A STUDY OF GLOBAL EXISTENCE TOWARD SOME CHEMOTAXIS MODELS

By

Minh Le

A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

Mathematics – Doctor of Philosophy

2024

ABSTRACT

The focus of this dissertation is on global existence of solutions of some chemotaxis systems with logistic sources, subject to both homogeneous and nonlinear Neumann boundary conditions. It is known that blow-up phenomena can be prevented for some chemotaxis models with quadratic logistic sources under homogeneous Neumann boundary conditions. However, it is shown in this research that quadratic logistic sources are not optimal for preventing blow-up phenomena in some chemotaxis systems. Indeed, our first result demonstrates that Keller-Segel systems can avoid blow-up solutions under nonlinear Neumann boundary conditions with quadratic logistic sources. Moreover, the second result shows that blow-up solutions can be prevented in two spatial dimensions with sub-logistic sources. Additionally, the third result shows that sub-logistic sources are even sufficient to avoid blow-up solutions under nonlinear Neumann boundary conditions in two spacial dimensions. Furthermore, we investigate some nonlinear nonlocal sources Keller systems and the effect of sub-logistic sources under nonlinear Neumann boundary conditions and two-species with two chemicals models in two spacial dimensions. Finally, we show that the presence of logistic sources can prevent blow-up phenomenon in superlinear cross diffusion chemotaxis and in superlinear signal production chemotaxis models. Mathematical tools utilized in the research include variational methods, Moser-Alikakos iterations, regularity theory for elliptic and parabolic equations in Sobolev spaces and in Orlicz spaces, some elemental inequalities in Sobolev spaces and differential inequalities.

ACKNOWLEDGEMENTS

I would like to express my heartfelt gratitude to my dear wife, Thao Vy Hoang Nguyen, for her unwavering love and support throughout my PhD journey. Without her, this achievement would not have been possible. I also extend my thanks to my father, mother, and brother for their support and trust in me to complete this work.

My sincerest appreciation goes to my advisor, Prof. Zhengfang Zhou, whose open-mindedness and broad vision inspired me to pursue this research topic. He provided me with invaluable guidance and encouragement throughout the process, dedicating his time to verify my work and engage in many discussions.

I am also grateful to Prof. Jun Kitagawa, who served on the committee and whose lecture on parabolic regularity theory laid the groundwork for my research, specifically the De Giorgi Moser iteration technique. Furthermore, I extend my thanks to Prof. Baisheng Yan, Prof. Wille Wong, and Prof. Olga Turanova for their participation in the committee.

Additionally, I am deeply grateful to Prof. Michael Winkler for his valuable comments, suggestions, and discussions in Chapter 2, Chapter 3, and Chapter 5.

Lastly, I wish to express my warm appreciation to my anonymous friends for their helpful suggestions and comments.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

My inspiration for this project stems from various papers and a book [50]. It is discovered that cross-diffusion systems, such as Keller-Segel, play a significant role in predicting the formation of aggregations, navigating an optimal path in a complex network, and even in physics, such as particle interaction. Many mathematical tools of nonlinear parabolic equation theory can be adapted and modified to tackle chemotaxis systems like Keller-Segel and its variations. As a result, this thesis aims to investigate solutions, including global existence of certain chemotaxis systems using these techniques.

The dissertation is structured as follows: Chapter 2 examines chemotaxis models with logistic sources under nonlinear Neumann boundary conditions. In Chapter 3, we analyze a chemotaxis model with sub-logistic sources in two spatial dimensions. In chapter 4, we investigate the global existence issues of several chemotaxis systems including Keller-Segel system in the limiting parameter, two species with two chemicals systems, nonlocal sources chemotaxis systems in any spacial dimensions $n \geq 3$. Furthermore, in two spacial dimensions the presence of sub-logistic sources can prevent blow-up even for chemotaxis system under nonlinear Neumann boundary conditions. Chapter 5 is devoted to investigating global existence issues in two chemotaxis models: one featuring superlinear cross-diffusion rates and sub-logistic sources, and another incorporating superlinear signal production with logistic sources. Finally, in appendix, we adopt and modify Moser-Alikakos iteration method for general class of chemotaxis systems with general Neumann boundary condition.

Each chapter is written as a self-contained unit, so readers can approach them independently and do not necessarily need to read them in order. This structure allows readers to focus on specific areas of interest and gain insights into particular aspects of chemotaxis models without having to read the entire dissertation.

1.1 Some background of chemotaxis systems

The chemotaxis phenomenon, which refers to the movement of cells toward chemical signal, has been intensively studied since 1880s. However until 1970s, the mathematical modelling for chemotaxis was first introduced in [26]. To be more specific, the general form of the model is described by the following PDEs:

$$\begin{cases} u_t = \nabla \cdot (D(u,v)\nabla u - S(u,v)u\nabla v) + f(u,v) \\ \tau v_t = d\Delta v + g(u,v) - h(u,v)v \end{cases} \tag{1.1.1}$$

where $u(x,t)$, $v(x,t)$ are functions correspondingly describing the cell density population and chemical signal concentration at a position x in an open smooth bounded domain $\Omega \subset \mathbb{R}^n$ and a instant of time t. The functions $D(u,v)$ and $S(u,v)$ represent diffusivity of the cells and the chemotactic sensitivity, respectively. The function $f(u,v)$ describes cell growth and death while the function $g(u,v)$ and $h(u,v)$ are kinetic functions that describe production and degradation of chemical signal, respectively. A step toward the derivation of the model as well as many interesting results can be found in [20]. Also, many developed techniques and results over decades to deal with the chemotaxis systems has been summarized in [4]. Moreover, [18] presents an extensive survey of various chemotaxis models, along with their corresponding biological background. Furthermore, [17] offers a summary of key techniques and recent findings regarding global existence and blow-up solutions, as well as a discussion of recent numerical studies and potential directions for future research.

Since the 1970s, the mathematical perspective of this phenomenon has rapidly evolved, fueled by its crucial applications and inherent mathematical elegance. Among the many areas of research in this field, one of the most fascinating aspects of chemotaxis systems is the critical mass phenomenon. Specifically, when $S \equiv D \equiv 1$, $g(u,v) = u$, $h \equiv 1$, $\tau = 0$, $d = 1$, and $\Omega = \mathbb{R}^2$, the simplicity and elegance of this system have captured the attention of numerous researchers, intrigued by the critical mass phenomenon it exhibits (see e.g. [12, 7, 42, 43, 52, 51, 45]...). Heuristically, the analysis picture of this critical can be roughly understood by differentiating the second momentum of the solution (see [19]). Specifically, the second equation of (1.1.1), the Poisson equation, gives us an explicit formula for v in terms of u.

$$v(x,t) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \ln(|x-y|)u(y)\,dy$$

By substituting this into the evolution equation for u, we obtain:

$$u_t = \Delta u + \frac{1}{2\pi}\nabla\cdot(u\nabla \ln|x| * u) \tag{1.1.2}$$

We can calculate the dissipation of the second momentum explicitly by using integration by parts

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^2} |x|^2 u(x,t)\,dx &= \int_{\mathbb{R}^2} |x|^2\left(\Delta u - \nabla\cdot(u\nabla v)\right)\,dx \\
&= \int_{\mathbb{R}^2} \Delta(|x|^2)u\,dx + \int_{\mathbb{R}^2}(2x\cdot\nabla v)u\,dx \\
&= 4m + 2\int_{\mathbb{R}^2}(x\cdot\nabla v)u\,dx \\
&= 4m - \frac{1}{\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} u(x,t)u(y,t)\frac{x\cdot(x-y)}{|x-y|^2}\,dxdy \\
&= 4m - \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} u(x,t)u(y,t)\frac{(x-y)\cdot(x-y)}{|x-y|^2}\,dxdy \\
&= 4m - \frac{m^2}{2\pi},
\end{aligned}$$

where $m = \int_{\mathbb{R}^2} u(x,t)\,dx$. If the initial second momentum is finite we have

$$\int_{\mathbb{R}^2}|x|^2 u(x,t)\,dx = \int_{\mathbb{R}^2}|x|^2 u(x,0)\,dx + 4m(1-\frac{m}{8\pi})t.$$

As a consequence, we find that solutions do not exist globally when $m > 8\pi$. Indeed, we have a rich literature concerning about this critical mass $m = 8\pi$. For more details, interested readers are referred to [19][Introduction].

In addition to the biological chemotaxis phenomena, the logistic term $f(u) = au - \mu u^2$ introduced in the evolution equation for u plays a role in describing the growth of the population. Specifically, the term au, with $a \in \mathbb{R}$, is the growth rate of population and the term $-\mu u^2$ models additional overcrowding effects. In a two-dimensional space, the system (KS) possesses a unique classical solution which is nonnegative and bounded in $\Omega\times(0,\infty)$ (see e.g. [47, 46]). In a higher-dimensional space, the similar results can be found in [21] for the parabolic-elliptic model, and in [65] for the parabolic-parabolic model with an additional largeness assumption of μ. In addition to the global classical solutions, the existence global weak solutions results for any arbitrary $\mu > 0$ were also obtained in [21] for the parabolic-elliptic models and in [28] for the parabolic-parabolic models in a three-dimensional system. Furthermore, a precise formula for a lower bound for μ was found in [69].

1.2 Problems and results

The main goal of this thesis is to investigate two crucial inquiries concerning the global boundedness of solutions in certain chemotaxis systems.

The first question revolves around the global existence of solutions under nonlinear Neumann boundary conditions, such as $\frac{\partial u}{\partial \nu} = |u|^p$ for $p > 1$. In Chapter 2, we establish the global existence of solutions when p is below a critical threshold. Specifically, Theorem 2.1.1 addresses parabolic-elliptic chemotaxis systems, while Theorems 2.1.2 and 2.1.3 tackle fully parabolic chemotaxis systems in two and three spatial dimensions, respectively. These findings have been published in [33]. Additionally, in Chapter 4, we extend these results to general nonlinear boundary conditions in 2D with sub-logistic sources [31].

The second question investigates the existence of a globally bounded solution for $f(u) = au - \mu u^k$ with $k \in (1, 2)$. Despite this question being open since 2002, progress has been made. In [71], it was demonstrated that logistic sources are not optimal for preventing blow-up solutions. Instead, sub-logistic sources of the form $f(u) = au - \frac{\mu u^2}{\ln^p(u+e)}$ with $0 < p < 1$ ensure global boundedness in 2D bounded domains. Chapter 3 extends this result to investigate sub-logistic sources in preventing blow-up phenomena for nondegenerate Keller-Segel systems (Theorem 3.2.1) and degenerate Keller-Segel systems (Theorem 3.2.2). This work is published in [30]. Furthermore, Chapter 4 establishes in Theorem 4.3.1 that sub-logistic sources also ensure the global existence of solutions in two-species chemotaxis models [32]. Chapter 5 demonstrates in Theorems 5.2.1 and 5.2.2 that sub-logistic sources prevent blow-up even in superlinear signal production chemotaxis systems. Finally, blow-up prevention by sub-logistic sources for chemotaxis systems with superlinear cross-diffusion rates is provided in Theorems 5.1.1 and 5.1.2 in Chapter 5.

In addition to addressing these fundamental questions, the appendix includes detailed proofs for the Moser iteration technique, along with important inequalities and fundamental results in regularity theory. These proofs are crucial for obtaining L^∞ bounds from L^p bounds with $p > 1$ sufficiently large for solutions to various chemotaxis systems, with or without nonlinear boundary conditions.

CHAPTER 2

CHEMOTAXIS WITH LOGISTIC SOURCES UNDER NONLINEAR NEUMANN BOUNDARY CONDITION

We consider classical solutions to the chemotaxis system with logistic source $f(u) := au - \mu u^2$ under nonlinear Neumann boundary conditions $\frac{\partial u}{\partial \nu} = |u|^p$ with $p > 1$ in a smooth convex bounded domain $\Omega \subset \mathbb{R}^n$ where $n \geq 2$. This chapter aims to show that if $p < \frac{3}{2}$, and $\mu > 0$, $n = 2$, or μ is sufficiently large when $n \geq 3$, then the parabolic-elliptic chemotaxis system admits a unique positive global-in-time classical solution that is bounded in $\Omega \times (0, \infty)$. The similar result is also true if $p < \frac{3}{2}$, $n = 2$, and $\mu > 0$ or $p < \frac{7}{5}$, $n = 3$, and μ is sufficiently large for the parabolic-parabolic chemotaxis system.

2.1 Introduction

We are concerned in this chapter with solutions to the chemotaxis model as follows:

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + au - \mu u^2 & x \in \Omega,\ t \in (0, T_{\max}), \\ \tau v_t = \Delta v + \alpha u - \beta v & x \in \Omega,\ t \in (0, T_{\max}), \end{cases} \tag{KS}$$

in a smooth, convex, bounded domain $\Omega \subset \mathbb{R}^n$ where $\alpha, \beta, a, \mu > 0$, $\tau \geq 0$, and $\chi \in \mathbb{R}$. The system (KS) is complemented with the nonnegative initial conditions in $C^{2+\gamma}(\Omega)$, where $\gamma \in (0, 1)$, not identically zero:

$$u(x, 0) = u_0(x), \qquad v(x, 0) = v_0(x), \qquad x \in \Omega, \tag{2.1.1}$$

and the nonlinear Neumann boundary conditions

$$\frac{\partial u}{\partial \nu} = |u|^p, \qquad \frac{\partial v}{\partial \nu} = 0, \qquad x \in \partial\Omega,\ t \in (0, T_{\max}). \tag{2.1.2}$$

where $p > 1$ and ν is the outward normal vector.

The logistic term, $au - \mu u^2$, introduced in the evolution equation for u plays a role in describing the growth of the population. Specifically, the term au, with $a \in \mathbb{R}$, is the growth rate of population and the term $-\mu u^2$ models additional overcrowding effects. It was investigated in [59] that the quadratic degradation term $-\mu u^2$ can prevent blow-up solutions. In fact, it was proven that if

$\mu > \frac{n-2}{n}\chi\alpha$, and $\tau = 0$, then the solutions exist globally and remain bounded at all time in a convex bounded domain with smooth boundary $\Omega \subset \mathbb{R}^n$, where $n \geq 2$. This result was later improved in [22, 25, 70] that $\mu = \frac{n-2}{n}\chi\alpha$ can prevent blow-up when $\tau = 0$. In a two-dimensional space with $\tau = 1$, the system (KS) possesses a unique classical solution which is nonnegative and bounded in $\Omega \times (0, \infty)$ (see e.g. [46, 47]). These results were later improved in [71, 72] by replacing the logistic sources by sub-logistic ones such as $au - \mu\frac{u^2}{\ln^p(u+e)}$ for $p \in (0, 1)$. In a higher-dimensional space with $\tau = 1$, the similar results can be found in [65] for the parabolic-parabolic model with an additional largeness assumption of μ. In addition to the global classical solutions, the existence global weak solutions results for any arbitrary $\mu > 0$ were also obtained in [59] for the parabolic-elliptic models and in [28] for the parabolic-parabolic models in a three-dimensional system. Furthermore, interested readers are referred to [24, 27, 34, 35, 38, 57, 63, 74, 75] to study more about qualitative and quantitative works of chemotaxis systems with logistic sources.

The problem becomes more interesting and challenging if the homogeneous Neumann boundary condition is replaced by the nonlinear Neumann boundary condition. The method in this chapter to obtain global boundedness results is first to establish a L^1 estimate, then for L^{p_0} for some $p_0 > 1$, and finally apply a Moser-type iteration to obtain for L^∞. Although this approach has been widely applied in treating global boundedness problems for reaction-diffusion equations ([1, 2, 13]), or for chemotaxis systems ([65]), the main difficulties rely heavily on tedious integral estimations. Unlike the homogeneous Neumann boundary conditions, it is not even straightforward to see whether the total mass of the cell density function is globally bounded or not due to the nonlinear boundary term. In fact, most of the technical challenges in this chapter are to deal with the nonlinear boundary term. Fortunately, the Sobolev's trace inequality enables us to solve a part of a problem:

Main Question: *"What is the largest value p so that logistic damping still avoids blow-up?"* This types of question for nonlinear parabolic equations has been intensively studied in 1990s. To be more precise, if we consider $\chi = 0$, our problem is similar to the following PDE:

$$\begin{cases} U_t = \Delta U - \mu U^Q & x \in \Omega,\ t \in (0, T_{\max}), \\ \frac{\partial U}{\partial \nu} = U^P & x \in \Omega,\ t \in (0, T_{\max}), \\ U(x,0) = U_0(x) & x \in \bar{\Omega}. \end{cases} \tag{NBC}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $Q, P > 1$, $\mu > 0$ and $U_0 \in W^{1,\infty}(\Omega)$ is a nonnegative function. The study concerning the global existence was first investigated in [10], and then improved in [49] for $n \geq 2$. Particularly, it was shown that $P = \frac{Q+1}{2}$ is critical for the blow up in the following sense:

1. if $P < \frac{Q+1}{2}$ then all solutions of (NBC) exist globally and are globally bounded,
2. If $P > \frac{Q+1}{2}$ (or $P = \frac{Q+1}{2}$ and μ is sufficiently small) then there exist initial functions U_0 such that the corresponding solutions of (NBC) blow-up in $L^\infty-$norm.

In comparison to our problem, we have $Q = 2$ and $P = \frac{3}{2}$ is the critical power. Indeed, we also obtain the similar critical power $p = \frac{3}{2}$ as in Theorem 2.1.1. Notice that the local existence of positive solution was not mentioned in [10, 49, 48], and it is not clear for us to define U^P without knowing U is nonnegative, so the presence of absolute sign in (2.1.2) is necessary to obtain local positive solutions from nonnegative, not identically zero initial data.

Heuristically, the analysis diagram can be presented as follows. In case $\tau = 0$, by substituting $-\Delta v = \alpha u - \beta v$ into the first equation of (KS), we obtain

$$u_t = \Delta u + au - \chi \nabla u \cdot \nabla v + (\chi - \mu)u^2 - \chi uv.$$

If μ is sufficiently large, then solutions might be bounded globally since the nonlinear term $(\chi - \mu)u^2$ might dominate other terms including the nonlinear boundary term. In case $\tau = 1$, we cannot substitute $\Delta v = \beta v - \alpha u$ directly into the first equation of (KS); however, we still have some certain controls of v by u from the second equation of (KS) thanks to Sobolev inequality. We expect that this intuition should be true in lower spacial dimension and "weaker" nonlinear boundary terms since the critical Sobolev exponent decreases if the spacial dimension increases. Indeed, our analysis does not work for $n \geq 4$ since we do not have enough rooms to control other positive nonlinear terms by using the term $(\chi-\mu)u^2$. One can also find similar ideas on sub-logistic source preventing 2D blow-up in [71].

We summarize the main results to answer a part of the main question. Let us begin with the following theorem for the parabolic-elliptic case.

Theorem 2.1.1. *Let Ω be a bounded, convex domain with smooth boundary in $\mathbb{R}^n$ where $n \geq 2$, and $\tau = 0$. If $\mu > \frac{n-2}{n}\chi\alpha$, and $1 < p < \frac{3}{2}$ or $\mu = \frac{n-2}{n}\chi\alpha$ with $n \geq 3$ and $1 < p < 1 + \frac{1}{n}$ then the system* (KS) *with initial conditions* (2.1.1) *and boundary condition* (2.1.2) *possesses a unique positive classical solution which remains bounded in $\Omega \times (0, \infty)$.*

Remark 2.1.1. *It is an open question whether there exists a classical finite time blow-up solution if $p \geq \frac{3}{2}$.*

Remark 2.1.2. *The proof of borderline boundedness in Theorem 2.1.1 when $\mu = \frac{n-2}{n}\chi\alpha$ is adopted and modified from the arguments in [22, 25, 70]. However, applying Lemma 2.3.2 to overcome challenges in boundary integral estimations was not possible. Instead, we had to derive an alternative and improved estimation to handle the boundary term, which necessitated the condition of $p < 1 + \frac{1}{n}$.*

The next theorem is for the parabolic-parabolic system in a two-dimensional space:

Theorem 2.1.2. *Let Ω be a bounded, convex domain with smooth boundary, and $\tau = 1$, $n = 2$, $1 < p < \frac{3}{2}$, then the system* (KS) *with initial conditions* (2.1.1) *and boundary condition* (2.1.2) *possesses a unique positive classical solution which remains bounded in $\Omega \times (0, \infty)$.*

Remark 2.1.3. *This theorem is an improvement of the result in [73, chapter 12] since not only the nonlinear boundary condition takes place but the smallness assumption of initial data also is no longer necessary.*

In three-dimensional space, we prove the following theorem for the parabolic-parabolic case.

Theorem 2.1.3. *Let Ω be a bounded, convex domain with smooth boundary, and $\tau = 1$, $n = 3$, $1 < p < \frac{7}{5}$, then there exists $\mu_0 > 0$ such that for every $\mu > \mu_0$, the system* (KS) *with initial conditions* (2.1.1) *and boundary condition* (2.1.2) *possesses a unique positive classical solution which remains bounded in $\Omega \times (0, \infty)$.*

Remark 2.1.4. *Here we expect $p = \frac{7}{5}$ may not be the threshold of global boundedness and blow-up solutions, but rather the limitation of our analysis tools.*

Remark 2.1.5. *We leave the open question whether for every $n \geq 4$, there exists $p_n > 1$ such that if $1 < p < p_n$ solutions remain bounded in $\Omega \times (0, \infty)$.*

Remark 2.1.6. *In case $p = 1$, one may adopt and modify the proof of Theorem 2.1.1, 2.1.2, and 2.1.3 to obtain similar results.*

The chapter is organized as follows. The local well-possedness of solutions toward the system (KS) including the short-time existence, positivity and uniqueness are established in Section 2.2. In Section 2.3, we recall some basic inequalities and provide some essential estimates on the boundary of the solutions, which will be needed in the sequel sections. Section 2.4 is devoted to establishing the $L \log L$, and L^r bounds for solutions. In Section 2.5, we firstly establish an L^∞ estimate from an L^r estimate for r sufficiently large and then prove the main theorems.

2.2 Local well-posedness

In this section, we prove the short-time existence, uniqueness and positivity of solutions to the system (KS) under certain conditions of initial data. Although the proof just follows a basic fixed point argument, however we cannot find any suitable reference for our system. For the sake of completeness, here we will provide a proof, which is a modification of the proof of Theorem 1.1 in [16]. Let us recall a useful result for the linear model. We consider the linear second order elliptic equation of non-divergence form:

$$Lu := u_t - \sum_{i,j} a^{ij} D_{ij} u + \sum_i b^i D_i u + cu = f \text{ in } \Omega_T. \tag{2.2.1}$$

Assume that there exists $\Lambda \geq \lambda > 0$ such that

$$\lambda|\xi|^2 \leq \sum_{i,j} a^{ij}(x,t)\xi_i\xi_j \leq \Lambda|\xi|^2, \qquad (x,t) \in \Omega_T, \xi \in \mathbb{R}^n, \tag{2.2.2}$$

where $a^{ij}, b^i, c \in C^\gamma(\bar{\Omega}_T)(0 < \gamma < 1)$ and

$$\frac{1}{\lambda}\left\{\sum_{i,j} \left\|a^{ij}\right\|_{C^\gamma(\bar{\Omega}_T)} + \sum_i \left\|b^i\right\|_{C^\gamma(\bar{\Omega}_T)} + \|c\|_{C^\gamma(\bar{\Omega}_T)}\right\} \leq \Lambda_\gamma. \tag{2.2.3}$$

Theorem 2.2.1 ([37], p. 79, Theorem 4.31)**.** *Let the assumptions* (2.2.2)*,* (2.2.3) *be in force, and* $\partial\Omega \in C^{2+\gamma}(0 < \gamma < 1)$. *Let* $f \in C^\gamma(\bar{\Omega_T})$*,* $g \in C^{1+\gamma}(\bar{\Omega}_T)$ *and* $u_0 \in C^{2+\gamma}(\bar{\Omega})$ *satisfying the first order compatibility condition:*

$$\frac{\partial u_0}{\partial \nu} = g(x,0) \text{ on } \partial\Omega. \tag{2.2.4}$$

Then there exists a unique solution $u \in C^{2+\gamma}(\bar{\Omega}_T)$ *to the problem* (2.2.1) *with the Neumann boundary condition* $\frac{\partial u}{\partial \nu} = g$ *on* $\partial\Omega \times (0,T)$*. Moreover, there exists a constant* C *independent of* g *and* u_0 *such that*

$$\|u\|_{C^{2+\gamma}(\bar{\Omega}_T)} \leq C\left(\frac{1}{\lambda}\|f\|_{C^\gamma(\bar{\Omega}_T)} + \|g\|_{C^{1+\gamma}(\bar{\Omega}_T)} + \|u_0\|_{C^{2+\gamma}(\bar{\Omega})}\right). \tag{2.2.5}$$

where C *is dependent only on* $n, \gamma, \Lambda/\lambda, \Lambda_\gamma$ *and* Ω*.*

This estimate, together with Leray-Schauder fixed point argument is the main tools to prove the following theorem.

Theorem 2.2.2. *If nonnegative functions* u_0, v_0 *are in* $C^{2+\gamma}(\bar{\Omega})$ *such that*

$$\frac{\partial u_0}{\partial \nu} = |u_0|^{1+\gamma} \qquad \text{on } \partial\Omega, \tag{2.2.6}$$

where $\gamma \in (0,1)$*. Then there exists* $T > 0$ *such that problem* (KS) *admits a unique nonnegative solution* u, v *in* $C^{2+\gamma}(\bar{\Omega}_T)$*. Moreover, if* u_0, v_0 *are not identically zero in* Ω *then* u, v *are strictly positive in* $\bar{\Omega_T}$*.*

Remark 2.2.1. *The convexity assumption of domain* Ω *is not necessary in this theorem.*

Remark 2.2.2. *By substituting* $\gamma = p - 1$ *into Theorem 2.2.2, we obtain local existence and uniqueness of positive solutions in Theorem 2.1.1, 2.1.2, and 2.1.3.*

Proof. From now to the end of this proof, we will use C as a universal notation for constants different from time to time. Firstly, the short-time existence of classical solution will be proved by a fixed point argument. Let $u \in C^{1+\gamma}(\bar{\Omega}_T)$ be such that $u(x,0) = u_0(x)$ in Ω. Then, the functions u_0 and $g(x,t) = |u(x,t)|^{1+\gamma}$ satisfy condition (2.2.4), and $g \in C^{1+\gamma}(\bar{\Omega}_T)$. We assume $T < 1$, and consider the set of functions given by

$$B_T(R) := \left\{ u \in C^{1+\gamma}(\bar{\Omega}_T) \text{ such that } \|u\|_{C^{1+\gamma}(\bar{\Omega}_T)} \leq R \right\}.$$

Now we define the map

$$A : B_T(R) \longrightarrow C^{1+\gamma}(\bar{\Omega}_T)$$

where $Au := U$ is a solution of

$$\begin{cases} U_t = \Delta U - \chi \nabla \cdot (u \nabla V) + au - \mu u^2 & x \in \Omega,\ t \in (0, T_{\max}), \\ \tau V_t = \Delta V + \alpha u - \beta V & x \in \Omega,\ t \in (0, T_{\max}), \end{cases} \tag{2.2.7}$$

under Neumann boundary condition:

$$\frac{\partial U}{\partial \nu} = |u|^{1+\gamma}, \qquad \frac{\partial V}{\partial \nu} = 0, \qquad x \in \partial\Omega,\ t \in (0, T_{\max}), \tag{2.2.8}$$

and initial data $(U(x,0), V(x,0)) = (u_0(x), v_0(x))$ in Ω. We first prove that A sends bounded sets into relative compact sets of $C^{1+\gamma}(\bar{\Omega}_T)$. Indeed, the inequality (2.2.5) implies there exists $R' > 0$ independent of T such that $\|Au\|_{C^{2+\gamma}(\bar{\Omega}_T)} \leq R'$ for all u in $B_T(R)$. As bounded sets in $C^{2+\gamma}(\bar{\Omega}_T)$ are relatively compact in $C^{1+\gamma}(\bar{\Omega}_T)$. We claim that A is continuous. In fact, let $u_n \to u$ in $C^{1+\gamma}(\bar{\Omega}_T)$, we need to prove $U_n := Au_n \to U := Au$ in $C^{1+\gamma}(\bar{\Omega}_T)$. Now we can see that $U_n - U$ satisfies

$$\begin{cases} (U_n - U)_t = \Delta(U_n - U) + f_n, & x \in \Omega,\ t \in (0, T_{\max}), \\ \tau(V_n - V)_t = \Delta(V_n - V) + \alpha(u_n - u) - \beta(V_n - V) & x \in \Omega,\ t \in (0, T_{\max}), \end{cases} \tag{2.2.9}$$

where $f_n := -\chi \nabla \cdot (u_n \nabla V_n - u \nabla V) + u_n(a - \mu u_n) - u(a - \mu u)$. One can verify that f_n satisfies the assumptions of Theorem 2.2.1. Plus, the boundary condition

$$\frac{\partial(U_n - U)}{\partial \nu} = |u_n|^{1+\gamma} - |u|^{1+\gamma}, \qquad \frac{\partial(V_n - V)}{\partial \nu} = 0, \qquad x \in \partial\Omega,\ t \in (0, T_{\max}). \tag{2.2.10}$$

We claim that $V_n \to V$ in $C^{2+\gamma}(\bar{\Omega_T})$ for $\tau \geq 0$. Indeed, when $\tau > 0$, we make use of (2.2.5) and when $\tau = 0$, we apply Schauder type estimate for elliptic equation to obtain that $V_n \to V$ in $C^{2+\gamma}(\bar{\Omega_T})$. This leads to $f_n \to 0$ in $C^{\gamma}(\bar{\Omega}_T)$, combine with inequality (2.2.5) entail that $U_n \to U$ in $C^{2+\gamma}(\bar{\Omega}_T)$. In order to apply the Leray-Schauder fixed point theorem we just have to prove that if T is sufficiently small, and $R \geq 2(1+d(\Omega)^{1-\gamma}) \|u_0\|_{C^{2+\gamma}(\bar{\Omega})}$, then $A(B_T(R)) \subset B_T(R)$. Indeed,

$$|Au(x,t)| \leq |Au(x,0)| + t \|D_t Au\|_{C^0(\bar{\Omega})} \leq \|u_0\|_{C^0(\bar{\Omega})} + TR'$$
$$\frac{|Au(x,t) - Au(x,s)|}{|t-s|^{\frac{1+\gamma}{2}}} \leq \|D_t Au\|_{C^0(\bar{\Omega}_T)} |t-s|^{\frac{1-\gamma}{2}} \leq R'T^{\frac{1-\gamma}{2}},$$

and,

$$\begin{aligned}\frac{|D_x Au(x,t) - D_x Au(y,s)|}{|x-y|^{\gamma} + |t-s|^{\frac{\gamma}{2}}} &\leq |t-s|^{1-\frac{\gamma}{2}} \left\|D_x^2 Au\right\|_{C^0(\bar{\Omega}_T)} + |x-y|^{1-\gamma}(s^{\frac{\gamma}{2}} + \left\|D^2 u_0\right\|_{C^0(\bar{\Omega})}) \\ &\leq T^{1-\frac{\gamma}{2}} R' + d(\Omega)^{1-\gamma} T^{\frac{\gamma}{2}} R' + d(\Omega)^{1-\gamma} \left\|D^2 u_0\right\|_{C^0(\bar{\Omega})}.\end{aligned}$$

These above estimates imply that

$$\|Au\|_{C^{1+\gamma}(\bar{\Omega}_T)} \leq \frac{R}{2} + R'T + R'T^{\frac{1-\gamma}{2}} + T^{1-\frac{\gamma}{2}} R' + d(\Omega)^{1-\gamma} T^{\frac{\gamma}{2}} R'.$$

Since R' is independent of T for all $T < 1$, we can choose T sufficiently small as to have

$$R'T + R'T^{\frac{1-\gamma}{2}} + T^{1-\frac{\gamma}{2}} R' + d(\Omega)^{1-\gamma} T^{\frac{\gamma}{2}} R' \leq \frac{R}{2}.$$

This further implies that

$$\|Au\|_{C^{1+\gamma}(\bar{\Omega}_T)} \leq R \text{ for all } u \in B_T(R).$$

Thus A has a fixed point in $B_T(R)$. Now if u is a fixed point of A, $u \in C^{2+\gamma}(\bar{\Omega}_T)$ and it is a solution of (KS).

Secondly, the nonnegativity of solutions will be proved by the truncation method: Letting

$$\phi := \min\{u, 0\}$$

and $\psi(t) := \frac{1}{2}\int_\Omega \phi^2 \, dx$, we see that ψ is continuously differentiable with the derivative

$$\begin{aligned}\psi'(t) &= -\int_\Omega |\nabla\phi|^2 + a\int_\Omega \phi^2 + \int_{\partial\Omega} \phi|u|^{1+\gamma} \, dS + \chi\int_\Omega \phi\nabla\phi\cdot\nabla v - \mu\int_\Omega \phi^3 \\ &\leq -\int_\Omega |\nabla\phi|^2 + a\int_\Omega \phi^2 + \chi\int_\Omega \phi\nabla\phi\cdot\nabla v - \mu\int_\Omega \phi^3. \qquad (2.2.11)\end{aligned}$$

We make use of Young's inequality combined with the global boundedness of $|\nabla v|$ in $\bar{\Omega}_T$ to obtain

$$\chi \int_\Omega \phi \nabla \phi \cdot \nabla v \leq \epsilon \int_\Omega |\nabla \phi|^2 + C \int_\Omega \phi^2, \tag{2.2.12}$$

for some $C > 0$. We also have $-\mu \int_\Omega \phi^3 \leq C \int_\Omega \phi^2$, where $C = \mu \sup_{\Omega_T} |u(x,t)|$. This together with (2.2.11), (2.2.12) implies that $\psi'(t) \leq C\psi(t)$ for all $0 < t < T$. By Gronwall's inequality and the initial condition $\psi(0) = 0$, we imply that $\psi \equiv 0$ or $u \geq 0$.

Thirdly, we will prove that if $u_0 \not\equiv 0$ then u is strictly positive in $\bar{\Omega}_T$ by a contradiction proof. Suppose that there exists $(x_0, t_0) \in \bar{\Omega}_T$ such that $\min_{\bar{\Omega}_T} u(x,t) = u(x_0, t_0) = 0$. By the strong parabolic maximum principle, we obtain $(x_0, t_0) \in \partial\Omega \times (0, T)$. However, it is a contradiction due to Hopf's lemma:

$$0 > \frac{\partial u}{\partial \nu}(x_0, t_0) = |u(x_0, t_0)|^{1+\gamma} = 0.$$

Thus, $u > 0$ and by similar arguments we also have $v > 0$.

Finally, the uniqueness of classical solutions will be proved by a contradiction proof. Assuming $(u_1, v_1$ and (u_2, v_2) are two positive classical solutions of the system (KS). Let $U := u_1 - u_2$, $V := v_1 - v_2$, then (U, V) is a solution of the following system:

$$\begin{cases} U_t = \Delta U + F, & x \in \Omega,\ t \in (0, T_{\max}), \\ \tau V_t = \Delta V + \gamma U - \beta V & x \in \Omega,\ t \in (0, T_{\max}), \end{cases} \tag{2.2.13}$$

where $F := -\chi \nabla(u_1 \nabla v_1 - u_2 \nabla v_2) + f(u_1) - f(u_2)$, and the boundary condition

$$\frac{\partial U}{\partial \nu} = |u_1|^{1+\gamma} - |u_2|^{1+\gamma}, \qquad \frac{\partial V}{\partial \nu} = 0, \qquad x \in \partial\Omega,\ t \in (0, T_{\max}). \tag{2.2.14}$$

By mean value theorem, there exists $z(x,t)$ between $u_1(x,t)$ and $u_2(x,t)$ such that

$$u_1(x,t) - u_2(x,t) = (u_1(x,t) - u_2(x,t)) f'(z(x,t)).$$

Multiplying the first equation of (2.2.13) by U implies

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} \int_\Omega U^2\, dx = &- \int_\Omega |\nabla U|^2 + \int_{\partial\Omega} U(|u_1|^{1+\gamma} - |u_2|^{1+\gamma})\, dS \\ &+ \chi \int_\Omega (u_1 \nabla v_1 - u_2 \nabla v_2) \cdot \nabla U + \int_\Omega U^2 f'(z). \end{aligned} \tag{2.2.15}$$

We make use of the global boundedness property of u_1, u_2 in $\bar{\Omega}_T$, thereafter apply Sobolev's trace theorem, and finally Young's inequality to have

$$\int_{\partial\Omega} U(|u_1|^{1+\gamma} - |u_2|^{1+\gamma})\, dS \leq C \int_{\partial\Omega} U^2\, dS \leq \epsilon \int_\Omega |\nabla U|^2 + C(\epsilon) \int_\Omega U^2. \tag{2.2.16}$$

Since,

$$u_1 \nabla v_1 - u_2 \nabla v_2 = U \nabla v_1 + u_2 \nabla V,$$

we have

$$\begin{aligned} \chi \int_\Omega \nabla U \cdot (u_1 \nabla v_1 - u_2 \nabla v_2) &\leq C \int_\Omega U|\nabla U| + |\nabla U||\nabla V| \\ &\leq \epsilon \int_\Omega |\nabla U|^2 + C \int_\Omega U^2 + C \int_\Omega |\nabla V|^2. \end{aligned} \tag{2.2.17}$$

We also have $\int_\Omega U^2 f'(z) \leq C \int_\Omega U^2$ where $C = \sup_{\min\{u_1,u_2\} \leq z \leq \max\{u_1,u_2\}} |f'(z)|$. Multiplying the second equation of (2.2.13) by V, and applying Young's inequality, we obtain

$$\frac{d}{dt} \int_\Omega V^2 + \int_\Omega |\nabla V|^2 \leq C \int_\Omega U^2. \tag{2.2.18}$$

From (2.2.15) to (2.2.18), we obtain

$$\frac{d}{dt} \left\{ \int_\Omega U^2 + \int_\Omega V^2 \right\} \leq C \left\{ \int_\Omega U^2 + \int_\Omega V^2 \right\}. \tag{2.2.19}$$

The initial conditions and Gronwall's inequality imply that $U \equiv V \equiv 0$, and thus there is a unique solution to the system (KS). $\square$

2.3 Preliminaries

The next lemma giving an useful estimate will later be applied in Section 2.4. Interested readers are referred to [56, 72] for more details about the proof.

Lemma 2.3.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary, and let $p > 1$ and $1 \leq r < p$. Then there exists $C > 0$ such that for each $\eta > 0$, one can pick $C(\eta) > 0$ such that*

$$\|u\|^p_{L^p(\Omega)} \leq \eta \, \|\nabla u\|^{p-r}_{L^2(\Omega)} \, \|u \ln |u|\|^r_{L^r(\Omega)} + C \, \|u\|^p_{L^r(\Omega)} + C(\eta) \tag{2.3.1}$$

holds for all $u \in W^{1,2}(\Omega)$.

The following lemma providing estimates on the boundary will be useful in Section 2.4.

Lemma 2.3.2. *If* $r \geq 1$, $p \in (1, \frac{3}{2})$, *and* $g \in C^1(\bar{\Omega})$, *then for every* $\epsilon > 0$, *there exists a constant* $C = C(\epsilon, \Omega, p, r)$ *such that*

$$\int_{\partial\Omega} |g|^{p+2r-1} \leq \epsilon \int_{\Omega} |g|^{2r+1} + \epsilon \int_{\Omega} |\nabla g^r|^2 + C. \tag{2.3.2}$$

Proof. Let $\phi := |g|^r$, we have $\phi^{2+\frac{p-1}{r}} \in W^{1,1}(\Omega)$. Trace theorem, $W^{1,1}(\Omega) \to L^1(\partial\Omega)$, yields

$$\begin{aligned} \int_{\partial\Omega} \phi^{2+\frac{p-1}{r}} &\leq c_1 \int_{\Omega} \phi^{2+\frac{p-1}{r}} + (2 + \frac{p-1}{r}) c_1 \int_{\Omega} \phi^{1+\frac{p-1}{r}} |\nabla \phi| \\ &\leq c_1 \int_{\Omega} \phi^{2+\frac{p-1}{r}} + 3c_1 \int_{\Omega} \phi^{1+\frac{p-1}{r}} |\nabla \phi| \end{aligned} \tag{2.3.3}$$

where $c_1 = c_1(n, \Omega) > 0$. By Young's inequality, the following holds for all $\epsilon > 0$

$$3c_1 \int_{\Omega} \phi^{1+\frac{p-1}{r}} |\nabla \phi| \leq \epsilon \int_{\Omega} |\nabla \phi|^2 + \frac{c_1^2}{\epsilon} \int_{\Omega} \phi^{2+\frac{2(p-1)}{r}}. \tag{2.3.4}$$

We apply Young's inequality again to obtain a further estimate

$$c_1 \int_{\Omega} \phi^{2+\frac{p-1}{r}} + \frac{c_1^2}{\epsilon} \int_{\Omega} \phi^{2+\frac{2(p-1)}{r}} \leq \epsilon \int_{\Omega} \phi^{2+\frac{1}{r}} + c_2. \tag{2.3.5}$$

where c_2 depending on $\epsilon, p, r, n, \Omega$. We complete the proof of (2.3.2) by collecting (2.3.3),(2.3.4) and (2.3.5) together. □

The following lemma is an essential estimate to obtain L^2 bounds from $L \ln L$ bounds.

Lemma 2.3.3. *If* $p \in (1, \frac{7}{5})$, $n = 3$, *and* (u, v) *are in* $C^1(\bar{\Omega} \times (0, T_{max}))$ *and*

$$\int_{\Omega} |\nabla v(\cdot, t)|^2 \leq A \tag{2.3.6}$$

holds for all $t \in (0, T_{max})$, *then for every* $\epsilon > 0$, *there exists a constant* $C = C(\epsilon, \Omega, p, A)$ *such that*

$$\int_{\partial\Omega} u^p |\nabla v|^2 \leq \epsilon \int_{\Omega} \left(u^3 + |\nabla u|^2 + u^2 |\nabla v|^2 + \left|\nabla |\nabla v|^2\right|^2 \right) + C. \tag{2.3.7}$$

holds for all $t \in (0, T_{max})$.

