APPROXIMATION AND INCOMPRESSIBLE LIMIT OF INHOMOGENEOUS POROUS MEDIUM EQUATIONS

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ABSTRACT

In this work we study the inhomogeneous porous medium equation (PME). In particular, we introduce a deterministic particle method that approximates the PME in chapter 2 and how the PME is used to model tumor growth and study the incompressible limit in chapter 3.

Particle methods aim to discretize a PDE into a system of ODEs for particles. In the absence of diffusion, particle methods can approximate solutions while maintaining the Wasserstein gradient flow structure. Diffusion smooths things out so that particles do not remain particles. This issue can be dealt with via regularization of the diffusion term. Much has been developed for convex and semi-convex energies. We would like to extend this particle method (blob method) to an energy with more general convexity (ω-convexity) containing nonlinear porous medium diffusion ($m = 2$). We connect this pde to chemotaxis via a Keller-Segel model for the Newtonian kernel and the Bessel kernel. Then, we perform numerical simulations.

The porous medium equation can be used to model tumor growth. We study the incompressible limit of an inhomogeneous porous medium equation (PME) with a cell division term that directly depends on space, time, and the pressure. The incompressible limit connects the PME to a Hele-Shaw free boundary problem (FBP). This relation is known as the complementarity condition. We first achieve convergence to the limiting problem along with uniqueness. Then, we gain enough compactness using L^3 bound of the pressure gradient and L^3 AB-estimate to get the complementarity condition. To finish the connection to the FBP, the velocity law to the boundary of the tumor is found. In particular, a novel inhomogeneous velocity of the free boundary is obtained.

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CHAPTER 1

INTRODUCTION

1.1 Approximating Inhomogeneous Porous Medium with Aggregation via Gradient Flow Methods

We study the weighted porous medium equation,

$$\text{(WPME)} \qquad \partial_t \mu = \underbrace{\nabla \cdot \left(\frac{a}{2} \nabla \left(\frac{\mu^2}{a^2} \right) \right)}_{\text{Diffusion}} + \underbrace{\nabla \cdot (\mu \nabla (V + V_k))}_{\text{Drift}} + \underbrace{\nabla \cdot (\mu (\nabla W * \mu))}_{\text{Aggregation}},$$

by using Wasserstein gradient flow theory. The main focus is to show that solutions to the regularized PDE,

$$\partial_t \mu = \nabla \cdot \left(\mu \left(\nabla \zeta_\epsilon * \left(\frac{\zeta_\epsilon * \mu}{a} \right) \right) + \mu (\nabla \zeta_\epsilon * V) + \mu (\nabla \zeta_\epsilon * \zeta_\epsilon * W * \mu) + \mu \nabla V_k \right), \tag{1.1}$$

converge to solutions of (WPME), where ζ_ϵ is a mollifier (see precise description in Assumption 2.1.1). Equivalently, we show that the gradient flow of the regularized energy functional converges to the gradient flow of the unregularized energy functional. This gives a convergence result for a deterministic particle method. Sometimes this particle method, developed by [9], is called blob method as the regularization involves convolving a mollifier (blob function) with the gradient flow. The particle method involves discretizing the initial data $\mu_0 = \mu(0)$ as a finite sum of Dirac masses. Moreover, μ in (1.1) is a finite sum of Dirac masses so that we obtain a system of ODEs for the particles. That is, the particle locations evolve based on this system of ODEs (written explicitly in Theorem 2.2.2). The gradient flow structure is preserved in the limit. As $\epsilon \to 0$, the gradient flows of (1.1) converge to gradient flows of (WPME). If there was no drift nor aggregation ($V \equiv 0 \equiv W$), then the target or weight $a(x)$ is the steady-state of (WPME).

In 2000, [22] study convergence results of the solutions to the regularized version of $\mu_t = \Delta \mu^2/2$. Due to assumptions made of the initial data, these result do not guarantee convergence of the particle method. In [9], they establish convergence of the deterministic particle method for

$$\partial_t \mu = \Delta \mu^m + \nabla \cdot (\mu \nabla V) + \nabla \cdot (\mu (\nabla W * \mu)), \quad m \geq 1,$$

where the coinciding energy functional is semi-convex. In [14], they improve on the results by considering the inhomogeneous PDE instead of the homogeneous. In particular, they studied

$$\partial_t \mu = \nabla \cdot \left(\frac{a}{2} \nabla \left(\frac{\mu}{a} \right)^2 \right) + \nabla \cdot (\mu \nabla V),$$

where the weight (or inhomogeneity) $a = a(x)$ is nice.

We will generalize this particle method from [14] by adding an aggregation term and more importantly generalizing the convexity property of the energy so that ω-convexity is sufficient rather than semi-convexity (or λ-convexity).

To preserve the gradient flow structure, we will show that the gradient flow of the regularized energy converges to the gradient flow of the unregularized energy. To get the Γ-convergence of gradient flows, we use Serfaty's sufficient conditions. The two main conditions that we require are

1. Γ-convergence of the energies
2. Γ-convergence (lower semi-continuity) of the local slopes.

Furthermore, an another important necessity is getting an H^1 bound on the mollified gradient flow of the regularized energy. The key idea for the H^1 bound is to use the flow interchange method. Suppose that we have two energy functionals and their respective gradient flows. Differentiating for a fixed time of the first energy at the gradient flow of the second is the same as differentiating for a fixed time the second energy at the gradient flow of the first. This allows us to deal with the "easier" energy functional. The PDE of interest has applications to chemotaxis as the Keller-Segel model and crowd-motion models. Two common kernels for the aggregation in this application are the Newtonian and Bessel kernel, in which, both satisfy the assumptions to get ω-convexity of the aggregation energy. Numerical simulations follow.

1.2 Incompressible Limit of Inhomogeneous Porous Medium Equations

The focus of our work is the inhomogeneous porous medium equation with reaction,

$$\partial_t u_m = \nabla \cdot \left(\frac{u_m}{a(x,t)} \nabla p_m \right) + \frac{u_m}{a(x,t)} \Phi(x,t,p_m), \quad \text{where } p_m = \frac{m}{m-1} \left(\frac{u_m}{b(x,t)} \right)^{m-1}. \tag{1.2}$$

In particular, we view (1.2) as a model for tumor growth and study the incompressible limit ($m \to \infty$). The cells tend to avoid over-crowding and move away from the congested regions. The density u_m

represents the cell population, in which, the pressure p_m is generated from. The cell division rate is controlled by Φ, where Φ depends on the pressure, space, and time. Given that the cells are less willing to divide in packed areas, the division rate will decrease as the pressure increases and will be zero once the pressure is high enough. We call this pressure the homeostatic pressure.

In terms of what laws to use, we can rewrite the PME so that we explicitly see the for the velocity we use Darcy's law and for the pressure we use the power law,

$$\text{(PME)} \qquad \begin{cases} \partial_t u_m = \nabla \cdot (u_m v) + \frac{u_m}{a(x,t)} \Phi(x, t, p_m), \\ v = \frac{\nabla p_m}{a(x,t)}, \, p_m = \frac{m}{m-1} \left(\frac{u_m}{b(x,t)} \right)^{m-1}. \end{cases}$$

From the power law, we see that taking the incompressible limit ($m \to \infty$), the stiffness of the pressure increases. The incompressible limit relates the PME and a Hele-Shaw free boundary problem (FBP). This link is called the complementarity condition and is one of the goals achieved here along with the inhomogeneous velocity law of the free boundary. The flow or velocity in both models (PME and FBP) are induced by Darcy's law. Now the free boundary velocity does not only depend on the pressure gradient but on $a(x, t)$ as well. The novelty being an inhomogeneous velocity law. The function $b(x, t)$ represents the max packing density of cells. That is, b is the largest the density can become. The ratio of the viscosity of the fluid and the permeability of the medium is represented by $a(x, t)$. In other words, $a(x, t)$ describes the ease in which a fluid can move through the medium.

We go back to 1981 where [4] established continuous dependence on φ of solutions of $\partial_t u = \Delta\varphi(u)$ (filtration equation). They achieved the first incompressible limit result by letting $m \to \infty$ for $\varphi(u) = u^m$. Caffarelli and Friedman in 1987 ([7]) studied the incompressible limit of $\partial_t u = \Delta u^m$ IVP on $\mathbb{R}^d$. They showed that the IVP coincides with motionless or stationary FBP. Gil and Quirós in 2001 ([19]) worked on the IVP on an open $\Omega \subset \mathbb{R}^d$ with boundary data depending on space. With boundary data idendically equal to zero and large enough Ω, they got the same results as Caffarelli and Friedman. In particular, a motionless or stationary boundary. With nontrivial boundary data, they get a nonstationary FBP. In more recent work, [23] includes the growth term $u\Phi(p)$. This results in a nonstationary free boundary in $\mathbb{R}^d$. Also they showed that the complementarity condition is equivalent to L^2 strong convergence of the pressure gradient. They used a L^∞ AB-estimate (Aronson, Bénilan) to get the result. In 2021, [17] included nutrient concentration in the growth term. They established a new way to get L^2 strong compactness for pressure gradient. They achieved a L^3 AB-estimate and a L^4 bound on the pressure gradient to gain enough compactness. In [21], they included a term that can be either a source or sink term. They prove the complementarity condition by connecting it to a obstacle problem, in which they show is true. In all of the above prior work, they consider $a = b \equiv 1$. We generalize this so that a, b are nice functions that depend not only on space but on time as well.

CHAPTER 2

APPROXIMATING INHOMOGENEOUS POROUS MEDIUM WITH AGGREGATION VIA GRADIENT FLOW METHODS

2.1 Notation and Assumptions

2.1.1 Definition of Energy Functionals

We study

$$\text{(WPME)} \qquad \partial_t \mu = \underbrace{\nabla \cdot \left(\frac{a}{2} \nabla \left(\frac{\mu^2}{a^2} \right) \right)}_{\text{Diffusion}} + \underbrace{\nabla \cdot (\mu \nabla (V + V_k))}_{\text{Drift}} + \underbrace{\nabla \cdot (\mu (\nabla W * \mu))}_{\text{Aggregation}}$$

by using the Wasserstein gradient flow, $\mu : [0, T] \to \mathcal{P}_2(\mathbb{R}^d)$, of the energy

$$\mathcal{F} : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R} \cup \{+\infty\}, \quad \mathcal{F}(\mu) = \mathcal{E}(\mu) + \mathcal{V}(\mu) + \mathcal{V}_\Omega(\mu) + \mathcal{W}(\mu)$$

where we have diffusion ($\mathcal{E}$), potential (or drift $\mathcal{V}$), confining potential ($\mathcal{V}_k$), and interaction (or aggregation $\mathcal{W}$) energies

$$\mathcal{E}(\mu) = \int_{\mathbb{R}^d} \frac{u^2(x)}{2a(x)} \, dx, \quad \mathcal{V}(\mu) = \int_{\mathbb{R}^d} V(x) \, d\mu(x),$$

$$\mathcal{V}_\Omega(\mu) = \begin{cases} 0, \ \operatorname{supp}(u) \subseteq \overline{\Omega} \\ \\ +\infty, \ \text{otherwise} \end{cases}, \quad \mathcal{W}(\mu) = \frac{1}{2} \int_{\mathbb{R}^d} W * \mu(x) \, d\mu(x),$$

respectively. Note $d\mu = \mu(x) \, dx$ and $V, W \colon \mathbb{R}^d \to \mathbb{R}$. Here, $\Omega \subseteq \mathbb{R}^d$ is nonempty, open, and convex. A probability measure $\mu \in \mathcal{P}(\mathbb{R}^d)$ lies in the domain of an energy $\mathcal{F}$, denoted $D(\mathcal{F})$, if $\mathcal{F}(\mu) < \infty$. The second moment of a probability measure μ is

$$M_2(\mu) = \int_{\mathbb{R}^d} |x|^2 \, d\mu(x).$$

We are interested in the space $\mathcal{P}_2(\mathbb{R}^d) = \mathcal{P}(\mathbb{R}^d) \cap D(M_2)$, probability measures with finite second moment, with the 2-Wasserstein distance W_2 (see Remark 2.3.4). The space $(\mathcal{P}_2(\mathbb{R}^d), W_2)$ is a metric space and in particular a geodesic space [24]. We will use the flow interchange method (see Remark 2.5.3) to prove a H^1 bound, in which, we define the heat entropy

$$\mathcal{S}(\mu) = \int_{\mathbb{R}^d} \mu \log \mu \, dx.$$

We assume that the weight $a \in C^1(\mathbb{R}^d)$ and there exists a constant $C > 0$ such that $1/C \leq a(x) \leq C$ for all $x \in \mathbb{R}^d$. For the well-posedness of the gradient flows of $\mathcal{E}$ (and therefore $\mathcal{F}$), $a(x)$ is log-concave on Ω. In particular, the functional $\mathcal{E}$ is convex and its local slope is a strong upper gradient. Note that if $\Omega = \mathbb{R}^d$, then $a(x)$ would be a constant. For $\mathcal{E}$ to be well-defined, we insist that $\|\mu\|_{L^2(\mathbb{R}^d)} \leq C$. This correlates well with Assumption 2.1.3. As in [9, Corollary 5.5], for particles to remain particles, regularization is used.

To illustrate why regularization is the solution [9] proposes to solve their issue, let us look at the continuity equation

$$\begin{cases} \partial_t \mu = \nabla \cdot (\mu v), \\ \mu_0(x) = \mu(x, 0). \end{cases}$$

If we define the velocity as $v = \left(\mu \nabla \left(\frac{\mu}{a}\right)\right) + \nabla V + \nabla W * \mu$, then the continuity equation is the same as (WPME) barring the confinement variable. If v is nice (that is, there is no diffusion term), then the particle method works without any regularization required. However, including diffusion makes v not nice. To make v nice or to give v stronger regularity, we can regularize.

In particular, we regularize the energies by convolving a mollifier with μ. That is,

$$\mathcal{F}_{\epsilon,k}(\mu) = \mathcal{E}_\epsilon(\mu) + \mathcal{V}_\epsilon(\mu) + \mathcal{V}_k(\mu) + \mathcal{W}_\epsilon(\mu)$$

where

$$\mathcal{E}_\epsilon(\mu) = \int_{\mathbb{R}^d} \frac{(\zeta_\epsilon * \mu(x))^2}{2a(x)}\, dx, \quad \mathcal{V}_\epsilon(\mu) = \int_{\mathbb{R}^d} V(x)\, d\zeta_\epsilon * \mu(x)$$
$$\mathcal{V}_k(\mu) = \int_{\mathbb{R}^d} V_k(x)\, d\mu(x), \quad \mathcal{W}_\epsilon(\mu) = \frac{1}{2}\int_{\mathbb{R}^d} W * \zeta_\epsilon * \mu(x)\, d\zeta_\epsilon * \mu(x).$$

The corresponding regularized PDE is

$$\partial_t \mu = \nabla \cdot \left(\mu\left(\nabla \zeta_\epsilon * \left(\frac{\zeta_\epsilon * \mu}{a}\right)\right) + \mu(\nabla \zeta_\epsilon * V) + \mu(\nabla \zeta_\epsilon * \zeta_\epsilon * W * \mu) + \mu \nabla V_k\right).$$

The particle method starts by approximating the initial data μ_0 as a finite sum of Dirac masses. We have a system of ODEs for the particles, where the particle locations evolve in time based on the regularized version of the velocity, say v_ϵ. So, we get the gradient flow, $\mu_\epsilon^N(t)$, of the regularized energy. Taking $\epsilon \to 0$ gives the gradient flow of unregularized energy, which corresponds to the original PDE.

2.1.2 Assumptions

There are various assumptions necessary of the functions a, V, W and the mollifier ζ_ϵ.

Assumption 2.1.1 (Mollifier)**.** We assume that the mollifier satisfies the following:

$$\zeta \in C^2(\mathbb{R}^d) \text{ is even, nonnegative, } \|\zeta\|_{L^1(\mathbb{R}^d)} = 1, D^2\zeta \in L^\infty(\mathbb{R}^d), \tag{2.1}$$

$$\zeta(x) \le C_\zeta |x|^{-q}, |\nabla\zeta(x)| \le C_\zeta |x|^{-q'}, \text{ for } C_\zeta > 0,\ q > d+1,\ q' > d. \tag{2.2}$$

Assumption 2.1.2 (Target function)**.** Let $\Omega \subseteq \mathbb{R}^d$ be nonempty, open, and convex. The weight $a \in C^1(\mathbb{R}^d)$ is log-concave on Ω and there exists a constant $C > 0$ such that $1/C \le a(x) \le C$ for all $x \in \mathbb{R}^d$.

We will consider V and W satisfying assumptions 4.1 and 4.2 from [13] so that the aggregation and drift functionals are ω-convex.

Assumption 2.1.3 (Aggregation/Interaction)**.** There exists a constant $C > 0$ (not necessarily the same constant) such that

1. For all $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ with $\|\mu\|_{L^p} \le C_p$, $W^- * \mu(x) \le C$.
2. For all $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ with $\|\mu\|_{L^p} \le C_p$, we have $\|\nabla W * \mu\|_{L^2(\nu)} \le C$.
3. For all $\|\mu\|_{L^p} \le C_p$, the kernel $W*\mu$ is continuously differentiable and there exists a continuous, nondecreasing, concave function $\psi : [0,\infty) \to [0,\infty)$ satisfying $\psi(0) = 0, \psi(x) \ge x$, and $\int_0^1 \frac{dx}{\psi(x)} = \infty$ so that
$$|\nabla W * \mu(x) - \nabla W * \mu(y)|^2 \le C^2 \psi(|x-y|^2).$$
4. For all $\|\mu\|_{L^p}, \|\nu\|_{L^p}, \|\rho\|_{L^p} \le C_p$,
$$\|\nabla W * \mu - \nabla W * \nu\|_{L^2(\rho)} \le C W_2(\mu, \nu).$$
5. W is lower semi-continuous.

6. $1 < p \leq +\infty$.

Assumption 2.1.4 (Drift Potential)**.** There exists a constant $C > 0$ (not necessarily the same constant) such that

1. $V \geq -C$.
2. For all $\mu \in D(\mathcal{V})$, we have $\|\nabla V\|_{L^2(\mu)} \leq C$.
3. The kernel V is continuously differentiable and there exists a continuous, nondecreasing, concave function $\psi : [0, \infty) \to [0, \infty)$ satisfying $\psi(0) = 0, \psi(x) \geq x$, and $\int_0^1 \frac{dx}{\psi(x)} = \infty$ so that
$$|\nabla V(x) - \nabla V(y)|^2 \leq C^2 \psi(|x - y|^2).$$

Assumption 2.1.5 (Confining Potential)**.** Let $\Omega \subseteq \mathbb{R}^d$ is nonempty, open, and convex. The confining potential $V_k(x) \geq 0$, for $k \in \mathbb{N}$ is convex and twice differentiable with $D^2 V_k \in L^\infty(\mathbb{R}^d)$. Furthermore, $V_k = 0$ on Ω and $\lim_{k\to\infty}(\inf_{x\in B} V_k(x)) = +\infty$ for all $B \subset\subset \Omega^c$.

Remark 2.1.6. Assumption 2.1.5 implies that $V_k \in L^1(\mu)$ and $\nabla V_k \in L^2(\mu)$ for $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ by taking the L^∞ norm. In particular, we get well-posedness of the gradient flow and the correct limiting dynamics as $k \to \infty$.

Assumption 2.1.7 (Additional Assumptions)**.** For any $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ with $\|\mu\|_{L^p(\mathbb{R}^d)} \leq C$, we require that either

1. $\exists R > 0$ such that, $\|\nabla W * \mu\|_{L^2(\mathbb{R}^d \setminus B_R)} \leq C_R$ and $\|\nabla V\|_{L^2(\mathbb{R}^d \setminus B_R)} \leq C_R$.
2. $\|D^2 W * \mu\|_{L^2(\mathbb{R}^d)} \leq C\|\mu\|_{L^2(\mathbb{R}^d)}$ and $\|D^2 V\|_{L^2(\mathbb{R}^d)} \leq C$.

Remark 2.1.8. Given Assumption 2.1.3, for any $R > 0$, choosing $\nu = \frac{1}{\mathscr{L}^d(B_R)}\mathscr{L}^d|_{B_R}$ gives us $\|\nabla W * \mu\|_{L^2(\mathscr{L}^d|_{B_R})} \leq C_R$. We have that item 1 in Assumption 2.1.7 gives $\|\nabla W * \mu\|_{L^2(\mathbb{R}^d)} \leq C$ after we fix R. Regarding item 2, recall that $\|\mu\|_{L^2(R^d)} \leq C$. Thus we have a bound of C after combining constants. Analogous arguments are made for the assumptions on V.

We can get the same bound if we convolve $\nabla W * \mu$ (or ∇V) with ζ_ϵ by the same method as Remark 2.4.10 or by using Young's convolution inequality. These additional assumptions are used in Proposition 2.5.10.

We give examples that satisfy the assumptions required for the aggregation kernel.

2.1.3 Newtonian Potential

There are multiple examples where ω-convex energies have the Newtonian potential,

$$\mathcal{N}(x) = \begin{cases} \frac{1}{2\pi}\log(|x|), d = 2 \\ \frac{|x|^{2-d}}{d(2-d)\alpha(d)}, d \neq 2 \end{cases}$$

as their kernel. It is only interesting when $d \geq 2$ as when $d = 1$ the Newtonian potential is convex and many energies (see example 2.19 of [13]) are ω-convex with $\lambda_\omega = 0$ (i.e. convex). For any extra assumptions added for W, we would like the Newtonian potential to satisfy them as well.

2.1.4 Kernels of Aggregation

There are two generalization of the Newtonian potential. The first is the Riesz potential

$$\mathcal{R}_{\beta,d}(x) = C_{d,\beta}|x|^{\beta-d}$$

with $2 \leq \beta < d$ and $d \geq 3$. See that the Newtonian and the Riesz kernels are equivalent when $\beta = 2$ ([8]). The Bessel kernel is given by

$$\mathcal{B}_{\alpha,d}(x) = \int_0^\infty \frac{1}{(4\pi t)^{d/2}} e^{\frac{-|x|^2}{4t} - \alpha t}\, dt$$

with $\alpha \geq 0$. The Newtonian and the Bessel kernels are equivalent when $\alpha = 0$ ([8]). Both of Riesz and Bessel kernels satisfy the assumptions for ω-convexity.

2.2 Main Results

Theorem 2.2.1 (Convergence of gradient flows as $k \to \infty, \epsilon = \epsilon(k) \to 0$)**.** *Suppose Assumptions 2.1.3, 2.1.4, 2.1.5, 2.1.7 hold. Fix $T > 0$ and $\mu(0) \in D(\mathcal{F}) \cap D(\mathcal{S}) \cap \mathcal{P}_2(\mathbb{R}^d)$. For $\epsilon > 0$ and $k \in \mathbb{N}$, let $\mu_{\epsilon,k} \in AC^2([0,T];\mathcal{P}_2(\mathbb{R}^d))$ be a gradient flow of $\mathcal{F}_{\epsilon,k}$ with the initial data $\mu(0)$. Then as $k \to \infty$, there exists a sequence $\epsilon = \epsilon(k) \to 0$ so that*

$$\lim_{k\to\infty} W_1(\mu_{\epsilon,k}(t), \mu(t)) = 0, \text{ uniformly for } t \in [0,T],$$

where $\mu \in AC^2([0,T];\mathcal{P}_2(\mathbb{R}^d))$ is the unique gradient flow of $\mathcal{F}$ with initial condition $\mu(0)$.

Before we give the result for convergence with particle initial data, we need to briefly discuss Osgood's criterion and the discretization of the PDE. The well-posedness of ω-convexity functionals were inspired by the Osgood's criterion of well-posedness of ODEs in Euclidean space. Furthermore, ω being an Osgood modulus of convexity (see [13]) ensures the ODE

$$\begin{cases} \frac{d}{dt}F_t(x) = \lambda_\omega \omega(F_t(x)), \\ F_0(x) = x, \end{cases}$$

is well-posed locally in time. We will review some properties of the ODE. The solutions to the ODE is

$$F_t(x) = \phi^{-1}(\phi(x) + t\lambda_\omega), \quad \phi(x) = \int_1^x \frac{dy}{\omega(y)}.$$

where $\phi : (0, \infty) \to \mathbb{R}$. If $\lambda_\omega \leq 0$, then $F_t(x)$ is a solution for all $t \geq 0$. If $\lambda_\omega > 0$, then $F_t(x)$ is a solution for $0 \leq t < (\phi(+\infty) - \phi(x))/\lambda_\omega$. The function $F_t(x)$ and its spatial inverse $F_t^{-1}(x) = F_{-t}(x)$ are continuous and strictly increasing in x. If $\lambda_\omega \leq 0$, then $F_t(x)$ is nonincreasing in t. If $\lambda_\omega > 0$, then $F_t(x)$ is nondecreasing in t. We also know that $\phi'(x) = 1/\omega(x)$ and $(\phi^{-1}(x))' = \omega(\phi^{-1}(x)) \geq 0$. In the particle method λ_ω is a function of ϵ, denoted $\lambda_{\omega,\epsilon}$. To this end $F_t(x)$ is also a function of ϵ, denoted $F_{t,\epsilon}(x) = \phi^{-1}(\phi(x) + t\lambda_{\omega,\epsilon})$.

After mollifying the gradient flow, the regularized PDE is

$$\partial_t \mu = \nabla \cdot \left(\mu \left(\nabla\zeta_\epsilon * \left(\frac{\zeta_\epsilon * \mu}{a} \right) \right) + \mu(\nabla\zeta_\epsilon * V) + \mu(\nabla\zeta_\epsilon * \zeta_\epsilon * W * \mu) + \mu\nabla V_k \right).$$

Given that the gradient flow of $\mathcal{F}_\epsilon$ solves the above PDE in the weak sense, we can discretize the problem by letting

$$\mu(t) = \sum_{i=1}^N \delta_{X^i(t)} m^i, \quad \mu_0 = \sum_{i=1}^N \delta_{X_0^i} m^i, \quad \sum_{i=1}^N m^i = 1,$$

where $\{X^i(t)\}_{i=1}^N$ are the location of the particles. We then obtain a system of ODEs for the particle,

$$\dot{X}^i(t) = -\sum_{j=1}^N f(X^i, X^j) m^j - \nabla\zeta_\epsilon * V(X^i) - \nabla V_k(X^i) - \sum_{j=1}^N m^j \nabla\zeta_\epsilon * \zeta_\epsilon * W(X^i - X^j)$$

with $X^i(0) = X_0^i$ where

$$f(X^i, X^j) = \int_{R^d} \frac{\nabla\zeta_\epsilon(X^i - z)\zeta_\epsilon(X^j - z)}{a(z)}\, dz.$$

Theorem 2.2.2 (Convergence with particle initial data)**.** *Suppose Assumptions 2.1.3, 2.1.4, 2.1.5, 2.1.7 hold. Let* $\lambda_{\omega,\epsilon} := \lambda_\omega^{\mathcal{F}_\epsilon}$*. Fix* $T > 0$ *and* $\mu(0) \in D(\mathcal{F}) \cap D(\mathcal{S}) \cap \mathcal{P}_2(\mathbb{R}^d)$*. For* $k, N \in \mathbb{N}, \epsilon > 0$*, and* $t \in [0,T]$*, consider the evolving empirical measure,*

$$\mu_{\epsilon,k}^N(t) = \sum_{i=1}^N \delta_{X_{\epsilon,k}^i(t)} m^i, \quad m^i \geq 0, \quad \sum_{i=1}^N m^i = 1,$$

where $X_{\epsilon,k}^i \in C^1([0,T];\mathbb{R}^d)$ *solves,*

$$\begin{cases} \dot{X}_{\epsilon,k}^i(t) & = -\sum_{j=1}^N m^j \int_{R^d} \nabla\zeta_\epsilon(X_{\epsilon,k}^i - z)\zeta_\epsilon(X_{\epsilon,k}^j - z)\frac{1}{a(z)}\, dz - \nabla\zeta_\epsilon * V(X_{\epsilon,k}^i) - \nabla V_k(X_{\epsilon,k}^i) \\ & \quad - \sum_{j=1}^N m^j \nabla\zeta_\epsilon * \zeta_\epsilon * W(X_{\epsilon,k}^i - X_{\epsilon,k}^j) \\ X_{\epsilon,k}^i(0) & = X_{0,\epsilon}^i. \end{cases}$$

Suppose that as $\epsilon \to 0$ *there exists* $N = N(\epsilon) \to \infty$*, such that, for all* $k \in \mathbb{N}$*,* $\mu_{\epsilon,k}^N(0) = \sum_{i=1}^N \delta_{X_{0,\epsilon}^i} m^i$ *converge to* $\mu(0)$ *with the rate,*

$$\lim_{k\to\infty} F_{-2t,\epsilon}(W_2^2(\mu_{\epsilon,k}^N(0), \mu(0))) = 0.$$

Then as $k \to \infty$*, there exists* $\epsilon = \epsilon(k) \to 0$ *and* $N = N(\epsilon) \to \infty$*, for which* $\mu_{\epsilon,k}^N(t) = \sum_{i=1}^N \delta_{X_{\epsilon,k}^i(t)} m^i$ *satisfies*

$$\lim_{k\to\infty} W_1(\mu_{\epsilon,k}^N(t), \mu(t)) = 0, \text{ uniformly for } t \in [0,T],$$

where $\mu \in AC^2([0,T];\mathcal{P}_2(\mathbb{R}^d))$ *is the unique weak solution of* (WPME) *with initial condition* $\mu(0)$.

Corollary 2.2.3 (Long time limit)**.** *Suppose Assumptions 2.1.3, 2.1.4, 2.1.5, 2.1.7 hold. Define the empirical measure* $\mu_{\epsilon,k}^N(t) = \sum_{i=1}^N \delta_{X_{\epsilon,k}^i(t)} m^i$*. Assume* $V = W = 0$*,* Ω *is bounded, and* $\int_\Omega a\, d\mathcal{L}^d = 1$*. Then there exists* $k = k(t) \to \infty$*,* $\epsilon = \epsilon(k) \to 0$*, and* $N = N(\epsilon) \to \infty$ *so that*

$$\lim_{t\to\infty} W_1(\mu_{\epsilon,k}^N(\cdot,t), a\mathbb{1}_{\overline{\Omega}}) = 0.$$

We can get the convergence results without diffusion on $\mathbb{R}^d$ instead of on Ω. That is, no confinement is necessary.

Theorem 2.2.4 (Convergence of gradient flow of Drift and Aggregation on $\mathbb{R}^d$)**.** *Suppose Assumptions 2.1.1, 2.1.3, 2.1.4, 2.1.7 hold. Define* $\mathcal{G}_\epsilon = \mathcal{V}_\epsilon + \mathcal{W}_\epsilon$ *and* $\mathcal{G} = \mathcal{V} + \mathcal{W}$*. Fix* $T > 0$, $\mu(0) \in D(\mathcal{G}) \cap \mathcal{P}_2(\mathbb{R}^d)$*. For* $\epsilon > 0$*, let* $\mu_\epsilon \in AC^2([0,T];\mathcal{P}_2(\mathbb{R}^d))$ *be the gradient flow of* $\mathcal{G}_\epsilon$ *with initial data* $\mu(0)$*. Then as* $\epsilon \to 0$, $\mu_\epsilon(t) \to \mu(t)$ *narrowly for* $t \in [0,T]$ *where* $\mu \in AC^2([0,T];\mathcal{P}_2(\mathbb{R}^d))$ *is the unique gradient flow of* $\mathcal{G}$ *with initial condition* $\mu(0)$*.*

2.2.1 Remarks on Convergence Rate

There are two interesting examples from [13, Example 2.17] worth discussing when it pertains to Theorem 2.2.2.

Remark 2.2.5 (Convergence when $\omega(x) = x$)**.** For $\omega(x) = x$, $F_t(x) = x\exp(\lambda_{\omega,\epsilon} t)$. Given the approximation via an empirical measure $W_2^2(\mu^N_{\epsilon,k}(0), \mu(0)) \leq \delta_\epsilon \sim 1/N_\epsilon$ ([14, Lemma A.4]), we want to analyze

$$\lim_{k\to\infty} F_{-2t,\epsilon}(\delta_\epsilon) = 0.$$

Using the semi-convexity of $\mathcal{E}_\epsilon$, $\lambda_{\omega,\epsilon} \sim -1/\epsilon^{d+2}$, as $\epsilon \to 0$,

$$0 \leftarrow F_{-2t,\epsilon}(\delta_\epsilon) = \delta_\epsilon \exp(-2t\lambda_{\omega,\epsilon}) = \frac{\delta_\epsilon}{\exp(-2t/\epsilon^{d+2})}.$$

Thus, we get the same convergence as [14] as $\epsilon \to 0$,

$$\frac{1}{N(\epsilon,k)} = o\left(\exp\left(\frac{-1}{\epsilon^{d+2}}\right)\right).$$

Remark 2.2.6 (Convergence for log-Lipschitz modulus of convexity)**.** Some explicit examples are mention later with the Newtonian potential and Bessel kernel for the aggregation energy. Both are ω-convex where ω is log-Lipschitz. In particular, $F_t(x) = x^{\exp(-t\lambda_{\omega,\epsilon})}$. Given the approximation via an empirical measure $W_2^2(\mu^N_{\epsilon,k}(0), \mu(0)) \leq \delta_\epsilon \sim 1/N_\epsilon$ ([14, Lemma A.4]), we want to analyze

$$\lim_{k\to\infty} F_{-2t,\epsilon}(\delta_\epsilon) = 0.$$

Using the semi-convexity of $\mathcal{E}_\epsilon$, $\lambda_{\omega,\epsilon} \sim -1/\epsilon^{d+2}$, as $\epsilon \to 0$,

$$0 \leftarrow F_{-2t,\epsilon}(\delta_\epsilon) = \exp\left(\frac{\log(\delta_e)}{\exp(2t/\epsilon^{d+2})}\right).$$

This requires that as $\epsilon \to 0$,

$$\frac{\log(\delta_e)}{\exp(2t/\epsilon^{d+2})} \to -\infty.$$

Equivalently, as $\epsilon \to 0$,

$$\frac{1/\log(N_e)}{\exp(-2t/\epsilon^{d+2})} \to 0.$$

Thus, we get what we expect based on the previous convergence result that as $\epsilon \to 0$,

$$\frac{1}{\log(N(\epsilon,k))} = o\left(\exp\left(\frac{-1}{\epsilon^{d+2}}\right)\right).$$

Neither of the previous convergence rates are desirable as in some numerical simulations in [14], $N \sim \epsilon^{-1.01}$. We cannot show this in particular, but do improve on the qualitative rates above.

Remark 2.2.7 (Improved Qualitative Convergence Rate)**.** Given the approximation via an empirical measure $W_2^2(\mu^N_{\epsilon,k}(0), \mu(0)) \leq \delta_\epsilon \sim 1/N_\epsilon$ ([14, Lemma A.4]), we want to analyze

$$\lim_{k\to\infty} F_{-2t,\epsilon}(\delta_\epsilon) = 0.$$

In particular, we want to have some convergence rate for $N(\epsilon)$ to achieve the above limit. Given that ω is nondecreasing and positive, we use a discrete approximation so that,

$$\begin{aligned}
\int_1^{\delta_\epsilon} \frac{dy}{\omega(y)} &\leq \frac{-(1-\delta_\epsilon)}{n}\sum_{i=1}^{n}\frac{1}{\omega(x_i)} + E_n \\
&= \frac{-(1-\delta_\epsilon)}{n}\frac{1}{\omega(\delta_\epsilon)} + \frac{-(1-\delta_\epsilon)}{n}\sum_{i=2}^{n}\frac{1}{\omega(x_i)} + E_n \\
&\leq \frac{-(1-\delta_\epsilon)}{n}\frac{1}{\omega(\delta_\epsilon)} + E_n,
\end{aligned}$$

for a fixed n, where E_n is the error. We use the fact that we know the explicit form of $F_t(x)$, ϕ^{-1} and ω are nondecreasing, and that the semi-convexity of $\mathcal{E}_\epsilon$ with $\lambda_{\omega,\epsilon} \sim -1/\epsilon^{d+2}$,

$$\begin{aligned}
F_{-2t,\epsilon}(\delta_\epsilon) &= \phi^{-1}(\phi(\delta_\epsilon) - 2t\lambda_{\omega,\epsilon}) \\
&= \phi^{-1}(\phi(\delta_\epsilon) + 2t/\epsilon^{d+2}) \\
&= \phi^{-1}\left(\int_1^{\delta_\epsilon}\frac{dy}{\omega(y)} + \frac{2t}{\epsilon^{d+2}}\right) \\
&\leq \phi^{-1}\left(\frac{-(1-\delta_\epsilon)}{n}\frac{1}{\omega(\delta_\epsilon)} + E_n + \frac{2t}{\epsilon^{d+2}}\right).
\end{aligned}$$

If $\int_0^1 \frac{dy}{\omega(y)} = \infty$ (see [13, Definition 2.15]), then $\phi(0) = -\infty$. Given that $\phi^{-1}(\phi(0)) = 0$,

$$\lim_{\epsilon \to 0} \frac{-(1-\delta_\epsilon)}{\omega(\delta_\epsilon)} + \frac{2t}{\epsilon^{d+2}} = -\infty.$$

Note that since n does not depend on ϵ and is fixed, we can ignore it as it is a constant. Taking $\log \exp$,

$$\log\left(\lim_{\epsilon \to 0} \frac{\exp\left(\frac{-(1-\delta_\epsilon)}{\omega(\delta_\epsilon)}\right)}{\exp\left(\frac{-2t}{\epsilon^{d+2}}\right)}\right) = -\infty.$$

It follows that,

$$\lim_{\epsilon \to 0} \frac{\exp\left(\frac{-(1-\delta_\epsilon)}{\omega(\delta_\epsilon)}\right)}{\exp\left(\frac{-2t}{\epsilon^{d+2}}\right)} = 0.$$

Thus as $\epsilon \to 0$,

$$\exp\left(\frac{-(1-\delta_\epsilon)}{\omega(\delta_\epsilon)}\right) = o\left(\exp\left(\frac{-2t}{\epsilon^{d+2}}\right)\right),$$

or with full generality,

$$\exp\left(\frac{-(1-\delta_\epsilon)}{\omega(\delta_\epsilon)}\right) = o\left(\exp\left(2t\lambda_{\omega,\epsilon}\right)\right).$$

As $\epsilon \to 0$, for $\omega(x) = x$,

$$\exp(1 - N(\epsilon, k)) = o\left(\exp\left(\frac{-1}{\epsilon^{d+2}}\right)\right),$$

and for $\omega(x) = x|\log(x)|$,

$$\exp\left(\frac{1 - N(\epsilon, k)}{|\log(N(\epsilon, k))|}\right) = o\left(\exp\left(\frac{-1}{\epsilon^{d+2}}\right)\right).$$

2.3 Background

2.3.1 Preliminaries

We discuss numerous definitions and lemmas used throughout the chapter. The first allows us to move the mollifier from the measure to the integrand.

Lemma 2.3.1 (mollifier exchange, [9] Lemma 2.2)**.** *Let $f : \mathbb{R}^d \to \mathbb{R}$ be Lipschitz continuous with constant $L_f > 0$, and let σ and ν be finite, signed Borel measures on $\mathbb{R}^d$. There is $p = p(q, d) > 0$ so that*

$$\left|\int \zeta_\epsilon * (f\nu)\, d\sigma - \int (\zeta_\epsilon * \nu) f\, d\sigma\right| \leq \epsilon^p L_f \left(\int (\zeta_\epsilon * |\nu|)\, d|\sigma| + C_\zeta |\sigma|(\mathbb{R}^d)|\nu|(\mathbb{R}^d)\right)$$

for all $\epsilon > 0$.

Narrow convergence is one of the main notions of convergence in this chapter.

Definition 2.3.2 (narrow convergence)**.** A sequence μ_n in $\mathcal{P}(\mathbb{R}^d)$ is said to narrowly converge to $\mu \in \mathcal{P}(\mathbb{R}^d)$ if $\int f\, d\mu_n \to \int f\, d\mu$ for all bounded and continuous functions f.

Lemma 2.3.3 (narrow convergence and mollifiers)**.** *Suppose* ζ_ϵ *is a mollifier satisfying Assumption 2.1.1 and let* μ_ϵ *be a sequence in* $\mathcal{P}(\mathbb{R}^d)$ *converging narrowly to* $\mu \in \mathcal{P}(\mathbb{R}^d)$*. Then* $\zeta_\epsilon * \mu_\epsilon$ *narrowly converges to* μ*.*

Another main notion of convergence used here is via distance. In particular, the 2-Wasserstein metric. This relates to optimal transport.

Remark 2.3.4 (optimal transport and Wasserstein metric)**.** For $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$, the set of transport plans from μ to ν is given by

$$\Gamma(\mu, \nu) = \{\gamma \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) |\ \pi^1_\# \gamma = \mu, \pi^2_\# \gamma = \nu\}$$

where $\pi^1, \pi^2 : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ are projections of $\mathbb{R}^d \times \mathbb{R}^d$ onto the first and second copy of $\mathbb{R}^d$, respectively. For $p \geq 1$, the p-Wasserstein distance between $\mu, \nu \in \mathcal{P}_p(\mathbb{R}^d)$ is given by

$$W_p(\mu, \nu) = \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p \, d\gamma(x, y) \right)^{1/p}$$

with $\gamma \in \Gamma_0(\mu, \nu)$ where $\Gamma_0(\mu, \nu)$ is the set of optimal transport plans. The pth moment is defined as $M_p(\mu) = \int_{\mathbb{R}^d} |x|^p \, d\mu$ and thus define the space

$$\mathcal{P}_p(\mathbb{R}^d) = \{\mu \in \mathcal{P}(\mathbb{R}^d) |\ M_p(\mu) < \infty\}.$$

Remark 2.3.5 (geodesics and generalized geodesics)**.** Given $\mu_0, \mu_1 \in \mathcal{P}_2(\mathbb{R}^d)$, a geodesic connecting μ_0 to μ_1 are curves of the form

$$\mu_\alpha = ((1 - \alpha)\pi^1 + \alpha \pi^2)_\# \gamma$$

for $\alpha \in [0, 1], \gamma \in \Gamma_0(\mu_0, \mu_1)$. Given $\mu_0, \mu_1, \mu_2 \in \mathcal{P}_2(\mathbb{R}^d)$, a generalized geodesic from μ_2 to μ_3 with base μ_1 is given by

$$\mu_\alpha^{2 \to 3} = ((1 - \alpha)\pi^2 + \alpha \pi^3)_\# \gamma$$

for $\alpha \in [0,1]$ and $\gamma \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d)$ such that $\pi^{1,2}_{\#}\gamma \in \Gamma_0(\mu_1, \mu_2)$ and $\pi^{1,3}_{\#}\gamma \in \Gamma_0(\mu_1, \mu_3)$. For short hand, sometimes μ_α will be used for the generalized geodesic.

Often times we require a curve to be absolutely continuous.

