

A FREE BOUNDARY PROBLEM, PROJECTIONS OF RANDOM CANTOR SETS, AND THE
GEOMETRY OF CURVES WITH SMALL INTERSECTION WITH MANY LINES

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ABSTRACT

Finding the geometric properties of a set is a very old problem. The present text consists of three chapters where we study such properties with techniques involving Complex and Harmonic Analysis, Probability, and Geometric Measure Theory. We specifically deal with a few considerations of free boundary problems, we calculate the decay rate of the projections of a certain random Cantor set, and we describe the shape of planar graphs which avoid having too many intersections with a positive cone of lines.

To begin with, we introduce Schwarz functions; holomorphic functions on open domains Ω satisfying $S(\zeta) = \bar{\zeta}$ on Γ , part of Ω 's boundary. Sakai in 1991 gave a complete characterization of the boundary of a domain admitting a Schwarz function. In fact, if Ω is simply connected and $\Gamma = \partial\Omega \cap D(\zeta, r)$, then Γ has to be regular real analytic. Here, we attempt to describe Γ when the boundary condition is slightly relaxed. In particular, three different scenarios over a simply connected domain Ω are treated: when $f_1(\zeta) = \bar{\zeta} f_2(\zeta)$ on Γ with f_1, f_2 holomorphic and continuous up to the boundary, when $\mathcal{U}/\mathcal{V}$ equals certain real analytic function on Γ with $\mathcal{U}, \mathcal{V}$ positive and harmonic on Ω and vanishing on Γ , and when $S(\zeta) = \Phi(\zeta, \bar{\zeta})$ on Γ with Φ a holomorphic function of two variables. It turns out that the boundary piece Γ can be, respectively, anything from real analytic to merely C^1 , regular except finitely many points, or regular except for a measure zero set.

For the second chapter, we consider a model of randomness for self-similar Cantor sets of finite and positive 1-Hausdorff measure. We find the sharp rate of decay of the probability that a Buffon needle lands δ -close to a Cantor set of this particular randomness. Two quite different models of randomness for Cantor sets, by Peres and Solomyak, and by Shiwen Zhang, appear to have the same order of decay for the Buffon needle probability: $\frac{c}{\log \frac{1}{\delta}}$. Here, we prove the same rate of decay for a third model of randomness, which asserts a vague feeling that any “reasonable” random Cantor set of positive and finite length will have Favard length of order $\frac{c}{\log \frac{1}{\delta}}$ for its δ -neighbourhood. The estimate from below was obtained long ago by Mattila.

In the last chapter, we show the local Lipschitz property for a graph avoiding multiple-point

intersection with lines directed in a given cone. The assumption is much stronger than those of the well-known Marstrand's theorem, but the conclusion is much stronger too. Additionally, we find that a continuous curve with a similar property is σ -finite with respect to Hausdorff length, and we give an estimate on the Hausdorff measure of each "piece".

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CHAPTER 1

FREE BOUNDARY PROBLEMS VIA SAKAI'S THEOREM

1.1 Introduction

Let $D(\zeta_0, r) \subset \mathbb{C}$ denote the open disk centred at $\zeta_0 \in \mathbb{C}$ and of radius $r > 0$. Let Ω be an open subset of $D(\zeta_0, r)$ where $\zeta_0 \in \Gamma = \partial\Omega \cap D(\zeta_0, r)$ is a non-isolated boundary point.

A Schwarz function of $\Omega \cup \Gamma$ is a function $S : \Omega \cup \Gamma \rightarrow \mathbb{C}$ holomorphic on Ω and continuous on $\Omega \cup \Gamma$ that satisfies

$$S(\zeta) = \bar{\zeta} \quad \text{on } \Gamma. \quad (1.1.1)$$

In his Acta Mathematica paper [13], Sakai proved that Schwarz functions completely characterize the shape of Γ . One of the technical tools used was the Phragmén–Lindelöf principle in the form below, but it is far from being the key to his proof; his paper is full of very subtle tricks.

Theorem 1.1.1. *Let Ω be an open set in $\mathbb{C}$ and let ζ_0 be a non-isolated boundary point of Ω . Let f be a holomorphic function on Ω and $D(\zeta_0, \delta)$ a ball satisfying the following:*

- (i) $\limsup |f(z)| \leq 1$ while $\Omega \ni z \rightarrow \zeta$ for every $\zeta \in \partial\Omega \cap D(\zeta_0, \delta) \setminus \{\zeta_0\}$ and
- (ii) $|f(z)| \leq \alpha|z - \zeta_0|^{-\beta}$ in $\Omega \cap D(\zeta_0, \delta)$ for some positive constants α and β .

Then,

$$\limsup |f(z)| \leq 1$$

while $\Omega \ni z \rightarrow \zeta_0$.

In particular, Sakai proved the following, see [13, Theorem 5.2].

Theorem 1.1.2. *Let $\Omega \subset D(\zeta_0, r)$ be a bounded open set in $\mathbb{C}$ and ζ_0 a non-isolated point of its boundary, $\Gamma = \partial\Omega \cap D(\zeta_0, r)$. Suppose S is a Schwarz function on $\Omega \cup \Gamma$, that is,*

- (i) S is holomorphic on Ω ,
- (ii) continuous on $\Omega \cup \Gamma$, and

(iii) $S(\zeta) = \bar{\zeta}$ on Γ .

Then, for some small $0 < \delta \leq r$ one of the following must occur (where we set $D = D(\zeta_0, \delta)$):

- (1) $\Omega \cap D$ is simply connected and $\Gamma \cap D$ is a regular real analytic simple arc through ζ_0 ;
- (2a) $\Gamma \cap D$ determines uniquely a regular real analytic arc through ζ_0 ; $\Gamma \cap D$ is either an infinite proper subset of this arc with ζ_0 as an accumulation point or equal to it; also, $\Omega \cap D = D \setminus \Gamma$;
- (2b) $\Omega \cap D = \Omega_1 \cup \Omega_2$ where Ω_1 and Ω_2 are (open) simply connected and $\partial\Omega_1 \cap D$ and $\partial\Omega_2 \cap D$ are regular real analytic simple arcs through ζ_0 and tangent at ζ_0 ;
- (2c) $\Omega \cap D$ is simply connected and $\Gamma \cap D$ is a regular real analytic simple arc except for a cusp at ζ_0 ; the cusp points into Ω .

Recall that a *regular arc* means a differentiable arc whose derivative never vanishes and *simple* means that it is parametrized by an injective continuous function.

Remarks 1.1.3. Here is an example of a cusp of (2c) at $\zeta_0 = 0$ with Schwarz function. There exist analytic functions T on $\{|z| \leq \eta\}$, for some $\eta > 0$, that have a zero of order 2 at 0, are univalent on closed upper half-disk $K_\eta \equiv \{|z| \leq \eta : \text{Im}(z) \geq 0\}$, and satisfy $\Gamma \cap D \subset T(-\eta, \eta)$ and $T(K_\eta) \subset \Omega \cup \Gamma$. In fact, it is easy to construct such functions. Every such T leads to a Schwarz function on the domain $\Omega = T(\{|z| < \eta, \text{Im } z > 0\})$, which has two analytic arcs forming a cusp Γ at 0. In order to have $S(\zeta) = \bar{\zeta}$ on Γ , it suffices to have a function analytic in $\{|z| < \eta, \text{Im } z > 0\}$ and continuous up to $(-\eta, \eta)$ such that $A(x) = \overline{T(x)}$, $x \in (-\eta, \eta)$. Having such an A we set $S = A \circ T^{-1}$ on Ω . On the other hand, using that T is analytic in the whole ball $\{|z| \leq \eta\}$, we can choose A as follows: $A(z) = \overline{T(\bar{z})}$. Moreover, Sakai [13] showed that every Schwarz function on a cusp domain appears because of an analytic function T as above.

The converse of this theorem also holds, in the sense that if any of the conditions (1), (2a), (2b), or (2c) is satisfied, then Ω admits a Schwarz function.

In order to distinguish between the cases, Sakai also showed an auxiliary result [13, Proposition 5.1], which we will also use here.

Theorem 1.1.4. *Set $D' = D(0, r)$. Let $\Omega' \subset D'$ be an open set and 0 an accumulation point of its boundary, $\Gamma' = \partial\Omega' \cap D'$. Then, for some $r' \leq r$, either*

(1) there exists a Schwarz function, S_t , of $(\Omega' \cup \Gamma') \cap D(0, r')$ at 0 if and only if there exists a function Φ_1 defined on $(\Omega' \cup \Gamma') \cap D(0, \delta)$ for some $\delta > 0$ such that

(i) Φ_1 is holomorphic and univalent in $\Omega' \cap D(0, \delta)$,

(ii) Φ_1 is continuous on $(\Omega' \cup \Gamma') \cap D(0, \delta)$,

(iii) $\Phi_1(\zeta) = |\zeta|^2$ on $\Gamma' \cap D(0, \delta)$

or

(2) there exists a Schwarz function, S_t , of $(\Omega' \cup \Gamma') \cap D(0, r')$ at 0 if and only if there exists a function Φ_2 defined on $(\Omega' \cup \Gamma') \cap D(0, \delta)$ for some $\delta > 0$ such that

(i') Φ_2 is holomorphic and univalent in $\Omega' \cap D(0, \delta)$,

(ii') Φ_2^2 is continuous on $(\Omega' \cup \Gamma') \cap D(0, \delta)$,

(iii') $\Phi_2^2(\zeta) = |\zeta|^2$ on $\Gamma' \cap D(0, \delta)$,

(iv') $\Phi_2(\Omega' \cap D(0, \delta)) \cup (-\epsilon, \epsilon)$ contains a neighbourhood of 0 for $\epsilon > 0$.

In particular, the functions Φ_1, Φ_2 are related to S_t by $\Phi_1(z) = zS_t(z)$ and $\Phi_2(z) = \sqrt{zS_t(z)}$.

Unfortunately, Theorem 1.1.4 is only valid around 0 in this form. Nevertheless, we can “translate” the setup of Theorem 1.1.2 by setting $\Omega' = \Omega - \zeta_0$, $\Gamma' = \Gamma - \zeta_0$ and $S_t(z) = S(z + \zeta_0) - \bar{\zeta}_0$ for $z \in \Omega'$. Then, S_t is a Schwarz function on $\Omega' \cup \Gamma'$ at 0. Cases (1) of the two theorems correspond with one another as do (2a), (2b), and (2c) with (2).

Sakai gave two applications of his results: the first one describes the local structure of the boundary of quadrature domains, while the second one deals with a free boundary problem of classical type, namely, what is the boundary of the set of positivity of a smooth *non-negative* function in the disk such that $\Delta u = 1$ on the set $\{u > 0\}$.

It is natural to wonder how one can derive similar results for other forms of (1.1.1). In this text, we examine three different scenarios for a simply connected domain Ω . In Sections 1.2 to 1.4 equation (1.1.1) is replaced by

$$f_1(\zeta) = \bar{\zeta} f_2(\zeta) \quad \text{for all } \zeta \in \partial\Omega \quad (1.1.2)$$

where f_1, f_2 are holomorphic functions continuous up to the boundary. This is closely related to the model subspaces K_θ and Nevanlinna domains, which will be important here. It is shown that there are domains so that (1.1.2) holds for which $\partial\Omega$ is C^∞ but not real analytic. Further, in Section 1.5 we replace the quantity $\bar{\zeta} f_2(\zeta)$ with $\Phi(\zeta, \bar{\zeta})$, where Φ is a holomorphic function of two variables, to find that the boundary is locally composed of real analytic arcs. Finally, in Section 1.6 we consider two positive harmonic functions $\mathcal{U}$ and $\mathcal{V}$ that are zero on a Jordan arc, Γ , of the boundary. If their ratio on Γ is equal to a real analytic function of the form $|A|^2$, where A is holomorphic, then Γ is real analytic itself with the possible exception of some cusps.

Our interests to the problems considered below also was spurred by an application, which originates from complex dynamics. A certain complex dynamics question naturally brought the second author to another *free boundary problem* described in Section 1.6. After that it was very natural to ask related questions, where the Sakai setup was generalized in yet two other ways. To our surprise the answers were quite different and required different techniques: from the use of Nevanlinna domains and pseudo-continuation to multivalued analytic functions.

1.2 Polynomials & analytic functions

Let Ω be an open domain, ζ_0 a non-isolated boundary point of Ω , and let $\Gamma = \partial\Omega \cap D(\zeta_0, r)$ for some $r > 0$. Suppose S is a holomorphic function on Ω continuous on $\Omega \cup \Gamma$. We start with a simple yet important case. Instead of (1.1.1), we consider

$$S(\zeta) = \bar{\zeta} p(\zeta) \quad \text{on } \Gamma, \quad (1.2.1)$$

where p is a polynomial. We will shortly show that $f(z) = \frac{S(z)}{p(z)}$ is, in fact, a Schwarz function on Γ .

Lemma 1.2.1. Assume that $S : \Omega \rightarrow \mathbb{C}$ is holomorphic on $\Omega \subset D(\zeta_0, r)$, continuous on $\Omega \cup \Gamma$, and that it satisfies

$$S(\zeta) = \bar{\zeta}(\zeta - \zeta_0)^n \quad \text{on } \Gamma.$$

Then, the function $S_t(z) = S(z + \zeta_0) - \bar{\zeta}_0 z^n$ is holomorphic on $\Omega - \zeta_0 \subset D(0, r)$, continuous on $(\Omega - \zeta_0) \cup (\Gamma - \zeta_0)$ and it satisfies

$$S_t(\zeta) = \bar{\zeta} \zeta^n \quad \text{on } \Gamma - \zeta_0.$$

Proposition 1.2.2. Assume $0 \in \Gamma$ is a non-isolated boundary point of $\Omega \subset D(0, r)$ and suppose S is a holomorphic function on Ω continuous on $\Omega \cup \Gamma$ and satisfying

$$S(\zeta) = \bar{\zeta} \zeta^n \quad \text{on } \Gamma.$$

Then, for any positive $\delta < r$ the function $\frac{S(z)}{z^n}$ is holomorphic on $\Omega \cap D(0, \delta)$ and continuous on $(\Omega \cup \Gamma) \cap D(0, \delta) \setminus \{0\}$. Moreover, the following holds while $z \in \Omega \cup \Gamma \setminus \{0\}$:

$$\lim_{z \rightarrow 0} \frac{S(z)}{z^n} = 0.$$

Proof. The function $\frac{S(z)}{z^n}$ is clearly holomorphic on $\Omega \cap D(0, \delta)$ and continuous on $(\Omega \cup \Gamma) \cap D(0, \delta) \setminus \{0\}$ for any $\delta \in (0, r)$. It remains to see what happens at 0.

Fix $\delta \in (0, r)$. Since S is bounded on $\Omega \cap D(0, r)$, say by m , we get

$$\left| \frac{S(z)}{z^n} \right| \leq m |z|^{-n} \quad \text{on } \Omega \cap D(0, \delta)$$

and additionally for any $\zeta \in \Gamma \cap D(0, \delta) \setminus \{0\}$ we have

$$\lim \left| \frac{S(z)}{z^n} \right| = |\bar{\zeta}| \leq \delta \quad \text{while } \Omega \ni z \rightarrow \zeta.$$

Hence, by the Phragmén-Lindelöf principle 1.1.1 we obtain

$$\limsup \left| \frac{S(z)}{z^n} \right| \leq \delta \quad \text{while } \Omega \ni z \rightarrow 0.$$

This last inequality is true for any positive $\delta < r$ and therefore $\lim_{z \rightarrow 0} \frac{S(z)}{z^n} = 0$ as $z \rightarrow 0$. □

Corollary 1.2.3. *Let p be a complex polynomial. Assume that $\zeta_0 \in \Gamma$ is a non-isolated boundary point of Ω other than zero and suppose S is a holomorphic function of $\Omega \subset D(\zeta_0, r)$ continuous on $\Omega \cup \Gamma$ and satisfying*

$$S(\zeta) = \bar{\zeta} p(\zeta) \quad \text{on } \Gamma.$$

Set $f(z) = S(z)/p(z)$ on $\Omega \cup \Gamma \setminus \{\zeta_0\}$ and $f(\zeta_0) = \bar{\zeta}_0$. Then, f is a Schwarz function of $\Omega \cup \Gamma$ on $D(\zeta_0, r)$ for sufficiently small $r > 0$.

Proof. Take r so small that p has no zeros on $\overline{D(\zeta_0, r)} \setminus \{\zeta_0\}$. If $p(\zeta_0) \neq 0$, the result is immediate.

If $p(\zeta_0) = 0$, we only need to show that f is continuous on $(\Omega \cup \Gamma) \cap D(\zeta_0, r)$. Denote by n the order of ζ_0 as a zero of p and consider the function

$$S_n(z) = S(z) \frac{(z - \zeta_0)^n}{p(z)}.$$

S_n is holomorphic on Ω , is continuous on $\Omega \cup \Gamma$, and satisfies

$$S_n(\zeta) = \bar{\zeta}(\zeta - \zeta_0)^n \quad \text{on } \Gamma.$$

From Lemma 1.2.1 we get

$$(S_n)_t(\zeta) = \bar{\zeta} \zeta^n \quad \text{on } \Gamma - \zeta_0$$

and from Proposition 1.2.2 we deduce that while

$$\begin{aligned} \lim_{z \rightarrow 0} \frac{(S_n)_t(z)}{z^n} = 0 &\implies \lim_{z \rightarrow 0} \frac{S_n(z + \zeta_0) - \bar{\zeta}_0 z^n}{z^n} = 0 \\ &\implies \lim_{z \rightarrow \zeta_0} \frac{S_n(z) - \bar{\zeta}_0 (z - \zeta_0)^n}{(z - \zeta_0)^n} = 0 \\ &\implies \lim_{z \rightarrow \zeta_0} f(z) = \lim_{z \rightarrow \zeta_0} \frac{S_n(z)}{(z - \zeta_0)^n} = \bar{\zeta}_0 \end{aligned}$$

$z \in \Omega$, and the conclusion follows. □

Notice that the same proof works with p replaced by any function F that is analytic in a neighbourhood of ζ_0 . This along with Lemma 1.2.1 give us the following corollary.

Corollary 1.2.4. Assume $\zeta_0 \in \Gamma$ is a non-isolated boundary point of $\Omega \subset D(\zeta_0, r)$. Suppose F is a function analytic around ζ_0 and S is a holomorphic function on Ω continuous on $\Omega \cup \Gamma$ and satisfying

$$S(\zeta) = \bar{\zeta}F(\zeta) \quad \text{on } \Gamma.$$

Set $f(z) = S(z)/F(z)$ on $(\Omega \cup \Gamma) \cap D(\zeta_0, \delta) \setminus \{\zeta_0\}$ for some sufficiently small $\delta > 0$ and $f(\zeta_0) = \bar{\zeta}_0$. Then, f is a Schwarz function of $\Omega \cup \Gamma$ on $D(\zeta_0, \delta)$.

The converse of this corollary also holds true in the sense that if Γ has certain shape, in particular, if it satisfies (1), (2a), (2b), or (2c) of 1.1.2, then there is a Schwarz function f of $\Omega \cup \Gamma$ at ζ_0 such that $S(\zeta) = \bar{\zeta}F(\zeta)$ on Γ where $S = Ff$.

In fact, we can slightly modify the same proof to get a little more, again through the Phragmén-Lindelöf principle 1.1.1.

Corollary 1.2.5. Let p be a polynomial, F a function analytic in a neighbourhood of $\bar{\Omega}$, and S a function holomorphic on the (bounded) set Ω and continuous on $\Omega \cup \Gamma$. Suppose that for all $\zeta \in \Gamma$ we have

$$S(\zeta) = p(\bar{\zeta})F(\zeta).$$

Then, for every non-isolated point ζ_0 of the boundary Γ for which $p'(\zeta_0) \neq 0$, there is some $\delta > 0$ such that the function $p^{-1}(S/F)$ is a Schwarz function of $\Omega \cup \Gamma$ on $D(\zeta_0, \delta)$.

We wish to examine what happens in the more general case where p in (1.2.1) is replaced with any analytic function of Ω continuous on its boundary, but not necessarily analytic on that boundary. More specifically, suppose that f_1 and f_2 are functions analytic on Ω , continuous on $\Omega \cup \Gamma$, and satisfying

$$f_1(\zeta) = \bar{\zeta}f_2(\zeta) \quad \text{on } \Gamma. \tag{1.2.2}$$

As above, if $f_2(\zeta_0) \neq 0$, the function $f = f_1/f_2$ is a Schwarz function around $\zeta_0 \in \Gamma$ and no issues arise. However, if $f_2(\zeta_0) = 0$, the situation is very complicated in general.

We start with a lemma analogous to Lemma 1.2.1:

Lemma 1.2.6. Assume that $f_1, f_2 : \Omega \rightarrow \mathbb{C}$ are holomorphic on $\Omega \subset D(\zeta_0, r)$, continuous on $\Omega \cup \Gamma$, and that they satisfy

$$f_1(\zeta) = \bar{\zeta} f_2(\zeta) \quad \text{on } \Gamma.$$

Then, there exist functions $(f_1)_t$ and $(f_2)_t$ holomorphic on $\Omega - \zeta_0$, continuous on $(\Omega - \zeta_0) \cup (\Gamma - \zeta_0)$ and such that

$$(f_1)_t(\zeta) = \bar{\zeta} (f_2)_t(\zeta) \quad \text{on } \Gamma - \zeta_0.$$

If additionally $f_2(\zeta_0) = 0$, then $(f_2)_t(0) = 0$.

Proof. Define $(f_1)_t$ by

$$(f_1)_t(z) = f_1(z + \zeta_0) - \bar{\zeta}_0 f_2(z + \zeta_0).$$

Then for $\zeta \in \Gamma - \zeta_0$ we have

$$\begin{aligned} (f_1)_t(\zeta) &= f_1(\zeta + \zeta_0) - \bar{\zeta}_0 f_2(\zeta + \zeta_0) \\ &= \overline{\zeta + \zeta_0} f_2(\zeta + \zeta_0) - \bar{\zeta}_0 f_2(\zeta + \zeta_0) \\ &= \bar{\zeta} f_2(\zeta + \zeta_0) \end{aligned}$$

Setting $(f_2)_t(z) = f_2(z + \zeta_0)$, we have the desired identity.

Clearly, $(f_1)_t(0) = 0$ and also if $f_2(\zeta_0) = 0$, $(f_2)_t(0) = 0$. □

Abusing the notation, we denote these new functions again by f_1 and f_2 .

It remains to show a result analogous to Corollary 1.2.3 with p replaced by f_2 . In particular, we would like to show that the function $f = f_1/f_2$ is holomorphic on Ω , continuous on Γ , and that it satisfies

$$f(\zeta) = \frac{f_1(\zeta)}{f_2(\zeta)} = \bar{\zeta} \quad \text{for all } \zeta \in \Gamma.$$

However, the limit of $f_1(z)/f_2(z)$ as $\Omega \ni z \rightarrow 0$ may even fail to exist when $f_2(0) = 0$, and we cannot apply the Phragmén-Lindelöf principle here. We will need to see this problem from a different scope.

1.3 Nevanlinna domains and inner functions

We recall that a bounded simply connected domain Ω is called a Nevanlinna domain if there exist bounded holomorphic functions f_1, f_2 in Ω such that

$$\overline{\varphi(z)} = \frac{f_1(\varphi(z))}{f_2(\varphi(z))}$$

for almost every $z \in \mathbb{T} = \{z : |z| = 1\}$, where φ is a conformal mapping of the unit disk onto Ω . Note that this definition does not imply any additional regularity (for instance, continuity) of the functions f_1, f_2 on $\partial\Omega$.

