CYCLIC SHUFFLE-COMPATIBILITY, CYCLIC PERMUTATION STATISTICS, CYCLIC QUASISYMMETRIC FUNCTIONS AND TORIC PARTITIONS

By

Jinting Liang

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ABSTRACT

A permutation statistic st is said to be shuffle-compatible if the distribution of st over the set of shuffles of two disjoint permutations π and σ depends only on $\mathrm{st}\,\pi$, $\mathrm{st}\,\sigma$, and the lengths of π and σ. Shuffle-compatibility is implicit in Stanley's early work on P-partitions, and was first explicitly studied by Gessel and Zhuang, who developed an algebraic framework for shuffle-compatibility centered around their notion of the shuffle algebra of a shuffle-compatible statistic. One of the places where shuffles are useful is in describing the product in the algebra of quasisymmetric functions. Recently Adin, Gessel, Reiner, and Roichman defined an algebra of cyclic quasisymmetric functions where a cyclic version of shuffling comes into play.

This dissertation focuses on the study of cyclic shuffle-compatibility. We began by showing a result called the "lifting lemma," which allows one (under certain nice conditions) to prove that a cyclic statistic is cyclic shuffle-compatible from the shuffle-compatibility of a related linear statistic. This lifting lemma can be used to prove the cyclic shuffle-compatibility of all four statistics cDes, cdes, cPk, and cpk. We then developed an algebraic framework for cyclic shuffle-compatibility centered around the notion of cyclic shuffle algebra of a cyclic shuffle-compatible statistic. Using this theory, we provide explicit descriptions for the cyclic shuffle algebras of various cyclic permutation statistics, which in turn gives algebraic proofs for their cyclic shuffle-compatibility. In particular, we developed the theory of enriched toric $[\vec{D}]$-partitions, which provides a characterization of the cyclic shuffle algebra of cPk.

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CHAPTER 1

INTRODUCTION

Denote by $\mathbb{N}$ and $\mathbb{P}$ the set of nonnegative integers and positive integers respectively. For $m, n \in \mathbb{N}$, define $[m, n] = \{m, m+1, \dots, n\}$ and, as an abbreviation, we write $[n] = [1, n]$ when $m = 1$.

1.1 Linear Permutations, Statistics, and Shuffles

A *linear permutation* of a finite set $A \subset \mathbb{P}$ is an arrangement $\pi = \pi_1\pi_2 \dots \pi_n$ of elements in A where each element is used exactly once. In this case, n is called the *length* of π, and we write it as $\#\pi = |\pi| = n$. The hash symbol and absolute value sign will also be used for the cardinality of a set. We let

$$L(A) = \{\pi \mid \pi \text{ is a linear permutation of } A\}.$$

For example,

$$L(\{1, 4, 6\}) = \{146, 164, 416, 461, 614, 641\}.$$

In particular, let $\mathfrak{S}_n$ be the symmetric group on $[n]$ viewed as linear permutations of $[n]$. We will often drop the descriptor "linear" if it is clear from context that we are referring to linear permutations, but it will be useful later to distinguish from cyclic permutations.

A (*linear*) *permutation statistic* is a function whose domain is the set of all linear permutations. There are several classical permutation statistics that we mainly concern with: the descent set Des, the descent number des, the peak set Pk, the peak number pk and the major index maj.

For a linear permutation $\pi = \pi_1\pi_2 \dots \pi_n$, a *descent* of π is a position i such that $\pi_i > \pi_{i+1}$. The *descent set* Des is defined by

$$\operatorname{Des}\pi = \{i \mid i \text{ is a descent of } \pi\} \subseteq [n-1].$$

The *descent number* of π is $\operatorname{des}\pi := |\operatorname{Des}\pi|$. A *peak* of π is a position i such that $\pi_{i-1} < \pi_i > \pi_{i+1}$. The *peak set* Pk is defined by

$$\operatorname{Pk}\pi = \{i \mid i \text{ is a peak of } \pi\} \subseteq [2, n-1].$$

The *peak number* is $\operatorname{pk}\pi := |\operatorname{Pk}\pi|$. The *major index*

$$\operatorname{maj}\pi = \sum_{i \in \operatorname{Des}\pi} i$$

is the sum of its descents.

We will often need to evaluate a statistic st on a set of permutations Π. So we define the *distribution* of st over Π to be

$$\operatorname{st}\Pi = \{\{\operatorname{st}\pi \mid \pi \in \Pi\}\}.$$

Note that this is a set with multiplicity since st may evaluate to the same thing on various members of Π. We will sometimes use exponents to denote multiplicities where, as usual, no exponent means multiplicity 1. For example,

$$\operatorname{des}\mathfrak{S}_3 = \{\{0, 1^4, 2\}\}$$

meaning that among the six permutations in $\mathfrak{S}_3$, only 123 has no descents, only 321 has two descents and the other four all have one descent.

If $\pi \in L(A)$ and $\sigma \in L(B)$ where $A \cap B = \emptyset$ then these permutations have *shuffle set*

$$\pi \sqcup\!\sqcup \sigma = \{\tau \in L(A \uplus B) \mid \pi, \sigma \text{ are subwords of } \tau\}$$

where a subwords of a permutation τ is a subsequence of (not necessarily consecutive) elements of τ. For instance,

$$64 \sqcup\!\sqcup 25 = \{6425, 6245, 6254, 2564, 2645, 2654\}.$$

All of the statistics defined above have a remarkable property related to shuffles, called "shuffle-compatibility". A permutation statistic st is *shuffle-compatible* if for all quadruples $\pi, \pi', \sigma, \sigma'$ with $\operatorname{st}\pi = \operatorname{st}\pi'$, $\operatorname{st}\sigma = \operatorname{st}\sigma'$, $|\pi| = |\pi'|$, and $|\sigma| = |\sigma'|$ we have

$$\operatorname{st}(\pi \sqcup\!\sqcup \sigma) = \operatorname{st}(\pi' \sqcup\!\sqcup \sigma'),$$

that is, the distribution of st on a set of shuffles depends only on the values of st on the permutations being shuffled and their lengths.

Shuffle-compatibility dates back to the early work of Stanley, as the shuffle-compatibility of the descent set, descent number, and major index are implicit consequences of the theory of P-partitions [Sta72]. Likewise, Stembridge's work on enriched P-partitions implies that the peak set and peak number are shuffle-compatible. Gessel and Zhuang coined the term "shuffle-compatibility" and initiated the study of shuffle-compatibility per se in 2018; in [GZ18], they developed an algebraic framework for shuffle-compatibility centered around the notion of the shuffle algebra of a shuffle-compatible permutation statistic, which is well-defined if and only if the statistic is shuffle-compatible and whose multiplication encodes the distribution of the statistic over sets of shuffles.

Gessel's [Ges84] quasisymmetric functions serve as natural generating functions for P-partitions. And for a special family of statistics called "descent statistics", Gessel and Zhuang used quasisymmetric functions to characterize shuffle algebras and prove shuffle-compatibility results.

Given two linear permutation statistics st_1 and st_2, we say that st_1 is a *refinement* of st_2 if for all permutations π and σ of the same length, $\mathrm{st}_1[\pi] = \mathrm{st}_1[\sigma]$ implies $\mathrm{st}_2[\pi] = \mathrm{st}_2[\sigma]$; when this is true, we also say that st_2 is a *coarsening* of st_1. A permutation statistic st is a *descent statistic* if $\mathrm{Des}\,\pi = \mathrm{Des}\,\sigma$ and $|\pi| = |\sigma|$ implies $\mathrm{st}\,\pi = \mathrm{st}\,\sigma$. Alternatively, a descent statistic is a coarsening of descent set. Note that Des itself, des, Pk, and pk are all descent statistics.

One of Gessel and Zhuang's main results is a necessary and sufficient condition for shuffle-compatibility of descent statistics which implies that the shuffle algebra of any shuffle-compatible descent statistic is isomorphic to a quotient algebra of the algebra of quasisymmetric functions.

1.2 Cyclic Permutations, Cyclic Statistics, and Cyclic Shuffles

Recently Adin, Gessel, Reiner, and Roichman [AGRR21] introduced a cyclic version of quasisymmetric functions with a corresponding cyclic shuffle operation. For a linear permutation $\pi = \pi_1\pi_2\ldots\pi_n$, we define the corresponding *cyclic permutation* $[\pi]$ to be the set of rotations of π, that is,

$$[\pi] = \{\pi_1\pi_2\ldots\pi_n,\ \ \pi_2\ldots\pi_n\pi_1,\ \ \ldots,\ \ \pi_n\pi_1\ldots\pi_{n-1}\}.$$

For example,

$$[2856] = \{2856, 8562, 5628, 6285\}.$$

Let $[\mathfrak{S}_n]$ denote the set of cyclic permutations on $[n]$. The *length* of a cyclic permutation $[\pi]$ refers to the length of π, which makes sense because all linear permutation representatives of $[\pi]$ have the same length. We let

$$C(A) = \{[\pi] \mid [\pi] \text{ is a cyclic permutation of } A\}.$$

For instance,

$$C(\{2, 3, 5, 8\}) = \{[2358], [2385], [2538], [2583], [2835], [2853]\}.$$

A *cyclic permutation statistic* is any function cst whose domain is cyclic permutations. We can lift the linear permutation statistics we have introduced to the cyclic realm as follows.

The *cyclic descent set* cDes of a linear permutation π is defined by

$$\operatorname{cDes} \pi = \{i \mid \pi_i > \pi_{i+1} \text{ where the subscripts are taken modulo } n\} \subseteq [n].$$

Also define the *cyclic descent number*

$$\operatorname{cdes} \pi := |\operatorname{cDes} \pi|.$$

This leads to the *cyclic descent set* of a cyclic permutation

$$\operatorname{cDes}[\pi] = \{\{\operatorname{cDes} \sigma \mid \sigma \in [\pi]\}\},$$

which is a multiset. The multiplicity comes into play since cDes may have the same value on different representatives σ in $[\pi]$. Now we can define the *cyclic descent number*

$$\operatorname{cdes}[\pi] := \operatorname{cdes} \pi.$$

This is well-defined since all elements of $[\pi]$ have the same number of cyclic descents. For example if $\pi = 3728$, then we have $\operatorname{cDes} \pi = \{2, 4\}$ and $\operatorname{cdes} \pi = 2$ which extend to cyclic permutations as

$$\operatorname{cDes}[3728] = \{\{\operatorname{cDes} 3728, \operatorname{cDes} 7283, \operatorname{cDes} 2837, \operatorname{cDes} 8372\}\} = \{\{\ \{1, 3\}^2, \{2, 4\}^2\ \}\},$$

and $\operatorname{cdes}[3728] = 2$. Similarly, if we define the *cyclic peak set* cPk of a linear permutation π by

$$\operatorname{cPk}\pi = \{i \mid \pi_{i-1} < \pi_i > \pi_{i+1} \text{ where the subscripts are taken modulo } n\} \subseteq [n],$$

and the *cyclic peak number*

$$\operatorname{cpk}\pi := |\operatorname{cPk}\pi|.$$

Then the cyclic counterpart of Pk, the *cyclic peak set* cPk of a cyclic permutation is defined as

$$\operatorname{cPk}[\pi] = \{\{\operatorname{cPk}\sigma \mid \sigma \in [\pi]\}\},$$

and the *cyclic peak number*

$$\operatorname{cpk}[\pi] := \operatorname{cpk}\pi.$$

Remark 1.2.1. By definition, $\operatorname{cDes}[\pi]$ carries the information for all representatives in $[\pi]$. Moreover, $\operatorname{cDes}\pi$ together with $|\pi|$ will be sufficient to determine $\operatorname{cDes}[\pi]$. In fact, $\operatorname{cDes}[\pi]$ is simply collecting all cyclic shifts of $\operatorname{cDes}\pi$ in $[n]$ where $n = |\pi|$, namely,

$$\operatorname{cDes}[\pi] = \{\{\, i + \operatorname{cDes}\pi \mid i \in [n] \,\}\}.$$

Here $i + \operatorname{cDes}\pi$ is the set defined by (2.1.1). Similarly, $\operatorname{cPk}[\pi]$ can be entirely determined by $\operatorname{cPk}\pi$ and $|\pi|$.

On the other hand, finding a suitable cyclic analogue of the major index statistic is challenging; we will address this in Section 4.4.3.

Given two cyclic permutation statistics cst_1 and cst_2, we say that cst_1 is a *refinement* of cst_2 if for all cyclic permutations $[\pi]$ and $[\sigma]$ of the same length, $\operatorname{cst}_1[\pi] = \operatorname{cst}_1[\sigma]$ implies $\operatorname{cst}_2[\pi] = \operatorname{cst}_2[\sigma]$; when this is true, we also say that cst_2 is a *coarsening* of cst_1. A cyclic permutations statistic cst is a *cyclic descent statistic* if cst is a coarensing of cDes. Alternatively, cst is a cyclic permutation statistic if $\operatorname{cDes}[\pi] = \operatorname{cDes}[\sigma]$ and $|\pi| = |\sigma|$ implies $\operatorname{cst}[\pi] = \operatorname{cst}[\sigma]$. We note that all four of our cyclic statistics are cyclic descent statistics.

The definitions for shuffles follow the same pattern already established. Given $[\pi] \in C(A)$ and $[\sigma] \in C(B)$ where $A \cap B = \emptyset$, we define their *cyclic shuffle set* to be

$$[\pi] \sqcup\!\sqcup [\sigma] = \{[\tau] \mid \ \tau = \pi' \sqcup\!\sqcup \sigma' \text{ where } \pi' \in [\pi] \text{ and } \sigma' \in [\sigma]\}.$$

To illustrate,

$$[14] \sqcup\!\sqcup [23] = \{[1234], [1243], [1324], [1342], [1423], [1432]\}. \tag{1.2.1}$$

We call a cyclic permutation statistic cst *cyclic shuffle-compatible* if for all quadruples $[\pi]$, $[\pi']$, $[\sigma]$, $[\sigma']$ with $\operatorname{cst}[\pi] = \operatorname{cst}[\pi']$, $\operatorname{cst}[\sigma] = \operatorname{cst}[\sigma']$, $|\pi| = |\pi'|$, and $|\sigma| = |\sigma'|$ we have

$$\operatorname{cst}([\pi] \sqcup\!\sqcup [\sigma]) = \operatorname{cst}([\pi'] \sqcup\!\sqcup [\sigma']).$$

The first results in cyclic shuffle-compatibility were implicit in the work of Adin et al. [AGRR21], which introduced toric $[\vec{D}]$-partitions (a toric poset analogue of P-partitions) and cyclic quasisymmetric functions (which are natural generating functions for toric $[\vec{D}]$-partitions). In particular, Adin et al. established a multiplication formula for fundamental cyclic quasisymmetric functions which implies that the cyclic descent set cDes is cyclic shuffle-compatible, and they also proved the formula

$$\sum_{[\tau]\in[\pi]\sqcup\!\sqcup[\sigma]} q^{\operatorname{cdes}\tau} = (1-q)^{|\pi|+|\sigma|} \sum_{k=0}^{\infty} \binom{k+|\pi|-\operatorname{cdes}\pi-1}{|\pi|-1}\binom{k+|\sigma|-\operatorname{cdes}\sigma-1}{|\sigma|-1} k q^k$$

which implies that the cyclic descent number cdes is cyclic shuffle-compatible.

1.3 Outline

This dissertation is devoted to the study of cyclic shuffle-compatibility, featuring on its correlations with cyclic permutations and cyclic quasisymmetric functions. In the next chapter, we recall some basic concepts such as quasisymmetric functions and cyclic quasisymmetric functions, with several concrete examples provided.

In Chapter 3 we study cyclic shuffle-compatibility through purely combinatorial means. In particular, we show how one can lift shuffle-compatibility results for linear permutations to cyclic ones. We then apply this result to cyclic descents and cyclic peaks. This chapter contains materials from Domagalski, Liang, Minnich, Sagan, Schmidt, and Sietsema [DLM+21].

In chapter 4, we define the cyclic shuffle algebra of a cyclic shuffle-compatible statistic, and develop an algebraic framework for cyclic shuffle-compatibility in which the role of quasisymmetric functions is replaced by the cyclic quasisymmetric functions recently introduced by Adin, Gessel, Reiner, and Roichman. We use our theory to provide explicit descriptions for the cyclic shuffle algebras of various cyclic permutation statistics, which in turn gives algebraic proofs for their cyclic shuffle-compatibility. This chapter contains materials from Liang, Sagan, and Zhuang [LSZ23].

Chapter 5 develops the theory of enriched toric $[\vec{D}]$-partitions. Whereas Stembridge's enriched P-partitions give rises to the peak algebra which is a subring of the ring of quasisymmetric functions QSym, our enriched toric $[\vec{D}]$-partitions will generate the cyclic peak algebra which is a subring of cyclic quasisymmetric functions cQSym. In the same manner as the peak set of linear permutations appears when considering enriched P-partitions, the cyclic peak set of cyclic permutations plays an important role in our theory. The associated order polynomial is discussed based on this framework.

CHAPTER 2

PRELIMINARIES

In this chapter, we provide the necessary background for this dissertation. In Section 2.1, we review the relations between sets and compositions which will be useful to index bases of (cyclic) quasisymmetric functions in later sections. In Section 2.2 and Section 2.3, we provide a terse introduction of quasisymmetric functions and cyclic quasisymmetric functions respectively.

2.1 Sets and Compositions

For $n \in \mathbb{P}$, let $2^{[n]}$ denote the set of all subsets of $[n]$, and $2_0^{[n]}$ be the set of all nonempty subsets of $[n]$. A *composition of n* is a tuple of positive integers that sum to n. Denote by Comp_n the set of all compositions of n and write $\alpha \vDash n$ for $\alpha \in \mathrm{Comp}_n$.

Definition 2.1.1 (Cyclic shift of a set)**.** Define a *cyclic shift* of a subset $E \subseteq [n]$ in $[n]$ to be a set of the form

$$i + E = \{i + e \ (\mathrm{mod}\ n) \mid e \in E\} \subseteq [n]. \tag{2.1.1}$$

For example if $E = \{2, 4, 5\} \subseteq [6]$, then $3 + E = \{1, 2, 5\}$. Note that sometimes we will use $E + i$ as well for the same concept. While using a negative shift, the reader should be careful to distinguish between $E - i$ and the set difference $E - \{i\} = E \setminus \{i\}$. We usually use $[S]$ to denote the equivalence class of S under cyclic shifts.

Definition 2.1.2 (Cyclic shift of a composition)**.** A *cyclic shift* of a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ is a composition of the form

$$(\alpha_k, \ldots, \alpha_m, \alpha_1, \ldots, \alpha_{k-1})$$

for some $k \in [m]$.

Definition 2.1.3 (Cyclic composition)**.** A *cyclic composition* of n is then the equivalence class of a composition of n under cyclic shift.

For example,

$$[2, 1, 3] = \{(2, 1, 3), (1, 3, 2), (3, 2, 1)\}$$

and

$$[1,2,1,2] = \{(1,2,1,2),(2,1,2,1)\}$$

are both cyclic compositions. By convention, we will also allow the empty set $\varnothing$ to be a cyclic composition. Furthermore, we adopt the notations from [AGRR21] and denote by $c2_0^{[n]}$ (respectively, cComp_n) the set of equivalence classes of elements of $2_0^{[n]}$ (respectively, Comp_n) under cyclic shifts. In another word, cComp_n is the set of cyclic compositions of n.

Now we recall two natural bijections which will play important roles when indexing two particular bases of (cyclic) quasisymmetric functions.

The first natural bijection is between $2^{[n-1]}$ and Comp_n. The map $\Phi : 2^{[n-1]} \to \mathrm{Comp}_n$ is defined by

$$\Phi(E) := (e_1 - e_0, e_2 - e_1, \dots, e_k - e_{k-1}, e_{k+1} - e_k) \tag{2.1.2}$$

for any given $E = \{e_1 < e_2 < \cdots < e_k\} \subseteq [n-1]$ with $e_0 = 0$ and $e_{k+1} = n$, where the inverse map is

$$\Phi^{-1}(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \dots, \alpha_1 + \alpha_2 + \cdots + \alpha_k\}$$

for any $\alpha = (\alpha_1, \dots, \alpha_{k+1}) \vDash n$. For example, $n = 6$ and $E = \{3,5\}$, then $\Phi(E) = (3,2,1)$ and $\Phi^{-1}(3,2,1) = \{3,5\} = E$.

Another bijection is between $c2_0^{[n]}$ and cComp_n, for the sake of which we need to consider the map $\psi : 2_0^{[n]} \to \mathrm{Comp}_n$ defined by

$$\psi(E) := (e_2 - e_1, \dots, e_k - e_{k-1}, e_1 - e_k + n) \tag{2.1.3}$$

where $E = \{e_1 < e_2 < \cdots < e_k\} \subseteq [n]$. Notice that if E' is a cyclic shift of E in $[n]$, then $\psi(E')$ is also a cyclic shift of $\psi(E)$. Therefore ψ induces a map $\Psi : c2_0^{[n]} \to \mathrm{cComp}_n$. Moreover, it is straightforward to check that the induced map Ψ is bijective.

Definition 2.1.4 (Non-Escher)**.** Let us call S a *non-Escher*[1] subset of $[n]$ if S is the cyclic descent set of some linear permutation of length n.

[1]We borrow the term "non-Escher" from [AGRR21] and other recent works on cyclic descent extensions. As explained there, this term is a reference to M. C. Escher's painting "Ascending and Descending".

When $n = 0$ or $n = 1$, only the empty set is non-Escher, and when $n \geq 2$, all subsets of $[n]$ are non-Escher except for the empty set and $[n]$ itself. We associate to each non-Escher subset $S \subseteq [n]$ a composition $\mathrm{cComp}\, S$ defined by

$$\mathrm{cComp}\, S := \begin{cases} (s_2 - s_1, \ldots, s_j - s_{j-1}, n - s_j + s_1), & \text{if } n \geq 2, \\ (1), & \text{if } n = 1, \\ \emptyset, & \text{if } n = 0. \end{cases}$$

It is easy to see that if S' is a cyclic shift of S, then $\mathrm{cComp}\, S'$ is a cyclic shift of $\mathrm{cComp}\, S$. So we can let $\mathrm{cComp}[S]$ be the cyclic composition defined by

$$\mathrm{cComp}[S] := [\mathrm{cComp}\, S].$$

We say that a cyclic composition is *non-Escher* if it is an image of this induced map cComp, and one can check that cComp is a bijection from equivalence classes of non-Escher subsets of $[n]$ under cyclic shift to non-Escher cyclic compositions of n.

Definition 2.1.5 (Cyclic descent composition)**.** If S is the cyclic descent set of a linear permutation π, then we call $\mathrm{cComp}[S]$ the *cyclic descent composition* of the cyclic permutation $[\pi]$. We denote the cyclic descent composition of $[\pi]$ simply as $\mathrm{cComp}[\pi]$.

For example, take $\pi = 179624$. Then π has cyclic descent set $S = \{3, 4, 6\}$, so the cyclic descent composition of $[\pi]$ is $\mathrm{cComp}[S] = [1, 2, 3]$, which we also denote by $\mathrm{cComp}[\pi]$.

2.2 Quasisymmetric Functions QSym

Definition 2.2.1 (Quasisymmetric function)**.** A *quasisymmetric function* is a formal power series $f \in \mathbb{Q}[[x_1, x_2, \ldots]]$ such that for any sequence of positive integers $a = (a_1, a_2, \ldots, a_s)$, and two increasing sequences $i_1 < i_2 < \cdots < i_s$ and $j_1 < j_2 < \cdots < j_s$ of positive integers,

$$[x_{i_1}^{a_1} x_{i_2}^{a_2} \ldots x_{i_s}^{a_s}]\, f = [x_{j_1}^{a_1} x_{j_2}^{a_2} \ldots x_{j_s}^{a_s}]\, f,$$

where $[x_{i_1}^{a_1} x_{i_2}^{a_2} \ldots x_{i_s}^{a_s}]\, f$ denotes the coefficient of monomial $x_{i_1}^{a_1} x_{i_2}^{a_2} \ldots x_{i_s}^{a_s}$ in the expression of f.

Let QSym_n be the set of all quasisymmetric functions which are homogeneous of degree n, and $\mathrm{QSym} = \oplus_{n\geq 0}\mathrm{QSym}_n$. Two bases of QSym are particularly important to our work: monomial quasisymmetric functions M_L and fundamental quasisymmetric functions F_L.

Definition 2.2.2 (Monomial quasisymmetric functions)**.** Given a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s) \vDash n$, the associated *monomial quasisymmetric function* indexed by α is

$$M_\alpha = \sum_{i_1<i_2<\cdots<i_s} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \ldots x_{i_s}^{\alpha_s}.$$

It is clear that $\{M_\alpha\}_{\alpha\vDash n}$ form a basis of QSym_n. From the natural bijection $\Phi : 2^{[n-1]} \to \mathrm{Comp}_n$ defined by (2.1.2), we can also index the monomial quasisymmetric functions by subsets $E \subseteq [n-1]$, and define $M_{n,E} := M_{\Phi(E)}$.

The fundamental quasisymmetric functions form another important basis of QSym.

Definition 2.2.3 (Fundamental quasisymmetric function)**.** The *fundamental quasisymmetric function* indexed by $E \subseteq [n-1]$ is

$$F_{n,E} = \sum_{\substack{i_1\leq\cdots\leq i_n \\ i_k<i_{k+1} \text{ if } k\in E}} x_{i_1} x_{i_2} \ldots x_{i_n}.$$

Similarly, we can define $F_\alpha := F_{\Phi^{-1}(\alpha)}$ indexed by compositions.

The relation between monomial and fundamental quasisymmetric functions is simple:

$$F_{n,E} = \sum_{L\supseteq E} M_{n,L}. \tag{2.2.1}$$

By the principle of inclusion and exclusion, $M_{n,E}$ can be expressed as a linear combination of the $F_{n,L}$, from which we can tell that $\{F_{n,L}\}_{L\subseteq[n-1]}$ spans QSym_n. By checking the cardinality of both sets $\{F_{n,L}\}_{L\subseteq[n-1]}$ and $\{M_{n,E}\}_{E\subseteq[n-1]}$, it follows that $\{F_{n,L}\}_{L\subseteq[n-1]}$ is indeed a basis of QSym_n.

Example 2.2.4. Consider $E = \{1, 3\}$, by definition we have

$$F_{4,\{1,3\}} = \sum_{i_1<i_2\leq i_3<i_4} x_{i_1}x_{i_2}x_{i_3}x_{i_4} = \sum_{i_1<i_2<i_3<i_4} x_{i_1}x_{i_2}x_{i_3}x_{i_4} + \sum_{i_1<i_2<i_4} x_{i_1}x_{i_2}^2 x_{i_4}.$$

There are only two choices for a set L satisfying that $E \subseteq L \subseteq [3]$: $\{1,3\}$ or $\{1,2,3\}$. Since $\Phi(\{1,3\}) = (1,2,1), \Phi(\{1,2,3\}) = (1,1,1,1)$, we get

$$M_{4,\{1,3\}} = M_{(1,2,1)} = \sum_{i_1<i_2<i_3} x_{i_1}x_{i_2}^2x_{i_3}, \;\; M_{4,\{1,2,3\}} = M_{(1,1,1,1)} = \sum_{i_1<i_2<i_3<i_4} x_{i_1}x_{i_2}x_{i_3}x_{i_4}.$$

The calculation above verifies that $F_{4,\{1,3\}} = M_{4,\{1,3\}} + M_{4,\{1,2,3\}} = \sum_{L\supseteq\{1,3\}} M_{4,L}$.

2.3 Cyclic Quasisymmetric Functions cQSym

In this subsection, we recall from [AGRR21] the theory of cyclic quasisymmetric functions.

Definition 2.3.1 (Cyclic quasisymmetric functions)**.** A *cyclic quasisymmetric function* is a formal power series $f \in \mathbb{Q}[[x_1, x_2, \ldots]]$ such that for any sequence of positive integers $a = (a_1, a_2, \ldots, a_s)$, a cyclic shift $(a'_1, a'_2, \ldots, a'_s)$ of a, and two increasing sequences $i_1 < i_2 < \cdots < i_s$ and $j_1 < j_2 < \cdots < j_s$ of positive integers,

$$[x_{i_1}^{a_1}x_{i_2}^{a_2}\ldots x_{i_s}^{a_s}]\, f = [x_{j_1}^{a'_1}x_{j_2}^{a'_2}\ldots x_{j_s}^{a'_s}]\, f,$$

namely the coefficients of $x_{i_1}^{a_1}x_{i_2}^{a_2}\ldots x_{i_s}^{a_s}$ and $x_{j_1}^{a'_1}x_{j_2}^{a'_2}\ldots x_{j_s}^{a'_s}$ in f are equal.

Denote by cQSym_n the set of all cyclic quasisymmetric functions which are homogeneous of degree n, and $\mathrm{cQSym} = \oplus_{n\geq 0} \mathrm{cQSym}_n$.

Remark 2.3.2. It is clear that there exists a strict inclusion relation $\mathrm{Sym} \subsetneq \mathrm{cQSym} \subsetneq \mathrm{QSym}$, where $\mathrm{Sym} = \oplus_{n\geq 0}\mathfrak{S}_n$ is the algebra of symmetric functions.

We have the following cyclic analogues of the concepts of monomial (fundamental) quasisymmetric functions.

Definition 2.3.3 (Monomial cyclic quasisymmetric function)**.**
Given a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s) \vDash n$, the associated *monomial cyclic quasisymmetric function* indexed by α is

$$M_\alpha^{\mathrm{cyc}} = \sum_{i=1}^{s} M_{(\alpha_i,\alpha_{i+1},\ldots,\alpha_{i-1})},$$

Table 2.1 Monomial cyclic quasisymmetric functions indexed by compositions of 4

$\alpha \vDash 4$	M_α^{cyc}
(4)	$M_{(4)} = M_{4,\emptyset}$
(1,3) or (3,1)	$M_{(1,3)} + M_{(3,1)} = M_{4,\{1\}} + M_{4,\{3\}}$
(2, 2)	$2M_{(2,2)} = 2M_{4,\{2\}}$
(1,1,2) or (1,2,1) or (2,1,1)	$M_{(1,1,2)} + M_{(1,2,1)} + M_{(2,1,1)} = M_{4,\{1,2\}} + M_{4,\{1,3\}} + M_{4,\{2,3\}}$
(1, 1, 1, 1)	$4M_{(1,1,1,1)} = 4M_{4,\{1,2,3\}}$

where the indices are interpreted modulo s, meaning $\alpha_j = \alpha_{j+s}$. In other words, M_α^{cyc} sums over all monomial quasisymmetric functions indexed by cyclic shifts of α. Therefore it is clear that $M_\alpha^{\mathrm{cyc}} = M_{\alpha'}^{\mathrm{cyc}}$ if α and α' only differ by a cyclic shift.

We can also index the monomial cyclic quasisymmetric function by sets. For a nonempty $E \subseteq [n]$, define $M_{n,E}^{\mathrm{cyc}} := M_{\psi(E)}^{\mathrm{cyc}}$ via the map $\psi : 2_0^{[n]} \to \mathrm{Comp}_n$ defined by (2.1.3), and set $M_{n,\emptyset}^{\mathrm{cyc}} := 0$. Similarly it can be shown that $M_{n,E}^{\mathrm{cyc}} = M_{n,E'}^{\mathrm{cyc}}$ if E' is a cyclic shift of E.

The following result gives the expression of monomial cyclic quasisymmetric functions in terms of monimial quasisymmetric functions.

