THE UNIQUENESS AND SMOOTHNESS OF CONFORMAL NORMAL METRICS

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ABSTRACT

It is known that every Riemannian metric on a closed manifold is conformal to a metric whose exponential map preserves the Euclidean volume near a point. This thesis concerns the classification problem of such "conformal normal metrics" on a conformal manifold $(X, [g])$ of dimension $n \geq 3$. We first prove the uniqueness of a conformal normal metric within a fixed 1-jet class of metrics. For the proof, we mainly follow Cao's method in [Cao91] by analyzing a non-linear singular elliptic equation in the framework of weighted Hölder spaces. Our second result concerns the smooth dependence of conformal normal metrics on parameters. As applications, we first construct a smooth Riemannian metric on $X \times X$ that is conformal normal near the diagonal on each fiber, and then use this metric to give a simplified proof of the regularity of Habermann's canonical metric.

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CHAPTER 1

INTRODUCTION

A general conformal structure on dimension $n \geq 3$ manifolds is locally nontrivial. Indeed, by the Weyl-Schouten Theorem, a conformal metric on manifolds of dimension $n \geq 4$ is locally conformally flat if and only if the Weyl tensor vanishes (Cotton tensor for 3-manifolds). In consequence, generic metrics on manifolds of dimension $n \geq 3$ do not admit isothermal coordinates. Nonetheless, within a given conformal class, one can always find a metric with the so-called conformal normal coordinates, a special coordinate system first introduced by Lee and Parker in [LP87]. Explicitly, we have:

Theorem 1.1 (cf.Theorem 5.1, [LP87])**.** *Let X be a smooth manifold together with a conformal class C. At a point $p \in X$, there is a conformal metric $g \in C$ such that for each $N \geqslant 2$*

$$\det g_{ij} = 1 + O\left(|x|^N\right)$$

in g-normal coordinates $\{x^i\}$ at p.

Lee and Parker's result has later on been improved by Cao [Cao91], and Günther [Gün93] independently. They proved the local existence of a conformal normal metric in a neighborhood U_p of a given point p.

On one hand, under conformal normal coordinates, local analysis on a conformal manifold can be simplified to a great extent. On the other hand, the set of conformal normal metrics is of interest on its own. By Theorem 5.6 in [LP87], on the jet-level, the conformal transformation group $\widehat{CO}(n)$ acts on the set of conformal normal coordinates free and transitively, which indicates a relation between the conformal normal metrics and the global conformal structure C(see Theorem 3.2.4).

Our first result concerns the germ level uniqueness of conformal normal metrics. In Chapter 5, we prove the following result:

Theorem 1.2. *Let $(X, [g])$ be a conformal manifold. At $p \in X$, any 1-jet class $j^1_p(g)$ of metrics in $[g]$ contains a conformal normal metric that is unique up to the germ level.*

The proof is based on Cao's approach to the existence theorem. In [Cao91], Cao rephrased the local existence of a conformal normal metric as the problem of finding solutions of a non-linear singular elliptic equation in a class of functions of weighted Hölder norms.

Briefly, fix a point $p \in X$ and a background metric g_0, and let $r_0 = \operatorname{dist}(p, \cdot)$ be the g_0-distance from p. Then a conformal metric $g = \Phi g_0$ is conformal normal at p if and only if its distance function r from p satisfies

$$\Delta_g r = \frac{n-1}{r}. \tag{1.1}$$

Let w be the function defined by $r = r_0 e^{w(x)}$. Then the conformal factor Φ and w determine each other by the formula

$$\Phi = \|dr\|^2_{g_0} = \left(1 + 2x^i w_i + r_0^2 \|dw\|^2_{g_0}\right) e^{2w}.$$

The Equation (1.1) can be converted into an equation for w in g_0 normal coordinates of the form

$$V(x, \partial w, \partial^2 w) = \mathcal{L}_0(w) + G(x, \partial w) + Q(x, w, \partial w) = -\frac{\partial_{r_0} \ln \sqrt{\det(g_0(x))}}{r_0}, \tag{1.2}$$

where V is a non-linear elliptic equation of w whose symbol is singular at the origin, and $\mathcal{L}_0$ is the scale-invariant linearization of V. See Chapter 4 for more details.

To understand Equation (1.2), we first study the linear operator $\mathcal{L}_0$. We show that $\ker \mathcal{L}_0 = \{0\}$ when restricted to functions that vanish to the infinite order at the origin (cf. Lemma 4.1.5). We use this fact to give a uniqueness theorem for the solution of Cao's equation (1.2) (cf. Theorem 5.2). This local analysis result is then used to prove that there is a unique germ of a conformal normal metric in each equivalence class in the set

$$J^1[g] = \left\{ j^1_p(g) \;\middle|\; g \in [g],\ p \in X \right\}$$

of 1-jets of metrics in the conformal class.

The organization of the thesis is as follows. In Chapter 2, we fix some notations and review background materials we will use throughout the discussion. This includes reviewing some basic facts about the relation between metric and exponential maps, jet spaces and jet bundles, and some basic facts about local conformal geometry.

In Chapter 3, we define conformal normal metrics and review Lee and Parker's work in [LP87] on the existence of conformal normal metrics up to the jet level. As an application of Lee and Parker's method, in Section 3.1, we give a jet level relation between conformal normal metrics and flat metrics on the conformally flat manifolds which will be generalized to a relation on the germ level by the uniqueness theorem.

In Chapter 4, we review Cao's approach to the local existence of conformal normal metrics and prove a unique continuation lemma for the $\mathcal{L}_0$ operator.

In Chapter 5, we state and prove the main theorem on the uniqueness of conformal normal metrics.

Finally, in Chapter 6, we show the smooth dependence of conformal normal metrics on the moduli space. This leads to the proof of our second main result: the existence of a metric $h = g \oplus \Phi^2 g$ on $X \times X$ such that, in a neighborhood of the diagonal, the restriction of h to the slice $S_p = \{(p, y) | y \in X\}$ is a conformal normal metric. As a further application, we use the metric h to give a simplified proof of the regularity of Habermann's canonical metric.

CHAPTER 2

PRELIMINARIES

2.1 Preliminaries on Exponential Maps

We will work at the level of germs.

Lemma 2.1.1. *Consider two conformal metrics g and $\tilde{g} = \Phi^2 g$ defined in a neighborhood of p with Levi-Civita connections ∇ and $\tilde{\nabla}$, distance functions $r = dist(p, \cdot)$ and $\tilde{r}$, and exponential maps $exp_p^g : T_pM \to M$ and $exp_p^{\tilde{g}}$. Then, at the level of germs at p, the following are equivalent:*

(a) $\nabla = \tilde{\nabla}$

(b) $exp_p^g = exp_p^{\tilde{g}}$

(c) $\tilde{r} = cr$ *where* $c = \Phi(p)$

(d) Φ *is constant.*

In particular, conformal metrics g and $\tilde{g}$ have the same distance function if and only if $g = \tilde{g}$.

Proof: The exponential map is defined by $\exp_p^g(v) = \gamma(1)$ where $\gamma : [0,1] \to M$ is the solution to the geodesic equation $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$ with $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. This solution is unique, so (a) implies (b).

If (b) holds, then the pullback functions $\rho = (\exp_p^g)^* r$ and $\tilde{\rho} = (\exp_p^{\tilde{g}})^* \tilde{r}$ are equal. But $\rho(v)$ is the norm $|v|_g$ for the inner product g_p on T_pM, and therefore $\tilde{\rho}^2(v) = \tilde{g}_p(v,v) = \Phi(p)^2 g_p(v,v) = \Phi(p)^2 \rho^2(v)$ for all $v \in T_pM$. These exponential maps are local diffeomorphisms, so (c) holds.

The distance function for g satisfies $|dr|_g = 1$, so if (c) holds then $1 = |d\tilde{r}|_{\tilde{g}} = |cdr|_{\Phi g} = c\Phi^{-1}|dr|_g = c\Phi^{-1}$, so Φ is constant. Finally, the implication $(d) \implies (a)$ is clear from the local coordinate formula for the Christoffel symbols. □

For an alternative proof that (a) and (d) are equivalent, consider the difference

$$D(X,Y) = \nabla_X Y - \tilde{\nabla}_X Y \tag{2.1.1}$$

for vector fields X and Y. The properties of the Levi-Civita connection show that $D(X,Y)$ is tensorial in both X and Y and is symmetric. Thus D is a tensor $D \in \Gamma(Sym^2(T^*M) \otimes TM)$ that vanishes if and only if $\nabla = \tilde{\nabla}$; by polarization this is also equivalent to the vanishing of $D(X,X)$. For conformal metrics g and $\tilde{g} = e^{2f} g$, the formula for the Christoffel symbols shows that

$$D(X,Y) \ = \ Xf \cdot Y + Yf \cdot X - g(X,Y) \cdot \nabla f.$$

If $\Phi = e^f$ is a constant, then D vanishes. Conversely, if $D = 0$ then, using inner products for g, we have

$$0 \ = \ \langle X, D(X,X) \rangle \ = \ 2Xf \cdot |X|^2 - |X|^2 \langle X, \nabla f \rangle \ = \ |X|^2 \cdot Xf$$

for all vector fields X, so f is constant.

As an aside, we record the following variation of Lemma 2.1.1 in which the metrics are not assumed to be conformal, but the exponential maps are assumed to be equal at all (or nearly all) points.

Proposition 2.1.2. *Suppose that g and $\tilde{g}$ are metrics on an open set U, that U is geodesically convex for both g and $\tilde{g}$ and that $S \subset U$ is a non-empty submanifold of codimension 1. Then the following are equivalent:*

(a) $\nabla = \tilde{\nabla}$ *(b)* $exp_p^g = exp_p^{\tilde{g}}$ *for all* $p \in U$ *(c)* $exp_p^g = exp_p^{\tilde{g}}$ *for all* $p \in S$.

Proof: The proof of Lemma 2.1.1 shows that (a) implies (b). Obviously (b) implies (c).

Now assume that (c) holds, and fix $p \in S$. Then $\exp_p^g$ is a diffeomorphism from a neighborhood V of 0 in T_pU to U. Because T_pS is a codimension 1 linear subspace of T_p, so

$$U^* \ = \ \left\{ \exp_p(v) \in U \,\middle|\, v \in T_pU \setminus T_pS \right\}$$

is an open dense subset of U. For each $q = \exp_p(v) \in U^*$, the path from p to q defined by $\gamma^q(t) = \exp_p(tv)$, $0 \leq t \leq 1$, is a geodesic for the metric g, and is transverse to S at the point p. By assumption, γ^q is also a geodesic for the metric $\tilde{g}$.

Reversing perspective, we can write $p = \exp_q(w)$, where $w = -\dot{\gamma}^q(1) \in T_qU$. Because γ^q is a geodesic for both metrics, we have $D(w,w) = 0$. Furthermore, for any $w' \in T_qU$ sufficiently close to w, there is a τ close to 1 such that the g-geodesic $\exp_q(\tau w')$ intersects S transversally at a point $p' = \exp_q(\tau w')$ close to p. Applying the previous argument with p replaced by p' shows that $D(\tau w, \tau w) = \tau^2 D(w',w') = 0$. Therefore $|D(w',w')|^4$, which is a quadric polynomial on T_qU, vanishes on a neighborhood of w so, by analyticity, is zero.

This is true at each point q in the dense set $U^* \subset U$. Therefore the tensor D vanishes on U, so $\nabla = \tilde{\nabla}$. Thus (c) implies (a). $\square$

The conditions of Proposition 2.1.2 are also equivalent to $\tilde{g} = cg$ if we impose one additional assumption. Recall that a Riemannian manifold (U, g) is *irreducible* if, for some $p \in U$, the action of the holonomy group $H_p(U, g)$ at p on T_pU is irreducible. This property is independent of p (cf. [KN63]).

Corollary 2.1.3. *Suppose that g and $\tilde{g}$ are metrics on an open set U and that (U, g) is irreducible. Then the Levi-Civita connections of g and $\tilde{g}$ are equal if and only if $\tilde{g} = cg$ for some constant c.*

Proof: As in the proof of Lemma 2.1.1, if $\tilde{g} = cg$ then $\nabla = \tilde{\nabla}$. For the converse, assume that $\nabla = \tilde{\nabla}$. This immediately means that the holonomy group $H_p(U, g)$, which is defined by parallel transport with respect to ∇, is equal to the holonomy group $H_p(U, \tilde{g})$ defined by $\tilde{\nabla}$.

Define a vector bundle map $A : TU \to TU$ by the condition

$$g(AX, Y) \;=\; \tilde{g}(X, Y) \tag{2.1.2}$$

for all vector fields X, Y. Because $\tilde{g}$ is symmetric, this implies that A is self-adjoint for the metric g, and hence is diagonalizable. Differentiating and noting that $\nabla g = 0$ and $\nabla \tilde{g} = \tilde{\nabla} \tilde{g} = 0$, one sees that $\nabla A = 0$.

Now suppose that $\gamma(t)$ is a path in U starting and ending at p with velocity vector T. If $X(t)$ is a vector field along γ that is parallel, i.e. $\nabla_T X = 0$, then $\nabla_T A(X) = (\nabla_T A)X + A(\nabla_T X) = 0$, so AX is also parallel. It follows that A commutes with the action of $H_p(U, g)$ on T_pU. By Schur's Lemma, A is a multiple of the identity. This is true for every $p \in U$, so (2.1.2) shows that $\tilde{g} = \Phi^2 g$ for some smooth function Φ. Applying Lemma 2.1.1, we conclude that $\tilde{g} = cg$ for a constant $c > 0$. $\square$

2.2 Preliminaries on Jet Spaces and Jet Bundles

Jet Spaces

Definition 2.2.1. The set $C_p^\infty(\mathbb{R}^n, \mathbb{R}^m)$ of germs of smooth functions $f : \mathbb{R}^n \to \mathbb{R}^m$ at a point $p \in \mathbb{R}^n$ is a module over $C^\infty(\mathbb{R}^n)$. Let $\mathfrak{m}_p \subseteq C^\infty(\mathbb{R}^n)$ be the ideal of functions that vanish at p. Define the *k-jet space at p* by:

$$J_p^k(\mathbb{R}^n, \mathbb{R}^m) = \frac{C_p^\infty(\mathbb{R}^n, \mathbb{R}^m)}{\mathfrak{m}_p^{k+1} \cdot C_p^\infty(\mathbb{R}^n, \mathbb{R}^m)}$$

The jet space of functions f such that $f(p) = q$ is

$$J_p^k(\mathbb{R}^n, \mathbb{R}^m)_q = \left\{ j_p^k(f) \in J_p^k(\mathbb{R}^n, \mathbb{R}^m) | f(p) = q \right\}$$

In particular, we denote $L_{n,m}^k = J_0^k(\mathbb{R}^n, \mathbb{R}^m)_0$, elements in $L_{n,m}^k$ can be identified as the k^{th} order Taylor polynomials of the generating functions. Explicitly, let $\{x^i\}$ and $\{y^j\}$ be coordinates on $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and α be a multi-index, then every element $j_p^k(f)$ in $L_{n,m}^k$ has a polynomial representative:

$$f(x) = (f^j(x)) = \left(\sum_{1 \le |\alpha| \le k} c_\alpha^j x^\alpha \right)$$

Define the following two natural operations on $L_{n,m}^k$:

(a) For $l \le k$, the jet projection map $\pi_l^k : L_{n,m}^k \to L_{n,m}^l$ is defined by the natural projection of modules:

$$\frac{C_0^\infty(\mathbb{R}^n, \mathbb{R}^m)_0}{\mathfrak{m}_0^{k+1} \cdot C_0^\infty(\mathbb{R}^n, \mathbb{R}^m)_0} \to \frac{C_0^\infty(\mathbb{R}^n, \mathbb{R}^m)_0}{\mathfrak{m}_0^{l+1} \cdot C_0^\infty(\mathbb{R}^n, \mathbb{R}^m)_0} \Big/ \frac{\mathfrak{m}_0^{l+1} \cdot C_0^\infty(\mathbb{R}^n, \mathbb{R}^m)_0}{\mathfrak{m}_0^{k+1} \cdot C_0^\infty(\mathbb{R}^n, \mathbb{R}^m)_0},$$

(b) The jet composition map $L_{n,m}^k \times L_{m,d}^k \to L_{n,d}^k$ is defined by composing the polynomial representatives and truncating to degree k.

Let $\mathit{Diff}_0(\mathbb{R}^n, \mathbb{R}^n)_0$ be the group of germs of diffeomorphisms of $\mathbb{R}^n$ fixing the origin. The k-jet space of $\mathit{Diff}_0(\mathbb{R}^n, \mathbb{R}^n)_0$, denoted as GL_n^k, is a Lie group concerning the jet composition operation. In particular, for $k = 1$, $GL_n^1 = GL(n)$, the general linear group.

For a Lie subgroup $G \subseteq \mathit{Diff}_0(\mathbb{R}^n, \mathbb{R}^n)_0$, the k-jet space of G is a Lie subgroup of GL_n^k, denoted as G^k. Denote $\mathfrak{g}^1$ the Lie algebra of G^1, then the manifold structure on G^k is inherited from the following identity:

$$G^k = \left\{ (A, \tau_1, \ldots, \tau_{k-1}) \mid A \in G^1, \tau_i \in Sym_{i+1}(\mathbb{R}^n), \tau_i(-, v_1, \ldots, v_i) \in \mathfrak{g}^1 \right\},$$

where $Sym_{i+1}(\mathbb{R}^n)$ is the space the symmetric $(i+1)$-linear maps $\mathbb{R}^{n\times(i+1)} \to \mathbb{R}^n$.

The jet projection $\pi_k^{k+1} : G^{k+1} \to G^k$ is given by $\pi_k^{k+1}(A, \tau_1, \dots, \tau_k) = (A, \tau_1, \dots, \tau_{k-1})$. The kernel of π_k^{k+1} is called the k-jet prolongation of $\mathfrak{g}^1$ and is denoted as $\mathfrak{g}^{k+1}$.

Define the order of G to be the smallest integer k such that π_k^{k+1} is an isomorphism, namely the smallest k such that $\mathfrak{g}^{k+1} = 0$. For a group G of order k, the Lie algebra $\mathfrak{g}$ of G decomposes as:

$$\mathfrak{g} = \mathfrak{g}^1 \oplus \cdots \oplus \mathfrak{g}^k,$$

which is a graded Lie algebra. For $\tau_p \in \mathfrak{g}^p$ and $\tau_q \in \mathfrak{g}^g$ the Lie bracket $[\tau_p, \tau_q] \in \mathfrak{g}^{p+q}$ and is given as follows (cf. [Kob12]):

$$\begin{aligned} [\tau_p, \tau_q](v_0, v_1, \dots, v_{p+q}) \; &= \frac{1}{p!(q+1)!} \sum \tau_p(\tau_q(v_{j_0}, \dots, v_{j_q}), v_{j_{q+1}}, \dots, v_{j_{p+q}}) \\ &\quad - \frac{1}{(p+1)!q!} \sum \tau_q(\tau_p(v_{k_0}, \dots, v_{k_p}), v_{k_{p+1}}, \dots, v_{k_{p+q}}) \end{aligned}$$

Definition 2.2.2. Let X be a smooth manifold of dimensions n and $U \subset \mathbb{R}^n$ be an open neighborhood of 0. We say two local diffeomorphisms $f, g : U \to X$ define the same *k-frame* at a point $p \in X$ if $f(0) = g(0) = p$ and $j_0^k(\varphi^{-1} \circ f) = j_0^k(\varphi^{-1} \circ g) \in GL_n^k$, where φ is an arbitrary local chart at p.

It is clear that a k-frame at $p \in X$ is well-defined independent of the choice of the local chart φ. The group GL_n^k acts on the set of k-frames at p free and transitively by the jet composition.

Definition 2.2.3. For a Lie subgroup $G \subseteq GL_n^k$, a principal G bundle $\pi : P \to X$ defines a *G-structure of order k* on X if for $\forall p \in X$ the fiber $F_p = \pi^{-1}(p)$ consists of k-frames at p and the principal bundle action of G on F_p is by the jet composition.

Classical examples of G-structures on a manifold X include: GL_n^+ structure defines an orientation on X; SL_n structure defines a volume element on X; $O(n)$ structure defines a Riemannian metric on X and so on. We will be focusing on the G-structure characterization of conformal structures, see Section 2.3 and [Kob12] for more details.

Jet Bundles

Definition 2.2.4 ([Par]). Let $\pi : E \to M$ be a smooth vector bundle, the set $\Gamma(E)$ of smooth sections is a module over $C^\infty(M)$. Denote $\mathfrak{m}_p = \{f \in C^\infty(M) \mid f(p) = 0\}$ the ideal of smooth functions vanishes at $p \in M$. We define the k-jets of sections of E at P by:

$$J^k(E)_p = \Gamma(E)/\mathfrak{m}_p^{k+1} \cdot \Gamma(E)$$

$J^k(E)_p$ is a vector space with vector summation as $[\xi]_k + [\eta]_k = [\xi + \eta]_k$ and scalar multiplication as $\lambda \cdot [\xi]_k = [\lambda \cdot \xi]_k$. $J^k(E) = \cup_{p\in M} J^k(E)_p$ is a vector bundle over M called the k^{th} *jet bundle of the vector bundle E.*

In particular, $J^0(E) \simeq E$. In coordinates $\{x^i\}$ near p and a basis $\{\sigma_\alpha\}$ of E_p, the k-jet of a section ξ is uniquely represented by its degree k Taylor polynomial

$$[\xi]_k = \sum_\alpha \left(a_0^\alpha + \sum_i a_i^\alpha (x-p)^i + \cdots + \sum a^\alpha_{i_1 i_2 \cdots i_k} (x-p)^{i_1 i_2 \cdots i_k} \right) \sigma_\alpha$$

We have the following exact sequence of jet bundles:

$$0 \to S^k(T^*M) \otimes E \to J^k(E) \to J^{k-1}(E) \to 0, \qquad (2.2.1)$$

where the map $\pi^k : J^k(E) \to J^{k-1}(E)$ is the natural jet projection, and $S^k(T^*M)$ is the k-fold symmetric tensor product of T^*M, the map $S^k(T^*M) \otimes E \to J^k(E)$ is defined by identifying the kernel of π^k with $S^k(T^*M) \otimes E$.

In particular, when $E = M \times \mathbb{R}^n$ being a trivial vector bundle, we have a canonical isomorphism

$$J^1(E) \simeq T^*M^{\oplus n} \oplus \mathbb{R}^n \qquad (2.2.2)$$

defined as follows: Let $s \in \Gamma(E)$ be a representative of an element $j_p^1(s) \in J^1(E)$. Take $s_2 = \pi_2 \circ s \in C^\infty(M, \mathbb{R}^n)$, where $\pi_2 : M \times \mathbb{R}^n \to \mathbb{R}^n$ is the projection to the $\mathbb{R}^n$ component. Define $\eta : J^1(E) \to T^*M^{\oplus n}$ by $\eta(j_p^1(s)) = d(s_2)(p)$. It is clear that η is well defined independent of the choice of the representative s and gives a splitting of the short exact sequence (2.2.1), and hence we have the isomorphism (2.2.2).

2.3 Preliminaries on Conformal Geometry

We use [Kob12] as a main reference for this section.

Let $\hat{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$ be the extended Euclidean space. Let $\widehat{CO}(n)$ be the group of conformal automorphism of $\hat{\mathbb{R}}^n$ having the origin fixed. A general element $\varphi \in \widehat{CO}(n)$ is of the form:

$$\varphi(x) = \lambda \frac{Ax - x^2\eta}{1 - 2\eta \cdot Ax + x^2\eta^2}, \tag{2.3.1}$$

where $\lambda > 0$, $\eta \in \mathbb{R}^n$, $A \in O(n)$.

Denote $\widehat{CO}^k(n)$ the group of k-jets of $\widehat{CO}(n)$ as a Lie subgroup of $Diff_0(\mathbb{R}^n, \mathbb{R}^n)_0$.