We will restrict the above situation, and suppose there are holomorphic functions $f_1, f_2 : \Omega \rightarrow \mathbb{C}$ continuous up to the boundary that satisfy

$$f_1(\zeta) = \overline{\zeta} f_2(\zeta) \quad \text{for } \zeta \in \Gamma. \quad (1.3.1)$$

In order to better understand the situation, we rewrite (1.3.1) as

$$\frac{f_1(\zeta)}{f_2(\zeta)} = \overline{\zeta}, \quad (1.3.1')$$

which is now fulfilled almost everywhere on Γ except for the closed set $\Gamma \cap f_2^{-1}\{0\}$, which has zero measure. Then, Ω is what we call a *strong Nevanlinna domain* and if such f_1 and f_2 exist, the ratio f_1/f_2 is unique thanks to the Lusin-Privalov uniqueness theorem.

Let $\phi : \mathbb{D} \rightarrow \Omega$ be a conformal map and consider the functions $F_1 = f_1 \circ \phi$ and $F_2 = f_2 \circ \phi$. Formulas (1.3.1) and (1.3.1') transform respectively to

$$F_1(\zeta) = \overline{\phi(\zeta)} F_2(\zeta) \quad (1.3.2)$$

and

$$\frac{F_1(\zeta)}{F_2(\zeta)} = \overline{\phi(\zeta)} \quad (1.3.2')$$

both of which hold true in the sense of angular boundary values almost everywhere on $\mathbb{T}$, because ϕ may fail to extend “nicely” to $\bar{\mathbb{D}}$. By the factorization theorem, we can write F_1 and F_2 in $\mathbb{D}$ as

$$F_1 = \theta_1 \mathcal{F}_1 \quad \text{and} \quad F_2 = \theta_2 \mathcal{F}_2 \quad (1.3.3)$$

where the $\mathcal{F}_i$ are the outer factors of F_i and the θ_i are their inner factors. Since $F_1, F_2 \in H^\infty$, also $\mathcal{F}_1, \mathcal{F}_2 \in H^\infty$, and from (1.3.2') we get

$$\frac{\theta_1(\zeta) \mathcal{F}_1(\zeta)}{\theta_2(\zeta) \mathcal{F}_2(\zeta)} = \overline{\phi(\zeta)}, \quad (1.3.4)$$

almost everywhere on $\mathbb{T}$ in the sense of angular boundary values. We distinguish between two cases: either θ_2 divides θ_1 , that is, $\theta_1/\theta_2 \in H^\infty$, or it does not.

1.3.1 $\theta_2 \mid \theta_1$.

Let $h = \theta_1/\theta_2 \in H^\infty$. Then, the function $(h\mathcal{F}_1)/\mathcal{F}_2$ belongs to the class N^+ , defined as

$$N^+ = \left\{ \frac{f}{g} : f, g \in H^\infty, g \text{ is an outer function} \right\},$$

and its (angular) boundary values are equal almost everywhere on $\mathbb{T}$ to the (angular) boundary values of $\bar{\phi}$. However, since Ω is bounded, we see that $\phi \in L^\infty(\mathbb{T}, m)$ where m is the normalized Lebesgue measure on $\mathbb{T}$. Smirnov's Theorem tells us that in fact $(h\mathcal{F}_1)/\mathcal{F}_2 \in H^\infty$. Therefore, we have a bounded holomorphic function on the disk that is equal to $\bar{\phi}$ almost everywhere on $\mathbb{T}$. This is impossible whenever ϕ is a bounded holomorphic function on $\mathbb{D}$.

We are necessarily left with the other case.

1.3.2 $\theta_2 \nmid \theta_1$.

We begin with some notation and definition which will be important for the rest of this text.

Let $\mathbb{D}_e = \widehat{\mathbb{C}} \setminus \bar{\mathbb{D}}$. For any function $h : \mathbb{D} \rightarrow \mathbb{C}$ we define $\tilde{h}$ as

$$\tilde{h}(z) = \overline{h(1/\bar{z})}.$$

The notation $\tilde{H}$ will stand for a function $\tilde{H} : \mathbb{D}_e \rightarrow \mathbb{C}$ and we will write H instead of $\tilde{H}$ for the function $\overline{\tilde{H}(1/\bar{z})}$. Observe that $h \in H^\infty$ if and only if $\tilde{h} \in H^\infty(\mathbb{D}_e)$, and $h(0) = 0$ if and only if $\tilde{h}(\infty) = 0$.

We will also consider the backward shift operator, $\mathcal{B} : H^p \rightarrow H^p$, for $p \in [1, \infty)$, that is

$$\mathcal{B} : f \mapsto \frac{f(z) - f(0)}{z}.$$

Definition 1.3.1. Let f be a meromorphic function on $\mathbb{D}$. We say that f admits pseudo-continuation (across $\mathbb{T}$) if there exists another meromorphic function g on $\mathbb{D}_e$ such that $f = g$ almost everywhere (on $\mathbb{T}$) in the sense of non-tangential limits.

The pseudo-continuation of f is called of bounded type or a Nevanlinna-type pseudo-continuation if g is of the form $g = h_1/h_2$ for some $h_1, h_2 \in H^\infty(\mathbb{D}_e)$.

Definition 1.3.2. A function $f \in H^p$ is called a cyclic vector for $\mathcal{B}$, or simply cyclic for $\mathcal{B}$ if the set $\{\mathcal{B}^n f\}_{n=0}^\infty$ spans the space H^p .

The following important result is due to Douglas, Shapiro, and Shields.

Theorem 1.3.3. Consider $1 \leq p < \infty$. A function $f \in H^p$ is not cyclic for $\mathcal{B}$ if and only if f has a pseudo-continuation of bounded type.

In the case when $p = 2$, it is known that any non-cyclic function of $\mathcal{B}$ belongs to a proper $\mathcal{B}$ -invariant subspace. As a consequence of Beurling's theorem, these spaces are of the form $(\theta H^2)^\perp$ and are known as *model spaces* and denoted by K_θ . Here we will need the fact that

$$K_\theta = (\theta H^2)^\perp = H^2(\mathbb{T}) \cap \overline{\theta H_0^2(\mathbb{T})},$$

where in the last identity we mean the boundary values of the corresponding functions and where $H_0^2 = \{f \in H^2 : f(0) = 0\}$.

Now, we can proceed with the case when $\theta_2 \nmid \theta_1$:

After dividing both θ_1 and θ_2 by their greatest common divisor, we may assume that θ_1 and θ_2 have no common zeros and that the Borel supports of their singular measures are disjoint. Much as above, we see that the function $F = (\theta_1 \mathcal{F}_1)/\mathcal{F}_2 = F_1/\mathcal{F}_2$ belongs the class N^+ and thus $F \in H^\infty$, because $\theta_2 \overline{\phi} \in L^\infty(\mathbb{T}, m)$. Then the following is true in the sense of angular boundary values for

almost every $\zeta \in \mathbb{T}$:

$$\begin{aligned} \overline{\phi(\zeta)} &= \frac{\theta_1(\zeta)\mathcal{F}_1(\zeta)}{\theta_2(\zeta)\mathcal{F}_2(\zeta)} = \frac{F(\zeta)}{\theta_2(\zeta)} \\ \iff \phi(\zeta) &= \theta_2(\zeta)\overline{F(\zeta)} \end{aligned} \tag{1.3.5}$$

$$\iff \phi(\zeta) = \frac{\tilde{F}(\zeta)}{\theta_2(\zeta)}. \tag{1.3.6}$$

Since $\tilde{F}, \tilde{\theta}_2 \in H^\infty(\mathbb{D}_e)$, we see that $\phi \in H^\infty \subset H^2$ admits pseudo-continuation across $\mathbb{T}$ of bounded type, and Theorem 1.3.3 shows that ϕ is not cyclic for $\mathcal{B}$. So, it has to belong to some model space K_θ . See [7, Theorem 1] for more details. In fact, from (1.3.5) and because we “need” to have $F(0) = 0$, it follows that either

$$\phi \in K_{\theta_2} \text{ if } \theta_1(0) = 0, \quad \text{or} \quad \phi \in K_{z\theta_2} \text{ if } \theta_1(0) \neq 0.$$

1.4 Boundary behaviour of conformal maps in K_θ

In this section we show that Theorem 1.1.2 fails when condition (iii) is replaced by (1.3.1). To this end, we will find a simply connected domain Ω and a conformal map $\phi : \mathbb{D} \rightarrow \Omega$ continuous up to the boundary that has a pseudo-continuation of bounded type and is smooth but not real analytic on $\mathbb{T}$. The functions participating in this pseudo-continuation will also be continuous on the boundary. First, we go one step back and work with Nevanlinna domains. Thanks to [7, Theorem 1] by Fedorovskiy, this is equivalent to studying the model subspaces, K_θ , for different inner functions θ .

If $\theta(z_0) = 0$ for some $z_0 \in \mathbb{D}$, the function

$$\phi(z) = \frac{1}{1 - \overline{z_0}z} \in K_\theta \cap C^\infty(\mathbb{T})$$

has bounded type pseudo-continuation across $\mathbb{T}$ and thus $\phi(\mathbb{D})$ is a Nevanlinna domain. In fact, ϕ can be analytically extended on the whole closed disk, $\bar{\mathbb{D}}$, and $\phi(\mathbb{T})$ is real analytic. On the other hand, in a series of papers, [10, 4, 7, 11, 12, 3], it has been shown that the boundary of a Nevanlinna domain can be “arbitrarily bad”. In particular, it can be nowhere analytic [10], of class C^1 but not in any $C^{1,\alpha}$ for no $\alpha > 0$ [7], or even non-rectifiable [11]. We refer also to the Belov-Fedorovskiy paper [2], where the description is given of model spaces that contain bounded univalent functions.

We mention that the Hausdorff dimension of the accessible boundary of a Nevanlinna domain can be any number between 1 and 2 as shown in [3], another construction can be found in [12].

However, in all the above work the inner function θ is a Blaschke product or has a Blaschke part. Moreover, in order to compare with Sakai's theorem, we have to consider the case where the functions $\tilde{F}_1, \tilde{F}_2 \in H^\infty(\mathbb{D}_e)$ for which $\phi = \tilde{F}_1/\tilde{F}_2$ on $\mathbb{T}$ are continuous up to $\mathbb{T}$. This is not always possible when θ is not purely singular (see [4, Example 5.8]).

Therefore, in this section θ will be a singular inner function of the form

$$\theta(z) = \exp \left(- \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\mu_\theta(\zeta) \right)$$

with μ_θ supported on a Carleson set, $E \subset \mathbb{T}$. We will show that there is a conformal map $\phi \in K_\theta$ continuous on $\bar{\mathbb{D}}$ which is in $C^\infty(\mathbb{T})$ but not real analytic on $\mathbb{T}$.

In view of [6, Theorem 2.1], since $\text{supp}(\mu_\theta)$ is Carleson, the space K_θ then contains a non-trivial function from some smoothness class, for example a function $g \in H^\infty \cap C^\infty(\mathbb{T})$ (or in a Bergman space, i.e., $g \in A^{p,1}$ for some $p > 1$). Since $g \in K_\theta$, it admits a bounded type pseudo-continuation of the form

$$g = \tilde{G}/\tilde{\theta} \quad \text{almost everywhere on } \mathbb{T},$$

where $\tilde{G} \in H^\infty(\mathbb{D}_e)$ vanishes at infinity (see [5, Theorem 5.1.4]). Additionally, g has an analytic continuation, say $\mathcal{G}$, to $\hat{\mathbb{C}} \setminus \text{supp}(\mu)$. Of course, $\mathcal{G} = \tilde{G}/\tilde{\theta}$ on $\mathbb{D}_e$ and observe that $\mathcal{G}$ cannot be bounded in $\mathbb{D}_e$; otherwise g would be constant, as $\mathcal{G}|_{\mathbb{D}} = g$ and $\mathcal{G}|_{\mathbb{D}_e}$ coincide almost everywhere on $\mathbb{T}$.

Now, consider $\alpha \in \mathbb{D}_e$ with $\theta(1/\bar{\alpha}) \neq 0$ and the following aggregate:

$$\phi(z) = \frac{\mathcal{G}(z) - \mathcal{G}(\alpha)}{z - \alpha}.$$

We will show that $\phi \in K_\theta \cap C^\infty(\mathbb{T})$ and ϕ is conformal in $\bar{\mathbb{D}}$.

Clearly, ϕ is inside $H^2(\mathbb{D})$ and also

$$\overline{\theta(\zeta)}\phi(\zeta) = \frac{\overline{\theta(\zeta)}g(\zeta) - \overline{\theta(\zeta)}\mathcal{G}(\alpha)}{\zeta - \alpha} = \frac{\tilde{G}(\zeta) - \tilde{\theta}(\zeta)\mathcal{G}(\alpha)}{\zeta - \alpha}.$$

For $z \in \mathbb{D}_e$ the function

$$\frac{\tilde{G}(z) - \tilde{\theta}(z)\mathcal{G}(\alpha)}{z - \alpha} = \frac{1}{z - \alpha} \left(\tilde{G}(z) - \frac{\tilde{\theta}(z)}{\tilde{\theta}(\alpha)} \tilde{G}(\alpha) \right)$$

is analytic around α and vanishes at infinity. Hence, $\phi \in K_\theta$.

Furthermore, ϕ is univalent in $\bar{\mathbb{D}}$. Indeed, suppose it is not. Then, there exist $z, w \in \bar{\mathbb{D}}$ with $z \neq w$ and $\phi(z) = \phi(w)$ or equivalently

$$\begin{aligned} \frac{g(z) - \mathcal{G}(\alpha)}{z - \alpha} &= \frac{g(w) - \mathcal{G}(\alpha)}{w - \alpha} \\ \iff \frac{g(z)}{z - \alpha} - \frac{g(w)}{w - \alpha} &= \frac{\mathcal{G}(\alpha)}{z - \alpha} - \frac{\mathcal{G}(\alpha)}{w - \alpha} = \mathcal{G}(\alpha) \frac{z - w}{(z - \alpha)(w - \alpha)} \\ \iff -\alpha \frac{g(z) - g(w)}{z - w} + w \frac{g(z) - g(w)}{z - w} - g(w) &= \mathcal{G}(\alpha). \end{aligned}$$

The left-hand side is bounded, because $g \in C^\infty(\bar{\mathbb{D}})$, whereas we can pick $1 < \alpha < 2$ so that $|\mathcal{G}(\alpha)|$ is arbitrarily large (recall $\mathcal{G}|_{\mathbb{D}_e}$ is not bounded), a contradiction, and therefore ϕ is univalent in $\bar{\mathbb{D}}$.

Consequently, if $g \in K_\theta \cap C^\infty(\mathbb{T})$ and θ is a singular inner function, then ϕ is univalent in $\bar{\mathbb{D}}$ and $\phi \in K_\theta \cap C^\infty(\mathbb{T})$. Also see [1, Section 4] for more details. At the same time, note that $\mathcal{G}$ cannot be analytically extended to the whole $\bar{\mathbb{D}}$, because it is unbounded near the unit circle, and thus neither can ϕ ; this fails exactly on the Carleson set E .

Now, since $\phi \in K_\theta$, we can write

$$\phi = \theta \bar{F} \iff \theta \bar{\phi} = F \tag{1.4.1}$$

almost everywhere on $\mathbb{T}$ for some function $F \in H^2$ with $F(0) = 0$. In fact, $F \in H^\infty$ because $\phi \in C^\infty(\bar{\mathbb{D}})$.

It is known that there exists some analytic function, $\mathcal{H}$, with $\mathcal{H}|_E = 0$ such that both $\mathcal{H}$ and $\mathcal{H}\theta$ are Lipschitz on $\bar{\mathbb{D}}$. In fact, we can further consider $\mathcal{H}$ to be an outer function in $C^\infty(\mathbb{T})$. Multiplying by $\mathcal{H}$ in (1.4.1), we get

$$(\mathcal{H}\theta)\bar{\phi} = \mathcal{H}F \tag{1.4.2}$$

almost everywhere on $\mathbb{T}$. In particular, the left-hand side is now smooth on the whole $\mathbb{T}$ and the same therefore holds true for the right-hand side. In a sense, $\mathcal{H}$ “annihilates” the singularities of θ as (1.4.1) fails exactly on the support, E , of μ_θ .

At this point, set $F_1 = \mathcal{H}F$, $F_2 = \mathcal{H}\theta$, and $f_j = F_j \circ \phi^{-1}$ for $j = 1, 2$. Then, (1.4.2) becomes

$$F_1 = \overline{\phi} F_2,$$

which now is fulfilled on the whole boundary $\mathbb{T}$, and in turn

$$f_1(\zeta) = \overline{\zeta} f_2(\zeta) \quad \text{for all } \zeta \in \Gamma.$$

This is exactly the setup we were looking for, albeit it contrasts with Sakai's result: Even though $\Gamma = \phi(\mathbb{T})$ is C^∞ -smooth, ϕ cannot be analytic on the Carleson set E and thus neither can Γ .

It is worth mentioning that there are examples of Nevanlinna domains that come from singular inner functions with particularly irregular boundaries. Namely, in [3] one can find examples of univalent functions in a Paley-Wiener space such that they map the upper half-plane onto a Nevanlinna domain whose boundary can have any dimension between 1 and 2.

1.5 Holomorphic functions in $\mathbb{C}^2$

In this section we attempt to replace the function $\overline{\zeta} f_0(\zeta)$ in (1.3.1) with a more general formula.

For some positive $r > 0$, let $\Omega \subset D(\zeta_0, r)$ be a simply connected open set, let $\Gamma = \partial\Omega \cap D(\zeta_0, r)$, and let $\zeta_0 \in \Gamma$. Here, we will also need the extra assumption that Γ is a Jordan arc (or possibly a union of Jordan arcs).

Let Φ be a holomorphic function of two variables, that is, a function of the form

$$\Phi(z, w) = \sum_{n,m=0}^{+\infty} b_{nm} z^n w^m$$

where each of the functions $\Phi(z, \cdot)$ and $\Phi(\cdot, w)$ is itself holomorphic. Suppose there exists a function R which is

- (i) holomorphic on Ω ,
- (ii) continuous on $\overline{\Omega}$, and
- (iii) satisfies $R(\zeta) = \Phi(\zeta, \overline{\zeta})$ on Γ .

In view of Lemma 1.2.6, we may assume that $\zeta_0 = 0$ and $b_{00} = 0$ so that $R(0) = \Phi(0, 0) = 0$. Notice that $R(z)$ and $\Phi(z, \bar{z})$ are bounded on $\bar{\Omega}$ and thanks to the Phragmén-Lindelöf Principle 1.1.1, we may assume without loss of generality that there exists some non-negative integer k for which

$$\Phi(0, 0) = \frac{\partial}{\partial w} \Phi(0, 0) = \cdots = \frac{\partial^{k-1}}{\partial w^{k-1}} \Phi(0, 0) = 0 \quad \text{and} \quad \frac{\partial^k}{\partial w^k} \Phi(0, 0) \neq 0 \quad (1.5.1)$$

otherwise Φ would be identically zero.

We would like to use the Weierstraß approximation theorem for the function $\Phi(z, w) - R(z)$ around 0, but R is not holomorphic on the boundary. But since it is continuous by (ii) and Γ is Jordan, we can use Mergelyan's theorem to get a sequence of polynomials p_n that converge to R uniformly on $\bar{\Omega}$. And we can pick this sequence so that $p_n(0) = 0$ for every $n = 0, 1, \dots$.

Next, we define the functions

$$\Psi(z, w) = \Phi(z, w) - R(z) \quad \text{and} \quad \Psi_n(z, w) = \Phi(z, w) - p_n(z).$$

The Ψ_n are holomorphic on $\mathbb{C}^2$ and converge uniformly to Ψ on $\bar{\Omega} \times \mathbb{C}$. Observe that for all n we have $\Psi_n(0, 0) = \Phi(0, 0) - p_n(0) = 0$ and also

$$\frac{\partial^\kappa}{\partial w^\kappa} \Psi_n = \frac{\partial^\kappa}{\partial w^\kappa} \Phi \quad \text{for all integers } \kappa \geq 1$$

and all points (z, w) . Then, from (1.5.1) and from the Weierstraß approximation theorem, there exist unique holomorphic functions $a_{0;n}, \dots, a_{k-1;n} : \mathbb{C} \rightarrow \mathbb{C}$ and $c_n : \mathbb{C}^2 \rightarrow \mathbb{C}$ with $a_{j;n}(0) = 0$ and $c_n(0, 0) \neq 0$ such that

$$\Psi_n(z, w) = c_n(z, w) \left(w^k + a_{k-1;n}(z)w^{k-1} + \cdots + a_{0;n}(z) \right).$$

Following the proof of the Weierstraß theorem and since the convergence $\Psi_n \rightarrow \Psi$ is uniform on $\bar{\Omega} \times \mathbb{C}$, we can find sufficiently small δ and ρ with $\rho \geq \delta > 0$ so that $a_{0;n}, \dots, a_{k-1;n}$ and the c_n converge uniformly on $\bar{\Omega} \cap D(0, \delta)$ and $(\bar{\Omega} \cap D(0, \delta)) \times D(0, \rho)$, respectively, to some functions $a_0, \dots, a_{k-1}$ and c with $a_j(0) = 0$ and $c(0, 0) \neq 0$. Note that the functions a_j are holomorphic on $\Omega \cap D(0, \delta)$ and continuous on $\bar{\Omega} \cap D(0, \delta)$. Subsequently, we get

$$\Phi(z, w) - R(z) = c(z, w) \left(w^k + a_{k-1}(z)w^{k-1} + \cdots + a_0(z) \right). \quad (1.5.2)$$

Let us write

$$P(z, w) = w^k + a_{k-1}(z)w^{k-1} + \cdots + a_0(z)$$

for the polynomial factor. From (iii), (1.5.2) and since $c(0, 0) \neq 0$, we have

$$P(\zeta, \bar{\zeta}) = \bar{\zeta}^k + a_{k-1}(\zeta)\bar{\zeta}^{k-1} + \cdots + a_0(\zeta) = 0 \quad \text{for all } \zeta \in \Gamma \cap D(0, \delta). \quad (1.5.3)$$

Remark. *Functions of the form*

$$P(z, \bar{z}) = \bar{z}^k + a_{k-1}(z)\bar{z}^{k-1} + \cdots + a_0(z),$$

where a_j are polynomials, are called *polyanalytic polynomials*. One can find more details on these in [8, 11] or [14].

We are interested in the roots of the polynomial $P(z, \cdot)$ when $z \in \bar{\Omega} \cap D(0, \delta)$. In other words, we will study the equation (in w)

$$P(z, w) = 0 \iff w^k + a_{k-1}(z)w^{k-1} + \cdots + a_0(z) = 0 \quad (1.5.4)$$

when $z \in \bar{\Omega} \cap D(0, \delta)$.