Lemma 2.3.4 ([AGRR21] Lemma 2.5, monomial to cyclic monomial)**.** *For any subset $E \subseteq [n]$*

$$M_{n,E}^{\mathrm{cyc}} = \sum_{e \in E} M_{n,(E-e)\cap[n-1]}, \tag{2.3.1}$$

where the set $E - e$ is defined as (2.1.1). □

Example 2.3.5. Table 2.1 computes all monomial cyclic quasisymmetric function indexed by compositions of 4, in terms of monomial quasisymmetric functions.

For the natural desire of establishing a similar relation as the one between monomial and fundamental quasisymmetric functions given by (2.2.1) in our cyclic situation, define

Definition 2.3.6 (Fundamental cyclic quasisymmetric function)**.** The *fundamental cyclic quasisymmetric function* indexed by $E \subseteq [n]$ is

$$F_{n,E}^{\mathrm{cyc}} := \sum_{L \supseteq E} M_{n,L}^{\mathrm{cyc}}. \tag{2.3.2}$$

Example 2.3.7. Consider $n = 4$ and $E = \{1, 3\}$. By definition

$$\begin{aligned}
F^{\mathrm{cyc}}_{4,\{1,3\}} &= \sum_{L\supseteq\{1,3\}} M^{\mathrm{cyc}}_{4,L} \\
&\overset{(i)}{=} M^{\mathrm{cyc}}_{4,\{1,3\}} + M^{\mathrm{cyc}}_{4,\{1,2,3\}} + M^{\mathrm{cyc}}_{4,\{1,3,4\}} + M^{\mathrm{cyc}}_{4,\{1,2,3,4\}} \\
&\overset{(ii)}{=} M^{\mathrm{cyc}}_{(2,2)} + M^{\mathrm{cyc}}_{(1,1,2)} + M^{\mathrm{cyc}}_{(2,1,1)} + M^{\mathrm{cyc}}_{(1,1,1,1)} \\
&\overset{(iii)}{=} 2M_{(2,2)} + 2\left(M_{(1,1,2)} + M_{(1,2,1)} + M_{(2,1,1)}\right) + 4M_{(1,1,1,1)} \\
&= 2\sum_{i_1<i_2} x_{i_1}^2 x_{i_2}^2 \\
&\quad + 2\sum_{i_1<i_2<i_3} (x_{i_1}x_{i_2}x_{i_3}^2 + x_{i_1}x_{i_2}^2x_{i_3} + x_{i_1}^2x_{i_2}x_{i_3}) \\
&\quad + 4\sum_{i_1<i_2<i_3<i_4} x_{i_1}x_{i_2}x_{i_3}x_{i_4}.
\end{aligned}$$

Equality (i) follows from the fact that the choices for $L \supseteq \{1, 3\}$ in $[4]$ are $\{1, 3\}$, $\{1, 2, 3\}$, $\{1, 3, 4\}$ and $\{1, 2, 3, 4\}$. Equality (ii) is obtained by changing indices under the map ψ defined by (2.1.3), equality (iii) is from Table 2.1.

The following transition from fundamental to cyclic fundamental quasisymmetric functions should come without surprise.

Lemma 2.3.8 ([AGRR21, Proposition 2.15], fundamental to cyclic fundamental)**.** *For any subset $E \subseteq [n]$,*

$$F^{\mathrm{cyc}}_{n,E} = \sum_{i\in[n]} F_{n,(E-i)\cap[n-1]},$$

with set $E - i$ defined by (2.1.1). □

Remark 2.3.9.

1. It follows directly from Lemma 2.3.8 that $F^{\mathrm{cyc}}_{n,E} = F^{\mathrm{cyc}}_{n,E'}$ if E' is a cyclic shift of E.

2. Clearly the set $\{M^{\mathrm{cyc}}_{n,E} : E \in c2^{[n]}_0\}$ spans cQSym_n and is linearly independent, as each monomial of degree n appears in $M^{\mathrm{cyc}}_{n,E}$ for exactly one $E \in c2^{[n]}_0$. Hence $\{M^{\mathrm{cyc}}_{n,E} : E \in c2^{[n]}_0\}$

is a basis of cQSym_n. Applying the principle of inclusion and exclusion on (2.3.2) we have

$$M^{\mathrm{cyc}}_{n,E} = \sum_{L \supseteq E} (-1)^{|L \setminus E|} F^{\mathrm{cyc}}_{n,L},$$

which implies that $\{F^{\mathrm{cyc}}_{n,E} : E \in c2^{[n]}_0\}$ also spans cQSym_n; together with the fact that the dimension of vector space cQSym_n is $\#c2^{[n]}_0$, $\{F^{\mathrm{cyc}}_{n,E} : E \in c2^{[n]}_0\}$ is also a basis of cQSym_n.

CHAPTER 3

CYCLIC SHUFFLE-COMPATIBILITY: COMBINATORIALLY

In this chapter we will provide a general method for proving cyclic shuffle-compatibility results as corollaries of linear ones. We note that in this chapter we will be using the set

$$\underline{E+i} = \{e+i \mid e \in E\}, \tag{3.0.1}$$

which is different from the set $E+i$ defined in equation (2.1.1).

3.1 The Lifting Lemma

Definition 3.1.1 (Standardization)**.** Suppose $A, B \subset \mathbb{P}$ with $|A| = |B| = n$ and $\pi = \pi_1\pi_2 \ldots \pi_n \in L(A)$. The *standardization of* π *to* B is

$$\operatorname{std}_B \pi = f(\pi_1)f(\pi_2)\ldots f(\pi_n) \in L(B)$$

where $f : A \to B$ is the unique order-preserving bijection between A and B. If $B = [n]$ then we write just $\operatorname{std}\pi$ for $\operatorname{std}_{[n]}\pi$ and call this the *standardization* of π.

For example, if $A = \{1, 4, 5, 8\}$ and $B = \{2, 3, 6, 9\}$, then $\operatorname{std}_B(5481) = 6392$. Standardization for cyclic permutations is defined in the analogous manner.

We first prove a result about cyclic descent statistics which is a cyclic analogue of one in the linear case [BJS20].

Lemma 3.1.2. *Let* cst *be a cyclic descent statistic. For any four cyclic permutations* $[\pi]$, $[\pi']$, $[\sigma]$, $[\sigma']$ *such that*

$$\operatorname{std}[\pi] = \operatorname{std}[\pi'] \quad \text{and} \quad \operatorname{std}[\sigma] = \operatorname{std}[\sigma']$$

we have

$$\operatorname{cst}([\pi] \sqcup\!\sqcup [\sigma]) = \operatorname{cst}([\pi'] \sqcup\!\sqcup [\sigma']).$$

Proof. Since cst is a cyclic descent statistic, its values only depends on the relative order of adjacent elements. So it suffices to prove the case when

$$[\pi] \uplus [\sigma] = [\pi'] \uplus [\sigma'] = [m+n]$$

where $m = |\pi| = |\pi'|$ and $n = |\sigma| = |\sigma'|$. For simplicity, let $A = [m]$ and $B = \underline{[n] + m}$.

We claim that it suffices to find, for any π and σ as in the previous paragraph, a cst-preserving bijection

$$[\pi] \sqcup [\sigma] \to \mathrm{std}_A[\pi] \sqcup \mathrm{std}_B[\sigma].$$

From this map and the hypothesis of the lemma we have

$$\mathrm{cst}([\pi] \sqcup [\sigma]) = \mathrm{cst}(\mathrm{std}_A[\pi] \sqcup \mathrm{std}_B[\sigma]) = \mathrm{cst}(\mathrm{std}_A[\pi'] \sqcup \mathrm{std}_B[\sigma']) = \mathrm{cst}([\pi'] \sqcup [\sigma']).$$

We will show the existence of this bijection by induction on the size of the set of what we will call out-of-order pairs

$$O = \{(i, j) \in \pi \times \sigma \mid i > j\}.$$

If $\#O = 0$ then $[\pi] \in C(A)$ and $[\sigma] \in C(B)$. It follows that $[\pi] = \mathrm{std}_A[\pi]$ and $[\sigma] = \mathrm{std}_B[\sigma]$ so the identity map will do.

For the induction step, let $\#O > 0$. Then there must be a pair $(i, i-1) \in O$. Let $\pi'' = (i-1, i)\pi$ and $\sigma'' = (i-1, i)\sigma$ where $(i-1, i)$ is the transposition which exchanges $i-1$ and i. We will be done if we can construct a cst preserving bijection

$$T_i : [\pi] \sqcup [\sigma] \to [\pi''] \sqcup [\sigma''].$$

This is because π'', σ'' have fewer out-of-order pairs and so, by induction, there is a cst-preserving bijection $[\pi''] \sqcup [\sigma''] \to \mathrm{std}_A[\pi''] \sqcup \mathrm{std}_B[\sigma'']$ which, when composed with T_i, will finish the construction.

Define T_i by

$$T_i[\tau] = \begin{cases} [(i-1, i)\tau] & \text{if } i-1, i \text{ are not cyclically adjacent in } [\tau], \\ [\tau] & \text{else.} \end{cases}$$

We must first check that T_i is well defined in that $T_i[\tau] \in [\pi''] \sqcup [\sigma'']$. This is true if $i-1$ and i are not cyclically adjacent since $i-1$ and i have been swapped in all three cyclic permutations involved. If they are adjacent then the relative order of the elements of $[\tau]$ corresponding to $[\pi]$

and $[\pi'']$ are the same, and similarly for $[\sigma]$ and $[\sigma'']$. So leaving $[\tau]$ fixed again gives a shuffle in the range.

Finally, we need to verify that T_i is cst preserving. Since cst is a cyclic descent statistic, it suffices to show that T_i is cDes preserving. Certainly this is true of $[\tau]$ is fixed. And if it is not, then $i-1, i$ are not cyclically adjacent in $[\tau]$. But switching $i-1$ and i could only change a cyclic descent into a cyclic ascent or vice-versa if these two elements were adjacent. So in this case $\mathrm{cDes}[(i-1,i)\tau] = \mathrm{cDes}[\tau]$ and we are done. □

Example 3.1.3. As an example of the map T_i, consider the shuffle set in (1.2.1). Here we can take $i = 4$ since $4 \in 14$ and $3 \in 23$. For $[\tau] = [1342]$ we have 3 and 4 cyclically adjacent so

$$T_4[1342] = [1342] \in [13] \sqcup\!\sqcup [24]$$

as desired. On the other hand, in $[1324]$ the 3 and 4 are not cyclically adjacent so

$$T_4[1324] = [1423].$$

Note that $[1423] \in [13] \sqcup\!\sqcup [24]$ and

$$\mathrm{cDes}[1423] = \{\{\ \{1,2\},\ \{2,3\},\ \{3,4\},\ \{1,4\}\ \}\} = \mathrm{cDes}[1324].$$

We can use the previous lemma to drastically cut down on the number of cases which need to be checked to obtain cyclic shuffle-compatibility. In particular, the component permutations in the shuffles to be considered can be on consecutive intervals of integers. And one can keep one component of the shuffle constant while the other varies.

Corollary 3.1.4. *Suppose that* cst *is cyclic descent statistic. The following are equivalent.*

(a) The cyclic statistic cst *is cyclic shuffle-compatible.*

(b) If $\mathrm{cst}([\pi]) = \mathrm{cst}([\pi'])$ *where* $[\pi], [\pi'] \in C[m]$ *and* $[\sigma] \in C(\underline{[n]+m})$ *for some* $m, n \in \mathbb{N}$ *then*

$$\mathrm{cst}([\pi] \sqcup\!\sqcup [\sigma]) = \mathrm{cst}([\pi'] \sqcup\!\sqcup [\sigma]).$$

(c) *If* $\operatorname{cst}([\sigma]) = \operatorname{cst}([\sigma'])$ *where* $[\sigma], [\sigma'] \in C(\underline{[n]+m})$ *and* $[\pi] \in C[m]$ *for some* $m, n \in \mathbb{N}$ *then*

$$\operatorname{cst}([\pi] \sqcup\!\sqcup [\sigma]) = \operatorname{cst}([\pi] \sqcup\!\sqcup [\sigma']).$$

Proof. We will prove the equivalence of (a) and (b) since the equivalence of (a) and (c) is similar. Clearly (a) implies (b). For the converse, let $\pi, \pi', \sigma, \sigma'$ be any four permutations satisfying the hypothesis of the cyclic shuffle-compatible definition and let $m = |\pi| = |\pi'|$, $n = |\sigma| = |\sigma'|$. Also let $A = [m]$, $B = \underline{[n]+m}$, $A' = \underline{[m]+n}$, and $B' = [n]$. Then, using Lemma 3.1.2 and (b) alternately

$$\begin{aligned}
\operatorname{cst}([\pi] \sqcup\!\sqcup [\sigma]) &= \operatorname{cst}(\operatorname{std}_A[\pi] \sqcup\!\sqcup \operatorname{std}_B[\sigma]) \\
&= \operatorname{cst}(\operatorname{std}_A[\pi'] \sqcup\!\sqcup \operatorname{std}_B[\sigma]) \\
&= \operatorname{cst}(\operatorname{std}_{A'}[\pi'] \sqcup\!\sqcup \operatorname{std}_{B'}[\sigma]) \\
&= \operatorname{cst}(\operatorname{std}_{A'}[\pi'] \sqcup\!\sqcup \operatorname{std}_{B'}[\sigma']) \\
&= \operatorname{cst}([\pi'] \sqcup\!\sqcup [\sigma'])
\end{aligned}$$

which is what we wished to prove. □

In order to prove the Lifting Lemma, we will need to define two functions.

Definition 3.1.5 (Splitting map)**.** For $i \in [n]$ we construct the *splitting map* $S_i : C[n] \to L[n]$ as follows. If $[\pi] \in C[n]$ then let $S_i[\pi]$ be the unique linear permutation in $[\pi]$ which starts with i.

For example,

$$S_3[45132] = 32451.$$

Definition 3.1.6 (Maximum removal map)**.** Define the *maximum removal* map $M : C[n] \to L[n-1]$ by first applying S_n to $[\pi]$ and then removing the initial n.

To illustrate

$$M[45132] = 1324.$$

Note that M is a bijection. We similarly define $M : C(\underline{[n] + m}) \to L(\underline{[n-1] + m})$ using $m + n$ in place of n, as in

$$M[67354] = 3546.$$

Finally, we say that y is *between* x and z in $[\pi]$ if this is true of any linear permutation in $[\pi]$ where x occurs to the left of z. If $[\pi] \in C[m]$, $i \in [m]$, and $[\sigma] \in C(\underline{[n] + m})$ define

$$[\pi] \sqcup\!\sqcup_i [\sigma] = \{[\tau] \in [\pi] \sqcup\!\sqcup [\sigma] \mid \text{only elements of } \sigma \text{ are between } m+n \text{ and } i \text{ in } [\tau]\}.$$

For example,

$$[12] \sqcup\!\sqcup [34] = \{[1234], [1243], [1324], [1423], [1342], [1432]\}$$

with

$$[12] \sqcup\!\sqcup_1 [34] = \{[1234], [1243], [1324]\}$$

and

$$[12] \sqcup\!\sqcup_2 [34] = \{[1423], [1342], [1432]\}.$$

Clearly $[\pi] \sqcup\!\sqcup [\sigma]$ is the disjoint union of the $[\pi] \sqcup\!\sqcup_i [\sigma]$ for $i \in [m]$.

Lemma 3.1.7 (Lifting Lemma)**.** *Let* cst *be a cyclic descent statistic and* st *be a shuffle-compatible linear descent statistic such that the following conditions hold.*

(a) For any $[\tau], [\tau']$ of the same length

$$\mathrm{st}(M[\tau]) = \mathrm{st}(M[\tau']) \quad \textit{implies} \quad \mathrm{cst}[\tau] = \mathrm{cst}[\tau'].$$

(b) Given any $[\pi], [\pi'] \in C[m]$ such that $\mathrm{cst}[\pi] = \mathrm{cst}[\pi']$, there exists a bijection $f : [m] \to [m]$ such that for all i and $j = f(i)$

$$\mathrm{st}(S_i[\pi]) = \mathrm{st}(S_j[\pi']).$$

Then cst *is cyclic shuffle-compatible.*

Proof. By Corollary 3.1.4, it suffices to show that if $[\pi], [\pi']$ are as given in (b) and $\sigma \in C(\underline{[n] + m})$ then $\text{cst}([\pi] \sqcup [\sigma]) = \text{cst}([\pi'] \sqcup [\sigma])$. The remarks preceding the lemma show that this reduces to proving

$$\text{cst}([\pi] \sqcup_i [\sigma]) = \text{cst}([\pi'] \sqcup_j [\sigma]) \tag{3.1.1}$$

for all $i \in [m]$ and $j = f(i)$.

It follows that $\text{st}(S_i[\pi]) = \text{st}(S_j[\pi'])$ by (b). So, since st is shuffle compatible, we have $\text{st}(S_i[\pi] \sqcup \sigma') = \text{st}(S_j[\pi'] \sqcup \sigma')$ where $\sigma' = M[\sigma]$. Thus there is an st-preserving bijection

$$\theta : S_i[\pi] \sqcup \sigma' \to S_j[\pi'] \sqcup \sigma'.$$

This gives rise to a map

$$\theta' : [\pi] \sqcup_i [\sigma] \xrightarrow{M} S_i[\pi] \sqcup \sigma' \xrightarrow{\theta} S_j[\pi'] \sqcup \sigma' \xrightarrow{M^{-1}} [\pi'] \sqcup_j [\sigma]$$

Since this is a bijection, to show (3.1.1) it suffices to prove that θ' is cst-preserving. So take $[\tau] \in [\pi] \sqcup_i [\sigma]$ and $[\tau'] = \theta'[\tau]$. Then $M[\tau'] = \theta \circ M[\tau]$. Since θ is st preserving, we have $\text{st}\, M[\tau'] = \text{st}\, M[\tau]$. But then hypothesis (a) implies that $\text{cst}[\tau] = \text{cst}[\tau']$ which is what we wished to prove. □

3.2 Applications

The hypotheses in the Lifting Lemma may seem strange at first glance, yet they can be quite easy to verify, making it a useful tool. We will see four instances of this by proving the cyclic shuffle-compatibility of cyclic statistics cDes, cdes, cPk and cpk using its aid.

Theorem 3.2.1. *The cyclic statistic* cDes *is cyclic shuffle-compatible.*

Proof. We will verify the hypotheses of Lemma 3.1.7 using st = Des. For (a), suppose that $|\tau| = |\tau'| = n$ and

$$\text{Des}(M[\tau]) = \text{Des}(M[\tau']). \tag{3.2.1}$$

Let us assume that τ, τ' were chosen from their cyclic equivalence class so that $\tau_1 = \max \tau$ and similarly for τ'. Then $M[\tau] = \tau_2\tau_3 \ldots \tau_n$. And by the choice of τ_1 we see that

$$\text{Des}\, \tau = \{1\} \uplus (\text{Des}\, M[\tau] + 1). \tag{3.2.2}$$

Since the same statement holds with τ' in place of τ and (3.2.1) holds, we have $\operatorname{Des}\tau = \operatorname{Des}\tau'$. But $\operatorname{Des}\tau$ is one set in the multiset $\operatorname{cDes}[\tau]$, and the others are gotten by adding each $i \in [n]$ to all elements of $\operatorname{Des}\tau$ modulo n. The same being true of $\operatorname{cDes}[\tau']$ shows that $\operatorname{cDes}[\tau] = \operatorname{cDes}[\tau']$ as desired.

For (b), we are given $[\pi], [\pi'] \in C[m]$ with $\operatorname{cDes}[\pi] = \operatorname{cDes}[\pi']$ and must construct the necessary bijection. From this assumption, we can choose π and π' such that

$$\operatorname{cDes}\pi = \operatorname{cDes}\pi' = A. \tag{3.2.3}$$

for some set A. Define $f : [m] \to [m]$ by $f(\pi_k) = \pi'_k$ for all $k \in [m]$. Let $i = \pi_k$ and $j = \pi'_k$. To show that $\operatorname{Des}(S_i[\pi]) = \operatorname{Des}(S_j[\pi'])$ there are two cases depending on whether $k-1 \in A$ or not, where $k-1$ is taken modulo m.

If $k - 1 \notin A$ then π_k, π'_k are not the second elements in cyclic descents of their respective permutations. From this and equation (3.2.3) it follows that

$$\operatorname{Des}(S_i[\pi]) = A - k + 1 = \operatorname{Des}(S_j[\pi']).$$

If $k-1 \in A$ then the same equalities hold with A replaced by A' which is A with $k-1$ removed. The completes the proof of (b) and of the theorem. □

Theorem 3.2.2. *The cyclic statistic* cdes *is cyclic shuffle-compatible.*

Proof. We proceed as in the previous proof with $\operatorname{st} = \operatorname{des}$. For (a) we assume $\operatorname{des}(M[\tau]) = \operatorname{des}(M[\tau'])$. But from equation (3.2.2) we see that $\operatorname{des}\tau = \operatorname{des}(M[\tau]) - 1$ and the same is true for τ'. Combining this with our initial assumption gives $\operatorname{des}\tau = \operatorname{des}\tau'$. But τ, τ' begin with their largest element so that

$$\operatorname{cdes}[\tau] = \operatorname{des}\tau = \operatorname{des}\tau' = \operatorname{cdes}[\tau']$$

which is what we wished to show.

To prove (b), we are given $\operatorname{cdes}[\pi] = \operatorname{cdes}[\pi']$. As in the preceding paragraph, choose π, π' to begin with their largest elements so that $\operatorname{cdes}[\pi] = \operatorname{des}\pi$ and similarly for π'. Since $\operatorname{des}\pi = \operatorname{des}\pi'$ there is a bijection $\theta : \operatorname{Des}\pi \to \operatorname{Des}\pi'$. Extend this map to $\theta : [m] \to [m]$ by using any bijection between the complements of $\operatorname{Des}\pi$ and $\operatorname{Des}\pi'$. The proof that θ has the desired property is now similar to that for Des and so is left to the reader. □

Theorem 3.2.3. *The cyclic statistic* cPk *is cyclic shuffle-compatible.*

Proof. This proof parallels the one for cDes, so we will only mention the highlights. For (a) one sees that $\operatorname{Pk}\tau = \operatorname{Pk} M[\tau] + 1$ since $\tau_1 = \max\tau$ is not a peak in τ and is removed in $M[\tau]$. But this still implies that $\operatorname{Pk}\tau = \operatorname{Pk}\tau'$ and the rest of this part of the demonstration goes through.

For (b), the map θ is constructed in exactly the same way using $A = \operatorname{cPk}\pi = \operatorname{cPk}\pi'$. The only difference with the remaining part of the proof is that there are two subcases when $k-1 \notin A$ depending on whether $k-2 \in A$ or not. If $k-2 \in A$ then one loses the peak which was at π_{k-2} in $S_i[\pi]$. On the other hand, the peak set stays the same modulo rotation if $k-2 \notin A$. But since the analogous statements hold for π', the manipulations in these subcases are as before. □

The proof of the next result is based on the proof of Theorem 3.2.3 in much the same way that the demonstrations of Theorems 3.2.1 and 3.2.2 are related. So the details are left to the reader.

Theorem 3.2.4. *The cyclic statistic* cpk *is cyclic shuffle-compatible.* □

Lest it appear that cyclic shuffle-compatibility follows exactly the same lines as the linear case, let us point out a place where they differ.

Definition 3.2.5. A *birun* of π is a maximal monotone factor (subsequence of consecutive elements). Let $\operatorname{bru}\pi$ be the number of biruns of π.

For example, bru 125346 = 3 because of the biruns 125, 53, and 346. It is easy to see [GZ18] that the birun statistic is not linearly shuffle-compatible. A *cyclic birun* of $[\pi]$ is defined in the obvious manner and denoted $\operatorname{cbru}[\pi]$. Returning to our example, $\operatorname{cbru}[125346] = 4$ because of the biruns above and 61.

Theorem 3.2.6. *The cyclic statistic* cbru *is cyclic shuffle-compatible.*

Proof. Clearly cyclic biruns always begin at a cyclic peak and end at a cyclic valley, or vice-versa. So $\text{cbru}[\pi] = 2\,\text{cpk}[\pi]$ and the result follows from Theorem 3.2.4. $\square$

CHAPTER 4

CYCLIC SHUFFLE-COMPATIBILITY: ALGEBRAICALLY

In Section 4.1, we review Gessel and Zhuang's definition of the shuffle algebra of a shuffle-compatible permutation statistic, and then we define the cyclic shuffle algebra of a cyclic shuffle-compatible statistic. We prove several general results about cyclic shuffle-compatibility via cyclic shuffle algebras, including a result (Theorem 4.1.13) allowing one to construct cyclic shuffle algebras from linear ones.

In Section 4.2, we review the role of quasisymmetric functions in the theory of (linear) shuffle-compatibility, and then we develop an analogous theory concerning cyclic quasisymmetric functions and cyclic shuffle-compatibility. We use Theorem 4.1.13 to construct the non-Escher subalgebra cQSym^{-} of cyclic quasisymmetric functions from the algebra QSym of quasisymmetric functions, which gives another proof that cDes is cyclic shuffle-compatible and shows that the cyclic shuffle algebra of cDes is isomorphic to cQSym^{-}. We then give a necessary and sufficient condition for cyclic shuffle-compatibility of cyclic descent statistics which implies that the cyclic shuffle algebra of any cyclic shuffle-compatible cyclic descent statistic is isomorphic to a quotient algebra of cQSym^{-}.

In Section 4.3, we use the theory developed in Section 4.2 to give explicit descriptions of the cyclic shuffle algebras of the statistics cPk, cpk cdes, and (cpk, cdes) which in turn yields algebraic proofs for their cyclic shuffle-compatibility.

In Section 4.4, we define a family of multiset-valued cyclic statistics induced from linear statistics, and investigate cyclic shuffle-compatibility for some of these statistics. This approach yields a definition of a cyclic major index which is different from the one proposed earlier by Ji and Zhang [JZ22]; unfortunately, neither of these cyclic major index statistics are cyclic shuffle-compatible.

4.1 Cyclic Shuffle Algebras

At the heart of Gessel and Zhuang's algebraic framework for shuffle-compatibility is the notion of a shuffle algebra. In this section, we review the definition of the shuffle algebra of a shuffle-compatible (linear) permutation statistic, define a cyclic analogue of shuffle algebras for cyclic shuffle-compatible statistics, and prove several general results about cyclic shuffle-compatibility through cyclic shuffle algebras, including one that can be used to construct cyclic shuffle algebras from shuffle algebras of linear permutation statistics.

4.1.1 Definitions

Definition 4.1.1 (st-equivalent)**.** Let st be a linear permutation statistic. We say that π and σ are st-*equivalent* if $\operatorname{st}\pi = \operatorname{st}\sigma$ and $|\pi| = |\sigma|$.

In this way, every permutation statistic induces an equivalence relation on permutations, and we write the st-equivalence class of π as π_{st}. Let $\mathcal{A}_{\text{st}}$ denote the $\mathbb{Q}$-vector space consisting of formal linear combinations of st-equivalence classes of permutations.

If st is shuffle-compatible, then we can turn $\mathcal{A}_{\text{st}}$ into a $\mathbb{Q}$-algebra by endowing it with the multiplication

$$\pi_{\text{st}}\sigma_{\text{st}} = \sum_{\tau \in \pi \sqcup\!\sqcup \sigma} \tau_{\text{st}}$$

for any disjoint representatives $\pi \in \pi_{\text{st}}$ and $\sigma \in \sigma_{\text{st}}$; this multiplication is well-defined (i.e., the choice of π and σ does not matter) precisely when st is shuffle compatible. The $\mathbb{Q}$-algebra $\mathcal{A}_{\text{st}}$ is called the (*linear*) *shuffle algebra* of st. Observe that $\mathcal{A}_{\text{st}}$ is graded by length, that is, π_{st} belongs to the nth homogeneous component of $\mathcal{A}_{\text{st}}$ if π has length n.

Our definition of cyclic shuffle algebras will be analogous to that of linear ones.

Definition 4.1.2 (cst-equivalent)**.** Let cst be a cyclic permutation statistic. Then the cyclic permutations $[\pi]$ and $[\sigma]$ are called cst-*equivalent* if $\operatorname{cst}[\pi] = \operatorname{cst}[\sigma]$ and $|\pi| = |\sigma|$, and we use the notation $[\pi]_{\text{cst}}$ to denote the cst-equivalence class of the cyclic permutation $[\pi]$.

We associate to cst a $\mathbb{Q}$-vector space $\mathcal{A}^{\text{cyc}}_{\text{cst}}$ by taking as a basis the set of all cst-equivalence classes of permutations, and then we give this vector space a multiplication by defining

$$[\pi]_{\text{cst}}[\sigma]_{\text{cst}} = \sum_{[\tau] \in [\pi] \sqcup\!\sqcup [\sigma]} [\tau]_{\text{cst}}$$

for any disjoint π and σ with $[\pi] \in [\pi]_{\mathrm{cst}}$ and $[\sigma] \in [\sigma]_{\mathrm{cst}}$; this multiplication is well-defined if and only if cst is cyclic shuffle-compatible. The resulting $\mathbb{Q}$-algebra $\mathcal{A}_{\mathrm{cst}}^{\mathrm{cyc}}$ is called the *cyclic shuffle algebra* of cst, and is also graded by length.

4.1.2 Two General Results on Cyclic Shuffle Algebras

We now give two general results on cyclic shuffle algebras, which are analogous to Theorems 3.2 and 3.3 of [GZ18] on linear shuffle algebras. We provide proofs for completeness, although they follow in essentially the same way as the proofs of the corresponding results in [GZ18].

Theorem 4.1.3. *Suppose that* cst_1 *is cyclic shuffle-compatible and is a refinement of* cst_2*. Let A be a* $\mathbb{Q}$*-algebra with basis* $\{v_\alpha\}$ *indexed by* cst_2*-equivalence classes* α*, and suppose that there exists a* $\mathbb{Q}$*-algebra homomorphism* $\phi\colon \mathcal{A}_{\mathrm{cst}_1}^{\mathrm{cyc}} \to A$ *such that for every* cst_1*-equivalence class* β*, we have* $\phi(\beta) = v_\alpha$ *where* α *is the* cst_2*-equivalence class containing* β*. Then* cst_2 *is cyclic shuffle-compatible and the map* $v_\alpha \mapsto \alpha$ *extends by linearity to an isomorphism from A to* $\mathcal{A}_{\mathrm{cst}_2}^{\mathrm{cyc}}$*.*

Proof. It suffices to show that for any disjoint π and σ, we have

$$v_{[\pi]_{\mathrm{cst}_2}} v_{[\sigma]_{\mathrm{cst}_2}} = \sum_{[\tau] \in [\pi] \sqcup\!\sqcup [\sigma]} v_{[\tau]_{\mathrm{cst}_2}}.$$

To that end, we have

$$\begin{aligned} v_{[\pi]_{\mathrm{cst}_2}} v_{[\sigma]_{\mathrm{cst}_2}} &= \phi([\pi]_{\mathrm{cst}_1})\phi([\sigma]_{\mathrm{cst}_1}) \\ &= \phi([\pi]_{\mathrm{cst}_1}[\sigma]_{\mathrm{cst}_1}) \\ &= \phi\Bigg(\sum_{[\tau] \in [\pi] \sqcup\!\sqcup [\sigma]} [\tau]_{\mathrm{cst}_1}\Bigg) \\ &= \sum_{[\tau] \in [\pi] \sqcup\!\sqcup [\sigma]} v_{[\tau]_{\mathrm{cst}_2}}, \end{aligned}$$

which completes the proof. □

We say that cst_1 and cst_2 are *equivalent* if cst_1 is a simultaneously a refinement and a coarsening of cst_2, that is, if for all cyclic permutations $[\pi]$ and $[\sigma]$ of the same length, $\mathrm{cst}_1[\pi] = \mathrm{cst}_1[\sigma]$ implies $\mathrm{cst}_2[\pi] = \mathrm{cst}_2[\sigma]$ and vice versa.

