SHOCK FORMATION IN THE BIG BANG

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ABSTRACT

This thesis aims to construct Big Bang models and to investigate their properties. Under the warped product spacetime ansatz, we classify all the physical Big Bang models that are spatially homogeneous and analyze their asymptotes. Furthermore, we prove that the Big Bang models with a positive blowup time $r_* > 0$ are dynamically unstable under non-homogeneous perturbations in Chapter 3.

In addition, we also prove the stability of specially relativistic fluids on a fixed Big Bang spacetime in Chapter 4. This work implies that Euler equations are not sufficient to generate shocks. One needs the feedback from fluids to metrics, together forming Einstein-Euler equations, to generate shocks.

Finally, we prove the global existence of the membrane equation on $\mathbb{R}^{1,2} \times \mathbb{T}^1$ for sufficiently small, compactly supported initial data in Chapter 5. This is a work independent of the previous ones. We use the standard vector field method to show that the energy remains small throughout the time, estabilishing the global existence for the equation.

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CHAPTER 1

CONSTRUCT BIG BANG MODELS

1.1 Classical Big Bang models

After Einstein proposed his General Relativity theory, people started to use the Einstein field equations to model the evolution of our universe. In cosmology, it is assumed that there exists a family of fundamental observers with timelike geodesics (timelines) which span a four-dimensional spacetime V with a Lorentzian metric g satisfying the Einstein field equations

$$Ric - \frac{1}{2}Sg + \Lambda g = T. \tag{1.1}$$

These fundamental observers have timelines orthogonal to space sections. In other words, the spacetime $V = \mathbb{R} \times M$ and the metric g decomposes to

$$g = -dt^2 + {}^{(3)}g.$$

Here M is a Riemannian manifold with the metric ${}^{(3)}g$ modeling our universe at a fixed time. In the cosmology scale, it is reasonable to assume the homogeneity and isotropy on M. Although there are also homogeneous non-isotropic models, such as Bianchi cosmologies, we only focus on the homogeneous isotropic models in this section. Intuitively, isotropy means that at each point in M, every direction looks the same for the observer. Homogeneity means that for every two points in M, there is an isometry of M that takes one point to the other. This section basically follows [4].

1.1.1 Riemannian manifolds with constant curvature

Mathematically, We interpret isotropy and homogeneity as follows. Recall that the definition of the sectional curvature of a Riemannian manifold M is

$$K(P) = \frac{R(X,Y,X,Y)}{g(X,X)g(Y,Y) - g(X,Y)^2}$$

where X, Y are two tangent vectors spanning the two sub-plane P. This quantity characterizes the geometry for each "direction" P. For a fixed point in M, isotropy means that the sectional curvature $K(P)$ is independent of the choice of P. On the other hand, homogeneity means that the sectional curvature K is furthermore independent of the choice of the point in M. In other words, K is constant throughout the whole manifold M. Indeed, Schûr proved that isotropy and the Bianchi identity imply the constancy of K on a Riemannian manifold whose dimension is greater than 2.

It is well-known in geometry that a Riemannian manifold with constant curvature K is locally isometric to a sphere of radius $\frac{1}{\sqrt{K}}$ (when $K > 0$), the Euclidean space (when $K = 0$), or a hyperbolic space with pseudo-radius $\frac{1}{\sqrt{|K|}}$ (when $K < 0$). In our context, the Riemannaian manifold M is 3-dimensional with the metric ${}^{(3)}g$ of the form

$$ {}^{(3)}g = \frac{1}{|K|}\gamma_\epsilon $$

where

$$ \gamma_\epsilon = \frac{dr^2}{1-\epsilon r^2} + r^2(d\theta^2 + \sin^2(\theta)d\phi^2), \quad \epsilon = sgn(K), $$

or more conveniently,

$$ \gamma_+ = d\alpha^2 + \sin^2(\alpha)(d\theta^2 + \sin^2(\theta)d\phi^2), \quad \epsilon = 1 $$

$$ \gamma_0 = dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2), \quad \epsilon = 0 $$

$$ \gamma_- = d\chi^2 + \sinh^2(\chi)(d\theta^2 + \sin^2(\theta)d\phi^2), \quad \epsilon = -1. $$

Note that in our context, the curvature K is constant over M but potentially depends on the time t. Therefore, the analysis shows that, under the assumptions homogeneity and isotropy of the universe M, the metric has the following form

$$ g = -dt^2 + R(t)^2\gamma_\epsilon, $$

where R is a function of t and $\epsilon = 1$, 0, or -1. From now on, we will assume our universe is $M = \mathbb{S}^3$, $\mathbb{R}^3$, or $\mathbb{H}^3$ supporting γ_+, γ_0, γ_- respectively. These spacetimes are called Robertson-Walker spacetimes.

1.1.2 Friedman equations

Consider the Einstein field equations for a Robertson-Walker spacetime with the perfect fluid as the source energy momentum tensor

$$T = (p + \rho)\xi \otimes \xi + pg,$$

where p is the fluid pressure, ρ is the fluid density, and ξ is the fluid velocity. Under the isotropy and homogeneity conditions, we may assume $p = p(t)$, $\rho = \rho(t)$, and $\xi = \partial_t$. Inserting this energy momentum tensor back to (1.1), one finds that the Einstein field equations are reduced to:

$$p = -\frac{2\ddot{R}}{R} - \frac{\epsilon}{R^2} - \frac{\dot{R}^2}{R^2} + \Lambda$$

$$\rho = \frac{3(\dot{R}^2 + \epsilon)}{R^2} - \Lambda.$$

These equations are called the Friedmann equations. They describe the evolution of the radius of our universe.