Let $\mathcal{D}(z)$ be the discriminant of $P(z, \cdot)$ (for any fixed z). Then, $\mathcal{D}(z)$ is a polynomial of the coefficients $a_0(z), \dots, a_{k-1}(z)$ and is equal to 0 if, and only if, $P(z, w)$ and $\frac{\partial}{\partial w}P(z, w)$ share a common factor. The roots of $P(z, \cdot)$ are given by a multivalued holomorphic function, $\mathcal{W}$, depending on $a_1, \dots, a_{k-1}$, and the points where $\mathcal{W}$ changes a branch inside $\Omega \cap D(0, \delta)$ are exactly the zeros of $\mathcal{D}$ (in $\Omega \cap D(0, \delta)$).

We distinguish between two cases: when $\mathcal{D}$ is identically 0 and when it is not.

Before moving on, let us note that the set $\mathcal{M}(\Omega, \Gamma, \delta)$ of all meromorphic functions on $\Omega \cap D(0, \delta)$ continuous up to $(\Omega \cup \Gamma) \cap D(0, \delta)$ except possibly a (closed) measure zero subset of Γ is a field with the usual operations of addition and multiplication.

1.5.1 $\mathcal{D} \neq 0$

Here $P(z, w)$ is irreducible over $\mathcal{M}(\Omega, \Gamma, \delta)$. Since $\mathcal{D}$ is continuous on $(\Omega \cup \Gamma) \cap D(0, \delta)$, the set $(\mathcal{D}^{-1}\{0\} \cap \Gamma) \cap D(0, \delta)$ is closed and of zero harmonic measure. Now, we decompose $(\Gamma \setminus \mathcal{D}^{-1}\{0\}) \cap D(0, \delta)$ into countably many open connected arcs.

Let γ be one of these arcs. Then, there exists a simply connected set $D \subset \Omega \cap D(0, \delta)$ such that $\partial D \cap \partial \Omega = \gamma$. Since $\mathcal{D}$ has no zeros on $D \cup \gamma$, by the monodromy theorem the multivalued function $\mathcal{W}$ “splits” into k distinct holomorphic functions, W_j ($j = 1, \dots, k$), and let $C_j = \{\zeta \in \gamma : W_j(\zeta) = \bar{\zeta}\}$. Notice that the C_j ’s are closed (in γ), they cover γ , and any two of them intersect at a (closed) set of zero harmonic measure.

Unfortunately, C_j need not be connected, but we can further decompose each $\mathring{C}_j$ (whenever it is non-empty) into countably many open arcs as in $\mathring{C}_j = \cup_i \gamma_j^i$, for $j = 1, \dots, k$. Again, around each γ_j^i we consider a neighbourhood $D_j^i \subset D$ with $\partial D_j^i \cap \partial D = \gamma_j^i$ (these can, but need not be simply connected) and let $W_j^i = W_j|_{D_j^i \cup \gamma_j^i}$.

Then, for each $j = 1, \dots, k$ and $i = 1, 2, \dots$ the functions W_j^i are holomorphic on D_j^i , continuous on $D_j^i \cup \gamma_j^i$ and satisfy $W_j^i(\zeta) = \bar{\zeta}$ for all $\zeta \in \gamma_j^i$; in other words, they are Schwarz functions on $D_j^i \cup \gamma_j^i$. Since Γ is Jordan, all γ_j^i are also Jordan and from Theorem 1.1.2 we conclude that each γ_j^i is, in fact, a regular real analytic simple arc except possibly some cusps.

1.5.2 $\mathcal{D} = 0$

In this case, $P(z, w)$ has to be reducible over $\mathcal{M}(\Omega, \Gamma, \delta)$. In particular, we can write $P(z, w) = P_1(z, w) \cdots P_{\tilde{k}}(z, w)$ for some $\tilde{k} \leq k$ where each $P_\kappa(z, w)$ has now coefficients in $\mathcal{M}(\Omega, \Gamma, \delta)$ and is irreducible, i.e., $\mathcal{D}_\kappa \neq 0$ where $\mathcal{D}_\kappa$ is the discriminant of $P_\kappa(z, \cdot)$. Since $P(\zeta, \bar{\zeta}) = 0$ for all $\zeta \in \Gamma$, we can split $(\Gamma \setminus E) \cap D(0, \delta)$, where E is some closed zero-(harmonic)-measure set, into open sets O_κ for $\kappa = 1, \dots, \tilde{k}$ so that $P_\kappa(\zeta, \bar{\zeta}) = 0$ for all $\zeta \in O_\kappa$. Notice that $O_\kappa \cap O_{\kappa'} = \emptyset$ when P_κ and $P_{\kappa'}$ are different.

Observe that, since $P(z, w)$ factors into the polynomials $P_\kappa(z, w)$ (over $\mathcal{M}(\Omega, \Gamma, \delta)$) and the roots of $P(z, \cdot)$ are given by the multivalued holomorphic function $\mathcal{W}$, the roots of each $P_\kappa(z, \cdot)$ are also given by a multivalued holomorphic function $\mathcal{W}_\kappa$ whose branches are comprised of branches of $\mathcal{W}$.

Working as above for each $\kappa = 1, \dots, \tilde{k}$, we separate $O_\kappa \setminus \mathcal{D}_\kappa^{-1}\{0\}$ into countably many open arcs and for each such arc, γ , we find some simply connected neighbourhood, $D \subset \Omega$, with $\partial D \cap \partial \Omega = \gamma$ so that $\mathcal{W}_\kappa$ “splits” into its different branches. Again following the above arguments,

we can decompose γ — minus a zero-measure set — into countably many open arcs over which $W_j(\zeta) = \bar{\zeta}$ for some branch W_j of $\mathcal{W}_\kappa$. Constructing appropriate neighbourhoods, we conclude that except a zero-measure set, γ is a countable union of regular real analytic simple arcs except possibly some cusps.

In either case, the cusps (if they exist) point into Ω and may only accumulate on the endpoints of each open arc.

Now we formulate the above results into a theorem.

Theorem 1.5.1. *Let Ω be a bounded simply connected domain such that $\Gamma = \partial\Omega \cap D(\zeta_0, r)$ is a (union of) Jordan arc(s). Also, let Φ be a (non-trivial) holomorphic function of two variables defined in $D(\zeta_0, r) \times D(\bar{\zeta}_0, r)$, and suppose there exists a function R*

- (i) *holomorphic on Ω ,*
- (ii) *continuous on $\bar{\Omega}$, and such that*
- (iii) *$R(\zeta) = \Phi(\zeta, \bar{\zeta})$ for all $\zeta \in \Gamma$.*

Then, there exists a closed set, $E \subset \Gamma$, of zero harmonic measure so that $\Gamma \setminus E$ is a countable union of regular real analytic simple arcs except possibly for some cusps. The cusps (if they exist) point into Ω and may only accumulate on E .

1.6 The $\mathcal{U}$ - $\mathcal{V}$ problem

In this section, we are interested in the following setup.

Let Ω be a simply connected open set in $\mathbb{C}$ and let $\zeta_0 \in \partial\Omega$ be a boundary point of Ω . Assume that for some $\rho > 0$ the connected component, Γ , of $\partial\Omega \cap D(\zeta_0, \rho)$ containing ζ_0 is a Jordan curve. Note that $\rho \geq \text{dist}(\zeta_0, \partial\Omega \setminus \Gamma) > 0$. For convenience we will write simply Ω to denote $\Omega \cap D(\zeta_0, \rho)$.

Let A be an analytic function in a neighbourhood, $D(\zeta_0, \epsilon)$, of ζ_0 and suppose we have two functions $\mathcal{U}$ and $\mathcal{V}$ defined on Ω that are not proportional and have the following properties:

- I) $\mathcal{U}$ and $\mathcal{V}$ are positive and harmonic on Ω ,

II) they are continuous on $\Omega \cup \Gamma$,

III) $\mathcal{U} = \mathcal{V} = 0$ on Γ , and

IV) $\frac{\mathcal{U}(\zeta)}{\mathcal{V}(\zeta)} = |A(\zeta)|^2 \neq \text{const}$ for $\zeta \in \Gamma$.

Notice that since $\mathcal{U} \neq c\mathcal{V}$, the function $|A|$ needs to be non-constant. Otherwise, we could have $\mathcal{U} = c\mathcal{V}$ and all our conditions work trivially for any Γ . Also, we may assume that $\rho < \epsilon$ without loss of generality (so that A is defined over the whole Ω) to avoid unnecessary technical difficulties.

Formula (IV) is to be understood in the sense of limits, i.e., the limit of $\mathcal{U}(z)/\mathcal{V}(z)$ as $\Omega \ni z \rightarrow \zeta \in \Gamma$ exists and is equal to $|A(\zeta)|^2$. In fact, this limit always exists when Ω is simply connected and Γ is Jordan (see Remark 1.6.1), so the only assumption here is the values it takes.

Consider a conformal map from the Poincaré plane to Ω , $\phi : \mathbb{H} \rightarrow \Omega$. Since Γ is connected and Jordan, Carathéodory's theorem implies that ϕ extends conformally to a function (abusing the notation) $\phi : \mathbb{H} \cup \gamma \rightarrow \Omega \cup \Gamma$ which we can pick so that $\gamma \subset \mathbb{R}$ is some bounded open interval with $\phi(\gamma) = \Gamma$ and $\phi(0) = \zeta_0$. Utilizing this ϕ , we can “transfer” the information about $\mathcal{U}$ and $\mathcal{V}$ over Ω to information over $\mathbb{H}$. Define

$$u \equiv \mathcal{U} \circ \phi, \quad v \equiv \mathcal{V} \circ \phi \quad \text{and} \quad a \equiv A \circ \phi$$

and note that a is analytic on $\mathbb{H}$ and continuous on $\mathbb{H} \cup \gamma$. As above, we have

i) u and v are positive and harmonic on $\mathbb{H}$,

ii) they are continuous on $\mathbb{H} \cup \gamma$,

iii) $u = v = 0$ on γ , and

iv) $\frac{u}{v} = |a|^2$ on γ .

Again, (iv) is to be understood in the sense of limits.

Now, harmonically extend u and v to $\mathbb{H} \cup \gamma \cup \mathbb{H}^-$ by

$$u^*(z) = \begin{cases} u(z), & z \in \mathbb{H} \\ 0, & z \in \gamma \\ -u(\bar{z}), & z \in \mathbb{H}^- \end{cases} \quad \text{and} \quad v^*(z) = \begin{cases} v(z), & z \in \mathbb{H} \\ 0, & z \in \gamma \\ -v(\bar{z}), & z \in \mathbb{H}^- \end{cases}$$

and let h be the function

$$h(z) = \begin{cases} \frac{u^*(z)}{v^*(z)}, & z \in \mathbb{H} \cup \mathbb{H}^-, \\ \frac{u_y^*(z)}{v_y^*(z)}, & z \in \gamma. \end{cases}$$

We claim that h is well defined and, in fact, real analytic on $\mathbb{H} \cup \gamma \cup \mathbb{H}^-$. Indeed, using Harnack's inequality, for any $(x, 0) \in \gamma$ there exists a constant $c > 0$ (dependent on v^*) such that

$$\begin{aligned} c \frac{y}{2-y} &\leq v^*(x, y) \leq c \frac{2-y}{y} \quad \text{for every } 0 < y < 1, \text{ or} \\ c \frac{1}{2-y} &\leq \frac{v^*(x, y)}{y} \leq c \frac{2-y}{y^2}. \end{aligned} \tag{1.6.1}$$

Recall that $v^*(x, 0) = 0$ and take limits as $y \rightarrow 0^+$. Since v^* is harmonic on $\mathbb{H} \cup \gamma \cup \mathbb{H}^-$, (1.6.1) guarantees that $v_y^* > 0$ on γ (the same holds true for u^*) and therefore the limit

$$\lim_{y \rightarrow 0} \frac{u^*(x, y)}{v^*(x, y)} = \frac{u_y^*(x, 0)}{v_y^*(x, 0)}$$

exists and is finite. Hence, h is a well-defined continuous function on $\mathbb{H} \cup \gamma \cup \mathbb{H}^-$. In fact, since u_y^* and v_y^* are real analytic and non-zero around γ , h is also real analytic on $\mathbb{H} \cup \gamma \cup \mathbb{H}^-$. What is more is that

$$h(\xi) = \frac{u_y^*(\xi)}{v_y^*(\xi)} = \lim_{\mathbb{H} \ni z \rightarrow \xi} \frac{u(z)}{v(z)} = |a(\xi)|^2 \quad \text{for any } \xi \in \gamma \tag{1.6.2}$$

because of (iv) and therefore $|a|^2$ is also real analytic on γ .

Remark 1.6.1. *The above is the reason why relation (IV) is meaningful. When we write $\frac{\mathcal{U}}{\mathcal{V}}$ on Γ , it really means the limit of $h \circ \phi^{-1}$ as we approach Γ from the inside of Ω . This limit always exist on a Jordan arc Γ when Ω is simply connected thanks to Harnack's inequality.*

It is worth mentioning the work of Jerison and Kenig who showed [9, Theorems 5.1 and 7.9] that equation (IV) makes sense whenever Ω is assumed to be a non-tangentially accessible (NTA) domain.

Next, consider $h|_\gamma$. Its power series around $0 \in \gamma$ is given by

$$h|_\gamma(x) = \sum_{n=0}^{\infty} b_n x^n$$

for some real numbers $b_0, b_1, \dots$. This readily extends to a complex analytic function, say r , on some open neighbourhood, $D(0, \epsilon')$:

$$r(z) = \sum_{n=0}^{\infty} b_n z^n,$$

where we can choose ϕ and ϵ' so that $\gamma \subset D(0, \epsilon')$. Of course, by construction and from (1.6.2) we get $r|_\gamma = h|_\gamma = |a|^2$.

At this point, we want to “shift” everything back at Ω . We set

$$V \equiv \phi(\mathbb{H} \cap D(0, \epsilon')) \subset \Omega$$

and observe that ∂V is a closed Jordan arc such that $\Gamma \subsetneq \partial V \cap D(\zeta_0, \rho)$. Define a new function

$$R \equiv r \circ (\phi^{-1})|_{V \cup \Gamma}, \tag{1.6.3}$$

which is holomorphic on V , continuous on $V \cup \Gamma$, and on Γ it satisfies $R(\zeta) = |A(\zeta)|^2$.

Now, consider the function $\Phi(z, w) = A(z)\overline{A(\overline{w})}$. Φ is holomorphic on $D(\zeta_0, \epsilon) \times D(\overline{\zeta}_0, \epsilon)$ and it satisfies $\Phi(\zeta, \overline{\zeta}) = A(\zeta)\overline{A(\overline{\zeta})} = |A(\zeta)|^2$ when $z = \overline{w} = \zeta \in \Gamma$. As a corollary to Theorem 1.5.1, the next theorem follows.

Theorem 1.6.2. *Let Ω be a bounded simply connected domain in $\mathbb{C}$ and let Γ be an open Jordan arc of its boundary with $\zeta_0 \in \Gamma$. Suppose there are two positive non-proportional harmonic functions $\mathcal{U}$ and $\mathcal{V}$ on Ω continuous on $\Omega \cup \Gamma$ and such that*

$$\mathcal{U}(\zeta) = \mathcal{V}(\zeta) = 0 \quad \text{and} \quad \frac{\mathcal{U}(\zeta)}{\mathcal{V}(\zeta)} = |A(\zeta)|^2 \quad \text{for all } \zeta \in \Gamma,$$

where A is a non-trivial analytic function on a neighbourhood of Ω .

Then, there exists some neighbourhood D of ζ_0 and a closed set $E \subset \Gamma$ of zero harmonic measure so that $(\Gamma \setminus E) \cap D$ is a countable union of regular real analytic simple arcs except possibly for some cusps. The cusps (if they exist) point into Ω and may only accumulate on $E \cap D$.

Of course, Theorems 1.5.1 and 1.6.2 are somewhat far from Sakai's result. Nevertheless, because of the special form of the function $\Phi(z, w) = A(z)\overline{A(\overline{w})}$, we can actually say more in this case.

Proposition 1.6.3. *Let Ω be a bounded simply connected domain in $\mathbb{C}$ and let Γ be an open Jordan arc of its boundary with $\zeta_0 \in \Gamma$. Suppose there are two positive non-proportional harmonic functions $\mathcal{U}$ and $\mathcal{V}$ on Ω continuous on $\Omega \cup \Gamma$ and such that*

$$\mathcal{U}(\zeta) = \mathcal{V}(\zeta) = 0 \quad \text{and} \quad \frac{\mathcal{U}(\zeta)}{\mathcal{V}(\zeta)} = |A(\zeta)|^2 \quad \text{for all } \zeta \in \Gamma$$

where A is a non-trivial analytic function on a neighbourhood of Γ .

Then, there exists a neighbourhood D of ζ_0 and a function R satisfying the following:

- (i) R is holomorphic on $\Omega \cap D$,
- (ii) R is continuous on $(\Omega \cup \Gamma) \cap D$ and
- (iii) $R(\zeta) = |A(\zeta)|^2$ for $\zeta \in \Gamma \cap D$.

Additionally, for any $\zeta_0 \in \Gamma$ with $A'(\zeta_0) \neq 0$ either

- (1) there exist a function Ψ_1 holomorphic and univalent on $\Omega \cap D$ such that Ψ_1 is continuous on $(\Omega \cup \Gamma) \cap D$, and $\Psi_1(\zeta) = |A(\zeta) - A(\zeta_0)|^2$ for $\zeta \in \Gamma \cap D$, or
- (2) there exist a function Ψ_2 holomorphic and univalent on $\Omega \cap D$ such that Ψ_2^2 is continuous on $(\Omega \cup \Gamma) \cap D$, and $\Psi_2^2(\zeta) = |A(\zeta) - A(\zeta_0)|^2$ for $\zeta \in \Gamma \cap D$.

Proof. We have already established the existence of such a function R in (1.6.3).

For the rest, $A'(\zeta_0) \neq 0$ and we may assume without loss of generality that A is conformal on a neighbourhood of $\bar{\Omega}$. Recall that V from the definition of R in (1.6.3) is such that ∂V is Jordan and $\Gamma \subsetneq \partial V \cap D(\zeta_0, \rho)$ when $\Omega \subset D(\zeta_0, \rho)$. Since A is continuous and injective on $\bar{V}$, there exists some small δ , $0 < \delta \leq \rho$, such that $\partial(A(V)) \cap D(\zeta'_0, \delta) \subset A(\Gamma)$.

Now, let $A(\zeta_0) = \zeta'_0$, $\Omega' = A(V) \cap D(\zeta'_0, \delta)$, and $\Gamma' = \partial\Omega' \cap D(\zeta'_0, \delta)$. The function

$$S(z) \equiv \frac{1}{z} R \circ A^{-1}(z) \quad (1.6.4)$$

is a Schwarz function of $\Omega' \cup \Gamma'$ in $D(\zeta'_0, \delta)$:

- (i) S is holomorphic on Ω' ,
- (ii) it is continuous on $\Omega' \cup \Gamma'$, and
- (iii) $S(\zeta) = \frac{1}{\zeta} R(A^{-1}(\zeta)) = \bar{\zeta}$ on Γ' .

Notice that from (1.6.1) the functions $a = A \circ \phi$ and A are always non-zero and thus S is a well-defined holomorphic function, because 0 cannot be a point of $\bar{\Omega}'$.

Finally, consider the function $S_t(z) = S(z + \zeta'_0) - \bar{\zeta}'_0$, which is a Schwarz function on $(\Omega' - \zeta'_0) \cup (\Gamma' - \zeta'_0)$ at 0. From Theorem 1.1.4, we know that one of the functions $\Phi_1(z) = zS_t(z)$ and $\Phi_2(z) = \sqrt{zS_t(z)}$ is univalent on $(\Omega' - \zeta'_0) \cap D(0, \delta')$ for some $\delta' \leq \delta$. Changing variables to get back to our initial domain Ω , we find that one of the following functions, Ψ_1 or Ψ_2 , has to be univalent on $\Omega \cap D'$:

$$\Psi_1(z) = (A(z) - A(\zeta_0)) \left(\frac{R(z)}{A(z)} - \overline{A(\zeta_0)} \right)$$

and

$$\Psi_2(z) = \sqrt{(A(z) - A(\zeta_0)) \left(\frac{R(z)}{A(z)} - \overline{A(\zeta_0)} \right)}$$

for $z \in \Omega \cap D'$, where $D' = A^{-1}(D(\zeta'_0, \delta'))$. The rest of the desired properties are obvious. $\square$

In the above proof, Γ' is the image of a Jordan arc under the (conformal) map A . Therefore, the existence of a Schwarz function, S , along with Theorem 1.1.2 imply that Γ' , and in turn Γ , satisfy (1) or (2c) of Theorem 1.1.2. Case (1) corresponds to (1) of Proposition 1.6.3 and (2c) to (2), that is, Γ' (respectively, Γ) has a cusp if, and only if, the function

$$\sqrt{z(S(z + \zeta'_0) - \bar{\zeta}'_0)}$$

is univalent on $(\Omega' - \zeta'_0) \cap D(0, \delta')$ (respectively, Ψ_2 on $\Omega \cap D$).

As a consequence, we have the following theorem, which is the main result of this section.

Theorem 1.6.4. *Let Ω be a bounded simply connected domain in $\mathbb{C}$ and let Γ be an open Jordan arc of its boundary with $\zeta_0 \in \Gamma$. Suppose there are two positive non-proportional harmonic functions $\mathcal{U}$ and $\mathcal{V}$ on Ω continuous on $\Omega \cup \Gamma$ and satisfying*

$$\mathcal{U}(\zeta) = \mathcal{V}(\zeta) = 0 \quad \text{and} \quad \frac{\mathcal{U}(\zeta)}{\mathcal{V}(\zeta)} = |A(\zeta)|^2 \quad \text{for all } \zeta \in \Gamma,$$

where A is a non-trivial analytic function on a neighbourhood of Γ .

Then, for all but possibly finitely many points $\zeta_0 \in \Gamma$ there exists some small neighbourhood D of ζ_0 such that the following holds:

$$\Gamma \cap D \text{ is a regular real analytic simple arc through } \zeta_0 \text{ except possibly a cusp at } \zeta_0. \quad (1.6.5)$$

The finitely many points around which (1.6.5) might fail are the points $\zeta \in \Gamma$ where $A'(\zeta) = 0$, i.e., where A might not be invertible.

There is a cusp at ζ_0 if and only if (2) of Proposition 1.6.3 holds true.

Of course, one can ask at this point whether it is possible to actually have a cusp. The answer is yes as the next example shows.

Example 1.6.5. *Let Ω be open and $\Gamma = \partial\Omega \cap D(0, \rho)$ (with $\rho \leq 1$ sufficiently small) be such that Γ has a cusp at 0 (i.e., $\zeta_0 = 0$). Then, from Remarks 1.1.3, for some $\eta > 0$, there is a holomorphic function T defined on $\{|z| \leq \eta\}$ that maps conformally the closed upper half-disk $K_\eta = \{|z| \leq \eta : \text{Im}(z) \geq 0\}$ into $\Omega \cup \Gamma$ and $\Gamma \cap D \subset T(-\eta, \eta)$ for some small neighbourhood D of 0. Also, $T(0) = 0$ with order 2. By dilating appropriately, we may assume that everything happens in the unit disk, that is, $\eta = 1$, T is defined on $\bar{\mathbb{D}}$ and is univalent on $K_1 = \{|z| \leq 1 : \text{Im}(z) \geq 0\}$, $T(K_1) \subset \Omega \cup \Gamma$, and $\Gamma \cap D(0, \rho) \subset T(-1, 1)$.*

Next, consider two positive harmonic functions, u and v , on the upper half-disk $\mathbb{D} \cup \mathbb{H}$ that are zero on $(-1, 1)$. As we saw in the beginning of this section, u and v can be extended on the whole

disk and the ratio u/v is a positive analytic function on $(-1, 1)$. Therefore, on $(-1, 1)$ we can write $u/v = |a|^2$ for some function a holomorphic on $\mathbb{D}$.