Proof. By trace theorem $W^{1,1}(\Omega) \to L^1(\partial\Omega)$,

$$\int_{\partial\Omega} |u|^p |\nabla v|^2 \le c_1 \int_\Omega |u|^p |\nabla v|^2 + c_1 \int_\Omega |u|^p \left|\nabla |\nabla v|^2\right| + c_1 p \int_\Omega |u|^{p-1} |\nabla u| |\nabla v|^2 \tag{2.3.8}$$

where $c_1 = c_1(\Omega) > 0$. Apply Young's inequality yields

$$\begin{aligned} c_1 \int_\Omega |u|^p |\nabla v|^2 &\le \frac{\epsilon}{2} \int_\Omega u^2 |\nabla v|^2 + \frac{2-p}{2} \left(\frac{p}{\epsilon c_1}\right)^{\frac{2-p}{p}} \int_\Omega |\nabla v|^2 \\ &\le \frac{\epsilon}{2} \int_\Omega u^2 |\nabla v|^2 + c_2 \end{aligned} \tag{2.3.9}$$

where $c_2 := \frac{2-p}{2} A \left(\frac{p}{\epsilon c_1}\right)^{\frac{2-p}{p}}$. Note that $2p < 3$, we apply Young's inequality to obtain

$$\begin{aligned} c_1 \int_\Omega |u|^p \left|\nabla |\nabla v|^2\right| &\le \frac{\epsilon}{2} \int_\Omega \left|\nabla |\nabla v|^2\right|^2 + \frac{1}{2\epsilon} \int_\Omega u^{2p} \\ &\le \frac{\epsilon}{2} \int_\Omega \left|\nabla |\nabla v|^2\right|^2 + \epsilon \int_\Omega u^3 + c_3 \end{aligned} \tag{2.3.10}$$

where $c_3 = c_3(\epsilon, \Omega, p)$. By Young's inequality,

$$\begin{aligned} c_1 p \int_\Omega |u|^{p-1} |\nabla u| |\nabla v|^2 &\le \frac{\epsilon}{2} \int_\Omega u^2 |\nabla v|^2 + c_4 \int_\Omega |\nabla u|^{\frac{2}{3-p}} |\nabla v|^2 \\ &\le \frac{\epsilon}{2} \int_\Omega u^2 |\nabla v|^2 + \epsilon \int_\Omega |\nabla u|^2 + c_5 \int_\Omega |\nabla v|^{\frac{6-2p}{2-p}} \end{aligned} \tag{2.3.11}$$

where c_4, c_5 are positive and dependent on ϵ, Ω, p. Here, we use the condition $1 < p < \frac{7}{5}$ to obtain $\frac{3-p}{2-p} < \frac{8}{3}$. By Young's inequality,

$$c_5 \int_\Omega |\nabla v|^{\frac{6-2p}{2-p}} \le \eta \int_\Omega |\nabla v|^{\frac{16}{3}} + c_6 \tag{2.3.12}$$

where $c_6 = c_6(\eta, \epsilon, p, \Omega)$. In light of Gagliardo-Nirenberg inequality,

$$\begin{aligned} \left\| |\nabla v|^2 \right\|_{L^{\frac{8}{3}}(\Omega)} &\le c_{GN} \left\| \nabla |\nabla v|^2 \right\|_{L^2(\Omega)}^{\frac{3}{4}} \left\| |\nabla v|^2 \right\|_{L^1(\Omega)}^{\frac{1}{4}} + c_{GN} \left\| |\nabla v|^2 \right\|_{L^1(\Omega)} \\ &\le c_{GN} A^{\frac{1}{4}} \left\| \nabla |\nabla v|^2 \right\|_{L^2(\Omega)}^{\frac{3}{4}} + c_{GN} A. \end{aligned} \tag{2.3.13}$$

Hence

$$\begin{aligned} \eta \int_\Omega |\nabla v|^{\frac{16}{3}} &\le \eta \left(c_{GN} A^{\frac{1}{4}} \left\| \nabla |\nabla v|^2 \right\|_{L^2(\Omega)}^{\frac{3}{4}} + c_{GN} A \right)^{\frac{8}{3}} \\ &\le 2^{5/3} (c_{GN} A^{\frac{1}{4}})^{\frac{8}{3}} \eta \int_\Omega \left| \nabla |\nabla v|^2 \right|^2 + 2^{5/3} (c_{GN} A)^{\frac{8}{3}} \eta. \end{aligned} \tag{2.3.14}$$

Choosing η such that $2^{5/3}(c_{GN}A^{\frac{1}{4}})^{\frac{8}{3}}\eta = \epsilon$, and plugging into (2.3.14), (2.3.12), and (2.3.11) respectively, we obtain

$$c_1 p \int_\Omega |u|^{p-1}|\nabla u||\nabla v|^2 \le \epsilon \int_\Omega \left|\nabla|\nabla v|^2\right|^2 + \frac{\epsilon}{2}\int_\Omega u^2|\nabla v|^2 + \epsilon\int_\Omega |\nabla u|^2 + c_7 \tag{2.3.15}$$

where $c_7 = c_6 + 2^{5/3}(c_{GN}A)^{\frac{8}{3}}\eta$. We finally complete the proof of (2.3.7) by substituting (2.3.9), (2.3.10) and (2.3.15) into (2.3.8). □

2.4 A priori estimates

Let us first give a priori estimate for the parabolic-elliptic system,

Lemma 2.4.1. *If $\mu > 0$ and $p \in (1, \frac{3}{2})$, for all $r \in (1, \frac{\chi\alpha}{(\chi\alpha-\mu)_+})$ then there exists $c = c(r, \|u_0\|_{L^r(\Omega)}) > 0$ such that*

$$\|u(\cdot,t)\|_{L^r(\Omega)} \le c, \qquad \forall t \in (0, T_{max}). \tag{2.4.1}$$

Proof. Multiplying the first equation in the system (KS) by u^{2r-1} yields

$$\begin{aligned}
\frac{1}{2r}\frac{d}{dt}\int_\Omega u^{2r} &= \int_\Omega u^{2r-1}u_t \\
&= \int_\Omega u^{2r-1}\left[\Delta u - \chi\nabla(u\nabla v) + f(u)\right] \\
&= -\frac{2r-1}{r^2}\int_\Omega |\nabla u^r|^2 - \chi\frac{2r-1}{2r}\int_\Omega u^{2r}\Delta v + \int_\Omega f(u)u^{2r-1} + \int_{\partial\Omega} u^{2r+p-1}\,dS \\
&= -\frac{2r-1}{r^2}\int_\Omega |\nabla u^r|^2 + \frac{2r-1}{2r}\int_\Omega u^{2r}(\chi\alpha u - \chi\beta v) \\
&\quad + \int_{\partial\Omega} u^{2r+p-1}\,dS + a\int_\Omega u^{2r} - \mu\int_\Omega u^{2r+1}.
\end{aligned} \tag{2.4.2}$$

Since $v \ge 0$, we have

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u^{2r} \le& -\frac{2(2r-1)}{r}\int_\Omega |\nabla u^r|^2 - [2r\mu - \chi\alpha(2r-1)]\int_\Omega u^{2r+1} \\
&+ 2r\int_{\partial\Omega} u^{2r+p-1}\,dS + 2ra\int_\Omega u^{2r}.
\end{aligned} \tag{2.4.3}$$

By Lemma 2.3.2, we obtain

$$2r\int_{\partial\Omega} u^{2r+p-1}\,dS \le 2r\epsilon\int_\Omega |\nabla u^r|^2 + 2r\epsilon\int_\Omega u^{2r+1} + c_2. \tag{2.4.4}$$

We make use of Young's inequality to obtain

$$(2ra+1)\int_\Omega u^{2r} \le \epsilon \int_\Omega u^{2r+1} + c_3. \tag{2.4.5}$$

Collecting (2.4.3), (2.4.4) and (2.4.5), we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega u^{2r} + \int_\Omega u^{2r} &\le \left[2r\epsilon - \frac{2(2r-1)}{r}\right]\int_\Omega |\nabla u^r|^2 \\ &\quad - \left[2r\mu - \chi\alpha(2r-1) - 2\epsilon\right]\int_\Omega u^{2r+1} + c_4. \end{aligned} \tag{2.4.6}$$

If $\frac{\chi\alpha}{(\chi\alpha-\mu)_+} > 2r > 1$, then selecting $\epsilon = \min\left\{\frac{2r-1}{r^2}, \frac{2r\mu-\chi\alpha(2r-1)}{2}\right\}$ and plugging into (2.4.6), we deduce

$$\frac{d}{dt}\int_\Omega u^{2r} + \int_\Omega u^{2r} \le c_2 \tag{2.4.7}$$

This yields (2.4.1), hence the proof is complete. □

In the parabolic-parabolic case $\tau = 1$, the following lemma gives us a priori bounds for solution of (KS) with initial data (2.1.1) and the boundary condition (2.1.2).

Lemma 2.4.2. *If $1 < p < \frac{3}{2}$, and (u, v) is a classical solution to* (KS) *with initial data* (2.1.1) *and the boundary condition* (2.1.2) *without the convexity assumption of Ω, and $n \ge 2$ then there exists a positive constant C such that*

$$\int_\Omega (u(\cdot,t)+1)\ln(u(\cdot,t)+1) + \int_\Omega |\nabla v(\cdot,t)|^2 \le C \tag{2.4.8}$$

for all $t \in (0, T_{max})$.

Proof. Let denote $y(t) := \int_\Omega (u(\cdot,t)+1)\ln(u(\cdot,t)+1) + \int_\Omega |\nabla v(\cdot,t)|^2$, we have

$$\begin{aligned} y'(t) &= \int_\Omega \left[\Delta u - \chi\nabla\cdot(u\nabla v) + f(u)\right]\left[\ln(u+1)+1\right] \\ &\quad + 2\int_\Omega \nabla v \cdot \nabla(\Delta v + \alpha u - \beta v) \\ &:= I_1 + I_2. \end{aligned} \tag{2.4.9}$$

By integration by parts, I_1 can be rewritten as

$$I_1 = -\int_\Omega \frac{|\nabla u|^2}{u+1} + \chi \int_\Omega \frac{u}{u+1}\nabla u \cdot \nabla v + a\int_\Omega u\left[\ln(u+1)+1\right] \\ - \mu \int_\Omega u^2\left[\ln(u+1)+1\right] + \int_{\partial\Omega} u^p \left[\ln(u+1)+1\right]\, dS \tag{2.4.10}$$

By integration by parts, Cauchy-Schwarz inequality and elementary inequality $\ln(u+1) \le u$, we have

$$\chi \int_\Omega \frac{u}{u+1}\nabla u \cdot \nabla v = \chi \int_\Omega \nabla\left(u - \ln(u+1)\right)\cdot \nabla v \\ = -\chi\int_\Omega \left(u - \ln(u+1)\right)\Delta v \le \frac{1}{2}\int_\Omega (\Delta v)^2 + \chi^2 \int_\Omega u^2. \tag{2.4.11}$$

One can verify that there exists $c_1(\mu, a) > 0$ satisfying

$$u\left[\ln(u+1)+1\right] \le \frac{\mu}{4a}u^2\left[\ln(u+1)+1\right] + c_1,$$

hence,

$$a\int_\Omega u\left[\ln(u+1)+1\right] \le \frac{\mu}{4}\int_\Omega u^2\left[\ln(u+1)+1\right] + c_1. \tag{2.4.12}$$

In light of Sobolev's trace theorem, $W^{1,1}(\Omega) \hookrightarrow L^1(\partial\Omega)$, there exists $c_2(\Omega) > 0$ such that

$$\int_{\partial\Omega} u^p\left[\ln(u+1)+1\right]\, dS \le c_2\int_\Omega u^p\left[\ln(u+1)+1\right] + pc_2\int_\Omega u^{p-1}|\nabla u|\left[\ln(u+1)+1\right] \\ + c_2\int_\Omega \frac{u^p}{u+1}|\nabla u|\left[\ln(u+1)+1\right]. \tag{2.4.13}$$

By Young's inequality, we have

$$pc_2\int_\Omega u^{p-1}|\nabla u|\left[\ln(u+1)+1\right] \le \frac{1}{4}\int_\Omega \frac{|\nabla u|^2}{u+1} + pc_2\int_\Omega u^{2p-2}(u+1)\left[\ln(u+1)+1\right]^2, \tag{2.4.14}$$

and

$$c_2\int_\Omega \frac{u^p}{u+1}|\nabla u|\left[\ln(u+1)+1\right] \le \frac{1}{4}\int_\Omega \frac{|\nabla u|^2}{u+1} + c_2^2\int_\Omega \frac{u^{2p}}{u+1}\left[\ln(u+1)+1\right]^2. \tag{2.4.15}$$

By the similar argument as in (2.4.12), there exists $c_3(p,\Omega,\mu)>0$ such that

$$c_2\int_\Omega u^p\left[\ln(u+1)+1\right]+pc_2\int_\Omega u^{2p-2}(u+1)\left[\ln(u+1)+1\right]^2 \\ +c_2^2\int_\Omega \frac{u^{2p}}{u+1}\left[\ln(u+1)+1\right]^2\le \frac{\mu}{4}\int_\Omega u^2\left[\ln(u+1)+1\right]+c_3. \tag{2.4.16}$$

From (2.4.13) to (2.4.16), we obtain

$$\int_{\partial\Omega} u^p\left[\ln(u+1)+1\right]\,dS\le \frac{1}{2}\int_\Omega \frac{|\nabla u|^2}{u+1}+\frac{\mu}{4}\int_\Omega u^2\left[\ln(u+1)+1\right]+c_3. \tag{2.4.17}$$

Now, we handle I_2 as follows:

$$I_2=-2\int_\Omega(\Delta v)^2-2\beta\int_\Omega|\nabla v|^2+2\alpha\int_\Omega\nabla u\cdot\nabla v. \tag{2.4.18}$$

By integration by part and Young's inequality, we have

$$2\alpha\int_\Omega\nabla u\cdot\nabla v\le\frac{1}{2}\int_\Omega(\Delta v)^2+2\alpha^2\int_\Omega u^2. \tag{2.4.19}$$

One can verify that there exists $c_4(\alpha,\beta,\chi,\Omega)>0$ such that

$$(\chi^2+2\alpha^2)\int_\Omega u^2+2\beta\int_\Omega(u+1)\ln(u+1)\le\frac{\mu}{4}\int_\Omega u^2\left[\ln(u+1)+1\right]+c_4. \tag{2.4.20}$$

Collecting (2.4.10), (2.4.12), (2.4.17) and from (2.4.18) to (2.4.20), we obtain

$$y'(t)+2\beta y(t)\le-\frac{1}{2}\int_\Omega\frac{|\nabla u|^2}{u+1}-\frac{\mu}{4}\int_\Omega u^2\left[\ln(u+1)+1\right]+c_5\le c_5, \qquad \forall t\in(0,T_{\max}), \tag{2.4.21}$$

where $c_5=2\beta$ and $c_5:=c_1+c_3+c_4$. This, together with the Gronwall's inequality, yields

$$y(t)\le e^{-2\beta t}y(0)+\frac{c_5}{2\beta}(1-e^{-2\beta t})\le C$$

where $C:=\max\left\{y(0),\frac{c_5}{2\beta}\right\}$, and the proof of (2.4.8) is complete. □

The following lemma gives an L^2-bound in two-dimensional space for the parabolic-parabolic system.

Lemma 2.4.3. *If* $\tau = 1$, $n = 2$, $1 < p < \frac{3}{2}$, *and* (u, v) *is a classical solution to* (KS) *with initial data* (2.1.1) *and the boundary condition then there exists a positive constant* C *such that*

$$\int_\Omega u^2(\cdot, t) + \int_\Omega |\nabla v(\cdot, t)|^4 \leq C \tag{2.4.22}$$

for all $t \in (0, T_{max})$.

Proof. Let denote

$$\phi(t) := \frac{1}{2}\int_\Omega u^2 + \frac{1}{4}\int_\Omega |\nabla v|^4,$$

we have

$$\begin{aligned}\phi'(t) &= \int_\Omega u\left[\Delta u - \chi\nabla\cdot(u\nabla v) + f(u)\right]\\ &\quad + \int_\Omega |\nabla v|^2\nabla v\cdot\nabla\left(\Delta v + \alpha u - \beta v\right)\\ &:= J_1 + J_2.\end{aligned} \tag{2.4.23}$$

By integration by parts, we obtain

$$J_1 = -\int_\Omega |\nabla u|^2 + \chi\int_\Omega u\nabla u\cdot\nabla v + a\int_\Omega u^2 - \mu\int_\Omega u^3 + \int_{\partial\Omega} u^{p+1}\,dS. \tag{2.4.24}$$

By Young's inequality, we have

$$\chi\int_\Omega u\nabla u\cdot\nabla v + a\int_\Omega u^2 \leq \frac{1}{2}\int_\Omega |\nabla u|^2 + \chi^2\int_\Omega u^2|\nabla v|^2. \tag{2.4.25}$$

In light of Sobolev's trace theorem, $W^{1,1}(\Omega) \hookrightarrow L^1(\partial\Omega)$, there exists $c_1 := c_1(\Omega) > 0$ such that

$$\int_{\partial\Omega} u^{p+1}\,dS \leq c_1\int_\Omega u^{p+1} + c_1(p+1)\int_\Omega u^p|\nabla u|. \tag{2.4.26}$$

Since $1 < p < \frac{3}{2}$, we apply Young's inequality to obtain

$$\begin{aligned}\int_{\partial\Omega} u^{p+1}\,dS &\leq \frac{\mu}{4}\int_\Omega u^3 + \frac{1}{4}\int_\Omega |\nabla u|^2 + c_1(p+1)^2\int_\Omega u^{2p} + c_2\\ &\leq \frac{1}{4}\int_\Omega |\nabla u|^2 + \frac{\mu}{2}\int_\Omega u^3 + c_3,\end{aligned} \tag{2.4.27}$$

where $c_2, c_3 > 0$ depending only on p, μ, Ω. To deal with J_2, we make use of the following pointwise identity

$$\nabla v \cdot \nabla \Delta v = \frac{1}{2}\Delta(|\nabla v|^2) - |D^2 v|^2$$

to obtain

$$\begin{aligned} J_2 = &-\frac{1}{2}\int_\Omega |\nabla|\nabla v|^2|^2 - \int_\Omega |\nabla v|^2|D^2 v|^2 \\ &+ \alpha \int_\Omega |\nabla v|^2 \nabla v \cdot \nabla u \\ &- \beta \int_\Omega |\nabla v|^4 + \frac{1}{2}\int_{\partial\Omega} \frac{\partial |\nabla v|^2}{\partial \nu}|\nabla v|^2. \end{aligned} \tag{2.4.28}$$

Applying Lemma B.0.6 and the pointwise inequality $(\Delta v)^2 \leq 2|D^2 v|^2$ to (2.4.28), we deduce

$$\begin{aligned} J_2 \leq &-\frac{1}{2}\int_\Omega |\nabla|\nabla v|^2|^2 - \beta \int_\Omega |\nabla v|^4 \\ &- \frac{1}{2}\int_\Omega |\nabla v|^2|\Delta v|^2 + \alpha \int_\Omega |\nabla v|^2 \nabla v \cdot \nabla u. \end{aligned} \tag{2.4.29}$$

By integral by parts and Young's inequality, we obtain

$$\begin{aligned} \alpha \int_\Omega |\nabla v|^2 \nabla v \cdot \nabla u &= -\alpha \int_\Omega u \nabla |\nabla v|^2 \cdot \nabla v - \alpha \int_\Omega u |\nabla v|^2 \Delta v \\ &\leq \frac{1}{4}\int_\Omega |\nabla|\nabla v|^2|^2 + \frac{1}{4}\int_\Omega |\nabla v|^2|\Delta v|^2 + 2\alpha^2 \int_\Omega u^2 |\nabla v|^2 \end{aligned} \tag{2.4.30}$$

Collecting from (2.4.23) to (2.4.30) yields

$$\phi' + 4\beta\phi \leq -\frac{1}{4}\int_\Omega |\nabla u|^2 - \frac{1}{2}\int_\Omega |\nabla|\nabla v|^2|^2 - \frac{\mu}{2}\int_\Omega u^3 + c_4 \int_\Omega u^2|\nabla v|^2 + c_5 \int_\Omega u^2 + c_3 \tag{2.4.31}$$

where c_3, c_4, c_5 are positive constants depending on α, β, χ, a. By Young's inequality

$$c_4 \int_\Omega u^2 |\nabla v|^2 \leq c_4 \epsilon \int_\Omega |\nabla v|^6 + \frac{c_4}{\sqrt{\epsilon}}\int_\Omega u^3 \tag{2.4.32}$$

By Gagliardo-Nirenberg inequality for $n = 2$ and (2.4.8), there exists $c_{GN} > 0$ such that

$$\begin{aligned} \int_\Omega |\nabla v|^6 &\leq c_{GN}\left(\int_\Omega |\nabla|\nabla v|^2|^2\right)\left(\int_\Omega |\nabla v|^2\right) + c_{GN}\left(\int_\Omega |\nabla v|^2\right)^3 \\ &\leq c_6 \left(\int_\Omega |\nabla|\nabla v|^2|^2\right) + c_7, \end{aligned} \tag{2.4.33}$$

where c_6, c_7 are positive constants depending on c_{GN} and $\sup_{t\in(0,T_{\max})}\int_\Omega |\nabla v|^2$. We make use of (2.3.1) for $n = 2$ and (2.4.8) to obtain

$$\begin{aligned}\int_\Omega u^3 &\le \epsilon\left(\int_\Omega |\nabla u|^2\right)\left(\int_\Omega u|\ln u|\right) + C\left(\int_\Omega u\right)^3 + c(\epsilon)\\ &\le c_8\epsilon\int_\Omega |\nabla u|^2 + c_9,\end{aligned}\tag{2.4.34}$$

where $c_8 := \sup_{t\in(0,T_{\max})}\int_\Omega u|\ln u|$ and $c_9 > 0$ depending on ϵ and $\sup_{t\in(0,T_{\max})}\int_\Omega u$. In light of Young's inequality

$$c_5\int_\Omega u^2 \le \frac{\mu}{4}\int_\Omega u^3 + c_{10}.\tag{2.4.35}$$

where $c_{10} > 0$ depending on $c_5, \mu, |\Omega|$. Combining from (2.4.31) to (2.4.35), we have

$$\phi' + 4\beta\phi \le \left(c_4c_8\sqrt{\epsilon} - \frac{1}{4}\right)\int_\Omega |\nabla u|^2 + \left(c_4c_6\epsilon - \frac{1}{4}\right)\int_\Omega |\nabla|\nabla v|^2|^2 + c_{11},\tag{2.4.36}$$

where $c_{11} > 0$ depending on ϵ. Choosing ϵ sufficiently small and substituting into (2.4.36), we obtain

$$\phi' + 4\beta\phi \le c_{11}.\tag{2.4.37}$$

This, together with Gronwall's inequality yields $\phi(t) \le C := \max\left\{\phi(0), \frac{c_{11}}{4\beta}\right\}$ for all $t \in (0, T_{\max})$, and the proof of Lemma 2.4.3 is complete. □

The next lemma is the key step in the proof of the parabolic-parabolic system in three-dimensional space.

Lemma 2.4.4. *Let (u, v) be a classical solution to* (KS) *in a convex bounded domain Ω with smooth boundary. If $\tau = 1$, $n = 3$, $1 < p < \frac{7}{5}$ and μ is sufficiently large, then there exists a positive constant C such that*

$$\int_\Omega u^2(\cdot,t) + \int_\Omega |\nabla v(\cdot,t)|^4 \le C\tag{2.4.38}$$

for all $t \in (0, T_{max})$.

Remark 2.4.1. *When we look at the proof of Theorem 2.1.2 carefully, $n = 2$ is utilized to estimate $\int_\Omega u^2|\nabla v|^2$. For $n = 3$, in order to eliminate this term we borrow the idea as in [53, 69] by* *introducing an extra term $\int_\Omega u|\nabla v|^2$, in the function ϕ for Theorem 2.1.2. Note that the term $u_t|\nabla v|^2$ will introduce $-\mu u^2|\nabla v|^2$, and a sufficiently large parameter μ will help the estimates.*

Proof. Let call $\psi(t) := \int_\Omega u^2 + \int_\Omega |\nabla v|^4 + \frac{1}{3}\int_\Omega u|\nabla v|^2$, we have

$$\begin{aligned}\psi'(t) &= 2\int_\Omega u\left[\Delta u - \chi\nabla\cdot(u\nabla v) + f(u)\right] \\ &\quad + 4\int_\Omega |\nabla v|^2\nabla v\cdot\nabla\left(\Delta v + \alpha u - \beta v\right) \\ &\quad + \frac{2}{3}\int_\Omega u\nabla v\cdot\nabla\left(\Delta v + \alpha u - \beta v\right) \\ &\quad + \frac{2}{3}\int_\Omega |\nabla v|^2\left[\Delta u - \chi\nabla\cdot(u\nabla v) + f(u)\right] \\ &:= K_1 + K_2 + K_3 + K_4. \end{aligned} \tag{2.4.39}$$

By integration by parts, K_1 can be written as:

$$\begin{aligned} K_1 &= -2\int_\Omega |\nabla u|^2 + 2\int_{\partial\Omega} u^{p+1}\,dS + 2\chi\int_\Omega u\nabla u\cdot\nabla v \\ &\quad + 2a\int_\Omega u^2 - 2\mu\int_\Omega u^3. \end{aligned} \tag{2.4.40}$$

By Lemma 2.3.2, we obtain

$$2\int_{\partial\Omega} u^{p+1}\,dS \le 2\epsilon_1\int_\Omega u^3 + 2\epsilon_1\int_\Omega|\nabla u|^2 + 2c_1. \tag{2.4.41}$$

By Young's inequality, we have

$$2\chi\int_\Omega u\nabla u\cdot\nabla v \le \chi\epsilon_1\int_\Omega|\nabla u|^2 + \frac{\chi}{\epsilon_1}\int_\Omega u^2|\nabla v|^2. \tag{2.4.42}$$

We also have

$$2a\int_\Omega u^2 \le 2a\epsilon_1\int_\Omega u^3 + c_2, \tag{2.4.43}$$

where $c_2 > 0$ depending on ϵ_1. From (2.4.40) to (2.4.43), we obtain

$$K_1 \le \left[(2+\chi)\epsilon_1 - 2\right]\int_\Omega|\nabla u|^2 + 2\left[(a+1)\epsilon_1 - \mu\right]\int_\Omega u^3 + \frac{\chi}{\epsilon_1}\int_\Omega u^2|\nabla v|^2 + c_3, \tag{2.4.44}$$

where $c_3 = 2 + c_2$. We choose $\epsilon_1 = \frac{1}{2}\min\left\{\frac{2}{2+\chi}, \frac{\mu}{a+1}\right\}$ and substitute into (2.4.44) to obtain

$$K_1 \le -\int_\Omega |\nabla u|^2 - \mu\int_\Omega u^3 + \frac{\chi}{\epsilon_1}\int_\Omega u^2|\nabla v|^2 + c_3. \tag{2.4.45}$$

To deal with K_2, we use similar estimates to (2.4.28) and (2.4.29) in estimating J_2 in Lemma 2.4.3 to obtain

$$K_2 + 4\beta\int_\Omega |\nabla v|^4 \le 2(\alpha\epsilon_2 - 1)\int_\Omega |\nabla|\nabla v|^2|^2 + \left(\frac{2\alpha}{\epsilon_2} + 3\alpha^2\right)\int_\Omega u^2|\nabla v|^2. \tag{2.4.46}$$

By substituting $\epsilon_2 = \frac{1}{2\alpha}$ into (2.4.46), we deduce

$$K_2 + 4\beta\int_\Omega |\nabla v|^4 \le -\int_\Omega |\nabla|\nabla v|^2|^2 + 7\alpha^2\int_\Omega u^2|\nabla v|^2 \tag{2.4.47}$$

To deal with K_3, we make use of the following identity

$$\nabla v \cdot \nabla\Delta v = \frac{1}{2}\Delta(|\nabla v|^2) - |D^2 v|^2.$$

Rewriting K_3 as

$$\begin{aligned} K_3 = &-\frac{1}{3}\int_\Omega \nabla u \cdot \nabla|\nabla v|^2 - \frac{2}{3}\int_\Omega u|D^2 v|^2 \\ &+ \frac{2\alpha}{3}\int_\Omega u\nabla v \cdot \nabla u - \frac{2\beta}{3}\int_\Omega u|\nabla v|^2 + \frac{1}{3}\int_{\partial\Omega} u\frac{\partial|\nabla v|^2}{\partial\nu}. \end{aligned} \tag{2.4.48}$$

We drop the last term due to Lemma B.0.6, neglect the second term and apply Cauchy-Schwartz inequality

$$ab \le \epsilon a^2 + \frac{1}{4\epsilon}b^2,$$

to the third and the forth terms with a sufficiently small ϵ to obtain

$$K_3 + \frac{2\beta}{3}\int_\Omega u|\nabla v|^2 \le \frac{1}{3}\int_\Omega |\nabla|\nabla v|^2|^2 + \frac{\alpha^2}{8}\int_\Omega u^3 + \frac{1}{3}\int_\Omega |\nabla u|^2. \tag{2.4.49}$$

By integration by parts, K_4 can be rewritten as:

$$\begin{aligned} K_4 = &-\frac{1}{3}\int_\Omega \nabla|\nabla v|^2 \cdot \nabla u + \frac{1}{3}\int_{\partial\Omega} u^p|\nabla v|^2 \\ &+ \frac{\chi}{3}\int_\Omega \nabla|\nabla v|^2 \cdot u\nabla v + \frac{a}{3}\int_\Omega u|\nabla v|^2 - \frac{\mu}{3}\int_\Omega u^2|\nabla v|^2. \end{aligned} \tag{2.4.50}$$

By Cauchy-Schwarz inequality, we obtain

$$-\frac{1}{3}\int_\Omega \nabla|\nabla v|^2\cdot\nabla u \le \frac{1}{6}\int_\Omega \big|\nabla|\nabla v|^2\big|^2 + \frac{1}{3}\int_\Omega |\nabla u|^2. \tag{2.4.51}$$

In light of Lemma 2.3.3 and Lemma 2.4.2, the following inequality

$$\frac{1}{3}\int_{\partial\Omega} u^p|\nabla v|^2 \le \frac{1}{3}\int_\Omega |\nabla u|^2 + u^2|\nabla v|^2 + \big|\nabla|\nabla v|^2\big|^2 + u^3 + c_1 \tag{2.4.52}$$

holds for all $t\in(0,T_{\max})$, with some positive constant c_1. By Young's inequality, we obtain

$$\frac{\chi}{3}\int_\Omega \nabla|\nabla v|^2\cdot u\nabla v \le \frac{1}{6}\int_\Omega \big|\nabla|\nabla v|^2\big|^2 + \frac{\chi^2}{3}\int_\Omega u^2|\nabla v|^2, \tag{2.4.53}$$

and

$$\frac{a}{3}\int_\Omega u|\nabla v|^2 \le \frac{1}{3}\int_\Omega u^2|\nabla v|^2 + c_2, \tag{2.4.54}$$

with some positive constant c_2. Combining from (2.4.50) to (2.4.54), we obtain

$$K_4 \le \frac{2}{3}\int_\Omega |\nabla u|^2 + \big|\nabla|\nabla v|^2\big|^2 + \frac{\chi^2-\mu+2}{3}\int_\Omega u^2|\nabla v|^2 + \frac{1}{3}\int_\Omega u^3 + c_3, \tag{2.4.55}$$

with some positive constant c_3. Collecting (2.4.45), (2.4.47), (2.4.49), and (2.4.55), we have

$$\begin{aligned}\psi' + 4\beta\int_\Omega |\nabla v|^4 + \frac{2\beta}{3}\int_\Omega u|\nabla v|^2 &\le \left(7\alpha^2 + \frac{\chi}{\epsilon_1} + \frac{\chi^2-\mu+2}{3}\right)\int_\Omega u^2|\nabla v|^2 \\ &\quad + \left(\frac{1}{3}-\mu\right)\int_\Omega u^3 + \frac{\alpha^2}{8}\int_\Omega u^2 + c_5. \end{aligned}\tag{2.4.56}$$

This leads to

$$\begin{aligned}\psi' + 2\beta\psi &\le \left(7\alpha^2 + \frac{\chi}{\epsilon_1} + \frac{\chi^2-\mu+2}{3}\right)\int_\Omega u^2|\nabla v|^2 \\ &\quad + \left(\frac{1}{3}-\mu\right)\int_\Omega u^3 + \frac{16\beta+\alpha^2}{8}\int_\Omega u^2 + c_5. \end{aligned}\tag{2.4.57}$$

By Young's inequality, for every $\epsilon>0$, there exists a positive constant $c_6 = c_6(\epsilon)$ such that:

$$\frac{16\beta+\alpha^2}{8}\int_\Omega u^2 \le \epsilon\int_\Omega u^3 + c_6. \tag{2.4.58}$$

Therefore, we need to choose μ_0 sufficiently large such that

$$\begin{cases} 7\alpha^2 + \frac{\chi}{\epsilon_1} + \frac{\chi^2-\mu_0+2}{3} & \leq 0 \\ \frac{1}{3} + \epsilon - \mu_0 & \leq 0 \\ \mu_0 & \geq \frac{2(a+1)}{2+\chi}. \end{cases}$$

Therefore, if $\mu > \mu_0$, where

$$\mu_0 := \max\left\{\frac{1}{3}, \frac{2(a+1)}{2+\chi}, 3\left(\frac{\chi}{2+\chi} + 7\alpha^2 + \frac{\chi^2+2}{2}\right)\right\} \tag{2.4.59}$$

and then (2.4.56) yields

$$\psi' + 2\beta\psi \leq c_6.$$

Applying Gronwall's inequality, we see that

$$\psi(t) \leq \max\left\{\psi(0), \frac{c_6}{2\beta}\right\} \tag{2.4.60}$$

and thereby conclude the proof. □

Remark 2.4.2. μ_0 *defined as in* (2.4.56) *is not sharp. We leave the open question to obtain an optimal formula* μ_0.

2.5 Global boundedness

In this section, we show that if u is uniformly bounded in time under $\|\cdot\|_{L^{r_0}(\Omega)}$, then it is also uniformly bounded in time under $\|\cdot\|_{L^\infty(\Omega)}$.

Theorem 2.5.1. *Let* $r_0 > \frac{n}{2}$ *and* (u, v) *be a classical solution of* (KS) *on* $\Omega \times (0, T_{max})$ *with maximal existence time* $T_{max} \in (0, \infty]$. *If*

$$\sup_{t\in(0,T_{max})} \|u(\cdot,t)\|_{L^{r_0}(\Omega)} < \infty,$$

then

$$\sup_{t\in(0,T_{max})} \left(\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)}\right) < \infty.$$

Proof. It is a direct consequence of Theorem C.2.1 for $f(u) = au - \mu u^2$ and $g(u) = u^q$ with $q \in \left(1, \frac{3}{2}\right)$. □

We are now ready to prove our main results. Let us begin with the proof of the parabolic-elliptic system.

Proof of Theorem 2.1.1. Throughout this proof, unless specified otherwise, the notation C represents constants that may vary from time to time. In case $\mu > \frac{n-2}{n}\chi\alpha$, we obtain $\frac{\chi\alpha}{(\chi\alpha-\mu)_+} > \frac{n}{2}$. We first apply Lemma 2.4.1 to have $u \in L^\infty\left((0, T_{\max}); L^q(\Omega)\right)$ for any $q \in \left(\frac{n}{2}, \frac{\chi\alpha}{(\chi\alpha-\mu)_+}\right)$ and thereby conclude that $u \in L^\infty\left((0, T_{\max}); L^\infty(\Omega)\right)$ thank to Theorem 2.5.1.
When $\mu = \frac{n-2}{n}\chi\alpha$, let $w = u^{\frac{n}{4}}$, and apply trace embedding Theorem $W^{1,1}(\Omega) \to L^1(\partial\Omega)$, we have

$$\begin{aligned}\int_{\partial\Omega} w^{2+\frac{4(p-1)}{n}} &\le C\int_\Omega w^{2+\frac{4(p-1)}{n}} + C\left(2+\frac{4(p-1)}{n}\right)\int_\Omega w^{1+\frac{4(p-1)}{n}}|\nabla w| \\ &\le C\int_\Omega w^{2+\frac{4(p-1)}{n}} + 3C\int_\Omega w^{1+\frac{4(p-1)}{n}}|\nabla w| \\ &\le C\int_\Omega w^{2+\frac{4(p-1)}{n}} + \frac{\epsilon}{2}\int_\Omega |\nabla w|^2 + C(\epsilon)\int_\Omega w^{2+\frac{8(p-1)}{n}}, \end{aligned} \tag{2.5.1}$$

where the last inequality comes from Young's inequality for any arbitrary $\epsilon > 0$. By Lemma B.0.2, we have

$$\int_\Omega w^{2+\frac{8(p-1)}{n}} \le C\left(\int_\Omega |\nabla w|^2\right)^{\frac{\bar{p}a}{2}}\left(\int_\Omega w^{\frac{4}{n}}\right)^{\frac{n\bar{p}(1-a)}{4}} + C\left(\int_\Omega w^{\frac{4}{n}}\right)^{\frac{n\bar{p}}{4}}, \tag{2.5.2}$$

where

$$\begin{aligned}\bar{p} &= 2 + \frac{8(p-1)}{n}, \\ a &= \frac{n^2(n+4(p-1)+2)}{(n+4(p-1))(n^2-2n+4)}, \\ \frac{\bar{p}a}{2} &= \frac{n^2+4(p-1)n-2n}{n^2-2n+4} < 1, \qquad \text{since } p < 1 + \frac{1}{n}.\end{aligned}$$

We make use of uniformly boundedness of $\int_\Omega u$ and then apply Young's inequality into (2.5.2) to obtain:

$$\int_\Omega w^{2+\frac{8(p-1)}{n}} \le \frac{\epsilon}{2}\int_\Omega |\nabla w|^2 + C(\epsilon), \tag{2.5.3}$$

for any $\epsilon > 0$. We apply Young's inequality in (2.5.1) and then use (2.5.2) to have

$$\int_{\partial\Omega} w^{2+\frac{4(p-1)}{n}} \leq \epsilon \int_{\Omega} |\nabla w|^2 + C(\epsilon). \tag{2.5.4}$$

This implies that

$$\int_{\partial\Omega} u^{\frac{n}{2}+p-1} \leq \epsilon \int_{\Omega} |\nabla u^{\frac{n}{4}}|^2 + C(\epsilon). \tag{2.5.5}$$

Substitute $r = \frac{n}{4}$ and $\mu = \frac{n-2}{n}\chi\alpha$ into (2.4.3), we have

$$\frac{d}{dt}\int_{\Omega} u^{\frac{n}{2}} \leq -\frac{4(n-2)}{n}\int_{\Omega} |\nabla u^{\frac{n}{4}}|^2 + \frac{n}{2}\int_{\partial\Omega} u^{\frac{n}{2}+p-1}\, dS + \frac{na}{2}\int_{\Omega} u^{\frac{n}{2}}. \tag{2.5.6}$$

We apply Lemma B.0.2 and uniform boundedness of $\int_{\Omega} u$ to obtain that

$$\int_{\Omega} u^{\frac{n}{2}} \leq \epsilon \int_{\Omega} |\nabla u^{\frac{n}{4}}|^2 + C(\epsilon). \tag{2.5.7}$$

This, together with (2.5.5) and (2.5.6) with ϵ sufficiently small implies that

$$\frac{d}{dt}\int_{\Omega} u^{\frac{n}{2}} + \int_{\Omega} u^{\frac{n}{2}} \leq C.$$

Thus, by Gronwall's inequality we obtain that

$$\sup_{t\in(0,T_{\max})} \int_{\Omega} u^{\frac{n}{2}} < \infty. \tag{2.5.8}$$

For any $\epsilon \in (0,1)$, we choose $\frac{n}{4} < r < \frac{n}{4} + \frac{n\epsilon}{4\chi\alpha}$ and substitute into (2.4.6) to obtain that

$$\frac{d}{dt}\int_{\Omega} u^{2r} + \int_{\Omega} u^{2r} \leq \left[2r\epsilon - \frac{2(2r-1)}{r}\right]\int_{\Omega} |\nabla u^{r}|^2 + 3\epsilon\int_{\Omega} u^{2r+1} + C. \tag{2.5.9}$$

Setting $z := u^r$ and applying interpolation inequality, we have

$$\int_{\Omega} z^{2+\frac{1}{r}} \leq \left(\int_{\Omega} z^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \left(\int_{\Omega} z^{\frac{n}{2r}}\right)^{\frac{2}{n}}. \tag{2.5.10}$$

By Sobolev's inequality and Poincare's inequality, we obtain

$$\begin{aligned}\left(\int_{\Omega} z^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} &\leq C\left(\int_{\Omega} |\nabla z|^2 + \int_{\Omega} z^2\right)\\ &\leq C\int_{\Omega} |\nabla z|^2 + C\left(\int_{\Omega} z\right)^2,\end{aligned}$$

where $C > 0$ independent of r. This, together with (2.5.10) implies that

$$\int_\Omega z^{2+\frac{1}{r}} \le C\left(\int_\Omega |\nabla z|^2\right)\left(\int_\Omega z^{\frac{n}{2r}}\right)^{\frac{2}{n}} + C\left(\int_\Omega z\right)^2\left(\int_\Omega z^{\frac{n}{2r}}\right)^{\frac{2}{n}}.$$

This is equivalent to

$$\int_\Omega u^{2r+1} \le C\left(\int_\Omega |\nabla u^r|^2\right)\left(\int_\Omega u^{\frac{n}{2}}\right)^{\frac{2}{n}} + C\left(\int_\Omega u^r\right)^2\left(\int_\Omega u^{\frac{n}{2}}\right)^{\frac{2}{n}}. \tag{2.5.11}$$

This, together with Lemma B.0.2 and (2.5.8) implies that

$$\int_\Omega u^{2r+1} \le c_1 \int_\Omega |\nabla u^r|^2 + c_2, \tag{2.5.12}$$

where $c_1 := C\sup_{t\in(0,T_{\max})}\left(\int_\Omega u^{\frac{n}{2}}\right)^{\frac{2}{n}}$, and $c_2 := C\sup_{t\in(0,T_{\max})}\left\{\left(\int_\Omega u^r\right)^2\left(\int_\Omega u^{\frac{n}{2}}\right)^{\frac{2}{n}}\right\}$. Therefore, there exists a positive constant c_3 such that

$$\frac{d}{dt}\int_\Omega u^{2r} + \int_\Omega u^{2r} \le \left[2r\epsilon + 3c_1\epsilon - \frac{2(2r-1)}{r}\right]\int_\Omega |\nabla u^r|^2 + c_3. \tag{2.5.13}$$

We now have to choose ϵ and $r > \frac{n}{4}$ such that

$$\frac{n}{4} < r < \min\left\{\frac{n}{4}\left(\frac{4(n-2)}{\frac{n}{2}+\frac{n}{2\chi\alpha}+3c_1}+1\right), \frac{n}{2}\right\} \tag{2.5.14}$$

which is possible for any r satisfying

$$\frac{n}{4} < r < \frac{n}{4}\left(\frac{4(n-2)}{\frac{n}{2}+\frac{n}{2\chi\alpha}+3c_1}+1\right).$$

This, together with (2.5.13), (2.5.14) implies that there exists some $r_0 > \frac{n}{2}$ such that

$$\frac{d}{dt}\int_\Omega u^{r_0} + \int_\Omega u^{r_0} \le c_3.$$

By Gronwall's inequality, we have $u \in L^\infty\left((0,T_{\max}); L^{r_0}(\Omega)\right)$. We finally complete the proof by applying Theorem 2.5.1. □

Next we prove the main theorems for the parabolic-parabolic system in two- and three-dimensional space.

Proof of Theorem 2.1.2 and Theorem 2.1.3. Theorem 2.1.2 and Theorem 2.1.3 are immediate consequences of Lemma 2.4.3, Lemma 2.4.4 and Theorem 2.5.1. □

CHAPTER 3

BLOW-UP PREVENTION BY SUB-LOGISTIC SOURCES IN 2D KELLER-SEGEL SYSTEM

The focus of this chapter is on solutions to a two-dimensional Keller-Segel system with sub-logistic sources. We show that the presence of sub-logistic terms is adequate to prevent blow-up phenomena even in strongly degenerate Keller-Segel systems. Our proof relies on several techniques, including parabolic regularity theory in Orlicz spaces, variational arguments, interpolation inequalities, and the Moser iteration method.

