and we may conclude that these six renormalization constants are sufficient to negate all divergences to all loop orders.

A couple remarks are now in order concerning the proof outlined above. First, there is a problem in defining sources at vanishing flow time for the fields B, χ, $\overline{\chi}$, and d, since they are constrained on the boundary and are thus not true degrees of freedom. In order to circumvent this ambiguity, the authors of Ref. [92] discretized flow time with a forward difference prescription, allowing them to omit the $t=0$ time slice from the relevant action integrals. This violates the Zinn-Justin equation, but these terms are shown to vanish in the continuum limit.

Second, the bulk gauge fields somehow need no renormalization in the bulk, while the fermions do. This disparity may be clarified by examining the boundary counterterms generated by the flow equations. Consider the (perfectly allowed) counterterm containing the Lagrange multipliers of the gauge sector on the boundary:

$$\tilde{\Gamma}_{L,\overline{d}} = \frac{1}{T_F} \int d^d x \ \mathrm{Tr}\left(z_1 L|_{t=0} A + z_2 \overline{d}|_{t=0} c \right), \tag{6.55}$$

with two divergent coefficients z_1 and z_2. Taking the variations of these fields to the boundary as well, the ZJ equation requires that

$$\begin{aligned} T_F \mathcal{D}_0 \tilde{\Gamma}_{L,\overline{d}} &= \int d^d x \ \mathrm{Tr}\left[z_1 \delta L|_{t=0} \cdot A + z_1 L|_{t=0} \cdot \delta A + z_2 \delta \overline{d}|_{t=0} \cdot c - z_2 \overline{d}|_{t=0} \cdot \delta c \right] \\ &= \int d^d x \ \mathrm{Tr}\left[(z_1 - z_2) L|_{t=0} \partial c - z_2 \left(L|_{t=0} [A, c] + \overline{d}|_{t=0} c^2 \right) \right] \end{aligned} \tag{6.56}$$

must vanish (ignoring the fermionic piece of $\delta\overline{d}$ without loss of generality). Since the BRST transformation is inhomogeneous, we must set $z_1 = z_2 = 0$; thus the gauge fields undergo no renormalization in the bulk. On the other hand, for the analogous fermionic counterterm with coefficient z,

$$\tilde{\Gamma}_{\overline{\lambda},\lambda} = z \int d^d x \ \left(\overline{\lambda}|_{t=0} \psi + \overline{\psi} \lambda|_{t=0} \right), \tag{6.57}$$

the variation is itself homogeneous:

$$\begin{aligned}\mathcal{D}_0\tilde{\Gamma}_{\overline{\lambda},\lambda} &= z\int d^dx\ \left(\overline{\lambda}|_{t=0}\psi + \overline{\psi}\lambda|_{t=0}\right) \\ &= z\int d^dx\ \left(\delta\overline{\lambda}|_{t=0}\psi - \overline{\lambda}|_{t=0}\delta\psi + \delta\overline{\psi}\lambda|_{t=0} - \overline{\psi}\delta\lambda|_{t=0}\right) \\ &= z\cdot 0.\end{aligned} \tag{6.58}$$

In this case, since the integral vanishes, there is no condition on z, so the fields λ and $\overline{\lambda}$ indeed produce a counterterm on the boundary (and so require wavefunction renormalization). Recalling that there may be no bulk counterterms like Eq. 6.23, we conclude that the fermions require the inverse renormalization.

Chapter 7

The Short-Flow-Time Expansion

7.1 Composite Operators

In the previous chapter, we found that, due to the Gaussian damping factors induced by the flow equations, the renormalization counterterms of a flowed theory reside exclusively on its boundary. As a result, all correlation functions of the form

$$\begin{aligned}&\langle B^{a_1}_{\alpha_1}(x_1;t_1)\cdots B^{a_n}_{\alpha_n}(x_n;t_n)\overline{\chi}(y_1;s_1)\cdots\overline{\chi}(y_m;s_m)\chi(z_1;u_1)\cdots\chi(z_m;u_m)\rangle\\&\quad = Z_\chi^{-m}\langle B^{a_1}_{\alpha_1}(x_1;t_1)\cdots B^{a_n}_{\alpha_n}(x_n;t_n)\overline{\chi}(y_1;s_1)\cdots\overline{\chi}(y_m;s_m)\chi(z_1;u_1)\cdots\chi(z_m;u_m)\rangle_0\end{aligned} \tag{7.1}$$

are strictly finite at positive flow times provided that the boundary theory is appropriately renormalized. Remarkably, this finiteness carries over to correlation functions as above for which any number of the spacetime coordinates coincide. This follows again from the association of a heat kernel to each flowed field. First observe that the flow does not affect the infrared regime, since all Gaussians tend to unity for small momenta and nonzero flow time, as they must in order to fulfill the boundary conditions. Then we may expect that any IR divergences originate on the boundary. Any other divergences will be ultraviolet, corresponding to the contact of any number of flowed fields at a single point. As the centers of the flowed distributions overlap, the momentum tends to infinity, driving the Gaussians to zero. Functions of the resulting local product of fields will then contain two types of loop integrals. The flowed integrals generated by direct contraction with the operator product are exponentially damped at UV scales by the flowed fields involved, so they converge absolutely. All other loops are radiative corrections to the boundary theory, which are exactly canceled by the boundary counterterms. It follows that for any bare operator

$$\mathcal{O}_0(x;t) = \Gamma B^n(x;t)\overline{\chi}_0^m(x;t)\chi_0^m(x;t), \tag{7.2}$$

where indices are suppressed and all tensor structure is generically represented by Γ, we need only renormalize the fermions (in addition to the boundary parameters, as usual). Then the renormalized operator is simply

$$\mathcal{O}(x;t) = Z_\chi^{-m}\mathcal{O}_0(x;t). \tag{7.3}$$

This allows us to define renormalized correlation functions of local operator products at finite flow time with a simple multiplicative prescription.

7.2 The Short-Flow-Time Expansion

We now have a straightforward and efficient method to renormalize composite operators. If this is to have any predictive power, the flowed matrix elements ought to be relatable to the physical theory at $t = 0$. Of course, as the flow time tends to zero, we expect that the contact divergences of the boundary theory should be recovered, so that all renormalized matrix elements of local operators will in general diverge in this limit. In Sec. 1.10, we saw that these divergences could be absorbed into a suitable renormalization of the composite operators by means of the OPE,

$$(\mathcal{O}_i)_0 = Z_{ij}\mathcal{O}_j, \tag{7.4}$$

where the implied sum over j runs over a basis of operators $\mathcal{O}_j$ restricted only by the quantum numbers of $\mathcal{O}_i$. In this case, the equivalence is meant to be interpreted in the limit that the coordinates of all fields in the operator coincide. The infinite constants Z_{ij} contain the contact divergences generated in this limit. Under the flow, the contact terms are smeared with the fields as functions of the flow time t. Following the same arguments, we may write an analogous asymptotic expansion for renormalized flowed operators near the boundary:

$$\mathcal{O}_i(x;t) \overset{t\to 0}{\sim} c_{ij}(t)\mathcal{O}_j(x), \tag{7.5}$$

called the short flow time expansion (SFTE). On the right-hand side, all flow-time-dependence is isolated within the Wilson coefficients $c_{ij}(t)$. By purely dimensional arguments, we may determine their leading-order scaling with the flow time:

$$[c_{ij}(t)] = [\mathcal{O}_i(x;t)] - [\mathcal{O}_j(x)] = d_i - d_j, \tag{7.6}$$

so that

$$c_{ij}(t) \propto t^{\frac{d_j - d_i}{2}}. \tag{7.7}$$

In the event that $d_i = d_j$, the Wilson coefficient diverges logarithmically with t. The more interesting cases, however, are when $d_i > d_j$, and the dependence on t of the mixing coefficients goes as an inverse power of the flow time. These power divergences are typically absent from perturbation theory with dimensional regularization, since they are generated by integrals which become scaleless on the boundary. This is particularly attractive to lattice applications, because the mixing coefficients are decoupled from the lattice regulator at leading order. Of course, in a discretized setting, there may be subleading corrections which depend on the lattice spacing, but these vanish in the continuum limit. In the remaining case, $d_i < d_j$, the Wilson coefficient is suppressed by some positive power of t and vanishes on the boundary. These terms correspond to irrelevant operators and will be hereafter neglected; we will truncate the sum at the logarithmic order and write the irrelevant contributions as an error of $\mathcal{O}(t)$.

7.3 Wilson Coefficients

Since the SFTE is an operator-level relation, we are afforded a considerable amount of freedom in choosing probes for the Wilson coefficients. Specifically, we may choose any external fields with any kinematics to construct matrix elements of the flowed operator. Choosing an operator as in Eq. 7.2 and some external state with generic flowed or unflowed fields Φ_k, we define the renormalized correlation function

$$\begin{aligned}\Gamma_i(x, y_1, ..., y_n; t, s_1, ..., s_n) &\equiv \langle \Phi_1(y_1, s_1) \cdots \Phi_n(y_n, s_n) \mathcal{O}_i(x; t) \rangle \\ &= Z_{\Phi_1}^{-1} \cdots Z_{\Phi_n}^{-1} Z_\chi^{-m} \langle \Phi_1(y_1, s_1) \cdots \Phi_n(y_n, s_n) \mathcal{O}_i(x; t) \rangle_0,\end{aligned} \tag{7.8}$$

where, in case $\Phi_k = B$ for some k, we write $Z_B = 1$ identically. With a suitable choice of external states depending on the field structure of the boundary operators, we may can choose which terms at any order contribute to the expansion of the correlation function. Particularly at next-to-leading order, the external states may often be chosen so that matrix elements of the form above vanish entirely for some j. Inserting this expression into the SFTE, we have

$$\langle \Phi_1(y_1, t_1) \cdots \Phi_n(y_n, t_n) \mathcal{O}_i(x; t) \rangle = \sum_j c_{ij}(t) \langle \Phi_1(y_1, t_1) \cdots \Phi_n(y_n, t_n) \mathcal{O}_j(x) \rangle. \tag{7.9}$$

Introducing the shorthand $\Gamma_i(t) = \Gamma_i(x, y_1, ..., y_n; t, s_1, ..., s_n)$ for $t \geq 0$, we may express the SFTE as a loop expansion. Writing

$$\Gamma_i(t) = \sum_{n=0}^{\infty} g^{2n} \Gamma_i^{(n)}(t), \qquad c_{ij}(t) = \sum_{n=0}^{\infty} g^{2n} c_{ij}^{(n)}, \tag{7.10}$$

the expansion assumes the form

$$\sum_{n=0}^{\infty} g^{2n} \Gamma_i^{(n)}(t) = \sum_j \sum_{n=0}^{\infty} g^{2n} c_{ij}^{(n)}(t) \sum_{m=0}^{\infty} g^{2m} \Gamma_j^{(m)}(0) = \sum_j \sum_{0 \geq m \geq n} g^{2n} c_{ij}^{(n-m)}(t) \Gamma_j^{(m)}(0). \tag{7.11}$$

Equating terms of the same order, we have

$$\Gamma_i^{(n)}(t) = \sum_j \sum_{m=0}^{n} c_{ij}^{(n-m)}(t) \Gamma_j^{(m)}(0). \tag{7.12}$$

On the right side, the boundary correlators may be further expanded in an OPE,

$$\Gamma_j(0) = Z_{jk}^{-1} [\Gamma_k]_0(0), \tag{7.13}$$

with renormalization constants likewise expanded in the coupling:

$$Z_{jk}^{-1} = \sum_{n=0}^{\infty} g^{2n} [Z_{jk}^{-1}]^{(n)}. \tag{7.14}$$

We may then write the n^{th} term of the SFTE as

$$\Gamma_i^{(n)}(t) = \sum_{j,k} \sum_{0 \leq \ell \leq m \leq n} c_{ij}^{(n-m)}(t) [Z_{jk}^{-1}]^{(m-\ell)} [\Gamma_k]_0^{(\ell)}(0). \tag{7.15}$$

The most useful cases within the scope of this work are the tree-level and one-loop expressions at $n = 0, 1$. In the former case, we have the trivial expression

$$\Gamma_i^{(0)}(t) = c_{ij}^{(0)}(t)[Z_{jk}^{-1}]^{(0)}[\Gamma_k]_0^{(0)}(0), \tag{7.16}$$

where the operator sums over j and k are once again made implicit. Using $Z_{jk}^{(0)} = \delta_{jk}$, and noting that the tree-level structures of the flowed and unflowed matrix elements are identical up to kernels attached to the flowed fields (therefore up to terms of at least order t),

$$\Gamma_i^{(0)}(t) = c_{ij}^{(0)}(t)[\Gamma_j]_0^{(0)}(0) = [\Gamma_i]_0^{(0)}(0) + \mathcal{O}(t), \tag{7.17}$$

we conclude that $c_{ij}^{(0)} = \delta_{ij} + \mathcal{O}(t)$. For $n = 1$, we have after some simplification

$$\Gamma_i^{(1)}(t) = \left\{c_{ij}^{(1)}(t) + [Z_{ij}^{-1}]^{(1)}\right\} \cdot [\Gamma_j]_0^{(0)}(0) + [\Gamma_i]_0^{(1)}(0) + \mathcal{O}(t), \tag{7.18}$$

which gives us an easy recipe for calculating the NLO Wilson coefficients. We will take two approaches in calculating the correlation functions in the next Part of this thesis. Many flowed diagrams will be exactly solvable by the same novel method used to calculate the renormalization constants in previous chapters. On the other hand, when we renormalize the topological charge density and gluon chromoelectric dipole moment operators, many of the integrals or sums will be unsolvable with current methods. When they are exactly solvable, we calculate every term above. After renormalizing the boundary parameters, the renormalization of any χ fields will take care of all remaining poles on the flowed side. On the expanded side, the poles from the bare one-loop matrix element are cancelled by the boundary counterterm. For the unsolvable cases, it is easiest to use the method of projectors.

To proceed, we first choose a set of external states, defining correlation functions Γ_j as above, and rotate to momentum space. We then define as many differential operators $\mathcal{P}_i$ satisfying

$$\mathcal{P}_i \Gamma_j^{(0)} = \delta_{ij}, \tag{7.19}$$

or, in other words, project out the tree level associated with j^{th} operator. The projectors generally contain derivatives with respect to masses and to any momenta related to derivative couplings and traces over all fermionic, Lorentz, and gauge group indices. After the derivatives are taken, all external scales are taken to zero. In order that these traces do not trivially vanish, we also insert appropriate elements of the spacetime and gauge algebras. To ensure orthogonality, we may diagonalize the operator basis. Finally, we normalize to one by dividing out various numerical constants (polynomials in d, group invariants, *etc.*). As an example, consider the qCMDM operator,

$$\mathcal{O}_{CM} = k_{CM} \overline{\psi} \sigma_{\mu\nu} G_{\mu\nu} \psi, \tag{7.20}$$

where k_{CM} is some unimportant normalization constant, and $\sigma_{\mu\nu} = \frac{i}{2}\gamma_{[\mu,\nu]}$. Choosing an external state of two fermions and a gluon, the amputated tree-level result is just the Feynman rule:

$$[\Gamma_{CM}]_0(p,q,r) = \langle \tilde{\psi}(r) \tilde{A}^a_\alpha(q) \tilde{\overline{\psi}}(p) \mathcal{O}_i \rangle_0 = -2ik_{CM} t^a \sigma_{\alpha\beta} q_\beta. \tag{7.21}$$

We now differentiate with respect to q_γ and multiply by $t^a \sigma_{\gamma\alpha}$ so that the traces do not vanish (t^a is traceless, and $\sigma_{\mu\nu}$ is antisymmetric), which determines the normalization:

$$\mathrm{Tr}\left\{ t^a \sigma_{\gamma\alpha} \frac{\partial}{\partial_{q_\gamma}} \left[-2ik_{CM} t^a \sigma_{\alpha\beta} q_\beta \right] \right\} = 2ik_{CM} d(d-1) n_f C_2(F), \tag{7.22}$$

The projector for the qCMDM is then:

$$\mathcal{P}_{CM}[X] = \frac{1}{2ik_{CM} d(d-1) n_f C_2(F)} \mathrm{Tr}\left\{ t^a \sigma_{\gamma\alpha} \frac{\partial}{\partial_{q_\gamma}} X \right\}. \tag{7.23}$$

We will not often worry about orthogonality. Indeed, there is another operator in Sec. 8.2 which will not vanish when acted upon by $\mathcal{P}_{CM}$, but the pieces are trivial to disentangle, and the derivative is the critical operation.

When we apply a projector to subleading diagrams, since all external scales are neglected, there may be nothing the regulate the infrared region of some loop integrals. At zero flow time, the loop integrals appearing in $[\Gamma_k]_0^{(\ell)}(0)$ for $\ell > 0$ will in general be of the form

$$I_n(0) = \int_p \frac{1}{(p^2)^n} = \frac{2(4\pi)^{-d/2}}{\Gamma(d/2)} \int_0^\infty dp\ p^{d-2n-1} = \frac{2(4\pi)^{-d/2}}{\Gamma(d/2)} \cdot \frac{p^{d-2n}}{d-2n}\bigg|_{p=0}^{\infty}. \tag{7.24}$$

In the radial form above, it is easy to see that the disjoint domains of convergence are defined by $d > 2n$ in the IR region ($p \to 0$) and $d < 2n$ in the UV ($p \to \infty$). There is no dimensionful parameter in the integrand, but the integral has a mass dimension of $d - 2n$, so we expect it to vanish. This may be achieved by introducing a factor of one into the formal (undefined) integral and expanding as such:

$$I_n(0) = \int_p \frac{1}{(p^2)^n} \left(\frac{p^2 + m^2}{p^2 + m^2} \right)^k = \sum_{\ell=0}^{k} \binom{k}{\ell} m^{2\ell} \int_p \frac{(p^2)^{k-\ell-n}}{(p^2+m^2)^k}. \tag{7.25}$$

for some integer k. Performing the integral then the sum, we arrive at

$$I_n(0) = \frac{m^{4-2n}}{(4\pi)^2}\left(\frac{4\pi}{m^2}\right)^{2-d/2} \cdot (-1)^n \frac{d-2k}{d-2n}\frac{\sin(k\pi)}{\pi}\frac{\Gamma(d/2-k)\Gamma(k-d/2)}{\Gamma(d/2)}. \tag{7.26}$$

Since k is an integer, the sine and therefore the integral both vanish.