Finally, construct the functions

$$\mathcal{U} = u \circ T^{-1}, \quad \mathcal{V} = v \circ T^{-1}, \quad \text{and} \quad A = a \circ T^{-1}.$$

Then, A is holomorphic around the cusp at 0, and $\mathcal{U}, \mathcal{V}$ are positive harmonic functions on $\Omega \cap D(0, \rho)$ and zero on the boundary Γ . Moreover, $\mathcal{U}$ and $\mathcal{V}$ satisfy $\mathcal{U}/\mathcal{V} = |A|^2$ on Γ .

1.7 Some open “free boundary” problems in the spirit of Sakai

All problems treated above are examples of the so-called free boundary problems (non-variational free boundary problems).

We would like to call the attention of the reader to one open question: what can one say for the boundary of a domain Ω that is not simply connected but admits positive harmonic functions vanishing on its boundary and whose ratio is “nice” on that boundary? Finitely connected situations present no difficulties, but what if, for example, Γ is a Cantor set and $\Omega = \mathbb{D} \setminus \Gamma$? Suppose we know that the ratio of two positive harmonic (non-proportional) functions $\mathcal{U}, \mathcal{V}$ in Ω vanishing on the Cantor set Γ has a well-defined ratio on Γ (this happens for a wide class of Cantor sets Γ ’s, for example for all regular Cantor sets of positive Hausdorff dimension). Suppose this ratio is equal to $|A(\zeta)|^2 \neq \text{const}$ for $\zeta \in \Gamma$, where A is a holomorphic function on $\mathbb{D}$. What we can say about the Cantor set Γ ? The “desired” answer is that this is impossible to happen on any Cantor set.

This type of problems (we may call them “one-phase free boundary problems”) appear naturally in certain problems of complex dynamics, see, e.g., [15]. If we would know the aforementioned answer (we conjecture that no Cantor set would allow such a triple $(\mathcal{U}, \mathcal{V}, A)$), then a long-standing problem about the dimension of harmonic measure on Cantor repellers would be solved.

Another similar one-phase boundary problem concerns functions in $\mathbb{R}^n$ for $n > 2$. Let Ω be a bounded domain in $\mathbb{R}^n$, $n > 2$, and let $\Gamma = \partial\Omega \cap D(x, r)$, where $x \in \partial\Omega$. Again, let $\mathcal{U}, \mathcal{V}$ be two positive (non-proportional) harmonic functions in Ω vanishing continuously on Γ . If Ω is assumed

to be a Lipschitz domain, then [9] claims that $\mathcal{U}/\mathcal{V}$ makes sense on Γ and is additionally a Hölder function on Γ (boundary Harnack principle).

Here is a question. Let R be a real analytic function on $D(x, r)$, $x \in \Gamma$, and let $\mathcal{U}/\mathcal{V} = R$ on $\Gamma \cap D(x, r)$. Is it true that $\Gamma \cap D(x, r)$ is real analytic, maybe with the exception of some lower dimensional singular set?

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CHAPTER 2

THE BUFFON'S NEEDLE PROBLEM FOR RANDOM PLANAR DISK-LIKE CANTOR SETS

2.1 Introduction

Let E be a subset of the unit disk, $\mathbb{D}$. The *Buffon needle problem* wants to determine the probability with which a random needle or line intersects E provided that it already intersects the unit disk. At the same time, let l_θ be the line passing through the origin and forming angle θ with the horizontal axis. The *Favard length* of E is the average length of the projection of E onto l_θ when averaging over all angles θ . It turns out these two quantities are proportional.

Now, consider the following picture: let us have L many ($L \geq 3$) disjoint closed disks ($D_1, \dots, D_L$) of diameter $1/L$ and strictly inside $\mathbb{D}$. These are disks of the first generation. Consider also a piecewise affine map $f = (f_1, \dots, f_L)$ from those disks onto $\mathbb{D}$. Then, $f^{-1}(\mathbb{D}) = D_1 \cup \dots \cup D_L$. Furthermore, $f^{-1}(D_1 \cup \dots \cup D_L)$ consists of L^2 disks (groups of L many disks in each D_i); we call those disks of the second generation. We can iterate this procedure: denoting by U_n the union of disks of the n -th generation, where $U_1 := D_1 \cup \dots \cup D_L$, we form the self-similar Cantor set $\mathcal{K} = \bigcap_{n=1}^{\infty} U_n$. This has positive and finite 1-dimensional Hausdorff measure; thus it is completely unrectifiable in the sense of Besicovitch [8]; and thus its Favard length is zero [8].

Of course, the disks can be replaced by other shapes. For example, U_1 can consist of L disjoint squares with side-length $1/L$ inside the unit square $[0, 1]^2$ (where the word “strictly” can be omitted but “disjoint” cannot). One of such Cantor sets is a rather “famous”, namely the 1/4-corner Cantor set, $\mathcal{K}_{1/4}$ (see [7]).

The L^{-n} -neighbourhood of such sets is roughly U_n , and therefore its Favard length

$$\text{Fav}(U_n) \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

But what is $\text{Fav}(U_n)$, or what is the speed with which $\text{Fav}(U_n)$ decreases? Nobody knows exactly, but there has been considerable interest in recent years. It is now clear that the answer may depend on several factors; the magnitude of L ; the geometry of U_1 ; the subtle algebraic and number

theoretic properties of a certain trigonometric sum built by the centres of the disks of the first generation. See [2, 3, 4, 6, 10] and the survey paper [5].

For the 1/4-corner Cantor set $\mathcal{K}_{1/4}$ in particular, the best known estimate from above for its 4^{-n} -neighbourhood is

$$\text{Fav}(N_{4^{-n}}(\mathcal{K}_{1/4})) \leq \frac{C_\epsilon}{n^{\frac{1}{6}-\epsilon}}, \quad \forall \epsilon > 0,$$

for all large n . We suspect that this estimate can be improved to

$$\text{Fav}(N_{4^{-n}}(\mathcal{K}_{1/4})) \leq \frac{C_\epsilon}{n^{1+\epsilon}}, \quad \forall \epsilon > 0,$$

but at this moment this is only a conjecture.

On the other hand, there is a universal estimate from below obtained in [9] for every self-similar Cantor set constructed as above:

$$\text{Fav}(N_{4^{-n}}(\mathcal{K})) \geq \frac{c}{n}. \quad (2.1.1)$$

For any concrete set, this bound from below could be improved. In fact, it is proven in [1] that for the same 1/4-corner Cantor set $\mathcal{K}_{1/4}$

$$\text{Fav}(N_{4^{-n}}(\mathcal{K}_{1/4})) \geq \frac{c \log n}{n}.$$

For random Cantor sets the situation should be simpler. With large probability, Mattila's lower estimate (2.1.1) is met by *the same* estimate from above (with a different constant). The problem is that in general there can be many different models of randomness.

In this note, we are interested in an analogue of the random Cantor set appearing in [11] and in [14]. In our case, this will come from the random Cantor disks constructed below at Section 2.2. The model of randomness presented here is somewhat different from the ones in the above two papers, but it amazingly exhibits the same behaviour, as we'll see below in our main Theorem 2.3.1, which we contrast with [11, Theorem 2.2] and [14, Theorem 1].

In particular, we prove an analogue of [14, Theorem 1]. Unfortunately, the randomness of the disk model we study here is not equivalent to that of the random (square) Cantor set $\mathcal{R} = \bigcap_{n=0}^{\infty} \mathcal{R}_n$

from [11], but it is nonetheless closer compared to the one constructed in [14]. The essential difference between [14] and our consideration are the angles $\omega_n^1, \omega_n^2, \dots, \omega_n^{4^{n-1}}$, which are here allowed to be distinct and independent whereas in [14] are all equal. So, our model is a little “more random” than the random Cantor sets of Zhang in [14].

We introduce our notations —some borrowed from [14]— in the next Section 2.2. The problem of interest, namely the *Favard length* of a random planar disk-like Cantor set, is explained in Section 2.3. Our results and their proofs are postponed to Sections 2.4 and 2.5. In Section 2.6, we compare the differences and difficulties between our work and that of Peres and Solomyak’s and Zhang’s.

2.2 Cantor Disks

Our work will be heavy on notation; without any ado let us introduce our basic “vocabulary”.

The letter n will stand for a (large) positive integer.

The letter ω will be used to denote angles with values inside the interval $[0, \frac{\pi}{2}]$. Now, let us consider a word of length n made of the alphabet of angles in $[0, \frac{\pi}{2}]$, i.e. a word of the form $\omega_1 \omega_2 \dots \omega_n$. The subscript in ω_k denotes the position of the angle ω_k within such a word of length n . We refer to the position of an angle within a word as the *depth* of that angle.

Our operators, which we will introduce below, are such that every choice of an angle, say, ω_1 necessitates four different independent choices for the angle ω_2 ; every choice of the angle ω_2 necessitates four different independent choices for the angle ω_3 ; and so on up until depth n where we will have 4^{n-1} different angles ω_n . In order to differentiate between all those, for each $j_k = 1, 2, \dots, 4^{k-1}$ we write $\omega_k^{j_k}$ for the j_k -th choice of an angle ω_k at depth k . Notice there are 4^{k-1} such choices. Therefore, a typical word from our alphabet of angles looks as follows, where we note that $\omega_k^{j_k} \in [0, \frac{\pi}{2}]$:

$$\omega_1^{j_1} \omega_2^{j_2} \dots \omega_k^{j_k} \dots \omega_n^{j_n} \quad \text{where} \quad \begin{aligned} j_1 &= 1, & j_k &= 1, 2, \dots, 4^{k-1}, \\ j_2 &= 1, 2, 3, 4, & \dots & \\ \dots & & j_n &= 1, 2, \dots, 4^{n-1}. \end{aligned}$$

At certain instances, we need to consider cumulatively all angles of a certain depth; given a collection of words of length n , for each $k = 1, 2, \dots, n$ let ω'_k be the collection of all 4^{k-1} many angles at depth k , that is $\omega'_k = (\omega_k^1, \dots, \omega_k^{4^{k-1}})$. With this notation, we may use the symbols ω_1 , $\omega_1^{j_1}$, ω_1^1 , and ω'_1 interchangeably as these all refer to the same single angle.

All the above give to our angles the structure of a rooted tree of height n with root ω_1 and such that each parent has four children. The vertexes have values in $[0, \frac{\pi}{2}]$, and are independent from each other and from their predecessors and ancestors. This tree we denote by $\omega'_1 \cdots \omega'_n$; the trimmed tree with root ω_1 and height k we denote as $\omega'_1 \cdots \omega'_k$ (for any $k = 1, 2, \dots, n$). For the subtree of height $n - k + 1$ with root $\omega_k^{j_k}$, which reaches up to the leaves (that is, from depth k till depth n with starting vertex $\omega_k^{j_k}$) we write $\bar{\omega}_k^{j_k}$. Later on, we will be working with rooted subtrees of the form $\bar{\omega}_{n-k+1}^{j_{n-k+1}}$. To reiterate, $\bar{\omega}_{n-k+1}^{j_{n-k+1}}$ consists of the angle $\omega_{n-k+1}^{j_{n-k+1}}$ (as its root) along with all the angles from depth $n - k + 1$ till depth n (which have $\omega_{n-k+1}^{j_{n-k+1}}$ as an ancestor). This has height k . Alternatively, $\bar{\omega}_{n-k+1}^{j_{n-k+1}}$ is the collection of all the words (from our alphabet of angles) which have depth k and the first letter is $\omega_{n-k+1}^{j_{n-k+1}}$. There are 4^{n-k} such words.

Next, we will need to introduce certain operators and sets. The main objects of interest will be the operators $\mathcal{D}_k$ ($k = 0, 1, \dots, n$) which will act on trees of angles of depth k . To understand these we need some auxiliary constructions first.

For any angle ω and for $\alpha = 0, 1, 2, 3$ consider the transformations

$$T_\alpha^\omega(z) = \frac{1}{4}z + \frac{3}{4}e^{(\alpha\frac{\pi}{2} - \omega)i}$$

where z is any number on the complex plane $\mathbb{C}$. Observe that if $\mathbb{D}$ is the unit disk, $T_0^0(\mathbb{D})$, $T_1^0(\mathbb{D})$, $T_2^0(\mathbb{D})$, and $T_3^0(\mathbb{D})$ are disks of radius $1/4$ centred respectively at $(3/4, 0)$, $(0, 3/4)$, $(-3/4, 0)$, and $(0, -3/4)$. Introducing an angle ω in $T_\alpha^\omega(\mathbb{D})$, rotates (about $(0, 0)$) the aforementioned disks by angle ω in the clockwise direction.

Moreover, given an angle $\omega_k^{j_k}$ from depth k let $\Omega_k^{j_k}$ be the set

$$\Omega_k^{j_k} = \bigcup_{\alpha=0}^3 \frac{1}{4^{k-1}} T_\alpha^{\omega_k^{j_k}}(\mathbb{D}).$$

That is, $\Omega_k^{j_k}$ is a collection of four disks of radius 4^{-k} with centres $(0, \pm 3/4^k)$ and $(\pm 3/4^k, 0)$ rotated clockwise by $\omega_k^{j_k}$.

We also give an enumeration to all the disks for all depths. We number the disks of $\Omega_k^{j_k}$ so that $\frac{1}{4^{k-1}}T_{\alpha}^{\omega_k^{j_k}}(\mathbb{D})$ is the $(4j_k - 3 + \alpha)$ -th disk at depth k . We call this the *k-depth enumeration* (of the disks lying at depth k). Illustratively, we note $\frac{1}{4^{k-1}}T_0^{\omega_k^1}(\mathbb{D})$, $\frac{1}{4^{k-1}}T_1^{\omega_k^1}(\mathbb{D})$, $\frac{1}{4^{k-1}}T_0^{\omega_k^{4^{k-1}}}(\mathbb{D})$, and $\frac{1}{4^{k-1}}T_3^{\omega_k^{4^{k-1}}}(\mathbb{D})$ are respectively the 1st, 2nd, $(4^k - 3)$ -th, and 4^k -th disks of depth k . We retain this enumeration as we translate these disks at different positions on the plane. This will be useful to track down each disk at each step so that our subsequent constructions make better sense.

Now, we are ready to introduce our main protagonists. The operator $\mathcal{D}_k$ acts on the collection of trees (of angles) of height k and for each such tree outputs a certain collection of 4^k disks of radius 4^{-k} . We define these inductively below.

To begin with, set $\mathcal{D}_0 = \mathbb{D}$ to be the unit disk.

Next, we define $\mathcal{D}_1$ by

$$\mathcal{D}_1(\omega'_1) = \Omega_1^1 = \bigcup_{\alpha=0}^3 T_{\alpha}^{\omega_1^1}(\mathcal{D}_0),$$

that is, $\mathcal{D}_1(\omega'_1)$ consists of four disks of radius $1/4$ centred at $(0, \pm 3/4)$ and $(\pm 3/4, 0)$ rotated clockwise by ω_1 . Recall these disks are enumerated as in Ω_1^1 .

For the operator $\mathcal{D}_2$, consider a tree of height 2, $\omega'_1\omega'_2$, which consists of the angles ω_1^1 , and $\omega_2^1, \omega_2^2, \omega_2^3, \omega_2^4$. Then, we define $\mathcal{D}_2(\omega'_1\omega'_2)$ to be the collection of disks constructed as follows: Replace the 1st, 2nd, 3rd and 4th disk of $\mathcal{D}_1(\omega'_1)$ respectively by $\Omega_2^1, \Omega_2^2, \Omega_2^3$ and Ω_2^4 . By replacing we mean the translation of Ω_2^j in such a way that $(0, 0)$ is translated to the centre of the j -th disk of $\mathcal{D}_1(\omega'_1)$.

Consequently, $\mathcal{D}_2(\omega'_1\omega'_2)$ consists of 4^2 disks of radius 4^{-2} translated appropriately so that each $\Omega_2^{j_2}$ replaces one the disks from $\mathcal{D}_1(\omega'_1)$. The set, say, Ω_2^1 is in fact a subset of the 1st disk of $\mathcal{D}_1(\omega')$; actually $\mathcal{D}_2(\omega'_1\omega'_2) \subset \mathcal{D}_1(\omega'_1)$. Again, the disks comprising $\mathcal{D}_2(\omega'_1\omega'_2)$ are enumerated to match $\Omega_2^1, \Omega_2^2, \Omega_2^3$ and Ω_2^4 as we described above. Also see the Figure 2.1 below.

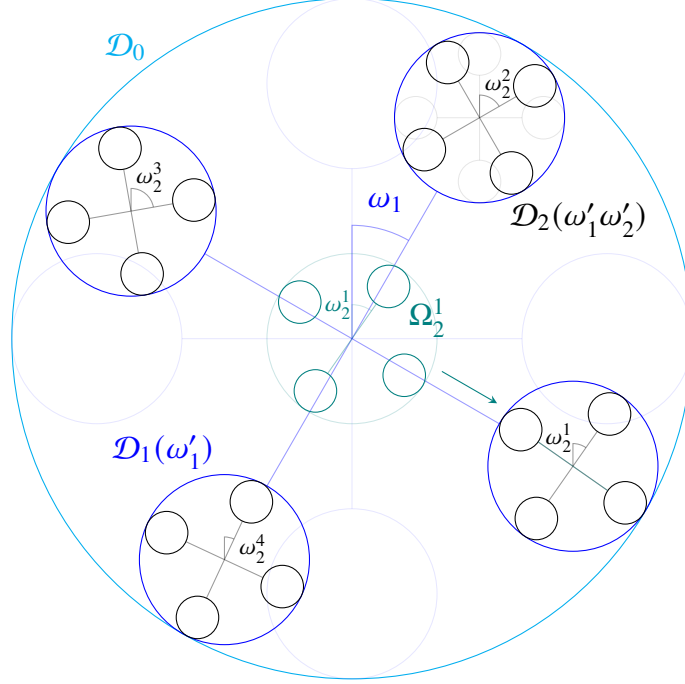


Figure 2.1 The collections $\mathcal{D}_0$, $\mathcal{D}_1(\omega'_1)$, $\mathcal{D}_2(\omega'_1 \omega'_2)$ and Ω_2^1 .

Continuing inductively, the operator $\mathcal{D}_k$ acts on the tree $\omega'_1 \cdots \omega'_k$ in this manner: Consider the collection $\mathcal{D}_{k-1}(\omega'_1 \cdots \omega'_{k-1})$. These are 4^{k-1} many (enumerated) disks. Replace the 1st of them by Ω_k^1 , the 2nd of them by Ω_k^2 , etc., until every disk of $\mathcal{D}_{k-1}(\omega'_1 \cdots \omega'_{k-1})$ has been replaced by four smaller ones. This replacement is done so that $(0, 0)$, as the “centre” of Ω_k^j , is translated to the centre of the j -th disk of $\mathcal{D}_{k-1}(\omega'_1 \cdots \omega'_{k-1})$. That is, we substitute the j -th disk (from depth $k-1$) with the $(4j-3)$ -, $(4j-2)$ -, $(4j-1)$ -, and $4j$ -th disks of depth k . The resulting collection, which has 4^k many disks of radius 4^{-k} , is $\mathcal{D}_k(\omega'_1 \cdots \omega'_k)$. It holds that $\mathcal{D}_k(\omega'_1 \cdots \omega'_k) \subset \mathcal{D}_{k-1}(\omega'_1 \cdots \omega'_{k-1})$.

In the present work, we will study the collection of disks $\mathcal{D}_n(\omega'_1 \cdots \omega'_n)$ where the angles $\omega_k^{j_k}$ (for $j_k = 1, \dots, 4^{k-1}$ and all $k = 1, 2, \dots, n$) of the tree $\omega'_1 \cdots \omega'_n$ are chosen randomly with uniform and independent distributions on the interval $[0, \frac{\pi}{2}]$. So, let us describe this picture once more before moving on further.

The set $\mathcal{D}_n(\omega'_1 \cdots \omega'_n)$ consists of 4^n disks of radius 4^{-n} . These can be separated into 4^{n-1} groups of four, which are copies of

$$\Omega_n^{j_n} = \bigcup_{\alpha=0}^3 \frac{1}{4^{n-1}} T_{\alpha}^{\omega_n^{j_n}}(\mathcal{D}_0)$$

(for $j_n = 1, \dots, 4^{n-1}$) translated appropriately within the unit disk.

2.3 Favard Length

Recall the *Favard length* of a planar set $E \subset \mathbb{C}$ is the integral

$$\text{Fav}(E) = \frac{1}{\pi} \int_0^\pi |\text{proj}_\theta E| d\theta$$

where $\text{proj}_\theta E$ is the projection of E onto the line with slope $\tan \theta$ passing through the origin, and $|A|$ is the (1-dimensional) Lebesgue measure of A .

Now, consider an infinite tree of angles from $[0, \frac{\pi}{2}]$ with root ω_1 and four branches at each vertex, and let $\mathcal{D}$ be the limit set

$$\mathcal{D} = \bigcap_{n=0}^{\infty} \mathcal{D}_n(\omega'_1 \cdots \omega'_n).$$

Notice that by construction, $\mathcal{D}$ a purely unrectifiable planar set. As such, $\text{Fav}(\mathcal{D}) = 0$ and by dominated convergence $\text{Fav}(\mathcal{D}_n(\omega'_1 \cdots \omega'_n)) \rightarrow 0$ while $n \rightarrow \infty$. In fact, if the angles are randomly chosen uniformly and independently over $[0, \frac{\pi}{2}]$, by dominated convergence and Fubini $\mathbb{E}[\text{Fav}(\mathcal{D})] = 0$ and $\mathbb{E}[\text{Fav}(\mathcal{D}_n(\omega'_1 \cdots \omega'_n))] \rightarrow 0$ as $n \rightarrow \infty$, where the expectation is taken over all such angles.

The question arises as to the rate with which $\mathbb{E}[\text{Fav}(\mathcal{D}_n(\omega'_1 \cdots \omega'_n))]$ goes to 0. This we answer in the following theorem:

Theorem 2.3.1. *Let $n \in \mathbb{N}$ and consider a tree of angles of height n with each vertex having four branches. Suppose that the angles $\omega_k^{j_k}$ (for all $j_k = 1, 2, \dots, 4^{k-1}$ and all $k = 1, 2, \dots, n$) are chosen randomly with uniform and independent distributions on the interval $[0, \frac{\pi}{2}]$. Also set $\omega'_k = (\omega_k^1, \omega_k^2, \dots, \omega_k^{4^{k-1}})$ for each $k = 1, 2, \dots, n$. Then, there exists a constant $c > 0$ such that for any $\theta \in [0, \frac{\pi}{2}]$ it holds that*

$$\mathbb{E}_{\omega'_1 \cdots \omega'_n} |\text{proj}_\theta \mathcal{D}_n(\omega'_1 \cdots \omega'_n)| \leq \frac{c}{n} \quad \forall n \in \mathbb{N}. \quad (2.3.1)$$

Consequently,

$$\mathbb{E}_{\omega'_1 \cdots \omega'_n} [\text{Fav}(\mathcal{D}_n(\omega'_1 \cdots \omega'_n))] \leq \frac{c}{n} \quad \forall n \in \mathbb{N} \quad (2.3.2)$$

and also

$$\liminf_{n \rightarrow \infty} n \text{Fav}(\mathcal{D}_n(\omega'_1 \cdots \omega'_n)) < \infty \quad \forall n \in \mathbb{N} \text{ almost surely.} \quad (2.3.3)$$

Clearly, (2.3.3) follows from (2.3.2) by an immediate application of Fatou's lemma, whereas (2.3.2) follows from (2.3.1) through Fubini.