Definition 2.3.6 (absolutely continuous)**.** We say $\mu(t)$ is absolutely continuous on $[0,T]$, and write $\mu \in AC^2_{\text{loc}}((0,T);\mathcal{P}_2(R^d))$, if there exists $f \in L^2_{\text{loc}}((0,T))$ so that,

$$W_2(\mu(t), \mu(s)) \leq \int_s^t f(r)\, dr$$

for all $t, s \in (0,T)$ with $s \leq t$.

The minimal f to satisfy this is the metric derivative of μ.

Definition 2.3.7 (metric derivative)**.** Given $\mu \in AC^2_{\text{loc}}((0,T);\mathcal{P}_2(R^d))$, the limit

$$|\mu'|(t) := \lim_{s \to t} \frac{W_2(\mu(t), \mu(s))}{|t-s|}$$

exists for a.e. $t \in (0,T)$ and is called the metric derivative of μ.

A main point of this work is to generalize results of [14] to ω-convex functionals. Here we provide a definition as well as recall other notions of convexity and how they relate.

Definition 2.3.8 (ω-convexity)**.** Given an energy $E : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R} \cup \{+\infty\}$, a curve $\mu_\alpha \in \mathcal{P}_2(\mathbb{R}^d)$, and a distance function $d : \mathcal{P}_2(\mathbb{R}^d) \times \mathcal{P}_2(\mathbb{R}^d) \to [0,\infty)$, we say that E is ω-convex along μ_α w.r.t. d if for some $\omega : [0,+\infty) \to [0,+\infty)$ and $\lambda_\omega \in \mathbb{R}$,

$$\begin{aligned} E(\mu_\alpha) \leq &(1-\alpha)E(\mu_0) + \alpha E(\mu_1) \\ &- \frac{\lambda_\omega}{2}\left((1-\alpha)\omega\left(\alpha^2 d(\mu_0,\mu_1)^2\right) + \alpha\omega\left((1-\alpha)^2 d(\mu_0,\mu_1)^2\right)\right). \end{aligned}$$

where the modulus of convexity $\omega(x)$ is continuous, nondecreasing, and vanishes only at $x = 0$. If μ_α is a geodesic, then $d = W_2$. If μ_α is a generalized geodesic from μ_2 to μ_3 with base μ_1, then $d = W_{2,\gamma}$ where

$$W_{2,\gamma}(\mu_2, \mu_3) = \left(\int |\pi^2 - \pi^3|^2\, d\gamma\right)^{1/2}.$$

Remark 2.3.9 (Discussion of notions of convexity)**.** In [13, Definition 2.4], convexity, semi-convexity, and ω-convexity are discussed. It is sometimes easier to compare the different notions of convexity using the negative part of λ, $\lambda^- = \max\{0, -\lambda\}$. It is clear that when $\lambda = 0$, convex and semi-convexity are equivalent. If $\lambda > 0$, then semi-convexity implies convexity. Conversely, if $\lambda < 0$, then convexity implies semi-convexity. We have similar implications for ω-convexity and semi-convexity (and therefore convexity). Semi-convexity and ω-convexity are equivalent when we have the identity map, $\omega(x) = x$. Semi-convexity implies ω-convexity when $\omega(x) \geq x$ and $\lambda_\omega^- \geq \lambda^-$. The first requirement is used so that

$$\begin{aligned}\alpha(1-\alpha)d^2(\mu_0,\mu_1) &= (1-\alpha)\alpha^2 d^2(\mu_0,\mu_1) + \alpha(1-\alpha)^2 d^2(\mu_0,\mu_1)\\ &\leq (1-\alpha)\omega(\alpha^2 d^2(\mu_0,\mu_1)) + \alpha\omega((1-\alpha)^2 d^2(\mu_0,\mu_1)).\end{aligned}$$

Requiring $\lambda_\omega^- \geq \lambda^-$ gives

$$\frac{-\lambda}{2}\alpha(1-\alpha)d^2(\mu_0,\mu_1) \leq \frac{-\lambda_\omega}{2}\left((1-\alpha)\omega(\alpha^2 d^2(\mu_0,\mu_1)) + \alpha\omega((1-\alpha)^2 d^2(\mu_0,\mu_1))\right).$$

Using the same reasoning, we get the converse. That is, if $\omega(x) \leq x$ and $\lambda_\omega^- \leq \lambda^-$, then ω-convexity implies semi-convexity.

It can be quite difficult to check if an energy functional is ω-convex by the definition. Here we have an criterion for ω-convexity using the above the tangent line property.

Proposition 2.3.10 (above the tangent line property and ω-convexity, [13] Proposition 2.7)**.** *Suppose that for all generalized geodesic μ_α from μ_0 to μ_1 with base ν such that $\mu_0, \mu_1 \in D(E)$, $E(\mu_\alpha)$ is differentiable for $\alpha \in [0,1]$, $\frac{d}{d\alpha}E(\mu_\alpha) \in L^1([0,1])$, and*

$$E(\mu_1) - E(\mu_0) - \frac{d}{d\alpha}E(\mu_\alpha)|_{\alpha=0} \geq \frac{\lambda_\omega}{2}\omega(W_{2,\gamma}^2(\mu_0,\mu_1)).$$

Then E is ω-convex along generalized geodesics. Furthermore, if E merely satisfies these assumptions in the specific case that $\nu = \mu_0$ or $\nu = \mu_1$, then E is ω-convex along geodesics.

Definition 2.3.11 (local slope)**.** Given $E : \mathcal{P}_2(\mathbb{R}^d) \to (-\infty, \infty]$, for any $\mu \in D(E)$, the local slope is

$$|\partial E|(\mu) = \limsup_{\nu\to\mu} \frac{(E(\mu) - E(\nu))_+}{W_2(\mu,\nu)}$$

where $\lambda_+ = \max\{\lambda, 0\}$ is the positive part of λ.

Many methods in this chapter take advantage of gradient flow. Moreover, the preservation of the gradient flow structure in the limit.

Definition 2.3.12 (gradient flow)**.** Suppose $E : (-\infty, \infty]$ is proper, lower semi-continuous, and ω-convex along generalized geodesics. A curve $\mu(t) \in AC^2([0,T]; \mathcal{P}_2(\mathbb{R}^d))$ is a gradient flow of E in the Wasserstein metric if $\mu(t)$ is a weak solution of the continuity equation

$$\partial_t \mu(t) + \nabla \cdot (v(t)\mu(t)) = 0$$

in the sense of distributions and $v(t) = -\nabla \frac{\delta \mathcal{F}}{\delta \mu}$ for $\mathcal{L}^1$-a.e. $t > 0$.

We state the characterization of the gradient flow, in which, are multiples results of [1].

Theorem 2.3.13 (well-posedness and characterization of gradient flow)**.** *Suppose $E : (-\infty, \infty]$ is proper, lower semi-continuous, and ω-convex along generalized geodesics and $\mu(0) \in D(E)$. Then there exists $\mu(t)$ an unique gradient flow of E such that as $t \to 0^+$, $W_2(\mu(t), \mu(0)) \to 0$. Moreover, $\mu(t) \in AC^2([0,T]; \mathcal{P}_2(\mathbb{R}^d))$ is a gradient flow of E if and only if $\mu(t)$ satisfies one the following equivalent conditions:*

1. *Curve of Maximal Slope: For all $0 < s \le t$,*

$$\frac{1}{2}\int_s^t |\mu'|^2(r)\, dr + \frac{1}{2}\int_s^t |\partial E|^2(\mu(r))\, dr \le E(\mu(s)) - E(\mu(t)).$$

2. *Evolution Variational Inequality: For all $\nu \in \mathcal{P}_2(\mathbb{R}^d)$ and for $\mathcal{L}^1$-a.e. $t \ge 0$,*

$$\frac{1}{2}\frac{d}{dt}W_2^2(\mu(t), \nu) + \frac{\lambda_\omega}{2}\omega(W_2^2(\mu(t), \nu)) \le E(\nu) - E(\mu(t)).$$

We will show that curves of maximal curves coincide with gradient flows of ω-convex energies in Theorem 2.3.23. The EVI condition comes from [13]. Note that the 1. is also referred to as the Energy Dissipation Inequality (EDI).

2.3.2 Sufficient Conditions via Serfaty's Theorem

Serfaty in [25], establishes a framework for convergence of gradient flows. We provide the definition of this type of convergence and adjust the framework of Serfaty for ω-convex functionals.

Definition 2.3.14 (Γ-convergence of energies)**.** We say that $\mathcal{G}_\epsilon : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R} \cup \{+\infty\}$ Γ-converges to $\mathcal{G} : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R} \cup \{+\infty\}$ if

1. For any $\mu_\epsilon \in \mathcal{P}(\mathbb{R}^d)$ converging narrowly to $\mu \in \mathcal{P}(\mathbb{R}^d)$,

$$\liminf_{\epsilon\to 0} \mathcal{G}_\epsilon(\mu_\epsilon) \geq \mathcal{G}(\mu). \tag{2.3}$$

2. For any $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, there exists $\mu_\epsilon \in \mathcal{P}(\mathbb{R}^d)$ converging narrowly to μ such that

$$\limsup_{\epsilon\to 0} \mathcal{G}_\epsilon(\mu) \leq \mathcal{G}(\mu). \tag{2.4}$$

The reason why Theorem 2.2.1 as uniform convergence in W_1 and not W_2 is because of the compactness of absolutely continuous curves.

Lemma 2.3.15 (compactness of absolutely continuous curves, [14] Lemma 2.15)**.** *Fix $T > 0$. Suppose we have a sequence $\{\mu_\epsilon\}_{\epsilon>0} \subset AC^2([0,T];\mathcal{P}_2(\mathbb{R}^d))$ and*

$$\sup_{\epsilon>0} \int_0^T |\mu_\epsilon'|^2(r)\, dr < \infty, \quad \sup_{\epsilon>0} M_2(\mu_\epsilon(0)) < \infty.$$

Then there exists $\mu \in AC^2([0,T];\mathcal{P}_2(\mathbb{R}^d))$ such that, along a subsequence $\epsilon \to 0$, $W_1(\mu_\epsilon(t), \mu(t)) \to 0$ uniformly in $t \in [0,T]$, and

$$\liminf_{\epsilon\to 0} \int_0^t |\mu_\epsilon'|^2(r)\, dr \geq \int_0^t |\mu'|^2(r)\, dr$$

for every $t \in [0,T]$.

Proof. We use Proposition 2.5.9 with the hypothesis of this lemma to get existence of a $C = C(T) > 0$, so that for all $t \in [0,T]$ and $\epsilon > 0$, $\mu_\epsilon(t)$ belongs to the set $\{\mu : M_2(\mu) \leq C\}$. This set is narrowly sequentially compact [1, Remark 5.1.5, Lemma 5.1.7] and uniformly integrable 1st moments [1, equation 5.1.20]. So it is relatively compact in the 1-Wasserstein metric [1, Proposition 7.1.5]. Thus pointwise in time, $\{\mu_\epsilon(t)\}_{\epsilon>0}$ is relatively compact with respect to the 1-Wasserstein metric. By

Hölder's inequality, for all $0 \le s \le t \le T$,

$$\begin{aligned}\sup_{\epsilon>0} W_1(\mu_\epsilon(s), \mu_\epsilon(t)) &\le \sup_{\epsilon>0} W_2(\mu_\epsilon(s), \mu_\epsilon(t)) \\ &\le \sup_{\epsilon>0} \int_s^t |\mu_\epsilon'|(r)\, dr \\ &\le \sqrt{t-s} \left(\sup_{\epsilon>0} \int_s^t |\mu_\epsilon'|^2(r)\, dr \right)^{1/2}.\end{aligned}$$

The equicontinuity with respect to the 1-Wasserstein metric means we can apply Arzelá-Ascoli so that there exits $\mu : [0,T] \to \mathcal{P}_2(\mathbb{R}^d)$, such that, up to a subsequence, $W_1(\mu_\epsilon(t), \mu(t)) \to 0$ uniformly in $t \in [0,T]$.

The hypothesis ensures $\{|\mu_\epsilon'|(r)\}_{\epsilon>0}$ is bounded in $L^2([0,T])$. Therefore, up to another subsequence, it is weakly convergent to some $v(r) \in L^2([0,T])$. For all $0 \le s \le t \le T$, using lower semi-continuity of 2-Wasserstein metric with respect to narrow convergence (and therefore 1-Wasserstein convergence),

$$W_2(\mu(s), \mu(t)) \le \liminf_{\epsilon\to 0} W_2(\mu_\epsilon(s), \mu_\epsilon(t)) \le \liminf_{\epsilon\to 0} \int_s^t |\mu_\epsilon'|(r)\, dr = \int_s^t v(r)\, dr.$$

This gives us $\mu \in AC^2([0,T]; \mathcal{P}_2(\mathbb{R}^d))$. By [1, Theorem 1.1.2], we have $|\mu'|(r) \le v(r)$ for a.e. $r \in [0,T]$. We finish by acknowledging that the $L^2([0,T])$ norm is lower semi-continuous with respect to weak convergence. □

Due to the fact that $a(x)$ is log-concave on Ω instead of $\mathbb{R}^d$, we require a weaker framework of Serfaty's result. Once we show that ω-convex functionals are regular in Theorem 2.3.23, then we obtain the following.

Theorem 2.3.16 (Weak Serfaty Framework for ω-convex functionals, [14] Proposition 2.16)**.** *Let $\mathcal{F}, \mathcal{F}_\epsilon : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ be functionals that are proper, lower semi-continuous, ω-convex along generalized geodesics, and bounded from below uniformly in ϵ and suppose $\mathcal{F}_\epsilon$ Γ-converges to $\mathcal{F}$ as $\epsilon \to 0$. Fix $T > 0$. Suppose that for all $\epsilon > 0$ there exists $\mu_\epsilon \in AC^2([0,T]; \mathcal{P}_2(\mathbb{R}^d))$ and for almost all $r \in [0,T]$ there exists $\eta_\epsilon(r) \in L^2(\mu_\epsilon(r))$ such that*

$$\frac{1}{2}\int_0^t |\mu_\epsilon'|^2(r)\, dr + \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} |\eta_\epsilon(r)|^2\, d\mu_\epsilon(r)\, dr \le \mathcal{F}_\epsilon(\mu_\epsilon(0)) - \mathcal{F}_\epsilon(\mu_\epsilon(t))$$

for all $0 \le t \le T$. *Suppose there exists* $\mu(0) \in D(\mathcal{F}) \cap \mathcal{P}_2(\mathbb{R}^d)$ *such that* $\sup_{\epsilon>0} M_2(\mu_\epsilon(0)) < \infty$ *and as* $\epsilon \to 0$

$$\mu_\epsilon(0) \to \mu(0) \text{ narrowly}, \quad \mathcal{F}_\epsilon(\mu_\epsilon(0)) \to \mathcal{F}(\mu(0)).$$

Then, there exists $\mu \in AC^2([0,T]; \mathcal{P}_2(\mathbb{R}^d))$ *so that up to a subsequence*

$$\lim_{\epsilon \to 0} W_1(\mu_\epsilon(t), \mu(t)) = 0, \text{ uniformly for } r \in [0,T].$$

Furthermore, we have

$$\frac{1}{2}\int_0^t |\mu'|^2(r)\, dr + \frac{1}{2}\int_0^t \liminf_{\epsilon \to 0} \int_{\mathbb{R}^d} |\eta_\epsilon(r)|^2\, d\mu_\epsilon(r)\, dr \le \mathcal{F}(\mu(0)) - \mathcal{F}(\mu(t))$$

for all $0 \le t \le T$.

Proof. As the initial data is well prepared, we may assume

$$\sup_{\epsilon>0} \mathcal{F}_\epsilon(\mu_\epsilon(0)) < \infty.$$

With the assumption that $\mathcal{F}_\epsilon$ is bounded from below uniformly in ϵ, then

$$\frac{1}{2}\int_0^t |\mu_\epsilon'|^2(r)\, dr + \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} |\eta_\epsilon(r)|^2\, d\mu_\epsilon(r)\, dr,$$

is bounded from above uniformly in ϵ. It follows that, $\sup_{\epsilon>0} \int_0^t |\mu_\epsilon'|^2(r)\, dr < \infty$. We may apply Lemma 2.3.15 so that,

$$\frac{1}{2}\int_0^t |\mu'|^2(r)\, dr + \frac{1}{2}\liminf_{\epsilon \to 0}\int_0^t \int_{\mathbb{R}^d} |\eta_\epsilon(r)|^2\, d\mu_\epsilon(r)\, dr \le \liminf_{\epsilon \to 0}(\mathcal{F}_\epsilon(\mu_\epsilon(0)) - \mathcal{F}_\epsilon(\mu_\epsilon(t))).$$

We use Fatou's Lemma to control the second term on the left-hand side. For the right-hand side, use use Γ-convergence of the energy functional and narrow convergence of the density to obtain the result. □

If the log-concavity of $a(x)$ were not required for the well-posedness of the gradient flows of $\mathcal{F}$ or if we did not have to restrict the log-concavity of $a(x)$ to Ω, then we could use the stronger result that would resemble [25, Theorem 2] much more. In fact, we can use this version to prove Theorem 2.2.4.

Theorem 2.3.17 (Serfaty's Sufficient Conditions for ω-convex functionals)**.** *Let $\mathcal{G}_\epsilon : \mathcal{P}_2(R^d) \to \mathbb{R} \cup \{+\infty\}$ and $\mathcal{G} : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R} \cup \{+\infty\}$ be proper, lower semi-continuous that are ω-convex along generalized geodesics. Let $|\partial \mathcal{G}_\epsilon|$ and $|\partial \mathcal{G}|$ be strong upper gradients of $\mathcal{G}_\epsilon$ and $\mathcal{G}$, respectively. For all $\epsilon > 0$, let $\mu_\epsilon \in AC^2([0,T];\mathcal{P}_2(\mathbb{R}^d))$ be a gradient flow of $\mathcal{G}_\epsilon$ and that there exists $\mu : [0,T] \to \mathcal{P}_2(R^d)$ such that, $\mu_\epsilon(t) \to \mu(t)$ narrowly for $t \in [0,T]$ and*

$$\mu(0) \in D(\mathcal{G}), \quad \lim_{\epsilon \to 0} \mathcal{G}_\epsilon(\mu_\epsilon(0)) = \mathcal{G}(\mu(0)). \tag{2.5}$$

Assume that (2.3) *holds and for almost every $t \in [0,T]$,*

$$\liminf_{\epsilon \to 0} \int_0^t |\mu_\epsilon'|^2(s)\, ds \geq \int_0^t |\mu'|(s)\, ds, \tag{2.6}$$

$$\liminf_{\epsilon \to 0} |\partial \mathcal{G}_\epsilon|^2(\mu_\epsilon(t)) \geq |\partial \mathcal{G}|^2(\mu(t)). \tag{2.7}$$

Then $\mu \in AC^2([0,T];\mathcal{P}_2(\mathbb{R}^d))$ and μ is a gradient flow of $\mathcal{G}$ with initial data $\mu(0)$.

Proof. Based on [25, Theorem 2], it suffices to show that the local slopes of ω-convex energies are strong upper gradients (Theorem 2.3.18) and that ω-convex energies are regular (Theorem 2.3.23). □

We wish to prove the Γ -convergence of the gradient flows of ω-convex energies using Serfaty's results [25, Theorem 2]. We must first show this holds for ω-convex energies instead of the usually semi-convex energies. The first thing to check is that local slopes are strong upper gradients.

Theorem 2.3.18 (Local slopes are strong upper gradients)**.** *Let E be ω-convex along geodesics. Then $|\partial E|$ is a strong upper gradient of E.*

Proof. Define the global slope $I_E(\mu) = \sup_{\nu \neq \mu} \frac{(E(\mu)-E(\nu))^+}{W_2(\mu,\nu)}$. By definition of local slope, $|\partial E|(\mu) \leq I_E(\mu)$. By the HWI inequality (see Proposition 2.5 of [14]),

$$|\partial E|(\mu) = \sup_{\nu \neq \mu} \left(\frac{E(\mu) - E(\nu)}{W_2(\mu,\nu)} + \frac{\lambda_\omega}{2} \frac{\omega(W_2^2(\mu,\nu))}{W_2(\mu,\nu)} \right)^+ .$$

Using that equality,

$$\begin{aligned}
I_E(\mu) &= \sup_{\nu\neq\mu}\left(\frac{E(\mu)-E(\nu)}{W_2(\mu,\nu)} + \frac{\lambda_\omega}{2}\frac{\omega(W_2^2(\mu,\nu))}{W_2(\mu,\nu)} - \frac{\lambda_\omega}{2}\frac{\omega(W_2^2(\mu,\nu))}{W_2(\mu,\nu)}\right)^+ \\
&\leq |\partial E|(\mu) + \frac{(-\lambda_\omega)^+}{2}\sup_{\nu\neq\mu}\frac{\omega(W_2^2(\mu,\nu))}{W_2(\mu,\nu)} \\
&= |\partial E|(\mu) + \frac{\lambda_\omega^-}{2}\sup_{\nu\neq\mu}\frac{\omega(W_2^2(\mu,\nu))}{W_2(\mu,\nu)}.
\end{aligned}$$

If $\lambda_\omega \geq 0$, then we get that the local slope is identical to the global slope. As the global slope is a strong upper gradient ([1, Theorem 1.2.5]), we get that the local slope is a strong upper gradient. Thus, we only need to consider when $\lambda_\omega < 0$. For $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$,

$$W_2^2(\mu,\nu) \leq M_2(\mu) + M_2(\nu) < \infty.$$

Since $\sup_{\nu\neq\mu}\frac{\omega(W_2^2(\mu,\nu))}{W_2(\mu,\nu)}$ is finite, then we can apply [1, Theorem 1.2.5]. We get that the local slope is a strong upper gradient as in [1, Corollary 2.4.10]. □

The next thing we must check is that curves of maximal slope coincide with gradient flows of ω-convex energies. It suffices to show that ω-convex energies are regular. In particular, we must show the subdifferential of an ω-convex energy is closed. We first characterize the subdifferential for ω-convex energies similarly to semi-convex energies.

Theorem 2.3.19 (subdifferential characterization of ω-convex functionals)**.** *Suppose $E : \mathcal{P}_2(\mathbb{R}^d) \to (-\infty, +\infty]$ that is proper, lower semi-continuous, and ω-convex along geodesics. Let $\mu \in D(E)$ and $\xi : \mathbb{R}^d \to \mathbb{R}^d$ with $\xi \in L^2(\mu)$. Then $\xi \in \partial E(\mu)$ if and only if for all $\nu \in \mathcal{P}_2(\mathbb{R}^d)$*

$$E(\nu) - E(\mu) \geq \int_{\mathbb{R}^d\times\mathbb{R}^d}\langle \xi(x), y-x\rangle\, d\gamma(x,y) + \frac{\lambda_\omega}{2}\omega(W_2^2(\mu,\nu)), \quad \forall \gamma \in \Gamma_0(\mu,\nu).$$

Proof. The ω-convexity characterization implies the definition ([1, Defintion 10.1.1]) as $\omega(W_2^2(\mu,\nu)) = o(W_2(\mu,\nu))$ as $\nu \to \mu$ in $\mathcal{P}_2(\mathbb{R}^d)$. The converse mostly follows from ([1, B in Section 10.1.1]). Conversely, suppose $\mu_0, \mu_1 \in \mathcal{P}_2(\mathbb{R}^d)$, $\gamma \in \Gamma_0(\mu_0,\mu_1)$, $\mu_\alpha = (\pi_\alpha^{1\to 2})_\# \gamma$ for

$t \in [0,1]$. By the definition of subdifferential at μ_0 for $\xi \in \partial E(\mu_0)$ and $\gamma_t \in \Gamma_0(\mu_0, \mu_t)$

$$\begin{aligned} E(\mu_t) - E(\mu_0) &\geq \int \langle \xi(x), y - x \rangle \, d\gamma_t(x,y) + o(W_2^2(\mu_0, \mu_t)) \\ &= \int \langle \xi(x), ty - tx \rangle \, d\gamma(x,y) + o(tW_2^2(\mu_0, \mu_1)) \\ &= t \int \langle \xi(x), y - x \rangle \, d\gamma(x,y) + o(t) \end{aligned}$$

where the first equality is a result of change of variables. Now we divide by t and compute the lim inf

$$\liminf_{t \to 0} \frac{E(\mu_t) - E(\mu_0)}{t} \geq \int \langle \xi(x), y - x \rangle \, d\gamma(x,y).$$

By the definition of ω-convexity,

$$E(\mu_t) \leq (1-t)E(\mu_0) + tE(\mu+1) - \frac{\lambda_\omega}{2}\left(((1-t)\omega(t^2 W_2^2(\mu_0, \mu_1)) + t\omega((1-t)^2 W_2^2(\mu_0, \mu_1))\right).$$

Rearranging and dividing by t we get a bound for the difference quotient

$$\frac{E(\mu_t) - E(\mu_0)}{t} \leq E(\mu_1) - E(\mu_0) - \frac{\lambda_\omega}{2}\left((\frac{1}{t} - 1)\omega(t^2 W_2^2(\mu_0, \mu_1)) + \omega((1-t)^2 W_2^2(\mu_0, \mu_1))\right).$$

We take the lim inf

$$\liminf_{t \to 0} \frac{E(\mu_t) - E(\mu_0)}{t} \leq E(\mu_1) - E(\mu_0) - \frac{\lambda_\omega}{2}\omega(W_2^2(\mu_0, \mu_1))$$

and get the right-hand side by using that ω is continuous, $\omega(0) = 0$, and $\omega(x) = o(\sqrt{x})$ as $x \to 0$. Rearranging and using the lower bound for the difference quotient gives the result. □

Definition 2.3.20 (weak convergence of varying measure)**.** Let μ_n be a sequence in $\mathcal{P}(\mathbb{R}^d)$ be narrowly converging to μ in $\mathcal{P}(\mathbb{R}^d)$ and let $v_n \in L^1(\mu_n; \mathbb{R}^m)$. We say that v_n weakly converges to $v \in L^1(\mu; \mathbb{R}^m)$ if for all $\zeta \in C_c^\infty(\mathbb{R}^d)$,

$$\lim_{n \to \infty} \int_{R^d} \zeta(x) v_n(x) \, d\mu_n(x) = \int_{R^d} \zeta(x) v(x) \, d\mu(x).$$

The main goal to show Theorem 2.3.21 is to take the lim inf as $n \to \infty$ of the inequality in the characterization of the subdifferential of an ω-convex energy. We have no issue with the energy term as E is lower semi-continuous and no issue with the ω-convexity term as it is continuous. The main issue is with the integral term, which has nothing to do with E being ω-convex. Thus, Theorem 2.3.21 follows from the standard result [1, Lemma 10.1.3] and noting that it is not restrictive to use a transport map as it can be adjusted so that no map is necessary (see [1, Remark 10.3.3]).

Theorem 2.3.21 (closure of subdifferential, [1] Lemma 10.1.3)**.** *Let E be proper, lower semi-continuous, and ω-convex functional. Let μ_n converge to $\mu \in D(E)$ in $\mathcal{P}_2(\mathbb{R}^d)$ and let $\xi \in \partial E(\mu_n)$ satisfying*

$$\sup_n \int |\xi(x)|^2 \, d\mu_n(x) < \infty,$$

and converge to ξ weakly (with varying measure). Then $\xi \in \partial E(\mu)$.

Definition 2.3.22 (regular functional)**.** A functional $E : \mathcal{P}_2(\mathbb{R}^d) \to (-\infty, \infty]$ is proper, lower semi-continuous, and $D(|\partial E|) \subseteq \mathcal{P}_2(\mathbb{R}^d)$. We say that E is regular if whenever the strong differentials $\xi_n \in \partial E(\mu_n)$, $\varphi_n = E(\mu_n)$ satisfy

$$\begin{cases} \mu_n \to \mu \text{ in } \mathcal{P}_2(\mathbb{R}^d), \quad \varphi_n \to \varphi, \quad \sup_n \|\xi_n\|_{L^2(\mu_n;\mathbb{R}^d)} < \infty \\ \xi_n \to \xi \quad \text{weakly (of varying measure),} \end{cases}$$

then $\xi \in \partial E(\mu)$ and $\varphi = E(\mu)$.

Theorem 2.3.23 (ω-convex functionals are regular)**.** *Suppose $E : \mathcal{P}_2(\mathbb{R}^d) \to (-\infty, +\infty]$ that is proper, lower semi-continuous, and $D(|\partial E|) \subseteq \mathcal{P}_2(\mathbb{R}^d)$. Then E is regular.*

Proof. By Theorem 2.3.21, we have $\xi \in \partial E(\mu)$. As E is lower semi-continuous, it suffices to show

$$\limsup_{n\to\infty} E(\mu_n) \le E(\mu),$$

which is equivalent to

$$\liminf_{n\to\infty} (E(\mu) - E(\mu_n)) \ge 0.$$

By Theorem 2.3.19,

$$E(\mu) - E(\mu_n) \ge \int_{\mathbb{R}^d\times\mathbb{R}^d} \langle \xi_n(x), y - x\rangle \, d\gamma_n(x,y) + \frac{\lambda_\omega}{2}\omega(W_2^2(\mu_n,\mu)), \quad \forall \gamma_n \in \Gamma_0(\mu_n,\mu).$$

As in Theorem 2.3.21, having μ_n converge to μ in $\mathcal{P}_2(R^d)$ (with respect to W_2), the right-hand side goes to zero. Thus,

$$\liminf_{n\to\infty}(E(\mu) - E(\mu_n)) \geq 0,$$

and moreover by lower semi-continuity $\lim_{n\to\infty} E(\mu_n) = E(\mu)$. □

2.4 Energy Properties

2.4.1 Lower semi-continuity of Energies

The first step in showing that the gradient flows of the energy functionals are well-posed is to show they are lower semi-continuous.

Proposition 2.4.1 (lower semi-continuity)**.** *Let $\epsilon > 0$ and V, W satisfy Assumptions 2.1.3 and 2.1.4, respectively. Then $\mathcal{V}_\epsilon$, $\mathcal{W}_\epsilon$ are lower semi-continuous with respect to narrow convergence in $\mathcal{P}(\mathbb{R}^d)$.*

Proof. By Lemma 5.1.7 of [1], $\mathcal{V}$ and $\mathcal{W}$ are lower semi-continuous with respect to narrow convergence. By definition $\mathcal{V}_\epsilon(\mu_n) = \mathcal{V}(\zeta_\epsilon * \mu_n)$ and $\mathcal{W}_\epsilon(\mu_n) = \mathcal{W}(\zeta_\epsilon * \mu_n)$. Thus we have,

$$\lim\inf_{n\to\infty} \mathcal{V}_\epsilon(\mu_n) \geq \mathcal{V}_\epsilon(\mu), \quad \lim\inf_{n\to\infty} \mathcal{W}_\epsilon(\mu_n) \geq \mathcal{W}_\epsilon(\mu).$$

This completes the proof. □

The following proposition is standard and follows from [1] and [14].

Proposition 2.4.2 (Lower semi-continuity of $\mathcal{E}, \mathcal{E}_\epsilon, \mathcal{V}, \mathcal{V}_k, \mathcal{V}_\Omega, \mathcal{W}$)**.** *Suppose Assumptions 2.1.3, 2.1.4, 2.1.5 hold. The for all $\epsilon > 0$, the functionals $\mathcal{E}, \mathcal{E}_\epsilon, \mathcal{V}, \mathcal{V}_k, \mathcal{V}_\Omega, \mathcal{W}$ are lower semi-continuous with respect to narrow convergence.*

2.4.2 Directional derivatives of Energies

The derivatives for the regularized drift and aggregation functionals follow from the first parts of the proofs of Proposition 4.6 and 4.7 in [13].

Proposition 2.4.3 (Directional Derivatives for $\mathcal{V}_\epsilon, \mathcal{W}_\epsilon$)**.** *Suppose Assumptions 2.1.3 and 2.1.4 holds. Fix $\epsilon > 0$, $\mu_0, \mu_1 \in \mathcal{P}_2(\mathbb{R}^d)$ satisfying $\|\mu_i\|_{L^2} \leq C$ for $i = 1, 2$, and $\gamma \in \Gamma(\mu_0, \mu_1)$ such that*

$\mu_\alpha = ((1-\alpha)\pi^1 + \alpha\pi^2)_\# \gamma$ *satisfying* $\|\mu_\alpha\|_{L^2} \le C$ *for all* $\alpha \in [0,1]$. *Then* $\mathcal{V}_\epsilon(\mu_\alpha), \mathcal{W}_\epsilon(\mu_\alpha)$ *are continuously differentiable and*

$$\frac{d}{d\alpha}\mathcal{V}_\epsilon(\mu_\alpha)|_{\alpha=0} = \int \langle \nabla(\zeta_\epsilon * V)(y_0), y_1 - y_0\rangle \, d\gamma,$$
$$\frac{d}{d\alpha}\mathcal{W}_\epsilon(\mu_\alpha)|_{\alpha=0} = \int \langle \nabla(\zeta_\epsilon * \zeta_\epsilon * W) * \mu_0(y_0), y_1 - y_0\rangle \, d\gamma.$$

Proof. As the calculations are similar, we elect to only show the one for the regularized drift energy. By Proposition 2.4.9, $\zeta_\epsilon * V$ is continuously differentiable. Define $\pi_\alpha = (1-\alpha)\pi^1 + \alpha\pi^2$ where π^1, π^2 are projections on the first and second axis, respectively. Then,

$$\begin{aligned}\frac{d}{d\alpha}\int \zeta_\epsilon * V \, d\mu_\alpha &= \lim_{h\to 0}\frac{1}{h}\left(\int \zeta_\epsilon * V \circ \pi_{\alpha+h}\, d\gamma - \int \zeta_\epsilon * V \circ \pi_\alpha \, d\gamma\right)\\ &= \int \langle \nabla(\zeta_\epsilon * V)\circ \pi_\alpha, y_1 - y_0\rangle \, d\gamma.\end{aligned}$$

Thus,

$$\frac{d}{d\alpha}\mathcal{V}_\epsilon(\mu_\alpha)|_{\alpha=0} = \int \langle \nabla(\zeta_\epsilon * V)(y_0), y_1 - y_0\rangle \, d\gamma.$$

This finishes the proof. □

Corollary 2.4.4 (Directional Derivatives for $\mathcal{V}, \mathcal{V}_k, \mathcal{W}$)**.** *There are analogous derivatives for* $\mathcal{V}, \mathcal{V}_k, \mathcal{W}$*,*

$$\frac{d}{d\alpha}\mathcal{V}_k(\mu_\alpha)|_{\alpha=0} = \int \langle \nabla V_k(y_0), y_1 - y_0\rangle \, d\gamma$$
$$\frac{d}{d\alpha}\mathcal{V}(\mu_\alpha)|_{\alpha=0} = \int \langle \nabla V(y_0), y_1 - y_0\rangle \, d\gamma,$$
$$\frac{d}{d\alpha}\mathcal{W}(\mu_\alpha)|_{\alpha=0} = \int \langle \nabla W * \mu_0(y_0), y_1 - y_0\rangle \, d\gamma.$$

Proof. As the calculations are similar, we elect to only show the one for the drift energy. By Assumption 2.1.4, V is continuously differentiable. Define $\pi_\alpha = (1-\alpha)\pi^1 + \alpha\pi^2$ where π^1, π^2 are projections on the first and second axis, respectively. Then,

$$\begin{aligned}\frac{d}{d\alpha}\int V \, d\mu_\alpha &= \lim_{h\to 0}\frac{1}{h}\left(\int V \circ \pi_{\alpha+h}\, d\gamma - \int V \circ \pi_\alpha \, d\gamma\right)\\ &= \int \langle \nabla V \circ \pi_\alpha, y_1 - y_0\rangle \, d\gamma.\end{aligned}$$

Thus,

$$\frac{d}{d\alpha}\mathcal{V}(\mu_\alpha)|_{\alpha=0} = \int \langle \nabla V(y_0), y_1 - y_0 \rangle \, d\gamma.$$

This completes the proof. □

We restate a result from [14].

Proposition 2.4.5 (Directional Derivatives of $\mathcal{E}_\epsilon$, [14] Proposition 3.4)**.** *Suppose Assumptions 2.1.1, 2.1.2 hold. Fix $\epsilon > 0$, $\nu_1, \nu_2, \nu_3 \in \mathcal{P}_2(\mathbb{R}^d)$ and $\gamma \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d)$ with $\pi^i_{\#}\gamma = \nu_i$. Consider the curve, $\mu_\alpha = \left((1-\alpha)\pi^2 + \alpha\pi^3\right)_{\#}\gamma$ for $\alpha \in [0,1]$. Then,*

$$\frac{d}{d\alpha}\mathcal{E}_\epsilon(\mu)|_{\alpha=0} = \frac{1}{2}\int \frac{\zeta_\epsilon * \nu_2(x)}{a(x)} \int \langle \nabla\zeta_\epsilon(x - y_2), y_3 - y_2 \rangle \, d\gamma(y_1, y_2, y_3) \, dx.$$

Proof. Let $x \in \mathbb{R}^d$ and $\alpha \in [0,1]$. By the dominated convergence theorem,

$$\lim_{\alpha \to 0} \frac{1}{\alpha}(\zeta_\epsilon * \mu_\alpha(x) - \zeta_\epsilon * \mu_0(x)) = \int \langle \nabla\zeta_\epsilon(x - y_2), y_3 - y_2 \rangle \, d\gamma(y_1, y_2, y_3),$$

as $\|\nabla\zeta_\epsilon\|_{L^\infty}|y_3 - y_2| \in L^1(\gamma)$ and $M_1(\gamma) \leq M_2(\gamma)^{1/2} < \infty$. Again by dominated convergence theorem,

$$\begin{aligned}
&\lim_{\alpha \to 0} \frac{1}{\alpha}(\mathcal{E}_\epsilon(\mu_\alpha) - \mathcal{E}_\epsilon(\mu_0)) \\
&= \int \lim_{\alpha \to 0} \frac{1}{2\alpha a(x)}(\zeta_\epsilon * \mu_\alpha(x) - \zeta_\epsilon * \mu_0(x))(\zeta_\epsilon * \mu_\alpha(x) + \zeta_\epsilon * \mu_0(x)) \, dx \\
&= \int \frac{\zeta_\epsilon * \mu_0(x)}{a(x)} \int \langle \nabla\zeta_\epsilon(x - y_2), y_3 - y_2 \rangle \, d\gamma(y_1, y_2, y_3) \, dx.
\end{aligned}$$

This finishes the proof. □

2.4.3 Convexity of Energies

The second step in showing that the gradient flows of the energy functionals are well-posed is to show they are convex. We start with the main purpose of [13]. Namely, the conditions in which an energy functional is ω-convex. We restate Proposition 4.6 and 4.7 in [13] as one result.

Proposition 2.4.6. *Let $\mu_0, \mu_1 \in \mathcal{P}_2(\mathbb{R}^d)$ satisfying $\|\mu_i\|_{L^2} \le C$ for $i = 1, 2$, and $\gamma \in \Gamma(\mu_0, \mu_1)$ such that $\mu_\alpha = ((1-\alpha)\pi^1 + \alpha\pi^2)_\# \gamma$ satisfying $\|\mu_\alpha\|_{L^2} \le C$ for all $\alpha \in [0,1]$. Then for $\omega(x) = \sqrt{x\psi(x)}$,*

$$\mathcal{V}(\mu_0) - \mathcal{V}(\mu_1) + \frac{d}{d\alpha}\mathcal{V}(\mu_\alpha)|_{\alpha=0} \le 2C\omega\left(\|x-y\|^2_{L^2(\gamma)}\right)$$
$$\left|\mathcal{W}(\mu_0) - \mathcal{W}(\mu_1) + \frac{d}{d\alpha}\mathcal{W}(\mu_\alpha)|_{\alpha=0}\right| \le 2C\omega\left(\|x-y\|^2_{L^2(\gamma)}\right)$$

Proof. We only show the proof of the drift as they are similar. Define $\pi_\alpha = (1-\alpha)\pi^1 + \alpha\pi^2$ where π^1, π^2 are projections on the first and second axis, respectively. As we know the directional derivative,

$$\mathcal{V}(\mu_1) = \mathcal{V}(\mu_0) + \frac{d}{d\alpha}\mathcal{V}(\mu_\alpha)|_{\alpha=0} + \iint \int_0^1 \langle \nabla V((1-\alpha x + \alpha y) - \nabla V(x), y - x\rangle \, d\alpha \, d\gamma.$$

So we only need to control the final term by the correct bound. By Hölder's inequality, item 3. of Assumption 2.1.4, and Jensen's inequality for concave $\psi(x)$,

$$\begin{aligned}
\iint |\nabla V \circ \pi_\alpha - \nabla V(x)| \, |x-y| \, d\gamma(x,y) &\le \|\nabla V \circ \pi_\alpha - \nabla V\|_{L^2(\gamma)} \|x-y\|_{L^2(\gamma)} \\
&\le 2C\|x-y\|_{L^2(\gamma)} \sqrt{\|\psi(|\pi_\alpha - \pi_0|^2)\|_{L^1(\gamma)}} \\
&\le 2C\|x-y\|_{L^2(\gamma)} \sqrt{\psi(\||\pi_\alpha - \pi_0|^2\|_{L^1(\gamma)})} \\
&= 2C\|x-y\|_{L^2(\gamma)} \sqrt{\psi(\alpha^2\|\pi_1 - \pi_0\|^2_{L^2(\gamma)})} \\
&\le 2C\omega\left(\|x-y\|^2_{L^2(\gamma)}\right).
\end{aligned}$$

Thus we have the result. □

By Propositions 2.3.10, 2.4.6, we achieve the ω-convexity of the drift and aggregation.

Proposition 2.4.7 (ω-convexity of $\mathcal{V}$, $\mathcal{W}$, [13] Theorem 4.3)**.** *Let W, V satisfy Assumptions 2.1.3 and 2.1.4, respectively. Then $\mathcal{V}, \mathcal{W}$ are ω-convex along generalized geodesics, with $\omega(x) = \sqrt{x\psi(x)}$, $\lambda_\omega = 4C$.*

Lemma 2.4.8. *Let $\tau_y : \mathbb{R}^d \to \mathbb{R}^d$ be the translation mapping $\tau_y(x) = x - y$ and $\nu \in \mathcal{P}_2(\mathbb{R}^d)$ with $\|\nu\|_{L^p} \le C_p$. Then, $\tau_{y\#}\nu \in \mathcal{P}_2(\mathbb{R}^d)$ and $\|\tau_{y\#}\nu\|_{L^p} \le C_p$.*

Proof. Given that $\tau_y^{-1}(\mathbb{R}^d) = \mathbb{R}^d$, we have the following

$$\tau_{y\#}\nu(\mathbb{R}^d) = \nu(\tau_y^{-1}(\mathbb{R}^d)) = \nu(\mathbb{R}^d) = 1.$$

So,

$$\int_{\mathbb{R}^d} |x|^2 \, d\tau_{y\#}\nu(x) = \int_{\tau_y^{-1}(\mathbb{R}^d)} |x|^2 \, d\nu(x) = \int_{\mathbb{R}^d} |x|^2 \, d\nu(x) < \infty.$$

Moreover,

$$\|\tau_{y\#}\nu\|_{L^p}^p = \|\nu\|_{L^p(\tau_y^{-1}(\mathbb{R}^d))}^p = \int_{\mathbb{R}^d} |\nu(x)|^p \, dx \leq C_p^p.$$

Thus we obtain the result. □

We use this lemma in a couple parts of the next proposition. Namely, that

$$\int |\nabla W * \mu(x-y)|^2 \, d\nu(x) = \int |\nabla W * \mu(x)|^2 \, d\tau_{y\#}\nu(x) \leq C^2.$$

We now show that the mollified versions of V and W satisfy Assumptions 2.1.3 and 2.1.4.