It is more practically useful to recover this result by analytically continuing the dimension. In this case we split the region of integration by some scale Λ instead of inserting a unit:

$$I_n(0) = \frac{2(4\pi)^{-d_+/2}}{\Gamma(d_+/2)}\int_0^\Lambda dp\, p^{d_+-2n-1} + \frac{2(4\pi)^{-d_-/2}}{\Gamma(d_-/2)}\int_\Lambda^\infty dp\, p^{d_--2n-1}. \tag{7.27}$$

in the first integral, we set $d_+ = 2n + \epsilon_{IR}$ with $\epsilon_{IR} > 0$ to regulate the infrared divergence. For the second integral, we contrariwise define $d_- = 2n - \epsilon_{UV}$ with $\epsilon_{UV} > 0$. The integrals evaluate to

$$I_n(0) = \frac{2(4\pi)^{-d_+/2}}{\Gamma(d_+/2)}\frac{\Lambda^{d_+-2n}}{d_+-2n} - \frac{2(4\pi)^{-d_-/2}}{\Gamma(d_-/2)}\frac{\Lambda^{d_--2n}}{d_--2n}, \tag{7.28}$$

which is easily expanded to leading order near $d_\pm = 2n$. For $n = 2$, we have

$$I_2(0) = \frac{1}{(4\pi)^2}\left[\frac{1}{\epsilon_{IR}} + \frac{1}{\epsilon_{UV}}\right] + \mathcal{O}(\epsilon_{IR}, \epsilon_{UV}), \tag{7.29}$$

so that continuing $\epsilon_{IR} \to -\epsilon_{UV}$ forces the integral to vanish. For any other value of n, the total trivially vanishes:

$$I_{n\neq 2}(0) = \frac{1}{(4\pi)^2}\left[-\frac{\Lambda^{4-2n}}{n-2} + \frac{\Lambda^{4-2n}}{n-2}\right] + \mathcal{O}(\epsilon_{IR}, \epsilon_{UV}) = \mathcal{O}(\epsilon_{IR}, \epsilon_{UV}). \tag{7.30}$$

This partition of the integrand is especially important for computing flowed integrals after projection. In Eq. 7.18, the correlators at zero flow time vanish by the above arguments.

The flowed correlation functions typically contain integrals of the form

$$I_n(u) = \int_p \frac{e^{-p^2 u}}{(p^2)^n} = \frac{2(4\pi)^{-d/2}}{\Gamma(d/2)} \int_0^\infty dp\ e^{-p^2 u} p^{d-2n-1}, \tag{7.31}$$

where u is some nonnegative parameter dependent on the flow time that endows the integral with a scale and damps the UV modes. For $d > 2n$, this is a simple gamma function:

$$I_{n<d/2}(u) = (4\pi u)^{2-d/2} \frac{u^{n-2}}{(4\pi)^2} \frac{\Gamma(d/2-n)}{\Gamma(d/2)}. \tag{7.32}$$

For $d \leq 2n$, however, we encounter another IR divergence, so we use $d = d_+$. Expanding the integral, we have

$$I_{n\geq d/2}(u) = -\frac{1}{(4\pi)^2} \left[\frac{1}{\epsilon_{IR}} - \log(4\pi u) - 1 + \mathcal{O}(\epsilon_{IR}) \right], \tag{7.33}$$

evincing the IR pole. These should cancel the the UV pole from any necessary renormalization factors and the boundary counterterm proportional to $[Z_{ij}^{-1}]^{(1)}$, but the signs are wrong. We may then define the boundary integrals as in Eq. 7.29 before analytically continuing, which reintroduces all UV poles and cancels the IR poles. Equivalently, we can simply set $\epsilon_{IR} = -\epsilon_{UV}$ in all calculations, and the poles manifestly cancel.

Now that we have two methods for calculating loop integrals, we can rearrange the SFTE at one-loop order, expressing the Wilson coefficient in terms of quantities we can now calculate:

$$c_{ij}^{(1)}(t)[\Gamma_j]_0^{(0)}(0) = \Gamma_i^{(1)}(t) - [\Gamma_i]_0^{(1)}(0) - [Z_{ij}^{-1}]^{(1)}[\Gamma_j]_0^{(0)}(0) + \mathcal{O}(t). \tag{7.34}$$

The mixing coefficients are then easily readable as the (finite) coefficients of the tree-levels for each operator j.

Part III

CP-Violating Operator Mixing

Chapter 8

Results

In this chapter, we discuss the mixing a few physically-relevant $\mathcal{CP}$ -odd operators. The three operators we will cover are the topological charge density (TCD), the quark chromoelectric dipole moment (qCEDM), and the gluon chromoelectric moment (gCEDM).

The TCD is defined as

$$\mathcal{O}_q = \frac{k_q}{g^2} \operatorname{Tr} G_{\mu\nu} \tilde{G}_{\mu\nu}, \tag{8.1}$$

where $\tilde{G}$ is the Hodge dual of the curvature G,

$$\tilde{G} = -\frac{1}{2} \epsilon_{\mu\nu\rho\sigma} G_{\rho\sigma} \tag{8.2}$$

and the Levi-Civita tensor ϵ ensures parity violation. As we mentioned in Sec. 1.5, renormalizability requires the most general BRST-invariant Lagrangian, which in the case of QCD should include the charge. Nevertheless, experimental evidence suggests that QCD preserves parity, so that the coupling of this term must be nearly zero. In light of this fact, the TCD is typically completely ignored when defining the QCD Lagrangian.

The second operator we consider is the quark chromoelectric dipole moment:

$$\mathcal{O}_{CE} = k_{CE} \bar{\psi} \tilde{\sigma}_{\mu\nu} G_{\mu\nu} \psi. \tag{8.3}$$

The dimension of the qCEDM is $d_{CE} = 3d/2 - 1$, which coincides with $d_{CE} = 5$ in four dimensions. This is an effective interaction formed by integrating out the squarks and gluinos from supersymmetric extensions of the Standard Model. The gCEDM is similarly constructed.

In n-Higgs doublet models with $n \geq 2$, one of the physical states is a pseudoscalar. In these models, a heavy quark loop with a virtual Higgs exchange can be reduced to a local interaction by integrating out all degrees of freedom heavier than the quarks. For a three-gluon external state mediated by such a loop, the result is the gCEDM, often called the Weinberg operator [95]:

$$\mathcal{O}_W = \frac{k_W}{g^2} \operatorname{Tr} G_{\mu\rho} G_{\nu\rho} \tilde{G}_{\mu\nu}, \tag{8.4}$$

with $d_{gC} = 5d/2 - 4$ ($d_{gC} = 6$ in four dimensions).

These operators may be renormalized by means of the OPE, Eq. 1.124. Granted that the latter two operators are both higher-dimensional ($d_i > d$) when $d > 2$, we expect that the corresponding Wilson coefficients will run as some inverse power in the chosen regulator. We use the gradient flow to regulate these divergences, transitioning to the SFTE,

$$\mathcal{O}_i(t) = c_{ij}(t) \mathcal{O}_j(0), \tag{8.5}$$

instead of the standard OPE. We then treat the flowed operators as insertions with unflowed external states, so that the overlap singularities alone are parametrized by the flow time. At leading order we can use Eq. 7.18 to extract the flowed Wilson coefficients, c_{ij}. Since the flow time has dimension $[t] = -2$, we expect that

$$c_{ij}(t) \sim t^{\frac{d_j - d_i}{2}}. \tag{8.6}$$

The unflowed renormalization matrix Z_{ij} in the $\overline{\text{MS}}$ scheme was determined for dimension-five operators in Ref. [96] and for dimension-six operators in Ref. [97]. In the flowed case, we need only renormalize the external fields and the flowed fermions. Then all we must calculate are the bare correlation functions at both $t = 0$ and $t > 0$. We proceed in dimensional regularization with the $\overline{\text{MS}}$ subtraction scheme.

8.1 Topological Charge Density

In QFT, the topological charge density counts instanton configurations, classical solutions to the Yang-Mills equations of motion with finite action. When we insert the TCD into the action, the resulting integral is always an integer (up to the normalization $k_q = 1/16\pi^2$) called the topological charge or instanton number. More precisely, the topological charge density is the divergence of the Chern-Simons current K:

$$Q = \int \mathcal{O}_q = \frac{k_q}{g^2} \int \operatorname{Tr} G_{\mu\nu} \tilde{G}_{\mu\nu} = \frac{k_q}{g^2} \int \operatorname{Tr} \partial_\mu K_\mu, \tag{8.7}$$

where

$$K_\mu = -\epsilon_{\mu\nu\rho\sigma} \left(A_\nu G_{\rho\sigma} + \frac{2}{3} A_{\nu\rho\sigma} \right). \tag{8.8}$$

The Feynman rule for the topological charge density is thus directly proportional to the total momentum of the operator. Without any injected momentum, we expect that all correlation functions including the TCD will vanish. Since the total charge is the relevant observable, the operator is always integrated, projecting the total momentum — and the operator itself — to zero. However, as we are only interested in renormalization and mixing in the present context, we can avoid this problem by injecting some momentum k into any correlation functions including the charge. After the Wilson coefficients are computed, we are free to discard k.

We write the SFTE for the topological charge density as

$$\mathcal{O}_q(t) = c_{q,P}(t)\mathcal{O}_P(0) + c_{q,\partial A}(t)\mathcal{O}_{\partial A}(0) + c_{q,q}(t)\mathcal{O}_q(0) + \mathcal{O}t, \tag{8.9}$$

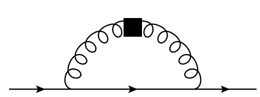
where the operators $\mathcal{O}_P$ and $\mathcal{O}_{\partial A}$ are respectively the pseudoscalar fermionic current and the divergence of the axial current:

$$\mathcal{O}_P = k_P \overline{\psi}\gamma_5\psi, \tag{8.10a}$$

$$\mathcal{O}_{\partial A} = \partial_\mu(\overline{\psi}\tilde{\gamma}_\mu\psi), \tag{8.10b}$$

where $\tilde{\gamma}_\mu = \frac{1}{2}[\gamma_\mu, \gamma_5]$ is a d-dimensional generalization of $\gamma_\mu\gamma_5$ which retains Hermiticity. Note that the coefficient $c_{q,P}$ has dimension $d_P - d_q = 1$, so that the mixing cannot be an integral power of the flow time. Instead, the difference of dimension is compensated by a factor of the mass, which ensures that both sides of Eq. 8.9 have the same chirality.

8.1.1 Mixing With Quark Bilinears: $c_{q,P}(t)$ and $c_{q,\partial A}(t)$



(a) $\Gamma_q^{(1)}(t \geq 0)$

Figure 8.1: The lone contribution to $\Gamma_q^{(1)}(t \geq 0)$

We can probe the quark bilinear operators by choosing a fermionic external state. Defining

$$\Gamma_i(k;t) = \int_p e^{-ip(y-x)} \int d^d z e^{-ikz} \langle \psi(y;0)\mathcal{O}_i(z;t)\overline{\psi}(x;0)\rangle, \tag{8.11}$$

the SFTE becomes

$$\Gamma_q(k;t) = c_{q,P}(t)\Gamma_P(t)(k;0) + c_{q,\partial A}(t)\Gamma_{\partial A}(k;0) + c_{q,q}(t)\Gamma_q(k;0) + \mathcal{O}t. \tag{8.12}$$

While we have suppressed the external momentum p in the above notation, the correlation functions depend on both p and k. Further, we need both momenta to extract the Feynman rules for each term. In this case, since the loop integrals quickly become very difficult, we can use the method of projectors, expanding the integrands to linear order in both p and k. As we discussed in Ch. 7, this allows us to discard the all subleading contributions to the $t = 0$ amplitudes. At one-loop order, the reduced SFTE is

$$\Gamma_q^{(1)}(k;t) = c_{q,P}^{(1)}(t)\Gamma_P^{(0)}(t)(k;0) + c_{q,\partial A}^{(1)}(t)\Gamma_{\partial A}^{(0)}(k;0) + c_{q,q}^{(1)}(t)\Gamma_q^{(0)}(k;0) + \mathcal{O}t. \tag{8.13}$$

On the left, we have only a single diagram, Fig. 8.1. On the right, the first two correlation functions are simply the Feynman rules for $\mathcal{O}_P$ and $\mathcal{O}_{\partial A}$, while the last term vanishes identically. Using the Feynman rules from earlier and solving the momentum integral as before we find

$$\Gamma_q^{(1)}(k;t) = ik_q T_F C_2(F)\frac{g^2}{(4\pi)^2}\left\{-\frac{1}{\epsilon_{IR}} + \log\left(2\bar{\mu}^2 t\right) + \gamma_E + \frac{3}{2}\right\} k_\mu \epsilon_{\mu\nu\rho\sigma}\gamma_{\nu\rho\sigma} + \mathcal{O}(p^2, k^2, m^2, t). \tag{8.14}$$

Since there are no terms linear in the mass, we can immediately see that the NLO Wilson coefficient $c_{q,P}$ vanishes to this order:

$$c_{q,P}^{(1)} = 0. \tag{8.15}$$

Using the identity

$$\epsilon_{\mu\nu\rho\sigma}\gamma_{\nu\rho\sigma} = 3\tilde{\gamma}_\mu, \tag{8.16}$$

and replacing the bare parameters with their renormalized counterparts, we have

$$g^2\Gamma_q^{(1)}(k;t) = 3ik_q T_F C_2(F)\frac{g^2}{(4\pi)^2}\left\{-\frac{1}{\epsilon_{IR}} + \log\left(2\bar{\mu}^2 t\right) + \gamma_E + \frac{3}{2}\right\}k_\mu\tilde{\gamma}_\mu + \mathcal{O}(p^2,k^2,m^2,t), \tag{8.17}$$

which is easily written in terms of $\Gamma_{\partial A}^{(0)}$:

$$g^2\Gamma_q^{(1)}(k;t) = 6T_F C_2(F)\frac{k_q}{k_{\partial A}}\frac{g^2}{(4\pi)^2}\left\{-\frac{1}{\epsilon_{IR}} + \log\left(2\bar{\mu}^2 t\right) + \gamma_E + \frac{3}{2}\right\}\Gamma_{\partial A}^{(0)} + \mathcal{O}(p^2,k^2,m^2,t). \tag{8.18}$$

The boundary diagram allows us to replace $-\epsilon_{IR} \to \epsilon_{UV} = \epsilon$, giving us renormalization constant

$$Z_{q,\partial A}^{(1)}{}^{-1} = -6T_F\frac{k_q}{k_{\partial A}}\frac{C_2(F)}{(4\pi)^2}\frac{1}{\epsilon} \tag{8.19}$$

and subsequently the Wilson coefficient:

$$c_{q,\partial A}^{(1)}(t) = 6\frac{k_q}{k_{\partial A}}\frac{T_F C_2(F)}{(4\pi)^2}\left\{\log\left(2\bar{\mu}^2 t\right) + \gamma_E + \frac{3}{2}\right\}. \tag{8.20}$$

8.1.2 Self-Mixing: $c_{q,q}(t)$

The self-mixing of the topological charge density, $c_{q,q}$ may be extracted with a two-gluon external state, since this choice projects the quark bilinear terms to zero at one-loop order. This time, the function

$$\Gamma_q(k;t) = \int_p e^{-ip(y-x)}\int d^d z e^{-ikz}\langle A_\beta^b(y;0)\mathcal{O}_q(z;t)A_\alpha^a(x;0)\rangle \tag{8.21}$$

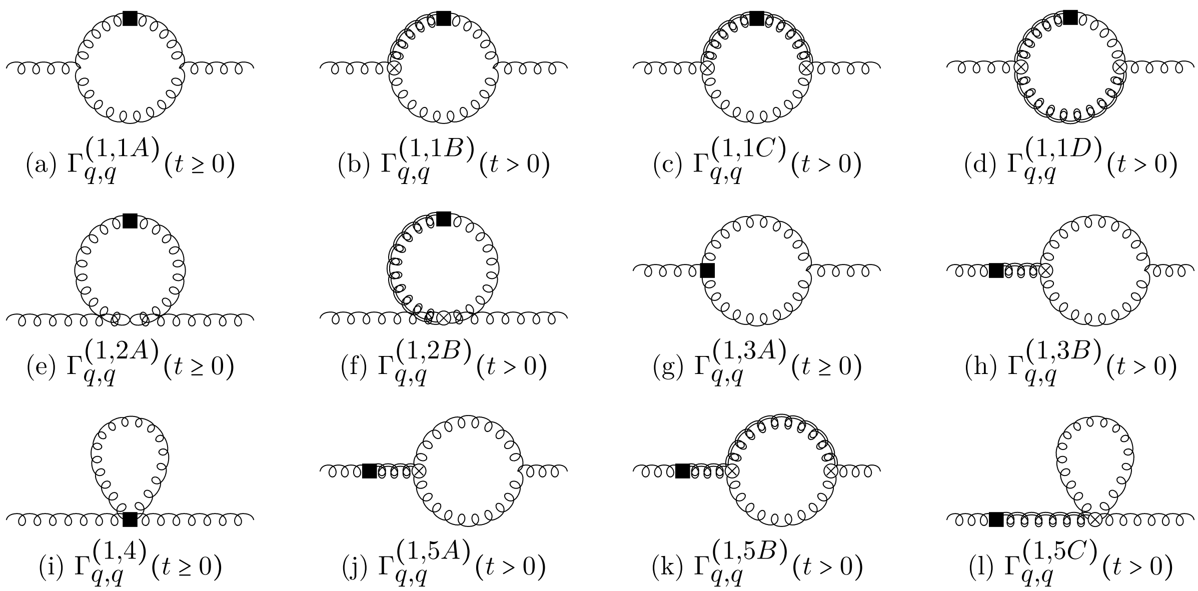


Figure 8.2: All topologically distinct contributions to $\Gamma^{(1)}_{q,q}(t \geq 0)$

is represented by twelve one-loop diagrams, shown in Fig. 8.2. Expanding again in k and p, the individual contributions can be evaluated as before:

$$\Gamma_q^{(1,1A)}(t) = 1 \cdot 5 \frac{T_A}{(4\pi)^2} \left\{ -\frac{1}{\epsilon_{IR}} + \log\left(2\bar{\mu}^2 t\right) + \gamma_E + \frac{3}{2} \right\} \Gamma_q^{(0)}(k;0) + \mathcal{O}(p^2, k^2, t), \tag{8.22a}$$

$$\Gamma_q^{(1,1B)}(t) = 1 \cdot \left(-\frac{9}{2}\right) \frac{T_A}{(4\pi)^2} \Gamma_q^{(0)}(k;0) + \mathcal{O}(p^2, k^2, t), \tag{8.22b}$$

$$\Gamma_q^{(1,1C)}(t) = 1 \cdot (-1) \frac{T_A}{(4\pi)^2} \Gamma_q^{(0)}(k;0) + \mathcal{O}(p^2, k^2, t), \tag{8.22c}$$

$$\Gamma_q^{(1,1D)}(t) = 1 \cdot \left(-\frac{9}{8}\right) \frac{T_A}{(4\pi)^2} \Gamma_q^{(0)}(k;0) + \mathcal{O}(p^2, k^2, t), \tag{8.22d}$$

$$\Gamma_q^{(1,2A)}(t) = \frac{1}{2} \cdot 0 + \mathcal{O}(p^2, k^2, t), \tag{8.22e}$$

$$\Gamma_q^{(1,2B)}(t) = 1 \cdot 0 + \mathcal{O}(p^2, k^2, t), \tag{8.22f}$$

$$\Gamma_q^{(1,3A)}(t) = \frac{1}{2} \cdot (-6) \frac{T_A}{(4\pi)^2} \left\{ -\frac{1}{\epsilon_{IR}} + \log\left(2\bar{\mu}^2 t\right) + \gamma_E + 1 \right\} \Gamma_q^{(0)}(k;0) + \mathcal{O}(p^2, k^2, t), \tag{8.22g}$$

$$\Gamma_q^{(1,3B)}(t) = 1 \cdot \frac{25}{8} \frac{T_A}{(4\pi)^2} \Gamma_q^{(0)}(k;0) + \mathcal{O}(p^2, k^2, t), \tag{8.22h}$$

$$\Gamma_q^{(1,4)}(t) \frac{1}{2} \cdot 0 + \mathcal{O}(p^2, k^2, t), \tag{8.22i}$$

$$\Gamma_q^{(1,5A)}(t) = \frac{1}{2}\cdot(-6)\frac{T_A}{(4\pi)^2}\left\{\frac{1}{\epsilon_{UV}} + \log\left(2\bar{\mu}^2 t\right) + \gamma_E + \frac{5}{6}\right\}\Gamma_q^{(0)}(k;0) + \mathcal{O}(p^2,k^2,t), \tag{8.22j}$$

$$\Gamma_q^{(1,5B)}(t) = 1\cdot(-2)\frac{T_A}{(4\pi)^2}\left\{\frac{1}{\epsilon_{UV}} + \log\left(2\bar{\mu}^2 t\right) + \gamma_E - 1\frac{1}{4}\right\}\Gamma_q^{(0)}(k;0) + \mathcal{O}(p^2,k^2,t), \tag{8.22k}$$

$$\Gamma_q^{(1,5C)}(t) = \frac{1}{2}\cdot 6\frac{T_A}{(4\pi)^2}\left\{\frac{1}{\epsilon_{UV}} + \log\left(2\bar{\mu}^2 t\right) + \gamma_E + \frac{1}{3}\right\}\Gamma_q^{(0)}(k;0) + \mathcal{O}(p^2,k^2,t). \tag{8.22l}$$