1.1.3 Big Bang

Assuming $\Lambda = 0$, the Friedmann equations can be rearranged to the following form.

$$\frac{\dot{R}^2}{R^2} = \frac{1}{3}\rho - \frac{\epsilon}{R^2}$$

$$\frac{\ddot{R}}{R} = -\frac{1}{2}\left(p + \frac{1}{3}\rho\right).$$

Assuming $3p + \rho > 0$ and assuming the current universe is expanding, General Relativity suggests that $\ddot{R} < 0$, so $\dot{R}$ is decreasing. This is surprising: denoting the current time by t_0, then less than $\frac{R(t_0)}{\dot{R}(t_0)} := \frac{1}{H(t_0)}$ time units ago, we have $R(0) = 0$ (see Figure 1.1). This is one motivation for the Big Bang conjecture. Note that H, the Hubble constant, is a function of time. We provide more details for the evolution of the universe in the next section.

1.1.4 Friedmann-Lemaître models

Assume $\Lambda = 0$. The above Friedmann equations imply

$$\dot{\rho} = -\frac{3\dot{R}}{R}(p + \rho).$$

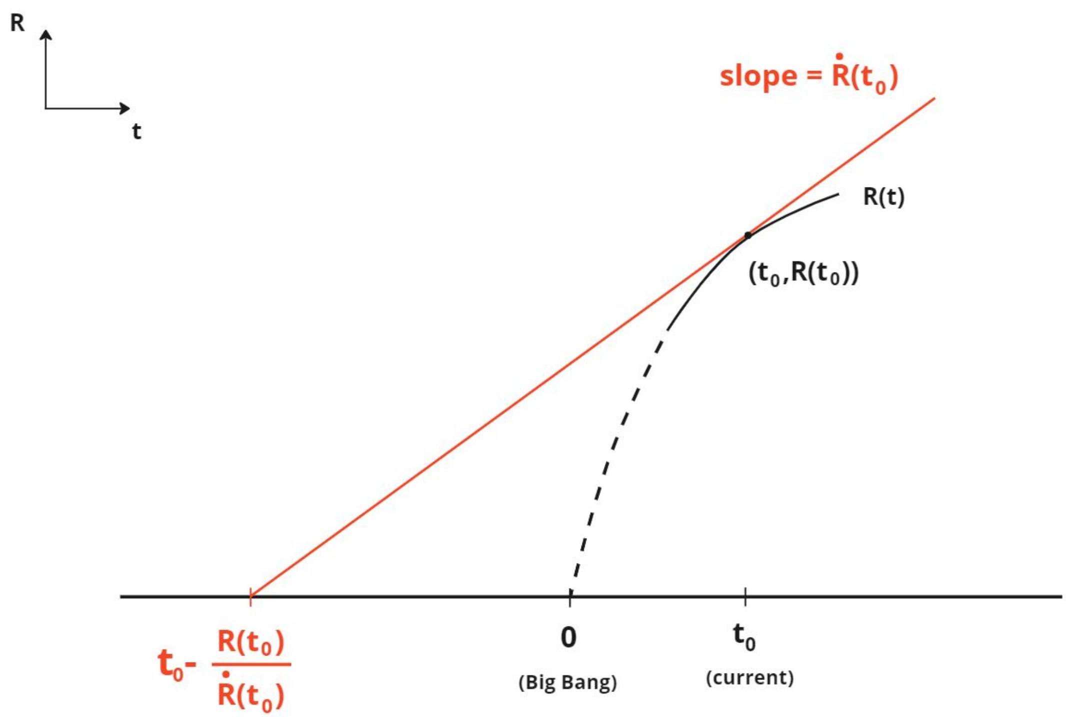

Figure 1.1 Big Bang conjecture for classical models.

Now we impose the equation of state

$$p = \gamma\rho$$

where $0 \leq \sqrt{\gamma} < 1$ denotes the sound speed. The models with the equation of state $p = \gamma\rho$ and $\Lambda = 0$ are usually called Friedmann-Lemaitr̂e models. Cosmologists believe that the early universe is dominated by radiation with $p = \frac{1}{3}\rho$, while the later universe is described by dust with $p = 0$. Using this equation of state, one derives the relation between ρ and R:

$$\rho \cdot R^{3(1+\gamma)} = M$$

where $M > 0$ is a constant. This suggests different asymptotes of ρ as $R \to 0$ for the dust and radiation cases.

- For the dust case, $\gamma = 0$, so $\rho \sim \frac{1}{R^3}$.
- For the radiation case, $\gamma = \frac{1}{3}$, so $\rho \sim \frac{1}{R^4}$.

For the early universe, R is close to 0. The above asymptotes imply that the density of radiation is greater than the density of dust. This explains why cosmologists believe that the radiation is a good model for the early universe. With this observation, one can simplify the Friedmann equation to

$$\dot{R}^2 = \frac{M}{3} R^{2-3(1+\gamma)} - \epsilon.$$

The evolution of the radius of our universe depends on the choice of ϵ.

- Hyperbolic case, $\epsilon = -1$. R will increase forever for this case.
- Euclidean case, $\epsilon = 0$. R keeps increasing with a slower rate compared with the previous case.
- Elliptic case, $\epsilon = 1$. R increases initially and decreases later because $\ddot{R} < 0$ if $\rho > 0$.

1.1.5 Einstein static universe

Historically, it was originally believed that $\epsilon = 1$ and $R(t) = R_0$ is independent of t for a Robertson-Walker cosmological model. In other words, the metric reads

$$g = -dt^2 + (R_0)^2 \gamma_+.$$

The Friedman equations reduce to

$$p = -\frac{1}{(R_0)^2} + \Lambda$$
$$\rho = \frac{3}{(R_0)^2} - \Lambda.$$

Since the pressure cannot be negative, Einstein introduced the cosmological constant $\Lambda > 0$ to save this. After a few years, Einstein accepted those cosmological models with R changing with time and abandoned the introduction of the cosmological constant Λ. The red-shift phenomenon can be explained if the universe is expanding. Later however, cosmologists reintroduced the cosmological term in a time-dependent form due to the observation that the universe's expansion is accelerating.

1.1.6 De Sitter and anti de Sitter spacetimes

Consider the vacuum Einstein field equations

$$Ric - \frac{1}{2}Sg + \Lambda g = 0$$

with a cosmological constant Λ. Using a Robertson-Walker spacetime model, one derives a system of evolution equations for the radius R:

$$\ddot{R} - \frac{\Lambda}{3}R = 0$$

$$\dot{R}^2 - \frac{\Lambda}{3}R^2 = -\epsilon.$$

We have three cases based on the sign of ϵ.