Proposition 2.4.9. *If W, V satisfies Assumptions 2.1.3 and 2.1.4, respectively. Then*

1. *for all $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ with$\|\mu\|_{L^p} \leq C_p$, $(\zeta_\epsilon * W)^- * \mu(x) \leq C$ and $\zeta_\epsilon * V \geq -C$*
2. *for all $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ with $\|\mu\|_{L^p} \leq C_p$, we have $\|\zeta_\epsilon * \nabla W * \mu\|_{L^2(\nu)} \leq C$ and with $\nu \in D(\mathcal{V})$, we have $\|\zeta_\epsilon * \nabla V\|_{L^2(\nu)} \leq C$*
3. *for all $\|\mu\|_{L^p} \leq C_p$, $\zeta_\epsilon * W * \mu$ and $\zeta_\epsilon * V$ are continuously differentiable and there exists a continuous, nondecreasing, concave function $\psi : [0,\infty) \to [0,\infty)$ satisfying $\psi(0) = 0, \psi(x) \geq x$, and $\int_0^1 \frac{dx}{\psi(x)} = \infty$ so that*

$$|\zeta_\epsilon * \nabla W * \mu(x) - \zeta_\epsilon * \nabla W * \mu(y)|^2 \leq C^2\psi(|x-y|^2),$$
$$|\zeta_\epsilon * \nabla V(x) - \zeta_\epsilon * \nabla V(y)|^2 \leq C^2\psi(|x-y|^2)$$

4. *for all $\|\mu\|_{L^p}, \|\nu\|_{L^p}, \|\rho\|_{L^p} \leq C_p$,*

$$\|\zeta_\epsilon * \nabla W * \mu - \zeta_\epsilon * \nabla W * \nu\|_{L^2(\rho)} \leq CW_2(\mu,\nu)$$

5. *$\zeta_\epsilon * W$ is lower semi-continuous*

Proof. We will only show the case for W as the case for V is similar when we choose $V := W * \mu$.

1. As $W^- * \mu$ is bounded above by C,

$$\zeta_\epsilon * W^- * \mu(x) = \int_{\mathbb{R}^d} \zeta_\epsilon(x-y)(W^- * \mu(y))\, dy \leq C.$$

   As $(\zeta_\epsilon * W)^- = \max\{0, -\zeta_\epsilon * W\}$ and

$$-\zeta_\epsilon * W * \mu \leq \zeta_\epsilon * \max\{0, -W\} * \mu = \zeta_\epsilon * W^- * \mu$$

   we get the result.

2. By Jensen's inequality and Assumption 2.1.3,

$$\begin{aligned}\int |\zeta_\epsilon * \nabla W * \mu|^2 d\nu(x) &= \int \left|\int \nabla W * \mu(x-y)\zeta_\epsilon(y)\, dy\right|^2 d\nu(x)\\ &\leq \int \int |\nabla W * \mu(x-y)|^2 \zeta_\epsilon(y)\, dy\, d\nu(x)\\ &\int \int |\nabla W * \mu(x-y)|^2\, d\nu(x)\, \zeta_\epsilon(y)\, dy\\ &\leq (C)^2.\end{aligned}$$

3. By Jensen's inequality and Assumption 2.1.3,

$$\begin{aligned}|\zeta_\epsilon * \nabla W * \mu(x) - \zeta_\epsilon * \nabla W * \mu(y)|^2 &= \left|\int (\nabla W * \mu(x-z) - \nabla W * \mu(y-z))\zeta_\epsilon(z)\, dz\right|^2\\ &\leq \int |(\nabla W * \mu(x-z) - \nabla W * \mu(y-z))|^2 \zeta_\epsilon(z)\, dz\\ &\leq C^2 \psi(|x-y|^2)\end{aligned}$$

4. By Jensen's inequality and Assumption 2.1.3,

$$\begin{aligned}&\int |\zeta_\epsilon * \nabla W * \mu - \zeta_\epsilon * \nabla W * \nu|^2 d\rho(x)\\ &= \int \left|\int (\nabla W * \mu(x-y) - \nabla W * \nu(x-y))\zeta_\epsilon(y)\, dy\right|^2 d\rho(x)\\ &\leq \int \int |\nabla W * \mu(x-y) - \nabla W * \nu(x-y)|^2 \zeta_\epsilon(y)\, dy\, d\rho(x)\\ &= \int \int |\nabla W * \mu(x-y) - \nabla W * \nu(x-y)|^2\, d\rho(x)\, \zeta_\epsilon(y)\, dy\\ &\leq (CW_2(\mu, \nu))^2.\end{aligned}$$

5. As W is lower semi-continuous $\forall \epsilon > 0\ \exists \delta > 0$ such that $W(x_0) < W(x) + \epsilon$ for all $x \in B_\delta(x_0)$.

   So we can translate by y so that,

$$\begin{aligned}\zeta_\epsilon * W(x_0) &= \int W(x_0 - y)\zeta_\epsilon(y)dy \\ &< \int W(x-y)\zeta_\epsilon(y)dy + \epsilon \int \zeta_\epsilon(y)dy \\ &= \zeta_\epsilon * W(x) + \epsilon.\end{aligned}$$

   Therefore, $\zeta_\epsilon * W$ is lower semi-continuous.

Thus we have the results. □

Remark 2.4.10. We can generalize this further having the same statement hold with another mollifier ζ_ϵ convolved against it by using the previous proposition and the proof along with it.

Now from Proposition 2.4.9, we obtained the conditions required to have $\mathcal{V}_\epsilon, \mathcal{W}_\epsilon$ are ω-convex via the directional derivatives (Proposition 2.4.3) and the above the tangent line property (Proposition 2.3.10).

Proposition 2.4.11 (ω-convexity of $\mathcal{V}_\epsilon$, $\mathcal{W}_\epsilon$)**.** *Let W, V satisfy Assumptions 2.1.3 and 2.1.4, respectively. Then $\mathcal{V}_\epsilon, \mathcal{W}_\epsilon$ are ω-convex along generalized geodesics, with $\omega(x) = \sqrt{x\psi(x)}$, $\lambda_\omega = 4C$.*

The next propositions follow from [1] and [14].

Proposition 2.4.12 (Convexity properties of $\mathcal{E}$)**.** *If Assumption 2.1.2 holds, then $\mathcal{E} + \mathcal{V}_\Omega$ convex along generalized geodesics. If Assumption 2.1.5 holds, then $\mathcal{V}_k$ is convex along generalize geodesics.*

Proposition 2.4.13 (Semi-convexity of $\mathcal{E}_\epsilon$, [14] Proposition 3.6)**.** *Suppose Assumptions 2.1.1, 2.1.2 hold. For all $\epsilon > 0$, $\mathcal{E}_\epsilon$ is λ_ϵ-convex along generalized geodesics, where,*

$$\lambda_\epsilon = -\epsilon^{-d-2}\|1/a\|_{L^\infty}\|D^2\zeta\|_{L^\infty}.$$

Proof. Let μ_α be a generalized geodesic with base $\mu_1 \in \mathcal{P}_2(\mathbb{R}^d)$ connect $\mu_2, \mu_3 \in \mathcal{P}_2(\mathbb{R}^d)$. As the mapping $x \mapsto x^2$ is convex, then the tangent inequality gives us,

$$\begin{aligned}
\mathcal{E}_\epsilon(\mu_3) - \mathcal{E}_\epsilon(\mu_2) &= \frac{1}{2}\int \frac{(\zeta_\epsilon * \mu_3(x))^2}{a(x)}\,dx - \frac{1}{2}\int \frac{(\zeta_\epsilon * \mu_2(x))^2}{a(x)}\,dx \\
&\geq \int \frac{\zeta_\epsilon * \mu_2(x)}{a(x)}(\zeta_\epsilon * \mu_3(x) - \zeta_\epsilon * \mu_2(x))\,dx \\
&= \int \frac{\zeta_\epsilon * \mu_2(x)}{a(x)} \int \zeta_\epsilon(x - y_3) - \zeta_\epsilon(x - y_2)\,d\gamma(y_1, y_2, y_3)\,dx.
\end{aligned}$$

By Proposition 2.4.5 and Taylor's theorem,

$$\begin{aligned}
&\mathcal{E}_\epsilon(\mu_3) - \mathcal{E}_\epsilon(\mu_2) - \frac{d}{d\alpha}\mathcal{E}_\epsilon(\mu_\alpha)|_{\alpha=0} \\
&\geq \int \frac{\zeta_\epsilon * \mu_2(x)}{a(x)} \int \zeta_\epsilon(x - y_3) - \zeta_\epsilon(x - y_2) - \langle \nabla\zeta_\epsilon(x - y_2), y_3 - y_2\rangle\,d\gamma(y_1, y_2, y_3)\,dx \\
&\geq -\frac{1}{2}\|D^2\zeta_\epsilon\|_{L^\infty} \int \frac{\zeta_\epsilon * \mu_2(x)}{a(x)} \int |y_2 - y_3|^2\,d\gamma(y_1, y_2, y_3)\,dx \\
&\geq -\frac{1}{2}\|1/a\|_{L^\infty}\|D^2\zeta_\epsilon\|_{L^\infty} W_{2,\gamma}^2(\mu_2, \mu_3).
\end{aligned}$$

The result follows from the above the tangent line property, Proposition 2.3.10. □

With the previous results above for the lower semi-continuity and convexity, the gradient flows of the energy functionals are well-posed. Now we move on to derivatives of the energies to obtain their subdifferentials.

2.4.4 Subdifferential of Energies

We use the directional derivatives to characterize the minimal elements of the subdifferentials of the energies.

Proposition 2.4.14 (Subdifferential of $\mathcal{V}_\epsilon, \mathcal{W}_\epsilon$)**.** *Suppose Assumption 2.1.3 holds. If $\mu \in D(\mathcal{W}_\epsilon)$, then $\nabla(\zeta_\epsilon * \zeta_\epsilon * W) * \mu \in \partial\mathcal{W}_\epsilon(\mu)$. If $\mu \in D(\mathcal{W})$, then $\nabla W * \mu \in \partial\mathcal{W}(\mu)$. Suppose Assumption 2.1.4 holds. If $\mu \in D(\mathcal{V}_\epsilon)$, then $\nabla(\zeta_\epsilon * V) \in \partial\mathcal{V}_\epsilon(\mu)$. If $\mu \in D(\mathcal{V})$, then $\nabla V \in \partial\mathcal{V}(\mu)$. Suppose Assumption 2.1.5 holds. If $\mu \in D(\mathcal{V}_k)$, then $\nabla V_k \in \partial\mathcal{V}_k(\mu)$.*

Proof. As all cases are similar, we will elect to only write down the case for $\mathcal{W}_\epsilon$. Fix $\epsilon > 0$, $\mu_0, \mu_1 \in \mathcal{P}_2(\mathbb{R}^d)$ satisfying $\|\mu_i\|_{L^2} \leq C$ for $i = 1, 2$, and $\gamma \in \Gamma(\mu_0, \mu_1)$ such that $\mu_\alpha = ((1-\alpha)\pi^1 + \alpha\pi^2)_{\#}\gamma$

satisfying $\|\mu_\alpha\| \le C$ for all $\alpha \in [0,1]$. Given that $\mathcal{W}_\epsilon$ is ω-convex, it satisfies the above the tangent line property,

$$\mathcal{W}_\epsilon(\mu_1) - \mathcal{W}_\epsilon(\mu_0) - \frac{d}{d\alpha}\mathcal{W}_\epsilon(\mu_\alpha)|_{\alpha=0} \ge \frac{\lambda_\omega}{2}\omega(W_2^2(\mu_0,\mu_1)).$$

As we calculated the directional derivatives (Proposition 2.4.3),

$$\mathcal{W}_\epsilon(\mu_1) - \mathcal{W}_\epsilon(\mu_0) \ge \int \langle \nabla(\zeta_\epsilon * \zeta_\epsilon * W) * \mu_0(y_0), y_1 - y_0\rangle \, d\gamma + \frac{\lambda_\omega}{2}\omega(W_2^2(\mu_0,\mu_1)).$$

The results follow from the the characterization of the subdifferential of ω-convex energies (Proposition 2.3.19). □

Proposition 2.4.15 (Subdifferential of $\mathcal{E}_\epsilon$, [14] Proposition 3.7 (i))**.** *Suppose Assumptions 2.1.1 and 2.1.2 hold. For all $\epsilon > 0$, and $\mu \in D(\mathcal{E}_\epsilon)$, we have*

$$\nabla \frac{\delta \mathcal{E}_\epsilon}{\delta \mu} \in \partial \mathcal{E}_\epsilon(\mu), \text{ where } \frac{\delta \mathcal{E}_\epsilon}{\delta \mu} = \zeta_\epsilon * \frac{\zeta_\epsilon * \mu}{a}.$$

Proof. Fix $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$ and $\gamma \in \Gamma_0(\mu,\nu)$. Let $\mu_\alpha = ((1-\alpha)\pi^1 + \alpha\pi^2)_\# \gamma$ be a geodesic from μ to ν. By Proposition 2.4.13, $\mathcal{E}_\epsilon$ is semi-convex along μ_α and by Proposition 2.3.10,

$$\mathcal{E}_\epsilon(\nu) - \mathcal{E}_\epsilon(\mu) - \frac{d}{d\alpha}\mathcal{E}_\epsilon(\mu_\alpha)|_{\alpha=0} \ge \frac{\lambda_\epsilon}{2} W_2^2(\mu,\nu).$$

Applying Proposition 2.4.5 with $\tilde{\gamma} = (\pi^1, \pi^1, \pi^2)_\# \gamma$,

$$\begin{aligned}
\mathcal{E}_\epsilon(\nu) - \mathcal{E}_\epsilon(\mu) &\ge \int \frac{\zeta_\epsilon * \mu(x)}{a(x)} \int \langle \nabla \zeta_\epsilon(x - y_2), y_3 - y_2 \rangle \, d\tilde{\gamma}(y_1,y_2,y_3)\, dx + \frac{\lambda_\epsilon}{2} W_2^2(\mu,\nu) \\
&= \int \frac{\zeta_\epsilon * \mu(x)}{a(x)} \int \langle \nabla \zeta_\epsilon(x - y_1), y_1 - y_2 \rangle \, d\gamma(y_1,y_2)\, dx + \frac{\lambda_\epsilon}{2} W_2^2(\mu,\nu) \\
&= \int \left\langle \nabla \zeta_\epsilon * \left(\frac{\zeta_\epsilon * \mu}{a}(y_1)\right), y_2 - y_1 \right\rangle \, d\gamma(y_1,y_2)\, dx + \frac{\lambda_\epsilon}{2} W_2^2(\mu,\nu) \\
&= \int \left\langle \nabla \frac{\delta \mathcal{E}_\epsilon}{\delta \mu}(y_1), y_2 - y_1 \right\rangle \, d\gamma(y_1,y_2)\, dx + \frac{\lambda_\epsilon}{2} W_2^2(\mu,\nu)
\end{aligned}$$

This completes the proof. □

Remark 2.4.16 (Minimal Selection and Chain rule)**.** As long as the energy functional is regular, then by [1, Lemma 10.1.5] the local slope is equivalent to the L^2 norm of the minimal element of the subdifferential. Therefore, ω-convex energies satisfy [1, Lemma 10.1.5]. Similarly, ω-convex energies satisfy the chain rule (E of section 10.1.2 in [1]).

We mention to minimality of the regularized subdifferential as in [14, Proposition 3.8].

Proposition 2.4.17 (Minimal subdifferential of $\mathcal{F}_{\epsilon,k}$)**.** *Suppose Assumptions 2.1.3 and 2.1.4 hold. For* $\epsilon > 0$ *and* $k \in \mathbb{N}$, $\mu \in D\mathcal{F}_{\epsilon,k}$,

$$\partial^\circ \mathcal{F}_{\epsilon,k}(\mu) = \nabla \frac{\delta \mathcal{F}_{\epsilon,k}}{\delta \mu} = \nabla \zeta_\epsilon * \frac{\zeta_\epsilon * \mu}{a} + \nabla \zeta_\epsilon * V + \nabla \zeta_\epsilon * \zeta_\epsilon * W * \mu + \nabla V_k.$$

Proof. For ease, let

$$\xi = \nabla \zeta_\epsilon * \frac{\zeta_\epsilon * \mu}{a} + \nabla \zeta_\epsilon * V + \nabla \zeta_\epsilon * \zeta_\epsilon * W * \mu + \nabla V_k.$$

By Propositions 2.4.14, 2.4.15 and additivity of the subdifferential, we have $\xi \in \partial \mathcal{F}_{\epsilon,k}(\mu)$. By [1, Lemma 10.1.5], it suffices to show that $\|\xi|_{L^2(\mu)} \leq |\partial \mathcal{F}_{\epsilon,k}|(\mu)$. Fix $\psi \in C^1(\mathbb{R}^d)$ so that $\nabla \psi \in L^2(\mu)$. Define $\mu_\alpha = (id + \alpha \nabla \psi)_{\#}\mu$, where id is the identity mapping. By definition of the 2-Wasserstein distance,

$$W_2(\mu_\alpha, \mu) \leq \|(id + \alpha \nabla \psi) - id\|_{L^2(\mu)} = \alpha \|\nabla \psi\|_{L^2(\mu)}.$$

By definition of local slope,

$$\begin{aligned} |\partial \mathcal{F}_{\epsilon,k}|(\mu) &\geq \limsup_{\alpha \to 0} \frac{(\mathcal{F}_{\epsilon,k}(\mu) - \mathcal{F}_{\epsilon,k}(\mu_\alpha))_+}{W_2(\mu, \mu_\alpha)} \\ &\geq \frac{1}{\|\nabla \psi\|_{L^2(\mu)}} \limsup_{\alpha \to 0} \frac{(\mathcal{F}_{\epsilon,k}(\mu) - \mathcal{F}_{\epsilon,k}(\mu_\alpha))_+}{\alpha}. \end{aligned}$$

Choosing $\nabla \psi = -\xi$ and applying the directional derivatives Propositions 2.4.3, 2.4.5, we get that $|\partial_{\epsilon,k} \mathcal{F}_{\epsilon,k}|(\mu) \|\nabla \psi\|_{L^2(\mu)} \geq \|\nabla \psi\|^2_{L^2(\mu)}$. Division by $\|\nabla \psi\|^2_{L^2(\mu)} = \|\xi\|^2_{L^2(\mu)}$ gives us the result. □

The minimal subdifferential of $\mathcal{F}$ is standard as seen in [1, Theorems 10.4.9-10.4.13].

Proposition 2.4.18 (Minimal subdifferential of $\mathcal{F}$)**.** *Suppose Assumptions 2.1.2, 2.1.3, 2.1.4, 2.1.5 hold. Given* $\mu \in D(\mathcal{F})$, *we have* $|\partial \mathcal{F}|(\mu) < \infty$ *if and only if* $(\mu/a)^2 \in W^{1,1}_{loc}(\Omega)$ *and there exists* $\xi \in L^2(\mu)$ *so that,*

$$\xi \mu = \frac{a}{2} \nabla \left(\frac{\mu}{a}\right)^2 + \mu \nabla V + \mu (\nabla W * \mu)$$

on Ω. *In particular,* ξ *is the minimal selection of* $\partial \mathcal{F}$. *That is,* $\xi = \partial^o \mathcal{F}(\mu)$.

Proposition 2.4.19 (Long time behavior, [1] Corollary 4.0.6)**.** *Suppose Assumption 2.1.2 holds, $V = W = 0$, and Ω bounded. Let $\mu_0 \in D(\mathcal{F})$ and let $\mu(t)$ be the gradient flow of μ of $\mathcal{F}$ with initial data μ_0. Then we have,*

$$\lim_{t\to\infty} W_2\left(\mu(t), \frac{a\mathbb{1}_{\overline{\Omega}}}{\int_\Omega a \, d\mathcal{L}^d}\right) = 0.$$

2.5 An H^1-type Bound

A key tool in the convergence of the gradient flows proof Proposition 2.6.6 is the H^1 bound of $\zeta_\epsilon * \mu_\epsilon$ by being an important hypothesis in the Γ-convergence (or lower semi-continuity) of the local slopes, Proposition 2.6.4. In pursuit of the H^1 bound of $\zeta_\epsilon * \mu_\epsilon$, we first start with the L^2 bound.

Lemma 2.5.1 (L^2 bound of convolved gradient flow)**.** *Suppose Assumptions 2.1.3, 2.1.4, 2.1.7 hold. For all $T > 0$ and $\epsilon > 0$, suppose that $\mu_\epsilon \in AC^2([0,T];\mathcal{P}_2(\mathbb{R}^d))$ is a gradient flow of $\mathcal{F}_{\epsilon,k}$. Then*

$$\|\zeta_\epsilon * \mu_\epsilon\|^2_{L^2(\mathbb{R}^d)} \leq 2\|a\|_{L^\infty(\mathbb{R}^d)} \left(\mathcal{F}_{\epsilon,k}(\mu_\epsilon(0)) + 2C\right).$$

Let $\mu_{\tau,\epsilon}$ be the piecewise constant interpolation in the minimizing movement scheme of $\mathcal{F}_\epsilon$. Then we get the same bound,

$$\|\zeta_\epsilon * \mu_{\tau,\epsilon}\|^2_{L^2(\mathbb{R}^d)} \leq 2\|a\|_{L^\infty(\mathbb{R}^d)} \left(\mathcal{F}_{\epsilon,k}(\mu_\epsilon(0)) + 2C\right).$$

Proof. For all $T > 0$ and $\epsilon > 0$, suppose that $\mu_\epsilon \in AC^2([0,T];\mathcal{P}(\mathbb{R}^d))$ is a gradient flow of $\mathcal{F}_{\epsilon,k}$. Recall that

$$\mathcal{E}_\epsilon(\mu_\epsilon) + \mathcal{W}_\epsilon(\mu_\epsilon) + \mathcal{V}_\epsilon(\mu_\epsilon) \leq \mathcal{F}_{\epsilon,k}(\mu_\epsilon) \leq \mathcal{F}_{\epsilon,k}(\mu_\epsilon(0)).$$

By definition,

$$\mathcal{E}_\epsilon(\mu_\epsilon) \geq \frac{1}{2\|a\|_{L^\infty(\mathbb{R}^d)}} \|\zeta_\epsilon * \mu_\epsilon\|^2_{L^2(\mathbb{R}^d)},$$

and by Assumption 2.1.3,

$$\mathcal{V}_\epsilon(\mu_\epsilon) \geq -C.$$

As $\mathcal{W}_\epsilon(\mu_\epsilon) = \mathcal{W}(\zeta_\epsilon * \mu_\epsilon)$ and $\mathcal{W}$ is lower semi-continuous, then we have

$$\liminf_{\epsilon\to 0} \mathcal{W}_\epsilon(\mu_\epsilon) \geq \mathcal{W}(\mu).$$

Moreover, $\sup_{\epsilon>0}\mathcal{W}_\epsilon(\mu_\epsilon) \geq \mathcal{W}(\mu)$. With Assumption 2.1.3,

$$\frac{1}{2\|a\|_{L^\infty(\mathbb{R}^d)}}\|\zeta_\epsilon * \mu_\epsilon\|^2_{L^2(\mathbb{R}^d)} - 2C \leq \mathcal{F}_{\epsilon,k}(\mu_\epsilon(0)).$$

Let $\mu_{\tau,\epsilon}$ be the piecewise constant interpolation in the minimizing movement scheme of $\mathcal{F}_{\epsilon,k}$. We get similar bounds for every term except for the interaction term. However, we can use the fact that $\mathcal{W}_\epsilon$ is lower semi-continuous and Assumption 2.1.3 so that

$$\sup_{\tau>0}\mathcal{W}_\epsilon(\mu_{\tau,\epsilon}) \geq \liminf_{\tau\to 0}\mathcal{W}_\epsilon(\mu_{\tau,\epsilon}) \geq \mathcal{W}_\epsilon(\mu_\epsilon) = \mathcal{W}(\zeta_\epsilon * \mu_\epsilon) \geq -C.$$

Moreover, we get the result as stated. □

2.5.1 H^1 bound on convolved gradient flow (formal/heuristic)

We first give a formal or heuristic argument of the H^1 bound. After the formal argument, we state some necessary definitions and lemmas to make the rigorous argument of the H^1 bound.

Proposition 2.5.2 (H^1 bound on $\zeta_\epsilon * \mu_\epsilon$)**.** *Let V, W satisfy Assumptions 2.1.3, 2.1.4, and Assumption 2.1.7. For all $T > 0$ and $\epsilon > 0$, suppose that $\mu_\epsilon \in AC^2([0,T];\mathcal{P}_2(\mathbb{R}^d))$ is a gradient flow of $\mathcal{F}_{\epsilon,k}$. Then,*

$$\int_0^T \|\nabla\zeta_\epsilon * \mu_\epsilon(s)\|^2_{L^2(\mathbb{R}^d)}\, ds \leq C\left(\mathcal{S}(\mu_\epsilon(0)) + M_2(\mu_\epsilon(0)) + \mathcal{F}_{\epsilon,k}(\mu_\epsilon(0)) + 1\right)$$

where $C = C(a, T, V, V_k, W)$.

Proof. We start by differentiating formally along gradient flow of μ_ϵ

$$\begin{aligned}
\frac{d}{dt}\left(\int_{\mathbb{R}^d}\mu_\epsilon\log\mu_\epsilon\, dx\right) &= \frac{d}{dt}\left(\int_{\mathbb{R}^d}\mu_\epsilon\log\mu_\epsilon\, dx - \int_{\mathbb{R}^d}\mu_\epsilon\, dx\right)\\
&= \int_{\mathbb{R}^d}\partial_t\mu_\epsilon\log\mu_\epsilon\, dx\\
&= \int_{\mathbb{R}^d}\nabla\cdot\left(\mu_\epsilon\nabla\frac{\delta\mathcal{F}_\epsilon}{\delta\mu_\epsilon}\right)\log\mu_\epsilon\, dx\\
&= -\int_{\mathbb{R}^d}\nabla\mu_\epsilon\cdot\nabla\frac{\delta\mathcal{F}_\epsilon}{\delta\mu_\epsilon}\, dx\\
&= \int_{\mathbb{R}^d} -\nabla\mu_\epsilon\cdot\nabla\zeta_\epsilon * \left(\frac{\zeta_\epsilon * \mu_\epsilon}{a}\right) - \nabla\mu_\epsilon\cdot\nabla\zeta_\epsilon * V - \nabla\mu_\epsilon\cdot\nabla\zeta_\epsilon * \zeta_\epsilon * W * \mu_\epsilon\\
&\qquad - \nabla\mu_\epsilon\nabla V_k\, dx\\
&=: I + I_V + I_W + I_{V_k}
\end{aligned}$$

where we use that μ_ϵ formally satisfies the continuity equation and integration by parts. Moving the gradient off ζ_ϵ and using the fact that ζ_ϵ is even,

$$
\begin{aligned}
I &= -\int_{\mathbb{R}^d} (\zeta_\epsilon * \nabla\mu_\epsilon)\nabla\left(\frac{\zeta_\epsilon * \mu_\epsilon}{a}\right)\, dx \\
&= -\int_{\mathbb{R}^d} (\nabla\zeta_\epsilon * \mu_\epsilon)\left(\frac{\nabla\zeta_\epsilon * \mu_\epsilon}{a} - \frac{\nabla a}{a^2}(\zeta_\epsilon * \mu_\epsilon)\right)\, dx \\
&= \int_{\mathbb{R}^d} \frac{-(\nabla\zeta_\epsilon * \mu_\epsilon)^2}{a}\, dx + \int_{\mathbb{R}^d} \frac{\nabla\zeta_\epsilon * \mu_\epsilon}{a^{1/2}}(\zeta_\epsilon * \mu_\epsilon)\frac{\nabla a}{a^{3/2}}\, dx \\
&\leq \int_{\mathbb{R}^d} \frac{-(\nabla\zeta_\epsilon * \mu_\epsilon)^2}{a}\, dx + \delta\int_{\mathbb{R}^d} \frac{(\nabla\zeta_\epsilon * \mu_\epsilon)^2}{a}\, dx + \int_{\mathbb{R}^d} \frac{(\zeta_\epsilon * \mu_\epsilon)^2}{4\delta}\left|\frac{\nabla a}{a^{3/2}}\right|^2\, dx \\
&\leq (\delta - 1)\int_{\mathbb{R}^d} \frac{(\nabla\zeta_\epsilon * \mu_\epsilon)^2}{a}\, dx + \frac{1}{4\delta}\left\|\left|\frac{\nabla a}{a^{3/2}}\right|^2\right\|_{L^\infty(\mathbb{R}^d)} \sup_{\epsilon>0}\int_{\mathbb{R}^d} |\zeta_\epsilon * \mu_\epsilon(0)|^2\, dx
\end{aligned}
$$

where Cauchy's inequality with $\delta > 0$ is used at the first inequality. For I_{V_k},

$$
\begin{aligned}
I_{V_k} &= \int_{\mathbb{R}^d} \mu_\epsilon \Delta V_k\, dx \\
&\leq \|D^2 V_k\|_{L^\infty}.
\end{aligned}
$$

Now we bound I_V, I_W using item 1 of Assumption 2.1.7. Moving the gradient on μ_ϵ to V by integration by parts and again using Cauchy's inequality with a δ,

$$
\begin{aligned}
I_V &= \int_{\mathbb{R}^d} (\nabla\zeta_\epsilon * \nabla V)\mu_\epsilon\, dx \\
&= -\int_{\mathbb{R}^d} (\nabla\zeta_\epsilon * \mu_\epsilon)\nabla V\, dx \\
&\leq \int_{\mathbb{R}^d} \frac{|\nabla\zeta_\epsilon * \mu_\epsilon|}{a^{1/2}}|\nabla V| a^{1/2}\, dx \\
&\leq \delta\int_{\mathbb{R}^d} \frac{|\nabla\zeta_\epsilon * \mu_\epsilon|^2}{a}\, dx + \frac{1}{4\delta}\int_{\mathbb{R}^d} |\nabla V|^2 a\, dx \\
&\leq \delta\int_{\mathbb{R}^d} \frac{|\nabla\zeta_\epsilon * \mu_\epsilon|^2}{a}\, dx + \frac{\|a\|_{L^\infty(\mathbb{R}^d)}}{4\delta}\int_{\mathbb{R}^d} |\nabla V|^2\, dx \\
&\leq \delta\int_{\mathbb{R}^d} \frac{|\nabla\zeta_\epsilon * \mu_\epsilon|^2}{a}\, dx + \frac{C\|a\|_{L^\infty(\mathbb{R}^d)}}{4\delta}.
\end{aligned}
$$

Using similar techniques with the intent of using item 1 of Assumption 2.1.7,

$$\begin{aligned}
I_W &= \int_{\mathbb{R}^d} (\nabla\zeta_\epsilon * \zeta_\epsilon * \nabla W * \mu_\epsilon)\mu_\epsilon \, dx \\
&= -\int_{\mathbb{R}^d} (\nabla\zeta_\epsilon * \mu_\epsilon)(\zeta_\epsilon * \nabla W * \mu_\epsilon) \, dx \\
&= -\int_{\mathbb{R}^d} \frac{(\nabla\zeta_\epsilon * \mu_\epsilon)}{a^{1/2}} a^{1/2} (\zeta_\epsilon * \nabla W * \mu_\epsilon) \, dx \\
&\le \delta \int_{\mathbb{R}^d} \frac{(\nabla\zeta_\epsilon * \mu_\epsilon)^2}{a} \, dx + \int_{\mathbb{R}^d} \frac{(\zeta_\epsilon * \nabla W * \mu_\epsilon)^2}{4\delta} a \, dx \\
&\le \delta \int_{\mathbb{R}^d} \frac{(\nabla\zeta_\epsilon * \mu_\epsilon)^2}{a} \, dx + \frac{C\|a\|_{L^\infty(\mathbb{R}^d)}}{4\delta}.
\end{aligned}$$

Thus combining the results above with $\delta = 1/6$ gives

$$\frac{d}{dt}\left(\int_{\mathbb{R}^d} \mu_\epsilon \log \mu_\epsilon \, dx\right) \le -\int_{\mathbb{R}^d} \frac{|\nabla\zeta_\epsilon * \mu_\epsilon|^2}{2a} \, dx + C\left(1 + \mathcal{F}_\epsilon(\mu_\epsilon(0))\right)$$

where C is the combination of constants. For ease we define

$$\mathcal{S}(\mu_\epsilon(t)) = \int_{\mathbb{R}^d} \mu_\epsilon \log \mu_\epsilon \, dx.$$

Integrating in time for $t \in [0, T]$ for $T > 0$

$$\mathcal{S}(\mu_\epsilon(t)) - \mathcal{S}(\mu_\epsilon(0)) \le -\int_0^t \int_{\mathbb{R}^d} \frac{(\nabla\zeta_\epsilon * \mu_\epsilon)^2}{2a} \, dx \, ds + tC\left(1 + \mathcal{F}_\epsilon(\mu_\epsilon(0))\right)$$

Given that we have the bound (from [9, Proposition 3.8])

$$\mathcal{S}(\nu) \ge -(2\pi)^{1/2} - M_2(\nu),$$

rearranging gives us

$$\int_0^t \int_{\mathbb{R}^d} \frac{(\nabla\zeta_\epsilon * \mu_\epsilon)^2}{2a} \, dx \, ds \le M_2(\mu_\epsilon(t)) + (2\pi)^{1/2} + \mathcal{S}(\mu_\epsilon(0)) + tC\left(1 + \mathcal{F}_\epsilon(\mu_\epsilon(0))\right) \tag{2.8}$$

We will briefly pause here and see what happens when we apply Assumption 2.1.7 item 2 to I_V, I_W.

Alternatively, moving the gradient from the mollifier to ∇V,

$$\begin{aligned}
I_V &= \int_{\mathbb{R}^d} (\zeta_\epsilon * \mu_\epsilon)\Delta V \, dx \\
&\le \|\zeta_\epsilon * \mu_\epsilon\|_{L^2(R^d)} \|D^2 V\|_{L^2(R^d)} \\
&\le C.
\end{aligned}$$

In a similar manner,

$$\begin{aligned}
I_W &= \int_{\mathbb{R}^d} (\zeta_\epsilon * \zeta_\epsilon * \Delta W * \mu_\epsilon)\mu_\epsilon \, dx \\
&= \int_{\mathbb{R}^d} (\zeta_\epsilon * \mu_\epsilon)(\zeta_\epsilon * \Delta W * \mu_\epsilon) \, dx \\
&\leq \|\zeta_\epsilon * \mu_\epsilon\|_{L^2(\mathbb{R}^d)} \|\zeta_\epsilon * D^2 W * \mu_\epsilon\|_{L^2(\mathbb{R}^d)} \\
&\leq C.
\end{aligned}$$

We instead choose $\delta = 1/2$, we get (2.8) with a different constant. What is left to show is that the second moment at $\mu_\epsilon(t)$ is uniformly bounded. We get this from Proposition 2.5.9. Therefore for any $t \in [0, T]$, we have an bound

$$\begin{aligned}
\int_0^t \int_{\mathbb{R}^d} \frac{(\nabla \zeta_\epsilon * \mu_\epsilon)^2}{2a} \, dx \, ds \leq (1 + Te^T)(M_2(\mu_\epsilon(0)) + \mathcal{F}_\epsilon(\mu_\epsilon(0))) \\
+ (2\pi)^{1/2} + \mathcal{S}(\mu_\epsilon(0)) + tC \left(1 + \mathcal{F}_\epsilon(\mu_\epsilon(0))\right).
\end{aligned}$$

We can take t as T and take the sup in ϵ to get a uniform bound in ϵ. □

2.5.2 H^1 bound on convolved gradient flow (rigorous)

Remark 2.5.3 (flow interchange method)**.** What makes the argument formal is the differentiation of the entropy, $\mathcal{S}$, along the gradient flow of the diffusion energy $\mathcal{F}_\epsilon$. Say that we have two energy functionals E_1, E_2 with gradient flows μ_1, μ_2 respectively. The flow interchange method says that the derivative with respect to time of E_1 along μ_2 at $t = 0$ is equivalent to the derivative with respect to time of E_2 along μ_1 at $t = 0$ as long as the gradient flow are equivalent at $t = 0$. We use the flow interchange method in discrete time via minimizing movement scheme (see [14, Definition A.1]) to make the argument rigorous by avoiding differentiating $\mathcal{S}(\mu_\epsilon)$ in time.

Definition 2.5.4 (Minimizing movement scheme)**.** Suppose that $\mathcal{G}$ is proper, lower semi-continuous, and ω-convex along generalized geodesics. Define the proximal operator J_τ by

$$J_\tau \mu = \text{argmin}_{\nu \in P_2(\mathbb{R}^d)} \frac{1}{2\tau} W_2^2(\mu, \nu) + \mathcal{G}(\nu),$$

and define the minimizing movement scheme $J_\tau^n \mu$ by

$$J_\tau^n \mu = \underbrace{J_\tau \circ J_\tau \circ \cdots \circ J_\tau}_{n \text{ times}} \mu.$$

Remark 2.5.5 (Minimizing movement scheme)**.** Given $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, let $J_{\tau,\epsilon}^n \mu$ denote the nth step of the minimizing movement scheme of $\mathcal{F}_{\epsilon,k}$ with time step τ and initial data $J_{\tau,\epsilon}^0 \mu = \mu$.

Definition 2.5.6 (Heat flow semigroup)**.** Given $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ and $h \geq 0$, we will let $S_h \mu$ denote the strongly continuous gradient flow of $\mathcal{S}$ with the initial data μ at time h. Moreover, S_h is the heat flow semigroup operator.

Note that in the following proofs, we use that for $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, $\zeta_\epsilon * S_h(\mu) = S_h(\zeta_\epsilon * \mu)$ as in [14].