Amazingly, the sum total of these diagrams reduces to

$$\Gamma_q^{(1)}(k;t) = -2\frac{T_A}{(4\pi)^2}\left[\frac{1}{\epsilon_{IR}} + \frac{1}{\epsilon_{UV}}\right] + \mathcal{O}(p^2,k^2,t) = \mathcal{O}(p^2,k^2,t), \tag{8.23}$$

so self-mixing coefficient of the topological charge density vanishes at next-to-leading order:

$$c_{q,q}(t) = \delta_{ij} + \mathcal{O}(g^4). \tag{8.24}$$

Actually, this should be expected. As we argued above, the topological charge density operator vanishes under momentum conservation in perturbation theory, so one cannot find any dependence on its coupling with any finite number of Feynman diagrams. There is therefore no perturbative anomalous dimension for the charge. The charge does, however, contribute to the path integral nonperturbatively [98].

The entire SFTE for the charge now reads

$$\mathcal{O}_q(t) = c^{(1)}_{q,\partial A}(t)\mathcal{O}^{(0)}_{\partial A}(0) + \mathcal{O}(g^2,t). \tag{8.25}$$

At least to one-loop order, the perturbative renormalization of the charge is simply a finite shift proportional to the divergence of the axial current [99].

8.2 Quark Chromoelectric Dipole Moment

We now consider renormalization of the quark chromoelectric- and chromomagnetic-dipole-moment operators (qCEDM and qCMDM) at one-loop order and at positive flow time. These operators are defined as

$$\mathcal{O}_{CE} = k_{CE}\bar{\psi}\tilde{\sigma}_{\mu\nu}G_{\mu\nu}\psi, \tag{8.26a}$$

$$\mathcal{O}_{CM} = k_{CM}\bar{\psi}\sigma_{\mu\nu}G_{\mu\nu}\psi, \tag{8.26b}$$

where k_i are generic normalization constants; F and G are, respectively, the $U(1)$ and $SU(N_C)$ curvature tensors:

$$F_{\mu\nu} = \partial_{[\mu}A_{\nu]}, \tag{8.27a}$$

$$G_{\mu\nu} = t^a G^a_{\mu\nu} = t^a\left\{\partial_{[\mu}G^a_{\nu]} + f^{abc}G^b_\mu G^c_\nu\right\}; \tag{8.27b}$$

and σ and $\tilde{\sigma}$ are tensor and pseudotensor elements of the d-dimensional spacetime algebra:

$$\sigma_{\mu\nu} = \frac{i}{2}\gamma_{[\mu,\nu]}, \tag{8.28a}$$

$$\tilde{\sigma}_{\mu\nu} = \frac{1}{2}\left\{\sigma_{\mu\nu},\gamma_5\right\} \xrightarrow{d\to 4^\pm} \sigma_{\mu\nu}\gamma_5; \tag{8.28b}$$

To avoid confusion in this section, we temporarily label the electric fields A and the chromoelectric fields G.

For the dipole-moment operators above, the short-flow-time expansions read

$$\mathcal{O}^R_{CE}(t) = c_{CE,P}(t)\mathcal{O}^R_P(0) + c_{CE,E}(t)\mathcal{O}^R_E(0) + c_{CE,CE}(t)\mathcal{O}^R_{CE}(0) + \mathcal{O}(m,t), \tag{8.29a}$$

$$\mathcal{O}^R_{CM}(t) = c_{CM,S}(t)\mathcal{O}^R_S(0) + c_{CM,M}(t)\mathcal{O}^R_M(0) + c_{CM,CM}(t)\mathcal{O}^R_{CM}(0) + \mathcal{O}(m,t), \quad (8.29b)$$

where the scalar, pseudoscalar, qEDM, and qMDM operators are defined by

$$\mathcal{O}_S = k_S\bar{\psi}\psi, \quad (8.30a)$$

$$\mathcal{O}_P = k_P\bar{\psi}\gamma_5\psi, \quad (8.30b)$$

$$\mathcal{O}_E = k_E\bar{\psi}\sigma_{\mu\nu}F_{\mu\nu}\psi, \quad (8.30c)$$

$$\mathcal{O}_M = k_M\bar{\psi}\tilde{\sigma}_{\mu\nu}F_{\mu\nu}\psi. \quad (8.30d)$$

For now, we neglect the quark mass. There are a handful of operators which contribute only at finite mass; these will be studied in a later section.

8.2.1 Mixing With the Pseudoscalar Density: $c_{CE,P}(t)$

Choosing two quark fields as external states, we define the correlation functions $\Gamma_i(p;t)$:

$$\Gamma_i(p;t) = \int_p e^{-ip(y-x)} \int d^dz \ \langle\psi(y;0)\mathcal{O}_i(z;t)\overline{\psi}(x;0)\rangle, \quad (8.31)$$

At zero electromagnetic coupling, expanding Eq. 8.29a to $\mathcal{O}(g^2)$, we have

$$\begin{aligned}\Gamma^{(0)R}_{CE,P}(t) + g^2\Gamma^{(1)R}_{CE,P}(t) =& \left[c^{(0)}_{CE,P}(t) + g^2 c^{(1)}_{CE,P}(t)\right]\left[\Gamma^{(0)R}_{P,P}(0) + g^2\Gamma^{(1)R}_{P,P}(0)\right] \\ &+ \left[c^{(0)}_{CE,CE}(t) + g^2 c^{(1)}_{CE,CE}(t)\right]\left[\Gamma^{(0)R}_{CE,P}(0) + g^2\Gamma^{(1)R}_{CE,P}(0)\right] + \mathcal{O}(g^4, t) \\ =&\left[c^{(0)}_{CE,P}(t)\Gamma^{(0)R}_{P,P}(0) + c^{(0)}_{CE,CE}(t)\Gamma^{(0)R}_{CE,P}(0)\right] \\ &+g^2\Big[c^{(0)}_{CE,P}(t)\Gamma^{(1)R}_{P,P}(0) + c^{(1)}_{CE,P}(t)\Gamma^{(0)R}_{P,P}(0) \\ &\quad +c^{(0)}_{CE,CE}(t)\Gamma^{(1)R}_{CE,P}(0) + c^{(1)}_{CE,CE}(t)\Gamma^{(0)R}_{CE,P}(0)\Big] \\ &+ \mathcal{O}(g^4, m, t)s\end{aligned} \tag{8.32}$$

Collecting like powers in the strong coupling and discarding correlation functions that vanish trivially, we have

$$0 = c^{(0)}_{CE,P}(t)\Gamma^{(0)R}_{P,P}(0) + \mathcal{O}(m, t), \tag{8.33a}$$

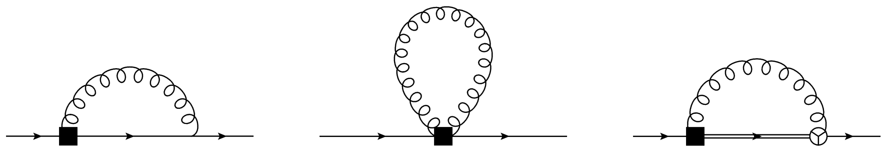
$$\Gamma^{(1)R}_{CE,P}(t) = c^{(0)}_{CE,P}(t)\Gamma^{(1)R}_{P,P}(0) + c^{(1)}_{CE,P}(t)\Gamma^{(0)R}_{P,P}(0) + c^{(0)}_{CE,CE}(t)\Gamma^{(1)R}_{CE,P}(0) + \mathcal{O}(m, t). \tag{8.33b}$$

Eq. 8.33a enforces

$$c^{(0)}_{CE,P}(t) = 0 + \mathcal{O}(m, t), \tag{8.34}$$

and, choosing external states as in Sec. 8.2.4, we can easily see that

$$c^{(0)}_{CE,CE}(t) = 1 + \mathcal{O}(m, t). \tag{8.35}$$



(a) $\Gamma^{(1,1)}_{CE,P}(t \geq 0)$ (b) $\Gamma^{(1,2)}_{CE,P}(t \geq 0)$ (c) $\Gamma^{(1,3)}_{CE,P}(t > 0)$

Figure 8.3: All topologically distinct contributions to $\Gamma^{(1)}_{CE,P}(t \geq 0)$

We are then left with

$$\Gamma^{(1)R}_{CE,P}(t) = c^{(1)}_{CE,P}(t)\Gamma^{(0)R}_{P,P}(0) + \Gamma^{(1)R}_{CE,P}(0) + \mathcal{O}(m,t). \tag{8.36}$$

On the lefthand side, there are three Feynman graphs which contribute to $\Gamma^{(1)}_{CE}(t)$, shown in Fig. 8.3:

$$\Gamma^{(1,1)}_{CE,P}(t) = 3i\frac{k_{CE}}{k_P}\frac{C_2(F)}{(4\pi)^2}\left\{\frac{1}{t} + p^2\left[\log\left(2p^2t\right) + \gamma_E - \frac{11}{4}\right]\right\}\gamma_5 + \mathcal{O}(m,t), \tag{8.37a}$$

$$\Gamma^{(1,2)}_{CE,P}(t) = 0 + \mathcal{O}(m,t), \tag{8.37b}$$

$$\Gamma^{(1,3)}_{CE,P}(t) = 0 + \mathcal{O}(m,t). \tag{8.37c}$$

Of course, diagrams 8.3a and 8.3c each have a twin diagram under the exchange of the position of the qCEDM vertex with the QCD or flow vertex. The results are identical under the interchange $p \leftrightarrow p'$, so that

$$\Gamma^{(1)}_{CE,P}(t) = 2\Gamma^{(1,1)}_{CE,P}(t) = 6ik_{CE}\frac{C_2(F)}{(4\pi)^2}\left\{\frac{1}{t} + p^2\left[\log\left(2p^2t\right) + \gamma_E - \frac{11}{4}\right]\right\}\gamma_5 + \mathcal{O}(m,t). \tag{8.38}$$

Notice the term proportional to p^2 in brackets. Since we are working at zero mass, we encounter the off-shell operator

$$\mathcal{O}_{\partial^2 P} = k_{\partial^2 P}\bar{\psi}\gamma_5 \overleftrightarrow{\partial}^2 \psi. \tag{8.39}$$

This term leads to the mixing of the qCEDM with the off-shell operator, but it clearly vanishes as we send p^2 to zero. Since the SFTE is insensitive to our kinematics, we may for now choose to put the quarks on shell, so that the subtraction of the pseudoscalar coefficient is cleaner.

For the righthand side of Eq. 8.36, there is a single graph for each term. The pseudoscalar term is tree-level and therefore trivial; it will be modded out of the final result to solve for the Wilson coefficient. The second term receives a contribution from a a pair of diagrams topologically identical to Fig. 8.3a. We find

$$\Gamma^{(0)}_{P,P}(0) = k_P\gamma_5, \tag{8.40a}$$

$$\Gamma^{(1)}_{CE,P}(0) = -6ik_{CE}\frac{C_2(F)}{(4\pi)^2}\left\{\left[\frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E}p^2}\right) + \frac{4}{3} + \frac{10}{9}\delta_{HV}\right]\right\}p^2\gamma_5 + \mathcal{O}(m,t). \tag{8.40b}$$

The second term is purely off-shell, so we set it to zero for now, and we may solve for the pseudoscalar mixing coefficient:

$$c^{(1)}_{CE,P}(t) = 6i\frac{k_{CE}}{k_P}\frac{C_2(F)}{(4\pi)^2}\frac{1}{t} + \mathcal{O}(t). \tag{8.41}$$

or

$$c_{CE,P}(t) = 6iC_2(F)\frac{k_{CE}}{k_P}\frac{g^2}{(4\pi)^2}\frac{1}{t} + \mathcal{O}(g^4,t). \tag{8.42}$$

Returning to the off-shell SFTE and subtracting the pseudoscalar piece from both sides, we are left with only the off-shell pieces of the correlation functions; taking their difference modulo the tree-level gives us the off-shell mixing coefficient:

$$c^{(1)}_{CE,\partial^2 P}(t) = 6i\frac{k_{CE}}{k_{\partial^2 P}}\frac{C_2(F)}{(4\pi)^2}\left\{\log\left(2\bar{\mu}^2 t\right) + \gamma_E - \frac{17}{12} + \frac{10}{9}\delta_{HV}\right\} + \mathcal{O}(t). \tag{8.43}$$

8.2.2 Mixing With the Topological Charge Density: $c_{CE,q}(t)$

If we include fermion masses in Eq. 8.29a, it is possible for the qCEDM to receive corrections from the topological charge density, as follows. Choosing two gluon fields as external states, we recycle our notation $\Gamma_i(q;t)$:

$$\Gamma_q(k;t) = \int_p e^{-ip(y-x)} \int d^d z e^{-ikz} \langle G^b_\beta(y;0)\mathcal{O}_q(z;t)G^a_\alpha(x;0)\rangle \tag{8.44}$$

At next-to-leading order, only the topological charge term survives on the right side of the short flow time expansion, so we need only compute the flowed correlator. There are three contributions to $\Gamma^{(1)}_{CE}(t)$, presented in Fig. 8.15. Expanding each Feynman diagram to first order in the mass and proceeding as before, we find

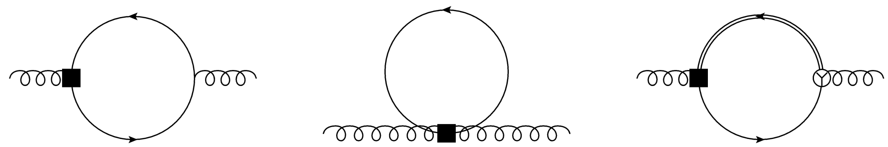
$$\Gamma^{(1,1)}_{CE}(k;t) = 4i\frac{k_{CE}}{k_q}\frac{n_f \dim(F)}{(4\pi)^2}\left\{\log\left(2k^2 t\right) + \gamma_E - 1\right\} \cdot m\Gamma^{(0)}_q(k;0) + \mathcal{O}(m^2, t), \tag{8.45a}$$

$$\Gamma^{(1,2)}_{CE}(k;t) = 0 + \mathcal{O}(m^2, t), \tag{8.45b}$$

$$\Gamma^{(1,3)}_{CE}(k;t) = 0 + \mathcal{O}(m^2, t). \tag{8.45c}$$

Any potential pole must then reside in the boundary correlator. Indeed, we find

$$\Gamma^{(1,1)}_{CE}(k;0) = -4i\frac{k_{CE}}{k_q}\frac{n_f \dim(F)}{(4\pi)^2}\left\{\frac{1}{\epsilon} + \log\left(\frac{\bar{\mu}^2}{k^2}\right) + 2\right\} \cdot m\Gamma^{(0)}_q(k;0) + \mathcal{O}(m^2, t), \tag{8.46}$$



(a) $\Gamma^{(1,1)}_{CE}(t \geq 0)$ (b) $\Gamma^{(1,2)}_{CE}(t \geq 0)$ (c) $\Gamma^{(1,3)}_{CE}(t > 0)$

Figure 8.4: All distinct contributions to $\Gamma^{(1)}_{CE}(t \geq 0)$

so that

$$Z^{(1)}_{CE,q}{}^{-1} = 4i\frac{k_{CE}}{k_q}\frac{n_f \dim(F)}{(4\pi)^2}\frac{1}{\epsilon}. \tag{8.47}$$

The mixing coefficient is then easily read off:

$$c^{(1)}_{CE,q}(t) = 4i\frac{k_{CE}}{k_q}\frac{n_f \dim(F)}{(4\pi)^2}\left\{\log\left(2\bar{\mu}^2 t\right) + \gamma_E + 1\right\} + \mathcal{O}(t). \tag{8.48}$$

8.2.3 Mixing With the Quark Electric Dipole Moment: $c_{CE,E}(t)$

Choosing now two quark fields and a single nondynamical photon as external states, we recycle our notation Γ:

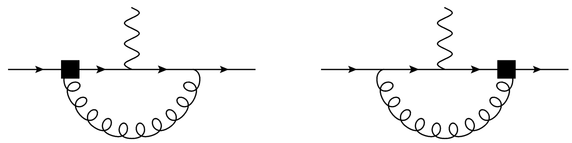
$$(2\pi)^d\delta^{(d)}(p+q-r)\Gamma^R_{i,E}(p,r;t) = \int_{wxyz} e^{-ipw}e^{-iqx}e^{iry}\langle\psi^R(y)\mathcal{O}^R_i(z;t)A^R_\alpha(x)\bar{\psi}^R(w)\rangle. \tag{8.49}$$

Repeating the expansion and reduction as in Sec. 8.2.1, we have

$$\Gamma^{(1)R}_{CE,E}(t) = c^{(1)}_{CE,P}(t)\Gamma^{(0)R}_{P,E}(0) + c^{(1)}_{CE,E}(t)\Gamma^{(0)R}_{E,E}(0) + \Gamma^{(1)R}_{CE,E}(0) + \mathcal{O}(t). \tag{8.50}$$