2.4 Statement and use of the main lemma

The present and the following sections are dedicated to the proof of (2.3.1). Towards this goal, we need to introduce Lemma 2.4.1 below, which describes the decay of the average projection when transitioning from depth k to depth $k + 1$. The main difficulty will come from obtaining the square factor appearing in (2.4.1), which emanates from the naturally occurring overlap of the projections.

From now on, suppose we are given a tree of angles of height n with four branches at each vertex where the angles are uniformly and independently distributed random variables on the interval $[0, \frac{\pi}{2}]$. Recall that given such a tree $\bar{\omega}_{n-k+1}^{j_{n-k+1}}$ is the subtree of height k with the vertex $\omega_{n-k+1}^{j_{n-k+1}}$ as its root. Observe that $\bar{\omega}_1^{j_1} = \omega'_1 \cdots \omega'_n$ is the full tree whilst $\bar{\omega}_n^{j_n} = \omega_n^{j_n}$ ($j_n = 1, 2, \dots, 4^{n-1}$) are the its leaves, i.e. trees of height 1.

For any $\theta \in [0, \frac{\pi}{2}]$ and all $k = 1, 2, \dots, n$, define the following quantities

$$\begin{aligned} D_1^{j_n} &= \mathbb{E}_{\bar{\omega}_n^{j_n}} \left| \text{proj}_\theta \mathcal{D}_1(\bar{\omega}_n^{j_n}) \right|, & j_n &= 1, 2, \dots, 4^{n-1} \\ D_k^{j_{n-k+1}} &= \mathbb{E}_{\bar{\omega}_{n-k+1}^{j_{n-k+1}}} \left| \text{proj}_\theta \mathcal{D}_k(\bar{\omega}_{n-k+1}^{j_{n-k+1}}) \right|, & j_{n-k+1} &= 1, 2, \dots, 4^{n-k} \\ D_n^{j_1} &= D_n^1 = \mathbb{E}_{\bar{\omega}_1^{j_1}} \left| \text{proj}_\theta \mathcal{D}_n(\bar{\omega}_1^{j_1}) \right|, & j_1 &= 1. \end{aligned}$$

Notice that, because we are averaging over the independent and identically distributed $\omega_k^{j_k}$,

$$D_k^1 = D_k^2 = \cdots = D_k^{4^{n-k}} \quad \text{for any } k = 1, 2, \dots, n.$$

Therefore, it suffices to work with D_k^1 ; the rest should be identical. Also, note that

$$D_n^1 = \mathbb{E}_{\omega'_1 \cdots \omega'_n} \left| \text{proj}_\theta \mathcal{D}_n(\omega'_1 \cdots \omega'_n) \right|.$$

Now, we are ready to state a simple but important lemma. Also, see [14, Lemma 2.1].

Lemma 2.4.1. *With notation as above, if $\omega_k^{j_k}$ are uniformly and independently distributed random variables on $[0, \frac{\pi}{2}]$, there exists a constant $c \geq 4$ such that for any $n \in \mathbb{N}$ (and any $\theta \in [0, \frac{\pi}{2}]$)*

$$D_{k+1}^1 \leq D_k^1 - c^{-1}(D_k^1)^2 \quad \text{for all } k = 1, \dots, n-1. \quad (2.4.1)$$

In fact, we will see below that $c = 8$.

Provided this holds true we can give a very compact proof of Theorem 2.3.1 using induction:

Proof of Theorem 2.3.1. Let c be as in Lemma 2.4.1 and note that $D_2^1 \leq D_1^1 < 2 \leq \frac{c}{2}$. Also, $D_1^1 < c$.

Next, assume $D_k^1 < \frac{c}{k}$ for some $2 \leq k \leq n-1$. From Lemma 2.4.1, and by the monotonicity of the function $x - x^2/c$ in $[0, \frac{c}{2}]$, we see that

$$D_{k+1}^1 \leq D_k^1 - c^{-1}(D_k^1)^2 < \frac{c}{k} - \frac{c}{k^2} = c \frac{k-1}{k^2} < \frac{c}{k+1}.$$

Therefore, $D_k^1 < \frac{c}{k}$ holds for all $1 \leq k \leq n-1$ and thus for $k = n$ we get

$$\mathbb{E}_{\omega'_1 \dots \omega'_n} |\text{proj}_\theta \mathcal{D}_n(\omega'_1 \dots \omega'_n)| = D_n^1 < \frac{c}{n}.$$

This is (2.3.1). Equation (2.3.2) follows after integrating with respect to θ , and (2.3.3) after applying Fatou's Lemma. $\square$

2.5 Proving the main lemma

Whatever follows is dedicated to the proof of (2.4.1).

First, we rewrite the length of the projection of a set in more convenient way. Let l_θ and $l_\theta^\perp$ be two lines through the origin so that l_θ forms an angle θ with the horizontal axis and $l_\theta^\perp$ is perpendicular to l_θ . Also, let $\mathbf{n}$ be the unit normal vector of $l_\theta^\perp$. The length of the projection of a planar set $E \subset \mathbb{C}$ onto the line l_θ can be written as

$$|\text{proj}_\theta E| = |\{t \in \mathbb{R} : (l_\theta^\perp + t\mathbf{n}) \cap E \neq \emptyset\}| = \int_{(l_\theta^\perp + t\mathbf{n}) \cap E \neq \emptyset} dt. \quad (2.5.1)$$

For brevity, we denote the line $l_\theta^\perp + t\mathbf{n}$ by $l_\theta^\perp(t)$ where $t \in \mathbb{R}$. Additionally, because of the symmetry of our considerations, we can assume without loss of generality that $\theta = 0$ —as we will average over all θ at the end. So, we can simply omit writing θ altogether from now on.

The idea behind Lemma 2.4.1 is to look at the collection $\mathcal{D}_n(\omega'_1 \cdots \omega'_n)$ at depth n but “zoomed in” so that it looks like depth 1. Then, we go one level up and look at the disks of depth $n - 1$ and n zooming in enough so that they to look like depth 2; and so forth. If we rewrite the projections in the form of (2.5.1), the average overlap at each level is of at least a square factor compared to the total average projection of the level above.

This last comparison is paramount to the proof. It will follow from the fact that the disks in our constructions never get too close to one another. In fact, this observation is not true in the case of the random square Cantor sets, which is the reason why we cannot directly apply the arguments here to the setting of [11].

Let us proceed with the proof of (2.4.1).

Fix some $k = 1, 2, \dots, n$ and recall that by construction

$$\mathcal{D}_1(\omega_{n-k+1}^1) = \bigcup_{\alpha=0}^3 T_{\alpha}^{\omega_{n-k+1}^1}(\mathcal{D}_0).$$

This means that each disk from the collection $\mathcal{D}_k(\bar{\omega}_{n-k+1}^1)$ lies inside one of the above four disks, and therefore we can separate $\mathcal{D}_k(\bar{\omega}_{n-k+1}^1)$ into four groups of disks depending on their positioning at depth 1.

More precisely, for each $\alpha = 0, 1, 2, 3$ define $\mathcal{T}_{\alpha}^k(\bar{\omega}_{n-k+1}^1)$ as

$$\mathcal{T}_{\alpha}^k(\bar{\omega}_{n-k+1}^1) = T_{\alpha}^{\omega_{n-k+1}^1}(\mathcal{D}_0) \cap \mathcal{D}_k(\bar{\omega}_{n-k+1}^1).$$

That is, the set $\mathcal{T}_{\alpha}^k(\bar{\omega}_{n-k+1}^1)$ consists of those disks of $\mathcal{D}_k(\bar{\omega}_{n-k+1}^1)$ which lie inside the $\frac{1}{4}$ -radius disk $T_{\alpha}^{\omega_{n-k+1}^1}(\mathcal{D}_0)$. We can think of $\mathcal{T}_{\alpha}^k(\bar{\omega}_{n-k+1}^1)$ as the East, North, West, and South parts of $\mathcal{D}_k(\bar{\omega}_{n-k+1}^1)$, respectively for $\alpha = 0, 1, 2, 3$. From this definition, it is also clear that

$$\mathcal{D}_k(\bar{\omega}_{n-k+1}^1) = \bigcup_{\alpha=0}^3 \mathcal{T}_{\alpha}^k(\bar{\omega}_{n-k+1}^1). \quad (2.5.2)$$

In fact, $\mathcal{T}_{\alpha}^k(\bar{\omega}_{n-k+1}^1)$ depends only on the angle ω_{n-k+1}^1 and the subtree $\bar{\omega}_{n-k+2}^{4 \cdot 1 - 3 + \alpha} = \bar{\omega}_{n-k+2}^{1+\alpha}$. (Recall our enumeration of the angles in Section 2.2.) Thus, we can write $\mathcal{T}_{\alpha}^k(\bar{\omega}_{n-k+1}^1)$ as

$$\mathcal{T}_{\alpha}^k(\bar{\omega}_{n-k+1}^1) = \mathcal{T}_{\alpha}^k(\omega_{n-k+1}^1, \bar{\omega}_{n-k+2}^{1+\alpha}).$$

2.5.1 Key observations

There are two key observations regarding the sets $\mathcal{T}_\alpha^k(\bar{\omega}_{n-k+1}^1)$. First, note that each point of the interval $(-1, 1)$ can be covered by at most two of the projections $\text{proj } T_\alpha^{\omega_{n-k+1}^1}(\mathcal{D}_0)$ for different α 's. Since $\mathcal{T}_\alpha^k(\bar{\omega}_{n-k+1}^1) \subset T_\alpha^{\omega_{n-k+1}^1}(\mathcal{D}_0)$, the same holds true for $\text{proj } \mathcal{T}_\alpha^k(\bar{\omega}_{n-k+1}^1)$; the intersection $\bigcap_\alpha \text{proj } \mathcal{T}_\alpha^k(\bar{\omega}_{n-k+1}^1)$ is empty when the intersection is over more than two values of α .

Second, we can compare the average projections of $\mathcal{T}_\alpha^k(\bar{\omega}_{n-k+1}^1) = \mathcal{T}_\alpha^k(\omega_{n-k+1}^1, \bar{\omega}_{n-k+2}^{1+\alpha})$ and $\mathcal{D}_{k-1}(\bar{\omega}_{n-k+2}^{1+\alpha})$. Notice that both these collections consist of 4^{k-1} many disks, which in fact have the same n -depth enumerations. This means that they correspond to same disks of the collection $\mathcal{D}_n(\omega'_1 \cdots \omega'_n)$. The difference is that the disks of the former are translated according to $\mathcal{D}_1(\omega_{n-k+1}^1)$ and have radius 4^{-k} , whereas the ones of the latter have radius $4^{-(k-1)}$.

Consequently, $\mathcal{T}_\alpha^k(\bar{\omega}_{n-k+1}^1)$ is a shifted copy of $\mathcal{D}_{k-1}(\bar{\omega}_{n-k+2}^{1+\alpha})$ dilated by a factor of $1/4$. As such, the (average of the) projections of $\mathcal{T}_\alpha^k(\bar{\omega}_{n-k+1}^1)$ and $\mathcal{D}_{k-1}(\bar{\omega}_{n-k+2}^{1+\alpha})$ should also differ by a factor of $1/4$. In other words, for any $\alpha = 0, 1, 2, 3$ we have

$$\begin{aligned} \mathbb{E}_{\bar{\omega}_{n-k+1}^1} |\text{proj } \mathcal{T}_\alpha^k(\bar{\omega}_{n-k+1}^1)| &= \mathbb{E}_{\omega_{n-k+1}^1} \mathbb{E}_{\bar{\omega}_{n-k+2}^{1+\alpha}} |\text{proj } \mathcal{T}_\alpha^k(\omega_{n-k+1}^1, \bar{\omega}_{n-k+2}^{1+\alpha})| \\ &= \frac{1}{4} \mathbb{E}_{\bar{\omega}_{n-k+2}^{1+\alpha}} |\text{proj } \mathcal{D}_{k-1}(\bar{\omega}_{n-k+2}^{1+\alpha})|. \end{aligned} \quad (2.5.3)$$

2.5.2 The estimates

Utilising the above, we can now estimate D_k^1 in terms of D_{k-1}^1 :

$$\begin{aligned} D_k^1 &= \mathbb{E}_{\bar{\omega}_{n-k+1}^1} |\text{proj } \mathcal{D}_k^1(\bar{\omega}_{n-k+1}^1)| \\ &\leq \mathbb{E}_{\bar{\omega}_{n-k+1}^1} \sum_{\alpha=0}^3 |\text{proj } \mathcal{T}_\alpha^k(\bar{\omega}_{n-k+1}^1)| - \mathbb{E}_{\bar{\omega}_{n-k+1}^1} |\text{proj } \mathcal{T}_0^k(\bar{\omega}_{n-k+1}^1) \cap \text{proj } \mathcal{T}_1^k(\bar{\omega}_{n-k+1}^1)| \\ &\stackrel{(2.5.3)}{=} \frac{1}{4} \sum_{\alpha=0}^3 \mathbb{E}_{\bar{\omega}_{n-k+2}^{1+\alpha}} |\text{proj } \mathcal{D}_{k-1}(\bar{\omega}_{n-k+2}^{1+\alpha})| \\ &\quad - \mathbb{E}_{\bar{\omega}_{n-k+1}^1} |\text{proj } \mathcal{T}_0^k(\bar{\omega}_{n-k+1}^1) \cap \text{proj } \mathcal{T}_1^k(\bar{\omega}_{n-k+1}^1)| \end{aligned}$$

where the inequality follows from the first observation above (see 2.5.1) and (2.5.2). This in turn gives

$$\begin{aligned} D_k^1 &\leq \frac{1}{4}(D_{k-1}^1 + D_{k-1}^2 + D_{k-1}^3 + D_{k-1}^4) - \mathbb{E}_{\bar{\omega}_{n-k+1}^1} \left| \text{proj } \mathcal{T}_0^k(\bar{\omega}_{n-k+1}^1) \cap \text{proj } \mathcal{T}_1^k(\bar{\omega}_{n-k+1}^1) \right| \\ &= D_{k-1}^1 - \mathbb{E}_{\bar{\omega}_{n-k+1}^1} \left| \text{proj } \mathcal{T}_0^k(\bar{\omega}_{n-k+1}^1) \cap \text{proj } \mathcal{T}_1^k(\bar{\omega}_{n-k+1}^1) \right|, \end{aligned} \quad (2.5.4)$$

since $D_{k-1}^{1+\alpha} = D_{k-1}^1$ for any $\alpha = 0, 1, 2, 3$.

The next step, is to estimate the overlap term $|\text{proj } \mathcal{T}_0^k(\bar{\omega}_{n-k+1}^1) \cap \text{proj } \mathcal{T}_1^k(\bar{\omega}_{n-k+1}^1)|$ from below. For this, recall $\mathcal{T}_0^k(\bar{\omega}_{n-k+1}^1)$ and $\mathcal{T}_1^k(\bar{\omega}_{n-k+1}^1)$ depend (aside from ω_{n-k+1}^1) respectively on $\bar{\omega}_{n-k+2}^1$ and $\bar{\omega}_{n-k+2}^2$.

First, we average with respect to the subtrees $\bar{\omega}_{n-k+2}^1$ and $\bar{\omega}_{n-k+2}^2$, and afterwards we integrate over their common ancestor ω_{n-k+1}^1 . To simplify the notation, let us write $\bar{\omega}_{n-k+2}^{1,2}$ for both the subtrees $\bar{\omega}_{n-k+2}^1$ and $\bar{\omega}_{n-k+2}^2$, and also ψ for the angle ω_{n-k+1}^1 . Then, we have

$$\begin{aligned} &\mathbb{E}_{\bar{\omega}_{n-k+2}^{1,2}} \left| \text{proj } \mathcal{T}_0^k(\bar{\omega}_{n-k+1}^1) \cap \text{proj } \mathcal{T}_1^k(\bar{\omega}_{n-k+1}^1) \right| \\ &= \mathbb{E}_{\bar{\omega}_{n-k+2}^{1,2}} \left| \text{proj } \mathcal{T}_0^k(\omega_{n-k+1}^1, \bar{\omega}_{n-k+2}^1) \cap \text{proj } \mathcal{T}_1^k(\omega_{n-k+1}^1, \bar{\omega}_{n-k+2}^2) \right| \\ &\stackrel{(2.5.1)}{=} \int \mathbb{P}_{\bar{\omega}_{n-k+2}^{1,2}} \left(l^\perp(t) \cap \mathcal{T}_0^k(\omega_{n-k+1}^1, \bar{\omega}_{n-k+2}^1) \neq \emptyset \right. \\ &\quad \left. \text{and } l^\perp(t) \cap \mathcal{T}_1^k(\omega_{n-k+1}^1, \bar{\omega}_{n-k+2}^2) \neq \emptyset \right) dt \\ &= \int \mathbb{P}_{\bar{\omega}_{n-k+2}^{1,2}} \left(l^\perp(t) \cap \mathcal{T}_0^k(\omega_{n-k+1}^1, \bar{\omega}_{n-k+2}^1) \neq \emptyset \right) \cdot \\ &\quad \cdot \mathbb{P}_{\bar{\omega}_{n-k+2}^{1,2}} \left(l^\perp(t) \cap \mathcal{T}_1^k(\omega_{n-k+1}^1, \bar{\omega}_{n-k+2}^2) \neq \emptyset \right) dt \\ &= \int \mathbb{P}_{\bar{\omega}_{n-k+2}^1} \left(l^\perp(t) \cap \mathcal{T}_0^k(\omega_{n-k+1}^1, \bar{\omega}_{n-k+2}^1) \neq \emptyset \right) \cdot \\ &\quad \cdot \mathbb{P}_{\bar{\omega}_{n-k+2}^2} \left(l^\perp(t) \cap \mathcal{T}_1^k(\omega_{n-k+1}^1, \bar{\omega}_{n-k+2}^2) \neq \emptyset \right) dt \\ &=: \mathcal{E}(\omega_{n-k+1}^1). \end{aligned}$$

The 3rd equality above holds because for a fixed angle ω_{n-k+1}^1 the events

$$\{l^\perp(t) \cap \mathcal{T}_\alpha^k(\omega_{n-k+1}^1, \bar{\omega}_{n-k+2}^{1+\alpha}) \neq \emptyset\}$$

are independent.

It would be very nice if these two events would have the same probability. Then at the end, we would use Hölder inequality to get

$$\begin{aligned}\mathcal{E}(\omega_{n-k+1}^1) &= \int \left[\mathbb{P}_{\bar{\omega}_{n-k+2}^1} \left(l^\perp(t) \cap \mathcal{T}_0^k(\omega_{n-k+1}^1, \bar{\omega}_{n-k+2}^1) \neq \emptyset \right) \right]^2 dt \\ &\geq C \left(\int \mathbb{P}_{\bar{\omega}_{n-k+2}^1} \left(l^\perp(t) \cap \mathcal{T}_0^k(\omega_{n-k+1}^1, \bar{\omega}_{n-k+2}^1) \neq \emptyset \right) dt \right)^2.\end{aligned}$$

However, this is not the case.

For brevity, let us temporarily denote

$$\psi := \omega_{n-k+1}^1 \quad \text{and} \quad s(\psi) := \frac{3}{4}(1 - \cos \psi). \quad (2.5.5)$$

Also, set

$$F(t) := \{l^\perp(t) \cap \mathcal{T}_0^k(0, \bar{\omega}_{n-k+2}^1) \neq \emptyset\}. \quad (2.5.6)$$

For fixed ψ , the events

$$\{l^\perp(t) \cap \mathcal{T}_0^k(\psi, \bar{\omega}_{n-k+2}^1) \neq \emptyset\} \quad \text{and} \quad \{l^\perp(t) \cap \mathcal{T}_1^k(\psi, \bar{\omega}_{n-k+2}^2) \neq \emptyset\}$$

do not have the same probability; one should take into consideration that the probability of the non-empty intersection with $l^\perp(t)$ for the first $\mathcal{T}$ has the same probability as the non-empty intersection with $l^\perp(t + s(\psi))$ with $\mathcal{T}_0^k(0, \bar{\omega}_{n-k+2}^1)$. (Notice what happens with ψ !) And the probability of the non-empty intersection with $l^\perp(t)$ for the second $\mathcal{T}$ has the same probability as the non-empty intersection with $l^\perp(t + s(\psi + \frac{\pi}{2}))$ for the event $\mathcal{T}_0^k(0, \bar{\omega}_{n-k+2}^1)$. In fact, a simple geometric consideration shows the following holds:

Lemma 2.5.1. *With notation as above we have that*

$$\mathbb{P}_{\bar{\omega}_{n-k+2}^1} \left(l^\perp(t) \cap \mathcal{T}_0^k(\psi, \bar{\omega}_{n-k+2}^1) \neq \emptyset \right) = \mathbb{P}_{\bar{\omega}_{n-k+2}^1} \left(l^\perp(t + s(\psi)) \cap \mathcal{T}_0^k(0, \bar{\omega}_{n-k+2}^1) \neq \emptyset \right). \quad (2.5.7)$$

In other words, Lemma 2.5.1 shows that

$$\begin{aligned}
\mathcal{E}(\psi) &= \\
&= \int \mathbb{P}_{\bar{\omega}_{n-k+2}^1} \left(l^\perp(t + s(\psi)) \cap \mathcal{T}_0^k(0, \bar{\omega}_{n-k+2}^1) \neq \emptyset \right) \cdot \\
&\quad \cdot \mathbb{P}_{\bar{\omega}_{n-k+2}^1} \left(l^\perp(t + s(\psi + \frac{\pi}{2})) \cap \mathcal{T}_0^k(0, \bar{\omega}_{n-k+2}^1) \neq \emptyset \right) dt \\
&= \int F(t + s(\psi)) \cdot F(t + s(\psi + \frac{\pi}{2})) dt.
\end{aligned}$$

Next, if we integrate over $\psi \in [0, \frac{\pi}{2}]$, we get that the

$$\text{Expectation of the overlap} = \int \mathcal{E}(\psi) d\psi = \int \int F(t + s(\psi)) \cdot F(t + s(\psi + \frac{\pi}{2})) d\psi dt.$$

Let's make this change of variables: $u = t + \frac{3}{4}(1 - \cos \psi)$ and $v = t + \frac{3}{4}(1 - \cos(\psi + \frac{\pi}{2}))$. The Jacobian of this change is at most $\frac{3}{4}$, and thus

$$\text{Expectation of the overlap} \geq \frac{4}{3} \int \int F(u)F(v) du dv = \frac{4}{3} \left(\int F(t) dt \right)^2.$$

Since there is no dependence on $\bar{\omega}_{n-k+1}^3$ or $\bar{\omega}_{n-k+1}^4$, we get

$$\begin{aligned}
&\mathbb{E}_{\bar{\omega}_{n-k+1}^1} \left| \text{proj } \mathcal{T}_0^k(\bar{\omega}_{n-k+1}^1) \cap \text{proj } \mathcal{T}_1^k(\bar{\omega}_{n-k+1}^1) \right| \\
&= \text{Expectation of the overlap} \\
&\geq \frac{4}{3} \left(\mathbb{E}_{\bar{\omega}_{n-k+1}^1} \left| \text{proj } \mathcal{T}_0^k(\bar{\omega}_{n-k+1}^1) \right| \right)^2 \\
&\stackrel{(2.5.3)}{=} \frac{4}{3} \cdot \frac{1}{16} \left(\mathbb{E}_{\bar{\omega}_{n-k+2}^1} \left| \text{proj } \mathcal{D}_{k-1}(\bar{\omega}_{n-k+1}^1) \right| \right)^2 \\
&= \frac{1}{12} (D_{k-1}^1)^2.
\end{aligned}$$

Finally, combining the fact that

$$\mathbb{E}_{\bar{\omega}_{n-k+1}^1} \left| \text{proj } \mathcal{T}_0^k(\bar{\omega}_{n-k+1}^1) \cap \text{proj } \mathcal{T}_1^k(\bar{\omega}_{n-k+1}^1) \right| \geq \frac{1}{12} (D_{k-1}^1)^2$$

with (2.5.4) and setting $c = 12$ we get

$$D_k^1 \leq D_{k-1}^1 - c^{-1} (D_{k-1}^1)^2$$

and Lemma 2.4.1 is proved.