- $\epsilon = 0$. In this case, $\Lambda > 0$ for non-trivial solutions. The first equation gives

  $$R = Ae^{kt} + Be^{-kt}$$

  where $k = \sqrt{\frac{\Lambda}{3}}$. From the second equation, $AB = 0$. Therefore, the spacetime metric is of the form

  $$-dt^2 + e^{k't}(dx^2 + dy^2 + dz^2)$$

  with $k' = \pm k$. The spatial metric is Euclidean up to a time-dependent factor.

- $\epsilon = 1$. In this case, $\Lambda > 0$ for non-trivial solutions. When R is time symmetric, the spacetime metric reduces to

  $$g_{\text{de Sitter}} = -dt^2 + \frac{\cosh^2(kt)}{k^2}\Big(d\alpha^2 + \sin^2(\alpha)(d\theta^2 + \sin^2(\theta)d\phi^2)\Big),$$

  the de Sitter spacetime where $k = \sqrt{\frac{\Lambda}{3}}$. The spatial metric is S^3 up to a $\frac{\cosh^2(kt)}{k^2}$ time-dependent factor. The de Sitter spacetime is conformal to the slice $-\pi < t' < \pi$ of the Einstein static universe via the change of variable $t' = 2\tan^{-1}(e^{kt})$.

- $\epsilon = -1$. In this case, Λ may be positive, negative, or zero. When $\Lambda = -3$ and R is time-symmetric, the spacetime metric becomes

$$-dt^2 + \cos^2(t)\Big(d\chi^2 + \sinh^2(\chi)(d\theta^2 + \sin^2(\theta)d\phi^2)\Big).$$

The standard anti de Sitter spacetime is an extension of the above spacetime

$$-\cosh^2(\chi)dt^2 + d\chi^2 + \sinh^2(\chi)(d\theta^2 + \sin^2(\theta)d\phi^2)$$

with $0 \le \chi < \infty$. The anti de Sitter spacetime is conformal to the Einstein cylinder $0 \le \alpha < \frac{\pi}{2}$ via the change of variable $\alpha = 2\tan^{-1}(e^\chi) - \frac{\pi}{2}$.

1.2 Warped product Big Bang models

It is difficult to deal with the Einstein field equations (1.1) directly. In order to simplify the equations, one usually put the spherical symmetry assumption on the spacetime metric:

$$\tilde{g} = g + r^2\big(d\theta^2 + \sin^2(\theta)d\phi^2\big)$$

where g is a metric on a $(1+1)$-Lorentzian manifold Q, r is a function on Q, and θ, ϕ are coordinates on the two sphere $\mathbb{S}^2$. Christodoulou and Dafermos are able to make progress using this ansatz ([5], [9]). Based on this, An and Wong proposed warped product spacetimes in [2] as the following definition. The reason why we put the warped function r as the time function is because we want to use this spacetime to model the Big Bang. Recall that in classical models, the radius R is an increasing function of time from the Big Bang to the current time. In section 1.2.2, we see that this is a generalization of the Friedmann-Lemaître-Robertson-Walker spacetime.

Definition 1.2.1. *A warped product spacetime is a spacetime $Q \times_r F$ with the metric*

$$\tilde{g} = g + r^2 h,$$

where (Q, g) is a simply-connected, 2-dimensional Lorentzian manifold, and (F, h) is an n-dimensional Riemannian manifold. Here $r : Q \to (0, \infty)$ is a positive function on Q. We further assume that r serves as a time function for the spacetime satisfying

$$\langle dr, dr\rangle_g < 0.$$

Since Q is simply-connected and dr is timelike, by setting $s = const$ along the integral curves for ∇r, one can assume $g = -\alpha dr^2 + \beta ds^2$ where $\alpha = \alpha(r, s), \beta = \beta(r, s)$ are functions on Q. Note that we have the freedom to choose where to set $s = 0$, but this point does not matter throughout our analysis. This means that our spacetime metric is

$$\tilde{g} = -\alpha dr^2 + \beta ds^2 + r^2 h.$$

1.2.1 Homogeneous solutions

If we put the homogeneity assumption on warped product spacetimes, the Einstein-Euler equations reduce to

$$\partial_r\Big(\frac{r^{n-1}}{\alpha} + \frac{S^{[h]}}{n(n-1)} r^{n-1} - \frac{2\Lambda}{n(n+1)} r^{n+1}\Big) = -\frac{2\gamma\rho_0}{n} \cdot \frac{1}{r^{n\gamma}\beta^{\frac{1+\gamma}{2}}} \tag{1.2}$$

$$\partial_r(\alpha\beta)^{\frac{1+\gamma}{2}} = \frac{(1+\gamma)^2\rho_0}{n} \cdot r^{1-n(1+\gamma)} \cdot \alpha^{\frac{3+\gamma}{2}}. \tag{1.3}$$

with $\rho \cdot r^{n(1+\gamma)} \cdot \beta^{\frac{1+\gamma}{2}} = \rho_0$, $S^{[h]}$ is the scalar curvature of h, and n is the dimension of the fiber h (refer to [2]). We are assuming the unknowns α and β are positive functions of r defined on $(r_*, r_0]$ where $0 \le r_* < r_0$,

$$0 < \alpha(r_0) < \infty, \quad 0 < \beta(r_0) < \infty,$$

and r_* is the first singularity; that is,

$$r_* = \inf\{r \ge 0 |\text{ There exist solutions } \alpha, \beta \text{ which are continuous over } (r, r_0] \\ \text{and differentiable over } (r, r_0)\}.$$

r_* is well defined by the Peano Existence Theorem. We also assume

$$0 < \gamma < 1$$

$$\rho_0 > 0$$

$$n \ge 2.$$

In a future paper, we will establish the mathematical definition for Big Bang singularites and classify all physically meaningful cosmological models with explicit asymptotes toward the Big Bang time.