3.1 Introduction

We consider the following nonlinear parabolic cross-diffusion partial differential equations arises from chemotaxis models

$$\begin{cases} u_t = \nabla \cdot (D(v)\nabla u) - \nabla \cdot (uS(v)\nabla v) + f(u) \\ v_t = \Delta v - v + u \end{cases} \tag{KS}$$

in a bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary, where

$$0 < D \in C^2([0,\infty)) \quad \text{and } S \in C^2([0,\infty)) \cap W^{1,\infty}((0,\infty)) \text{ such that } S' \geq 0, \tag{3.1.1}$$

and f is a smooth function generalizing the sub-logistic and signal production source respectively,

$$f(u) = ru - \mu \frac{u^2}{\ln^p(u+e)}, \quad \text{with } r \in \mathbb{R}, \mu > 0, \text{and } p > 0, \tag{3.1.2}$$

The system (KS) is complemented with nonnegative initial conditions in $W^{1,\infty}(\Omega)$ not identically zero:

$$u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), \qquad \text{with } x \in \mathbb{R}, \tag{3.1.3}$$

and homogeneous Neumann boundary condition are imposed as follows:

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \qquad x \in \partial\Omega,\ t \in (0, T_{\max}), \tag{3.1.4}$$

where ν denotes the outward normal vector. In more general conditions for D and S, as described by (3.1.1) without the non-decreasing requirement of S, were studied in [61]. It was proven that the terms $-\mu u^2$ are sufficient in preventing blow-up solutions by using variational techniques and parabolic regularity theory in Orlicz spaces. So, it is worth it to investigate whether the term "$-\mu u^k$" for some $k \in (1,2)$ is sufficient to prevent blow-up solutions. This question has been remained open since 2000s, however, previous studies found that the term "$-\mu u^2$" is not optimal to prevent blow-up solutions. Indeed, it was proved in [71] that when D and S are constant functions, sub-logistic sources $f(u) = ru - \frac{\mu u^2}{\ln^p(u+e)}$ where $0 < p < 1$ are sufficient to avoid blow-up solutions. In this chapter, we apply parabolic regularity results in Orlicz spaces to obtain the similar result that the term $-\mu u^2$ is not optimal in preventing blow-up solutions. Indeed, our results indicate that solutions to the system (KS) under the conditions from (3.1.1) to (3.1.4) exist globally when the terms $-\mu u^2$ are replaced by $\frac{-\mu u^2}{\ln^p(u+e)}$, where $0 < p < 1$, with an extra assumption that $\inf_{s\geq 0} D(s) > 0$ or $0 < p < 1/2$ without it. It was studied in [71] that when D and S are constant functions weaker terms $\frac{-\mu u^2}{\ln^p(u+e)}$ where $0 < p < 1$ are sufficient to avoid blow-up solutions.

The selling point of the chapter is the introduction of the energy functional

$$y(t) := \int_\Omega u \ln^k(u+e) + \int_\Omega |\nabla v|^2,$$

where the value of k is determined later. To establish an appropriate differential inequality for y, we perform a tedious analysis calculation, utilize interpolation inequalities in Sobolev spaces, and employ Moser iteration arguments. Our approach is a combination of two previous ideas: the first, proposed in [71, Lemma 3.2], offers a method to obtain a uniform bound for $\|u\ln(u)\|_{L^1(\Omega)}$, while the second, described in [61, Lemma 4.5], provides an additional argument for obtaining a uniform bound for $\left\|u\ln^2(u)\right\|_{L^1(\Omega)}$. It is important to note that using only one of these ideas is insufficient to obtain any $\left\|u\ln^k(u)\right\|_{L^1(\Omega)}$ bounds for solutions.

The chapter is organized as follows. Section 3.2 briefly contains our main results. The local well-posedness of solutions and some interpolation inequalities are presented in Section 3.3. In Section 3.4, we establish a priori estimates including $L\ln^k(L+e)$, and L^2 bounds for solutions. Finally, the main theorems are proved in Section 3.5.

3.2 Main theorems

In this section, we summarize two main theorems for the existence of global solutions to non-degenerate and degenerate chemotaxis systems. Let us begin with the nondegenate case:

Theorem 3.2.1 (Nondegenerate)**.** *In addition to the conditions from* (3.1.1) *to* (3.1.4)*, we assume that* $p < 1$*, and* $\inf_{s \geq 0} D(s) > 0$*. The system* (KS) *possesses a global classical bounded solution at all time.*

Remark 3.2.1. *Our theorem aligns with and strengthens the outcomes of [71, Theorem 1.1] by allowing* S *and* D *to be arbitrary functions satisfying* (3.1.1) *rather than being restricted to constant functions.*

The degenerate case are presented as follows:

Theorem 3.2.2 (Degenerate)**.** *If* $p < 1/2$*, then the system* (KS) *with the conditions from* (3.1.1) *to* (3.1.4) *admits a global classical bounded solution in* $\Omega \times (0, \infty)$.

Remark 3.2.2. *The theorem represents an advancement over the findings of [61, Theorem 1.4] as it incorporates sub-logistic sources instead of logistic ones. However, it should be noted that our result assumes the non-decreasing property of* S*, whereas [61, Theorem 1.4] does not require this condition.*

3.3 Preliminaries

The local existence and uniqueness of non-negative classical solutions to the system (KS) can be established by adapting and adjusting the fixed point argument and standard parabolic regularity theory. For further details, we refer the reader to [21, 58, 29]. For convenience, we adopt Lemma 4.1 from [61].

Lemma 3.3.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary, and suppose $r \in \mathbb{R}$ and $\mu > 0$ and that* (3.1.1)*,* (3.1.3)*, and* (3.1.4) *hold. Then there exist $T_{max} \in (0, \infty]$ and functions*

$$\begin{cases} u \in C^0\left(\bar{\Omega} \times (0, T_{max})\right) \cap C^{2,1}\left(\bar{\Omega} \times (0, T_{max})\right) \text{ and} \\ v \in \bigcap_{q>2} C^0\left([0, T_{max}); W^{1,q}(\Omega)\right) \cap C^{2,1}\left(\bar{\Omega} \times (0, T_{max})\right) \end{cases} \tag{3.3.1}$$

such that $u > 0$ and $v > 0$ in $\bar{\Omega} \times (0, \infty)$, that (u, v) solves (KS) *classically in $\Omega \times (0, T_{max})$, and that*

$$\text{if } T_{max} < \infty, \quad \text{then } \limsup_{t \to T_{max}} \left\{ \|u\|_{L^\infty(\Omega)} + \|u\|_{W^{1,\infty}(\Omega)} \right\} = \infty. \tag{3.3.2}$$

3.4 A priori estimates

In this section, we assume that the system (KS) admits a classical solution (u, v) and a maximal existence time $T_{\max}$, subject to conditions given by (3.1.1) to (3.1.4), as established in Lemma 3.3.1. To prove our main theorems, we rely heavily on a bound of the form $L \ln^k(L + e)$ for solutions to the system of equations in (KS). The proof utilizes standard variational arguments and fundamental functional inequalities. It is worth noting that the logistic degradation terms in the first equation of (KS), given by $-\frac{\mu u^2}{\ln^p(u+e)}$, effectively handle the corresponding cross-diffusion contribution. To precisely state this result, we present the following lemma:

Lemma 3.4.1. *If $p \leq 1 < k < 2 - p$, then*

$$\sup_{t \in (0, T_{max})} \int_\Omega u \ln^k(u + e) + |\nabla v|^2 < \infty,$$

and

$$\sup_{t \in (0, T_{max} - \tau)} \int_t^{t+\tau} \int_\Omega u^2 \ln^{k-p}(u + e) + (\Delta v)^2 < \infty,$$

where $\tau = \min\left\{1, \frac{T_{max}}{2}\right\}$.

Proof. We define

$$y(t) := \int_\Omega u \ln^k(u + e) + \frac{1}{2} \int_\Omega |\nabla v|^2,$$

and differentiate $y(\cdot)$ to obtain

$$
\begin{aligned}
y'(t) &= \int_\Omega \left(\ln^k(u+e) + ku\frac{\ln^{k-1}(u+e)}{u+e} \right) u_t + \int_\Omega \nabla v \cdot \nabla v_t \\
&:= I + J. \qquad (3.4.1)
\end{aligned}
$$

Now we make use of the first equation of (KS) to deal with I

$$
\begin{aligned}
I &= \int_\Omega \left(\ln^k(u+e) + ku\frac{\ln^{k-1}(u+e)}{u+e} \right) (\nabla \cdot (D(v)\nabla u - uS(v)\nabla v) + f(u)) \\
&= -k\int_\Omega \frac{D(v)\ln^{k-1}(u+e)}{u+e}|\nabla u|^2 - k(k-1)\int_\Omega \frac{D(v)u\ln^{k-2}(u+e)}{(u+e)^2}|\nabla u|^2 \\
&- k\int_\Omega \frac{eD(v)\ln^{k-1}(u+e)}{(u+e)^2}|\nabla u|^2 + k\int_\Omega \frac{S(v)u\ln^{k-1}(u+e)}{u+e}\nabla u \cdot \nabla v \\
&+ k(k-1)\int_\Omega \frac{S(v)u^2\ln^{k-2}(u+e)}{(u+e)^2}\nabla u \cdot \nabla v + k\int_\Omega \frac{eS(v)u\ln^{k-1}(u+e)}{(u+e)^2}\nabla u \cdot \nabla v \\
&+ \int_\Omega \left(\ln^k(u+e) + ku\frac{\ln^{k-1}(u+e)}{u+e} \right) f(u) \\
&:= \sum_{i=1}^{7} I_i. \qquad (3.4.2)
\end{aligned}
$$

To estimate I_4, I_5, and I_6 from above, we aim to bound them by using two terms $\int_\Omega u^2 \ln^{k-p}(u+e)$, and $\int_\Omega (\Delta v)^2$. Achieving this requires a meticulous application of integral by parts and Young's inequality. Specifically, we handle I_4 in the following manner:

$$
\begin{aligned}
I_4 &:= k\int_\Omega \frac{eS(v)u\ln^{k-1}(u+e)}{(u+e)^2}\nabla u \cdot \nabla v \\
&= k\int_\Omega S(v)\nabla\phi_1(u) \cdot \nabla v, \qquad (3.4.3)
\end{aligned}
$$

where

$$
\phi_1(l) := \int_0^l \frac{s\ln^{k-1}(s+e)}{s+e} \le l\ln^{k-1}(l+e).
$$

We utilize the integration by parts on equation (3.4.3), taking into account the condition $S' \geq 0$

and applying Young's inequality to obtain

$$
\begin{aligned}
I_4 &= -k \int_\Omega S(v)\phi_1(u)\Delta v - k \int_\Omega S'(v)\phi_1(u)|\nabla v|^2 \\
&\le c_1 \int_\Omega \phi_1(u)|\Delta v| \\
&\le \epsilon \int_\Omega (\Delta v)^2 + c_2 \int_\Omega \phi_1^2(u) \\
&\le \epsilon \int_\Omega (\Delta v)^2 + c_2 \int_\Omega u^2 \ln^{2k-2}(u+e) \\
&\le \epsilon \int_\Omega (\Delta v)^2 + \epsilon \int_\Omega u^2 \ln^{k-p}(u+e) + c_3, \qquad (3.4.4)
\end{aligned}
$$

where $c_1 = k\,\|S\|_{L^\infty(0,\infty)}$, $\epsilon > 0$, $c_2 > 0$ depends on ϵ, and the last inequality comes from the fact that for any $\delta > 0$, there exist a positive constant c_3 depending on δ such that

$$
c_2 u^2 \ln^{2k-2}(u+e) \le \delta u^2 \ln^{k-p}(u+e) + c_3 \qquad 2k-2 < k-p.
$$

We apply a similar reasoning to handle I_5 and I_6. To be more specific, we have:

$$
\begin{aligned}
I_5 &:= k(k-1) \int_\Omega \frac{S(v)u^2 \ln^{k-2}(u+e)}{(u+e)^2} \nabla u \cdot \nabla v \\
&= k(k-1) \int_\Omega S(v)\nabla\phi_2(u)\cdot\nabla v, \qquad (3.4.5)
\end{aligned}
$$

where

$$
\phi_2(l) := \int_0^l \frac{s^2 \ln^{k-2}(s+e)}{(s+e)^2} \le \int_0^l \ln^{k-2}(s+e) \le \phi_1(l).
$$

By using the same procedure to (3.4.4), it follows that for any $\epsilon > 0$, there exist $c_4 > 0$ depending on ϵ such that

$$
I_5 \le \epsilon \int_\Omega (\Delta v)^2 + \epsilon \int_\Omega u^2 \ln^{k-p}(u+e) + c_4. \qquad (3.4.6)
$$

The term I_6 can be handled as follows

$$
\begin{aligned}
I_6 &:= k \int_\Omega \frac{euS(v)\ln^{k-1}(u+e)}{(u+e)^2} \nabla u \cdot \nabla v \\
&= -k \int_\Omega S(v)\nabla\phi_3(u)\cdot\nabla v, \qquad (3.4.7)
\end{aligned}
$$

where

$$\phi_3(l) := \int_0^l \ln^{k-1}(s+e)\frac{es}{(s+e)^2} \leq \frac{1}{4}\int_0^l \ln^{k-1}(s+e) \leq l\ln^{k-1}(l+e)$$

As the right-hand side of (3.4.7) resembles that of (3.4.3), we employ the same reasoning to deduce that for any $\epsilon > 0$, there exist $c_5 > 0$ depending on ϵ such that

$$I_6 \leq \epsilon\int_\Omega (\Delta v)^2 + \epsilon\int_\Omega u^2\ln^{k-p}(u+e) + c_5. \tag{3.4.8}$$

To handle I_7, we make use of the fact that for any $\epsilon > 0$, there exist $c(\epsilon) > 0$ such that

$$u^{a_1}\ln^{b_1}(u+e) \leq \epsilon u^{a_2}\ln^{b_2}(u+e) + c(\epsilon),$$

where a_1, a_2, b_1, b_2 are real numbers such that $a_1 < a_2$. This implies that for any $\epsilon > 0$, there exist a positive constant c_7 depending on ϵ such that

$$\begin{aligned}\left(\ln^k(u+e) + k\frac{\ln^{k-1}(u+e)}{u+e}\right)f(u) &\leq ru\ln^k(u+e)\\ &\quad + rk\ln^{k-1}(u+e) - \mu u^2\ln^{k-p}(u+e)\\ &\leq (\epsilon-\mu)\int_\Omega u^2\ln^{k-p}(u+e) + c_7. \end{aligned}\tag{3.4.9}$$

Therefore, we obtain:

$$I_7 \leq (\epsilon-\mu)\int_\Omega u^2\ln^{k-p}(u+e) + c_7. \tag{3.4.10}$$

Since $I_i \leq 0$ for $i = 1, 2, 3$, and combine with (3.4.2), (3.4.4), (3.4.6), (3.4.8) and (3.4.10), for any $\epsilon > 0$, there exist a positive constant c_8 depending on ϵ such that

$$I \leq 3\epsilon\int_\Omega (\Delta v)^2 + (4\epsilon-\mu)\int_\Omega u^2\ln^{k-p}(u+e) + c_8. \tag{3.4.11}$$

By integration by parts and elemental inequalities, it follows that for any $\epsilon > 0$, there exist $c_9 > 0$ depending on ϵ such that

$$\begin{aligned} J &:= \int_\Omega \nabla v\cdot\nabla v_t\\ &= -\int_\Omega (\Delta v)^2 - \int_\Omega |\nabla v|^2 - \int_\Omega u\Delta v\\ &\leq -\frac{1}{2}\int_\Omega (\Delta v)^2 - \int_\Omega |\nabla v|^2 + \frac{1}{2}\int_\Omega u^2\\ &\leq -\frac{1}{2}\int_\Omega (\Delta v)^2 - \int_\Omega |\nabla v|^2 + \epsilon\int_\Omega u^2\ln^{k-p}(u+e) + c_9. \end{aligned}\tag{3.4.12}$$

For any $\epsilon > 0$, there exist a positive constant c_{10} such that

$$\int_\Omega u \ln^k(u+e) \le \epsilon \int_\Omega u^2 \ln^{k-p}(u+e) + c_{10}. \tag{3.4.13}$$

By combining (3.4.1), (3.4.11), (3.4.12), and (3.4.13), we obtain that for any $\epsilon > 0$, there exist $c_{11} > 0$ depending on ϵ such that

$$\begin{aligned} y'(t) + y(t) + \frac{1}{4}\int_\Omega (\Delta v)^2 + \frac{\mu}{2}\int_\Omega u^2 \ln^{k-p}(u+e) &\le (3\epsilon - \frac{1}{4})\int_\Omega (\Delta v)^2 \\ &+ (6\epsilon - \frac{\mu}{2})\int_\Omega u^2 \ln^{k-p}(u+e) + c_{11}, \end{aligned} \tag{3.4.14}$$

Choose ϵ sufficiently small, we have

$$y'(t) + y(t) \le c_{11}.$$

Using Gronwall's inequality with the previous equation, it follows that $y(t) \le \max\{y(0), c_{11}\}$. Additionally, we also have:

$$\frac{1}{4}\int_\Omega (\Delta v)^2 + \frac{\mu}{2}\int_\Omega u^2 \ln^{k-p}(u+e) \le c_{11} - y'(t). \tag{3.4.15}$$

By integrating the previous inequality from t to $t+\tau$ and using the fact that y is bounded, we can conclude the proof. □

Remark 3.4.1. *The non-decreasing assumption of S allows us to obtain a uniform bound for $\left\|u \ln^k(u+e)\right\|_{L^1(\Omega)}$ without using a uniform bound $\|u\|_{L^1(\Omega)}$ as in [61] and [71].*

The logistic degradation term $-\frac{\mu u^2}{\ln^p(u+e)}$ can ensure the boundedness of chemical density functions, even in the presence of strongly degenerate diffusion terms. To state this result precisely, we present the following lemma.

Lemma 3.4.2. *If $1 + p < k$, and*

$$\sup_{t\in(0,T_{max}-\tau)} \int_t^{t+\tau}\int_\Omega u^2 \ln^{k-p}(u+e) < \infty,$$

where $\tau = \min\left\{1, \frac{T_{max}}{2}\right\}$, then v is globally bounded in time.

Proof. This is a direct application of Proposition C.1.1 with $\alpha = k - p > 1$. □

We examine the nondegenerate diffusion mechanism and obtain bounds for u and ∇v through a standard testing procedure.

Lemma 3.4.3. *If $p < 1$, $q \geq 2$, $S' \geq 0$, $\inf_{s\geq 0} D(s) > 0$ and (u, v) is a classical solution to* (KS) *in $\Omega \times (0, T_{max})$ then there exists a positive constant C such that*

$$\int_\Omega u^q(\cdot, t) + \int_\Omega |\nabla v(\cdot, t)|^{2q} \leq C \tag{3.4.16}$$

for all $t \in (0, T_{max})$.

Proof. We define

$$\phi(t) := \frac{1}{q}\int_\Omega u^q + \frac{1}{2q}\int_\Omega |\nabla v|^{2q},$$

and differentiate ϕ to obtain:

$$\begin{aligned}\phi'(t) &= \int_\Omega u^{q-1}\left[\nabla\cdot(D(v)\nabla u) - \nabla\cdot(S(v)u\nabla v) + f(u)\right]\\ &+ \int_\Omega |\nabla v|^{2q-2}\nabla v\cdot\nabla\left(\Delta v + u - v\right)\\ &:= J_1 + J_2.\end{aligned} \tag{3.4.17}$$

By integration by parts, we have

$$\begin{aligned}J_1 &= -c_1\int_\Omega D(v)|\nabla u^{\frac{q}{2}}|^2 + c_2\int_\Omega S(v)u^{\frac{q}{2}}\nabla u^{\frac{q}{2}}\cdot\nabla v + r\int_\Omega u^q - \mu\int_\Omega \frac{u^{q+1}}{\ln^p(u+e)}\\ &:= J_{11} + J_{12} + J_{13} + J_{14},\end{aligned} \tag{3.4.18}$$

where positive constants c_1, c_2 depends on q. Since $\inf_{(x,t)\in\Omega\times(0,T)} D(v(x,t)) > 0$, we obtain

$$J_{11} \leq -c_3\int_\Omega |\nabla u^{\frac{q}{2}}|^2, \tag{3.4.19}$$

for some $c_3 = c_1 \inf_{(x,t)\in\Omega\times(0,T)} D(v(x,t))$. For any $\epsilon > 0$, there exist a positive constant c_4 depending on ϵ such that

$$J_{12} \leq \epsilon\int_\Omega |\nabla u^{\frac{q}{2}}|^2 + c_4\,\|S\|_{L^\infty(0,\infty)}\int_\Omega u^q|\nabla v|^2. \tag{3.4.20}$$

Choosing ϵ sufficiently small implies that

$$J_1 \leq -c_5 \int_\Omega |\nabla u^{\frac{q}{2}}|^2 + c_6 \int_\Omega u^q |\nabla v|^2 + r \int_\Omega u^q - \mu \int_\Omega \frac{u^{q+1}}{\ln^p(u+e)}, \tag{3.4.21}$$

where $c_5 = c_3/2$ and $c_6 = c_4 \|S\|_{L^\infty(0,\infty)}$. In treating J_2, we make use of the following pointwise identity

$$\nabla v \cdot \nabla \Delta v = \frac{1}{2}\Delta(|\nabla v|^2) - |D^2 v|^2$$

to obtain

$$\begin{aligned} J_2 = &-c_7 \int_\Omega |\nabla |\nabla v|^q|^2 - \int_\Omega |\nabla v|^{2q-2} |D^2 v|^2 \\ &+ \int_\Omega |\nabla v|^{2q-2} \nabla v \cdot \nabla u \\ &- \int_\Omega |\nabla v|^{2q} + c_8 \int_{\partial\Omega} \frac{\partial |\nabla v|^2}{\partial \nu} |\nabla v|^{2q-2}, \end{aligned} \tag{3.4.22}$$

where c_7, c_8 are positive constants depending on q. The inequality $\frac{\partial |\nabla v|^2}{\partial \nu} \leq c|\nabla v|^2$ for some $c > 0$ depending only on Ω implies that

$$\int_{\partial\Omega} \frac{\partial |\nabla v|^2}{\partial \nu} |\nabla v|^{2q-2} \, dS \leq c \int_{\partial\Omega} |\nabla v|^{2q} \, dS.$$

Let $g := |\nabla v|^q$ and apply Trace Imbedding Theorem $W^{1,1}(\Omega) \longrightarrow L^1(\partial\Omega)$ together with Young's inequality, we obtain the following

$$\begin{aligned} c \int_{\partial\Omega} g^2 \, dS &\leq C \int_\Omega g|\nabla g| + C \int_\Omega g^2 \\ &\leq \epsilon \int_\Omega |\nabla g|^2 + c_9 \int_\Omega g^2, \end{aligned} \tag{3.4.23}$$

for any $\epsilon > 0$ and $c_9 > 0$ depending on ϵ. Therefore, we have

$$\int_{\partial\Omega} |\nabla v|^{2q} \, dS \leq \epsilon \int_\Omega |\nabla |\nabla v|^q|^2 + c_9 \int_\Omega |\nabla v|^{2q}. \tag{3.4.24}$$

Applying the pointwise inequality $(\Delta v)^2 \leq 2|D^2 v|^2$ to (3.4.22) and choosing $\epsilon = c_7/2$ yields

$$\begin{aligned} J_2 &\leq -c_{10} \int_\Omega |\nabla |\nabla v|^q|^2 - \frac{1}{2} \int_\Omega |\nabla v|^{2q-2} |\Delta v|^2 \\ &\quad + \int_\Omega |\nabla v|^{2q-2} \nabla v \cdot \nabla u + c_{11} \int_\Omega |\nabla v|^{2q} \\ &= J_{21} + J_{22} + J_{23} + J_{24}, \end{aligned} \tag{3.4.25}$$

where $c_{10} = c_7/2$ and $c_{11} = cc_8c_9$. By integration by parts and elemental inequalities, we obtain that for any $\epsilon > 0$, there exist a positive constant c_{12} depending on ϵ such that

$$J_{23} = \int_\Omega u|\nabla v|^{2q-2}\nabla v \cdot \nabla v = -\int_\Omega |\nabla v|^{2q-2}\Delta u - c\int_\Omega u|\nabla v|^{q-1}\nabla|\nabla v|^q \cdot \frac{\nabla v}{|\nabla v|}$$
$$\leq \epsilon \int_\Omega (\Delta v)^2|\nabla v|^{2q-2} + \epsilon\int_\Omega |\nabla|\nabla v|^q|^2$$
$$+ c_{12}\int_\Omega u^2|\nabla v|^{2q-2}, \tag{3.4.26}$$

where c is a positive constant depending on q only. Choosing ϵ sufficiently small, we obtain

$$J_2 \leq -c_{13}\int_\Omega |\nabla|\nabla v|^q|^2 + c_{11}\int_\Omega |\nabla v|^{2q} + c_{12}\int_\Omega u^2|\nabla v|^{2q-2}, \tag{3.4.27}$$

where $c_{13} = c_{10}/2$. By Young inequality, for any $\epsilon > 0$, there exist $c_{14} > 0$ depending on ϵ such that:

$$c_6\int_\Omega u^q|\nabla v|^2 + c_12\int_\Omega u^2|\nabla v|^{2q-2} \leq \epsilon\int_\Omega |\nabla v|^{2q+2} + c_{14}\int_\Omega u^{q+1}. \tag{3.4.28}$$

Using the Gagliardo-Nirenberg inequality in Lemma B.0.2 for $n = 2$ and Lemma 3.4.1, we can conclude that there exists a positive constant c_{GN} such that:

$$\int_\Omega |\nabla v|^{2q+2} \leq c_{GN}\int_\Omega |\nabla|\nabla v|^q|^2\int_\Omega |\nabla v|^2 + c_{GN}\left(\int_\Omega |\nabla v|^2\right)^{q+1}$$
$$\leq c_{15}\int_\Omega |\nabla|\nabla v|^q|^2 + c_{16}, \tag{3.4.29}$$

where $c_{15} = c_{GN}\sup_{t>0}\int_\Omega |\nabla v|^2$ and $c_{16} = c_{GN}\left(\sup_{t>0}\int_\Omega |\nabla v|^2\right)^{q+1}$. The condition $0 < p < 1$ enables us to choose $k \in (p, 2-p)$, particularly we select $k = 1$ and apply Lemma 3.4.1 to obtain the uniformly boundedness of $\|u\ln(u+e)\|_{L^1(\Omega)}$. This together with Lemma B.0.9 imply that for any $\epsilon > 0$, there exist a positive constant c depending on ϵ satisfying

$$\int_\Omega u^{q+1} \leq \epsilon\int_\Omega |\nabla u^{\frac{q}{2}}|^2\int_\Omega u\ln(u+e) + c\left(\int_\Omega u\right)^{q+1} + c$$
$$\leq c_{17}\epsilon\int_\Omega |\nabla u^{\frac{q}{2}}|^2 + c_{18}, \tag{3.4.30}$$

where $c_{17} = \sup_{t>0}\int_\Omega u\ln(u+e)$ and $c_{18} > 0$ depend on ϵ. Combining (3.4.17), (3.4.21), and from (3.4.27) to (3.4.30), and choosing ϵ sufficiently small, we obtain

$$\phi'(t) \leq -c_{19}\int_\Omega |\nabla|\nabla v|^q|^2 + c_{20}\int_\Omega |\nabla v|^{2q} + r\int_\Omega u^q - \mu\int_\Omega \frac{u^{q+1}}{\ln^p(u+e)} + c_{21}. \tag{3.4.31}$$

For any $\epsilon > 0$, there exist a positive constant c depending on ϵ such that

$$x^q \leq \frac{\epsilon x^{q+1}}{\ln^p(x+e)} + c.$$

This implies that

$$\int_\Omega u^q \leq \epsilon \int_\Omega \frac{u^{q+1}}{\ln^p(u+e)} + c_{22}, \tag{3.4.32}$$

where $c_{22} = c|\Omega|$. By applying Lemma B.0.2 and using the fact that $\|\nabla v\|_{L^2(\Omega)}$ is uniformly bounded, and combining with Young inequality we obtain that for any $\epsilon > 0$, there exist a positive constant c_{23} depending on ϵ and $\|\nabla v\|_{L^2(\Omega)}$ such that

$$\begin{aligned}
\int_\Omega |\nabla|\nabla v|^q|^2 &\leq c_{GN} \left(\int_\Omega |\nabla|\nabla v|^q|^2 \right)^{\frac{q-1}{q}} \int_\Omega |\nabla v|^2 + c_{GN} \left(\int_\Omega |\nabla v|^2 \right)^q \\
&\leq c_{GN} \sup_{t>0} \int_\Omega |\nabla v|^2 \left(\int_\Omega |\nabla|\nabla v|^q|^2 \right)^{\frac{q-1}{q}} + c_{GN} \left(\sup_{t>0} \int_\Omega |\nabla v|^2 \right)^q \\
&\leq \epsilon \int_\Omega |\nabla|\nabla v|^q|^2 + c_{23}.
\end{aligned} \tag{3.4.33}$$

By combining (3.4.31), (3.4.32), and (3.4.33), and selecting an appropriate value for ϵ, we can find a postive constant c_{24} depending on ϵ such that $\phi'(t) + \phi(t) \leq c_{24}$. The proof is completed by applying Gronwall's inequality. □

When the chemical concentration function v is bounded, the degeneracies in the diffusion mechanism are eliminated, thus enabling us to derive bounds for u and ∇v. Specifically, we present the following lemma.

Lemma 3.4.4. *If $p < 1/2$, $q \geq 2$, $S' \geq 0$, and (u, v) is a classical solution to* (KS) *in $\Omega \times (0, T_{max})$ then there exists a positive constant C such that*

$$\int_\Omega u^q(\cdot, t) + \int_\Omega |\nabla v(\cdot, t)|^{2q} \leq C \tag{3.4.34}$$

for all $t \in (0, T_{max})$.

Proof. Since $0 < p < \frac{1}{2}$, we can select a constant $k \in (1+p, 2-p)$. By utilizing Lemma 3.4.1, we obtain

$$\sup_{t \in (0, T-\tau)} \int_t^{t+\tau} \int_\Omega u^2 \ln^{k-p}(u+e) < \infty.$$

Then, applying Lemma 3.4.2, we deduce that v is globally bounded in time, implying that $D(v) \geq c > 0$. Using the same argument as in the proof of Lemma 3.4.3, we can conclude the proof. □

It is possible to obtain an L^∞ bound for solutions of equation (KS) by using Lemma A.1 in [53], provided that we have L^{q_0} bounds for some $q_0 > 2$. However, for the sake of completeness, we present a proof that uses the Moser iteration method [2, 1] to establish the iteration process from L^{q_0} to L^∞. To this end, we rely on the following lemma:

Lemma 3.4.5. *Let* (u, v) *be a classical solution of* (KS) *on* $(0, T_{max})$ *and*

$$U_q := \max\left\{ \|u_0\|_{L^\infty(\Omega)}, \sup_{t\in(0,T_{max})} \|u(\cdot,t)\|_{L^q(\Omega)} \right\}.$$

If $\sup_{t\in(0,T_{max})} \|u(\cdot,t)\|_{L^q(\Omega)} < \infty$ *for some* $q > n$*, then there exists constants* $A, B > 0$ *independent of* q *such that*

$$U_{2q} \leq (Aq^B)^{\frac{1}{2q}} U_q. \tag{3.4.35}$$

Proof. The primary objective is to initially establish an inequality of the form:

$$\frac{d}{dt}\int_\Omega u^{2q} + \int_\Omega u^{2q} \leq Aq^B \left(\int_\Omega u^q\right)^2, \tag{3.4.36}$$

where A and B are positive constants. We then proceed to apply the Moser iteration technique. It is crucial to note that the dependence of all the constants on q is tracked carefully. Multiplying the first equation in the system (KS) by u^{2q-1} we obtain

$$\begin{aligned}
\frac{1}{2q}\frac{d}{dt}\int_\Omega u^{2q} &= \int_\Omega u^{2q-1}u_t \\
&= \int_\Omega u^{2q-1}\left[\nabla\cdot(D(v)\nabla u) - \nabla\cdot(S(v)u\nabla v) + ru - \frac{\mu u^2}{\ln^p(u+e)}\right] \\
&:= I + J + K.
\end{aligned} \tag{3.4.37}$$

Since there exist $C > 0$ such that $\int_\Omega u^q(\cdot,t) < C$ for all $t \in (0, T_{\max})$, Lemma C.1.2 entails that v is globally bounded, which further implies $\inf_{(x,t)\in\Omega\times(0.T_{\max})} D(v(x,t)) := c_1 > 0$. Thus, we have

$$I := -\frac{2q-1}{q^2}\int_\Omega D(v)|\nabla u^q|^2 \leq -c_1\frac{2q-1}{q^2}\int_\Omega |\nabla u^q|^2. \tag{3.4.38}$$

In treating J,

$$J := \int_\Omega u^{2q-1}\nabla\cdot(S(v)u\nabla v)$$
$$= \chi\frac{2q-1}{2q}\int_\Omega S(v)\nabla u^{2q}\cdot\nabla v \tag{3.4.39}$$
$$= \chi\frac{2q-1}{q}\int_\Omega S(v)u^q\nabla u^q\cdot\nabla v \tag{3.4.40}$$

Lemma (C.1.2) asserts that v is in $L^\infty\left((0,T);W^{1,\infty}(\Omega)\right)$, which entails that

$$\sup_{0<t<T}\|\nabla v\|_{L^\infty}^2 := c_2 < \infty,$$

Apply Young inequality yields

$$J \le \epsilon\int_\Omega|\nabla u^q|^2 + \frac{(2q-1)^2}{4q^2\epsilon}\|S\|_{L^\infty(0,\infty)}\int_\Omega u^{2q}|\nabla v|^2$$
$$\le \epsilon\int_\Omega|\nabla u^q|^2 + \frac{(2q-1)^2}{4q^2\epsilon}\|S\|_{L^\infty(0,\infty)}c_2\int_\Omega u^{2q}. \tag{3.4.41}$$

It follows from (3.4.37) and (3.4.41) that

$$\frac{d}{dt}\int_\Omega u^{2q} + \int_\Omega u^{2q} \le 2q\left(-\frac{2q-1}{q^2}+\epsilon\right)\int_\Omega|\nabla u^q|^2 - 2q\mu\int_\Omega u^{2q+1}$$
$$+\left[\frac{(2q-1)^2}{2q\epsilon}\chi^2c_2\|S\|_{L^\infty(0,\infty)} + 2qr + 1\right]\int_\Omega u^{2q}. \tag{3.4.42}$$

Substitute $\epsilon = \min\left\{\frac{q-1}{q^2},\mu\right\}$ into (3.4.42) we obtain

$$\frac{d}{dt}\int_\Omega u^{2q} + \int_\Omega u^{2q} \le -2\int_\Omega|\nabla u^q|^2 + c_3q^2\int_\Omega u^{2q} \tag{3.4.43}$$

where c_3 are independent of q. Apply Lemma B.0.3, and plug into (3.4.43) entails the following inequality for all $\eta\in(0,1)$

$$\frac{d}{dt}\int_\Omega u^{2q} + \int_\Omega u^{2q} \le (c_3q^2\eta-2)\int_\Omega|\nabla u^q|^2 + \frac{c_4q^2}{\eta^{\frac{n}{2}}}\left(\int_\Omega u^q\right)^2, \tag{3.4.44}$$

where $c_4>0$ independent of r,η. Substitute $\eta = \min\left\{\frac{1}{c_3q^2},1\right\}$ into this yields

$$\frac{d}{dt}\int_\Omega u^{2q} + \int_\Omega u^{2q} \le c_5q^{n+2}\left(\int_\Omega u^q\right)^2, \tag{3.4.45}$$

where c_5 independent of q. Apply Gronwall inequality yields

$$\int_\Omega u^{2q}(\cdot,t) \leq \max\left\{c_5 q^{n+2} U_q^{2q}, \int_\Omega u_0^{2q}\right\}$$

This entails

$$\|u(\cdot,t)\|_{L^{2q}(\Omega)} \leq \max\left\{(c_5 q^{n+2})^{\frac{1}{2q}} U_q, |\Omega|^{\frac{1}{2q}} \|u_0\|_{L^\infty(\Omega)}\right\},$$

and further implies that

$$U_{2q} \leq (Aq^B)^{\frac{1}{2q}} U_q$$

where $A = \max\{c_5, |\Omega|\}$ and $B = n+2$. The proof of (3.4.35) is complete. □

3.5 Proof of main theorems

This section focuses on proving our main theorems, starting with the non-degenerate case.

Proof of Theorem 3.2.1. From Lemma 3.4.1 and Lemma 3.4.3, for some fixed $q_0 > 2$

$$\sup_{t\in(0,T_{\max})} \int_\Omega u^{q_0} + |\nabla v|^{2q_0} \leq C < \infty. \tag{3.5.1}$$

By using Lemma C.1.2, we can conclude that v belongs to $L^\infty((0,T_{\max}); W^{1,\infty}(\Omega))$. Furthermore, Lemma 3.4.5 implies that the following inequality holds

$$U_{2^{k+1}q_0} \leq \left(A(2^k q_0)^B\right)^{\frac{1}{2^{k+1}q_0}} U_{2^k q_0} \tag{3.5.2}$$

for all integers $k \geq 0$. After taking the log of the above inequality, we can use Lemma B.0.10 for the following sequence.

$$a_k = \frac{\ln A}{2^{k+1}q_0} + \frac{Bk\ln 2}{2^{k+1}q_0} + \frac{B\ln q_0}{2^{k+1}q_0}$$

One can verify that

$$\sum_{k=0}^\infty a_k = \frac{\ln\left(A(2q_0)^B\right)}{q_0}.$$

Thus, we obtain

$$U_{2^{k+1}q_0} \le A^{\frac{1}{q_0}} (2q_0)^{\frac{B}{q_0}} U_{q_0} \tag{3.5.3}$$

for all $k \ge 1$. Send $k \to \infty$ yields

$$U_\infty \le A^{\frac{1}{q_0}} (2q_0)^{\frac{B}{q_0}} U_{q_0}. \tag{3.5.4}$$

This implies that $u \in L^\infty\left((0, T_{\text{max}}); L^\infty(\Omega)\right)$. □

The proof of Theorem 3.2.2 is similar to that of Theorem 3.2.1, with the additional requirement of showing that the diffusion mechanism remains non-degenerate throughout the evolution of the system.

Proof of Theorem 3.2.2. By using Lemma 3.4.2 and Lemma 3.4.4, it follows that for a fixed $q_0 > 2$, we have

$$\sup_{t\in(0,T_{\text{max}})} \int_\Omega u^{q_0} + |\nabla v|^{2q_0} \le C < \infty. \tag{3.5.5}$$

We can now repeat the same arguments from (3.5.2) to (3.5.4) to establish L^∞ bounds for u and v. □

CHAPTER 4

NONLOCAL; TWO SPECIES WITH TWO CHEMICALS; AND NONLINEAR BOUNDARY PROBLEMS

This chapter aims to extend the previous research on the global existence of solutions for chemotaxis systems by presenting four main results. The first result focuses on the global existence of solutions for elliptic-parabolic chemotaxis systems with logistic sources in the limiting case. The second result examines the global existence of solutions in two species with two chemical chemotaxis models. The third result is to investigate the global solutions of chemotaxis systems with nonlocal sources. Finally, we show that the quadradic degradation is sufficiently strong to prevent blow-up even for nonlinear Neumann boundary condition. These results contribute to our understanding of the global existence of solutions for chemotaxis systems and highlight important areas of research within the field.

4.1 A priori estimate in the limiting cases

In this section, we investigate on the a priori estimate for the chemotaxis system with the logistic source $f(u) = ru - \mu u^2$ in $\mathbb{R}^n$ where $n \geq 3$.

$$\begin{cases} u_t = \nabla \cdot (D(v)\nabla u) - \nabla \cdot (uS(v)\nabla u) + f(u) \\ v_t = \Delta v + u - v \end{cases} \tag{4.1.1}$$

We also have the global boundedness property for the solution to the parabolic-elliptic system when $\mu = \frac{n-2}{n}\chi\alpha$, $n \geq 3$ and $f(u) = au - \mu u^2$. Here we provide a shorter proof but similar to the result in [23].

Theorem 4.1.1. *If $\mu = \frac{n-2}{n}\chi\alpha$ and (u, v) is a classical solution of* (4.1.1) *on $\Omega \times (0, T_{max})$ with maximal existence time $T_{max} \in (0, \infty]$, then*

$$\sup_{t \in (0, T_{max})} \Big(\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} \Big) < \infty.$$

Proof. Multiplying the first equation in the system (4.1.1) by u^{2r-1} yields

$$\begin{aligned}
\frac{1}{2r}\frac{d}{dt}\int_\Omega u^{2r} &= \int_\Omega u^{2r-1}u_t \\
&= \int_\Omega u^{2r-1}\left[\Delta u - \chi\nabla(u\nabla v) + f(u)\right] \\
&= -\frac{2r-1}{r^2}\int_\Omega |\nabla u^r|^2 - \chi\frac{2r-1}{2r}\int_\Omega u^{2r}\Delta v + \int_\Omega u^{2r-1}f(u) \\
&= -\frac{2r-1}{r^2}\int_\Omega |\nabla u^r|^2 + \frac{2r-1}{2r}\int_\Omega u^{2r}(\chi\alpha u - \chi\beta v) \\
&+ a\int_\Omega u^{2r} - \mu\int_\Omega u^{2r+1}. \qquad (4.1.2)
\end{aligned}$$

Since $v \geq 0$, we have

$$\frac{d}{dt}\int_\Omega u^{2r} \leq -\frac{2(2r-1)}{r}\int_\Omega |\nabla u^r|^2 - [2r\mu - \chi\alpha(2r-1)]\int_\Omega u^{2r+1} + 2ra\int_\Omega u^{2r}. \qquad (4.1.3)$$

Plug $\mu = \frac{n-2}{n}\chi\alpha$ into the last term of (2.4.3), we have

$$-[2r\mu - \chi\alpha(2r-1)]\int_\Omega u^{2r+1} = \chi\alpha\left(\frac{4}{n}r - 1\right)\int_\Omega u^{2r+1}$$

Now, we choose $r = \frac{n}{4}$ to obtain

$$\frac{d}{dt}\int_\Omega u^{\frac{n}{2}} + \int_\Omega u^{\frac{n}{2}} \leq -\frac{4(n-2)}{n}\int_\Omega |\nabla u^{\frac{n}{4}}|^2 + \frac{na+2}{2}\int_\Omega u^{\frac{n}{2}},$$

By applying GN inequality, then Young inequality and finally making use of $\sup_{t\in(0,T)}\int_\Omega u < \infty$, we obtain that for every arbitrary small $\epsilon > 0$, there exists $c = c(\epsilon) > 0$ such that

$$\int_\Omega u^{\frac{n}{2}} \leq \epsilon\int_\Omega |\nabla u^{\frac{n}{4}}|^2 + c.$$

Therefore, we choose ϵ sufficiently small to obtain

$$\frac{d}{dt}\int_\Omega u^{\frac{n}{2}} + \int_\Omega u^{\frac{n}{2}} \leq c.$$

By Gronwall inequality, we have

$$\sup_{t\in(0,T_{\max})}\int_\Omega u^{\frac{n}{2}}(x,t)\,dx \leq c$$

For every $\epsilon > 0$, there exists r such that $\chi\alpha\left(\frac{4}{n}r - 1\right) < \epsilon$, we have

$$\frac{d}{dt}\int_\Omega u^{2r} \leq -\frac{2(2r-1)}{r}\int_\Omega |\nabla u^r|^2 + \epsilon\int_\Omega u^{2r+1}. \tag{4.1.4}$$

We apply GN-inequality to obtain

$$\begin{aligned}\int_\Omega u^{2r+1} &\leq C\left(\int_\Omega |\nabla u^r|^2\right)\left(\int_\Omega u^{\frac{n}{2}}\right)^{\frac{2}{n}} + \left(\int_\Omega u\right)^{2r+1} \\ &\leq C\int_\Omega |\nabla u^r|^2 + C,\end{aligned} \tag{4.1.5}$$

where C is independent of r. Thus from (4.1.4) and (4.1.5), we have

$$\frac{d}{dt}\int_\Omega u^{2r} \leq \left(C\epsilon - \frac{2(2r-1)}{r}\right)\int_\Omega |\nabla u^r|^2$$

Now, we choose ϵ such that

$$C\epsilon - \frac{2(2r-1)}{r} \leq 0$$

which is possible since the inequality

$$\chi\alpha\left(\frac{4}{n}r - 1\right) < \epsilon < \frac{2(2r-1)}{Cr}$$

holds when

$$\frac{n}{4} < r < \frac{n}{4}\left(\frac{4(n-2)}{Cn\chi\alpha} + 1\right).$$

Therefore, there exists $p > \frac{n}{2}$ such that

$$\sup_{t\in(0,T_{\max})}\int_\Omega u^p < \infty$$

which further implies $u \in L^\infty\left((0,\infty); L^\infty(\Omega)\right)$. □

4.2 Nonlocal problems

In this section, we study some chemotaxis models involving nonlocal terms as the following

$$\begin{cases} u_t = \Delta u - \nabla\cdot(u\nabla v) + f(u) \\ \tau v_t = \Delta v - v + u \end{cases} \tag{4.2.1}$$

where $f(u) = ru - \mu u\left(\int_\Omega u^p\right)^q$ with positive parameters r, μ, p, q .