This time there are extra diagrams on the left that exactly cancel the pseudoscalar term on the right. These are non-1PI, so we may study an equivalent equation where the correlators are strictly 1PI;

$$\Gamma^{(1)R}_{CE,E}(t) = c^{(1)}_{CE,E}(t)\Gamma^{(0)R}_{E,E}(0) + \Gamma^{(1)R}_{CE,E}(0) + \mathcal{O}(t). \tag{8.51}$$



(a) $\Gamma^{(1,1a)}_{CE,E}(t \geq 0)$ (b) $\Gamma^{(1,1b)}_{CE,E}(t \geq 0)$

Figure 8.5: All 1PI contributions to $\Gamma^{(1)}_{CE,E}(t \geq 0)$

There is one pair of diagrams for each term, shown in Fig. 8.5. Since the mixing is linear in the photon momentum q_α, we are free to discard other soft scales; the IR divergences will be regulated by this momentum so long as the outgoing quark is kept off-shell. This, however, breaks the symmetry under the exchange of quark indices, and we must evaluate the graphs in both Figs. 8.5a and 8.5b independently. Further, this choice of kinematics introduces another "nuisance" operator:

$$\mathcal{O}_N = k_N \bar{\psi}_E \gamma_5 \psi_E, \tag{8.52}$$

where the equation-of-motion fields

$$\psi_E = (\not{D} + m)\psi, \tag{8.53a}$$

$$\bar{\psi}_E = \bar{\psi}(\overleftarrow{\not{D}} - m) \tag{8.53b}$$

vanish on the mass shell. At $(p, r) = (0, q)$, there are two tensors which appear within this calculation, $t^a q_\alpha \gamma_5$ and $t^a \sigma_{\alpha\beta}\gamma_5 q_\beta$, which are related to the tree-level correlators by

$$t^a q_\alpha \gamma_5 = \frac{1}{2k_E}\Gamma_E^{(0)} + \frac{i}{k_N}\Gamma_N^{(0)}, \tag{8.54a}$$

$$t^a \sigma_{\alpha\beta}\gamma_5 q_\beta = \frac{i}{2k_E}\Gamma_E^{(0)}. \tag{8.54b}$$

At positive flow time,

$$\Gamma_{CE,E}^{(1,1a)}(t) = 2\frac{C_2(F)}{(4\pi)^2}\Big\{\left[\log\left(2q^2 t\right) + \gamma_E - 1\right]\Gamma_E^{(0)} + \frac{3}{2}i\frac{k_{CE}}{k_N}\left[\log\left(2q^2 t\right) + \gamma_E - \frac{3}{2}\right]\Gamma_N^{(0)}\Big\} + \mathcal{O}(m,\epsilon), \tag{8.55a}$$

$$\Gamma_{CE,E}^{(1,1b)}(t) = 2\frac{C_2(F)}{(4\pi)^2}\Big\{\left[\log\left(2q^2 t\right) + \gamma_E - 1\right]\Gamma_E^{(0)} + \frac{3}{2}i\frac{k_{CE}}{k_N}\left[\log\left(2q^2 t\right) + \gamma_E - 1\right]\Gamma_N^{(0)}\Big\} + \mathcal{O}(m,\epsilon). \tag{8.55b}$$

On the boundary,

$$\Gamma_{CE,E}^{(1,1a)}(0) = -2\frac{C_2(F)}{(4\pi)^2}\Bigg\{\left[\frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E}p^2}\right) + \frac{3}{2} + \frac{1}{3}\delta_{HV}\right]\Gamma_E^{(0)} + \frac{3}{2}i\frac{k_{CE}}{k_N}\left[\frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E}q^2}\right) + \frac{4}{3} + \frac{10}{9}\delta_{HV}\right]\Gamma_N^{(0)}\Bigg\} + \mathcal{O}(m,\epsilon), \tag{8.56a}$$

$$\Gamma_{CE,E}^{(1,1b)}(0) = -2\frac{C_2(F)}{(4\pi)^2}\Bigg\{\left[\frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E}p^2}\right) + 2 + \frac{1}{3}\delta_{HV}\right]\Gamma_E^{(0)} + \frac{3}{2}i\frac{k_{CE}}{k_N}\left[\frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E}q^2}\right) + \frac{4}{3} + \frac{10}{9}\delta_{HV}\right]\Gamma_N^{(0)}\Bigg\} + \mathcal{O}(m,\epsilon), \tag{8.56b}$$

and the Wilson coefficients are

$$c_{CE,E}(t) = 4C_2(F)\frac{g^2}{(4\pi)^2}\left\{\log\left(2\bar{\mu}^2 t\right) + \gamma_E + \frac{3}{2} + \frac{2}{3}\delta_{HV}\right\} + \mathcal{O}(g^4,m,t), \tag{8.57a}$$

$$c_{CE,N}(t) = 4C_2(F)\frac{g^2}{(4\pi)^2}\left\{\log\left(2\bar{\mu}^2 t\right) + \gamma_E + \frac{1}{6} + \frac{20}{9}\delta_{HV}\right\} + \mathcal{O}(g^4,m,t). \tag{8.57b}$$

8.2.4 Self-Mixing: $c_{CE,CE}(t)$

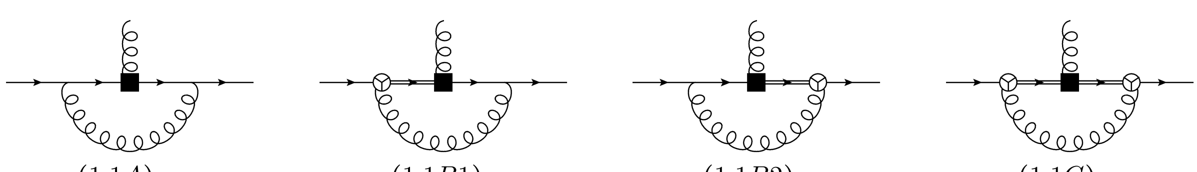
We now choose two quark fields and a single gluon as external states and again redefine Γ:

$$(2\pi)^d \delta^{(d)}(p+q-r)\Gamma^R_{i,CE}(p,r;t) = \int_{wxyz} e^{-ipw} e^{-iqx} e^{iry} \langle \psi^R(y) \mathcal{O}^R_i(z;t) G^{aR}_\alpha(x) \bar{\psi}^R(w) \rangle. \tag{8.58}$$

Reducing the SFTE, we may subtract all one-particle reducible diagrams from both sides, so that all pseudoscalar and qEDM terms cancel, leaving us with

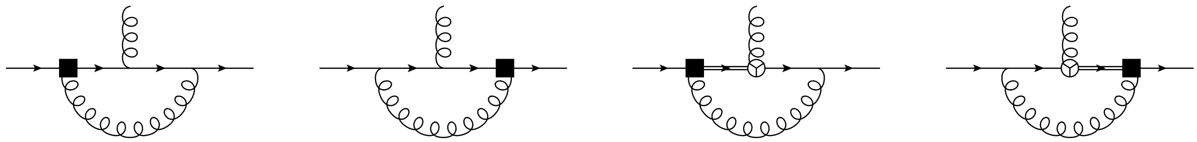
$$\Gamma^{(1)R}_{CE,CE}(t) = c^{(1)}_{CE,CE}(t)\Gamma^{(0)R}_{CE,CE}(0) + \Gamma^{(1)R}_{CE,CE}(0) + \mathcal{O}(t). \tag{8.59}$$

The one-loop flowed correlator produces thirty-four 1PI diagrams, which are shown in Figs. 8.6-8.12, as well as a handful of unique topologies related to the renormalization of the strong coupling and the fermion fields at positive flow time, shown in Figs. 8.13 and 8.14. These latter diagrams have poles at $d = 4$ which are renormalized away; however, they also contain logarithms and finite pieces which ultimately contribute to the self-mixing coefficient for the qCEDM. The 1PI diagrams are collected into classes by the structure of their radiative corrections. Classes 1 – 5 consist of the topologies that exist on the boundary, along with their corrections derived from higher-order terms in the flow equations. Classes 6 and 7 are 1PI diagrams that exist only in the bulk.

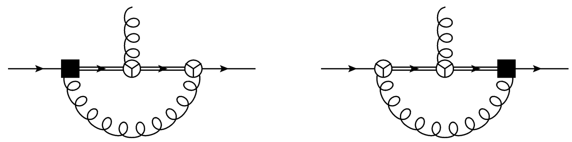


(a) $\Gamma^{(1,1A)}_{CE,CE}(t \geq 0)$ (b) $\Gamma^{(1,1B1)}_{CE,CE}(t > 0)$ (c) $\Gamma^{(1,1B2)}_{CE,CE}(t > 0)$ (d) $\Gamma^{(1,1C)}_{CE,CE}(t > 0)$

Figure 8.6: All contributions to $\Gamma^{(1,1)}_{CE,CE}(t \geq 0)$

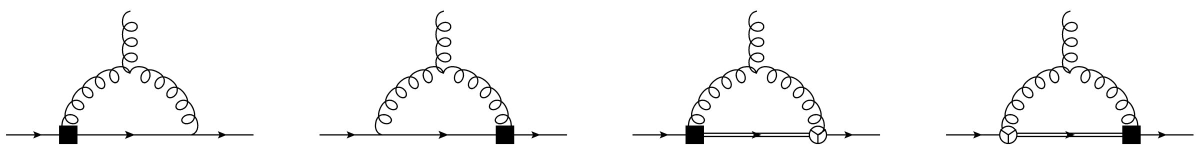


(a) $\Gamma^{(1,2A1)}_{CE,CE}(t \geq 0)$ (b) $\Gamma^{(1,2A2)}_{CE,CE}(t \geq 0)$ (c) $\Gamma^{(1,2B1)}_{CE,CE}(t > 0)$ (d) $\Gamma^{(1,2B2)}_{CE,CE}(t > 0)$

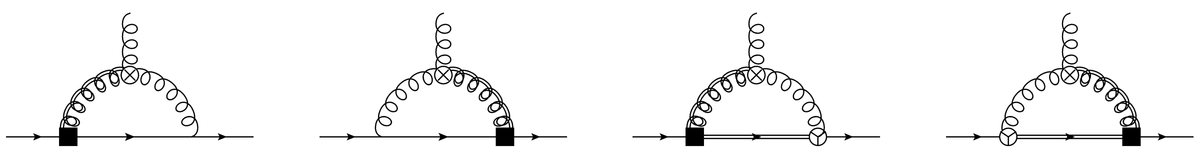


(e) $\Gamma^{(1,2C1)}_{CE,CE}(t > 0)$ (f) $\Gamma^{(1,2C2)}_{CE,CE}(t > 0)$

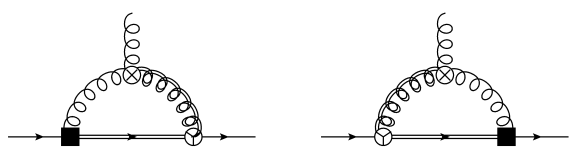
Figure 8.7: All contributions to $\Gamma^{(1,2)}_{CE,CE}(t \geq 0)$



(a) $\Gamma^{(1,3A1)}_{CE,CE}(t \geq 0)$ (b) $\Gamma^{(1,3A2)}_{CE,CE}(t \geq 0)$ (c) $\Gamma^{(1,3B1)}_{CE,CE}(t > 0)$ (d) $\Gamma^{(1,3B2)}_{CE,CE}(t > 0)$

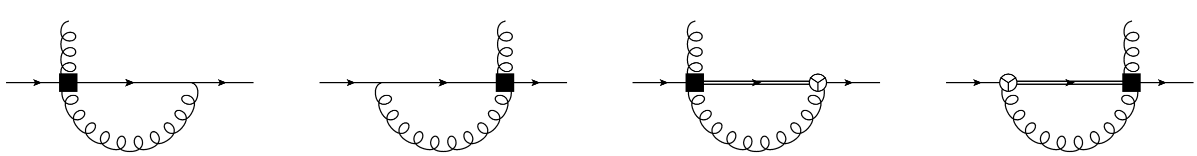


(e) $\Gamma^{(1,3C1)}_{CE,CE}(t > 0)$ (f) $\Gamma^{(1,3C2)}_{CE,CE}(t > 0)$ (g) $\Gamma^{(1,3D1)}_{CE,CE}(t > 0)$ (h) $\Gamma^{(1,3D2)}_{CE,CE}(t > 0)$



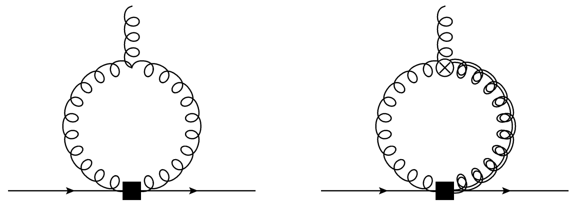
(i) $\Gamma^{(1,3E1)}_{CE,CE}(t > 0)$ (j) $\Gamma^{(1,3E2)}_{CE,CE}(t > 0)$

Figure 8.8: All contributions to $\Gamma^{(1,3)}_{CE,CE}(t \geq 0)$



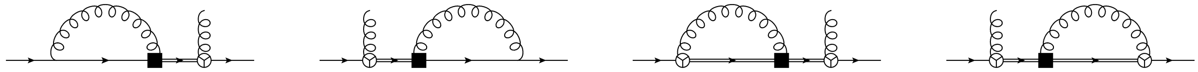
(a) $\Gamma^{(1,4A1)}_{CE,CE}(t \geq 0)$ (b) $\Gamma^{(1,4A2)}_{CE,CE}(t \geq 0)$ (c) $\Gamma^{(1,4B1)}_{CE,CE}(t > 0)$ (d) $\Gamma^{(1,4B2)}_{CE,CE}(t > 0)$

Figure 8.9: All contributions to $\Gamma^{(1,4)}_{CE,CE}(t \geq 0)$



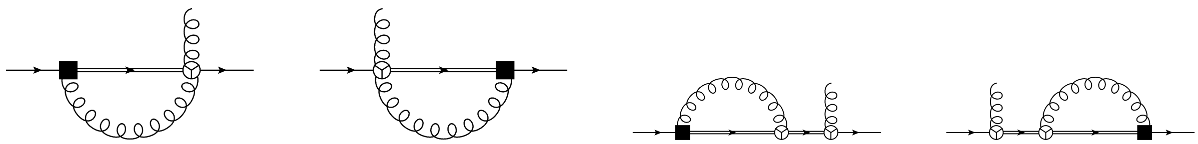
(a) $\Gamma^{(1,5A)}_{CE,CE}(t \geq 0)$ (b) $\Gamma^{(1,5B)}_{CE,CE}(t > 0)$

Figure 8.10: All contributions to $\Gamma^{(1,5)}_{CE,CE}(t \geq 0)$



(a) $\Gamma^{(1,6A1)}_{CE,CE}(t > 0)$ (b) $\Gamma^{(1,6A2)}_{CE,CE}(t > 0)$ (c) $\Gamma^{(1,6B1)}_{CE,CE}(t > 0)$ (d) $\Gamma^{(1,6B2)}_{CE,CE}(t > 0)$

Figure 8.11: All contributions to $\Gamma^{(1,6)}_{CE,CE}(t \geq 0)$



(a) $\Gamma^{(1,7A1)}_{CE,CE}(t > 0)$ (b) $\Gamma^{(1,7A2)}_{CE,CE}(t > 0)$ (c) $\Gamma^{(1,7B1)}_{CE,CE}(t > 0)$ (d) $\Gamma^{(1,7B2)}_{CE,CE}(t > 0)$

Figure 8.12: All contributions to $\Gamma^{(1,7)}_{CE,CE}(t \geq 0)$

As with the qEDM mixing, if we set $(p, r) = (0, q)$, we have a simpler calculation with no loss of information on the mixing of on- or off-shell operators at the expense of having to calculate all diagrams, including mirror-images. The diagrams are labeled $\Gamma^{(1,XYZ)}_{CE,CE}$: X is the class; Y is the diagram within that class; and Z is the orientation, with 1 having the operator directly connected to the $\bar{\psi}$ field and 2 having the operator insertion on the ψ field. On the flowed side of Eq. 8.59, the momentum of the incoming gluon q regulates all IR divergences, while the flow time regulates the UV for all 1PI diagrams, and no regulator is needed; *viz.*, all diagrams are evaluated directly at $d = 4$. The reducible diagrams (Figs. 8.13-8.14) are equal to their counterparts from Secs. 6.1 and 6.2 (modulo their tree-levels) times the tree-level qCEDM. Thus, the definition of σ is purely four-dimensional within $\Gamma^{(1)}_{CE,CE}(t > 0)$. The $t = 0$ side loses its UV regulator, and must be evaluated at $d = 4 - 2\epsilon$, where $1 > |\epsilon| > 0$.