2.6 Comparison with the other random models

The random Cantor set in [14] is a very close relative of the random Cantor set in this note, the difference is that Zhang's random construction of n generations has n independent rotations involved, whereas our construction has $1 + \dots + 4^{n-1}$ independent rotations. There the disks of generation k are rotated by the same angle ω_k , while in this note we have 4^{k-1} independent rotations of disks of generation k . Naturally, it is more difficult to work in a more chaotic model such as ours, and the techniques here use independence in a more involved way than in [14]. It is just a little harder to make sense of the combinatorics involved in our model.

On the other hand, there are many “common places”: the use of overlap as the way to see the rate of decays of successive approximations of the random Cantor set, the use of Lemma 2.4.1, as well as the technical Lemma 2.5.1.

Concerning [11], there are two main differences which create difficulties. The first is the fact that at most two of the projections $\text{proj}_\theta \mathcal{T}_\alpha^k(\bar{\omega}_{n-k+1}^1)$ can intersect at each point on the line l_θ . This is equivalent to line $l_\theta^\perp(t)$ intersecting at most two of the disks for any t , and is key to the square factor appearing in our calculations.

However, this is simply not true in the case of squares. In fact, in the Peres and Solomyak case the corresponding line $l_\theta^\perp(t)$ can simultaneously intersect 3 squares of generation k for any k and any t . Because of this, the inequalities appearing here cannot be translated directly in the square setting.

But even if this wasn't an obstacle, the reader should pay attention to Lemma 2.5.1. Let's pretend that we can repeat everything before this lemma for the model of Peres and Solomyak. The role of the angle ω_{n-k+1}^1 will be played by the “Favard angle” θ , the shift function $s(\omega_{n-k+1}^1)$ will be replaced by

$$S(\theta) = \frac{1}{2} \sin \theta,$$

and all seems to be following smoothly along the same lines. Also, the following equality

$$\int \mathbb{E} \mathbf{1}_{\{l_\theta^\perp(t) \cap \mathcal{T}_0^k(\bar{\omega}_{n-k+2}^1) \neq \emptyset\}} d\theta = \int \mathbb{E} \mathbf{1}_{\{l_\theta^\perp(t+s(\theta)) \cap \mathcal{T}_1^k(\bar{\omega}_{n-k+2}^1) \neq \emptyset\}} d\theta, \quad (2.6.1)$$

which would be the analogue of (2.5.7), makes sense in principle if we understand ω 's as the random variables in the Peres–Solomyak model, which assume the values 0, 1, 2, 3 (instead of values in the interval $[0, \frac{\pi}{2}]$ as in our's and Zhang's models).

But, there is a caveat. We reduced the function of two variables

$$G(\psi, t) := P_{\bar{\omega}_{n-k+2}^1} \left(l^\perp(t) \cap \mathcal{T}_0^k(\psi, \bar{\omega}_{n-k+2}^1) \neq \emptyset \right)$$

to the composition with a function of one variable and the shift (see (2.5.6) for the definition of F):

$$G(\psi, t) = G(0, t + s(\psi)) = F(t + s(\psi)) \quad (2.6.2)$$

thanks to (2.5.7). But looking at (2.6.1), we can notice that the function

$$\mathcal{G}(\theta, t) := \mathbb{E} \mathbf{1}_{\{l_\theta^\perp(t) \cap \mathcal{T}_0^k(\bar{\omega}_{n-k+2}^1) \neq \emptyset\}}$$

cannot be written as some $\mathcal{F}(t + S(\theta))$.

As a result of this misfortune, we cannot write

$$\text{Expectation of the overlap} = \int \mathcal{E}(\theta) d\theta = \int \int \mathcal{F}(t + S(\theta)) \cdot \mathcal{F}(t) d\theta dt$$

as before. Working similarly, this would in turn bring about the term $(\int \mathcal{F} dt)^2$. Instead, we only have that

$$\text{Expectation of the overlap} = \int \mathcal{E}(\theta) d\theta = \int \int \mathcal{G}(\theta, t) \cdot \mathcal{G}(\theta, t + S(\theta)) d\theta dt,$$

and it is not clear (at least to us) how to estimate this integral from below as no change of variables seems to be of help.

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CHAPTER 3

GEOMETRY OF PLANAR CURVES INTERSECTING MANY LINES IN A FEW POINTS

3.1 The statement of the problem

The problem at hand is to better understand the structure of Borel sets in $\mathbb{R}^2$ that have a small intersection with parallel shifts of lines from a whole cone. Here, we work only with sets that are graphs and continuous curves. So we have strong assumptions. But the results claim some estimate on the Hausdorff measure (not merely the Hausdorff dimension).

Initially, we show that a function's graph intersecting all parallel shifts of lines from a nondegenerate cone in at most two points is locally Lipschitz and also present a counter-example showing this fails if more intersection points are allowed.

Next, we prove that any curve that has finitely many intersections with a cone of lines is σ -finite with respect to Hausdorff length and we find a bound on the Hausdorff measure of each "piece."

On the other hand, in [1] it was shown that, given countably many graphs of functions, there is another function whose graph has only one intersection with all shifts of the given graphs but whose graph has dimension 2.

This result shows that there is a "thick" graph having only one intersection with all shifts of countably many other graphs. In our turn, we show that the graph having finitely many intersection with shifts of the whole cone of linear functions must be in fact very "thin".

Proposition 3.1.1. *Let $\lambda > 0$ be a fixed number and consider all the cones of lines with slopes between λ and $-\lambda$ (containing the vertical line). If $f: (0, 1) \rightarrow \mathbb{R}$ is a continuous function such that any line of these cones intersects its graph at at most two points, then f is locally Lipschitz.*

Notice that our hypothesis implies that no three points of the graph of f can lie on the same line that is a parallel shift of a line from a given cone.

For the proof we will need the following lemmas.

Lemma 3.1.2. *Every convex (or concave) function on an open interval is locally Lipschitz.*

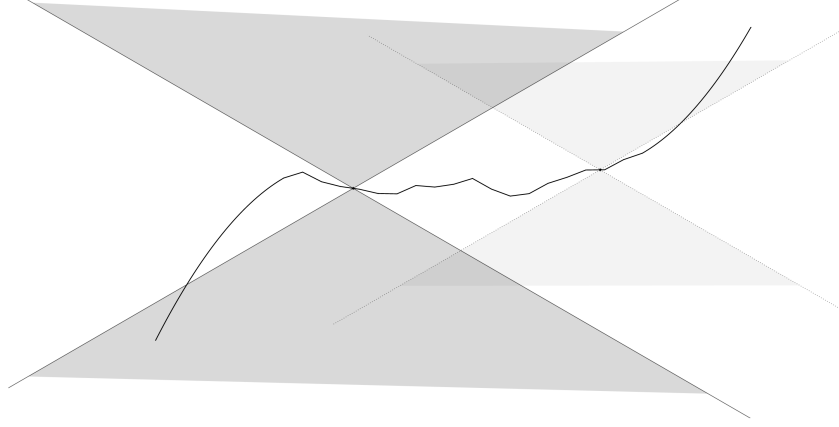


Figure 3.1 Each line from any cone intersects the graph at at most two points.

Lemma 3.1.3. *If a function $g: (0, 1) \rightarrow \mathbb{R}$ is continuous and has a unique local extremum, $\tilde{x}$, inside $(0, 1)$, then it is strictly monotone in $(0, \tilde{x}]$ and $[\tilde{x}, 1)$ with opposite monotonicity on each interval.*

Proof of Lemma 3.1.3. Suppose $\tilde{x}$ is a local minimum for g . We will show that g is strictly monotone increasing in $[\tilde{x}, 1)$. Assume the contrary, i.e., consider two points $x_1 < x_2 \in [\tilde{x}, 1)$ such that $g(x_1) \geq g(x_2)$. On the compact interval $[x_1, x_2]$, the function g has to attain a minimum and a maximum, which respectively are at x_2 and x_1 otherwise the uniqueness of $\tilde{x}$ is contradicted. If $x_1 = \tilde{x}$, the point $\tilde{x}$ is not a local minimum and so $\tilde{x} < x_1$. Again, $\tilde{x}$ and x_1 must be the minimum and maximum, respectively, of g in $[\tilde{x}, x_1]$, which in turn says x_1 is a local maximum contradicting the uniqueness of $\tilde{x}$. Therefore, $g(x_1) < g(x_2)$ and g is strictly monotone increasing on $[\tilde{x}, 1)$. Similarly, on $(0, \tilde{x}]$ g is (strictly) monotone decreasing and the same arguments work for when $\tilde{x}$ is local maximum. $\square$

Proof. Consider the slope function of f , $S(x, y) = \frac{f(x)-f(y)}{x-y}$, and note that

$$S(x, y) = \frac{f(x) - f(y)}{x - y} = \zeta \iff f(x) - \zeta x = f(y) - \zeta y.$$

If for any two points $x < y \in (0, 1)$ we have $|S(x, y)| < \lambda$, then f is Lipschitz (with Lipschitz constant at most λ).

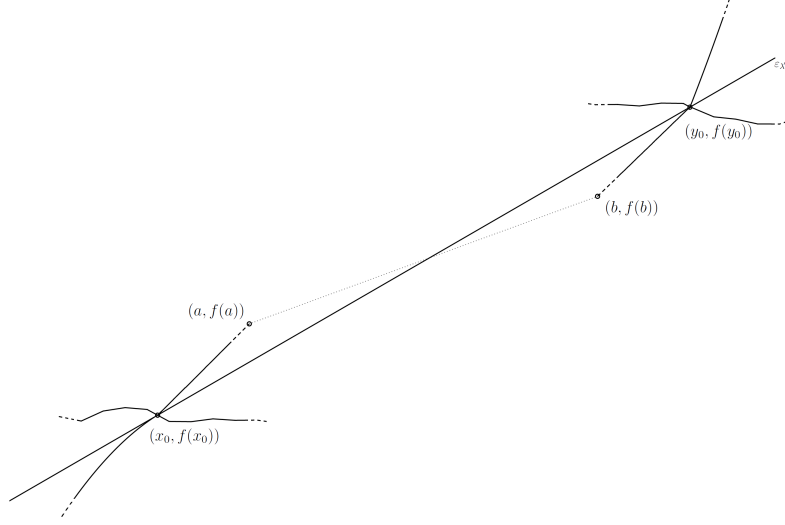


Figure 3.2 $S(x_0, y_0) = \lambda' \geq \lambda$; The part of the graph of f between x_0 and y_0 cannot lie on different sides of $\epsilon_{\lambda'}$.

Now suppose that there exist $x_0, y_0 \in (0, 1)$ for which $|S(x_0, y_0)| \geq \lambda$ and consider the case where $S(x_0, y_0) = \lambda' \geq \lambda$. Since $S(x, y) = S(y, x)$, we may assume that $x_0 < y_0$. We will denote the line passing through $(x_0, f(x_0))$ and $(y_0, f(y_0))$ by $\epsilon_{\lambda'}$.

If there are numbers $x_0 < a < b < y_0$ such that

$$(S(x_0, a) - \lambda')(S(x_0, b) - \lambda') \leq 0,$$

then by the continuity of $S(x, \cdot)$ there has to exist a number $c \in [a, b]$ such that $\frac{f(x_0) - f(c)}{x_0 - c} = \lambda' = \frac{f(x_0) - f(y_0)}{x_0 - y_0}$. But this means that $(x_0, f(x_0))$, $(c, f(c))$ and $(y_0, f(y_0))$ are colinear, which contradicts our hypothesis and therefore $S(x_0, y)$ has to be constantly greater or constantly less than λ' for $x_0 < y < y_0$ (see Figure 3.2). For the same reasons $S(x_0, y)$ has to be constantly greater or constantly less than λ' also for $y > y_0$ and the same holds for $S(x, y_0)$ for $x < x_0$.

Graphically, this means that $\epsilon_{\lambda'}$ separates f in three parts that do not intersect $\epsilon_{\lambda'}$; one before x_0 , one over (x_0, y_0) , and one after y_0 . We proceed to show that the part over (x_0, y_0) lies on a different side of $\epsilon_{\lambda'}$ from the other two.

Let us consider the case when $S(x_0, y) < \lambda'$ for $x_0 < y < y_0$. Then, the function $f(x) - \lambda'x$ defined on $[x_0, y_0]$ attains a maximum at x_0 and at y_0 (which also implies that $S(x, y_0) > \lambda'$ for $x_0 < x < y_0$) and let $\tilde{y} \in (x_0, y_0)$ be the point where $f(x) - \lambda'x$ attains a minimum (see Figure 3.4).



Figure 3.3 The two cases when $x_0 < y < y_0$.

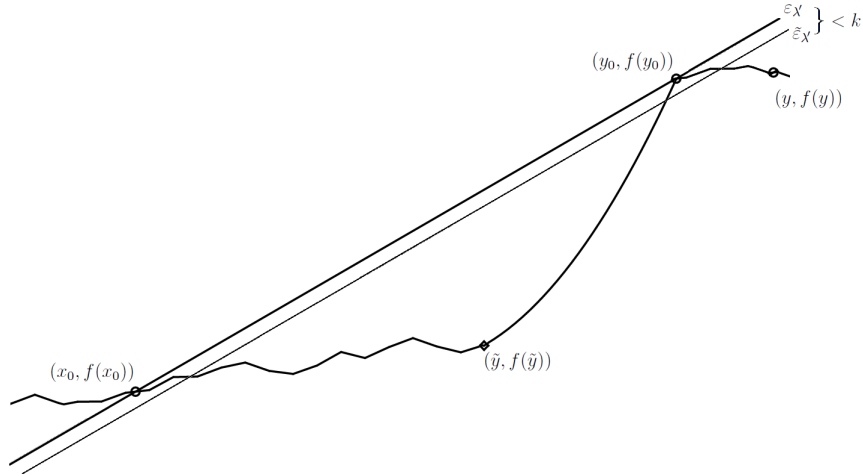


Figure 3.4 If $S(x_0, y) < \lambda'$ for every $y \notin (x_0, 1) \setminus \{y_0\}$, by moving the line $\epsilon_{\lambda'}$ slightly down, we get three points of intersection.

Now, suppose additionally that $S(x_0, y) < \lambda'$ also for $y > y_0$.

Pick a number k with $f(x_0) - \lambda'x_0 > k > \max\{f(\tilde{y}) - \lambda'\tilde{y}, f(y) - \lambda'y\}$ for some $y > y_0$. Then, we have simultaneously

$$f(\tilde{y}) - \lambda'\tilde{y} < k < f(x_0) - \lambda'x_0,$$

$$f(\tilde{y}) - \lambda'\tilde{y} < k < f(y_0) - \lambda'y_0,$$

$$f(y) - \lambda'y < k < f(y_0) - \lambda'y_0.$$

The continuity of f and the above inequalities imply that there must exist numbers a, b , and c in $(x_0, \tilde{y})$, $(\tilde{y}, y_0)$, and (y_0, y) respectively such that

$$f(a) - \lambda'a = f(b) - \lambda'b = f(c) - \lambda'c = k$$

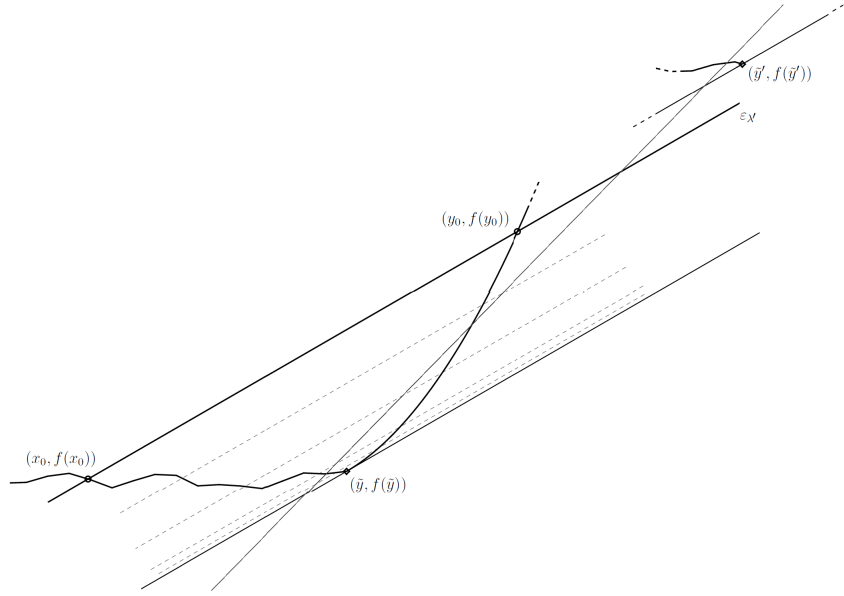


Figure 3.5 If f attains a local minimum at another point $\tilde{y}' > \tilde{y}$, we can find a line of slope greater than λ' intersecting f at three points.

which implies that $(a, f(a))$, $(b, f(b))$, and $(c, f(c))$ are colinear, a contradiction, and therefore $S(x_0, y)$ has to be greater than λ' for $y > y_0$. Working similarly, we see that $S(x, y_0) < \lambda'$ for $x < x_0$.

An identical argument gives us that $\tilde{y}$ is the only point in $[x_0, y_0]$, and eventually in $[x_0, 1)$, where $f(x) - \lambda'x$ attains a local minimum (see Figure 3.5) and from Lemma 3.1.3 we deduce that $f(x) - \lambda'x$ has to be monotone increasing in $[\tilde{y}, 1)$. Hence, for any $x, y \geq \tilde{y}$ we have:

$$x < y \iff f(x) - \lambda'x < f(y) - \lambda'y \stackrel{x < y}{\iff} S(x, y) > \lambda'.$$

However, observe that for any x and y for which $S(x, y) > \lambda'$, the function $S(x, \cdot)$ has to be 1-1 otherwise our hypothesis fails in a similar way as above and, since it is continuous, it has to be monotone in $(x, 1)$ for every $x \in [\tilde{y}, 1)$. Therefore, f is either convex or concave in $[\tilde{y}, 1)$ and thus locally Lipschitz in $(\tilde{y}, 1)$ thanks to Lemma 3.1.2.

In particular, f has to be convex in $[\tilde{y}, 1)$. Indeed, assume f is concave and let x be any number in $(\tilde{y}, y_0)$, see Figure 3.6. By concavity, the point $(\tilde{y}, f(\tilde{y}))$ has to lie below the line passing through $(y_0, f(y_0))$ with slope $\zeta = S(x, y_0)$ and, since $\zeta = S(x, y_0) > S(x_0, y_0) = \lambda' \geq \lambda$, the point $(x_0, f(x_0))$ lies above. Hence, this line will intersect the graph of f at some point $(c, f(c))$ with

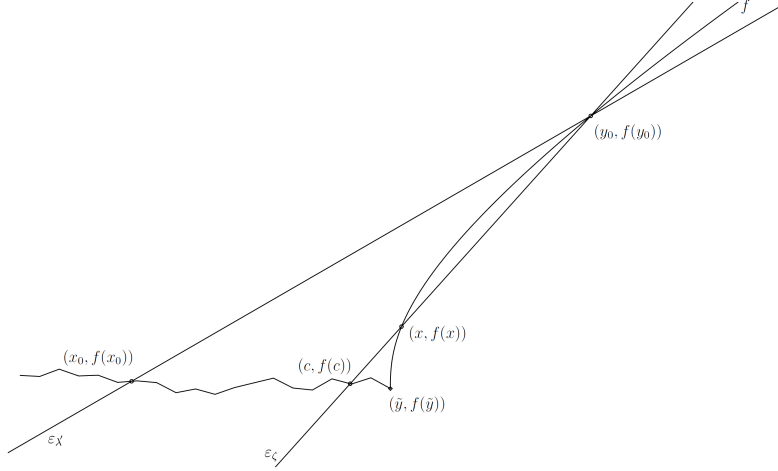


Figure 3.6 $S(x_0, y)$ has to be strictly monotone increasing in $(y_0, 1)$.

$c \in (x_0, \tilde{y})$ and the points $(c, f(c))$, $(x, f(x))$, and $(y_0, f(y_0))$ are colinear, a contradiction.

If we instead assume that $S(x_0, y) > \lambda'$ for $x_0 < y < y_0$, working similarly we conclude that there must exist $\tilde{y} \in [x_0, y_0]$ such that f is concave in $(0, \tilde{y}]$.

The case when there exist $x_0, y_0 \in (0, 1)$ for which $S(x_0, y_0) = \lambda' \leq -\lambda$ is identical and gives us the reverse implications.

To sum up, we conclude that there are points $\tilde{x}, \tilde{y} \in (0, 1)$ such that f has some particular convexity on $(0, \tilde{x}]$ and on $[\tilde{y}, 1)$. These intervals cannot overlap, because otherwise f would be a line segment of slope at least λ (or at most $-\lambda$) on $[\tilde{y}, \tilde{x}]$, which contradicts our hypothesis and so $\tilde{x} \leq \tilde{y}$. Let $\tilde{x}$ be the maximal point so that f is, for instance, convex on $(0, \tilde{x}]$, and $\tilde{y}$ the minimal so that f is convex on $[\tilde{y}, 1)$. When $\tilde{x} \neq \tilde{y}$, for every points $x, y \in [\tilde{x}, \tilde{y}]$ we have $|S(x, y)| \leq \lambda$ and f is Lipschitz in $[\tilde{x}, \tilde{y}]$ with Lipschitz constant λ .

This concludes the proof. □

Of course, any continuous function that satisfies the condition of the proposition and has different convexity on $(a, \tilde{x}]$ and on $[\tilde{y}, b)$ has to additionally satisfy $\lim_{x \rightarrow a^+, y \rightarrow b^-} |S(x, y)| < \lambda$.