Theorem 4.2.1. *The problem* (4.2.1) *with parameters* $\tau = 0$*,* $p > \frac{n}{2}$ *and* $q > 1 + \frac{4}{2p-n}$ *possesses a global classical solution* (u, v)*.*

Proof. Multiplying the first equation in the system (4.1.1) by u^{p-1} yields

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega u^p &= \int_\Omega u^{p-1}u_t \\ &= \int_\Omega u^{p-1}\left[\Delta u - \chi\nabla(u\nabla v) + f(u)\right] \\ &= -\frac{4(p-1)}{p^2}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 - \chi\frac{p-1}{p}\int_\Omega u^p\Delta v + \int_\Omega u^{p-1}f(u) \\ &= -\frac{4(p-1)}{p^2}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + \frac{p-1}{p}\int_\Omega u^p(u-v) \\ &+ r\int_\Omega u^p - \mu\left(\int_\Omega u^p\right)^{q+1}. \end{aligned} \tag{4.2.2}$$

Since $v \geq 0$, we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega u^p \leq -\frac{p-1}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + (p-1)\int_\Omega u^{p+1} \\ + pr\int_\Omega u^p - \mu\left(\int_\Omega u^p\right)^{q+1}. \end{aligned} \tag{4.2.3}$$

Now we make use of Gigliardo-Neirenberg inequality to obtain

$$\int_\Omega u^{p+1} \leq C_{GN}\left(\int_\Omega |\nabla u^{\frac{p}{2}}|^2\right)^{\frac{n}{2p}}\left(\int_\Omega u^p\right)^{\frac{2p-n+2}{2}} + C_{GN}\left(\int_\Omega u\right)^{p+1}. \tag{4.2.4}$$

Since $\frac{n}{2p} < 1$, we apply Young's inequality to the first term of (4.2.4), to obtain

$$(p-1)\int_\Omega u^{p+1} \leq \epsilon\int_\Omega |\nabla u^{p/2}|^2 + c\left(\int_\Omega u^p\right)^{\frac{2(2p-n+4)}{2p-n}} + C_{GN}M^{p+1}, \tag{4.2.5}$$

where $\epsilon > 0$ will be determined later, $c = c(\epsilon) > 0$, and

$$M := \sup_{t\in(0,T)}\int_\Omega u(x,t)\,dx < \infty.$$

From (4.2.3) and (4.2.5), we imply

$$\begin{aligned}\frac{d}{dt}\int_\Omega u^p \leq \left(\epsilon - \frac{p-1}{p}\right)\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + pr\int_\Omega u^p \\ + c\left(\int_\Omega u^p\right)^{\frac{2(2p-n+4)}{2p-n}} - \mu\left(\int_\Omega u^p\right)^{q+1} + C_{GN}M^{p+1}. \end{aligned} \tag{4.2.6}$$

Now we choose $\epsilon < \frac{p-1}{p}$ and denote

$$y(t) := \int_\Omega u^p,$$

and $g(s) := prs + s^{\frac{2(2p-n+4)}{2p-n}} - \mu s^{q+1} + C_{GN} M^{p+1}$. We have the following differential inequality

$$y'(t) \le g(y(t)).$$

The condition

$$q + 1 > \frac{2(2p - n + 4)}{2p - n}$$

is equivalent to

$$q > 1 + \frac{4}{2p - n}.$$

Thus, the equation $g(s) = 0$ has a positive solution s_0 such that for all $s > s_0$ we have $g(s) < 0$. Finally, we find that $y'(t) < 0$ when $y(t) > s_0$, and therefore maximum principle implies that $y(t)$ is bounded globally. □

4.3 Two-species chemotaxis system with two chemicals with sub-logistic sources in 2d

This section aims to study the global existence and boundedness of solutions in a two-species chemotaxis system with two chemicals and sub-logistic sources. The appearance of a sub-logistic source in only one cell density equation effectively prevents the occurrence of blow-up solutions, even in fully parabolic chemotaxis systems.

4.3.1 Introduction

We consider a model involving the interaction of two species through chemotaxis, where each species emits a signal that influences the movement of the other species. Specifically, we study the following PDE in a open bounded domain $\Omega \subset \mathbb{R}^2$

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) + f(u) \\ \tau v_t = \Delta v - v + w, \\ w_t = \Delta w - \nabla \cdot (w \nabla z) \\ \tau z_t = \Delta z - z + u \end{cases} \tag{4.3.1}$$

where $\tau \in \{0,1\}$, and $f(u) = ru - \frac{\mu u^2}{\ln^p(u+e)}$, with $r \in \mathbb{R}$, $\mu \geq 0$, and $p \geq 0$, under the homogeneous Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \qquad (x,t) \in \partial\Omega \times (0, T_{\max}), \tag{4.3.2}$$

where $T_{\max} \in (0, \infty]$ is the maximal existence time for classical solutions.

The effect of logistic sources or sub-logistic sources on blow-up prevention in two-species chemotaxis models is quite limited, especially when the logistic sources appear only in the first equation of the system (4.3.1). In [60], the authors study the global existence and long time behavior of solutions to the following system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) + r_1 u - \mu_1 u^2 \\ v_t = \Delta v - v + w, \\ w_t = \Delta w - \nabla \cdot (w \nabla z) + r_2 w - \mu_2 w^2 \\ 0 = \Delta z - z + u, \end{cases} \tag{4.3.3}$$

with $r_1, r_2 \in \mathbb{R}$ and $\mu_1, \mu_2 > 0$. It was shown that if $\mu_1\mu_2$ is sufficiently large then all solutions to (4.3.3) are global and bounded for any $n \geq 1$. For further studies on global existence and equilibrium solutions for two species with logistic sources appearing in two cell density equations, readers can refer to [3, 55, 11]. In fact, the analysis framework to prove the global existence of solutions to (4.3.3) for sufficiently large $\mu_1\mu_2$ is similar to the one for one species [64]. However, there has no result so far considering the presence of a logistic source in only one species. Our purpose is to address that the appearance of the sub-logistic sources in one species can effectively eliminate the occurrence of finite time or infinite time blow-up solutions in two dimensional domains. Precisely, we have the following theorem:

Theorem 4.3.1. *Assume that* $\tau \in \{0,1\}$, $f(u) = ru - \frac{\mu u^2}{\ln^p(u+e)}$, *where* $r, \mu > 0$, *and* $p \in [0,1)$ *and nonnegative initial data* $u_0, w_0 \in C^{0,\alpha}(\bar{\Omega})$ *for some* $\alpha \in (0,1)$ *when* $\tau = 0$ *and* $u_0, w_0 \in C^0(\bar{\Omega})$ *and* $v_0, z_0 \in W^{1,\infty}(\Omega)$ *when* $\tau = 1$. *Then there exists a unique quadruple* (u, v, w, z) *of nonnegative* *functions*

$$
\begin{aligned}
u &\in C^0\left(\bar{\Omega} \times [0,\infty)\right) \cap C^{2,1}\left(\bar{\Omega} \times (0,\infty)\right), \\
v &\in C^{2,1}\left(\bar{\Omega} \times (0,\infty)\right), \\
w &\in C^0\left(\bar{\Omega} \times [0,\infty)\right) \cap C^{2,1}\left(\bar{\Omega} \times (0,\infty)\right) \qquad \textit{and} \\
z &\in C^{2,1}\left(\bar{\Omega} \times (0,\infty)\right),
\end{aligned}
$$

which solve the system (4.3.1) *in the classical pointwise sense in* $\Omega \times (0,\infty)$. *Moreover,*

$$
\sup_{t\in(0,\infty)} \left\{ \|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|w(\cdot,t)\|_{L^\infty(\Omega)} + \|z(\cdot,t)\|_{W^{1,\infty}(\Omega)} \right\} < \infty. \quad (4.3.4)
$$

In the following sections, we will briefly recall the local well-posedness results for solutions to the system (4.3.1) in Section 4.3.2, and explore the mechanisms behind blow-up prevention by sub-logistic sources in Section 4.3.3.

4.3.2 Local existence

The local existence of solutions to the system (4.3.1) under homogeneous Neumann boundary conditions can be proved by adapting approaches that are well-established in the context chemotaxis models with logistic sources. Firstly, we establish the local existence of solutions for parabolic-elliptic chemotaxis models by adapting [59][Theorem 2.1].

Lemma 4.3.2. *Suppose that* $\tau = 0$, $\alpha \in (0,1)$ *and* u_0 *and* w_0 *are nonnegative functions in* $C^{0,\alpha}(\bar{\Omega})$. *Then there exist* $T_{max} \in (0,\infty]$ *and a unique quadruple* (u,v,w,z) *of nonnegative functions from* $C^0(\bar{\Omega}\times(0,T_{max}))\cap C^{2,1}(\bar{\Omega}\times(0,T_{max}))$ *solving* (4.3.1) *under boundary condition* (4.3.2) *classically in* $\Omega \times (0,T_{max})$. *Moreover, if* $T_{max} < \infty$, *then*

$$
\limsup_{t\to T_{max}} \left(\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|w(\cdot,t)\|_{L^\infty(\Omega)} \right) = \infty. \quad (4.3.5)
$$

Secondly, one can adapt and modify the proof of [64][Lemma 1.1] to obtain the local existence of solutions for fully parabolic models.

Lemma 4.3.3. *Suppose that* $\tau = 1$, *and* $(u_0,v_0,w_0,z_0) \in C^0(\bar{\Omega}) \times W^{1,\infty}(\Omega) \times C^0(\bar{\Omega}) \times W^{1,\infty}(\Omega)$ *such that* u_0, v_0, w_0, z_0 *are nonnegative. Then there exist* $T_{max} \in (0,\infty]$ *and a unique quadruple*

(u, v, w, z) *of nonnegative functions*

$$u, w \in C^0(\bar{\Omega} \times (0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max})) \tag{4.3.6}$$

$$v, z \in C^0(\bar{\Omega} \times (0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max})) \times L^\infty_{loc}\left([0, T_{max}); W^{1,\infty}(\Omega)\right)$$

solving (4.3.1) *under boundary condition* (4.3.2) *classically in* $\Omega \times (0, T_{max})$. *Moreover, if* $T_{max} < \infty$*, then*

$$\limsup_{t \to T_{max}} \Big(\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|w(\cdot, t)\|_{L^\infty(\Omega)} + \|z(\cdot, t)\|_{W^{1,\infty}(\Omega)} \Big) = \infty. \tag{4.3.7}$$

4.3.3 Global boundedness with sub-logistic sources. Proof of Theorem 4.3.1

The subsequent lemma holds a central position in this section, serving as a cornerstone of our work. A noteworthy innovation introduced herein is the functional described by (4.3.9), with the specific values of the positive parameters A and B to be determined subsequently in the analysis. It is notable that, within the context of inequality (4.3.24), we identify a unique choice for A that depends on the parameter ϵ, resulting in the nonpositivity of the first term on the right-hand side of (4.3.24).

Lemma 4.3.4. *Under the assumptions as in Theorem 4.3.1, there exists a positive constant* C *such that*

$$\int_\Omega u(\cdot, t) \ln(u(\cdot, t) + e) + \int_\Omega w(\cdot, t) \ln(w(\cdot, t) + e) + \tau \int_\Omega |\nabla v(\cdot, t)|^2 + \tau \int_\Omega |\nabla z(\cdot, t)|^2 < C, \tag{4.3.8}$$

for all $t \in (0, T_{max})$.

Remark 4.3.1. *Lemma 4.3.4 continues to hold in smooth bounded domains with arbitrary dimension.*

Proof. We define

$$y(t) := \int_\Omega u \ln(u + e) + \int_\Omega w \ln(w + e) + \frac{A}{2} \int_\Omega |\nabla v|^2 + \frac{B}{2} \int_\Omega |\nabla z|^2, \tag{4.3.9}$$

where $A := 2\epsilon$, $B := \epsilon + \frac{1}{4\epsilon}$ and $\epsilon := \min\left\{\frac{\mu}{4}, \frac{1}{3G_{GN}}\left(\int_\Omega w_0 + e|\Omega|\right)^{-1}\right\}$. Differentiating y in time, we obtain

$$
\begin{aligned}
y'(t) + y(t) = &\int_\Omega \left(\ln(u+e) + \frac{u}{u+e}\right)\left(\Delta u - \nabla\cdot(u\nabla v) + ru - \frac{\mu u^2}{\ln^p(u+e)}\right) \\
&\int_\Omega \left(\ln(w+e) + \frac{w}{w+e}\right)(\Delta w - \nabla\cdot(w\nabla z)) \\
&+ \tau A \int_\Omega \nabla v \cdot \nabla(\Delta v - v + w) \\
&+ \tau B \int_\Omega \nabla z \cdot \nabla(\Delta z - z + u) \\
&:= I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{4.3.10}
$$

In case $\tau = 1$, we use integration by parts to obtain:

$$
\begin{aligned}
I_1 = &-\int_\Omega \frac{|\nabla u|^2}{u+e} - \int_\Omega \frac{e|\nabla u|^2}{(u+e)^2} + \int_\Omega \left(\frac{u}{u+e} + \frac{eu}{(u+e)^2}\right)\nabla u\cdot\nabla v \\
&+ \int_\Omega \left(\ln(u+e) + \frac{u}{u+e}\right)\left(ru - \frac{\mu u^2}{\ln^p(u+e)}\right).
\end{aligned} \tag{4.3.11}
$$

Let us define

$$
\phi(u) := \int_0^u \left(\frac{s}{s+e} + \frac{es}{(s+e)^2}\right) ds,
$$

we see that $\phi(u) \le u$. When $\tau = 0$, by integration by parts and elementary inequality, we make use of the second equations to obtain that:

$$
\begin{aligned}
\int_\Omega \left(\frac{u}{u+e} + \frac{eu}{(u+e)^2}\right)\nabla u\cdot\nabla v &= \int_\Omega \nabla\phi(u)\cdot\nabla v = -\int_\Omega \phi(u)\Delta v \\
&= \int_\Omega \phi(u)(w-v) \le \int_\Omega uw \\
&\le \epsilon\int_\Omega w^2 + \frac{1}{4\epsilon}\int_\Omega u^2 \\
&\le \epsilon\int_\Omega w^2 + \epsilon\int_\Omega u^2\ln^{1-p}(u+e) + c,
\end{aligned} \tag{4.3.12}
$$

where the last inequality comes from the fact that for any $\delta > 0$, there exists $C > 0$ depending on δ such that

$$
u^2 \le \delta u^2 \ln^{1-p}(u+e) + C(\delta), \qquad 0 \le p < 1.
$$

We apply similar argument in case $\tau = 1$ to obtain

$$\begin{aligned}\int_\Omega \left(\frac{u}{u+e} + \frac{eu}{(u+e)^2}\right) \nabla u \cdot \nabla v &= -\int_\Omega \phi(u)\Delta v \\ &\le \epsilon \int_\Omega (\Delta v)^2 + \frac{1}{4\epsilon}\int_\Omega \phi^2(u) \\ &\le \epsilon \int_\Omega (\Delta v)^2 + \frac{1}{4\epsilon}\int_\Omega u^2 \\ &\le \epsilon \int_\Omega (\Delta v)^2 + \epsilon \int_\Omega u^2 \ln^{1-p}(u+e) + c, \end{aligned} \tag{4.3.13}$$

One can verify that there exists $c > 0$ depending on ϵ such that

$$\int_\Omega \left(\ln(u+e) + \frac{u}{u+e}\right)\left(ru - \frac{\mu u^2}{\ln^p(u+e)}\right) \le (\epsilon - \mu)\int_\Omega u^2 \ln^{1-p}(u+e) + c. \tag{4.3.14}$$

From (4.3.11) to (4.3.14), we obtain that

$$I_1 \le (2\epsilon - \mu)\int_\Omega u^2 \ln^{1-p}(u+e) + \epsilon \int_\Omega w^2 + c, \quad \text{for } \tau = 0, \tag{4.3.15}$$

and,

$$I_1 \le (2\epsilon - \mu)\int_\Omega u^2 \ln^{1-p}(u+e) + \epsilon \int_\Omega (\Delta v)^2 + c, \quad \text{for } \tau = 1. \tag{4.3.16}$$

By similar arguments, one can also obtain that

$$I_2 \le -\int_\Omega \frac{|\nabla w|^2}{w+e} + \epsilon \int_\Omega w^2 + \epsilon \int_\Omega u^2 \ln^{1-p}(u+e) + c, \quad \text{for } \tau = 0, \tag{4.3.17}$$

and

$$I_2 \le -\int_\Omega \frac{|\nabla w|^2}{w+e} + \epsilon \int_\Omega w^2 + \frac{1}{4\epsilon}\int_\Omega (\Delta z)^2, \quad \text{for } \tau = 1. \tag{4.3.18}$$

By integration by parts and elementary inequalities, we have

$$\begin{aligned} I_3 &= -\tau A \int_\Omega (\Delta v)^2 - \tau A \int_\Omega |\nabla v|^2 - \tau A \int_\Omega w \Delta v \\ &\le \tau\left(\frac{A^2}{4\epsilon} - A\right)\int_\Omega (\Delta v)^2 - \tau A \int_\Omega |\nabla v|^2 + \epsilon\tau \int_\Omega w^2, \end{aligned} \tag{4.3.19}$$

and

$$
\begin{aligned}
I_4 &= -\tau B \int_\Omega (\Delta z)^2 - \tau B \int_\Omega |\nabla z|^2 - \tau B \int_\Omega u \Delta z \\
&\leq \tau(\epsilon - B) \int_\Omega (\Delta z)^2 - \tau B \int_\Omega |\nabla z|^2 + \tau \frac{B^2}{4\epsilon} \int_\Omega u^2 \\
&\leq \tau(\epsilon - B) \int_\Omega (\Delta z)^2 - \tau B \int_\Omega |\nabla z|^2 + \tau\epsilon \int_\Omega u^2 \ln^{1-p}(u+e) + c. \qquad (4.3.20)
\end{aligned}
$$

One can verify that

$$
\int_\Omega u \ln(u+e) + \int_\Omega w \ln(w+e) \leq \epsilon \int_\Omega u^2 \ln^{1-p}(u+e) + \epsilon \int_\Omega w^2 + c. \qquad (4.3.21)
$$

From (4.3.15) to (4.3.21), we have

$$
\begin{aligned}
y'(t) + y(t) \leq& - \int_\Omega \frac{|\nabla w|^2}{w+e} + (4\epsilon - \mu) \int_\Omega u^2 \ln^{1-p}(u+e) + 3\epsilon \int_\Omega w^2 \\
&+ \tau \left(\frac{A^2}{4\epsilon} + \epsilon - A \right) \int_\Omega (\Delta v)^2 + \tau \left(\epsilon + \frac{1}{4\epsilon} - B \right) \int_\Omega (\Delta z)^2 + c \\
\leq& - \int_\Omega \frac{|\nabla w|^2}{w+e} + (4\epsilon - \mu) \int_\Omega u^2 \ln^{1-p}(u+e) + 3\epsilon \int_\Omega w^2, \qquad (4.3.22)
\end{aligned}
$$

where the last inequality comes from the fact that $\frac{A^2}{4\epsilon} + \epsilon - A = 0$ and $B = \epsilon + \frac{1}{4\epsilon}$. The third term can be controlled by Gagliardo–Nirenberg interpolation inequality

$$
\begin{aligned}
3\epsilon \int_\Omega w^2 &\leq 3C_{GN}\epsilon \int_\Omega \frac{|\nabla w|^2}{w+e} \int_\Omega (w+e) + 3C_{GN}\epsilon \left(\int_\Omega (w+e) \right)^2 \\
&\leq 3C_{GN}\epsilon \left(\int_\Omega w_0 + e|\Omega| \right) \int_\Omega \frac{|\nabla w|^2}{w+e} + 3C_{GN}\epsilon \left(\int_\Omega w_0 + e|\Omega| \right)^2. \qquad (4.3.23)
\end{aligned}
$$

From (4.3.22) and (4.3.23), we have

$$
\begin{aligned}
y'(t) + y(t) \leq& \left[3C_{GN}\epsilon \left(\int_\Omega w_0 + e|\Omega| \right) - 1 \right] \int_\Omega \frac{|\nabla w|^2}{w+e} \\
&+ (4\epsilon - \mu) \int_\Omega u^2 \ln^{1-p}(u+e) + c. \qquad (4.3.24)
\end{aligned}
$$

Given the inequalities $3C_{GN}\epsilon \left(\int_\Omega w_0 + e|\Omega| \right) - 1 \leq 0$ and $4\epsilon - \mu \leq 0$, we can deduce from (4.3.24) that $y'(t) + y(t) \leq c$. Finally we make use of Gronwall's inequality to complete the proof. □

We are now ready to prove the main theorem.

Proof of Theorem 4.3.1. We employ the arguments from [54][Lemma 4.2] with some modifications, and leverage Lemma 4.3.4 to derive the following inequality for all $t \in (0, T_{\max})$

$$\|u(\cdot,t)\|_{L^2(\Omega)} + \|w(\cdot,t)\|_{L^2(\Omega)} < C.$$

Subsequently, we apply Moser-type iterations, akin to [55][Lemma 3.2] to establish the boundedness of u and w in $\Omega \times (0, T_{\max})$. Combining this with (4.3.5) when $\tau = 0$ and (4.3.7) when $\tau = 1$, we conclude that $T_{\max} = \infty$. Employing elliptic regularity for $\tau = 0$, and parabolic regularity for $\tau = 1$, we have that $\sup_{t>0}\left(\|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|z(\cdot,t)\|_{W^{1,\infty}(\Omega)}\right) < \infty$. Consequently, we derive (4.3.4), thereby completing the proof. □

4.4 Blow-up prevention by sub-logistic sources under vanishing Neumann boundary condition

This section investigates the global existence of solutions to Keller-Segel systems with sub-logistic sources using the test function method. Prior work by [71] demonstrated that sub-logistic sources $f(u) = ru - \mu\frac{u^2}{\ln^p(u+e)}$ with $p \in (0,1)$ can prevent blow-up solutions for the 2D minimal Keller-Segel chemotaxis model. Our study extends this result by showing that when $p = 1$, sub-logistic sources can still prevent the occurrence of finite time blow-up solutions. Additionally, we provide a concise proof for a result previously proven in [9] that the equi-integrability of $\left\{\int_\Omega u^{\frac{n}{2}}(\cdot,t)\right\}_{t\in(0,T_{\max})}$ can avoid blow-up.

4.4.1 Introduction

In this section, we consider the following chemotaxis model with sub-logistic sources in a bounded domain with smooth boundary $\Omega \subset \mathbb{R}^n$, where $n \geq 2$:

$$\begin{cases} u_t = \Delta u - \nabla\cdot(u\nabla v) + f(u) \\ 0 = \Delta v + u - v, \end{cases} \tag{4.4.1}$$

where f is a smooth function generalizing the sub-logistic source,

$$f(u) = ru - \mu\frac{u^2}{\ln^p(u+e)}, \quad \text{with } r \in \mathbb{R}, \mu > 0, \text{and } p > 0. \tag{4.4.2}$$

The system (4.4.1) is complemented with nonnegative initial conditions in $W^{1,\infty}(\Omega)$ not identically zero:

$$u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), \qquad \text{with } x \in \Omega, \tag{4.4.3}$$

and homogeneous Neumann boundary condition are imposed as follows:

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \qquad x \in \partial\Omega,\ t \in (0, T_{\max}), \tag{4.4.4}$$

where ν denotes the outward normal vector.

The logistic sources, $f(u) := ru - \mu u^2$, was introduced and studied in [58] that if $\mu > \frac{n-2}{n}$ then solutions exist globally and are bounded at all time in a convex open bounded domain $\Omega \subset \mathbb{R}^n$ where $n \geq 2$. In order word, if μ is sufficiently large, then the quadratic term $-\mu u^2$ ensures no occurrence of blow-up solutions in two spacial dimensional domain. This leads to a natural question that whether the term "$-\mu u^2$" is optimal to prevent blow-up solutions. However, it has been discovered in [71] that the answer is negative. To be specific, the "weaker" term $-\frac{\mu u^2}{\ln^p(u+e)}$ for $0 < p < 1$ is sufficient to avoid blow-up solutions for both elliptic-parabolic and fully parabolic minimal Keller-Segel chemotaxis models in a two spacial dimensional domain. Our main work improve the previous finding by showing that $p = 1$ can prevent blow-up solutions of the system (4.4.1).

Our analysis relies on a test function method and Moser iteration technique. It is proved in [9] that if the family of $\left\{\int_\Omega u^{\frac{n}{2}}(\cdot,t)\right\}_{t\in(0,T_{\max})}$ is equi-integrable, then solutions of (4.4.1) when $f \equiv 0$ exist globally and remain bounded at all time. In this section, we give another shorter proof in Proposition 4.4.1 for that result as well as indicate that the equi-integrability is not optimal to prevent blow-up thank to de la Vallée-Poussin Theorem. Thereafter, we try to find a suitable functional and establish a differential inequality to obtain a priori estimate for solutions of (4.4.1) thank to the presence of the sub-logistic quadratic degradation term "$-\mu\frac{u^2}{\ln(u+e)}$". Indeed, the key milestone in this study is the choice of the following functional:

$$y(t) = \int_\Omega u(\cdot,t)\ln(\ln(u(\cdot,t)+e)), \tag{4.4.5}$$

which enables us to control the integral $\int_\Omega \frac{u^2}{\ln(u+e)}$ to establish a appropriate differential inequality. One can also try to examine a functional

$$y_k(t) = \int_\Omega u(\cdot,t)\ln^k(u(\cdot,t)+e)$$

to find an appropriate k, however, there is no suitable k satisfying the conditions that μ can be arbitrary small. In order word, this method leads to the choice of k, but it does require the largeness assumption for μ. So the functional (4.4.5) enables us to overcome that obstacle to prove our main theorem as follows:

Theorem 4.4.1. *Let $\mu > 0$, and $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. The system* (4.4.1) *under the assumptions* (4.4.2), (4.4.3), *and* (4.4.4) *admits a global bounded solution in* $\Omega \times (0,\infty)$.

Remark 4.4.1. *Theorem 4.4.1 is a special case of [72][Remark 1.1(ii)]*

.

4.4.2 Preliminaries

The local existence and uniqueness of non-negative classical solutions to the system (4.4.1) can be established by adapting and adjusting the fixed point argument and standard parabolic regularity theory. For further details, we refer the reader to [21, 29, 58]. For convenience, we adopt Lemma 4.1 from [61].

Lemma 4.4.2. *Let $\Omega \subset \mathbb{R}^n$, where $n \geq 2$ be a bounded domain with smooth boundary, and suppose $r \in \mathbb{R}$, $\mu > 0$, the conditions* (4.4.3), *and* (2.1.2) *hold. Then there exist $T_{max} \in (0,\infty]$ and functions*

$$\begin{cases} u \in C^0\left(\bar\Omega \times [0,T_{max})\right) \cap C^{2,1}\left(\bar\Omega \times (0,T_{max})\right) \text{ and} \\ v \in \bigcap_{q>2} C^0\left([0,T_{max}); W^{1,q}(\Omega)\right) \cap C^{2,1}\left(\bar\Omega \times (0,T_{max})\right) \end{cases} \tag{4.4.6}$$

such that $u > 0$ and $v > 0$ in $\bar{\Omega} \times (0, \infty)$, that (u, v) solves (4.4.1) *classically in $\Omega \times (0, T_{max})$, and that if $T_{max} < \infty$, then*

$$\limsup_{t \to T_{max}} \left\{ \|u\|_{L^\infty(\Omega)} + \|u\|_{W^{1,\infty}(\Omega)} \right\} = \infty. \tag{4.4.7}$$

4.4.3 A priori estimates and proof of main theorem

In this section, (u, v) is a classical solutions as defined in Lemma 4.4.2 to the system (4.4.1) with $p = 1$. Our aim is to establish a priori estimate for the solutions. While the method in [47] and [71] relies on the L^1-estimate of u and the absorption of $-\int_\Omega |\nabla u^{\frac{1}{2}}|^2$ to obtain a $L \ln L$ uniform bound, we take advantage of the term $-\mu \frac{u^2}{\ln(u+e)}$ to obtain a weaker $L \ln \ln L$ uniform bound. This result is a special case of [72][Remark 1.1(ii)]. Notice that we have adopted and modified the argument of [72] for the global existence of solutions in case $p = 1$.

Lemma 4.4.3. *There exists $C = C(u_0, v_0, |\Omega|, \mu) > 0$ such that*

$$\sup_{t \in (0, T_{max})} \int_\Omega u(\cdot, t) \ln(\ln(u(\cdot, t) + e)) \leq C. \tag{4.4.8}$$

Proof. We define $y(t) = \int_\Omega u \ln(\ln(u + e))$ and differentiate y to obtain

$$\begin{aligned}
y'(t) &= \int_\Omega \left[\ln(\ln(u+e)) + \frac{u}{(u+e)\ln(u+e)} \right] u_t \\
&= \int_\Omega \left[\ln(\ln(u+e)) + \frac{u}{(u+e)\ln(u+e)} \right] \left(\Delta u - \nabla \cdot (u \nabla v) + ru - \mu \frac{u^2}{\ln(u+e)} \right) \\
&= -\int_\Omega \nabla \left[\ln(\ln(u+e)) + \frac{u}{(u+e)\ln(u+e)} \right] \cdot \nabla u \\
&+ \int_\Omega u \nabla \left(\ln(\ln(u+e)) + \frac{u}{(u+e)\ln(u+e)} \right) \cdot \nabla v \\
&+ \int_\Omega \left[\ln(\ln(u+e)) + \frac{u}{(u+e)\ln(u+e)} \right] \left(ru - \mu \frac{u^2}{\ln(u+e)} \right) \\
&:= I + J + K
\end{aligned} \tag{4.4.9}$$

By integration by parts, we have

$$
\begin{aligned}
I &= -\int_\Omega \nabla \left[\ln(\ln(u+e)) + \frac{u}{(u+e)\ln(u+e)} \right] \cdot \nabla u \\
&= -\int_\Omega \left[\frac{1}{(u+e)\ln(u+e)} + \frac{e\ln(u+e) - u}{(u+e)^2 \ln^2(u+e)} \right] |\nabla u|^2 \\
&= -\int_\Omega \frac{u\ln(u+e) + 2e\ln(u+e) - u}{(u+e)^2 \ln^2(u+e)} |\nabla u|^2 \le 0. \qquad (4.4.10)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
J &= \int_\Omega u\nabla \left(\ln(\ln(u+e)) + \frac{u}{(u+e)\ln(u+e)} \right) \cdot \nabla v \\
&= \int_\Omega \frac{u^2(\ln(u+e) - 1) + 2eu\ln(u+e)}{(u+e)^2 \ln^2(u+e)} \nabla u \cdot \nabla v \\
&= \int_\Omega \nabla\phi(u) \cdot \nabla v = \int_\Omega \phi(u)(u - v) \le \int_\Omega u\phi(u), \qquad (4.4.11)
\end{aligned}
$$

where

$$
0 \le \phi(u) := \int_0^u \frac{s^2(\ln(s+e) - 1) + 2es\ln(s+e)}{(s+e)^2 \ln^2(s+e)}\, ds \le \int_0^u \frac{1}{\ln(s+e)}\, ds. \qquad (4.4.12)
$$

Thus, we obtain

$$
J \le \int_\Omega u \int_0^u \frac{1}{\ln(s+e)}\, ds. \qquad (4.4.13)
$$

By L'Hospital lemma, we have

$$
\lim_{u\to\infty} \frac{\int_0^u \frac{1}{\ln(s+e)}\, ds}{\frac{u\ln(\ln(u+e))}{\ln(u+e)}} = \lim_{u\to\infty} \frac{\ln(u+e)}{\ln(u+e)\ln(\ln(u+e)) + \frac{u}{u+e} - \frac{u}{u+e}\ln(\ln(u+e))} = 0. \qquad (4.4.14)
$$

Therefore, for any $\epsilon > 0$, there exist N depending on ϵ such that for $u > N$, we have

$$
\int_0^u \frac{1}{\ln(s+e)}\, ds \le \epsilon u \frac{\ln(\ln(u+e))}{\ln(u+e)}. \qquad (4.4.15)
$$

This leads to

$$
\begin{aligned}
\int_\Omega u \int_0^u \frac{1}{\ln(s+e)}\, ds &= \int_{u\le N} u \int_0^u \frac{1}{\ln(s+e)}\, ds + \int_{u>N} u \int_0^u \frac{1}{\ln(s+e)}\, ds \\
&\le \epsilon \int_\Omega u^2 \frac{\ln(\ln(u+e))}{\ln(u+e)} + c \qquad (4.4.16)
\end{aligned}
$$

where $c = N^2|\Omega|$. From (4.4.13) and (4.4.16), we have

$$J \leq \epsilon \int_\Omega u^2 \frac{\ln(\ln(u+e))}{\ln(u+e)} + c. \tag{4.4.17}$$

One can verify that for any $\epsilon > 0$, there exist $C(\epsilon) > 0$ such that

$$\begin{aligned} K &= \int_\Omega \left[\ln(\ln(u+e)) + \frac{u}{(u+e)\ln(u+e)}\right]\left(ru - \mu\frac{u^2}{\ln(u+e)}\right) \\ &\leq (\epsilon - \mu)\int_\Omega u^2 \frac{\ln(\ln(u+e))}{\ln(u+e)} + c \end{aligned} \tag{4.4.18}$$

and

$$y(t) \leq \epsilon \int_\Omega u^2 \frac{\ln(\ln(u+e))}{\ln(u+e)} + c. \tag{4.4.19}$$

Collect (4.4.9), (4.4.10), (4.4.13), (4.4.17),(4.4.18), and (4.4.19), we have

$$y'(t) + y(t) \leq (3\epsilon - \mu)\int_\Omega u^2 \frac{\ln(\ln(u+e))}{\ln(u+e)} + c. \tag{4.4.20}$$

We choose ϵ sufficiently small and apply Gronwall's inequality to imply $y(t) \leq C$ for all $t > 0$. $\square$

Let us recall de la Vallée-Poussin Theorem

Theorem 4.4.4 (de la Vallée-Poussin)**.** *The family $\{X_\alpha\}_{\alpha\in A} \subset L^1(\mu)$ is equi-integrable if and only if there exists a non-negative increasing convex function $G(t)$ such that*

$$\lim_{t\to\infty} \frac{G(t)}{t} = \infty \qquad \textit{and } \sup_\alpha \int_\Omega G(X_\alpha) < \infty.$$

Thank to Theorem 4.4.4, the equi-integrability of $\left\{\int_\Omega u^{\frac{n}{2}}(\cdot,t) < \infty\right\}_{t\in(0,T_{\max})}$ is equivalent to $\sup_{t\in(0,T_{\max})} \int_\Omega G(u^{\frac{n}{2}}(\cdot,t)) < \infty$ for some non-negative increasing convex function such that

$$\lim_{s\to\infty} \frac{G(s)}{s} = \infty.$$

However, the convexity condition is not necessary, which means that the equi-integrable condition can be relaxed. Indeed, following proposition gives us the L^q bounds, where $q > \frac{n}{2}$ for solutions without the convexity assumption.

Proposition 4.4.1. *Let $\Omega \subset \mathbb{R}^n$, where $n \geq 2$, be a bounded domain with smooth boundary, and $f \in C^2([0,\infty))$ such that $f(s) \leq c(s^2+1)$ for all $s \geq 0$, where $c > 0$. Assume that (u,v) is a classical solution as in Lemma 4.4.2 of* (4.4.1) *on $\Omega \times (0, T_{max})$ with maximal existence time $T_{max} \in (0,\infty]$. If there exists a nonnegative increasing function G such that*

$$\lim_{t\to\infty} \frac{G(s)}{s} = \infty \qquad \textit{and} \sup_{t\in(0,T_{max})} \int_\Omega G(u^{\frac{n}{2}}(\cdot,t)) < \infty,$$

then for any $q > \frac{n}{2}$ we have

$$\sup_{t\in(0,T_{max})} \int_\Omega u^q(\cdot,t) < \infty.$$

Proof. We define

$$\phi(t) := \frac{1}{q}\int_\Omega u^q,$$

where $q > \frac{n}{2}$, and differentiate ϕ to obtain

$$\begin{aligned}\phi'(t) &= \int_\Omega u^{q-1}[\Delta u - \nabla\cdot(u\nabla v) + f(u)] \\ &= -c_1\int_\Omega |\nabla u^{\frac{q}{2}}|^2 + c_2\int_\Omega u^{\frac{q}{2}}\nabla u^{\frac{q}{2}}\cdot\nabla v + c\int_\Omega u^{q+1} + u^{q-1} \\ &= I + J + K, \end{aligned} \tag{4.4.21}$$

where c_1, c_2 are positive depending only on q. We make use of integration by parts and the second equation of (4.4.1) to obtain

$$\begin{aligned} J := c_2\int_\Omega u^{\frac{q}{2}}\nabla u^{\frac{q}{2}}\cdot\nabla v &= -c_3\int_\Omega u^q\Delta v \\ &= -c_3\int_\Omega u^q(v-u) \leq c_3\int_\Omega u^{q+1}, \end{aligned} \tag{4.4.22}$$

where c_3 is positive depending only on q. From (4.4.21), (4.4.22), together with Young inequality, imply that there exists $c_4 = c_4(q) > 0$, and $c_5 = c_5(q, |\Omega|) > 0$ such that

$$\phi'(t) + \phi(t) \leq -c_1\int_\Omega |\nabla u^{\frac{q}{2}}|^2 + c_4\int_\Omega u^{q+1} + c_5. \tag{4.4.23}$$

We make use of Lemma B.0.9 to obtain that there exist $C > 0$ such that for any $\epsilon > 0$, there exists $c_6 = c_6(\epsilon) > 0$ such that

$$c_5\int_\Omega u^{q+1} \leq \epsilon\int_\Omega |\nabla u^{\frac{q}{2}}|^2 \left(\int_\Omega G(u^{\frac{n}{2}})\right)^{\frac{2}{n}} + C\left(\int_\Omega u\right)^{q+1} + c_6\int_\Omega u$$

This, together with the uniform bounded condition of $\int_\Omega G(u^{\frac{n}{2}}(\cdot,t))$ imply that

$$c_4 \int_\Omega u^{q+1} \le c_7\epsilon \int_\Omega |\nabla u^{\frac{q}{2}}|^2 + c_8, \tag{4.4.24}$$

where c_7 is positive independent of ϵ and $c_8 = c_8(\epsilon) > 0$. From (4.4.21) to (4.4.24), we obtain that

$$\phi'(t) + \phi(t) \le (c_7\epsilon - c_1) \int_\Omega |\nabla u^{\frac{q}{2}}|^2 + c_9, \tag{4.4.25}$$

where $c_9 = c_5 + c_8$. The proof is now completed by choosing $\epsilon < \frac{c_1}{c_7}$ and applying Gronwall's inequality. □

We are now ready to prove the main result.

Proof of Theorem 4.4.1. From Lemma 4.4.3, we obtain that there exists $C_1 > 0$ such that

$$\sup_{t\in(0,T_{\max})} \int_\Omega G(u(\cdot,t)) \le C_1,$$

where $G(s) := s\ln(\ln(s+e))$, satisfying all conditions of Proposition 4.4.1. Therefore, we can apply Proposition 4.4.1 to deduce that for any $q > 1$ there exists $C_2 = C_2(q) > 0$ such that

$$\sup_{t\in(0,T_{\max})} \int_\Omega u^q(\cdot,t) \le C_2.$$

This, together with the second equation and elliptic regularity theory imply that

$$\sup_{t\in(0,T_{\max})} \int_\Omega |\nabla v(\cdot,t)|^{2q} \le C_3,$$

for some $C_3 = C_3(q) > 0$. By applying Moser iteration procedure as in [1], [2] and [53], we obtain that

$$\sup_{t\in(0,T_{\max})} \|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} < \infty.$$

This, combined with (4.4.7), implies that $T_{\max} = \infty$ and uniform boundedness of (u,v). □

4.5 Blow-up prevention by sub-logistic sources in 2d chemotaxis system under nonlinear Neumann boundary conditions

This section deals with classical solutions to the chemotaxis system with sub-logistic sources, $ru - \frac{\mu u^2}{\ln^p(u+e)}$, where $r, p \geq 0$ and $\mu > 0$ under nonlinear Neumann boundary condition in a smooth bounded domain $\Omega \subset \mathbb{R}^2$. It is shown that if $p < 1$ in fully parabolic systems and $p \leq 1$ in parabolic- elliptic systems, then solutions exist globally and remain bounded in time. Our proof relies on several techniques, including parabolic regularity in Sobolev spaces, variational arguments, interpolation inequalities in Sobolev spaces, Trace Sobolev embedding theorem and Moser iteration method.

4.5.1 Introduction

In this section, we consider the following chemotaxis model with sub-logistic sources in a bounded domain with smooth boundary $\Omega \subset \mathbb{R}^2$:

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) + ru - \frac{\mu u^2}{\ln^p(u+e)} \\ v_t = \Delta v - v + u, \end{cases} \tag{4.5.1}$$

where $r, q \geq 0$, $\mu > 0$. The system (4.5.1) is complemented with nonnegative initial conditions in $C^{2+\gamma}(\Omega)$, where $\gamma \in (0,1)$, not identically zero:

$$u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), \qquad \text{with } x \in \Omega, \tag{4.5.2}$$

and nonlinear Neumann boundary condition are imposed as follows:

$$\frac{\partial u}{\partial \nu} = g(u), \qquad \frac{\partial v}{\partial \nu} = 0, \qquad x \in \partial\Omega,\ t \in (0, T_{\max}), \tag{4.5.3}$$

where ν is the outward normal vector and g is nonnegative in $C^1([0,\infty])$.