In particular, the Big Bang time may be zero ($r_* = 0$) or nonzero ($r_* > 0$). We investigate the nonhomogeneous instability of the Big Bang for $r_* > 0$ in Chapter 3. This is also the main topic this thesis intends to focus on.

1.2.2 Recovering the FLRW spacetime - $r_* = 0$

In a future paper, we do the asymptote analysis for $r_* = 0$ case. In this section, we only emphasize that a special case for $r_* = 0$ recovers the Friedmann-Lemaître-Robertson-Walker spacetime. Recall that our warped product spacetime has the metric

$$\tilde{g} = -\alpha dr^2 + \beta ds^2 + r^2 h.$$

Setting $\sqrt{\alpha}dr = dt$, $\beta = C^2 r^2$, $h = C^2(dy^2 + dz^2)$, $R = Cr$, we recover the Friedmann-Lemaître-Robertson-Walker spacetime for $\Lambda = 0$, $\epsilon = 0$:

$$\tilde{g} = -dt^2 + R^2(ds^2 + dy^2 + dz^2).$$

Notice that when

$$\alpha = \frac{27C^2}{4M} \cdot (1+\gamma)^2 \cdot r^{1+3\gamma}, \quad \beta = C^2 r^2, \quad \rho = \frac{M}{C^{3(1+\gamma)} r^{3(1+\gamma)}},$$

where $C^{1+3\gamma} = \frac{9}{4}(1+\gamma)^2$, $M = \rho \cdot R^{3(1+\gamma)}$, $R = Cr$, one can verify that α, β, ρ defined above satisfy the equations (1.2), (1.3) , the reduced Einstein field equations for homogeneous solutions. In other words, the warped product spacetimes can be regarded as an extension of the classical cosmological models.

1.2.3 Classification of singularities - $r_* > 0$ case

Our goal is to classify all the possibilities when $r_* > 0$ (Proposition 1.2.1). We begin with an observation.

Lemma 1.2.1. *Suppose there exists a pair of solution* (α, β) *to the equations* (1.2) *and* (1.3) *having a singularity at* $r = r_* > 0$. *Then*

$$\lim_{r \to r_*^+} \frac{r^{n-1}}{\alpha(r)} = \infty.$$

Remark. *Actually in this case, we would have*

$$\lim_{r\to r_*^+}(\alpha(r)\beta(r)) = 0,$$

as we will see later.

Proof. Firstly, we claim that

$$\limsup_{r\to r_*^+}\frac{r^{n-1}}{\alpha(r)} = \liminf_{r\to r_*^+}\frac{r^{n-1}}{\alpha(r)}.$$

This is because (1.2) implies that the quantity in the parenthesis on the left hand side

$$\frac{r^{n-1}}{\alpha} + \frac{S^{[h]}}{n(n-1)}r^{n-1} - \frac{2\Lambda}{n(n+1)}r^{n+1}$$

is monotonic, and hence it has a limit (may be infinity) when $r \to r_*^+$. Since both $\frac{S^{[h]}}{n(n-1)}r^{n-1}$ and $\frac{2\Lambda}{n(n+1)}r^{n+1}$ have a **finite** limit as $r \to r_*^+$, the above claim follows. Here we use the assumption $n \geq 2$.

Secondly, we show that

$$\lim_{r\to r_*^+}\frac{r^{n-1}}{\alpha(r)} \neq 0.$$

Suppose it is zero. Since we can rewrite (1.2) as

$$\begin{aligned}\partial_r\Big(\frac{r^{n-1}}{\alpha}\Big) &= -\frac{2\gamma\rho_0}{nr^{n\gamma}\beta^{\frac{1+\gamma}{2}}} - \frac{S^{[h]}}{n}r^{n-2} + \frac{2\Lambda}{n}r^n \\ &= -\frac{2\gamma\rho_0}{nr^{n\gamma}\sigma\cdot\big(\frac{r^{n-1}}{\alpha}\big)^{\frac{1+\gamma}{2}}}\cdot r^{(n-1)\cdot\frac{1+\gamma}{2}} - \frac{S^{[h]}}{n}r^{n-2} + \frac{2\Lambda}{n}r^n \qquad (1.4)\end{aligned}$$

where $\sigma = \big(\alpha\beta\big)^{\frac{1+\gamma}{2}}$, and σ cannot go to infinity as $r \to r_*^+$ by (1.3) (because $\sigma(r_0)$ is finite), we find that the first term on the right hand side of (1.4) will go to negative infinity as $r \to r_*^+$ (because both σ and $\frac{r^{n-1}}{\alpha}$ have a limit). This is impossible if we require

$$\lim_{r\to r_*^+}\frac{r^{n-1}}{\alpha(r)} = 0$$

and $\frac{r^{n-1}}{\alpha(r)} > 0$ for $r_* < r \leq r_0$, a contradiction.

Lastly, we show that it is impossible to have

$$0 < \lim_{r \to r_*^+} \frac{r^{n-1}}{\alpha(r)} < \infty.$$

This will require the following lemma.

Lemma 1.2.2. *Suppose the solution (α, β) has a singularity at $r = r_* > 0$. It is impossible that $\lim_{r \to r_*^+} \sigma(r) = 0$ but $0 < \lim_{r \to r_*^+} \alpha(r) < \infty$, where*

$$\sigma = (\alpha\beta)^{\frac{1+\gamma}{2}}$$

as in the proof of Lemma 1.2.1.