For $k = 1$, $\widehat{CO}^1(n) = CO(n)$, the linear conformal group on $\mathbb{R}^n$. Explicitly, we have:

$$CO(n) = \{A \in GL(n); A^t A = \lambda I, \lambda \in \mathbb{R}^+\} \cong O(n) \times \mathbb{R}^+.$$

For $k = 2$, we have the following short exact sequence:

$$1 \to \mathfrak{co}_1 \xrightarrow{\tau} \widehat{CO}^2(n) \longrightarrow \widehat{CO}^1(n) \to 1, \tag{2.3.2}$$

where $\mathfrak{co}_1(n)$ is the first prolongation of the Lie algebra $\mathfrak{co}(n)$ of $CO(n)$.

By definition $\mathfrak{co}_1(n) = \{f \in Sym_2(\mathbb{R}^n) \mid \forall v \in R^n, f(-,v) \in \mathfrak{co}(n)\}$. $\mathfrak{co}_1(n)$ is isomorphic to $\mathbb{R}^n$ by the following map:

$$t : \mathbb{R}^n \to \mathfrak{co}_1(n)$$
$$v \mapsto t_v(a,b) = \frac{1}{2}(\langle v,a\rangle b + \langle v,b\rangle a - \langle a,b\rangle v).$$

The map τ in (2.3.2) can be given explicitly as $\tau(t)(x) = x + t(x,x)$.

Let $\varphi \in \widehat{CO}(n)$ be a general element as given in(2.3.1). By taking the 2-jet of φ at the origin, we have:

$$j_0^2(\varphi) = \varphi_i^k x^i + \varphi_{ij}^k x^i x^j = \lambda(Ax + t_\eta(Ax, Ax)). \tag{2.3.3}$$

The coefficient matrix $C = \varphi_i^k$ of the linear terms of $j_0^2(\varphi)$ equals the matrix λA, hence $\lambda = (\det(C))^{\frac{1}{n}}$ and $A = \frac{C}{(\det(C))^{\frac{1}{n}}}$. Let $\{e_i\}$ be the standard basis of $\mathbb{R}^n$, then the quadratic term of $j_0^2(\varphi)$ at $A^{-1}(e_i)$ equals $\eta/2$. Hence the data $\{\lambda, A, \eta\}$ is uniquely determined by $j_0^2(\varphi)$. Therefore elements in $\widehat{CO}(n)$ are uniquely determined by $j_0^2(\varphi)$, namely $\widehat{CO}(n)$ is a subgroup of GL_n^2.

By the fact that $\widehat{CO}(n)$ is a subgroup of GL_n^2, we define the $\widehat{CO}(n)$ structure of order 2 on a smooth manifold X as in definition 2.2.3, to be a principal $\widehat{CO}(n)$ bundle $\pi : P \to X$ of 2-frames with $\widehat{CO}(n)$ acts on each fiber by the jet composition.

Explicitly, in local coordinates $\{x\}$ at $p \in X$, let $f(x) = c^l_i x^i + c^l_{ij} x^i x^j$ be the polynomial representative of a 2-frame at p. The action of $\varphi \in \widehat{CO}(n)$ on f is given as

$$f \cdot \varphi = j^2_0(f \circ \varphi) = (c^l_k \varphi^k_i) x^i + (c^l_k \varphi^k_{ij} + c^l_{ks} \varphi^k_i \varphi^s_j) x^i x^j, \tag{2.3.4}$$

where φ^k_i and φ^k_{ij} are the coefficients of $j^2_0(\varphi)$ given in (2.3.3).

In fact, a $\widehat{CO}(n)$ structure P is uniquely determined by a conformal structure $[g]$ and vice versa. We argue as follows (cf. [Kob12]):

On one hand, suppose P is a $\widehat{CO}(n)$ structure on X. Projecting P to the corresponding 1-jet bundle P^1 by the jet projection map, we obtain P^1 as a principal $CO(n)$ bundle. Since $CO(n) = O(n) \times \mathbb{R}^+$, a section s of the orbit bundle $P^1/O(n)$ defines a principal bundle reduction of P^1 to a principal $O(n)$ bundle $H = H(s)$. At each point $p \in X$ choose a frame $\theta \in H_p$, define a metric on T_pX as $g(p)(v,w) = \sum v^i w^i$, where v^i and w^i are components of v and w with respect to the frame θ. Let $\theta' \in H_p$ be a different frame such that $\theta' = A\theta$ with $A \in O(n)$. With respect to θ', the metric $g'(p)(v,w) = \sum (Av)^i (A^t w)^i = \sum v^i w^i = g(p)$. Hence the metric g is well defined independent of the choice of $\theta \in H_p$. Different choices of section s will give metrics conformal to g, hence defining a conformal structure $[g]$.

On the other hand, suppose $[g]$ is a conformal structure on X. Denote $\mathcal{O}^g(n)$ the orthonormal frame bundle with respect to $g \in [g]$, define

$$P^1 = \bigcup_{p \in X, g \in [g]} \mathcal{O}^g_p(n).$$

P^1 is a principal $CO(n)$ bundle. The corresponding $\widehat{CO}(n)$ structure P on X is defined as the first prolongation of P^1 as follows:

We first embed $\mathfrak{co}_1(n)$ as a subgroup of $End(\mathbb{R}^n \oplus \mathfrak{co}(n))$ by the following map:

$$\iota : \mathfrak{co}_1(n) \to End(\mathbb{R}^n \oplus \mathfrak{co}(n))$$

$$t \mapsto \begin{cases} \bar{t}(v) = v + t(-,v), & \text{for } v \in R^n \\ \bar{t}(A) = A, & \text{for } A \in \mathfrak{co}(n). \end{cases}$$

Let $\Lambda^2 \mathbb{R}^{n*} \otimes \mathbb{R}^n$ be the space of skew-symmetric bilinear mappings, define a linear map $\partial : \mathfrak{co}(n) \otimes \mathbb{R}^{n*} \to \Lambda^2 \mathbb{R}^{n*} \otimes \mathbb{R}^n$ by

$$(\partial f)(v_1, v_2) = f(v_1)v_2 - f(v_2)v_1,$$

where $f \in \mathfrak{co}(n) \otimes \mathbb{R}^{n*}$, v_1, $v_2 \in R^{n*}$.

We choose once and for all a direct sum complement of $\partial(\mathfrak{co}(n) \otimes \mathbb{R}^{n*})$ in $\Lambda^2\mathbb{R}^{n*} \otimes \mathbb{R}^n$, denoted as C.

Let $\theta \in \Omega^1(P^1, \mathbb{R}^n)$ be the canonical 1 form on P^1. At a point $u \in P^1$, a n-dimensional subspace H of T_uP^1 is called horizontal if $\theta : H \to \mathbb{R}^n$ is an isomorphism. A horizontal space H is called C admissible if $d\theta(TH \oplus TH) \in C$. Every C admissible space H determines a linear frame of T_uP^1 as follows:

Let $f : \mathfrak{co}(n) \to T_uP^1$ be the map that sends $A \in \mathfrak{co}(n)$ to A^*_u, where A^* is the fundamental vector field on P^1 with respect to A. The direct sum of f with the map $\theta^{-1} : \mathbb{R}^n \to H \subseteq T_uP^1$ defines a linear frame $\theta^{-1} \oplus f : \mathbb{R}^n \oplus \mathfrak{co}(n) \to T_uP^1$.

The union of all linear frames induced from C admissible horizontal spaces is a principal $\iota(\mathfrak{co}_1(n))$ bundle of 1-frames P over P^1. As bundle over X, P is a principal $\widehat{CO}(n)$ bundle of 2-frames defining a $\widehat{CO}(n)$ structure on X.

CHAPTER 3

CONFORMAL NORMAL METRICS AT JET LEVELS

Let (X, g) be a Riemannian manifold (all manifolds are assumed to be smooth of dimension $n \geq 3$). We first give a coordinate-free definition of the determinant of the metric g near a point $p \in X$.

Let $U \subseteq T_pX$ be an open neighborhood of the origin on which the Riemannian exponential map is a well-defined diffeomorphism onto an open neighborhood V of p. Let dv_g be the g-Riemannian volume form which can defined in a coordinate-free way. The pullback $exp_{g,p}^* dv_g|_V$ by the exponential map at p is a volume form on U. On the other hand, the inner product space $(T_pX, g(p))$ has a canonical volume form dv_p. Both volume forms are nonvanishing sections of $\Lambda^{top}(T_pU)$ which is a real line bundle, and hence the division $\frac{exp^* dv_g}{dv_p}$ is a well-defined smooth function on U. Define $\det(\exp^* g) := (\frac{\exp^* dv_g}{dv_p})^2$.

Definition 3.1. A metric g is called *conformal normal* at a point p, if there exists an open neighborhood V of p such that $\det(\exp^* g) = 1$ on $U = \exp^{-1}(V)$. Also, we say g is k^{th} *order conformal normal* at p if the k-jet $j_0^k(\det(\exp^* g)) = 1$.

An arbitrary Riemannian metric is not necessarily conformal normal, indeed the word "conformal" indicates the following fact (cf. [Cao91, Corollary 0.1]):

Theorem 3.2. *For any Riemannian metric g on X, and a point $p \in X$, there exists a conformal metric $\tilde{g} = \Phi g$ which is conformal normal in a small neighborhood of p.*

Given two conformal metrics g and $\tilde{g} = e^{2f} g$, assume that they are both conformal normal and define the same exponential map at a point p, we then have:

$$\det(\exp^*(\tilde{g})) = \det(\exp^*(e^{2f} g)) = e^{2nf} \det(\exp^* g) = e^{2nf} = 1. \tag{3.1}$$

Thus $g = \tilde{g}$ within the injective radius, and conformal normal metrics at p are uniquely determined by the exponential maps on T_pX. In fact, as we will show in Theorem 5.1, a conformal normal metric is locally uniquely determined by its 1-jet class.

In this chapter, we work on jet levels and fix the following notations:

On a conformal manifold $(X, [g])$ at a point $p \in X$, for $k = 1, 2, \ldots, \infty$, define the set of k-jets of Riemannian metrics at p with k^{th} order conformal normal determinant as

$$\mathcal{CN}^k(p) = \{j_p^k(g) | g \in [g], j_0^k(\det(\exp_p^* g)) = 1\}. \tag{3.2}$$

And correspondingly, define the conformal normal k frames as:

$$\mathcal{CNC}^k(p) = \{j_0^k(\varphi) \mid \varphi_{g,\beta} = \exp_{g,p} \circ \beta, \beta \in O(T_pX, g_p), g \in [g] \text{ s.t. } j_0^k(\det(\exp_p^* g)) = 1\}.$$

For $k = 1$, $j_0^1(\det(\exp_p^* g)) = 1$ holds for any metric g. Thus we have

$$\mathcal{CN}^1(X) = J^1[g] = \left\{ j_p^1(g) \mid g \in [g], p \in X \right\}.$$

Denote $C^\infty(\mathrm{Sym}^2 T^*X)$ the space of smooth symmetric $(0,2)$ tensors, which is a Fréchet space with C^∞ topology. Fix a background Riemannian metric g^0, by Theorem 1.5 in [FM77], the conformal class $[g^0]$ as the orbit of the $C^\infty(X)$ action on g^0 is a smooth sub-manifold of $C^\infty(\mathrm{Sym}^2 T^*X)$, i.e, the map $\Phi^0 : C^\infty(X) \to C^\infty(\mathrm{Sym}^2 T^*X)$ by $\Phi^0(f) = e^f g^0$ is a smooth embedding. Define the descending of Φ^0 to 1-jets by the following diagram:

$$\begin{array}{ccc}
C^\infty(X) & \xrightarrow{\Phi^0} & C^\infty(\mathrm{Sym}^2 T^*X) \\
\downarrow{\scriptstyle \pi^1} & & \downarrow{\scriptstyle \pi^1} \\
J^1(\mathbb{R}) & \xrightarrow{J^1\Phi^0} & J^1(\mathrm{Sym}^2 T^*X) \\
j_p^1(f) & \longmapsto & j_p^1(e^f g^0)
\end{array}$$

$J^1\Phi^0$ is a bundle isomorphism onto $J^1[g]$. Indeed, the image

$$j_p^1(e^f g^0) = e^{f(p)}(j_p^1(g^0) + j_p^1(f - f(p))g^0(p))$$

is uniquely determined by $j_p^1(f)$.

In summary, we have the following lemma:

Lemma 3.3. *A metric $g^0 \in [g]$ induces a smooth bundle isomorphism :*

$$\eta := J^1\Phi^0 \circ \iota^{-1} : \mathbb{R} \oplus T^*X \longrightarrow J^1[g] = \mathcal{CN}^1,$$

where ι is the bundle isomorphism defined in (2.2.2)*:*

$$\iota : J^1(\mathbb{R}) \longrightarrow \mathbb{R} \oplus T^*X$$
$$j_p^1(f) \longmapsto (f(p), df(p)).$$

3.1 A Jet Level Relation of Flat, Conformally Flat and Conformal Normal Metrics

First, recall the following theorem on flatness and conformal flatness of Riemannian metrics (cf. [Lee18, Theorem 7.10 & Theorem 7.37]). As usual, a Riemannian manifold (X, g) is flat if and only if its Riemann curvature tensor vanishes identically.

Weyl-Schouten Theorem. A manifold (X, g) of dimension $n \geq 4$ is locally conformally flat if and only if its Weyl tensor is identically zero. A 3-manifold is locally conformally flat if and only if its Cotton tensor is identically zero. The Cotton and Weyl tensors are defined in equations (3.1.13) and (3.1.12) below.

Inspired by the Weyl-Schouten Theorem, we give the following definition of jet-level flatness and conformal flatness:

Definition 3.1.1. We say a Riemannian metric g is *flat up to order* r at a point $p \in X$ if its Riemann curvature tensor vanishes to order r at p. Similarly, we say that a conformal structure $[g]$ on a manifold of dimension $n \geq 4$ is *conformally flat up to order* r if the Weyl tensor W vanishes to order r at p. A 3-manifold is conformally flat up to order r at p if its Cotton tensor vanishes to order $r - 1$ at p.

To be brief, we fixed some notations. We work under a g-normal coordinate chart and use the Einstein summation convention. Let $\mu = (\mu_1 \dots \mu_k)$ be a k multi-index, with $1 \leq \mu_i \leq n$ and $|\mu| = k \geq 2$. The permutation group S_k acts on μ by permuting μ_i and the orbits of this action define an equivalent relation $\mu \sim \mu'$. Each equivalence class has a unique representative $\bar{\mu} = (\mu_1 \dots \mu_k)$ in ascending order $1 \leq \mu_1 \cdots \leq \mu_k \leq n$. When we have a list of multi-indices, we use upper indices: $\mu^1, \dots, \mu^l$.

Let $Riem_\mu$ denote the matrix whose ij^{th} entry is the covariant derivative $R^i_{\mu_1\mu_2 j;\mu_3\dots\mu_k}$. Then define

$$Riem_{\mu\nu} := Riem_\mu Riem_\nu^t + Riem_\nu Riem_\mu^t, \tag{3.1.1}$$

where the upper t denotes the transpose matrix. Define $Riem_{\mu^1\dots\mu^l}$ by this formula and induction, i.e. take μ to be μ^1 and ν to be $\mu^2 \cdots \mu^l$ in (3.1.1).

Denote $\partial Riem_\mu = \partial_{\mu_3 \dots \mu_k} R_{i\mu_1\mu_2 j}$ for the partial derivatives. Since we have

$$R_{j_1j_2j_3j_4;i} = \partial_i R_{j_1j_2j_3j_4} - \Gamma^k_{j_1i} R_{kj_2j_3j_4} - \cdots - \Gamma^k_{j_4i} R_{j_1j_2j_3k}, \tag{3.1.2}$$

$Riem_\mu$ and $\partial Riem_\mu$ differ by derivatives of order less than $|\mu| - 2$ at the origin in normal coordinates.

Denote the covariant derivatives and partial derivatives of the Ricci curvature tensor Ric similarly as above.

With this notation, one can give closed formulas for the Taylor expansions of g and $\det(g)$ under the normal coordinates. This was neatly done by Schubert and van de Ven in [MSvdV99]. Specifically,

$$g = I + \sum_{l=1}^{\infty} \sum_{|\mu^i|=2}^{\infty} c_{\mu^1 \dots \mu^l} Riem_{\mu^1 \dots \mu^l}(0)\ x^{\mu^1 + \dots + \mu^l}, \tag{3.1.3}$$

where the coefficients $c_{\mu^1 \dots \mu^l}$ are constants depend only on absolute values $|\mu^i|$ of μ^i. In particular, for $l = 1$, and $|\mu| = k$, we have $c_k = \frac{2k-2}{(k+1)!}$.

Remark 3.1.2. The expansion (3.1.3) is obtained in [MSvdV99] by writing

$$g(x) = e^t(x)e(x),$$

Then, as shown in [MSvdV99], the coefficients in the Taylor series

$$e(x) = I + \sum_{|\mu| \geq 2} e_\mu(0) x^\mu$$

are given, for each μ with $|\mu| = k$, by

$$(k+1)e_\mu(0) = (k-1)Riem_\mu(0) + \sum_{n=2}^{k-2} (n+1) \binom{k-1}{n+1} Riem_{(\mu_1 \dots \mu_{k-n})}(0) e_{(\mu_{k-n+1} \dots \mu_k)}(0)$$

Combining the above three formulas yields (3.1.3) and also defines the coefficients $c_{\mu^1 \dots \mu^l}$ recursively.

To calculate $\det(g)$, write $g(x) = \exp(A(x))$ with $A(x) = A_{\bar{\mu}}\, x^{\bar{\mu}}$. Then we have

$$\operatorname{Tr}(A(x)) = \operatorname{Tr}(A_{\bar{\mu}}) x^{\bar{\mu}}, \qquad \det(g) = \det(\exp(A(x))) = \exp(\operatorname{Tr}(A(x))).$$

Hence

$$\exp(\operatorname{Tr}(A(x))) = 1 + \operatorname{Tr}(A_{\bar{\mu}}) x^{\bar{\mu}} + \cdots + \frac{1}{l!} \prod_{i=1}^{l} \operatorname{Tr}(A_{\bar{\mu}^i}) x^{\bar{\mu}^i} + \cdots \tag{3.1.4}$$

Since $g = I + O(r^2)$ in normal coordinates, we have $A_{\bar{\mu}} = 0$ for $|\bar{\mu}| \leqslant 1$; thus the summation in $A_{\bar{\mu}}x^{\bar{\mu}}$ is taken over $\bar{\mu}$ with $|\bar{\mu}| \geq 2$.

Denote the Taylor expansion of $\exp(A(x))$ as:

$$\exp(A(x)) = I + A_{\bar{\mu}}x^{\bar{\mu}} + \frac{1}{2!}A_{\bar{\mu}^1}A_{\bar{\mu}^2}x^{\bar{\mu}^1+\bar{\mu}^2} + \cdots + \frac{1}{l!}\prod_{i=1}^{l} A_{\bar{\mu}^i}x^{\bar{\mu}^i} + \cdots \tag{3.1.5}$$

For each $\bar{\mu}$, comparing (3.1.3) and (3.1.5), we have

$$\sum_{\mu^1+\cdots+\mu^l\sim\bar{\mu}} c_{\mu^1\ldots\mu^l} Riem_{\mu^1\ldots\mu^l}(0) = A_{\bar{\mu}} + \frac{1}{2!}\sum_{\bar{\mu}^1+\bar{\mu}^2\sim\bar{\mu}} A_{\bar{\mu}^1}A_{\bar{\mu}^2} + \cdots \tag{3.1.6}$$

By induction on $|\bar{\mu}|$ for the equalities (3.1.6), we can calculate $A_{\bar{\mu}}(0)$ recursively as follows:

$$\begin{aligned} A_{\bar{\mu}}(0) &= c_2 \sum_{\mu\sim\bar{\mu}} Riem_{\mu}(0), \quad \text{for} |\bar{\mu}| = 2; \\ A_{\bar{\mu}}(0) &= c_3 \sum_{\mu\sim\bar{\mu}} Riem_{\mu}(0), \quad \text{for} |\bar{\mu}| = 3; \\ A_{\bar{\mu}}(0) &= \sum_{\mu^1+\cdots+\mu^l\sim\bar{\mu}} c_{\mu^1\ldots\mu^l} Riem_{\mu^1\ldots\mu^l}(0) + \\ &\qquad - \sum_{k=2}^{[\frac{|\mu|}{2}]} \frac{1}{k!} \sum_{\bar{\mu}^1+\cdots+\bar{\mu}^k\sim\bar{\mu}} \prod_{i=1}^{k} A_{\bar{\mu}^i}, \quad \text{for } |\bar{\mu}| \geq 4. \end{aligned} \tag{3.1.7}$$

For $|\bar{\mu}| = k$, by (3.1.2) and (3.1.7), we have:

$$A_{\bar{\mu}}(0) = c_k \sum_{\mu\sim\bar{\mu}} \partial Riem_{\mu}(0) + P(Riem_{\mu'}(0)), \tag{3.1.8}$$

where $P(Riem_{\mu'}(0))$ is a polynomial of $Riem_{\mu'}$ with $2 \leq |\mu'| \leq k-2$.

Take trace of the identities (3.1.8) and substitute back into (3.1.4), we obtain a closed formula for the Taylor expansion of $\det(g)$:

$$\det(g) = \exp(\mathrm{Tr}(A(x))) = 1 + \sum_{|\bar{\mu}|=2}^{\infty} \left(\mathrm{Tr}\left(\sum_{\mu^1+\cdots+\mu^l\sim\bar{\mu}} c_{\mu^1\ldots\mu^l} Riem_{\mu^1\ldots\mu^l} \right) - T_{\bar{\mu}} \right) x^{\bar{\mu}}, \tag{3.1.9}$$

where $T_{\bar{\mu}}$ is a symmetric tensor defined by a polynomial of $\mathrm{Tr}(A_{\bar{\nu}})$ with $2 \leq |\nu| \leq |\mu| - 2$, in particular $T_{\bar{\mu}} = 0$ for $|\mu| = 2, 3$. The general $T_{\bar{\mu}}$ can be calculated recursively by the formula (3.1.7).

Based on the above discussions, we can now give a jet-level condition for metrics being flat, conformally flat, and conformal normal at a point p by the theorem below:

Theorem 3.1.3. *Suppose that a conformal structure $[g]$ on X is k^{th} order conformally flat at a point p. Then $g \in [g]$ is $(k+2)^{th}$ order conformal normal at p if and only if g is k^{th}*

order flat at p.

Proof: Suppose that g is k^{th} order flat at p. Then (3.1.8) shows that $A_{\bar{\mu}}(p)$ vanishes for $2 \leqslant |\bar{\mu}| \leqslant k+2$. Hence $\mathrm{Tr}(A_{\bar{\mu}}(p))$ vanishes to the same order, and hence $\det g = 1 + O(r^{k+3})$ by (3.1.4). Thus g is $(k+2)^{\text{th}}$ order conformal normal at p.

On the other hand, assume that $g \in \mathcal{CN}^{k+2}(p)$, i.e. $j_p^{k+2}\det(g) = j_p^{k+2}(\exp(\mathrm{Tr}(A))) = 1$.

Hence, by induction on $|\bar{\mu}|$ in (3.1.4), we have:

$$\mathrm{Tr}(A_{\bar{\mu}}(p)) = 0, \text{ for } |\bar{\mu}| \leq k+2 \tag{3.1.10}$$

Take trace of (3.1.8) on both sides up to $|\bar{\mu}| = k+2$, along with (3.1.10), we have:

$$Ric_{ij}(p) = 0, \text{ for } \bar{\mu} = (ij), \tag*{$(3.1.11)_0$}$$

$$Ric_{ij,k}(p) + Ric_{ik,j}(p) + Ric_{jk,i}(p) = 0, \text{ for } \bar{\mu} = (ijk), \tag*{$(3.1.11)_1$}$$

$$\cdots$$

$$c_{k+2}\sum_{\mu\sim\bar{\mu}} Ric_{\mu}(p) + P(Riem_{\mu'}(p)) = 0, \tag*{$(3.1.11)_k$}$$

where $|\bar{\mu}| = k+2$ and $P(Riem_{\mu'})$ is a polynomial of $Riem_{\mu'}$ with $|\mu'| \leq k$.