Theorem 4.1.4. *Let* cst_1 *and* cst_2 *be equivalent statistics. If* cst_1 *is shuffle-compatible with cyclic shuffle algebra* $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cst}_1}$*, then* cst_2 *is also cyclic shuffle-compatible with cyclic shuffle algebra* $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cst}_2}$ *isomorphic to* $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cst}_1}$.

Proof. Because equivalent statistics have the same equivalence classes on cyclic permutations, we know that $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cst}_1}$ and $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cst}_2}$ have the same basis elements. Since cst_1 and cst_2 are equivalent, we have

$$[\pi]_{\mathrm{st}_2}[\sigma]_{\mathrm{st}_2} = [\pi]_{\mathrm{st}_1}[\sigma]_{\mathrm{st}_1} = \sum_{[\tau]\in[\pi]\sqcup\!\sqcup[\sigma]} [\tau]_{\mathrm{st}_1} = \sum_{[\tau]\in[\pi]\sqcup\!\sqcup[\sigma]} [\tau]_{\mathrm{st}_2},$$

which proves the result. □

4.1.3 Symmetries and Cyclic Shuffle Algebras

Many permutation statistics—both linear and cyclic—are related via various symmetries, such as reversal, complementation, and reverse-complementation. For a linear permutation $\pi = \pi_1\pi_2\cdots\pi_n \in \mathfrak{S}_n$, we define the *reversal* π^r of π by $\pi^r := \pi_n\pi_{n-1}\cdots\pi_1$, the *complement* π^c of π to be the permutation obtained by (simultaneously) replacing the ith smallest letter in π with the ith largest letter in π for all $1 \le i \le n$, and the *reverse-complement* π^{rc} of π by $\pi^{rc} := (\pi^r)^c = (\pi^c)^r$. For example, given $\pi = 318269$, we have $\pi^r = 962813$, $\pi^c = 692831$, and $\pi^{rc} = 138296$.

More generally, let f be an involution on linear permutations which preserves the length, i.e., $|f(\pi)| = |\pi|$ for all π. We shall write π^f in place of $f(\pi)$. For a set Π of permutations, let

$$\Pi^f := \{\, \pi^f : \pi \in \Pi \,\},$$

so f induces an involution on sets of permutations as well. In particular, this lets us define $[\pi]^f$ for a cyclic permutation $[\pi]$. Going further, if C is a set of cyclic permutations, then

$$C^f := \{\, [\pi]^f : [\pi] \in C \,\}.$$

Following Gessel and Zhuang [GZ18],

Definition 4.1.5 (Shuffle-compatibility-preserving)**.** We say that f is *shuffle-compatibility-preserving* if for any pair of disjoint permutations π and σ, there exist disjoint permutations $\hat{\pi}$ and $\hat{\sigma}$ with the same relative order as π and σ, respectively, such that $(\pi\sqcup\!\sqcup\sigma)^f = \hat{\pi}^f\sqcup\!\sqcup\hat{\sigma}^f$ and $(\hat{\pi}\sqcup\!\sqcup\hat{\sigma})^f = \pi^f\sqcup\!\sqcup\sigma^f$.

This definition implies that π^f and σ^f are disjoint, and similarly with $\hat{\pi}^f$ and $\hat{\sigma}^f$.

Definition 4.1.6 (f-equivalent)**.** Two linear permutation statistics st_1 and st_2 are called *f-equivalent* if $\mathrm{st}_1 \circ f$ is equivalent to st_2—that is, $\mathrm{st}_1\,\pi^f = \mathrm{st}_1\,\sigma^f$ if and only if $\mathrm{st}_2\,\pi = \mathrm{st}_2\,\sigma$. In other words, st_1 and st_2 are f-equivalent if and only if $(\pi^f)_{\mathrm{st}_1} = (\pi_{\mathrm{st}_2})^f$ for all π.

It is easy to see that, if $\mathrm{st}_1\,\pi^f = \mathrm{st}_2\,\pi$ for all π, then st_1 and st_2 are f-equivalent (although this is not a necessary condition).

Example 4.1.7. For example, the peak set Pk is c-equivalent to the *valley set* Val defined in the following way. We call $i \in \{2, 3, \ldots, n-1\}$ a *valley* of $\pi \in \mathfrak{S}_n$ if $\pi_{i-1} > \pi_i < \pi_{i+1}$, and we let $\mathrm{Val}\,\pi$ be the set of valleys of π. We also define $\mathrm{val}\,\pi$ to be the number of valleys of π; then, pk and val are c-equivalent as well.

Despite its name, f-equivalence is not an equivalence relation (although it is symmetric). However, it turns out that if the statistics involved are shuffle-compatible, then f-equivalences induce isomorphisms on the corresponding shuffle algebras. This idea is expressed in the following theorem of Gessel and Zhuang.

Theorem 4.1.8 ([GZ18] Theorem 3.5)**.** *Let f be shuffle-compatibility-preserving, and suppose that st_1 and st_2 are f-equivalent (linear) permutation statistics. If st_1 is shuffle-compatible with shuffle algebra $\mathcal{A}_{\mathrm{st}_1}$, then st_2 is also shuffle-compatible, and the linear map defined by $\pi_{\mathrm{st}_1} \mapsto \pi^f_{\mathrm{st}_2}$ is a $\mathbb{Q}$-algebra isomorphism between their shuffle algebras $\mathcal{A}_{\mathrm{st}_1}$ and $\mathcal{A}_{\mathrm{st}_2}$.*

Gessel and Zhuang proved that reversal, complementation, and reverse-complementation are all shuffle-compatibility-preserving. Thus, they were able to use Theorem 4.1.8 to prove a collection of shuffle-compatibility results for statistics that are r-, c-, or rc-equivalent to another statistic whose shuffle-compatibility had already been established. For example, it follows from the shuffle-compatibility of the peak set Pk that the valley set Val is shuffle-compatible with shuffle algebra $\mathcal{A}_{\mathrm{Val}}$ isomorphic to $\mathcal{A}_{\mathrm{Pk}}$.

Moving onto the cyclic setting, we call f *rotation-preserving* if $[\pi]^f = [\pi^f]$ for all π.

Lemma 4.1.9. *If f is shuffle-compatibility-preserving and rotation-preserving, then for any pair of disjoint permutations π and σ, there exist disjoint permutations $\hat{\pi}$ and $\hat{\sigma}$ with the same relative order as π and σ, respectively, for which $([\pi] \sqcup [\sigma])^f = [\hat{\pi}^f] \sqcup [\hat{\sigma}^f]$ and $([\hat{\pi}] \sqcup [\hat{\sigma}])^f = [\pi^f] \sqcup [\sigma^f]$.*

Proof. Let $[\tau] \in [\pi] \sqcup [\sigma]$, so that $\tau \in \bar{\pi} \sqcup \bar{\sigma}$ for some $\bar{\pi} \in [\pi]$ and $\bar{\sigma} \in [\sigma]$, and thus $\tau^f \in (\bar{\pi} \sqcup \bar{\sigma})^f$. Since f is shuffle-compatibility-preserving, we have that $\tau^f \in \hat{\bar{\pi}}^f \sqcup \hat{\bar{\sigma}}^f$ where $\hat{\bar{\pi}}$ and $\hat{\bar{\sigma}}$ are disjoint permutations with the same relative order as $\bar{\pi}$ and $\bar{\sigma}$, respectively. Since $\bar{\pi}$ is a rotation of π and $\hat{\bar{\pi}}$ has the same relative order as $\bar{\pi}$, it follows that $\hat{\bar{\pi}}$ is a rotation of a permutation $\hat{\pi}$ with the same relative order as π, and similarly $\hat{\bar{\sigma}}$ is a rotation of a permutation $\hat{\sigma}$ with the same relative order as σ. Clearly, $\hat{\pi}$ and $\hat{\sigma}$ are disjoint because $\hat{\bar{\pi}}$ and $\hat{\bar{\sigma}}$ are disjoint. Because f is rotation-preserving, $\hat{\bar{\pi}} \in [\hat{\pi}]$ and $\hat{\bar{\sigma}} \in [\hat{\sigma}]$ imply $\hat{\bar{\pi}}^f \in [\hat{\pi}^f]$ and $\hat{\bar{\sigma}} \in [\hat{\sigma}^f]$. Therefore, $\tau^f \in \hat{\bar{\pi}}^f \sqcup \hat{\bar{\sigma}}^f$ implies $[\tau]^f = [\tau^f] \in [\hat{\pi}^f] \sqcup [\hat{\sigma}^f]$.

We have shown that $([\pi] \sqcup [\sigma])^f$ is a subset of $[\hat{\pi}^f] \sqcup [\hat{\sigma}^f]$, but since these two sets have the same cardinality, they are in fact equal. We omit the proof of $([\hat{\pi}] \sqcup [\hat{\sigma}])^f = [\pi^f] \sqcup [\sigma^f]$ as it is similar. □

Lemma 4.1.10. *Reversal, complementation, and reverse-complementation are all rotation-preserving.*

Proof. Let $\pi = \pi_1\pi_2 \cdots \pi_n$ be a (linear) permutation. We have

$$\begin{aligned}
[\pi]^r &= \{\pi_1\pi_2 \cdots \pi_n, \pi_n\pi_1 \cdots \pi_{n-1}, \ldots, \pi_2 \cdots \pi_n\pi_1\}^r \\
&= \{\pi_n \cdots \pi_2\pi_1, \pi_{n-1} \cdots \pi_1\pi_n, \ldots, \pi_1\pi_n \cdots \pi_2\} \\
&= [\pi^r],
\end{aligned}$$

so reversal is rotation-preserving. Moreover, it is clear that taking the complement of the permutation $\pi_{i+1} \cdots \pi_n\pi_1 \cdots \pi_i$ (obtained by rotating the last $n-i$ letters of π to the front) yields the same result as first taking the complement of π and then rotating the last $n-i$ letters of π^c to the front, so complementation is rotation-preserving. Lastly, since we have established that $[\pi^c] = [\pi]^c$ for all permutations π, we can replace π by π^r to obtain $[\pi^{rc}] = [\pi^r]^c = [\pi]^{rc}$, so reverse-complementation is rotation-preserving as well. □

In analogy with f-equivalence of linear permutation statistics, let us call two cyclic permutation statistics cst_1 and cst_2 *f-equivalent* if $\mathrm{cst}_1 \circ f$ is equivalent to cst_2, or equivalently, if $[\pi^f]_{\mathrm{cst}_1} = ([\pi]_{\mathrm{cst}_2})^f$. The following is a cyclic version of Theorem 4.1.8.

Theorem 4.1.11. *Let f be shuffle-compatibility-preserving and rotation-preserving, and let* cst_1 *and* cst_2 *be f-equivalent cyclic permutation statistics. If* cst_1 *is cyclic shuffle-compatible, then* cst_2 *is cyclic shuffle-compatible with $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cst}_2}$ isomorphic to $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cst}_1}$.*

Proof. Let $[\pi]$ and $[\tilde{\pi}]$ be cyclic permutations in the same cst_2-equivalence class, and similarly with $[\sigma]$ and $[\tilde{\sigma}]$, such that π and σ are disjoint and $\tilde{\pi}$ and $\tilde{\sigma}$ are disjoint. We know from Lemma 4.1.9 that there exist permutations $\hat{\pi}$, $\hat{\sigma}$, $\hat{\tilde{\pi}}$, and $\hat{\tilde{\sigma}}$—having the same relative order as π, σ, $\tilde{\pi}$, and $\tilde{\sigma}$, respectively—satisfying $([\pi] \sqcup [\sigma])^f = [\hat{\pi}^f] \sqcup [\hat{\sigma}^f]$, $([\hat{\pi}] \sqcup [\hat{\sigma}])^f = [\pi^f] \sqcup [\sigma^f]$, $([\tilde{\pi}] \sqcup [\tilde{\sigma}])^f = [\hat{\tilde{\pi}}^f] \sqcup [\hat{\tilde{\sigma}}^f]$, and $([\hat{\tilde{\pi}}] \sqcup [\hat{\tilde{\sigma}}])^f = [\tilde{\pi}^f] \sqcup [\tilde{\sigma}^f]$.

Because $\hat{\pi}$ and $\hat{\tilde{\pi}}$ have the same relative order as π and $\tilde{\pi}$, respectively, we have

$$[\hat{\pi}]_{\mathrm{cst}_2} = [\pi]_{\mathrm{cst}_2} = [\tilde{\pi}]_{\mathrm{cst}_2} = [\hat{\tilde{\pi}}]_{\mathrm{cst}_2}.$$

Then, because cst_1 and cst_2 are f-equivalent, we have

$$[\hat{\pi}^f]_{\mathrm{cst}_1} = ([\hat{\pi}]_{\mathrm{cst}_2})^f = ([\hat{\tilde{\pi}}]_{\mathrm{cst}_2})^f = [\hat{\tilde{\pi}}^f]_{\mathrm{cst}_1},$$

so $[\hat{\pi}^f]$ and $[\hat{\tilde{\pi}}^f]$ are cst_1-equivalent. The same reasoning shows that $[\hat{\sigma}^f]$ and $[\hat{\tilde{\sigma}}^f]$ are also cst_1-equivalent.

By cyclic shuffle-compatibility of cst_1, we have the multiset equality

$$\{\{\, \mathrm{cst}_1[\tau] : [\tau] \in [\hat{\pi}^f] \sqcup [\hat{\sigma}^f] \,\}\} = \{\{\, \mathrm{cst}_1[\tau] : [\tau] \in [\hat{\tilde{\pi}}^f] \sqcup [\hat{\tilde{\sigma}}^f] \,\}\},$$

which—by f-equivalence of cst_1 and cst_2—is equivalent to

$$\{\{\, \mathrm{cst}_2[\tau^f] : [\tau] \in [\hat{\pi}^f] \sqcup [\hat{\sigma}^f] \,\}\} = \{\{\, \mathrm{cst}_2[\tau^f] : [\tau] \in [\hat{\tilde{\pi}}^f] \sqcup [\hat{\tilde{\sigma}}^f] \,\}\},$$

which is in turn equivalent to

$$\{\{\, \mathrm{cst}_2[\tau] : [\tau]^f \in [\hat{\pi}^f] \sqcup [\hat{\sigma}^f] \,\}\} = \{\{\, \mathrm{cst}_2[\tau] : [\tau]^f \in [\hat{\tilde{\pi}}^f] \sqcup [\hat{\tilde{\sigma}}^f] \,\}\}$$

because f is rotation-preserving. Since $([\pi] \sqcup [\sigma])^f = [\hat{\pi}^f] \sqcup [\hat{\sigma}^f]$ and $([\tilde{\pi}] \sqcup [\tilde{\sigma}])^f = [\hat{\tilde{\pi}}^f] \sqcup [\hat{\tilde{\sigma}}^f]$, we have

$$\{\{\,\mathrm{cst}_2[\tau] : [\tau] \in [\pi] \sqcup [\sigma]\,\}\} = \{\{\,\mathrm{cst}_2[\tau] : [\tau] \in [\tilde{\pi}] \sqcup [\tilde{\sigma}]\,\}\},$$

which shows that cst_2 is cyclic shuffle-compatible.

It remains to prove that $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cst}_2}$ is isomorphic to $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cst}_1}$. Define the linear map $\lambda \colon \mathcal{A}^{\mathrm{cyc}}_{\mathrm{cst}_2} \to \mathcal{A}^{\mathrm{cyc}}_{\mathrm{cst}_1}$ by $[\pi]_{\mathrm{cst}_2} \mapsto [\pi^f]_{\mathrm{cst}_1}$. Observe that

$$\sum_{[\tau]\in[\pi]\sqcup[\sigma]} [\tau]_{\mathrm{cst}_2} = \sum_{[\tau]\in[\hat{\pi}]\sqcup[\hat{\sigma}]} [\tau]_{\mathrm{cst}_2}$$

because cst_2 is cyclic shuffle-compatible, and thus we have

$$\begin{aligned}
\lambda([\pi]_{\mathrm{cst}_2}[\sigma]_{\mathrm{cst}_2}) &= \lambda\Big(\sum_{[\tau]\in[\pi]\sqcup[\sigma]} [\tau]_{\mathrm{cst}_2}\Big) \\
&= \lambda\Big(\sum_{[\tau]\in[\hat{\pi}]\sqcup[\hat{\sigma}]} [\tau]_{\mathrm{cst}_2}\Big) \\
&= \sum_{[\tau]\in[\hat{\pi}]\sqcup[\hat{\sigma}]} [\tau^f]_{\mathrm{cst}_1} \\
&= \sum_{[\tau]^f\in[\hat{\pi}]\sqcup[\hat{\sigma}]} [\tau]_{\mathrm{cst}_1} \\
&= \sum_{[\tau]\in[\pi^f]\sqcup[\sigma^f]} [\tau]_{\mathrm{cst}_1} \\
&= [\pi^f]_{\mathrm{cst}_1}[\sigma^f]_{\mathrm{cst}_1} \\
&= \lambda([\pi]_{\mathrm{cst}_2})\lambda([\sigma]_{\mathrm{cst}_2}).
\end{aligned}$$

Hence, λ is a $\mathbb{Q}$-algebra isomorphism from $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cst}_2}$ to $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cst}_1}$. □

Corollary 4.1.12. *Suppose that the cyclic permutation statistics* cst_1 *and* cst_2 *are* r*-equivalent,* c*-equivalent, or* rc*-equivalent. If* cst_1 *is cyclic shuffle-compatible, then* cst_2 *is cyclic shuffle-compatible with cyclic shuffle algebra* $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cst}_2}$ *isomorphic to* $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cst}_1}$.

4.1.4 Constructing Cyclic Shuffle Algebras from Linear Ones

The following theorem—one of the main results of this paper—allows us to construct cyclic shuffle algebras from shuffle algebras of shuffle-compatible (linear) permutation statistics.

Theorem 4.1.13. *Let* cst *be a cyclic permutation statistic and let* st *be a shuffle-compatible* (*linear*) *permutation statistic. Given a cyclic permutation* $[\pi]$*, let*

$$v_{[\pi]} = \sum_{\bar{\pi}\in[\pi]} \bar{\pi}_{\mathrm{st}} \in \mathcal{A}_{\mathrm{st}}.$$

Suppose that $v_{[\pi]} = v_{[\sigma]}$ *whenever* $[\pi]$ *and* $[\sigma]$ *are* cst*-equivalent, and that* $\{v_{[\pi]}\}$ (*ranging over all* cst*-equivalence classes*) *is linearly independent. Then* cst *is cyclic shuffle-compatible and the map* $\psi_{\mathrm{cst}}\colon \mathcal{A}^{\mathrm{cyc}}_{\mathrm{cst}} \to \mathcal{A}_{\mathrm{st}}$ *given by*

$$\psi_{\mathrm{cst}}([\pi]_{\mathrm{cst}}) = v_{[\pi]}$$

extends linearly to a $\mathbb{Q}$*-algebra isomorphism from* $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cst}}$ *to the span of* $\{v_{[\pi]}\}$*, a subalgebra of* $\mathcal{A}_{\mathrm{st}}$*.*

Proof. Since $v_{[\pi]} = v_{[\sigma]}$ whenever $[\pi]$ and $[\sigma]$ are cst-equivalent, we know that ψ_{cst} is a well-defined linear map on $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cst}}$. (We do not yet know whether $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cst}}$ is an algebra; here we are only considering $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cst}}$ as a vector space.) Furthermore, because $\{v_{[\pi]}\}$ is linearly independent, the linear map ψ_{cst} is a vector space isomorphism from $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cst}}$ to a subspace of $\mathcal{A}_{\mathrm{st}}$.

To show that cst is cyclic shuffle-compatible, we show that

$$[\pi]_{\mathrm{cst}}[\sigma]_{\mathrm{cst}} = \sum_{[\tau]\in[\pi]\sqcup[\sigma]} [\tau]_{\mathrm{cst}}$$

is a well-defined multiplication in $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cst}}$. Let $[\pi'], [\pi''] \in [\pi]_{\mathrm{cst}}$ and let $[\sigma'], [\sigma''] \in [\sigma]_{\mathrm{cst}}$, where π' and σ' are disjoint and so are π'' and σ''. Then

$$\begin{aligned}\psi_{\mathrm{cst}}\Bigg(\sum_{[\tau]\in[\pi']\sqcup[\sigma']} [\tau]_{\mathrm{cst}}\Bigg) &= \sum_{[\tau]\in[\pi']\sqcup[\sigma']} v_{[\tau]} \\ &= \sum_{[\tau]\in[\pi']\sqcup[\sigma']} \sum_{\bar{\tau}\in[\tau]} \bar{\tau}_{\mathrm{st}} \\ &= \sum_{\bar{\pi}\in[\pi']} \sum_{\bar{\sigma}\in[\sigma']} \sum_{\bar{\tau}\in\bar{\pi}\sqcup\bar{\sigma}} \bar{\tau}_{\mathrm{st}} \\ &= v_{[\pi']}v_{[\sigma']}\end{aligned}$$

and similarly

$$\psi_{\mathrm{cst}}\Bigg(\sum_{[\tau]\in[\pi'']\sqcup[\sigma'']} [\tau]_{\mathrm{cst}}\Bigg) = v_{[\pi'']}v_{[\sigma'']}.$$

Since $[\pi']$ and $[\pi'']$ are cst-equivalent and similarly with $[\sigma']$ and $[\sigma'']$, we have

$$\psi_{\text{cst}}\left(\sum_{[\tau]\in[\pi']\sqcup\!\sqcup[\sigma']}[\tau]_{\text{cst}}\right) = v_{[\pi']}v_{[\sigma']} = v_{[\pi'']}v_{[\sigma'']} = \psi_{\text{cst}}\left(\sum_{[\tau]\in[\pi'']\sqcup\!\sqcup[\sigma'']}[\tau]_{\text{cst}}\right)$$

and thus

$$\sum_{[\tau]\in[\pi']\sqcup\!\sqcup[\sigma']}[\tau]_{\text{cst}} = \sum_{[\tau]\in[\pi'']\sqcup\!\sqcup[\sigma'']}[\tau]_{\text{cst}}$$

due to injectivity of ψ_{cst}. We have shown that the multiplication of the cyclic shuffle algebra $\mathcal{A}^{\text{cyc}}_{\text{cst}}$ is well-defined, and therefore cst is shuffle-compatible.

Finally, we have

$$\begin{aligned}\psi_{\text{cst}}([\pi]_{\text{cst}}[\sigma]_{\text{cst}}) &= \psi_{\text{cst}}\left(\sum_{[\tau]\in[\pi]\sqcup\!\sqcup[\sigma]}[\tau]_{\text{cst}}\right)\\ &= v_{[\pi]}v_{[\sigma]}\\ &= \psi_{\text{cst}}([\pi]_{\text{cst}})\psi_{\text{cst}}([\sigma]_{\text{cst}}),\end{aligned}$$

so ψ_{cst} is a $\mathbb{Q}$-algebra isomorphism from $\mathcal{A}^{\text{cyc}}_{\text{cst}}$ to the span of $\{v_{[\pi]}\}$. □

4.2 Shuffle-compatibility and Quasisymmetric Functions

The focus of this section is the relationship between cyclic shuffle-compatibility and cyclic quasisymmetric functions.

4.2.1 Backgrounds

First we recall that in the linear case, the product of two quasisymmetric functions is again quasisymmetric. The multiplication rule for the fundamental basis is given by the following theorem, which can be proved using P-partitions; see [Sta24, Exercise 7.93].

Theorem 4.2.1. *Let m and n be non-negative integers, and let $A \subseteq [m-1]$ and $B \subseteq [n-1]$. Then*

$$F_{m,A}F_{n,B} = \sum_{\tau\in\pi\sqcup\!\sqcup\sigma} F_{m+n,\operatorname{Des}\tau}$$

where π is any permutation of length m with descent set A and σ is any permutation (disjoint from π) of length n with descent set B.

It follows directly from this theorem that the descent set shuffle algebra $\mathcal{A}_{\mathrm{Des}}$ is isomorphic to QSym; this is of [GZ18, Corollary 4.2].

Recall the definition of non-Escher sets from Definition 2.1.4. Let cQSym^- denote the span of $\{F^{\mathrm{cyc}}_{n,[S]}\}$ over all $n \geq 0$ and all equivalence classes $[S]$ of non-Escher subsets $S \subseteq [n]$. The following theorem, proven by Adin et al. [AGRR21, Theorem 3.22], gives a multiplication rule for the fundamental cyclic quasisymmetric functions in cQSym^-, which also implies that the cyclic descent set cDes is cyclic shuffle-compatible and has cyclic shuffle algebra isomorphic to cQSym^-.

Theorem 4.2.2. *Let m and n be non-negative integers, and let $A \subseteq [m]$ and $B \subseteq [n]$ be non-Escher subsets. Then*

$$F^{\mathrm{cyc}}_{m,[A]} F^{\mathrm{cyc}}_{n,[B]} = \sum_{[\tau] \in [\pi] \sqcup [\sigma]} F^{\mathrm{cyc}}_{m+n,\mathrm{cDes}[\tau]} \tag{4.2.1}$$

where $[\pi]$ is any cyclic permutation of length m with cyclic descent set $[A]$ and $[\sigma]$ is any cyclic permutation (with σ disjoint from π) of length n with cyclic descent set $[B]$.

Adin et al. proved Theorem 4.2.2 using toric $[\vec{D}]$-partitions; we now supply an alternative proof using Theorem 4.1.13.

Proof. We know that the descent set Des is shuffle-compatible and its shuffle algebra $\mathcal{A}_{\mathrm{Des}}$ is isomorphic to the algebra of quasisymmetric functions, QSym, through the isomorphism $\phi_{\mathrm{Des}}(\pi_{\mathrm{Des}}) = F_{|\pi|,\mathrm{Des}(\pi)}$. Then, using the notation of Theorem 4.1.13, we have

$$\phi_{\mathrm{Des}}(\nu_{[\pi]}) = \phi_{\mathrm{Des}}\Big(\sum_{\bar{\pi} \in [\pi]} \bar{\pi}_{\mathrm{Des}} \Big) = \sum_{i \in [n]} F_{n,(\mathrm{cDes}\,\pi + i) \cap [n-1]} = F^{\mathrm{cyc}}_{n,\mathrm{cDes}[\pi]}$$

where $n = |\pi|$. If $[\pi]$ and $[\sigma]$ are cDes-equivalent, then both $\phi_{\mathrm{Des}}(\nu_{[\pi]})$ and $\phi_{\mathrm{Des}}(\nu_{[\sigma]})$ are equal to $F^{\mathrm{cyc}}_{n,[S]}$ where $n = |\pi| = |\sigma|$ and $[S] = \mathrm{cDes}[\pi] = \mathrm{cDes}[\sigma]$, so $\nu_{[\pi]} = \nu_{[\sigma]}$. The linear independence of the $F^{\mathrm{cyc}}_{n,[S]}$ can be established by showing that the monomial cyclic quasisymmetric functions are linearly independent and expressing each $F^{\mathrm{cyc}}_{n,[S]}$ in terms of monomial cyclic quasisymmetric functions; see [AGRR21, Section 2] for details. Theorem 4.1.13 implies that cDes is cyclic shuffle-compatible and that $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cDes}}$ is isomorphic to cQSym^- via the isomorphism $[\pi]_{\mathrm{cDes}} \mapsto F^{\mathrm{cyc}}_{|\pi|,\mathrm{cDes}[\pi]}$, from which the multiplication rule (4.2.1) follows. □

As a direct consequence of Theorem 4.2.2, we have that cQSym^- is a graded $\mathbb{Q}$-subalgebra of QSym. Adin et al. also show that the span of

$$\{F_{0,\emptyset}^{\mathrm{cyc}}, F_{1,\emptyset}^{\mathrm{cyc}}, F_{1,\{1\}}^{\mathrm{cyc}}\} \cup \{F_{n,[S]}^{\mathrm{cyc}}\}_{n\geq 2,\, \emptyset\neq S\subsetneq[n]},$$

denoted cQSym, is a graded $\mathbb{Q}$-subalgebra of QSym, although this result is less relevant to cyclic shuffle-compatibility. Thus we have the subalgebra relations

$$\mathrm{cQSym}^- \subseteq \mathrm{cQSym} \subseteq \mathrm{QSym},$$

and cQSym^- is called the *non-Escher subalgebra* of cQSym.

Before moving on, let us explicitly state the cyclic shuffle-compatibility of cDes as a corollary of the preceding theorem.

Corollary 4.2.3 (Cyclic shuffle-compatibility of cDes)**.** *The cyclic descent set* cDes *is cyclic shuffle-compatible, and the linear map on* $\mathcal{A}_{\mathrm{cDes}}^{\mathrm{cyc}}$ *defined by* $[\pi]_{\mathrm{cDes}} \mapsto F_{|\pi|,\mathrm{cDes}[\pi]}^{\mathrm{cyc}}$ *is a* $\mathbb{Q}$*-algebra isomorphism from* $\mathcal{A}_{\mathrm{cDes}}^{\mathrm{cyc}}$ *to* cQSym^-.

4.2.2 A General Cyclic Shuffle-compatibility Criterion for Cyclic Descent Statistics

The theorem below is [GZ18, Theorem 4.3], which provides a necessary and sufficient condition for shuffle-compatibility of descent statistics in terms of quasisymmetric functions, and implies that the shuffle algebra of any shuffle-compatible descent statistic is a quotient algebra of QSym.

Theorem 4.2.4. *A descent statistic* st *is shuffle-compatible if and only if there exists a* $\mathbb{Q}$*-algebra homomorphism* $\phi_{\mathrm{st}}\colon \mathrm{QSym} \to A$*, where* A *is a* $\mathbb{Q}$*-algebra with basis* $\{u_\alpha\}$ *indexed by* st*-equivalence classes* α *of compositions, such that* $\phi_{\mathrm{st}}(F_L) = u_\alpha$ *whenever* $L \in \alpha$*. In this case, the linear map on* $\mathcal{A}_{\mathrm{st}}$ *defined by*

$$\pi_{\mathrm{st}} \mapsto u_\alpha,$$

where $\mathrm{Comp}\,\pi \in \alpha$*, is a* $\mathbb{Q}$*-algebra isomorphism from* $\mathcal{A}_{\mathrm{st}}$ *to* A*.*

We now prove our main result of this section: a cyclic analogue of Theorem 4.2.4.