We would postpone the proof of Lemma 1.2.2 . □

This lemma implies that $\frac{r^{n-1}}{\alpha(r)}$ will stay away from 0 when r is close to $r_* > 0$. It gives a hint to do the following change of variable

$$\tau = \frac{\alpha}{r^{n-1}}$$

$$\sigma = (\alpha\beta)^{\frac{1+\gamma}{2}}.$$

The two equations (1.2) and (1.3) become

$$\begin{aligned} \partial_r \tau &= -\tau^2 \Big(-\frac{2\gamma\rho_0}{n} \cdot r^{\frac{1}{2}n - \frac{1}{2} - \frac{\gamma}{2}n - \frac{\gamma}{2}} \cdot \frac{\tau^{\frac{1+\gamma}{2}}}{\sigma} - \frac{S^{[h]}}{n} r^{n-2} + \frac{2\Lambda}{n} r^n \Big) \qquad (1.5) \\ &= \frac{2\gamma\rho_0}{n} \cdot r^{\frac{1}{2}n - \frac{1}{2} - \frac{\gamma}{2}n - \frac{\gamma}{2}} \cdot \frac{\tau^{\frac{5+\gamma}{2}}}{\sigma} + \frac{S^{[h]}}{n} r^{n-2} \cdot \tau^2 - \frac{2\Lambda}{n} r^n \cdot \tau^2 \\ \partial_r \sigma &= \frac{(1+\gamma)^2 \rho_0}{n} \cdot r^{\frac{1}{2}n - \frac{1}{2} - \frac{\gamma}{2}n - \frac{\gamma}{2}} \cdot \tau^{\frac{3+\gamma}{2}}, \qquad (1.6) \end{aligned}$$

which imply

$$\partial_r(\tau^a \sigma) = \big(a \cdot 2\gamma + (1+\gamma)^2\big) \cdot \frac{\rho_0}{n} r^{\frac{1}{2}n - \frac{1}{2} - \frac{\gamma}{2}n - \frac{\gamma}{2}} \cdot \tau^{\frac{3+\gamma}{2} + a} + a \Big(\frac{S^{[h]}}{n} r^{n-2} - \frac{2\Lambda}{n} r^n \Big) \cdot \tau^{a+1} \sigma$$

for any real number $a \in \mathbb{R}$. In order to eliminate the first term on the right hand side, we calculate

$$\partial_r \Big(\frac{\tau^{\frac{(1+\gamma)^2}{2\gamma}}}{\sigma} \Big) = \frac{(1+\gamma)^2}{2\gamma} \Big(\frac{S^{[h]}}{n} r^{n-2} - \frac{2\Lambda}{n} r^n \Big) \tau \cdot \Big(\frac{\tau^{\frac{(1+\gamma)^2}{2\gamma}}}{\sigma} \Big)$$

$$= \frac{(1+\gamma)^2}{2\gamma} G(r)\tau \cdot \Big(\frac{\tau^{\frac{(1+\gamma)^2}{2\gamma}}}{\sigma}\Big),$$

where we denote $\big(\frac{S[h]}{n} r^{n-2} - \frac{2\Lambda}{n} r^n\big)$ by $G(r)$. This implies

Lemma 1.2.3. *Suppose there exists a pair of solution (α, β) to the equations (1.2) and (1.3) having a singularity at $r = r_* > 0$ and τ, σ are defined as above. If τ satisfies*

$$\tau(r) \le M, \quad r_* < r \le r_0$$

for some constant $M < \infty$, then the following limit

$$0 < \lim_{r \to r_1^+} \frac{\tau^{\frac{(1+\gamma)^2}{2\gamma}}}{\sigma} < \infty$$

exists.

Note that this lemma proves Lemma 1.2.2 and the remark after Lemma 1.2.1. The above lemma suggests doing the following change of variable

$$\eta = \Big(\frac{\tau^{\frac{(1+\gamma)^2}{2\gamma}}}{\sigma}\Big)$$

and then we have the new system of equations

$$\begin{aligned} \partial_r \sigma &= \frac{(1+\gamma)^2 \rho_0}{n} \cdot r^b \cdot \Big(\frac{\tau^{\frac{(1+\gamma)^2}{2\gamma}}}{\sigma}\Big)^{\frac{\gamma(3+\gamma)}{(1+\gamma)^2}} \cdot \sigma^{\frac{\gamma(3+\gamma)}{(1+\gamma)^2}} \\ &= \frac{(1+\gamma)^2 \rho_0}{n} \cdot r^b \cdot \eta^{\frac{\gamma(3+\gamma)}{(1+\gamma)^2}} \cdot \sigma^{\frac{\gamma(3+\gamma)}{(1+\gamma)^2}} \end{aligned} \tag{1.7}$$

$$\begin{aligned} \partial_r \eta &= \frac{(1+\gamma)^2}{2\gamma} G(r) \cdot \Big(\frac{\tau^{\frac{(1+\gamma)^2}{2\gamma}}}{\sigma}\Big)^{\frac{2\gamma}{(1+\gamma)^2}} \cdot \sigma^{\frac{2\gamma}{(1+\gamma)^2}} \cdot \Big(\frac{\tau^{\frac{(1+\gamma)^2}{2\gamma}}}{\sigma}\Big) \\ &= \frac{(1+\gamma)^2}{2\gamma} G(r) \cdot \eta^{\frac{2\gamma}{(1+\gamma)^2}+1} \cdot \sigma^{\frac{2\gamma}{(1+\gamma)^2}}, \end{aligned} \tag{1.8}$$

where $b = \frac{1}{2}n - \frac{1}{2} - \frac{\gamma}{2}n - \frac{\gamma}{2}$. If we try to separate σ and η, we find

$$\partial_r\big(\sigma^{\frac{1-\gamma}{(1+\gamma)^2}}\big) = \frac{1-\gamma}{(1+\gamma)^2} \cdot \frac{(1+\gamma)^2 \rho_0}{n} \cdot r^b \cdot \eta^{\frac{\gamma(3+\gamma)}{(1+\gamma)^2}}$$

$$= \frac{1-\gamma}{(1+\gamma)^2} \cdot \frac{(1+\gamma)^2 \rho_0}{n} \cdot r^b \cdot \left(\eta^{-\frac{2\gamma}{(1+\gamma)^2}}\right)^{-\frac{3+\gamma}{2}}$$

$$\partial_r\left(\eta^{-\frac{2\gamma}{(1+\gamma)^2}}\right) = -\frac{2\gamma}{(1+\gamma)^2} \cdot \frac{(1+\gamma)^2}{2\gamma} G(r) \cdot \sigma^{\frac{2\gamma}{(1+\gamma)^2}}$$

$$= -\frac{2\gamma}{(1+\gamma)^2} \cdot \frac{(1+\gamma)^2}{2\gamma} G(r) \cdot \left(\sigma^{\frac{1-\gamma}{(1+\gamma)^2}}\right)^{\frac{2\gamma}{1-\gamma}}.$$