Lemma 2.5.7 (derivatives along the heat semigroup)**.** *Suppose Assumptions 2.1.3, 2.1.4, 2.1.5, 2.1.7 hold. Let $\mu \in \mathcal{P}_2(\mathbb{R}^d)$. We have,*

$$\begin{aligned}
\limsup_{h \to 0^+} \frac{\mathcal{W}_\epsilon(J_{\tau,\epsilon}^n \mu) - \mathcal{W}_\epsilon(S_h(J_{\tau,\epsilon}^n \mu))}{h} &= \int_{\mathbb{R}^d} \langle \zeta_\epsilon * \nabla W * J_{\tau,\epsilon}^n \mu, \nabla \zeta_\epsilon * J_{\tau,\epsilon}^n \mu \rangle \, d\mathcal{L}^d, \\
\limsup_{h \to 0^+} \frac{\mathcal{V}_\epsilon(J_{\tau,\epsilon}^n \mu) - \mathcal{V}_\epsilon(S_h(J_{\tau,\epsilon}^n \mu))}{h} &= \int_{\mathbb{R}^d} \langle \nabla V, \nabla \zeta_\epsilon * J_{\tau,\epsilon}^n \mu \rangle \, d\mathcal{L}^d, \\
\limsup_{h \to 0^+} \frac{\mathcal{V}_k(J_{\tau,\epsilon}^n \mu) - \mathcal{V}_k(S_h(J_{\tau,\epsilon}^n \mu))}{h} &= -\int_{\mathbb{R}^d} \Delta V_k \, dJ_{\tau,\epsilon}^n \mu, \\
\limsup_{h \to 0^+} \frac{\mathcal{E}_\epsilon(J_{\tau,\epsilon}^n \mu) - \mathcal{E}_\epsilon(S_h(J_{\tau,\epsilon}^n \mu))}{h} &= -\int_{\mathbb{R}^d} \frac{1}{a} (\Delta \zeta_\epsilon * J_{\tau,\epsilon}^n \mu)(\zeta_\epsilon * J_{\tau,\epsilon}^n \mu) \, d\mathcal{L}^d.
\end{aligned}$$

Proof. Using the fact that W, ζ_ϵ are both even functions combined with the definition of convolution,

$$\begin{aligned}
&\frac{\mathcal{W}_\epsilon(J^n_{\tau,\epsilon}\mu) - \mathcal{W}_\epsilon(S_h(J^n_{\tau,\epsilon}\mu))}{h} \\
&= \frac{1}{2h}\int \zeta_\epsilon * W * \zeta_\epsilon * J^n_{\tau,\epsilon}\mu \, dJ^n_{\tau,\epsilon}\mu - \frac{1}{2h}\int \zeta_\epsilon * W * \zeta_\epsilon * S_h(J^n_{\tau,\epsilon}\mu) \, dS_h(J^n_{\tau,\epsilon}\mu) \\
&= \frac{1}{2h}\int (\zeta_\epsilon * W * J^n_{\tau,\epsilon}\mu)(\zeta_\epsilon * J^n_{\tau,\epsilon}\mu) \, d\mathcal{L}^d \\
&\quad - \frac{1}{2h}\int (\zeta_\epsilon * W * S_h(J^n_{\tau,\epsilon}\mu))(\zeta_\epsilon * S_h(J^n_{\tau,\epsilon}\mu)) \, d\mathcal{L}^d \\
&= \frac{1}{2h}\int (\zeta_\epsilon * W * J^n_{\tau,\epsilon}\mu)(\zeta_\epsilon * J^n_{\tau,\epsilon}\mu) - (\zeta_\epsilon * W * J^n_{\tau,\epsilon}\mu)(\zeta_\epsilon * S_h(J^n_{\tau,\epsilon}\mu)) \, d\mathcal{L}^d \\
&\quad + \frac{1}{2h}\int (\zeta_\epsilon * W * J^n_{\tau,\epsilon}\mu)(\zeta_\epsilon * S_h(J^n_{\tau,\epsilon}\mu))\mathcal{L}^d \\
&\quad - \frac{1}{2h}\int (\zeta_\epsilon * W * S_h(J^n_{\tau,\epsilon}\mu))(\zeta_\epsilon * S_h(J^n_{\tau,\epsilon}\mu)) \, d\mathcal{L}^d \\
&= \frac{1}{2h}\int (\zeta_\epsilon * W * J^n_{\tau,\epsilon}\mu)\left((\zeta_\epsilon * J^n_{\tau,\epsilon}\mu) - (\zeta_\epsilon * S_h(J^n_{\tau,\epsilon}\mu))\right) \, d\mathcal{L}^d \\
&\quad + \frac{1}{2h}\int (\zeta_\epsilon * S_h(J^n_{\tau,\epsilon}\mu))\left((\zeta_\epsilon * W * J^n_{\tau,\epsilon}\mu) - (\zeta_\epsilon * W * S_h(J^n_{\tau,\epsilon}\mu))\right) \, d\mathcal{L}^d \\
&=: J_1 + J_2
\end{aligned}$$

where we add and subtract a term in the third equality. As $S_h(J^n_{\tau,\epsilon}\mu)$ satisfies the heat equation classically, then by the fundamental theorem of calculus

$$J_1 = -\int (\zeta_\epsilon * W * J^n_{\tau,\epsilon}\mu)\frac{1}{2h}\int_0^h \Delta S_t(\zeta_\epsilon * J^n_{\tau,\epsilon}\mu) \, dt \, d\mathcal{L}^d.$$

To get J_2 to look similar to J_1, we use that both $\zeta_\epsilon * W, \zeta_\epsilon$ are even functions,

$$\begin{aligned}
J_2 &= \frac{1}{2h}\int (\zeta_\epsilon * S_h(J^n_{\tau,\epsilon}\mu))\left(\zeta_\epsilon * W * (J^n_{\tau,\epsilon}\mu - S_h(J^n_{\tau,\epsilon}\mu))\right) \, d\mathcal{L}^d \\
&= \frac{1}{2h}\int (\zeta_\epsilon * W * S_h(J^n_{\tau,\epsilon}\mu))\left(\zeta_\epsilon * (J^n_{\tau,\epsilon}\mu - S_h(J^n_{\tau,\epsilon}\mu))\right) \, d\mathcal{L}^d \\
&= \frac{1}{2h}\int (\zeta_\epsilon * W * S_h(J^n_{\tau,\epsilon}\mu))\left((\zeta_\epsilon * J^n_{\tau,\epsilon}\mu - S_h(\zeta_\epsilon * J^n_{\tau,\epsilon}\mu))\right) \, d\mathcal{L}^d \\
&= -\int (\zeta_\epsilon * W * S_h(J^n_{\tau,\epsilon}\mu))\frac{1}{2h}\int_0^h \Delta S_t(\zeta_\epsilon * J^n_{\tau,\epsilon}\mu) \, dt \, d\mathcal{L}^d.
\end{aligned}$$

From here we want to say that either item 1 or item 2 of Assumption 2.1.7 is enough to justify passing the limit in h. Let us recall that $S_t(\mu) = K_t * \mu$ where K_t is the heat kernel. Therefore by

Young's convolution inequality,

$$\|S_t(\mu)\|_{L^p(\mathbb{R}^d)} \leq \|K_t\|_{L^1(\mathbb{R}^d)}\|\mu\|_{L^p(\mathbb{R}^d)} = \|\mu\|_{L^p(R^d)} \leq C.$$

Let us first start by using item 1. By integration by parts, Hölder's inequality, and Jensen's inequality

$$\begin{aligned}
J_1 &\leq \int (\zeta_\epsilon * \nabla W * J^n_{\tau,\epsilon}\mu)\frac{1}{2h}\int_0^h \nabla S_t(\zeta_\epsilon * J^n_{\tau,\epsilon}\mu)\, dt\, d\mathcal{L}^d \\
&\leq \frac{1}{2}\|\zeta_\epsilon * \nabla W * J^n_{\tau,\epsilon}\mu\|_{L^2(\mathbb{R}^d)}\left(\int \frac{1}{h}\int_0^h |\nabla S_t(\zeta_\epsilon * J^n_{\tau,\epsilon}\mu)|^2\, dt\, d\mathcal{L}^d\right)^{1/2} \\
&\leq \frac{1}{2}\|\zeta_\epsilon * \nabla W * J^n_{\tau,\epsilon}\mu\|_{L^2(\mathbb{R}^d)}\left(\int \frac{1}{h}\int_0^h |\Delta S_t(\zeta_\epsilon * J^n_{\tau,\epsilon}\mu)||S_t(\zeta_\epsilon * J^n_{\tau,\epsilon}\mu)|\, dt\, d\mathcal{L}^d\right)^{1/2} \\
&\leq \frac{1}{2}\|\zeta_\epsilon * \nabla W * J^n_{\tau,\epsilon}\mu\|_{L^2(\mathbb{R}^d)}C_{\tau,n,\epsilon}\left(\int \frac{1}{h}\int_0^h |S_t(\zeta_\epsilon * J^n_{\tau,\epsilon}\mu)|\, dt\, d\mathcal{L}^d\right)^{1/2} \\
&\leq \frac{1}{2}\|\zeta_\epsilon * \nabla W * J^n_{\tau,\epsilon}\mu\|_{L^2(\mathbb{R}^d)}C_{\tau,n,\epsilon}\left(\int \frac{1}{h}\int_0^h |\zeta_\epsilon * J^n_{\tau,\epsilon}\mu|\, dt\, d\mathcal{L}^d\right)^{1/2} \\
&\leq \frac{1}{2}\|\zeta_\epsilon * \nabla W * J^n_{\tau,\epsilon}\mu\|_{L^2(\mathbb{R}^d)}C_{\tau,n,\epsilon}\left(\int |\zeta_\epsilon * J^n_{\tau,\epsilon}\mu|\, d\mathcal{L}^d\right)^{1/2} \\
&\leq C_{\tau,n,\epsilon}
\end{aligned}$$

where contractivity of the semigroup was used, $\|\Delta S_t(\zeta_\epsilon * J^n_{\tau,\epsilon}\mu)\|_{L^\infty(\mathbb{R}^d)} \leq C_{\tau,n,\epsilon}$ for all t, and item 1 of Assumption 2.1.7. The calculation for J_2 is similar. Since we have a bound independent of h, we may pass the limit.

Now let us try to get a bound of again J_1, J_2 independent of h by using item 2 of Assumption 2.1.7. We use integration by parts twice, Hölder's inequality, and contractivity of the heat semigroup so that

$$\begin{aligned}
J_1 &\leq \int |\zeta_\epsilon * \Delta W * J^n_{\tau,\epsilon}\mu|\frac{1}{2h}\int_0^h |S_t(\zeta_\epsilon * J^n_{\tau,\epsilon}\mu)|\, dt\, d\mathcal{L}^d \\
&\leq \frac{1}{2}\|\zeta_\epsilon * D^2 W * J^n_{\tau,\epsilon}\mu\|_{L^2(\mathbb{R}^d)}\|\zeta_\epsilon * J^n_{\tau,\epsilon}\mu\|_{L^2(\mathbb{R}^d)} \\
&\leq \frac{1}{2}C_{\tau,n,\epsilon}\|J^n_{\tau,\epsilon}\mu\|_{L^2(\mathbb{R}^d)}\|\zeta_\epsilon * J^n_{\tau,\epsilon}\mu\|_{L^2(\mathbb{R}^d)}
\end{aligned}$$

which is finite and independent of h. The calculation for J_2 is similar. Since we have a bound independent of h, we may pass the limit. Taking lim sup gives

$$\limsup_{h\to 0^+} \frac{\mathcal{W}_\epsilon(J^n_{\tau,\epsilon}\mu) - \mathcal{W}_\epsilon(S_h(J^n_{\tau,\epsilon}\mu))}{h} = -\int (\zeta_\epsilon * W * J^n_{\tau,\epsilon}\mu)\Delta(\zeta_\epsilon * J^n_{\tau,\epsilon}\mu)\ dt\ d\mathcal{L}^d$$
$$= \int \langle \zeta_\epsilon * \nabla W * J^n_{\tau,\epsilon}\mu, \nabla\zeta_\epsilon * J^n_{\tau,\epsilon}\mu\rangle\ d\mathcal{L}^d.$$

We can justify getting the same result for $\mathcal{V}_\epsilon$ by using the same techniques used for $\mathcal{W}_\epsilon$. The results of $\mathcal{V}_k, \mathcal{E}_\epsilon$ carry over from [14, Lemma 4.4]. □

The next lemma is one small piece in the H^1 bound proof.

Lemma 2.5.8 (Mollified nth step of minimizing movement scheme, [14] Lemma 4.5)**.** *For* $J^n_{\tau,\epsilon} \in D(\mathcal{E}_\epsilon)$*,*

$$-\int_{\mathbb{R}^d} \frac{1}{a}\Delta(\zeta_\epsilon * J^n_{\tau,\epsilon}\mu)(\zeta_\epsilon * J^n_{\tau,\epsilon}\mu)\ d\mathcal{L}^d \geq C_a\|\nabla\zeta_\epsilon * J^n_{\tau,\epsilon}\mu\|^2_{L^2(\mathbb{R}^d)} - C'_a\|\zeta_\epsilon * J^n_{\tau,\epsilon}\mu\|^2_{L^2(\mathbb{R}^d)}.$$

We seek the control the second moment of $\mu_{\epsilon,k}$ so that it is independent of ϵ by controlling the second moment along a curve at any time by the second moment along the same curve at the initial time.

Proposition 2.5.9 (M_2 bound for AC^2 curves, [14] Proposition A.3)**.** *Suppose* $\mu \in AC^2([0,T]; \mathcal{P}_2(\mathbb{R}^d))$*. Then for all* $t \in [0,T]$*,*

$$M_2(\mu(t)) \leq (1 + te^t)\left(M_2(\mu(0)) + \int_0^T |\mu'|^2(r)\ dr\right).$$

Proof. It suffices to show for any $\rho \in \mathcal{P}_2(\mathbb{R}^d)$,

$$W_2^2(\mu(t), \rho) \leq (1 + te^t)\left(W_2^2(\mu(0), \rho) + \int_0^T |\mu'|(r)\ dr\right)$$

for all $t \in [0,T]$ by taking $\rho = \delta_0$. Define $\mathcal{H}(\mu) = -\frac{1}{2}W_2^2(\mu, \rho)$. By [1, Proposition 9.3.12] $\mathcal{H}$ is (-1)-convex and lower semi-continuous and by [1, Definition 1.2.1, Corollary 2.4.10] the local slope $|\partial\mathcal{H}|(\mu)$ is a strong upper gradient for $\mathcal{H}$. Thus,

$$|\mathcal{H}(\mu(t)) - \mathcal{H}(\mu(0)) \leq \int_0^t |\partial\mathcal{H}|(\mu(s))|\mu'|(s)\ ds.$$

Using the definition of local slope and triangle inequality,

$$
\begin{aligned}
|\partial \mathcal{H}|(\mu) &= \limsup_{\nu\to\mu} \frac{W_2^2(\nu,\rho) - W_2^2(\mu,\rho)}{2W_2(\mu,\nu)} \\
&= \limsup_{\nu\to\mu} \frac{(W_2(\nu,\rho) - W_2(\mu,\rho))(W_2(\nu,\rho) + W_2(\mu,\rho))}{2W_2(\mu,\nu)} \\
&\le \limsup_{\nu\to\mu} \frac{W_2(\nu,\mu)(W_2(\nu,\rho) + W_2(\mu,\rho))}{2W_2(\mu,\nu)} \\
&= W_2(\mu,\rho).
\end{aligned}
$$

Combining the previous two estimates,

$$
\begin{aligned}
\frac{1}{2}W_2^2(\mu(t),\rho) - \frac{1}{2}W_2^2(\mu(0),\rho) &\le |\mathcal{H}(\mu(t)) - \mathcal{H}(\mu(0))| \\
&\le \int_0^t W_2(\mu(s),\rho)|\mu'|(s)\, ds \\
&\le \frac{1}{2}\int_0^t W_2^2(\mu(t),\rho)\, ds + \frac{1}{2}\int_0^T |\mu'|^2(s)\, ds.
\end{aligned}
$$

Applying Gröwall's inequality gives us the result. □

Proposition 2.5.10 (H^1 bound on convolved gradient flow)**.** *Let V, W satisfy Assumptions 2.1.3, 2.1.4, and Assumption 2.1.7. For all $T > 0$ and $\epsilon > 0$, suppose that $\mu_\epsilon \in AC^2([0,T]; \mathcal{P}_2(\mathbb{R}^d))$ is a gradient flow of $\mathcal{F}_{\epsilon,k}$. Then,*

$$
\int_0^T \|\nabla \zeta_\epsilon * \mu_\epsilon(s)\|_{L^2(\mathbb{R}^d)}^2\, ds \le C\left(\mathcal{S}(\mu_\epsilon(0)) + M_2(\mu_\epsilon(0)) + \mathcal{F}_{\epsilon,k}(\mu_\epsilon(0)) + 1\right)
$$

where $C = C(a, T, V, V_k, W)$.

Proof. By definition of minimizing movement scheme for any $\mu \in D(\mathcal{F}_{\epsilon,k})$,

$$
\mathcal{F}_{\epsilon,k}(J_{\tau,\epsilon}^n\mu) - \mathcal{F}_{\epsilon,k}(S_h(J_{\tau,\epsilon}^n\mu)) \le \frac{1}{2h}\left(W_2^2(S_h(J_{\tau,\epsilon}^n\mu), J_{\tau,\epsilon}^{n-1}\mu) - W_2^2(J_{\tau,\epsilon}^n\mu, J_{\tau,\epsilon}^{n-1}\mu)\right).
$$

By the EVI condition in Theorem 2.3.13, we get

$$
\begin{aligned}
\limsup_{h\to 0^+} \frac{\mathcal{F}_{\epsilon,k}(J_{\tau,\epsilon}^n\mu) - \mathcal{F}_{\epsilon,k}(S_h(J_{\tau,\epsilon}^n\mu))}{h} &\le \frac{1}{2\tau}\frac{d^+}{dh}W_2^2(S_h(J_{\tau,\epsilon}^n\mu), J_{\tau,\epsilon}^{n-1}\mu)|_{h=0} \\
&\le \frac{\mathcal{S}(J_{\tau,\epsilon}^{n-1}\mu) - \mathcal{S}(J_{\tau,\epsilon}^n\mu)}{\tau}.
\end{aligned}
$$

Using the derivatives along the heat semigroup from Lemma 2.5.7,

$$\begin{aligned}
&\limsup_{h\to 0^+} \frac{\mathcal{F}_{\epsilon,k}(J^n_{\tau,\epsilon}\mu) - \mathcal{F}_{\epsilon,k}(S_h(J^n_{\tau,\epsilon}\mu))}{h} \\
&= \int_{\mathbb{R}^d} \langle \zeta_\epsilon * \nabla W * J^n_{\tau,\epsilon}\mu, \nabla \zeta_\epsilon * J^n_{\tau,\epsilon}\mu\rangle \, d\mathcal{L}^d + \int_{\mathbb{R}^d} \langle \nabla V, \nabla\zeta_\epsilon * J^n_{\tau,\epsilon}\mu\rangle \, d\mathcal{L}^d - \int_{\mathbb{R}^d} \Delta V_k \, dJ^n_{\tau,\epsilon}\mu \\
&\quad - \int_{\mathbb{R}^d} \frac{1}{a}(\Delta\zeta_\epsilon * J^n_{\tau,\epsilon}\mu)(\zeta_\epsilon * J^n_{\tau,\epsilon}\mu)\, d\mathcal{L}^d.
\end{aligned}$$

Bounding the difference quotient of the heat entropy from below,

$$\begin{aligned}
\frac{\mathcal{S}(J^0_{\tau,\epsilon}\mu) - \mathcal{S}(J^n_{\tau,\epsilon}\mu)}{\tau} &= \sum_{i=1}^n \frac{\mathcal{S}(J^{i-1}_{\tau,\epsilon}\mu) - \mathcal{S}(J^i_{\tau,\epsilon}\mu)}{\tau} \\
&\geq \int_{\mathbb{R}^d} \langle \zeta_\epsilon * \nabla W * J^n_{\tau,\epsilon}\mu, \nabla\zeta_\epsilon * J^n_{\tau,\epsilon}\mu\rangle \, d\mathcal{L}^d + \int_{\mathbb{R}^d} \langle \nabla V, \nabla\zeta_\epsilon * J^n_{\tau,\epsilon}\mu\rangle \, d\mathcal{L}^d \\
&\quad - \int_{\mathbb{R}^d} \Delta V_k \, dJ^n_{\tau,\epsilon}\mu - \int_{\mathbb{R}^d} \frac{1}{a}(\Delta\zeta_\epsilon * J^n_{\tau,\epsilon}\mu)(\zeta_\epsilon * J^n_{\tau,\epsilon}\mu)\, d\mathcal{L}^d.
\end{aligned}$$

Choose $\tau = T/n$ and let $\mu_{\tau,\epsilon}(t)$ be the piecewise constant interpolation of the minimizing movement scheme $J^n_{\tau,\epsilon}$. Therefore,

$$\begin{aligned}
&\mathcal{S}(\mu_{\tau,\epsilon}(0)) - \mathcal{S}(\mu_{\tau,\epsilon}(T)) \\
&\geq \int_0^T \int_{\mathbb{R}^d} \frac{-1}{a}\Delta(\zeta_\epsilon * \mu_{\tau,\epsilon}(s))(\zeta_\epsilon * \mu_{\tau,\epsilon}(s)) + \langle \nabla V, \nabla(\zeta_\epsilon * \mu_{\tau,\epsilon}(s))\rangle \, d\mathcal{L}^d \, ds \\
&\quad + \int_0^T \int_{\mathbb{R}^d} \langle \zeta_\epsilon * \nabla W * \mu_{\tau,\epsilon}(s), \nabla\zeta_\epsilon * \mu_{\tau,\epsilon}(s)\rangle \, d\mathcal{L}^d \, ds - \int_0^T \int_{\mathbb{R}^d} \Delta V_k \, d\mu_{\tau,\epsilon}(s)\, ds.
\end{aligned}$$

The last term can easily be bounded from below by $T\|\Delta V_k\|_{L^\infty}$. By Lemma 2.5.8,

$$\int_{\mathbb{R}^d} \frac{-1}{a}\Delta(\zeta_\epsilon * \mu_{\tau,\epsilon}(s))(\zeta_\epsilon * \mu_{\tau,\epsilon}(s))\, d\mathcal{L}^d \geq C_a \|\nabla\zeta_\epsilon * \mu_{\tau,\epsilon}(s)\|^2_{L^2(\mathbb{R}^d)} - C'_a \|\zeta_\epsilon * \mu_{\tau,\epsilon}(s)\|^2_{L^2(\mathbb{R}^d)}$$

It follows by Lemma 2.5.1 that

$$\begin{aligned}
\int_0^T \int_{\mathbb{R}^d} \frac{-1}{a}\Delta(\zeta_\epsilon * \mu_{\tau,\epsilon}(s))(\zeta_\epsilon * \mu_{\tau,\epsilon}(s))\, d\mathcal{L}^d \, ds &\geq C_a \int_0^T \|\nabla\zeta_\epsilon * \mu_{\tau,\epsilon}(s)\|^2_{L^2(\mathbb{R}^d)}\, ds \\
&\quad - C'_a T(\mathcal{F}_{\epsilon,k}(\mu_\epsilon(0)) + 2C).
\end{aligned}$$

From here we want to say that either item 1 or item 2 of Assumption 2.1.7 is enough to get a lower bound on the other terms. If we have item 1 of Assumption 2.1.7, then we can use Cauchy inequality,

$$\int_0^T \int_{\mathbb{R}^d} \langle \nabla V, \nabla(\zeta_\epsilon * \mu_{\tau,\epsilon}(s)) \rangle \, d\mathcal{L}^d \, ds \geq \frac{-C_a}{2} \int_0^T \|\nabla \zeta_\epsilon * \mu_{\tau,\epsilon}(s)\|^2_{L^2(\mathbb{R}^d)} \, ds - C_a' T \|\nabla V\|_{L^2(\mathbb{R}^d)}$$
$$\geq \frac{-C_a}{2} \int_0^T \|\nabla \zeta_\epsilon * \mu_{\tau,\epsilon}(s)\|^2_{L^2(\mathbb{R}^d)} \, ds - C_a' T C$$

and

$$\int_0^T \int_{\mathbb{R}^d} \langle \zeta_\epsilon * \nabla W * \mu_{\tau,\epsilon}(s), \nabla \zeta_\epsilon * \mu_{\tau,\epsilon}(s) \rangle \, d\mathcal{L}^d \, ds \geq \frac{-C_a}{2} \int_0^T \|\nabla \zeta_\epsilon * \mu_{\tau,\epsilon}(s)\|^2_{L^2(\mathbb{R}^d)} \, ds$$
$$- C_a' \int_0^T \|\zeta_\epsilon * \nabla W * \mu_{\tau,\epsilon}(s)\|_{L^2(\mathbb{R}^d)} \, ds$$
$$\geq \frac{-C_a}{2} \int_0^T \|\nabla \zeta_\epsilon * \mu_{\tau,\epsilon}(s)\|^2_{L^2(\mathbb{R}^d)} \, ds - C_a' T C.$$

Notice that last inequality is because of Lemma 2.5.1 allows us to apply Assumption 2.1.7 (and 2.4.10). To use item 2 instead, we first use integration by parts and then Young's inequality,

$$\int_0^T \int_{\mathbb{R}^d} \langle \nabla V, \nabla(\zeta_\epsilon * \mu_{\tau,\epsilon}(s)) \rangle \, d\mathcal{L}^d \, ds \geq \frac{-1}{2} \int_0^T \|\zeta_\epsilon * \mu_{\tau,\epsilon}(s)\|^2_{L^2(\mathbb{R}^d)} \, ds - T \|\Delta V\|_{L^2(\mathbb{R}^d)}$$
$$\geq \frac{-TC_a}{2} (\mathcal{F}_{\epsilon,k}(\mu_\epsilon(0)) + 2C) - TC$$

and

$$\int_0^T \int_{\mathbb{R}^d} \langle \zeta_\epsilon * \nabla W * \mu_{\tau,\epsilon}(s), \nabla(\zeta_\epsilon * \mu_{\tau,\epsilon}(s)) \rangle \, d\mathcal{L}^d \, ds \geq \frac{-1}{2} \int_0^T \|\zeta_\epsilon * \mu_{\tau,\epsilon}(s)\|^2_{L^2(\mathbb{R}^d)} \, ds$$
$$- \int_0^T \|\zeta_\epsilon * \Delta W * \mu_{\tau,\epsilon}(s)\|_{L^2(\mathbb{R}^d)} \, ds$$
$$\geq \frac{-TC_a}{2} (\mathcal{F}_{\epsilon,k}(\mu_\epsilon(0)) + 2C) - TC.$$

In either case,

$$\mathcal{S}(\mu_{\tau,\epsilon}(0)) - \mathcal{S}(\mu_{\tau,\epsilon}(T)) \geq \frac{-C}{2} \int_0^T \|\nabla \zeta_\epsilon * \mu_{\tau,\epsilon}(s)\|^2_{L^2(\mathbb{R}^d)} \, ds - C(\mathcal{F}_{\epsilon,k}(\mu_\epsilon(0)) + 1)$$

where $C = C(a, T)$. Using $\mu_{\tau,\epsilon}(t) \to \mu_\epsilon(t)$ narrowly for all $t \geq 0$ ([14, Theorem A.2]), then for $f \in L^2(\mathbb{R}^d)$ and $s \in [0, T]$, we have that $\nabla \zeta_\epsilon * \mu_{\tau,\epsilon}(s) \to \nabla \zeta_\epsilon * \mu_\epsilon(s)$ weakly in $L^2(\mathbb{R}^d)$ and $s \in [0, T]$. That is as $n \to \infty$,

$$\int_{\mathbb{R}^d} f \nabla \zeta_\epsilon * \mu_{\tau,\epsilon}(s) = \int_{\mathbb{R}^d} (\nabla \zeta_\epsilon * f) \mu_{\tau,\epsilon}(s) \to \int_{\mathbb{R}^d} (\nabla \zeta_\epsilon * f) \mu_\epsilon(s) = \int_{\mathbb{R}^d} f \nabla \zeta_\epsilon * \mu_\epsilon(s).$$

By lower semi-continuity of $L^2(\mathbb{R}^d)$ norm with respect to weak convergence and Fatou's lemma, taking $n \to \infty$

$$\limsup_{n\to\infty} \mathcal{S}(\mu_{\tau,\epsilon}(0)) - \mathcal{S}(\mu_{\tau,\epsilon}(T)) \geq \frac{C}{2}\int_0^T \|\nabla\zeta_\epsilon * \mu_\epsilon(s)\|^2_{L^2(\mathbb{R}^d)}\, ds - C(\sup_{\epsilon>0}\mathcal{F}_\epsilon(\mu_\epsilon(0)) + 1)$$

On the left-hand side we use the initial data from the minimizing movement scheme $\mathcal{S}(\mu_{\tau,\epsilon}(0)) = \mathcal{S}(\mu_\epsilon(0))$ for all $\tau > 0$, the lower semi-continuity of the entropy with respect to narrow convergence $\limsup_{n\to\infty} -\mathcal{S}(\mu_{\tau,\epsilon}(T)) \leq -\mathcal{S}(\mu_\epsilon(T))$, a Carleman-type bound (from [9, Proposition 3.8]) $-\mathcal{S}(\mu_\epsilon(T)) \leq (2\pi)^{1/2} + M_2(\mu_\epsilon(T))$, and Proposition 2.5.9

$$\limsup_{n\to\infty} \mathcal{S}(\mu_{\tau,\epsilon}(0)) - \mathcal{S}(\mu_{\tau,\epsilon}(T)) \leq \mathcal{S}(\mu_\epsilon(0)) + (2\pi)^{1/2} + (1+Te^T)(M_2(\mu_\epsilon(0)) + \mathcal{F}_{\epsilon,k}(\mu_\epsilon(0))).$$

Thus,

$$\begin{aligned}\frac{C}{2}\int_0^T \|\nabla\zeta_\epsilon * \mu_\epsilon(s)\|^2_{L^2(\mathbb{R}^d)}\, ds - C(\mathcal{F}_{\epsilon,k}(\mu_\epsilon(0)) + 1) &\leq \mathcal{S}(\mu_\epsilon(0)) + (2\pi)^{1/2} \\ &\quad + (1+Te^T)(M_2(\mu_\epsilon(0)) + \mathcal{F}_{\epsilon,k}(\mu_\epsilon(0))).\end{aligned}$$

The results follow. □

2.6 Convergence of Gradient Flow

2.6.1 Γ-convergence of the energies

We establish one of the key hypotheses of Serfaty's Theorem 2.3.16.

Proposition 2.6.1 (Γ-convergence of energies)**.** *Suppose Assumptions 2.1.3, 2.1.4 holds. Fix $k \in \mathbb{N}$. Then $\mathcal{F}_{\epsilon,k}$ Γ-converges to $\mathcal{F}_k$ as $\epsilon \to 0$.*

Proof. As $\mu_\epsilon \to \mu$ narrowly as $\epsilon \to 0$, then $\zeta_\epsilon * \mu_\epsilon \to \mu$ narrowly as $\epsilon \to 0$ by Lemma 2.3.3. Since $\mathcal{V}$ is lower semi-continuous with respect to narrow convergence,

$$\liminf_{\epsilon\to 0} \mathcal{V}_\epsilon(\mu_\epsilon) = \liminf_{\epsilon\to 0} \mathcal{V}(\zeta_\epsilon * \mu_\epsilon) \geq \mathcal{V}(\mu).$$

Similarly, we get that

$$\liminf_{\epsilon\to 0} \mathcal{W}_\epsilon(\mu_\epsilon) = \liminf_{\epsilon\to 0} \mathcal{W}(\zeta_\epsilon * \mu_\epsilon) \geq \mathcal{W}(\mu),$$

$$\liminf_{\epsilon\to 0} \mathcal{E}_\epsilon(\mu_\epsilon) = \liminf_{\epsilon\to 0} \mathcal{E}(\zeta_\epsilon * \mu_\epsilon) \geq \mathcal{E}(\mu).$$

It remains to show that

$$\limsup_{\epsilon \to 0} E_\epsilon(\mu) \leq E(\mu)$$

for $E = \mathcal{V}, \mathcal{W}$ as the case for $\mathcal{E}$ is documented in [14, Theorem 5.1]. Let us look at when $E = \mathcal{W}$. The inequality is trivially true for $\mathcal{W}(\mu) = \infty$, so we can assume it is finite. We might as well assume that there exists a $C > 0$ such that

$$\mathcal{W}(\mu) = \int W * \mu \, d\mu \leq C.$$

Thus,

$$\int \zeta_\epsilon * W * \mu \, d\mu \leq \iint W * \mu(x-y) \, d\mu(x) \, \zeta_\epsilon(y) \, dy \leq C$$

where the C may of changed. Then,

$$\mathcal{W}_\epsilon(\mu) \leq \iint \zeta_\epsilon * W * \mu(x-y) \, d\mu(x) \, \zeta_\epsilon(y) \, dy \leq C$$

Therefore we can take the lim sup as $\epsilon \to 0$ and interchange the lim sup and the integral to get the result. Similarly, it holds for when $E = \mathcal{V}$. □

2.6.2 LSC of the local slope of the energies

Lemma 2.6.2 (Upgraded convergence)**.** *Suppose Assumptions 2.1.1, 2.1.2, 2.1.3, 2.1.4, 2.1.5 hold. Fix $k \in \mathbb{N}$. Consider any sequence μ_ϵ in $\mathcal{P}(\mathbb{R}^d)$ and $\mu_k \in \mathcal{P}(\mathbb{R}^d)$ such that μ_ϵ narrowly converges to μ_k and*

$$\sup_{\epsilon > 0} \mathcal{F}_{\epsilon,k}(\mu_\epsilon), \quad \liminf_{\epsilon \to 0} \|\nabla \zeta_\epsilon * \mu_\epsilon\|_{L^2(\mathbb{R}^d)}$$

are all finite. Then, $\mu_k \in L^2(\mathbb{R}^d)$ and there exists a subsequence (denoted μ_ϵ) along which we have,

$$\sup_{\epsilon > 0} \|\zeta_\epsilon * \mu_\epsilon\|_{H^1(\mathbb{R}^d)} < \infty$$

and $\zeta_\epsilon * \mu_\epsilon$ converges to μ_k in $L^2_{loc}(\mathbb{R}^d)$.

Proof. By Proposition 2.6.6 and (2.3),

$$\mathcal{F}_k(\mu_k) \le \sup_{\epsilon>0} \mathcal{F}_{\epsilon,k}(\mu_\epsilon) < \infty.$$

Using the ideas of the proof of Lemma 2.5.1, we get that $\mu_k \in L^2(\mathbb{R}^d)$. Combining the results of Lemma 2.5.1 and $\liminf_{\epsilon\to 0} \|\nabla\zeta_\epsilon * \mu_\epsilon\|_{L^2(\mathbb{R}^d)} < \infty$, then up to a subsequence we have

$$\sup_{\epsilon>0} \|\zeta_\epsilon * \mu_\epsilon\|_{H^1(\mathbb{R}^d)} < \infty.$$

By Rellich-Kondrachov, up to another subsequence, $\zeta_\epsilon * \mu_\epsilon$ converges in $L^2_{loc}(\mathbb{R}^d)$. By Lemma 2.3.3, $\zeta_\epsilon * \mu_\epsilon$ narrowly converges to μ_k. Uniqueness of limits gives us the convergence result in $L^2_{loc}(\mathbb{R}^d)$. □

We require the weak limit of the subdifferentials to achieve to Γ-convergence (lower semi-continuity) of the local slopes.

Lemma 2.6.3 (Weak limit of subdifferentials, [14] Lemma 5.5)**.** *Suppose Assumptions 2.1.1, 2.1.2, 2.1.3, 2.1.4, 2.1.5 hold. Fix $k \in \mathbb{N}$. Consider any sequence μ_ϵ in $\mathcal{P}(\mathbb{R}^d)$ and $\mu_k \in \mathcal{P}(\mathbb{R}^d)$ such that μ_ϵ narrowly converges to μ_k and*

$$\sup_{\epsilon>0} \mathcal{F}_{\epsilon,k}(\mu_\epsilon), \quad \liminf_{\epsilon\to 0} \|\nabla\zeta_\epsilon * \mu_\epsilon\|_{L^2(\mathbb{R}^d)}$$

are all finite. For all $\epsilon > 0$ and $f \in C_c^\infty(\mathbb{R}^d)$, define,

$$L_\epsilon(f) = \int_{\mathbb{R}^d} f\nabla\zeta_\epsilon * \left(\frac{\zeta_\epsilon * \mu_\epsilon}{a}\right) d\mu_\epsilon, \quad L(f) = -\int_{\mathbb{R}^d} \frac{\mu_k^2}{2}\nabla\left(\frac{f}{a}\right) dx + \int_{\mathbb{R}^d} f\mu_k^2\nabla\left(\frac{1}{a}\right) dx.$$

There exists a subsequence, denoted by ϵ, such that for any $f \in C_c^\infty(\mathbb{R}^d)$, we have

$$\lim_{\epsilon\to 0} L_\epsilon(f) = L(f).$$

Furthermore, L is a bounded linear operator on $L^2(\mu_k)$.

Here we establish another key hypothesis of Serfaty's Theorem.

Proposition 2.6.4 (lower semi-continuity of local slopes)**.** *Suppose Assumptions 2.1.3, 2.1.4 hold. Fix $k \in \mathbb{N}$. Let $\mu_\epsilon \in \mathcal{P}_2(\mathbb{R}^d)$ such that*

$$\sup_{\epsilon>0} \mathcal{F}_{\epsilon,k}(\mu_\epsilon), \quad \liminf_{\epsilon\to 0} \|\nabla\zeta_\epsilon * \mu_\epsilon\|_{L^2(\mathbb{R}^d)}, \quad \liminf_{\epsilon\to 0} \int \left|\nabla\zeta_\epsilon * \left(\frac{\zeta_\epsilon * \mu_\epsilon}{a}\right)\right|^2 \, d\mu_\epsilon$$

are all finite. Suppose $\exists\, \mu_k \in \mathcal{P}(\mathbb{R}^d)$ such that μ_ϵ converges to μ_k. Then $\mu_k^2 \in W^{1,1}(\mathbb{R}^d)$ and $\exists\, \eta_k \in L^2(\mu_k)$ where

$$\eta_k \mu_k = \frac{a}{2}\nabla\left(\left(\frac{\mu_k}{a}\right)^2\right) + \mu_k(\nabla W * \mu_k) + \mu_k \nabla(V + V_k)$$

and

$$\liminf_{\epsilon\to 0} \int \left|\nabla\zeta_\epsilon * \left(\frac{\zeta_\epsilon * \mu_\epsilon}{a}\right) + \nabla(\zeta_\epsilon * \zeta_\epsilon * W) * \mu_\epsilon + \nabla(\zeta_\epsilon * V) + \nabla V_k\right|^2 \, d\mu_\epsilon \geq \int |\eta_k|^2 \, d\mu_k.$$

Proof. We may choose a subsequence, denoted μ_ϵ, so that

$$\lim_{\epsilon\to 0} |\partial\mathcal{F}_{\epsilon,k}|(\mu_\epsilon) = \liminf_{\epsilon\to 0} |\partial\mathcal{F}_{\epsilon,k}|(\mu_\epsilon).$$

By [1, Theorem 5.4.4(ii)] (or [9, Proposition B.2(ii)]), it is now sufficient to show there exists $\eta_k \in L^2(\mu)$ satisfying the hypothesis above and up to another subsequence,

$$\lim_{\epsilon\to 0} \int f\left(\nabla\zeta_\epsilon * \left(\frac{\zeta_\epsilon * \mu_\epsilon}{a}\right) + \nabla(\zeta_\epsilon * \zeta_\epsilon * W) * \mu_\epsilon + \nabla(\zeta_\epsilon * V) + \nabla V_k\right) \, d\mu_\epsilon = \int f\eta_k \, d\mu_k$$

for all $f \in C_c^\infty(\mathbb{R}^d)$. Given the continuity of $\nabla V, \nabla V_k$,

$$\lim_{\epsilon\to 0} \int_{\mathbb{R}^d} f\nabla(\zeta_\epsilon * V) \, d\mu_\epsilon = \lim_{\epsilon\to 0} \int_{\mathbb{R}^d} \nabla V(\zeta_\epsilon * (f\mu_\epsilon)) \, dx = \int_{\mathbb{R}^d} f\nabla V \, d\mu_k$$
$$\lim_{\epsilon\to 0} \int_{\mathbb{R}^d} f\nabla V \, d\mu_\epsilon = \int_{\mathbb{R}^d} f\nabla V_k \, d\mu_k.$$

By using $L_\epsilon(f), L(f)$ from Lemma 2.6.3 as well as the lemma itself, we may apply the Riesz Representation Theorem on $L^2(\mu_k)$, so that there exists $\tilde{\eta}_k \in L^2(\mu_k)$ such that

$$\lim_{\epsilon\to 0} \int_{\mathbb{R}^d} f\nabla\zeta_\epsilon * \left(\frac{\zeta_\epsilon * \mu_\epsilon}{a}\right) \, d\mu_\epsilon = -\int_{\mathbb{R}^d} \frac{\mu_k^2}{2}\nabla\left(\frac{f}{a}\right) \, dx + \int_{\mathbb{R}^d} f\mu_k^2\nabla\left(\frac{1}{a}\right) \, dx = \int_{\mathbb{R}^d} f\tilde{\eta}_k \, d\mu_k.$$

By rearranging we get,

$$-\int_{\mathbb{R}^d} \frac{\mu_k^2}{2}\nabla\left(\frac{f}{a}\right) \, dx = \int_{\mathbb{R}^d} \frac{f}{a}\left(\tilde{\eta}_k\mu_k a - a\mu_k^2\nabla(1/a)\right) \, dx.$$

We find that $\mu_k^2 \in W^{1,1}(\mathbb{R}^d)$ and its weak derivative is

$$\nabla\left(\frac{\mu_k^2}{2}\right) = \tilde{\eta}_k \mu_k a - a\mu_k^2 \nabla(1/a).$$

By the chain rule for $W^{1,1}(\mathbb{R}^d)$ functions, we obtain

$$\tilde{\eta}_k \mu_k = \frac{a}{2}\nabla\left(\left(\frac{\mu_k}{a}\right)^2\right).$$

We have to check the aggregation portion and verify that η_k is $L^2(\mu_k)$. First, we check that $\eta_k = \tilde{\eta}_k + \nabla V + \nabla W * \mu_k + \nabla V_k$ is $L^2(\mu_k)$. We get this easily by Assumption 2.1.3, 2.1.4 and 2.1.5. Namely, that $\|\nabla W * \mu_k\|_{L^2(\mu)} \le C$, $\|\nabla V\|_{L^2(\mu_k)} \le C$, and $\|\nabla V_k\|_{L^2(\mu_k)} \le C$. Second, we require that for $f \in C_c^\infty(\mathbb{R}^d)$,

$$\int f(\zeta_\epsilon * \zeta_\epsilon * \nabla W * \mu_\epsilon)\, d\mu_\epsilon \to \int f(\nabla W * \mu_k)\, d\mu_k$$

as $\epsilon \to 0$. Let us look at the difference and write

$$\begin{aligned}&\left|\int f(\zeta_\epsilon * \zeta_\epsilon * \nabla W * \mu_\epsilon)\, d\mu_\epsilon - \int f(\nabla W * \mu_k)\, d\mu_k\right|\\ &\quad\le \|f\|_{L^\infty}\left|\int (\zeta_\epsilon * \zeta_\epsilon * \nabla W * \mu_\epsilon)\mu_\epsilon - (\nabla W * \mu_k)\mu_k\, dx\right|.\end{aligned}$$

Thus it is sufficient to look at

$$\begin{aligned}&\left|\int (\zeta_\epsilon * \zeta_\epsilon * \nabla W * \mu_\epsilon)\, d\mu_\epsilon - \int (\nabla W * \mu_k)\, d\mu_k\right|\\ &\le \left|\int (\nabla W * \zeta_\epsilon * \mu_\epsilon)\, d\zeta_\epsilon * \mu_\epsilon - \int (\nabla W * \mu_k)\, d\zeta_\epsilon * \mu_\epsilon\right|\\ &\quad + \left|\int (\nabla W * \mu_k)\, d\zeta_\epsilon * \mu_\epsilon - \int (\nabla W * \mu_k)\, d\mu_k\right|\\ &=: J_1 + J_2.\end{aligned}$$

We have $J_2 \to 0$ as $\epsilon \to 0$ as $\zeta_\epsilon * \mu_\epsilon$ converges narrowly to μ_k along with $f \in C_c^\infty(\mathbb{R}^d)$ and $\nabla W * \mu_k$ is continuous. For J_1, we apply Assumption 2.1.3 (item 4),

$$J_1 \le C W_2((\zeta_\epsilon * \mu_\epsilon)\mathbb{1}_{\mathrm{supp}(f)}, \mu_k \mathbb{1}_{\mathrm{supp}(f)}) \to 0$$

as $\zeta_\epsilon * \mu_\epsilon \to \mu_k$ narrowly and $\mathrm{supp}(f)$ is compact (namely bounded). This gives us the result. Note that due to the minimality of the subdifferential of $\mathcal{F}_{\epsilon,k}$ and $\mathcal{F}_k$ and properties of convergence of varying measure ([9, Proposition B.2(ii)]), we get the lower semi-continuity of the local slopes. □

2.6.3 Convergence as ϵ limit

One would expect that gradient flows of $\mathcal{F}_{\epsilon,k}$ to convergence to gradient flows of $\mathcal{F}_k$. Unfortunately, due to the triviality of having $a(x)$ be log-concave on all of $\mathbb{R}^d$, we lack the regularity to define some notion of gradient flows of $\mathcal{F}_k$. However, we are able to identify to limit and have it "almost" satisfy the conditions of gradient flow of $\mathcal{F}_k$.

Definition 2.6.5 ("almost" curves of maximal slope of $\mathcal{F}_k$)**.** A curve $\mu_k \in AC^2([0,T];\mathcal{P}_2(\mathbb{R}^d))$ is an "almost" curve of maximal slope of $\mathcal{F}_k$ if it satisfies,

$$\frac{1}{2}\int_0^t |\mu_k'|^2(r)\,dr + \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} |\eta_k(r)|^2\,d\mu_k(r)\,dr \le \mathcal{F}_k(\mu_k(0)) - \mathcal{F}_k(\mu_k(t))$$

for all $t \in [0,T]$. Here $\mu_k^2(t) \in W^{1,1}(\mathbb{R}^d)$ and $\eta_k(t) \in L^2(\mu_k(t))$ satisfies

$$\eta_k\mu_k = \frac{a}{2}\nabla\left(\left(\frac{\mu_k}{a}\right)^2\right) + \mu_k(\nabla W * \mu_k) + \mu_k\nabla(V + V_k)$$

for almost every $t \in [0,T]$.

It is worth noting that if $a(x)$ was log-concave on all $\mathbb{R}^d$, then $\int_{\mathbb{R}^d} |\eta_k(r)|^2\,d\mu_k(r)$ would be a strong upper gradient of $\mathcal{F}_k$ so that an almost curve of maximal slope would actually be a curve of maximal slope.