The problem (4.5.1) with nonlinear boundary condition (4.5.3) introduced and studied in [33] indicates that the quadratic degradation term can prevent blow-up in a smooth convex bounded domain $\Omega \subset \mathbb{R}^n$ with $n \geq 2$ when $g(s) = s^q$ for $q \geq 1$. To more specific, if $q \in \left(1, \frac{3}{2}\right)$ then solutions exist globally and remain bounded when $\mu > \frac{n-2}{n}$, with $n \geq 2$ and the borderline case when

$\mu = \frac{n-2}{n}$, and $p \in \left(1, 1+\frac{1}{n}\right)$ with $n \geq 3$ for parabolic-elliptic chemotaxis system. Moreover, similar result was also obtain for fully parabolic system when $n = 2, 3$. Especially, in two-dimensional domain for any $\mu > 0$ and $q \in \left(1, \frac{3}{2}\right)$, the system (4.5.1) with $p = 0$ possesses a unique positive classical solution which remains bounded at all time. Therefore, it is natural to ask:

Main Question: *"Can sub-logistic sources still avoid blow-up in a nonlinear Neumann boundary condition? "*

In this section, our objective is to address this question by employing modified arguments from [33] to handle the nonlinear term and drawing upon techniques from [71, 72] to handle the sub-logistic sources. We summarize our findings as follows:

Theorem 4.5.1. *Assume that* (u, v) *is a local classical solution of the system* (4.5.1) *under the conditions* (4.5.2)*, and* (4.5.3) *in* $\Omega \times (0, T_{max})$. *If* g *satisfies the following conditions:*

$$\lim_{s\to\infty} \frac{|g'(s)|}{\sqrt{s}} \ln^{\frac{p+1}{2}}(s+e) = 0, \qquad \textit{with } p < 1, \tau = 1, \tag{4.5.4}$$

or

$$\lim_{s\to\infty} \frac{|g'(s)| \ln(s+e) \ln^{\frac{1}{2}}(\ln(s+e))}{\sqrt{s}} = 0, \qquad \textit{with } p = 1, \tau = 0, \tag{4.5.5}$$

then $T_{max} = \infty$ *and* (u, v) *remains bounded at all time in the sense that*

$$\sup_{t\in(0,\infty)} \left\{ \|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} \right\} < \infty. \tag{4.5.6}$$

Remark 4.5.1. *The main results in [33] is a special case of Theorem 4.5.1 when replacing* $p = 0$*, and* $g(s) = |s|^q$ *for* $1 < q < \frac{3}{2}$ *by condition* (4.5.4) *for fully parabolic systems or* (4.5.5) *for parabolic-elliptic systems.*

Remark 4.5.2. *Our analysis does not work when replacing condition* (4.5.4) *by a weaker one*

$$\limsup_{s\to\infty} \frac{|g'(s)|}{\sqrt{s}} \ln^{\frac{p+1}{2}}(s+e) < \infty.$$

As a immediate consequence, we have the following result

Corollary 4.5.2. *Assume that $p < 1$, and (u, v) is a local classical solution of the system* (4.5.1) *under the conditions* (4.5.2)*, and* (4.5.3) *in $\Omega \times (0, T_{max})$. If g satisfies*

$$g(s) = s^q, \qquad or \qquad g(s) = \frac{s^{\frac{3}{2}}}{\ln^k(s+e)} \quad \text{for all } s \geq 0, \tag{4.5.7}$$

for any $1 < q < \frac{3}{2}$ and $k > \frac{p+1}{2}$, then $T_{max} = \infty$ and (u, v) remains bounded at all time in the sense that

$$\sup_{t \in (0,\infty)} \left\{ \|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} \right\} < \infty. \tag{4.5.8}$$

Remark 4.5.3. *We leave the open question whether the logistic sources can still prevent blow-up for $g(u) = \delta u^{\frac{3}{2}}$ for δ sufficiently small.*

Remark 4.5.4. *One can also adopt and modify the proof of Lemma 4.5.3 to obtain the global boundedness result for parabolic-elliptic case when*

$$g(s) = \delta \frac{s^{\frac{3}{2}}}{\ln^{\frac{p+1}{2}}(s+e)} \qquad \text{for all } s \geq 0,$$

for $\delta > 0$ sufficiently small.

The section is organized in three subsections. The key estimates, comprising $L \ln L$ and L^2 estimates, are provided in Subsection 4.5.2. Finally, we introduce the Moser iteration procedure to obtain an L^∞ bound for the solution, and then apply it to prove the main results in Subsection 4.5.3. Let us introduce local wellposedness results, which was established in [33].

Proposition 4.5.1. *If nonnegative functions u_0, v_0 are in $C^{2+\gamma}(\bar{\Omega})$ such that*

$$\frac{\partial u_0}{\partial \nu} = |u_0|^{1+\gamma} \qquad \text{on } \partial\Omega, \tag{4.5.9}$$

where $\gamma \in (0, 1)$. Then there exists $T_{max} \in (0, \infty]$ such that problem (4.5.1) *admits a unique nonnegative solution u, v in $C^{2+\gamma, 1+\gamma/2}(\bar{\Omega} \times (0, T_{max}))$. Moreover, if u_0, v_0 are not identically zero in Ω then u, v are strictly positive in $\Omega \times (0, T_{max})$. If $T_{max} < \infty$, then*

$$\limsup_{t \to T_{max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} = \infty. \tag{4.5.10}$$

4.5.2 A priori estimates

In this section, we denote $a := e^e$ and "c" as a universal constant that can vary depending on different parameters and may change over time. We also assume that (u, v) is a local classical solution of the system (4.5.1) under the conditions (4.5.2), and (4.5.3) in $\Omega \times (0, T_{\max})$. Our aim is to establish a priori estimate for the solutions. While the method in [47] and [71] relies on the L^1-estimate of u and the absorption of $-\int_\Omega |\nabla u^{\frac{1}{2}}|^2$ to obtain a $\int_\Omega u \ln u$, we take advantage of both terms$-\int_\Omega |\nabla u^{\frac{1}{2}}|^2$ and $-\mu \frac{u^2}{\ln(u+e)}$. Let us begin with an estimate of $\int_\Omega u \ln(u+e)$ as follows:

Lemma 4.5.3. *If then there exists $C > 0$ such that for all $t \in (0, T_{max})$, we have*

$$\int_\Omega u(\cdot, t) \ln(u(\cdot, t) + e) + |\nabla v(\cdot, t)|^2 < C. \tag{4.5.11}$$

Proof. Let call

$$y(t) := \int_\Omega u \ln(u+e) + \frac{1}{2}|\nabla v|^2,$$

we have

$$\begin{aligned} y'(t) &= \int_\Omega \left(\ln(u+e) + \frac{u}{u+e} \right) \left(\Delta u - \nabla \cdot (u \nabla v) + ru - \frac{\mu u^2}{\ln^p(u+e)} \right) \\ &\quad + \int_\Omega \nabla v \cdot \nabla (\Delta v - v + u) \\ &:= I + J. \end{aligned} \tag{4.5.12}$$

By integration by parts, I can be rewritten as:

$$\begin{aligned} I = &- \int_\Omega \frac{|\nabla u|^2}{u+e} - \int_\Omega \frac{e|\nabla u|^2}{(u+e)^2} + \int_\Omega \left(\frac{u}{u+e} + \frac{eu}{(u+e)^2} \right) \nabla u \cdot \nabla v \\ &+ \int_\Omega \left(\ln(u+e) + \frac{u}{u+e} \right) \left(ru - \frac{\mu u^2}{\ln^p(u+e)} \right) + \int_{\partial\Omega} \left(\ln(u+e) + \frac{u}{u+e} \right) g(u)\, dS. \end{aligned} \tag{4.5.13}$$

Let denote

$$\phi(u) := \int_0^u \frac{s}{s+e} + \frac{es}{(s+e)^2}\, ds$$

we see that $\phi(u) \le u$. By integration by parts and elementary inequality, we have that for any $\epsilon > 0$ there exists $c > 0$ depending on ϵ such that

$$\begin{aligned}\int_\Omega \left(\frac{u}{u+e} + \frac{eu}{(u+e)^2}\right) \nabla u \cdot \nabla v = \int_\Omega \nabla\phi(u) \cdot \nabla v = -\int_\Omega \phi(u)\Delta v \\ \le \epsilon \int_\Omega (\Delta v)^2 + \frac{1}{4\epsilon}\int_\Omega \phi^2(u) \\ \le \epsilon \int_\Omega (\Delta v)^2 + \frac{1}{4\epsilon}\int_\Omega u^2 \\ \le \epsilon \int_\Omega (\Delta v)^2 + \epsilon \int_\Omega u^2 \ln^{1-p}(u+e) + c, \end{aligned} \tag{4.5.14}$$

where the last inequality comes from the fact that for any $\delta > 0$, there exists $C > 0$ depending on δ such that

$$u^2 \le \delta u^2 \ln^{1-p}(u+e) + C(\delta), \qquad 0 \le p < 1.$$

This also implies that for any $\epsilon > 0$, there exists $c > 0$ depending on ϵ such that

$$\int_\Omega \left(\ln(u+e) + \frac{u}{u+e}\right)\left(ru - \frac{\mu u^2}{\ln^p(u+e)}\right) \le (\epsilon - \mu)\int_\Omega u^2 \ln^{1-p}(u+e) + c. \tag{4.5.15}$$

By trace Sobolev's embedding theorem $W^{1,1}(\Omega) \to L^1(\partial\Omega)$, we have

$$\begin{aligned}\int_{\partial\Omega} \left(\ln(u+e) + \frac{u}{u+e}\right) g(u)\, dS \le C \int_\Omega \left(\ln(u+e) + \frac{u}{u+e}\right)|g(u)| \\ + C\int_\Omega \left(\frac{1}{u+e} + \frac{e}{(u+e)^2}\right)|g(u)||\nabla u| \\ + C\int_\Omega \left(\ln(u+e) + \frac{u}{u+e}\right)|g'(u)||\nabla u|. \end{aligned} \tag{4.5.16}$$

The conditions (4.5.4) implies that

$$\lim_{s\to\infty} \frac{|g(s)|}{s^{\frac{3}{2}}} = 0.$$

This, together with elementary inequalities deduces that for any $\epsilon > 0$, there exists $c > 0$ depending on ϵ such that

$$C\int_\Omega \left(\ln(u+e) + \frac{u}{u+e}\right)|g(u)| \le \epsilon \int_\Omega u^2 \ln^{1-p}(u+e) + c. \tag{4.5.17}$$

By similar argument, one can also verify that

$$
\begin{aligned}
C\int_\Omega\left(\frac{1}{u+e}+\frac{e}{(u+e)^2}\right)|g(u)||\nabla u| &\le 2C\int_\Omega\frac{1}{u+e}|g(u)||\nabla u|\\
&\le \epsilon\int_\Omega\frac{|\nabla u|^2}{u+e}+\frac{C^2}{\epsilon}\int_\Omega\frac{g^2(u)}{u+e}\\
&\le \epsilon\int_\Omega\frac{|\nabla u|^2}{u+e}+\epsilon\int_\Omega u^2\ln^{1-p}(u+e)+c, \qquad (4.5.18)
\end{aligned}
$$

where $c>0$ depending on ϵ. From the condition (4.5.4), we have

$$
\begin{aligned}
C\int_\Omega\left(\ln(u+e)+\frac{u}{u+e}\right)|g'(u)||\nabla u| &\le 2C\int_\Omega\ln(u+e)|g'(u)||\nabla u|\\
&\le \epsilon\int_\Omega\frac{|\nabla u|^2}{u+e}+\frac{C^2}{\epsilon}\int_\Omega(u+e)\ln^2(u+e)|g'(u)|^2\\
&\le \epsilon\int_\Omega\frac{|\nabla u|^2}{u+e}+\epsilon\int_\Omega u^2\ln^{1-p}(u+e)+c, \qquad (4.5.19)
\end{aligned}
$$

for any $\epsilon>0$, and $c=c(\epsilon)>0$. Collecting from (4.5.16) to (4.5.19) and replacing ϵ by $\epsilon/3$, we have

$$
\int_{\partial\Omega}\left(\ln(u+e)+\frac{u}{u+e}\right)g(u)\,dS\le\epsilon\int_\Omega\frac{|\nabla u|^2}{u+e}+\epsilon\int_\Omega u^2\ln^{1-p}(u+e)+c, \qquad (4.5.20)
$$

for any $\epsilon>0$, and $c=c(\epsilon)>0$. Combining (4.5.13), (4.5.14), (4.5.15), and (4.5.20), and choosing ϵ sufficiently small, we obtain

$$
I\le-\frac{1}{2}\int_\Omega\frac{|\nabla u|^2}{u+e}-\frac{\mu}{2}\int_\Omega u^2\ln^{1-p}(u+e)+\epsilon\int_\Omega(\Delta v)^2+c. \qquad (4.5.21)
$$

By integration by parts and elemental inequalities, we obtain that for any $\epsilon>0$, there exist $c>0$ depending on ϵ such that

$$
\begin{aligned}
J &= -\int_\Omega(\Delta v)^2-\int_\Omega|\nabla v|^2-\int_\Omega u\Delta v\\
&\le -\frac{1}{2}\int_\Omega(\Delta v)^2-\int_\Omega|\nabla v|^2+\frac{1}{2}\int_\Omega u^2\\
&\le -\frac{1}{2}\int_\Omega(\Delta v)^2-\int_\Omega|\nabla v|^2+\epsilon\int_\Omega u^2\ln^{1-p}(u+e)+c. \qquad (4.5.22)
\end{aligned}
$$

For any $\epsilon>0$, there exist a positive constant $c>0$ depending on ϵ such that

$$
\int_\Omega u\ln(u+e)\le\epsilon\int_\Omega u^2\ln^{1-p}(u+e)+c. \qquad (4.5.23)
$$

This, together with (4.5.12), (4.5.21), and (4.5.22), implies that

$$y'(t) + y(t) \le (\epsilon - \frac{1}{2}) \int_\Omega |\nabla u|^2 - \frac{1}{2} \int_\Omega |\nabla v|^2 + (\epsilon - \frac{\mu}{2}) \int_\Omega u^2 \ln^{1-p}(u+e) + c. \tag{4.5.24}$$

We choose ϵ sufficiently small and apply Gronwall's inequality to complete the proof. □

The following lemma provides us an essential estimate on the boundary. The following lemma provides us an essential estimate on the boundary.

Lemma 4.5.4. *Assume that g satisfies the condition (4.5.5), then for any $\epsilon > 0$, there exists a positive constant C depending on ϵ such that for any $u \in C^1(\bar{\Omega})$:*

$$\int_{\partial\Omega} \left[\ln(\ln(u+a)) + \frac{u}{(u+a)\ln(u+a)} \right] g(u) \le \epsilon \int_\Omega \frac{|\nabla u|^2}{(u+a)\ln(u+a)} + \epsilon \int_\Omega \frac{u^2 \ln(\ln(u+a))}{\ln(u+a)} + C. \tag{4.5.25}$$

Proof. By trace Sobolev embedding theorem $W^{1,1}(\Omega) \to L^1(\partial\Omega)$, we have

$$\begin{aligned}
&\int_{\partial\Omega} \left[\ln(\ln(u+a)) + \frac{u}{(u+a)\ln(u+a)} \right] g(u) \le C \int_\Omega \left[\ln(\ln(u+a)) + \frac{u}{(u+a)\ln(u+a)} \right] g(u) \\
&+ C \int_\Omega \left[\ln(\ln(u+a)) + \frac{u}{(u+a)\ln(u+a)} \right] |g'(u)||\nabla u| \\
&+ C \int_\Omega \left[\frac{1}{(u+a)\ln(u+a)} + \frac{(u+a)\ln(u+a) - u\ln(u+a) - u}{(u+a)^2 \ln^2(u+a)} \right] |\nabla u||g(u)| \\
&:= I + J + K.
\end{aligned} \tag{4.5.26}$$

The condition (4.5.5) entails that for any $\epsilon > 0$

$$I \le \epsilon \int_\Omega \frac{u^2 \ln(\ln(u+a))}{\ln(u+a)} + c(\epsilon). \tag{4.5.27}$$

By applying Young's inequality and then using condition (4.5.5), we have

$$\begin{aligned}
J &\le \epsilon \int_\Omega \frac{|\nabla u|^2}{(u+a)\ln(u+a)} + c(\epsilon) \int_\Omega (u+a)\ln(u+a)\ln(\ln(u+a))|g'(u)|^2 \\
&\le \epsilon \int_\Omega \frac{|\nabla u|^2}{(u+a)\ln(u+a)} + \epsilon \int_\Omega \frac{u^2 \ln(\ln(u+a))}{\ln(u+a)} + c(\epsilon).
\end{aligned} \tag{4.5.28}$$

One can verify that

$$K \le C \int_\Omega \frac{1}{(u+a)\ln(u+a)} |\nabla u||g(u)|.$$

Applying Young's inequality to the right, we obtain

$$\begin{aligned} K &\le \epsilon \int_\Omega \frac{|\nabla u|^2}{(u+a)\ln(u+a)} + c(\epsilon)\int_\Omega \frac{|g(u)|^2}{(u+a)\ln(u+a)} \\ &\le \epsilon \int_\Omega \frac{|\nabla u|^2}{(u+a)\ln(u+a)} + \epsilon \int_\Omega \frac{u^2\ln(\ln(u+a))}{\ln(u+a)} + c(\epsilon), \end{aligned} \tag{4.5.29}$$

where the last inequality comes from a consequence of condition (4.5.5):

$$\lim_{s\to\infty} \frac{|g(s)|}{s^{3/2}} = 0.$$

Collecting from (4.5.26) to (4.5.29), we obtain (4.5.25). □

Next, we derive a priori estimate for solutions of parabolic-elliptic systems.

Lemma 4.5.5. *If $\tau = 0$ and $p = 1$ then there exists $C > 0$ such that for all $t \in (0, T_{max})$, we have*

$$\int_\Omega u(\cdot,t)\ln(\ln(u(\cdot,t)+a)) \le C. \tag{4.5.30}$$

Proof. We define $y(t) = \int_\Omega u\ln(\ln(u+a))$ and differentiate y to obtain

$$\begin{aligned} y'(t) &= \int_\Omega \left[\ln(\ln(u+a)) + \frac{u}{(u+a)\ln(u+a)}\right] u_t \\ &= \int_\Omega \left[\ln(\ln(u+a)) + \frac{u}{(u+a)\ln(u+a)}\right]\left(\Delta u - \nabla\cdot(u\nabla v) + ru - \mu\frac{u^2}{\ln(u+a)}\right) \\ &= -\int_\Omega \nabla\left[\ln(\ln(u+a)) + \frac{u}{(u+a)\ln(u+a)}\right]\cdot\nabla u \\ &+ \int_\Omega u\nabla\left(\ln(\ln(u+a)) + \frac{u}{(u+a)\ln(u+a)}\right)\cdot\nabla v \\ &+ \int_\Omega \left[\ln(\ln(u+a)) + \frac{u}{(u+a)\ln(u+a)}\right]\left(ru - \mu\frac{u^2}{\ln(u+a)}\right) \\ &+ \int_{\partial\Omega} \left[\ln(\ln(u+a)) + \frac{u}{(u+a)\ln(u+a)}\right] g(u) \\ &:= I + J + K + L \end{aligned} \tag{4.5.31}$$

By integration by parts, we have

$$\begin{aligned} I &= -\int_\Omega \nabla\left[\ln(\ln(u+a)) + \frac{u}{(u+a)\ln(u+a)}\right]\cdot\nabla u \\ &= -\int_\Omega \left[\frac{1}{(u+a)\ln(u+a)} + \frac{a\ln(u+a) - u}{(u+a)^2\ln^2(u+a)}\right]|\nabla u|^2 \end{aligned}$$

$$
\begin{aligned}
&= -\int_\Omega \frac{u\ln(u+a) + 2a\ln(u+a) - u}{(u+a)^2\ln^2(u+a)}|\nabla u|^2 \\
&\le -\frac{1}{2}\int_\Omega \frac{|\nabla u|^2}{(u+a)\ln(u+a)}. \qquad (4.5.32)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
J &= \int_\Omega u\nabla\left(\ln(\ln(u+a)) + \frac{u}{(u+a)\ln(u+a)}\right)\cdot\nabla v \\
&= \int_\Omega \frac{u^2(\ln(u+a)-1) + 2au\ln(u+a)}{(u+a)^2\ln^2(u+a)}\nabla u\cdot\nabla v \\
&= \int_\Omega \nabla\phi(u)\cdot\nabla v = \int_\Omega \phi(u)(u-v) \le \int_\Omega u\phi(u), \qquad (4.5.33)
\end{aligned}
$$

where

$$
0 \le \phi(u) := \int_0^u \frac{s^2(\ln(s+a)-1) + 2as\ln(s+a)}{(s+a)^2\ln^2(s+a)}\,ds \le \int_0^u \frac{1}{\ln(s+a)}\,ds. \qquad (4.5.34)
$$

Thus, we obtain

$$
J \le \int_\Omega u\int_0^u \frac{1}{\ln(s+a)}\,ds. \qquad (4.5.35)
$$

By L'Hospital lemma, we have

$$
\lim_{u\to\infty} \frac{\int_0^u \frac{1}{\ln(s+a)}\,ds}{\frac{u\ln(\ln(u+a))}{\ln(u+a)}} = \lim_{u\to\infty} \frac{\ln(u+a)}{\ln(u+a)\ln(\ln(u+a)) + \frac{u}{u+a} - \frac{u}{u+a}\ln(\ln(u+a))} = 0. \qquad (4.5.36)
$$

Therefore, for any $\epsilon > 0$, there exist N depending on ϵ such that for $u > N$, we have

$$
\int_0^u \frac{1}{\ln(s+a)}\,ds \le \epsilon u\frac{\ln(\ln(u+a))}{\ln(u+a)}. \qquad (4.5.37)
$$

This leads to

$$
\begin{aligned}
\int_\Omega u\int_0^u \frac{1}{\ln(s+a)}\,ds &= \int_{u\le N} u\int_0^u \frac{1}{\ln(s+a)}\,ds + \int_{u>N} u\int_0^u \frac{1}{\ln(s+a)}\,ds \\
&\le \epsilon\int_\Omega u^2\frac{\ln(\ln(u+a))}{\ln(u+a)} + c \qquad (4.5.38)
\end{aligned}
$$

where $c = N^2|\Omega|$. From (4.5.35) and (4.5.38), we have

$$
J \le \epsilon\int_\Omega u^2\frac{\ln(\ln(u+a))}{\ln(u+a)} + c. \qquad (4.5.39)
$$

One can verify that for any $\epsilon > 0$, there exist $C(\epsilon) > 0$ such that

$$\begin{aligned} K &= \int_\Omega \left[\ln(\ln(u+a)) + \frac{u}{(u+a)\ln(u+a)}\right]\left(ru - \mu\frac{u^2}{\ln(u+a)}\right) \\ &\le (\epsilon - \mu)\int_\Omega u^2\frac{\ln(\ln(u+a))}{\ln(u+a)} + c(\epsilon). \end{aligned} \tag{4.5.40}$$

From (4.5.25), we have

$$L \le \epsilon\int_\Omega \frac{|\nabla u|^2}{(u+a)\ln(u+a)} + \epsilon\int_\Omega \frac{u^2\ln(\ln(u+a))}{\ln(u+a)} + c(\epsilon). \tag{4.5.41}$$

Furthermore, we have

$$y(t) \le \epsilon\int_\Omega u^2\frac{\ln(\ln(u+a))}{\ln(u+a)} + c. \tag{4.5.42}$$

Collect (4.5.31), (4.5.32), (4.5.35), (4.5.39),(4.5.40), (4.5.41) and (4.5.42), we have

$$y'(t) + y(t) \le (4\epsilon - \mu)\int_\Omega u^2\frac{\ln(\ln(u+a))}{\ln(u+a)} + c. \tag{4.5.43}$$

We choose ϵ sufficiently small and apply Gronwall's inequality to imply $y(t) \le C$ for all $t > 0$. □

Consequently, an L^2-estimate of u is derived as follows:

Lemma 4.5.6. *If g satisfies* (4.5.4) *and inequality* (4.5.11) *holds then there exists $C > 0$ such that*

$$\int_\Omega u^2(\cdot,t) + \tau\int_\Omega |\Delta v(\cdot,t)|^2 < C, \tag{4.5.44}$$

for all $t \in (0, T_{max})$.

Proof. Let call

$$y(t) = \frac{1}{2}\int_\Omega u^2 + \frac{\tau}{2}\int_\Omega (\Delta v)^2,$$

we have

$$\begin{aligned} y'(t) &= \int_\Omega u\left(\Delta u - \nabla\cdot(u\nabla v) + ru - \frac{\mu u^2}{\ln^p(u+e)}\right) \\ &+ \tau\int_\Omega \Delta v\Delta v_t \end{aligned}$$

$$= I + J. \tag{4.5.45}$$

By integration by parts, I can be rewritten as

$$I = -\int_\Omega |\nabla u|^2 + \int_\Omega u\nabla u \cdot \nabla v + r\int_\Omega u^2 - \mu\int_\Omega \frac{u^3}{\ln^p(u+e)} + \int_{\partial\Omega} ug(u)\, dS. \tag{4.5.46}$$

In case $\tau = 0$, we use the second equation of (4.5.1) to obtain

$$\int_\Omega u\nabla u \cdot \nabla v = -\int_\Omega u^2 \Delta v = \int_\Omega u^2(u-v) \le \int_\Omega u^3. \tag{4.5.47}$$

When $\tau = 1$, we use Young's inequality to obtain

$$\int_\Omega u\nabla u \cdot \nabla v = -\int_\Omega u^2 \Delta v \le \epsilon \int_\Omega (\Delta v)^3 + c\int_\Omega u^3, \tag{4.5.48}$$

where $c > 0$ depending on ϵ. By trace Sobolev's embedding Theorem $W^{1,1}(\Omega) \to L^1(\partial\Omega)$, we obtain

$$\int_{\partial\Omega} ug(u)\, dS \le C\int_\Omega u|g(u)| + C\int_\Omega |g(u)||\nabla u| + C\int_\Omega u|g'(u)||\nabla u|.$$

When g satisfies condition (4.5.4), we apply Young's inequality to the last two terms of the right hand side to obtain

$$\int_{\partial\Omega} ug(u)\, dS \le \epsilon\int_\Omega |\nabla u|^2 + \epsilon\int_\Omega u^3 + c, \tag{4.5.49}$$

where $c > 0$ depending on ϵ. We make use of the following inequality established in [30][Lemma 3.3] for any $\delta > 0$

$$\int_\Omega u^3 \le \delta\int_\Omega |\nabla u|^2 \int_\Omega G(u) + C\left(\int_\Omega u\right)^3 + c(\delta)\left(\int_\Omega u\right),$$

where

$$G(u) = \begin{cases} u\ln(u+e), & \text{when } \tau = 1 \\ u\ln(\ln(u+a)), & \text{when } \tau = 0. \end{cases}$$

We apply Lemma 4.5.3 to obtain that

$$\int_\Omega u^3 \le \epsilon\int_\Omega |\nabla u|^2 + c, \tag{4.5.50}$$

where $c > 0$ depending on ϵ. Collecting (4.5.46), (4.5.48), (4.5.49)and (4.5.50), we have

$$I \leq (\epsilon - 1)\int_\Omega |\nabla u|^2 + \epsilon \int_\Omega (\Delta v)^3 + c,$$

for any $\epsilon > 0$, and $c > 0$ depending on ϵ. Since $\sup_{t\in(0,T_{\max})}\int_\Omega |\nabla v|^2 < \infty$, the following inequality (see [68]) holds

$$\int_\Omega (\Delta v)^3 \leq C\int_\Omega |\nabla \Delta v|^2 + C.$$

This leads to

$$I \leq (\epsilon - 1)\int_\Omega |\nabla u|^2 + \epsilon \int_\Omega |\nabla \Delta v|^2 + c, \tag{4.5.51}$$

for any $\epsilon > 0$, and $c > 0$ depending on ϵ. By integration by parts, and Young's inequality, we have

$$\begin{aligned} J &= -\tau \int_\Omega |\nabla\Delta v|^2 - \tau\int_\Omega (\Delta v)^2 - \tau \int_\Omega \nabla\Delta v \cdot \nabla u \\ &\leq -\frac{\tau}{2}\int_\Omega |\nabla \Delta v|^2 - \tau \int_\Omega (\Delta v)^2 + \frac{\tau}{2}\int_\Omega |\nabla u|^2. \end{aligned} \tag{4.5.52}$$

Collecting (4.5.51),(4.5.52), and using the following inequality

$$\int_\Omega u^2 \leq \epsilon \int_\Omega |\nabla u|^2 + c(\epsilon),$$

thereafter choosing ϵ sufficiently small, we obtain that

$$y'(t) + y(t) \leq C, \tag{4.5.53}$$

for some positive constant C. This, together with Gronwall's inequality asserts that
$y(t) \leq \max\{y(0), C\}$ for all $t \in (0, T_{\max})$. □

4.5.3 Regularity and proof of main results

In this section, we show that if u is uniformly bounded in time under $\|\cdot\|_{L^2(\Omega)}$ then it is also uniformly bounded in time under $\|\cdot\|_{L^\infty(\Omega)}$. We consider the following equation, which is more general than (4.5.1)

$$\begin{cases} u_t = \Delta u - \nabla\cdot(u\nabla v) + f(u) \\ v_t = \Delta v - v + u, \end{cases} \tag{4.5.54}$$

where f is continuous such that $f(s) \leq c + cs^2$ for all $s \geq 0$ with $c \geq 0$.

Theorem 4.5.7. *Assume that*

$$g(s) \leq \alpha s^{\frac{3}{2}}, \qquad s \geq 0, \tag{4.5.55}$$

where $\alpha > 0$. *Let* (u, v) *be a classical solution of* (4.5.54) *under conditions* (4.5.2) *and* (4.5.3) *in* $\Omega \times (0, T_{max})$ *with maximal existence time* $T_{max} \in (0, \infty]$. *If*

$$\sup_{t \in (0, T_{max})} \|u(\cdot, t)\|_{L^2(\Omega)} < \infty,$$

then

$$\sup_{t \in (0, T_{max})} \left(\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} \right) < \infty.$$

Proof of Theorem 2.5.1. It is the immediate consequence of Theorem C.2.1 when $n = 2$, $g(u) = \alpha u^{\frac{3}{2}}$ and $f(u) = ru - \frac{\mu u^2}{\ln^q(u+e)} \leq ru$. □

We are now ready to prove the main theorem

Proof of Theorem 4.5.1. It is the immediate consequence of Lemma 4.5.3, 4.5.5, 4.5.6 and Theorem 4.5.7. □

CHAPTER 5

SUPERLINEAR CROSS-DIFFUSION ; SUPERLINEAR SIGNAL PRODUCTION

In this chapter, we investigate the global existence of solutions to some chemotaxis models with superlinear cross diffusion rates and superlinear signal production. It is shown that the appreence of the quadratic degradation terms can ensure to exclude blow-up phenomenon in those models. The pivotal analysis tool is the regularity theory in Orlicz spaces for elliptic and parabolic equations, which enables us to eliminate degeneracies of the diffusion terms. Subsequently, we can apply the well-established framework in previous chapters to obtain the global existence and boundedness of solutions.

5.1 Degenerate chemotaxis systems with superlinear growth in cross-diffusion rates and logistic sources

The objective is to investigate the global existence of solutions for degenerate chemotaxis systems with logistic sources in a two-dimensional domain. It is demonstrated that the inclusion of logistic sources can exclude the occurrence of blow-up solutions, even in the presence of superlinear growth in the cross-diffusion rate. Our proof relies on the application of elliptic and parabolic regularity in Orlicz spaces and variational approach.

5.1.1 Introduction

We consider the following system arising from chemotaxis in a smooth bounded domain $\Omega \subset \mathbb{R}^2$:

$$\begin{cases} u_t &= \nabla \cdot (D(v)\nabla(u)) - \nabla \cdot (S(v)u \ln^{\alpha}(u+e)\nabla v) + ru - \mu u^2 \\ \eta v_t &= \Delta v - v + u, \end{cases} \tag{5.1.1}$$

where $\eta \in \{0,1\}$, $r \in \mathbb{R}$, $\mu \geq 0$, $\alpha \geq 0$, and

$$0 < D \in C^2([0,\infty)) \quad \text{and } S \in C^2([0,\infty)) \cap W^{1,\infty}((0,\infty)) \text{ such that } S' \geq 0. \tag{5.1.2}$$

The system (5.1.1) is complemented with nonnegative, initial conditions in $W^{1,\infty}(\Omega)$ not identically zero:

$$u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), \qquad \text{with } x \in \Omega, \tag{5.1.3}$$

and homogeneous Neumann boundary condition are imposed as follows:

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \qquad x \in \partial\Omega,\ t \in (0, T_{\max}), \tag{5.1.4}$$

where ν denotes the outward normal vector.

Our main goal is to show that the quadratic logistic degradation term can effectively prevent blow-up for both elliptic-parabolic and fully parabolic degenerate chemotaxis models with superlinear growth in the cross-diffusion rate where $\alpha > 0$. To be more precise, our main result reads as follows:

Theorem 5.1.1. *Suppose that* $\eta = 0$ *and* $\alpha \in (0, 1)$ *then the system* (5.1.1) *under the assumptions* (5.1.2)*,* (5.1.3) *and* (5.1.4) *admits a global classical solution* (u, v) *in* $\left[C^0\left(\bar{\Omega} \times [0, \infty)\right) \cap C^{2,1}\left(\bar{\Omega} \times (0, \infty)\right)\right]^2$ *such that* $u > 0$ *and* $v > 0$ *in* $\bar{\Omega} \times (0, \infty)$*. Furthermore, this solution is bounded in the sense that*

$$\sup_{t>0} \left\{ \|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} \right\} < \infty. \tag{5.1.5}$$

For fully parabolic cases, we have the following theorem:

Theorem 5.1.2. *Suppose* $\eta = 1$ *and* $\alpha \in (0, \frac{1}{2})$ *then the system* (5.1.1) *under the assumptions* (5.1.2)*,* (5.1.3) *and* (5.1.4) *admits a global classical solution* (u, v) *with*

$$\begin{cases} u \in C^0\left(\bar{\Omega} \times [0, \infty)\right) \cap C^{2,1}\left(\bar{\Omega} \times (0, \infty)\right) & \textit{and} \\ v \in \cap_{q>2} C^0\left([0, \infty); W^{1,q}(\Omega)\right) \cap C^{2,1}\left(\bar{\Omega} \times (0, \infty)\right), \end{cases}$$

such that $u > 0$ *and* $v > 0$ *in* $\bar{\Omega} \times (0, \infty)$*. Furthermore, this solution is bounded in the sense that*

$$\sup_{t>0} \left\{ \|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} \right\} < \infty. \tag{5.1.6}$$

The proof of the main results can be summarized into three steps:

1. **Derive an initial estimate for solutions:**

$$\sup_{t \in (0, T_{\max})} \int_\Omega u \ln^k(u + e) + \eta |\nabla v|^2 < \infty, \qquad \text{for some } k \geq 1.$$

To accomplish this, we adapt and modify the argument presented in [30][Lemma 4.1] for the proof of Lemma 5.1.7 and 5.1.10.

2. **Address the degeneracy of the diffusion term:** Eliminate the degeneracy of the diffusion term by employing elliptic and parabolic regularity in Orlicz spaces. The proof of the elliptic part is provided in Lemma 5.1.6, and we apply the parabolic part as established in [61].

3. **Establish L^p bounds for the solution:** Lemma 5.1.8 and 5.1.11 establish L^p bounds for the solution for any $p > 1$. The primary challenge lies in incorporating the term $\int_\Omega u^p \ln^\alpha(u+e)$ into the diffusion term. Overcoming this difficulty involves the utilization of logarithmically refined Gagliardo-Nirenberg interpolation inequalities, as established in [66].

This section is structured as follows. In Subsection 5.1.2, we revisit local existence results for both elliptic-parabolic and fully parabolic models, along with key inequalities used in subsequent sections. We also provide results on regularity in Orlicz spaces. Subsection 5.1.3 presents a priori estimates, including $L \ln^k L$ and L^p estimates for solutions of elliptic-parabolic models when $\eta = 0$, and includes the proof of Theorem 5.1.1. Subsection 5.1.4 follows a similar framework but addresses the fully parabolic case when $\eta = 1$ and includes the proof of Theorem 5.1.2.

5.1.2 Preliminaries

By employing fixed point arguments and applying standard theories of elliptic and parabolic regularity, we can establish the local existence and uniqueness of non-negative classical solutions to the system (5.1.1). Our initial step involves establishing the local existence of solutions for parabolic-elliptic chemotaxis models, and we achieve this by adapting the method presented in [59][Theorem 2.1].

Lemma 5.1.3. *Let $\eta = 0$ and $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and that* (5.1.2)*,* (5.1.3)*, and* (5.1.4) *hold. Then there exist $T_{max} \in (0, \infty]$ and functions (u, v) in* $\left[C^0\left(\bar{\Omega} \times [0, T_{max})\right) \cap C^{2,1}\left(\bar{\Omega} \times (0, T_{max})\right)\right]^2$ *such that $u > 0$ and $v > 0$ in $\bar{\Omega} \times (0, \infty)$, that (u, v)*

solves (5.1.1) *classically in* $\Omega \times (0, T_{max})$*, and that*

$$\text{if } T_{max} < \infty, \quad \text{then } \limsup_{t \to T_{max}} \left\{ \|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} \right\} = \infty. \tag{5.1.7}$$

The local existence of solutions for fully parabolic models can be attained by modifying and adjusting the proof in [64][Lemma 1.1] or referring to [21, 29].

Lemma 5.1.4. *Let* $\eta = 1$ *and* $\Omega \subset \mathbb{R}^2$ *be a bounded domain with smooth boundary and that* (5.1.2)*,* (5.1.3)*, and* (5.1.4) *hold. Then there exist* $T_{max} \in (0, \infty]$ *and functions*

$$\begin{cases} u \in C^0\left([0, T_{max}); C^0(\bar{\Omega})\right) \cap C^{2,1}\left(\bar{\Omega} \times (0, T_{max})\right) \text{ and} \\ v \in \bigcap_{q>2} C^0\left([0, T_{max}); W^{1,q}(\Omega)\right) \cap C^{2,1}\left(\bar{\Omega} \times (0, T_{max})\right) \end{cases} \tag{5.1.8}$$

such that $u > 0$ *and* $v > 0$ *in* $\bar{\Omega} \times (0, \infty)$*, that* (u, v) *solves* (5.1.1) *classically in* $\Omega \times (0, T_{max})$*, and that*

$$\text{if } T_{max} < \infty, \quad \text{then } \limsup_{t \to T_{max}} \left\{ \|u(\cdot, t)\|_{L^\infty(\Omega)} + \|u(\cdot, t)\|_{W^{1,\infty}(\Omega)} \right\} = \infty. \tag{5.1.9}$$

The following Lemma [56][Lemma A.1] provides a useful pointwise estimate for Green's function of $-\Delta + 1$.

Lemma 5.1.5. *Suppose that* $\Omega \subset \mathbb{R}^2$ *is a bounded domain with smooth boundary, and let* G *denote Green's function of* $-\Delta + 1$ *in* Ω *subject to Neumann boundary conditions. Then there exist* $A > diam(\Omega)$ *and* $K > 0$ *such that*

$$|G(x, y)| \leq K \ln \frac{A}{|x - y|} \qquad \text{for all } x, y \in \Omega \text{ with } x \neq y. \tag{5.1.10}$$

By the pointwise estimate for the Green's function and Legendre transform, we can derive a L^∞ bound for solutions of (5.1.1) when $\eta = 0$, and therefore eliminate the degeneracy of diffusion term.

Lemma 5.1.6. *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded domain with smooth boundary. Suppose that the non-negative function* f *in* $L^2(\Omega)$ *satisfies*

$$\int_\Omega f \ln(f + e) \leq M \tag{5.1.11}$$

and w is a solutions of

$$
\begin{cases}
-\Delta w + w = f, \quad x \in \Omega \\
\frac{\partial u}{\partial \nu} = 0, \quad x \in \partial\Omega,
\end{cases}
\tag{5.1.12}
$$

then we have

$$
\|w\|_{L^\infty(\Omega)} \leq C,
$$

where $C = C(M) > 0$

Proof. By using the Green's function G of $-\Delta + 1$ in Ω, Lemma 5.1.5 and the inequality that

$$
ab \leq a \ln a + e^{b-1}, \qquad \text{for all } a, b \geq 0,
$$

we deduce that

$$
\begin{aligned}
w(x) &= \int_\Omega G(x,y) f(y)\, dy \\
&\leq K \int_\Omega \ln \frac{A}{|x-y|} f(y)\, dy \\
&\leq K \int_\Omega f(y) \ln f(y) + K \int_\Omega e^{\ln \frac{A}{|x-y|} - 1} \\
&\leq KM + \frac{AK}{e} \int_\Omega \frac{1}{|x-y|}\, dy \\
&\leq KM + \frac{AK}{e} diam(\Omega).
\end{aligned}
\tag{5.1.13}
$$

□

5.1.3 Elliptic-Parabolic system

Let us begin this section with an $L \ln^k L$ estimate for solutions of (5.1.1). The key approach in the proof is grounded in the Lyapunov functional method. While a standard estimate in two-dimensional domains is often considered when $k = 1$, we aim to enhance it by exploring the case where $k \geq 1$. The inspiration is drawn from the construction of a Lyapunov functional in an unconventional manner, as introduced in [72]. This idea has been adapted and refined in [30] for addressing two-dimensional chemotaxis models with a degenerate diffusion term, and in [32] for two-species with two chemicals, although the logistic source appears only in one of the two density population equations.

Lemma 5.1.7. *Under the assumptions in Theorem 5.1.1, for any $k \geq 1$, we have that*

$$\sup_{t\in(0,T_{max})} \int_\Omega u(\cdot,t)\ln^k(u(\cdot,t)+e) < \infty. \tag{5.1.14}$$

Proof. We define

$$I(t) := \int_\Omega u \ln^k(u+e)$$

and differentiate $I(\cdot)$ to obtain

$$\begin{aligned}
I'(t) &= \int_\Omega \left\{\ln^k(u+e) + ku\frac{\ln^{k-1}(u+e)}{u+e}\right\}(\nabla\cdot(D(v)\nabla u - uS(v)\ln^\alpha(u+e)\nabla v) + f(u)) \\
&= -k\int_\Omega \frac{D(v)\ln^{k-1}(u+e)}{u+e}|\nabla u|^2 - k(k-1)\int_\Omega \frac{D(v)u\ln^{k-2}(u+e)}{(u+e)^2}|\nabla u|^2 \\
&- k\int_\Omega \frac{eD(v)\ln^{k-1}(u+e)}{(u+e)^2}|\nabla u|^2 + \int_\Omega S(v)\nabla\phi(u)\cdot\nabla v \\
&+ \int_\Omega \left\{\ln^k(u+e) + ku\frac{\ln^{k-1}(u+e)}{u+e}\right\}(ru-\mu u^2) \\
&\leq \int_\Omega S(v)\nabla\phi(u)\cdot\nabla v + \int_\Omega \left\{\ln^k(u+e) + ku\frac{\ln^{k-1}(u+e)}{u+e}\right\}(ru-\mu u^2)
\end{aligned} \tag{5.1.15}$$

where

$$\begin{aligned}
\phi(l) &:= \int_0^l \left\{\frac{ks\ln^{k+\alpha-1}(s+e)}{s+e} + \frac{k(k-1)u^2\ln^{k+\alpha-2}(s+e)}{(s+e)^2} + \frac{kes\ln^{k+\alpha-1}}{(s+e)^2}\right\} ds \\
&\leq c_1 l \ln^{k+\alpha-1}(l+e), \qquad \text{for all } l \geq 0,
\end{aligned} \tag{5.1.16}$$

with $c_1 = k^2 + k$. By using integration by parts, taking into account the condition $S' \geq 0$ and applying elementary inequalities, we obtain that

$$\begin{aligned}
\int_\Omega S(v)\nabla\phi(u)\cdot\nabla v &= -\int_\Omega S(v)\phi(u)\Delta v - \int_\Omega S'(v)\phi(u)|\nabla v|^2 \\
&\leq \|S\|_{L^\infty((0,\infty))}\int_\Omega \phi(u)u \\
&\leq c_1\|S\|_{L^\infty((0,\infty))}\int_\Omega u^2\ln^{k+\alpha-1}(u+e) \\
&\leq \frac{\mu}{4}\int_\Omega u^2\ln^k(u+e) + c_2,
\end{aligned} \tag{5.1.17}$$

where $c_2 = C(\mu, \alpha, k) > 0$ and the last inequality comes from the fact that $k + \alpha - 1 < k$ when $\alpha < 1$ and the inequality that for any $\delta > 0$, there exist $A = c(\delta) > 0$ such that

$$s^{a_1} \ln^{b_1}(s+e) \leq \delta s^{a_2} \ln^{b_2}(s+e) + A, \qquad \text{for all } s \geq 0, \tag{5.1.18}$$

where a_1, a_2, b_1, b_2 are positive numbers such that $a_1 < a_2$. To handle the last term of (5.1.15) , we make use of again (5.1.18) to obtain

$$\begin{aligned}
\int_\Omega \left(\ln^k (u+e) + k \frac{\ln^{k-1}(u+e)}{u+e} \right) (ru - \mu u^2) &\leq r \int_\Omega u \ln^k(u+e) \\
&\quad + r \int_\Omega k \ln^{k-1}(u+e) - \mu \int_\Omega u^2 \ln^k(u+e) \\
&\leq -\frac{\mu}{4} \int_\Omega u^2 \ln^k(u+e) + c_3,
\end{aligned} \tag{5.1.19}$$

where $c_3 = C(\mu) > 0$ .The inequality (5.1.18) also implies that there exists $c_4 = C(\mu) > 0$ such that

$$\int_\Omega u \ln^k(u+e) \leq \frac{\mu}{4} \int_\Omega u^2 \ln^k(u+e) + c_4. \tag{5.1.20}$$

Collecting (5.1.15), (5.1.17), (5.1.19), and (5.1.20) yields

$$I'(t) + I(t) \leq c_5, \tag{5.1.21}$$

where $c_5 = c_2 + c_3 + c_4$. Finally, we apply Gronwall's inequality to prove (5.1.14). □

We will establish an L^p estimate for the solution in the following lemma. When employing the standard testing approach commonly used in chemotaxis, controlling the term $\int_\Omega u^{p+1} \ln^\alpha(u+e)$ proves challenging using the diffusion term $-\int_\Omega |\nabla u^{\frac{p}{2}}|^2$ and the global boundedness of $\int_\Omega u$. To overcome this difficulty, the key idea is to utilize the bound $\int_\Omega u \ln^k(u+e)$ instead of $\int_\Omega u$ and the logarithmically refined Gagliardo-Nirenberg interpolation inequality in Lemma B.0.5.