This means if we do another change of variable

$$u = \sigma^{\frac{1-\gamma}{(1+\gamma)^2}}$$

$$v = \eta^{-\frac{2\gamma}{(1+\gamma)^2}},$$

the two evolution equations become

$$\partial_r u = \frac{(1-\gamma)\rho_0}{n} \cdot r^b \cdot v^{-\frac{3+\gamma}{2}}$$

$$\partial_r v = -G(r) \cdot u^{\frac{2\gamma}{1-\gamma}}.$$

Since the initial data of u is going to be zero in our application and it is the only difficulty to extend the differential equation over $r = r_*$, we could consider the following better evolution equations instead.

$$\partial_r u = \frac{(1-\gamma)\rho_0}{n} \cdot |r|^b \cdot v^{-\frac{3+\gamma}{2}} \tag{1.9}$$

$$\partial_r v = -G(r) \cdot |u|^{\frac{2\gamma}{1-\gamma}}. \tag{1.10}$$

This does not change what (u, v) is because $u(r_*) = 0$ and the right hand side of (1.9) is positive. By the Peano existence theorem, for ϵ small enough, there exists a solution (u, v) solving (1.9) and (1.10) on $[r_* - \epsilon, r_* + \epsilon]$, with initial data $u(r_*) = 0$, $v(r_*) > 0$, where we are assuming $r_1 > 0$.

Lemma 1.2.4. *Let $r_* > 0$, $A > 0$, and $v_* > 0$ be given. There exists an $\epsilon = \epsilon(r_*, A, v_*) > r_*$ so that there exists a solution $(u(r), v(r))$ to the system of equations* (1.9) *and* (1.10) *with the initial data*

$$u(r_*) = 0$$

$$v(r_*) = v_*$$

and the bounds

$$0 < u(r) \leq A$$

$$\frac{1}{2}v_* \leq v(r) \leq 2v_*$$

for $r_* - \epsilon \leq r \leq r_* + \epsilon$. Notice that with these initial data, $\lim_{r\to r_*^+} \alpha(r) = 0$, so that corresponds to a singularity of α.

Proposition 1.2.1. *Let $r_* > 0$. We have the following.*

1. *If there exists a solution (α, β) to the equations* (1.2) *and* (1.3)*, having a singularity at $r = r_* > 0$, then*

$$\lim_{r\to r_*^+} \frac{r^{n-1}}{\alpha(r)} = \infty$$

$$\lim_{r\to r_*^+} (\alpha(r) \cdot \beta(r)) = 0.$$

2. *Conversely, given any $A > 0$ and $v_* > 0$, there exists an $r_0 = r_0(r_*, A, v_*) > r_*$ so that there exists a solution (α, β) (on $(r_*, r_0]$) to the system of equations* (1.2) *and* (1.3) *with*

$$\lim_{r\to r_*^+} \left(\alpha(r)\beta(r)\right)^{\frac{1-\gamma}{2(1+\gamma)}} = 0$$

$$\lim_{r\to r_*^+} r^{n-1} \cdot \frac{1}{\alpha} \cdot \left(\alpha\beta\right)^{\frac{\gamma}{1+\gamma}} = v_*,$$

which implies $\lim_{r\to r_*^+} \alpha(r) = 0$, and with the bounds

$$0 \leq \left(\alpha(r)\beta(r)\right)^{\frac{1-\gamma}{2(1+\gamma)}} \leq A$$

$$\frac{1}{2}v_* \leq r^{n-1} \cdot \frac{1}{\alpha} \cdot \left(\alpha\beta\right)^{\frac{\gamma}{1+\gamma}} \leq 2v_*.$$

Remark. *The relation between* (u, v) *and the original unknowns* (α, β) *is*

$$u = \left(\alpha\beta\right)^{\frac{1-\gamma}{2(1+\gamma)}}$$
$$v = r^{n-1} \cdot \frac{1}{\alpha} \cdot \left(\alpha\beta\right)^{\frac{\gamma}{1+\gamma}},$$

or

$$\alpha = u^{\frac{2\gamma}{1-\gamma}} \cdot \frac{1}{v} \cdot r^{n-1}$$
$$\beta = u^{\frac{2}{1-\gamma}} \cdot v \cdot \frac{1}{r^{n-1}}.$$

Since $\lim_{r\to r_*^+} u = 0$ *and* $0 < \lim_{r\to r_*^+} v < \infty$*, we have*

$$\alpha \approx (r - r_*)^{\frac{2\gamma}{1-\gamma}}$$
$$\beta \approx (r - r_*)^{\frac{2}{1-\gamma}},$$

where the implicit constant depends on $0 < \gamma < 1, \rho_0, n, G(r_*)$*, and* $r_* > 0$*.*

CHAPTER 2

TECHNIQUES FOR SHOCKS

This chapter aims to introduce the dynamics of compressible fluids (Section 2.1 and 2.2) and the techniques to deal with shocks (Section 2.3 and 2.4).

2.1 Newtonian fluids

This section aims to derive the conservation laws for classical Newtonian fluids. We will basically follow Chapter 1 of Toro's book [26]. We use $\rho : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ as the fluid mass density, $v : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ as the fluid velocity, $p : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ as the fluid pressure, e as the specific internal energy, s as the specific entropy.