Let C be the Cotton tensor defined for the metric g, in local coordinates we have

$$\begin{aligned} C_{ijk} &= Ric_{ij,k} - Ric_{ik,j} + \frac{1}{2(n-1)}(S_j g_{ik} - S_k g_{ij}) \\ &= P_{ij,k} - P_{ik,j}, \end{aligned} \tag{3.1.12}$$

where $P = Ric - \frac{1}{2(n-1)}Sg$ is the so-called Schouten tensor.

We claim the following lemma with the detailed proof given in Appendix A.

Lemma A.3. *Equations $(3.1.11)_0$ to $(3.1.11)_k$, together with the assumption $j_p^{k-1}(C) = 0$, imply that $j_p^k(Ric) = 0$.*

Since conformally flatness is defined by two cases: Namely, dimension $n = 3$ and $n \geq 4$, we discuss accordingly below:

First, recall the Ricci decomposition of the Riemannian curvature tensor:

$$\begin{aligned} R_{ijkl} = W_{ijkl} &+ \frac{S}{n(n-1)}\big(g_{il}g_{jk} - g_{ik}g_{jl}\big) \\ &+ \frac{1}{n-2}\big(Z_{il}g_{jk} - Z_{jl}g_{ik} - Z_{ik}g_{jl} + Z_{jk}g_{il}\big), \end{aligned} \tag{3.1.13}$$

where $Z_{ij} = Ric_{ij} - \frac{1}{n}Sg_{ij}$.

In dimension 3, the Weyl tensor W vanishes identically, and hence by (3.1.13), the Riemannian curvature tensor is determined by the Ricci curvature.

In dimension 3, conformally flatness to order k means $j_p^{k-1}(C) = 0$. And hence by Lemma A.3, we have $j_p^k(Ric) = 0$. And $j_p^k(Riem) = 0$ in turn by (3.1.13).

In dimension $n \geq 4$, conformally flatness to order k means $j_p^k(W) = 0$. The relation between the Weyl tensor and the Cotton tensor is given below:

$$\nabla_a W^a{}_{bcd} = (n-3)C_{bcd}.$$

And thus $j_p^k(W) = 0 \Rightarrow j_p^{k-1}(C) = 0$. Again by Lemma A.3, we have $j_p^k(Ric) = 0$.

Thus, by the Ricci decomposition (3.1.13), $j_p^k(W) = 0$ together with $j_p^k(Ric) = 0$ imply $j_p^k(Riem) = 0$. $\square$

3.2 Jet Level Existence of Conformal Normal Metrics

The jet level existence of conformal normal metrics is proved by Lee and Parker in [LP87] using a Graham normalization process. In this section, we review their proof with a focus on the conformal factors, the construction of which will be applied later.

Lemma 3.2.1. *Let (X, g) be an n-dimensional Riemannian manifold, and $x : \mathbb{R}^n \to X$ be the g-normal coordinate chart at a point $p \in X$. For a smooth function $f \in C^\infty(\mathbb{R}^n)$ such that $f = O(|x|^k)$, take the conformal metric $g^f := e^{2f}x^*g$ and let $\tilde{x} : \mathbb{R}^n \to \mathbb{R}^n$ be the corresponding coordinate transformation to the g^f-normal coordinate system, then we have:*

$$\tilde{x}(x) - x = O(|x|^k).$$

Proof: In the g-normal coordinates x, let Γ^a_{bc} and $\tilde{\Gamma}^a_{bc}$ be the Christoffel symbols for g and g^f respectively.

For each vector x, the radial ray $\gamma = x \cdot t$ is a g-geodesic from the point p, hence satisfies the g-geodesic equation:

$$\Gamma^a_{bc}(x \cdot t)x^b x^c = 0. \tag{3.2.1}$$

We also have the conformal transformation formula for $\tilde{\Gamma}^a_{bc}$ below:

$$\tilde{\Gamma}^a_{bc} = \Gamma^a_{bc} + f_b\delta^a_c + f_c\delta^a_b - f_d g^{da} g_{bc}. \tag{3.2.2}$$

Hence $f = O(|x|^k) \Rightarrow T^a_{bc} = \tilde{\Gamma}^a_{bc} - \Gamma^a_{bc} = O(|x|^{k-1})$.

Let $\gamma_x(t)$ be the g^f-geodesic satisfying the initial conditions $\gamma_x(0) = 0, \gamma_x'(0) = x$. Then by definition we have $\tilde{x}(x) = \gamma_x(1)$ and $\gamma_x(t)$ satisfies the g^f-geodesic equation:

$$\frac{d^2}{dt^2}\gamma_x^a + \tilde{\Gamma}_{bc}^a(\gamma)\frac{d}{dt}\gamma_x^b\frac{d}{dt}\gamma_x^c = 0 \tag{3.2.3}$$

Take the Taylor expansion of $\gamma_x(t)$ at the origin and evaluate at $t = 1$, we have:

$$\tilde{x}(x) = \gamma_x(1) = x + \cdots + \frac{\gamma_x^{(k)}(0)}{k!} + \cdots,$$

where $\frac{\gamma_x^{(k)}(0)}{k!}$ is the degree k term in the Taylor expansion of $\tilde{x}(x)$.

Hence to prove $\tilde{x}(x) - x = O(|x|^k)$ is to prove $\gamma_x^{(i)}(0) = 0$ for all $i = 2, \ldots, k-1$, which we prove below by induction on k.

Since $\gamma_x(0) = 0$, the statement is true for $k = 1$. Assume the statement is true for $k \leq m+1$.

For $k = m+2$, take the $(m+1)^{\text{th}}$ order derivative of $\gamma(x)$, by Equation (3.2.3) at the origin, we have:

$$\begin{aligned} -\gamma_x^{a(m+1)}(0) &= \frac{d^{(m-1)}}{dt^{(m-1)}}(\tilde{\Gamma}_{bc}^a(\gamma)\frac{d}{dt}\gamma_x^b\frac{d}{dt}\gamma_x^c)|_{t=0} \\ &= \frac{d^{(m-1)}}{dt^{(m-1)}}(\Gamma_{bc}^a(\gamma_x)\dot{\gamma}_x^b\dot{\gamma}_x^c)|_{t=0} + \frac{d^{(m-1)}}{dt^{(m-1)}}(T_{bc}^a(\gamma_x)\dot{\gamma}_x^b\dot{\gamma}_x^c)|_{t=0}, \end{aligned}$$

where the second equality follows from (3.2.2). By the fact that $T_{bc}^a = O(|x|^{m+1})$, the second term vanishes at the origin. By the induction assumption, we have $\gamma_x^{(i)}(0) = 0$ for $2 \leq i \leq m$,, and hence the first term equals:

$$0 = \sum_{\alpha,b,c}\Gamma_{bc}^{a(m-1)}(0)\prod_{i=1}^{n}(\dot{\gamma}_x^i(0))^{\alpha_i}\dot{\gamma}_x^b(0)\dot{\gamma}_x^c(0) = \frac{d^{(m-1)}}{dt^{(m-1)}}(\Gamma_{bc}^a(x\cdot t)x^bx^c)|_{t=0},$$

where α are multi-indices of absolute value $m-1$ and the last equality follows from the euqation (3.2.1). □

Theorem 3.2.2 (Lee-Parker)**.** *Let $(X, [g])$ be a conformal manifold of dimension $n \geq 3$. At a point $p \in X$, for $\forall g \in [g]$ and $l = 1, \ldots, \infty$, there is a unique formal polynomial h_l in n variables and of degree $\leq l$ such that for any smooth function f satisfying $j_p^l(f) = h_l$ with respect to the g-normal coordinates, the conformal metric $\tilde{g} = e^{2f}g$ has the following properties :*

(a) $j_p^1(\tilde{g}) = j_p^1(g)$,

(b) $\det(exp^*\tilde{g})(p) = 1 + O(r^{l+1})$ *in the* $\tilde{g}$ *normal coordinates, namely* $\tilde{g}$ *is* k^{th} *order conformal normal at* p.

Proof: The proof is by induction on the jet level l. For $l = 1$, it is clear that a smooth function f preserves the 1-jet of g at p if and only if $j^1_p(f) = 0$, and hence $h_1(x) = 0$ and is unique. For the same reason, we see that the property (a) is true if and only if $j^1_p(h_l) = 0$ for any l. We hence assume h_l has quadratic leading terms and prove the unique existence of such h_l satisfying the property (b) by induction.

Assume the statement is true for $l = k$, namely, there's a unique formal polynomial h_k of degree $\leq k$ such that for any smooth function f with $j^k_p(f) = h_k$ under the g-normal coordinates, the conformal metric $g_k := e^{2f}g$ is conformal normal to the k^{th} order at p.

Explicitly, let $g_k = e^{2h_k} \cdot g$, with $\{x_k = (x^i_k)\}$ being the g_k normal coordinates at p. With respect to $\{x_k\}$ we have:

$$\det(g_k) = 1 + \sum_{|\mu|=k+1} \{c_{k+1} \sum_{\mu\sim\bar{\mu}} \partial Ric_\mu(0)\} x^\mu_k + S_{\bar{\mu}} x^{\bar{\mu}}_k + O(r^{k+2}),$$

where $S_{\bar{\mu}}$ is a symmetric tensor defined by a polynomial of $Riem_\nu$ with $2 \leq |\nu| \leq k-1$.

For $l = k+1$, let $e^{2f_{k+1}}$ be a conformal factor such that $g_{k+1} := e^{2f_{k+1}} g_k$ is conformal normal to $(k+1)^{\text{th}}$ order at p.

On one hand, if $j^k_p(f_{k+1}) \neq 0$, then by assumption $g_{k+1} = e^{2(f_{k+1}+h_k)} \cdot g$ satisfies (b) for $l = k$, however $j^k_p(f_{k+1} + h_k) = j^k_p(f) + h_k \neq h_k$ contradicts with the uniqueness of h_k in the induction assumption. Hence $j^k_p(f_{k+1}) = 0$ and the $S_{\bar{\mu}}$ term is invariant under the conformal change by $e^{f_{k+1}}$.

Let x_{k+1} be the g_{k+1} normal coordinates, by Lemma 3.2.1, we have $x_{k+1} - x_k = O(|x_k|^{k+1})$. Hence for $|\mu| = k+1$, $\partial(Ric_{g_{k+1}})_\mu(0)$ have the same value under both coordinates x_{k+1} and x_k.

On the other hand, for an arbitrary conformal metric $g_f = e^{2f} g_k$, the conformal transformation formula of the Ricci curvature is:

$$Ric_{g_f} = Ric_{g_k} - (n-2)(d^2 f - df^{\otimes 2}) + (\Delta f - (n-2)|df|^2) g_k \tag{3.2.4}$$

By (3.2.4), we see that if $j_p^k(f) = 0$, then conformal change $\partial(Ric_{g_f})_\mu(0)$ with $|\mu| = k+1$ depends only on $j_p^{k+1}(f)$.

In summary, $\partial(Ric_{g_{k+1}})_\mu(0)$ depends only on the homogeneous degree $k+1$ terms of f_{k+1} and has the same value with respect to both $\{x_{k+1}\}$ and $\{x_k\}$ coordinates.

The Taylor expansion of g_{k+1} in the $\{x_{k+1}\}$ coordinates is

$$\det(g_{k+1}) = 1 + \sum_{|\mu|=k+1} c_{k+1}(\partial Ric_{g_{k+1}\mu}(0))x_{k+1}^{\bar{\mu}} + S_{\bar{\mu}}x_{k+1}^{\bar{\mu}} + O(r^{k+2}).$$

Thus g_{k+1} is conformal normal to the order $(k+1)$ if and only if for any $\bar{\mu}$ with $|\bar{\mu}| = k+1$, we have:

$$\sum_{|\mu|=k+1} c_{k+1}(\partial Ric_{g_{k+1}\mu}(0)) + S_{\bar{\mu}})x_{k+1}^{\bar{\mu}} = 0.$$

Since $\partial(Ric_{g_{k+1}})_\mu(0)$ have the same value with respect to both x_{k+1} and x_k coordinates, by a coordinate transformation to the $\{x_k\}$ coordinate system, we have:

$$\sum_{|\mu|=k+1} c_{k+1}(\partial Ric_{g_{k+1}\mu}(0))x_k^{\mu} + \tilde{S}_{\bar{\mu}})x_k^{\bar{\mu}} = 0. \tag{3.2.5}$$

We work with $\{x_k\}$ coordinates and let

$$f_{k+1} = \sum c_{i_1\cdots i_{k+1}} x_k^{i_1} \cdots x_k^{i_{k+1}}. \tag{3.2.6}$$

Take the $(k+1)^{\text{th}}$ order derivative of (3.2.4) in the $\{x_k\}$ coordinates, we have:

$$\sum_{|\mu|=k+1} c_{k+1}(\partial Ric_{g_{k+1}\mu}(0))x_k^{\mu} + \tilde{S}_{\bar{\mu}}x_k^{\bar{\mu}} =$$
$$\sum_{|\mu|=k+1} c_{k+1}\partial(Ric_{g_k})_\mu(0)x_k^{\mu} - (n-2)d^2 f_{k+1}(x_k, x_k) + r^2\Delta_0 f_{k+1} + \tilde{S}_{\bar{\mu}}x_k^{\bar{\mu}}.$$

By Euler's formula, $d^2 f_{k+1}(x_k, x_k) = (x_k d)^2 f_{k+1} - (x_k d) f = k(k+1) f_{k+1}$.

Thus Equation (3.2.5) is equivalent to the following equation of f_{k+1}:

$$(r^2\Delta - (n-2)k(k+1))f_{k+1} = -c_{k+1}(\sum_{|\mu|=k+1} \partial Ric_\mu(0) + \tilde{S}_{\bar{\mu}})x_k^{\mu}. \tag{3.2.7}$$

Comparing the coefficients of $x_k^{i_1\cdots i_{k+1}}$ on both sides of (3.2.7) and by Lemma 5.3 in [LP87], we obtain a non-degenerate linear system of equations for the unknowns, i.e. the coefficients $c_{i_1\cdots i_{k+1}}$ of f_{k+1}. Thus gives a unique solution for $c_{i_1\cdots i_{k+1}}$.

Let $h_{k+1}(y) = h_k(y) + \sum c_{i_1\cdots i_{k+1}} y^{i_1} \cdots y^{i_{k+1}}$ with y being the formal variables. Denote $\{x\}$ for the g normal coordinates, then any function $\tilde{f}_{k+1}$ satisfying $j_p^{k+1}(\tilde{f}_{k+1}) =$

$\sum c_{i_1\cdots i_{k+1}} x^{i_1}\cdots x^{i_{k+1}}$ in the $\{x\}$ coordinates, by changing to the $\{x_k\}$ coordinates, we have:

$$\begin{aligned}\tilde{f}_{k+1} &= \sum c_{i_1\cdots i_{k+1}} x^{i_1}\cdots x^{i_{k+1}} + O(r^{k+2}) \\ &= \sum c_{i_1\cdots i_{k+1}} x_k^{i_1}\cdots x_k^{i_{k+1}} + O(r^{k+2}) = f_{k+1} + O(r^{k+2})\end{aligned} \tag{3.2.8}$$

Hence $j_p^{k+1}(\tilde{f}_{k+1}) = j_p^{k+1}(f_{k+1})$ and the metric $\tilde{g}_{k+1} := e^{2\tilde{f}_{k+1}}\cdot g_k = e^{2\tilde{f}_{k+1}+2h_k(x)}\cdot g$ with $j_p^{k+1}(\tilde{f}_{k+1}+h_k) = h_{k+1}(x)$ is conformal normal to the $(k+1)^{\text{th}}$ order. We thereby proved the theorem for all $k<\infty$ by induction.

For $k=\infty$, we first recall Borel's Lemma below (cf. [Hör15]):

Lemma 3.2.3 (Borel's Lemma)**.** *The canonical map from the ring of germs of C^∞ function at $0\in\mathbb{R}^n$ to the ring of formal power series obtained by taking the Taylor series at 0 is surjective.*

Explicitly, let $\underline{f} = \sum_{|\alpha|=2}^{\infty} c_\alpha x^\alpha$ be the unique formal power series obtained by the above algorithm, where α is a multi-index with absolute value $|\alpha|$. Fix ψ a smooth bump function on $\mathbb{R}$ such that $\psi=1$ on B_1, and $\mathrm{supp}(\psi)\subseteq B_2$. For $|\alpha|=m$, let

$$H_\alpha = \max_{0\le l\le k<m}\left(\frac{2^{2m-l} m!(k+1)!|c_\alpha|\|\psi^{(k-l)}\|_\infty}{(m-k)!(k-l)!l!}\right)^{\frac{1}{m-k}}.$$

Then

$$f := \sum_{|\alpha|=2}^{\infty} c_\alpha \psi(H_\alpha|x|)x^\alpha \tag{3.2.9}$$

is a smooth function in $C^\infty(\mathbb{R}^n)$ of which the Taylor series at the origin equals $\underline{f}$.

For any finite integer $k\ge 2$, let $\underline{f}_k$ be the truncation of $\underline{f}$ up to degree k. Then $e^{2f}g = e^{2f-2\underline{f}_k}(e^{2\underline{f}_k}g)$, where by definition the function $2f-2\underline{f}_k = O(r^{k+1})$ and the conformal metric $(e^{2\underline{f}_k}g)$ is of k^{th} order conformal normal at p. Let x^f and $x^{\underline{f}_k}$ be the normal coordinates of $e^{2f}g$ and $e^{2\underline{f}_k}g$ respectively, then by Lemma 3.2.1, we have $x^f - x^{\underline{f}_k} = O(|x^{\underline{f}_k}|^{k+1})$. Hence the coordinate transformation to the x^f coordinates preserves the k^{th} order conformal normal property. Hence $e^{2f}g$ is conformal normal at p to any finite order k and thus the statement is true for $k=\infty$. $\square$

As we have seen in Section 2.3, given a conformal structure $[g]$, one can define the corresponding $\widehat{CO}(n)$ structure P which is a principal 2-frame bundle over X. By applying the conformal normal coordinates, we can give a geometric construction of P as the principal $\widehat{CO}(n)$ bundle of conformal normal 2-frames:

Theorem 3.2.4. *Let $\mathcal{CNC}^2(X) = \sqcup_{p\in X}\mathcal{CNC}^2(p)$ be the bundle of 2-jets of the second order conformal normal coordinates, then $\mathcal{CNC}^2(X)$ is a principal $\widehat{CO}(n)$ bundle on X which characterizes the conformal structure $[g]$ as the $\widehat{CO}(n)$ structure.*

Proof: $\mathcal{CNC}^2(X)$ is a sub-bundle of the general linear 2-frame bundle P^2, and hence has a local trivialization induced from P^2. To see $\mathcal{CNC}^2(X)$ is a principal $\widehat{CO}(n)$ bundle, we define the $\widehat{CO}(n)$ group action on $\mathcal{CNC}^2(X)$ as follows:

For an element $j_0^2(\varphi) \in \mathcal{CNC}^2(p)$ defined by a metric $g \in [g]$ such that $j_p^2(g) \in \mathcal{CN}^2(p)$, and $\beta \in O(T_pX, g_p)$, the action of $\widehat{CO}(n)$ on $j_0^2(\varphi)$ is defined as follows:

By (2.3.1), a general element of $\widehat{CO}(n)$ is of the form:

$$h(x) = \lambda\frac{Ax - x^2\eta}{1 - 2\eta\cdot Ax + x^2\eta^2},$$

where $\lambda > 0$, $\eta \in \mathbb{R}^n$, $A \in O(n)$.

We now focus on $\mathbb{R}^n$ and denote φ^*g briefly as g, by a direct calculation we have:

$$j_0^1(h^*g) = \lambda^2(1 + 4\eta Ax)j^1(g).$$

Hence $\widehat{CO}(n)$ acts free and transitively on $\mathcal{CN}^1(p)$. By the Lemma (3.2.2), for any $\tilde{g} \in j_0^1(h^*g)$, there is a unique homogeneous degree 2 polynomial f such that $\det(e^{2f}\tilde{g})(p) = 1 + o(r^3)$ and notice that $h^*\beta$ is an orthonormal frame of $h^*g(p) = e^{2f}\tilde{g}(p)$. Hence the normal coordinates $\tilde{\varphi}$ define by $e^{2f}\tilde{g}$ and $h^*\beta$ gives an element $j_0^2(\tilde{\varphi}) \in \mathcal{CNC}^2$.

We define the action of $\widehat{CO}(n)$ on $\mathcal{CNC}^2(p)$ as

$$h \cdot j_0^2(\varphi) = j_0^2(\tilde{\varphi}). \tag{3.2.10}$$

Since the action of $\widehat{CO}(n)$ on $\mathcal{CN}^1 = J^1([g])$ is free and transitive and $O(n) \subseteq \widehat{CO}(n)$ acts on the orthonormal frames free and transitive, the $\widehat{CO}(n)$ action on $\mathcal{CNC}^2(X)$ is free and transitive.

To show $\mathcal{CNC}^2(X)$ defines a $\widehat{CO}(n)$ structure, we need to show the $\widehat{CO}(n)$ action defined above is the jet composition action (2.3.4). Let $\iota : \mathcal{CNC}^2(X) \to P^2$ be the natural inclusion map, it is sufficient to show ι is $\widehat{CO}(n)$ equivariant.

Recall near a point $p \in P^2$, we have natural local coordinates $u = (u_i; u^i_j; u^i_{jk})$ and a natural local coordinates for $\widehat{CO}(n)$: For a general element $h = h_{\lambda,A,\eta} \in \widehat{CO}(n)$, the corresponding coordinates is

$$h = (h^i_j, h^i_{jk}) = (\lambda a_{ij}, \frac{1}{2}(\eta_j \delta^i_k + \eta_k \delta^i_j - \eta^i \delta^{pq})\lambda^2 a_{pj} a_{qk})$$

Let $u = (0, \delta_{ij}, 0)$, then the GL^2_n action on u in local coordinates is:

$$h \cdot u = (h^i_j, h^i_{jk}) \cdot (0, \delta_{ij}, 0) = (0, \lambda a_{ij}, \frac{1}{2}(\eta_j \delta^i_k + \eta_k \delta^i_j - \eta^i \delta^{pq})\lambda^2 a_{pj} a_{qk}). \tag{3.2.11}$$

On the other hand, the action of $\widehat{CO}(n)$ on $\mathcal{CNC}^2$ is defined by exponential maps, and thus to obtain a local expression of the action (3.2.10), consider the initial value problem of the geodesic equation:

$$\begin{cases} \ddot{\gamma}^i + \widetilde{\Gamma}^i_{jk} \dot{\gamma}^j \dot{\gamma}^k = 0, \\ \dot{\gamma}(0) = dh(X), \\ \gamma(0) = p, \end{cases} \tag{3.2.12}$$

where $\widetilde{\Gamma}^i_{jk}$ is the Christoffel symbol of the conformal normal metric $\tilde{g} = e^{2f} g$, with $j^1_0(e^{2f}) = \lambda^2(1 + 4\eta \cdot Ax)$.

The conformal transformation formula of Christoffel symbols is given as below:

$$\widetilde{\Gamma}^i_{jk} = \Gamma^i_{jk} + \frac{\partial f}{\partial x^j} \delta^i_k + \frac{\partial f}{\partial x^j} \delta^i_k - \frac{\partial f}{\partial x^l} g^{li} g_{jk}.$$

Thus $\widetilde{\Gamma}^i_{jk} = -(b_j \delta^i_k + b_k \delta^i_j - b^i \delta^{jk}) + o(r)$. Taking the Taylor series iteration of (3.2.12), at step 1, we have $(\dot{\gamma}^i)_1 = \lambda a_{ij} x^j + (b_j \delta^i_k + b_k \delta^i_j - b^i \delta^{jk})\lambda^2 a_{ip} x^p a_{jq} x^q(t)$.

By taking the integral of t on $[0, 1]$, we have

$$j^2_0(h \cdot \tilde{\varphi}) = \lambda a_{ij} x^j + \frac{1}{2}(b_j \delta^i_k + b_k \delta^i_j - b^i \delta^{jk})\lambda^2 a_{ip} x^p a_{jq} x^q \tag{3.2.13}$$

The inclusion map ι is $\widehat{CO}(n)$ equivariant by comparing (3.2.11) and (3.2.13). $\square$

CHAPTER 4

LOCAL EXISTENCE RESULTS

In this chapter, we discuss linear PDE results which will be applied in the proof of Theorem 5.1 and review Cao's proof of the local existence of conformal normal metrics with careful tracking of the background metrics and constants in estimations.