Theorem 4.2.5. *A cyclic descent statistic* cst *is cyclic shuffle-compatible if and only if there exists a* $\mathbb{Q}$*-algebra homomorphism* $\phi_{\mathrm{cst}}\colon \mathrm{cQSym}^{-} \to A$*, where* A *is a* $\mathbb{Q}$*-algebra with basis* $\{v_\alpha\}$ *indexed by* cst*-equivalence classes* α *of non-Escher cyclic compositions, such that* $\phi_{\mathrm{cst}}(F^{\mathrm{cyc}}_{[L]}) = v_\alpha$ *whenever* $[L] \in \alpha$*. In this case, the linear map on* $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cst}}$ *defined by*

$$[\pi]_{\mathrm{cst}} \mapsto v_\alpha,$$

where $\mathrm{cComp}[\pi] \in \alpha$*, is a* $\mathbb{Q}$*-algebra isomorphism from* $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cst}}$ *to* A*.*

Proof. Suppose that the cyclic descent statistic cst is cyclic shuffle-compatible. Let $A = \mathcal{A}^{\mathrm{cyc}}_{\mathrm{cst}}$ be the cyclic shuffle algebra of cst, and let $v_\alpha = [\pi]_{\mathrm{cst}}$ for any $[\pi]$ satisfying $\mathrm{cComp}[\pi] \in \alpha$, so that

$$v_\beta v_\gamma = \sum_\alpha c^\alpha_{\beta,\gamma} v_\alpha$$

where $c^\alpha_{\beta,\gamma}$ is the number of cyclic permutations with cyclic descent composition in α that are obtained as a cyclic shuffle of two disjoint cyclic permutations, one with cyclic descent composition in β and the other with cyclic descent composition in γ. Observe that $c^\alpha_{\beta,\gamma} = \sum_{[L]\in\alpha} c^L_{J,K}$ for any choice of $[J] \in \beta$ and $[K] \in \gamma$, where $c^L_{J,K}$ is the number of cyclic permutations with cyclic descent composition $[L]$ that are obtained as a cyclic shuffle of two disjoint cyclic permutations, one with cyclic descent composition $[J]$ and the other with cyclic descent composition $[K]$.

Define the linear map $\phi_{\mathrm{cst}}\colon \mathrm{cQSym}^{-} \to A$ by $\phi_{\mathrm{cst}}(F^{\mathrm{cyc}}_{[L]}) = v_\alpha$ for $[L] \in \alpha$. Then any $[J] \in \beta$ and $[K] \in \gamma$ satisfy

$$\begin{aligned}
\phi_{\mathrm{cst}}(F^{\mathrm{cyc}}_{[J]} F^{\mathrm{cyc}}_{[K]}) &= \phi_{\mathrm{cst}}\Big(\sum_{[L]} c^L_{J,K} F^{\mathrm{cyc}}_{[L]}\Big) \\
&= \sum_\alpha \sum_{[L]\in\alpha} c^L_{J,K} v_\alpha \\
&= \sum_\alpha c^\alpha_{\beta,\gamma} v_\alpha \\
&= v_\beta v_\gamma \\
&= \phi_{\mathrm{cst}}(F^{\mathrm{cyc}}_{[J]})\phi_{\mathrm{cst}}(F^{\mathrm{cyc}}_{[K]}),
\end{aligned}$$

so ϕ_{cst} is a $\mathbb{Q}$-algebra homomorphism, thus completing one direction of the proof.

The converse follows from Theorem 4.1.3, where we take cst_1 to be cDes (which is cyclic shuffle-compatible by Corollary 4.2.3) and cst_2 to be cst. □

Corollary 4.2.6. *If* cst *is a cyclic shuffle-compatible descent statistic, then* $\mathcal{A}_{\mathrm{cst}}^{\mathrm{cyc}}$ *is isomorphic to a quotient algebra of* cQSym^{-}.

To conclude this section, we state a special case of Theorem 4.2.5 in which the homomorphism ϕ_{cst} is given in terms of the homomorphism ϕ_{st} of a related (linear) descent statistic; c.f. Theorem 4.1.13. We will use this theorem to prove cyclic shuffle-compatibility results for cyclic analogues of shuffle-compatible descent statistics.

Theorem 4.2.7. *Let* cst *be a cyclic descent statistic and let* st *be a shuffle-compatible* (*linear*) *descent statistic, so that there exists a* $\mathbb{Q}$*-algebra homomorphism* $\phi_{\mathrm{st}}\colon \mathrm{QSym} \to A$ *satisfying the conditions in Theorem 4.2.4. Define the* $\mathbb{Q}$*-algebra homomorphism* $\phi_{\mathrm{cst}}\colon \mathrm{cQSym}^{-} \to A$ *by*

$$\phi_{\mathrm{cst}}(F_{n,S}^{\mathrm{cyc}}) = \sum_{i\in[n]} \phi_{\mathrm{st}}(F_{n,S+i}).$$

Suppose that $\phi_{\mathrm{cst}}(F_{n,S}^{\mathrm{cyc}}) = \phi_{\mathrm{cst}}(F_{n,T}^{\mathrm{cyc}})$ *whenever* $\mathrm{cComp}[S]$ *and* $\mathrm{cComp}[T]$ *are* cst*-equivalent cyclic compositions—so that we can write* $\phi_{\mathrm{cst}}(F_{n,S}^{\mathrm{cyc}}) = v_{\alpha}$ *whenever* $\mathrm{cComp}[S] \in \alpha$*—and suppose that* $\{v_{\alpha}\}$ *is linearly independent. Then* cst *is cyclic shuffle-compatible and the linear map on* $\mathcal{A}_{\mathrm{cst}}^{\mathrm{cyc}}$ *defined by*

$$[\pi]_{\mathrm{cst}} \mapsto v_{\alpha},$$

where $\mathrm{cComp}[\pi] \in \alpha$*, is a* $\mathbb{Q}$*-algebra isomorphism from* $\mathcal{A}_{\mathrm{cst}}^{\mathrm{cyc}}$ *to the span of* $\{v_{\alpha}\}$*, a subalgebra of* A.

4.3 Characterizations of Cyclic Shuffle Algebras

Our next goal is to use the theory developed in the previous section to give explicit descriptions of cyclic shuffle algebras. First, we will characterize the cyclic shuffle algebras of cPk, (cpk, cdes), cpk, and cdes. This yields new proofs for the cyclic shuffle-compatibility of the statistics cPk, cpk, and cdes, as well as the first proof for (cpk, cdes).

4.3.1 The Cyclic Descent Number and Cyclic Peak Number

When we characterize the $(\mathrm{cpk}, \mathrm{cdes})$ cyclic shuffle algebra, we shall need to determine all values that the $(\mathrm{cpk}, \mathrm{cdes})$ statistic can take, which we can do with the help of two lemmas. The first of these lemmas is Proposition 2.5 of [GZ18], so we omit its proof.

Lemma 4.3.1. *Let* $n \geq 1$.

(a) *If* $\pi \in \mathfrak{S}_n$*, then* $0 \leq \operatorname{pk}\pi \leq \lfloor (n-1)/2 \rfloor$ *and* $\operatorname{pk}\pi \leq \operatorname{des}\pi \leq n - \operatorname{pk}\pi - 1$.

(b) *If* j *and* k *are integers satisfying* $0 \leq j \leq \lfloor (n-1)/2 \rfloor$ *and* $j \leq k \leq n - j - 1$*, then there exists* $\pi \in \mathfrak{S}_n$ *with* $\operatorname{pk}\pi = j$ *and* $\operatorname{des}\pi = k$.

Lemma 4.3.2. *Let* $n \geq 2$*. If* $\pi \in \mathfrak{S}_{n-1}$ *and* m *is greater than the largest letter of* π*, then* $\operatorname{cpk}[\pi m] = \operatorname{pk}\pi + 1$ *and* $\operatorname{cdes}[\pi m] = \operatorname{des}\pi + 1$*, where* πm *is the permutation in* $\mathfrak{S}_n$ *obtained by appending the letter* m *to* π.

Proof. Every peak of π is a cyclic peak of πm, and every cyclic peak of πm is either m or a peak of π. The same relationship is true for descents of π and cyclic descents of πm. □

Corollary 4.3.3. *Let* $n \geq 2$.

(a) *If* $\pi \in \mathfrak{S}_n$*, then* $1 \leq \operatorname{cpk}\pi \leq \lfloor n/2 \rfloor$ *and* $\operatorname{cpk}\pi \leq \operatorname{cdes}\pi \leq n - \operatorname{cpk}\pi$.

(b) *If* j *and* k *are integers satisfying* $1 \leq j \leq \lfloor n/2 \rfloor$ *and* $j \leq k \leq n - j$*, then there exists* $\pi \in \mathfrak{S}_n$ *with* $\operatorname{cpk}\pi = j$ *and* $\operatorname{cdes}\pi = k$.

Proof. Fix $\pi \in \mathfrak{S}_n$. Let m be the largest letter of π, let $\bar{\pi}$ be the unique representative of $[\pi]$ which ends with m, and let π' be the permutation of length $n-1$ obtained from $\bar{\pi}$ upon removing its last letter m. Applying Lemma 4.3.2, we obtain

$$\operatorname{cpk}\pi = \operatorname{cpk}[\bar{\pi}] = \operatorname{pk}\pi' + 1 \qquad \text{and} \qquad \operatorname{cdes}\pi = \operatorname{cdes}[\bar{\pi}] = \operatorname{des}\pi' + 1.$$

Then part (a) follows from these equations and Lemma 4.3.1 (a).

To prove part (b), let j and k be integers in the specified ranges. By Lemma 4.3.1 (b), we know there exists a permutation $\pi' \in \mathfrak{S}_{n-1}$ with $\operatorname{pk}\pi' = j-1$ and $\operatorname{des}\pi' = k-1$. Let $m \in \mathbb{P}$ be greater than the largest letter of π'; then it follows from Lemma 4.3.2 that πm is a permutation in $\mathfrak{S}_n$ satisfying $\operatorname{cpk}\pi = j$ and $\operatorname{cdes}\pi = k$. □

4.3.2 The Cyclic Shuffle Algebra of cPk

We will construct the cyclic shuffle algebra $\mathcal{A}^{\text{cyc}}_{\text{cPk}}$ from the linear shuffle algebra $\mathcal{A}_{\text{Pk}}$. The latter is known to be isomorphic to a subalgebra Π of QSym—introduced by Stembridge [Ste97]—called the *algebra of peaks*, which is spanned by the *peak quasisymmetric functions* $K_{n,S}$ where n ranges over all non-negative integers and S over all possible peak sets of permutations in $\mathfrak{S}_n$. We won't need the precise definition of $K_{n,S}$ here, only that the isomorphism from $\mathcal{A}_{\text{Pk}}$ to Π sends π_{Pk} to $K_{|\pi|,\operatorname{Pk}\pi}$. We state this fact in the following theorem, which appears as Theorem 4.7 of [GZ18]. (For a detailed description of the algebra of peaks, see Section 5.1.)

Theorem 4.3.4 (Shuffle-compatibility of Pk)**.** *The peak set* Pk *is shuffle-compatible, and the linear map on $\mathcal{A}_{\text{Pk}}$ defined by $\pi_{\text{Pk}} \mapsto K_{|\pi|,\operatorname{Pk}\pi}$ is a $\mathbb{Q}$-algebra isomorphism from $\mathcal{A}_{\text{Pk}}$ to* Π.

The analogue of Stembridge's quasisymmetric peak functions in the cyclic setting are the *cyclic peak quasisymmetric functions* $K^{\text{cyc}}_{n,S}$ which will be discussed in Section 5.2.4. Here, we shall define the cyclic peak functions $K^{\text{cyc}}_{n,S}$ in terms of the $K_{n,S}$. For brevity, let us say that S is a *cyclic peak set* of $[n]$ if S is the cyclic peak set of some permutation of length n. Then, if S is a cyclic peak set of $[n]$, let

$$K^{\text{cyc}}_{n,S} \coloneqq \sum_{i\in[n]} K_{n,(S+i)\setminus\{1,n\}} = \sum_{\bar{\pi}\in[\pi]} K_{n,\operatorname{Pk}\bar{\pi}}$$

where π is any permutation in $\mathfrak{S}_n$ with cyclic peak set S. We can also write $K^{\text{cyc}}_{n,[S]} \coloneqq K^{\text{cyc}}_{n,S}$ since the $K^{\text{cyc}}_{n,S}$ are invariant under cyclic shift.

The following theorem gives a multiplication rule for the $K^{\text{cyc}}_{n,[S]}$, which implies that cPk is cyclic shuffle-compatible. This provides another proof different from the bijective one in Theorem 3.2.3.

Theorem 4.3.5. *Let m and n be non-negative integers, let A be a cyclic peak set of $[m]$, and let B be a cyclic peak set of $[n]$. Then*

$$K^{\mathrm{cyc}}_{m,[A]} K^{\mathrm{cyc}}_{n,[B]} = \sum_{[\tau]\in[\pi]\sqcup[\sigma]} K^{\mathrm{cyc}}_{m+n,\mathrm{cPk}[\tau]} \tag{4.3.1}$$

where $[\pi]$ is any cyclic permutation of length m with cyclic peak set $[A]$ and $[\sigma]$ is any cyclic permutation (with σ disjoint from π) of length n with cyclic peak set $[B]$.

Proof. First, we take $\phi_{\mathrm{Pk}}\colon \mathrm{QSym} \to \Pi$ to be the composition of the map $F_L \mapsto \pi_{\mathrm{Pk}}$ with the map $\pi_{\mathrm{Pk}} \mapsto K_{|\pi|,\mathrm{Pk}\,\pi}$ from Theorem 4.3.4 where π is any permutation with $\mathrm{Pk}\,\pi = \mathrm{Pk}\,L$; then ϕ_{Pk} satisfies the conditions in Theorem 4.2.4.

Let S be a non-Escher subset of $[n]$, and let $[P]$ be the cyclic peak set of any cyclic permutation $[\pi]$ of length n with cyclic descent set $[S]$. Note that the sets $S+i$ where i ranges from 1 to n are precisely the descent sets of the n linear permutations in $[\pi]$. Hence, we have

$$\phi_{\mathrm{cPk}}(F^{\mathrm{cyc}}_{n,S}) = \sum_{i\in[n]} \phi_{\mathrm{Pk}}(F_{n,S+i}) = \sum_{\bar{\pi}\in[\pi]} \phi_{\mathrm{Pk}}(F_{n,\mathrm{Des}\,\bar{\pi}}) = \sum_{\bar{\pi}\in[\pi]} K_{n,\mathrm{Pk}\,\bar{\pi}} = K^{\mathrm{cyc}}_{n,[P]}.$$

Clearly, $\phi_{\mathrm{cPk}}(F^{\mathrm{cyc}}_{n,S})$ depends only on the cPk-equivalence class of the cyclic composition $\mathrm{cComp}[S]$, and we know that the $K^{\mathrm{cyc}}_{n,[P]}$ are linearly independent. Applying Theorem 4.2.7, we conclude that cPk is cyclic shuffle-compatible and that $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cPk}}$ is isomorphic to Λ via the isomorphism $[\pi]_{\mathrm{cPk}} \mapsto K^{\mathrm{cyc}}_{|\pi|,\mathrm{cPk}[\pi]}$, from which the multiplication rule (4.3.1) follows. □

Corollary 4.3.6 (Cyclic shuffle-compatibility of cPk)**.** *The cyclic peak set* cPk *is cyclic shuffle-compatible, and the linear map on $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cPk}}$ defined by $[\pi]_{\mathrm{cPk}} \mapsto K^{\mathrm{cyc}}_{|\pi|,\mathrm{cPk}[\pi]}$ is a $\mathbb{Q}$-algebra isomorphism from $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cPk}}$ to Λ.*

4.3.3 The Cyclic Shuffle Algebra of (cpk, cdes)

We will now use Theorem 4.2.7 to construct the cyclic shuffle algebra $\mathcal{A}^{\mathrm{cyc}}_{(\mathrm{cpk},\mathrm{cdes})}$ from the linear shuffle algebra $\mathcal{A}_{(\mathrm{pk},\mathrm{des})}$. We begin by recalling the following result about $\mathcal{A}_{(\mathrm{pk},\mathrm{des})}$, which is Theorem 5.9 of Gessel and Zhuang [GZ18]. Below, we will use the notation $\mathbb{Q}[[t*]]$ to denote the

$\mathbb{Q}$-algebra of formal power series in t where the multiplication is given by the *Hadamard product* $*$, defined by

$$\Big(\sum_{n=0}^{\infty} a_n t^n\Big) * \Big(\sum_{n=0}^{\infty} b_n t^n\Big) \coloneqq \sum_{n=0}^{\infty} a_n b_n t^n.$$

Theorem 4.3.7 (Shuffle-compatibility of (pk, des))**.**

(a) *The pair* (pk, des) *is shuffle-compatible.*

(b) *Let*

$$u_{n,j,k}^{(\mathrm{pk},\mathrm{des})} = \frac{t^{j+1}(y+t)^{k-j}(1+yt)^{n-j-k-1}(1+y)^{2j+1}}{(1-t)^{n+1}} x^n.$$

Then the linear map on $\mathcal{A}_{(\mathrm{pk},\mathrm{des})}$ *defined by*

$$\pi_{(\mathrm{pk},\mathrm{des})} \mapsto \begin{cases} u_{|\pi|,\mathrm{pk}\,\pi,\mathrm{des}\,\pi}^{(\mathrm{pk},\mathrm{des})}, & \text{if } |\pi| \geq 1, \\ 1/(1-t), & \text{if } |\pi| = 0, \end{cases}$$

is a $\mathbb{Q}$*-algebra isomorphism from* $\mathcal{A}_{(\mathrm{pk},\mathrm{des})}$ *to the span of*

$$\Big\{\frac{1}{1-t}\Big\} \bigcup \{u_{n,j,k}^{(\mathrm{pk},\mathrm{des})}\}_{\substack{n\geq 1, \\ 0\leq j\leq \lfloor (n-1)/2 \rfloor, \\ j\leq k\leq n-j-1,}},$$

a subalgebra of $\mathbb{Q}[[t*]][x,y]$.

We note that, in the definition of $u_{n,j,k}^{(\mathrm{pk},\mathrm{des})}$, all products should be interpreted as ordinary multiplication; the Hadamard product in t is only used when multiplying elements in the span of the $u_{n,j,k}^{(\mathrm{pk},\mathrm{des})}$. The same is true in Theorems 4.3.8, 4.3.9, and 4.3.10 presented later in this section.

Theorem 4.3.8 (Cyclic shuffle-compatibility of (cpk, cdes))**.**

(a) *The pair* (cpk, cdes) *is cyclic shuffle-compatible.*

(b) *Let*

$$\begin{aligned} v^{(\mathrm{cpk},\mathrm{cdes})}_{n,j,k} &= j u^{(\mathrm{pk},\mathrm{des})}_{n,j-1,k} + j u^{(\mathrm{pk},\mathrm{des})}_{n,j-1,k-1} + (k-j) u^{(\mathrm{pk},\mathrm{des})}_{n,j,k-1} + (n-j-k) u^{(\mathrm{pk},\mathrm{des})}_{n,j,k} \\ &= [j(y+t)(1+yt)(1+y+t+yt) \\ &\qquad + ((k-j)(1+yt) + (n-j-k)(y+t))t(1+y)^2] \\ &\qquad \times \frac{t^j (y+t)^{k-j-1}(1+yt)^{n-j-k-1}(1+y)^{2j-1}}{(1-t)^{n+1}} x^n. \end{aligned}$$

Then the linear map on $\mathcal{A}^{\mathrm{cyc}}_{(\mathrm{cpk},\mathrm{cdes})}$ *defined by*

$$[\pi]_{(\mathrm{cpk},\mathrm{cdes})} \mapsto \begin{cases} v^{(\mathrm{cpk},\mathrm{cdes})}_{|\pi|,\mathrm{cpk}[\pi],\mathrm{cdes}[\pi]}, & \text{if } |\pi| \geq 1, \\ 1/(1-t), & \text{if } |\pi| = 0, \end{cases}$$

is a $\mathbb{Q}$*-algebra homomorphism from* $\mathcal{A}^{\mathrm{cyc}}_{(\mathrm{cpk},\mathrm{cdes})}$ *to the span of*

$$\left\{ \frac{1}{1-t}, \frac{t(1+y)}{(1-t)^2} x \right\} \bigcup \{ v^{(\mathrm{cpk},\mathrm{cdes})}_{n,j,k} \}_{n \geq 2,\ 1 \leq j \leq \lfloor n/2 \rfloor,\ j \leq k \leq n-j},$$

a subalgebra of $\mathbb{Q}[[t*]][x,y]$.

(c) *For all* $n \geq 2$*, the* n*th homogeneous component of* $\mathcal{A}^{\mathrm{cyc}}_{(\mathrm{cpk},\mathrm{cdes})}$ *has dimension* $\lfloor n^2/4 \rfloor$.

Proof. We shall apply Theorem 4.2.7 using $\mathrm{st} = (\mathrm{pk}, \mathrm{des})$. In doing so, we take $\phi_{(\mathrm{pk},\mathrm{des})}$ to be the composition of the map $F_L \mapsto \pi_{(\mathrm{pk},\mathrm{des})}$ with the map from Theorem 4.3.7 (b), where π is any permutation with $\mathrm{pk}\,\pi = \mathrm{pk}\,L$ and $\mathrm{des}\,\pi = \mathrm{des}\,L$.

Let π be a permutation of length $n \geq 2$ with cyclic descent set S, and let $j = \mathrm{cpk}[\pi]$ and $k = \mathrm{cdes}[\pi]$ (which only depend on S and not the specific choice of π). Let us consider the n linear permutations in $[\pi]$, whose descent sets are given by $S + i$ where i ranges from 1 to n. Among these n permutations, the following hold:

- Exactly j of these permutations have $\mathrm{cpk}[\pi] - 1$ peaks and $\mathrm{cdes}[\pi]$ descents, which are those that have a cyclic peak in the first position.

- Exactly j of these permutations have $\mathrm{cpk}[\pi] - 1$ peaks and $\mathrm{cdes}[\pi] - 1$ descents, which are those that have a cyclic peak in the last position.

- Exactly $k - j$ of these permutations have $\mathrm{cpk}[\pi]$ peaks and $\mathrm{cdes}[\pi] - 1$ descents, which are those that have a cyclic descent in the last position which is not a cyclic peak.

- The remaining $n - j - k$ permutations have $\mathrm{cpk}[\pi]$ peaks and $\mathrm{cdes}[\pi]$ descents.

Therefore, we have

$$\begin{aligned}\phi_{(\mathrm{cpk},\mathrm{cdes})}(F^{\mathrm{cyc}}_{n,S}) &= \sum_{i\in[n]} \phi_{(\mathrm{pk},\mathrm{des})}(F_{n,S+i}) \\ &= ju^{(\mathrm{pk},\mathrm{des})}_{n,j-1,k} + ju^{(\mathrm{pk},\mathrm{des})}_{n,j-1,k-1} + (k-j)u^{(\mathrm{pk},\mathrm{des})}_{n,j,k-1} + (n-j-k)u^{(\mathrm{pk},\mathrm{des})}_{n,j,k} \\ &= v^{(\mathrm{cpk},\mathrm{cdes})}_{n,j,k}.\end{aligned}$$

For $n = 0$ and $n = 1$, we have

$$\phi_{(\mathrm{cpk},\mathrm{cdes})}(F^{\mathrm{cyc}}_{0,\emptyset}) = \frac{1}{1-t} \quad \text{and} \quad \phi_{(\mathrm{cpk},\mathrm{cdes})}(F^{\mathrm{cyc}}_{1,\emptyset}) = \frac{t(1+y)}{(1-t)^2}x.$$

Clearly, $\phi_{(\mathrm{cpk},\mathrm{cdes})}(F^{\mathrm{cyc}}_{n,S})$ depends only on the $(\mathrm{cpk},\mathrm{cdes})$-equivalence class of $\mathrm{cComp}[S]$.

To prove linear independence, let us order monomials in the variables t and y lexicographically by the exponent of t followed by the exponent of y, that is, $t^a y^b > t^c y^d$ if and only if either $a > c$, or if $a = c$ and $b > d$. Since Corollary 4.3.3 implies $j \geq 1$, it is readily verified that the least monomial in $(1-t)^{n+1} v^{(\mathrm{cpk},\mathrm{cdes})}_{n,j,k}/x^n$ is $t^j y^{k-j}$; thus

$$\left\{ \frac{(1-t)^{n+1}}{x^n} v^{(\mathrm{cpk},\mathrm{cdes})}_{n,j,k} \right\}_{\substack{1\leq j\leq \lfloor n/2\rfloor \\ j\leq k\leq n-j}}$$

is linearly independent for each $n \geq 2$, and this in turn implies that

$$\left\{ \frac{1}{1-t}, \frac{t(1+y)}{(1-t)^2}x \right\} \bigcup \{ v^{(\mathrm{cpk},\mathrm{cdes})}_{n,j,k} \}_{\substack{n\geq 2 \\ 1\leq j\leq \lfloor n/2\rfloor \\ j\leq k\leq n-j}}$$

is linearly independent. Corollary 4.3.3 ensures that we have the correct limits on j and k, so we can use Theorem 4.2.7 to conclude that parts (a) and (b) hold.

From Corollary 4.3.3, we know that for $n \geq 2$, the number of $(\mathrm{cpk},\mathrm{cdes})$-equivalence classes of cyclic permutations of length n is

$$\sum_{j=1}^{\lfloor n/2\rfloor} ((n-j) - j + 1) = \sum_{j=1}^{\lfloor n/2\rfloor} (n - 2j + 1),$$

and it is straightforward to show that this is equal to $\lfloor n^2/4 \rfloor$. Thus, part (c) follows. □

4.3.4 The Cyclic Shuffle Algebras of cpk and cdes

Next, we use our characterization of the cyclic shuffle algebra $\mathcal{A}^{\mathrm{cyc}}_{(\mathrm{cpk},\mathrm{cdes})}$ along with Theorem 4.1.3 to characterize $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cpk}}$ and $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cdes}}$, which also provides an alternative proof for the cyclic shuffle-compatibility of cpk and cdes.

In the theorems below, we use the notation $\mathbb{Q}[x]^{\mathbb{N}}$ to denote the algebra of functions $\mathbb{N} \to \mathbb{Q}[x]$ in the non-negative integer variable p. For example, the map $p \mapsto \binom{p}{2}x + p^3$—which we write simply as $\binom{p}{2}x + p^3$ for brevity—is an element of $\mathbb{Q}[x]^{\mathbb{N}}$. Moreover, in Theorem 4.3.9 below, $\left(\!\binom{n}{k}\!\right)$ is the number of k-element multisubsets of $[n]$.

Theorem 4.3.9 (Cyclic shuffle-compatibility of cpk)**.**

(a) *The cyclic peak number* cpk *is cyclic shuffle-compatible.*

(b) *The linear map on $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cpk}}$ defined by*

$$[\pi]_{\mathrm{cpk}} \mapsto \begin{cases} \dfrac{(\mathrm{cpk}[\pi](1+t)^2 + 2(|\pi| - 2\,\mathrm{cpk}[\pi])t)(4t)^{\mathrm{cpk}[\pi]}(1+t)^{|\pi|-2\,\mathrm{cpk}[\pi]-1}}{(1-t)^{|\pi|+1}} x^{|\pi|}, & \text{if } |\pi| \geq 1, \\ 1/(1-t), & \text{if } |\pi| = 0, \end{cases}$$

is a $\mathbb{Q}$-algebra isomorphism from $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cpk}}$ to the span of

$$\left\{ \frac{1}{1-t}, \frac{tx}{(1-t)^2} \right\} \bigcup \left\{ \frac{(j(1+t)^2 + 2(n-2j)t)(4t)^j (1+t)^{n-2j-1}}{(1-t)^{n+1}} x^n \right\}_{\substack{n \geq 2, \\ 1 \leq j \leq \lfloor n/2 \rfloor}},$$

a subalgebra of $\mathbb{Q}[[t*]][x]$.

(c) *For all $n \geq 2$, the nth homogeneous component of $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cpk}}$ has dimension $\lfloor n/2 \rfloor$.*

Proof. Let $\phi\colon \mathcal{A}^{\mathrm{cyc}}_{(\mathrm{cpk},\mathrm{cdes})} \to \mathbb{Q}[[t*]][x]$ be the composition of the map from Theorem 4.3.8 (b) and the $y = 1$ evaluation map. Since

$$v^{(\mathrm{cpk},\mathrm{cdes})}_{n,j,k}\Big|_{y=1} = \frac{(j(1+t)^2 + 2(n-2j)t)(4t)^j (1+t)^{n-2j-1}}{(1-t)^{n+1}} x^n$$

for all $n \geq 1$, we see that ϕ is precisely the map in part (b) of this theorem. Note that $v^{(\mathrm{cpk},\mathrm{cdes})}_{n,j,k}|_{y=1}$ depends only on n and j, so the $v^{(\mathrm{cpk},\mathrm{cdes})}_{n,j,k}|_{y=1}$ correspond to cpk-equivalence classes. Furthermore, it is straightforward to verify that the $v_{n,j,k}^{(\mathrm{cpk},\mathrm{cdes})}|_{y=1}$ are linearly independent, so we may apply Theorem 4.1.3 to complete the proof. □

Theorem 4.3.10 (Cyclic shuffle-compatibility of cdes)**.**

(a) *The cyclic descent number* cdes *is cyclic shuffle-compatible.*

(b) *The linear map on* $\mathcal{A}_{\mathrm{cdes}}^{\mathrm{cyc}}$ *defined by*

$$[\pi]_{\mathrm{cdes}} \mapsto \begin{cases} \dfrac{\mathrm{cdes}[\pi] t^{\mathrm{cdes}[\pi]} + (|\pi| - \mathrm{cdes}[\pi]) t^{\mathrm{cdes}[\pi]+1}}{(1-t)^{|\pi|+1}} x^{|\pi|}, & \text{if } |\pi| \geq 1, \\ 1/(1-t), & \text{if } |\pi| = 0, \end{cases}$$

is a $\mathbb{Q}$*-algebra isomorphism from* $\mathcal{A}_{\mathrm{cdes}}^{\mathrm{cyc}}$ *to the span of*

$$\left\{ \frac{1}{1-t}, \frac{tx}{(1-t)^2} \right\} \bigcup \left\{ \frac{kt^k + (n-k)t^{k+1}}{(1-t)^{n+1}} x^n \right\}_{\substack{n \geq 2, \\ 1 \leq k \leq n-1}},$$

a subalgebra of $\mathbb{Q}[[t*]][x]$.

(c) *The linear map on* $\mathcal{A}_{\mathrm{cdes}}^{\mathrm{cyc}}$ *defined by*

$$[\pi]_{\mathrm{cdes}} \mapsto \begin{cases} \dbinom{p + |\pi| - \mathrm{cdes}[\pi] - 1}{|\pi| - 1} p x^{|\pi|}, & \text{if } |\pi| \geq 1, \\ 1, & \text{if } |\pi| = 0, \end{cases}$$

is a $\mathbb{Q}$*-algebra isomorphism from* $\mathcal{A}_{\mathrm{cdes}}^{\mathrm{cyc}}$ *to the span of*

$$\{1, px\} \bigcup \left\{ \binom{p+n-k-1}{n-1} p x^n \right\}_{\substack{n \geq 2, \\ 1 \leq k \leq n-1}},$$

a subalgebra of $\mathbb{Q}[x]^{\mathbb{N}}$.

(d) *For all* $n \geq 2$*, the nth homogeneous component of* $\mathcal{A}_{\mathrm{cdes}}^{\mathrm{cyc}}$ *has dimension* $n-1$.