Furthermore, notice that the fact that the cone is vertical (or at least that it contains the vertical line) is essential to get the locally Lipschitz property. Indeed, if C is a cone avoiding the vertical line, we can restrict the function $\sqrt[3]{x}$ to a sufficiently small interval around 0 so that it intersects all the lines of the cone at at most two points. But $\sqrt[3]{x}$ is clearly not Lipschitz around 0. However, we

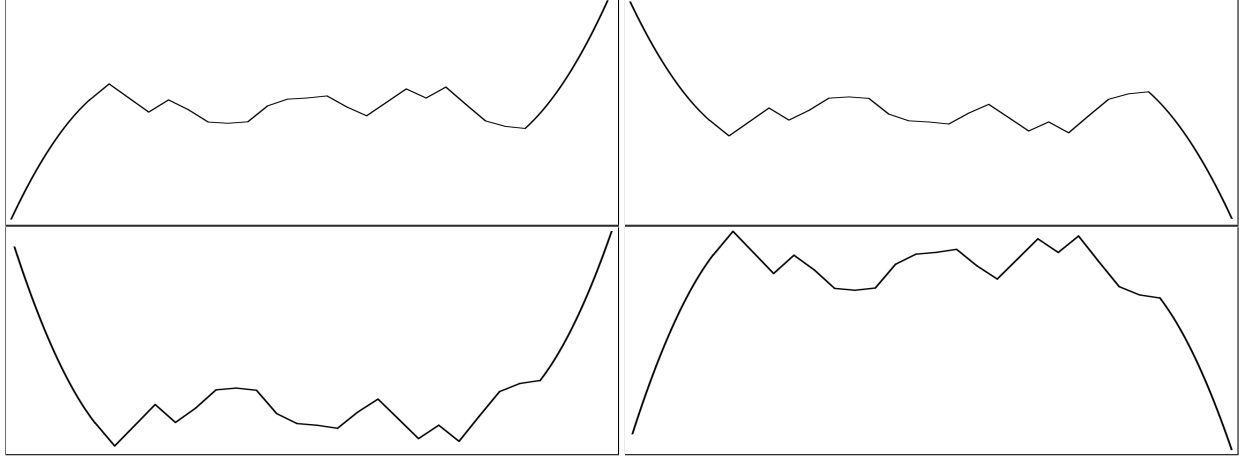


Figure 3.7 All the possible ways the graph of f can look like.

do have the following corollary.

Corollary. *Let $\lambda_1 > 0 > \lambda_2$ be some fixed numbers and consider all the cones of lines with slopes between λ_1 and λ_2 (containing the vertical line). If $f: (0, 1) \rightarrow \mathbb{R}$ is a continuous function satisfying the same condition as above, then it is locally Lipschitz.*

Proof. The inequalities $|S(x, y)| < \lambda$ and $|S(x, y)| \geq \lambda$ in this case correspond to $\lambda_2 < S(x, y) < \lambda_1$ and $S(x, y) \geq \lambda_1$ or $S(x, y) \leq \lambda_2$, respectively. The proof is the same as before and on the regions where f is not convex or concave it is Lipschitz with Lipschitz constant the maximum of λ_1 and $-\lambda_2$. $\square$

Remark. *All the above remains true for any interval (a, b) . It is not hard to see that the same proof also works in the case where f is defined on a closed interval, but Lemma 3.1.2 cannot be used in this setting. However, if $f: [0, 1] \rightarrow \mathbb{R}$, its restriction $f|_{(0,1)}$ is locally Lipschitz.*

3.2 An example

It is natural then to ask whether our assumption still gives us the locally Lipschitz property when we allow more points of intersection. It turns out this fails even for at most 3 points of intersection in the sense that there can be infinitely many points around where the function cannot be locally Lipschitz. Here, we construct such a function whose graph intersects a certain cone of lines at at most three points.

Consider the sequence $a_k = \frac{1}{2} - \frac{1}{2^k}$ for $k \geq 1$, and on the each of the intervals $[a_k, a_{k+1}]$ define a continuous function f_k with the following properties:

- i) $f_1(0) = 0, f_1(\frac{1}{4}) = f_2(\frac{1}{4}) = \frac{\lambda}{4}$;
- ii) $f_{k+1}(a_{k+1}) = f_k(a_{k+1})$;
- iii) $f_k(a_{k+1}) = \frac{1}{2} (f_k(a_k) + f_{k-1}(a_{k-1}))$;
- iv) f_{2k} is monotone decreasing and convex on $[a_{2k}, a_{2k+1}]$ and f_{2k-1} is monotone increasing and concave on $[a_{2k-1}, a_{2k}]$;
- v) the tangent line to f_k at $(a_k, f_k(a_k))$ is vertical.

Let $f: [0, 1] \rightarrow \mathbb{R}$ be the function given by

$$f(x) = \begin{cases} f_k(x) & \text{if } x \in [a_k, a_{k+1}), \\ f_k(1 - x) & \text{if } x \in (1 - a_{k+1}, 1 - a_k], \\ \frac{\lambda}{6} & \text{if } x = \frac{1}{2} \end{cases}$$

for all $k \geq 1$ (Figure 3.8), which is clearly continuous in $(0, 1) \setminus \{\frac{1}{2}\}$ because of (ii). Observe that the sequence $(b_k) = (f_k(a_k))$ is recursively defined by $b_{k+1} = \frac{b_k + b_{k-1}}{2}$ (through property (iii)) and it converges. In particular, we have $\frac{b_{k+1} - b_k}{b_k - b_{k-1}} = -\frac{1}{2}$ and therefore

$$b_{k+1} = b_k + \left(\frac{-1}{2}\right)^{k-1} (b_2 - b_1) \implies b_{k+1} = b_2 - \frac{1}{3} \left(1 - \left(\frac{-1}{2}\right)^{k-1}\right) (b_2 - b_1). \quad (3.2.1)$$

In our case, we have $b_1 = f_1(0) = 0, b_2 = f_2(\frac{1}{4}) = \frac{\lambda}{4}$, and also

$$f_k(a_k) = \frac{\lambda}{6} \left(1 - \left(\frac{-1}{2}\right)^{k-1}\right),$$

hence $\lim_{k \rightarrow +\infty} f_k(a_k) = \frac{\lambda}{6}$. But note that for every $x \in (0, \frac{1}{2})$ there is an $n \geq 1$ for which $x \in [a_n, a_{n+1})$ and, since each f_k is monotone in $[a_k, a_{k+1})$ for every k , we get

$$\min \{f_n(a_n), f_{n+1}(a_{n+1})\} \leq f(x) \leq \max \{f_n(a_n), f_{n+1}(a_{n+1})\}.$$

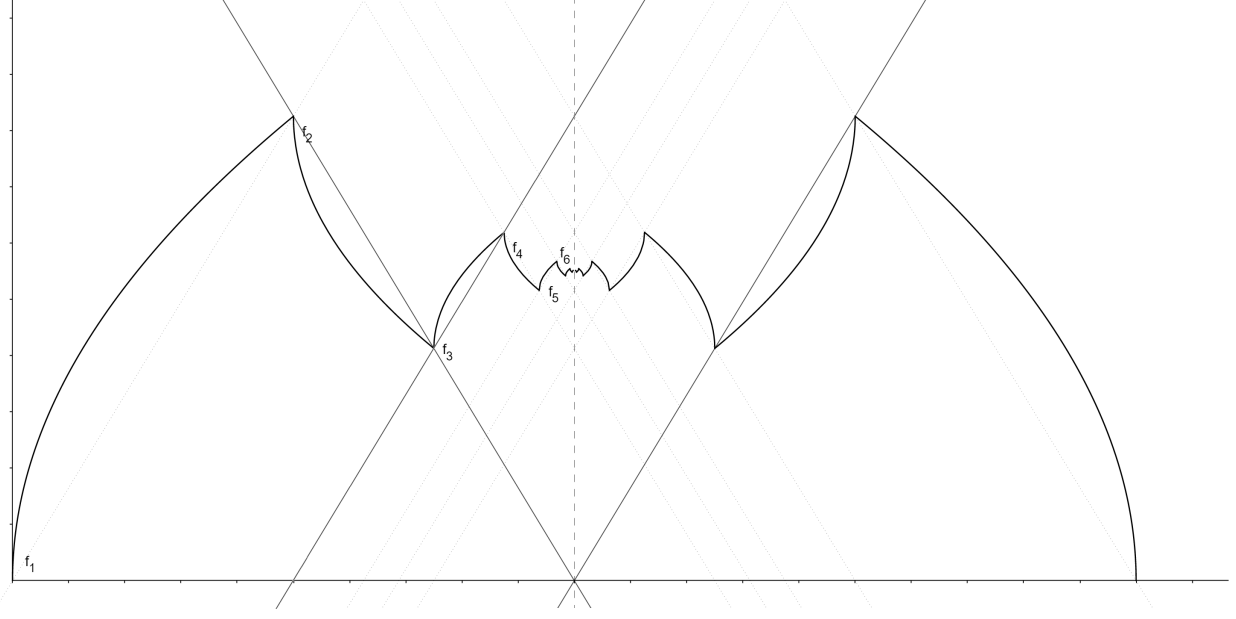


Figure 3.8 At most 3 points of intersection with any line inside the cones.

Therefore, we have $\lim_{x \rightarrow \frac{1}{2}^-} f(x) = \frac{\lambda}{6} = f(\frac{1}{2})$, and similarly for $x \in (\frac{1}{2}, 1)$, which means that f is also continuous at $\frac{1}{2}$.

However, by construction f is locally Lipschitz on $(0, 1) \setminus \{\frac{1}{2}\}$ except at around a_k and $1 - a_k$, $k \geq 1$, and therefore it is not locally Lipschitz around $\frac{1}{2}$ either, because $a_k \rightarrow \frac{1}{2}$ as $k \rightarrow +\infty$.

Now we proceed to show the graph of f has at most 3 intersection points with any line inside a vertical cone with slopes between λ and $-\lambda$.

Each f_k is monotone and has certain concavity on $[a_k, a_{k+1}]$, hence its graph is contained inside the triangle T_k with vertices $(a_k, f(a_k))$, $(a_{k+1}, f(a_{k+1}))$, and $(a_k, f(a_{k+1}))$ (see Figure 3.9) and therefore any line intersecting the graph of f (at at least two points) has to pass through some of these triangles. Notice, however, that if a line passes through two nonconsecutive triangles, say T_k and T_{k+j} ($j > 1$), then it falls outside the admissible cone of lines. In particular, (because of properties (ii) through (iv)) each T_{k+1} is half the size of T_k and they are placed in such a way that the maximum and minimum slope a line through them can have are respectively the maximum and the minimum of the quantities

$$\frac{f_{k+j}(a_{k+j}) - f_k(a_k)}{a_{k+j} - a_k} \quad \text{and} \quad \frac{f_{k+j}(a_{k+j+1}) - f_k(a_{k+1})}{a_{k+j} - a_{k+1}},$$

when one of the numbers k and $k + j$ is even and the other is odd, and the maximum and minimum

of the quantities

$$\frac{f_{k+j}(a_{k+j+1}) - f_k(a_k)}{a_{k+j} - a_k} \quad \text{and} \quad \frac{f_{k+j}(a_{k+j}) - f_k(a_{k+1})}{a_{k+j} - a_{k+1}},$$

when k and $k + j$ are both even or both odd. Using (3.2.1) we can see that each of the above is bounded in absolute value by λ whenever $j > 1$.

For the same reasons any admissible line passing through $(\frac{1}{2}, \frac{\lambda}{6})$ intersects the graph only at that point, because

$$\left| \frac{f_k(a_k) - \frac{\lambda}{6}}{a_k - \frac{1}{2}} \right| = \frac{\lambda}{3} < \lambda.$$

Therefore, the admissible lines intersecting the graph necessarily pass through two (or maybe only one) consecutive triangles and each such line intersects the graph of f_k at at most two points because of (iv). Furthermore, due to the difference in concavity of f_k and f_{k+1} , a line cannot intersect both of their graphs at two points, because then it would need to have both negative and positive slope, which is absurd.

An example of a sequence (f_k) of functions with the above properties is the following:

$$f_k(x) = \frac{\lambda}{6} \left(1 - \left(\frac{-1}{2} \right)^{k-1} \right) + \frac{(-1)^{k+1} \lambda}{2^{\frac{k+1}{2}}} \sqrt{x - a_k}.$$

3.3 Hausdorff measure

Marstrand in [5, Theorem 6.5.III] proved that if a Borel set on the plane has the property that if the lines in a positive measure of directions intersect this Borel set at a set of Hausdorff dimension zero, then the Hausdorff dimension of this Borel set is at most 1. (3.3.1)

In particular, this happens if the intersections are at most countable. The Borel assumption is essential.

That said, Marstrand's theorem does not in general guarantee the Hausdorff measure of the Borel set is finite. Our next goal will be to deal with the Hausdorff measure of a continuous curve and also generalise to arbitrarily many points of intersection with our cones (still finitely many, though). It turns out that the curve has to always be σ -finite with respect to the $\mathcal{H}^1$ measure.

In order to proceed we need set up things more rigorously:

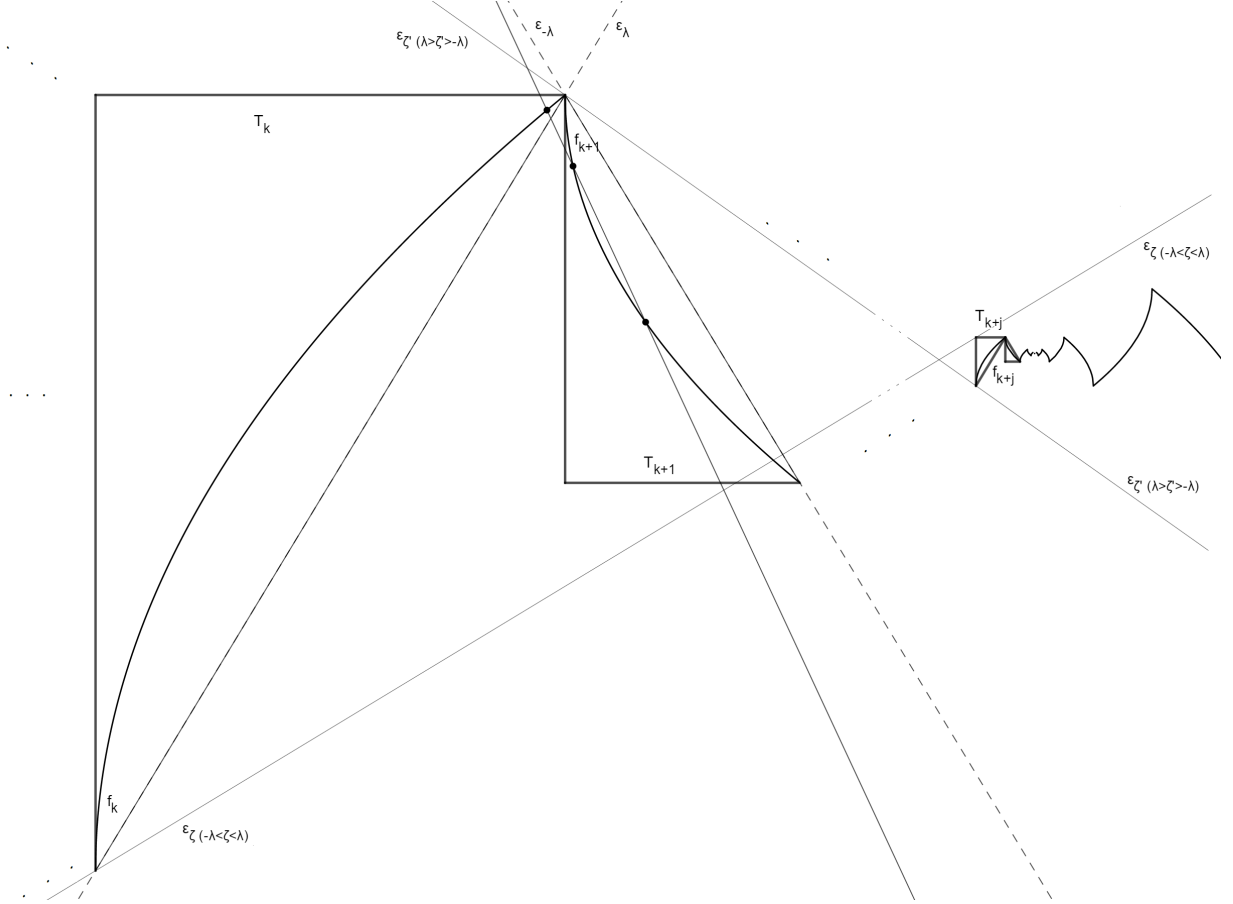


Figure 3.9 The case when k and $k + j$ are both odd.

Notations. Let $C(\phi, 0) = \{(x, y) \in \mathbb{R}^2 : |y| \geq \tan(\phi) |x|\}$ denote the vertical closed cone in between the lines through the point $(0, 0)$ with slopes $\tan(\phi)$ and $-\tan(\phi)$ (where $0 < \phi < \frac{\pi}{2}$). By $C_+(\phi, 0)$ we will denote the upper half of the cone $C(\phi, 0)$, that is $C_+(\phi, 0) = \{(x, y) \in \mathbb{R}^2 : |y| \geq \tan(\phi) |x|, y \geq 0\}$, and by $C_-(\phi, 0)$ its lower half. Let $C(\phi, \rho)$ be the cone's counter-clockwise rotation by angle ρ , $C(\phi, 0, h) = B_0(h) \cap C(\phi, 0)$, where $B_x(r) = B(x, r)$ is the closed ball centred at x with radius r , and $C_P(\phi, 0)$ the translation of $C(\phi, 0)$ so that its vertex is the point P . Finally, C^* will denote the dual cone of C , that is $C^*(\phi, 0) = \overline{C(\phi, 0)^C}$. We will be combining different notation in a natural way, for example $C_+(\phi, \rho, h)$ is the upper half of the truncated and rotated cone with vertex at 0.

$\gamma: [0, 1] \rightarrow \mathbb{R}^2$ will be a continuous curve.

3.3.1 The main hypothesis

Fix an integer $k \geq 2$. Fix an angle $\phi \in (0, \frac{\pi}{2})$ and a rotation $\rho \in [0, 2\pi)$. A line contained inside the cone $C_P(\phi, \rho)$ for any point $P \in \mathbb{R}^2$ intersects the curve γ at at most k points. (3.3.2)

Any such line will be called *admissible*. A cone consisting of only admissible lines will also be called *admissible*.

3.3.2 γ is σ -finite

For simplicity and without loss of generality we will assume the the curve $\gamma: [0, 1] \rightarrow \mathbb{R}^2$ is bounded inside the unit square and that $(0, 0), (1, 1) \in \gamma$. We additionally assume that the cones of our hypothesis are vertical, i.e., that $\rho = 0$.

Theorem 3.3.1. *γ can be split into countably many sets γ_n with finite $\mathcal{H}^1$ measure. In particular, γ is 1-rectifiable.*

The following lemma plays a key role in the proof of this theorem, but we will postpone its proof until later.

Lemma 3.3.2. *For every point $P \in \gamma$ there exists an admissible cone $C_P(\theta, \rho, h)$ that avoids the curve γ except at P , that is $C_P(\theta, \rho, h) \cap \gamma = \{P\}$.*

In view of Lemma 3.3.2 — by slightly tilting ρ , enlarging θ and monotone decreasing h — we may assume the triplet (θ, ρ, h) consists of rational numbers. If $\{(\theta_n, \rho_n, h_n)\}$ is an enumeration of all rational triples that still lie within our admissible set, then we can decomposed γ into the countably many sets

$$\gamma_n = \{P \in \gamma : C_P(\theta_n, \rho_n, h_n) \cap \gamma = \{P\}\}$$

(see Figure 3.10). Note that γ_n are not necessarily disjoint for different values of n .

We proceed to prove each one of them has finite $\mathcal{H}^1$ measure. Note that this is not new knowledge and it can be found, for example, in [2, Lemma 3.3.5] or [6, Lemma 15.13] in a more general setup. Nevertheless, we present it here for completeness.

For the rest of this section n will be fixed.



Figure 3.10 The curve γ and its part γ_n for $\theta_n, \rho_n = 0$, and h_n .

Lemma 3.3.3. $\mathcal{H}^1(\gamma_n) < \frac{2k}{\cos(\theta_n)}$.

Proof. Without loss of generality we may assume the cone $C_P(\theta_n, \rho_n, h_n)$ is vertical, i.e., that $\rho_n = 0$. Let us now split the unit square into N vertical strips, S_j ($j = 1, 2, \dots, N$), of base length $\frac{1}{N}$ with N sufficiently large so that $\frac{1}{N} < \cos(\theta_n) h_n$. Let J be the set of indices j for which

$$S_j \cap \gamma_n \neq \emptyset$$

and for any point $P \in \gamma$ denote the connected component of γ inside S_j through $P \in S_j \cap \gamma$ by $\Gamma_P^*(j)$.