Let $U(t) \subset \mathbb{R}^3$ (called control volume in the Physics context) be a family of open, bounded, connected regions, bounded by the smooth boundary $\partial U(t)$ moving with the fluid. For any quantity

$$\Psi(t) = \int_{U(t)} \psi(t,x)dx$$

with $\psi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$, we have the material derivative is

$$\frac{d\Psi}{dt} = \int_{U(t)} \partial_t \psi(t,x)dx + \int_{\partial U(t)} \psi(t,x)(v \cdot n)dS$$

where $v : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ is the fluid velocity, the velocity of the boundary ∂U, and n is the outward normal vector on ∂U. Intuitively, the second surface integral says that if the boundary tends to expand ($v \cdot n > 0$), ψ should contribute to Ψ at the points ∂U is expanding. Applying Divergence theorem to this surface integral, we derive

$$\frac{d\Psi}{dt} = \int_{U(t)} \partial_t \psi(t,x)dx + \int_{U(t)} div(\psi v)dx.$$

One can take $\psi(t,x)$ to be the density ρ, the momentum ρv, or the energy E to derive the following conservation laws:

$$(\partial_t \rho) + div(\rho v) = 0 \tag{2.1}$$

$$\partial_t(\rho v_i) + div(\rho v_i v) + (\partial_i p) = 0, \quad 1 \le i \le 3 \tag{2.2}$$

$$(\partial_t E) + div(Ev) + div(pv) = 0 \tag{2.3}$$

where p is the pressure coming from the stress tensor, $E = \frac{1}{2}\rho|v|^2 + \rho e$, and e is the specific internal energy. The first equation (2.2) comes from the conservation of mass. For (2.2), we are assuming that the stress tensor is diagonal and a multiple of the identity matrix $I \in M_{3\times 3}(\mathbb{R})$. Therefore, the source term for the rate of change of the momentum, coming from the stress tensor acting on the boundary ∂U, is

$$-\int_{\partial U} (pI)ndS = -\int_U (\nabla p)dx,$$

which appears as the last term $(\partial_i p)$ in (2.2). In (2.3), we are assuming the stress tensor is pI again and moreover there is no net heat flowing across the boundary. The stress energy will provide the source term for the rate of change of energy:

$$-\int_{\partial U} (pI)n \cdot vdS = -\int_U div(pv)dx.$$

The term $(pI)n \cdot v$ comes from the fact that power = force $\cdot$ velocity. This explains the last term $div(pv)$ in (2.3). These three conservation laws combined with equation of state give the following observation:

Observation 1. *In the isentropic case, the conservation law of energy (2.3) is redundant.*

Proof. To simplify the notation, we assume the fluid is in $\mathbb{R}$ rather than $\mathbb{R}^3$, but the result still holds in $\mathbb{R}^3$. The first two conservation laws now become

$$(\partial_t \rho) + \partial_x(\rho v) = 0$$

$$\partial_t(\rho v) + \partial_x(\rho v^2) + (\partial_x p) = 0.$$

Using the Leibniz rule, we can convert them to the following equations

$$(\partial_t \rho) + \partial_x(\rho v) = 0 \tag{2.4}$$

$$(\partial_t v) + v(\partial_x v) + \frac{1}{\rho}(\partial_x p) = 0. \tag{2.5}$$

On the other hand, the isentropic condition means $ds = 0$ along the fluid line (the $\partial_t + v\partial_x$ direction), where s denotes the specific entropy. This implies that

$$0 = Tds = de + pd\Big(\frac{1}{\rho}\Big) = de - \frac{1}{\rho^2}pd\rho \tag{2.6}$$

from the first law of Thermodynamics. Here e denotes the specific internal energy, p is the pressure, and $\frac{1}{\rho}$ is the specific volume. In order to show that the energy conservation law is redundant, we calculate

$$\begin{aligned}
(\partial_t E) + \partial_x(Ev) + \partial_x(pv) &= \partial_t\Big(\frac{1}{2}\rho v^2 + \rho e\Big) + \partial_x\Big(\frac{1}{2}\rho v^3 + \rho v e\Big) + \partial_x(pv) \\
&= \rho\partial_t\Big(\frac{1}{2}v^2\Big) + \rho(\partial_t e) + \rho v\partial_x\Big(\frac{1}{2}v^2\Big) + \rho v(\partial_x e) + \partial_x(pv) \\
&= \rho v(\partial_t + v\partial_x)v + \rho(\partial_t + v\partial_x)e + \partial_x(pv) \\
&= \rho v\Big(-\frac{1}{\rho}(\partial_x p)\Big) + \frac{p}{\rho}(\partial_t + v\partial_x)\rho + \partial_x(pv) \\
&= \frac{p}{\rho}(\partial_t + v\partial_x)\rho + p(\partial_x v) = 0,
\end{aligned}$$

where we make use of the definition of E, (2.4), (2.5), and (2.6).

□

Indeed, the isentropic condition and the energy conservation law are equivalent by the same computation. A natural question arises:

What if the entropy is no longer constant in time?

Historically ([8]), when Riemann considered one dimensional fluids, he discovered that even starting from smooth initial data, shocks can appear in finite time. Before the formation of shocks, the solution is smooth, and therefore the energy conservation law is equivalent to the adiabatic condition Riemann was considering. However, after the shock formation, the two are no longer equivalent to each other. Thus, if one wants to continue the solution after the shock formation, one must choose between energy equations and the adiabatic condition. Riemann made the wrong choice before the concept of entropy was introduced. After Clausius introduced the concept of entropy, it is clear that one should let the entropy increase across the shock boundary while maintaining the energy conservation law. The correct jump condition across the shock boundary is the Rankine–Hugoniot conditions.