4.1 Linear PDE Results

Let $\mathbb{R}^n$ be the standard Euclidean n-space with coordinates $\{x = (x_i)\}$.

Denote $R = |x| = (\sum x_i^2)^{\frac{1}{2}}$, $\theta = \frac{x}{|x|}$, $B_\rho = \{|x| \leq \rho | x \in \mathbb{R}^n\}$. Take $\Delta_0 = \sum \frac{\partial^2}{\partial x_i^2}$ the standard Laplacian and Δ^* the spherical Laplacian. Two Laplacian operators are related by:

$$\Delta_0(v) = \frac{\partial^2 v}{\partial R^2} + \frac{n-1}{R}\frac{\partial v}{\partial R} + \frac{1}{R^2}\Delta^* v. \tag{4.1.1}$$

Consider the following linear differential operator:

$$\mathcal{L}_0(v) = \Delta_0(v) + (n-2)\frac{\partial^2}{\partial R^2} \tag{4.1.2}$$

Define the following weighted Hölder norms and spaces on which the operator $\mathcal{L}_0$ is applied:

For $0 < \alpha < 1$, $3 \leq k < N$, define $C_{k,\alpha;N,\rho}$ to be the space of functions in the Hölder space $C^{k,\alpha}(B_\rho)$ for which the following norm is finite:

$$\begin{aligned} |||v|||_{k,\alpha;N,\rho} &= \sup_{0<r\leq\rho}\{r^{-N}|v|_{C^{k,\alpha}(B_r - B_{r/2})}\} \\ &= \sup_{0<r\leq\rho}\{r^{-N}(\sum_{|\beta|=0}^{k} r^{|\beta|}\sup_{\frac{r}{2}\leqslant|x|\leqslant r}\{|\partial_\beta v(x)|\} + \\ &\qquad r^{k+\alpha}\sup\{\frac{|\partial_\beta v(x) - \partial_\beta v(y)|}{|x-y|^\alpha} \mid x \neq y, \frac{r}{2} \leqslant x, y \leqslant r, |\beta| \leqslant k\})\} \end{aligned} \tag{4.1.3}$$

Remark 4.1.1. In [Cao91], Cao generalized the operator $\mathcal{L}_0$ to a collection of singular elliptic differential operators, see also [PR00] for a semi-linear generalization. In [PR00], the weighted Hölder norm $|||\,|||_{k,\alpha;N,\rho}$ is generalized to define a family of Banach spaces $C^{k,\alpha}_{\nu,\sigma}(\bar{\Omega}\backslash\Sigma)$, see [PR00] page 23 Lemma 2.1 for details. In particular, for $\sigma = \rho, \bar{\Omega} = B_\rho, \Sigma = \{0\}, \nu = N \geqslant 2k$, we have a Banach space

$$C^{k,\alpha}_{N,\rho}(B_\rho\backslash\{0\}) = \{v \mid v \in C^{k,\alpha}_{loc}(B_\rho\backslash\{0\}), |||v|||_{k,\alpha;N,\rho} < \infty\}.$$

On the other hand, for an element $v \in C^{k,\alpha}_{N,\rho}(B_\rho\backslash\{0\})$, by taking the term with $|\beta| = 0$ in the $(k, \alpha; N, \rho)$ norm, we have $|x|^{-N}\sup_{0<|x|\leqslant\rho}|v(x)| < \infty$. Therefore $v(x) = O(|x|^N)$ near the origin, and thus $v(x) \in C^{k,\alpha}_{loc}(B_\rho) = C^{k,\alpha}(B_\rho)$ and $C_{k,\alpha;N,\rho}$ is a Banach space.

Lemma 4.1.2. *Let* $0 < \rho \leq 1$*, and* $v \in C^{k,\alpha}(B_1)$*, then*

$$|||v(\rho x)|||_{k,\alpha;N,1} \leq \rho^{N-k}|||v(x)|||_{k,\alpha;N,\rho}. \tag{4.1.4}$$

Proof: By definition

$$|||v(\rho x)|||_{k,\alpha;N,1} = \sup_{0<r\leq 1}\{r^{-N}(\sum_{|\beta|=0}^{k} r^{|\beta|} \sup_{\frac{r}{2}\leqslant|x|\leqslant r}\{|\partial_\beta v(\rho x)|\}+ r^{k+\alpha}\sup\{\frac{|\partial_\beta v(\rho x) - \partial_\beta v(\rho x')|}{|x-x'|^\alpha}|x \neq x', \frac{r}{2} \leqslant x, x' \leqslant r, |\beta| \leqslant k\})\}$$

Substituting $\lambda = \rho r$ and $y = \rho x$ to the above equation, we have

$$\begin{aligned} |||v(\rho x)|||_{k,\alpha;N,1} = \sup_{0<\lambda\leq\rho}\Bigg\{&\frac{\rho^N}{\lambda^N}(\sum_{|\beta|=0}^{k}\lambda^{|\beta|}\sup_{\frac{\lambda}{2}\leqslant|x|\leqslant\lambda}\{|\partial_\beta v(y)|\}+ \\ &\lambda^{k+\alpha}\sup\{\rho^{(|\beta|-k)}\frac{|\partial_\beta v(y)-\partial_\beta v(y')|}{|y-y'|^\alpha} \mid y\neq y', \frac{\lambda}{2}\leqslant x,x'\leqslant\lambda, |\beta|\leqslant k\})\Bigg\} \\ \leq \rho^{N-k}\sup_{0<\lambda\leq\rho}\Bigg\{&\lambda^{-N}(\sum_{|\beta|=0}^{k}\lambda^{|\beta|}\sup_{\frac{\lambda}{2}\leqslant|x|\leqslant\lambda}\{|\partial_\beta v(y)|\}+ \\ &\lambda^{k+\alpha}\sup\{\frac{|\partial_\beta v(y)-\partial_\beta v(y')|}{|y-y'|^\alpha} \mid y\neq y', \frac{\lambda}{2}\leqslant x,x'\leqslant\lambda, |\beta|\leqslant k\})\Bigg\} \\ = \rho^{N-k}&|||v(x)|||_{k,\alpha;N,\rho}. \end{aligned}$$

□

Define $Z_\rho := \{f \in C^\infty(B_\rho) | j^\infty_0(f) = 0\}$, we have the following lemma:

Lemma 4.1.3.

$$Z_\rho = \bigcap_{k,N}^{\infty} C_{k,\alpha;N,\rho}$$

Proof: Without loss of generality, we prove for $\rho = 1$ and omit the index ρ.

For a smooth function $f \in C^\infty(B_1)$, we clearly have the following equivalence:

$$\sup_{0<r\leq 1}\sup_{\frac{r}{2}\leq|x|\leq r}\frac{|f(x)|}{r^N} < \infty \Leftrightarrow f(x) = O(|x|^N). \tag{4.1.5}$$

To see $\bigcap_{k,N}^{\infty} C_{k,\alpha;N} \subseteq Z_1$, for each $v \in C_{k,\alpha;N}$, take the $|\beta| = 0$ term in its weighted norm, we have $\sup_{0<r<1}\sup_{\frac{r}{2}\leq|x|\leq r}\frac{|v(x)|}{r^N} < \infty$, thus by (4.1.5) we have $v \in O(|x|^N)$. Let $N \to \infty$, we have $v \in Z_1$.

Conversely, for $v(x) \in Z_1$, for each ∂_β term in the norm $\|\|\|_{k,\alpha;N}$, we have:

$$O(|x|^{N-|\beta|}) = \sup_{0<r<1} r^{-N+|\beta|} \sup_{\frac{r}{2}\leq|x|\leq r} |\partial_\beta v(x)| < \infty \Leftrightarrow \partial_\beta v(x).$$

For the α continuous terms, we have

$$\begin{aligned}
&\sup_{0<r<1} r^{k+\alpha-N} \sup\{\frac{|\partial_\beta v(x) - \partial_\beta v(y)|}{|x-y|^\alpha} \mid x \neq y, \frac{r}{2} \leq |x|, |y| < r, |\beta| \leq k\} \\
&< 2 \sup_{0<r<1} r^{k+\alpha-N} \sup\{\frac{|\partial_\beta v(x) - \partial_\beta v(y)|}{|x-y|} \mid x \neq y, \frac{r}{2} \leq |x|, |y| < r, |\beta| \leq k\} \\
&< 2 \sup_{0<r<1} r^{k+\alpha-N} \sup_{\frac{r}{2}\leq|x|\leq r} \{\partial_{\beta'} v(x) \mid |\beta'| \leq k+1\} < \infty \Rightarrow \partial_{\beta'} v(x) = O(|x|^{N-k-\alpha})
\end{aligned}$$

Thus $v \in \bigcap_{k,N}^{\infty} C_{k,\alpha;N}$. $\square$

$\mathcal{L}_0$ maps $C_{k,\alpha;N,\rho}$ to $C_{k-2,\alpha;N-2,\rho}$, Z_ρ to Z_ρ and has a bounded linear right inverse operator $\mathscr{S}$ on Z_ρ defined as follows:

The eigenvalues of $-\Delta^*$ are $\lambda_l = l(l+n-2)$ with $l = 0, 1, 2, \dots$. The corresponding eigenspace of λ_l is of dimension $n_l = \binom{n+l-1}{n-1} - \binom{n+l-3}{n-1}$. Denote $\{\varphi_m\}$ the orthonormal basis of $L^2(S^{n-1})$ such that each φ_m is an eigenvector of $\lambda_{l(m)}$, with $l(m) = l$ for $1 + \sum_{i=0}^{l-1} n_i \leq m \leq \sum_{i=0}^{l} n_i$.

For a function $f \in Z_\rho$, consider the inhomogeneous equation

$$\mathcal{L}_0 v = f. \tag{4.1.6}$$

In spherical coordinates (R, θ), take the Fourier series

$$f(R, \theta) = \sum_{m=0}^{\infty} f_m(R) \varphi_m(\theta),$$

where $f_m(R) = \int_{S^{n-1}} f(R, \theta) \varphi_m(\theta) d\theta$.

By the separation of variables method, a formal Fourier series $v(R, \theta) = \sum_{m=0}^{\infty} \beta_m(R) \varphi_m(\theta)$ solves (4.1.6) iff β_m satisfies the following inhomogeneous Cauchy-Euler equation for each m:

$$(n-1)\beta_m'' + \frac{n-1}{R}\beta_m' - \frac{\lambda_{l(m)}}{R^2}\beta_m = f_m. \tag{4.1.7$_m$}$$

By variation of parameters, we obtain the following particular solutions of $(4.1.7)_m$:

$$\beta_0(f) = \frac{1}{n-1} \int_0^R r \cdot \ln \frac{R}{r} f_0(r) dr, \quad \beta_m(f) = \mathrm{Re} \sum_{i=1,2} R^{\gamma_m^i} \int_0^R \frac{f_m(r) r^{1-\gamma_m^i}}{2(n-1)\gamma_m^i} dr \text{ for } m \geq 1,$$

where $\gamma_m^i = (-1)^i \sqrt{\frac{\lambda_{l(m)}}{n-1}}$.

Take a smooth cut-off function $\zeta \in C^\infty(\mathbb{R})$ such that :

$$\zeta(R) = \begin{cases} 1, & R \leqslant \frac{3}{4}\rho; \\ 0, & R \geqslant \frac{4}{5}\rho. \end{cases}$$

We can now define the promised operator $\mathscr{S}$ as:

$$\mathscr{S}(f) = \zeta(R) \sum_{m=0}^{\infty} \beta_m(f) \cdot \varphi_m.$$

Theorem 4.1.4 (cf. Corollary 2.14, [Cao91])**.** *$\mathscr{S}$ is a linear operator on Z_ρ such that:*

a) $\mathcal{L}_0(\mathscr{S} f)(y) = f(y)$ for all $|y| \leq \frac{3}{4}\rho$.

b) If $f^t(y) = f(ty), 0 < t \leq 1$ and $|y| \leq \frac{3}{4}\rho$, then $(\mathscr{S} f^t)(y) = t^{-2}(\mathscr{S} f)(ty)$.

c) For any k, α, N, ρ, there is a constant number K_1 such that

$$|||\mathscr{S} \circ f|||_{k,\alpha;N,\rho} \leq K_1\{|||f|||_{0,\alpha;N,\rho} + |||f|||_{k-2,\alpha;N-2,\rho}\}. \tag{4.1.8}$$

A fact: Given (X, μ), (Y, ν) two σ-finite measure spaces. Let $\{f_n\}, \{g_k\}$ be two countable orthonormal basis for the Hilbert spaces $L^2(X)$ and $L^2(Y)$ respectively. Then $\{f_n g_k\}$ is a complete orthonormal basis for $L^2(X \times Y)$.

Proof of the fact: $< f_n g_k, f_m g_l >=< f_n, f_m >< g_k, g_l >= \delta_{nm}\delta_{kl}$, hence $\{f_n g_k\}$ are orthonormal.

Let $h \in L^2(X \times Y)$ such that $< h, f_n g_k >= 0$ for any n, k.

Namely, we have

$$\int_X \left(\int_Y h g_k d\nu \right) f_n d\mu = 0. \tag{4.1.9}$$

Denote $u_k(x) := \int_Y h(x, y) g_k d\nu$, by the Hölder inequality, we have

$$\|u_k\|_{L^2}^2 = \int_X \left(\int_Y h g_k d\nu \right)^2 d\mu \leq \int_X \left(\int_Y h^2 d\nu \right) \left(\int_Y g_k^2 d\nu \right) d\mu = \int_X \int_Y h^2 d\nu d\mu = \|h\|_{L^2}^2 < \infty.$$

And hence $u_k \in L^2(X)$, and by (3.1) $u_k = 0$ almost everywhere on X.

Denote $E_k = \{x \in X | u_k(x) \neq 0\}$, $\mu(E_k) = 0$, thus the countable union $E = \bigcup_0^\infty E_k$ has $\mu(E) = 0$.

Thus $u_k = 0$ on $X \backslash E$ for all k, namely $\int_Y h(x, y) g_k(y) d\nu = 0$.

Thus $h(x, y) = 0$ almost everywhere on Y, for all $x \in X \backslash E$.

$$\|h\|_{L^2}^2 = \int_X \int_Y h^2 d\nu d\mu = \int_{X \backslash E} \int_Y h^2 d\nu d\mu = 0.$$

Thus $h = 0$, and $\{f_n g_k\}$ is a complete orthonormal basis for $L^2(X \times Y)$. $\square$

Lemma 4.1.5. *For k, N big enough, the kernel of $\mathcal{L}_0$ in $C_{k,\alpha;N,\rho}$ in polar coordinates consists of*

$$u = \sum_{m=m_N}^{\infty} c_m R^{\gamma_{l(m)}} \varphi_m(\theta), \tag{4.1.10}$$

where $\gamma_{l(m)} = \sqrt{\frac{\lambda_{l(m)}}{n-1}}$, $\lambda_{l(m)}$ and φ_m are given as above. m_N is the smallest integer such that $\gamma_{l(m)} \geq N$. In particular, $Ker\mathcal{L}_0 \cap Z_\rho = 0$.

Proof. By Sturm-Liouville Theorem, the countable set

$$\left\{\psi_0 = \sqrt{\frac{1}{\rho}}, \psi_l = \sqrt{\frac{2}{\rho}} sin\left(\frac{n\pi r}{\rho}\right)\right\}_{l=1}^{\infty}$$

is a complete orthonormal basis of $C^0[0,\rho]$, thus of $L^2[0,\rho]$.

By the fact proved above, $\{\psi_l \cdot \varphi_m\}$ is a complete orthonormal basis of $L^2([0,\rho] \times S^{n-1})$.

For $k > 3$, take $u \in C_{k,\alpha;N,\rho}$ such that $\mathcal{L}_0 u = 0$. Under spherical coordinates we have

$$u(r,\theta) \in C^0([0,\rho] \times S^{n-1}) \subset L^2([0,\rho] \times S^{n-1}).$$

Thus

$$u(r,\theta) = \sum_{m=0}^{\infty}(\sum_{l=0}^{\infty} c_{ml}\psi_l(r))\varphi_m(\theta) = \sum_{m=0}^{M}(\sum_{l=0}^{L} c_{ml}\psi_l(r))\varphi_m(\theta) + \xi_{ML},$$

where the remainder ξ_{ML} satisfies:

$$\lim_{M,L\to\infty} \|\xi_{ML}\|_{L^2} = 0. \tag{4.1.11}$$

For each index m, denote $\beta_m = \sum_{l=0}^{\infty} c_{ml}\psi_l(r)$, then $u(r,\theta) = \sum_{m=0}^{\infty} \beta_m(r)\varphi_m(\theta)$.

Since $\{\varphi_m\}_0^{\infty}$ is a complete orthonormal basis of $L^2(S^{n-1})$, we have

$$\beta_m(r) = \int_{S_r^{n-1}} u(r,\theta)\varphi_m(\theta) dvol_{S_1^{n-1}}.$$

Thus $\beta_m(r) \in C^k[0,\rho]$, with $\sum_{l=0}^{\infty} c_{ml}\psi_l(r)$ its Fourier expansion.

Let $g(r) \in C_0^{\infty}([0,\rho])$ be an arbitrary test function.

For each index m_0, and $M > m_0$, consider

$$\begin{aligned} 0 &= \int_{B_\rho} \mathcal{L}_0 u \cdot g(r)\varphi_{m_0} \\ &= \int_{B_\rho} \mathcal{L}_0(\sum_{m=0}^{M}(\sum_{l=0}^{L} c_{ml}\psi_l(r))\varphi_m(\theta) + \xi_{ML})g(r)\varphi_{m_0} \\ &= \int_{B_\rho} \mathcal{L}_0(\sum_{m=0}^{M}(\sum_{l=0}^{L} c_{ml}\psi_l(r))\varphi_m(\theta))g(r)\varphi_{m_0} + \int_{B_\rho} \mathcal{L}_0(\xi_{ML})g(r)\varphi_{m_0} \end{aligned}$$

On one hand, integration by parts and using Hölder inequality, we have:

$$\begin{aligned} \left|\int_{B_\rho} \mathcal{L}_0(\xi_{ML})g(r)\varphi_{m_0}\right| &= \left|\int_{B_\rho} \xi_{ML}\mathcal{L}_0(g(r)\varphi_{m_0})\right| \\ &\leq \int_{B_\rho} |\xi_{ML}| \cdot |\mathcal{L}_0(g(r)\varphi_{m_0}| \\ &\leq \|\xi_{ML}\|_{L^2}\|\mathcal{L}_0(g(r)\varphi_{m_0}\|_{L^2} \leq C\|\xi_{ML}\|_{L^2} \end{aligned}$$

Thus by(4.1.11), we have

$$\lim_{M\to\infty, L\to\infty} \int_{B_\rho} \mathcal{L}_0(\xi_{ML})g(r)\varphi_{m_0} = 0.$$

On the other hand,

$$\begin{aligned} \int_{B_\rho} \mathcal{L}_0\left(\sum_{m=0}^{M}(\sum_{l=0}^{L} c_{ml}\psi_l(r))\varphi_m(\theta)\right) g(r)\varphi_{m_0} &= \int_{B_\rho}\left(\sum_{m=0}^{M}\mathcal{L}_0(\beta_{Lm})\varphi_m(\theta)\right) g(r)\varphi_{m_0} \\ &= \int_{B_\rho}((n-1)\beta''_{Lm_0} + \frac{n-1}{r}\beta'_{Lm_0} - \frac{\lambda_{m_0}}{r^2})g(r)\varphi^2_{m_0} \\ &= \int_0^\rho((n-1)\beta''_{Lm_0} + \frac{n-1}{r}\beta'_{Lm_0} - \frac{\lambda_{m_0}}{r^2})r^{n-1}gdr. \end{aligned}$$

Thus in summary

$$\int_0^\rho \lim_{L\to\infty}((n-1)\beta''_{Lm_0} + \frac{n-1}{r}\beta'_{Lm_0} - \frac{\lambda_{m_0}}{r^2})g(r)r^{n-1}dr = 0$$

Since $g(r)$ is arbitrary, we have

$$\lim_{L\to\infty}((n-1)\beta''_{Lm_0} + \frac{n-1}{r}\beta'_{Lm_0} - \frac{\lambda_{m_0}}{r^2}) = 0 \tag{4.1.12}$$

Since $\beta_{m_0} \in C^k$ with $k > 3$, by the standard Fourier series fact, β''_{Lm_0} converges uniformly to β''_{m_0}, thus (4.1.12) gives

$$(n-1)\beta''_{m_0} + \frac{n-1}{r}\beta'_{m_0} - \frac{\lambda_{m_0}}{r^2} = 0,$$

which is the Cauchy-Euler equation with characteristic polynomial

$$\gamma^2 = \frac{\lambda_{m_0}}{n-1}.$$

Thus $\pm\gamma_{m_0} = \pm\sqrt{\frac{\lambda_{m_0}}{n-1}}$. and the general solution is of the form:

$$u_{\text{gen}} = \sum_0^\infty (c_m r^{\gamma_m} + d_m r^{-\gamma_m})\varphi_m.$$

Since $u \in C_{k,\alpha;N,\rho}$, by definition, we have

$$\frac{\sup_{x\in S^{n-1}_R} |u(x)|}{R^N} \leq \frac{\sup_{x\in B_R\setminus B_{\frac{R}{2}}} |u(x)|}{R^N} \leq \| \, |u| \, \|_{k,\alpha;N,\rho} < \infty,$$

hence $|u(x)| \leq c_N |x|^N$, and

$$u = \sum_{m_N}^\infty c_m R^{\gamma_m}\varphi_m(\theta),$$

where m_N the smallest integer such that $\gamma_m \geq N$, and the equality holds in L^2 sense.

If $u \in Z_\rho$, we then have

$$U(r) = \int_{S_r} |u|^2 dvol_{S_r} \leq C_N r^{2N} Vol(S^{n-1}_r) = Cr^{2N+n-1} \tag{4.1.13}$$

holds for any N.

Assume, by contradiction, that there exists a smallest m_0 such that the coefficient $c_{m_0} \neq 0$, then

$$\begin{aligned} U(r) = \int_{S_r} |u|^2 dvol_{S_r} &= \int_{S_r} \sum (c_m^2 r^{2\gamma_m}\varphi_m^2) dvol_{S_r} \\ &\geq \int_{S_r} c_{m_0}^2 r^{2\gamma_{m_0}}\varphi_{m_0}^2 dvol_{S_r} = Cr^{2\gamma_{m_0}+n-1}, \end{aligned}$$

contradicts with (4.1.13) when $N > \gamma_{m_0}$. Thus $u = 0$. □

Corollary 4.1.6. *On the function space, Z_ρ, $\mathscr{S}$ is an inverse operator of $\mathcal{L}_0$ on both sides.*

Proof: For $v \in Z_\rho$, since $\mathscr{S}$ is a right inverse of $\mathcal{L}_0$, we have:

$$\mathcal{L}_0(\mathscr{S}(\mathcal{L}_0(v)) - v) = (\mathcal{L}_0 \circ \mathscr{S})(\mathcal{L}_0(v)) - \mathcal{L}_0(v) = 0.$$

Thus $\mathscr{S}(\mathcal{L}_0(v)) - v \in ker(\mathcal{L}_0) \cap Z_\rho$, by Lemma 4.1.5, we have $\mathscr{S}(\mathcal{L}_0(v)) = v$, hence $\mathscr{S}$ is a left inverse of $\mathcal{L}_0$ on Z_ρ. □

4.2 Local Existence of Conformal Normal Metrics

Theorem 4.2.1. *Let (X, g_0) be a C^∞ Riemannian manifold and let p be a point on X. Then there exists a conformal metric $g = \Phi g_0$ such that $\det g_{ij}(y) = 1$ for all sufficiently small $\|y\|$, i.e., the exponential map of g at p, exp_p, is a local volume preserving map in a small neighborhood of p.*

The above theorem is proved as Corollary 0.1 in [Cao91], we give a brief review of Cao's work focusing on conformal normal metrics. We first rephrase the existence problem of a conformal normal metric into solving a singular elliptic equation as follows:

Let (X, g_0), $p \in X$ be given as in the theorem above. Within a g_0 normal coordinate chart at p, multiplying g_0 with a conformal factor e^f defined in (3.2.9), we obtain a conformal metric $\tilde{g} = e^f g^0$ such that $j_p^\infty(\det(\exp^*\tilde{g})) = 1$. We can therefore assume in the beginning that $j_p^\infty(\det(\exp^* g_0)) = 1$ and also up to a constant rescaling, we assume that the injective radius of g_0 at p is greater than 1.