Proposition 2.6.6 (Convergence as $\epsilon \to 0$)**.** *Suppose Assumptions 2.1.3, 2.1.4, 2.1.5, 2.1.7 hold. Fix $T > 0$ and $k \in \mathbb{N}$. For $\epsilon > 0$, let $\mu_{\epsilon,k} \in AC^2([0,T];\mathcal{P}_2(\mathbb{R}^d))$ be a gradient flow of $\mathcal{F}_{\epsilon,k}$ satisfying,*

$$\sup_{\epsilon>0} \mathcal{S}(\mu_{\epsilon,k}(0)) < \infty, \quad \sup_{\epsilon>0} M_2(\mu_{\epsilon,k}(0)) < \infty.$$

Suppose there exists $\mu_k(0) \in D(\mathcal{F}) \cap \mathcal{P}_2(\mathbb{R}^d)$ that is an "almost" curve of maximal slope of $\mathcal{F}_k$ in the sense of Definition 2.6.5, and a subsequence $\epsilon_n^{(k)}$, depending on k, such that

$$\lim_{n\to\infty} W_1(\mu_{\epsilon_n^{(k)},k}(t), \mu_k(t)) = 0$$

uniformly for $t \in [0,T]$.

Proof. By Theorem 2.3.13, $\mu_{\epsilon,k}$ is a curve of maximal slope of $\mathcal{F}_{\epsilon,k}$ so that for all $0 \le t \le T$,

$$\frac{1}{2}\int_0^t |\mu_{\epsilon,k}'|^2(r)\,dr + \frac{1}{2}\int_0^t |\partial\mathcal{F}_{\epsilon,k}|^2(\mu_{\epsilon,k}(r))\,dr \le \mathcal{F}_{\epsilon,k}(\mu_{\epsilon,k}(0)) - \mathcal{F}_{\epsilon,k}(\mu_{\epsilon,k}(t)).$$

We check the hypotheses of the weak Serfaty framework, Theorem 2.3.16, to apply the Theorem. Proposition 2.6.1 gives us the required Γ-convergence. Proposition 2.4.17 gives an explicit characterization of the local slope, $|\partial \mathcal{F}_{\epsilon,k}|$ and therefore, an explicit characterization of $\eta_{\epsilon,k}(r) \in L^2(\mu_{\epsilon,k}(r))$. The hypotheses of this proposition ensure the initial data is well prepared. We may now apply Theorem 2.3.16. There exists $\mu_k \in AC^2([0,T];\mathcal{P}_2(\mathbb{R}^d))$ and a subsequence $\epsilon_n^{(k)}$, depending on k, so that

$$\lim_{n\to\infty} W_1(\mu_{\epsilon_n^{(k)},k}(t), \mu_k(t)) = 0$$

and for all $t \in [0,T]$,

$$\frac{1}{2}\int_0^t |\mu_k'|^2(r)\, dr + \frac{1}{2}\int_0^t \liminf_{n\to\infty} \int_{\mathbb{R}^d} |\eta_{\epsilon_n^{(k)}}(r)|^2\, d\mu_{\epsilon_n^{(k)},k}(r)\, dr \leq \mathcal{F}_k(\mu_k(0)) - \mathcal{F}_k(\mu_k(t)).$$

To conclude we must show the μ_k is an "almost" curve of maximal slope (Definition 2.6.5). To do so, we seek to apply Proposition 2.6.4. As the right-hand side above is finite, then the left-hand side is finite for almost every $r \in [0,T]$. In particular, for almost every $r \in [0,T]$,

$$\liminf_{n\to\infty} \int_{\mathbb{R}^d} |\eta_{\epsilon_n^{(k)}}(r)|^2\, d\mu_{\epsilon_n^{(k)},k}(r)\, dr < \infty.$$

Given that

$$\sup_{n\in\mathbb{N}} \mathcal{F}_{\epsilon_n^{(k)},k}(\mu_{\epsilon_n^{(k)},k}(0)) < \infty,$$

and the hypotheses that the heat entropy and second moment is finite at the initial data we apply the H^1 bound (Proposition 2.5.10) so that,

$$\liminf_{n\to\infty} \int_0^T \|\nabla \zeta_{\epsilon_n^{(k)}} * \mu_{\epsilon_n^{(k)},k}(r)\|_{L^2(\mathbb{R}^d)}^2\, dr < \infty.$$

Fatou's lemma gives the integrand must be finite for almost every $r \in [0,T]$. As the energy decreases along its gradient flow in time,

$$\sup_{n\in\mathbb{N}} \mathcal{F}_{\epsilon_n^{(k)},k}(\mu_{\epsilon_n^{(k)},k}(t)) < \infty.$$

Now we may apply Proposition 2.6.4 and obtain the result. □

2.7 Convergence of the Confining Limit

Now after taking the limit in ϵ, we take the limit in k.

Proposition 2.7.1 (Γ-convergence of confining energy, [14] Theorem 6.1)**.** *Suppose Assumptions 2.1.3, 2.1.4, 2.1.5, 2.1.7 hold. Then $\mathcal{V}_k, \mathcal{F}_k$ Γ-converge to $\mathcal{V}_\Omega, \mathcal{F}$ respectively as $k \to \infty$. In particular,* $\lim_{k\to\infty} \mathcal{V}_k(\mu) = \mathcal{V}_\Omega(\mu)$ *for any $\mu \in \mathcal{P}_2(\mathbb{R}^d)$.*

Proof. Notice that the Γ-convergence of $\mathcal{F}_k$ to $\mathcal{F}$ follows from the Γ-convergence of $\mathcal{V}_k$ to $\mathcal{V}_\Omega$. We may assume that $\liminf_{k\to\infty} \mathcal{V}_k(\mu_k) < \infty$, so up to a subsequence, $\sup_{k\in\mathbb{N}} \mathcal{V}_k(\mu_k) < \infty$. To show (2.3), it suffices to show that $\operatorname{supp} \mu \subseteq \overline{\Omega}$, since $\mathcal{V}_k(\mu_k)$ is nonnegative and $\mathcal{V}_\Omega(\mu)$ would be zero. By contradiction, suppose that $\operatorname{supp} \mu \nsubseteq \overline{\Omega}$, so that there exists $x \in \overline{\Omega}^c$ and an open ball B containg x so that $B \subset\subset \overline{\Omega}^c$ and $\mu(B) > 0$. By equivalent definitions of weak convergence of sequence measures (Portmanteau Theorem), $\mu_k \longrightarrow \mu$ narrowly implies $\liminf_{k\to\infty} \mu_k(B) \geq \mu(B) > 0$. So up to another subsequence, we may assume that there exists $\delta > 0$ so that $\mu_k(B) \geq \delta$ for all $k \in \mathbb{N}$. By Assumption 2.1.5,

$$\begin{aligned}
\liminf_{k\to\infty} \int_{\mathbb{R}^d} V_k \, d\mu_k &\geq \liminf_{k\to\infty} \int_B V_k \, d\mu_k \\
&\geq \liminf_{k\to\infty} \left(\inf_{x\in B} V_k(x) \right) \mu_k(B) \\
&\geq \delta \liminf_{k\to\infty} \left(\inf_{x\in B} V_k(x) \right) \\
&= \infty,
\end{aligned}$$

a contradiction. To show the $\Gamma - \limsup$ convergence, we use Assumption 2.1.5, so that

$$\limsup_{k\to\infty} \mathcal{V}_k(\mu) = \limsup_{k\to\infty} \int V_k \, d\mu \leq \int V_\Omega \, d\mu = \mathcal{V}_\Omega(\mu).$$

This ends the proof. $\square$

We take the limit in the confining variable so that the "almost" gradient flows of $\mathcal{F}_k$ converge to the gradient flows of $\mathcal{F}$.

Proposition 2.7.2 (Convergence as $k \to \infty$)**.** *Suppose Assumptions 2.1.3, 2.1.4, 2.1.5, 2.1.7 hold. Fix $T > 0$. For $k \in \mathbb{N}$, let $\mu_k \in AC^2([0,T];\mathcal{P}_2(\mathbb{R}^d))$ be an "almost" curve of maximal slope of $\mathcal{F}_k$, in the sense of Definition 2.6.5, and suppose there exists $\mu(0) \in D(\mathcal{F}) \cap \mathcal{P}_2(\mathbb{R}^d)$ such that* $\sup_{k\in\mathbb{N}} M_2(\mu_k(0)) < \infty$ *and as $k \to \infty$*

$$\mu_k(0) \to \mu(0) \textit{ narrowly}, \quad \mathcal{F}_k(\mu_k(0)) \to \mathcal{F}(\mu(0)).$$

Then,

$$\lim_{k\to\infty} W_1(\mu_k(t), \mu(t)) = 0, \textit{ uniformly for } t \in [0,T],$$

where $\mu \in AC^2([0,T];\mathcal{P}_2(\mathbb{R}^d))$ is the unique gradient flow of $\mathcal{F}$ with initial condition $\mu(0)$.

Proof. Given that the proof here is similar to the proof of [14, Proposition 6.2], we will only provided the necessary updates. Recall that Proposition 2.7.1 gives us Γ-convergence of $\mathcal{F}_k$ to $\mathcal{F}$. So we can apply Theorem 2.3.16. There exists $\mu \in AC^2([0,T];\mathcal{P}_2(\mathbb{R}^d))$ so that up to a subsequence,

$$\lim_{k\to\infty} W_1(\mu_k(t), \mu(t)) = 0, \text{ uniformly for } t \in [0,T],$$

holds and for all $t \in [0,T]$,

$$\frac{1}{2}\int_0^t |\mu'|^2(r)\, dr + \frac{1}{2}\int_0^t \liminf_{k\to\infty} \int_{\mathbb{R}^d} |\eta_k(r)|^2\, d\mu_k(r)\, dr \leq \mathcal{F}(\mu(0)) - \mathcal{F}(\mu(t)).$$

By definition of "almost" curve of maximal slope, $\mathcal{F}_k(\mu_k(t)) \leq \mathcal{F}_k(\mu_k(0))$ for all $k \in \mathbb{N}, t \in [0,T]$. As the initial data is well prepared,

$$\sup_{t\in[0,T],k\in\mathbb{N}} \mathcal{F}_k(\mu_k(t)) \leq \sup_{k\in\mathbb{N}} \mathcal{F}_k(\mu_k(0)) < \infty.$$

As we have W_1 convergence of the density, then the density converges narrowly for all $t \in [0,T]$. By Γ-convergence,

$$\sup_{t\in[0,T]} \mathcal{F}(\mu(t)) \leq \liminf_{k\to\infty} \mathcal{F}_k(\mu_k(t)) \leq \sup_{t\in[0,T],k\in\mathbb{N}} \mathcal{F}_k(\mu_k(t)) < \infty.$$

It suffices to show that for almost every $t \in [0,T]$, we have

$$\liminf_{k\to\infty} \int_{R^d} |\eta_k(t)|^2\, d\mu_k(t) \geq \int_{\mathbb{R}^d} |\eta(t)|^2\, d\mu(t) \tag{2.9}$$

for μ and η satisfying on Ω,

$$\eta\mu = \frac{a}{2}\nabla\left(\left(\frac{\mu}{a}\right)^2\right) + \mu(\nabla W * \mu) + \mu\nabla V, \quad \left(\frac{\mu(t)}{a}\right)^2 \in W^{1,1}(\Omega).$$

Since that $\mathcal{F}(\mu(0)) - \mathcal{F}(\mu(t))$ is finite, then the left-hand side above is finite and up to a subsequence in k, we may assume $\sup_{k\in\mathbb{N}} \|\eta_k(t)\|_{L^2(\mu_k(t))} < \infty$. Given that we can bound the drift and the aggregation from below (say by $2C$), then

$$\sup_{k\in\mathbb{N}} \frac{\|\mu_k(t)\|_{L^2(\mathbb{R}^d)}}{2\|a\|_{l^\infty}} \leq \sup_{k\in\mathbb{N}} \int_{\mathbb{R}^d} \frac{|\mu_k(t)|^2}{2a} = \sup_{k\in\mathbb{N}} \mathcal{E}(\mu_k(t)) \leq \sup_{k\in\mathbb{N}} \mathcal{F}_k(\mu_k(t)) + 2C < \infty.$$

Moreover, as $\sup_{t\in[0,T]} \mathcal{F}(\mu(t)) < \infty$, supp $\mu(t) \subseteq \overline{\Omega}$. By Assumptions 2.1.3, 2.1.4, we have $\nabla V, \nabla W * \mu_k \in L^2(\mu_k)$. Therefore,

$$\begin{aligned}
\sup_k \int \left|\frac{a}{2}\nabla\left(\left(\frac{\mu_k}{a}\right)^2\right) + \nabla V_k\mu_k\right| &= \sup_k \|\eta_k - \nabla V - \nabla W * \mu_k\|_{L^1(\mu_k)} \\
&\leq \sup_k \|\eta_k - \nabla V - \nabla W * \mu_k\|_{L^2(\mu_k)} \\
&\leq \sup_k \left(\|\eta_k\|_{L^2(\mu_k)} + \|\nabla W * \mu_k\|_{L^2(\mu_k)} + \|\nabla V\|_{L^2(\mu_k)}\right) \\
&< \infty.
\end{aligned}$$

By [1, Theorem 5.4.4(ii)] (or [9, Proposition B.2(ii)]), to get the lower semi-continuity of the η, as in (2.9), it suffices to show weak convergence with varying measure,

$$\begin{aligned}
\lim_{k\to\infty} \int_{\mathbb{R}^d} f\left(\frac{a}{2}\nabla\left(\left(\frac{\mu_k}{a}\right)^2\right) + \mu_k(\nabla W * \mu_k) + \mu_k\nabla(V + V_k)\right) \\
= \int_{\Omega} f\left(\frac{a}{2}\nabla\left(\left(\frac{\mu}{a}\right)^2\right) + \mu(\nabla W * \mu) + \mu\nabla V\right)
\end{aligned}$$

for all $f \in C_c^\infty(\mathbb{R}^d)$ where we recall that $\mu = 0$ a.e. on Ω^c.

By Assumptions 2.1.3, 2.1.4, we have $V, W*\mu_k \in C^1(\mathbb{R}^d)$. Moreover, $f\nabla V, f\nabla W*\mu_k \in C_b(\mathbb{R}^d)$. By narrow convergence,

$$\liminf_{k\to\infty} \int_{\mathbb{R}^d} f\nabla V\, d\mu_k = \int_{\Omega} f\nabla V\, d\mu, \quad \liminf_{k\to\infty} \int_{\mathbb{R}^d} f\nabla W * \mu_k\, d\mu_k = \int_{\Omega} f\nabla W * \mu\, d\mu$$

for a.e $t \in [0, T]$. Thus is suffices to show for all $f \in C_c^\infty(\mathbb{R}^d)$,

$$\lim_{k\to\infty} \int_{\mathbb{R}^d} f\left(\frac{a}{2}\nabla\left(\left(\frac{\mu_k}{a}\right)^2\right) + \mu_k\nabla V_k\right) = \int_{\Omega} f\left(\frac{a}{2}\nabla\left(\left(\frac{\mu}{a}\right)^2\right)\right).$$

What remains is to first show it is true for $f \in C_c^\infty(\Omega)$ then generalized to $f \in C_c^\infty(\mathbb{R}^d)$. We define the operator

$$L(f) := \int_\Omega f \frac{a}{2} \nabla \left(\frac{\mu}{a}\right)^2 = \liminf_{k\to\infty} \int_\Omega f \left(\frac{a}{2} \nabla \left(\frac{\mu_k}{a}\right)^2 + \nabla V_k \mu_k\right),$$

show that it is bounded, and apply the Riesz Representation Theorem. To extend the results to $f \in C_c^\infty(\mathbb{R}^d)$, a cut off function is used. This is exactly the same as [14, Proposition 6.2]. □

2.8 Proofs of Main Results

Proof of Theorem 2.2.1. Let $\mu \in AC^2([0,T]; \mathcal{P}_2(\mathbb{R}^d))$ be the unique gradient flow of $\mathcal{F}$ with the initial condition $\mu(0)$. By Proposition 2.6.1, for all $k \in \mathbb{N}$,

$$\lim_{\epsilon\to 0} \mathcal{F}_{\epsilon,k}(\mu(0)) = \mathcal{F}_k(\mu(0)).$$

By Proposition 2.6.6, there exists an "almost" curve of maximal slope $\mu_k \in AC^2([0,T]; \mathcal{P}_2(\mathbb{R}^d))$ and a subsequence $\{\epsilon_j^{(k)}\}_{j=1}^\infty$, depending on k, such that

$$\lim_{j\to\infty} W_1(\mu_{\epsilon_j^{(k)},k}(t), \mu_k(t)) = 0$$

uniformly for $t \in [0,T]$. In particular, for each $k \in \mathbb{N}$, there exists $\epsilon_k > 0$ so that $\lim_{k\to\infty} \epsilon_k = 0$ and

$$W_1(\mu_{\epsilon_k,k}(t), \mu_k(t)) < \frac{1}{k},$$

for all $t \in [0,T]$. By Proposition 2.7.1,

$$\lim_{k\to\infty} \mathcal{F}_k(\mu(0)) = \mathcal{F}(\mu(0)).$$

So by Proposition 2.7.2,

$$\lim_{k\to\infty} W_1(\mu_k(t), \mu(t)) = 0,$$

uniformly for $t \in [0,T]$. Fix $\delta > 0$. Choose $K_\delta > 0$ such that, for all $k \geq K_\delta$, $W_1(\mu_k(t), \mu(t)) < \delta/2$ for all $t \in [0,T]$. Then, for all $k \geq \max\{2/\delta, K_\delta\}$,

$$W_1(\mu_{\epsilon_k,k}(t), \mu(t)) \leq W_1(\mu_k(t), \mu(t)) + W_1(\mu_{\epsilon_k,k}(t), \mu_k(t)) \leq \frac{\delta}{2} + \frac{1}{k} \leq \delta,$$

for all $t \in [0,T]$. □

Proof of Theorem 2.2.2. Let $\mu_{\epsilon,k}(t)$ be the gradient flow of $\mathcal{F}_{\epsilon,k}$ with initial data $\mu(0)$. By Theorem 2.2.1, as $k \to \infty$, $\epsilon = \epsilon(k) \to 0$,

$$\lim_{k\to\infty} W_1(\mu_{\epsilon,k}(t), \mu(t)) = 0$$

uniformly for $t \in [0, T]$, where $\mu(t)$ is the gradient flow of $\mathcal{F}$ with initial data $\mu(0)$. Recall that $\mathcal{F}_{\epsilon,k}$ is lower semi-continuous and ω-convex along generalized geodesics with $\lambda_{\omega,\epsilon} = -\epsilon^{-d-2}\|D^2\zeta/a\|_{L^\infty} + 8C$. Note that $\lambda_{\omega,\epsilon}$ is nonpositive for sufficiently small ϵ. The empirical measure, $\mu^N_{\epsilon,k}(t)$ as defined in the hypothesis, is the unique gradient flow of $\mathcal{F}_{\epsilon,k}$ with initial data $\mu^N_{\epsilon,k}(0)$. By Theorem 2.2.1, it suffices to show that as $k \to \infty$, $\epsilon = \epsilon(k) \to 0$, $N = N(\epsilon) \to \infty$,

$$\lim_{k\to\infty} W_1(\mu^N_{\epsilon,k}(t), \mu_{\epsilon,k}(t)) = 0$$

uniformly for $t \in [0, T]$. As both $\mu_{\epsilon,k}(t), \mu^N_{\epsilon,k}(t)$ are both gradient flows of the ω-convex energy functional $\mathcal{F}_{\epsilon,k}$, we may apply [13, Theorem 3.11(iii)]. That is, given that $\mu_{\epsilon,k}(t)$ has initial data $\mu(0)$,

$$W_2^2(\mu^N_{\epsilon,k}(t), \mu_{\epsilon,k}(t)) \le F_{-2t,\epsilon}(W_2^2(\mu^N_{\epsilon,k}(0), \mu(0))).$$

By hypothesis as $k \to \infty$,

$$W_1(\mu^N_{\epsilon,k}(t), \mu_{\epsilon,k}(t)) \le W_2(\mu^N_{\epsilon,k}(t), \mu_{\epsilon,k}(t)) \le \sqrt{F_{-2t,\epsilon}(W_2^2(\mu^N_{\epsilon,k}(0), \mu(0)))} \to 0$$

uniformly in $t \in [0, T]$. □

Proof of Corollary 2.2.3. By Proposition 2.4.19,

$$\lim_{t\to\infty} W_1\left(\mu(t), a\mathbb{1}_{\overline{\Omega}}\right) \le \lim_{t\to\infty} W_2\left(\mu(t), a\mathbb{1}_{\overline{\Omega}}\right) = 0.$$

The result follows from Theorem 2.2.2. □

Proof of Theorem 2.2.4. We would like to use Theorem 2.3.17 to conclude the results. By Lemma 2.3.15, up to a subsequence, we get that $\mu_\epsilon \to \mu$ narrowly (as convergence in distance implies convergence narrowly) and the Γ-convergence of the metric derivatives. By Proposition 2.6.1, the Γ-convergence of the energy holds and $\lim_{\epsilon\to 0} \mathcal{G}_\epsilon(\mu_\epsilon(0)) = \mathcal{G}(\mu(0))$. Finally, by Proposition 2.6.4, the Γ-convergence (or lower semi-continuity) of the local slopes hold. Now we may apply Theorem 2.3.17 and gain the results immediately. □

2.9 Applications

2.9.1 Keller-Segel Chemotaxis Model

Next we look at the general Keller-Segel Model for chemotaxis in [3],

$$\begin{cases} \partial_t u = \nabla \cdot (\phi(u,v)\nabla u - \psi(u,v)\nabla v) + f(u,v) \\ \tau\partial_t v = d\Delta v + g(u,v) - h(u,v)v. \end{cases}$$

We have u represents the cell density on $\Omega \subset \mathbb{R}^d$, v represents the concentration of the chemical signal, ϕ represents the diffusivity of the cells, and ψ represents the chemotactic sensitivity. Chemotaxis is when cells movements are affected by chemicals in their environment. In this model the aggregation term and the diffusion term compete. When aggregation wins, it can be studied in mathematics as blow-up in finite time; however, this is not realistic from a biological viewpoint. Moreover with a more general diffusion, the system has solutions based on the inequalities that involve the diffusion exponent and the dimension ([5], [15]). Here we choose

$$\phi(u,v) = \psi(u,v) = g(u,v) = u, \quad d = 1, \quad \tau = f(u,v) = 0, \quad h(u,v) = \alpha \geq 0.$$

Most choices above are relatively common as seen in table 1 of [3]. With that we obtain that $v = \mathcal{B}_{\alpha,d} * u$ ([8]) where $\mathcal{B}_{\alpha,d}$ is the Bessel kernel

$$\mathcal{B}_{\alpha,d}(x) = \int_0^\infty \frac{1}{(4\pi t)^{d/2}} e^{\frac{-|x|^2}{4t} - \alpha t}\, dt.$$

Notice that $\mathcal{B}_{0,d} = -\mathcal{N}$ (see [3]), where we get the original PDE that we are considering with a(x) removed (or $a(x) \equiv 1$). Note that for potential future work choosing $\tau = 1$ implies that we get the heat kernel for $\alpha = 0$ and the integrand of the Bessel kernel for $\alpha > 0$. The Newtonian potential satisfies Assumption 2.1.3 by [13, Proposition 4.4], however, we need to show that it satisfies the additional assumptions.

Remark 2.9.1 (The Newtonian potential satisfies Assumption 2.1.7)**.** As $\|\mu\|_{L^p(\mathbb{R}^d)} \leq C$ for $p = 1, 2$, by interpolation it is true for all $p \in [1, 2]$. Using Young's convolution inequality, as long as $\|\nabla W\|_{L^p(\mathbb{R}^d \setminus B_R)} \leq C_R$ for some $p \in [1, 2]$, then we get item 1. For the Newtonian potential, we only have this for $d \geq 3$ (see Lemma A.0.4). However, the Newtonian potential does satisfy item 2 for all d (see [13] equation 61).

2.9.2 Bessel Kernel

Naturally, we will now show that the Bessel kernel satisfies Assumption 2.1.3 and Assumption 2.1.7.

Proposition 2.9.2. *The Bessel kernel, $-\mathcal{B}_{\alpha,d}$, with $d \geq 3$ satisfies Assumption 2.1.3. Furthermore, it satisfies Assumption 2.1.7 item 1 for $d \geq 3$ and item 2 for all d.*

Proof. Given that most of the assumptions involve a norm or modulus, it suffices to show $\mathcal{B}_{\alpha,d}$ satisfies the above assumptions (barring the lower semi-continuity). Furthermore $\alpha = 0$ is already shown as $\mathcal{B}_{0,d} = -\mathcal{N}$, so we will assume $\alpha > 0$. We first start with the items of Assumption 2.1.3.

1. Since we have the chain of inequalities $\mathcal{B}^-_{\alpha,d} \leq \mathcal{B}^-_{\alpha,d} + \mathcal{B}^+_{\alpha,d} = \mathcal{B}_{\alpha,d} \leq \mathcal{B}_{0,d} \leq -\mathcal{N} \leq \mathcal{N}^-$, we get $|\mathcal{B}^-_{\alpha,d} * \mu| \leq |\mathcal{N}^- * \mu| \leq C$.
2. It is sufficient to show that $|\nabla \mathcal{B}_{\alpha,d} * \mu| \leq C$ as ν is a probability measure. By computation or [8, Lemma 2.4], $|\nabla \mathcal{B}_{\alpha,d}(x)| \leq C_d |x|^{1-d} g_\alpha(|x|)$ where $g_\alpha(|x|)$ is a positive radial function exponentially decreasing from 1 to 0 as $|x| \to \infty$. Thus, $\exists c$ such that $|\nabla \mathcal{B}_{\alpha,d}| \leq c$ on $\mathbb{R}^d \backslash B_1$ where B_1 is the unit ball centered at the origin. By computation, $\|\nabla \mathcal{B}_{\alpha,d}\|_{L^p(B_1)} \leq c$ for $p < \frac{d}{d-1}$ (see Lemma A.0.4). Using Hölder's inequality,

$$|\nabla \mathcal{B}_{\alpha,d} * \mu| \leq \|\nabla \mathcal{B}_{\alpha,d}\|_{L^{p'}(B_1)} \|\mu\|_{L^p(\mathbb{R}^d)} + c\|\mu\|_{L^1(\mathbb{R}^d)} \leq c.$$

3. This results from Proposition 2.1 of [10].
4. Using the ideas of [13, Proposition 4.4], it suffices to show that $\|D^2 \mathcal{B}_{\alpha,d} * \mu\|_{L^2(\mathbb{R}^d)} \leq C\|\mu\|_{L^2(\mathbb{R}^d)}$.

   Define $\nu = \mathcal{B}_{\alpha,d} * \mu$ so that ν satisfies $-\Delta \nu + \alpha \nu = \mu$. By interchanging derivatives via integration by parts twice (or see [20, Theorem 9.9, Corollary 9.10]),

$$\begin{aligned}\int_{B_R} |D^2 v|^2 &= \int_{B_R} (\Delta v)^2 \\ &= \int_{B_R} (\alpha v - \mu)^2 \\ &\leq 2\int_{B_R} \alpha^2 v^2 + 2\int_{B_R} \mu^2.\end{aligned}$$

Using Young's convolution inequality,

$$\begin{aligned}\alpha^2 \|\mathcal{B}_{\alpha,d} * \mu\|^2_{L^2(B_R)} &\leq \|\mathcal{B}_{\alpha,d}\|^2_{L^1(B_R)} \alpha^2 \|\mu\|^2_{L^2(B_R)} \\ &\leq \frac{1}{\alpha^2}\alpha^2 \|\mu\|^2_{L^2(B_R)} \\ &= \|\mu\|^2_{L^2(B_R)}.\end{aligned}$$

Taking $R \to \infty$ and square roots we get $\|D^2 \mathcal{B}_{\alpha,d} * \mu\|_{L^2(\mathbb{R}^d)} \leq 2\|\mu\|_{L^2(\mathbb{R}^d)}$.

5. Define $g(x,t) = \frac{-1}{(4\pi t)^{d/2}} e^{-|x|^2/(4t)}$, $f(x,t) = e^{-\alpha t} g(x,t)$. Then,

$$F(x) := \int_0^\infty f(x,t)\, dt = -\mathcal{B}_{\alpha,d}(x).$$

As g is lower semi-continuous in x, $\forall \epsilon > 0\ \exists \delta > 0$ such that $g(x_0,t) < g(x,t) + \epsilon$ for all $x \in B_\delta(x_0)$. So, $f(x_0,t) < f(x,t) + \epsilon e^{-\alpha t}$ holds. So, $F(x_0) < F(x) + \epsilon/\alpha$ holds for $\alpha > 0$. Thus, $F(x) = -\mathcal{B}_{\alpha,d}(x)$ is lower semi-continuous in x.

Now we address the items in Assumption 2.1.7.

(i) As with the Newtonian potential, using Young convolution inequality and L^p bounds (Lemma A.0.4) gives the result.

(ii) This is proven in item 4 of this proof.

All of the items are complete and this finishes the proof. □

Corollary 2.9.3. *The Bessel kernel, $-\mathcal{B}_{\alpha,d}$, is convex for $d = 1$.*

Proof. Taking $g(x,t), f(x,t), F(x)$ in item 5 of the previous proof. We have that $g(x,t)$ is convex in x and so therefore $F(x) = -\mathcal{B}_{\alpha,1}(x)$ is convex in x by linearity of the integral. □

2.10 Numerical Simulations

We now implement the particle method discussed earlier. Here we have simulations in dimension $d = 1$. We explore different targets, aggregation kernels, and initial conditions. We also calculate the L^1 error for the convergence rate in N for a fixed final time. We start with some of the basic details.

We define the domain $\Omega = [-1, 1]$ where we define a confining potential

$$V_k(x) = \begin{cases} \frac{k}{2}(x+1)^2 & \text{if } x < -1, \\ \frac{k}{2}(x-1)^2 & \text{if } x > 1, \\ 0 & \text{otherwise.} \end{cases}$$

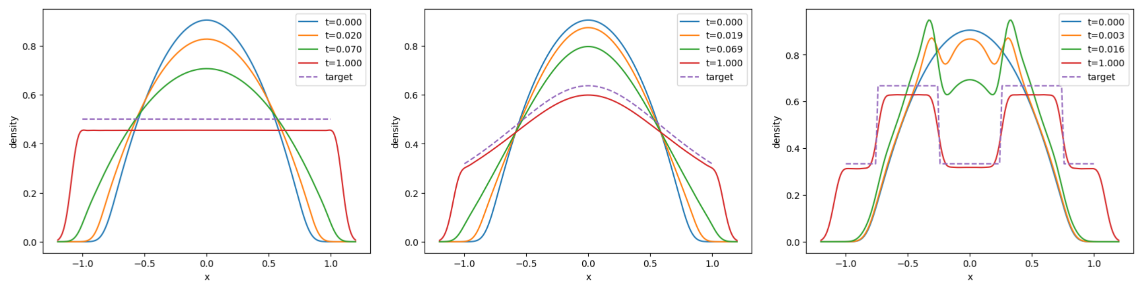
The confinement strength is controlled by the value of $k \in \mathbb{N}$. Most simulation we set $k = 100$, which is a medium strength confinement. Medium strength confinement is preferred here because the strong the confinement, the lower the convergence rate in N is (see [14]). We choose to use the Gaussian mollifier $\zeta_\epsilon(x) = \exp(-x^2/2\epsilon^2)/\sqrt{2\pi\epsilon^2}$ where we define $\epsilon = 4/N^{0.99}$. The relationship between ϵ and N is better than the expected qualitative results previously mentioned. From Theorem 2.2.2, we define the empirical measure

$$\mu^N_{\epsilon,k}(t) = \sum_{i=1}^N \delta_{X^i_{\epsilon,k}(t)} m^i, \quad m^i \geq 0, \quad \sum_{i=1}^N m^i = 1.$$

Moreover, we define the approximate initial condition as $\mu^N_{\epsilon,k}(0) = \sum_{i=1}^N \delta_{X^i_{0,\epsilon}} m^i$. The particles are uniformly spread out in the domain Ω and the mass m^i is computed from the initial condition $\mu(0)$. In most cases, we choose the initial condition as the Barenblatt profile $\psi_\tau(x) = \frac{\tau^{-1/3}}{12}\left(3^{4/3} - \frac{|x|^2}{\tau^{2/3}}\right)_+$ with $\tau = 0.0625$. This corresponds to a more general profile in [9]. The function is chosen because the Barenblatt profile is a solution to the homogeneous (non-weighted) PME. We typically stick with three weights: uniform $a(x) = 1/2$, log-cave $a(x) = 2/\pi/(1 + |x|^2)$, and piecewise $a(x) = 2/3$ for $x \in [-0.75, -0.25) \cup [0.25, 0.75)$ and $a(x) = 1/3$ otherwise. To visualize $\mu^N_{\epsilon,k}$ and compute L^1 errors of μ, we use $\zeta_\epsilon * \mu^N_{\epsilon,k}$ instead.

2.10.1 Demonstrating the Particle Method

In most simulations, we choose $N = 100$. We start by demonstrating the particle method in the case when we only have diffusion ($V = W = 0$) in Figure 2.1. We have three different weights: uniform $a(x) = 1/2$, log-cave $a(x) = 2/\pi/(1+|x|^2)$, and piecewise $a(x) = 2/3$ for $x \in [-0.75, -0.25) \cup [0.25, 0.75)$ and $a(x) = 1/3$ otherwise. This echos the simulations in [14].



(a) Uniform weight (b) Log-concave weight (c) Piecewise weight

Figure 2.1 Density of diffusion energy with different targets evolves in time.

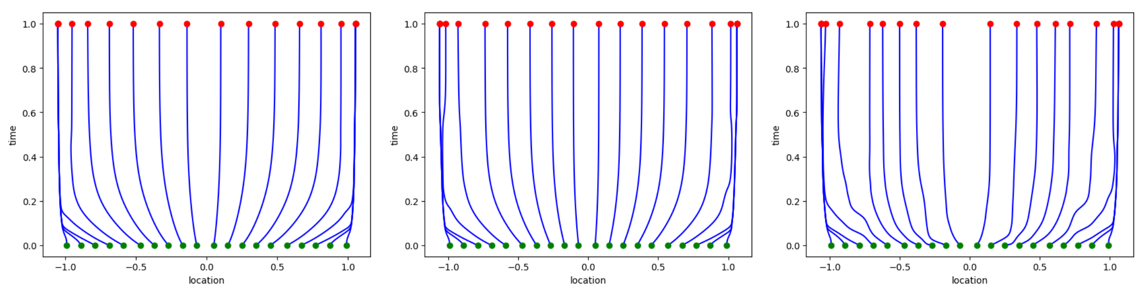
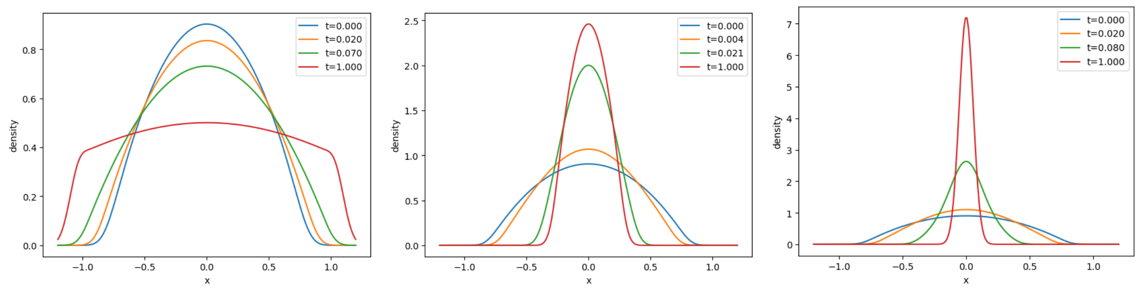


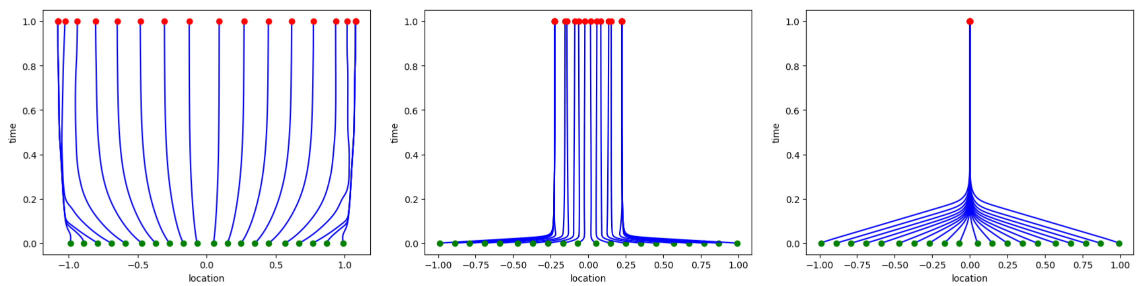
Figure 2.2 Particles location evolution in time of Figure 2.1.

In Figure 2.3, we look at the diffusion (with uniform weight) and aggregation energies. Here the attractive Newtonian potential is used for the kernel of the aggregation. Thus, the aggression wants to bring the particles together while the diffusion wants to spread the particles out. In Figure 2.3a and 2.4a, the coefficients for both the aggregation and diffusion is one. We see that the diffusion wins and particles separate. In Figure 2.3b and 2.4b, the coefficient for the aggregation is increased while keeping the coefficient for the diffusion one. Very quickly the particles come together, however, the diffusion is strong enough to keep the the particles from moving to the origin. The particles are located between -0.25 and 0.25. In Figure 2.3c and 2.4c, The aggregation strength falls between the previous two cases and the diffusion is dramatically decreased. As expected, we see the particles come together and meet at the origin.



(a) (b) (c)

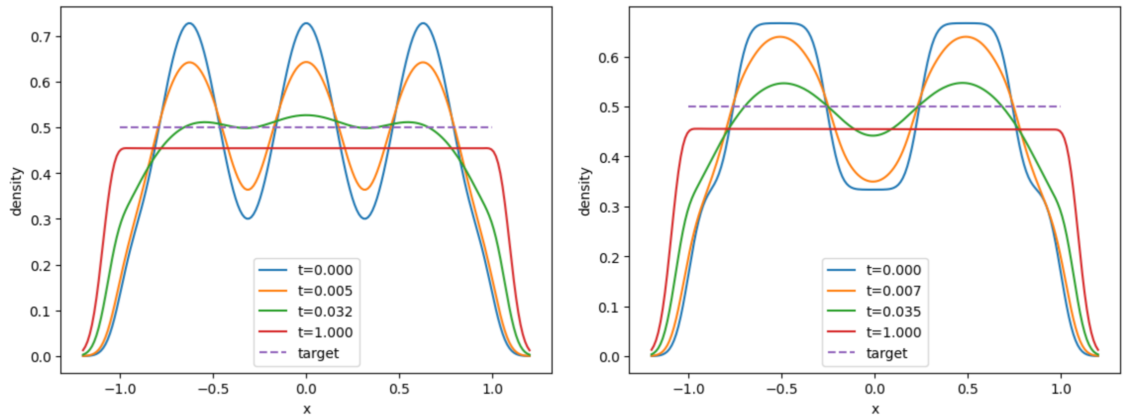
Figure 2.3 Density of aggregation and diffusion energy with uniform target evolution in time.



(a) (b) (c)

Figure 2.4 Particles location evolution in time of Figure 2.3.

In the previous figures, the initial data considered is the Barenblatt profile. An obvious question rises that would another initial data be sufficient to get convergence to the steady-state solution. We would still expect convergence with another initial data. Indeed, Figure 2.5 shows this. Figure 2.5a, a cosine function is used as the initial condition. In particular, $(\cos(10x) + 2)/(hC)$ where $C = 2(\sin(10)/10 + 2)$ is the normalization term and $h = |\Omega|/N$. In Figure 2.5b, the piecewise function $\mu_0 = 2/3$ for $x \in [-0.75, -0.25) \cup [0.25, 0.75)$ and $\mu_0 = 1/3$ otherwise, is the initial condition. The particle evolution in time in Figure 2.6 is different than the previous figures. The particles are still spreading out, however, there is a noticeable partitioning of the particles corresponding to the number of peaks of the initial data. This is most likely because the particles initially near the peaks have more mass than the surrounding particles.



(a) (b)

Figure 2.5 Density of diffusion and aggregation energy with different initial data evolution in time.

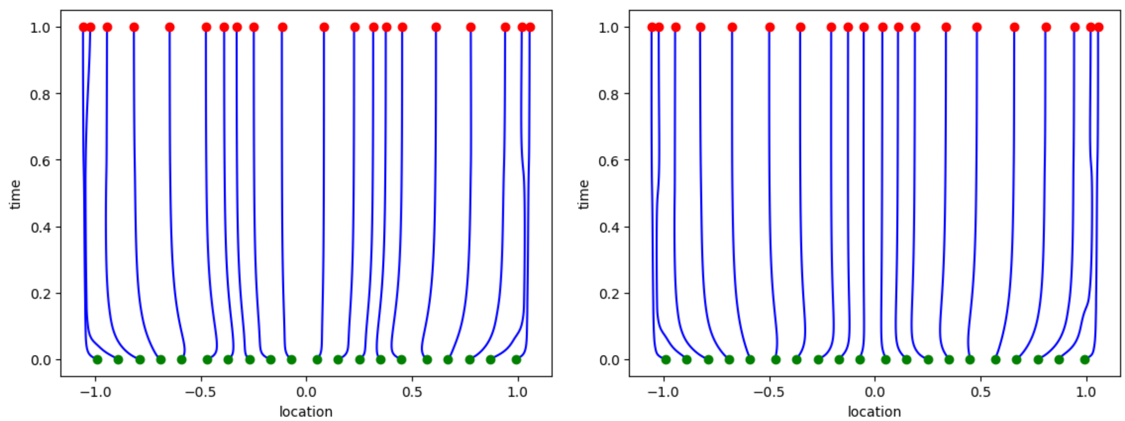
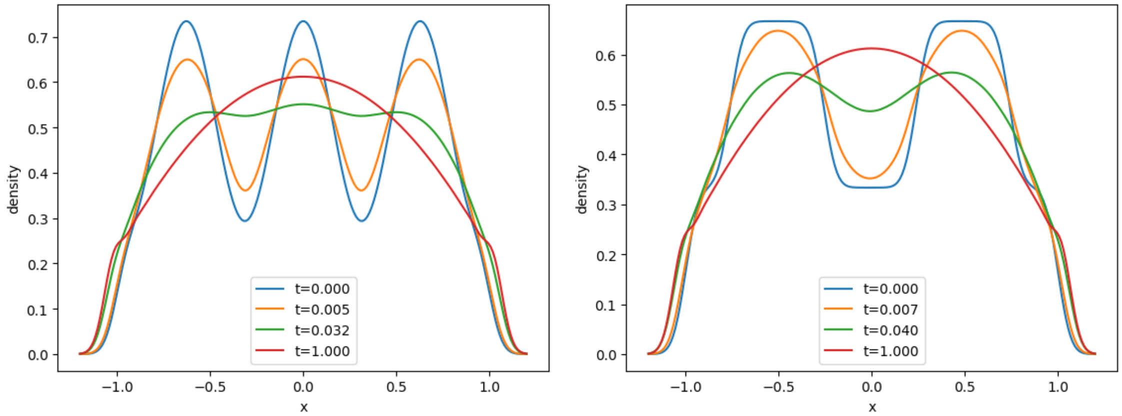


Figure 2.6 Particles location evolution in time of Figure 2.5.