Proof. The proofs for parts (a), (b), and (d) follow in the same way as in for Theorem 4.3.9, except that we evaluate at $y = 0$ as opposed to $y = 1$. Part (c) follows from part (b) and the identity

$$\frac{kt^k + (n-k)t^{k+1}}{(1-t)^{n+1}} = \sum_{p=0}^{\infty} \binom{p+n-k-1}{n-1} p t^p,$$

which was established in [AGRR21, Lemma 5.8]. □

4.4 Cyclic Permutation Statistics Induced by Linear Permutation Statistics

Recall that the cyclic permutation statistics cDes and cPk are defined by

$$\mathrm{cDes}[\pi] := \{\{\,\mathrm{cDes}\,\bar{\pi} : \bar{\pi} \in [\pi]\,\}\} \quad \text{and} \quad \mathrm{cPk}[\pi] := \{\{\,\mathrm{cPk}\,\bar{\pi} : \bar{\pi} \in [\pi]\,\}\}.$$

In other words, $\mathrm{cDes}[\pi]$ is simply the distribution of the linear permutation statistic cDes over all linear permutations in $[\pi]$, and similarly with $\mathrm{cPk}[\pi]$. In fact, any linear permutation statistic st induces a multiset-valued cyclic permutation statistic (which we also denote st by a slight abuse of notation) if we let

$$\mathrm{st}[\pi] := \{\{\,\mathrm{st}\,\bar{\pi} : \bar{\pi} \in [\pi]\,\}\}.$$

In this section, we study these multiset-valued cyclic statistics induced from various linear permutation statistics.

4.4.1 The Cyclic Statistics Des, des, Pk, and pk

To begin, we note that the cyclic statistics induced from the linear statistics Des, des, Pk, and pk are equivalent to cDes, cdes, cPk, and cpk, respectively.

Lemma 4.4.1. *The cyclic permutation statistics* Des *and* cDes *are equivalent.*

Proof. Let $\pi \in \mathfrak{S}_n$. For any $\bar{\pi} \in [\pi]$, we have $\mathrm{Des}\,\bar{\pi} = \mathrm{cDes}\,\bar{\pi} \setminus \{n\}$ if $n \in \mathrm{cDes}\,\bar{\pi}$ and $\mathrm{Des}\,\bar{\pi} = \mathrm{cDes}\,\bar{\pi}$ otherwise. Therefore, we can obtain $\mathrm{Des}[\pi]$ from $\mathrm{cDes}[\pi]$ by removing every n from the cyclic descent sets in $\mathrm{cDes}[\pi]$, and we can obtain $\mathrm{cDes}[\pi]$ from $\mathrm{Des}[\pi]$ by adding n to each descent set in $\mathrm{Des}[\pi]$ with one fewer element than the others. □

Lemma 4.4.2. *The cyclic permutation statistics* des *and* cdes *are equivalent.*

Proof. Let $\pi \in \mathfrak{S}_n$. For any $\bar{\pi} \in [\pi]$, we have $\mathrm{des}\,\bar{\pi} = \mathrm{cdes}[\pi] - 1$ if $n \in \mathrm{cDes}\,\bar{\pi}$ and $\mathrm{des}\,\bar{\pi} = \mathrm{cdes}[\pi]$ otherwise. The unique permutation in $[\pi]$ beginning with its largest letter does not have n as a cyclic descent, so we can determine $\mathrm{cdes}[\pi]$ from the multiset $\mathrm{des}[\pi]$ by taking the largest value in $\mathrm{des}[\pi]$.

Conversely, among the n rotations of π, there are exactly $\operatorname{cdes}[\pi]$ permutations with a cyclic descent in the last position; this implies that $\operatorname{des}[\pi]$ is the multiset with $\operatorname{cdes}[\pi]$ copies of $\operatorname{cdes}[\pi]-1$ and $n-\operatorname{cdes}[\pi]$ copies of $\operatorname{cdes}[\pi]$, so we can determine $\operatorname{des}[\pi]$ from $\operatorname{cdes}[\pi]$ as well. □

Lemma 4.4.3. *The cyclic permutation statistics* Pk *and* cPk *are equivalent.*

Proof. Let $\pi \in \mathfrak{S}_n$. For any $\bar{\pi} \in [\pi]$, we have $\operatorname{Pk}\bar{\pi} = \operatorname{cPk}\bar{\pi}\backslash\{1\}$ if $1 \in \operatorname{cPk}\bar{\pi}$, $\operatorname{Pk}\bar{\pi} = \operatorname{cPk}\bar{\pi}\backslash\{n\}$ if $n \in \operatorname{cPk}\bar{\pi}$, and $\operatorname{Pk}\bar{\pi} = \operatorname{cPk}\bar{\pi}$ otherwise. (Note that $\operatorname{cPk}\bar{\pi}$ cannot simultaneously contain 1 and n.) Hence, we can obtain $\operatorname{Pk}[\pi]$ from $\operatorname{cPk}[\pi]$ by removing every 1 and n from the cyclic peak sets in $\operatorname{cPk}[\pi]$.

Conversely, suppose that we are given $\operatorname{Pk}[\pi]$ and wish to recover $\operatorname{cPk}[\pi]$. Let $i \in [n]$ be arbitrary. Notice that, among all n representatives of $[\pi]$, the index of π_i spans the entire range $\{1, 2, \ldots, n\}$. If i is a cyclic peak of π in particular, this means that the index of π_i will be a peak of all n representatives of $[\pi]$ except for the linear permutation beginning with π_i and the one ending with π_i; hence, if one adds up $\operatorname{pk}\bar{\pi}$ over all $\bar{\pi} \in [\pi]$, then each of these π_i will contribute $n-2$ to the summation. It follows that the sum of the sizes of all peak sets in $\operatorname{Pk}[\pi]$ is equal to $(n-2)\operatorname{cpk}[\pi]$; in other words, we can determine $\operatorname{cpk}[\pi]$ from $\operatorname{Pk}[\pi]$. It remains to show that we can recover $\operatorname{cPk}[\pi]$ from $\operatorname{cpk}[\pi]$ and $\operatorname{Pk}[\pi]$. To do so, we divide into two cases:

- *Case 1*: Suppose that there exists a peak set $\operatorname{Pk}\bar{\pi}$ in $\operatorname{Pk}[\pi]$ with $\operatorname{cpk}[\pi]$ elements. Then $\operatorname{Pk}\bar{\pi} = \operatorname{cPk}\bar{\pi}$, and we can recover the entire multiset $\operatorname{cPk}[\pi]$ by taking all n cyclic shifts of $\operatorname{Pk}\bar{\pi}$.

- *Case 2*: Suppose instead that all peak sets in $\operatorname{Pk}[\pi]$ have $\operatorname{cpk}[\pi]-1$ elements. Then, every linear permutation in $[\pi]$ has either 1 or n as a cyclic peak. In general, among the n representatives of $[\pi]$, there are exactly $2\operatorname{cpk}[\pi]$ of them with a cyclic peak at one end. This means that $2\operatorname{cpk}[\pi] = n$, and since cyclic peak sets cannot contain two consecutive indices, it follows that every cyclic peak set in $\operatorname{cPk}[\pi]$ is of the form $\{1, 3, \ldots, n-1\}$ or $\{2, 4, \ldots, n\}$. More precisely, we must have

$$\operatorname{cPk}[\pi] = \{\{\, \{1, 3, \ldots, n-1\}^{n/2}, \{2, 4, \ldots, n\}^{n/2} \,\}\}.$$

Since $\mathrm{cPk}[\pi]$ can be recovered from $\mathrm{Pk}[\pi]$ in both cases, we are done. □

Lemma 4.4.4. *The cyclic permutation statistics* pk *and* cpk *are equivalent.*

Proof. Let $\pi \in \mathfrak{S}_n$. As shown in the proof of Lemma 4.4.3, the sum of the sizes of all peak sets in $\mathrm{Pk}[\pi]$ is equal to $(n-2)\,\mathrm{cpk}[\pi]$, but this is the same as the sum of all elements of the multiset $\mathrm{pk}[\pi]$. Thus, $\mathrm{cpk}[\pi]$ can be determined from $\mathrm{pk}[\pi]$.

For the converse, we use the observation (also used in the proof of Lemma 4.4.3) that among the n representatives of a cyclic permutation $[\pi]$, there are exactly $2\,\mathrm{cpk}[\pi]$ of them with a cyclic peak at one end. This implies that the multiset $\mathrm{pk}[\pi]$ has $2\,\mathrm{cpk}[\pi]$ copies of $\mathrm{cpk}[\pi]-1$ and $n-2\,\mathrm{cpk}[\pi]$ copies of $\mathrm{cpk}[\pi]$. Hence, $\mathrm{cpk}[\pi]$ completely determines $\mathrm{pk}[\pi]$. □

Since cDes, cdes, cPk, and cpk are cyclic shuffle-compatible, it follows from these equivalences and Theorem 4.1.4 that the cyclic statistics Des, des, Pk, and pk are as well.

Theorem 4.4.5 (Cyclic shuffle-compatibility of Des, des, Pk, and pk)**.** *The cyclic statistics* Des*,* des*,* Pk*, and* pk *are cyclic shuffle-compatible, and we have the* $\mathbb{Q}$*-algebra isomorphisms*

$$\mathcal{A}^{\mathrm{cyc}}_{\mathrm{Des}} \cong \mathcal{A}^{\mathrm{cyc}}_{\mathrm{cDes}}, \quad \mathcal{A}^{\mathrm{cyc}}_{\mathrm{des}} \cong \mathcal{A}^{\mathrm{cyc}}_{\mathrm{cdes}}, \quad \mathcal{A}^{\mathrm{cyc}}_{\mathrm{Pk}} \cong \mathcal{A}^{\mathrm{cyc}}_{\mathrm{cPk}}, \quad \textit{and} \quad \mathcal{A}^{\mathrm{cyc}}_{\mathrm{pk}} \cong \mathcal{A}^{\mathrm{cyc}}_{\mathrm{cpk}}.$$

4.4.2 Symmetries Revisited

Let f be a length-preserving involution on permutations that is both shuffle-compatibility-preserving and rotation-preserving. In Section 4.1.3, we proved that if the cyclic permutation statistics cst_1 and cst_2 are f-equivalent and if cst_1 is cyclic shuffle-compatible, then cst_2 is also cyclic shuffle-compatible with cyclic shuffle algebra isomorphic to that of cst_1. We now show that f-equivalence of two linear permutation statistics induces f-equivalence of their induced cyclic statistics.

Lemma 4.4.6. *Let f be rotation-preserving. If* st_1 *and* st_2 *are f-equivalent linear permutation statistics, then their induced cyclic permutation statistics* st_1 *and* st_2 *are f-equivalent.*

Proof. Since st_1 and st_2 are f-equivalent linear permutation statistics, we have $\mathrm{st}_1\,\pi^f = \mathrm{st}_1\,\sigma^f$ if and only if $\mathrm{st}_2\,\pi = \mathrm{st}_2\,\sigma$. Suppose that $\mathrm{st}_2[\pi] = \mathrm{st}_2[\sigma]$. Then, there is a bijective correspondence $g\colon [\pi] \to [\sigma]$ satisfying $\mathrm{st}_2\,\bar{\pi} = \mathrm{st}_2\,g(\bar{\pi})$ for all $\bar{\pi} \in [\pi]$, so $\mathrm{st}_1\,\bar{\pi}^f = \mathrm{st}_1\,g(\bar{\pi})^f$ for all $\bar{\pi} \in [\pi]$. Because f is rotation-preserving, the permutations $\bar{\pi}^f$ and $g(\bar{\pi})^f$ over all $\bar{\pi} \in [\pi]$ are precisely the rotations of π^f and σ^f, respectively. Thus, we have $\mathrm{st}_1[\pi^f] = \mathrm{st}_1[\sigma^f]$. The converse follows from similar reasoning, so we have $\mathrm{st}_1[\pi^f] = \mathrm{st}_1[\sigma^f]$ if and only if $\mathrm{st}_2[\pi] = \mathrm{st}_2[\sigma]$—in other words, the cyclic permutation statistics st_1 and st_2 are f-equivalent. □

Theorem 4.4.7. *Let f be shuffle-compatibility-preserving and rotation-preserving, and let* st_1 *and* st_2 *be f-equivalent linear permutation statistics. If the induced cyclic statistic* st_1 *is cyclic shuffle-compatible, then the induced cyclic statistic* st_2 *is also cyclic shuffle-compatible and* $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{st}_2}$ *is isomorphic to* $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{st}_1}$.

Proof. This is an immediate consequence of Theorem 4.1.11 and Lemma 4.4.6. □

Corollary 4.4.8. *Suppose that the linear permutation statistics* st_1 *and* st_2 *are r-equivalent, c-equivalent, or rc-equivalent. If the induced cyclic statistic* st_1 *is cyclic shuffle-compatible, then the induced cyclic statistic* st_2 *is also cyclic shuffle-compatible and its cyclic shuffle algebra* $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{st}_2}$ *is isomorphic to* $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{st}_1}$.

Given $\pi \in \mathfrak{S}_n$, recall that the valley set Val statistic is defined by

$$\mathrm{Val}\,\pi := \{\, i \in [n] : \pi_{i-1} > \pi_i < \pi_{i+1}, \},$$

and let us also define the *cyclic valley set* cVal by

$$\mathrm{cVal}\,\pi := \{\, i \in [n] : \pi_{i-1} > \pi_i < \pi_{i+1} \text{ where } i \text{ is considered modulo } n \,\}.$$

As a sample application of Corollary 4.4.8, observe that Val is c-equivalent to Pk (as linear permutation statistics) and similarly with cVal and cPk. Combining this with Lemma 4.4.3, we immediately obtain the following.

Theorem 4.4.9 (Cyclic shuffle-compatibility of Val and cVal)**.** *The cyclic statistics* Val *and* cVal *are cyclic shuffle-compatible, and we have the* $\mathbb{Q}$*-algebra isomorphisms*

$$\mathcal{A}^{\mathrm{cyc}}_{\mathrm{Val}} \cong \mathcal{A}^{\mathrm{cyc}}_{\mathrm{Pk}} \cong \mathcal{A}^{\mathrm{cyc}}_{\mathrm{cPk}} \cong \mathcal{A}^{\mathrm{cyc}}_{\mathrm{cVal}}.$$

4.4.3 Cyclic Major Index

A natural question to ask is whether there is a nice cyclic analogue of the major index. This question was raised in [AGRR21]. One first needs to explain what one means by "nice."

If $\pi \in \mathfrak{S}_m$ and $\sigma \in \mathfrak{S}_n$, then

$$|\pi \sqcup \sigma| = \binom{m+n}{m}.$$

From Stanley's theory of P-partitions [Sta72], one gets the q-analogue

$$\sum_{\tau \in \pi \sqcup \sigma} q^{\operatorname{maj} \tau} = q^{\operatorname{maj} \pi + \operatorname{maj} \sigma} \begin{bmatrix} m+n \\ m \end{bmatrix} \tag{4.4.1}$$

where $\left[\begin{smallmatrix} m+n \\ m \end{smallmatrix}\right]$ is a q-binomial coefficient. Note that (4.4.1) implies that maj is shuffle-compatible.

It can be shown that

$$|[\pi] \sqcup [\sigma]| = (m+n-1)\binom{m+n-2}{m-1},$$

so one could ask that the cyclic major index give a q-analogue of this identity, similar to (4.4.1), or at least for the cyclic major index to be cyclic shuffle-compatible.

Stanley also refined Equation (4.4.1) as follows. Let

$$\pi \sqcup_k \sigma = \{\, \tau \in \pi \sqcup \sigma : \operatorname{des} \tau = k \,\}.$$

If $\operatorname{des} \pi = i$ and $\operatorname{des} \sigma = j$, then

$$\sum_{\tau \in \pi \sqcup_k \sigma} q^{\operatorname{maj} \tau} = q^{\operatorname{maj} \pi + \operatorname{maj} \sigma + (k-i)(k-j)} \begin{bmatrix} m-j+i \\ k-j \end{bmatrix} \begin{bmatrix} n-i+j \\ k-i \end{bmatrix}; \tag{4.4.2}$$

in particular, this implies that (des, maj) is shuffle-compatible, and so we would like a cmaj statistic for which cmaj and (cdes, cmaj) are both cyclic shuffle-compatible.

In [AGRR21], Adin et al. computed the cardinality of

$$[\pi] \sqcup_k [\sigma] = \{\, [\tau] \in [\pi] \sqcup [\sigma] : \operatorname{cdes}[\tau] = k \,\}$$

which inspired Ji and Zhang [JZ22] to define a cmaj statistic which gives a q-analogue of this count. They proved a generating function formula analogous to (4.4.2), but unfortunately, the formula does not simplify into single product, and one could hope for a different cyclic major index whose generating function would do so. Furthermore, their formula does not actually show that their (cdes, cmaj) is cyclic shuffle-compatible; in fact, neither of their cmaj and (cdes, cmaj) are cyclic shuffle-compatible.

Each of the cyclic statistics cDes, cdes, cPk, and cpk is (or is equivalent to) a multiset-valued cyclic statistic induced by a corresponding linear permutation statistic, so a natural alternative definition for a cyclic major index would be to define cmaj first on linear permutations and then consider the multiset-valued statistic induced by the linear cmaj. To that end, given a linear permutation π, let

$$\operatorname{cmaj} \pi := \sum_{k \in \operatorname{cDes} \pi} k.$$

Unfortunately, the induced statistics cmaj and (cdes, cmaj) are not cyclic shuffle-compatible. As a counterexample, take $\pi = 1\,4\,7\,6\,9\,10\,8\,2\,5\,3$, $\sigma = 1\,3\,5\,4\,7\,6\,9\,10\,8\,2$, and $\rho = 11$. Then $\operatorname{cdes}[\pi] = \operatorname{cdes}[\sigma] = 5$ and $\operatorname{cmaj}[\pi] = \operatorname{cmaj}[\sigma] = \{\{20, 25^4, 30^4, 35\}\}$, but $\operatorname{cmaj}([\pi] \sqcup [\rho]) \neq \operatorname{cmaj}([\sigma] \sqcup [\rho])$. For instance, the multiset $\{\{22, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35\}\}$ is an element of $\operatorname{cmaj}([\pi] \sqcup [\rho])$ but not $\operatorname{cmaj}([\sigma] \sqcup [\rho])$.

Another option is to consider the cyclic statistic induced by the usual major index maj, as opposed to cmaj. Even if cmaj and (cdes, cmaj) are not cyclic shuffle-compatible, it's conceivable that maj and (des, maj) are. It turns out that maj is equivalent to cmaj and similarly with (des, maj) and (cdes, cmaj), so by Theorem 4.1.4, neither maj nor (des, maj) are cyclic shuffle-compatible.

Lemma 4.4.10. *The cyclic permutation statistics* (des, maj) *and* (cdes, cmaj) *are equivalent.*

Proof. Fix a cyclic permutation $[\pi] = \{\pi = \pi^{(1)}, \pi^{(2)}, \dots, \pi^{(n)}\}$ of length n where, for each $i \in [n]$, $\pi^{(i+1)}$ is obtained from $\pi^{(i)}$ by rotating its last element to the front of the permutation and i is taken modulo n. We claim that, for all $i \in [n]$,

$$\operatorname{cmaj} \pi^{(i+1)} = \begin{cases} \operatorname{cmaj} \pi^{(i)} + \operatorname{cdes}[\pi] - n, & \text{if } n \in \operatorname{cDes} \pi^{(i)}, \\ \operatorname{cmaj} \pi^{(i)} + \operatorname{cdes}[\pi], & \text{if } n \notin \operatorname{cDes} \pi^{(i)}. \end{cases} \tag{4.4.3}$$

To prove (4.4.3), first assume that $n \in \operatorname{cDes} \pi^{(i)}$, and let $k = \operatorname{cdes}[\pi]$. Then

$$\operatorname{cDes} \pi^{(i)} = \{j_1 < j_2 < \cdots < j_k = n\}$$

whereas

$$\operatorname{cDes} \pi^{(i+1)} = \{1 < j_1 + 1 < j_2 + 1 < \cdots < j_{k-1} + 1\}.$$

So

$$\operatorname{cmaj} \pi^{(i)} - \operatorname{cmaj} \pi^{(i+1)} = n - k,$$

which is equivalent to the first case of (4.4.3). The second case is proven using a similar computation.

Observe that Equation (4.4.3) is equivalent to

$$\operatorname{cmaj} \pi^{(i+1)} = \operatorname{maj} \pi^{(i)} + \operatorname{cdes}[\pi], \tag{4.4.4}$$

which allows us to determine $\operatorname{cmaj}[\pi]$ from $\operatorname{maj}[\pi]$ and $\operatorname{cdes}[\pi]$. Moreover, $\operatorname{cdes}[\pi]$ can be determined from $\operatorname{des}[\pi]$ by Lemma 4.4.2, so $(\operatorname{cdes}, \operatorname{cmaj})[\pi]$ can be determined from $(\operatorname{des}, \operatorname{maj})[\pi]$.

Conversely, we can use (4.4.4) to determine $\operatorname{maj}[\pi]$ from $\operatorname{cmaj}[\pi]$ and $\operatorname{cdes}[\pi]$, and $\operatorname{des}[\pi]$ can be determined from $\operatorname{cdes}[\pi]$ by Lemma 4.4.2; altogether, this means that we can also determine $(\operatorname{des}, \operatorname{maj})[\pi]$ from $(\operatorname{cdes}, \operatorname{cmaj})[\pi]$. □

Lemma 4.4.11. *The cyclic permutation statistics* maj *and* cmaj *are equivalent.*

Proof. Let $\pi \in \mathfrak{S}_n$. We first claim that $\operatorname{cdes}[\pi]$ can be determined either from $\operatorname{maj}[\pi]$ or from $\operatorname{cmaj}[\pi]$. Fix $i \in \operatorname{cDes} \pi$. Among all n representatives of $[\pi]$, the index of π_i spans the entire range $\{1, 2, \ldots, n\}$. Hence, if one adds up $\operatorname{maj} \bar{\pi}$ over all $\bar{\pi} \in [\pi]$, then π_i will contribute $1 + 2 + \cdots + (n-1) = \binom{n}{2}$ to the summation. Similarly, in taking the sum of all $\operatorname{cmaj} \bar{\pi}$, each π_i will contribute $1 + 2 + \cdots + n = \binom{n+1}{2}$. Thus, the sum of all elements of the multiset $\operatorname{maj}[\pi]$ is equal to $\binom{n}{2}\operatorname{cdes}[\pi]$ and the sum of all elements of $\operatorname{cmaj}[\pi]$ is equal to $\binom{n+1}{2}\operatorname{cdes}[\pi]$, and it follows that $\operatorname{cdes}[\pi]$ can be determined from $\operatorname{maj}[\pi]$ or $\operatorname{cmaj}[\pi]$.

Now we are ready to prove the equivalence between maj and cmaj. For one direction, $\operatorname{maj}[\pi]$ completely determines $\operatorname{cdes}[\pi]$ and hence determines $\operatorname{des}[\pi]$ by Lemma 4.4.2. Furthermore, $\operatorname{maj}[\pi]$ and $\operatorname{des}[\pi]$ together determine $(\operatorname{cdes},\operatorname{cmaj})[\pi]$ by Lemma 4.4.10, so $\operatorname{cmaj}[\pi]$ can be determined from $\operatorname{maj}[\pi]$. One can similarly prove the other direction using the above claim and Lemma 4.4.10. □

Definition 4.4.12 (Comajor index)**.** The *comajor index* comaj, defined by

$$\operatorname{comaj}\pi := \sum_{k\in\operatorname{Des}\pi}(n-k)$$

for $\pi \in \mathfrak{S}_n$, is a classical variation of the major index statistic.

Because the linear permutation statistics maj and comaj are rc-equivalent and the induced cyclic statistic maj is not cyclic shuffle-compatible, it follows from Corollary 4.4.8 that the induced cyclic statistic comaj is not cyclic shuffle-compatible either.

Definition 4.4.13 (Cyclic comajor index)**.** Define the *cyclic comajor index* ccomaj by

$$\operatorname{ccomaj}\pi := \sum_{k\in\operatorname{cDes}\pi}(n-k)$$

for $\pi \in \mathfrak{S}_n$.

It follows similarly that the induced cyclic statistics ccomaj, (des, comaj), and (cdes, ccomaj) are not cyclic shuffle-compatible either.

Perhaps surprisingly, adding just a little bit of structure to our cmaj statistic gives a statistic which is equivalent to cDes. As in the proof of Lemma 4.4.10, given $\pi \in \mathfrak{S}_n$, let us write

$$[\pi] = \{\pi = \pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(n)}\}$$

where $\pi^{(i+1)}$ is obtained from $\pi^{(i)}$ by rotating its last element to the front of the permutation and i is taken modulo n.

Definition 4.4.14 (Ordered cyclic major index)**.** Define the *ordered cyclic major index* of $[\pi]$ to be the cyclic word

$$\operatorname{ocmaj}[\pi] := [\operatorname{cmaj}\pi^{(1)}, \operatorname{cmaj}\pi^{(2)}, \dots, \operatorname{cmaj}\pi^{(n)}],$$

i.e., the equivalence class of the sequence $(\operatorname{cmaj}\pi^{(1)}, \operatorname{cmaj}\pi^{(2)}, \dots, \operatorname{cmaj}\pi^{(n)})$ under cyclic shift.

Theorem 4.4.15. *The cyclic permutation statistics* cDes *and* ocmaj *are equivalent.*

Proof. Let us assume throughout this proof that $n \geq 2$, as the cases $n = 0$ and $n = 1$ are trivial. To see that cDes is a refinement of ocmaj, suppose $\operatorname{cDes}[\pi] = \operatorname{cDes}[\sigma]$ where π and σ have the same length n. So, we can write $[\sigma] = \{\sigma^{(1)}, \sigma^{(2)}, \dots, \sigma^{(n)}\}$ where $\operatorname{cDes}\pi^{(i)} = \operatorname{cDes}\sigma^{(i)}$ for all $i \in [n]$. It follows that

$$\operatorname{cmaj}\pi^{(i)} = \sum_{k \in \operatorname{cDes}\pi^{(i)}} k = \sum_{k \in \operatorname{cDes}\sigma^{(i)}} k = \operatorname{cmaj}\sigma^{(i)}$$

for all i, so $\operatorname{ocmaj}[\pi] = \operatorname{ocmaj}[\sigma]$.

For the converse, it is sufficient to show that the cyclic descent composition $\operatorname{cComp}[\pi]$ can be reconstructed from $\operatorname{ocmaj}[\pi]$. First, recall Equation (4.4.3):

$$\operatorname{cmaj}\pi^{(i+1)} = \begin{cases} \operatorname{cmaj}\pi^{(i)} + \operatorname{cdes}[\pi] - n, & \text{if } n \in \operatorname{cDes}\pi^{(i)}, \\ \operatorname{cmaj}\pi^{(i)} + \operatorname{cdes}[\pi], & \text{if } n \notin \operatorname{cDes}\pi^{(i)}. \end{cases}$$

Since $n \geq 2$, we have $1 \leq \operatorname{cdes}[\pi] \leq n - 1$, and together with the above equation, we have that $n \in \operatorname{cDes}\pi^{(i)}$ if and only if $\operatorname{cmaj}\pi^{(i)} > \operatorname{cmaj}\pi^{(i+1)}$. A similar argument shows that we can never have $\operatorname{cmaj}\pi^{(i)} = \operatorname{cmaj}\pi^{(i+1)}$.

Now, suppose we are given $\operatorname{ocmaj}[\pi] = [m_1, m_2, \dots, m_n]$ where $m_i = \operatorname{cmaj}\pi^{(i)}$. Let s and t be two consecutive cyclic descents of $\operatorname{ocmaj}[\pi]$, i.e.,

$$m_s > m_{s+1} < m_{s+2} < \cdots < m_t > m_{t+1}$$

where subscripts are considered modulo n as usual. From the previous paragraph, it follows that n is in both $\operatorname{cDes}\pi^{(s)}$ and $\operatorname{cDes}\pi^{(t)}$, and that the penultimate descent in $\operatorname{cDes}\pi^{(s)}$ becomes the descent $n \in \operatorname{cDes}\pi^{(t)}$ with n never being a descent for any of the intermediate cyclic descent sets.

So $t - s$ (modulo n) is a part of the cyclic composition $\operatorname{cComp}[\pi]$. Therefore, all the parts of $\operatorname{cComp}[\pi]$ can be determined, and their order will be the same as that induced by the consecutive cyclic descents in $\operatorname{ocmaj}[\pi]$. Thus we have reconstructed $\operatorname{cComp}[\pi]$ from $\operatorname{ocmaj}[\pi]$, completing the proof. □

Corollary 4.4.16 (Cyclic shuffle-compatibility of ocmaj)**.** *The ordered cyclic major index* ocmaj *is cyclic shuffle-compatible, and its cyclic shuffle algebra* $\mathcal{A}^{\text{cyc}}_{\text{ocmaj}}$ *is isomorphic to* $\mathcal{A}^{\text{cyc}}_{\text{cDes}}$.

Of course, one could wonder if the unordered multiset of cmaj values is also equivalent to cDes for cyclic permutations, but this is not the case. Indeed, if the cyclic permutation statistics cmaj and cDes were equivalent, then the cyclic shuffle-compatibility of cDes would imply that cmaj is cyclic shuffle-compatible as well, which we know to be false.

4.4.4 Other Descent Statistics

To conclude this section, let us consider the cyclic permutation statistics induced by the following linear descent statistics:

- The *valley number* val, which we defined earlier to be the number of valleys of a permutation.

- The *double descent set* Ddes and the *double descent number* ddes. We call $i \in \{2, 3, \ldots, n-1\}$ a *double descent* of $\pi \in \mathfrak{S}_n$ if $\pi_{i-1} > \pi_i > \pi_{i+1}$. Then $\operatorname{Ddes} \pi$ is the set of double descents of π, and $\operatorname{ddes} \pi$ the number of double descents of π.

- The *left peak set* Lpk and the *left peak number* lpk. We call $i \in [n-1]$ a *left peak* of $\pi \in \mathfrak{S}_n$ if i is a peak of π, or if $i = 1$ and $\pi_1 > \pi_2$. Then $\operatorname{Lpk} \pi$ is the set of left peaks of π, and $\operatorname{lpk} \pi$ the number of left peaks of π.

- The *right peak set* Rpk and the *right peak number* rpk. We call $i \in \{2, 3, \ldots, n\}$ a *right peak* of $\pi \in \mathfrak{S}_n$ if i is a peak of π, or if $i = n$ and $\pi_{n-1} < \pi_n$. Then $\operatorname{Rpk} \pi$ is the set of right peaks of π, and $\operatorname{rpk} \pi$ the number of right peaks of π.

- The *exterior peak set* Epk and the *exterior peak number* epk. We call $i \in [n]$ an *exterior peak* of $\pi \in \mathfrak{S}_n$ if i is a left peak or right peak of π. Then $\operatorname{Epk}\pi$ is the set of exterior peaks of π, and $\operatorname{epk}\pi$ the number of exterior peaks of π.

- The *number of biruns* bru and the *number of up-down runs* udr. Recall that s *birun* of π is a maximal consecutive monotone subsequence of π; an *up-down run* of π is a birun of π, or the first letter π_1 of π if $\pi_1 > \pi_2$. Then $\operatorname{bru}\pi$ and $\operatorname{udr}\pi$ are the number of biruns and the number of up-down runs, respectively, of π.

For example, take $\pi = 713942658$. Then we have $\operatorname{val}\pi = 3$, $\operatorname{Ddes}\pi = \{5\}$, $\operatorname{ddes}\pi = 1$, $\operatorname{Lpk}\pi = \{1,4,7\}$, $\operatorname{lpk}\pi = 3$, $\operatorname{Rpk}\pi = \{4,7,9\}$, $\operatorname{rpk}\pi = 3$, $\operatorname{Epk}\pi = \{1,4,7,9\}$, $\operatorname{epk}\pi = 4$, $\operatorname{bru}\pi = 6$, and $\operatorname{udr}\pi = 7$.