Let $g = \Phi g_0$ be a metric conformal to g_0. The fact that g is conformal normal in a neighborhood of p is equivalent to the fact that $\det(g) = |g|$ is a solution of the initial value problem of the following ordinary differential equation:

$$\begin{cases} \partial_r \ln |g| = 0, \\ \ln |g(p)| = 0 \end{cases}$$

Namely $|g| = 1 \Leftrightarrow \partial_r \ln \sqrt{|g|} = 0$.

Under the g_0-normal coordinates $\{x_i\}$, we take $r(x) = \mathrm{dist}_g(p, x)$ the g-distance function from a point x to p, for which we have:

$$\begin{aligned} \Delta_g r &= \frac{1}{\sqrt{|g|}} \partial_j (\sqrt{|g|} g^{ij} \partial_i r) \\ &= \partial_j (g^{ij} \partial_i r) + \frac{1}{\sqrt{|g|}} \partial_j (\sqrt{|g|}) g^{ij} \partial_i r \\ &= \partial_j (dr) + \partial_r \ln(\sqrt{|g|}) = \frac{n-1}{r} + \partial_r \ln(\sqrt{|g|}) \end{aligned}$$

In summary, g is locally conformal normal in a neighborhood U of p if and only if g satisfies the following equation on U:

$$\Delta_g r = \frac{n-1}{r}. \tag{4.2.1}$$

Denote $r_0(x)$ the g_0 distance function from a point x to p, we have

$$1 = \|dr\|_g^2 = \|dr\|_{\Phi g_0}^2 = \frac{1}{\Phi} \|dr\|_{g_0}^2.$$

Hence

$$\Phi = \|dr\|_{g_0}^2. \tag{4.2.2}$$

Define the function $w(x)$ by $r(x) = r_0(x)e^{w(x)}$ and substitute w into (4.2.2), we have

$$\Phi = \|dr\|_g^2 = (1 + 2x^i w_i + r_0^2\|dw\|_{g_0}^2)e^{2w}. \tag{4.2.3}$$

It is clear by definition that $\Phi = 1 \Leftrightarrow r = r_0 \Leftrightarrow w = 0$. Furthermore, by comparing partial derivatives, we have $j_0^k(w) = 0 \Leftrightarrow j_0^k(\Phi) = 1$.

Thus g and g_0 have the same k-jet at p iff $j_0^k\Phi = 1$ iff $j_0^k w = 0$. By (4.2.3) we can rewrite (4.2.1) as an equation of $w(x)$:

$$V(x, \partial w, \partial^2 w) = f, \tag{4.2.4}$$

where

$$\begin{gathered} f = -\frac{\partial_{r_0} \ln \sqrt{\det(g_0(x))}}{r_0} \in Z_1 \\ V(x, \partial w, \partial^2 w) = \mathcal{L}_0(w) + G(x, \partial w) + Q(x, w, \partial w), \end{gathered} \tag{4.2.5}$$

where $\mathcal{L}_0$ is the operator (4.1.2) defined above, and G and Q are smooth functions satisfying:

$$G = \sum \tfrac{x^i x^j}{|x|^2} G_{ij}(x, \partial w) \text{ and } G_{ij}(x, 0) = 0; \tag{4.2.6}$$

$$Q = \textstyle\sum_{ij} Q_{ij}(x, \partial w) w_{ij} \text{ and } Q_{ij} = \sum x^k \partial_k Q_{ij} \tag{4.2.7}$$

Theorem 4.2.1 above is then a corollary of the following result:

Theorem 4.2.2 ([Cao91], Corollary B)**.** *Given $f \in Z_1$, then for a small enough constant $0 < \rho \leq 1$, there exists a function $w \in Z_\rho$ solving Equation* (4.2.4).

Proof: Fix $* = (2n, \frac{1}{2}; 4n, 1)$, we will prove the existence of a solution $w \in C_{2n,\frac{1}{2};4n,\rho}$ of (4.2.4). For the regularity of $w \in Z_\rho$, see Corollary B in [Cao91].

Take the complete metric space $D_{K_0} = \{v \in C_* \mid |||v|||_* \leq K_0\}$, where $K_0 = 8|||\mathscr{S}(f)|||_* + 1$.

For $v \in D_{K_0}$ and $\rho > 0$, define

$$F_\rho(v) = \mathscr{S}[\mathcal{L}_0(v) - V(\rho x, \rho\partial v, \partial^2 v) + f(\rho x)]. \tag{4.2.8}$$

By the estimate (4.1.8) in Theorem 4.1.4, we see that for two functions v_1 and v_2 in D_{K_0}, there exists a constant K_1 depends on n and K_0 such that:

$$|||F_\rho(v_1) - F_\rho(v_2)|||_* \leq K_1\rho|||v_1 - v_2|||_* \tag{4.2.9}$$

Take $\rho = \rho_f = \frac{1}{4(K_1+K_0)}$, then $K_1\rho < \frac{1}{4}$ and

$$|||F_\rho(v)|||_* \leq |||F_\rho(v) - F_\rho(0)|||_* + |||F_\rho(0)|||_* \leq \frac{K_0}{4} + |||\mathscr{S}(f(\rho x))|||_*$$
$$= \frac{K_0}{4} + \frac{|||(\mathscr{S}f)(\rho x)|||_*}{\rho^2} \leq \frac{K_0}{4} + |||(\mathscr{S}f)(x)|||_* \leq K_0,$$

where the equality follows from Theorem 4.1.4.(b) and the third inequality follows from Lemma 4.1.4. Thus, F_{ρ_f} is a contraction mapping on D_{K_0}. Hence by the Banach fixed point theorem, F_{ρ_f} has a fixed point in D_{K_0}, denoted as v.

Apply $\mathcal{L}_0$ on both sides of $F_{\rho_f}(v) = v$ and by Theorem 4.1.4.(a), we see that v is a solution of $V(\rho x, \rho\partial v, \partial^2 v) = f(\rho x)$ in C_*, and hence $w(x) = \rho^2 v(\frac{x}{\rho})$ is a solution of (4.2.4) in $C_{2n,\frac{1}{2};4n,\rho}$. $\square$

CHAPTER 5

UNIQUENESS OF CONFORMAL NORMAL METRICS AT THE GERM LEVEL

Let $(X, [g])$ be a conformal manifold. At a point $p \in X$, the germs at p of metrics g in the conformal class $[g]$ is a set

$$\mathscr{G}_p[g] \ = \{\mathrm{germ}_{\,p}(g) \mid g \in [g] \ \} \,.$$

By Theorem 4.2.1, this set contains at least one conformal normal metric. Let

$$\mathcal{CN}(p) \ = \ \left\{\mathrm{germ}_{\,p}(g) \in \mathscr{G}_p[g] \mid \ \det(\exp{}_p^* g) = 1\right\}$$

denote the subset of a (germs of) conformal normal metrics in the conformal class.

Let $J_p^1[g] = \left\{j_p^1(g) \mid g \in [g] \ \right\}$ and $\pi^1 : \mathscr{G}_p[g] \to J_p^1[g]$ with $\pi^1(\mathrm{germ}_{\,p}(g)) = j_p^1(g)$ be the projection map to 1-jets.

The main theorem is stated as follows: Fix $p \in X$. For each metric g, the conformal class $[g]$, the 1-jet class $j_p^1(g)$ contains a unique conformal normal metric.

Theorem 5.1 (main theorem)**.** *At $p \in X$, fix an arbitrary 1-jet class of the conformal metrics, there is a conformal normal metric g at p within the 1-jet class and the metric g is unique up to the germ level. Namely, the jet projection map π^1 restricted to $\mathcal{CN}(p)$:*

$$\pi^1|_{\mathcal{CN}(p)} : \mathcal{CN}(p) \to J_p^1[g]$$

is a bijection onto $J_p^1[g]$.

We first prove a uniqueness theorem for solutions of equations of type (4.2.4) which is a corollary of Lemma 4.1.5.

Theorem 5.2. *Given the inhomogeneous equation*

$$V(x, \partial w, \partial^2 w) = \mathcal{L}_0(w) + G(w) + Q(w) = f, \tag{5.1}$$

where $\mathcal{L}_0$ is defined as (4.1.2)*, G and Q satisfies* (4.2.6) *and* (4.2.7) *respectively, and $f \in Z_\lambda$ for some $\lambda > 0$. Then there exists a positive constant $0 < \tau \le \lambda$ small enough such that there exists a unique solution of* (5.1) *in Z_τ.*

Proof: For $\rho > 0$, define the operator F_ρ as in (4.2.8). Let $K_0 = 8|||\mathscr{S}(f)|||_{2n,\frac{1}{2};4n,\lambda} + 1$, then by the same argument as Theorem 4.2.2, there exists a constant $K > 0$ depends on K_0

such that for $\rho \leq \frac{1}{4(K+K_0)}$, F_ρ is a contraction mapping on

$$D_{K_0} = \{v \in C_{2n,\frac{1}{2};4n,\lambda} \mid |||v|||_{2n,\frac{1}{2};4n,\lambda} \leq K_0\}$$

For such a $\rho > 0$, on one hand, if $v \in Z_\lambda$ is a fixed point of F_ρ, then by applying $\mathcal{L}_0$ on both sides of $F_\rho(v) = v$, we see that v is a solution of the equation:

$$V(\rho y, \rho\partial v, \partial^2 v) = f(\rho y). \tag{5.2}$$

On the other hand, if $v \in Z_\lambda$ solves (5.2), then $F_\rho(v) = \mathscr{S}(\mathcal{L}_0(v)) = v$, where the last equality follows from Corollary 4.1.6.

Hence, by the uniqueness property of Banach's fixed point theorem, Equation (5.2) has a unique solution in Z_λ.

We clearly have a bijection between solutions of Equation (5.2) in Z_λ and solutions of Equation (5.1) in $Z_{\rho\lambda}$ by letting $w(x) = \rho^2 v(\frac{x}{\rho})$. Set $\tau = \rho\lambda$, and the conclusion follows. □

Proof of the main theorem:: We prove this by constructing a map

$$s_p : J^1_p[g] \to \mathcal{CN}_p,$$

and prove it is a well-defined inverse map of $\pi^1|_{\mathcal{CN}(p)}$ on both sides.

Fix a background metric $g^0 \in [g]$ together with an othonormal frame θ^0 of $(T_pX, g^0(p))$.

By Lemma 3.3, g^0 induces an isomorphism

$$\eta : \mathbb{R} \oplus T^*_pX \to J^1_p[g].$$

Let $(\varphi_0, x = (x_i))$ be the g^0 normal coordinate chart at p specified by θ^0. With respect to which, we can write η explicitly:

For each $\alpha = (\lambda, v) \in \mathbb{R} \oplus T^*_pX$, take $\alpha(x) = \lambda + \sum_{i=1}^n c_i x_i$, where $c_i = v(\theta^0_i)$.

Let $g^\alpha_{ij}(x) = e^{\alpha(x)} g^0_{ij}(x)$, then $\eta(\alpha) = j^1_p((\varphi^{-1})^*(g^\alpha_{ij} dx_i dx_j))$.

Let $\theta^\lambda = e^{-\frac{\lambda}{2}}\theta^0$, then θ^λ is an orthonormal frame of $(T_pX, g^\alpha(p))$. Let $(\varphi_\alpha, x^\alpha = (x^\alpha_i))$ be the g^α normal coordinate chart at p specified by θ^λ.

We now work with $\{x^\alpha\}$: By Lemma 3.2.2, we see that by choosing a smooth bump function ψ on $\mathbb{R}$, one can construct a smooth function $h = h(\alpha)$ in the x^α chart defined as (3.2.9) such that $h(x^\alpha) = O(r^2)$ and $j^\infty_0(\det(\exp^*(e^h g^\alpha))) = 1$. Denote $g^h = e^h g^\alpha$, it is clear that θ^λ is an orthonormal frame of $(T_pX, g^h(p))$. Again let $(\varphi_h, x^h = (x^h_i))$ be the g^h

normal coordinate chart at p specified by θ^{λ}.

We now work with $\{x^h\}$: Let $\rho = \rho(h)$ be the injective radius of g^h at p, then the function

$$f = -\frac{\partial_r \ln \sqrt{\det(g^h_{ij}(x^h))}}{r} \in Z_\rho.$$

Let $w \in Z_\tau$ be the unique solution of Equation (4.2.4) with f being the inhomogeneous term. The existence and uniqueness of w is ensured by Theorem 4.2.2 and Theorem 5.2.

Let $g^w = \Phi(w)g^h$, with $\Phi(w) = (1 + 2\sum x^h_i \partial_i w + |x^h|^2 \|dw\|^2_{g^h})e^{2w}$, then g^w is a conformal normal metric on B_τ.

Define the map s_p as follows:

$$s_p(j^1_p(g)) = \text{germ}_p(g^w) = \text{germ}_p((\Phi(w) \circ \varphi_h^{-1}) \cdot (e^{h\circ\varphi_\alpha^{-1} + \alpha\circ\varphi_0^{-1}}) \cdot g^0) \tag{5.3}$$

The map s_p is well-defined independent of the choice of the frame θ_p and the smooth bump function ψ.

For θ_p independency, we check the following:

1. For $\alpha = (\lambda, \nu) \in \mathbb{R} \oplus T^*_p X$, $\alpha \circ \varphi_0^{-1}(v) = \lambda + \nu(v)$, hence α is well-defined independent of θ.

2. To show the function h is independent of the choice of θ_p, it is sufficient to show its Taylor series at the origin is independent of the choice of θ. By choosing a different orthonormal frame, we have a linear coordinate transformation $y = Ax$, with $A \in O(n)$. Therefore at the origin, we have
$$\frac{\partial^n h}{\partial x_{i_1} \cdots \partial x_{i_n}}(0) = \frac{\partial^n h}{\partial y_{j_1} \cdots \partial y_{j_n}} \frac{\partial y_{j_1}}{\partial x_{i_1}} \cdots \frac{\partial y_{j_n}}{\partial x_{i_n}}(0).$$
Hence the Taylor series of h at the origin is independent of θ.

3. The function $\Phi(w)$ is a solution of the equation:
$$\Delta_{g^h} r + \frac{n-2}{2}\langle d(\ln \Phi), dr\rangle_{g^h} = -\frac{\partial_r \ln \sqrt{\det(\exp^* g^h)}}{r}.$$
By 1 and 2 above, the metric g^h is independent of the choice of θ, and hence the equation of Φ is independent of θ and so is the solution Φ.

For ψ independency: Let $\tilde{h}$ be a different Borel extension of the formal power series in Lemma 3.2.2 and $g^{\tilde{h}}$ the corresponding conformal metric. By solving Cao's equation (4.2.4), we obtain a conformal normal metric $\tilde{g}$. Let $\tilde{g} = \Phi(\tilde{w})g^w$, with Φ the conformal factor and $\tilde{w}$ defined in (4.2.3). Then under the g^w normal coordinates, $\tilde{w} \in Z_\rho$ for some $\rho > 0$ small enough and satisfies the equation

$$V(x, \partial\tilde{w}, \partial^2\tilde{w}) = 0. \tag{5.4}$$

By Theorem 5.2, the solution of (5.4) is unique in a small neighborhood of p, hence $\tilde{w} = 0$ and $\mathrm{germ}\,(g^w) = \mathrm{germ}\,(\tilde{g})$.

To see s_p is a right inverse of $\pi^1|_{\mathcal{CN}_p}$, observe that the conformal factors $\Phi(w)$ and h both vanish to the second order and thus do not affect the 1-jet of g^w. Hence we have

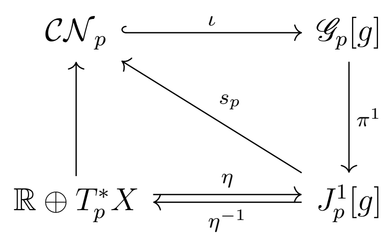
$$\pi^1 \circ s_p(j^1_p(g)) = j^1_p(g^w) = j^1_p(e^{\alpha\circ\varphi_0^{-1}}g^0) = \eta^{-1}(\eta(j^1_p(g))) = j^1_p(g).$$

To see s_p is a left inverse of $\pi^1|_{\mathcal{CN}_p}$, take $g \in [g]$ such that $\mathrm{germ}_p(g) \in \mathcal{CN}_p$, and let g^w be the corresponding conformal metric obtained as above that is also conformal normal at p. Hence we can write $g = \Phi(x)g^w$ with respect to g^w-normal coordinates x at p, and apply the same argument as ψ independency above, we see that $\Phi(x) = 1$, hence $\mathrm{germ}_p(g) = \mathrm{germ}_p(g^w)$. Thus, the left inverse follows:

$$s_p \circ \pi^1(\mathrm{germ}_p(g)) = s_p(j^1_p(g)) = \mathrm{germ}_p(g^w) = \mathrm{germ}_p(g).$$

□

In summary, we have the following diagram of maps:



And by the section map s, we see that $\mathbb{R} \oplus T^*X$ serves as a rough moduli of the set of germs of conformal normal metric in class $[g]$.

5.1 Conformal Normal Metrics on Locally Conformally Flat Manifolds

In section 3.1, we proved Theorem 3.1.3, a jet level relation among conformal flatness, flatness, and conformal normal. By applying Theorem 5.1, we can obtain a similar result in the germ level as follows:

Corollary 5.1.1. *Let $(X, [g])$ be a smooth manifold with a conformal structure $[g]$. If $[g]$ is locally conformally flat on an open neighborhood U of $p \in X$, then for a conformal metric $g \in [g]$, $germ_p(g)$ is flat if and only if $germ_p(g)$ is conformal normal.*

Proof. It is clear by definition that a flat metric is conformal normal.

Conversely, let $g \in [g]$ be a conformal normal metric on U. Since $[g]$ is conformally flat on U, there exists a conformal factor e^{2f} such that $g^f = e^{2f} \cdot g$ is a flat metric on U. With respect to the g^f-normal coordinates $\{x\}$, $g^f(x) = \delta_{ij}$ is the standard Euclidean metric. Pullback δ_{ij} by the conformal mapping $\varphi = \varphi_{\lambda,A,\eta}$ defined in (2.3.1), we have

$$\varphi^*\delta_{ij} = \frac{\lambda^2}{(1 - 2Ax \cdot \eta + x^2\eta^2)^2}\delta_{ij}.$$

$\varphi^*\delta_{ij}$ has a pole at $x = \frac{A^t\eta}{\eta^2}$. Since $\varphi^* Riem = 0$, $\varphi^*\delta_{ij}$ is a flat metric on $U \cap B(0, |\eta|^{-1})$. Up to pulling back by a conformal mapping φ, we can assume $j^1_p(\varphi^* g^f) = j^1_p(g)$, with both metrics being conformal normal at p. Hence by Theorem 5.1, we have $\text{germ}_p(g^f) = \text{germ}_p(g)$. □

Remark 5.1.2. By Corollary 5.1.1, we see that the germ of a flat metric is uniquely determined by its 1-jet. This fact can be proved directly without referring to Theorem 5.1:

Proof: Without loss of generality, let $h \in [g]$ be a flat metric on U. Let φ be a smooth function such that $\tilde{h} = e^{2\varphi}h$ remains flat on some open neighborhood $V \subseteq U$.

By the conformal transformation formula for the Riemann curvature tensor, we have

$$\begin{aligned} 0 = \widetilde{\text{Riem}} &= e^{2\varphi}\text{Riem} - e^{2\varphi}h \owedge (\text{Hess}\,\varphi - d\varphi \otimes d\varphi + \frac{1}{2}|d\varphi|^2 h) \\ &= -e^{2\varphi}h \owedge (\text{Hess}\,\varphi - d\varphi \otimes d\varphi + \frac{1}{2}|d\varphi|^2 h). \end{aligned}$$

Hence we have

$$h \owedge (\text{Hess}\,\varphi - d\varphi \otimes d\varphi + \frac{1}{2}|d\varphi|^2 h) = 0, \tag{5.1.1}$$

where $\owedge$ is the Kulkarni-Nozumi product.

Write (5.1.1) in the h normal coordinates, we have

$$\begin{aligned} &\delta_{ik}(\varphi_{jl} - \varphi_j\varphi_l + \frac{1}{2}|d\varphi|^2\delta_{jl}) + \delta_{jl}(\varphi_{ik} - \varphi_1\varphi_k + \frac{1}{2}|d\varphi|^2\delta_{ik}) \\ &\qquad - \delta_{jk}(\varphi_{il} - \varphi_i\varphi_l + \frac{1}{2}|d\varphi|^2\delta il) - \delta_{il}(\varphi_{jk} - \varphi_j\varphi_k + \frac{1}{2}|d\varphi|^2\delta jk) = 0. \end{aligned}$$

This is an overdetermined system of equations of φ, among which we have the following non-trivial relations for $i \neq j$:

$$\begin{cases} \varphi_{ii} + \varphi_{jj} = \varphi_i^2 + \varphi_j^2 - |d\varphi|^2, \\ \varphi_{ij} = \varphi_i \varphi_j, \end{cases} \tag{5.1.2}$$

where the first equation is for the cyclic pairs $(i, j) = (1, 2), (2, 3), \ldots, (n, 1)$. Denote A the coefficient matrix of $\varphi_i i$. It is clear by elementary row operation that A is nonsingular.

View the first order terms φ_i as constants and solve for φ_{ii}, we get

$$\varphi_{ii} = \varphi_i^2 - \frac{1}{2}|d\varphi|^2.$$

Since A is nonsingular, the above solution is unique.

Define $u : \mathbb{R}^n \to \mathbb{R}^n$ as $u = (\varphi_i)_{i=1}^n$, and

$$\begin{aligned} \alpha : \mathbb{R}^n &\to M_{n\times n}(\mathbb{R}) \\ z \mapsto (\alpha_i^j) &= \begin{cases} (z^i)^2 - \frac{1}{2}z^2, & \text{for } i = j \\ z^i z^j, & \text{for } i \neq j \end{cases} \end{aligned}$$

As one can check, we have $\sum_{l=1}^n \alpha_k^l \frac{\partial \alpha_j^i}{\partial z^l} = \sum_{l=1}^n \alpha_j^l \frac{\partial \alpha_k^i}{\partial z^l}$.

Hence the functions u and α satisfy the compatible conditions in the following lemma for an over-determined system which is the proposition 19.29 in [Lee12].

Lemma 5.1.3. *Suppose W is an open subset of $\mathbb{R}^n \times \mathbb{R}^m$, and $\alpha = (\alpha_j^l) : W \to \mathrm{M}(m \times n, \mathbb{R})$ is a smooth matrix-valued function satisfying*

$$\frac{\partial \alpha_j^i}{\partial x^k} + \alpha_k^l \frac{\partial \alpha_j^i}{\partial z^l} = \frac{\partial \alpha_k^i}{\partial x^j} + \alpha_j^l \frac{\partial \alpha_k^i}{\partial z^l} \quad \textit{for all } i, j, k,$$

where we denote a point in $\mathbb{R}^n \times \mathbb{R}^m$ by $(x, z) = (x^1, \ldots, x^n, z^1, \ldots, z^m)$. For any $(x_0, z_0) \in W$, there is a neighborhood U of x_0 in $\mathbb{R}^n$ and a unique smooth function $u : U \to \mathbb{R}^m$ such that $u(x_0) = z_0$ and the Jacobian of u satisfies

$$\frac{\partial u^i}{\partial x^j}\left(x^1, \ldots, x^n\right) = \alpha_j^i\left(x^1, \ldots, x^n, u^1(x), \ldots, u^m(x)\right)$$

Hence for any fixed initial condition $\varphi(0) = c$ and $d\varphi(0) = v$, we have a unqiue conformal factor φ such that $e^{2\varphi}g$ is flat in some small open neighborhood V of p. ■

CHAPTER 6

SMOOTH DEPENDENCE OF CONFORMAL NORMAL METRICS ON PARAMETERS

In Chapter 5, we see that for a smooth background metric g, there exists a unique conformal normal metric $\tilde{g}$ at any point $p \in X$. In this chapter, we prove the smooth dependence of conformal normal metrics on a family of background metrics and give an application to the regularity of the canonical metric g_C in a Yamabe-positive conformal class C introduced by Habermann and Jost in [HJ99].