We must then make a choice of prescription for γ_5, or more specifically, $\tilde{\sigma}_{\mu\nu}$. We present results in two γ_5 schemes with three definitions of $\tilde{\sigma}_{\mu\nu}$: naïve dimensional regularization (NDR) and the t'Hooft-Veltman-Breitenlohner-Maison (HVBM) scheme, with the definitions

$$\tilde{\sigma}_{\mu\nu} = \begin{cases} \frac{1}{2}\left\{\sigma_{\mu\nu}, \gamma_5\right\}, & \text{scheme 1;} \quad (8.60a) \\ -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\sigma_{\rho\sigma}, & \text{scheme 2;} \quad (8.60b) \\ -\frac{1}{(d-2)(d-3)}\epsilon_{\mu\nu\rho\sigma}\sigma_{\rho\sigma}, & \text{scheme 3;} \quad (8.60c) \end{cases}$$

all of which coincide in NDR, for which

$$\tilde{\sigma}_{\mu\nu} = \sigma_{\mu\nu}\gamma_5. \quad (8.61)$$

Of course, all conventions yield the same logarithms, but the finite parts are scheme-dependent. To that end, we define

$$\delta^i_{HV} = \begin{cases} 1, & \text{scheme } i, \\ 0, & \text{else.} \end{cases} \quad (8.62)$$

For $t > 0$, we find

$$\Gamma^{(1,1A)}_{CE,CE}(t) = \mathcal{O}(m), \quad (8.63a)$$

$$\Gamma^{(1,1B1)}_{CE,CE}(t) = \mathcal{O}(m), \quad (8.63b)$$

$$\Gamma^{(1,1B2)}_{CE,CE}(t) = \mathcal{O}(m), \quad (8.63c)$$

$$\Gamma^{(1,1C)}_{CE,CE}(t) = \mathcal{O}(m), \quad (8.63d)$$

$$\Gamma^{(1,2A1)}_{CE,CE}(t) = -\frac{C_2(A) - 2C_2(F)}{(4\pi)^2} \cdot \Big\{ \left[\log\left(2q^2 t\right) + \gamma_E - 1\right] \Gamma^{(0)}_{CE} + \frac{3}{2} i \frac{k_{CE}}{k_N} \left[\log\left(2q^2 t\right) + \gamma_E - \frac{3}{2}\right] \Gamma^{(0)}_{N} \Big\} + \mathcal{O}(m,t), \tag{8.63e}$$

$$\Gamma^{(1,2A2)}_{CE,CE}(t) = -\frac{C_2(A) - 2C_2(F)}{(4\pi)^2} \cdot \Big\{ \left[\log\left(2q^2 t\right) + \gamma_E - 1\right] \Gamma^{(0)}_{CE} + \frac{3}{2} i \frac{k_{CE}}{k_N} \left[\log\left(2q^2 t\right) + \gamma_E - 1\right] \Gamma^{(0)}_{N} \Big\} + \mathcal{O}(m,t), \tag{8.63f}$$

$$\Gamma^{(1,2B1)}_{CE,CE}(t) = \frac{C_2(A) - 2C_2(F)}{(4s\pi)^2} \cdot \left\{ \frac{13}{16} \Gamma^{(0)}_{CE} + \frac{15}{8} i \frac{k_{CE}}{k_N} \Gamma^{(0)}_{N} \right\} + \mathcal{O}(m,t), \tag{8.63g}$$

$$\Gamma^{(1,2B2)}_{CE,CE}(t) = \frac{C_2(A) - 2C_2(F)}{(4\pi)^2} \cdot \left\{ -\frac{13}{16} \Gamma^{(0)}_{CE} - \frac{3}{8} i \frac{k_{CE}}{k_N} \Gamma^{(0)}_{N} \right\} + \mathcal{O}(m,t), \tag{8.63h}$$

$$\Gamma^{(1,2C1)}_{CE,CE}(t) = \frac{C_2(A) - 2C_2(F)}{(4\pi)^2} \cdot \frac{1}{16} \Gamma^{(0)}_{CE} + \mathcal{O}(m,t), \tag{8.63i}$$

$$\Gamma^{(1,2C2)}_{CE,CE}(t) = \mathcal{O}(m), \tag{8.63j}$$

$$\Gamma^{(1,3A1)}_{CE,CE}(t) = -\frac{C_2(A)}{(4\pi)^2} \cdot \Big\{ \frac{7}{4} \left[\log\left(2q^2 t\right) + \gamma_E - \frac{3}{14}\right] \Gamma^{(0)}_{CE} + \frac{3}{4} i \frac{k_{CE}}{k_N} \left[\log\left(2q^2 t\right) + \gamma_E + \frac{3}{2}\right] \Gamma^{(0)}_{N} \Big\} + \mathcal{O}(m,t), \tag{8.63k}$$

$$\Gamma^{(1,3A2)}_{CE,CE}(t) = -\frac{C_2(A)}{(4\pi)^2} \cdot \Big\{ \frac{5}{4} \left[\log\left(2q^2 t\right) + \gamma_E - \frac{1}{5}\right] \Gamma^{(0)}_{CE} - \frac{3}{2} i \frac{k_{CE}}{k_N} \left[\log\left(2q^2 t\right) + \gamma_E - \frac{1}{2}\right] \Gamma^{(0)}_{N} \Big\} + \mathcal{O}(m,t), \tag{8.63l}$$

$$\Gamma^{(1,3B1)}_{CE,CE}(t) = \frac{C_2(A)}{(4\pi)^2} \cdot \frac{3}{8} \Gamma^{(0)}_{CE} + \mathcal{O}(m,t), \tag{8.63m}$$

$$\Gamma^{(1,3B2)}_{CE,CE}(t) = \mathcal{O}(m), \tag{8.63n}$$

$$\Gamma^{(1,3C1)}_{CE,CE}(t) = 2\frac{C_2(A)}{(4\pi)^2} \cdot \left\{ \frac{3}{32} \Gamma^{(0)}_{CE} - \frac{9}{16} i \frac{k_{CE}}{k_N} \Gamma^{(0)}_{N} \right\} + \mathcal{O}(m,t), \tag{8.63o}$$

$$\Gamma^{(1,3C2)}_{CE,CE}(t) = 2\frac{C_2(A)}{(4\pi)^2} \cdot \left\{ \frac{11}{32} \Gamma^{(0)}_{CE} - \frac{3}{16} i \frac{k_{CE}}{k_N} \Gamma^{(0)}_{N} \right\} + \mathcal{O}(m,t), \tag{8.63p}$$

$$\Gamma^{(1,3D1)}_{CE,CE}(t) = 2\frac{C_2(A)}{(4\pi)^2} \cdot \frac{1}{16} \Gamma^{(0)}_{CE} + \mathcal{O}(m,t), \tag{8.63q}$$

$$\Gamma^{(1,3D2)}_{CE,CE}(t) = \mathcal{O}(m), \tag{8.63r}$$

$$\Gamma^{(1,3E1)}_{CE,CE}(t) = 2\frac{C_2(A)}{(4\pi)^2} \cdot \frac{3}{64} \Gamma^{(0)}_{CE} + \mathcal{O}(m,t), \tag{8.63s}$$

$$\Gamma^{(1,3E2)}_{CE,CE}(t) = \mathcal{O}(m), \tag{8.63t}$$

$$\Gamma^{(1,4A1)}_{CE,CE}(t) = \frac{C_2(A)}{(4\pi)^2} \cdot \left\{ \frac{1}{2} \left[\log\left(2q^2 t\right) + \gamma_E - 1 \right] \Gamma^{(0)}_{CE} + \frac{3}{2} i \frac{k_{CE}}{k_N} \left[\log\left(2q^2 t\right) + \gamma_E - 1 \right] \Gamma^{(0)}_{N} \right\} + \mathcal{O}(m,t), \tag{8.63u}$$

$$\Gamma^{(1,4A2)}_{CE,CE}(t) = \mathcal{O}(m), \tag{8.63v}$$

$$\Gamma^{(1,4B1)}_{CE,CE}(t) = -\frac{C_2(A)}{(4\pi)^2} \cdot \frac{1}{2} \Gamma^{(0)}_{CE} + \mathcal{O}(m,t), \tag{8.63w}$$

$$\Gamma^{(1,4B2)}_{CE,CE}(t) = \mathcal{O}(m), \tag{8.63x}$$

$$\Gamma^{(1,5A)}_{CE,CE}(t) = \frac{1}{2} \frac{C_2(A)}{(4\pi)^2} \cdot 3 \left[\log\left(2q^2 t\right) + \gamma_E - 1 \right] \Gamma^{(0)}_{CE} + \mathcal{O}(m,t), \tag{8.63y}$$

$$\Gamma^{(1,5B)}_{CE,CE}(t) = -2 \frac{C_2(A)}{(4\pi)^2} \cdot \frac{25}{32} \Gamma^{(0)}_{CE} + \mathcal{O}(m,t), \tag{8.63z}$$

$$\Gamma^{(1,6A1)}_{CE,CE}(t) = \mathcal{O}(m), \tag{8.63aa}$$

$$\Gamma^{(1,6A2)}_{CE,CE}(t) = \mathcal{O}(m), \tag{8.63ab}$$

$$\Gamma^{(1,6B1)}_{CE,CE}(t) = -\frac{C_2(F)}{(4\pi)^2} \cdot \left\{ 3\Gamma^{(0)}_{CE} + 6i \frac{k_{CE}}{k_N} \Gamma^{(0)}_{N} \right\} + \mathcal{O}(m,t), \tag{8.63ac}$$

$$\Gamma^{(1,6B2)}_{CE,CE}(t) = \mathcal{O}(m), \tag{8.63ad}$$

$$\Gamma^{(1,7A1)}_{CE,CE}(t) = \mathcal{O}(m), \tag{8.63ae}$$

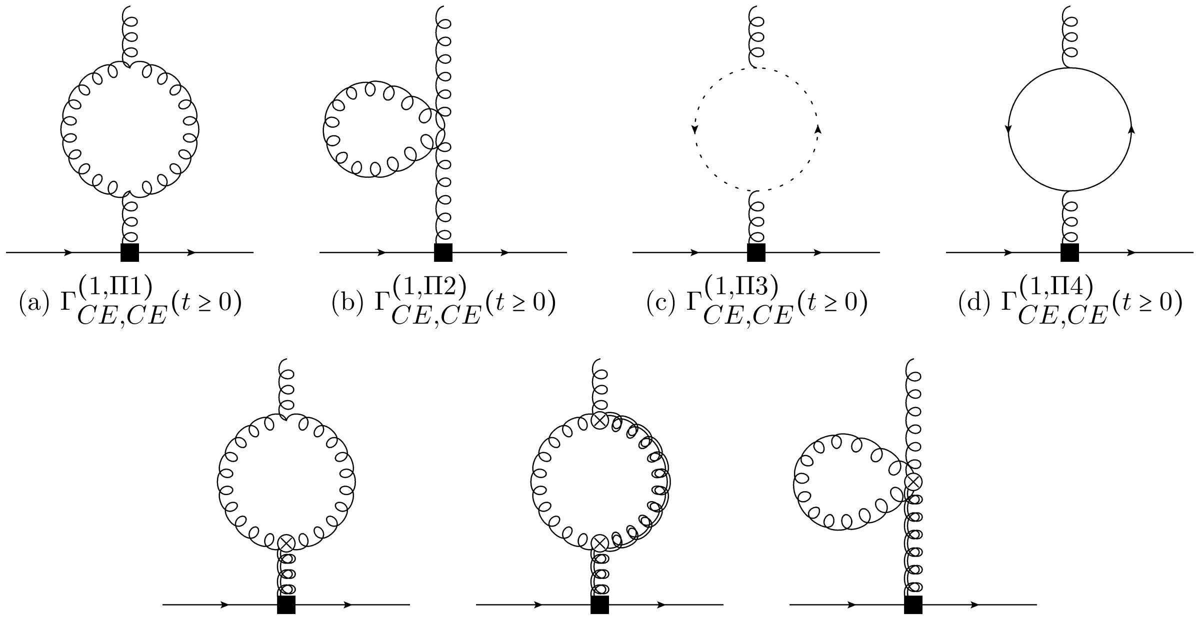
$$\Gamma^{(1,7A2)}_{CE,CE}(t) = 2 \frac{C_2(A) - 4C_2(F)}{(4\pi)^2} \cdot \frac{1}{64} \Gamma^{(0)}_{CE} + \mathcal{O}(m,t), \tag{8.63af}$$

$$\Gamma^{(1,7B1)}_{CE,CE}(t) = \mathcal{O}(m), \tag{8.63ag}$$

$$\Gamma^{(1,7B2)}_{CE,CE}(t) = \mathcal{O}(m). \tag{8.63ah}$$

The diagrams related to the coupling renormalization and quark field renormalization are easily evaluated:

$$\Gamma^{(1,\Pi 1)}_{CE,CE}(t \geq 0) = \frac{19}{12} \frac{C_2(A)}{(4\pi)^2} \cdot \left\{ \frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E} q^2} \right) + \frac{116}{57} \right\} \Gamma^{(0)}_{CE} + \mathcal{O}(\epsilon, t), \tag{8.64a}$$



(a) $\Gamma^{(1,\Pi1)}_{CE,CE}(t \geq 0)$ (b) $\Gamma^{(1,\Pi2)}_{CE,CE}(t \geq 0)$ (c) $\Gamma^{(1,\Pi3)}_{CE,CE}(t \geq 0)$ (d) $\Gamma^{(1,\Pi4)}_{CE,CE}(t \geq 0)$

(e) $\Gamma^{(1,\Pi5)}_{CE,CE}(t > 0)$ (f) $\Gamma^{(1,\Pi6)}_{CE,CE}(t > 0)$ (g) $\Gamma^{(1,\Pi7)}_{CE,CE}(t > 0)$

Figure 8.13: Gluon Leg Corrections

$$\Gamma^{(1,\Pi2)}_{CE,CE}(t \geq 0) = \mathcal{O}(\epsilon), \tag{8.64b}$$

$$\Gamma^{(1,\Pi3)}_{CE,CE}(t \geq 0) = \frac{1}{12}\frac{C_2(A)}{(4\pi)^2} \cdot \left\{ \frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E} q^2}\right) + \frac{8}{3} \right\} \Gamma^{(0)}_{CE} + \mathcal{O}\epsilon, (t), \tag{8.64c}$$

$$\Gamma^{(1,\Pi4)}_{CE,CE}(t \geq 0) = \frac{4}{3}\frac{T_F n_f}{(4\pi)^2} \cdot \left\{ \frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E} q^2}\right) + \frac{5}{3} \right\} \Gamma^{(0)}_{CE} + \mathcal{O}(\epsilon, t), \tag{8.64d}$$

$$\Gamma^{(1,\Pi5)}_{CE,CE}(t > 0) = \frac{3}{2}\frac{C_2(A)}{(4\pi)^2} \cdot \left\{ \frac{1}{\epsilon} + \log(8\pi t) + \frac{5}{6} \right\} \Gamma^{(0)}_{CE} + \mathcal{O}(\epsilon, t), \tag{8.64e}$$

$$\Gamma^{(1,\Pi6)}_{CE,CE}(t > 0) = \frac{C_2(A)}{(4\pi)^2} \cdot \left\{ \frac{1}{\epsilon} + \log(8\pi t) - \frac{1}{4} \right\} \Gamma^{(0)}_{CE} + \mathcal{O}(\epsilon, t), \tag{8.64f}$$

$$\Gamma^{(1,\Pi7)}_{CE,CE}(t > 0) = -\frac{3}{2}\frac{C_2(A)}{(4\pi)^2} \cdot \left\{ \frac{1}{\epsilon} + \log(8\pi t) + \frac{1}{3} \right\} \Gamma^{(0)}_{CE} + \mathcal{O}(\epsilon, t), \tag{8.64g}$$

$$\Gamma^{(1,\Sigma1a)}_{CE,CE}(t \geq 0) = -\frac{C_2(F)}{(4\pi)^2} \cdot \left\{ \frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E} q^2}\right) - 1 \right\} \Gamma^{(0)}_{CE} + \mathcal{O}(\epsilon, m, t), \tag{8.64h}$$

$$\Gamma^{(1,\Sigma1b)}_{CE,CE}(t \geq 0) = -\frac{C_2(F)}{(4\pi)^2} \cdot \left\{ \frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E} p^2}\right) - 1 \right\} \Gamma^{(0)}_{CE} + \mathcal{O}(\epsilon, m, t), \tag{8.64i}$$

$$\Gamma^{(1,\Sigma2a)}_{CE,CE}(t > 0) = \frac{C_2(F)}{(4\pi)^2} \cdot \left\{ \frac{1}{\epsilon} + \log(8\pi t) + \gamma_E + 1 \right\} \Gamma^{(0)}_{CE} + \mathcal{O}(\epsilon, m, t), \tag{8.64j}$$

$$\Gamma^{(1,\Sigma2b)}_{CE,CE}(t > 0) = \frac{C_2(F)}{(4\pi)^2} \cdot \left\{ \frac{1}{\epsilon} + \log(8\pi t) + \gamma_E + 1 \right\} \Gamma^{(0)}_{CE} + \mathcal{O}(\epsilon, m, t), \tag{8.64k}$$

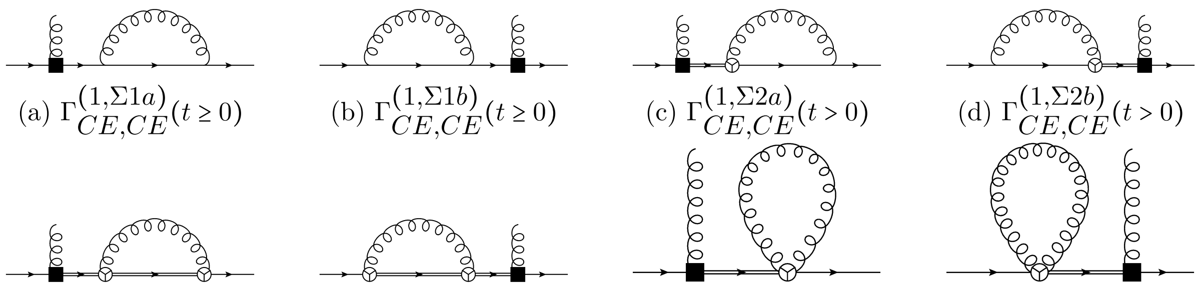
$$\Gamma^{(1,\Sigma 3a)}_{CE,CE}(t>0) = \mathcal{O}(m,t), \tag{8.64l}$$

$$\Gamma^{(1,\Sigma 3b)}_{CE,CE}(t>0) = \mathcal{O}(m,t), \tag{8.64m}$$

$$\Gamma^{(1,\Sigma 4a)}_{CE,CE}(t>0) = -2\frac{C_2(F)}{(4\pi)^2}\cdot\left\{\frac{1}{\epsilon} + \log(8\pi t) + \gamma_E + \frac{1}{2}\right\}\Gamma^{(0)}_{CE} + \mathcal{O}(\epsilon,m,t), \tag{8.64n}$$

$$\Gamma^{(1,\Sigma 4b)}_{CE,CE}(t>0) = -2\frac{C_2(F)}{(4\pi)^2}\cdot\left\{\frac{1}{\epsilon} + \log(8\pi t) + \gamma_E + \frac{1}{2}\right\}\Gamma^{(0)}_{CE} + \mathcal{O}(\epsilon,m,t). \tag{8.64o}$$

The diagram in Fig. 8.64i vanishes for $p = 0$, since the loop becomes scaleless. We leave it here, however, so that the pole will be explicitly renormalized by Z_ψ on both sides of the flow equation, taking the $p \to 0$ limit of the Wilson coefficient. (The p-dependence cancels precisely between the two sides.) At $t = 0$, we have



(a) $\Gamma^{(1,\Sigma 1a)}_{CE,CE}(t \geq 0)$ (b) $\Gamma^{(1,\Sigma 1b)}_{CE,CE}(t \geq 0)$ (c) $\Gamma^{(1,\Sigma 2a)}_{CE,CE}(t > 0)$ (d) $\Gamma^{(1,\Sigma 2b)}_{CE,CE}(t > 0)$

(e) $\Gamma^{(1,\Sigma 3a)}_{CE,CE}(t > 0)$ (f) $\Gamma^{(1,\Sigma 3b)}_{CE,CE}(t > 0)$ (g) $\Gamma^{(1,\Sigma 4a)}_{CE,CE}(t > 0)$ (h) $\Gamma^{(1,\Sigma 4b)}_{CE,CE}(t > 0)$