Lemma 5.1.8. *Under the assumptions in Theorem 5.1.1, for any $p > 1$, we have that*

$$\sup_{t \in (0, T_{max})} \int_\Omega u^p(x,t) dx < \infty. \tag{5.1.22}$$

Proof. By integration by parts, we have

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p = \int_\Omega u^{p-1}\left(\nabla(D(v)\nabla u) - \nabla\cdot(S(v)u\ln^\alpha(u+e)\nabla v) + ru - \mu u^2\right)$$
$$= -\frac{2(p-1)}{p}\int_\Omega D(v)|\nabla u^{\frac{p}{2}}|^2 + (p-1)\int_\Omega S(v)u^{p-1}\ln^\alpha(u+e)\nabla u\cdot\nabla v$$
$$+ r\int_\Omega u^p - \mu\int_\Omega u^{p+1} \tag{5.1.23}$$

From Lemma 5.1.7, there exist a constant $M > 0$ such that

$$\int_\Omega u(\cdot,t)\ln(u(\cdot,t)+e) \le M, \qquad \text{for all } t\in(0,T_{\max}) \tag{5.1.24}$$

This, together with Lemma 5.1.6 implies that

$$\|v(\cdot,t)\|_{L^\infty(\Omega)} \le C, \qquad \text{for all } t\in(0,T_{\max}) \tag{5.1.25}$$

for some $C = C(M) > 0$. This implies that $\inf_{(x,t)} D(v(x,t) > 0$ and therefore the degeneracy of the diffusion term is now eliminated. It follows that

$$-\frac{2(p-1)}{p}\int_\Omega D(v)|\nabla u^{\frac{p}{2}}|^2 \le -c_1\int_\Omega |\nabla u^{\frac{p}{2}}|^2, \tag{5.1.26}$$

where $c_1 = \frac{2p-2}{p}\inf_{(x,t)\in\Omega\times(0,T)} D(v(x,t))$. By integration by parts and the condition $S' \ge 0$, we have

$$(p-1)\int_\Omega S(v)u^{p-1}\ln^\alpha(u+e)\nabla u\cdot\nabla v = -c_2\int_\Omega S(v)\phi(u)\Delta v - c_2\int_\Omega S'(v)\phi(u)|\nabla v|^2$$
$$\le c_3\int_\Omega u\phi(u)$$
$$\le c_3\int_\Omega u^{p+1}\ln^\alpha(u+e). \tag{5.1.27}$$

where

$$\phi(l) := \int_0^l s^{p-1}\ln^\alpha(s+e)\,ds \le l^p\ln^\alpha(l+e), \qquad \text{for all } l\ge 0. \tag{5.1.28}$$

From Lemma 5.1.7, we obtain that $\sup_{t\in(0,T_{\max})}\int_\Omega u\ln(u+e) < \infty$. Now applying Lemma B.0.5 with

$$\epsilon = \frac{c_1}{2c_3\sup_{t\in(0,T_{\max})}\int_\Omega u\ln(u+e)}$$

yields

$$c_3 \int_\Omega u^{p+1} \ln^\alpha(u+e) \le c_3\epsilon \int_\Omega |\nabla u^{\frac{p}{2}}|^2 \cdot \int_\Omega u\ln(u+e) + \epsilon c_3 \int_\Omega u \cdot \int_\Omega u \ln(u+e) + c_4$$
$$\le \frac{c_1}{2}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + c_5 \tag{5.1.29}$$

where $c_4 = C(\epsilon) > 0$ and $c_5 = C(\epsilon) > 0$. By Young's inequality, we obtain that

$$\left(r + \frac{1}{p}\right)\int_\Omega u^p \le \frac{\mu}{2}\int_\Omega u^{p+1} + c_6. \tag{5.1.30}$$

where $c_6 = C(r,p,\mu) > 0$. Collecting (5.1.23), (5.1.26), (5.1.27) and (5.1.30) yields

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + \frac{1}{p}\int_\Omega u^p \le c_7, \tag{5.1.31}$$

where $c_7 = C(\epsilon, p, \mu, r) > 0$. Finally, we apply Gronwall's inequality to complete the proof. □

We are now ready to prove the main theorem.

Proof of Theorem 5.1.1. By using Lemma 5.1.8 for a fixed $p > 2$, it follows that

$$\sup_{t\in(0,T_{\max})} \|u(\cdot,t)\|_{L^p(\Omega)} < \infty.$$

By elliptic regularity theory in Sobolev spaces, we obtain that

$$\sup_{t\in(0,T_{\max})} \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} < \infty. \tag{5.1.32}$$

Applying Moser-Alikakos iteration (see e.g [53, 2, 1]) yields

$$\sup_{t\in(0,T_{\max})} \|u(\cdot,t)\|_{L^\infty(\Omega)} < \infty.$$

This, together with Lemma 5.1.4 implies that $T_{\max} = \infty$, which finishes the proof. □

5.1.4 Fully Parabolic system

We will follow the framework established in the previous section to prove Theorem 5.1.2. However, for the fully parabolic system, we cannot directly use the equation $\Delta v = u - v$ for estimations, as done in Lemmas 5.1.7 and 5.1.8. Instead, we need to establish an intermediate estimate to connect the two equations of (5.1.1). Let us commence this section with the following lemma.

Lemma 5.1.9. *For any $p > 1$, there exist positive constants A_1, A_2, A_3 depending only on p such that*

$$\frac{1}{2p}\frac{d}{dt}\int_\Omega |\nabla v|^{2p} + A_1\int_\Omega |\nabla|\nabla v|^p|^2 + \int_\Omega |\nabla v|^{2p} \le A_2\int_\Omega u^2|\nabla v|^{2p-2} + A_3\int_\Omega |\nabla v|^{2p} \quad (5.1.33)$$

Proof. We make use of the following point-wise identity

$$\nabla v \cdot \nabla\Delta v = \frac{1}{2}\Delta(|\nabla v|^2) - |D^2 v|^2$$

to obtain

$$\begin{aligned}\frac{1}{2p}\frac{d}{dt}\int_\Omega |\nabla v|^{2p} + \int_\Omega |\nabla v|^{2p} = &-c_1\int_\Omega |\nabla|\nabla v|^p|^2 - \int_\Omega |\nabla v|^{2p-2}|D^2 v|^2\\ &+ \int_\Omega |\nabla v|^{2p-2}\nabla v\cdot\nabla u\\ &+ c_2\int_{\partial\Omega}\frac{\partial|\nabla v|^2}{\partial\nu}|\nabla v|^{2p-2}, \end{aligned}\quad (5.1.34)$$

where c_1, c_2 are positive constants depending only on p. The inequality $\frac{\partial|\nabla v|^2}{\partial\nu} \le M|\nabla v|^2$, (see [41][Lemma 4.2]) for some $M > 0$ depending only on Ω, implies that

$$c_2\int_{\partial\Omega}\frac{\partial|\nabla v|^2}{\partial\nu}|\nabla v|^{2p-2}\,dS \le c_2 M\int_{\partial\Omega}|\nabla v|^{2p}\,dS.$$

Let $g := |\nabla v|^p$ and apply trace embedding theorem $W^{1,1}(\Omega) \longrightarrow L^1(\partial\Omega)$ together with Young's inequality, there exist positive constants C and c_3 such that

$$\begin{aligned}c_2 M\int_{\partial\Omega} g^2\,dS &\le C\int_\Omega g|\nabla g| + C\int_\Omega g^2\\ &\le \frac{c_1}{2}\int_\Omega |\nabla g|^2 + c_3\int_\Omega g^2,\end{aligned}\quad (5.1.35)$$

Therefore, we have

$$c_2 M\int_{\partial\Omega}|\nabla v|^{2p}\,dS \le \frac{c_1}{2}\int_\Omega |\nabla|\nabla v|^p|^2 + c_3\int_\Omega |\nabla v|^{2p}. \quad (5.1.36)$$

Applying the pointwise inequality $(\Delta v)^2 \le 2|D^2 v|^2$ to (5.1.34) yields

$$\frac{1}{2p}\frac{d}{dt}\int_\Omega |\nabla v|^{2p} + \int_\Omega |\nabla v|^{2p} \le -\frac{c_1}{2}\int_\Omega |\nabla|\nabla v|^p|^2 - \frac{1}{2}\int_\Omega |\nabla v|^{2p-2}|\Delta v|^2$$

$$+\int_\Omega |\nabla v|^{2p-2}\nabla v\cdot\nabla u + c_3\int_\Omega |\nabla v|^{2p} \tag{5.1.37}$$

By integration by parts and elemental inequalities, there exist constants $c_4 = C(p) > 0$ and $c_5 = C(p) > 0$ in such a way that

$$\begin{aligned}\int_\Omega |\nabla v|^{2p-2}\nabla v\cdot\nabla u &= -\int_\Omega u|\nabla v|^{2p-2}\Delta v - c_4\int_\Omega u|\nabla v|^{p-1}\nabla|\nabla v|^p\cdot\frac{\nabla v}{|\nabla v|}\\ &\le \frac{1}{2}\int_\Omega(\Delta v)^2|\nabla v|^{2p-2} + \frac{c_1}{4}\int_\Omega|\nabla|\nabla v|^p|^2\\ &+ c_5\int_\Omega u^2|\nabla v|^{2p-2},\end{aligned} \tag{5.1.38}$$

From (5.1.37) and (5.1.38), we finally prove (5.1.33).

□

The following lemma, akin to Lemma 5.1.7, provides a crucial a priori estimate for solutions. However, the constant k is now bounded from above due to the structure of parabolic equations. In addition to the estimate for $\int_\Omega u\ln^k(u+e)$, we also require a uniform bound in time for $\int_t^{t+\tau}\int_\Omega u^2\ln^k(u+e)$ to cooperate with Proposition C.1.1 in order to obtain uniform bounds for v.

Lemma 5.1.10. *Under the assumptions in Theorem 5.1.2, for any $k\in(1,2-2\alpha)$, we have that*

$$\sup_{t\in(0,T_{max})}\int_\Omega\left\{u\ln^k(u+e)+|\nabla v|^2\right\} + \sup_{t\in(0,T_{max}-\tau)}\int_t^{t+\tau}\int_\Omega u^2\ln^k(u+e) < \infty, \tag{5.1.39}$$

where $\tau = \min\left\{1, \frac{T_{max}}{2}\right\}$.

Proof. We define

$$y(t) := \int_\Omega u\ln^k(u+e) + \frac{1}{2}\int_\Omega|\nabla v|^2,$$

and differentiate $y(\cdot)$ to obtain

$$\begin{aligned}y'(t) &= \int_\Omega\left(\ln^k(u+e) + ku\frac{\ln^{k-1}(u+e)}{u+e}\right)u_t + \int_\Omega\nabla v\cdot\nabla v_t\\ &:= I'(t) + J'(t).\end{aligned} \tag{5.1.40}$$

Where I is given in Lemma 5.1.7. We now just reuse estimations from (5.1.15) to (5.1.19) for I' except for (5.1.17). By using integration by parts, taking into account the condition $S'\ge 0$ and applying elementary inequalities, we obtain that

$$\begin{aligned}
\int_\Omega S(v)\nabla\phi(u)\cdot\nabla v &= -\int_\Omega S(v)\phi(u)\Delta v - \int_\Omega S'(v)\phi(u)|\nabla v|^2 \\
&\le \|S\|_{L^\infty(0,\infty)}\int_\Omega \phi(u)|\Delta v| \\
&\le \frac{1}{2}\int_\Omega (\Delta v)^2 + \frac{\|S\|^2_{L^\infty(0,\infty)}}{2}\int_\Omega \phi^2(u) \\
&\le \frac{1}{2}\int_\Omega (\Delta v)^2 + \frac{c_1\,\|S\|^2_{L^\infty(0,\infty)}}{2}\int_\Omega u^2\ln^{2k+2\alpha-2}(u+e) \\
&\le \frac{1}{2}\int_\Omega (\Delta v)^2 + \frac{\mu}{4}\int_\Omega u^2\ln^k(u+e) + c_2, \qquad (5.1.41)
\end{aligned}$$

where ϕ is defined in (5.1.16), $c_2 = C(\mu) > 0$ and the last inequality comes from the fact that $2k+2\alpha-2 < k$ and the inequality (5.1.18). Collecting (5.1.15), (5.1.41) and (5.1.19), we have

$$I'(t) \le \frac{1}{2}\int_\Omega (\Delta v)^2 - \frac{\mu}{2}\int_\Omega u^2\ln^k(u+e) + c_4, \qquad (5.1.42)$$

where $c_4 = C(\mu) > 0$. By integration by parts and elemental inequalities, it follows that there exist $c_5 = C(\mu) > 0$ such that

$$\begin{aligned}
J'(t) &:= \int_\Omega \nabla v\cdot\nabla v_t \\
&= -\int_\Omega (\Delta v)^2 - \int_\Omega |\nabla v|^2 - \int_\Omega u\Delta v \\
&\le -\frac{1}{2}\int_\Omega (\Delta v)^2 - \int_\Omega |\nabla v|^2 + \frac{1}{2}\int_\Omega u^2 \\
&\le -\frac{1}{2}\int_\Omega (\Delta v)^2 - \int_\Omega |\nabla v|^2 + \frac{\mu}{4}\int_\Omega u^2\ln^k(u+e) + c_5. \qquad (5.1.43)
\end{aligned}$$

The inequality (5.1.18) implies that there exists $c_6 = C(\mu) > 0$ such that

$$\int_\Omega u\ln^k(u+e) \le \frac{\mu}{8}\int_\Omega u^2\ln^k(u+e) + c_6. \qquad (5.1.44)$$

Collecting (5.1.40), (5.1.42), (5.1.43), and (5.1.44) we obtain

$$y'(t) + y(t) + \frac{\mu}{8}\int_\Omega u^2\ln^k(u+e) \le c_7 \qquad (5.1.45)$$

for some $c_7 = C\left(r, k, \mu, \|S\|_{L^\infty((0,\infty))}\right) > 0$. Applying Gronwall's inequality to this leads to $y(t) \le \max\{y(0), c_7\}$. Additionally, we also have:

$$\frac{\mu}{8}\int_\Omega u^2\ln^k(u+e) \le c_{11} - y'(t). \qquad (5.1.46)$$

By integrating the previous inequality from t to $t+\tau$ and using the fact that y is non-negative and bounded, we can conclude the proof. $\square$

Now, we can establish L^p bounds for solutions in the following lemma, akin to Lemma 5.1.8.

Lemma 5.1.11. *Under the assumption in Theorem 5.1.2, for any* $p > \max\left\{\frac{\alpha}{1-2\alpha}, 1\right\}$*, we have that*

$$\sup_{t\in(0,T_{max})} \int_\Omega \left\{u^p(\cdot,t) + |\nabla v(\cdot,t)|^{2p}\right\} dx < \infty. \tag{5.1.47}$$

Proof. We define

$$\phi(t) := \frac{1}{p}\int_\Omega u^p + \frac{1}{2p}\int_\Omega |\nabla v|^{2p},$$

and differentiate $\phi(\cdot)$ to obtain:

$$\begin{aligned}
\phi'(t) &= \int_\Omega u^{p-1}\left[\nabla\cdot(D(v)\nabla u) - \nabla\cdot(S(v)u\ln^\alpha(u+e)\nabla v) + ru - \mu u^2\right] \\
&\quad + \int_\Omega |\nabla v|^{2p-2}\nabla v\cdot\nabla\left(\Delta v + u - v\right) \\
&:= M_1 + M_2.
\end{aligned} \tag{5.1.48}$$

By integration by parts, we have

$$\begin{aligned}
M_1 = &-\frac{2(p-1)}{p}\int_\Omega D(v)|\nabla u^{\frac{p}{2}}|^2 + (p-1)\int_\Omega S(v)u^{p-1}\ln^\alpha(u+e)\nabla u\cdot\nabla v \\
&+ r\int_\Omega u^p - \mu\int_\Omega u^{p+1}.
\end{aligned} \tag{5.1.49}$$

From Lemma 5.1.10, we find that

$$\sup_{t\in(0,T_{\max}-\tau)} \int_\Omega u^2\ln^k(u+e) < \infty$$

for any $k\in(1, 2-2\alpha)$. This, together with Proposition C.1.1 implies that

$$\sup_{t\in(0,T_{\max})} \|v(\cdot,t)\|_{L^\infty(\Omega)} < \infty. \tag{5.1.50}$$

Therefore, it follows that $\inf_{(x,t)\in\Omega\times(0,T)} D(v(x,t)) > 0$ and

$$-\frac{2(p-1)}{p}\int_\Omega D(v)|\nabla u^{\frac{p}{2}}|^2 \le -c_1\int_\Omega |\nabla u^{\frac{p}{2}}|^2, \tag{5.1.51}$$

where $c_1 = \frac{2p-2}{p} \inf_{(x,t)\in\Omega\times(0,T)} D(v(x,t))$. Since $p > \max\left\{1, \frac{\alpha}{1-2\alpha}\right\}$ we can fix

$$k \in \left(\max\left\{\frac{2\alpha(p+1)}{p}, 1\right\}, 2-2\alpha\right)$$

and Lemma 5.1.10 allows us to choose

$$\epsilon = \min\left\{\frac{A_1}{2C_{GN}\sup_{t\in(0,T_{\max})}\int_\Omega |\nabla v|^2}; \frac{c_1}{2\sup_{t\in(0,T_{\max})}\int_\Omega u\ln^k(u+e)}\right\}, \tag{5.1.52}$$

where A_1 is the constant defined in Lemma 5.1.9. By Young's inequality, we obtain

$$\begin{aligned}(p-1)\int_\Omega S(v)u^{p-1}\ln^\alpha(u)\nabla u\cdot\nabla v &= \frac{2(p-1)}{p}\int_\Omega S(v)u^{\frac{p}{2}}\ln^\alpha(u+e)\nabla u^{\frac{p}{2}}\cdot\nabla v\\ &\le \frac{c_1}{4}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + \frac{4(p-1)^2}{p^2c_1}\int_\Omega u^p\ln^{2\alpha}(u+e)|\nabla v|^2\\ &\le \frac{c_1}{4}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + \frac{\epsilon}{2}\int_\Omega |\nabla v|^{2p+2}\\ &+ c_2\int_\Omega u^{p+1}\ln^{\frac{2\alpha(p+1)}{p}}(u+e),\end{aligned} \tag{5.1.53}$$

where $c_2 = \frac{8(p-1)^2}{p^2c_1\epsilon}$. Combining (5.1.49), (5.1.51), and (5.1.53) yields

$$M_1 \le -\frac{3c_1}{4}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + \frac{\epsilon}{2}\int_\Omega |\nabla v|^{2p+2} + c_2\int_\Omega u^{p+1}\ln^{\frac{2\alpha(p+1)}{p}}(u+e) + r\int_\Omega u^p. \tag{5.1.54}$$

From Lemma 5.1.9, we have

$$M_2 + A_1\int_\Omega |\nabla|\nabla v|^p|^2 + \int_\Omega |\nabla v|^{2p} \le A_2\int_\Omega u^2|\nabla v|^{2p-2} + A_3\int_\Omega |\nabla v|^{2p}. \tag{5.1.55}$$

By elementary inequalities, we obtain that

$$\begin{aligned}\left(r+\frac{1}{p}\right)\int_\Omega u^p + A_2\int_\Omega u^2|\nabla v|^{2p-2} + A_3\int_\Omega |\nabla v|^{2p} &\le \frac{\epsilon}{2}\int_\Omega |\nabla v|^{2p+2}\\ &+ c_3\int_\Omega u^{p+1}\ln^{\frac{2\alpha(p+1)}{p}}(u+e) + c_4,\end{aligned} \tag{5.1.56}$$

where $c_3 = C(\epsilon) > 0$ and $c_4 = C(\epsilon) > 0$. Collecting (5.1.48), (5.1.54), (5.1.55), and (5.1.56) yields

$$\phi'(t) + \phi(t) \le -\frac{3c_1}{4}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 - A_1\int_\Omega |\nabla|\nabla v|^p|^2 + \epsilon\int_\Omega |\nabla v|^{2q+2}$$

$$+ c_5 \int_\Omega u^{p+1} \ln^{\frac{2\alpha(p+1)}{p}}(u+e) + c_4, \tag{5.1.57}$$

where $c_5 = c_2 + c_3$. Using the Gagliardo-Nirenberg interpolation inequality for $n = 2$ and the fact that $\sup_{t\in(0,T_{\max})} \int_\Omega |\nabla v(\cdot,t)|^2\, dx < \infty$ from Lemma 5.1.10, there exists a positive constant C_{GN} such that:

$$\begin{aligned}
\epsilon \int_\Omega |\nabla v|^{2p+2} &\le \epsilon C_{GN} \int_\Omega |\nabla |\nabla v|^p|^2 \int_\Omega |\nabla v|^2 + \epsilon C_{GN} \left(\int_\Omega |\nabla v|^2 \right)^{p+1} \\
&\le c_6 \epsilon \int_\Omega |\nabla |\nabla v|^p|^2 + c_7,
\end{aligned} \tag{5.1.58}$$

where $c_6 = C_{GN} \sup_{t\in(0,T_{\max})} \int_\Omega |\nabla v(\cdot,t)|^2$ and $c_7 = \epsilon C_{GN} \sup_{t\in(0,T_{\max})} \left(\int_\Omega |\nabla v(\cdot,t)|^2\right)^{p+1}$. The condition $\frac{2\alpha(p+1)}{p} < k < 2 - 2\alpha$ when $p > \max\left\{\frac{\alpha}{1-2\alpha}, 1\right\}$ enables us to apply Lemma B.0.5 to obtain

$$\begin{aligned}
c_5 \int_\Omega u^{p+1} \ln^{\frac{2\alpha(p+1)}{p}}(u+e) &\le \epsilon \int_\Omega |\nabla u^{\frac{p}{2}}|^2 \int_\Omega u \ln^k(u+e) + \epsilon \left(\int_\Omega u \right)^p \int_\Omega u \ln^k(u+e) + c_7 \\
&\le c_8 \epsilon \int_\Omega |\nabla u^{\frac{p}{2}}|^2 + c_9
\end{aligned} \tag{5.1.59}$$

where $c_8 = \sup_{t\in(0,T_{\max}} \int_\Omega u \ln^k(u+e))$, and $c_9 = c(\epsilon) > 0$. From (5.1.57), (5.1.58), and (5.1.59), we have

$$\phi'(t) + \phi(t) \le \left(c_8 \epsilon - \frac{3c_1}{4} \right) \int_\Omega |\nabla u^{\frac{p}{2}}|^2 + (c_6 \epsilon - A_1) \int_\Omega |\nabla |\nabla v|^p|^2 \int_\Omega |\nabla v|^2 + c_{10},$$

where $c_{10} = c_4 + c_9$. From (5.1.52), we find that $c_8\epsilon - \frac{3c_1}{4} \le 0$, and $c_6 \epsilon - A_1 \le 0$. It follows that $\phi'(t) + \phi(t) \le c_{10}$. The proof is finished by applying Gronwall's inequality. □

Proof of Theorem 5.1.2. By using Lemma 5.1.11 for a fixed $p > 2$, it follows that

$$\sup_{t\in(0,T_{\max})} \|u(\cdot,t)\|_{L^p(\Omega)} < \infty.$$

By Lemma C.1.2, we have that

$$\sup_{t\in(0,T_{\max})} \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} < \infty. \tag{5.1.60}$$

Now by applying Moser-Alikakos iteration procedure, we obtain

$$\sup_{t\in(0,T_{\max})} \|u(\cdot,t)\|_{L^\infty(\Omega)} < \infty. \tag{5.1.61}$$

This, together with (5.1.60) and Lemma 5.1.4 implies that $T_{\max} = \infty$, which completes the proof.

□

5.2 Chemotaxis system with superlinear signal production

This section focuses on studying blow-up prevention of sub-logistic sources for 2d Keller-Segel chemotaxis systems with superlinear signal production. An application of a result on parabolic gradient regularity for parabolic equations in Orlicz spaces shows that the presence of sub-logistic sources are indeed sufficiently strong to ensure the global existence and boundedness of solutions. Our proof also relies on several techniques, including parabolic regularity in Sobolev spaces, variational arguments, interpolation inequalities in Sobolev spaces and Moser iteration method.

5.2.1 Introduction

We consider the following chemotaxis model with sub-logistic sources and superlinear signal production in a bounded domain with smooth boundary $\Omega \subset \mathbb{R}^2$:

$$\begin{cases} u_t = \nabla\cdot(D(v)\nabla u) - \nabla\cdot(uS(v)\nabla v) + f(u) \\ \kappa v_t = \Delta v - v + g(u), \end{cases} \tag{5.2.1}$$

where $\kappa \in \{0, 1\}$, and

$$0 < D \in C^2([0,\infty)) \quad \text{and } S \in C^2([0,\infty)) \cap W^{1,\infty}((0,\infty)) \text{ such that } S' \geq 0, \tag{5.2.2}$$

and the logistic source

$$f(u) = ru - \mu\frac{u^2}{\ln^p(u+e)}, \quad \text{with } r \in \mathbb{R},\ \mu > 0, \text{ and } p \in \left[0, 1 - \frac{\kappa}{2}\right), \tag{5.2.3}$$

and the superlinear signal production

$$g(u) = u\ln^q(u+e), \quad \text{with } q \in \left[0, 1 - \frac{\kappa}{2} - p\right). \tag{5.2.4}$$

The system (5.2.1) is complemented with nonnegative initial conditions in $W^{1,\infty}(\Omega)$ not identically zero:

$$u(x,0) = u_0(x), \qquad \kappa v(x,0) = \kappa v_0(x), \qquad \text{with } x \in \Omega, \tag{5.2.5}$$

and homogeneous Neumann boundary condition are imposed as follows:

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \qquad x \in \partial\Omega,\, t \in (0, T_{\max}), \tag{5.2.6}$$

where ν denotes the outward normal vector.

We aim to show that the presence of sub-logistic sources is sufficiently strong to avoid blow-up solutions in a superlinear signal production chemotaxis system. Precisely, we have the following theorems.

Theorem 5.2.1. *Let* $\kappa = 0$ *and the system* (5.2.1) *satisfy the assumptions from* (5.2.2) *to* (5.2.6). *There exists a unique pair of nonnegative functions* (u, v) *with*

$$\begin{cases} u \in C^0\left(\bar{\Omega} \times [0,\infty)\right) \cap C^{2,1}\left(\bar{\Omega} \times (0,\infty)\right) \qquad \textit{and} \\ v \in \cap_{q>2} C^0\left([0,\infty); W^{1,q}(\Omega)\right) \cap C^{2,1}\left(\bar{\Omega} \times (0,\infty)\right), \end{cases}$$

solving the system (5.2.1) *in the classical sense. Furthermore, this solution is bounded in the sense that*

$$\sup_{t>0} \left\{ \|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} \right\} < \infty. \tag{5.2.7}$$

The next theorem asserts the global existence and boundedness of solutions to the fully parabolic system (5.2.1) when $\kappa = 1$.

Theorem 5.2.2. *Let* $\kappa = 1$ *and the system* (5.2.1) *satisfy the assumptions from* (5.2.2) *to* (5.2.6). *There exists a unique pair of nonnegative functions* (u, v) *with*

$$\begin{cases} u \in C^0\left(\bar{\Omega} \times [0,\infty)\right) \cap C^{2,1}\left(\bar{\Omega} \times (0,\infty)\right) \qquad \textit{and} \\ v \in \cap_{q>2} C^0\left([0,\infty); W^{1,q}(\Omega)\right) \cap C^{2,1}\left(\bar{\Omega} \times (0,\infty)\right), \end{cases}$$

solving the system (5.2.1) *in the classical sense. Furthermore, this solution is bounded in the sense that*

$$\sup_{t>0}\left\{\|u(\cdot,t)\|_{L^\infty(\Omega)}+\|v(\cdot,t)\|_{W^{1,\infty}(\Omega)}\right\}<\infty. \tag{5.2.8}$$

The major difficulties to obtain a uniform bound for solutions of (5.2.1) come from the superlinear signal production of the second equation. It is not clear that whether $\|v(\cdot,t)\|_{L^1(\Omega)}$ is uniformly bounded in time. Indeed, by integrating the second equation

$$\frac{d}{dt}\int_\Omega v=\int_\Omega v-\int_\Omega u\ln^q(u+e),$$

we see that the presence of $\int_\Omega u\ln^q(u+e)$ has not known to be uniformly bounded in time. Moreover, the equi-integrability of the family $\left\{\int_\Omega u(\cdot,t)\right\}_{t\in(0,T_{\max})}$ is not sufficient to prevent blow-up due to the superlinear signal production. In this section, we overcome these challenges by introducing the following functional:

$$y(t)=\int_\Omega u(\cdot,t)\ln^k(u(\cdot,t)+e),$$

where $k>0$ will be determined later. This functional, which has been used in [30], is the adaptation and modification of a well-known functional called entropy,

$$y(t)=\int_\Omega u(\cdot,t)\ln(u(\cdot,t)+e),$$

which has been used in various papers such as [8, 39, 44, 71, 33, 32].

The paper is structured in four sections. Section 2 establishes a local-wellposedness result for solutions as well as recall some vital inequalities, which will be frequently used in sequel sections. Section 3 includes a priori estimates for solutions to the parabolic-elliptic system when $\kappa=0$ and the proof of Theorem 5.2.1. In Section 4, we establish a priori estimates for solutions to the fully parabolic system when $\kappa=1$ and prove Theorem 5.2.2

5.2.2 Preliminaries

The local existence and uniqueness of non-negative classical solutions to the system (5.2.1) can be established by adapting and adjusting the fixed point argument and standard parabolic regularity theory. For further details, we refer the reader to [21, 29, 58]. For convenience, we adopt Lemma 4.1 from [61].

Lemma 5.2.3. *Let $\Omega \subset \mathbb{R}^n$, where $n \geq 2$ be a bounded domain with smooth boundary, and the system* (5.2.1) *satisfy the conditions from* (5.2.2) *to* (5.2.6)*. Then there exist $T_{max} \in (0, \infty]$ and functions*

$$\begin{cases} u \in C^0\left(\bar{\Omega} \times [0, T_{max})\right) \cap C^{2,1}\left(\bar{\Omega} \times (0, T_{max})\right) \text{ and} \\ v \in \bigcap_{q>2} C^0\left([0, T_{max}); W^{1,q}(\Omega)\right) \cap C^{2,1}\left(\bar{\Omega} \times (0, T_{max})\right) \end{cases} \tag{5.2.9}$$

such that $u > 0$ and $v > 0$ in $\bar{\Omega} \times (0, \infty)$, that (u, v) solves (5.2.1) *classically in $\Omega \times (0, T_{max})$, and that if $T_{max} < \infty$, then*

$$\limsup_{t \to T_{max}} \left\{ \|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} \right\} = \infty. \tag{5.2.10}$$

The following lemma is essential to obtain an $L^2 \ln(L + e)$ estimate for the solution of (5.2.1) when $\kappa = 1$.

Lemma 5.2.4. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary, and $k > \frac{1}{2}$. Then there exists $C > 0$ such that for each $\epsilon > 0$, one can pick $C(\epsilon) > 0$ such that*

$$\begin{aligned} \int_\Omega u^3 \ln^{3/2}(u + e) \leq \epsilon \int_\Omega |\nabla u|^2 \ln(u + e) \int_\Omega u \ln^k(u + e) \\ + \epsilon \int_\Omega u \ln^k(u + e) + c \left(\int_\Omega u \ln^{1/2}(u + e) \right)^3. \end{aligned} \tag{5.2.11}$$

holds for all $u \in C^2(\bar{\Omega})$.

Proof. We apply Lemma B.0.9 with $G(s) := s \ln^{k-1/2}(s + e)$, to deduce that for any $\epsilon > 0$, there exists $c = c(\epsilon) > 0$ such that

$$\int_\Omega \left(u \ln^{1/2}(u + e) \right)^3 \leq \epsilon \int_\Omega |\nabla (u \ln^{1/2}(u + e))|^2 \int_\Omega u \ln^k(u + e)$$

$$+ c\left(\int_\Omega u\ln^{1/2}(u+e)\right)^3. \tag{5.2.12}$$

Notice that

$$\begin{aligned}\int_\Omega |\nabla(u\ln^{1/2}(u+e))|^2 &\le \int_\Omega 2|\nabla u|^2\ln(u+e) + \frac{u}{(u+e)\ln^{1/2}(u+e)}|\nabla u|^2\\ &\le c\int_\Omega |\nabla u|^2\ln(u+e),\end{aligned} \tag{5.2.13}$$

Where the last inequality comes from

$$\int_\Omega \frac{u}{(u+e)\ln^{1/2}(u+e)}|\nabla u|^2 \le \int_\Omega |\nabla u|^2\ln(u+e)$$

Finally, we make use of (5.2.12) and (5.2.13) to complete the proof. □

5.2.3 Parabolic-elliptic

In this section, we assume that (u, v) is a solution of the system (5.2.1) with $\kappa = 0$, under the conditions from (5.2.2) to (5.2.6). Let us begin with the following priori estimate, which will be used to obtain L^m bounds for the solution.

Lemma 5.2.5. *For any $k \ge 1$, then*

$$\sup_{t\in(0,T_{max})}\int_\Omega u\ln^k(u+e) < \infty. \tag{5.2.14}$$

Proof. We make use of the first equation and integration by parts to obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega u\ln^k(u+e) &= \int_\Omega\left(\ln^k(u+e) + ku\frac{\ln^{k-1}(u+e)}{u+e}\right)(\nabla\cdot(D(v)\nabla u - uS(v)\nabla v) + f(u))\\ &= -k\int_\Omega \frac{D(v)\ln^{k-1}(u+e)}{u+e}|\nabla u|^2 - k(k-1)\int_\Omega \frac{D(v)u\ln^{k-2}(u+e)}{(u+e)^2}|\nabla u|^2\\ &- k\int_\Omega \frac{eD(v)\ln^{k-1}(u+e)}{(u+e)^2}|\nabla u|^2 + \int_\Omega S(v)\nabla\phi(u)\cdot\nabla v\\ &+ \int_\Omega\left(\ln^k(u+e) + ku\frac{\ln^{k-1}(u+e)}{u+e}\right)f(u)\\ &\le \int_\Omega S(v)\nabla\phi(u)\cdot\nabla v + \int_\Omega\left(\ln^k(u+e) + ku\frac{\ln^{k-1}(u+e)}{u+e}\right)f(u)\end{aligned} \tag{5.2.15}$$

where

$$\phi(l) = \int_0^l \left\{ k\frac{s\ln^{k-1}(s+e)}{s+e} + k(k-1)\frac{s^2\ln^{k-2}(s+e)}{(s+e)^2} + k\frac{eu\ln^{k-1}(s+e)}{(s+e)^2} \right\} ds$$
$$\le c_1 l \ln^{k-1}(l+e), \tag{5.2.16}$$

with $c_1 := k^2 + k$. This, together with integration by parts, the condition $S' \ge 0$ and elementary inequalities, follows that

$$\begin{aligned}\int_\Omega S(v)\nabla\phi(u)\cdot\nabla v &= -\int_\Omega S(v)\phi(u)\Delta v - \int_\Omega S'(v)\phi(u)|\nabla v|^2 \\ &\le \|S\|_{L^\infty([0,\infty))}\int_\Omega \phi(u)u\ln^q(u+e) \\ &\le c_2\int_\Omega u^2\ln^{k+q-1}(u+e) \\ &\le \frac{\mu}{2}\int_\Omega u^2\ln^{k-p}(u+e) + c_3,\end{aligned} \tag{5.2.17}$$

where $c_2 = c_1\|S\|_{L^\infty([0,\infty))}$, $c_3 = C(\mu, k, \|S\|_{L^\infty([0,\infty))}) > 0$ and the last inequality comes from the fact that $p < 1-q$ and for any $\epsilon > 0$, there exist $c(\epsilon) > 0$ such that

$$u^{a_1}\ln^{b_1}(u+e) \le \epsilon u^{a_2}\ln^{b_2}(u+e) + c(\epsilon),$$

where a_1, a_2, b_1, b_2 are real numbers such that $a_1 < a_2$. This also implies that there exists a positive constant c_4 depending on μ such that

$$\begin{aligned}\left(\ln^k(u+e) + k\frac{\ln^{k-1}(u+e)}{u+e}\right)f(u) + u\ln^k(u+e) &\le (r+1)u\ln^k(u+e) \\ &\quad + rk\ln^{k-1}(u+e) - \mu u^2\ln^{k-p}(u+e) \\ &\le \frac{\mu}{4}\int_\Omega u^2\ln^{k-p}(u+e) + c_4.\end{aligned} \tag{5.2.18}$$

Integrating (5.2.18) both sides entails that

$$\int_\Omega \left(\ln^k(u+e) + k\frac{\ln^{k-1}(u+e)}{u+e}\right)f(u) + \int_\Omega u\ln^k(u+e) \le \frac{\mu}{4}\int_\Omega u^2\ln^{k-p}(u+e) + c_5, \tag{5.2.19}$$

where $c_5 = c_4|\Omega|$. Collecting (5.2.15), (5.2.17) and (5.2.19) yields

$$\frac{d}{dt}\int_\Omega u\ln^k(u+e) + \int_\Omega u\ln^k(u+e) \le c_6, \tag{5.2.20}$$

where $c_6 = c_3 + c_5$. This, together with Gronwall's inequality completes the proof. □

The next lemma provides L^m bounds for the solution of (5.2.1) for any $m \geq 1$.

Lemma 5.2.6. *For any* $m > 1$ *, then*

$$\sup_{t\in(0,T_{max})} \int_\Omega u^m < \infty. \tag{5.2.21}$$

Proof. Multiplying the first equation of (5.2.1) by u^{m-1} and using integration by parts yields

$$\begin{aligned}\frac{1}{m}\frac{d}{dt}\int_\Omega u^m &= \int_\Omega u^{m-1}\left(\nabla\cdot(D(v)\nabla u - uS(v)\nabla v) + f(u)\right)\\ &= -\frac{4(m-1)}{m^2}\int_\Omega D(v)|\nabla u^{\frac{m}{2}}|^2 + \frac{m-1}{m}\int_\Omega S(v)\nabla u^m\cdot\nabla v + \int_\Omega u^{m-1}f(u)\end{aligned} \tag{5.2.22}$$

Let $f = u\ln^q(u+e)$, we obtain

$$\begin{aligned}\int_\Omega f\ln(f+e) &= \int_\Omega u\ln^q(u+e)\ln\left(u\ln^q(u+e)+e\right)\\ &\leq \int_\Omega u\ln^q(u+e)\ln\left((u+e)\ln^q(u+e)\right)\\ &\leq (q+1)\int_\Omega u\ln^k(u+e),\end{aligned} \tag{5.2.23}$$

for any $k > q+1$. Now by applying Lemma 5.2.5 with a fixed $k > q+1$, and Lemma 5.1.6 , we find that v is uniformly bounded in time. Therefore, degeneracy of the diffusion terms is eliminated since $\inf_{(x,t)\in\Omega\times(0,T_{\max})} D(v(x,t)) := c_1 > 0$. Thus, we have

$$-\frac{4(m-1)}{m^2}\int_\Omega D(v)|\nabla u^{\frac{m}{2}}|^2 \leq -c_2\int_\Omega|\nabla u^{\frac{m}{2}}|^2, \tag{5.2.24}$$

where $c_2 = \frac{4(m-1)c_1}{m^2}$. Using integration by parts, the second equation of (5.2.1) and nonnegativity of S' and v deduces that

$$\begin{aligned}\frac{m-1}{m}\int_\Omega S(v)\nabla u^m\cdot\nabla v &= -\frac{m-1}{m}\int_\Omega S(v)u^m\Delta v - \frac{m-1}{m}\int_\Omega S'(v)u^m|\nabla v|^2\\ &\leq -\frac{m-1}{m}\int_\Omega S(v)u^m\left(u\ln^q(u+e)-v\right)\\ &\leq c_3\int_\Omega u^{m+1}\ln^q(u+e),\end{aligned} \tag{5.2.25}$$

where $c_3 = -\frac{(m-1)\|S\|_{L^\infty([0,\infty))}}{m}$. Applying Lemma 5.2.5 with $k = 1$ and Lemma B.0.5 with

$$\epsilon = -\frac{c_2}{2c_3 \sup_{t\in(0,T_{\max})} \int_\Omega u \ln(u+e)},$$

yields

$$\begin{aligned}
c_3 \int_\Omega u^{m+1} \ln^q(u+e) &\le c_3\epsilon \int_\Omega |\nabla u^{\frac{m}{2}}|^2 \cdot \int_\Omega u \ln(u+e) + \epsilon \left(\int_\Omega u\right)^m \cdot \int_\Omega u \ln(u+e) + c_4 \\
&\le \frac{c_2}{2} \int_\Omega |\nabla u^{\frac{m}{2}}|^2 + c_5,
\end{aligned} \tag{5.2.26}$$

where $c_4 = C(\epsilon) > 0$ and $c_5 = c_4 + \sup_{t\in(0,T_{\max})} \int_\Omega u \ln(u+e) \cdot \sup_{t\in(0,T_{\max})} \left(\int_\Omega u\right)^m$. By elementary inequalities, there exists a positive constant $c_6 = C(r, m, p, \mu)$ such that

$$\frac{1}{m} \int_\Omega u^m + \int_\Omega u^{m-1} f(u) \le -\frac{\mu}{2} \int_\Omega \frac{u^{m+1}}{\ln^p(u+e)} + c_6. \tag{5.2.27}$$

Collecting (5.2.22), (5.2.25), (5.2.26), and (5.2.27) entails that

$$\frac{1}{m}\frac{d}{dt} \int_\Omega u^m + \frac{1}{m} \int_\Omega u^m \le c_7,$$

where $c_7 = c_5 + c_6$. This, together with Gronwall's inequality proves (5.2.21), which finishes the proof. □

We are now ready to prove Theorem 5.2.1.