Aside from Ddes, ddes, and bru, all of the above statistics (as linear permutation statistics) are shuffle-compatible. Also, because these are all descent statistics, each of the induced cyclic statistics are cyclic descent statistics. Indeed, if we are given $\operatorname{cDes}[\pi]$ and the length of π, then we can determine $\operatorname{Des}[\pi]$ by Lemma 4.4.1, and we can then use the descent sets in $\operatorname{Des}[\pi]$ to obtain the multiset $\operatorname{st}[\pi]$ for any descent statistic st.

Let us begin by examining the double descent statistics Ddes and ddes. Since neither Ddes nor ddes are shuffle-compatible as linear permutation statistics, it is perhaps unsurprising that their induced cyclic statistics are not cyclic shuffle-compatible. As a counterexample, let $\pi = 1234$, $\sigma = 1324$, and $\rho = 5$. Then both $\operatorname{Ddes}[\pi] = \operatorname{Ddes}[\sigma]$ and $\operatorname{ddes}[\pi] = \operatorname{ddes}[\sigma]$, but we have $\operatorname{Ddes}([\pi] \sqcup [\rho]) \neq \operatorname{Ddes}([\sigma] \sqcup [\rho])$ and $\operatorname{ddes}([\pi] \sqcup [\rho]) \neq \operatorname{ddes}([\sigma] \sqcup [\rho])$. For instance, $\{\{\emptyset^5\}\}$ appears three times in $\operatorname{Ddes}([\pi] \sqcup [\rho])$ but only twice in $\operatorname{Ddes}([\sigma] \sqcup [\rho])$, and accordingly $\{\{0^5\}\}$ appears three times in $\operatorname{ddes}([\pi] \sqcup [\rho])$ but only twice in $\operatorname{ddes}([\sigma] \sqcup [\rho])$.

While the linear statistic bru is not shuffle-compatible, in Theorem 3.2.6 we defined the cyclic statistic cbru which gives the number of *cyclic biruns*—maximal consecutive monotone cyclic subsequences—is cyclic shuffle-compatible as it is precisely twice the number of cyclic peaks.

Theorem 4.4.17 (Cyclic shuffle-compatibility of cbru and (cbru, cdes))**.** *The cyclic statistics* cbru *and* (cbru, cdes) *are cyclic shuffle-compatible, and we have the* $\mathbb{Q}$*-algebra isomorphisms*

$$\mathcal{A}^{\text{cyc}}_{\text{cbru}} \cong \mathcal{A}^{\text{cyc}}_{\text{cpk}} \quad \textit{and} \quad \mathcal{A}^{\text{cyc}}_{(\text{cbru},\text{cdes})} \cong \mathcal{A}^{\text{cyc}}_{(\text{cpk},\text{cdes})}.$$

Because des and cdes are equivalent as cyclic permutation statistics and similarly with pk and cpk, one might expect the cyclic statistics bru and cbru to be equivalent as well, but this is not the case because bru is not actually cyclic shuffle-compatible. For instance, consider $\pi = 25673489$, $\sigma = 24567389$, and $\rho = 1$. Then $\text{bru}[\pi] = \text{bru}[\sigma]$, but the multiset $\{\{5^4, 6^4, 7\}\}$ appears four times in $\text{bru}([\pi] \sqcup [\rho])$ but only twice in $\text{bru}([\sigma] \sqcup [\rho])$. One can also use the same permutations π, σ, and ρ to show that (bru, des) is not cyclic shuffle-compatible.

Even though the linear statistics Lpk and Epk are shuffle-compatible, their induced cyclic statistics are not cyclic shuffle-compatible. As a counterexample, take

$$\begin{aligned} \pi &= 11\,6\,3\,7\,1\,4\,12\,10\,2\,9\,6\,8, & \sigma &= 13, \\ \pi' &= 13\,7\,2\,9\,5\,3\,10\,4\,8\,12\,6\,11, \text{ and} & \sigma' &= 1. \end{aligned}$$

Then we have $\text{Lpk}[\pi] = \text{Lpk}[\pi']$, $\text{Lpk}[\sigma] = \text{Lpk}[\sigma']$, $\text{Epk}[\pi] = \text{Epk}[\pi']$, and $\text{Epk}[\sigma] = \text{Epk}[\sigma']$, yet $\text{Lpk}([\pi] \sqcup [\sigma]) \neq \text{Lpk}([\pi'] \sqcup [\sigma'])$ and $\text{Epk}([\pi] \sqcup [\sigma]) \neq \text{Epk}([\pi'] \sqcup [\sigma'])$ as the multiset

$$\begin{aligned} \{\{\,&\{1,5,8,11\}, \{2,6,9,12\}, \{3,7,10\}, \{1,4,8,11\}, \{2,5,9,12\}, \{1,3,6,10\}, \\ &\{1,4,7,11\}, \{2,5,8,12\}, \{3,6,9\}, \{1,4,7,10\}, \{2,5,8,11\}, \{1,3,6,9,12\}, \{1,4,7,10\}\,\}\} \end{aligned}$$

belongs to $\text{Lpk}([\pi] \sqcup [\sigma])$ but not $\text{Lpk}([\pi'] \sqcup [\sigma'])$, and the multiset

$$\begin{aligned} \{\{\,&\{1,4,7,10\}, \{1,4,8,11\}, \{2,5,8,11\}, \{2,5,8,12\}, \{2,5,9,12\}, \\ &\{2,6,9,12\}, \{3,6,9,13\}, \{3,7,10,13\}, \{1,3,6,9,12\}, \\ &\{1,3,6,10,13\}, \{1,4,7,10,13\}, \{1,4,7,11,13\}, \{1,5,8,11,13\}\,\}\} \end{aligned}$$

belongs to $\text{Epk}([\pi] \sqcup [\sigma])$ but not $\text{Epk}([\pi'] \sqcup [\sigma'])$.

The left peak number lpk, number of up-down runs udr, and the pairs (lpk, des) and (udr, des) are also shuffle-compatible linear statistics whose induced cyclic statistics are not cyclic shuffle-compatible. For example, take $\pi = 87516439$, $\sigma = 53187649$, and $\rho = 2$. Then $(\text{lpk}, \text{des})[\pi] =$

$(\mathrm{lpk},\mathrm{des})[\sigma]$ and $(\mathrm{udr},\mathrm{des})[\pi] = (\mathrm{udr},\mathrm{des})[\sigma]$ (and thus $\mathrm{lpk}[\pi] = \mathrm{lpk}[\sigma]$ and $\mathrm{udr}[\pi] = \mathrm{udr}[\sigma]$). However:

- $\{\{(3,5)^6,(3,6)^3\}\}$ is in $(\mathrm{lpk},\mathrm{des})([\pi] \sqcup [\rho])$ but not $(\mathrm{lpk},\mathrm{des})([\sigma] \sqcup [\rho])$,
- $\{\{(6,5)^3,(6,6)^3,(7,5)^3\}\}$ is in $(\mathrm{udr},\mathrm{des})([\pi] \sqcup [\rho])$ but not $(\mathrm{udr},\mathrm{des})([\sigma] \sqcup [\rho])$,
- $\{\{3^9\}\}$ is in $\mathrm{lpk}([\pi] \sqcup [\rho])$ but not $\mathrm{lpk}([\sigma] \sqcup [\rho])$,
- and $\{\{6^6,7^3\}\}$ is in $\mathrm{udr}([\pi] \sqcup [\rho])$ but not $\mathrm{udr}([\sigma] \sqcup [\rho])$.

Observe that Rpk is r-equivalent to Lpk and rpk is r-equivalent to lpk. Hence, by Corollary 4.4.8, neither Rpk nor rpk are cyclic shuffle-compatible. One can also define "left", "right", and "exterior" versions of the valley set and valley number; by similar symmetry arguments, none of these are cyclic shuffle-compatible either.

In contrast, the exterior peak number epk and the pair (epk, des) are cyclic shuffle-compatible because they are equivalent to cpk and (cpk, cdes), respectively. To prove these equivalences, we will also need to consider the *cyclic valley number* statistic cval: we say that $i \in [n]$ is a *cyclic valley* of $\pi \in \mathfrak{S}_n$ if $\pi_{i-1} > \pi_i < \pi_{i+1}$ with the indices considered modulo n, and $\mathrm{cval}[\pi]$ is defined to be the number of cyclic valleys of any permutation in $[\pi]$. Equivalently, $\mathrm{cval}[\pi]$ is the cardinality of the cyclic valley set $\mathrm{cVal}[\pi]$ defined in Section 4.4.2.

Lemma 4.4.18. *The cyclic permutation statistics* val *and* cval *are equivalent.*

Proof. We have $\mathrm{val}[\pi] = \mathrm{pk}[\pi^c]$ for all π—that is, val and pk are c-equivalent—and similarly with cval and cpk. By Lemma 4.4.4, pk and cpk are equivalent, so the same is true of val and cval. □

Lemma 4.4.19. *For any cyclic permutation* $[\pi]$*, we have* $\mathrm{cval}[\pi] = \mathrm{cpk}[\pi]$.

Proof. Each cyclic birun starts with a cyclic peak and ends with a cyclic valley or vice-versa. So $2\,\mathrm{cpk}[\pi] = \mathrm{cbru}[\pi] = 2\,\mathrm{cval}[\pi]$. □

Lemma 4.4.20. *The cyclic permutation statistics* epk *and* cpk *are equivalent.*

Proof. For any linear permutation π, we have $\operatorname{epk}\pi = \operatorname{val}\pi + 1$ [GZ18, Lemma 2.1 (e)], so epk and val are equivalent as linear permutation statistics and thus as cyclic permutation statistics. (We can obtain $\operatorname{val}[\pi]$ from $\operatorname{epk}[\pi]$ by subtracting 1 from each element in the multiset, and $\operatorname{epk}[\pi]$ from $\operatorname{val}[\pi]$ by adding 1 to each element.) Moreover, val is equivalent to cval (Lemma 4.4.18) which is in turn equivalent to cpk (Lemma 4.4.19); hence, epk is equivalent to cpk. □

Theorem 4.4.21 (Cyclic shuffle-compatibility of val, cval, epk, $(\operatorname{val}, \operatorname{des})$, $(\operatorname{cval}, \operatorname{cdes})$, and $(\operatorname{epk}, \operatorname{des})$)**.** *The cyclic statistics* val*,* cval*,* epk*,* $(\operatorname{val}, \operatorname{des})$*,* $(\operatorname{cval}, \operatorname{cdes})$*, and* $(\operatorname{epk}, \operatorname{des})$ *are cyclic shuffle-compatible, and we have the* $\mathbb{Q}$*-algebra isomorphisms*

$$\mathcal{A}^{\text{cyc}}_{\text{val}} \cong \mathcal{A}^{\text{cyc}}_{\text{cval}} \cong \mathcal{A}^{\text{cyc}}_{\text{epk}} \cong \mathcal{A}^{\text{cyc}}_{\text{cpk}} \quad \textit{and} \quad \mathcal{A}^{\text{cyc}}_{(\text{val},\text{des})} \cong \mathcal{A}^{\text{cyc}}_{(\text{cval},\text{cdes})} \cong \mathcal{A}^{\text{cyc}}_{(\text{epk},\text{des})} \cong \mathcal{A}^{\text{cyc}}_{(\text{cpk},\text{cdes})}.$$

Proof. The cyclic shuffle-compatibility of val, cval, and epk, and the corresponding isomorphisms, follow from the cyclic shuffle-compatibility of cpk and the equivalences between these four statistics. Furthermore, $(\operatorname{val}, \operatorname{des})$ is equivalent to $(\operatorname{cpk}, \operatorname{cdes})$ because val is equivalent to cpk and des is equivalent to cdes, and similarly $(\operatorname{cval}, \operatorname{cdes})$ and $(\operatorname{epk}, \operatorname{des})$ are equivalent to $(\operatorname{cpk}, \operatorname{cdes})$ as well. Because $(\operatorname{cpk}, \operatorname{cdes})$ is cyclic shuffle-compatible, the results for $(\operatorname{val}, \operatorname{des})$, $(\operatorname{cval}, \operatorname{cdes})$, and $(\operatorname{epk}, \operatorname{des})$ follow. □

Finally, we provide counterexamples showing that neither $(\operatorname{Pk}, \operatorname{Val})$ nor $(\operatorname{pk}, \operatorname{val})$ are cyclic shuffle-compatible. Let $\pi = 214$, $\sigma = 536$, $\pi' = 123$, and $\sigma' = 546$. Then $(\operatorname{Pk}, \operatorname{Val})[\pi] = (\operatorname{Pk}, \operatorname{Val})[\pi']$ and $(\operatorname{Pk}, \operatorname{Val})[\sigma] = (\operatorname{Pk}, \operatorname{Val})[\sigma']$, which imply $(\operatorname{pk}, \operatorname{val})[\pi] = (\operatorname{pk}, \operatorname{val})[\pi']$ and $(\operatorname{pk}, \operatorname{val})[\sigma] = (\operatorname{pk}, \operatorname{val})[\sigma']$ as well. However,

$$\{\{\, (\emptyset, \emptyset), (\emptyset, \{5\}), (\{2\}, \emptyset), (\{3\}, \{2\}), (\{4\}, \{3\}), (\{5\}, \{4\}) \,\}\}$$

is an element of $(\operatorname{Pk}, \operatorname{Val})([\pi] \sqcup\!\sqcup [\sigma])$ but not $(\operatorname{Pk}, \operatorname{Val})([\pi'] \sqcup\!\sqcup [\sigma'])$, and

$$\{\{\, (0,0), (0,1), (1,0), (1,1)^3 \,\}\}$$

is an element of $(\operatorname{pk}, \operatorname{val})([\pi] \sqcup\!\sqcup [\sigma])$ but not $(\operatorname{pk}, \operatorname{val})([\pi'] \sqcup\!\sqcup [\sigma'])$.

CHAPTER 5

ENRICHED TORIC $[\vec{D}]$-PARTITIONS

In section 5.1 we introduce enriched $\vec{D}$-partitions in terms of directed acyclic graphs (DAGs), and review the weight enumerators of enriched $\vec{D}$-partitions from Stembridge. Section 5.2 will focus on defining enriched toric $[\vec{D}]$-partitions and developing some of their properties. In particular, the weight enumerators corresponding to different cyclic peak sets generate a subring of cQSym and we call it the algebra of cyclic peaks. We also compute the order polynomial of enriched toric $[\vec{D}]$-partitions. In section 5.3, we further discuss the implication on cyclic shuffle-compatibility from the order polynomial in previous section. In this chapter, we will use $\leq$ with no subscript to denote the ordinary total order on $\mathbb{Z}$, the set of integers.

5.1 Enriched Partitions on Posets

In [Ste97], Stembridge defined enriched P-partitions for a poset P. In contrast to having $\mathbb{P}$ with the ordinary order $\leq$ as the range for ordinary P-partitions, enriched P-partitions are obtained by imposing another total order on the range, $\mathbb{P}'$, which is the set of nonzero integers with an unusual order. We review the basic theory from Stembridge and naturally extend enriched P-partitions to enriched $\vec{D}$-partitions where $\vec{D}$ is a directed acyclic graph which is not necessarily transitive.

5.1.1 DAGs and Posets

A *directed acyclic graph* (DAG) is a digraph with no directed cycles. Suppose $\vec{D}$ is a DAG with vertex set $[n]$. A DAG $\vec{D}$ is *transitive* if $i \to j$ and $j \to k$ implies $i \to k$ for $i, j, k \in [n]$. If $\vec{P}$ is a transitive DAG, it will naturally induce a partial order $<_{\vec{P}}$ on the vertex set $[n]$, defined so that for two vertices i and j, one has $i <_{\vec{P}} j$ if and only if $i \to j$ in $\vec{P}$; in that case, we also use $\vec{P}$ to denote this poset.

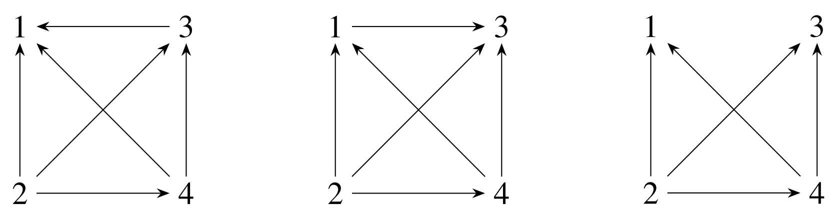
In general, we can associate a partial order with any DAG $\vec{D}$. For this purpose, define the *transitive closure* $\vec{P}$ of $\vec{D}$ as the directed graph obtained from $\vec{D}$ by adding in $i \to k$ if one has both $i \to j$ and $j \to k$ in $\vec{D}$. Such $\vec{P}$ is unique. Moreover, it is straightforward to verify that the transitive closure $\vec{P}$ is both acyclic and transitive. This implies that $\vec{P}$ is actually a transitive DAG, hence has a partial order structure. We will maintain the use $\rightarrow$ for the relation between vertices in a general DAG $\vec{D}$, while using the partial order $\leq_{\vec{P}}$ on a poset $\vec{P}$.

Definition 5.1.1 (Total linear order)**.** A poset $\vec{P}$ is a *total linear order* if it is a complete DAG, i.e., there is a directed edge between every pair of vertices in $\vec{P}$.

There is a bijection between the set of total linear orders $\vec{P}$ on the vertex set $[n]$ and $\mathfrak{S}_n$. For a total linear order $\vec{P}$ on $[n]$, there exists a unique directed path $\pi_1 \rightarrow \pi_2 \rightarrow \cdots \rightarrow \pi_n$ in $\vec{P}$, hence $\vec{P}$ can be identified with the permutation $\pi = \pi_1\pi_2 \ldots \pi_n \in \mathfrak{S}_n$. Conversely, given a permutation $\pi = \pi_1\pi_2 \ldots \pi_n \in \mathfrak{S}_n$, we can construct a DAG $\vec{P}$ by putting arrows $\pi_i \rightarrow \pi_j$ for all $1 \leq i < j \leq n$ on the vertex set $[n]$, and it is easy to check that the resulting DAG $\vec{P}$ is actually a total linear order. In this case, we usually use a permutation π to denote the corresponding total linear order $\vec{P}$.

For two DAGs $\vec{D}_1$ and $\vec{D}_2$ on the same vertex set $[n]$, we say $\vec{D}_2$ *extends* $\vec{D}_1$ if $\vec{D}_1$ is a subgraph of $\vec{D}_2$, written as $\vec{D}_1 \subseteq \vec{D}_2$. If, furthermore, $\vec{D}_2$ is a total linear order corresponding to the permutation $\pi \in \mathfrak{S}_n$, we also say that π *linearly extends* $\vec{D}_1$. Denote by $\mathcal{L}(\vec{D})$ the set of all permutations $\pi \in \mathfrak{S}_n$ which linearly extend $\vec{D}$.

Example 5.1.2. Here are several DAGs on the vertex set $[4] = \{1, 2, 3, 4\}$.



$\vec{D}_1 = 2431$ $\vec{D}_2 = 2413$ $\vec{D}_3$

Note that $\vec{D}_1$ is the total linear order with permutation 2431, $\vec{D}_2$ is the total linear order for 2413. Both $\vec{D}_1$ and $\vec{D}_2$ extend $\vec{D}_3$, hence 2431 and 2413 linearly extend $\vec{D}_3$. Moreover, $\vec{D}_1$ and $\vec{D}_2$ are the only total linear orders which extend $\vec{D}_3$, therefore $\mathcal{L}(\vec{D}_3) = \{2431, 2413\}$.

5.1.2 Enriched $\vec{D}$-partition

Now we are in a good position to define enriched $\vec{D}$-partitions for a DAG $\vec{D}$. Stembridge originally defined the enriched P-partitions when P is a poset. This definition can be easily extended to the cases when $\vec{D}$ is simply a DAG, as the definition does not rely on the transitivity of P.

Stembridge defines $\mathbb{P}'$ to be the set of nonzero integers, totally ordered as

$$-1 \prec 1 \prec -2 \prec 2 \prec -3 \prec 3 \prec \cdots .$$

Definition 5.1.3 (Enriched $\vec{D}$-partition)**.** Let $\vec{D}$ be a DAG on $[n]$. An *enriched $\vec{D}$-partition* is a function $f : [n] \to \mathbb{P}'$ such that for all $i \to j$ in $\vec{D}$,

(a) $f(i) \preceq f(j)$,

(b) $f(i) = f(j) > 0$ implies $i < j$,

(c) $f(i) = f(j) < 0$ implies $i > j$.

Denote by $\mathcal{E}(\vec{D})$ the set of all enriched $\vec{D}$-partitions f.

Remark 5.1.4.

1. In this definition we are using two order structures on the domain $[n]$: the order $\to$ induced by DAG $\vec{D}$ and the ordinary total order $\leq$ on integers in (b) and (c). Both of them will impose restrictions on the possible choices for f. As for the range $\mathbb{P}'$, we also use two order structures: the total order $\preceq$ defined by Stembridge in (a) and the usual order $\leq$ on the integers in (b) and (c).

2. If $\vec{D} = \pi$ is a total linear order, the structure of the set of enriched π-partitions is quite simple:

$$\begin{aligned}\mathcal{E}(\pi) = \{\, f : [n] \to \mathbb{P}' \mid \quad & f(\pi_1) \preceq \cdots \preceq f(\pi_n), \\ & f(\pi_i) = f(\pi_{i+1}) > 0 \Rightarrow i \notin \mathrm{Des}(\pi), \\ & f(\pi_i) = f(\pi_{i+1}) < 0 \Rightarrow i \in \mathrm{Des}(\pi) \,\}.\end{aligned} \tag{5.1.1}$$

The following fundamental lemma is a straightforward analogue of Stembridge [Ste97, Lemma 2.1].

Lemma 5.1.5 (Fundamental lemma of enriched $\vec{D}$-partitions)**.** *For any DAG $\vec{D}$ with vertex set $[n]$, one has a decomposition of $\mathcal{E}(\vec{D})$ as the following disjoint union:*

$$\mathcal{E}(\vec{D}) = \bigsqcup_{\pi \in \mathcal{L}(\vec{D})} \mathcal{E}(\pi).$$

Proof. Given an enriched $\vec{D}$-partition f. First we arrange the elements of $[n]$ in a weakly increasing order of f-values with respect to the total order $\preceq$ on the range. Then if some elements in $[n]$ have the same f-value $-k$ (respectively, $+k$) for some positive integer k, we arrange them in a decreasing (respectively, increasing) order with respect to the usual order $\leq$ on the domain. The resulting permutation π is unique with $f \in \mathcal{E}(\pi)$, and π linearly extends $\vec{D}$. On the other hand, for $\pi \in \mathcal{L}(\vec{D})$, every enriched π-partition is also an enriched $\vec{D}$-partition. Therefore the conclusion follows. □

Example 5.1.6. Returning to Example 5.1.2, $\mathcal{L}(\vec{D}_3) = \{2431, 2413\}$, hence by Lemma 5.1.5, $\mathcal{E}(\vec{D}_3) = \mathcal{E}(2431) \uplus \mathcal{E}(2413)$.

5.1.3 Weight Enumerators

Suppose $\vec{D}$ is a DAG on $[n]$. Define the *weight enumerator for enriched $\vec{D}$-partitions* to be the formal power series

$$\Delta_{\vec{D}} := \sum_{f \in \mathcal{E}(\vec{D})} \prod_{i \in [n]} x_{|f(i)|},$$

where $\mathcal{E}(\vec{D})$ is the set of enriched $\vec{D}$-partitions. By the Fundamental Lemma 5.1.5, one has

$$\Delta_{\vec{D}} = \sum_{\pi \in \mathcal{L}(\vec{D})} \Delta_{\pi}. \tag{5.1.2}$$

It is clear from equation (5.1.1) that Δ_π is a homogeneous quasisymmetric function. More generally, $\Delta_{\vec{D}}$ is a homogeneous quasisymmetric function in QSym.

It also follows from equation (5.1.1) that the weight enumerator Δ_π depends only on the descent set $\operatorname{Des}\pi$. A less obvious but important observation, that Δ_π depends only on the peak set $\operatorname{Pk}\pi$, will follow directly from the following proposition, proved by Stembridge in [Ste97].

Proposition 5.1.7 ([Ste97, Proposition 2.2]). *As a quasisymmetric function, Δ_π has the following expansion in terms of monomial quasisymmetric functions:*

$$\Delta_\pi = \sum_{\substack{E\subseteq[n-1]:\\ \mathrm{Pk}\,\pi\subseteq E\cup(E+1)}} 2^{|E|+1} M_{n,E}, \tag{5.1.3}$$

where the set $E+1$ is defined by (2.1.1). □

As a counterpart, the weight enumerator Δ_π also has an expansion in terms of another basis: the fundamental quasisymmetric functions.

Proposition 5.1.8 ([Ste97, Proposition 3.5]). *As a quasi-symmetric function, Δ_π has the following expansion in terms of fundamental quasisymmetric functions:*

$$\Delta_\pi = 2^{\mathrm{pk}\,\pi+1} \sum_{D\subseteq[n-1]:\ \mathrm{Pk}\,\pi\subseteq D\triangle(D+1)} F_{n,D}.$$

Here $\triangle$ denotes symmetric difference, that is, $D\triangle E = (D\cup E)\setminus(D\cap E)$.

Definition 5.1.9 (Peak set). A subset $S\subseteq[n]$ is a *peak set in* $[n]$ if $\mathrm{Pk}\,\pi = S$ for some $\pi\in\mathcal{S}_n$.

For any peak set S in $[n]$, from the above proposition we can define an associated quasisymmetric function by

$$K_{n,S} := \Delta_\pi,$$

where π is any permutation with peak set S. It follows that $\Delta_\pi = K_{n,\mathrm{Pk}\,\pi}$ and one can rewrite equation (5.1.2) as

$$\Delta_{\vec{D}} = \sum_{\pi\in\mathcal{L}(\vec{D})} K_{n,\mathrm{Pk}\,\pi}.$$

Let Π_n denote the space of quasisymmetric functions spanned by $K_{n,S}$, taken over all peak sets in $[n]$, and set $\Pi = \oplus_{n\geq 0}\Pi_n$. In [Ste97] Stembridge referred to Π as the *algebra of peaks*, and proved that Π is a graded subring of QSym.

5.1.4 Order Polynomial

Given a DAG $\vec{D}$, we can define the *order polynomial of enriched $\vec{D}$-partitions*, $\Omega(\vec{D}, m)$, by

$$\Omega(\vec{D}, m) = \Delta_{\vec{D}}(1^m),$$

where $\Delta_{\vec{D}}(1^m)$ means that we set $x_1 = \cdots = x_m = 1$, and $x_k = 0$ for $k > m$. In fact, $\Omega(\vec{D}, m)$ counts the number of enriched $\vec{D}$-partitions with absolute value at most m.

Stembridge computed the corresponding generating function as follows:

Theorem 5.1.10 ([Ste97, Theorem 4.1])**.** *For a given $\pi \in \mathcal{S}_n$, one has*

$$\sum_m \Omega(\pi, m)t^m = \frac{1}{2}\left(\frac{1+t}{1-t}\right)^{n+1}\left(\frac{4t}{(1+t)^2}\right)^{1+\operatorname{pk}\pi}. \tag{5.1.4}$$

It is not hard to see that $\Omega(\vec{D}, t)$ is indeed a polynomial in t. If $\vec{D}$ has vertex set V and $|V| = n$, then

$$\Omega(\vec{D}, m) = \sum_{k=1}^{n} c_k \binom{m}{k},$$

where c_k denotes the number of $f \in \mathcal{E}(\vec{D})$ such that $\{|f(x)| \colon x \in V\} = [k]$. For any fixed k, $\binom{m}{k}$ is a polynomial in m of degree k. Since c_k and n are constants, it follows that the summation is also a polynomial in m. This verifies that $\Omega([\vec{D}], t)$ is a polynomial.

Clearly, one can get a formula for $\Omega(\pi, m)$ by taking the coefficient of t^m on both sides of equation (5.1.4). Here we provide a combinatorial explanation for this formula of $\Omega(\pi, m)$. We define the (π, m)-marking for a permutation π and a positive integer m, and show that each (π, m)-marking corresponds to exactly $2^{2\operatorname{pk}\pi+1}$ enriched π-partitions with absolute value at most m.

Suppose $\pi \in \mathcal{S}_n$, $m \in \mathbb{P}$. One can naturally extend $\pi = \pi_1 \dots \pi_n$ to $\pi' = \pi_0\pi_1 \dots \pi_n\pi_{n+1}$ where $\pi_0 = \pi_{n+1} = \infty$. Let R_1 be the longest decreasing initial factor of π'. Now let τ denote π' with R_1 deleted, and let R_2 be the longest increasing initial factor of τ. Continue in this way, alternating between decreasing and increasing factors to get a factorization of π'. We call the factors *runs* and the corresponding indices *run indices*. We denote by I_j the set of run indices corresponding to a factor set R_j. Note that any extension π' will start with a decreasing run and end with an increasing run, which implies that the number of runs is always even.

Take

$$\pi = \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \mathbf{1} & 4 & \mathbf{3} & \mathbf{2} & 5 & 6 \end{array}$$

as an example. The corresponding natural extension

$$\pi' = \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \infty & \mathbf{1} & 4 & \mathbf{3} & \mathbf{2} & 5 & 6 & \infty \end{array}$$

has four runs

$$R_1 = \infty 1, \ R_2 = 4, \ R_3 = 32, \ R_4 = 56\infty,$$

where the decreasing runs are in bold, and the corresponding set of run indices are

$$I_1 = \{0, 1\}, \ \ I_2 = \{2\}, \ I_3 = \{3, 4\}, \ I_4 = \{5, 6, 7\}.$$

Suppose that the permutation π' has r runs. We have the following observations:

1. The parity of i indicates the type of the run R_i. If i is even, then R_i is increasing. If i is odd, then R_i is decreasing.

2. The total number of runs r is closely related to the peak number $\operatorname{pk}\pi$:

$$r = 2\operatorname{pk}\pi + 2.$$

We now decorate permutations with bars and marks. Bars can be inserted between adjacent columns in the two-line notation (including the space before the first column and the space after the last), whereas a column of π with index $i \in [n]$ can be marked if and only if $i, i+1 \in I_j$ for some j; in other words, if i and $i+1$ are in the same run index set. There can be multiple bars between two adjacent columns and we count them with multiplicity, while each column can be marked at most once. We will denote by $\mathcal{M}_\pi$ the set of indices of the columns that can be marked,

$$\mathcal{M}_\pi := \{i \in [n] \mid \ i, i+1 \in I_j \text{ for some } j\}.$$

We note that for a given π, the cardinality of the complement of the set $\mathcal{M}_\pi$ in $[n]$ is $2\operatorname{pk}\pi + 1$. Equivalently, one has $|\mathcal{M}_\pi| = n - 2\operatorname{pk}\pi - 1$.

$$\begin{array}{cc||cccc} 1 & 2 & 3 & 4 & \mathbf{5} & 6 \\ 1 & 4 & 3 & 2 & \mathbf{5} & 6 \end{array} \qquad\qquad \begin{array}{cc|cccc|} 1 & 2 & \mathbf{3} & 4 & 5 & 6 \\ 1 & 4 & \mathbf{3} & 2 & 5 & 6 \end{array}$$

Figure 5.1 Two examples of $(\pi, 5)$-markings

Definition 5.1.11 ((π, m)-marking)**.** Suppose that π is a linear permutation and m is a positive integer. A (π, m)*-marking* is a marking of π using b bars and d marked columns, satisfying that $b + d = m - 1 - \operatorname{pk}\pi$.