Fix a $j \in J$ and consider a point $P \in S_j \cap \gamma_n$. Since $\frac{1}{N} < \cos(\theta_n) h_n$, the sides of S_j necessarily intersect both sides of the cone $C_P(\theta_n, 0, h_n)$ creating thus two triangles both contained inside the ball $B_P(\frac{1}{N \cos(\theta_n)})$ (see Figure 3.11). For any point $P' \in S_j \cap \gamma_n$ other than P there are two

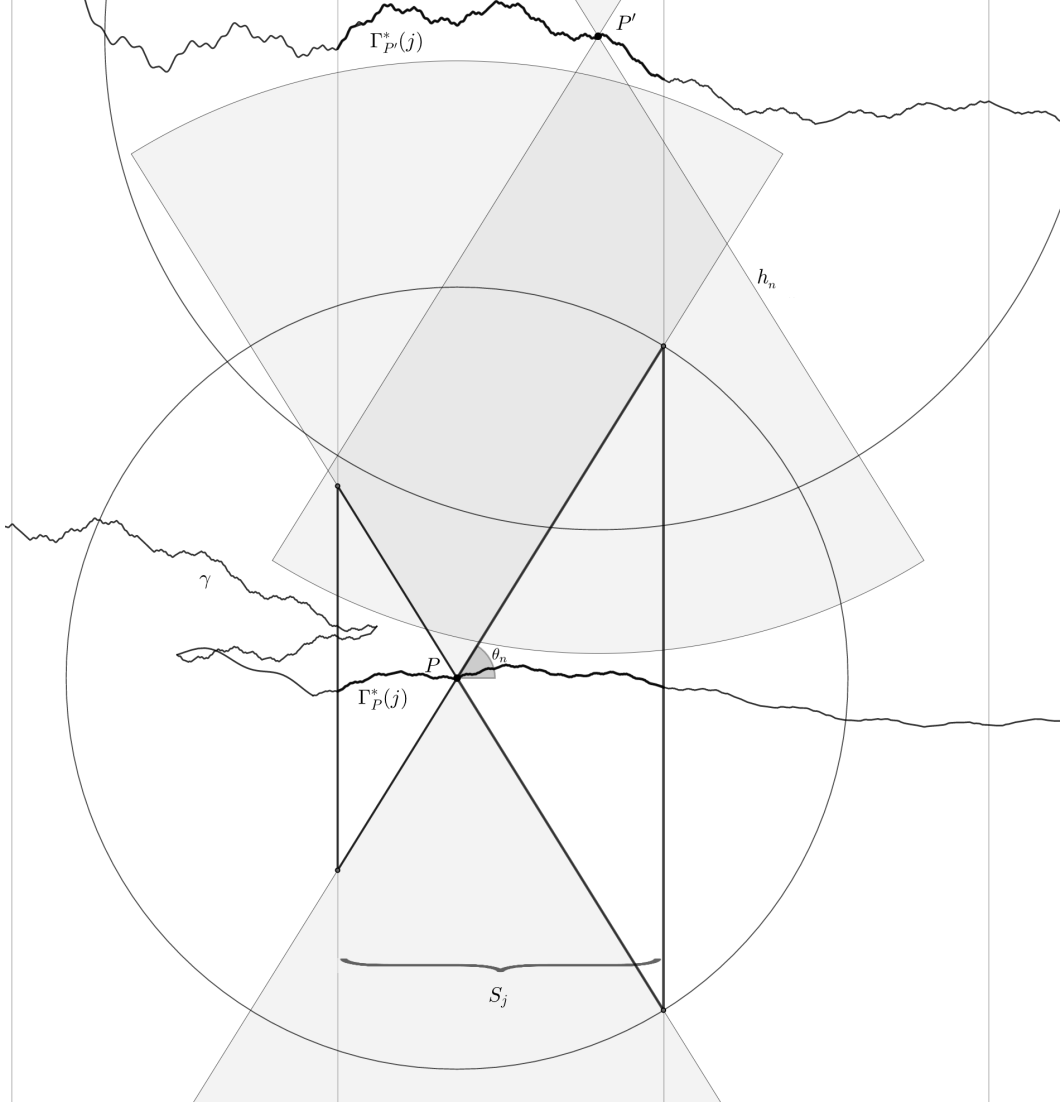


Figure 3.11 Each cone intersects a strip of length $\frac{1}{N} < \cos(\theta_n) h_n$.

cases: either $|P - P'| \leq h_n$ or $|P - P'| > h_n$. In the first case, the sets $\Gamma_P^*(j)$ and $\Gamma_{P'}^*(j)$ are both contained inside the two triangles $C_P^*(\theta_n, 0) \cap S_j$. In the second, they are necessarily disjoint, because $C_P(\theta_n, 0, h_n)$ is free from points of γ (other than P). These additionally imply that there can be no more than $\frac{1}{\sin(\theta_n) h_n}$ such distinct paths inside S_j . In particular,

$$P \in \Gamma_P^*(j) \subset S_j \cap \gamma \cap B_P(h_n) \subset C_P^*(\theta_n, 0, h_n) \cap S_j \subset B_P\left(\frac{1}{N \cos(\theta_n)}\right).$$

Now, let $\mathcal{P}_j$ be a maximal set of points in $S_j \cap \gamma_n$ such that the sets $\Gamma_P^*(j)$ for $P \in \mathcal{P}_j$ are all disjoint and observe that $S_j \cap \gamma_n$ is covered by the balls $B_P(\frac{1}{N \cos(\theta_n)})$ with $P \in \mathcal{P}_j$. Indeed, if $P_0 \in S_j \cap \gamma_n$ is not inside the set $\bigcup_{P \in \mathcal{P}_j} B_P(\frac{1}{N \cos(\theta_n)})$, then by construction it is also outside

$\bigcup_{P \in \mathcal{P}_j} B_P(h_n)$ and therefore $\Gamma_{P_0}^*(j)$ and $\Gamma_P^*(j)$ are disjoint for all $P \in \mathcal{P}_j$, which contradicts the maximality of $\mathcal{P}_j$. Moreover, due to the connectedness of γ , the set $\{P\}$ has to be path-connected with $(0, 0)$ and $(1, 1)$ and therefore each $\Gamma_P^*(j)$ has to intersect at least one side of the strip S_j . Hence, because of (3.3.2), there can be at most $2k$ of these paths, i.e., $\#(\mathcal{P}_j) \leq \min\{2k, \frac{1}{\sin(\theta_n) h_n}\} \leq 2k$ for every $j \in J$. Therefore,

$$\gamma_n \cap S_j \subset \bigcup_{P \in \mathcal{P}_j} B_P\left(\frac{1}{N \cos(\theta_n)}\right) \implies \gamma_n \subset \bigcup_{j \in J} \bigcup_{P \in \mathcal{P}_j} B_P\left(\frac{1}{N \cos(\theta_n)}\right)$$

and the total sum of the radii of these balls is at most

$$2k \frac{1}{N \cos(\theta_n)} \#(J) \leq \frac{2k}{\cos(\theta_n)}.$$

Finally, if $\tilde{\gamma}_n = \{P \in \gamma : C_P(\theta_n, 0, h_n/2) \cap \gamma = \{P\}\}$, then $\gamma_n \subset \tilde{\gamma}_n$. Repeating the above construction with $\frac{1}{N} < \cos(\theta_n) \frac{h_n}{2}$, we get a cover of $\tilde{\gamma}_n$ — and thus of γ_n — consisting of balls with a total sum of radii at most $\frac{2k}{\cos(\theta_n)}$. The result follows. $\square$

Remark. *In the above construction we are in fact able to cover the whole part of γ inside $\bigcup_{j \in J} S_j$ with the same balls, and not merely γ_n .*

Eventually, the curve γ has to be σ -finite.

3.3.3 Cones free of γ

Here we prove Lemma 3.3.2.

Fix $P \in \gamma$. Since γ is bounded, there must exist an $\tilde{h} > 0$ such that $C_P(\phi, 0) \cap \gamma = C_P(\phi, 0, \tilde{h}) \cap \gamma$.

If

$$C_P(\phi', 0) \cap \gamma = \{P\} \quad \text{or} \quad C_P(\phi', 0, h) \cap \gamma = \{P\}$$

for some $\phi' \in [\phi, \frac{\pi}{2})$ and some $h > 0$, then we are done.

Suppose this does not happen. Then, for all $\phi' \in [\phi, \frac{\pi}{2})$ and for all sufficiently small $h > 0$ we have

$$C_P(\phi', 0, h) \cap \gamma \setminus \{P\} \neq \emptyset. \tag{3.3.3}$$

Lemma 3.3.4. *For any $P \in \gamma$ the set $C_P(\phi, 0) \cap \gamma$ has finitely many (closed) connected components.*

Proof. Since γ is connected, every point of $C_P(\phi, 0) \cap \gamma$ has to be path-connected with the point P through some part of the curve γ . There are two possibilities: either that path is entirely contained inside $C_P(\phi, 0)$ or it has to pass through its sides. If a path does not intersect the sides, then it necessarily has to pass through P otherwise γ would not be connected. This yields precisely one connected component — the one containing P — and all the rest (if any) have to intersect the sides of the cone. If these components are infinitely many, there have to exist also infinitely many points of intersection on the sides of the cone; at least one for each connected component. But this contradicts (3.3.2). $\square$

Remark. *The connected components of Lemma 3.3.4 total at most $2k$ and P need not be a point of the curve. This lemma is still valid regardless of the cone we are working with as soon as it is in our admissible family of cones.*

Let $\Gamma_P(\phi, 0)$ be the connected component of $C_P(\phi, 0) \cap \gamma$ that contains the point P , which because of (3.3.3) cannot be precisely the point set $\{P\}$. Because of Lemma 3.3.4, the set $C_P(\phi, 0) \cap \gamma \setminus \Gamma_P(\phi, 0)$ is compact and thus there exists $h_0 > 0$ such that $C_P(\phi, 0, h_0) \cap \gamma \subset \Gamma_P(\phi, 0)$. Observe that $C_P(\phi, 0) \cap \gamma \setminus \Gamma_P(\phi, 0)$ could be empty in general in which case $h_0 = \infty$, however, we can always assume that $h_0 \leq \tilde{h}$.

Next, we bisect our cone into two new identical cones sharing one common side

$$C_P(\phi, 0) = C_P(\phi_1, \rho_1) \cup C_P(\phi_1, -\rho_1),$$

where $\phi_1 = \frac{\pi}{4} + \frac{\phi}{2}$ and $\rho_1 = \frac{\pi}{4} - \frac{\phi}{2}$, and repeat the above arguments for each new cone: If

$$C_P(\phi', \rho_1) \cap \gamma = \{P\} \quad \text{or} \quad C_P(\phi', \rho_1, h) \cap \gamma = \{P\}$$

for some $\phi' \in [\phi_1, \frac{\pi}{2})$ and some $h > 0$, then we are done. Similarly for $-\rho_1$ in place of ρ_1 .

Suppose none of these happen. Then, for all $\phi' \in [\phi_1, \frac{\pi}{2})$ and for all sufficiently small h and h' we have

$$C_P(\phi', \rho_1, h) \cap \gamma \setminus \{P\} \neq \emptyset \quad \text{and} \quad C_P(\phi', -\rho_1, h') \cap \gamma \setminus \{P\} \neq \emptyset. \quad (3.3.4)$$

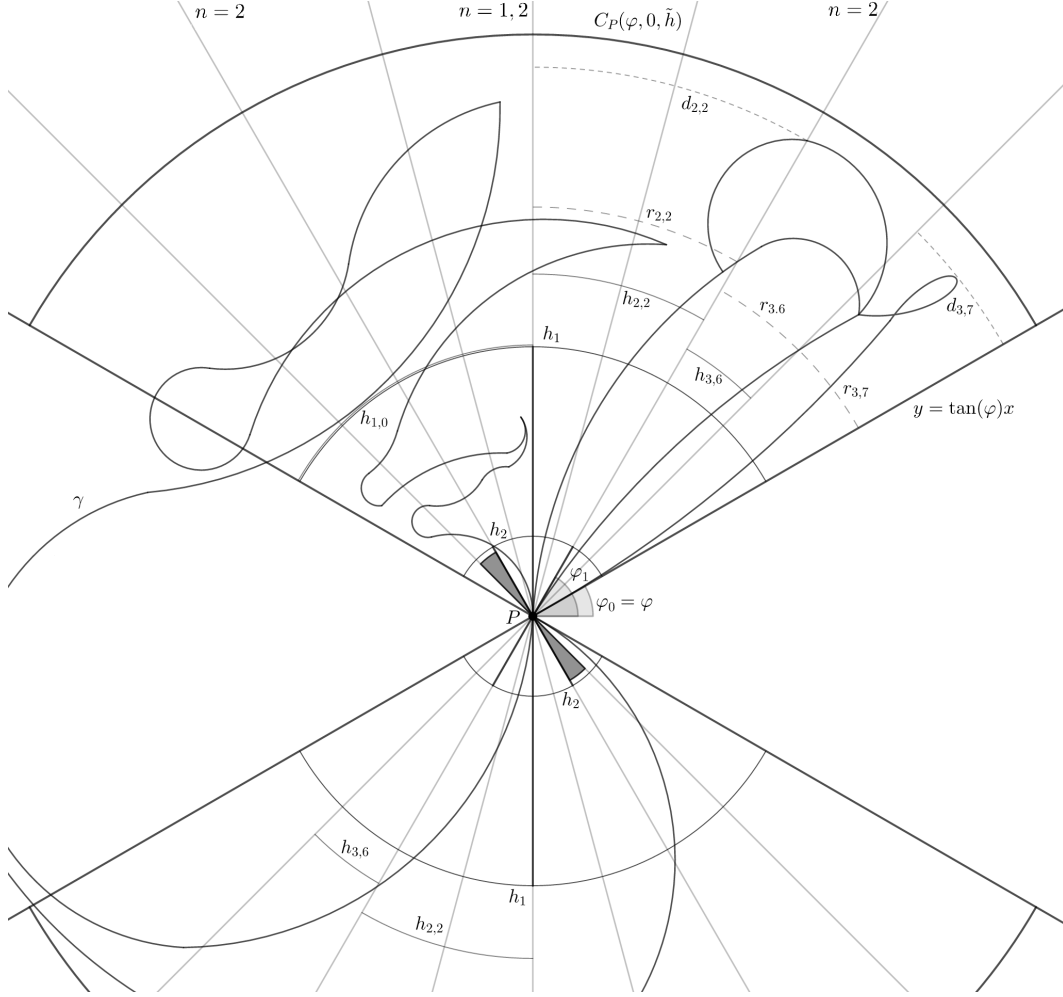


Figure 3.12 Finding a cone free from points of γ . The parameters r , d , and h determine the radius.

We denote by $\Gamma_P(\phi_1, \rho_1)$ and $\Gamma_P(\phi_1, -\rho_1)$ the connected component of

$$C_P(\phi_1, \rho_1) \cap \gamma \quad \text{and} \quad C_P(\phi_1, -\rho_1) \cap \gamma$$

containing P , respectively. Then, the sets $C_P(\phi_1, \rho_1) \cap \gamma \setminus \Gamma_P(\phi_1, \rho_1)$ and $C_P(\phi_1, -\rho_1) \cap \gamma \setminus \Gamma_P(\phi_1, -\rho_1)$ are compact (thanks to Lemma 3.3.4) and thus there exist $h_{1,0}, h_{1,1} \in (0, \tilde{h}]$ such that $C_P(\phi_1, \rho_1, h_{1,0}) \cap \gamma \subset \Gamma_P(\phi_1, \rho_1)$ and $C_P(\phi_1, -\rho_1, h_{1,1}) \cap \gamma \subset \Gamma_P(\phi_1, -\rho_1)$.

We iterate this construction indefinitely (Figure 3.12). If at any step we get

$$C_P(\phi', \rho, h) \cap \gamma = \{P\} \tag{3.3.5}$$

for some ϕ' , ρ , and h , then we have found our desired cone and we stop. Otherwise, we get an

infinite sequence of smaller and smaller cones satisfying the following:

$$\begin{aligned} \{P\} \subsetneq C_P(\phi_n, \rho_{n,i}, h_{n,i}) \cap \gamma \subset \Gamma_P(\phi_n, \rho_{n,i}) \subset C_P(\phi_n, \rho_{n,i}) \\ \text{for all } i = 0, 1, \dots, 2^n - 1 \end{aligned}$$

for all $n \geq 0$ where

$$\begin{aligned} \phi_0 &= \phi & \phi_1 &= \frac{\pi}{4} + \frac{\phi}{2} & \phi_n &= \frac{\pi}{4} + \frac{\phi_{n-1}}{2} \\ \rho_{0,0} &= 0 & \rho_{1,0} &= \rho_1 = \frac{\pi}{4} - \frac{\phi}{2} & \rho_{1,1} &= -\rho_1 & \rho_{n,i} &= (\phi_n - \phi) - i \frac{2(\phi_n - \phi)}{2^n - 1} \\ h_{0,0} &= h_0 & & & & & 0 < h_{n,i} &\leq \tilde{h}. \end{aligned}$$

Note that at the n th iteration we have exactly 2^n truncated closed cones separated by the lines

$$l_{n,i} = P + \{(x, y) : y = \tan(\pi - \phi_n + \rho_{n,i})x\}$$

through P . The sets $\Gamma_P(\phi_n, \rho_{n,i})$ might intersect these lines, but this can happen at at most k may points due to (3.3.2). Let $r_{n,i}$ be the smallest distance between these points of intersection (if any) and P , that is

$$r_{n,i} = \text{dist}(P, l_{n,i} \cap \Gamma_P(\phi_n, \rho_{n,i}) \setminus \{P\})$$

(again we can arbitrarily set some $0 < r_{n,i} \leq \tilde{h}$ if $l_{n,i} \cap \Gamma_P(\phi_n, \rho_{n,i}) \setminus \{P\} = \emptyset$) and let

$$d_{n,i} = \min \left\{ \sup\{d(P, \Gamma_{P+}(t) \setminus P) : t \in (0, 1]\}, \sup\{d(P, \Gamma_{P-}(t) \setminus P) : t \in (0, 1]\} \right\}$$

where $\Gamma_{P+}(t)$ and $\Gamma_{P-}(t)$ are parametrisations of the sets $\Gamma_P(\phi_n, \rho_{n,i}) \cap C_{P+}(\phi_n, \rho_{n,i})$ and $\Gamma_P(\phi_n, \rho_{n,i}) \cap C_{P-}(\phi_n, \rho_{n,i})$ respectively (which in general could be precisely the point set $\{P\}$) with $\Gamma_{P+}(0) = \Gamma_{P-}(0) = P$. Finally, we set

$$h_n = \min\{r_{n,i}, d_{n,i}, h_{n,i} : i = 0, 1, \dots, 2^n - 1\}.$$

Since the above set is finite, $h_n > 0$. From this construction for every $n \geq 0$ we get a collection of truncated cones $C_P(\phi_n, \rho_{n,i}, h_n)$, for $i = 0, 1, \dots, 2^n - 1$, (see Figure 3.12) that have the following

property.

There is a path (part of γ) lying inside the cone that connects the point P with at least one of the two arcs of length $(\pi - 2\phi_n)h_n$ which bound the cone $C_P(\phi_n, \rho_{n,i}, h_n)$. Moreover, these paths avoid any other intersections with that cone's boundary aside P and the (closed) arc(s). (3.3.6)

Now, fix n sufficiently large so that $2^n \geq 2k + 3$. Then, we can find at least $k + 2$ of the cones $C_P(\phi_n, \rho_{n,i}, h_n)$ that contain some path of those mentioned at (3.3.6) all lying on the same half-cone, say on $C_{P+}(\phi, 0, h_n)$. Consider one of the sides of our initial cone $C_P(\phi, 0)$, say $l = P + \{(x, y) : y = \tan(\phi)x\}$, fix $0 < \epsilon < h_n \sin(\pi - 2\phi_n)$ and translate l vertically by ϵ : $l_\epsilon = l + (0, \epsilon)$. Then, l_ϵ necessarily intersects all the 2^n different sectors of the ball $B_P(h_n)$ inside $C_{P+}(\phi, 0, h_n)$, but only the right-most one, $C_{P+}(\phi_n, \rho_{n,2^n-1}, h_n)$, at its arc-like part of the boundary. In particular, l_ϵ has to intersect the sides of at least $k + 1$ sectors that contain the paths described in (3.3.6) and therefore also intersects these paths. Hence, l_ϵ is one of our admissible lines that has at least $k + 1$ intersections with γ , a contradiction.

Lemma 3.3.2 is proved. □

Remarks. *i) In the definition of h_n , three different parameters occur, $r_{n,i}$, $d_{n,i}$, and $h_{n,i}$. Without $h_{n,i}$, (3.3.5) automatically fails; $d_{n,i}$ is to ensure $\Gamma_P(\phi_n, \rho_{n,i})$ will always intersect the boundary of the corresponding cone and $r_{n,i}$ forces this intersection to avoid the sides.*

ii) In the above construction we bisected the initial cone into 2, 4, 8 etc. smaller cones every time. However, any possible way to cut the cones would still work as soon as it eventually yields an infinite sequence.

iii) The same proof can be applied to any cone within our admissible set of directions.

3.4 Higher dimensions

Mattila in [7, Lemma 6.4] generalised Marstrand's results from [5] and showed the following.

Lemma 3.4.1 (Mattila). *Let E be an $\mathcal{H}^s$ measurable subset of $\mathbb{R}^n$ with $0 < \mathcal{H}^s(E) < \infty$. Then,*

$$\dim(E \cap (V + x)) \geq s + m - n$$

for almost all $(x, V) \in E \times G(n, m)$.

In particular, for a Borel set in, say, $\mathbb{R}^2$ we have:

if any 2-dimensional plane in a positive measure of directions intersects this Borel set at a set of Hausdorff dimension at most 1, then the Hausdorff dimension of this Borel set is at most 2.

Furthermore, if every line in the direction of some 2-dimensional cone intersects a Borel set (not merely the graph of some continuous function) at at most countably many points, then any 2-dimensional plane in a positive measure of directions intersects this Borel set by a set of Hausdorff dimension at most 1 (Marstrand) and then the Hausdorff dimension of this Borel set is at most 2 (Mattila).

Of course, the same is also true in $\mathbb{R}^n$, that is, if a Borel set has countable intersection with a certain cone of lines, then its dimension does not exceed $n - 1$.

Now, we restrict our attention to what happens with only 2 points of intersection in higher dimensions and we would like to generalize Proposition 3.1.1 to $\mathbb{R}^n$.

Suppose we have a continuous function $z = f(x, y)$, say, on a square in $\mathbb{R}^2$, satisfying the property that

any line in the direction of a certain open cone with axis along a vector $\mathbf{v} \in \mathbb{R}^3$ intersects the graph at at most two points. (3.4.1)

Then, we would want f to obey the same rule. Namely we ask the following:

Question. *Is a continuous function on $(-1, 1)^2$ having property (3.4.1) locally Lipschitz?*

3.5 Relationships with perturbation theory

The problem we consider in this note grew from a question in perturbation theory of self-adjoint operators (see [4]). The question was to better understand the structure of Borel sets in $\mathbb{R}^n$ that have a small intersection with a whole cone of lines. Marstrand's and Mattila's theorems in [5] and [7], respectively, give a lot of information about the exceptional set of finite-rank perturbations of a given self-adjoint operator. The exception happens when singular parts of unperturbed and

perturbed operators are *not mutually singular*. It is known that this is a rare event in the sense that its measure is zero among all finite-rank perturbations. The paper [4] proves a stronger claim: the dimension of a bad set of perturbations actually drops.

Let us explain what was the thrust from [4] and why that paper naturally gives rise to the questions considered above: what is the structure of Borel sets in $\mathbb{R}^n$ that have a small intersection with all the lines filling a whole cone and their parallel shifts?

In [4], a family of finite rank (self-adjoint) perturbations, A_α , of a self-adjoint (suppose bounded for simplicity) operator A in a Hilbert space $\mathcal{H}$ is considered:

$$A_\alpha := A + B\alpha B^*$$

parametrized by self-adjoint operators $\alpha: \mathbb{C}^d \rightarrow \mathbb{C}^d$ (i.e., Hermitian matrices). The operator $B: \mathbb{C}^d \rightarrow \mathcal{H}$ is a fixed injective and bounded operator. It is also assumed that range of B is cyclic with respect to A . In the case when $d = 1$ (rank-one perturbations), the Aronszajn-Donoghue theorem states that the singular parts of the spectral measures of A and A_α are always mutually singular. However, it is known that for $d > 1$ the singular parts of the spectral measures of unperturbed and perturbed operators are not always mutually singular.

Notice that the space of perturbations, that is the space $H(d)$ of Hermitian $(d \times d)$ matrices, has dimension d^2 . In [3], it was proved that, given a singular measure ν , the scalar spectral measure μ_α of the perturbation A_α is *not* singular with respect to ν for the set of α 's having zero Lebesgue measure in $H(d)$. Such α 's are called *exceptional*, and this result shows that even though the set of exceptional α 's can be non-empty (for $d > 1$), it is a thin set. But is it maybe thinner?

In fact, the following result was proved in [3]. Fix $\alpha_0, \alpha_1 \in H(d)$ where α_1 is in the cone of positive Hermitian matrices and consider $\alpha(t) = \alpha_0 + t\alpha_1$. Then, for any such α_0, α_1 there are at most countably many $t \in \mathbb{R}$ such that the $\alpha(t)$ is exceptional. This extra information allowed the authors in [4] to prove that the Hausdorff dimension of exceptional perturbations is actually at most $d^2 - 1$.

The reader might have noticed an underlying geometric measure theory fact: a Borel set in $\mathbb{R}^n$ (here $n = d^2$) that has an at most countable intersection with a whole cone of lines and their parallel

shifts is, in fact, of dimension $n - 1$.

Thus the dimension drop detected in Marstrand's and Mattila's theorems was instrumental for the drop in dimension for exceptional perturbations.

It seems enticing to understand the structure of the sets that have even less than countable intersection with all parallel shifts of all lines from a fixed cone. Suppose the Borel set under investigation intersects only at at most two, or at most $k < \infty$, points with these lines. What additional knowledge one can obtain about this set? This question motivated the work presented in the previous sections.

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