Figure 8.14: Quark Leg Corrections

$$\Gamma^{(1,1A)}_{CE,CE}(0) = \mathcal{O}(m), \tag{8.65a}$$

$$\begin{aligned}\Gamma^{(1,2A1)}_{CE,CE}(0) = \frac{C_2(A)-2C_2(F)}{(4\pi)^2}\Bigg\{&\left[\frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E}q^2}\right) + \frac{3}{2} + \frac{1}{3}\delta^1_{HV} + \frac{1}{2}\delta^2_{HV} + \frac{7}{2}\delta^3_{HV}\right]\Gamma^{(0)}_{CE} \\ &+ \frac{3}{2}i\frac{k_{CE}}{k_N}\left[\frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E}q^2}\right) + \frac{4}{3} + \frac{10}{9}\delta^1_{HV} + \delta^2_{HV} + 4\delta^3_{HV}\right]\Gamma^{(0)}_N\Bigg\} \\ &+ \mathcal{O}(\epsilon,m),\end{aligned} \tag{8.65b}$$

$$\Gamma^{(1,2A2)}_{CE,CE}(0) = \frac{C_2(A) - 2C_2(F)}{(4\pi)^2}\Bigg\{\left[\frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E} q^2}\right) + 2 + \frac{1}{3}\delta^1_{HV} + \frac{1}{2}\delta^2_{HV} + \frac{7}{2}\delta^3_{HV}\right]\Gamma^{(0)}_{CE} + \frac{3}{2} i \frac{k_{CE}}{k_N}\left[\frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E} q^2}\right) + \frac{4}{3} + \frac{10}{9}\delta^1_{HV} + \delta^2_{HV} + 4\delta^3_{HV}\right]\Gamma^{(0)}_N\Bigg\} + \mathcal{O}(\epsilon, m), \tag{8.65c}$$

$$\Gamma^{(1,3A1)}_{CE,CE}(0) = \frac{C_2(A)}{(4\pi)^2}\Bigg\{\frac{7}{4}\left[\frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E} q^2}\right) + \frac{11}{7} + \frac{8}{21}\delta^1_{HV} + \frac{3}{7}\delta^2_{HV} + \frac{24}{7}\delta^3_{HV}\right]\Gamma^{(0)}_{CE} + \frac{3}{4} i \frac{k_{CE}}{k_N}\left[\frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E} q^2}\right) + \frac{4}{3} - \frac{2}{9}\delta^1_{HV} + 3\delta^3_{HV}\right]\Gamma^{(0)}_N\Bigg\} + \mathcal{O}(\epsilon, m), \tag{8.65d}$$

$$\Gamma^{(1,3A2)}_{CE,CE}(0) = \frac{C_2(A)}{(4\pi)^2}\Bigg\{\frac{5}{4}\left[\frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E} q^2}\right) + \frac{4}{5} + \frac{8}{15}\delta^1_{HV} + \frac{3}{5}\delta^2_{HV} + \frac{18}{5}\delta^3_{HV}\right]\Gamma^{(0)}_{CE} - \frac{3}{2} i \frac{k_{CE}}{k_N}\left[\frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E} q^2}\right) + \frac{4}{3} + \frac{10}{9}\delta^1_{HV} + \delta^2_{HV} + 4\delta^3_{HV}\right]\Gamma^{(0)}_N\Bigg\} + \mathcal{O}(\epsilon, m), \tag{8.65e}$$

$$\Gamma^{(1,4A1)}_{CE,CE}(0) = \frac{C_2(A)}{(4\pi)^2}\Bigg\{-\frac{1}{2}\left[\frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E} q^2}\right) + 2 + 3\delta^3_{HV}\right]\Gamma^{(0)}_{CE} - \frac{3}{2} i \frac{k_{CE}}{k_N}\left[\frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E} q^2}\right) + \frac{4}{3} + \frac{2}{3}\delta^1_{HV} + \frac{2}{3}\delta^2_{HV} + \frac{11}{3}\delta^3_{HV}\right]\Gamma^{(0)}_N\Bigg\} + \mathcal{O}(\epsilon, m), \tag{8.65f}$$

$$\Gamma^{(1,4A2)}_{CE,CE}(0) = \mathcal{O}(m), \tag{8.65g}$$

$$\Gamma^{(1,5A)}_{CE,CE}(0) = \frac{C_2(A)}{(4\pi)^2}\Bigg\{-\frac{3}{2}\left[\frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E} q^2}\right) + 2 + 3\delta^3_{HV}\right]\Gamma^{(0)}_{CE}\Bigg\} + \mathcal{O}(\epsilon, m), \tag{8.65h}$$

Summing all contributions on either side, we find the bare correlators:

$$
\begin{aligned}
\Gamma^{(1)}_{CE,CE}(t) = \frac{1}{(4\pi)^2}\Bigg\{ \Bigg[& (2C_2(F) - 2C_2(A)) \log(8\pi t) \\
& + \left(\frac{14}{3}C_2(A) - 5C_2(F) + \frac{4}{3}T_F n_f\right) \log\left(\frac{4\pi}{e^{\gamma_E} q^2}\right) \\
& - C_2(F) \log\left(\frac{4\pi}{e^{\gamma_E} p^2}\right) + \frac{169}{36}C_2(A) - \frac{13}{2}C_2(F) + \frac{20}{9}T_F n_f \Bigg] \Gamma^{(0)}_{CE} \\
& + \frac{3}{8} i \frac{k_{CE}}{k_N} \Bigg[(16C_2(F) - 2C_2(A)) \log(8\pi t) \\
& \qquad + (2C_2(A) - 16C_2(F)) \log\left(\frac{4\pi}{e^{\gamma_E} q^2}\right) \\
& \qquad + C_2(A) - 44C_2(F) \Bigg] \Gamma^{(0)}_{N} \Bigg\} + \mathcal{O}(m, t),
\end{aligned}
\tag{8.66a}
$$

$$
\begin{aligned}
\Gamma^{(1)}_{CE,CE}(0) = \frac{1}{(4\pi)^2}\Bigg\{ \Bigg[& \left(\frac{14}{3}C_2(A) - 5C_2(F) + \frac{4}{3}T_F n_f\right)\left(\frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E} q^2}\right)\right) \\
& - C_2(F)\left(\frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E} p^2}\right)\right) + \frac{241}{36}C_2(A) - 5C_2(F) + \frac{20}{9}T_F n_f \\
& + \left(2C_2(A) - \frac{4}{3}C_2(F)\right)\delta^1_{HV} + \left(\frac{5}{2}C_2(A) - 2C_2(F)\right)\delta^2_{HV} \\
& + \left(\frac{23}{2}C_2(A) - 14C_2(F)\right)\delta^3_{HV} \Bigg] \Gamma^{(0)}_{CE} \\
& + \frac{3}{8} i \frac{k_{CE}}{k_N} \Bigg[(2C_2(A) - 16C_2(F))\left(\frac{1}{\epsilon} + \log\left(\frac{4\pi}{e^{\gamma_E} q^2}\right)\right) \\
& \qquad + \frac{8}{3}C_2(A) - \frac{64}{3}C_2(F) + \left(\frac{4}{3}C_2(A) - \frac{160}{9}C_2(F)\right)\delta^1_{HV} \\
& \qquad + \left(\frac{4}{3}C_2(A) - 16C_2(F)\right)\delta^2_{HV} \\
& \qquad + \left(\frac{22}{3}C_2(A) - 64C_2(F)\right)\delta^3_{HV} \Bigg] \Gamma^{(0)}_{N} \Bigg\} + \mathcal{O}(\epsilon, m).
\end{aligned}
\tag{8.66b}
$$

These renormalize as

$$\Gamma^R_{CE,CE}(0) = Z_\psi^{-1} Z_A^{-1} Z_{CE}^{-1} \Gamma_{CE,CE}(0), \tag{8.67}$$

$$\Gamma^R_{CE,CE}(t) = Z_\psi^{-1} Z_A^{-1} Z_\chi^{-1} \Gamma_{CE,CE}(t), \tag{8.68}$$

where we have implicitly renormalized the coupling with

$$g_0^2 = Z_g \mu^{2\epsilon} g^2, \tag{8.69}$$

and the $\overline{\text{MS}}$ Z-factors are

$$Z_g = 1 + \frac{g^2}{(4\pi)^2} \cdot \left[-\frac{11}{3} C_2(A) - \frac{4}{3} T_F n_f \right] \frac{1}{\epsilon} + \mathcal{O}(g^4), \tag{8.70a}$$

$$Z_\xi = 1 + \frac{g^2}{(4\pi)^2} \cdot \left[\frac{13 - 3\xi}{6} C_2(A) + \frac{4}{3} T_F n_f \right] \frac{1}{\epsilon} + \mathcal{O}(g^4), \tag{8.70b}$$

$$Z_\psi = 1 + \frac{g^2}{(4\pi)^2} \cdot \left[-C_2(F) \right] \frac{1}{\epsilon} + \mathcal{O}(g^4), \tag{8.70c}$$

$$Z_\chi = 1 + \frac{g^2}{(4\pi)^2} \cdot \left[-3C_2(F) \right] \frac{1}{\epsilon} + \mathcal{O}(g^4), \tag{8.70d}$$

$$Z_{CE} = 1 + \frac{g^2}{(4\pi)^2} \cdot \left[-C_2(A) - C_2(F) \right] \frac{1}{\epsilon} + \mathcal{O}(g^4), \tag{8.70e}$$

$$Z_A = Z_g^{1/2} Z_\xi^{1/2}. \tag{8.70f}$$

Then, in the Feynman gauge, $\xi = 1$, we have

$$\begin{aligned} c_{CE,CE}(t) = 1 + \frac{g^2}{(4\pi)^2} \cdot \Bigg\{ & \left[2C_2(F) - 2C_2(A)\right] \log\left(2e^{\gamma_E} \bar{\mu}^2 t\right) - 2C_2(A) - \frac{3}{2} C_2(F) \\ & - \left(2C_2(A) - \frac{4}{3} C_2(F) \right) \delta^1_{HV} - \left(\frac{5}{2} C_2(A) - 2C_2(F) \right) \delta^2_{HV} \\ & - \left(\frac{23}{2} C_2(A) - 14 C_2(F) \right) \delta^3_{HV} \Bigg\} + \mathcal{O}(g^4, m, t) \end{aligned} \tag{8.71}$$

for the self-mixing coefficient and

$$c_{CE,N}(t) = 1 + \frac{g^2}{(4\pi)^2} \cdot i\frac{k_{CE}}{k_N}\Bigg\{ \Bigg[6C_2(F) - \frac{3}{4}C_2(A)\Bigg]\log\left(2e^{\gamma_E}\bar{\mu}^2 t\right) - \frac{5}{8}C_2(A) - \frac{17}{2}C_2(F) - \left(\frac{1}{2}C_2(A) - \frac{20}{3}C_2(F)\right)\delta^1_{HV} - \left(\frac{1}{2}C_2(A) - 6C_2(F)\right)\delta^2_{HV} - \left(\frac{11}{4}C_2(A) - 24C_2(F)\right)\delta^3_{HV}\Bigg]\Gamma^{(0)}_N\Bigg\} + \mathcal{O}(\epsilon, m, t) \tag{8.72}$$

for the coefficient of the purely $SU(N_C)$ nuisance operator. (We have implicitly taken $g_e \to 0$ for this calculation, so the photon term drops out of the covariant derivative.)

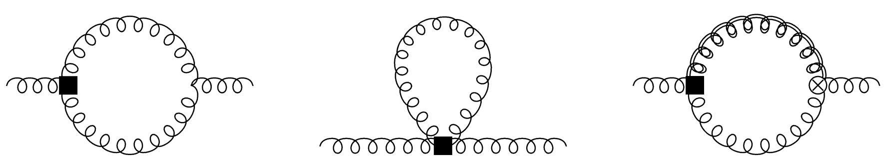
8.3 Gluon Chromoelectric Dipole Moment

8.3.1 Mixing With the Topological Charge Density: $c_{W,q}(t)$

The gluon CEDM is treated just as before. To extract its mixing with the topological charge density, we define once again a two-gluon correlation function:

$$\Gamma_i(k;t) = \int_p e^{-ip(y-x)} \int d^d z e^{-ikz} \langle A^b_\beta(y;0)\mathcal{O}_i(z;t)A^a_\alpha(x;0)\rangle. \tag{8.73}$$

There are three contributions, Fig. 8.15, one of which vanishes. The results are



(a) $\Gamma^{(1,1)}_W(t \geq 0)$ (b) $\Gamma^{(1,2)}_W(t \geq 0)$ (c) $\Gamma^{(1,3)}_W(t > 0)$

Figure 8.15: All distinct contributions to $\Gamma^{(1)}_W(t \geq 0)$

$$\Gamma^{(1,1)}_W(k;t) = -\frac{9}{4}\frac{k_W}{k_q}\frac{T_A}{(4\pi)^2}\frac{1}{t}\Gamma^{(0)}_q(k;0) + \mathcal{O}(p^2, t), \tag{8.74a}$$

$$\Gamma_W^{(1,2)}(k;t) = 0 + \mathcal{O}(p^2, t), \tag{8.74b}$$

$$\Gamma_W^{(1,3)}(k;t) = -\frac{9}{8}\frac{k_W}{k_q}\frac{T_A}{(4\pi)^2}\frac{1}{t}\Gamma_q^{(0)}(k;0) + \mathcal{O}(p^2, t), \tag{8.74c}$$

and the power-divergent mixing coefficient is

$$c_{W,q}^{(1)}(t) = -\frac{27}{8}\frac{k_W}{k_q}\frac{T_A}{(4\pi)^2}\frac{1}{t} + \mathcal{O}(t). \tag{8.75}$$

8.3.2 Self-Mixing: $c_{W,W}(t)$

The final mixing coefficient we consider is the self-mixing of the gCEDM. We calculate the coefficient by again expanding in all external scales. There are fifteen diagrams contributing to the flowed correlator, displayed in Fig. 8.16, and none on the unflowed side. There is another nuisance operator to consider:

$$\mathcal{O}_N = k_N \operatorname{Tr}\left\{\tilde{G}_{\mu\nu}\partial_\mu D_\rho G_{\rho\nu}\right\}. \tag{8.76}$$

The results are

$$\Gamma_W^{(1,1)}(k;t) = \frac{1}{2}\cdot\left(\frac{1}{4}\right)\frac{T_A}{(4\pi)^2}\left\{7\left[L_{IR} + \frac{299}{252}\right]\Gamma_W^{(0)}(k;0) - \frac{k_W}{k_N}\left[L_{IR} + \frac{3}{4}\right]\Gamma_N^{(0)}(k;0)\right\} + \mathcal{O}(t), \tag{8.77a}$$

$$\Gamma_W^{(1,2)}(k;t) = 1\cdot\left(-\frac{1}{8}\right)\frac{T_A}{(4\pi)^2}\left\{\frac{17}{3}\Gamma_W^{(0)}(k;0) - \frac{k_W}{k_N}\Gamma_N^{(0)}(k;0)\right\} + \mathcal{O}(t), \tag{8.77b}$$

$$\Gamma_W^{(1,3)}(k;t) = 1\cdot\left(-\frac{1}{16}\right)\frac{T_A}{(4\pi)^2}\left\{3\Gamma_W^{(0)}(k;0) - \frac{11}{6}\frac{k_W}{k_N}\Gamma_N^{(0)}(k;0)\right\} + \mathcal{O}(t), \tag{8.77c}$$

$$\Gamma_W^{(1,4)}(k;t) = 1\cdot\left(-\frac{1}{8}\right)\frac{T_A}{(4\pi)^2}\left\{\frac{7}{3}\Gamma_W^{(0)}(k;0) + \frac{k_W}{k_N}\Gamma_N^{(0)}(k;0)\right\} + \mathcal{O}(t), \tag{8.77d}$$

$$\Gamma_W^{(1,5)}(k;t) = \frac{1}{2}\cdot\left(-\frac{1}{2}\right)\frac{T_A}{(4\pi)^2}\left\{\left[L_{IR}+\frac{7}{12}\right]\Gamma_W^{(0)}(k;0)+3\frac{k_W}{k_N}\left[L_{IR}+\frac{13}{36}\right]\Gamma_N^{(0)}(k;0)\right\}+\mathcal{O}(t), \tag{8.77e}$$

$$\Gamma_W^{(1,6)}(k;t) = 1\cdot\left(-\frac{35}{96}\right)\frac{T_A}{(4\pi)^2}\left\{\frac{1}{3}\Gamma_W^{(0)}(k;0)+\frac{k_W}{k_N}\Gamma_N^{(0)}(k;0)\right\}+\mathcal{O}(t), \tag{8.77f}$$

$$\Gamma_W^{(1,7)}(k;t) = \frac{1}{2}\cdot(-2)\frac{T_A}{(4\pi)^2}\left\{\left[L_{IR}+\frac{7}{12}\right]\Gamma_W^{(0)}(k;0)-3\frac{k_W}{k_N}\left[L_{IR}+\frac{13}{36}\right]\Gamma_N^{(0)}(k;0)\right\}+\mathcal{O}(t), \tag{8.77g}$$

$$\Gamma_W^{(1,8)}(k;t) = 1\cdot\left(\frac{7}{8}\right)\frac{T_A}{(4\pi)^2}\left\{\Gamma_W^{(0)}(k;0)-3\frac{k_W}{k_N}\Gamma_N^{(0)}(k;0)\right\}+\mathcal{O}(t), \tag{8.77h}$$

$$\Gamma_W^{(1,9)}(k;t) = 0+\mathcal{O}(t), \tag{8.77i}$$

$$\Gamma_W^{(1,10)}(k;t) = 0+\mathcal{O}(t), \tag{8.77j}$$

$$\Gamma_W^{(1,11)}(k;t) = 0+\mathcal{O}(t), \tag{8.77k}$$

$$\Gamma_W^{(1,12)}(k;t) = 0+\mathcal{O}(t), \tag{8.77l}$$

$$\Gamma_W^{(1,13)}(k;t) = \frac{1}{2}\cdot(-3)\frac{T_A}{(4\pi)^2}\left[L_{UV}+\frac{5}{6}\right]\Gamma_W^{(0)}(k;0)+\mathcal{O}(t), \tag{8.77m}$$

$$\Gamma_W^{(1,14)}(k;t) = 1\cdot(-1)\frac{T_A}{(4\pi)^2}\left[L_{UV}-\frac{1}{4}\right]\Gamma_W^{(0)}(k;0)+\mathcal{O}(t), \tag{8.77n}$$

$$\Gamma_W^{(1,15)}(k;t) = \frac{1}{2}\cdot\left(\frac{3}{2}\right)\frac{T_A}{(4\pi)^2}\left[L_{UV}+\frac{1}{3}\right]\Gamma_W^{(0)}(k;0)+\mathcal{O}(t), \tag{8.77o}$$

where we have employed the shorthand notation:

$$L_{IR} = -\frac{1}{\epsilon_{IR}}+\log(8\pi t), \qquad \text{and} \qquad L_{UV} = \frac{1}{\epsilon_{UV}}+\log(8\pi t). \tag{8.78}$$