We can also use different initial data when the energy in consideration is the sum of the diffusion, aggregation, and drift. We see the evolution of the density of the energy in Figure 2.7.

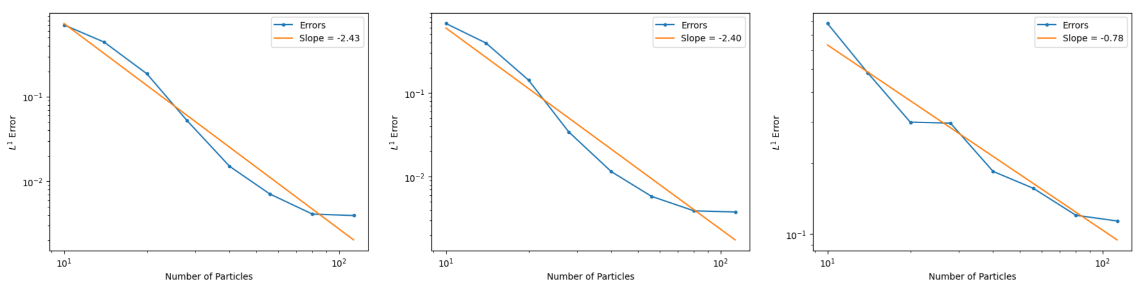


(a) Cosine initial data (b) Piecewise initial data

Figure 2.7 Density of all energies (diffusion, aggregation, drift) with different initial data evolution in time.

2.10.2 Convergence rate in N

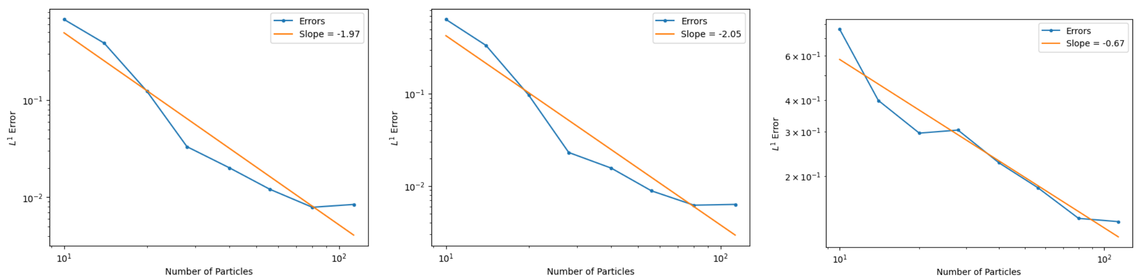
In Figure 2.8, we examine the L^1 error of the density (for N between 10 and 113) and the density at $N = 226$. In this case, we are only focusing on the diffusion energy with different weights. We get the expected results that the convergence rate is faster for the smoother weights (uniform, and log-concave) and slower for the discontinuous weight (piecewise).



(a) Uniform weight (b) Log-concave weight (c) Piecewise weight

Figure 2.8 L^1 errors of the density of the diffusion energy for three different targets demonstrating convergence rate in N.

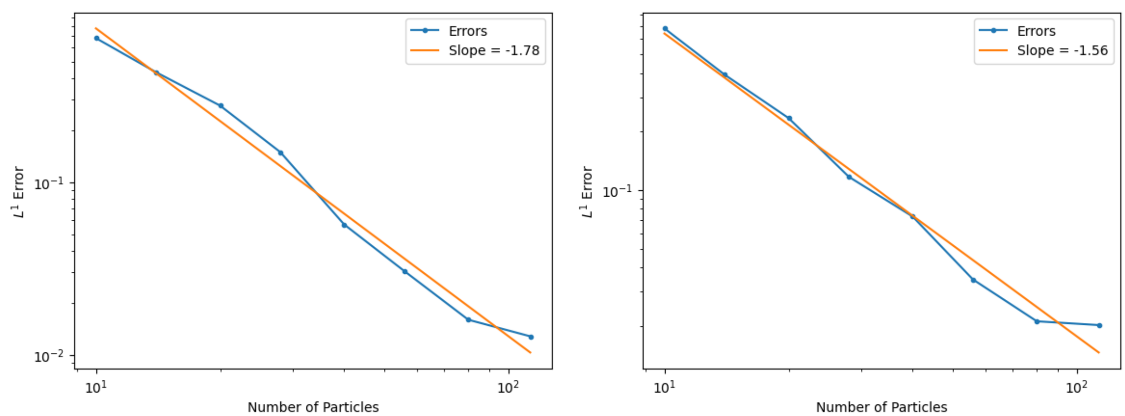
Now we do the same experiment now including the aggregation and drift energies in Figure 2.9. The addition of the energies slightly slowed down the convergence rate in each scenario.



(a) Uniform weight (b) Log-concave weight (c) Piecewise weight

Figure 2.9 L^1 errors of the density of the diffusion, aggregation, and drift energies for different targets demonstrating convergence rate in N.

In Figure 2.10, we see if changing the initial data degrades the convergence rate in N. Indeed, we see a convergence rate below quadratic with the change of initial data, compared to quadratic convergence with the Barenblatt profile initial condition and more than quadratic convergence with only the diffusion energy.



(a) Cosine initial data (b) Piecewise initial data

Figure 2.10 L^1 errors of the density of the diffusion, aggregation, and drift energies for different initial data demonstrating convergence rate in N.

2.10.3 Computational Complexity

Based on the ODE (velocity law) of Theorem 2.2.2, we would expect that for a general mollifier the computation complexity for the drift is $O(N)$, the diffusion is $O(N^2)$, and the aggregation is $O(N^3)$. However, using properties of a specific mollifier, we can do better. Recall that we use a Gaussian mollifier in these numerical simulations. Using the Fourier transform, one can show that the convolution of two Gaussians is Gaussian. Namely, $\varphi_\epsilon := \zeta_\epsilon * \zeta_\epsilon$ is Gaussian. In particular, the mean of the new Gaussian is the sum of the means of the Gaussians and the variance of the new Gaussian is the sum of the variances of the Gaussians.

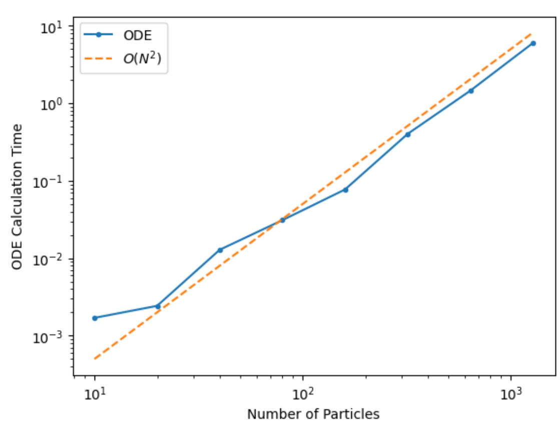
Thus, we can reduce the aggregation to $O(N^2)$,

$$\dot{X}^i_{\epsilon,k}(t) = -\sum_{j=1}^{N} m^j \int_{R^d} \nabla\zeta_\epsilon(X^i_{\epsilon,k} - z)\zeta_\epsilon(X^j_{\epsilon,k} - z)\frac{1}{a(z)}\,dz - \nabla\zeta_\epsilon * V(X^i_{\epsilon,k}) - \nabla V_k(X^i_{\epsilon,k})$$
$$- \sum_{j=1}^{N} m^j \nabla\varphi_\epsilon * W(X^i_{\epsilon,k} - X^j_{\epsilon,k}).$$

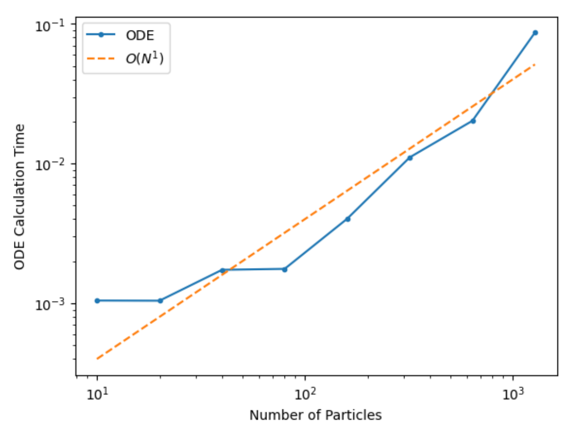
This in turn reduces the computation complexity of the ODE from $O(N^3)$ to $O(N^2)$. This can be seen in Figure 2.11a. In specific scenarios, we can reduce the complexity ever further. If $a(z)$ is a scalar (such as the uniform weight), then we can reduce the diffusion to $O(N)$,

$$\dot{X}^i_{\epsilon,k}(t) = -\sum_{j=1}^{N} m^j \frac{1}{a}\nabla\varphi_\epsilon(X^i_{\epsilon,k} - X^j_{\epsilon,k}) - \nabla\zeta_\epsilon * V(X^i_{\epsilon,k}) - \nabla V_k(X^i_{\epsilon,k})$$
$$- \sum_{j=1}^{N} m^j \nabla\varphi_\epsilon * W(X^i_{\epsilon,k} - X^j_{\epsilon,k}).$$

Thus, having the diffusion energy with the uniform weight and the drift (with no aggregation, $W = 0$), we have a $O(N)$ complexity of the ODE. We can observe this in Figure 2.11b.



(a) The complexity of the ODE of the diffusion, aggregation, and drift.



(b) The complexity of the ODE of the diffusion (with the uniform weight) and drift.

Figure 2.11 The computational complexity of the ODE (velocity law).

CHAPTER 3

INCOMPRESSIBLE LIMIT OF INHOMOGENEOUS POROUS MEDIUM EQUATIONS

3.1 Introduction

The focus of our work is the $m \to \infty$ limit (called the incompressible limit or stiff pressure limit) of the inhomogeneous porous medium equation with reaction,

$$\partial_t u_m = \nabla \cdot \left(\frac{u_m}{a(x,t)} \nabla p_m \right) + \frac{u_m}{a(x,t)} \Phi(x,t,p_m) \text{ on } \mathbb{R}^d \times (0,\infty), \tag{3.1}$$

where the pressure p_m is given in terms of the density u_m by the power law,

$$p_m = \frac{m}{m-1} \left(\frac{u_m}{b(x,t)} \right)^{m-1}, \quad m \geq 2.$$

It is sometimes useful to rewrite (3.1) as

$$\partial_t u_m = \nabla \cdot \left(\frac{b}{a} \nabla \left(\frac{u_m}{b} \right)^m \right) + \frac{u_m}{a} \Phi(x,t,p_m). \tag{3.2}$$

The coefficients, a and b, which are assumed to be bounded from above and strictly away from zero, represent heterogeneity in the underlying medium and in the cellular packing density, respectively [26]. It is also assumed that the growth term Φ is strictly decreasing in p and that there exists $p_M > 0$ with $\Phi(x,t,p_M) = 0$, which corresponds to a ceiling on the maximum pressure that the medium can support.

The aim of our work is to study the limit $m \to \infty$ of (3.1). In our first result we establish that the density and pressure converge to the pair (u_∞, p_∞), which is the (unique) weak solution of

$$\begin{aligned} &\partial_t u_\infty = \nabla \cdot \left(\frac{b}{a} \nabla p_\infty \right) + \frac{u_\infty}{a} \Phi(x,t,p_\infty) \\ &p_\infty(x,t) \in P_\infty(u_\infty(x,t), b(x,t)) \text{ almost everywhere }, \end{aligned} \tag{3.3}$$

where we use the notation $P_\infty(u,b)$ for the Hele-Shaw graph: for any $u, b \in [0,\infty)$,

$$P_\infty(u,b) = \begin{cases} 0, & 0 \leq u < b, \\ [0,\infty), & u = b. \end{cases}$$

In our next two main results we provide more detail on the behavior of the limiting density and pressure. The heuristics for this can be seen by examining the equation that the pressure satisfies,

$$\partial_t p_m - \frac{|\nabla p_m|^2}{a} = (m-1)\frac{p_m}{a}\left(\nabla p_m \cdot \nabla \log\left(\frac{b}{a}\right) + \Delta p_m + \Phi(x,t,p_m) - a\partial_t \log(b)\right), \tag{3.4}$$

which is obtained by multiplying (3.2) by $\frac{m}{b}\left(\frac{u_m}{b}\right)^{m-2}$ and performing standard manipulations. For some calculations, it is easier to evaluate the normalized density $v_m = \frac{u_m}{b}$, which solves

$$\partial_t v_m + v_m \partial_t \log(b) = \frac{1}{a}\left(\nabla v_m^m \cdot \nabla \log\left(\frac{b}{a}\right) + \Delta v_m^m + v_m \Phi(x,t,p_m)\right). \tag{3.5}$$

It is natural to guess that, in the $m \to \infty$ limit of (3.4), the limit of the term on the right-hand side of (3.4) should be zero:

$$\frac{p_\infty}{a}\left(\nabla p_\infty \cdot \nabla \log\left(\frac{b}{a}\right) + \Delta p_\infty + \Phi(x,t,p_\infty) - a\partial_t \log(b)\right) = 0. \tag{3.6}$$

This is the so-called complementarity condition. In Theorem 3.3.3, we prove that (u_∞, p_∞) does indeed satisfy (3.6) in the sense of distributions. This means that, for each time t, there are two regions of interest: the region where p_∞ is zero, and the region where p_∞ is positive (and therefore the term in the parentheses of (3.6) is identically zero). Thus, it is natural to attempt to characterize the evolution of the boundary between these two regions.

Examining (3.4) suggests that, on this boundary, the limit as $m \to \infty$ of the left-hand side of (3.4) should be identically zero:

$$\frac{\partial_t p_\infty}{|\nabla p_\infty|} = \frac{|\nabla p_\infty|}{a}.$$

The left-hand side of the previous line is exactly the normal velocity of the zero level set of p_∞. Thus, the guess is that this normal velocity is exactly $\frac{|\nabla p_\infty|}{a}$. It turns out that this guess is correct, in the absence of an external density; this follows from Proposition 3.3.4, our third main result, in which we characterize the normal velocity of $\partial\{x : p_\infty(x,t) = 0\}$ in the sense of comparison with barriers [21].

3.2 Notation and Assumptions

Throughout, we use the notation $Q_T = \mathbb{R}^d \times (0,T)$ and $Q = \mathbb{R}^d \times (0,\infty)$. We will also use the notation u_+ and u_- to denote the positive and negative part of u, respectively. Throughout, we use C

to denote any positive constant that is independent of m but may depend on d and the constants Λ, λ, and p_M in Assumption 3.2.1. Note that the constant C will only depend on d in the AB-estimate, Proposition 3.7.3.

Assumption 3.2.1. Suppose that $a, b \in C^3(\mathbb{R}^d \times (0,\infty))$ and there exists $\Lambda > 0$ such that $1/\Lambda \leq a, b \leq \Lambda$ for all $(x,t) \in Q_T$. Suppose $\Phi \in C^2(\mathbb{R}^d \times (0,\infty) \times [0, p_M])$ satisfies $\partial_p \Phi < -\lambda$ and $\Phi(x,t,p_M) = 0$, for some $p_M > 0$ and $\lambda > 0$.

Example 1 (Example of Φ)**.** One choice of Φ is where the pressure is separate from the space and time such as $\Phi(x,t,p) = g(p)h(x,t)$. Here g is decreasing, $g(p_M) = 0$, and h is a positive function that is bounded. A standard choice, as in [23, Fig 1], is to choose a linear function $g(p) = C(p_M - p)$, where $C > 0$.

Assumption 3.2.2 (Initial Data)**.** For some $u^0 \in L^1(\mathbb{R}^d)$, suppose that the initial data u_m^0 satisfies,

$$(3.7)\qquad \begin{cases} u_m^0 \geq 0, & \frac{m}{m-1}\left(\frac{u_m^0}{b}\right)^{m-1} \leq p_M, \\ \|u_m^0 - u^0\|_{L^1(\mathbb{R}^d)} \to 0 \text{ as } m \to \infty, & \|\partial_{x_i} u_m^0\|_{L^1(\mathbb{R}^d)} \leq C, i = 1, \ldots, d, \end{cases}$$

and $\operatorname{supp}(u_m^0) \subset \Omega_0$ for $\Omega_0 \subset \mathbb{R}^d$ compact. There exists a constant $C > 0$ such that

$$(3.8)\qquad \|\nabla p_m^0\|_{L^2(\mathbb{R}^d)} + \|\Delta p_m^0\|_{L^2(\mathbb{R}^d)} + \|\partial_t p_m^0\|_{L^1(\mathbb{R}^d)} \leq C.$$

Assumptions 3.2.1 and 3.2.2 are similar to standard assumptions in the literature, such as in [12, 16, 17, 21, 23].

Assumption 3.2.3 (Construction of Supersolution)**.** Suppose that either,

1. $a(x,t) = a(|x|,t), b(x,t) = b(|x|,t)$ (a, b are radial in space),
2. For all $t \in [0,T]$, there exists $R(t) > 0$, such that if $|x| \geq R$, then we can find an $0 < \epsilon \leq \frac{d-\frac{1}{2}}{\Lambda^2}$ such that
$$\left|\nabla\left(\frac{b}{a}\right)\right| \leq \frac{\epsilon}{|x|}.$$

We make precise our notion of weak solution, as in [26, Definition 5.4].

Definition 3.2.4 (Notion of weak solution to (3.1))**.** A non-negative $u \in L^1(Q_T)$ is a weak solution of (3.1) with initial data $u^0 \in L^1(\mathbb{R}^d)$ if

1. $\left(\frac{u}{b}\right)^m \in L^2(0,T;H^1(\mathbb{R}^d))$
2. for $p = \frac{m}{m-1}\left(\frac{u}{b}\right)^{m-1}$, u satisfies

$$\iint_{Q_T} u\partial_t\zeta - \frac{b}{a}\nabla\left(\frac{u}{b}\right)^m \cdot \nabla\zeta + \frac{u}{a}\Phi(x,t,p)\zeta = -\int_{\mathbb{R}^d} u^0(x)\zeta(x,0),$$

for any $\zeta \in C^1(\overline{Q}_T)$ that vanishes for $t = T$.

Existence can be achieved by approximating the weak solution with smooth functions that solve the PME with strictly positive initial data such as in [26, Theorem 5.14] or [26, Section 9.3].

Definition 3.2.5 (Solution to liming problem)**.** Let $T > 0$. We say

$$(u_\infty, p_\infty) \in C([0,T);L^1(\mathbb{R}^d)) \times L^1([0,T);L^1(\mathbb{R}^d)) \cap L^2((0,T);H^1(\mathbb{R}^d))$$

is a weak subsolution (resp. supersolution) of (3.3) if for all test functions $\zeta \in C_c^1(\mathbb{R}^d \times [0,T))$ we have

$$-\iint_{Q_T} u_\infty\partial_t\zeta + \frac{b}{a}\nabla\left(\frac{u_\infty}{b}\right)^m \cdot \nabla\zeta - \frac{u_\infty}{a}\Phi(x,t,p_\infty)\zeta \le \int_{\mathbb{R}^d} u^0(x)\zeta(x,0) \quad \text{(resp. } \ge\text{)},$$

and if $p_\infty(x,t) \in P_\infty(u_\infty(x,t), b(x,t))$ and $0 \le u_\infty(x,t) \le b(x,t)$ hold almost everywhere in Q_T. We say (u_∞, p_∞) is a weak solution of (3.3) if it is both a weak subsolution and a weak supersolution.

3.3 Main Results

Theorem 3.3.1 (Convergence to limiting problem)**.** *Suppose Assumptions 3.2.1 and 3.2.2 hold and fix $T > 0$. Then, up to a subsequence, the density u_m and the pressure p_m solution pair of* (3.1) *converge strongly in $L^1(Q_T)$ as $m \to \infty$ to u_∞, p_∞ respectively, which satisfy*

$$u_\infty, p_\infty \in BV(Q_T),$$
$$u_\infty \in C^s([0,\infty);H^{-1}(\mathbb{R}^d)) \cap C([0,\infty);L^1(\mathbb{R}^d)) \;\forall s < \frac{1}{2},\; p_\infty \in L^2(0,T;H^1(\mathbb{R}^d)),$$
$$0 \le p_\infty \le p_M,$$

and solve (3.3).

Theorem 3.3.2 (Uniqueness of solutions to (3.3))**.** *Suppose Assumptions 3.2.1, 3.2.2, and 3.2.3 hold. Let $T > 0$. Then, there exists a unique pair*

$$(u_\infty, p_\infty) \in C([0,T); L^1(\mathbb{R}^d)) \times L^1((0,T); L^1(\mathbb{R}^d)) \cap L^2((0,T); H^1(\mathbb{R}^d))$$

such that $u(\cdot,t)$ is compactly supported for each $0 < t < T$, which solves (3.3) in the sense of Definition 3.2.5.

Theorem 3.3.3 (Complementarity condition)**.** *Suppose Assumptions 3.2.1, 3.2.2, and 3.2.3 hold. Then (3.6) holds in the sense of distributions. More precisely, for any $\zeta \in C^1(\overline{Q}_T)$ that vanishes for $t = T$,*

$$\begin{aligned} 0 = &\iint_{Q_T} \zeta \frac{p_\infty}{a} \nabla p_\infty \cdot \nabla \log\left(\frac{b}{a}\right) - |\nabla p_\infty|^2 \frac{\zeta}{a} - \nabla p_\infty \cdot \nabla \left(\frac{\zeta}{a}\right) p_\infty \\ &+ \iint_{Q_T} \frac{p_\infty}{a} \zeta \Phi(x,t,p_\infty) - p_\infty \zeta \partial_t \log(b). \end{aligned}$$

Let D be a ball in $\mathbb{R}^d$. For a time interval $[t_1, t_2] \subset [0, \infty)$, consider a function (that represent the pressure) $\zeta \in C_c(\bar{D} \times [t_1, t_2])$ such that the initial density $u_1(x)$ satisfies $u_1(x) = b(x, t_1)$ in $\{\zeta(t_1) > 0\}$. For all $x \notin \{\zeta(t_1) > 0\}$, we define $t(x)$ as the last time that $\zeta(x,t) = 0$ (with $t(x) = t_2$ is $\zeta(x, t_2 = 0)$) and define the external density

$$u^E_\zeta(x,t) = u_1(x) \exp\left(\int_{t_1}^{t} \frac{\Phi(x,s,\zeta(x,s))}{a(x,s)}\, ds\right)$$

for all $t < t(x)$. We assume that the external density satisfies,

$$u^E_\zeta(x,t) < b(x,t) \text{ in } \{\zeta = 0\}.$$

The external density solves $\partial_t u = \frac{u}{a}\Phi(x,t,\zeta)$ in $\{\zeta = 0\}$. The density in $D \times (t_1, t_2)$ is defined by

$$\begin{aligned} u_\zeta(x,t) &= b(x,t)\chi_{\{\zeta>0\}}(x) + u^E_\zeta(x,t)(1 - \chi_{\{\zeta>0\}}(x)) \\ &= \begin{cases} b(x,t), \text{in } \{\zeta > 0\} \\ u^E_\zeta(x,t), \text{in } \{\zeta = 0\}. \end{cases} \end{aligned}$$

Proposition 3.3.4 (Velocity Law)**.** *The external density, u_∞^E, is the limit of the density from outside the saturated region $\{x : p_\infty(x,t) > 0\}$. The normal velocity, V_∞, of the free boundary $\partial\{x : p_\infty(x,t) > 0\}$ satisfies in a viscosity sense*

$$\left(1 - \frac{u_\infty^E}{b}\right) V_\infty = \frac{|\nabla p_\infty|}{a}.$$

3.4 Estimates

This section is devoted to estimates for solutions of (3.1); these estimates will allow us to take the incompressible limit and obtain our first main result, Theorem 3.3.1.

Remark 3.4.1. We often manipulate the equation pointwise and/or differentiate the equation. These manipulations are justified by approximating the solution by the solution with initial data $u_m^0 + \epsilon$ (that solution is uniformly positive and thus smooth), establishing the desired estimate, and then taking the limit $\epsilon \to 0$. See, for example, [26, Section 9.3].

3.4.1 L^∞ bound and compact support

We begin by using the comparison principle for (3.1) to establish that the solutions (u_m, p_m) of (3.1) are uniformly bounded and have a (uniformly) finite speed of propagation. For the latter, we construct appropriate supersolutions to (3.1); our construction is similar to [12, Lemma 3.1].

Lemma 3.4.2 (L^∞ bounds and compact support)**.** *Suppose that Assumptions 3.2.1, 3.2.2, and 3.2.3 hold and let (u_m, p_m) solve (3.1). Then, there exits a constant $C > 0$ such that,*

$$0 \le p_m \le p_M, \quad 0 \le u_m(x,t) \le Cb(x,t), \quad 0 \le v_m \le C \quad a.e.\ Q_T,$$
$$supp(u_m(\cdot,t)) \subset B_{Ct} \quad a.e.\ 0 \le t \le T.$$

Proof. By assumption on the initial data and the comparison principle, $0 \le p_m \le \frac{m}{m-1}\left(\frac{u_m^0}{b}\right)^{m-1} \le p_M$ a.e. in Q_T. By definition of the density,

$$0 \le \frac{u_m}{b} \le \left(\frac{m-1}{m} p_M\right)^{1/(m-1)} \to 1,$$

as $m \to \infty$. Thus, $u_m \le Cb$ a.e. in Q_T. Moreover, $v_m \le C$ a.e. in Q_T. Now, let $Z(x,t)$ be as in Lemma B.0.2 (if Assumption 3.2.3(i) holds) or as in Lemma B.0.3 (if Assumption 3.2.3(ii) holds).

In either case, let the constant α be chosen large enough, depending on the initial data, to ensure $Z(x,0) \geq u_m^0(x)$ on $\mathbb{R}^d$. The comparison principle for (3.1) thus ensures $u_m(x,t) \leq Z(x,t)$ for all $t > 0$. By construction, $Z(x,t)$ is compactly supported in x; therefore, so are u_m and p_m. □

Now that we have compact support and L^∞ bounds, we immediately get L^1 bounds.

Corollary 3.4.3 (L^1 bounds for u_m, p_m). *Suppose Assumptions 3.2.1 and 3.2.2 hold and let (u_m, p_m) solve (3.1). There exists a constant $C > 0$ such that for $t \in [0,T]$ and $m \geq 2$,*

$$\|u_m(t)\|_{L^1(\mathbb{R}^d)} + \|v_m(t)\|_{L^1(\mathbb{R}^d)} + \|p_m(t)\|_{L^1(\mathbb{R}^d)} \leq C.$$

3.4.2 Derivative bounds

In the next two lemmas, we establish integral bounds for the time and spatial derivatives of the pressure and density. The techniques are similar to those of [23]; however, more care has to be taken, especially in the proof of the time derivative bounds, due to the coefficients' dependence on space and time.

Lemma 3.4.4 (L^1 and L^2 bound for ∇p_m). *Suppose Assumptions 3.2.1 and 3.2.2 hold, and let (u_m, p_m) solve (3.1). There exists a constant $C > 0$ such that for $m > 3$,*

$$\|\nabla p_m\|_{L^1(Q_T)} + \|\nabla p_m\|_{L^2(Q_T)} \leq C.$$

Proof. First we point out the identity,

$$|\nabla p_m|^2 + p_m \Delta p_m = \nabla \cdot (p_m \nabla p_m) = \Delta(p_m^2).$$

Rearranging the equation for the pressure (3.4), and using this identity, yields,

$$|\nabla p_m|^2 = \frac{m-1}{2(m-2)}\Delta(p_m^2) - \frac{a\partial_t p_m}{m-2} + \frac{m-1}{m-2} p_m \left(\Phi(x,t,p_m) - a\partial_t \log(b) + \nabla p_m \cdot \nabla \log\left(\frac{b}{a}\right)\right).$$

We integrate over Q_T and note that, since p_m is compactly supported in x, the integral of the Laplacian term vanishes upon integrating by parts. Thus we find,

$$\begin{aligned}
\iint_{Q_T} |\nabla p_m|^2 &\leq \frac{\|\partial_t a\|_{L^\infty}}{m-2} \iint_{Q_T} p_m + \frac{m-1}{m-2} C \iint_{Q_T} p_m + \frac{m-1}{m-2} C \iint_{Q_T} p_m |\nabla p_m| \\
&\leq \frac{m-1}{m-2}\left(C + \frac{C}{m-1}\right) \iint_{Q_T} p_m + \left(\frac{m-1}{m-2} C\right)^2 \iint_{Q_T} p_m^2 + \frac{1}{2} \iint_{Q_T} |\nabla p_m|^2,
\end{aligned}$$

where we've also used Young's inequality. Therefore,

$$\|\nabla p_m\|_{L^2(Q_T)} \leq C\|p_m\|_{L^1(Q_T)} + C\|p_m\|_{L^2(Q_T)}.$$

According to Lemma 3.4.2, we have $p_m \in L^1(Q_T) \cap L^\infty(Q_T)$. The L^2 bound follows. Using the compact support of p_m (Lemma 3.4.2) along with Hölder's inequality yields the L^1 bound as well. □

Corollary 3.4.5. *Suppose that Assumptions 3.2.1, and 3.2.2 hold. For* $T > 0$*, there exists a constant* $C > 0$ *such that for* $m > 3$*,*

$$\|\nabla v_m^m\|_{L^1(Q_T)} + \|\nabla v_m^m\|_{L^2(Q_T)} \leq C.$$

Proof. By Lemma 3.4.2,

$$|\nabla v_m^m| = v_m|\nabla p_m| \leq C|\nabla p_m|.$$

Integrate in Q_T and apply Lemma 3.4.4 to get the result. □

Using the previous corollary, we establish the L^1 bound for ∇u_m.

Lemma 3.4.6 (L^1 bound for ∇u_m)**.** *Suppose Assumptions 3.2.1 and 3.2.2 hold and let* (u_m, p_m) *solve (3.1). For* $T > 0$*, there exists a constant* $C > 0$ *such that for* $m > 3$*,*

$$\|\partial_{x_i} u_m\|_{L^1(Q_T)} + \|\partial_{x_i} v_m\|_{L^1(Q_T)} \leq C.$$

Proof. Let $\lambda = \min_{p\in[0,p_M]} |\partial_p \Phi(x,t,p)| > 0$. We differentiate (3.5) with respect to x_i, multiply by $\mathrm{sgn}(\partial_{x_i} v_m) = \mathrm{sgn}(\partial_{x_i} p_m) = \mathrm{sgn}(\partial_{x_i} v_m^m)$ and using Kato's inequality,

$$\begin{aligned}
&\partial_t|\partial_{x_i} v_m| + v_m \partial_{x_i}\partial_t \log(b)\mathrm{sgn}(\partial_{x_i} v_m) + |\partial_{x_i} v_m|\partial_t \log(b) \\
&\leq \mathrm{sgn}(\partial_{x_i} v_m)\partial_{x_i}(1/a)\left(\nabla v_m^m \cdot \nabla \log\left(\frac{b}{a}\right) + \Delta v_m^m + v_m \Phi(x,t,p_m)\right) \\
&+ (1/a)(\mathrm{sgn}(\partial_{x_i} v_m)\nabla v_m^m \cdot \partial_{x_i}\nabla \log(b/a) + \nabla|\partial_{x_i} v_m^m| \cdot \nabla \log(b/a) + \Delta|\partial_{x_i} v_m^m| + |\partial_{x_i} v_m|\Phi(x,t,p_m) \\
&\quad + v_m(\Phi_{x_i}\mathrm{sgn}(\partial_{x_i} v_m) + \Phi_p|\partial_{x_i} p_m|)).
\end{aligned}$$

Integrating in space, using integration by parts, and rearranging,

$$\frac{d}{dt}\int_{\mathbb{R}^d}|\partial_{x_i}v_m| \le C\int_{\mathbb{R}^d}v_m + C\int_{\mathbb{R}^d}|\nabla v_m^m| + C\int_{\mathbb{R}^d}|\partial_{x_i}v_m| - \lambda\int_{\mathbb{R}^d}\frac{v_m}{a}|\partial_{x_i}p_m|.$$

By Gr onwall's inequality and Corollary 3.4.5,

$$\begin{aligned}\|\partial_{x_i}v_m\|_{L^1(\mathbb{R}^d)} + \lambda\iint_{Q_T}\frac{v_m}{a}|\partial_{x_i}p_m| &\le e^{tC}(\|\partial_{x_i}v_m^0\|_{L^1(\mathbb{R}^d)} + C\|v_m\|_{L^1(Q_T)} \\ &\quad + C\|\nabla v_m^m\|_{L^1(Q_T)}) \\ &\le C.\end{aligned}$$

Notice that each term on the left-hand side is bounded by this constant. Then,

$$\|\partial_{x_i}v_m\|_{L^1(\mathbb{R}^d)} \le C, \quad \frac{\lambda}{\|a\|_{L^\infty}}\iint_{Q_T}v_m|\partial_{x_i}p_m| \le C.$$

Using the fact that

$$\partial_{x_i}u_m = b\partial_{x_i}v_m + u_m\partial_{x_i}\log(b),$$

we achieve the L^1 bound for the density. □

We continue with L^1 bounds for $\partial_t u_m, \partial_t p_m$.

Lemma 3.4.7 (L^1 bounds for $\partial_t u_m, \partial_t p_m$)**.** *Suppose Assumptions 3.2.1 and 3.2.2 hold and let (u_m, p_m) solve (3.1). For $T > 0$, there exists a constant $C > 0$ such that for $m > 3$,*

$$\|\partial_t u_m\|_{L^1(Q_T)} + \|\partial_t v_m\|_{L^1(Q_T)} + \|\partial_t p_m\|_{L^1(Q_T)} \le C.$$

Proof. Let $\lambda = \min_{p\in[0,p_M]}|\partial_p\Phi(x,t,p)| > 0$. We rearrange (3.5) so that

$$b\partial_t v_m + v_m b\partial_t\log(b) = \nabla\cdot\left(\frac{b}{a}\nabla v_m^m\right) + v_m\frac{b}{a}\Phi(x,t,p_m).$$

We differentiate with respect to t, multiply by $\operatorname{sgn}(\partial_t v_m) = \operatorname{sgn}(\partial_t p_m) = \operatorname{sgn}(\partial_t v_m^m)$ and using Kato's inequality,

$$\begin{aligned}&\partial_t(b|\partial_t v_m|) + |\partial_t v_m|b\partial_t\log(b) + v_m\partial_t b\partial_t\log(b)\operatorname{sgn}(\partial_t v_m) + v_m b\partial_t\partial_t\log(b)\operatorname{sgn}(\partial_t v_m) \le \\ &\le |\partial_t\nabla(b/a)||\nabla v_m^m| + \nabla\cdot\left((b/a)\nabla|\partial_t v_m^m|\right) + |\partial_t v_m|(b/a)\Phi(x,t,p_m) \\ &\quad + v_m\partial_t(b/a)\Phi(x,t,p_m)\operatorname{sgn}(\partial_t v_m) + v_m(b/a)\Phi_t\operatorname{sgn}(\partial_t v_m) + v_m(b/a)\Phi_p|\partial_t p_m|.\end{aligned}$$

Integrating in space, using integration by parts, and rearranging,

$$\frac{d}{dt}\int_{\mathbb{R}^d} b|\partial_t v_m| \leq C\int_{\mathbb{R}^d} v_m + C\int_{\mathbb{R}^d} |\nabla v_m^m| + C\int_{\mathbb{R}^d} b|\partial_t v_m| - \lambda\int_{\mathbb{R}^d} v_m \frac{b}{a}|\partial_t p_m|.$$

By Gr onwall's inequality and Corollary 3.4.5,

$$\begin{aligned}
\|b\partial_t v_m\|_{L^1(\mathbb{R}^d)} + \lambda \iint_{Q_T} \frac{b}{a} v_m|\partial_t p_m| &\leq e^{tC}(C\|\partial_t v_m^0\|_{L^1(\mathbb{R}^d)} + C\|v_m\|_{L^1(Q_T)} \\
&\quad + C\|\nabla v_m^m\|_{L^1(Q_T)}) \\
&\leq C.
\end{aligned}$$

Notice that each term on the left-hand side is bounded by this constant. Then,

$$\|\partial_t v_m\|_{L^1(\mathbb{R}^d)} \leq C, \quad \iint_{Q_T} v_m|\partial_t p_m| \leq C.$$

Bounding the pressure by using the estimates above,

$$\begin{aligned}
\|\partial_t p_m\|_{L^1(Q_T)} &\leq \iint_{Q_T\cap\{v_m\leq\frac{1}{2}\}} m v_m^{m-2}|\partial_t v_m| + \iint_{Q_T\cap\{v_m\geq\frac{1}{2}\}} 2v_m|\partial_t p_m| \\
&\leq m\left(\frac{1}{2}\right)^{m-2} C + C \\
&\leq C,
\end{aligned}$$

where we use the fact that $m\left(\frac{1}{2}\right)^{m-2} \leq 2$ for all $m \geq 2$ and λ is absorbed into the constant. Using the fact that

$$\partial_t u_m = b\partial_t v_m + u_m \partial_t \log(b),$$

we achieve the L^1 bound for the density. □

3.5 Proof of Convergence to Limiting Problem

Before we establish uniqueness of the limit in a bounded region, we let u_∞ be any subsequential limit of the density with the same initial condition when establishing the regularity and initial data of u_∞ below.

Lemma 3.5.1. *Suppose Assumptions 3.2.1 and 3.2.2 hold. Let u_∞ be any subsequential limit of the density with the same initial condition. The sequence $\{u_m\}$ is relatively compact in $C^s([0,T); H^{-1}(\mathbb{R}^d))$ for $s \in (0, 1/2)$. Furthermore, $u_\infty \in C^s([0,\infty); H^{-1}(\mathbb{R}^d))$ for $s \in (0, 1/2)$.*

Proof. Suppose Assumptions 3.2.1 and 3.2.2 hold. From Lemma 3.4.2, $u_m \in L^\infty(0,T;L^1(\mathbb{R}^d)) \cap L^\infty(0,T;L^\infty(\mathbb{R}^d))$ and by interpolation have $u_m \in L^\infty(0,T;L^2(\mathbb{R}^d))$. Taking the test function $\varphi \in H_0^1$ and using (3.2),

$$\begin{aligned}\int_{\mathbb{R}^d} \partial_t u_m \varphi &= -\int_{\mathbb{R}^d} \frac{u_m}{a}\nabla\varphi \cdot \nabla p_m + \int_{\mathbb{R}^d} \Phi(x,t,p_m)\frac{u_m}{a}\varphi \\ &\leq \left\|\frac{b}{a}\right\|_{L^\infty} \|\nabla\varphi\|_{L^2(\mathbb{R}^d)}\|\nabla p_m(t)\|_{L^2(\mathbb{R}^d)} + \left\|\frac{\Phi}{a}\right\|_{L^\infty} \|\varphi\|_{L^2(\mathbb{R}^d)}\|u_m(t)\|_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Thus,

$$\|\partial_t u_m\|^2_{L^2(0,T;H^{-1}(\mathbb{R}^d))} \leq \|\nabla\varphi\|^2_{L^2(\mathbb{R}^d)}\|\nabla p_m\|^2_{L^2(Q_T)} + C\|u_m\|^2_{L^\infty(0,T;L^2(\mathbb{R}^d))}\|\varphi\|^2_{L^2(\mathbb{R}^d)}.$$

Since $H^{-1}(\mathbb{R}^d)$ is compactly embedded in $L^2(\mathbb{R}^d)$, by Lions-Aubin (for reference see [16, Proposition 1.2.5]) we have that $\{u_m\}$ is relatively compact in $C^s([0,T);H^{-1}(\mathbb{R}^d))$ for $s \in (0,1/2)$. By compactness, the result on u_∞ follows. □

3.5.1 Time Continuity

Lemma 3.5.2. *Suppose Assumptions 3.2.1 and 3.2.2 hold. Let u_∞ be any subsequential limit of the density with the same initial condition. Then the limiting density is continuous in time. In particular, $u_\infty \in C([0,\infty);L^1(\mathbb{R}^d))$.*

Proof. For times $0 < t_1 < t_2 \leq T$, given that u_∞ solves the limiting PDE and using integration by parts,

$$\begin{aligned}\int_{\mathbb{R}^d} |u_\infty(t_2) - u_\infty(t_1)| &= \int_{t_1}^{t_2}\int_{\mathbb{R}^d} \nabla\cdot\left(\frac{b}{a}\nabla p_\infty\right) + \frac{u_\infty}{a}\Phi(x,t,p_\infty) \\ &= \int_{t_1}^{t_2}\int_{\mathbb{R}^d} \frac{u_\infty}{a}\Phi(x,t,p_\infty) \\ &\leq C(t_2 - t_1).\end{aligned}$$

Thus, $u_\infty \in C([0,\infty);L^1(\mathbb{R}^d))$. □

3.5.2 Initial Condition

Lemma 3.5.3. *Suppose Assumptions 3.2.1 and 3.2.2 hold. Let u_∞ be any subsequential limit of the density with the same initial condition. Then the limiting density at $t = 0$ coincides with the the limiting initial condition in L^1. That is, $u_\infty(0) = u^0$ in $L^1(\mathbb{R}^d)$*

Proof. For time $0 < t \leq T$, given that u_m is a solution to the PME,

$$\int_{\mathbb{R}^d} u_m(t) - \int_{\mathbb{R}^d} u_m^0 = \int_0^t \int_{\mathbb{R}^d} \nabla \cdot \left(\frac{b}{a} \nabla \left(\frac{u_m}{b} \right)^m \right) + \frac{u_m}{a} \Phi(x, t, p_m).$$

Letting $m \to \infty$,

$$\int_{\mathbb{R}^d} u_\infty(t) - \int_{\mathbb{R}^d} u_\infty^0 = \int_0^t \int_{\mathbb{R}^d} \nabla \cdot \left(\frac{b}{a} \nabla p_\infty \right) + \frac{u_\infty}{a} \Phi(x, t, p_\infty).$$

Using integration by parts and letting $t \to 0$,

$$\int_{\mathbb{R}^d} u_\infty(0) - \int_{\mathbb{R}^d} u^0 = 0.$$

Thus, $u_\infty(0) = u^0$ in $L^1(\mathbb{R}^d)$. □

3.5.3 Proof of convergence to limiting problem

Proof of Theorem 3.3.1. From previous estimates Lemmas 3.4.2, 3.4.4, 3.4.7, we have that u_m, p_m are bounded in $W^{1,1}(Q_T)$. By Rellich-Kondrachov, u_m, p_m converge (up to a subsequence) strongly in $L^1(Q_T)$.