6.1 Smooth Dependence of Conformal Normal Metrics

We first work over the background Euclidean space $\mathbb{R}^n$ and give a local smooth dependence result(Lemma 6.1.7).

Suppose $B_1 \subseteq \mathbb{R}^n$ is the unit ball centered at the origin and denote the space of smooth metrics on B_1 as $\mathrm{Met}^{\infty}(B_1)$ which is an open cone in the Fréchet space $C^{\infty}(B_1, Sym^2\mathbb{R}^n)$.

Consider a l-parameter family of smooth metrics on $\mathrm{Met}^{\infty}(B_1)$ as follows:

$$\begin{aligned} \gamma : \mathbb{R}^l &\to \mathrm{Met}^{\infty}(B_1) \\ t &\mapsto \gamma(t) = g_t(x), \end{aligned} \tag{6.1.1}$$

with $g_t(x) = (g_{ij}(t,x))$ satisfies $g_{ij}(t,x) \in C^{\infty}(\mathbb{R}^l \times B_1)$.

Recall the following well-known result on the smooth dependence of solutions of a system of ordinary differential equations, see Sec1.6 in [Tay96] for details:

Theorem 6.1.1. *Suppose $D = \{(s,x,t)\} \subseteq \mathbb{R} \oplus \mathbb{R}^n \oplus \mathbb{R}^m$ is open and $f : D \to \mathbb{R}^n$ is a smooth vector valued function. Consider the initial value problem:*

$$\begin{cases} \frac{dx}{ds} = f(s, x(s), t) \\ x(s_0) = v, \end{cases} \tag{6.1.2}$$

with $(s_0, v, t) \in D$.

Then there exists a constant $\delta > 0$ such that on $[-\delta, \delta] \times D$ the initial value problem (6.1.2) *has a unique solution $x = x(s; s_0, v, t) \in C^{\infty}([-\delta, \delta] \times D)$.*

Lemma 6.1.2. *Suppose γ is an arbitrary l-parameter family of metrics on $B_1 \subseteq \mathbb{R}^n$ given as* (6.1.1)*. For any $K \subseteq \mathbb{R}^l$ compact, there exists $\delta > 0$ small enough such that $\tilde{g}_t = exp^*_{g_t}(g_t)$*

satisfies $\tilde{g}_{ij}(t,x) \in C^\infty(K \times B_\delta)$*. In particular, the coordinates* $\{x^i\}$ *on* B_δ *are normal coordinates for* $\tilde{g}_{ij}(t,x)$ *for each* t*.*

Proof: Apply Theorem 6.1.1 to the geodesic equations with metrics in $\gamma(K) \subseteq \mathrm{Met}^\infty(B_1)$. Namely, consider the following system of equations:

$$\begin{cases} \frac{du^a}{ds} = -\Gamma^a_{bc}(x(s),t)y^b y^c \\ u(0) = (p,v), \end{cases}$$

where $u(s) = (x(s), y(s))$, $y(s) = \frac{dx(s)}{ds}$ and $\Gamma^a_{bc}(x(s),t)$ is the Christoffel symbol of $g_t(x)$. Suppose $(p,v,t) \in B_1 \times B_1 \times K$. We see that, up to restricting to a smaller ball $B_\delta \subseteq B_1$, the Riemannian exponential map is uniformly defined on B_δ and is smooth with respect to (p,v,t). $\square$

We give a parameter-dependent version of Borel's Lemma of asymptotic expansions below:

Lemma 6.1.3. *Suppose* $\{a_k(t) \in C^\infty(K) \mid K \subseteq \mathbb{R} \text{ compact}, k \in \mathbb{N}\}$ *is an arbitrary collection of smooth functions. There exists a smooth function* $f(t,x) \in C^\infty(K \times \mathbb{R})$ *such that* $\partial_x^k f(t,0) = a_k(t)$ *for any* $k \in \mathbb{N}$*.*

Proof: Let $\rho(x)$ be a smooth bump function such that $\rho = 1$ on $|x| \leqslant \frac{3}{4}$ and $\mathrm{supp}(\rho) = B_1$. For each $k \in \mathbb{N}$, take

$$A_k = \|a_k(t)\|_{C^k(K)}, \qquad B_k = \|\rho(x)\|_{C^k(\mathbb{R})}, \qquad M_k = \sup_{i \le k}\left\{\sum_{j=0}^{i}\binom{i}{j}\frac{1}{(k-j)!}\right\}$$

Define

$$f(t,x) = \sum_{k=0}^{\infty} a_k(t)\frac{x^k}{k!}\rho\left(h_k x\right),$$

with $h_k = (2^k A_k B_k M_k)^{\frac{2}{k}}$.

We claim that the function $f(t,x)$ defined above satisfies the requirements. Formally we have $\partial_x^k f(t,0) = a_k(t)$. Hence it is sufficient to prove that $f(t,x) \in C^\infty(K \times \mathbb{R})$. We prove by check the following fact: For each $n, m \in \mathbb{N}$, the series

$$\sum_{k \ge 2(n+m)}^{\infty}\left|\partial_x^n \partial_t^m\left(a_k(t)\frac{x^k}{k!}\rho\left(h_k x\right)\right)\right|$$

converges on $K \times \mathbb{R}$. Indeed, we have:

$$\sum_{k\geq 2(n+m)}^{\infty}\left|\partial_x^n\partial_t^m\left(a_k(t)\frac{x^k}{k!}\rho\left(h_kx\right)\right)\right| = \sum_{k\geq 2(n+m)}^{\infty}\left|a_k^{(m)}(t)\sum_{j=0}^{n}\binom{n}{j}\frac{x^{k-j}}{(k-j)!}\cdot h_k^{n-j}\cdot\rho^{(n-j)}\left(h_kx\right)\right|$$
$$\leq \sum_{k\geq 2(n+m)}^{\infty} A_kB_kM_kh_k^{j-k}\cdot h_k^{n-j} \leq \sum_{k\geq 2(n+m)}^{\infty}\frac{A_kM_kB_k}{h_k^{2/k}} = \sum_{k\geq 2(n+m)}\frac{1}{2^k}\leq 1,$$

where the first inequality follows from the definition of A_k, B_k, M_k, and the fact that $\rho(h_kx) = 0$ for $|x| > \frac{1}{h_k}$, the second inequality is by $-\frac{k}{2} \geqslant n-k$. □

Remark 6.1.4. By a completely similar argument, one can show that Lemma 6.1.3 holds for the multivariable cases.

Theorem 6.1.5. *For $\{\tilde{g}_t \in Met^\infty(B_\delta) \mid t \in K\}$, the smooth l-parameter family of metrics obtained in Lemma 6.1.2, there exists a function $\Psi(t,x)$ over $K\times B_\delta$ such that:*

a) $\Psi(t,x) \in C^\infty(K\times B_\delta)$,

b) $\Psi_t(x) = 1 + O(|x|^2)$,

c) For each $t\in K$, the metric $g_t^\infty(x) := \Psi(t,x)\tilde{g}_t(x)$ is ∞-order conformal normal at the origin in $\{t\}\times B_\delta$.

Proof: Apply Lemma 3.2.1(cf. [LP87, Theorem 5.]) to each $\tilde{g}_t(x)$, we obtain a unique formal power series:

$$h_t^\infty(x) = \sum_{|\alpha|=2}^{\infty} c_\alpha(t)x^\alpha,$$

with $c_\alpha(t)$ a polynomial of $\partial_\alpha Ricg_t(0)$ and $\partial_\beta Riem_{g_t}(0)$ with $|\beta| < |\alpha|-2$, such that for $h_t(x)$, an arbitrary Borel's extension of $h_t^\infty(x)$, the function $\Psi_t(x) = e^{h_t(x)}$ satisfies requirements *b)* and *c)*.

By definition $c_\alpha(t)$ is a smooth function of t over $\mathbb{R}^l$. On $\forall K \subseteq \mathbb{R}^l$ compact, by Lemma 6.1.3, we can choose $h(t,x)\in C^\infty(K\times B_\delta)$, hence $\Psi(t,x) = e^{h(t,x)} \in C^\infty(K\times B_\delta)$. □

In summary, we obtain a smooth l-parameter family of metrics:

$$\{g_t^\infty \in \text{Met}^{\infty}(B_\delta) \mid t\in K\} \tag{6.1.3}$$

such that for each $t\in K$, g_t^∞ is ∞-order conformal normal at the origin and $j_0^1(g_t^\infty) = j_0^1(g_t)$.

By applying Lemma 6.1.2 to the family $\{g_t^\infty\}$, we can assume the coordinates $\{x^i\}$ on B_δ are normal coordinates for g_t^∞ for each $t \in K$.

For each $t \in K$, we then correct the metric g_t^∞ to a conformal normal metric by Cao's PDE approach. On B_δ, write down Cao's equation (4.2.4) with respect to g_t^∞:

$$\mathcal{L}_0(w) + \sum_{i,j} \frac{x^i x^j}{x^2} \cdot G_{ij}(x, \partial w, t) + \sum_{ijk} x^k \cdot w_{ij} \cdot Q_{ijk}(x, \partial w, t) = f(x,t), \tag{6.1.4$_t$}$$

where the explicit formula for $f(x,t), G_{ij}(x,\zeta,t)$, and $Q_{ijk}(x,\zeta,t)$ are given as follows:

$$f(x,t) = \frac{-\partial_r \det g_t^\infty(x)}{2r \det g_t^\infty(x)} \tag{6.1.5}$$

Denote $\Phi_1(x,\partial w) = 1 + 2\sum x_i w_i + x^2 \sum (g_t^\infty)^{ij} w_i w_j$, we have (using Einstein's convention below):

$$\begin{aligned} G_{ij}(x,\zeta,t) &= (g_t^\infty)^{ab} \cdot \Gamma^c_{ab}(x,t)\zeta_c \cdot \delta_{ij} + \frac{n-2}{\Phi_1}(2\zeta_i\zeta_j + (g_t^\infty)^{ab} \cdot \zeta_a\zeta_b\zeta_i x_j) + \\ &\quad \frac{n-2}{\Phi_1}(2(g_t^\infty)^{ab}\zeta_a\zeta_b(1 + x^c\zeta_c) + \frac{1}{2}\partial_c(g_t^\infty)^{ab}\zeta_a\zeta_b(x^c + x^2(g_t^\infty)^{cd}\zeta_d))\delta_{ij} \end{aligned} \tag{6.1.6}$$

$$\begin{aligned} Q^{ijk}(x,\zeta,t) &= \int_0^1 \partial_k((g_t^\infty)^{ij} - \delta^{ij})(xs)ds + \\ &\quad \frac{n-2}{\Phi_1}(2x^k(g_t^\infty)^{ia}(g_t^\infty)^{jb}\zeta_a\zeta_b + (g_t^\infty)^{ia}\zeta_a\delta^{jk}) \end{aligned} \tag{6.1.7}$$

Upto choosing a smaller compact set K and $\delta > 0$ if necessary, we can assume that $\frac{1}{2} < \Phi_1(x,\zeta,t) < 2$ on $B_\delta \times B_\delta \times K$ and $\frac{1}{2} < \det(g^\infty(x,t)) < 2$ on $B_\delta \times K$. Hence by the construction, $f(x,t)$ is a smooth function in x, t and $f_t(x) \in Z_\delta$ with respect to x, $G_{ij}(x,\zeta,t)$ and $Q_{ijk}(x,\zeta,t)$ are smooth functions with respect to x, ζ, t.

Let $K_0 = 8|||\mathscr{S}(f)|||_{2n,\frac{1}{2};4n,\lambda} + 1$, then by the same argument as Theorem 4.2.2, there exists a constant $K_3 > 0$ depends on K_0 such that for $\rho \le \frac{1}{4(K_3+K_0)}$, F_ρ is a contraction mapping on $D_{K_0} = \{v \in C_{2n,\frac{1}{2};4n,\lambda} \mid |||v|||_{2n,\frac{1}{2};4n,\lambda} \le K_0\}$.

As in the proof of Theorem 5.2, fix $* = (2n, \frac{1}{2}; 4n, \delta)$. With respect to each g_t^∞, take $K_0(t) = 8|||\mathscr{S}(f_t)|||_* + 1$, by construction $K_0(t)$ is continuous in t, hence for $t \in K$ compact, we have a maximum $\bar{K}_0$. Denote the convex set $D_{\bar{K}_0} = \{v \in C_* \mid |||v|||_* \le \bar{K}_0\}$.

Denote $K_3(t)$ the contraction constant in (4.2.9) for the map $F_{\rho,t}(v)$. Namely, we have:

$$|||F_{\rho,t}(v) - F_{\rho,t}(\tilde{v})|||_* \le K_3(t)\rho|||v - \tilde{v}|||_*.$$

The following lemma is proved in Appendix B:

Lemma B.1. *For $K_0 > 0$, and functions in $D_{K_0} = \{v \mid |||v|||_{k,\alpha;N,\delta} \le K_0\}$, define*

$$T_\rho(v) = G_\rho(v) + Q_\rho(v) = \sum_{i,j} \frac{x^i x^j}{x^2} \cdot G_{ij}(\rho x, \rho\partial v) + \sum_{ijk} \rho x^k \cdot v_{ij} \cdot Q_{ijk}(\rho x, \rho\partial v),$$

where $Q_{ijk}(x,\zeta)$ and $G_{ij}(x,\zeta)$ are smooth functions with respect to x, ζ.

There exists a constant K_2 such that for $0 < \rho < 1$ and any pair of functions $v_1, v_2 \in D_{K_0}$, we have

$$\|T_\rho(v_2) - T_\rho(v_1)\|_{k-2,\alpha;N-2,\delta} + \|T_\rho(v_2) - T_\rho(v_1)\|_{0,\alpha;N-1,\delta} \le K_2\rho\|\tilde{v} - v\|_{k,\alpha;N,\delta}.$$

In fact, $K_2 = C(\alpha)P(K_0)M$, where $C(\alpha)$ is a constant depends on α, $P(K_0)$ is a polynomial of K_0 and

$$M = \max_{ijk}\left\{\|G_{ij}(x,\zeta)\|_{C^k(D(\delta,K_0))}, \|x^k Q_{ijk}(x,\zeta)\|_{C^k(D(\delta,K_0))}\right\},$$

with $D = D(\delta, K_0) = B_\delta \times B_{K_0} \subseteq \mathbb{R}^n \oplus \mathbb{R}^n$.

Let K_1 be the constant in (4.1.8) for $* = (2n, \frac{1}{2}; 4n, \delta)$:

$$|||\mathscr{S} \circ f|||_* \le K_1\{|||f|||_{0,1/2;4n,\delta} + |||f|||_{2n-2,1/2;4n-2,\delta}\}.$$

Apply Lemma B.1 to the functional $T_{\rho,t}$ defined by the metric g_t^∞, for a pair of functions $v_1, v_2 \in D_{K_0}$, we have:

$$|||F_{\rho,t}(v_1) - F_{\rho,t}(v_2)|||_* = |||\mathscr{S} \circ (T_{\rho,t}(v_1) - T_{\rho,t}(v_2))\,|||_* \le K_1 \cdot K_2(t)\rho|||v - \tilde{v}|||_*.$$

We see that $K_3(t) = K_1 \cdot K_2(t)$. Since G_{ij} and Q_{ijk} depend smoothly on the metrics g_t^∞, hence so do their C^k norms. We obtain an upper bound of $K_3(t)$ by take the maximum values for $C^{2n}(D)$ norms of G_{ij} and Q_{ijk}.

On one hand, let $\tau(t) = \frac{1}{8(K_3(t)+K_0)}$, then by the argument above $\tau(t)$ is uniformly bounded below by $\tau = \tau(K)$ for all $t \in K$, the solution $w(x,t)$ of Cao's equation $(6.1.4)_t$ exists on B_τ; on the other hand, the map $F_{\rho,t}$ is a smooth family of contraction maps over $D_{\bar{K}_0} \times K$ with a uniform contraction constant. Recall the following classical result of uniform contractions(cf. [CH12, Theorem 2.2]):

Lemma 6.1.6 (Uniform Contraction Principle)**.** *Let U, V be open sets in Banach spaces X, Y respectively. Let $\bar{U}$ be the closure of U, $T : \bar{U} \times V \to \bar{U}$ a uniform contraction on $\bar{U}$ and let $g(y)$ be the unique fixed point of $T(\cdot, y)$ in $\bar{U}$. If T is smooth over $\bar{U} \times V$, then $g(\cdot) \in C^\infty(V, X)$.*

Apply Lemma 6.1.6 to $F_{\rho,t}$, we see that the solution $w(x,t)$ of Cao's equation $(6.1.4)_t$ with respect to the metric $g_t^\infty(x)$ depends smoothly on t. In summary, we have the following result:

Lemma 6.1.7. *Give a smooth family of metrics $\gamma : \mathbb{R}^l \to Met^\infty(B_\epsilon)$, for any $K \subset \mathbb{R}^l$ compact, there exist $0 < \delta < \varepsilon$ small enough such that there is an unique function $\Phi \in C^\infty(K \times B_\delta, \mathbb{R}^+)$ such that $\Phi_t(x) = 1 + O(r^2)$ and $\Phi_t^2 g$ is conformal normal on B_δ for any $t \in K$.*

Apply Lemma 6.1.7, we construct a smooth Riemannian metric on $X \times X$ which is conformal normal near the diagonal in the following sense:

Corollary 6.1.8. *Given (X, g), there exists a smooth Riemannian metric $g \oplus \Phi^2 g$ on $X \times X$, where Φ is a function satisfies the following conditions:*

(a) $\Phi = \Phi(p,q) \in C^\infty(X \times X, \mathbb{R}^+)$.

(b) Near the diagonal $\Delta \subset X \times X$, $\Phi = 1 + O(r^2)$, where r is the distance from the diagonal.

(c) There exists an open neighborhood U of the diagonal Δ such that for $\forall p \in X$, $\Phi_p^2 g$ is conformal normal on $U_p = (\{p\} \times X) \cap U$.

Proof: Let $\varphi_0 : U_0 \to X$ be a chart centered at a point $p_0 \in X$. Fix $\theta_0 \in O_{p_0}(X)$, where $O(X)$ is the g-orthonormal frame bundle. For $\delta > 0$ small enough, and the δ ball $B_\delta \subset \mathbb{R}^n$ at the origin, we define a smooth family of metrics

$$\gamma : U_0 \to \text{Met}^\infty(B_\delta)$$

as follows: For any $x \in U_0$, let $\theta_x \in O_{\varphi(x)}(X)$ be the orthonormal frame obtained by the parallel transportation from θ_0 with respect to the metric g. We identify $T_{\varphi(x)}X$ with $\mathbb{R}^n$ by $\theta_x : \mathbb{R}^n \to T_{\varphi(x)}X$ as $\theta_x(v) = \sum v^i (\theta_x)_i$.

Let $\delta > 0$ small enough such that $\theta_x(B_\delta)$ is in the injectivity domain of the exponential map on $T_{\varphi(x)}X$ for any $x \in U_0$.

Define $\gamma(x) = (exp_{\varphi(x)} \circ \theta_x)^* g \in \text{Met}^\infty(B_\delta)$. It is clear by definition that γ is a smooth family of metrics.

Apply Lemma 6.1.7 to γ, we obtain a smooth family of conformal normal metric $\tilde{g} = \Phi^2 g$, with $\Phi \in C^\infty(U_0 \times B^\delta)$. Let $\tilde{\varphi} : U_0 \times B_\delta \to \exp(TX|_{\varphi(U_0)}) \subset X \times X$ be the map to an open neighborhood of the diagonal Δ obtained by θ_x and the exponential map. Then by construction, the metric $g \oplus \tilde{\varphi}^* \tilde{g}$ is a smooth metric on $\tilde{\varphi}(U_0 \times B_\delta)$.

For another point, $p_1 \in X$ together with an open neighborhood U_1, by the same argument, we obtain a smooth family of conformal normal metrics parametrized over U_1 and by the uniqueness of conformal normal metrics, the two families of metrics coincide on the overlap $U_0 \cap U_1$ and hence we obtain a smooth family of metrics $g \oplus \Phi^2 g$ over X in an open neighborhood U_Δ of the diagonal.

Take $K \subset U_\Delta$ compact and $V \supset K$ open. Respectively, let μ be a bump function on $X \times X$ such that $\mu = 1$ on K and $supp(\mu) \subseteq V$. Then the metric $g \oplus (\Phi^2\mu + 1 - \mu)g$ satisfies the requirements. $\square$

6.2 An Application of Smooth Family of Conformal Normal Metrics

In this section, we apply the smooth family of conformal normal metrics in Corollary 6.1.8 to give a shorter proof of the regularity of the canonical metric g_C in a Yamabe-positive conformal class C introduced by Habermann and Jost in [HJ99]. We begin by reviewing some background knowledge on Green's functions for Riemannian manifolds and the mass of asymptotically flat manifolds.

Conformal Laplacian and Green's function.

On a Riemannian manifold (X, g) of dimension $n \geq 3$, by adding a multiple of the scalar curvature S_g to the Laplace-Beltrami operator Δ_g, we obtain the so-called conformal Laplacian operator:

$$L_g = \frac{4(n-1)}{(n-2)}\Delta_g + S_g.$$

L_g is conformally covariant in the following sense. Suppose $\tilde{g} = u^{\frac{4}{n-2}} g$, then the conformal Laplacian changes correspondingly as:

$$u^{\frac{n+2}{n-2}} \circ L_{\tilde{g}} = L_g \circ u. \tag{6.2.1}$$

Remark 6.2.1. L_g is the most famous example of a general hierarchy of conformally covariant operators. Let $a, b \in \mathbb{R}$, a linear differential operator P_g of order m on (X, g) is called *conformally covariant of bi-degree* (a, b) if with respect to a conformal change $\tilde{g} = \varphi^2 g$, we have:

$$\varphi^b \circ P_{\tilde{g}} = P_g \circ \varphi^a.$$

L_g can be generalized to the conformal powers $P_{2N,g}$ of Laplacian (also known as GJMS operators):

$$P_{2N,g} = \Delta_g^N + LOT,$$

where "LOT" indicates terms of order lower than $2N$. $P_{2N,g}$ is of bi-degree $(\frac{n-2N}{2}, \frac{n+2N}{2})$. In particular $P_{2,g} = L_g$, $P_{4,g}$ is the so-called Paneitz operator which was discovered independently by Paneitz [Pan08], Eastwood-Singer [ES85] and Riegert [Rie84]. Explicitly:

$$P_{4,g} = \Delta_g^2 + \delta\left(\frac{n-2}{2(n-1)} S_g \cdot g - 4P\right) d + \frac{n-4}{2} Q,$$

where δ is the formal adjoint of d, P is the Schouten tensor, which is defined by $(n-2)P = Ric_g - \frac{1}{2(n-1)} S_g \cdot g$, and $Q = \frac{n}{4(n-1)} S_g^2 - 2|P|^2 - \frac{1}{2(n-1)} \Delta_g S_g$ is the so-called Q-curvature tensor. See [Juh09] for more details.

The Green's function G of the operator L_g is a smooth function on $X \times X - \Delta$. Following [LP87], we will normalize it by requiring that

$$\int_X G(p,q) L_g(\varphi(q))\, \mathrm{dvol}_g(q) \;=\; (n-2)\, \omega_{n-1}\, \varphi(p) \tag{6.2.2}$$

for all $\varphi \in C_0^\infty(X)$. Applying Equation (6.2.1) to (6.2.2), we see that under the conformal change $\tilde{g} = u^{\frac{4}{n-2}} g$, the Green's function transforms as

$$\widetilde{G}(p,q) = \frac{1}{u(p)u(q)}\; G(p,q). \tag{6.2.3}$$

Next, recall that the Yamabe constant $\mathcal{Y}(C)$ of a conformal class C is defined by

$$\mathcal{Y}(C) = \inf_{g \in C} \frac{\int_X S_g\; dvol_g}{\left(\int_X dvol_g\right)^{\frac{n-2}{n}}}.$$

The proof of Lemma 6.1 in [LP87] (see also Prop 2.2.9 in [Hab00]) shows that $\mathcal{Y}(C) > 0$ if and only if there exists a metric $g \in C$ with positive scalar curvature. Furthermore, for each $g \in C$, the smallest eigenvalue λ_1 of L_g has the same sign as $\mathcal{Y}(C)$, so in the case that $\mathcal{Y}(C) > 0$, L_g is invertible and we have the following result.