Proof of Theorem 5.2.1. It follows from Lemma 5.2.6 that $u \in L^\infty((0, T_{\max}); L^m(\Omega))$ for any $m > 1$. By standard elliptic regularity theory, we obtain that $v \in L^\infty((0, T_{\max}); W^{1,\infty}(\Omega))$. By applying Moser-Alikakos iteration (see e.g [53, 2, 1]), it follows that $u \in L^\infty((0, T_{\max}); L^\infty(\Omega))$. Now applying the extensibity of solutions (5.2.10) yields that $T_{\max} = \infty$. Therefore, (5.2.7) is proved, which finishes the proof. □

5.2.4 Fully parabolic

In this section, we consider a solution (u, v) of the system (5.2.1) with $\kappa = 1$, under the conditions from (5.2.2) to (5.2.6). Let us commence with the following priori estimate, similar to Lemma 5.2.5.

Lemma 5.2.7. *If* $p < 1 < k < 2 - p$, *and* $2q + p < 1$ *then*

$$\sup_{t\in(0,T_{max})} \int_\Omega \left(u \ln^k (u+e) + |\nabla v|^2\right) + \sup_{t\in(0,T_{max}-\tau)} \int_t^{t+\tau} \int_\Omega u^2 \ln^{k-p}(u+e) < \infty, \tag{5.2.28}$$

where $\tau = \min\left\{1, \frac{T_{max}}{2}\right\}$.

Proof. We define

$$y(t) := \int_\Omega u \ln^k (u+e) + \frac{1}{2}\int_\Omega |\nabla v|^2,$$

and differentiate $y(\cdot)$ to obtain

$$y'(t) = \int_\Omega \left(\ln^k (u+e) + ku\frac{\ln^{k-1}(u+e)}{u+e}\right) u_t + \nabla v \cdot \nabla v_t \tag{5.2.29}$$

By integration by parts, the first term of (5.2.29) is expressed as in (5.2.15). We have

$$\begin{aligned}\int_\Omega S(v)\nabla\phi(u)\cdot\nabla v &= -\int_\Omega S(v)\phi(u)\Delta v - \int_\Omega S'(v)\phi(u)|\nabla v|^2 \\ &\le \|S\|_{L^\infty([0,\infty))}\int_\Omega \phi(u)|\Delta v|,\end{aligned}$$

where the last inequality comes from the assumption $S' \ge 0$ and ϕ is defined in (5.2.16). Recalling the upper bound for ϕ as in (5.2.16), we have

$$\phi(l) \le c_1 l \ln^{k-1}(l+e).$$

This, together with Young's inequality to this yields

$$\begin{aligned}\int_\Omega S(v)\nabla\phi(u)\cdot\nabla v &\le \frac{1}{2}\int_\Omega(\Delta v)^2 + \frac{\|S\|_{L^\infty([0,\infty))}}{2}\int_\Omega \phi^2(u) \\ &\le \frac{1}{2}\int_\Omega(\Delta v)^2 + c_2\int_\Omega u^2\ln^{2k-2}(u+e) \\ &\le \frac{1}{2}\int_\Omega(\Delta v)^2 + \frac{\mu}{4}\int_\Omega u^2\ln^{k-p}(u+e) + c_3, \end{aligned} \tag{5.2.30}$$

where $c_2 = \frac{c_1\|S\|_{L^\infty([0,\infty))}}{2}$ and $c_3 = C(\mu.k,p) > 0$. Collecting (5.2.15),(5.2.17),(5.2.19) and (5.2.30) implies that

$$\frac{d}{dt}\int_\Omega u\ln^k(u+e) + \int_\Omega u\ln^k(u+e) + \frac{3\mu}{4}\int_\Omega u^2\ln^{k-p}(u+e) \le \frac{1}{2}\int_\Omega(\Delta v)^2 + c_4, \tag{5.2.31}$$

where $c_4 = C(\mu, r, k, p, \|S\|_{L^\infty([0,\infty))}) > 0$. By integration by parts and elementary inequalities, the second term of (5.2.29) can be handled as follows:

$$\begin{aligned}\int_\Omega \nabla v \cdot \nabla v_t &= -\int_\Omega (\Delta v)^2 - \int_\Omega |\nabla v|^2 - \int_\Omega u \ln^q(u+e)\Delta v \\ &\le -\frac{1}{2}\int_\Omega (\Delta v)^2 - \int_\Omega |\nabla v|^2 + \frac{1}{2}\int_\Omega u^2 \ln^{2q}(u+e) \\ &\le -\frac{1}{2}\int_\Omega (\Delta v)^2 - \int_\Omega |\nabla v|^2 + \frac{\mu}{2}\int_\Omega u^2 \ln^{k-p}(u+e) + c_5, \quad 2q < k-p. \quad (5.2.32)\end{aligned}$$

where $c_5 = C(\mu, k, p, q) > 0$. From (5.2.31) and (5.2.32), we obtain that

$$y'(t) + y(t) + \frac{\mu}{2}\int_\Omega u^2 \ln^{k-p}(u+e) \le c_6, \tag{5.2.33}$$

where $c_6 = c_4 + c_5$. By applying Gronwall's inequality, we deduce that $y(t) \le \max\{y(0), c_6\}$. Additionally, we have

$$\frac{\mu}{2}\int_\Omega u^2 \ln^{k-p}(u+e) \le c_6 - y'(t). \tag{5.2.34}$$

By integrating the previous inequality from t to $t+\tau$ and using the fact that y is bounded, we can conclude the proof. □

Thanks to Lemma 5.2.7 and C.1.1, we can now establish an L^∞ bound for v, which is subsequently used to eliminate the degeneracy of the diffusion term.

Lemma 5.2.8. *If* $1+p+2q<k$, *and* (u,v) *is a solution of the system* (5.2.1) *such that*

$$\sup_{t\in(0,T-\tau)} \int_t^{t+\tau}\int_\Omega u^2 \ln^{k-p}(u+e) < \infty$$

then v *is globally bounded in time.*

Proof. We let $f := u\ln^q(u+e)$ and $L(s) := \ln^\lambda(s+e)$, where $\lambda > 1$ will be determined later. Notice that $u\ln^q(u+e) \le (u+e)^2$, we have

$$L(f) = \ln^\lambda(u\ln^q(u+e)) \le 2^\lambda \ln^\lambda(u+e).$$

Now, we want

$$\sup_{t\in(0,T-\tau)} \int_t^{t+\tau}\int_\Omega f^2 L(f) < \infty, \tag{5.2.35}$$

which is indeed true when $2q+\lambda \le k-p$, since

$$\int_t^{t+\tau}\int_\Omega f^2 L(f) \le c\int_t^{t+\tau}\int_\Omega u^2 \ln^{2q+\lambda}(u+e) \le c\int_t^{t+\tau}\int_\Omega u^2 \ln^{k-p}(u+e). \tag{5.2.36}$$

Now we fix where $\lambda := k-p-2q > 1$ and complete the proof by applying Proposition C.1.1
. □

In contrast to the parabolic-elliptic scenario where $\kappa = 0$, the direct derivation of L^m bounds for u from the a priori estimate in Lemma 5.2.7 is not feasible. Instead, we rely on the assistance of the subsequent lemma, which functions as an intermediate estimate facilitating the connection between the two equations presented in (5.2.1).

Lemma 5.2.9. *There exist positive constants A_1, A_2, A_3 such that*

$$\frac{1}{4}\frac{d}{dt}\int_\Omega |\nabla v|^4 + A_1\int_\Omega \left|\nabla|\nabla v|^2\right|^2 + \int_\Omega |\nabla v|^4 \le A_2\int_\Omega u^2\ln^{2q}(u+e)|\nabla v|^2 + A_3\int_\Omega |\nabla v|^4 \tag{5.2.37}$$

Proof. We make use of the following point-wise identity

$$\nabla v\cdot\nabla\Delta v = \frac{1}{2}\Delta(|\nabla v|^2) - |D^2 v|^2$$

to obtain

$$\begin{aligned}\frac{1}{4}\frac{d}{dt}\int_\Omega |\nabla v|^4 + \int_\Omega |\nabla v|^4 &= -c_1\int_\Omega |\nabla|\nabla v|^2|^2 - \int_\Omega |\nabla v|^2|D^2 v|^2\\ &\quad+\int_\Omega |\nabla v|^2\nabla v\cdot\nabla(u\ln^q(u+e))\\ &\quad+c_2\int_{\partial\Omega}\frac{\partial|\nabla v|^2}{\partial\nu}|\nabla v|^2,\end{aligned} \tag{5.2.38}$$

where c_1, c_2 are positive constants. The inequality $\frac{\partial|\nabla v|^2}{\partial\nu} \le M|\nabla v|^2$, (see [41][Lemma 4.2]) for some $M>0$ depending only on Ω, implies that

$$c_2\int_{\partial\Omega}\frac{\partial|\nabla v|^2}{\partial\nu}|\nabla v|^2\,dS \le c_2 M\int_{\partial\Omega}|\nabla v|^4\,dS.$$

Let $g := |\nabla v|^2$ and apply trace embedding theorem $W^{1,1}(\Omega) \longrightarrow L^1(\partial\Omega)$ together with Young's inequality, there exist positive constants C and c_3 such that

$$c_2 M\int_{\partial\Omega} g^2\,dS \le C\int_\Omega g|\nabla g| + C\int_\Omega g^2$$

$$\leq \frac{c_1}{2}\int_\Omega |\nabla g|^2 + c_3\int_\Omega g^2, \tag{5.2.39}$$

Therefore, we have

$$c_2 M\int_{\partial\Omega} |\nabla v|^4\,dS \leq \frac{c_1}{2}\int_\Omega |\nabla|\nabla v|^2|^2 + c_3\int_\Omega |\nabla v|^4. \tag{5.2.40}$$

Applying the pointwise inequality $(\Delta v)^2 \leq 2|D^2 v|^2$ to (5.2.38) yields

$$\begin{aligned}\frac{1}{4}\frac{d}{dt}\int_\Omega |\nabla v|^4 + \int_\Omega |\nabla v|^4 \leq& -\frac{c_1}{2}\int_\Omega |\nabla|\nabla v|^2|^2 - \frac{1}{2}\int_\Omega |\nabla v|^2|\Delta v|^2\\ &+ \int_\Omega |\nabla v|^2\nabla v\cdot\nabla(u\ln^q(u+e)) + c_3\int_\Omega |\nabla v|^4\end{aligned} \tag{5.2.41}$$

By integration by parts and elemental inequalities, there exist constants $c_4 > 0$ and $c_5 > 0$ in such a way that

$$\begin{aligned}\int_\Omega |\nabla v|^2\nabla v\cdot\nabla(u\ln^q(u+e)) =& -\int_\Omega u\ln^q(u+e)|\nabla v|^2\Delta v - c_4\int_\Omega u\ln^q(u+e)\nabla|\nabla v|^2\cdot\nabla v\\ \leq& \frac{1}{2}\int_\Omega (\Delta v)^2|\nabla v|^2 + \frac{c_1}{4}\int_\Omega |\nabla|\nabla v|^2|^2 + c_5\int_\Omega u^2\ln^{2q}(u+e)|\nabla v|^2.\end{aligned} \tag{5.2.42}$$

Combining (5.2.41) and (5.2.42) yields (5.2.37). □

One may employ Lemma B.0.5 to establish an L^m bound for u, where $m \in (2, 3-2q)$. However, in this context, an alternative methodology is adopted, relying on Lemma 5.2.4. This approach involves initially deriving an $L^2\ln(L+e)$ bound and subsequently utilizing it to establish an L^4 bound for u.

Lemma 5.2.10. *If $k > p + 2q + 1$ such that*

$$\sup_{t\in(0,T_{max})}\int_\Omega u\ln^k(u+e) < \infty,$$

then we have

$$\sup_{t\in(0,T_{max})}\int_\Omega \left(u^2\ln(u+e) + |\nabla v|^4\right) < \infty.$$

Proof. We denote

$$y(t) := \int_\Omega u^2 \ln(u+e) + \frac{1}{4}\int_\Omega |\nabla v|^4,$$

and differentiate y to obtain

$$y'(t) = \int_\Omega \left(2u\ln(u+e) + \frac{u^2}{u+e}\right) u_t + \int_\Omega |\nabla v|^2 \nabla v \cdot \nabla v_t := I + J. \tag{5.2.43}$$

We use the first equation of (5.2.1), and integration by parts to estimate I

$$\begin{aligned}
I &:= \int_\Omega \left(2u\ln(u+e) + \frac{u^2}{u+e}\right)(\nabla\cdot(D(v)\nabla u - uS(v)\nabla v + f(u))) \\
&= -\int_\Omega D(v)\left(2\ln(u+e) + \frac{2u}{u+e} + \frac{u^2+2ue}{(u+e)^2}\right)|\nabla u|^2 \\
&+ \int_\Omega uS(v)\left(2\ln(u+e) + \frac{2u}{u+e} + \frac{u^2+2ue}{(u+e)^2}\right)\nabla u\cdot\nabla v \\
&+ \int_\Omega \left(2u\ln(u+e) + \frac{u^2}{u+e}\right) f(u) \\
&:= I_1 + I_2 + I_3.
\end{aligned} \tag{5.2.44}$$

Lemma 5.2.8 implies that v is bounded at all time, which entails that $\inf_{(x,t)\in\Omega\times(0,T)} D(v(x,t)) := \alpha > 0$. Therefore, I_1 is bounded by

$$I_1 \le -\alpha \int_\Omega \ln(u+e)|\nabla u|^2. \tag{5.2.45}$$

Now, I_2 can be controlled by using elementary inequalities

$$\begin{aligned}
I_2 &:= \int_\Omega uS(v)\left(2\ln(u+e) + \frac{2u}{u+e} + \frac{u^2+2ue}{(u+e)^2}\right)\nabla u\cdot\nabla v \\
&\le c_1 \int_\Omega u\ln(u+e)|\nabla u||\nabla v| \\
&\le \frac{\alpha}{2}\int_\Omega |\nabla u|^2 \ln(u+e) + c_2 \int_\Omega u^2\ln(u+e)|\nabla v|^2
\end{aligned} \tag{5.2.46}$$

where $c_1 = 5\,\|S\|_{L^\infty([0,\infty))}$ and $c_2 = \frac{c_1^2}{4\alpha}$. Using Young's inequality with $\epsilon > 0$ yields

$$c_2 \int_\Omega u^2 \ln(u+e)|\nabla v|^2 \le \epsilon \int_\Omega |\nabla v|^6 + c_3 \int_\Omega u^3 \ln^{\frac{3}{2}}(u+e), \tag{5.2.47}$$

where $c_3 = C(\epsilon) > 0$. By using elementary inequalities, one can find a positive constant $c_4 = C(\mu, r, p)$ such that

$$I_3 + \int_\Omega u^2\ln(u+e) = \int_\Omega \left(2u\ln(u+e) + \frac{u^2}{u+e}\right)\left(ru - \frac{\mu u^2}{\ln^p(u+e)}\right) + \int_\Omega u^2 \ln(u+e)$$

$$\leq c_4. \tag{5.2.48}$$

Collecting from (5.2.45) to (5.2.48) yields

$$I + \int_\Omega u^2 \ln(u+e) + \frac{\alpha}{2}\int_\Omega |\nabla u|^2 \ln(u+e) \leq \epsilon \int_\Omega |\nabla v|^6 + c_3 \int_\Omega u^3 \ln^{3/2}(u+e) + c_4. \tag{5.2.49}$$

From Lemma 5.2.9 and applying Young's inequality, we have

$$\begin{aligned}\frac{1}{4}\frac{d}{dt}\int_\Omega |\nabla v|^4 + A_1 \int_\Omega \left|\nabla |\nabla v|^2\right|^2 + \int_\Omega |\nabla v|^4 &\leq A_2 \int_\Omega u^2 \ln^{2q}(u+e)|\nabla v|^2 + A_3 \int_\Omega |\nabla v|^4 \\ &\leq \epsilon \int_\Omega |\nabla v|^6 + c_5 \int_\Omega u^3 \ln^{3q}(u+e) + c_6 \\ &\leq \epsilon \int_\Omega |\nabla v|^6 + c_5 \int_\Omega u^3 \ln^{3/2}(u+e) + c_6,\end{aligned} \tag{5.2.50}$$

where $c_5 = C(\epsilon) > 0$, $c_6 = C(\epsilon) > 0$, and the last inequality comes from the fact that $2q < 1$. Collecting (5.2.49) and (5.2.50) yields

$$\begin{aligned}y'(t) + y(t) + A_1 \int_\Omega \left|\nabla |\nabla v|^2\right|^2 + \frac{\alpha}{2}\int_\Omega |\nabla u|^2 \ln(u+e) &\leq 2\epsilon \int_\Omega |\nabla v|^6 \\ &\quad + c_7 \int_\Omega u^3 \ln^{3/2}(u+e) + c_8\end{aligned} \tag{5.2.51}$$

where $c_7 = c_3 + c_5$ and $c_8 = c_4 + c_6$. From Lemma B.0.2, there exists a positive constant c_9 such that

$$\int_\Omega |\nabla v|^6 \leq c_9 \int_\Omega |\nabla |\nabla v|^2|^2 \cdot \int_\Omega |\nabla v|^2 + c_9 \left(\int_\Omega |\nabla v|^2\right)^3. \tag{5.2.52}$$

Lemma 5.2.7 asserts that $\int_\Omega |\nabla v|^2$ is uniformly bounded in time, therefore from (5.2.52) we obtain

$$2\epsilon \int_\Omega |\nabla v|^6 \leq \epsilon c_{10} \int_\Omega |\nabla |\nabla v|^2|^2 + c_{11}, \tag{5.2.53}$$

where $c_{10} = 2\epsilon c_9 \sup_{t\in(0,T_{\max})} \int_\Omega |\nabla v|^2$ and $c_{11} = \epsilon c_9 \left(\sup_{t\in(0,T_{\max})} \int_\Omega |\nabla v|^2\right)^3$. Now we fix $\epsilon = \frac{A_1}{2c_{10}}$ and apply Lemma 5.2.4 with

$$\delta = \frac{\alpha}{4c_7 \sup_{t\in(0,T_{\max})} \int_\Omega u \ln^k(u+e)}$$

to deduce that

$$\begin{aligned}
c_7 \int_\Omega u^3 \ln^{3/2}(u+e) &\leq c_7\delta \int_\Omega |\nabla u|^2 \ln(u+e) \cdot \int_\Omega u \ln^k(u+e) \\
&\quad + c_7\delta \int_\Omega u \ln^k(u+e) + c_{12} \left(\int_\Omega u \ln^{1/2}(u+e) \right)^3 \\
&\leq \frac{\alpha}{4} \int_\Omega |\nabla u|^2 \ln(u+e) + c_{13}, \qquad (5.2.54)
\end{aligned}$$

where $c_{12} = C(\delta) > 0$ and

$$c_{13} = c_7\delta \sup_{t\in(0,T_{\max})} \int_\Omega u \ln^k(u+e) + c_{12} \left(\sup_{t\in(0,T_{\max})} \int_\Omega u \ln^{1/2}(u+e) \right)^3.$$

Collecting (5.2.51), (5.2.53), and (5.2.54) yields

$$y'(t) + y(t) \leq c_{14}, \qquad (5.2.55)$$

where $c_{14} = c_8 + c_{13}$. This, together with Gronwall's inequality completes the proof. □

We are read to derive an L^4 bound for u by employing a standard testing procedure.

Lemma 5.2.11. *If* (u, v) *is a solution of the system* (5.2.1) *such that*

$$\sup_{t\in(0,T)} \int_\Omega u^2 \ln(u+e) < \infty,$$

then

$$\sup_{t\in(0,T)} \int_\Omega u^4 < \infty.$$

Proof. Multiplying the first equation of (5.2.1) to u^3 and using integration by parts, we have

$$\frac{d}{dt} \int_\Omega \frac{u^4}{4} = -3 \int_\Omega D(v)u^2|\nabla u|^2 + 3 \int_\Omega S(v)u^3 \nabla u \cdot \nabla v + \int_\Omega u^3 f(u). \qquad (5.2.56)$$

From Lemma 5.2.8 entails that v is bounded at all time, we have $\inf_{(x,t)\in\Omega\times(0,T)} D(v(x,t)) := c_1 > 0$. Therefore, we obtain

$$-3 \int_\Omega D(v)u^2|\nabla u|^2 \leq -3c_1 \int_\Omega u^2|\nabla u|^2. \qquad (5.2.57)$$

By using Holder's inequality, we have

$$3c_1\int_\Omega S(v)u^3\nabla u\cdot\nabla v \le c_1\int_\Omega u^2|\nabla u|^2+\frac{9c_1}{4}\int_\Omega u^4|\nabla v|^2. \tag{5.2.58}$$

Now we find that $g(u)\in L^\infty\left((0,T_{\max});L^2(\Omega)\right)$ since

$$\int_\Omega g^2(u)=\int_\Omega u^2\ln^{2q}(u+e)\le\int_\Omega u^2\ln(u+e)\le C$$

for all $t\in(0,T_{\max})$. Therefore, Lemma C.1.2 implies that $v\in L^\infty\left((0,T_{\max});W^{1,\lambda}(\Omega)\right)$ for any $\lambda\in[1,\infty)$, which means that $\sup_{t\in(0,T)}\int_\Omega|\nabla v|^\lambda<\infty$. Now, we estimate the last term of the right hand side of (5.2.58) by using Holder's inequality and then Young's inequality as follows

$$\begin{aligned}\frac{9c_1}{4}\int_\Omega u^4|\nabla v|^2 &\le \frac{9c_1}{4}\left(\int_\Omega u^{9/2}\right)^{8/9}\left(\int_\Omega|\nabla v|^{18}\right)^{1/9}\\ &\le c_2\left(\int_\Omega u^{9/2}\right)^{8/9}\\ &\le c_2\int_\Omega u^{9/2}+c_3\\ &\le \frac{\mu}{2}\int_\Omega\frac{u^5}{\ln^p(u+e)}+c_4.\end{aligned} \tag{5.2.59}$$

where $c_2=\frac{9c_1}{4}\sup_{t\in(0,T)}\int_\Omega|\nabla v|^{18}$, $c_3>0$, and $c_4=C(\mu,p)>0$ By applying Young's inequality again, one can verify that

$$\int_\Omega u^3 f(u)+\frac{1}{4}\int_\Omega u^4\le-\frac{\mu}{2}\int_\Omega\frac{u^5}{\ln^p(u+e)}+c_5, \tag{5.2.60}$$

where $c_5=C(\mu,p)>0$. Collecting (5.2.59) and (5.2.60) yields

$$\frac{1}{4}\frac{d}{dt}\int_\Omega u^4+\frac{1}{4}\int_\Omega u^4\le c_6,$$

where $c_6=c_4+c_5$. The proof is finished by applying Gronwall's inequality. □

Now we are in the position to prove Theorem 5.2.2.

Proof of Theorem 5.2.2. First, from the assumption $2q+p<k<2-p$, we obtain

$$\sup_{t\in(0,T-\tau)}\int_t^{t+\tau}\int_\Omega u^2\ln^{k-p}(u+e)<\infty.$$

Furthermore, the condition $0 < q < 1/2 - p$ enables us to choose $k \in (1+p+2q, 2-p)$ and then Lemma 5.2.8 implies that v is globally bounded in time. Thus, we have $\inf_{(x,t)\in\Omega\times(0,T)} D(v(x,t)) > 0$. Now, we use Lemma 5.2.10 and Lemma 5.2.11 to obtain

$$\sup_{t\in(0,T)} \int_\Omega u^4 < \infty.$$

This, together with $\int_\Omega g^3(u) \leq \int_\Omega u^4$, entails that $g(u) \in L^\infty\left((0, T_{\max}); L^3(\Omega)\right)$. Therefore, by applying Lemma C.1.2, we deduce that $v \in L^\infty\left((0, T_{\max}); W^{1,\infty}(\Omega)\right)$. Finally, we can routinely apply Moser-Alikakos iteration (see e.g [53, 2, 1]) to deduce that $u \in L^\infty\left((0, T_{\max}); L^\infty(\Omega)\right)$. By extensibility of solutions (5.2.10), it follows that $T_{\max} = \infty$. Therefore, we have

$$\sup_{t\in(0,\infty)} \left\{ \|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} \right\} < \infty,$$

which finishes the proof. □

BIBLIOGRAPHY

[1] N.D. Alikakos. "An application of the invariance principle to reaction diffusion equations". In: *J. Differential Equations* 33 (1979), pp. 201–225.

[2] N.D. Alikakos. "Lp bounds of solutions of reaction-diffusion equations". In: *Comm. Partial Differential Equations* 4 (1979), pp. 827–868.

[3] X. Bai and M. Winkler. "Equilibration in a Fully Parabolic Two-Species Chemotaxis System with Competitive Kinetics". In: *Indiana University Mathematics Journal* 65 (2016), pp. 553–583.

[4] N. Bellomo et al. "Towards a Mathematical Theory of Keller-Segel Models of Pattern Formation in Biological Tissues". In: *Mathematical Models and Methods in Applied Sciences* 25 (Mar. 2015), p. 150324201437000.

[5] N. Bellomo et al. "Towards a Mathematical Theory of Keller-Segel Models of Pattern Formation in Biological Tissues". In: *Mathematical Models and Methods in Applied Sciences* 25 (Mar. 2015), pp. 1663–1763.

[6] P. Biler, W. Hebisch, and T. Nadzieja. "The Debye system: existence and large time behavior of solutions". In: *Nonlinear Analysis, Theory, Methods, and Applications* 23 (1994), pp. 1189–1209.

[7] A. Blanchet, J. Dolbeault, and B. Perthame. "Two-dimensional Keller-Segel model: Optimal critical mass and qualitative properties of the solutions". In: *Electronic Journal of Differential Equations,* (May 2006).

[8] V. Calvez and J.A. Carrillo. "Volume effects in the Keller–Segel model: energy estimates preventing blow-up". In: *J. Math. Pures Appl.* 86 (2006), pp. 155–175.

[9] X. Cao. "An interpolation inequality and its application in Keller-Segel model". In: *arXiv e-prints*, arXiv:1707.09235 (July 2017), arXiv:1707.09235. DOI: 10.48550/arXiv.1707.09235. arXiv: 1707.09235 [math.AP].

[10] M. Chipot, M. Fila, and P. Quittner. "Stationary solutions, blow up and convergence to stationary solutions for semilinear parabolic equations with nonlinear boundary conditions". In: *Acta Math. Univ. Comenianae* 60 (1991), pp. 35–103.

[11] E. Cruz, M. Negreanu, and J. I. Tello. "Asymptotic behavior and global existence of solutions to a two-species chemotaxis system with two chemicals". In: *Z. Angew. Math. Phys.* 69 (Aug. 2018).

[12] J. Dolbeault and B. Perthame. "A Optimal critical mass in the two dimensional Keller–Segel model in R2". In: *C. R. Acad. Sci. Paris, Ser. I* 339 (2004), pp. 611–616.

[13] J. Filo. "Uniform bounds for solutions of a degenerate diffusion equation with nonlinear boundary conditions". eng. In: *Commentationes Mathematicae Universitatis Carolinae* 030.3 (1989), pp. 485–495. URL: `http://eudml.org/doc/17762`.

[14] M. Freitag. "Blow-up profiles and refined extensibility criteria in quasilinear Keller–Segel systems". In: *J. Math.Anal.Appl.* 463 (Mar. 2018), pp. 964–988.

[15] D. Gilbarg and N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. 1998.

[16] J. L. Gómez, V. Márquez, and N. Wolanski. "Blow up results and localization of blow up points for the heat equation with a nonlinear boundary condition". In: *Journal of Differential Equations* 92 (1991), pp. 384–401.

[17] A. Gurusamy and T. Jagmohan. "Keller-Segel Chemotaxis Models: A Review". In: *Acta Applicandae Mathematicae* 171 (Feb. 2021).

[18] T. Hillen and K.J. Painter. "A user's guide to PDE models for chemotaxis". In: *J. Math. Biol.* 58 (2009), pp. 183–217.

[19] F. K. O. Hoffmann. *Keller-Segel-Type Models and Kinetic Equations for Interacting Particles: Long-Time Asymptotic Analysis(Doctoral thesis)*. 2017.

[20] D. Horstmann. "From 1970 until present: the Keller-Segel model in chemotaxis and its consequences". In: *I. Jahresber. Deutsch. Math.-Verein* 105 (2003), pp. 103–165.

[21] D. Horstmann and M. Winkler. "Boundedness vs. blow-up in a chemotaxis system". In: *J. Differential Equations* 215 (2005), pp. 52–107.

[22] B. Hu and Y. Tao. "Boundedness in a parabolic–elliptic chemotaxis-growth system under a critical parameter condition". In: *Nonlinear Analysis* 64 (2017), pp. 1–7.

[23] B. Hu and Y. Tao. "Boundedness in a parabolic–elliptic chemotaxis-growth system under a critical parameter condition". In: *Applied Mathematics Letters* 64 (2017), pp. 1–7.

[24] H. Jina and T. Xiang. "Chemotaxis effect vs. logistic damping on boundedness in the2-D minimal Keller–Segel model". In: *C. R. Acad. Sci. Paris, Ser.I* 356 (2018), pp. 875–885.

[25] K. Kan and A. Stevens. "Blowup and global solutions in a chemotaxis–growth system". In: *Nonlinear Analysis* 135 (2016), pp. 57–72.

[26] E. F. Keller and L. A. Segel. "Initiation of Slime Mold Aggregation Viewed as an Instability". In: *Journal of Theoretical Biology* 26 (1970), pp. 399–415.

[27] J. Lankeit. "Chemotaxis can prevent thresholds on population density". In: *Discrete and*

Continuous Dynamical Systems - B 20 (2015), pp. 1499–1527.

[28] J. Lankeit. “Eventual smoothness and asymptotics in a three-dimensional chemotaxis system with logistic source”. In: *Journal of Differential Equations* 258 (2015), pp. 1158–1191.

[29] J. Lankeit. “Locally bounded global solutions to a chemotaxis consumption model with singular sensitivity and nonlinear diffusion,” in: *Journal of Differential Equations* 262 (2017), pp. 4052–4048.

[30] M. Le. “Blow-up prevention by sub-logistic sources in Keller-Segel cross diffusion type system”. In: *Discrete and Continuous Dynamical Systems - Series B* 29 (2024), pp. 796–811.

[31] M. Le. “Global existence of solutions in some chemotaxis systems with sub-logistic source under nonlinear Neumann boundary conditions in 2d”. In: *Nonlinear Analysis* 241 (2024), p. 113491.

[32] M. Le. “Global existence of solutions in two-species chemotaxis system with two chemicals with sub-logistic sources in 2d”. In: *Applied Mathematics Letter* 149 (Mar. 2024), p. 108925.

[33] M. Le. “Global existence of solutions to the chemotaxis system with logistic source under nonlinear Neumann boundary condition”. In: *Journal of Differential Equations* 377 (Dec. 2023), pp. 1–37.

[34] D. Li et al. “Large time behavior of solution to an attraction–repulsion chemotaxis system with logistic source in three dimensions”. In: *Journal of Mathematical Analysis and Applications* 448 (2017), pp. 914–936.

[35] X. Li and Y. Wang. “On a fully parabolic chemotaxis system with Lotka–Volterra competitive kinetics”. In: *Journal of Mathematical Analysis and Applications* 471 (Mar. 2019), pp. 584–598.

[36] Y. Li and J. Lankeit. “Boundedness in a chemotaxis-haptotaxis model with nonlinear diffusion”. In: *Nonlinearity.* 29(6) (2016), pp. 1564–1595.

[37] G. M. Lieberman. *Second order parabolic differential equations*. World Scientific, 1996.

[38] K. Lin, C. Mu, and H. Zhong. “A blow-up result for a quasilinear chemotaxis system with logistic source in higher dimensions”. In: *Journal of Mathematical Analysis and Applications* 464 (Aug. 2018), pp. 435–455.

[39] D. Liu and Y. Tao. “Global boundedness in a fully parabolic attraction–repulsion chemotaxis model”. In: *Math. Meth. Appl. Sci.* 38 (2015), pp. 2537–2546.

[40] H. Matano. “Asymptotic Behavior and Stability of Solutions of Semilinear Diffusion Equa-

tions". In: *Publ. Res. Inst. Math. Sci.* 15 (1979), pp. 401–454.

[41] N. Mizoguchi and P. Souplet. "Nondegeneracy of blow-up points for the parabolic Keller–Segel system". In: *Annales de l'Institut Henri Poincare (C) Non Linear Analysis* 31 (Jan. 2013).

[42] T. Nagai. "Blow-up of radially symmetric solutions to a chemotaxis system". In: *Adv. Math. Sci. Appl.* 5 (1995), pp. 581–601.

[43] T. Nagai. "Global existence and blowup of solutions to a chemotaxis system". In: *Nonlinear Analysis* 47 (2001), pp. 777–787.

[44] T. Nagai, T. Senba, and K. Yoshida. "Application of the Trudinger-Moser inequality to a Parabolic System of Chemotaxis," in: *Funkcilaj Ekvacioj* 40 (1997), pp. 411–433.

[45] T. Nagai and T. Yoshida. "Global Existence of Solutions to a Parabolic-Elliptic System of Drift-Diffusion Type in R2," in: *Funkcilaj Ekvacioj* 59 (2016), pp. 67–112.

[46] K. Osaki and A. Yagi. "Global existence for a chemotaxis-growth system in $\mathbb{R}^2$". In: *Adv. Math. Sci. Appl.* 2 (Dec. 2002), pp. 587–606.

[47] K. Osaki et al. "Exponential attractor for a chemotaxis-growth system of equations". In: *Nonlinear Analysis* 51 (2002), pp. 119–144.

[48] P. Quittner. "Global existence of solutions of parabolic problems with nonlinear boundary conditions". In: *Banach Center Publ.* 33 (1996), pp. 309–314.

[49] P. Quittner. "On global existence and stationary solutions for two classes of semilinear parabolic problems Pavol Quittner". In: *Comment.Math.Univ.Carolin.* 34 (1993), pp. 105–124.

[50] P. Quittner and P. Souplet. *Superlinear Parabolic Problems: Blow-up, Global Existence and Steady States*. Springer, 2019. ISBN: 978-3-030-18220-5.

[51] T. Senba T. Nagai and T. Suzuki. "Blowup of nonradial solutions to parabolic-elliptic systems modeling chemotaxis in two-dimensional domains". In: *J. Inequal. Appl.* 6 (2001), pp. 37–55.

[52] T. Senba T. Nagai and T. Suzuki. "Chemotaxis collapse in a parabolic system of mathematical biology". In: *HiroshimaMath. J.* 20 (2000), pp. 463–497.

[53] Y. Tao and M. Winkler. "Boundedness in a quasilinear parabolic-parabolic Keller-Segel system with subcritical sensitivity". In: *Journal of Differential Equations* 252 (June 2011).

[54] Y. Tao and M. Winkler. "Boundedness vs. blow-up in a two-species chemotaxis system

with two chemicals". In: *Discrete and Continuous Dynamical Systems - Series B* 20 (2015), pp. 3165–3183.

[55] Y. Tao and M. Winkler. "Dominance of chemotaxis in a chemotaxis–haptotaxis model". In: *Nonlinearity* 27 (May 2014), pp. 1225–1239.

[56] Y. Tao and M. Winkler. "Energy-type estimates and global solvability in a two-dimensional chemotaxis–haptotaxis model with remodeling of non-diffusible attractant". In: *Journal of Differential Equations* 257 (Aug. 2014), pp. 784–815.

[57] Y. Tao and M. Winkler. "Persistence of mass in a chemotaxis system with logistic source". In: *Journal of Differential Equations* 259 (Dec. 2015), pp. 6142–6161.

[58] J. Tello and M. Winkler. "A chemotaxis system with logistic source," in: *Comm. Partial Differential Equations* 32 (2007), pp. 849–877.

[59] J. I. Tello and M. Winkler. "A Chemotaxis System with Logistic Source". In: *Communications in Partial Differential Equations* 32 (June 2007), pp. 849–877.

[60] M. Tian, X. He, and S. Zheng. "Global attractors in a two-species chemotaxis system with two chemicals and logistic sources". In: *Journal of Mathematical Analysis and Applications* 508 (2022), p. 125861.

[61] M. Winkler. "A result on parabolic gradient regularity in Orlicz spaces and application to absorption-induced blow-up prevention in a Keller-Segel type cross-diffusion system". In: *Preprint* 1 (2022).

[62] M. Winkler. "Aggregation vs. global diffusive behavior in the higher-dimensional Keller–Segel model". In: *J. Differential Equations* 248(12) (2010), pp. 2889–2905.

[63] M. Winkler. "Blow-up in a higher-dimensional chemotaxis system despite logistic growth restriction". In: *Journal of Mathematical Analysis and Applications* 384 (Dec. 2011), pp. 261–272.

[64] M. Winkler. "Boundedness in the Higher-Dimensional Parabolic-Parabolic Chemotaxis System with Logistic Source". In: *Communications in Partial Differential Equations* 35 (2010), pp. 1516–1537.

[65] M. Winkler. "Finite-time blow-up in the higher-dimensional parabolic–parabolic Keller–Segel system". In: *Journal de Mathématiques Pures et Appliquées* 100.5 (2013), pp. 748–767.

[66] M. Winkler. "Logarithmically refined Gagliardo-Nirenberg interpolation and application to blow-up exclusion in a two-dimensional chemotaxis-consumption system". In: *Preprint* (2023).

[67] T. Xiang. “Boundedness and global existence in the higher-dimensional parabolic–parabolic chemotaxis system with/without growth source”. In: *Journal of Differential Equations* 258 (June 2015), pp. 4275–4323.

[68] T. Xiang. “Boundedness and global existence inthehigher-dimensional parabolic–parabolic chemotaxis system with/without growth source”. In: *J. Differential Equations* 258 (July 2015), pp. 4275–4323.

[69] T. Xiang. “Chemotactic Aggregation versus Logistic Damping on Boundedness in the 3D Minimal Keller-Segel Model”. In: *SIAM J. APPL. MATH.* 78 (2018), pp. 2420–2438.

[70] T. Xiang. “Dynamics in a parabolic-elliptic chemotaxis system with growth source and nonlinear secretion”. In: *Communications on Pure and Applied Analysis* 18 (2019), pp. 255–284.

[71] T. Xiang. “Sub-logistic source can prevent blow-up in the 2D minimal Keller-Segel chemotaxis system”. In: *J. Math. Phys.* 59 (Aug. 2018), p. 081502.

[72] T. Xiang and J. Zheng. “A new result for 2D boundedness of solutions to a chemotaxis–haptotaxis model with/without sub-logistic source”. In: *Nonlinearity* 32 (Dec. 2019), pp. 4-890–4911.

[73] A. Yagi. *Abstract Parabolic Evolution Equations and their Applications*. Springer, 2010. ISBN: 978-3-642-04630-8.

[74] C. Yang et al. “Boundedness in a quasilinear fully parabolic Keller–Segel system of higher dimension with logistic source”. In: *Journal of Mathematical Analysis and Applications* 430 (Oct. 2015), pp. 585–591.

[75] J. Zheng et al. “A new result for global existence and boundedness of solutions to a parabolic–parabolic Keller–Segel system with logistic source”. In: *Journal of Mathematical Analysis and Applications* 462 (June 2018), pp. 1–25.

APPENDIX A

NOTATIONS

We introduce some notations used throughout this book.

Ω is an open bounded set in $\mathbb{R}^n$ with smooth boundary.

$\Omega_T := \Omega \times (0,T)$.

ν is the outward normal vector.

We follow some definitions of Holder continuous spaces given in [37]. For $k \in \mathbb{N}$, and $\gamma \in (0,1]$, we define the following norms and seminorms:

$$
\begin{aligned}
[f]_{k+\gamma,\Omega_T} &:= \sum_{|\beta|+2j=k} \sup_{(x,t)\neq(y,s)\in\Omega_T} \frac{|D_x^\beta D_t^j\left(f(x,t)-f(y,s)\right)|}{\|x-y\|^\gamma + |t-s|^{\frac{\gamma}{2}}}, \\
\langle f\rangle_{k+\gamma,\Omega_T} &:= \sum_{|\beta|+2j=k-1} \sup_{(x,t)\neq(x,s)\in\Omega_T} \frac{|D_x^\beta D_t^j\left(f(x,t)-f(x,s)\right)|}{|t-s|^{\frac{1+\gamma}{2}}}, \\
|f|_{k+\gamma,\Omega_T} &:= [f]_{k+\gamma,\Omega_T} + \langle f\rangle_{k+\gamma,\Omega_T}, \\
\|f\|_{C^k(\Omega_T)} &:= \sum_{|\beta|+2j\leq k} \sup_{(x,t)\in\Omega_T} |D_x^\beta D_t^j f(x,t)| \\
\|f\|_{C^{k+\gamma}(\Omega_T)} &:= \|f\|_{C^k(\Omega_T)} + |f|_{k+\gamma,\Omega_T}
\end{aligned}
$$

Now we define the following functional spaces

$$
\begin{aligned}
C^k(\Omega_T) &:= \left\{f : D_x^\beta D_t^j f \text{ is continuous in } \Omega_T \text{ for } |\beta|+2j=k\right\} \\
C^{k+\gamma}(\Omega_T) &:= \left\{f \in C^k(\Omega_T) : \|f\|_{C^{k+\gamma}(\Omega_T)} < \infty\right\}.
\end{aligned}
$$

One can verify that $C^k(\bar{\Omega}_T)$ and $C^{k+\gamma}(\bar{\Omega}_T)$ are Banach spaces. The smoothness condition of boundary is necessary to guarantee the inclusion $C^a(\bar{\Omega}_T) \subset C^b(\bar{\Omega}_T)$ for $b > a \geq 0$, since it is not true in general domain Ω. Moreover, $C^a(\bar{\Omega}_T)$ is compactly embedding in $C^b(\bar{\Omega}_T)$ for any $a \geq 1$ and $a > b \geq 0$ (see [15][Lemma 6.36]).