Integrating (3.2) against a test function $\psi \in C^1(\overline{Q}_T)$ that vanishes for $t = T$,

$$\iint_{Q_T} u_m \partial_t \psi + \left(\frac{u_m}{b} \right)^m \nabla \cdot \left(\frac{b}{a} \nabla \psi \right) + \frac{u_m}{a} \Phi(x, t, p_m) \psi = - \int_{\mathbb{R}^d} u_m^0 \psi(x, 0).$$

As u_m, p_m converges strongly in $L^1(Q_T)$, we obtain that

$$\partial_t u_\infty = \nabla \cdot \left(\frac{b}{a} \nabla p_\infty \right) + \frac{u_\infty}{a} \Phi(x, t, p_\infty) \quad \text{in } \mathcal{D}'(Q).$$

Using the definition of p_m and rearranging

$$\frac{u_m}{b} = \left(\frac{m-1}{m} p_m \right)^{1/(m-1)} \Rightarrow \frac{u_m}{b} p_m = \left(\frac{m-1}{m} \right)^{1/(m-1)} p_m^{m/(m-1)}.$$

Up to a subsequence, we can pass to the limit a.e. so that $u_\infty p_\infty / b = p_\infty$ or equivalently,

$$\left(1 - \frac{u_\infty}{b}\right) p_\infty = 0$$

This gets us $p_\infty \in P_\infty(u_\infty, b)$. We also obtain almost everywhere in Q_T that $0 \le u_\infty \le b, 0 \le p_\infty \le p_M$, and $u_\infty, p_\infty \in BV(Q_T)$ for all $T > 0$. □

3.6 Uniqueness of the Limit

To establish uniqueness of solutions to (3.3), we follow the Hilbert's duality method, which was used for the homogeneous version of (3.3) in [23].

For the remainder of this section, let us fix two non-negative densities u_1, u_2 with corresponding pressures p_1, p_2, solving (3.3). Let Ω be a bounded domain containing the supports of both solutions for all time $t \in [0, T]$ and $\Omega_T = \Omega \times (0, T)$. For ease of notation, we will abbreviate $\Phi(x, t, p)$ as $\Phi(p)$.

First we shall prove that the densities must agree. To this end, we use the definition of weak solution to find,

$$0 = \iint_{\Omega_T} (u_1 - u_2)\partial_t \psi + (p_1 - p_2)\nabla \cdot \left(\frac{b}{a}\nabla\psi\right) + \frac{\psi}{a}(u_1 \Phi(p_1) - u_2 \Phi(p_2)). \tag{3.9}$$

We shall denote $Z := u_1 - u_2 + p_1 - p_2$ and define

$$A := \frac{u_1 - u_2}{Z}, \quad B := \frac{p_1 - p_2}{Z}, \quad C := (-u_2)\frac{\Phi(p_1) - \Phi(p_2)}{p_1 - p_2}.$$

We define $A = 0$ when $u_1 = u_2$ (even when $p_1 = p_2$) and similarly set $B = 0$ when $p_1 = p_2$ (even when $u_1 = u_2$), which yields the bounds,

$$0 \le A \le 1, \quad 0 \le B \le 1, \quad 0 \le C \le \nu. \tag{3.10}$$

With this in hand, we see that (3.9) may be rewritten as,

$$0 = \iint_{\Omega_T} Z\left(A\partial_t\psi + B\nabla \cdot \left(\frac{b}{a}\nabla\psi\right) + A\Phi(p_1)\frac{\psi}{a} - CB\frac{\psi}{a}\right). \tag{3.11}$$

If, given any smooth G, we could find ψ solving the dual problem,

$$\begin{cases} A\partial_t\psi + B\nabla\cdot\left(\frac{b}{a}\nabla\psi\right) + A\Phi(p_1)\frac{\psi}{a} - CB\frac{\psi}{a} = AG & \text{in } \Omega_T, \\ \psi = 0 \quad \text{in } \partial\Omega\times(0,T), \quad \psi(\cdot,T) = 0 \quad \text{in } \Omega, \end{cases}$$

then, by taking ψ as the test function in (3.11), we would obtain

$$0 = \iint_{\Omega_T}(u_1 - u_2 + p_1 - p_2)AG = \iint_{\Omega_T}(u_1 - u_2)G.$$

From this we have uniqueness for the density, as the smooth function G in the previous line is arbitrary.

However, since we may not be able to solve the dual problem due to the degeneracy of the coefficients A, B, C, we proceed by an approximation argument. Let $\{A_n\}, \{B_n\}, \{C_n\}, \{\Phi_{1,n}\}$ be sequences of smooth bounded functions such that

$$\begin{cases} \|A - A_n\|_{L^2(\Omega_T)} < K/n, \quad 1/n < A_n \le 1, \\ \|B - B_n\|_{L^2(\Omega_T)} < K/n, \quad 1/n < B_n \le 1, \\ \|C - C_n\|_{L^2(\Omega_T)} < K/n, \quad 0 \le C_n < K, \quad \|\partial_t C_n\|_{L^1(\Omega_T)} \le K \\ \|\Phi(p_1) - \Phi_{1,n}\|_{L^2(\Omega_T)} < K/n, \quad |\Phi_{1,n}| < K, \quad \|\nabla\Phi_{1,n}\|_{L^2(\Omega_T)} \le K \end{cases}$$

where K is a positive constant. Fix a smooth function G. Standard theory for parabolic PDEs yields that there exists a unique solution ψ_n to the regularized dual problem

(RDP) $$\begin{cases} \partial_t\psi_n + \frac{B_n}{A_n}\nabla\cdot\left(\frac{b}{a}\nabla\psi_n\right) + \Phi_{1,n}\frac{\psi_n}{a} - C_n\frac{B_n}{A_n}\frac{\psi_n}{a} = G & \text{in } \Omega_T, \\ \psi_n = 0 \quad \text{in } \partial\Omega\times(0,T), \quad \psi_n(\cdot,T) = 0 \quad \text{in } \Omega. \end{cases}$$

We shall establish some estimates on the solution ψ_n:

Lemma 3.6.1. *There are constants $\kappa_i = \kappa_i(a, b, T, G, |\Omega|)$ for $i = 1, 2, 3$ such that*

$$\|\psi_n\|_{L^\infty(\Omega_T)} \le \kappa_1, \quad \sup_{0\le t\le T}\|\nabla\psi_n(t)\|_{L^2(\Omega)} \le \kappa_2$$
$$\left\|\left(\frac{B_n}{A_n}\right)^{1/2}\left(\nabla\cdot\left(\frac{b}{a}\nabla\psi_n\right) - \frac{C_n}{a}\psi_n\right)\right\|_{L^2(\Omega_T)} \le \kappa_3.$$

Proof. We will follow the proof of [23, Lemma 3.1], which the first bound is obtained in the same way (also see [12, Lemma 4.1]). Multiplying (RDP) by $\nabla \cdot \left(\frac{b}{a}\nabla\psi_n\right) - \frac{C_n}{a}\psi_n$,

$$\begin{aligned}
\int_\Omega \frac{b(t)}{2a(t)}|\nabla\psi_n(t)|^2 + \int_t^T \int_\Omega \frac{B_n}{A_n}\left|\nabla \cdot \left(\frac{b}{a}\nabla\psi_n\right) - \frac{C_n}{a}\psi_n\right|^2 \\
= -\int_t^T \int_\Omega \partial_t\left(\frac{b}{a}\right)\frac{1}{2}|\nabla\psi_n|^2 - \int_t^T \int_\Omega \partial_t\left(\frac{C_n}{a}\right)\frac{1}{2}\psi_n^2 - \int_\Omega \left(\frac{C_n}{2a}\psi_n^2\right)(t) \\
+ \int_t^T \int_\Omega \left(\frac{b}{a}\psi_n\nabla\left(\frac{\Phi_{1,n}}{a}\right)\cdot\nabla\psi_n + \frac{b}{a}\frac{\Phi_{1,n}}{a}|\nabla\psi_n|^2 + \Phi_{1,n}C_n\left(\frac{\psi_n}{a}\right)^2\right) \\
+ \int_t^T \int_\Omega \left(\frac{b}{a}\nabla G\cdot\nabla\psi_n + \frac{C_n}{a}G\psi_n\right).
\end{aligned}$$

Bounding the right-hand side,

$$\int_\Omega |\nabla\psi_n(t)|^2 + \int_t^T \int_\Omega \frac{B_n}{A_n}\left|\nabla \cdot \left(\frac{b}{a}\nabla\psi_n\right) - \frac{C_n}{a}\psi_n\right|^2 \leq K\left(1 + t + \int_t^T \int_\Omega |\nabla\psi_n|^2\right),$$

where K is independent of n but contains various constants. From here, we use Grönwall's inequality to get the second bound. The third bound is obtained when using the equation above and the second bound. □

With these estimates in hand, we proceed with the uniqueness proof. Combining (3.11) and (RDP),

$$\begin{aligned}
\iint_{\Omega_T}(u_1 - u_2)G &= (u_1 - u_2)\left(\partial_t\psi_n + \frac{B_n}{A_n}\nabla\cdot\left(\frac{b}{a}\nabla\psi_n\right) + \Phi_{1,n}\frac{\psi_n}{a} - C_n\frac{B_n}{A_n}\frac{\psi_n}{a}\right) \\
&\quad - Z\left(A\partial_t\psi_n + B\nabla\cdot\left(\frac{b}{a}\nabla\psi_n\right) + A\Phi(p_1)\frac{\psi_n}{a} - CB\frac{\psi_n}{a}\right) \\
&= I_n^1 - I_n^2 - I_n^3 + I_n^4,
\end{aligned}$$

where

$$\begin{aligned}
I_n^1 &= \iint_{\Omega_T} Z\frac{B_n}{A_n}(A - A_n)\left(\nabla\cdot\left(\frac{b}{a}\nabla\psi_n\right) - \frac{C_n}{a}\psi_n\right) \\
I_n^2 &= \iint_{\Omega_T} Z(B - B_n)\left(\nabla\cdot\left(\frac{b}{a}\nabla\psi_n\right) - \frac{C_n}{a}\psi_n\right) \\
I_n^3 &= \iint_{\Omega_T} \frac{u_1 - u_2}{a}(\Phi(p_1) - \Phi_{1,n})\psi_n \\
I_n^4 &= \iint_{\Omega_T} ZB\left(\frac{C}{a} - \frac{C_n}{a}\right)\psi_n.
\end{aligned}$$

Our goal is to show that $I_n^i \to 0$ as $n \to 0$ for $i = 1, 2, 3, 4$. Now we bound the integrals above by using Hölder's inequality, bounds of the coefficients (namely $(A_n/B_n)^{1/2}, (B_n/A_n)^{1/2} \leq n^{1/2}$), and the convergence of the coefficients,

$$\begin{aligned}
|I_n^1| &\leq K \iint_{\Omega_T} \frac{B_n}{A_n}|A - A_n| \left| \nabla \cdot \left(\frac{b}{a} \nabla \psi_n \right) - \frac{C_n}{a}\psi_n \right| \leq \frac{K}{n^{1/2}}, \\
|I_n^2| &\leq K \iint_{\Omega_T} |B - B_n| \left| \nabla \cdot \left(\frac{b}{a} \nabla \psi_n \right) - \frac{C_n}{a}\psi_n \right| \\
&\leq K \|(A_n/B_n)^{1/2}(B - B_n)\|_{L^2(\Omega_T)} \\
&\leq \frac{K}{n^{1/2}}, \\
|I_n^3| &\leq \iint_{\Omega_T} \frac{|u_1 - u_2|}{a} |\Phi(p_1) - \Phi_{1,n}||\psi_n| \\
&\leq K \iint_{\Omega_T} |u_1 - u_2||\Phi(p_1) - \Phi_{1,n}| \\
&\leq K \|\Phi(p_1) - \Phi_{1,n}\|_{L^2(\Omega_T)} \\
&\leq \frac{K}{n}, \\
|I_n^4| &\leq K \iint_{\Omega_T} |C_n - C||\psi_n| \\
&\leq K \|C_n - C\|_{L^2(\Omega_T)} \\
&\leq \frac{K}{n},
\end{aligned}$$

where $K = K(a, b, T, G, |\Omega_T|)$.

For uniqueness of the pressure, using uniqueness of the density ($u_1 = u_2$) and defining $\psi := p_1 - p_2$,

$$\begin{aligned}
0 &= \iint_{\Omega_T} (u_1 - u_2)\partial_t \psi + (p_1 - p_2)\nabla \cdot \left(\frac{b}{a} \nabla \psi \right) + \frac{\psi}{a}(u_1 \Phi(p_1) - u_2 \Phi(p_2)) \\
&= - \iint_{\Omega_T} \frac{b}{a} |\nabla (p_1 - p_2)|^2 + \iint_{\Omega_T} \frac{p_1 - p_2}{a}(u_1 \Phi(p_1) - u_1 \Phi(p_2)).
\end{aligned}$$

Recalling that Φ strictly decreasing in p implies $\operatorname{sgn}(p_1 - p_2) = -\operatorname{sgn}(\Phi(p_1) - \Phi(p_2))$, the density is non-negative, and a, b are strictly positive,

$$\begin{aligned}0 &\le \iint_{\Omega_T} \frac{b}{a}|\nabla(p_1 - p_2)|^2 \\ &= \iint_{\Omega_T} \frac{u_1}{a}(p_1 - p_2)(\Phi(p_1) - \Phi(p_2)) \\ &\le 0.\end{aligned}$$

Given that the pressures have the same initial condition, we achieve the uniqueness of the pressure. Now that we have uniqueness for the limiting solution, we get a comparison principle.

Corollary 3.6.2 (Comparison principle)**.** *Suppose Assumptions 3.2.1, 3.2.2, and 3.2.3 hold. Let* (u_∞, p_∞) *be the limit solution of* (3.1) *in* Q_T. *Let* (u_1, p_1) *be a weak solution of* (3.1) *in* $\Omega \times [t_1, t_2]$. *If* $p_\infty \le p_1$ *on* $\partial\Omega \times [t_1, t_2]$ *and* $u_\infty \le u_1$ *when* $t = t_1$, *then* $p_\infty \le p_1$ *and* $u_\infty \le u_1$ *in* $\Omega \times [t_1, t_2]$.

3.7 Complementarity Condition

By Lemma 3.4.2, let Ω be a compact domain containing the support of p_m for almost every time $t \in [0, T]$ and $\Omega_T = \Omega \times (0, T)$. As shown in [23], the complementarity condition is equivalent to strong L^2 convergence of the pressure gradient. We use ideas in [18]. In particular, an L^3 AB-estimate is obtained to get space compactness, instead of the classical L^∞ AB-estimate. The L^3 bound of the pressure gradient gives enough compactness needed to pass the limit. Indeed, to get the L^3 AB-estimate we first require the L^3 bound for the pressure gradient.

3.7.1 L^3 bound for ∇p_m

Proposition 3.7.1 (L^3 bound for ∇p_m)**.** *Suppose Assumptions 3.2.1, 3.2.2, and 3.2.3 hold. For* $T > 0$ *and* $m > 4$, *there exists* $C > 0$, *independent of m, such that*

$$\iint_{\Omega_T} p_m(\Delta p_m + \Phi(x, t, p_m))^2 \le C(T, a, b, p_M, \Phi)$$

and

$$\iint_{\Omega_T} |\nabla p_m|^3 \le C(T, a, b, p_M, \Phi).$$

Proof. By integration by parts, Young's inequality, and Lemma 3.4.4,

$$\begin{aligned}\iint_{\Omega_T} |\nabla p_m|^3 &= -2 \iint_{\Omega_T} p_m \Delta p_m |\nabla p_m| \\ &\leq \iint_{\Omega_T} p_m^2 |\Delta p_m|^2 + \iint_{\Omega_T} |\nabla p_m|^2 \\ &\leq p_M \iint_{\Omega_T} p_m |\Delta p_m|^2 + C.\end{aligned}$$

It is sufficient to show that the right hand-side is bounded. By triangle inequality, it is enough to show that $\iint_{\Omega_T} p_m(\Delta p_m + \Phi(x,t,p_m))^2$ is controlled. Multiplying the pressure equation (3.4) by $-(\Delta p_m + \Phi(x,t,p_m))$, and integrating in space and time,

$$\begin{aligned}&\int_0^T \frac{d}{dt} \int_\Omega \frac{|\nabla p_m|^2}{2} + (m-1) \iint_{\Omega_T} \frac{p_m}{a} (\Delta p_m + \Phi(x,t,p_m))^2 \\ &\quad + I_1 + I_2 + I_3 + I_4 = -\iint_{\Omega_T} \frac{|\nabla p_m|^2}{a} \Phi(x,t,p_m)\end{aligned}$$

where

$$\begin{aligned}I_1 &= -(m-1) \iint_{\Omega_T} p_m (\Delta p_m + \Phi(x,t,p_m)) \partial_t \log(b) \\ I_2 &= (m-1) \iint_{\Omega_T} \frac{p_m}{a} (\Delta p_m + \Phi(x,t,p_m)) \left(\nabla p_m \cdot \nabla \log\left(\frac{b}{a}\right)\right) \\ I_3 &= -\iint_{\Omega_T} \partial_t p_m \Phi(x,t,p_m) \\ I_4 &= \iint_{\Omega_T} \Delta p_m \frac{|\nabla p_m|^2}{a}.\end{aligned}$$

Integrating by parts,

$$\begin{aligned}I_4 &= \iint_{\Omega_T} p_m \left(\frac{\Delta |\nabla p_m|^2}{a} + |\nabla p_m|^2 \Delta\left(\frac{1}{a}\right) + 2\nabla\left(\frac{1}{a}\right) \cdot \nabla |\nabla p_m|^2 \right) \\ &= I_{4,1} + I_{4,2} + I_{4,3}.\end{aligned}$$

Using integration by parts

$$\begin{aligned}
I_{4,1} &= 2\iint_{\Omega_T} \frac{p_m}{a} \sum_{i,j} (\partial^2_{i,j} p_m)^2 + 2\iint_{\Omega_T} \frac{p_m}{a} \nabla p_m \nabla \Delta p_m \\
&= 2\iint_{\Omega_T} \frac{p_m}{a} \sum_{i,j} (\partial^2_{i,j} p_m)^2 - 2\iint_{\Omega_T} \frac{p_m}{a} |\Delta p_m|^2 - 2\iint_{\Omega_T} \frac{\Delta p_m}{a} |\nabla p_m|^2 \\
&\quad - 2\iint_{\Omega_T} p_m \Delta p_m \nabla p_m \cdot \nabla \left(\frac{1}{a}\right) \\
&= 2\iint_{\Omega_T} \frac{p_m}{a} \sum_{i,j} (\partial^2_{i,j} p_m)^2 - 2\iint_{\Omega_T} \frac{p_m}{a} |\Delta p_m|^2 - 2I_4 - 2\iint_{\Omega_T} p_m \Delta p_m \nabla p_m \cdot \nabla \left(\frac{1}{a}\right).
\end{aligned}$$

Thus combining the I_4 terms,

$$\begin{aligned}
I_4 = &-\frac{2}{3}\iint_{\Omega_T} \frac{p_m}{a} |\Delta p_m|^2 + \frac{2}{3}\iint_{\Omega_T} \frac{p_m}{a} \sum_{i,j} (\partial^2_{i,j} p_m)^2 - \frac{2}{3}\iint_{\Omega_T} p_m \Delta p_m \nabla p_m \cdot \nabla \left(\frac{1}{a}\right) \\
&+ \frac{1}{3} I_{4,2} + \frac{1}{3} I_{4,3}.
\end{aligned}$$

So,

$$\begin{aligned}
&\int_0^T \frac{d}{dt} \int_\Omega \frac{|\nabla p_m|^2}{2} + (m-1)\iint_{\Omega_T} \frac{p_m}{a} (\Delta p_m + \Phi(x,t,p_m))^2 \\
&\quad + I_1 + I_2 + I_3 - \frac{2}{3}\iint_{\Omega_T} \frac{p_m}{a} |\Delta p_m|^2 + \frac{2}{3}\iint_{\Omega_T} \sum_{i,j} \frac{p_m}{a} (\partial^2_{i,j} p_m)^2 \\
&\le -\iint_{\Omega_T} \frac{|\nabla p_m|^2}{a} \Phi(x,t,p_m) - \frac{1}{3} I_{4,2} - \frac{1}{3} I_{4,3} + \frac{2}{3}\iint_{\Omega_T} p_m \Delta p_m \nabla p_m \cdot \nabla \left(\frac{1}{a}\right).
\end{aligned}$$

Lemmas 3.4.2 and 3.4.4 imply

$$-\iint_{\Omega_T} \frac{|\nabla p_m|^2}{a} \Phi(x,t,p_m) - \frac{1}{3} I_{4,2} \le \left(\|\frac{\phi}{a}\|_{L^\infty} + C \|\Delta(1/a)\|_{L^\infty} p_M \right) \iint_{\Omega_T} |\nabla p_m|^2 \le C.$$

Young's inequality and Lemma 3.4.4 imply,

$$\begin{aligned}
\frac{2}{3}\iint_{\Omega_T} p_m \Delta p_m \nabla p_m \cdot \nabla \left(\frac{1}{a}\right) &\le \frac{1}{3}\iint_{\Omega_T} \frac{p_m}{a} |\Delta p_m|^2 + C\iint_{\Omega_T} \frac{p_m}{a} |\nabla p_m|^2 \frac{|\nabla a|^2}{a^2} \\
&\le \frac{1}{3}\iint_{\Omega_T} \frac{p_m}{a} |\Delta p_m|^2 + C.
\end{aligned}$$

Calculating we have,

$$I_{4,3} = -4\iint_{\Omega_T} \frac{p_m}{a} \sum_{i,j} (\partial^2_{i,j} p_m) \frac{\nabla p_m \cdot \nabla a}{a}$$

and

$$\frac{-1}{3} I_{4,3} \leq \frac{1}{3} \iint_{\Omega_T} \frac{p_m}{a} \sum_{i,j} (\partial^2_{i,j} p_m)^2 + C \iint_{\Omega_T} \frac{p_m}{a} |\nabla p_m|^2 \frac{|\nabla a|^2}{a^2}$$
$$\leq \frac{1}{3} \iint_{\Omega_T} \frac{p_m}{a} \sum_{i,j} (\partial^2_{i,j} p_m)^2 + C.$$

Combining like terms,

$$\int_0^T \frac{d}{dt} \int_\Omega \frac{|\nabla p_m|^2}{2} + (m-1) \iint_{\Omega_T} \frac{p_m}{a} (\Delta p_m + \Phi(x,t,p_m))^2$$
$$+ I_1 + I_2 + I_3 - \iint_{\Omega_T} \frac{p_m}{a} |\Delta p_m|^2 + \frac{1}{3} \iint_{\Omega_T} \sum_{i,j} \frac{p_m}{a} (\partial^2_{i,j} p_m)^2 \leq C.$$

Now using Young's inequality in each of $-I_1$ and $-I_2$, then adding, and finally using Assumption 3.2.1 and Lemma 3.4.4 yields,

$$-I_1 - I_2 \leq \frac{(m-1)}{2} \iint_{\Omega_T} \frac{p_m}{a} (\Delta p_m + \Phi(x,t,p_m))^2$$
$$+ (m-1)C \iint_{\Omega_T} \frac{p_m}{a} a^2 |\partial_t \log(b)|^2 + (m-1)C \iint_{\Omega_T} \frac{p_m}{a} |\nabla p_m|^2 |\nabla \log(b/a)|^2$$
$$\leq \frac{(m-1)}{2} \iint_{\Omega_T} \frac{p_m}{a} (\Delta p_m + \Phi(x,t,p_m))^2 + (m-1)C.$$

Thus,

$$\int_0^T \frac{d}{dt} \int_\Omega \frac{|\nabla p_m|^2}{2} + I_3 + I_5 \leq (m-1)C,$$

where

$$I_5 = \frac{(m-1)}{2} \iint_{\Omega_T} \frac{p_m}{a} (\Delta p_m + \Phi(x,t,p_m))^2 - \iint_{\Omega_T} \frac{p_m}{a} |\Delta p_m|^2 + \frac{1}{3} \iint_{\Omega_T} \frac{p_m}{a} \sum_{i,j} (\partial^2_{i,j} p_m)^2.$$

Using the fact that $\Phi(x,t,p) \geq 0$ yields,

$$I_5 \geq \frac{(m-3)}{2} \iint_{\Omega_T} \frac{p_m}{a} (\Delta p_m + \Phi(x,t,p_m))^2 + \iint_{\Omega_T} \frac{p_m}{a} (\Delta p_m)^2 - \iint_{\Omega_T} \frac{p_m}{a} |\Delta p_m|^2$$
$$= \frac{(m-3)}{2} \iint_{\Omega_T} \frac{p_m}{a} (\Delta p_m + \Phi(x,t,p_m))^2.$$

Define $\Psi(x,t,p_m) = \int_0^{p_m} \Phi(x,t,q)\, dq$. Then, $\partial_t \Psi = \Phi(x,t,p_m) \partial_t p_m + \int_0^{p_m} \Phi_t(x,t,q)\, dq$. Given that

$$\int_0^{p_m} \Phi_t(x,t,q)\, dq \leq \|\Phi_t\|_{L^\infty} p_M,$$

then

$$I_3 \geq -\int_0^T \frac{d}{dt}\int_\Omega \Psi - C.$$

Therefore,

$$\int_0^T \frac{d}{dt}\int_\Omega \frac{|\nabla p_m|^2}{2} - \int_{\Omega(T)} \Psi(x,T,p_m) + \frac{(m-3)}{2}\iint_{\Omega_T} \frac{p_m}{a}(\Delta p_m + \Phi(x,t,p_m))^2 \leq (m-1)C.$$

Moreover by using previous bounds and Assumption 3.2.2,

$$\frac{(m-3)}{2}\iint_{\Omega_T} \frac{p_m}{a}(\Delta p_m + \Phi(x,t,p_m))^2 \leq (m-1)C.$$

Therefore we get,

$$\iint_{\Omega_T} \frac{p_m}{a}(\Delta p_m + \Phi(x,t,p_m))^2 \leq C,$$

which implies that

$$\iint_{\Omega_T} p_m(\Delta p_m + \Phi(x,t,p_m))^2 \leq C.$$

This finishes the proof. □

Remark 3.7.2. Integrals I_1, I_2 are new relative to [17, Theorem 3.2]. They do interfere with achieving the L^4 bound by introducing an m on the right hand-side. This prevents us from bounding the second derivative (Hessian) of the pressure by a constant independent of m. One may be able to achieve the L^4 bound in this setting by rearranging the diffusion portion as (3.13). Thus, we settle for the L^3 bound, which is sufficient. The most interesting terms are I_3 and I_4. Both appear in [17, Theorem 3.2] where Φ only depends on the pressure and $a \equiv 1$. Due to our generalization of these terms, bounding each becomes a little more difficult due to having more sub-integrals (such as $I_{4,1}, I_{4,2}, I_{4,3}$), however; the strategy is similar.

3.7.2 L^3 Aronson-Bénilan Estimate

Proposition 3.7.3 (L^3 AB-Estimate)**.** *Suppose Assumptions 3.2.1, 3.2.2, and 3.2.3 hold. For* $T > 0$ *and* $m > \max\{2, 5 - \frac{4}{d}\}$*, there exists* $C > 0$*, independent of* m*, such that*

$$\iint_{\Omega_T} (\Delta p_m + \Phi(x,t,p_m))_-^3 \leq C(T, a, b, p_M, \Phi, |\Omega_T|)$$

and

$$\iint_{\Omega_T} |\Delta p_m| \leq C(T, a, b, p_M, \Phi, |\Omega_T|).$$

Proof. For sake of simplicity, we drop the subscript m and denote $\Phi(x,t,p)$ as Φ. Define $w = \Delta p + \Phi$. Taking the time derivative of w,

$$\partial_t w = \Delta \partial_t p + \Phi_t + \Phi_p \partial_t p. \tag{3.12}$$

Given that we know (3.4), we compute

$$\begin{aligned}
\Delta \partial_t p &= \Delta\left(\frac{|\nabla p|^2}{a}\right) + (m-1)\Delta\left(\frac{p}{a}\left(\nabla p \cdot \nabla \log\left(\frac{b}{a}\right)\right)\right) + (m-1)\Delta\left(\frac{p}{a}w\right) \\
&\quad - (m-1)\Delta\left(p\partial_t \log(b)\right) \\
&= D_1 + (m-1)(D_2 + D_3 + D_4).
\end{aligned}$$

For ease, we will denote $\alpha = \frac{\nabla \log(b/a)}{a}$ and $\beta = \partial_t \log(b)$. Computing D_1,

$$\begin{aligned}
D_1 &= \frac{1}{a}\Delta|\nabla p|^2 + 2\nabla|\nabla p|^2 \cdot \nabla\left(\frac{1}{a}\right) + |\nabla p|^2 \Delta\left(\frac{1}{a}\right) \\
&= D_{1,1} + D_{1,2} + D_{1,3}.
\end{aligned}$$

Using Young's inequality (with $\epsilon = 1/4$),

$$\begin{aligned}
D_{1,2} &= -4\sum_{i,j}(\partial_{i,j}^2 p)\frac{\nabla p \cdot \nabla a}{a^2} \\
&\geq -\frac{1}{a}\sum_{i,j}(\partial_{i,j}^2 p)^2 - 4\frac{|\nabla a|^2}{a^3}|\nabla p|^2.
\end{aligned}$$

and so

$$\begin{aligned}
D_{1,1} + D_{1,2} &\geq \frac{2}{a}\sum_{i,j}(\partial_{i,j}^2 p)^2 + \frac{2}{a}\nabla p \cdot \nabla \Delta p - \frac{1}{a}\sum_{i,j}(\partial_{i,j}^2 p)^2 - 4\frac{|\nabla a|^2}{a^3}|\nabla p|^2 \\
&\geq \frac{1}{ad}(\Delta p)^2 + \frac{2}{a}\nabla p \cdot \nabla \Delta p - 4\frac{|\nabla a|^2}{a^3}|\nabla p|^2 \\
&= \frac{1}{ad}(w - \Phi)^2 + \frac{2}{a}\nabla p \cdot \nabla w - \Phi_{x_i} - \Phi_p \nabla p - 4\frac{|\nabla a|^2}{a^3}|\nabla p|^2.
\end{aligned}$$

Going back to D_4,

$$\begin{aligned} D_4 &= \Delta p \beta + p \Delta \beta + 2\nabla p \cdot \nabla \beta \\ &= (w - \Phi)\beta + p\Delta\beta + 2\nabla p \cdot \nabla \beta. \end{aligned}$$

Now computing D_2,

$$\begin{aligned} D_2 &= \Delta(p\nabla p \cdot \alpha) \\ &= \Delta p \nabla p \cdot \alpha + p\nabla \Delta p \cdot \alpha + p \nabla p \Delta(\alpha) + 2\nabla p \Delta p \alpha \\ &\quad + 2p\Delta p \nabla(\alpha) + 2|\nabla p|^2 \nabla(\alpha) \\ &= 3\Delta p \nabla p \cdot \alpha + 2p\Delta p \nabla(\alpha) + p\nabla\Delta p \cdot \alpha + p\nabla p \Delta(\alpha) + 2|\nabla p|^2 \nabla(\alpha) \\ &= 3(w-\Phi)\nabla p \cdot \alpha + 2(w - \Phi) p \nabla(\alpha) \\ &\quad + (\nabla w - \Phi_{x_i} - \Phi_p \nabla p) p \alpha + p \nabla p \Delta(\alpha) + 2|\nabla p|^2 \nabla(\alpha) \\ &= D_{2,1} + D_{2,2} + D_{2,3} + D_{2,4} + D_{2,5}. \end{aligned}$$

Multiplying (3.12) by $-(w)_-$,

$$\frac{1}{2}\partial_t (w)_-^2 = -\partial_t w (w)_- = -\Delta \partial_t p (w)_- - \Phi_t (w)_- - \Phi_p \partial_t p (w)_-.$$

Focusing on the last term,

$$\begin{aligned} -\Phi_p \partial_t p (w)_- &= -\Phi_p (w)_- \frac{|\nabla p|^2}{a} \\ &\quad - (m-1)\left(\Phi_p \frac{p}{a}(w)_- \nabla p \cdot \nabla \log(b/a) - \Phi_p \frac{p}{a}(w)_-^2 - \Phi_p p (w)_- \beta\right). \end{aligned}$$

Updating $-\Delta\partial_t p(w)_-$,

$$
\begin{aligned}
-D_{1,1}(w)_- - D_{1,2}(w)_- &\leq \frac{-1}{da}(w)_-^3 - \frac{2}{da}(w)_-^2\Phi - \frac{1}{da}(w)_-\Phi^2 + \frac{1}{a}\nabla p \cdot \nabla (w)_-^2 \\
&\quad + \frac{2}{a}(\nabla p(w)_-\Phi_{x_i} + \Phi_p|\nabla p|^2(w)_-) + 4\frac{|\nabla a|^2}{a^3}|\nabla p|^2(w)_-, \\
-D_{1,3}(w)_- &= -(w)_-|\nabla p|^2\Delta\left(\frac{1}{a}\right), \\
-D_{2,1}(w)_- &= 3((w)_-^2 + \Phi(w)_-)\nabla p \cdot \alpha, \\
-D_{2,2}(w)_- &= 2((w)_-^2 + \Phi(w)_-)p\nabla(\alpha), \\
-D_{2,3}(w)_- &= \frac{1}{2}\nabla(w)_-^2 p\alpha + (\Phi_{x_i}(w)_- + \Phi_p\nabla p(w)_-)p\alpha, \\
-D_{2,4}(w)_- &= -p\nabla p\Delta(\alpha)(w)_-, \\
-D_{2,5}(w)_- &= -2|\nabla p|^2\nabla(\alpha)(w)_-, \\
-D_3(w)_- &= \Delta\left(\frac{p}{a}(w)_-\right)(w)_-, \\
-D_4(w)_- &= ((w)_-^2 + \Phi(w)_-)\beta - (w)_- p\Delta\beta - 2(w)_-\nabla p \cdot \nabla\beta.
\end{aligned}
$$

Integrating in space and time,

$$
\int_0^T \frac{d}{dt}\int_\Omega \frac{(w)_-^2}{2} \leq I_1 + I_2 + I_3 + I_4 + I_5 + I_6,
$$

where

$$
\begin{aligned}
I_1 &= \frac{-1}{d}\left(\iint_{\Omega_T} \frac{(w)_-^3}{a} + 2\frac{(w)_-^2}{a}\Phi + \frac{(w)_-}{a}\Phi^2\right),\\
I_2 &= \iint_{\Omega_T} \frac{1}{a}\nabla p\cdot\nabla(w)_-^2 + (m-1)\iint_{\Omega_T} \Delta\left(\frac{p}{a}(w)_-\right)(w)_- + (m-1)\iint_{\Omega_T} \frac{\nabla(w)_-^2}{2}p\alpha,\\
I_3 &= \iint_{\Omega_T} \left(\frac{\Phi_p}{a} - \Delta\left(\frac{1}{a}\right) + 4\frac{|\nabla a|^2}{a^3}\right)|\nabla p|^2(w)_- - (m-1)\iint_{\Omega_T} 2\nabla(\alpha)\,|\nabla p|^2(w)_-,\\
I_4 &= 3(m-1)\iint_{\Omega_T} \nabla p(w)_-^2\alpha,\\
I_5 &= \iint_{\Omega_T} \nabla p(w)_-\frac{2}{a}\Phi_{x_i}\\
&\quad + (m-1)\iint_{\Omega_T} \nabla p(w)_-\left(3\Phi\alpha - p\Delta(\alpha) - 2\nabla\beta\right),\\
I_6 &= -\iint_{\Omega_T} \Phi_t(w)_- + (m-1)\iint_{\Omega_T} 2((w)_-^2 + \Phi(w)_-)p\nabla(\alpha) + \Phi_{x_i}(w)_-p\alpha\\
&\quad + (m-1)\iint_{\Omega_T} ((w)_-^2 + \Phi(w)_-)\beta - (w)_-p\Delta\beta\\
&\quad + (m-1)\iint_{\Omega_T} \Phi_p\frac{p}{a}(w)_-^2 + \Phi_p p(w)_-\beta
\end{aligned}
$$

We start with the most interesting integral $I_2 = I_{2,1} + I_{2,2} + I_{2,3}$. Using integration by parts for each integral,

$$
\begin{aligned}
I_{2,1} &= -\iint_{\Omega_T} \frac{\Delta p}{a}(w)_-^2 + \nabla p\cdot\nabla(1/a)(w)_-^2\\
&= \iint_{\Omega_T} \frac{(w)_-^3}{a} + \iint_{\Omega_T} \frac{(w)_-^2}{a}\Phi - \iint_{\Omega_T} \nabla p\cdot\nabla(1/a)(w)_-^2,\\
I_{2,2} &= -(m-1)\iint_{\Omega_T} \nabla\left(\frac{p}{a}(w)_-\right)\cdot\nabla(w)_-\\
&= -(m-1)\iint_{\Omega_T} \nabla\left(\frac{p}{2a}\right)\cdot\nabla(w)_-^2 + \frac{p}{a}|\nabla(w)_-|^2\\
&= \frac{(m-1)}{2}\iint_{\Omega_T} \Delta\left(\frac{p}{a}\right)(w)_-^2 - \frac{2p}{a}|\nabla(w)_-|^2\\
&\le \frac{(m-1)}{2}\iint_{\Omega_T} (w)_-^2\left(\frac{\Delta p}{a} + p\Delta\left(\frac{1}{a}\right) + 2\nabla p\cdot\nabla\left(\frac{1}{a}\right)\right)\\
&= \frac{(1-m)}{2}\iint_{\Omega_T} \frac{(w)_-^3}{a} + \frac{(w)_-^2}{a}\Phi + \frac{(m-1)}{2}\iint_{\Omega_T} p\Delta\left(\frac{1}{a}\right)(w)_-^2 + 2\nabla p\cdot\nabla\left(\frac{1}{a}\right)(w)_-^2,\\
I_3 &= \frac{(1-m)}{2}\iint_{\Omega_T} (w)_-^2\nabla p\cdot\alpha + (w)_-^2 p\nabla(\alpha)\,.
\end{aligned}
$$

Combining the sub-integrals above,

$$I_2 \le \frac{(3-m)}{2} \iint_{\Omega_T} \frac{(w)_-^3}{a} + \frac{(w)_-^2}{a}\Phi + (m-2) \iint_{\Omega_T} \nabla p \cdot \nabla \left(\frac{1}{a}\right)(w)_-^2$$
$$+ \frac{(m-1)}{2} \iint_{\Omega_T} p\Delta \left(\frac{1}{a}\right)(w)_-^2 - (w)_-^2 \nabla p \cdot \alpha - (w)_-^2 p \nabla (\alpha).$$

Define

$$J_1 = \frac{(3-m)}{2} \iint_{\Omega_T} \frac{(w)_-^3}{a} + \frac{(w)_-^2}{a}\Phi,$$
$$J_2 = (m-2) \iint_{\Omega_T} \nabla p \cdot \nabla \left(\frac{1}{a}\right)(w)_-^2 - \frac{(m-1)}{2} \iint_{\Omega_T} (w)_-^2 \nabla p \cdot \alpha,$$
$$J_3 = \frac{(m-1)}{2} \iint_{\Omega_T} p\Delta \left(\frac{1}{a}\right)(w)_-^2 - \frac{(m-1)}{2} \iint_{\Omega_T} (w)_-^2 p \nabla (\alpha).$$

Combing I_1 and J_1,

$$I_1 + J_1 = \left(\frac{(3-m)}{2} - \frac{1}{d}\right) \iint_{\Omega_T} \frac{(w)_-^3}{a} + \left(\frac{(3-m)}{2} - \frac{2}{d}\right) \iint_{\Omega_T} \frac{(w)_-^2}{a}\Phi$$
$$- \frac{1}{d} \iint_{\Omega_T} \frac{(w)_-}{a}\Phi^2$$
$$\le \left(\frac{(3-m)}{2} - \frac{1}{d}\right) \iint_{\Omega_T} \frac{(w)_-^3}{a}.$$

By Young's inequality with ϵ and Proposition 3.7.1,

$$J_2 \le \left((m-2) + \frac{(m-1)}{2}\right)\epsilon \iint_{\Omega_T} \frac{(w)_-^3}{a} + \left((m-2) + \frac{(m-1)}{2}\right) C \iint_{\Omega_T} |\nabla p|^3$$
$$\le \frac{(3m-5)\epsilon}{2} \iint_{\Omega_T} \frac{(w)_-^3}{a} + \frac{(3m-5)}{2} C.$$

This type of bound is similar to what we will do for I_3 and I_4. By previous L^∞ bounds,

$$J_3 \le \frac{(m-1)}{2} C \iint_{\Omega_T} \frac{(w)_-^2}{a}.$$

This type of bound is similar to what we will do for I_6. By Young's inequality with ϵ and Proposition 3.7.1,

$$I_3 \le m\epsilon \iint_{\Omega_T} \frac{(w)_-^3}{a} + mC \iint_{\Omega_T} |\nabla p|^3$$
$$\le m\epsilon \iint_{\Omega_T} \frac{(w)_-^3}{a} + mC.$$

Again using Young's inequality with ϵ and Proposition 3.7.1,

$$I_4 \leq 3(m-1)\epsilon \iint_{\Omega_T} \frac{(w)_-^3}{a} + (m-1)C \iint_{\Omega_T} |\nabla p|^3$$
$$\leq 3(m-1)\epsilon \iint_{\Omega_T} \frac{(w)_-^3}{a} + (m-1)C.$$

Using Cauchy's inequality,

$$I_5 \leq m \iint_{\Omega_T} \frac{(w)_-^2}{a} + mC \iint_{\Omega_T} |\nabla p|^2$$
$$\leq m \iint_{\Omega_T} \frac{(w)_-^2}{a} + mC.$$

Using various L^∞ bounds,

$$I_6 \leq mC \iint_{\Omega_T} \frac{(w)_-^2}{a} + mC \iint_{\Omega_T} \frac{(w)_-}{a} + mC.$$

Collecting the bounds,

$$\int_0^T \frac{d}{dt} \int_\Omega \frac{(w)_-^2}{2} \leq \left(\frac{(3-m)}{2} - \frac{1}{d} + \frac{(3m-5)\epsilon}{2} + m\epsilon \right) \iint_{\Omega_T} \frac{(w)_-^3}{a}$$
$$+ mC \iint_{\Omega_T} \frac{(w)_-^2}{a} + mC \iint_{\Omega_T} \frac{(w)_-}{a} + mC.$$

Rearranging, choosing $\epsilon = 1/10$, using Assumption 3.2.2 and Hölder's inequality,

$$\left(\frac{(m-5)}{4} + \frac{1}{d} \right) \iint_{\Omega_T} \frac{(w)_-^3}{a} \leq mC \iint_{\Omega_T} \frac{(w)_-^2}{a} + mC \iint_{\Omega_T} \frac{(w)_-}{a} + mC + \int_\Omega \frac{(w^0)_-^2}{2}$$
$$\leq mC \left(\iint_{\Omega_T} \frac{(w)_-^3}{a} \right)^{2/3} + mC \left(\iint_{\Omega_T} \frac{(w)_-^3}{a} \right)^{1/3} + mC$$

Given the hypothesis on m,

$$\iint_{\Omega_T} \frac{(w)_-^3}{a} \leq C.$$

Moreover,

$$\iint_{\Omega_T} (w)_-^3 \leq C.$$

This gives us the first bound. For the second, using integration by parts,

$$\iint_{\Omega_T} (\Delta p + \Phi) \leq C.$$

So,

$$\begin{aligned}\iint_{\Omega_T} |\Delta p + \Phi| &= \iint_{\Omega_T} (\Delta p + \Phi) + 2 \iint_{\Omega_T} (w)_- \\ &\leq C + C \left(\iint_{\Omega_T} (w)_-^3 \right)^{1/3} \\ &\leq C.\end{aligned}$$

As Φ is bounded,

$$\begin{aligned}\iint_{\Omega_T} |\Delta p| &\leq \iint_{\Omega_T} |\Delta p + \Phi| + \iint_{\Omega_T} |\Phi| \\ &\leq C.\end{aligned}$$

This completes the proof. □

Remark 3.7.4. Compared to [17, Theorem 3.1], we have approximately double the number of terms because a, b are non-constant functions of space and time. The integrals I_3, I_4 are completely new and I_5, I_6 are mostly new. Integrals I_5, I_6 do not present an issue, however; integrals I_3, I_4 require us to have an L^3 bound on the gradient of the pressure. The term I_2 is similar to one in [17, Theorem 3.1], but is more extensive. Though a similar strategy is used here, we require more. In particular, as $a \not\equiv 1$, we require L^3 bound on the gradient of the pressure for I_2 as well.