Theorem 6.2.2. *If $\mathcal{Y}(C) > 0$, then for each $g \in C$, there is a unique Green's function G*

for L_g. Moreover, G is symmetric and positive.

We work with $\mathcal{Y}(C) > 0$ throughout this section. For convenience, we adopt the following notations as in [LP87].

Notation. *We write $f = O'\left(r^k\right)$ to mean $f = O\left(r^k\right)$ and $\nabla f = O\left(r^{k-1}\right)$. O'' is defined similarly.*

Near the diagonal of $X \times X$, G has the following asymptotic expansion under the conformal normal coordinates.

Lemma 6.2.3 ([LP87], Lemma 6.4)**.** *Let X be a smooth manifold of dimension n with a conformal structure C such that $\mathcal{Y}(C) > 0$ and either $n = 3, 4, 5$ or C is conformally flat. Fix $p \in X$. Then, in conformal normal coordinates $\{x^i\}$ at p, the Green's function $G_p = G(p, \cdot)$ has an asymptotic expansion of the form*

$$G_p(x) = |x|^{2-n} + \alpha(p) + O''(|x|)$$

for some constant $\alpha(p)$.

In fact, the regular part $\alpha(p) + O''(|x|)$ of $G_p(x)$ can be expressed in terms of the heat kernel k of L_g. Recall that the *heat kernel* k of L_g is a smooth function $k : X \times X \times \mathbb{R}^+ \to \mathbb{R}^+$ such that:

(a) $k_p(q, t) = k(p, q, t)$ as a function of q and $t > 0$ solves the L_g heat equation:

$$(\partial_t + L_g) k_p(q, t) = 0.$$

(b) For $\forall p \in X$ and $\forall \varphi \in C^\infty(X)$,

$$\lim_{t \to 0} \int_M k(p, q, t) \varphi(q) \ \mathrm{dvol}_g(q) = \varphi(p).$$

It is a well-known fact that the heat kernel k for L_g is uniquely determined by the metric g. Furthermore, k depends smoothly on the metric in the following sense (cf. [PR87, Lemma 1.1] or [BGV03, Theorem 2.48]).

Lemma 6.2.4. *Suppose $\gamma : \mathbb{R}^l \to Met^\infty(X)$ with $\gamma(s) = g_s$ is a smooth family of smooth metrics, then the corresponding family of heat kernels $k(s, p, q, t)$ is a smooth function on $\mathbb{R}^l \times X \times X \times \mathbb{R}^+$.*

We will use the following relation between Green's function G and the heat kernel k (cf. [BGV03, Theorem 2.38]).

Lemma 6.2.5. *For each $(p,q) \in X \times X$, we have*

$$G(p,q) = (n-2)\omega_{n-1} \int_0^\infty k(p,q,t)\mathrm{d}t, \tag{6.2.4}$$

where ω_{n-1} is the volume of the unit sphere S^{n-1}.

In particular, the standard Euclidean heat kernel centered at the origin on $\mathbb{R}^n$ is $k_0 = (4\pi t)^{\frac{-n}{2}} \exp\left(\frac{-|x|^2}{4t}\right)$, and direct integration using the Gamma function shows that

$$(n-2)\omega \int_0^\infty k_0 \ \mathrm{d}t = |x|^{2-n}. \tag{6.2.5}$$

Proposition 6.2.6. *If the manifold X is of dimension $n = 3, 4, 5$ or C is conformally flat, then in conformal normal coordinates $\{x^i\}$ on a neighborhood U_p of a point p,*

$$G_p(x) \ = \ |x|^{2-n} + \Phi_0(p,x) \tag{6.2.6}$$

where

$$\Phi_0(p,x) \ = \ (n-2)\omega_{n-1} \int_0^\infty (k - k_0) dt \tag{6.2.7}$$

is a bounded function on U_p.

Proof. By (6.2.4), (6.2.5) and (6.2.7), we have

$$G_p(x) = (n-2)\omega_{n-1} \left(\int_0^\infty k_0(p,x,t) \ \mathrm{d}t + \int_0^\infty (k_p(x,t) - k_0(p,x,t)) \, \mathrm{d}t \right) = |x|^{2-n} + \Phi_0(p,x).$$

To see that Φ_0 is bounded on U_p, we introduce the function $k_1 = k_0(1 + a_1(x)t)$, where

$$a_1(x) = \int_0^1 S(xt) \ dt \tag{6.2.8}$$

is the integral of the scalar curvature S. By Theorem 2.2 in [PR87], k_1 is a parametrix for k in dimension ≤ 5, and there are bounded functions φ_j such that

$$\Phi_0(p,x) \ = \ \begin{cases} \varphi_1 + \varphi_3 - \varphi_4 & n = 3 \text{ or } C \text{ locally conformally flat} \\ \varphi_1' + \varphi_3 - \varphi_4 + a_1(x)\big(\varphi_2 - \ln(|x|)\big), & n = 4 \\ \varphi_1' + \varphi_3 - \varphi_4' - \frac{a_1(x)}{|x|} & n = 5. \end{cases} \tag{6.2.9}$$

Specifically, with the same labeling as in [PR87],

$$\varphi_1 = (n-2)\omega_{n-1}\int_0^{\frac{1}{4}}(k-k_0)\ dt, \qquad \varphi_1' = (n-2)\omega_{n-1}\int_0^{\frac{1}{4}}(k-k_1)\ dt,$$

$$\varphi_2 = \int_1^\infty \lambda^{-1}e^{-\lambda}d\lambda + \int_{|x|^2}^1 \lambda^{-1}(e^{-\lambda}-1)\ d\lambda,$$

$$\varphi_3 = (n-2)\omega_{n-1}\int_{\frac{1}{4}}^\infty k\ dt,$$

$$\varphi_4 = (n-2)\omega_{n-1}\int_{\frac{1}{4}}^\infty k_0\ \mathrm{dt}, \qquad \varphi_4' = (n-2)\omega_{n-1}\int_{\frac{1}{4}}^\infty k_1\ dt$$

The Proposition follows because, in conformal normal coordinates at p, $S(x) = O(|x|^2)$ (cf. [LP87, Theorem 5.1]), and hence $a_1(x) = O(|x|^2)$. □

Remark 6.2.7. The restriction of Φ_0 to the diagonal is a smooth function $\Phi(p,p)$ on X. This can be seen from (6.2.9). By (6.2.8), $a_1(x)$ vanishes on the diagonal, and φ_3, φ_4 and φ_4' are clearly smooth. As for φ_1 and φ_1', the arguments in Section 2.5 of [BGV03] show that the functions $\kappa(p,t) = k(p,p,t) - k_0(p,p,t)$ and $\kappa'(p,t) = k(p,p,t) - k_1(p,p,t)$ satisfy

$$|\partial_p^l \kappa(p,t)| \le \frac{c_l}{\sqrt{t}}, \text{ for dim } n = 3,$$

$$|\partial_p^l \kappa'(p,t)| \le \frac{c_l'}{\sqrt{t}}, \text{ for dim } n = 4, 5.$$

for any $l \in \mathbb{N}^+$. Because κ and κ' are smooth for $t > 0$ and the function $1/\sqrt{t}$ is integrable, the standard theorem on differentiating under the integral shows that φ_1 and φ_1' are smooth.

Mass of An Asymptotically Flat Manifold

Definition 6.2.8. An n-dimensional Riemannian manifold (X,h) is called *asymptotically flat of order* $\tau > 0$ if there exist a compact subset $K \subset X$ and a diffeomorphism $\Psi : X\backslash K \to \{z \in \mathbb{R}^n : |z| > 1\}$ such that, in the coordinates $z^1, \dots, z^n$ induced on $X\backslash K$,

$$h_{ij}(z) - \delta_{ij} = O''\left(\rho^{-\tau}\right)$$

as $\rho := |z| \to \infty$. The coordinates $\{z^i\}$ are called *asymptotic coordinates.*

Given an asymptotically flat manifold (X,h), let S_r denote the sphere of radius $r \gg 0$ in the asymptotic coordinate system $\{x^i\}$. We can then define the following fundamental quantity.

Definition 6.2.9. *The mass of an asymptotically flat manifold* (X, h) *is the number*

$$mass(h) = \frac{1}{\omega_{n-1}} \lim_{r\to\infty} \int_{S_r} \sum_{i,j=1}^{n} (h_{ij,i} - h_{ii,j})\,(-1)^{j+1} dz^1 \wedge \cdots \wedge \widehat{dz^j} \wedge \cdots \wedge dz^n, \qquad (6.2.10)$$

where ω_{n-1} *is the volume of the unit sphere* S^{n-1}.

By this definition, the mass is a measure of how quickly the metric approaches the Euclidean metric near infinity. The following theorem of R. Bartnik (cf. [Bar86]) shows that the mass depends only on the metric h.

Theorem 6.2.10 (Bartnik)**.** *If* (X, h) *is asymptotically flat of order* $\tau > \frac{n-2}{2} > 0$, *then* $mass(h)$ *is independent of the choices of asymptotic coordinates, so is an invariant of the Riemannian metric* h.

We shall later apply the following version of the n-dimensional Positive Mass Theorem of Schoen and Yau (cf. [SY79], [SY81], [LP87]).

Theorem 6.2.11. *Let* (X, h) *be a Riemannian manifold of dimension* $n \geq 3$ *that is asymptotically flat of order* $\tau > \frac{n-2}{2}$. *If* (X, h) *has non-negative scalar curvature, then* $mass(h) \geq 0$, *with equality if and only if* (X, h) *is isometric to the Euclidean* $\mathbb{R}^n$.

Conformal blowup

Unless specifically stated otherwise, we work with the cases $n = 3, 4, 5$ or C is conformally flat, and we assume that $\mathcal{Y}(C) > 0$.

In [Sch84], Schoen introduced the idea of conformally blowing up a metric $g \in C$ by the Green's function of L_g to turn X into an asymptotically flat manifold (cf. [LP87]). Explicitly, the conformal blowup of (X, g) at a point $p \in X$ of g is the manifold $X \setminus \{p\}$ with the Riemannian metric h_p defined by

$$h_p = h(p, q) \;:=\; \big(G(p, g)\big)^{\frac{4}{n-2}} g(q). \qquad (6.2.11)$$

Note that h_p is a smooth metric by Theorem 6.2.2.

Suppose that $\{x^i\}$ are conformal normal coordinates centered at p defined on a neighborhood U of p as in Lemma 6.2.3. On $U \setminus \{p\}$ define "inverted conformal normal coordinates" by $z^i = \frac{x^i}{|x|^2}$. By the asymptotic expansion of G in Lemma 6.2.3 (cf. [LP87, Theorem 6.5]), we have:

$$h_{ij}(z) = \gamma^{\frac{4}{n-2}}(z)\left(\delta_{ij} + O''\left(|z|^{-2}\right)\right), \tag{6.2.12}$$

where

$$\gamma(z) = 1 + \alpha(p)|z|^{2-n} + O''\left(|z|^{1-n}\right).$$

This shows that h_p is an asymptotically flat metric of order 1 if $n = 3$, order 2 if $n = 4, 5$, and order $n-2$ if g is conformally flat near p. Hence, in each of these cases, Bartnik's Theorem implies that the mass $m(h_p)$ is well-defined and depends only on the metric $g \in C$.

Lemma 6.2.12. *(cf. [LP87, Lemma 10.5])*

$$mass(h_p) = \lim_{q \to p}(G(p,q) - r_p(q)^{2-n}) = 4(n-1)\alpha(p). \tag{6.2.13}$$

Proof. Take the spherical coordinates, $\rho = |z|$ and $\xi = \frac{z}{|z|}$. With respect to (ρ, ξ), the definition (6.2.10) of mass gives:

$$mass(h_p) = \frac{1}{\omega_{n-1}} \lim_{\lambda \to \infty} \lambda^{n-2} \int_{S_\lambda} \sum_{i,j=1}^{n} \left(h_{ij,i} - h_{ii,j}\right) z^j d\xi. \tag{6.2.14}$$

By the asymptotic formula (6.2.12), we have

$$h_{ij}(z) = (1 + \frac{4\alpha(p)}{n-2}\rho^{2-n})\delta_{ij} + O''(\rho^{1-n}).$$

Hence

$$\sum_{i,j=1}^{n}(h_{ij,i} - h_{ii,j})z^j = (1-n)\sum_{j=1}^{n} z^j \partial_j(1 + \frac{4\alpha(p)}{n-2}\rho^{2-n} + O''(\rho^{1-n}))$$

$$= (1-n)\rho\partial_\rho(1 + \frac{4\alpha(p)}{n-2}\rho^{2-n} + O''(\rho^{1-n}))$$

$$= 4(n-1)\alpha(p)\rho^{2-n} + O(\rho^{1-n}).$$

The result follows by substituting the above result into (6.2.14). ☐

Regularity of Habermann's Canonical Metric

Definition 6.2.13. For a Riemannian metric g on X, define the corresponding *mass function* as $m_g : X \to \mathbb{R}$

$$m_g(p) = \frac{mass(h_p)}{4(n-1)}, \tag{6.2.15}$$

where h_p is the asymptotically flat metric at $p \in X$ with respect to g by formula (6.2.11).

Theorem 6.2.14. *The mass function* (6.2.15) *satisfies the following properties :*

(a) $m = 0$ *if* (X, g) *conformal to the standard n-sphere, and* $m > 0$ *in all other cases.*

(b) $m_{f^*g} = f^* m_g$ *for* $f: X \to X$ *a diffeomorphism.*

(c) $u^2 \cdot m_{u^{\frac{4}{n-2}}g} = m_g$ *for* $u \in C^\infty(X)$.

(d) $m \in C^\infty(X)$.

Proof: (a) For each point p, the Positive Mass Theorem 6.2.11 shows that $m(p) \geq 0$, with equality if and only if $(X \backslash \{p\}, h_p)$ is isometric to euclidean $\mathbb{R}^n$, in which case (X, C) is conformally equivalent to the sphere S^n with its standard metric. Hence for all the other cases, m is strictly positive on X.

(b) An isometry f preserves the distance function: $r_g(f(p), f(q)) = r_{f^*g}(p,q)$. It also preserves the Green's function, as follows:

$$\begin{aligned}&\int_X G(f(p), f(q); g) L_{f^*g}(\varphi(q)) \,\mathrm{dvol}_{f^*g}(q)\\ &= \int_X G(f(p), f(q); g) L_g(\varphi(f(q))) \,\mathrm{dvol}_g(f(q)) = (n-2)\,\omega_{n-1}\,(\varphi \circ f)(p)\end{aligned}$$

Hence by the uniqueness of Green's function, we have $G(f(p), f(q); g) = G(p, q; f^*g)$.

By (6.2.13), we have

$$\begin{aligned}f^* m_g = \alpha_g(f(p)) &= \lim_{q \to p} \left| G(f(p), f(q); g) - r_g^{2-n}(f(p), f(q)) \right| \\ &= \lim_{q \to p} \left| G(p, q; f^*g) - r_{f^*g}^{2-n}(p,q) \right| \;=\; \alpha_{f^*g}(p) \;=\; m_{f^*g}.\end{aligned}$$

(c) By (6.2.3)

$$\widetilde{G}(p,q) \;=\; \frac{1}{u(p)u(q)}\, G(p,q).$$

Temporarily setting $\gamma = \frac{4}{n-2}$, the definition (6.2.11) shows that

$$\tilde{h}_p(q) \;=\; \left(\widetilde{G}(p,q)\right)^\gamma \tilde{g} \;=\; \left(\frac{G(p,q)}{u(p)u(q)}\right)^\gamma u(q)^\gamma g(q) \;=\; u(p)^{-\gamma} h_p(q).$$

It is clear that if $\{z^i\}$ are asymptotic coordinates of h, then $\{\lambda z^i\}$ are asymptotic coordinates for a constant rescaling $\lambda^2 h$, and by a coordinate changing in (6.2.10), we have $m(\lambda^2 h) = \lambda^{n-2} m(h)$. The conclusion follows by taking $\lambda = u(p)^{-\gamma/2} = u(p)^{\frac{-2}{n-2}}$.

(d) Let $\Phi \in C^\infty(X \times X, \mathbb{R}^+)$ be the conformal factor defined in Lemma 6.1.8. Notice that proposition (c) is pointwise true, hence at a point p, let $u^{\frac{4}{n-2}}(q) = \Phi^2(p,q)$, we have

$$\Phi^{1/(n-2)} m_{\Phi^2 g}(p) = m_g(p). \tag{6.2.16}$$

By Lemma 6.1.8, $g_p = \Phi_p^2 g$ is the smooth family of metrics that are conformal normal near p.

Let $k : X \times X \times \mathbb{R}^+ \times X \to \mathbb{R}^+$ be the family of heat kernels parametrized by the smooth family of metrics $\{g_p \mid p \in X\}$, i.e. $k(q_1, q_2, t; g_p) = k_p(q_1, q_2, t)$ is the heat kernel of L_{g_p}. By Lemma 6.2.4, $k(q_1, q_2, t; g_p)$ is smooth in all the variables. Let $G_p = G(q_1, q_2; g_p)$ denote the Green's function related to the heat kernel of g_p by (6.2.4). Since g_p are conformal normal for each p, by (6.2.13) and (6.2.6), we have

$$\begin{aligned} mass(h_p(g_p)) &= \lim_{x\to 0} \; G_p(p,x) - |x|^{2-n} \\ &= \lim_{x\to 0} \; \Phi_0(p,x;g_p) + O(|x|) \;=\; \Phi_0(p,p;g_p). \end{aligned}$$

By equation (6.2.7),

$$\Phi_0 \;=\; (n-2)\int_0^\infty (k(p,p,t;g_p) - k_0(p,p,t))dt,$$

where $k(p,p,t;g_p)$ by Lemma 6.2.4 is a smooth function on $X \times \mathbb{R}^+$.

By Lemma 6.2.6 and Remark 6.2.7, Φ_0 is a smooth function on X. Hence by (6.2.16) m_g is a smooth function on X. □

In [HJ99], Habermann and Jost observed that each conformal class C on X has a canonically associated metric, defined as follows.

Definition 6.2.15. For a conformal class C on X, suppose $\psi : X \to C$ is a smooth map, the *canonical metric* κ_C is the $(0,2)$ tensor below:

$$\kappa_C(p) \;=\; m_g(p)^{\frac{2}{n-2}} g(p).$$

By Theorem 6.2.14, κ_C is well defined independent of the choice of $g \in C$ and is preserved by the pullback map by isometry, and κ_C vanishes identically if and only if (X, C) is conformally equivalent to the sphere S^n with its standard metric. Otherwise, $\kappa_C \in C$ is a smooth Riemannian metric on X.

Remark 6.2.16. On a Yamabe positive manifold (X, C) of dimension n, consider P_{2N} the conformal N^{th}-power of Laplacian with $2N+1 \le n \le 2N+3$. Let G be the Green's function of P_{2N}. By a completely similar argument as above, one can see that the $(0,2)$ tensor g_C defined by

$$g_C(p) \;=\; mass\left(G_p^{\frac{4}{n-2N}} g\right)^{\frac{2}{n-2N}} g(p)$$

is smooth and depends only on the conformal class of the metric g. This was proved by

B. Michel in [Mic10]. Again, the proof of regularity ([Mic10, Prop. 3.3]) can be simplified using Lemma 6.1.8.

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APPENDIX A

PROOF OF LEMMA A.3

This appendix supplies the proof of Lemma A.3. The proof is completely algebraic and begins with the following preliminary lemma.

Let $(\mathbb{R}^n, \delta)$ be the standard n-dimensional Euclidean space with dimension $n \geq 3$ and let

$$V = \mathrm{Sym}^2(\mathbb{R}^n) \otimes \mathrm{Sym}^l(\mathbb{R}^n).$$

Consider the following linear operators on V:

$$\begin{aligned} \mathrm{Sym} : V &\to \mathrm{Sym}^k(\mathbb{R}^n) \\ \alpha_\mu &\mapsto \frac{1}{k!} \sum_{\sigma \in S^k} \alpha_{\sigma \cdot \mu} \end{aligned} \tag{A.1}$$

where $k = 2 + l$.

$$\begin{aligned} \mathrm{Tr} : V &\to \mathrm{Sym}^l(\mathbb{R}^n) \\ a_{ij} \otimes b_\nu &\mapsto \sum_{i,j} \delta^{ij} a_{ij} b_\nu \end{aligned} \tag{A.2}$$

The symmetrization operator Sym gives a direct sum decomposition:

$$V = \mathrm{Sym}^k(\mathbb{R}^n) \oplus \ker(\mathrm{Sym}).$$

For $\varepsilon > 0$ small enough, define the following operator:

$$\begin{aligned} P_\varepsilon(\alpha) : V &\to V \\ \alpha &\mapsto \alpha - \varepsilon \cdot \delta \otimes \mathrm{Tr}(\alpha) \end{aligned} \tag{A.3}$$

We say $\alpha \in V$ is *ε-symmetric* if $P_\varepsilon(\alpha) \in \mathrm{Sym}^k(\mathbb{R}^n)$, and denote the ε-symmetric subspace of V as:

$$\mathrm{Sym}^k_\varepsilon(\mathbb{R}^n) := \{\alpha \in V | P_\varepsilon(\alpha) \in \mathrm{Sym}^k(\mathbb{R}^n)\} \tag{A.4}$$

Lemma A.1. *For $\varepsilon \leq \frac{1}{2(n-1)}$, we have following the direct sum decomposition:*

$$V = \mathrm{Sym}^k_\varepsilon(\mathbb{R}^n) \oplus \ker(\mathrm{Sym}).$$

Proof. If $P_\varepsilon(\alpha) = 0$, then we have $\mathrm{Tr}(P_\varepsilon(\alpha)) = (1 - n\varepsilon)\,\mathrm{Tr}(\alpha) = 0$. By assumption, $\varepsilon \leq \frac{1}{2(n-1)} < \frac{1}{n}$, hence $\mathrm{Tr}(\alpha) = 0$ and $0 = P_\varepsilon(\alpha) = \alpha - \varepsilon \cdot \delta_{ij} \otimes \mathrm{Tr}(\alpha) = \alpha$. Thus P_ε is an isomorphism on V and $\dim(\mathrm{Sym}^k_\varepsilon(\mathbb{R}^n)) = \dim(\mathrm{Sym}^k(\mathbb{R}^n))$.

We are left to show that $\mathrm{Sym}^k_\varepsilon(\mathbb{R}^n) \cap \ker(\mathrm{Sym}) = \{0\}$. Assume $\alpha \in V$ such that $\mathrm{Sym}(\alpha) = 0$ and $P_\varepsilon(\alpha) \in \mathrm{Sym}^k(\mathbb{R}^n)$. By taking $\mathrm{Tr} \circ \mathrm{Sym}$ on $P_\varepsilon(\alpha)$, we can obtain the following equation on $\mathrm{Sym}^l(\mathbb{R}^n)$:

$$(1-\varepsilon\cdot n)\,\mathrm{Tr}(\alpha) = \mathrm{Tr}(P_\varepsilon(\alpha)) = Tr(\mathrm{Sym}(P_\varepsilon(\alpha))) = -\varepsilon\,\mathrm{Tr}(\mathrm{Sym}(\delta\otimes\mathrm{Tr}(\alpha))). \tag{A.5}$$

To be brief, denote $\mathrm{Tr}(\alpha) = S_\nu$ with $\nu = (i_1\cdots i_l)$. Since $S_\nu \in \mathrm{Sym}^l(\mathbb{R}^n)$, we may assume the indices are nondecreasing: $i_1 \le \cdots \le i_l$. For each $1 \le i \le n$, let ν_i be the number of i's in ν, then it is clear that the index ν is determined by the numbers $\nu_1,\ldots,\nu_n$, and hence in other words by the Young diagram $(\nu_1,\ldots,\nu_n)$ of n rows with ν_i many boxes on the i^{th} row such that $\sum_{i=1}^n \nu_i = l$.