Example 5.1.12. If we set $\pi = 143256$ as before, and take $m = 5$, then a $(\pi, 5)$-marking has b bars and d marked columns, such that $b + d = 3$. Two $(\pi, 5)$-markings are provided in Figure 5.1.12. Both have two bars and one marked column, where the marked column is in blue.

Proposition 5.1.13. *For a given $\pi \in \mathfrak{S}_n$, one has*

$$\Omega(\pi, m) = 2^{2\operatorname{pk}\pi+1} \sum_{k=0}^{m-1-\operatorname{pk}\pi} \left(\!\!\binom{n+1}{k}\!\!\right) \binom{n - 2\operatorname{pk}\pi - 1}{m - 1 - \operatorname{pk}\pi - k}, \tag{5.1.5}$$

where $\left(\!\binom{n+1}{k}\!\right)$ denotes the number of multisets on $[n+1]$ with cardinality k.

Proof. It is clear by definition that the number of possible choices for (π, m)-markings is

$$\sum_{b+d=m-1-\operatorname{pk}\pi} \left(\!\!\binom{n+1}{b}\!\!\right) \binom{n - 2\operatorname{pk}\pi - 1}{d} = \sum_{k=0}^{m-1-\operatorname{pk}\pi} \left(\!\!\binom{n+1}{k}\!\!\right) \binom{n - 2\operatorname{pk}\pi - 1}{m - 1 - \operatorname{pk}\pi - k}.$$

Therefore, it suffices to construct a $2^{2\operatorname{pk}\pi+1}$-to-one map from the set of enriched π-partitions with absolute value at most m to the set of all (π, m)-markings.

Given an enriched π-partition f with absolute value at most m, we can inductively associate to it a unique (π, m)-marking as follows: We first determine, for each $k \in \mathcal{M}_\pi$, whether column k gets marked. Suppose $k \in I_i$ for some $i \in [r]$. We mark column k if and only if $\delta_k + \gamma_k = 1$ where

$$\delta_k := \delta\,(i \text{ is even and } f(\pi_k) < 0), \qquad \gamma_k := \delta\,(i \text{ is odd and } f(\pi_k) > 0).$$

Here the *Kronecker function* on a statement R is defined by

$$\delta(R) = \begin{cases} 1 & \text{if } R \text{ is true,} \\ 0 & \text{if } R \text{ is false.} \end{cases}.$$

As for the placement of bars, we start by putting $|f(\pi_1)| - 1$ bars before the first column. Inductively, suppose that $k \in I_i$ for some $k \in [2, n]$, $i \in [r]$ and we have already constructed the markings and bars on and before the $(k-1)$st column. Then the number of marks and bars strictly before the kth column is constructed to be

$$|f(\pi_k)| - \lceil i/2 \rceil - \delta_k. \tag{5.1.6}$$

Finally we add bars after the last column so that the total number of marks and bars is $m - \operatorname{pk} \pi - 1$. In this manner, we can inductively define a unique (π, m)-marking for f.

We must verify that the constructed marking is indeed a (π, m)-marking. Firstly we will verify that (5.1.6) is a weakly increasing function of k, and strictly increasing from the kth to the $(k+1)$st term if column k is marked. In other words, if $k \in I_i$ and $k+1 \in I_j$, it suffices to show that

$$|f(\pi_k)| - \lceil i/2 \rceil - \delta_k \leq |f(\pi_{k+1})| - \lceil j/2 \rceil - \delta_{k+1},$$

for all k, and that

$$|f(\pi_k)| - \lceil i/2 \rceil - \delta_k < |f(\pi_{k+1})| - \lceil j/2 \rceil - \delta_{k+1},$$

if column k is marked. Notice that

$$(\delta_k + \gamma_k) \cdot \delta(k \in \mathcal{M}_\pi) = 1$$

if column k is marked and 0 otherwise. Hence it suffices to show that

$$|f(\pi_k)| - \lceil i/2 \rceil - \delta_k + (\delta_k + \gamma_k) \cdot \delta(k \in \mathcal{M}_\pi) \leq |f(\pi_{k+1})| - \lceil j/2 \rceil - \delta_{k+1}. \tag{5.1.7}$$

By the definition of enriched P-partitions, we have

$$|f(\pi_k)| \leq |f(\pi_{k+1})|. \tag{5.1.8}$$

Let us consider the following cases:

(a) If $j = i$, it follows that $k \in \mathcal{M}_\pi$. Hence inequality (5.1.7) simplifies to

$$|f(\pi_k)| + \gamma_k \leq |f(\pi_{k+1})| - \delta_{k+1}.$$

If i is even, then $\gamma_k = 0$ and $\pi_k < \pi_{k+1}$. By inequality (5.1.8), one only needs to consider whether the inequality holds when $\delta_{k+1} = 1$, which implies $f(\pi_{k+1}) < 0$. It follows from the definition of enriched P-partitions that $f(\pi_k) \preceq f(\pi_{k+1})$, namely $|f(\pi_k)| < |f(\pi_{k+1})|$ or $f(\pi_k) = f(\pi_{k+1}) < 0$, but the second possibility contradicts $\pi_k < \pi_{k+1}$. This proves (5.1.7) in this case. The proof is similar when i is odd.

(b) If $j = i + 1$, then $k \notin \mathcal{M}_\pi$. Inequality (5.1.7) reduces to

$$|f(\pi_k)| - \lceil i/2 \rceil - \delta_k \leq |f(\pi_{k+1})| - \lceil (i+1)/2 \rceil - \delta_{k+1}.$$

If i is even, then $\pi_k > \pi_{k+1}$, $\lceil i/2 \rceil + 1 = \lceil (i+1)/2 \rceil$ and j is odd, hence $\delta_{k+1} = 0$. Therefore one only needs to prove

$$|f(\pi_k)| - \delta_k \leq |f(\pi_{k+1})| - 1.$$

The inequality clearly holds when $\delta_k = 1$, so it suffices to consider the case when $\delta_k = 0$. In this case, $f(\pi_k) > 0$, hence by the definition of enriched P-partitions, or equivalently $|f(\pi_{k+1})| \geq |f(\pi_k)| + 1$, proves (5.1.7). The case when i is odd is similar and left to the reader.

Secondly, one also needs to check that it is possible to add bars (possibly 0) after the last column so that the total number of marks and bars is $m - \operatorname{pk}\pi - 1$. In other words, the number of bars we add after the nth column is nonnegative. Notice that $\lceil \frac{i}{2} \rceil - 1$ counts the number of peaks before the k-th column. Since $n \in I_r$, this implies that $\lceil \frac{r}{2} \rceil = \operatorname{pk}\pi + 1$. By (5.1.6) the total number of bars and marked columns before the nth column is $|f(\pi_n)| - \operatorname{pk}\pi - 1 - \delta_n$. Together with the fact that the nth column is marked if and only if $\delta_n + \gamma_n = 1$ and $n \in \mathcal{M}_\pi$, the total number of bars that should be added after the nth column is

$$\begin{aligned} & m - \operatorname{pk}\pi - 1 - (|f(\pi_n)| - \operatorname{pk}\pi - 1 - \delta_n) - (\delta_n + \gamma_n) \cdot \delta(n \in \mathcal{M}_\pi) \\ = & m - |f(\pi_n)| + \delta_n - (\delta_n + \gamma_n) \cdot \delta(n \in \mathcal{M}_\pi). \end{aligned}$$

Hence the nonnegativity condition becomes

$$m - |f(\pi_n)| + \delta_n - (\delta_n + \gamma_n) \cdot \delta(n \in \mathcal{M}_\pi) \geq 0,$$

or equivalently,

$$m - |f(\pi_n)| \geq (\delta_n + \gamma_n) \cdot \delta(n \in \mathcal{M}_\pi) - \delta_n. \tag{5.1.9}$$

By assumption f has absolute value at most m, hence $m - |f(\pi_n)| \geq 0$. Therefore, one only needs to check that the inequality holds when $(\delta_n + \gamma_n) \cdot \delta(n \in \mathcal{M}_\pi) - \delta_n = 1$, namely $\delta_n + \gamma_n = 1$, $\delta(n \in \mathcal{M}_\pi) = 1$ and $\delta_n = 0$, or equivalently, $n \in I_i$ where i is odd, $f(\pi_n) > 0$ and $n \in \mathcal{M}_\pi$. This is a contradiction since $n + 1$ must be in a run where the index is even by definition, which implies that n and $n + 1$ cannot be in the same run index set, hence $n \notin \mathcal{M}_\pi$. This completes the proof of inequality (5.1.9). It therefore follows that the marking we constructed is indeed a (π, m)-marking.

Now we need to show that for a given (π, m)-marking, there are $2^{2\,\mathrm{pk}\,\pi+1}$ different associated functions. From the above construction we notice that for each $k \in \mathcal{M}_\pi$, $f(\pi_k)$ is uniquely determined. More precisely, whether column k gets marked determines δ_k and γ_k as k determines the parity of i, hence the sign of $f(\pi_k)$ is determined by the definitions of δ_k and γ_k. Suppose $k \in I_i$. The number of marks and bars strictly before the kth column determines the value $|f(\pi_k)| - \lceil i/2 \rceil - \delta_k$, therefore it determines $f(\pi_k)$ as well. The only ambiguity is about $f(\pi_k)$ for $k \notin \mathcal{M}_\pi$. As the number of marks and bars strictly before the kth column fixes the value $L = |f(\pi_k)| - \lceil i/2 \rceil - \delta_k$, there are two possible choices of the value $f(\pi_k)$ for each $k \notin \mathcal{M}_\pi$: if i is even, then either $f(\pi_k) = -(L + \lceil i/2 \rceil + 1)$ or $f(\pi_k) = L + \lceil i/2 \rceil$; if i is odd, then either $f(\pi_k) = L + \lceil i/2 \rceil$ or $f(\pi_k) = -(L + \lceil i/2 \rceil)$. It follows that there are $2^{2\,\mathrm{pk}\,\pi+1}$ different functions corresponding to a given (π, m)-marking. □

Example 5.1.14. Consider the permutation $\pi = 143256$, and an example of enriched π-partitions f is defined as follows:

$$f(1) = 1,\ f(4) = -2,\ f(3) = -4,\ f(2) = -4,\ f(5) = -5,\ f(6) = 5.$$

The corresponding marking is the first one in the Figure 5.1.12.

5.2 Enriched Partitions for Toric DAGs

In this section, we review the toric DAGs and toric posets as cyclic analogues of DAGs and posets. Then we define enriched toric $[\vec{D}]$-partitions and develop some of their properties. The concept of toric poset was originally defined and studied by Develin, Macauley and Reiner in [DMR16]. Here we follow the presentation from Adin, Gessel, Reiner and Roichman [AGRR21].

Just like a linear permutation has a corresponding cyclic permutation as the equivalence class under the equivalence of rotation, for a DAG, we will define an equivalence relation and consider the equivalence class to be the corresponding toric DAG. It turns out that if π is a linear extension of the DAG $\vec{D}$, then $[\pi]$ is a toric extension of the corresponding toric DAG $[\vec{D}]$.

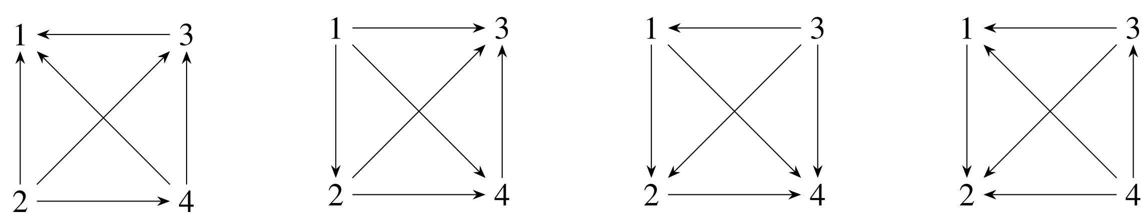
5.2.1 Toric DAGs and Toric Posets

A DAG $\vec{D}$ on $[n]$ has $i_0 \in [n]$ as a *source* (respectively, *sink*) if $\vec{D}$ does not contain $j \to i_0$ (respectively, $i_0 \to j$) for any $j \in [n]$. Suppose i_0 is a source or a sink in $\vec{D}$, we say $\vec{D}'$ is obtained from $\vec{D}$ by a *flip* at i_0 if $\vec{D}'$ is obtained by reversing all arrows containing i_0. We define the equivalence relation $\equiv$ on DAGs as follows: $\vec{D}' \equiv \vec{D}$ if and only if $\vec{D}'$ is obtained from $\vec{D}$ by a sequence of flips. A *toric* DAG is the equivalence class $[\vec{D}]$ of a DAG $\vec{D}$.

In particular, if $\vec{D} = \pi = \pi_1\pi_2 \dots \pi_n$ is a total linear order, the next proposition claims that the corresponding toric DAG $[\vec{D}]$ can be identified with the cyclic permutation $[\pi]$.

Proposition 5.2.1 ([DMR16, Proposition 4.2])**.** *If $\vec{D} = \pi$ is a total linear order with $\pi = \pi_1 \dots \pi_n$, then there is a bijection between toric DAG $[\vec{D}]$ and cyclic permutation $[\pi]$.* □

Example 5.2.2. The total linear order $\vec{D}_1 = 2431$ from Example 5.1.2 has a corresponding toric DAG $[\vec{D}_1]$:



$\vec{D}_1 = 2431$ $\quad$ $\vec{D}_1' = 1243$ $\quad$ $\vec{D}_1'' = 3124$ $\quad$ $\vec{D}_1''' = 4312$

They can be obtained by a sequence of flips:

$$\vec{D}_1 \xrightarrow{\text{flip at 1}} \vec{D}_1' \xrightarrow{\text{flip at 3}} \vec{D}_1'' \xrightarrow{\text{flip at 4}} \vec{D}_1''' \xrightarrow{\text{flip at 2}} \vec{D}_1.$$

Therefore it is easy to see that $[\vec{D}_1]$ can be identified with $[2431] = \{ 2431, 1243, 3124, 4312 \}$.

As transitivity turns a DAG into a poset, we now introduce the definition of toric transitivity for a DAG, which will turn the corresponding toric DAG into a toric poset.

A DAG $\vec{D}$ with vertex set $[n]$ is *toric transitive* if the existence of a directed path $i_1 \to i_2 \to \cdots \to i_k$ and $i_1 \to i_k$ implies the existence of $i_a \to i_b$ in $\vec{D}$ for all $1 \leq a < b \leq k$.

Definition 5.2.3 (Toric poset)**.** A toric DAG $[\vec{D}]$ is a *toric poset* if $\vec{D}'$ is toric transitive for some $\vec{D}' \in [\vec{D}]$, or equivalently from [AGRR21, Proposition 3.10] , for all representatives $\vec{D}'$.

In this paper, we adopt the definition of toric posets from [AGRR21], which is not quite the same as it was originally defined in [DMR16], but they are essentially equivalent by [DMR16, Theorem 1.4].

Definition 5.2.4 (Total cyclic order)**.** A toric poset is a *total cyclic order* if one (or equivalently, all, according to Proposition 5.2.1) representative is a total linear order.

In this case, we usually use the corresponding cyclic permutation $[\pi]$ to denote the total cyclic order $[\vec{D}]$.

For two toric DAGs $[\vec{D}_1], [\vec{D}_2]$ on the same vertex set $[n]$, we say $[\vec{D}_2]$ *extends* $[\vec{D}_1]$ if there exist $\vec{D}_i' \in [\vec{D}_i]$ for $i = 1, 2$ such that $\vec{D}_2'$ extends $\vec{D}_1'$. If, furthermore, $[\vec{D}_2]$ is a total cyclic order corresponding to the cyclic permutation $[\pi]$, we also say that $[\pi]$ torically extends $[\vec{D}_1]$. Let $\mathcal{L}^{\text{tor}}([\vec{D}])$ denote the set of cyclic permutations $[\pi]$ which torically extend $[\vec{D}]$.

Example 5.2.5. In Example 5.1.2, both $[\vec{D}_1] = [2431]$ and $[\vec{D}_2] = [2413]$ torically extend $[\vec{D}_3]$. Moreover, they are the only total cyclic orders that torically extend $[\vec{D}_3]$, namely $\mathcal{L}^{\text{tor}}([\vec{D}_3]) = \{[2431], [2413]\}$.

In fact, Figure 5.2 lists all representatives of $[\vec{D}_3]$, and it is straightforward to check that every total linear order linearly extending some DAG in $[\vec{D}_3]$ is in either $[2431]$ or $[2413]$.

5.2.2 Enriched Toric $[\vec{D}]$-Partition

Definition 5.2.6 (Enriched toric $[\vec{D}]$-partition)**.** An *enriched toric* $[\vec{D}]$*-partition* is a function $f : [n] \to \mathbb{P}'$ which is an enriched $\vec{D}'$-partition for at least one DAG $\vec{D}'$ in $[\vec{D}]$. Let $\mathcal{E}^{\text{tor}}([\vec{D}])$ denote

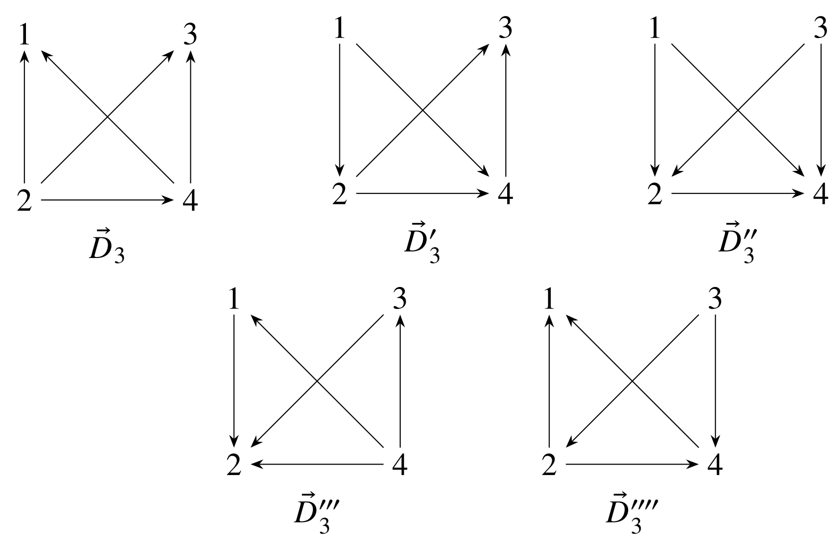


Figure 5.2 All representatives of $[\vec{D}_3]$

the set of all enriched toric $[\vec{D}]$-partitions.

If $[\vec{D}] = [\pi]$ is a total cyclic order, the set of enriched toric $[\pi]$-partitions is the union of the set of enriched π'-partitions for all representatives π' of $[\pi]$:

$$\mathcal{E}^{\mathrm{tor}}([\pi]) = \bigcup_{\pi' \in [\pi]} \mathcal{E}(\pi'). \tag{5.2.1}$$

As in the linear case, we have the following fundamental lemma for the decomposition of enriched toric $[\vec{D}]$-partitions. The proof is analogous to [AGRR21, Lemma 3.15].

Lemma 5.2.7 (Fundamental lemma of enriched toric $[\vec{D}]$-partitions)**.** *For a DAG $\vec{D}$, the set of all enriched toric $[\vec{D}]$-partitions is a disjoint union of the set of enriched toric $[\pi]$-partitions of all toric extensions $[\pi]$ of $[\vec{D}]$:*

$$\mathcal{E}^{\mathrm{tor}}([\vec{D}]) = \bigsqcup_{[\pi] \in \mathcal{L}^{\mathrm{tor}}([\vec{D}])} \mathcal{E}^{tor}([\pi]).$$

Proof. By the definition of $\mathcal{E}^{\mathrm{tor}}([\vec{D}])$, one has

$$\mathcal{E}^{\mathrm{tor}}([\vec{D}]) = \bigcup_{\vec{D}' \in [\vec{D}]} \mathcal{E}(\vec{D}').$$

In particular when $[\vec{D}] = [\pi]$ is a total cyclic order, it follows from Proposition 5.2.1 that,

$$\mathcal{E}^{\text{tor}}([\pi]) = \bigcup_{\pi' \in [\pi]} \mathcal{E}(\pi').$$

Hence,

$$\begin{aligned}
\mathcal{E}^{\text{tor}}([\vec{D}]) &= \bigcup_{\vec{D}' \in [\vec{D}]} \mathcal{E}(\vec{D}') \\
&\overset{(i)}{=} \bigcup_{\vec{D}' \in [\vec{D}]} \bigcup_{\pi' \in \mathcal{L}(\vec{D}')} \mathcal{E}(\pi') \\
&\overset{(ii)}{=} \bigcup_{[\pi] \in \mathcal{L}^{\text{tor}}([\vec{D}])} \bigcup_{\pi' \in [\pi]} \mathcal{E}(\pi') \\
&= \bigcup_{[\pi] \in \mathcal{L}^{\text{tor}}([\vec{D}])} \mathcal{E}^{\text{tor}}([\pi]).
\end{aligned}$$

To justify these steps, first note that equality (i) follows from Lemma 5.1.5.

As for equality (ii), it suffices to show that $\pi' \in \mathcal{L}(\vec{D}')$ for some $\vec{D}' \in [\vec{D}]$ if and only if $\pi' \in [\pi]$ for some $[\pi] \in \mathcal{L}^{\text{tor}}([\vec{D}])$. For the forward direction, if $\pi' \in \mathcal{L}(\vec{D}')$ for some $\vec{D}' \in [\vec{D}]$, then $[\pi']$ torically extends $[\vec{D}'] = [\vec{D}]$. While for the reverse implication, given $\pi' \in [\pi] \in \mathcal{L}^{\text{tor}}([\vec{D}])$, pick $\vec{D}'' \in [\vec{D}]$ and $\pi'' \in [\pi]$ with π'' linearly extending $\vec{D}''$, then $[\pi''] = [\pi'] = [\pi]$. It follows that there exists a sequence of flips which takes π'' to π'. Now applying the same sequence of flips to $\vec{D}''$ will result in some $\vec{D}'$. One then has $\vec{D}' \in [\vec{D}''] = [\vec{D}]$ and $\pi' \in \mathcal{L}(\vec{D}')$ as desired.

The assertion of disjointness follows directly from the fact that every function $f : [n] \to \mathbb{P}'$ has a unique linear permutation $\pi \in \mathcal{S}_n$ such that f is also an enriched π-partition, hence an enriched toric $[\pi]$-partition. Such a linear permutation π can be similarly constructed as in the proof of Lemma 5.1.5, so the details are omitted. This completes the proof. □

5.2.3 Weight Enumerator

Definition 5.2.8 (Weight enumerator)**.** For a given toric poset $[\vec{D}]$ with vertex set $[n]$, we define the *weight enumerator for enriched toric* $[\vec{D}]$*-partitions* by the formal power series

$$\Delta^{\text{cyc}}_{[\vec{D}]} := \sum_{f \in \mathcal{E}^{\text{tor}}([\vec{D}])} \prod_{i \in [n]} x_{|f(i)|}.$$

Namely, for integer $k > 0$ we assign the weight x_k to both f-values k and $-k$.

As a direct consequence of the Fundamental Lemma 5.2.7, one has

$$\Delta^{\text{cyc}}_{[\vec{D}]} = \sum_{[\pi]\in\mathcal{L}^{\text{tor}}([\vec{D}])} \Delta^{\text{cyc}}_{[\pi]}. \tag{5.2.2}$$

Therefore, it suffices to discuss $\Delta^{\text{cyc}}_{[\pi]}$ for cyclic permutations $[\pi]$. It follows from the formula (5.2.1) that $\Delta^{\text{cyc}}_{[\pi]}$ can be expressed in terms of the weight enumerators $\{\Delta_\tau\}$ as

$$\Delta^{\text{cyc}}_{[\pi]} = \sum_{\tau\in[\pi]} \Delta_\tau. \tag{5.2.3}$$

Moreover, $\Delta^{\text{cyc}}_{[\pi]}$ has the following expression.

Proposition 5.2.9. *For any given cyclic permutation $[\pi]$ of length n, we have*

$$\Delta^{cyc}_{[\pi]} = \sum_{\substack{E\subseteq[n]:\\ \text{cPk}(\pi)\subseteq E\cup(E+1)}} 2^{|E|} M^{cyc}_{n,E}, \tag{5.2.4}$$

with $E+1$ defined by (2.1.1)*. The sum is independent of the choice of representative π of* $[\pi]$.

Proof. The independence of the choice of representatives is a result of the following two observations:

(a) If E and E' only differ by a cyclic shift, one has $|E| = |E'|$ and $M^{cyc}_{n,E} = M^{cyc}_{n,E'}$.

(b) For two representatives π and π' of $[\pi]$,

$$\{E \subseteq [n] : \text{cPk}(\pi) \subseteq E \cup (E+1)\} = \{E' \subseteq [n] : \text{cPk}(\pi') \subseteq E' \cup (E'+1)\} + i,$$

for some $i \in [n]$, namely, the two sets only differ by a cyclic shift.

Now fix a representative π of $[\pi]$. We rewrite both sides of equation (5.2.4) as follows:

$$\text{RHS} \overset{(i)}{=} \sum_{\substack{F\subseteq[n]:\\ \text{cPk}(\pi)\subseteq F\cup(F+1)}} 2^{|F|} \sum_{f\in F} M_{n,(F-f)\cap[n-1]} \overset{(ii)}{=} \sum_{E\subseteq[n-1]} 2^{|E|+1}\alpha_E M_{n,E} \tag{5.2.5}$$

where $\alpha_E = \#A_E$, with

$$A_E = \{(F, f) : f \in F \subseteq [n] \text{ with } \mathrm{cPk}(\pi) \subseteq F \cup (F+1),\ E = (F - f) \cap [n-1]\}.$$

Here equality (i) is a result of applying equation (2.3.1) to move from cyclic monomial to monomial quasisymmetric functions, while equality (ii) is obtained by calculating the coefficient of $M_{n,E}$. It is noted that in equality (ii), for each pair $(F, f) \in A_E$, we have $n \in F - f$. As a result, $E = (F - f) \cap [n-1] = (F - f) \setminus \{n\}$. Therefore their cardinalities satisfy $|F| = |E| + 1$.

$$\begin{aligned}
\mathrm{LHS} &\overset{(i)'}{=} \sum_{\tau \in [\pi]} \Delta_\tau \\
&\overset{(ii)'}{=} \sum_{\tau \in [\pi]} \sum_{\substack{E \subseteq [n-1]: \\ \mathrm{Pk}(\tau) \subseteq E \cup (E+1)}} 2^{|E|+1} M_{n,E} \\
&\overset{(iii)'}{=} \sum_{E \subseteq [n-1]} \beta_E 2^{|E|+1} M_{n,E}
\end{aligned} \tag{5.2.6}$$

where $\beta_E = \#B_E$ with

$$B_E = \{i \in [n] : (\mathrm{cPk}\,\pi - i) \cap [2, n-1] \subseteq E \cup (E+1)\}.$$

Equality $(i)'$ follows from equation (5.2.3). Equality $(ii)'$ is obtained by applying equation (5.1.3) to express the weight enumerator Δ_τ in terms of monomial quasisymmetric functions.

Equality $(iii)'$ follows from the observation

$$\{\{\, \mathrm{Pk}(\tau) : \tau \in [\pi] \,\}\} = \{\{\, (\mathrm{cPk}\,\pi - i) \cap [2, n-1] : i \in [n] \,\}\}.$$

By comparing equations (5.2.5) and (5.2.6), it suffices to prove that $\alpha_E = \beta_E$ for every $E \subseteq [n-1]$, or equivalently, to construct a bijection between A_E and B_E.

Set $\theta_E : A_E \to B_E$ as $\theta_E(F, f) = f$. To prove that this map is well-defined, it suffices to show that for each $(F, f) \in A_E$, we have $f \in B_E$. It follows from the definition of A_E that $F - f = E \cup \{n\}$. Applying the operation on both sides of the inclusion $\mathrm{cPk}(\pi) \subseteq F \cup (F+1)$, we get

$$\mathrm{cPk}(\pi) - f \subseteq (F - f) \cup (F - f + 1) = E \cup (E+1) \cup \{1, n\}.$$

Hence $(\mathrm{cPk}\,\pi - f) \cap [2, n-1] \subseteq E \cup (E+1)$ and $f \in B_E$.

Conversely, define $\sigma_E : B_E \to A_E$ by $\sigma_E(f) = (\,(E+f) \cup \{f\}, f\,)$. One can similarly check that this map is well-defined.

It is straightforward to verify that θ_E and σ_E are inverse to each other, hence we get a bijection between A_E and B_E. This finishes the proof. □

Remark 5.2.10. As a direct corollary of the above proposition, $\Delta^{\mathrm{cyc}}_{[\vec{D}]}$ is a homogeneous cyclic quasi-symmetric function of degree n, and that $\Delta^{\mathrm{cyc}}_{[\pi]}$ depends only on $\mathrm{cPk}[\pi]$, or equivalently (by Remark 1.2.1) on $\mathrm{cPk}\,\pi$ for any representative π.

Example 5.2.11. Let $\tau = 1243$ so that $[\tau] = \{1243, 3124, 4312, 2431\}$ and $\mathrm{cPk}(\tau) = \{3\}$; $\pi = 1324$, $[\pi] = \{1324, 4132, 2413, 3241\}$ and $\mathrm{cPk}(\pi) = \{2, 4\}$.

To use Proposition 5.2.9 for calculation, we first need all possible choices of $E \subseteq [4]$ satisfying $\{3\} = \mathrm{cPk}\,\tau \subseteq E \cup (E+1)$, which are

$$\{1,2,3,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2\}, \{1,3\}, \{2,3\}, \{2,4\}, \{3,4\}, \{2\}, \{3\},$$

with corresponding monomial cyclic quasisymmetric functions

$$M^{cyc}_{(1,1,1,1)} = M^{cyc}_{4,\{1,2,3,4\}},$$

$$M^{cyc}_{(2,1,1)} = M^{cyc}_{4,\{1,2,3\}} = M^{cyc}_{4,\{1,2,4\}} = M^{cyc}_{4,\{1,3,4\}} = M^{cyc}_{4,\{2,3,4\}},$$

$$M^{cyc}_{(3,1)} = M^{cyc}_{4,\{1,2\}} = M^{cyc}_{4,\{2,3\}} = M^{cyc}_{4,\{3,4\}},$$

$$M^{cyc}_{(2,2)} = M^{cyc}_{4,\{1,3\}} = M^{cyc}_{4,\{2,4\}},$$

$$M^{cyc}_{(4)} = M^{cyc}_{4,\{2\}} = M^{cyc}_{4,\{3\}}.$$

Applying equation (5.2.4), we have

$$\Delta^{cyc}_{[\tau]} = 2^4 M^{cyc}_{(1,1,1,1)} + 4 \cdot 2^3 M^{cyc}_{(2,1,1)} + 3 \cdot 2^2 M^{cyc}_{(3,1)} + 2 \cdot 2^2 M^{cyc}_{(2,2)} + 2 \cdot 2 M^{cyc}_{(4)}.$$

Similarly for π, all possible choices of $E \subseteq [4]$ satisfying $\{2,4\} = \operatorname{cPk}\pi \subseteq E \cup (E+1)$ are as follows:

$$\{1,2,3,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\},$$

hence by equation (5.2.4),

$$\Delta^{cyc}_{[\pi]} = 2^4 M^{cyc}_{(1,1,1,1)} + 4 \cdot 2^3 M^{cyc}_{(2,1,1)} + 2 \cdot 2^2 M^{cyc}_{(3,1)} + 2 \cdot 2^2 M^{cyc}_{(2,2)}.$$

It follows from the calculation above that

$$\Delta^{cyc}_{[\tau]} + \Delta^{cyc}_{[\pi]} = 2 \cdot 2^4 M^{cyc}_{(1,1,1,1)} + 8 \cdot 2^3 M^{cyc}_{(2,1,1)} + 5 \cdot 2^2 M^{cyc}_{(3,1)} + 4 \cdot 2^2 M^{cyc}_{(2,2)} + 2 \cdot 2 M^{cyc}_{(4)}.$$

5.2.4 Algebra of Cyclic Peaks

Recall from Section 4.3.2 that S is a cyclic peak set in $[n]$ if there exists some $\pi \in \mathfrak{S}_n$ with $\operatorname{cPk}\pi = S$. It follows from Proposition 5.2.9 that, for any cyclic peak set S in $[n]$, we can define an associated cyclic quasisymmetric function by

$$K^{\text{cyc}}_{n,S} := \Delta^{\text{cyc}}_{[\pi]},$$

for any permutation π with $\operatorname{cPk}\pi = S$. One can observe that $\Delta^{\text{cyc}}_{[\pi]} = K^{\text{cyc}}_{n,\operatorname{cPk}\pi}$ and rewrite equation (5.2.2) as

$$\Delta^{\text{cyc}}_{[\vec{D}]} = \sum_{[\pi] \in \mathcal{L}^{\text{tor}}([\vec{D}])} K^{\text{cyc}}_{n,\operatorname{cPk}\pi}.$$

Moreover, it follows from formula (5.2.1) that $K^{\text{cyc}}_{n,S}$ can be expressed in terms of the quasisymmetric functions $\{K_{n,T}\}$ as

$$K^{\text{cyc}}_{n,S} = \sum_{\bar{\pi} \in [\pi]} K_{n,\operatorname{Pk}\bar{\pi}},$$

where $\operatorname{cPk}\pi = S$.