Summing these diagrams and replacing $\epsilon = -\epsilon_{IR} = \epsilon_{UV}$, we have

$$\begin{aligned}\Gamma_W^{(1)}(k;t) = -\frac{17}{8}\frac{T_A}{(4\pi)^2}\Big\{&\left[\frac{1}{\epsilon}+\log\left(2\bar{\mu}^2\right)+\gamma_E+\frac{91}{306}\right]\Gamma_W^{(0)}(k;0)\\ &-\frac{k_W}{k_N}\left[\frac{1}{\epsilon}+\log\left(2\bar{\mu}^2\right)+\gamma_E-\frac{137}{204}\right]\Gamma_N^{(0)}(k;0)\Big\}+\mathcal{O}(t),\end{aligned} \tag{8.79}$$

and we can read off both the renormalization constants and the Wilson coefficients:

$$ {Z_{W,W}^{(1)}}^{-1} = -\frac{17}{8} \frac{T_A}{(4\pi)^2} \frac{1}{\epsilon}, \tag{8.80a} $$

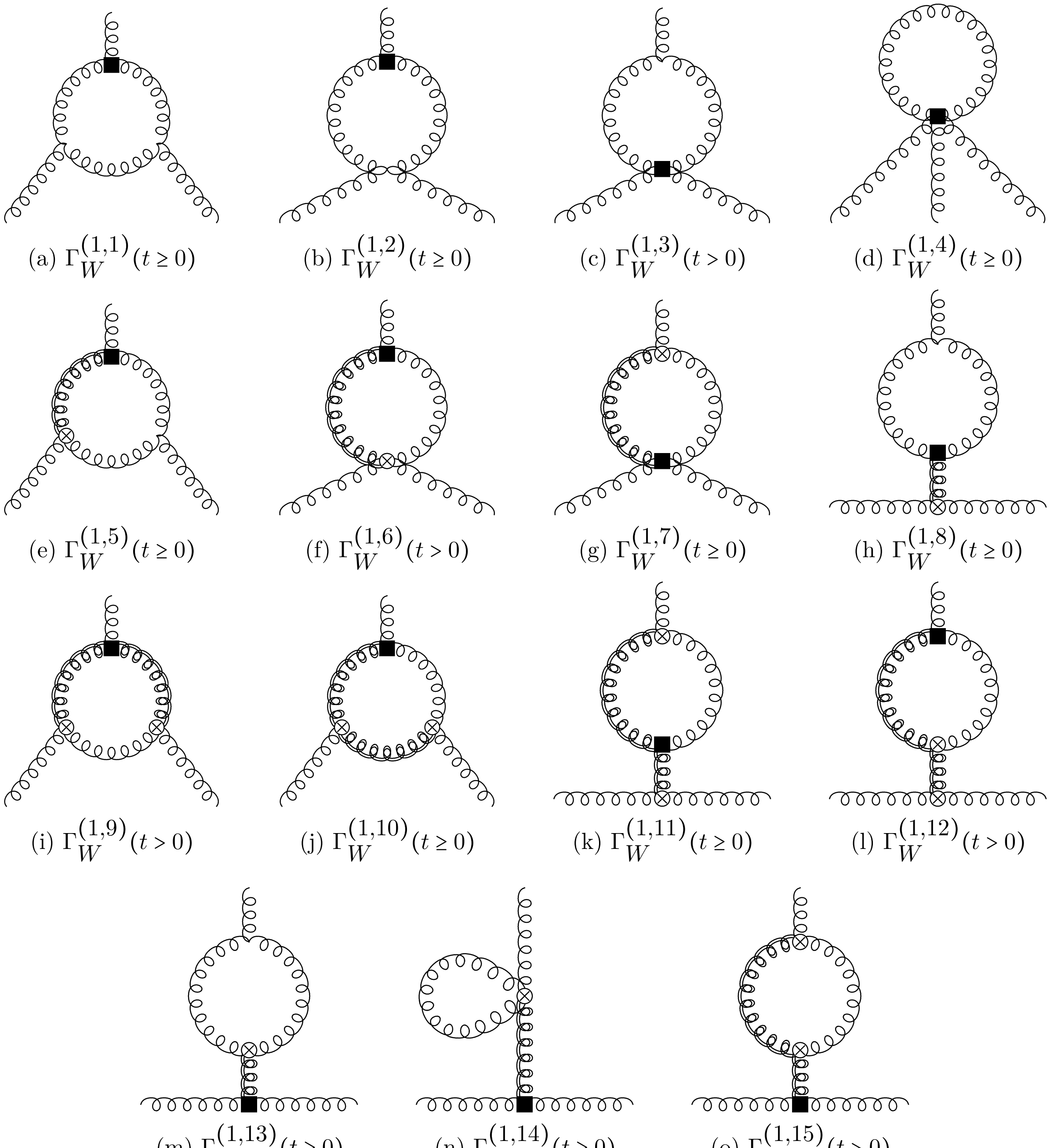
$$ {Z_{W,N}^{(1)}}^{-1} = \frac{17}{8} \frac{k_W}{k_N} \frac{T_A}{(4\pi)^2} \frac{1}{\epsilon}, \tag{8.80b} $$

$$ \tag{8.80c} $$

and

$$ c_{W,W}^{(1)}(t) = -\frac{17}{8} \frac{T_A}{(4\pi)^2} \left[\log\left(2\bar{\mu}^2\right) + \gamma_E + \frac{91}{306} \right] + \mathcal{O}(t), \tag{8.81a} $$

$$ c_{W,N}^{(1)}(t) = \frac{17}{8} \frac{k_W}{k_N} \frac{T_A}{(4\pi)^2} \left[\log\left(2\bar{\mu}^2\right) + \gamma_E - \frac{137}{204} \right] + \mathcal{O}(t). \tag{8.81b} $$



(a) $\Gamma_W^{(1,1)}(t \geq 0)$ (b) $\Gamma_W^{(1,2)}(t \geq 0)$ (c) $\Gamma_W^{(1,3)}(t > 0)$ (d) $\Gamma_W^{(1,4)}(t \geq 0)$

(e) $\Gamma_W^{(1,5)}(t \geq 0)$ (f) $\Gamma_W^{(1,6)}(t > 0)$ (g) $\Gamma_W^{(1,7)}(t \geq 0)$ (h) $\Gamma_W^{(1,8)}(t \geq 0)$

(i) $\Gamma_W^{(1,9)}(t > 0)$ (j) $\Gamma_W^{(1,10)}(t \geq 0)$ (k) $\Gamma_W^{(1,11)}(t \geq 0)$ (l) $\Gamma_W^{(1,12)}(t > 0)$

(m) $\Gamma_W^{(1,13)}(t \geq 0)$ (n) $\Gamma_W^{(1,14)}(t \geq 0)$ (o) $\Gamma_W^{(1,15)}(t > 0)$

Figure 8.16: All distinct contributions to $\Gamma_W^{(1)}(t \geq 0)$

Part IV

Discussion

This thesis has covered two major projects in renormalization. First, we studied the short-flow-time expansion of CP-violating operators. In doing so we were able to linearly relate physical operators on the boundary of the flowed theory to those in the bulk. Because the flow reparametrizes the Wilson coefficients, the continuum limit is free of local divergences. On the lattice, one solves the SFTE for the physical operators, effectively subtracting the divergences which have plagued all past explorations of the neutron electric dipole moment on the lattice. Since the coefficients contain the divergent mixings, subtracting the flowed matrix elements exactly cancels the poles at zero flow time, and physical predictions may be made as we extrapolate toward the boundary. Due also to this reparametrization, the corresponding perturbative treatment was seen to be free from artifacts of the lattice action. Consequently, our SFTE was restricted only by continuum symmetries, reducing the dimension of the operator basis.

The entire SFTE was calculated for three CP-violating operators with ostensibly large contributions to the nEDM. The topological charge density was briefly treated, confirming that it renormalizes by a simple shift proportional to the divergence of the axial vector current. We then treated two effective operators, the qCEDM and the gCEDM. The former represents the hypothetical effect of supersymmetric CP-violating interactions at the hadronic scale. The latter is the result after integrating out heavy quark loops containing a BSM Higgs exchange. Its contribution is potentially very large, since there is no suppression by light quark masses. These operators represent the foremost BSM candidates for CP violation during baryogenesis. This work represents the first steps taken toward a complete nonperturbative renormalization of EDM operators.

The perturbative methods presented in this thesis are brand new. Since the Gaussian damping factors in the flowed formalism precluded the use of standard integral representations for Feynman integrals, we developed a method which brings any analytic integrand to a spherically symmetric form. Expanding the integrand in a MacLaurin series about the angular terms, the integral is reduced to a scalar integral and an infinite series of potential tensor decompositions. The possible contractions of these tensors is an exercise in combinatorics for which many low-order solutions were derived. The author hopes to expand these arguments to arbitrary order with a more robust treatment in invariant theory, currently in development, which should be readily automated.

The second project was the development of a new nonperturbative renormalization scheme for lattice operators. In this section, we showed how a renormalization group flow can be induced by using the flow time as a scaling parameter. This is particularly useful in that it allows us to define renormalization conditions with no reference to the lattice spacing while respecting the symmetries of the discretized action. Similarly to the first project, this permits a smooth continuum limit. Additionally, we defined the scheme with manifestly gauge-invariant vacuum correlation functions. As applied to the lattice, this eliminates the need for gauge fixing, so the Gribov ambiguity is completely avoided. This construction is computationally inexpensive and applicable over a large range of scales accessible only on the lattice. In order to minimize cutoff effects, the relevant operator and its probe were fixed at some physical separation in Euclidean time much larger than the inverse flow-time radius. As a result, the perturbative treatment in the natural time-momentum representation was seemingly impossible by brute force. Instead we proposed to inject some nonzero momentum into the correlators, which may then calculated purely in reciprocal space. Taking the Fourier transform in the limit of vanishing spatial momenta returned us to coordinate space with the necessary time separation. As a pilot study we rederived the anomalous dimensions of the fermion bilinears at one loop-order with these vacuum correlators. For now, the difficulties of flowed perturbation theory stand as the single disadvantage of our scheme, although the techniques are currently being simplified and extended to higher loop order with the ultimate goal of complete automation. This will allow the nonperturbative results found on the lattice to be integrated up to high energies through the renormalization group, at which point they may be compared directly to phenomenological results.

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APPENDIX

COMBINATORIAL TENSOR DECOMPOSITION

In the perturbation theory associated with the Yang-Mills gradient flow, one regularly encounters integrals of the form

$$I_{ijk}^{abc}(p,\mu;t) = \int_q \frac{e^{-ap^2}e^{-bq^2}e^{-c(p+q)^2}}{\left(p^2\right)^i\left(q^2\right)^j\left((p+q)^2\right)^k}; \quad a,b,c,i,j,k \in \mathbb{R}, \tag{A.1}$$

where we define the shorthand

$$\int_q \equiv \mu^{4-d}\int_{\mathbb{R}^d}\frac{d^dq}{(2\pi)^d}. \tag{A.2}$$

The energy scale μ and the dimension $d = 4 - 2\epsilon$ are included to prepare the integral for dimensional regularization and renormalization by construction. We will, for full generality, solve all integrals in d-dimensions, which will allow the reader to modify the integrals without worrying about regulators.

A.1 Standard Integrals in Dimensional Regularization

Without the gradient flow, loop integrals are relatively simple to generalize to d-dimensions, where they generally take the form

$$I_{\mu_I}^{n_I}(p_I;m_I) = \int_q \frac{q_{\mu_1}\cdots q_{\mu_n}}{\prod_{i=1}^N\left(r_i^2+m_i^2\right)^{n_i}} \tag{A.3}$$

with muti-index $I = \{1,\ldots,N\}$, where the product in the denominator runs over all propagators in the loop with their respective masses and momenta indexed by i, and the vectors in the numerator are the result of n derivative couplings (non-scalar vertices). Each r_i in the denominator has the form $r_i = q + s_i$, where $s_i = p_1 + \cdots + p_i$ for external momenta $\{p_i\}_{i=1}^N$. The dimension of the integral is undetermined and generically non-integral, so we cannot integrate over each component directly. If the integrand is spherically symmetric, however, we may transform to spherical coordinates, where the $(d-1)$-dimensional spherical shell is readily integrated out, as follows. First we extract the solid angle:

$$\int_q f(q^2) = \frac{\mu^{4-d}}{(2\pi)^d} \int_\Omega d\Omega \int_0^\infty r^{d-1} dr \; f(r^2), \tag{A.4}$$

where

$$d\Omega = \prod_{k=1}^{d-1} \sin^{d-k-1}(\phi_k) d\phi_k. \tag{A.5}$$

and where the angular domain Ω is defined by

$$\begin{aligned} &\phi_k \in [0, \pi); \quad && k < d-1 \\ &\phi_k \in [0, 2\pi); && k = d-1. \end{aligned} \tag{A.6}$$

Symmetry allows us to write

$$\int_0^\pi d\phi_k \, \sin^{d-k-1}(\phi_k) = 2 \int_0^{\pi/2} d\phi_k \, \sin^{d-k-1}(\phi_k); \quad k < d-1 \tag{A.7}$$

and

$$\int_0^{2\pi} d\phi_k \, \sin^{d-k-1}(\phi_k) = 4 \int_0^{\pi/2} d\phi_k; \quad k = d-1, \tag{A.8}$$

so that

$$\begin{aligned}\int_\Omega d\Omega =&2\prod_{k=1}^{d-1}\left[2\int_0^{\pi/2} d\phi_k\ \sin^{d-k-1}(\phi_k)\right]\\ =&2\prod_{k=1}^{d-1} B\left(\frac{d-k}{2},\frac{1}{2}\right)\\ =&2\Gamma^{d-1}\left(\frac{1}{2}\right)\prod_{k=1}^{d-1}\Gamma\left(\frac{d-k}{2}\right)\Big/\Gamma\left(\frac{d-k+1}{2}\right).\end{aligned} \tag{A.9}$$

The numerator of each factor cancels the denominator of the next, and we are left with

$$\int_\Omega d\Omega = 2\pi^{\frac{d-1}{2}}\frac{\Gamma(1/2)}{\Gamma(d/2)} = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \tag{A.10}$$

Thus

$$\int_q f(q^2) = \frac{2\mu^{4-d}}{(4\pi)^{d/2}\Gamma(d/2)}\int_0^\infty\ r^{d-1}dr\ f(r^2). \tag{A.11}$$

If the integrand is not even, we must transform it to a spherical form, the standard for which is Feynman parameterization. The identity

$$\frac{1}{\prod_{i=1}^N\left(r_i^2+m_i^2\right)^{n_i}} = \frac{1}{B(n_1,\ldots n_N)}\int_0^\infty dz_1\cdots\int_0^\infty dz_N\frac{\delta\left(1-\sum_{i=1}^N z_i\right)\prod_{i=1}^N z_i^{n_i-1}}{\left[\sum_{i=1}^N z_i\left(r_i^2+m_i^2\right)\right]^{\sum_{i=1}^N n_i}} \tag{A.12}$$

allows the denominator to be expressed as a sum, so that we can complete the square in the momentum of integration q:

$$\begin{aligned}I_{\mu_I}^{n_I}(p_I;m_I) =& \frac{1}{B(n_1,\ldots n_N)}\int_0^\infty\prod_{i=1}^N\left(z_i^{n_i-1}dz_i\right)\int_q\frac{\delta\left(1-\sum_{i=1}^N z_i\right)}{\left[\sum_{i=1}^N z_i\left((q+s_i)^2+m_i^2\right)\right]^{\sum_{i=1}^N n_i}}q_{\mu_1}\cdots q_{\mu_n}\\ =&\frac{1}{B(n_1,\ldots n_N)}\int_0^\infty\prod_{i=1}^N\left(z_i^{n_i-1}dz_i\right)\int_q\frac{\delta\left(1-\sum_{i=1}^N z_i\right)}{\left[(q+Q)^2+\Delta\right]^{\sum_{i=1}^N n_i}}q_{\mu_1}\cdots q_{\mu_n},\end{aligned} \tag{A.13}$$

where

$$\Delta = \sum_{i=1}^{N} z_i(s_i^2 + m_i^2) - Q^2, \tag{A.14}$$

and

$$Q_\mu = \sum_{i=1}^{N} z_i(s_i)_\mu. \tag{A.15}$$

Under the change of variables $k_\mu = q_\mu + Q_\mu$, we have,

$$I_{\mu_I}^{n_I}(p_I; m_I) = \frac{1}{B(n_1, \ldots n_N)} \int_0^\infty \prod_{i=1}^{N} \left(z_i^{n_i - 1} dz_i \right) \int_k \frac{\delta\left(1 - \sum_{i=1}^N z_i\right)}{\left[k^2 + \Delta\right]^{\sum_{i=1}^N n_i}} (k-Q)_{\mu_1} \cdots (k-Q)_{\mu_n}, \tag{A.16}$$

and the evenness or oddness of the integrand is more obvious. The product of vectors $(k - Q)_{\mu_i}$ is a polynomial in k, so the even-degree terms will survive integration, and the odd terms will vanish. Since the fraction above is even, the momentum integral is a sum over integrals of the form

$$\int_q f(q^2) q_{\mu_1} \cdots q_{\mu_{2n}}, \tag{A.17}$$

for some n. The $2n$-fold product ensures that the integral does not trivially vanish, but we now must discern the tensor structure.