Lemma A.0.1. *Assume that $\gamma \in (0,1]$, the functions $f(x) = |x|^\gamma$ and $g(x) = |x|^{1+\gamma}$ belong to $C^\gamma(\mathbb{R})$ and $C^{1+\gamma}(\mathbb{R})$ respectively.*

Proof. We have

$$|x| \leq |y| + |x - y| \leq (|y|^\gamma + |x - y|^\gamma)^{\frac{1}{\gamma}}, \tag{A.0.1}$$

where the last inequality comes from

$$a^s + b^s \leq (a + b)^s, \qquad s \geq 1,\ a, b \geq 0.$$

From (A.0.1), we have

$$|f(x) - f(y)| = ||x|^\gamma - |y|^\gamma| \leq |x - y|^\gamma, \tag{A.0.2}$$

which implies that f is in $C^\gamma(\mathbb{R})$. Now we show that g is in $C^{1+\gamma}(\mathbb{R})$. Differentiating g, we obtain

$$g'(x) = \begin{cases} (1+\gamma)|x|^\gamma, & x > 0, \\ 0, & x = 0, \\ -(1+\gamma)|x|^\gamma, & x < 0. \end{cases} \tag{A.0.3}$$

When $xy = 0$, one can verify that

$$|g'(x) - g'(y)| \leq (1 + \gamma)|x - y|^\gamma. \tag{A.0.4}$$

When $xy > 0$, we make use of the fact that $f \in C^\gamma(\mathbb{R})$ to have

$$|g'(x) - g'(y)| \leq |x - y|^\gamma. \tag{A.0.5}$$

When $xy < 0$, we have

$$|g'(x) - g'(y)| = (1 + \gamma)(|x|^\gamma + |y|^\gamma) \leq 2(1 + \gamma)|x - y|^\gamma. \tag{A.0.6}$$

From (A.0.4), (A.0.5), and (A.0.6), we conclude that $g \in C^{1+\gamma}(\mathbb{R})$. □

Lemma A.0.2. *If $u \in C^{1+\gamma}(\bar{\Omega}_T)$, and $f \in C^{1+\gamma}(\mathbb{R})$ with $\gamma \in (0, 1]$, then $f(u) \in C^{1+\gamma}(\bar{\Omega}_T)$.*

Proof. It is straightforward to verify that $f(u)$ is in $C^1(\bar{\Omega}_T)$. For $|\beta| = 1$, we have

$$|D_x^\beta (f(u(x,t)) - f(u(y,s)))| \leq |D_x^\beta (f(u(x,t)) - f(u(x,s)))|$$

$$+|D_x^\beta\left(f(u(x,s))-f(u(y,s))\right)|. \tag{A.0.7}$$

Since $f \in C^{1+\gamma}(\mathbb{R})$ and $u \in C^{1+\gamma}(\bar{\Omega}_T)$, we obtain

$$\begin{aligned}
|D_x^\beta\left(f(u(x,t))-f(u(x,s))\right)| &= |f'(u(x,t))D_x^\beta u(x,t)-f'(u(x,s))D_x^\beta u(x,s)| \\
&\le |f'(u(x,t))||D_x^\beta(u(x,t)-u(x,s))| \\
&+ |f'(u(x,t))-f'(u(x,s))||D_x^\beta u(x,s)| \\
&\le C_1|t-s|^{\gamma/2}+C_2|u(x,t)-u(x,s)|^\gamma \\
&\le C_1|t-s|^{\gamma/2}+C_2|t-s|^\gamma \\
&\le (C_1+C_2T^{\gamma/2})|t-s|^{\gamma/2}. \qquad \text{(A.0.8)}
\end{aligned}$$

Similarly, we obtain

$$|D_x^\beta\left(f(u(x,s))-f(u(y,s))\right)| \le C\left\|x-y\right\|^\gamma. \tag{A.0.9}$$

From (A.0.7), (A.0.8), and (A.0.9), we have

$$\sup_{(x,t)\neq(y,s)\in\bar{\Omega}_T} \frac{|D_x^\beta\left(f(u(x,t))-f(u(y,s))\right)|}{\|x-y\|^\gamma+|t-s|^{\gamma/2}} < \infty. \tag{A.0.10}$$

By similar arguments, we also have

$$\sup_{(x,t)\neq(y,s)\in\bar{\Omega}_T} \frac{|D_x^\beta\left(f(u(x,t))-f(u(x,s))\right)|}{|t-s|^{(1+\gamma)/2}} < \infty. \tag{A.0.11}$$

Finally, (A.0.10) and (A.0.11) imply $f(u) \in C^{1+\gamma}(\bar{\Omega}_T)$. $\square$

For any $p \in [1,\infty)$, we define L^p spaces

$$L^p(\Omega) := \left\{ f \text{ is measurable in } \Omega : \|f\|_{L^p(\Omega)} := \left(\int_\Omega |f|^p\, dx\right)^{\frac{1}{p}} < \infty \right\},$$

and

$$L^\infty(\Omega) := \left\{ f \text{ is measurable in } \Omega : \|f\|_{L^\infty(\Omega)} := \text{ess-sup}_{x\in\Omega}|f(x)| < \infty \right\}.$$

We define Sobolev spaces for any $1 \le p \le \infty$

$$W^{k,p}(\Omega) := \left\{ f \in L^p(\Omega) : \|D^\alpha f\|_{L^p(\Omega)} < \infty \right\}.$$

where α is a multi-index such that $|\alpha| \le k$, and $D^\alpha f$ is a weak derivative of f.

APPENDIX B

INEQUALITIES

We collect some useful inequalities frequently used through this thesis. Let us begin with Young's inequality:

Lemma B.0.1 (Young's inequality)**.** *For any $\epsilon > 0$, $p > 1$, and $a, b > 0$, the following inequality holds*

$$ab \leq \epsilon a^s + \frac{s-1}{s}(s\epsilon)^{\frac{1}{1-s}} b^{\frac{s}{s-1}}. \tag{B.0.1}$$

Next, let us introduce an extended version of the Gagliardo-Nirenberg interpolation inequality, which was established in [36][Lemma 2.3].

Lemma B.0.2 (Gagliardo-Nirenberg interpolation inequality)**.** *Let Ω be a bounded and smooth domain of $\mathbb{R}^n$ with $n \geq 1$. Let $r \geq 1$, $0 < q \leq p < \infty$, $s \geq 1$. Then there exists a constant $C_{GN} > 0$ such that*

$$\|f\|^p_{L^p(\Omega)} \leq C_{GN}\left(\|\nabla f\|^{pa}_{L^r(\Omega)} \|f\|^{p(1-a)}_{L^q(\Omega)} + \|f\|^p_{L^s(\Omega)}\right)$$

for all $f \in L^q(\Omega)$ with $\nabla f \in (L^r(\Omega))^n$, and $a = \frac{\frac{1}{q}-\frac{1}{p}}{\frac{1}{q}+\frac{1}{n}-\frac{1}{r}} \in [0,1]$.

Consequently, the next lemma is derived as follows:

Lemma B.0.3. *If Ω be a bounded and smooth domain of $\mathbb{R}^n$ with $n \geq 1$, then there exists a positive constant C depending only on Ω such that for any $f \in W^{1,2}(\Omega)$ the following inequality*

$$\int_\Omega f^2 \leq C\eta \int_\Omega |\nabla f|^2 + \frac{C}{\eta^{\frac{n}{2}}}\left(\int_\Omega |f|\right)^2 \tag{B.0.2}$$

holds for all $\eta \in (0,1)$.

Proof. The Lemma follows from Lemma B.0.2 by choosing $p = r = 2$ and $q = s = 1$ and Young's inequality. □

The next lemma provides an essential inequality used to absorb nonlinear chemo-attractants term into the diffusion term. It is a direct consequence of [66][Corollary 1.2], however for the convenience, we provide the detail proof here.

Lemma B.0.4. *If* $\Omega \subset \mathbb{R}^2$ *is a bounded domain with smooth boundary, then for each* $m > 0$ *and* $\gamma \geq 0$ *there exists* $C = C(m, \gamma) > 0$ *with the property that whenever* $\phi \in C^1(\bar{\Omega})$ *is positive in* $\bar{\Omega}$

$$\int_\Omega \phi^{m+1} \ln^\gamma(\phi + e) \leq C \left(\int_\Omega \phi \ln^\gamma(\phi + e) \right) \left(\int_\Omega |\nabla \phi^{\frac{m}{2}}|^2 \right) + C \left(\int_\Omega \phi \right)^m \left(\int_\Omega \phi \ln^\gamma(\phi + e) \right). \tag{B.0.3}$$

Proof. By applying Sobolev's inequality when $n = 2$, there exists a postive constant c_1 such that

$$\int_\Omega \phi^{m+1} \ln^\gamma(\phi + e) \leq c_1 \left(\int_\Omega \left| \nabla \left(\phi^{\frac{m+1}{2}} \ln^{\frac{\gamma}{2}}(\phi + e) \right) \right| \right)^2 + c_1 \left(\int_\Omega \phi \ln^{\frac{\gamma}{m+1}}(\phi + e) \right)^{m+1} \tag{B.0.4}$$

By using elementary inequalities, one can verify that

$$\left| \nabla \left(\phi^{\frac{m+1}{2}} \ln^{\frac{\gamma}{2}}(\phi + e) \right) \right| \leq c_2 \phi^{\frac{1}{2}} \ln^{\frac{\gamma}{2}}(\phi + e) |\nabla \phi^{\frac{m}{2}}|,$$

where $c_2 = C(m, \gamma) > 0$. This, together with Holder's inequality leads to

$$c_1 \left(\int_\Omega \left| \nabla \left(\phi^{\frac{m+1}{2}} \ln^{\frac{\gamma}{2}}(\phi + e) \right) \right| \right)^2 \leq c_3 \int_\Omega |\nabla \phi^{\frac{m}{2}}|^2 \cdot \int_\Omega \phi \ln^\gamma(\phi + e), \tag{B.0.5}$$

where $c_3 = c_1 c_2$. By Holder's inequality, we deduce that

$$c_1 \left(\int_\Omega \phi \ln^{\frac{\gamma}{m+1}}(\phi + e) \right)^{m+1} \leq c_1 \left(\int_\Omega \phi \right)^m \left(\int_\Omega \phi \ln^\gamma(\phi + e) \right). \tag{B.0.6}$$

Collecting (B.0.4), (B.0.5) and (B.0.6) implies (B.0.3), which finishes the proof. ☐

As a consequence, we have the following interpolation inequality with arbitry ϵ parameters.

Lemma B.0.5. *Assume that* $\Omega \subset \mathbb{R}^2$ *is a bounded domain with smooth boundary and* $p > 0$, $\gamma > \xi \geq 0$. *For each* $\epsilon > 0$, *there exists* $C = C(\epsilon, \xi, \gamma) > 0$ *such that*

$$\int_\Omega \phi^{m+1} \ln^\xi(\phi + e) \leq \epsilon \left(\int_\Omega \phi \ln^\gamma(\phi + e) \right) \left(\int_\Omega |\nabla \phi^{\frac{m}{2}}|^2 \right) + \epsilon \left(\int_\Omega \phi \right)^m \left(\int_\Omega \phi \ln^\gamma(\phi + e) \right) + C. \tag{B.0.7}$$

Proof. Since $\gamma > \xi \geq 0$, one can verify that for any $\delta > 0$, there exists $c_1 = c(\delta, \xi, \gamma) > 0$ such that for any $a \geq 0$ we have

$$a^{m+1} \ln^{\xi}(a+e) \leq \delta a^{m+1} \ln^{\gamma}(a+e) + c_1. \tag{B.0.8}$$

This entails that

$$\int_{\Omega} \phi^{m+1} \ln^{\xi}(\phi+e) \leq \delta \int_{\Omega} \phi^{m+1} \ln^{\gamma}(\phi+e) + c_1|\Omega|. \tag{B.0.9}$$

Now for any fixed ϵ, we choose $\delta = \frac{\epsilon}{C}$ where C as in Lemma B.0.4, and apply (B.0.3) to have the desire inequality (B.0.7). □

The following lemma provides estimates on the boundary (see Lemma 5.3 in [40]):

Lemma B.0.6. *Assume that Ω is a convex bounded domain, and that $f \in C^2(\bar{\Omega})$ satisfies $\frac{\partial f}{\partial \nu} = 0$ on $\partial\Omega$. Then*

$$\frac{\partial |\nabla f|^2}{\partial \nu} \leq 0 \qquad \text{on } \partial\Omega.$$

The next lemma provides estimates on the boundary of nonconvex bounded domain (see [41][Lemma 4.2]).

Lemma B.0.7. *Assume that Ω is bounded and let $w \in C^2(\bar{\Omega})$ satisfy $\frac{\partial w}{\partial \nu} = 0$ on $\partial\Omega$. Then we have*

$$\frac{\partial |\nabla w|^2}{\partial \nu} \leq 2\kappa |\nabla w|^2, \tag{B.0.10}$$

where $\kappa = \kappa(\Omega) > 0$ is an upper bound for the curvatures of $\partial\Omega$.

Next let us derive an estimate for a particular boundary integral that enables us to cover possibly non-convex domains.

Lemma B.0.8. *Let Ω be a bounded domain with smooth boundary, let $q \in [1, \infty)$. Then for any $\theta > 0$ there is $C_\theta > 0$ such that for any $f \in C^2(\bar{\Omega})$ satisfying $\frac{\partial f}{\partial \nu} = 0$ on $\partial\Omega$ and , the inequality*

$$\int_{\partial\Omega} |\nabla f|^{2(q-1)} \nabla(|\nabla f|^2) \cdot \nu \leq \epsilon \int_{\Omega} |\nabla(|\nabla f|^q)|^2 + c(\epsilon) \int_{\Omega} |\nabla f|^{2q}.$$

holds.

Proof. From Lemma B.0.7, it follows that

$$\int_{\partial\Omega} |\nabla f|^{2(q-1)} \nabla(|\nabla f|^2) \le 2\kappa \int_{\partial\Omega} |\nabla f|^{2q}. \tag{B.0.11}$$

By trace's Sobolev embedding theorem $W^{1,1}(\Omega) \to L^1(\partial\Omega)$, we obtain

$$\begin{aligned} 2\kappa \int_{\partial\Omega} |\nabla f|^{2q} &\le c \int_\Omega |\nabla f|^{2q} + c \int_\Omega \left|\nabla\left(|\nabla f|^{2q}\right)\right| \\ &\le \epsilon \int_\Omega |\nabla(|\nabla f|^q)|^2 + c(\epsilon) \int_\Omega |\nabla f|^{2q}. \end{aligned} \tag{B.0.12}$$

The proof is complete. □

In [9], an interpolation inequality of Ehrling-type is utilized to show that the equi-integrability of the family $\left\{\int_\Omega u^{\frac{n}{2}}(\cdot,t)\right\}_{t\in(0,T_{\max})}$ implies the uniform boundedness of solutions. Here we present an interpolation inequality that is similar to [9, Lemma 2.1] or [72][Lemma 3.4], and which will be employed to obtain an L^q estimate with $q \ge 2$ for the solutions of the system (5.2.1). To prove this inequality, we adapt the argument used in the proof of inequality (22) in [6], with some modifications. We include a complete proof of this interpolation inequality below for the reader's convenience.

Lemma B.0.9. *Let $\Omega \subset \mathbb{R}^n$, with $n \ge 2$ be a bounded domain with smooth boundary and $q > \frac{n}{2}$. Then one can find $C > 0$ such that for each $\epsilon > 0$, there exists $c = c(\epsilon) > 0$ such that*

$$\int_\Omega |w|^{q+1} \le \epsilon \int_\Omega |\nabla w^{\frac{q}{2}}|^2 \left(\int_\Omega G(|w|^{\frac{n}{2}})\right)^{\frac{2}{n}} + C\left(\int_\Omega |w|\right)^{q+1} + c\int_\Omega |w| \tag{B.0.13}$$

holds for all $w^{\frac{q}{2}} \in W^{1,2}(\Omega)$, and $\int_\Omega G(|w|^{\frac{n}{2}}) < \infty$ where G is continuous, strictly increasing and nonnegative in $[0,\infty)$ such that $\lim_{s\to\infty} \frac{G(s)}{s} = \infty$.

Proof. We call

$$\xi(s) = \begin{cases} 0 & |s| \le N \\ 2(|s|-N) & N < |s| \le 2N \\ |s| & |s| > 2N. \end{cases} \tag{B.0.14}$$

One can verify that

$$\int_\Omega \big||w| - \xi(w)\big|^{q+1} \le (2N)^q \int_\Omega |w| \tag{B.0.15}$$

and,

$$\int_\Omega \xi(w)^{\frac{n}{2}} \le \frac{N^{\frac{n}{2}}}{G(N^{\frac{n}{2}})} \int_\Omega G(|w|^{\frac{n}{2}}). \tag{B.0.16}$$

Notice that $|\nabla\left(\xi(w)\right)^{\frac{q}{2}}|^2 \le c|w|^{q-2}|\nabla w|^2$, for some $c > 0$, and combine with Lemma B.0.2, we obtain

$$\begin{aligned}\int_\Omega (\xi(w))^{q+1} &\le c\int_\Omega |\nabla(\xi(w))^{\frac{q}{2}}|^2 \left(\int_\Omega \xi(w)^{\frac{n}{2}}\right)^{\frac{2}{n}} + C\left(\int_\Omega \xi(w)\right)^{q+1} \\ &\le c\left(\frac{N^{\frac{n}{2}}}{G(N^{\frac{n}{2}})}\right)^{\frac{2}{n}} \int_\Omega |\nabla w^{\frac{q}{2}}|^2 \left(\int_\Omega G(|w|^{\frac{n}{2}})\right)^{\frac{2}{n}} + C\left(\int_\Omega |w|\right)^{q+1}.\end{aligned} \tag{B.0.17}$$

This leads to

$$\begin{aligned}\int_\Omega |w|^{q+1} &\le c\left(\int_\Omega |\xi(w)|^{q+1} + \int_\Omega |\xi(w) - |w||^{q+1}\right) \\ &\le \left(\frac{N^{\frac{n}{2}}}{G(N^{\frac{n}{2}})}\right)^{\frac{2}{n}} \int_\Omega |\nabla w^{\frac{q}{2}}|^2 \left(\int_\Omega G(|w|^{\frac{n}{2}})\right)^{\frac{2}{n}} + C\left(\int_\Omega |w|\right)^{q+1} + (2N)^q \int_\Omega |w|.\end{aligned} \tag{B.0.18}$$

We finally complete the proof by choosing N sufficiently large such that $c\left(\frac{N^{\frac{n}{2}}}{G(N^{\frac{n}{2}})}\right)^{\frac{2}{n}} \le \epsilon$. □

The following Lemma is useful in iteration procedure to obtain L^∞ bounds from L^q bounds for some $q > 1$.

Lemma B.0.10. *Suppose that the positive sequences $(a_k, b_k, u_k)_{k\ge 1}$ satisfy the following conditions:*

$$\begin{cases} u_{k+1} \le a_k + b_k u_k, \\ \sum_{k=1}^\infty a_k = a < \infty, \\ \prod_{k=1}^\infty b_k = b < \infty, \\ b_k \ge 1, \end{cases} \tag{B.0.19}$$

for all $k \in \mathbb{N}$, then $\sup_k u_k \le ab + bu_1$.

Proof. We have

$$u_{k+1} \le a_k + b_k u_k \le a_k + a_{k-1}b_k + b_k b_{k-1} u_{k-1}$$

$$
\begin{aligned}
&\leq a_k + \sum_{i=0}^{k-2} a_{k-1-i} \prod_{j=0}^{i} b_{k-j} + u_1 \prod_{i=1}^{k} b_i \\
&\leq b \left(\sum_{i=1}^{k} a_i \right) + bu_1 \leq ab + bu_1.
\end{aligned}
$$

□

APPENDIX C

REGULARITY THEORY

C.1 Parabolic regularity

In order to obtain $L^p - L^q$ estimates for solutions to parabolic equations, we need some estimates on the heat semigroup under Neumann boundary conditions. Interested readers are referred to [62][Lemma 1.3] for more details about the proof.

Lemma C.1.1. *Let $\left(e^{t\Delta}\right)_{t\geq 0}$ be the Neumann heat semigroup in Ω, and let $\lambda_1 > 0$ denote the first nonzero eigenvalue of $-\Delta$ in Ω under Neumann boundary conditions. Then there exist constants $C_1, ..., C_4$ depending on Ω only which have the following properties.*

1. *If $1 \leq q \leq p \leq \infty$ then*

$$\left\|e^{t\Delta}w\right\|_{L^p(\Omega)} \leq C_1 \left(1 + t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\right) e^{-\lambda_1 t} \left\|w\right\|_{L^q(\Omega)} \text{ for all } t > 0 \tag{C.1.1}$$

holds for all $w \in L^q(\Omega)$ satisfying $\int_\Omega w = 0$.

2. *If $1 \leq q \leq p \leq \infty$ then*

$$\left\|\nabla e^{t\Delta}w\right\|_{L^p(\Omega)} \leq C_2 \left(1 + t^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\right) e^{-\lambda_1 t} \left\|w\right\|_{L^q(\Omega)} \text{ for all } t > 0 \tag{C.1.2}$$

is true for each $w \in L^q(\Omega)$.

3. *If $2 \leq p < \infty$ then*

$$\left\|\nabla e^{t\Delta}w\right\|_{L^p(\Omega)} \leq C_3 e^{-\lambda_1 t} \left\|\nabla w\right\|_{L^p(\Omega)} \text{ for all } t > 0 \tag{C.1.3}$$

is valid for all $w \in W^{1,p}(\Omega)$.

4. *Let $1 < q \leq p < \infty$. Then*

$$\left\|e^{t\Delta}\nabla \cdot w\right\|_{L^p(\Omega)} \leq C_4 \left(1 + t^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\right) e^{-\lambda_1 t} \left\|w\right\|_{L^q(\Omega)} \text{ for all } t > 0 \tag{C.1.4}$$

holds for all $w \in \left(C_0^\infty(\Omega)\right)^n$. Consequently, for all $t > 0$ the operator $e^{t\Delta}\nabla\cdot$ possesses a unique determined extension to an operator from $L^q(\Omega)$ into $L^p(\Omega)$, with norm controlled according to (C.1.4).

Consequently, we have the following lemma, which derives estimates on solutions of the parabolic equations. For more details, see Lemma 2.1 in [14].

Lemma C.1.2. *Let $\Omega \subset \mathbb{R}^n$, with $n \geq 2$ be open bounded with smooth boundary, $p \geq 1$ and $q \geq 1$ satisfy*

$$\begin{cases} q & < \frac{np}{n-p}, \qquad \text{when } p < n, \\ q & < \infty, \qquad \text{when } p = n, \\ q & = \infty, \qquad \text{when } p > n. \end{cases}$$

Assuming that $g_0 \in W^{1,q}(\Omega)$, $f \in C\left(\bar{\Omega} \times [0,T)\right)$, and $g \in C\left(\bar{\Omega} \times [0,T)\right) \cap C^{2,1}\left(\bar{\Omega} \times (0,T)\right) \cap C\left([0,T); W^{1,q}(\Omega)\right)$ is a classical solution to the following system

$$\begin{cases} g_t = \Delta g - g + f & \text{in } \Omega \times (0,T), \\ \frac{\partial g}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0,T), \\ g(\cdot, 0) = g_0 & \text{in } \Omega \end{cases} \tag{C.1.5}$$

for some $T \in (0, \infty]$. If $f \in L^\infty\left((0,T); L^p(\Omega)\right)$, then $g \in L^\infty\left((0,T); W^{1,q}(\Omega)\right)$.

Proof. We have

$$g(\cdot, t) = e^{t(\Delta-1)} g_0 + \int_0^t e^{(t-s)(\Delta-1)} f(\cdot, s)\, ds. \tag{C.1.6}$$

We apply ∇ to both sides and make use of Lemma C.1.1 to obtain that

$$\begin{aligned} \|\nabla g(\cdot,t)\|_{L^q(\Omega)} &\leq \left\|\nabla e^{t(\Delta-1)} g_0\right\|_{L^q(\Omega)} + \int_0^t \left\|\nabla e^{(t-s)(\Delta-1)} f(\cdot,s)\right\|_{L^q(\Omega)} ds \\ &\leq c e^{-(\lambda_1+1)t} \|\nabla g_0\|_{L^q(\Omega)} \\ &\quad + c \sup_{t>0} \|f(\cdot,t)\|_{L^p(\Omega)} \int_0^\infty (1 + (t-s)^{-\frac{1}{2} - \frac{n}{2}(\frac{1}{p} - \frac{1}{q})}) e^{-(\lambda_1+1)(t-s)}\, ds \end{aligned} \tag{C.1.7}$$

The conditions of p, q imply that $-\frac{1}{2} - \frac{n}{2}(\frac{1}{p} - \frac{1}{q}) > -1$, which makes the integral on the right convergent. Therefore, we obtain

$$\sup_{t>0} \|\nabla g(\cdot,t)\|_{L^q(\Omega)} \leq c \|\nabla g_0\|_{L^q(\Omega)} + c \sup_{t>0} \|f(\cdot,t)\|_{L^p(\Omega)}, \tag{C.1.8}$$

which concludes the proof. □

The following parabolic regularity result plays a important role in the strongly degenerate case where $\inf_{s\geq 0} D(s) = 0$. Indeed, it was proved that equation (C.1.5) possesses a global bounded solution under a suitable slow growth condition of f. Precisely, we have the following proposition, which is a direct application of Corollary 1.3 in [61] with $n = 2$.

Proposition C.1.1. *For each $a > 0$, $q > n$, $K > 0$ and $\tau > 0$, there exist $C(a, q, K, \tau) > 0$ such that if $T \geq 2\tau$, $f \in C^0(\bar{\Omega}\times[0, T])$, and $V \in C^0(\bar{\Omega}\times[0, T])\cap C^{2,1}(\bar{\Omega}\times(0, T))\cap C^0([0, T); W^{1,q}(\Omega))$ are such that* (C.1.5) *is satisfied with*

$$\int_t^{t+\tau}\int_\Omega |f|^2 \ln^\alpha(|f| + e) < K \textit{ for all } t \in (0, T - \tau). \tag{C.1.9}$$

and

$$\|V_0\|_{W^{1,q}(\Omega)} < K,$$

then

$$|V(x, t)| \leq C(a, q, K, \tau) \quad \textit{for all } (x, t) \in \Omega \times (0, T). \tag{C.1.10}$$

C.2 Regularity for chemotaxis systems

In this chapter, we shall apply Moser-Akikos iteration procedure (see [2, 1]) to obtain L^∞ bounds from L^p-bounds for some $p > 1$ for various chemotaxis models with homogenuous Neumann boundary condition or general nonlinear Neumann boundary condition.

C.2.1 Introduction

We consider the following chemotaxis system in a open, bounded domain with smooth boundary $\Omega \subset \mathbb{R}^n$, with $n \geq 2$

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + f(u) & x \in \Omega,\ t \in (0, T_{\text{max}}), \\ \tau v_t = \Delta v + u - v & x \in \Omega,\ t \in (0, T_{\text{max}}), \end{cases} \tag{KS}$$

where $f \in C([0, \infty))$ such that $f(u) \leq c(1+u^p)$ with $c, p \geq 0$ under nonlinear Neumann boundary condition:

$$\frac{\partial u}{\partial \nu} = g(u), \qquad \frac{\partial v}{\partial \nu} = 0, \qquad x \in \partial\Omega, t \in (0, T_{\text{max}}), \tag{C.2.1}$$

where ν is the outward normal vector and $g(u) \leq cu^q$ with $c, q \geq 0$. The system (KS) is complemented with the nonnegative initial conditions in $C^{2+\gamma}(\Omega)$, where $\gamma \in (0,1)$, not identically zero:

$$u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), \qquad x \in \Omega, \tag{C.2.2}$$

In this chapter, we assume that (u,v) is a classical solution of (KS) with initial condition (C.2.2) under nonlinear boundary conditions (C.2.1) in $\Omega \times (0, T_{\max})$, where $T_{\max} \in (0,\infty]$ is the maximal existence time.

Theorem C.2.1. *If $u \in L^\infty((0,T_{max}), L^{r_0}(\Omega))$ for some $r_0 > \max\left\{\frac{n}{2}, \frac{n(p-1)}{2}, n(q-1)\right\}$, then $u \in L^\infty((0,T_{max}), L^\infty(\Omega))$.*

Remark C.2.1. *One can also follow the argument as in [48] to prove the above theorem.*

Remark C.2.2. *The $L^{\frac{n}{2}+}$-criterion for homogeneous Neumann boundary conditions has been studied in [5] for general chemotaxis systems and in [67] for the fully parabolic chemotaxis system, both with and without a logistic source. However, Theorem C.2.1 not only addresses nonlinear Neumann boundary conditions, but also employs a different analysis approach compared to [5, 67]. Instead of utilizing the semigroup estimate, the analysis in Theorem 2.5.1 relies on the L^p-regularity theory for parabolic equations.*

C.2.2 An reverse Holder's inequality

We rely on the following reverse Holder's inequality, which is the milestone in establishing Moser-Akikos iteration procedure.

Lemma C.2.2. *Let (u,v) be a classical solution of* (KS) *on $(0,T_{max})$ and*

$$U_r := \max\left\{1, \|u_0\|_{L^\infty(\Omega)}, \sup_{t\in(0,T_{max})} \|u(\cdot,t)\|_{L^r(\Omega)}\right\}.$$

If $\sup_{t\in(0,T_{max})} \|u(\cdot,t)\|_{L^r(\Omega)} < \infty$ for some $r > \max\left\{n, \frac{n(p-1)}{2}, n(q-1)\right\}$, then there exists constant $C > 0$ independent of r such that

$$U_{2r} \leq c^{\frac{1}{2r-nk}} r^{\frac{3}{2r}} U_r^{1+\frac{(n+2)k}{4r-2nk}} \tag{C.2.3}$$

where $k = \max\{p-1, 2q-2\}$.

Proof. Through out this proof, the notation c, unless being specified, represents a positive constant independent of ϵ, r. Multiplying u^{2r-1} to the first equation of (KS), we obtain

$$\begin{aligned}
\frac{1}{2r}\frac{d}{dt}\int_\Omega u^{2r} &= \int_\Omega u^{2r-1}\left[\Delta u - \chi\nabla(u\nabla v) + f(u)\right] \\
&= -\frac{2r-1}{r^2}\int_\Omega |\nabla u^r|^2\,dx + \frac{2r-1}{r}\int_\Omega u^r\nabla u^r\cdot\nabla v \\
&+ \int_\Omega f(u)u^{2r-2} + \int_{\partial\Omega} g(u)u^{2r-1}\,dS, \\
&\leq -\frac{2r-1}{r^2}\int_\Omega |\nabla u^r|^2\,dx + \frac{2r-1}{r}\int_\Omega u^r\nabla u^r\cdot\nabla v \\
&+ c\int_\Omega u^{2r-1} + c\int_\Omega u^{2r-1+p} + c\int_{\partial\Omega} u^{2r-1+q}\,dS \qquad \text{(C.2.4)}
\end{aligned}$$

By Lemma C.1.2, we deduce that $v \in L^\infty\left((0, T_{\text{max}}), W^{1,\infty}(\Omega)\right)$. This, together with Holder's inequality implies

$$\begin{aligned}
\frac{2r-1}{r}\int_\Omega u^r\nabla u^r\cdot\nabla v &\leq c\int_\Omega u^r|\nabla u^r| \\
&\leq \epsilon\int_\Omega |\nabla u^r|^2 + \frac{c}{\epsilon}\int_\Omega u^{2r}. \qquad \text{(C.2.5)}
\end{aligned}$$

We make use of Trace Sobolev's Embedding Theorem and Young's inequality to handle the boundary integral as follows:

$$\int_\Omega u^{2r+q-1}\,dS \leq c\int_\Omega u^{2r+q-1} + \epsilon\int_\Omega |\nabla u^r|^2 + \frac{c}{\epsilon}\int_\Omega u^{2r+2q-2}, \qquad \text{(C.2.6)}$$

for any $\epsilon > 0$. From (C.2.5) and (C.2.6), we have

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u^{2r} + \int_\Omega u^{2r} &\leq 2r\left(c\epsilon - \frac{2r-1}{r^2}\right)\int_\Omega |\nabla u^r|^2 + \left(\frac{cr}{\epsilon} + 1\right)\int_\Omega u^{2r} \\
&+ cr\int_\Omega u^{2r-1} + cr\int_\Omega u^{2r+p-1} + cr\int_\Omega u^{2r+q-1} + \frac{cr}{\epsilon}\int_\Omega u^{2r+2q-2}
\end{aligned}$$

Substituting $c\epsilon = \frac{r-1}{r^2}$ into this, and noticing that $r > n \geq 2$, we have

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u^{2r} + \int_\Omega u^{2r} &\leq -2\int_\Omega |\nabla u^r|^2 + cr^3\int_\Omega u^{2r} + cr\int_\Omega u^{2r+p-1} \\
&+ cr\int_\Omega u^{2r-1} + cr\int_\Omega u^{2r+q-1} + cr^3\int_\Omega u^{2r+2q-2}
\end{aligned}$$

$$\leq -2\int_\Omega |\nabla u^r|^2 + cr^3 \int_\Omega u^{2r+k} + cr^3, \tag{C.2.7}$$

where the last inequality comes from

$$\int_\Omega u^{2r+l} = \int_{u\leq 1} u^{2r+l} + \int_{u>1} u^{2r+l} \leq \int_\Omega u^{2r+k} + |\Omega| \tag{C.2.8}$$

for any $l \leq k$. We set $w = u^r$, and apply Lemma B.0.2 to obtain

$$\begin{aligned} \int_\Omega u^{2r+k} &= \int_\Omega w^{2+\frac{k}{r}} := \int_\Omega w^{\bar{p}} \\ &\leq c\left(\int_\Omega |\nabla w|^2\right)^{\frac{\bar{p}a}{2}} \left(\int_\Omega w\right)^{\bar{p}(1-a)} + c\left(\int_\Omega w\right)^{\bar{p}}, \end{aligned} \tag{C.2.9}$$

where $\frac{2r}{n} > k > -r$ and

$$\bar{p} = 2 + \frac{k}{r}, \qquad a = \frac{1 - \frac{r}{2r+k}}{1 + \frac{1}{n} - \frac{1}{2}} = \frac{2n(r+k)}{(n+2)(2r+k)} < 1. \tag{C.2.10}$$

This implies that

$$\frac{\bar{p}a}{2} = \frac{n(r+k)}{r(n+2)} < 1 \qquad \text{when } r > \frac{nk}{2}.$$

This, together with (C.2.7) and Young's inequality leads to

$$\begin{aligned} \int_\Omega u^{2r+k} &\leq c\left(\int_\Omega |\nabla u^r|^2\right)^{\frac{\bar{p}a}{2}} \left(\int_\Omega u^r\right)^{\bar{p}(1-a)} + c\left(\int_\Omega u^r\right)^{\bar{p}} \\ &\leq c\epsilon \int_\Omega |\nabla u^r|^2 + c\epsilon^{-\frac{(n+2)r}{2r-nk}} \left(\int_\Omega u^r\right)^{2+\frac{(n+2)k}{2r-nk}} + c\left(\int_\Omega u^r\right)^{2+\frac{k}{r}}, \end{aligned} \tag{C.2.11}$$

for any $\epsilon > 0$. From (C.2.7) and (C.2.11), we obtain

$$\begin{aligned} \frac{d}{dt}\int_\Omega u^{2r} + \int_\Omega u^{2r} \leq (c\epsilon - 2)\int_\Omega |\nabla u^r|^2 + cr^3 \epsilon^{-\frac{(n+2)r}{2r-nk}} \left(\int_\Omega u^r\right)^{2+\frac{(n+2)k}{2r-nk}} \\ + cr^3\left(\int_\Omega u^r\right)^{2+\frac{k}{r}} + cr^3, \end{aligned}$$

Substituting $c\epsilon = 1$ into this, we have

$$\begin{aligned} \frac{d}{dt}\int_\Omega u^{2r} + \int_\Omega u^{2r} &\leq c^{\frac{(n+2)r}{2r-nk}} r^3 U_r^{2r+\frac{r(n+2)k}{2r-nk}} + cr^3 U_r^{2r+k} + cr^3 \\ &\leq c^{\frac{(n+2)r}{2r-nk}} r^3 U_r^{2r+\frac{r(n+2)k}{2r-nk}} \end{aligned}$$

with some $c > 1$ independent of r. This, together with Gronwall's inequality implies that

$$\int_\Omega u^{2r} \leq \max\left\{ c^{\frac{(n+2)r}{2r-nk}} r^3 U_r^{2r+\frac{r(n+2)k}{2r-nk}}, \int_\Omega u_0^{2r}, \right\}$$

which further entails (C.2.3) $\square$

C.2.3 Proof of main results

Before proving our main theorem, we rely on the following lemma:

Lemma C.2.3. *Let* (u, v) *be the classical solution to* (KS) *on* $\Omega \times (0, T_{max})$ *with maximal existence time* $T_{max} \in (0, \infty]$. *If* $u \in L^\infty((0, T_{max}); L^r(\Omega))$ *for some* $r > \max\{\frac{n}{2}, \frac{n(p-1)}{2}, n(q-1)\}$, *then* $u \in L^\infty((0, T_{max}); L^{2r}(\Omega))$.

Proof. If $r > \max\{\frac{n}{2}, \frac{n(p-1)}{2}, n(q-1)\} > n$, then Lemma C.2.2 asserts that $u \in L^\infty((0, T_{\max}); L^{2r}(\Omega))$. Now we just need to consider $\max\{\frac{n}{2}, \frac{n(p-1)}{2}, n(q-1)\} < r \leq n$. By Lemma C.1.2 we see that v is in $L^\infty((0, T); W^{1,q}(\Omega))$ for $q < \frac{rn}{n-r}$ if $r < n$ and any $q < \infty$ if $r = n$. We denote

$$\lambda := \begin{cases} \frac{2n}{n-2} & \text{if } n \geq 3 \\ \frac{1}{r-1} & \text{if } n = 2, \end{cases} \tag{C.2.12}$$

and apply Holder's inequality to deduce that

$$\int_\Omega u^{2r} |\nabla v|^2 \leq \left(\int_\Omega u^{2r+\lambda} \right)^{\frac{2r}{2r+\lambda}} \left(\int_\Omega |\nabla v|^{\frac{2(2r+\lambda)}{\lambda}} \right)^{\frac{\lambda}{2r+\lambda}}. \tag{C.2.13}$$

Since $\frac{n}{2} < r < n$, we find that

$$\frac{2(2r+\lambda)}{\lambda} < \frac{rn}{n-r},$$

therefore v belongs to $L^\infty\left((0, T); W^{1,\frac{2(2r+\lambda)}{\lambda}}(\Omega)\right)$. Clearly, v is in $L^\infty\left((0, T); W^{1,\frac{2(2r+\lambda)}{\lambda}}(\Omega)\right)$ when $r = n$. That leads to $\sup_{0<t<T} \|\nabla v\|^2_{L^{\frac{2(2r+\lambda)}{\lambda}}} < \infty$, which further entails that

$$\int_\Omega u^{2r} |\nabla v|^2 \leq c \left(\int_\Omega u^{2r+\lambda} \right)^{\frac{2r}{2r+\lambda}}. \tag{C.2.14}$$

Using the estimates (C.2.4), (C.2.6), and (C.2.14) we have

$$\frac{d}{dt} \int_\Omega u^{2r} + \int_\Omega u^{2r} \leq 2r \left(c\epsilon - \frac{2r-1}{r^2} \right) \int_\Omega |\nabla u^r|^2 + c \int_\Omega u^{2r+1}$$

$$
\begin{aligned}
&+ c\left(\int_\Omega u^{2r+\lambda}\right)^{\frac{2r}{2r+\lambda}} + c\int_\Omega u^{2r} + c\int_\Omega u^{2r-1} \\
&+ c\int_\Omega u^{2r+p-1} + c\int_\Omega u^{2r+q-1} + c\int_\Omega u^{2r+2q-2},
\end{aligned}
$$

with some $c > 0$ depending on ϵ, r. Substituting $c\epsilon = \frac{r-1}{r^2}$ into this yields

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega u^{2r} + \int_\Omega u^{2r} \leq -2\int_\Omega |\nabla u^r|^2 + c\int_\Omega u^{2r+1} \\
+ c\left(\int_\Omega u^{2r+\lambda}\right)^{\frac{2r}{2r+\lambda}} + c\int_\Omega u^{2r} + c\int_\Omega u^{2r-1} \\
+ c\int_\Omega u^{2r+p-1} + c\int_\Omega u^{2r+q-1} + c\int_\Omega u^{2r+2q-2},
\end{aligned}
\tag{C.2.15}
$$

with some $c > 0$ depending on r. Using GN inequality, then Young's inequality with $\epsilon > 0$ and noticing that $U_r < \infty$, we obtain

$$
\begin{aligned}
\left(\int_\Omega u^{2r+\lambda}\right)^{\frac{2r}{2r+\lambda}} &\leq c\left(\int_\Omega |\nabla u^r|^2\right)^s \left(\int_\Omega u^r\right)^{2(1-s)} + c\left(\int_\Omega u^r\right)^2 \\
&\leq \epsilon U_r^{2r(1-s)}\left(\int_\Omega |\nabla u^r|^2\right)^s + c(\epsilon)C_{GN}U_r^{2r} \\
&\leq \epsilon\int_\Omega |\nabla u^r|^2 + c(\epsilon),
\end{aligned}
\tag{C.2.16}
$$

where $s = \frac{2n(r+\lambda)}{(n+2)(2r+\lambda)} \in (0,1)$. Substituting $\epsilon = 1$ into this, we have

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega u^{2r} + \int_\Omega u^{2r} &\leq -\int_\Omega |\nabla u^r|^2 + c\int_\Omega u^{2r-1} + u^{2r} \\
&\quad + u^{2r+1} + u^{2r+p-1} + u^{2r+q-1} + u^{2r+2q-2} + c, \\
&\leq -\int_\Omega |\nabla u^r|^2 + \int_\Omega u^{2r+k} + c,
\end{aligned}
\tag{C.2.17}
$$

where $k = \max\{p-1, 2q-2\}$ and the last inequality comes from (C.2.8). By using (C.2.11) and noticing that $r > \frac{kn}{2}$ and $U_r < \infty$, we obtain

$$
\int_\Omega u^{2r+k} \leq c\epsilon\int_\Omega |\nabla u^r|^2 + c(\epsilon). \tag{C.2.18}
$$

From (C.2.17), (C.2.18), we choose ϵ sufficiently small to obtain

$$
\frac{d}{dt}\int_\Omega u^{2r} + \int_\Omega u^{2r} \leq c, \tag{C.2.19}
$$

with some $c > 0$. Applying Gronwall's inequality to this, we obtain

$$\sup_{t\in(0,T_{\max})} \|u(\cdot,t)\|_{L^{2r}(\Omega)} \le \max\left\{\|u_0\|_{L^\infty(\Omega)}, c\right\}.$$

The proof is complete. □

Proof of Theorem C.2.1. When $r_0 > \max\left\{\frac{n}{2}, \frac{n(p-1)}{2}, n(q-1)\right\}$, Lemma C.2.3 deduces that $u \in L^\infty((0,T_{\max}), L^{2r_0}(\Omega))$. Thus, we can assume that $r_0 > n$. Since $r_0 > \max\left\{n, \frac{n(p-1)}{2}, n(q-1)\right\}$, Lemma C.2.2 implies that the following inequality

$$U_{2^{j+1}r_0} \le c^{\frac{1}{2^{j+1}r_0 - nk}} \left((2^j r_0)^3\right)^{\frac{1}{2^{j+1}r_0}} U_{2^j r_0}^{1+\frac{(n+2)k}{2^{j+2}r_0-2nk}}.$$

for all integers $k \ge 1$. We take log of the above inequality to obtain

$$\ln U_{2^{j+1}r_0} \le a_j + \left(1 + \frac{(n+2)k}{2^{j+2}r_0 - 2nk}\right) \ln U_{2^j r_0},$$

where

$$a_j = \frac{\ln C}{2^{j+1}r_0 - nk} + \frac{3j\ln 2}{2^{j+1}r_0} + \frac{3\ln r_0}{2^{j+1}r_0},$$
$$b_j = 1 + \frac{(n+2)k}{2^{j+2}r_0 - 2nk}.$$

One can verify that

$$\sum_{j=1}^{\infty} a_j := A < \infty, \text{ and } \qquad \prod_{j=1}^{\infty} b_j := B < \infty.$$

Thus, we obtain

$$U_{2^{k+1}r} \le e^A U_{r_0}^B \tag{C.2.20}$$

for all $k \ge 1$. Sending $k \to \infty$ yields

$$U_\infty \le e^A U_{r_0}^B. \tag{C.2.21}$$

This asserts that $u \in L^\infty((0,T_{\max}); L^\infty(\Omega))$, and thereafter Lemma C.1.2 yields that $v \in L^\infty((0,T_{\max}); W^{1,\infty}(\Omega))$. □