Let Λ_n denote the vector space of cyclic quasisymmetric functions spanned by $K^{\text{cyc}}_{n,S}$ where S ranges over cyclic peak sets in $[n]$, and set $\Lambda = \oplus_{n \geq 0} \Lambda_n$. We will show that Λ is an algebra by proving that the product of $K^{\text{cyc}}_{n,U}$ and $K^{\text{cyc}}_{n,T}$ is a linear combination of $K^{\text{cyc}}_{n,S}$'s. We call Λ the *algebra of cyclic peaks*.

Lemma 5.2.12. *The $K^{\mathrm{cyc}}_{n,S}$ are linearly independent, where S are, up to cyclic shift, distinct cyclic peak sets.*

Proof. For this proof, we totally order the subsets of $[n-1]$, first by cardinality, then by the lexicographic order. We therefore have

$$\emptyset \lhd \{1\} \lhd \{2\} \lhd \cdots \lhd \{n-1\} \lhd \{1,2\} \lhd \{1,3\} \lhd \cdots \lhd \{1,n-1\} \lhd \{2,3\} \lhd \{2,4\} \lhd \cdots .$$

One can similarly order the compositions of n as

$$(n) \lhd (1,n-1) \lhd (2,n-2) \lhd \cdots \lhd (1,1,n-2) \lhd (1,2,n-3) \lhd \cdots \lhd (1,n-2,1) \lhd \cdots .$$

Noting that $K^{\mathrm{cyc}}_{n,S} = K^{\mathrm{cyc}}_{n,S'}$ if the sets S and S' only differ by a cyclic shift, we will always assume the index set S to be the least among all its cyclic shifts. It is not hard to see that $\psi(S)$ is also the least composition among all its cyclic shifts, where ψ is defined by equation (2.1.3).

We now show that the matrix of $\{K^{\mathrm{cyc}}_{n,S}\}$ with respect to the basis $\{M^{\mathrm{cyc}}_{n,L}\}$ has full rank. Then the linear independence of $\{K^{\mathrm{cyc}}_{n,S}\}$ will follow immediately from the fact that the monomial cyclic quasi-symmetric functions form a basis of cQSym.

Let us fix n and suppose $\{S_1 \lhd S_2 \lhd \ldots \lhd S_m\}$ is the set of all distinct cyclic peak sets in $[n]$. Given a $K^{\mathrm{cyc}}_{n,S}$, suppose $|S| = k$ and $S = \{1 = s_1 < s_2 < \ldots < s_k\}$ for some $n \geq 2$ and $k \geq 1$. Here all indices are taken modulo k unless otherwise noted. To each K^{cyc}_{S} we associate a corresponding monomial quasi-symmetric function by $F(K^{\mathrm{cyc}}_{n,S}) = M^{\mathrm{cyc}}_{n,f(S)}$, where

$$f(S) = \{s_1, s_2 - 1, \ldots, s_k - 1\}.$$

Since S is a cyclic peak set, elements in $f(S)$ are distinct. So $f(S)$ is a set and f is well-defined. If one denotes $\psi(S)$ by $(\alpha_1, \alpha_2, \ldots, \alpha_k)$, then

$$\psi(f(S)) = (\alpha_1 - 1, \alpha_2, \ldots, \alpha_{k-1}, \alpha_k + 1).$$

Also notice that the assumption of S being the least among its cyclic shifts (in particular, $\alpha_1 = \min\{\alpha_i\}_{i=1}^{n}$) ensures that $f(S)$ is also the least among all its cyclic shifts. It follows that f is injective.

Claim: Consider the matrix of $\{K^{\text{cyc}}_{n,S}\}$ with respect to the basis $\{M^{\text{cyc}}_{n,E}\}$. The square submatrix with columns restricted to $\{M^{\text{cyc}}_{n,f(S)}\}$ is upper triangular with nonzero diagonal entries. In particular, it is invertible.

Let $A_{i,j}$ denote the coefficient of $M^{\text{cyc}}_{n,f(S_j)}$ in the expression of K^{cyc}_{n,S_i} in terms of monomial cyclic quasi-symmetric functions and set $A = (A_{i,j})$ to be the $m \times m$ matrix that we need to consider. To prove that A is an upper triangular matrix, it suffices to show that $A_{i,j} = 0$ if $i > j$, or equivalently, the term $M^{\text{cyc}}_{n,f(S_j)}$ does not appear in the expression of K^{cyc}_{n,S_i} if $S_j \lhd S_i$.

Suppose towards a contradiction that $A_{i,j} \neq 0$ for some $i > j$. It then follows from Proposition 5.2.9 that there exists some $E \subseteq [n]$ such that

$$S_i \subseteq E \cup (E+1) \quad \text{and} \quad M^{\text{cyc}}_{n,f(S_j)} = M^{\text{cyc}}_{n,E}.$$

Since S_i is a cyclic peak set, e and $e+1$ cannot be in S_i at the same time. It then follows from the assumption $S_i \subseteq E \cup (E+1)$ that $|E| \geq |S_i|$. The condition $i > j$ implies $S_i \rhd S_j$, hence $|S_i| \geq |S_j| = |f(S_j)| = |E|$. Therefore, we only need to consider the cases when S_i and S_j have the same cardinality.

Suppose

$$\psi(S_i) = (\alpha_1, \alpha_2, \ldots, \alpha_k), \quad \psi(S_j) = (\beta_1, \beta_2, \ldots, \beta_k), \quad \psi(E) = (\beta'_1, \beta'_2, \ldots, \beta'_k).$$

Then $\psi(E) = (\beta'_1, \beta'_2, \ldots, \beta'_k)$ is a cyclic shift of

$$\psi(f(S_j)) = (\beta_1 - 1, \beta_2, \ldots, \beta_{k-1}, \beta_k + 1).$$

Assume

$$(\beta'_q, \beta'_{q+1}, \ldots, \beta'_{q-1}) = (\beta_1 - 1, \beta_2, \ldots, \beta_{k-1}, \beta_k + 1),$$

for some $q \in [k]$. Since $S_i \subseteq E \cup (E+1)$, $|S_i| = |E|$, and S_i does not have cyclically consecutive elements, each $a \in S_i$ corresponds to a unique $a' \in E$ where a' equals either a or $a-1$ (considered modulo n). A crucial observation therefore follows:

$$|(\alpha_r + \cdots + \alpha_s) - (\beta'_r + \cdots + \beta'_s)| \leq 1 \quad \text{for } r, s \in [k]. \tag{5.2.7}$$

In particular,

$$\alpha_1 \leq \alpha_q \leq \beta'_q + 1 = (\beta_1 - 1) + 1 = \beta_1,$$

where the first inequality comes from the fact that S_i is the least among its cyclic shifts, and the second one follows from (5.2.7) by taking $r = s = q$. Note that $S_j \lhd S_i$ implies $\psi(S_j) \lhd \psi(S_i)$, so $\beta_1 \leq \alpha_1$. Thus, necessarily,

$$\alpha_1 = \alpha_q = \beta'_q + 1 = \beta_1.$$

It then follows from the inequality (5.2.7) that for any $r \in [k-1]$,

$$\alpha_q + \cdots + \alpha_{q+r} \leq \beta'_q + \cdots + \beta'_{q+r} + 1 = \beta_1 + \cdots + \beta_{1+r} + \delta_{1+r,k},$$

where $\delta_{1+r,k} = 1$ if $1 + r = k$ and 0 otherwise.

If $\alpha_{q+r} = \beta_{1+r}$ for every $r \in [k-2]$, then $\psi(S_i)$ is a cyclic shift of $\psi(S_j)$, which contradicts our assumption $S_i \rhd S_j$. Now assume that $t \in [k-2]$ is the smallest index such that $\alpha_{q+t} \neq \beta_{1+t}$. Then necessarily $\alpha_{q+t} < \beta_{1+t}$, and

$$(\alpha_q, \ldots, \alpha_{q+t}) \lhd (\beta_1, \ldots, \beta_{1+t}).$$

Combined with the inequality $(\alpha_1, \ldots, \alpha_{1+t}) \unlhd (\alpha_q, \ldots, \alpha_{q+t})$, as $(\alpha_1, \ldots, \alpha_k)$ is the least among all its cyclic shifts, one has

$$(\alpha_1, \ldots, \alpha_{1+t}) \lhd (\beta_1, \ldots, \beta_{1+t}).$$

This implies that $\psi(S_i) \lhd \psi(S_j)$, which is a contradiction to the assumption $S_i \rhd S_j$. Moreover, it is not hard to see that $A_{i,i} \neq 0$ as $S_i \subseteq f(S_i) \cup (f(S_i) + 1)$, therefore the diagonal entries are nonzero. This proves the claim, hence the lemma. □

Define *the union of digraphs* $\vec{D} \uplus \vec{E}$ to be the digraph with vertices $V(\vec{D}) \cup V(\vec{E})$ and arcs $A(\vec{D}) \cup A(\vec{E})$. The next result follows easily from the definition.

Proposition 5.2.13. *If $\vec{D}$ and $\vec{E}$ are two DAGs on disjoint subsets of $\mathbb{P}$, then*

$$\Delta^{\mathrm{cyc}}_{[\vec{D} \uplus \vec{E}]} = \Delta^{\mathrm{cyc}}_{[\vec{D}]} \cdot \Delta^{\mathrm{cyc}}_{[\vec{E}]}. \tag{5.2.8}$$

The proposition above yields a combinatorial proof that Λ is an algebra.

Proposition 5.2.14. *Λ is a graded subring of* cQSym.

Proof. As a subspace of cQSym, Λ naturally inherits the addition and multiplication operations from cQSym.

To show that Λ is closed under multiplication, take $K^{\text{cyc}}_{n,U} \in \Lambda_m$ and $K^{\text{cyc}}_{n,T} \in \Lambda_n$, where U and T are cyclic peak sets in $[m]$ and $[n]$ respectively, namely, there exist $\pi \in \mathfrak{S}_m$ and $\tau \in \mathfrak{S}_n$ such that $\operatorname{cPk}\pi = U,\ \operatorname{cPk}\tau = T$. For the purpose of constructing two corresponding disjoint DAGs, we standardize $\tau = \tau_1\tau_2\ldots\tau_n \in \mathfrak{S}_n$ to $\{m+1, m+2, \ldots, m+n\}$, that is, construct $\tau' = \tau'_1\tau'_2\ldots\tau'_n$ where $\tau'_i = \tau_i + m$ for $i \in [n]$. As a consequence of equation (5.2.8), we have

$$K^{\text{cyc}}_{n,U} \cdot K^{\text{cyc}}_{n,T} = \Delta^{\text{cyc}}_{[\pi]} \cdot \Delta^{\text{cyc}}_{[\tau']} = \Delta^{\text{cyc}}_{[\pi \uplus \tau']} = \sum_{\sigma \in \mathcal{L}^{\text{tor}}([\pi \uplus \tau'])} K^{\text{cyc}}_{n,\operatorname{cPk}\sigma}. \tag{5.2.9}$$

This completes the proof. □

It follows from the Proposition 5.2.14 that the theory of enriched toric $[\vec{D}]$-partitions can be used to supply an alternative proof for the cyclic shuffle-compatibility of cPk, aside from the Theorem 3.2.3 in combinatorial means and Theorem 4.3.5 using cyclic shuffle algebras.

In terms of another basis of cQSym, the fundamental cyclic quasisymmetric functions, $K^{\text{cyc}}_{n,S}$ has the following expansion.

Proposition 5.2.15. *For any cyclic peak set S in $[n]$, we have*

$$2^{-|S|}K^{\text{cyc}}_{n,S} = \sum_{\substack{E \subseteq [n]: \\ S \subseteq E \triangle (E+1)}} F^{\text{cyc}}_{n,E},$$

where $\triangle$ denotes symmetric difference.

Proof. Since $F^{\text{cyc}}_{n,E} = \sum_{F \supseteq E} M^{\text{cyc}}_{n,F}$, the coefficient of $M^{\text{cyc}}_{n,F}$ on the right-hand side of the above expansion is $|\{E \subseteq F : S \subseteq E\triangle(E+1)\}|$.

To count this set, we need the following observations. For each $k \in S \subseteq E\triangle(E+1)$, exactly one of $k-1$ and k is in E. It follows from $E \subseteq F$ that at least one of $k-1$ and k is in F. So one has the following two cases:

(1) If both $k, k-1 \in F$, then E must contain exactly one of k or $k-1$.

(2) If only one of $k, k-1 \in F$, then E must contain this element.

Note that the restrictions above only involve two adjacent numbers.

Also notice that if $k \in F$ but neither k nor $k+1$ is in S, then k is free to be in E or not. Denote

$$S_1 = \{k \in S \mid \text{both } k, k-1 \text{ are in } F\},$$

$$S_2 = \{k \in S \mid k \in E, k-1 \notin F\},$$

$$S_3 = \{k \in S \mid k \notin E, k-1 \in F\}.$$

Then we have a set partition $S = S_1 \cup S_2 \cup S_3$. Therefore if we denote $s_i = \#S_i$ for $i \in \{1,2,3\}$, we have $|S| = s_1 + s_2 + s_3$.

Denote now

$$T = \{k \in F \mid \text{none of } k, k+1 \text{ is in } S\}.$$

By the definition of a peak set, numbers in S are not adjacent. Hence we have the following partition of F into disjoint sets:

$$F = S_1 \cup (S_1 - 1) \cup S_2 \cup (S_3 - 1) \cup T.$$

It follows that $|F| = 2s_1 + s_2 + s_3 + t$ with $t = |T|$. Hence, the number of choices for E is

$$2^{s_1+t} = 2^{s_1+|F|-2s_1-s_2-s_3} = 2^{|F|-(s_1+s_2+s_3)} = 2^{|F|-|S|}.$$

So we have

$$\sum_{\substack{E\subseteq[n]:\\ S\subseteq E\triangle(E+1)}} F^{cyc}_{n,E} = \sum_{\substack{F\subseteq[n]:\\ S\subseteq F\cup(F+1)}} 2^{|F|-|S|} M^{cyc}_{n,F}.$$

By equation (5.2.4), this quantity is $2^{-|S|}K^{cyc}_{n,S}$. □

5.2.5 Order Polynomials

Definition 5.2.16 (Order polynomial)**.** Define the *order polynomial of enriched toric* $[\vec{D}]$*-partitions*, $\Omega^{\text{cyc}}([\vec{D}], m)$, by

$$\Omega^{\text{cyc}}([\vec{D}], m) = \Delta^{\text{cyc}}_{[\vec{D}]}(1^m).$$

The following result is the toric analogue of formula (5.1.4).

Proposition 5.2.17. *Given $\pi \in \mathfrak{S}_n$, then*

$$\sum_m \Omega^{\mathrm{cyc}}([\pi], m)t^m = \left(\frac{4t}{(1+t)^2}\right)^{\operatorname{cpk}\pi} \left(\frac{1+t}{1-t}\right)^{n-1} \left(\operatorname{cpk}\pi + \frac{2nt}{(1-t)^2}\right). \tag{5.2.10}$$

This right side of the equation does not depend on the choice of representative π, as they all have the same cyclic peak number.

Proof. By the definition of order polynomial,

$$\begin{aligned}
\sum_m \Omega^{\mathrm{cyc}}([\pi], m)t^m &= \sum_m \Delta^{\mathrm{cyc}}_{[\pi]}(1^m)t^m \\
&= \sum_m \sum_{\tau\in[\pi]} \Delta_\tau(1^m)t^m \\
&= \sum_{\tau\in[\pi]} \sum_m \Delta_\tau(1^m)t^m \\
&= \sum_{\tau\in[\pi]} \sum_m \Omega(\tau, m)t^m \\
&= \sum_{\tau\in[\pi]} \frac{1}{2}\left(\frac{1+t}{1-t}\right)^{n+1} \left(\frac{4t}{(1+t)^2}\right)^{1+\operatorname{pk}\tau},
\end{aligned}$$

where the last equality is obtained by applying (5.1.4).

Observe that each representative of $[\pi]$ will either start with a cyclic peak, end with a cyclic peak, or none of the two ends are cyclic peaks, which will yield peak number $\operatorname{cpk}\pi - 1$, $\operatorname{cpk}\pi - 1$ or $\operatorname{cpk}\pi$ respectively. The number of those representatives with a cyclic peak at one end is $2\operatorname{cpk}\pi$. It follows that

$$\begin{aligned}
\sum_m \Omega^{\mathrm{cyc}}([\pi], m)t^m &= \frac{2\operatorname{cpk}\pi}{2}\left(\frac{1+t}{1-t}\right)^{n+1}\left(\frac{4t}{(1+t)^2}\right)^{\operatorname{cpk}\pi} \\
&\qquad + \frac{n-2\operatorname{cpk}\pi}{2}\left(\frac{1+t}{1-t}\right)^{n+1}\left(\frac{4t}{(1+t)^2}\right)^{1+\operatorname{cpk}\pi} \\
&= \left(\frac{4t}{(1+t)^2}\right)^{\operatorname{cpk}\pi}\left(\frac{1+t}{1-t}\right)^{n-1}\left(\operatorname{cpk}\pi + \frac{2nt}{(1-t)^2}\right).
\end{aligned}$$

This completes the proof. □

We note that by taking the coefficient of t^m on both sides of equation (5.2.10), one can get an expression for the order polynomial of enriched toric $[\vec{D}]$-partitions $\Omega^{\mathrm{cyc}}([\pi], m)$ in an algebraic manner. It would be desirable to derive $\Omega^{\mathrm{cyc}}([\pi], m)$ combinatorially.

We now give a totally combinatorial derivation of the order polynomial $\Omega^{\mathrm{cyc}}([\pi], m)$ by using the Proposition 5.1.13, which is also proved in a combinatorial way.

Corollary 5.2.18. *For a given $[\pi] \in [\mathfrak{S}_n]$, one has*

$$\Omega^{\mathrm{cyc}}([\pi], m) = (n - 2\operatorname{cpk}\pi) \cdot 2^{2\operatorname{cpk}\pi+1} \sum_{k=0}^{m-1-\operatorname{cpk}\pi} \left(\!\binom{n+1}{k}\!\right) \binom{n-2\operatorname{cpk}\pi-1}{m-1-\operatorname{cpk}\pi-k}$$
$$+ \operatorname{cpk}\pi \cdot 2^{2\operatorname{cpk}\pi} \sum_{k=0}^{m-\operatorname{cpk}\pi} \left(\!\binom{n+1}{k}\!\right) \binom{n-2\operatorname{cpk}\pi+1}{m-\operatorname{cpk}\pi-k}.$$

This right side of the equation does not depend on the choice of representative π, as they all have the same cyclic peak number.

Proof. Notice that any representative π' of $[\pi]$ satisfies $\operatorname{pk}\pi' = \operatorname{cpk}[\pi] - 1$ if π' starts or ends with a cyclic peak. Otherwise, $\operatorname{pk}\pi' = \operatorname{cpk}[\pi]$. Among the n representatives of $[\pi]$, there are $2\operatorname{cpk}[\pi]$ of them with a peak element at one end. Therefore, applying equation (5.2.3) we have

$$\Omega^{\mathrm{cyc}}([\pi], m) = \Delta^{\mathrm{cyc}}_{[\pi]}(1^m) = \sum_{\tau\in[\pi]} \Delta_\tau(1^m) = \sum_{\tau\in[\pi]} \Omega(\tau, m),$$

and by the previous observation and the Proposition 5.1.13, one has

$$\sum_{\tau\in[\pi]} \Omega(\tau, m) = \sum_{\tau\in[\pi]} 2^{2\operatorname{pk}\tau+1} \sum_{k=0}^{m-1-\operatorname{pk}\tau} \left(\!\binom{n+1}{k}\!\right) \binom{n-2\operatorname{pk}\tau-1}{m-1-\operatorname{pk}\tau-k}$$
$$= (n - 2\operatorname{cpk}[\pi]) \cdot 2^{2\operatorname{cpk}[\pi]+1} \sum_{k=0}^{m-1-\operatorname{cpk}[\pi]} \left(\!\binom{n+1}{k}\!\right) \binom{n-2\operatorname{cpk}[\pi]-1}{m-1-\operatorname{cpk}[\pi]-k}$$
$$+ 2\operatorname{cpk}[\pi] \cdot 2^{2(\operatorname{cpk}[\pi]-1)+1} \sum_{k=0}^{m-1-(\operatorname{cpk}[\pi]-1)} \left(\!\binom{n+1}{k}\!\right) \binom{n-2(\operatorname{cpk}[\pi]-1)-1}{m-1-(\operatorname{cpk}[\pi]-1)-k}$$
$$= (n - 2\operatorname{cpk}\pi) \cdot 2^{2\operatorname{cpk}\pi+1} \sum_{k=0}^{m-1-\operatorname{cpk}\pi} \left(\!\binom{n+1}{k}\!\right) \binom{n-2\operatorname{cpk}\pi-1}{m-1-\operatorname{cpk}\pi-k}$$
$$+ \operatorname{cpk}\pi \cdot 2^{2\operatorname{cpk}\pi} \sum_{k=0}^{m-\operatorname{cpk}\pi} \left(\!\binom{n+1}{k}\!\right) \binom{n-2\operatorname{cpk}\pi+1}{m-\operatorname{cpk}\pi-k}$$

The conclusion follows immediately. □

We note that the generating function of order polynomials in Proposition 5.2.17 can also be deduced from the corollary above. More precisely,

$$\begin{aligned}
\sum_m \Omega^{\mathrm{cyc}}([\pi], m)t^m &= (n - 2\operatorname{cpk}\pi)\cdot 2^{2\operatorname{cpk}\pi+1}\,\frac{(1+t)^{n-2\operatorname{cpk}\pi-1}}{(1-t)^{n+1}}\,t^{\operatorname{cpk}\pi+1} \\
&\quad + \operatorname{cpk}\pi\cdot 2^{2\operatorname{cpk}\pi}\,\frac{(1+t)^{n-2\operatorname{cpk}\pi+1}}{(1-t)^{n+1}}\,t^{\operatorname{cpk}\pi} \\
&= \left(\frac{4t}{(1+t)^2}\right)^{\operatorname{cpk}\pi}\left(\frac{1+t}{1-t}\right)^{n+1}\left((n-2\operatorname{cpk}\pi)\cdot\frac{2t}{(1+t)^2}+\operatorname{cpk}\pi\right) \\
&= \left(\frac{4t}{(1+t)^2}\right)^{\operatorname{cpk}\pi}\left(\frac{1+t}{1-t}\right)^{n-1}\left(\operatorname{cpk}\pi+\frac{2nt}{(1-t)^2}\right).
\end{aligned}$$

5.3 Shuffle-compatibility Revisited

In this section, we will use the order polynomials for both peaks and cyclic peaks to provide another characterization of the shuffle algebra $\mathcal{A}_{\mathrm{pk}}$ and the cyclic shuffle algebra $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cpk}}$. First recall from [GZ18]:

Theorem 5.3.1 ([GZ18, Theorem 4.8 (b)])**.** *The linear map on $\mathcal{A}_{\mathrm{pk}}$ defined by*

$$\pi_{\mathrm{pk}} \mapsto \begin{cases} \dfrac{2^{2\operatorname{pk}\pi+1}t^{\operatorname{pk}\pi+1}(1+t)^{|\pi|-2\operatorname{pk}\pi-1}}{(1-t)^{|\pi|+1}}x^{|\pi|}, & \text{if } |\pi|\geq 1; \\ \dfrac{1}{1-t}, & \text{if } |\pi| = 0, \end{cases}$$

is a $\mathbb{Q}$-algebra isomorphism from $\mathcal{A}_{\mathrm{pk}}$ to the span of

$$\left\{\frac{1}{1-t}\right\}\bigcup\left\{\frac{2^{2j+1}t^{j+1}(1+t)^{n-2j-1}}{(1-t)^{n+1}}x^n\right\}_{n\geq 1,\,0\leq j\leq\lfloor\frac{n-1}{2}\rfloor}$$

a subalgebra of $\mathbb{Q}[[t*]][x]$, where multiplication of formal power series in t is by Hadamard product. □

Now we give another characterization of $\mathcal{A}_{\mathrm{pk}}$ which follows from our Proposition 5.1.13.

Theorem 5.3.2. *The linear map on $\mathcal{A}_{\mathrm{pk}}$ defined by*

$$\pi_{\mathrm{pk}} \mapsto 2^{2\operatorname{pk}\pi+1}\sum_{k=0}^{m-1-\operatorname{pk}\pi}\left(\!\!\binom{|\pi|+1}{k}\!\!\right)\binom{|\pi|-2\operatorname{pk}\pi-1}{m-1-\operatorname{pk}\pi-k}x^{|\pi|}$$

is a $\mathbb{Q}$-algebra isomorphism from $\mathcal{A}_{\mathrm{pk}}$ to the span of

$$\{1\} \bigcup \left\{ 2^{2j+1} \sum_{k=0}^{m-1-j} \left(\!\binom{n+1}{k}\!\right) \binom{n-2j-1}{m-1-j-k} x^n \right\}_{n \geq 1,\, 0 \leq j \leq \lfloor \frac{n-1}{2} \rfloor}$$

a subalgebra of $\mathbb{Q}[x]^{\mathbb{N}}$, the algebra of functions $\mathbb{N} \to \mathbb{Q}[x]$ in the non-negative integer value m.

Proof. Define a map $\kappa_m : \mathrm{QSym} \to \mathbb{Q}[x]$ by

$$\kappa_m(F_L) = K_{\mathrm{Pk}(L)}(1^m) x^n,$$

linearly extended to all of QSym. The following equation, [Ste97, Equation (3.1)],

$$K_{\mathrm{Pk}\,\pi} \cdot K_{\mathrm{Pk}\,\sigma} = \sum_{\tau \in \pi \sqcup\!\sqcup \sigma} K_{\mathrm{Pk}\,\tau},$$

implies that κ_m is a $\mathbb{Q}$-algebra homomorphism. The map that takes F_L to the function $\theta_L : m \mapsto \kappa_m(F_L)$ is therefore a homomorphism from QSym to $\mathbb{Q}[x]^{\mathbb{N}}$. It follows from Proposition 5.1.13 that

$$\kappa_m(F_L) = 2^{2\,\mathrm{pk}(L)+1} \sum_{k=0}^{m-1-\mathrm{pk}(L)} \left(\!\binom{n+1}{k}\!\right) \binom{n-2\,\mathrm{pk}(L)-1}{m-1-\mathrm{pk}(L)-k} x^n.$$

Moreover, from Theorem 5.1.10 one has

$$\sum_{m=0}^{\infty} 2^{2\,\mathrm{pk}(L)+1} \sum_{k=0}^{m-1-\mathrm{pk}(L)} \left(\!\binom{n+1}{k}\!\right) \binom{n-2\,\mathrm{pk}(L)-1}{m-1-\mathrm{pk}(L)-k} t^m = \frac{1}{2} \left(\frac{1+t}{1-t} \right)^{n+1} \left(\frac{4t}{(1+t)^2} \right)^{1+\mathrm{pk}(L)},$$

which is the generating function of θ_L and only depends on n and $\mathrm{pk}\,L$. They are clearly linearly independent for L with distinct peak numbers. The result then follows immediately from Theorem 4.2.4. □

The following theorem append to Theorem 4.3.9 in a different flavor.

Theorem 5.3.3. *Let*

$$w_{n,j}^{\mathrm{cpk}} = j4^j \sum_{k=0}^{p-j} \left(\!\binom{n+1}{k}\!\right) \binom{n-2j+1}{p-j-k} x^n + 2(n-2j)4^j \sum_{k=0}^{p-1-j} \left(\!\binom{n+1}{k}\!\right) \binom{n-2j-1}{p-j-k-1} x^n.$$

Then the linear map on $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cpk}}$ defined by

$$[\pi]_{\mathrm{cpk}} \mapsto \begin{cases} w^{\mathrm{cpk}}_{|\pi|,\mathrm{cpk}[\pi]}, & \text{if } |\pi| \geq 1, \\ 1, & \text{if } |\pi| = 0, \end{cases}$$

is a $\mathbb{Q}$-algebra isomorphism from $\mathcal{A}^{\mathrm{cyc}}_{\mathrm{cpk}}$ to the span of

$$\{1\} \cup \{w^{\mathrm{cpk}}_{n,j}\}_{\substack{n\geq 1,\\ 1\leq j\leq \lfloor n/2\rfloor}},$$

a subalgebra of $\mathbb{Q}[x]^{\mathbb{N}}$.

Proof. It follows from Theorem 4.3.9 (b) and the identity

$$\begin{aligned}\sum_{p=0}^{\infty} w^{\mathrm{cpk}}_{|\pi|,\mathrm{cpk}[\pi]} t^p &= \left(\frac{4t}{(1+t)^2}\right)^{\mathrm{cpk}[\pi]} \left(\frac{1+t}{1-t}\right)^{|\pi|-1} \left(\mathrm{cpk}[\pi] + \frac{2\,|\pi|\,t}{(1-t)^2}\right) x^{|\pi|} \\ &= \frac{(\mathrm{cpk}[\pi](1+t)^2 + 2(|\pi| - 2\,\mathrm{cpk}[\pi])t)(4t)^{\mathrm{cpk}[\pi]}(1+t)^{|\pi|-2\,\mathrm{cpk}[\pi]-1}}{(1-t)^{|\pi|+1}} x^{|\pi|},\end{aligned}$$

where the first equality follows from Proposition 5.2.17 and Corollary 5.2.18. □

For convenience, we always assume the label set to be $[n]$. However, the definitions of (toric) DAGs and (toric) posets can be extended to any finite subsets of $\mathbb{P}$ as the set of labels, with all consequent conclusions continuing to hold.

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