A.2 Reduction of Tensor Integrals

The solution of the integral must have the same symmetry as the integrand, so it must be proportional to some tensor with such symmetry:

$$\int_q f(q^2) \prod_{m=1}^{2n} q_{\mu_m} = A \cdot T_{\mu_1 \cdots \mu_{2n}}. \tag{A.18}$$

Since the product is commutative, it is entirely symmetric with respect to any permutation of the $2n$ indices μ_m. The only tensor with such a symmetry is the symmetrized sum of products of n metric tensors over all unordered partitions of the $2n$ indices into n pairs. Let $\sigma_r(s)$ denote the r^{th} permutation on the index s of this form. Note that under these restrictions, the following partitions are all equivalent:

$$\{1,2\},\{3,4\},\{5,6\}\{2,1\},\{3,4\},\{5,6\}\{3,4\},\{1,2\},\{5,6\} \tag{A.19}$$

We must first count the number of ways we may group $2n$ indices into n pairs. Choosing an index generically, there are $2n-1$ remaining indices available for pairing. Continuing in this manner, there are $2n-2$ indices we may choose to begin the second pair, with $2n-3$ partners remaining, and so on to $(2n)!$. Since the ordering of the pairs doesn't matter, we divide by $n!$. Moreover, each pair is itself unordered with respect to its two elements, so we divide again by 2^n. There are, then, $\frac{(2n)!}{2^n n!} = (2n-1)!!$ distributions of the indices, and the sum over each permutation $\sigma_r(s)$ runs from $r=1$ to $r=(2n-1)!!$ with $s \in [1,n] \cap \mathbb{N}$, so

$$T_{\mu_1 \cdots \mu_{2n}} = \sum_{r=1}^{(2n-1)!!} \prod_{s=1}^{n} g_{\mu_{\sigma_r(2s-1)}\mu_{\sigma_r(2s)}} \tag{A.20}$$

To find the constant of proportionality A, contract both sides of equation (A.18) with any term of the sum over permutations; any term may be chosen due to its symmetrical construction. Without loss of generality, we make the natural choice $g_{\mu_1\mu_2} \cdots g_{\mu_{2n-1}\mu_{2n}}$:

$$\prod_{k=1}^{n} g_{\mu_{2k-1}\mu_{2k}} \int_q f(q^2) \prod_{m=1}^{2n} q_{\mu_m} = A \prod_{k=1}^{n} g_{\mu_{2k-1}\mu_{2k}} \sum_{r=1}^{(2n-1)!!} \prod_{s=1}^{n} g_{\mu_{\sigma_r(2s-1)}\mu_{\sigma_r(2s)}}. \tag{A.21}$$

Commutativity and associativity under addition allow us to rearrange the products and contract all indices first, resulting in a scalar expression. On the left, the components of the momentum q are simply paired into a product of n squares, leaving $(q^2)^n$ in place of the integrand's product. The right side is far less trivial, and it will require a bit of care. Before we tackle this problem, however, we note that the integral has been indeed reduced to a scalar, and we are left with a combinatorial problem on the right-hand side:

$$\frac{1}{A}\int_q f(q^2)\cdot(q^2)^n = \prod_{k=1}^{n} g_{\mu_{2k-1}\mu_{2k}} \sum_{r=1}^{(2n-1)!!} \prod_{s=1}^{n} g_{\mu_{\sigma_r(2s-1)}\mu_{\sigma_r(2s)}}. \tag{A.22}$$

A.3 Normalizing the Totally Symmetric Tensor with Graphs

The product on the right-hand side of equation (A.22),

$$s_n = \prod_{k=1}^{n} g_{\mu_{2k-1}\mu_{2k}} \sum_{r=1}^{(2n-1)!!} \prod_{s=1}^{n} g_{\mu_{\sigma_r(2s-1)}\mu_{\sigma_r(2s)}}, \tag{A.23}$$

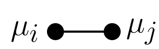
where we have introduced the shorthand s_n, is most easily illustrated by examining the $n = 2$ case, where it reads

$$s_2 = \prod_{k=1}^{2} g_{\mu_{2k-1}\mu_{2k}} \sum_{r=1}^{3} \prod_{s=1}^{2} g_{\mu_{\sigma_r(2s-1)}\mu_{\sigma_r(2s)}} = g_{\mu\nu}g_{\rho\sigma}\cdot\left(g_{\mu\nu}g_{\rho\sigma}+g_{\mu\rho}g_{\nu\sigma}+g_{\mu\sigma}g_{\nu\rho}\right). \tag{A.24}$$

Contracting all indices gives $s_2 = d^2 + d + d$, following the order of the parenthetical term. We notice that only the first term shares the ordering of the indices $g_{\mu\nu}g_{\rho\sigma}$, while the other two do not. Since the first term shares this pairing, there is an n-fold product over traces $g_{\mu\nu}g_{\mu\nu}$, each of which evaluates to the value of the dimension d. The second and third terms differ by a transposition, so the factor outside the parentheses serves to connect the permuted indices, *e.g.*:

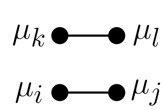
$$g_{\mu\nu}g_{\rho\sigma} \cdot g_{\mu\rho}g_{\nu\sigma} = g_{\mu\sigma}g_{\mu\sigma} = g_{\mu\mu} = d, \tag{A.25}$$

but in doing so, we lose two powers of the metric tensor, so the result will be correspondingly reduced by a trace, or, in other words, one power of the dimension. Since in the preceding case n was very small, there is no need for more advanced machinery. Recall, however, that the number of terms in the parentheses will grow as $(2n-1)!!$. Even for the $n = 3$ case, there are 15 terms, and the result is not trivial. At $n = 4, 5, \ldots$, there are $105, 945, \ldots$ terms, and the number of contractions becomes intractable. Fortunately, this problem is mapped very cleanly to graphs. Let each of the $2n$ indices μ_i represent a vertex on a graph $\mathcal{G}^I_{2n}$, where the multi-index $I = \{1, 2, \ldots, 2n\}$ represents the ordered set of indices being mapped to the graph's vertices. Then let each metric tensor represent an edge connecting the vertices corresponding to its indices. Since each index appears once and only once in each term, and since the metric tensor connects only two indices, we have the mapping

$$g_{\mu_i\mu_j} \mapsto$$ 

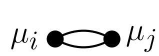
. (A.26)

Since each metric tensor only connects two points, we can write the n-fold product of (un-contracted) metric tensors as a (disjoint) graph union:

$$g_{\mu_i\mu_j}g_{\mu_k\mu_l} \mapsto \mathcal{G}^{ij}_2 \oplus \mathcal{G}^{kl}_2 =$$ 

. (A.27)

If we have a product of metric tensors with repeated indices, then we take a simple graph union:

$$g_{\mu_i\mu_j}g_{\mu_i\mu_j} \mapsto \mathcal{G}^{ij}_2 \cup \mathcal{G}^{ij}_2 =$$ 

. (A.28)

When a cycle appears as above, we recognize the trace of a metric tensor; since every edge corresponds to a metric tensor, and there are no 1-valent vertices, every index is contracted until we are left with a trace over a single metric tensor. These cycles, then, map back to powers of the dimension d, and a graph with k cycles corresponds to d^k. Then we have a correspondence:

$$\prod_{s=1}^{n} g_{\mu_{\sigma_r(2s-1)}\mu_{\sigma_r(2s)}} \sim \mathcal{G}_{2n}^{I_r}, \tag{A.29}$$

where the multi-index I_r is defined by $I_r = \{\sigma_r(1), \ldots, \sigma_r(2n)\}$, and edges are meant to exist between every two vertices as they are ordered in I_r. These graphs are 1-regular, since there are no repeated indices in the product of metric tensors, but there is a metric tensor (edge) pairing each index (vertex) to one other. We define $\mathcal{G}_{2n}^{I_i I_j} = \mathcal{G}_{2n}^{I_i} \cup \mathcal{G}_{2n}^{I_j}$ to be the 2-regular union of 1-regular graphs with edges defined by the i^{th} and j^{th} permutations on the indices. Each term in the sum s_n then maps to a graph $\mathcal{G}_{2n}^{I_i I_j}$ for some $i, j \in \{1, 2, \ldots, (2n-1)!!\}$. Since $\mathcal{G}_{2n}^{I_i I_j}$ is 2-regular, it must be a union of cycles, each of which evaluates to a power of d. Summarily: each term $\left(g_{\mu_1\mu_2} \cdots g_{\mu_{2n-1}\mu_{2n}}\right)\left(g_{\mu_{\sigma_r(1)}\mu_{\sigma_r(2)}} \cdots g_{\mu_{\sigma_r(2n-1)}\mu_{\sigma_r(2n)}}\right)$ in s_n maps to a 2-regular graph $\mathcal{G}_{2n}^{I I_r}$ which contains k cycles, and maps back to d^k. Thus s_n has the form

$$s_n = \sum_{k=1}^{n} G(n,k) d^k, \tag{A.30}$$

where $G(n,k)$ is the number of 2-regular graphs containing the 1-regular subgraph $\mathcal{G}_{2n}^{I}$, which decompose into k cycles. Note that this number is invariant under the choice of multi-index I. We are now left to the problem of counting these graphs. Fortunately, we may construct them recursively. Consider any such 2-regular graph on $2n$ indices. If we wish to create another 2-regular graph on $2n+2$ vertices, we may add the vertices μ_{2n+1} and μ_{2n+2} in $d + 2n$ ways, as follows. First, note that there must be an edge between μ_{2n+1}

and μ_{2n+2}. This comes from the restriction that we must recover the 1-regular graph $\mathcal{G}^{I}_{2n+2}$ as a subgraph, and such a subgraph must contain every edge from μ_{2i+1} and μ_{2i+2} for all $i \in \{0, 1, \dots, n-1\}$ and no more. For the same reason, every graph we create by adding μ_{2n+1} and μ_{2n+2} also must contain all edges of this form. Therefore, we can add the two new vertices as a disjoint 2-cycle, or we may break any existing cycle and insert the new vertices, increasing the size of the cycle by two. Thus the first case corresponds to a disjoint union and its natural mapping back to a polynomial in d:

$$\mathcal{G}_2^{\{\mu_{2n+1},\mu_{2n+2}\}\{\mu_{2n+1},\mu_{2n+2}\}} \oplus \left(\bigcup_{r=1}^{(2n-1)!!} \mathcal{G}^{II_r}_{2n} \right) \mapsto d \cdot s_n. \tag{A.31}$$

Onto the latter case, since each graph is 2-regular, then $|E| = |V|$ necessarily. Then, for the graph on $2n$ vertices, we may cut each of the $2n$ edges and insert the new vertices. We can insert this new edge in 2 ways for each cut, but we must retain the subgraph $\mathcal{G}^{I}_{2n}$, so half of the cuts produce unusable graphs, and we must divide by two. Then we have $2n$ graphs for each of the graphs in s_n, which adds $2n \cdot s_n$ to our total. This gives us a recursion:

$$s_{n+1} = (d + 2n) \cdot s_n. \tag{A.32}$$

Finally, induction on n gives us

$$s_n = d(d+2)(d+4)\cdots(d+2(n-1)) = (d)_{n,2}, \tag{A.33}$$

where

$$(d)_{n,2} = \frac{2^n \Gamma(d/2+n)}{\Gamma(d/2)} \tag{A.34}$$

is the Pochhammer 2-symbol. Using the identity

$$(x)_{n,k} = k^n \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} \left(\frac{x}{k}\right)^j, \tag{A.35}$$

we find that

$$G(n,k) = 2^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}. \tag{A.36}$$

In fact, there is a reason for the Stirling numbers of the first kind to appear, and we present an alternative proof of equation (A.33) via explicit permutations on the indices in Appendix A.4. Finally, we have

$$\int_q f(q^2) \prod_{m=1}^{2n} q_{\mu_m} = \frac{1}{(d)_{n,2}} T_{\mu_1 \cdots \mu_{2n}} \cdot \int_q f(q^2) \cdot (q^2)^n. \tag{A.37}$$

The case for $n = 3$ is illustrated in Appendix A.5

A.4 Normalizing the Symmetric Tensor (Alternative Method)

Whichever term has the identical arrangement of indices scompared to the term with which we chose to contract will evaluate to d^n, where d is the dimension, since the result is a product of n traced metric tensors, each equal to d by definition. For each remaining term in the sum over distributions, we wish to find the number of interchanges of indices which will return the ordering to the arbitrary arrangement with which we are contracting; in our case, we want to return each permutation to the natural numerical order $(1,2);(3,4);\ldots;(2n-1,2n)$. Begin by fixing the first element of the chosen permutation (the odd numbers in our scenario); this leaves n free indices, so we consider the permutation group S_n. We now decompose each element into k disjoint cycles, where k ranges from 1 to n. These may be counted using the unsigned Stirling numbers of the first kind $\begin{bmatrix} n \\ k \end{bmatrix} = |S_1(n,k)|$; specifically, there are $\begin{bmatrix} n \\ k \end{bmatrix}$ elements of S_n which may be decomposed as the composition of k disjoint cycles. The unsigned Stirling numbers of the first kind are recursively defined as

$$\begin{bmatrix} n+1 \\ k \end{bmatrix} = n\begin{bmatrix} n \\ k \end{bmatrix} + \begin{bmatrix} n \\ k-1 \end{bmatrix} \tag{A.38}$$

for $k > 0$ with the initial conditions that

$$\begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ n \end{bmatrix} = 0 \text{ and } \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1 \tag{A.39}$$

for $n > 0$. These may be further decomposed into $n - k$ transpositions. Transpositions are functionally equivalent to contraction with a metric tensor indexed by the two indices to be transposed. For each contraction, the exponent of d will be reduced by one, since we have one fewer square of a metric tensor. Since there are $n - k$ transpositions for some term, we have $d^{n-(n-k)} = d^k$. We now consider the weight factor for each k, since we have obviously ignored many (namely $(2n-1)!! - n!$) distributions of pairs by fixing the first index of each pair. The remaining distributions may be constructed by transposing the indices for the pairs. The term with which we choose to contract is insensitive to such transpositions, so the powers of d on the left should be as well. We are simply counting multiplicities of each power of d. For each k of our fixed-index permutations under S_n, we can construct 2^{n-k} permutations with the pairwise transposition symmetry of our chosen term, since each permutation has been decomposed into $n - k$ transpositions. Thus for each k we have $2^{n-k}\begin{bmatrix} n \\ k \end{bmatrix}$ terms which evaluate to d^k. We now sum over the permutations, which has been shown to be equivalent to summing over powers of the dimension with the aforementioned weighting:

$$\sum_{r=1}^{(2n-1)!!} \prod_{k=1}^{n} g_{\mu_{2k-1}\mu_{2k}} \prod_{s=1}^{n} g_{\mu_{\sigma_r(2s-1)}\mu_{\sigma_r(2s)}} = \sum_{k=1}^{n} 2^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} d^k. \tag{A.40}$$

This may be further simplified by noting that

$$\sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k = x^{\bar{n}}, \tag{A.41}$$

called the rising factorial or Pochhammer symbol. In our case, we find

$$\sum_{k=1}^{n} 2^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} d^k = 2^n \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} \left(\frac{d}{2}\right)^k = 2^n \left(\frac{d}{2}\right)^{\bar{n}}, \tag{A.42}$$

Which is the definition of the Pochhammer k-symbol $(x)_{n,k}$ in the case that $x = d$ and $k = 2$. Note that setting $d = 1$, which is tantamount to ignoring contractions and simply counting our permutations, we have

$$\sum_{k=1}^{n} 2^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} = 2^n \left(\frac{1}{2}\right)^{\bar{n}} = 2^n \frac{\Gamma(n+1/2)}{\Gamma(1/2)} = 2^n \frac{\sqrt{\pi}(2n-1)!!}{2^n} \frac{1}{\sqrt{\pi}} = (2n-1)!!, \tag{A.43}$$

which is exactly the number of ways we may split the set of $2n$ indices into n pairs, which provides a nice sanity check.

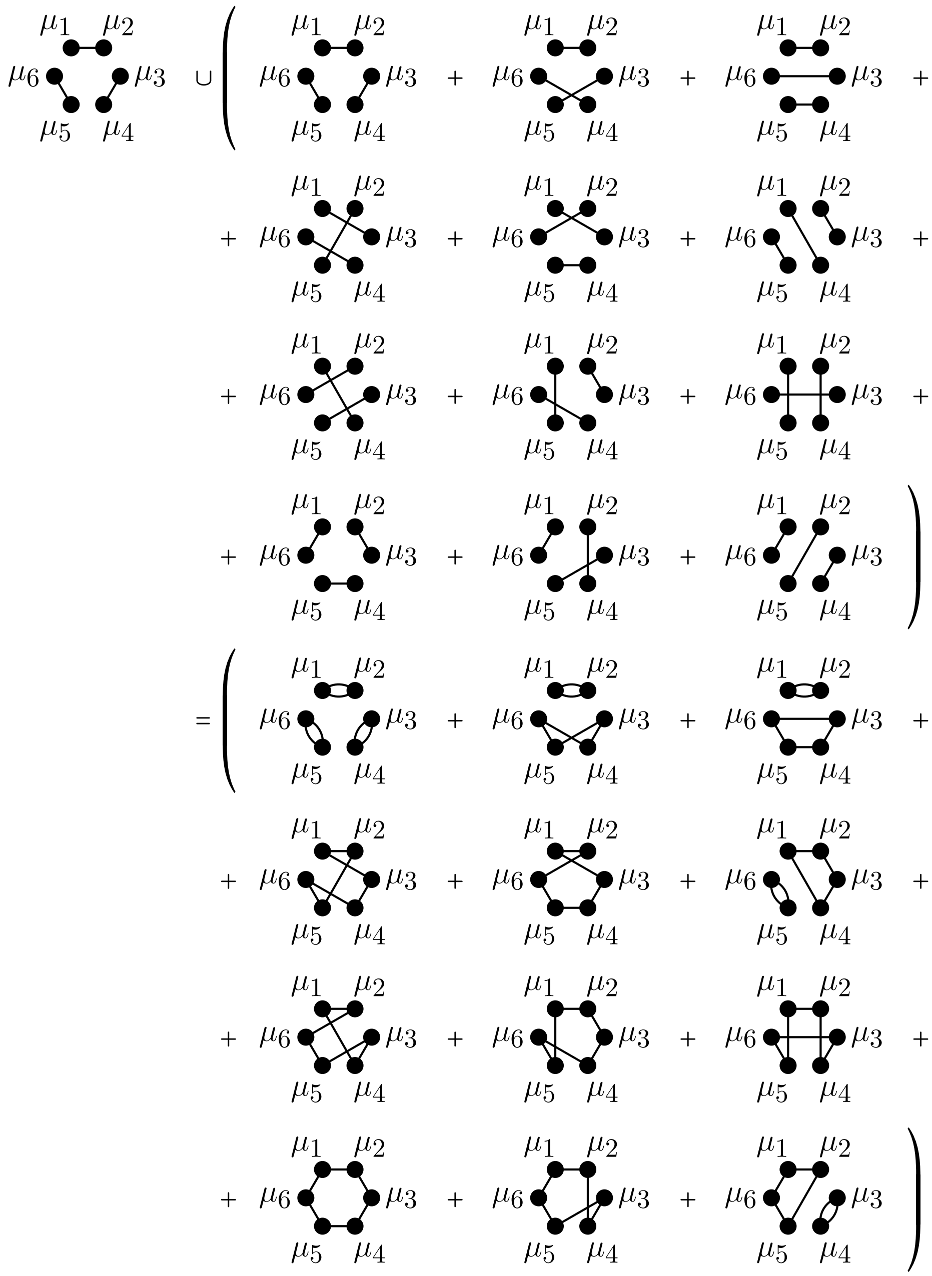
We may now solve for our constant of proportionality A:

$$\int_q \frac{e^{-(b+c+z)q^2}}{\left(q^2\right)^j} (q^2)^n = (d)_{n,2} A \Rightarrow A = \frac{\Gamma(d/2)}{2^n \Gamma(d/2+n)} \int_q \frac{e^{-(b+c+z)q^2}}{\left(q^2\right)^j} (q^2)^n, \tag{A.44}$$

where we have used the the identity $(d)_{n,2} = 2^n \frac{\Gamma(d/2+n)}{\Gamma(d/2)}$. We finally evaluate the momentum integral in its scalar, spherically symmetric form.

A.5 Graphs for $n = 3$

The following is an explicit example of the graph representation of the isotropic tensors for the case $n = 3$. Note the presence of the $n = 2$ subgraphs of the disconnected graphs, for example, on vertices $3 - 6$ of the first three graphs.



$$\mapsto \quad d^3 + d^2 + d^2 + d^2 + d + d + d^2 + d + d + d + d + d^2 + d + d + d^2$$

$$= \quad 1 \cdot d^3 + 6 \cdot d^2 + 8 \cdot d$$

$$= \quad d(d+2)(d+4). \tag{A.45}$$