3.7.3 Complementarity Condition

Proposition 3.7.5 (Strong convergence of the pressure gradient)**.** *Suppose Assumptions 3.2.1, 3.2.2, and 3.2.3 hold. For $T > 0$, $\nabla p_m \to \nabla p_\infty$ strongly in $L^2(Q_T)$.*

Proof. We first start by showing spatial compactness of the pressure gradient. For $\epsilon > 0$, define the continuous function,

$$\psi(s) = \begin{cases} -\epsilon, & \text{for } s < -\epsilon, \\ s, & \text{for } -\epsilon \leq s \leq \epsilon, \\ \epsilon, & \text{for } s > \epsilon. \end{cases}$$

Using integration by parts for $m, n > 1$,

$$\iint_{\Omega_T} |\nabla p_m - \nabla p_n|^2 \psi'(p_m - p_n)\, dx\, dt = -\iint_{\Omega_T} (\Delta p_m - \Delta p_n)\psi(p_m - p_n)\, dx\, dt.$$

Defining the domain,

$$\Omega_{T,\epsilon} = \{(x,t) \in \Omega_T : |p_m(x,t) - p_n(x,t)| \leq \epsilon\},$$

and using Proposition 3.7.3,

$$\iint_{\Omega_{T,\epsilon}} |\nabla p_m - \nabla p_n|^2 \leq C\epsilon.$$

Thus we can use H older's inequality so that

$$\begin{aligned}\iint_{\Omega_T} |\nabla p_m - \nabla p_n| &= \iint_{\Omega_{T,\epsilon}} |\nabla p_m - \nabla p_n| + \iint_{\Omega_{T,\epsilon}^c} |\nabla p_m - \nabla p_n| \\ &\leq C\epsilon^{1/2} + C\|p_m\|_{L^2(Q_T)} |\Omega_{T,\epsilon}^c|^{1/2}.\end{aligned}$$

As p_m has compact support, Lemma 3.4.2, p_m is Cauchy and there exists $N(\epsilon)$ large enough such that for $m, n > N(\epsilon)$,

$$\iint_{\Omega_T} |\nabla p_m - \nabla p_n| \leq C\epsilon^{1/2} + C\epsilon.$$

This implies that ∇p_m is Cauchy in $L^1(Q_T)$ and thus, up to a subsequence, we have almost everywhere convergence. By Proposition 3.7.1, we have, possibly up to a subsequence, ∇p_m weakly converges to ∇p_∞ in $L^3(Q_T)$. So, the pressure gradient is compact in space for any $L^q(Q_T)$ for $1 \leq q < 3$.

Now we move on to time compactness. Let $\phi_\alpha = \alpha^{-d}\phi(x/\alpha)$, with $\alpha > 0$ be a nonnegative, smooth mollifier where $\int_{\mathbb{R}^d} \phi = 1$. We will comput the time shift of ∇p_m with $h > 0$. In particular, we add and subtract $\nabla p_m * \phi_\alpha$ at time $t + h$ and t and use the triangle inequality,

$$\begin{aligned}\int_0^{T-h} \|\nabla p_m(t+h) - \nabla p_m(t)\|_{L^1(\Omega)}\, dt \leq &\int_0^{T-h} \|\nabla p_m(t+h) - \nabla p_m(t+h) * \phi_\alpha\|_{L^1(\Omega)}\, dt \\ &+ \int_0^{T-h} \|(\nabla p_m(t+h) - \nabla p_m(t)) * \phi_\alpha\|_{L^1(\Omega)}\, dt \\ &+ \int_0^{T-h} \|\nabla p_m(t) * \phi_\alpha - \nabla p_m(t)\|_{L^1(\Omega)}\, dt.\end{aligned}$$

Let us focus on controlling the middle term first. Using integration by parts in space and Young's convolution inequality,

$$\begin{aligned}
&\int_0^{T-h} \|(\nabla p_m(t+h) - \nabla p_m(t)) * \phi_\alpha\|_{L^1(\Omega)}\, dt \\
= \quad & \int_0^{T-h} \left\| \int_0^h \partial_t p_m(t+S) * \nabla \phi_\alpha \, ds \right\|_{L^1(\Omega)} dt \\
& \leq \int_0^{T-h} \int_0^h \int_\Omega |\partial_t p_m(t+s) * \nabla \phi_\alpha| \; dx\, ds\, dt \\
& \leq \|\nabla \phi_\alpha\|_{L^1(\mathbb{R}^d)} \int_0^{T-h} \int_0^h \int_\Omega |\partial_t p_m(x, t+s)| \; dx\, ds\, dt
\end{aligned}$$

By Lemma 3.4.7,

$$\begin{aligned}
&\|\nabla \phi_\alpha\|_{L^1(\mathbb{R}^d)} \int_0^{T-h} \int_0^h \int_\Omega |\partial_t p_m(x, t+s)| \; dx\, ds\, dt \\
&= \|\nabla \phi_\alpha\|_{L^1(\mathbb{R}^d)} \int_0^{T-h} \int_t^{t+h} \int_\Omega |\partial_t p_m(x, s)| \; dx\, ds\, dt \\
&= \|\nabla \phi_\alpha\|_{L^1(\mathbb{R}^d)} \int_0^{T-h} \int_{\max(0,s-h)}^{\min(s,T-h)} \int_\Omega |\partial_t p_m(x, s)| \; dx\, dt\, ds \\
&\leq C \frac{|h|}{\alpha}.
\end{aligned}$$

Now we deal with the first (and third) term. By Fréchet-Kolmogorov Theorem, the space shifts of the pressure gradient converge as well. In particular, there exists a funtion $\omega : \mathbb{R} \to \mathbb{R}_{\geq 0}$, such that

$$\iint_{\Omega_T} |\nabla p_m(x+k, t) - \nabla p_n(x, t)| \leq \omega(|k|),$$

with $\omega(|k|) \to 0$ as $|k| \to 0$. So,

$$\begin{aligned}
&\int_0^{T-h} \|\nabla p_m(t) * \phi_\alpha - \nabla p_m(t)\|_{L^1(\Omega)}\, dt \\
&\leq \int_0^{T-h} \int_\Omega \left| \int_{\mathbb{R}^d} \phi(y)(\nabla p_m(x - \alpha y, t) - \nabla p_m(x, t)) \right| dy\, dx\, dt \\
&\leq \int_{\mathbb{R}^d} \phi(y) \int_{\Omega_T} |\nabla p_m(x - \alpha y, t) - \nabla p_m(x, t)| \; dx\, dt\, dy \\
&\leq \int_{\mathbb{R}^d} \phi(y) \omega(\alpha |y|) \, dy.
\end{aligned}$$

Combining the estimates,

$$\int_0^{T-h} \|\nabla p_m(t+h) - \nabla p_m(t)\|_{L^1(\Omega)}\, dt \leq C \frac{|h|}{\alpha} + 2 \int_{\mathbb{R}^d} \phi(y) \omega(\alpha |y|) \, dy.$$

Choosing $\alpha = |h|^{1/2}$ and taking $\alpha \to 0$,

$$\int_0^{T-h} \|\nabla p_m(t+h) - \nabla p_m(t)\|_{L^1(\Omega)}\, dt \to 0.$$

This gives us time compactness by Aubin-Lions lemma. Thus, $\nabla p_m \to \nabla p_\infty$ strongly in $L^q(Q_T)$ for $1 \leq q < 3$ and in particular for $q = 2$. □

Proof of Theorem 3.3.3. By Proposition 3.7.5, $\nabla p_m \to \nabla p_\infty$ in $L^2(Q_T)$. Integrating the PDE that the pressure satisfies (3.4) and rearranging gives us,

$$\begin{aligned}\frac{1}{m-1}\iint_{Q_T}\left(\partial_t p_m - \frac{|\nabla p_m|^2}{a}\right)\zeta &= \iint_{Q_T} p_m\frac{\zeta}{a}\left(\nabla p_m\cdot\nabla\log\left(\frac{b}{a}\right) + \Delta p_m\right)\\ &\quad + \iint_{Q_T} p_m\frac{\zeta}{a}\left(\Phi(x,t,p_m) - a\partial_t\log(b)\right).\end{aligned}$$

Integrating by parts,

$$\begin{aligned}&\frac{-1}{m-1}\iint_{Q_T} p_m\partial_t\zeta + \frac{|\nabla p_m|^2}{a}\zeta - \frac{1}{m-1}\int_{\mathbb{R}^d} p_m(x,0)\zeta(x,0)\\ &= \iint_{Q_T}\zeta\frac{p_m}{a}\nabla p_m\cdot\nabla\log\left(\frac{b}{a}\right) - \iint_{Q_T}|\nabla p_m|^2\frac{\zeta}{a} + \nabla p_m\cdot\nabla\left(\frac{\zeta}{a}\right)p_m\\ &\quad + \iint_{Q_T}\frac{p_m}{a}\zeta\Phi(x,t,p_m) - p_m\zeta\partial_t\log(b).\end{aligned}$$

Taking the limit as $m \to \infty$ gives the result. □

3.8 Velocity Law

Notice that

$$\frac{a}{b}\nabla\cdot\left(\frac{b}{a}\nabla p\right) = \nabla\log\left(\frac{b}{a}\right)\cdot\nabla p + \Delta p.$$

Thus, the complementarity condition in (3.6) can be rewritten as

$$-\nabla\cdot\left(\frac{b}{a}\nabla p_\infty\right) = \frac{b}{a}\Phi(x,t,p_\infty) - \partial_t b. \tag{3.13}$$

We will use this representation in the upcoming proposition.

3.8.1 Comparison with barriers

Let D be a ball in $\mathbb{R}^d$. For a time interval $[t_1, t_2] \subset [0, \infty)$, consider a function (that represent the pressure) $\zeta \in C_c(\bar{D} \times [t_1, t_2])$ such that the initial density $u_1(x)$ satisfies $u_1(x) = b(x, t_1)$ in $\{\zeta(t_1) > 0\}$. For all $x \notin \{\zeta(t_1) > 0\}$, we define $t(x)$ as the last time that $\zeta(x, t) = 0$ (with $t(x) = t_2$ is $\zeta(x, t_2 = 0)$) and define the external density

$$u_\zeta^E(x, t) = u_1(x) \exp\left(\int_{t_1}^{t} \frac{\Phi(x, s, \zeta(x, s))}{a(x, s)}\, ds\right)$$

for all $t < t(x)$. We assume that the external density satisfies,

$$u_\zeta^E(x, t) < b(x, t) \text{ in } \{\zeta = 0\}.$$

The external density solves $\partial_t u = \frac{u}{a}\Phi(x, t, \zeta)$. The density in $D \times (t_1, t_2)$ is defined by

$$\begin{aligned} u_\zeta(x, t) &= b(x, t)\chi_{\{\zeta>0\}}(x) + u_\zeta^E(x, t)(1 - \chi_{\{\zeta>0\}}(x)) \\ &= \begin{cases} b(x, t), \text{in } \{\zeta > 0\} \\ u_\zeta^E(x, t), \text{in } \{\zeta = 0\}. \end{cases} \end{aligned}$$

Proposition 3.8.1. *Suppose that (u_ζ, ζ) are such that*

1. $\zeta \in C^1(\overline{\{\zeta > 0\}}) \cap C^2_{loc}(\{\zeta > 0\})$ *and* $\Gamma = \partial\{\zeta > 0\}$ *is* C^2 *in space and* C^1 *in time.*
2. ζ *satisfies*

$$\begin{cases} -\nabla \cdot \left(\frac{b}{a}\nabla\zeta\right) \le \frac{b}{a}\Phi(x, t, \zeta) - \partial_t b, \textit{in } \{\zeta > 0\} \\ \left(1 - \frac{u_\zeta^E}{b}\right) V_\zeta \le \frac{|\nabla\zeta|}{a}, \textit{on } \partial\{\zeta > 0\}, \end{cases}$$

where V_ζ denotes the normal velocity of $\partial\{\zeta > 0\}$. Then (u_ζ, ζ) is a weak subsolution of the limiting problem

$$\partial_t u_\zeta \le \nabla \cdot \left(\frac{b}{a}\nabla\zeta\right) + \frac{u_\zeta}{a}\Phi(x, t, \zeta) \textit{ in } D \times (t_1, t_2), \zeta \in P_\infty(u_\zeta, b) \textit{ a.e. in } D \times (t_1, t_2)$$

where the PDE holds in the sense that for every smooth, compactly supported test function $\psi : D \times (t_1, t_2) \to \mathbb{R}$ with $\psi(\cdot, t_2) = 0$ and $\psi(\cdot, t) = 0$ on $\partial D \times [t_1, t_2]$, we have

$$\int_{D\times[t_1,t_2]} u_\zeta \partial_t \psi - \frac{b}{a}\nabla\zeta \cdot \nabla\psi + \frac{u_\zeta}{a}\Phi(x, t, \zeta)\psi \ge -\int_D u_1(x)\psi(x, t_1).$$

Proof. Let us denote $S(t) = \{\zeta(\cdot,t) > 0\} = \{u(\cdot,t) = b(\cdot,t)\}$ and $\Gamma(t) = \partial S(t) \cap D$. We also have ν as the outward normal of the boundary of either $\Gamma(t)$ or ∂D with respect to $S(t)$. Using integration by parts,

$$\begin{aligned}
-\int_D \frac{b}{a}\nabla\zeta\cdot\nabla\psi &= -\int_{S(t)} \frac{b}{a}\nabla\zeta\cdot\nabla\psi \\
&= \int_{S(t)} \nabla\cdot\left(\frac{b}{a}\nabla\zeta\right)\psi - \int_{\partial S(t)} \frac{b}{a}\psi\nabla\zeta\cdot\nu\, dS \\
&\geq -\int_{S(t)} \frac{b}{a}\Phi(x,t,\zeta)\psi + \int_{S(t)} \psi\partial_t b + \int_{\Gamma(t)} \frac{b}{a}|\nabla\zeta|\psi,
\end{aligned}$$

where it is used that $\zeta = 0$ and $\nabla\zeta = |\nabla\zeta|\nu$ on $\Gamma(t)$. Using the definition of u_ζ,

$$\int_D u_\zeta\partial_t\psi = \int_{S(t)} b\partial_t\psi + \int_{D\setminus S(t)} u_\zeta^E\partial_t\psi.$$

By product rule and differentiating moving regions,

$$\begin{aligned}
\int_{S(t)} b\partial_t\psi &= \int_{S(t)} \partial_t(b\psi) - \int_{S(t)} \psi\partial_t b \\
&= \frac{d}{dt}\int_{S(t)} b\psi - \int_{\Gamma(t)} b\psi V_\zeta - \int_{S(t)} \psi\partial_t b.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\int_{D\setminus S(t)} u_\zeta^E\partial_t\psi &= \int_{D\setminus S(t)} \partial_t(u_\zeta^E\psi) - \int_{D\setminus S(t)} \psi\partial_t(u_\zeta^E) \\
&= \frac{d}{dt}\int_{D\setminus S(t)} u_\zeta^E\psi - \int_{D\setminus S(t)} -V_\zeta u_\zeta^E\psi - \int_{D\setminus S(t)} \psi\partial_t(u_\zeta^E).
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_D u_\zeta\partial_t\psi &= \frac{d}{dt}\int_D u_\zeta\psi - \int_{\Gamma(t)} (b - u_\zeta^E)\psi V_\zeta - \int_{S(t)} \psi\partial_t b - \int_{D\setminus S(t)} \psi\partial_t(u_\zeta^E) \\
&\geq \frac{d}{dt}\int_D u_\zeta\psi - \int_{\Gamma(t)} \frac{b}{a}|\nabla\zeta|\psi - \int_{S(t)} \psi\partial_t b - \int_{D\setminus S(t)} \psi\partial_t(u_\zeta^E).
\end{aligned}$$

Using the estimates above and recalling that the external density solves $\partial_t u = \frac{u}{a}\Phi(x,t,\zeta)$,

$$\begin{aligned}
\int_D u_\zeta\partial_t\psi - \int_D \frac{b}{a}\nabla\zeta\cdot\nabla\psi &\geq \frac{d}{dt}\int_D u_\zeta\psi - \int_{S(t)} \frac{b}{a}\Phi(x,t,\zeta)\psi - \int_{D\setminus S(t)} \psi\partial_t(u_\zeta^E) \\
&= \frac{d}{dt}\int_D u_\zeta\psi - \int_{S(t)} \frac{b}{a}\Phi(x,t,\zeta)\psi - \int_{D\setminus S(t)} \frac{u_\zeta^E}{a}\Phi(x,t,\zeta)\psi \\
&= \frac{d}{dt}\int_D u_\zeta\psi - \int_D \frac{u_\zeta}{a}\Phi(x,t,\zeta)\psi.
\end{aligned}$$

Integrating in time from t_1 to t_2 we obtain the result

$$\int_{D\times[t_1,t_2]} u_\zeta \partial_t \psi - \frac{b}{a}\nabla\zeta\cdot\nabla\psi + \frac{u_\zeta}{a}\Phi(x,t,\zeta)\psi \geq -\int_D u_1(x)\psi(x,t_1).$$

Thus the proof is complete. □

Proposition 3.8.2. *Suppose that (u_ζ, ζ) are such that*

1. $\zeta \in C^1(\overline{\{\zeta > 0\}}) \cap C^2_{loc}(\{\zeta > 0\})$ *and* $\Gamma = \partial\{\zeta > 0\}$ *is* C^2 *in space and* C^1 *in time.*
2. ζ *satisfies*

$$\begin{cases} -\nabla\cdot\left(\frac{b}{a}\nabla\zeta\right) \geq \frac{b}{a}\Phi(x,t,\zeta) - \partial_t b, \text{ in } \{\zeta > 0\} \\ \left(1 - \frac{u^E_\zeta}{b}\right) V_\zeta \geq \frac{|\nabla\zeta|}{a}, \text{ on } \partial\{\zeta > 0\}, \end{cases}$$

where V_ζ denotes the normal velocity of $\partial\{\zeta > 0\}$. Then (u_ζ, ζ) is a weak supersolution of the limiting problem

$$\partial_t u_\zeta \geq \nabla\cdot\left(\frac{b}{a}\nabla\zeta\right) + \frac{u_\zeta}{a}\Phi(x,t,\zeta) \text{ in } D\times(t_1,t_2), \zeta \in P_\infty(u_\zeta, b) \text{ a.e. in } D\times(t_1,t_2)$$

where the PDE holds in the sense that for every smooth, compactly supported test function $\psi : D\times(t_1,t_2) \to \mathbb{R}$ with $\psi(\cdot,t_2) = 0$ and $\psi(\cdot,t) = 0$ on $\partial D\times[t_1,t_2]$, we have

$$\int_{D\times[t_1,t_2]} u_\zeta \partial_t \psi - \frac{b}{a}\nabla\zeta\cdot\nabla\psi + \frac{u_\zeta}{a}\Phi(x,t,\zeta)\psi \leq -\int_D u_1(x)\psi(x,t_1).$$

By Proposition 3.8.1 and the comparison principle for weak solutions of the limiting problem Corollary 3.6.2, we get the following corollary.

Corollary 3.8.3. *Let (u_ζ, ζ) be a sub-solution as Proposition 3.8.1. If*

1. $u_\zeta(\cdot,t_1) \leq u_\infty(\cdot,t_1)$ *in* D,
2. $\zeta \leq p_\infty$ *on* $\partial D\times[t_1,t_2]$,

then $u_\zeta \leq u_\infty$ in $D\times[t_1,t_2]$.

In the viscosity sense or comparison with barriers, we have the motion law

$$\left(1 - \frac{u^E_\infty}{b}\right) V_\infty \leq \frac{|\nabla p_\infty|}{a}.$$

In a similar manner, we have the analogous corollary.

Corollary 3.8.4. *Let (u_ζ, ζ) be a super-solution as Proposition 3.8.2. If*

1. $u_\zeta(\cdot, t_1) \geq u_\infty(\cdot, t_1)$ *in* D*,*
2. $\zeta \geq p_\infty$ *on* $\partial D \times [t_1, t_2]$*,*

then $u_\zeta \geq u_\infty$ *in* $D \times [t_1, t_2]$.

In the viscosity sense or comparison with barriers, we have the motion law

$$\left(1 - \frac{u_\infty^E}{b}\right) V_\infty \geq \frac{|\nabla p_\infty|}{a}.$$

Thus, we have achieved the velocity law in the viscosity sense. Moreover, we have proven the following proposition.

Proposition 3.8.5 (Velocity Law)**.** *The external density,* u_∞^E*, is the limit of the density from outside the saturated region* $\{x : p_\infty(x,t) > 0\}$*. The normal velocity,* V_∞*, of the free boundary* $\partial\{x : p_\infty(x,t) > 0\}$ *satisfies in a viscosity sense*

$$\left(1 - \frac{u_\infty^E}{b}\right) V_\infty = \frac{|\nabla p_\infty|}{a}.$$

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APPENDIX A

MOLLIFIER AND AGGREGATION KERNELS

Extra Details About the Mollifier

We first discuss some extra details about the mollifier.

Remark A.0.1. Notice the $L^1(\mathbb{R}^d)$ norm of the positive mollifier ζ_ϵ is one by change of variables,

$$\int_{\mathbb{R}^d} \zeta_\epsilon(y)\, dy = \int_{\mathbb{R}^d} \frac{1}{\epsilon^d} \zeta\left(\frac{y}{\epsilon}\right)\, dy = \int_{\mathbb{R}^d} \zeta(z)\, dz = 1.$$

Remark A.0.2 (the first moment of ζ is finite)**.** First we split the integral where we focus inside the unit ball and outside the unit ball,

$$M_1(\zeta) = \int_{(\mathbb{R}^d)} |y|\zeta(y)\, dy = \int_{B_1} |y|\zeta(y) + \int_{\mathbb{R}^d \setminus B_1} |y|\zeta(y) =: J_1 + J_2.$$

Inside the unit ball we have

$$J_1 \le \int_{B_1} \zeta(y)\, dy \le 1.$$

Outside the unit ball we can use polar coordinates along with $|\zeta|(y) \le C_\zeta |y|^{-q}$ for $q > d+1$,

$$\begin{aligned} J_2 &\le C_\zeta \int_{\mathbb{R}^d \setminus B_1} |y|^{1-q}\, dy \\ &= C_{\zeta,d} \int_1^\infty r^{d-q}\, dr \\ &= C_{\zeta,d} \frac{-1}{d-q+1} \\ &< \infty \end{aligned}$$

where using the power rule is justified as $d - q < -1$. Note that as $r \to \infty$, then $r^{d-q+1} \to 0$ as $d - q + 1 < 0$.

Lemma A.0.3. *If $|\zeta| < C_\zeta |x|^{-q}$ for $q > d + p$, then $M_p(\zeta) < \infty$. In particular,*

$$M_p(\zeta_\epsilon * \mu) \le 2^{p-1} \left(M_p(\mu) + \epsilon^p M_p(\zeta)\right)$$

is finite.

Proof. The first statement follows from generalizing the $M_1(\zeta)$ calculation, namely,

$$\begin{aligned} C_\zeta \int_{\mathbb{R}^d \backslash B_1} |y|^{p-q} \, dy &= C_{\zeta,d} \int_1^\infty r^{d-1+p-q} \, dr \\ &= C_{\zeta,d} \frac{-1}{d+p-q} \\ &< \infty \end{aligned}$$

where the integration is justified as $0 > d + p - q$. Let us focus on the second statement. We can rewrite

$$M_p(\zeta_\epsilon * \mu) = \iint \zeta_\epsilon(y)|x-y|^p \, dy \, d\mu(x).$$

Recall that $|x-y|^p \leq 2^{p-1}(|x|^p + |y|^p)$ and μ, ζ_ϵ are probability measures. As $M_p(\zeta_\epsilon) = \epsilon^p M_p(\zeta)$ by change of variables, then we get

$$M_p(\zeta_\epsilon * \mu) \leq 2^{p-1} \left(M_p(\mu) + \epsilon^p M_p(\zeta) \right).$$

Thus as long ϵ is finite then so is $M_p(\zeta_\epsilon * \mu)$. □

Bounds for Newtonian and Bessel Kernels

We now move on to some aggregation kernels. In particular, bounds for the Newtonian and Bessel kernels.

Lemma A.0.4 (L^p norm of the Newtonian and Bessel Kernels)**.** *Let $R > 0$. For $d \geq 3$, $\mathcal{N} \in L^p(R^d \backslash B_R)$ for $p > \frac{d}{d-2}$ and $\mathcal{N} \in L^p(B_R)$ for $p < \frac{d}{d-2}$. Similarly for $d \geq 3$, $\nabla \mathcal{N} \in L^p(R^d \backslash B_R)$ for $p > \frac{d}{d-1}$ and $\nabla \mathcal{N} \in L^p(B_R)$ for $p < \frac{d}{d-1}$. For $\alpha > 0$, $\|\mathcal{B}_{\alpha,d}\|_{L^1(B_R)} \leq \frac{1}{\alpha}$. Furthermore, $\nabla \mathcal{B}_{\alpha,d}$ has the same L^p-ness as $\nabla \mathcal{N}$ for $d \geq 3$.*

Proof. Let $0 < \epsilon < R$. Let A_ϵ^R be an annulus with inner radius ϵ and outer radius R. Thus, as $\epsilon \to 0$, $A_\epsilon^R \to B_R$ and as $R \to \infty$, $A_\epsilon^R \to \mathbb{R}^d \backslash B_\epsilon$ (this is written as $\mathbb{R}^d \backslash B_R$ in the statement). The first gives us information locally while the second gives us information away from the origin. For $d \geq 3$, computing via polar coordinates,

$$\|\mathcal{N}\|_{L^p(A_\epsilon^R)}^p = C_d^p d\alpha_d \int_\epsilon^R r^{d-1} r^{(2-d)p} \, dr = C_d^p d\alpha_d \int_\epsilon^R r^{d-1+(2-d)p} \, dr.$$

Away from the origin, we require $d-1+(2-d)p < -1$. So, $p > \frac{d}{d-2}$. Locally, we require $d-1+(2-d)p > -1$. So, $p < \frac{d}{d-2}$. For $d \geq 3$,

$$\|\nabla \mathcal{N}\|^p_{L^p(A^R_\epsilon)} = C^p_d d\alpha_d \int_\epsilon^R r^{d-1+(1-d)p}\, dr.$$

Away from the origin, we require $d-1+(1-d)p < -1$. So, $p > \frac{d}{d-1}$. Locally, we require $d-1+(1-d)p > -1$. So, $p < \frac{d}{d-1}$.

Define $g(x,t) = \frac{1}{(4\pi t)^{d/2}} e^{-|x|^2/(4t)}$, $f(x,t) = e^{-\alpha t} g(x,t)$. Then,

$$\mathcal{B}_{\alpha,d}(x) = \int_0^\infty f(x,t)\, dt.$$

Given that g is the heat kernel and its $L^1(\mathbb{R}^d)$ norm is one, then

$$\|\mathcal{B}_{\alpha,d}\|_{L^1(\mathbb{R}^d)} \leq \int_0^\infty e^{-\alpha t}\, dt = \frac{1}{\alpha}$$

By lemma 2.4 of [8], $|\nabla \mathcal{B}_{\alpha,d}(x)| \leq C_d |x|^{1-d} g_\alpha(|x|)$ where $g_\alpha(|x|)$ is a positive radial function exponentially decreasing from 1 to 0 as $|x| \to \infty$. Moreover, $|\nabla \mathcal{B}_{\alpha,d}(x)| \leq C_d |x|^{1-d}$. Therefore, $|\nabla \mathcal{B}_{\alpha,d}(x)|$ is proportional to $|\nabla \mathcal{N}(x)|$. Thus, $\nabla \mathcal{B}_{\alpha,d}$ has the same L^p-ness as $\nabla \mathcal{N}$ for $d \geq 3$. □

APPENDIX B

CONSTRUCTION OF SUPERSOLUTION AND HEURISTICS

Construction of Supersolution

Here we will briefly talk about the construction of the barrier used for lemma 3.4.2. More details can be seen in [12, Lemmas 8.1 - 8.3].

Lemma B.0.1 ([12] Lemma 8.1)**.** *Assume a, b are radial in space. Then,*

$$\varphi(|x|, t) := \frac{1}{d}\int_0^{|x|} \frac{a(r,t)}{b(r,t)} r \, dr$$

solves $\nabla \cdot \left(\frac{b}{a}\nabla\varphi\right) = 1$ in $\mathbb{R}^d$ where there exists constants $\kappa_i > 0$ for $i = 1, 2, 3$ such that $\kappa_1|x|^2 \leq \varphi(|x|, t) \leq \kappa_2|x|^2$ and $|\nabla\varphi(|x|, t)| \leq \kappa_3|x|$.

Lemma B.0.2 (Supersolutions to pressure equation for radial assumption, [12] Lemma 8.2)**.** *Let a, b be radial in space and define φ as above. Define*

$$Z(x,t) = \alpha\,|R(t) - \varphi(|x|,t)|_+, \quad R(t) = \alpha \exp\left((C_2 + \|(1/a)\|_{L^\infty} C_1\alpha)\alpha t\right),$$

for constants $C_1, C_2 > 0$ and $\alpha > \|\Phi b/a\|_{L^\infty} + \|\partial_t b\|_{L^\infty}$. Then, Z is a supersolution for (3.4).

Proof. Note that we are only interested in the region $\{R(t) \geq \varphi(|x|, t)\}$ by the definition of Z. We can rewrite (3.4) as

$$\partial_t p_m = \frac{|\nabla p_m|^2}{a} + (m-1)\frac{p_m}{b}\left(\nabla \cdot \left(\frac{b}{a}\nabla p_m\right) + \frac{b}{a}\Phi(x,t,p_m) - \partial_t b\right).$$

By the previous lemma and the hypothesis of this lemma,

$$\begin{aligned}\nabla \cdot \left(\frac{b}{a}\nabla Z\right) + \frac{b}{a}\Phi(x,t,0) - \partial_t b &= -\alpha + \frac{b}{a}\Phi(x,t,0) - \partial_t b \\ &< 0.\end{aligned}$$

Thus, it is left to show that

$$\partial_t Z \geq \frac{\alpha^2|\nabla\varphi|^2}{a}.$$

By the bounds in Lemma B.0.1,

$$|\nabla\varphi|^2 \leq \kappa_3^2|x|^2 \leq \frac{\kappa_3^2}{\kappa_1}|\varphi| \leq \frac{\kappa_3^2}{\kappa_1}R(t),$$

and

$$|\partial_t\varphi| \leq \kappa_4|\varphi| \leq \kappa_4 R(t),$$

for some constant κ_4. So for some constants $C_1, C_2 > 0$,

$$\partial_t Z - \frac{\alpha^2|\nabla\varphi|^2}{a} \geq \alpha\partial_t R - (\alpha C_2 + \|(1/a)\|_{L^\infty}\alpha^2 C_1)R \geq 0.$$

This ends the proof. □

Lemma B.0.3 (Supersolutions to pressure equation for fast enough decay assumption, [12] Lemma 8.3)**.** *Suppose Assumption 3.2.3 holds. Define*

$$Z(x,t) = \alpha\,|R(t) - \varphi(|x|,t)|_+, \quad R(t) = \alpha\exp\left((C_2 + \|(1/a)\|_{L^\infty}C_1\alpha)\alpha t\right),$$

for constants $C_1, C_2 > 0$ and $\alpha > \|\Phi b/a\|_{L^\infty} + \|\partial_t b\|_{L^\infty}$. Then, Z is a supersolution for (3.4)*. Moreover, Z is bounded in $L^\infty(Q_T)$ and is compactly supported for an fixed time.*

Proof. We construct a positive subsolution of $\nabla\cdot\left(\frac{b}{a}\nabla u\right) = 1$ in $\mathbb{R}^d$. We start with the construction inside a ball centered at the origin B_R. Let us examine the unique solution of

$$\begin{cases} \nabla\cdot\left(\frac{b}{a}\nabla\phi\right) = 1 \quad \text{in } B_R\times[0,T], \\ \phi(x,t) = 1 \quad \text{on } \partial B_R. \end{cases}$$

By standard estimate for uniformly elliptic PDEs ([20]), there is a constant $C = C(a,b,R,d)$ such that

$$\sup_{(x,t)\in B_R\times[0,T]} |\phi(x,t)| \leq C.$$

As $\phi(x,t) + 2C$ is positive and solves the same PDE with constant boundary data, we may assume that $\phi(x,t) > 0$ on $B_R\times[0,T]$. Now we focus on the regularity in time. By estimates in [20], there is a constant $C = C(a,b,R,d)$ such that

$$\sup_{(x,t)\in B_R\times[0,T]}\sum_{i=1}^d|\partial_i\phi(x,t)| + \sup_{(x,t)\in B_R\times[0,T]}\sum_{i,j=1}^d|\partial_{i,j}^2\phi(x,t)| \leq C.$$

Differentiating the elliptic equation in time,

$$\begin{cases} \nabla \cdot \left(\partial_t \left(\frac{b}{a} \right) \nabla \phi \right) + \nabla \cdot \left(\frac{b}{a} \nabla \partial_t \phi \right) = 0 \quad \text{in } B_R \times [0, T], \\ \partial_t \phi(x, t) = 0 \quad \text{on } \partial B_R. \end{cases}$$

As $\nabla \cdot \left(\partial_t \left(\frac{b}{a} \right) \nabla \phi \right)$ is smooth in space and bounded $B_R \times [0, T]$, again by uniformly elliptic estimates in [20], there is a constant $C = C(a, b, R, d)$ such that

$$|\partial_t \phi(x, t)| \leq C.$$

For $|x| \geq R$ we have by Assumption 3.2.3,

$$\nabla \cdot \left(\frac{b}{a} \nabla |x|^2 \right) = 2d\frac{b}{a} + 2\nabla \left(\frac{b}{a} \right) \cdot x \geq \frac{2d}{\Lambda^2} - 2\epsilon \geq \frac{1}{\Lambda^2}.$$

Define $w(x) = 1 + C(|x|^2 - R^2(t))$. From the regularity of ϕ, we can choose $C \geq \Lambda^2$ large enough so that for if $x \in \partial B_R$, then

$$\frac{\partial w}{\partial |x|}(x) > \frac{\partial \phi}{\partial |x|}(x).$$

Thus if we define

$$\varphi(x, t) = \begin{cases} \phi(x, t) \quad \text{for } |x| \leq R, \\ w(x) \quad \text{for } |x| \geq R, \end{cases}$$

then we have that φ is a viscosity solution of $\nabla \cdot \left(\frac{b}{a} \nabla u \right) \geq 1$. It follows that $\partial_t \varphi$ is bounded as $|\partial_t \phi| \leq C$ when $|x| \leq R$ and $\partial_t w = 0$ for $|x| \geq R$. The results on Z are achieved in the same way as in the case when a, b are radial in space. □

Complementarity Condition Heuristics

We first examine the heuristics of the complementarity condition. We will find the equation for the pressure using density equation (3.1). From there, we will see the complementarity condition.

Given the definition of p_m,

$$\nabla p_m = m \left(\frac{u_m}{b} \right)^{m-2} \left(\frac{\nabla u_m}{b} - \frac{u_m \nabla b}{b^2} \right) \Rightarrow m \left(\frac{u_m}{b} \right)^{m-2} \frac{\nabla u_m}{b} = \nabla p_m + (m-1) p_m \frac{\nabla b}{b}.$$

Similarly,

$$\partial_t p_m = m\left(\frac{u_m}{b}\right)^{m-2}\left(\frac{\partial_t u_m}{b} - \frac{u_m \partial_t b}{b^2}\right) \Rightarrow m\left(\frac{u_m}{b}\right)^{m-2}\frac{\partial_t u_m}{b} = \partial_t p_m + (m-1)p_m\frac{\partial_t b}{b}.$$

We multiply (3.1) by $\frac{m}{b}\left(\frac{u_m}{b}\right)^{m-2}$ to achieve,

$$\partial_t p_m + (m-1)p_m\frac{\partial_t b}{b} = \frac{m}{b}\left(\frac{u_m}{b}\right)^{m-2}\left(\nabla\left(\frac{u_m}{a}\right)\cdot\nabla p_m\right) + (m-1)\frac{p_m}{a}(\Delta p_m + \Phi(x,t,p_m)).$$

Focusing on the first term on the right-hand side,

$$\begin{aligned}
\frac{m}{b}\left(\frac{u_m}{b}\right)^{m-2}\left(\nabla\left(\frac{u_m}{a}\right)\cdot\nabla p_m\right) &= \frac{m\nabla p_m}{b}\left(\frac{u_m}{b}\right)^{m-2}\left(\frac{\nabla u_m}{a} - \frac{u_m\nabla a}{a^2}\right)\\
&= \frac{\nabla p_m}{a}\left(\nabla p_m + (m-1)p_m\frac{\nabla b}{b}\right) - (m-1)\frac{p_m}{a}\frac{\nabla p_m\cdot\nabla a}{a}\\
&= \frac{|\nabla p_m|^2}{a} + (m-1)\frac{p_m\nabla p_m}{a}\left(\frac{\nabla b}{b} - \frac{\nabla a}{a}\right).
\end{aligned}$$

As an aside, we could write $\frac{\nabla b}{b} - \frac{\nabla a}{a} = \nabla\log\left(\frac{b}{a}\right)$ if desired. Therefore, the equation for the pressure is

$$\partial_t p_m + (m-1)p_m\partial_t\log(b) = \frac{|\nabla p_m|^2}{a} + (m-1)\frac{p_m}{a}\left(\nabla p_m\cdot\nabla\log\left(\frac{b}{a}\right) + \Delta p_m + \Phi(x,t,p_m)\right).$$

Formally, letting $m\to\infty$ we find the complementarity condition

$$\frac{p_\infty}{a}\left(\nabla p_\infty\cdot\nabla\log\left(\frac{b}{a}\right) + \Delta p_\infty + \Phi(x,t,p_\infty) - a\partial_t\log(b)\right) = 0 \quad \text{in} \quad \{p_\infty(x,t)>0\}.$$

Furthermore, we get the Hele-Shaw free boundary problem,

$$\text{(FBP)} \qquad \begin{cases} -\Delta p_\infty = \nabla p_\infty\cdot\nabla\log\left(\frac{b}{a}\right) + \Phi(x,t,p_\infty) - a\partial_t\log(b) & \text{in } \{x : p_\infty(x,t)>0\},\\ V = \frac{\partial_t p_\infty}{|\nabla p_\infty|} = \frac{1}{a}|\nabla p_\infty| & \text{on } \partial\{x : p_\infty(x,t)>0\}. \end{cases}$$

For some calculations it is easier to evaluate the normalized density $v_m = \frac{u_m}{b}$. Similar to the calculation above, we have the equation for the normalized density

$$\partial_t v_m + v_m\partial_t\log(b) = \frac{1}{a}\left(\nabla v_m^m\cdot\nabla\log\left(\frac{b}{a}\right) + \Delta v_m^m + v_m\Phi(x,t,p_m)\right).$$

In particular, see that

$$\begin{aligned}
\partial_t v_m &= \frac{\partial_t u_m}{b} - \frac{u_m}{b}\frac{\partial_t b}{b} \\
&= \frac{\partial_t u_m}{b} - v_m \partial_t \log(b).
\end{aligned}$$

Dividing (3.2) by b,

$$\begin{aligned}
\partial_t v_m + v_m \partial_t \log(b) &= \frac{1}{b}\nabla \cdot \left(\frac{b}{a}\nabla v_m^m\right) + \frac{v_m}{a}\Phi(x,t,p_m) \\
&= \frac{\Delta v_m^m}{a} + \frac{1}{b}\nabla\left(\frac{b}{a}\right)\cdot \nabla v_m^m + \frac{v_m}{a}\Phi(x,t,p_m) \\
&= \frac{\Delta v_m^m}{a} + \frac{1}{a}\nabla v_m^m \cdot \nabla \log\left(\frac{b}{a}\right) + \frac{v_m}{a}\Phi(x,t,p_m).
\end{aligned}$$

Velocity Law Heuristics

We now move on to the heuristics of the velocity law. The heuristics here is similar to the heuristics in [21]. Denote u_∞^I, u_∞^E as the internal and external density of $\Omega(t) = \{x : p_\infty(x,t) > 0\}$, respectively. Starting with (3.2) (after formally taking $m \to \infty$),

$$\begin{aligned}
\int_{\mathbb{R}^d} \frac{u_\infty}{a}\Phi &= \frac{d}{dt}\int_{\mathbb{R}^d} u_\infty \\
&= \frac{d}{dt}\left(\int_{\Omega(t)} u_\infty + \int_{\mathbb{R}^d\setminus\Omega(t)} u_\infty\right) \\
&= \int_{\Omega(t)} \partial_t u_\infty^I + \int_{\mathbb{R}^d\setminus\Omega(t)} \partial_t u_\infty^E + \int_{\partial\Omega(t)} V(u_\infty^I - u_\infty^E) \\
&= \int_{\Omega(t)} \nabla \cdot \left(\frac{b}{a}\nabla p_\infty\right) + \int_{\mathbb{R}^d} \frac{u_\infty}{a}\Phi + \int_{\partial\Omega(t)} V(u_\infty^I - u_\infty^E) \\
&= \int_{\partial\Omega(t)} \frac{b}{a}\nabla p_\infty \cdot \nu + V(b - u_\infty^E) + \int_{\mathbb{R}^d} \frac{u_\infty}{a}\Phi.
\end{aligned}$$

This suggests that in the presence of the mushy region (where $0 < u_\infty < b$) the normal boundary velocity of $\partial\Omega(t)$ satisfies $(1 - u_\infty^E/b)V = -(1/a)\nabla p_\infty \cdot \nu$ since $b > 0$. When the external density vanishes on the boundary, then we get the velocity law as expected.