Let $N = \varepsilon^{-1}$, multiply Equation (A.5) with $(l+2)(l+1)N$ and write with indices, we have:

$$[(l+2)(l+1)(N-n) + \sum_{i=1}^n (v_i+1)(v_i+2)]S_\nu + \sum_{\substack{j\neq i\\ v_j>1, v_i<l-1}} v_j(v_j-1)S_{v'_{ij}} = 0, \tag{A.6$_\nu$}$$

where the index v'_{ij} is determined by the Young diagram $(\nu_1,\ldots,\nu_i+2,\ldots,\nu_j-2,\ldots,\nu_n)$.

We see that two elements S_ν and S'_ν are correlated by Equation ((A.6)$_\nu$) if there exist $1 \le i \ne j \le n$ such that $\nu'_i = \nu_i + 2$, $\nu'_j = \nu_j - 2$ and $\nu'_k = \nu_k$ for $k \notin \{i,j\}$. We thus define an equivalence relation between the indices ν and ν' as follows: we say ν and ν' are elementary correlated if there exist $1 \le i \ne j \le n$ such that $\nu'_i = \nu_i + 2$, $\nu'_j = \nu_j - 2$ and $\nu'_k = \nu_k$ for $k \notin \{i,j\}$. And ν, ν' are equivalent if ν' can be obtained from ν by finitely many steps of elementary correlations.

Denote $\bar\nu$ for an equivalent class of ν, let $|\bar\nu|$ be its cardinality. For each $\nu \in \bar\nu$, we have the corresponding Equation (A.6)$_\nu$, and hence we have $|\bar\nu|$ many homogenous linear equations for $|\bar\nu|$ many unknowns. In matrix form, we have

$$A\cdot (S_\nu)_{\nu\in\bar\mu} = 0, \tag{A.7}$$

where A is a $|\bar\nu|\times|\bar\nu|$ square matrix defined as

$$A_{\nu\nu'} = \begin{cases} (l+2)(l+1)(N-n) + \sum_{i=1}^n(\nu_i+1)(\nu_i+2), & \nu' = \nu \\ \nu_j(\nu_j-1), & i\neq j, \nu' = \nu'_{ij} \\ 0, & \text{otherwise.} \end{cases} \tag{A.8}$$

We claim that A is a diagonally dominated matrix. Indeed, for the ν^{th} row, the difference between the absolute value of the diagonal and the sum of absolute values of elements of the diagonal is:

$$\begin{aligned}
&(l+2)(l+1)(N-n)+\sum_{i=1}^{n}(\nu_i+1)(\nu_i+2)-\sum_{i\neq j}\nu_j(\nu_j-1)\\
&=(N-n)l^2+l^2-\sum_{i\neq j}(\nu_i+\nu_j)\nu_j+(3(N-n)+n+2)l+2(N+1-n)\\
&\geq (N-n)l^2+l^2-l\sum_{i\neq j}\nu_j+(3(N-n)+n+2)l+2(N+1-n)\\
&=(N-2n+2)l^2+(3(N-n)+n+2)l+2(N+1-n)\\
&\geq (3(N-n)+n+2)l+2(N+1-n)>0,
\end{aligned}\tag{A.9}$$

where the first equality is by the fact:

$$\sum_{i=1}^{n}\nu_i^2=(\sum_{i=1}^{n}\nu_i)^2-\sum_{i\neq j}\nu_i\nu_j=l^2-\sum_{i\neq j}\nu_i\nu_j,$$

the first inequality is by $\nu_i+\nu_j\leq l$ and the second inequality is by the assumption that $N=\varepsilon^{-1}\geq 2n-2$. Hence the coefficient matrix A is diagonally dominated. It is a classical fact that diagonally dominant matrices are invertible, and hence $(S_\nu)_{\nu\in\bar{\nu}}=A^{-1}(A(S_\nu))=0$.

Since the class $\bar{\nu}$ is arbitrarily taken, we have $S_\nu=0$ for any index ν, and hence $\mathrm{Tr}(\alpha)=0$. Hence $P_\varepsilon(\alpha)=\alpha\in\mathrm{Sym}^k(\mathbb{R}^n)$ and $\mathrm{Sym}(\alpha)=0$, which means $\alpha=0$. □

Remark A.2. Diagonal dominance is a sufficient but not necessary condition for the matrix A being nonsingular, and hence $\frac{1}{2n-2}$ is not a sharp bound for transversality. Indeed, if we formally write the unknowns S_ν in the equations $(\mathrm{A.6})_\nu$ as coefficients of a homogeneous degree l polynomial f, we can then write the system $(\mathrm{A.6})_\nu$ in a compact way as:

$$((l+2)(l+1)(N-n)+4l+2n)f+r^2\Delta(f)=0.\tag{A.10}$$

See Lemma 5.3 in [LP87] for the following fact: The eigenvalues of $r^2\Delta$ on the space of homogeneous degree l polymonials are

$$\{\lambda_j=-2j(n-2+2l-2j):j=0,\dots,[l/2]\}.$$

Hence to ensure $(S_\nu)=0$, it is sufficient to require the following set does not contain 0:

$$\Lambda=\{((l+2)(l+1)(N-n)+4l+2n)-2j(n-2+2l-2j)|j=0,\dots,[l/2]\}.$$

Hence transversality result holds for a general ε away from a countable subset in $\mathbb{R}$. In particular, for $N = n = 3$, we have $0 \notin \Lambda$. For this case, if we take $\alpha = Ric$, then $P_{1/3}(Ric)$ is the traceless Ricci.

Lemma A.3. *For $k \geq 0$, equations (3.1.11)$_0$, ... ,(3.1.11)$_k$ together with $j_p^{k-1}(C) = 0$ deduce $j_p^k(Ric) = 0$.*

Proof. We prove by induction on k:

For $k = 0$, the claim is true by (3.1.11)$_0$. Assume the claim is true for $k \leq m$.

For $k = m+1$, by the induction assumption, we have $j_p^m(Ric) = 0$.

In Equation (3.1.11)$_k$ with $k = m+1$, we have:

$$\frac{2m+4}{(m+4)!}\sum_{\mu\sim\bar{\mu}} Ric_\mu(p) + P(R_{\mu'}(p)) = 0,$$

where $|\mu| = m+3$ and $|\mu'| \leq m+1$ and the $P(R_{\mu'}(p))$ term consists of derivatives of the Riemannian curvature R of order less than m and therefore vanishes by the induction assumption.

We thus have:

$$\sum_{\mu\sim\bar{\mu}} Ric_\mu(p) = 0.$$

On the other hand, by the condition $j_p^m(C) = 0$, for any index ν with $|\nu| = m$, we have:

$$0 = C_\nu(p) = P_{ij,k\nu} - P_{ik,j\nu}, \tag{A.11}$$

Where $P = Ric - \frac{1}{2(n-1)}Sg$ is the Schouten tensor.

Since $S = g^{ij}Ric_{ij}$, and $j_p^m(Ric) = 0$, we have $j_p^m(S) = 0$.

Hence

$$(Sg_{ij})_{k\nu}(p) = \delta_{ij}(p)S_{k\nu}(p).$$

For $l = m+1$, and $\mu = (ij\nu)$ with $|\nu| = m+1$, in local coordinates we have

$$Ric_\mu(p) \in \mathrm{Sym}^2(\mathbb{R}^n) \otimes \mathrm{Sym}^l(\mathbb{R}^n) = V.$$

Let $\varepsilon = \frac{1}{2n-2}$ in Lemma A.1, we have

$$P_{\frac{1}{2n-2}}(Ric_\mu(p)) = Ric_\mu(p) - \frac{1}{2n-2}\delta(p) \otimes S_\nu(p) = P_\nu,$$

which is symmetric by (A.11).

Hence $Ric_\mu(p)$ satisfies the conditions of Lemma A.1, by which we have $Ric_\mu(p) = 0$, for $|\mu| = m + 1$, namely $j_p^{m+1}(Ric) = 0$. $\square$

APPENDIX B

PROOF OF LEMMA B.1

This appendix supplies the proof of Lemma B.1.

For $0 < \alpha, \delta < 1$, $k \leq N$, and denote A_r the annulus $B_r - B_{r/2}$, recall the definition of the $\|\cdot\|_{k,\alpha;N,\delta}$ norm:

$$\|f\|_{k,\alpha;N,\delta} = \sup_{0<r\leq\delta} r^{-N}\left(\sum_{|\beta|=0}^{k} r^{|\beta|}\sup_{A_r}\{|\partial_\beta f(x)|\} + r^{k+\alpha}\sup_{\substack{x\neq y\in A_r\\ |\beta|\leq k}}\frac{|\partial_\beta f(x)-\partial_\beta f(y)|}{|x-y|^\alpha}\right).$$

For $K_0 > 0$, let $D_{K_0} = \{v \mid \|v\|_{k,\alpha;N,\delta} \leq K_0\}$, on which define the functional:

$$T_\rho(x,v) = G_\rho(x,v) + Q_\rho(x,v) = \sum_{i,j}\frac{x^ix^j}{x^2}\cdot G_{ij}(\rho x,\rho\partial v) + \sum_{ijk}\rho x^k Q_{ijk}(\rho x,\rho\partial v)\partial_{ij}v,$$

where $Q_{ijk}(x,\zeta)$ and $G_{ij}(x,\zeta)$ are smooth functions with respect to x, ζ.

Lemma B.1. *There exists a constant K_2 such that for $0 < \rho < 1$ and any pair of functions $v_1, v_2 \in D_{K_0}$, we have*

$$\|T_\rho(v_2) - T_\rho(v_1)\|_{k-2,\alpha;N-2,\delta} + \|T_\rho(v_2) - T_\rho(v_1)\|_{0,\alpha;N-1,\delta} \leq K_2\rho\|v_2 - v_1\|_{k,\alpha;N,\delta}. \qquad \text{(B.1)}$$

In fact, $K_2 = C(\alpha)P(K_0)M$, where $C(\alpha)$ is a constant depends on α, $P(K_0)$ is a polynomial of K_0 and $M = \max_{ijk}\left\{\|G_{ij}\|_{C^k(D)}, \|x^kQ_{ijk}\|_{C^k(D)}\right\}$, with $D = D(\delta, K_0) = B_\delta \times B_{K_0} \subseteq \mathbb{R}^n \oplus \mathbb{R}^n$.

Proof: Write $T_\rho\Big|_1^2 = T(\rho x, \rho\partial v_2) - T(\rho x, \rho\partial v_1)$ and similarly for G_ρ and Q_ρ terms.

First, consider the case $k = 2$. In this case, the second term in (B.1) dominates the first term, so it suffices to bound $\|T_\rho\Big|_1^2\|_{0,\alpha;N-1,\delta}$. Since

$$\begin{aligned}\|T_\rho(v)\Big|_1^2\|_{0,\alpha;N-1,\delta} &= \|G_\rho(v)\Big|_1^2 + Q_\rho(v)\Big|_1^2\|_{0,\alpha;N-1,\delta}\\ &\leq \|G_\rho(v)\Big|_1^2\|_{0,\alpha;N-1,\delta} + \|Q_\rho(v)\Big|_1^2\|_{0,\alpha;N-1,\delta}.\end{aligned}$$

We will bound the G and Q norms separately.

For the G part, we have:

$$\begin{aligned}|G_\rho\Big|_1^2| &= |\sum_{ij}\frac{x^ix^j}{x^2}(G_{ij}(\rho x,\rho\partial v_2) - G_{ij}(\rho x,\rho\partial v_1))|\\ &\leq \sum_{ij}|G_{ij}(\rho x,\rho\partial v_2) - G_{ij}(\rho x,\rho\partial v_1)| \leq \|G_{ij}\|_{C^1(D)}\rho|\partial v\Big|_1^2|,\end{aligned}$$

where the last inequality is by the Mean Value Theorem.

By induction on m, we have $\partial^m \frac{x^i x^j}{x^2} \leq C\frac{1}{r^m}$, where C depends m and the dimension n. Hence

$$|\partial^m G_\rho\Big|_1^2| \leq C\sum_{ij}\sum_{l=0}^{m} r^{l-m}|\partial^l G_{ij}(\rho x, \rho\partial v))\Big|_1^2|. \tag{B.2}$$

Apply the composition rule to $\partial^l G_{ij}(\rho x, \rho\partial v)$, we obtain a linear combination of the following terms:

$$\partial^{m_1+\cdots+m_l} G_{ij}(\rho x, \rho\partial v)\prod_{j=1}^{l}(\partial^{j+1}v)^{m_j},$$

where $\sum_{j=1}^{l} jm_j = l$.

Evaluate each term at v_1 and v_2 and by the Mean Value Theorem similar to above, we have:

$$|\partial^l G_{ij}(\rho x, \rho\partial v))\Big|_1^2| \leq \rho\|G_{ij}\|_{C^{l+1}(D)} P_1(K_0)(\sum_{l'=1}^{l+1}|\partial^{l'}v\Big|_1^2|), \tag{B.3}$$

Combining (B.2) and (B.3), we have

$$|\partial^m G_\rho\Big|_1^2| \leq \rho P(K_0)\max_{ij}\left\{\|G_{ij}\|_{C^{m+1}(D)}\right\}\sum_{l=0}^{m} r^{l-m}|\partial^{l+1}v\Big|_1^2| \tag{B.4}$$

Hence for the C^α term we have:

$$\frac{|\partial^m G_\rho(x,v)\Big|_1^2 - \partial^m G_\rho(y,v)\Big|_1^2|}{|x-y|^\alpha} = \frac{|\partial^m G_\rho(x,v)\Big|_1^2 - \partial^m G_\rho(y,v)\Big|_1^2|}{|x-y|}|x-y|^{1-\alpha}$$

$$\leq C(\alpha)r^{1-\alpha}\sup_{A_r}|\partial^{m+1}G_\rho\Big|_1^2|$$

$$\leq \rho C(\alpha)r^{1-\alpha}P(K_0)\max_{ij}\left\{\|G_{ij}\|_{C^{m+2}(D)}\right\}\sup_{A_r}\sum_{l=0}^{m+1}|r^{l-m}\partial^{l+1}v\Big|_1^2| \tag{B.5}$$

For $m = 0$, apply (B.4) and (B.5), we have:

$$\|G_\rho(v)\Big|_1^2\|_{0,\alpha;N-1,\delta} \leq C(\alpha)(K_0+1)\max_{ij}\|G_{ij}\|_{C^2(D)}\rho\|v_1 - v_2\|_{2,\alpha;N,\delta}. \tag{B.6}$$

Similarly, for the Q part, by the Mean Value Theorem, we have:

$$\begin{aligned}
|Q_\rho(v)\Big|_1^2| &= \rho\cdot\sum_{ijk}|x^k Q_{ijk}(\rho x, \rho\partial v)\partial_{ij}v\Big|_1^2| \\
&\leq \rho\sum_{ijk}|x^k|\cdot|Q_{ijk}(\rho\partial v_1)\partial_{ij}v_2 - Q_{ijk}(\rho\partial v_2)\partial_{ij}v_1 + Q_{ijk}(\rho\partial v_2)\partial_{ij}v_1 - Q_{ijk}(\rho\partial v_2)\partial_{ij}v_2| \\
&\leq \rho\sum_{ijk}|x^k Q_{ijk}(\rho x, \rho\partial v)\Big|_1^2|\cdot K_0 + |x^k Q_{ijk}(\rho x, \rho v_2)||\partial^2 v\Big|_1^2| \\
&\leq \rho\sum_{ijk}|x^k Q_{ijk}|_{C^1(D)}(K_0|\partial v\Big|_1^2| + |\partial^2 v\Big|_1^2|)
\end{aligned}$$

$$\frac{|Q_\rho(x,v)\Big|_1^2 - Q_\rho(y,v)\Big|_1^2|}{|x-y|^\alpha} \le \rho \cdot \sum_{ijk} \frac{|x^k Q_{ijk}(\rho x, \rho\partial v)\partial_{ij}v\Big|_1^2 - y^k Q_{ijk}(\rho y, \rho\partial v)\partial_{ij}v\Big|_1^2|}{|x-y|^\alpha}$$

To be brief, in the numerator denote $x^k Q_{ijk}(\rho x, \rho\partial v_1)\partial_{ij}v_1(x)$ as $Q(x,v_1)\partial^2(x,v_1)$, similarly for y and v_2. We then have:

$$|x^k Q_{ijk}(\rho x, \rho\partial v)\partial_{ij}v\Big|_1^2 - y^k Q_{ijk}(\rho y, \rho\partial v)\partial_{ij}v\Big|_1^2)| = A + B + C + D,$$

where

$$A = Q(x,v)\Big|_1^2 \cdot \left(\partial^2(x,v_1) - \partial^2(y,v_1)\right), B = Q\left(x,v_2\right)\left(\partial^2(x,v)\Big|_1^2 - \partial^2(y,v)\Big|_1^2\right),$$

$$C = \partial^2(y,v_1)\left(Q(x,v)\Big|_1^2 - Q(y,v)\Big|_1^2\right), D = \left(Q(x,v_2) - Q(y,v_2)\right)\left(\partial^2(y,v)\Big|_1^2\right).$$

Respectively, we have:

$$\frac{|A|}{|x-y|^\alpha} = |x^k Q_{ijk}(\rho x, \rho\partial v)\Big|_1^2| \cdot \frac{|\partial^2(x,v_1) - \partial^2(y,v_1)|}{|x-y|^\alpha} \le K_0 \|x^k Q_{ijk}\|_{C^1(D)}(|\partial v\Big|_1^2),$$

$$\frac{|B|}{|x-y|^\alpha} \le r \cdot \|x^k Q_{ijk}\|_{C^0(D)} \cdot \frac{|\partial^2(x, v_1 - v_2) - \partial^2(y, v_2 - v_2)\ |}{|x-y|^\alpha},$$

$$\frac{|C|}{|x-y|^\alpha} \le K_0 r \frac{|Q(x,v)\Big|_1^2 - Q(y,v)\Big|_1^2|}{|x-y|}|x-y|^\alpha \le C(\alpha) K_0 \sup_{A_r} |\partial x^k Q_{ijk}(\rho x, \rho\partial v)\Big|_1^2|$$

$$\le C(\alpha) K_0 (1 + K_0) r^{N-1} \|x^k Q_{ijk}\|_{C^2(D)}(|\partial v\Big|_1^2|) + |\partial^2 v\Big|_1^2|),$$

$$\frac{|D|}{|x-y|^\alpha} \le C(\alpha) r^{1-\alpha} \|x^k Q_{ijk}\|_{C^1(D)}(|\partial^2 v\Big|_1^2|).$$

In summary, for the Q term, we have:

$$\|Q_\rho(v)\Big|_1^2\|_{0,\alpha;N-1,\delta} \le C(\alpha)(K_0^2 + K_0 + 1) \max_{ijk} \|x^k Q_{ijk}\|_{C^2(D)} \rho \|v_1 - v_2\|_{2,\alpha;N,\delta}. \quad \text{(B.7)}$$

Combining (B.6) and (B.7), we conclude the proof for $k = 2$.

Assume the statement is true for $k = m - 1$. For $k = m$, we have:

$$\|T_\rho\Big|_1^2\|_{m-2,\alpha;N-2,\delta} + \|T_\rho\Big|_1^2\|_{0,\alpha;N-1,\delta}$$

$$\le \|T_\rho\Big|_1^2\|_{m-3,\alpha;N-2,\delta} + \|T_\rho\Big|_1^2\|_{0,\alpha;N-1,\delta} +$$

$$\sup_{0<r\le\delta} r^{m-N}\left(\sup_{A_r}|\partial^{m-2}T_\rho\Big|_1^2| + \sup_{x\ne y\in A_r} r^\alpha \frac{|\partial^{m-2}T_\rho(x,\partial v)\Big|_1^2| - \partial^{m-2}T_\rho(y,\partial v)\Big|_1^2|}{|x-y|^\alpha}\right)$$

Hence, by the induction assumption, it is sufficient to show that:

$$\sup_{0<r<\delta} r^{m-N}\left(\sup_{A_r}|\partial^{m-2}T_\rho(v)\Big|_1^2| + \sup_{x\neq y\in A_r} r^\alpha \frac{|\partial^{m-2}T_\rho(x,v)\Big|_1^2| - \partial^{m-2}T_\rho(y,v)\Big|_1^2|}{|x-y|^\alpha}\right)$$

$$\leq C(\alpha)P(K_0)M\rho\|v_2 - v_1\|_{m,\alpha;N,\delta}.$$

For the G terms, apply (B.4) to $m-2$, we have:

$$\sup_{0<r<\delta} r^{m-N}\left(\sup_{A_r}\{|\partial^{m-2}G_\rho(v)\Big|_1^2|\}\right)$$

$$\leq \rho P(K_0)\max_{ij}\left\{\|G_{ij}\|_{C^{m-1}(D)}\right\}\sup_{0<r<\delta} r^{-N}\sup_{A_r}\left\{\sum_{l=0}^{m-2} r^{l+1}|\partial^{l+1}v\Big|_1^2|\right\} \tag{B.8}$$

Apply (B.5) to $m-2$, we have:

$$\sup_{0<r<\delta} r^{m-N}\left\{\sup_{x\neq y\in A_r} r^\alpha \frac{|\partial^{m-2}G_\rho(x,v)\Big|_1^2 - \partial^{m-2}G_\rho(y,v)\Big|_1^2|}{|x-y|^\alpha}\right\}$$

$$\leq \rho C(\alpha)P(K_0)\max_{ij}\left\{\|G_{ij}\|_{C^m(D)}\right\}\sup_{0<r<\delta} r^{-N}\sup_{A_r}\left\{\sum_{l=0}^{m}|r^{l+1}\partial^{l+1}v\Big|_1^2|\right\} \tag{B.9}$$

For the Q terms, by the Leibniz rule:

$$\partial^{m-2}(Q_\rho) = \partial^{m-2}(\rho x^k Q_{ijk,\rho}\partial_{ij}v) = \rho\sum_{a+b=m-2}\binom{m-2}{a}\partial^a(\rho x^k Q_{ijk,\rho})\partial_{ij}(\partial^b v)$$

For each term $\partial^a(\rho x^k Q_{ijk,\rho})\partial_{ij}(\partial^b v)$, denote $\tilde{Q}_{ijk} = \partial^a(\rho x^k Q_{ijk,\rho})$ and $\tilde{v} = \partial^b v$, apply (B.7), we have:

$$\sup_{0<r<\delta} r^{1-N}\left(\sup_{A_r}|\partial^{m-2}\tilde{Q}_{ijk}\Big|_1^2| + \sup_{x\neq y\in A_r} r^\alpha \frac{|\partial^{m-2}\tilde{Q}_{ijk}(x,v)\Big|_1^2| - \partial^{m-2}\tilde{Q}_{ijk}(y,v)\Big|_1^2|}{|x-y|^\alpha}\right)$$

$$\leq C(\alpha)P(K_0)\max_{ijk}\|\tilde{Q}_{ijk}\|_{C^2(D)}\rho\|\tilde{v}_1 - \tilde{v}_2\|_{2,\alpha;N,\delta}.$$

Multiply by r^{m-1} on both sides of the inequality above. By definition of the $\|\cdot\|_{k,\alpha;N,\delta}$ norm, we have $r^{m-1}\|\tilde{v}_1 - \tilde{v}_2\|_{2,\alpha;N,\delta} \leq \|v_1 - v_2\|_{m,\alpha;N,\delta}$, and hence we have:

$$\sup_{0<r<\delta} r^{m-N}\left(\sup_{A_r}|\partial^{m-2}Q_\rho(v)\Big|_1^2| + \sup_{x\neq y\in A_r} r^\alpha \frac{|\partial^{m-2}Q_\rho(x,v)\Big|_1^2| - \partial^{m-2}Q_\rho(y,v)\Big|_1^2|}{|x-y|^\alpha}\right)$$

$$\leq C(\alpha)P(K_0)\max_{ijk}\{\|x^k Q_{ijk}\|_{C^m(D)}\}\rho\|v_1 - v_2\|_{m,\alpha;N,\delta}. \tag{B.10}$$

Combining (B.8), (B.9) and (B.10), we conclude the $k = m$ step. □