Example 2.1.1. *(RH condition) This example follows Section 3.4 in Evans' book [10]. Consider the partial differential equation*

$$\partial_t u + \partial_x(F(u)) = 0 \quad \text{in } [0,\infty)\times\mathbb{R}. \tag{2.7}$$

Let C be a regular curve cutting through $[0,\infty)\times\mathbb{R}$ described by

$$C = \{(t, x(t)) \mid t \geq 0\}.$$

Set the left part of $[0,\infty)\times\mathbb{R}$ to be Ω_l and the right part to be Ω_r. In other words,

$$[0,\infty)\times\mathbb{R} = \Omega_l \cup C \cup \Omega_r.$$

The formulation for weak solutions to (2.7) is as follows. Given a test function $v \in C_c^1([0,\infty)\times\mathbb{R})$, we should have

$$-\int_0^\infty \int_{\mathbb{R}} \Big(u(\partial_t v) + F(u)(\partial_x v)\Big) dx dt - \int_{\mathbb{R}} g v dx = 0, \tag{2.8}$$

where $g(x) = u(0,x)$ for all $x \in \mathbb{R}$. Chossing the test function v to be compactly supported in Ω_l and Ω_r respectively, we conclude that

$$\partial_t u + \partial_x(F(u)) = 0 \quad \text{in } \Omega_l$$

and

$$\partial_t u + \partial_x(F(u)) = 0 \quad \text{in } \Omega_r.$$

Chossing the test function v that does not vanish on C, (2.8) gives

$$-\iint_{\Omega_l} \Big(u(\partial_t v) + F(u)(\partial_x v)\Big) dx dt - \iint_{\Omega_r} \Big(u(\partial_t v) + F(u)(\partial_x v)\Big) dx dt - \int_{\mathbb{R}} g v dx = 0.$$

Using the divergence theorem, we have

$$-\int_C (u_l v \vec{n}_t + F(u_l) v \vec{n}_x) ds + \int_C (u_r v \vec{n}_t + F(u_r) v \vec{n}_x) ds = 0,$$

where $\vec{n}$ is the normal vector on C pointing from Ω_l to Ω_r, $ds = \sqrt{(dt)^2 + (dx)^2}$, u_l denotes the limit of u from the left of C, and u_r denotes the limit of u from the right of C. This implies that the velocity of C

$$(1, \dot{x}(t))$$

is parallel to

$$(u_l - u_r, F(u_l) - F(u_r)).$$

In other words, the velocity of C is the jump of $F(u)$ divided by the jump of u:

$$\dot{x}(t) = \frac{F(u_l) - F(u_r)}{u_l - u_r}.$$

This is called Rankine-Hugoniot condition.

2.2 Relativisitic fluids

Our goal in this section is to explain the connection between relativsitc fluids and Newtonian fluids. This section basically follows Chapter 4 of [21]. We use $\mathbb{R}^{1,3}$ as the Minkowski spacetime, $\rho : \mathbb{R}^{1,3} \to \mathbb{R}$ as the fluid proper mass density, ξ as the fluid velocity, a $(4,0)$-tensor on the Minkowski spacetime $\mathbb{R}^{1,3}$ with $\langle \xi, \xi \rangle_\eta = -c^2$, $p : \mathbb{R}^{1,3} \to \mathbb{R}$ as the fluid pressure, ϵ as the internal energy density (different from the specific internal energy $e = \frac{\epsilon}{\rho}$ from the previous section), s as the specific entropy, c as the light speed, $\mu = \rho c^2 + \epsilon$ as the energy density. The energy momentum tensor for a perfect fluid in this section is

$$T = \frac{\mu + p}{c^2} \xi \otimes \xi + p\eta,$$

where $\eta = -d(x^0)^2 + d(x^1)^2 + d(x^2)^2 + d(x^3)^2$ with $x^0 = ct$. We write $\xi = \xi^a \partial_a$ as

$$\xi^a = \frac{1}{\sqrt{1 - \frac{|v|^2}{c^2}}}(c, v) \tag{2.9}$$

where $v = (v^1, v^2, v^3)$. The idea is that when $|v| \ll c$, the relativistic conservation law for fluids should reduce to Newtonian conservation law. The results in this section for the Minkowski spacetime can be generalized to a general curved spacetime.

The conservation laws for relativistic fluids are

$$div(\rho\xi) \coloneqq \nabla_a(\rho\xi^a) = 0 \tag{2.10}$$

$$div(T) \coloneqq \nabla_a T^{ab} = 0. \tag{2.11}$$

Using (2.9), one can show that the first conservation law (2.10) reduces to (2.1) when $|v| \ll c$. The second conservation law actually implies (2.2) and (2.3) as we show below. Expanding (2.11) gives

$$\underbrace{\frac{1}{c^2}(\nabla_\xi \mu)\xi + \frac{\mu+p}{c^2}(\nabla_a \xi^a)\xi}_{\xi \text{ direction}} + \underbrace{\frac{\mu+p}{c^2}\nabla_\xi \xi + \Pi \cdot \nabla p}_{\xi^\perp \text{ direction}}$$

where $\Pi = \frac{1}{c^2}(\xi \otimes \xi) + \eta$ is the orthogonal projection to the spatial hyperplane. The first two terms are in the ξ direction, and the latter two terms are orthogonal to the ξ direction. Therefore, we arrive at

$$(\nabla_\xi \mu) + (\mu + p)(\nabla_a \xi^a) = 0 \tag{2.12}$$

$$\frac{\mu+p}{c^2}\nabla_\xi \xi + \Pi \cdot \nabla p = 0. \tag{2.13}$$

The first equation (2.12) gives

$$\nabla_\xi \epsilon - \frac{\epsilon + p}{\rho}\nabla_\xi \rho = 0, \tag{2.14}$$

where we use $\mu = \rho c^2 + \epsilon$ and $\nabla_a \xi^a = -\frac{1}{\rho}\nabla_\xi \rho$ from (2.10). This equation (2.14) is precisely the isentropic condition along the fluid velocity ξ: one can rewrite (2.14) as

$$\nabla_\xi\left(\frac{\epsilon}{\rho}\right) + p\nabla_\xi\left(\frac{1}{\rho}\right) = 0,$$

which implies $\nabla_\xi s = 0$ by the first law of Thermodynamics, identifying $\frac{\epsilon}{\rho}$ as the specific internal energy and $\frac{1}{\rho}$ as the specific volume. The second equation (2.13) can be reduced to (2.3) using $\mu = \rho c^2 + \epsilon$ and $|v| \ll c$.

2.3 Total variation: John's technique

This section aims to prove Proposition 2.3.1. This is a more transparent version of John's technique (refer to [15]). In John's original paper, he incorporated a lot of ingredients to the bootstrap mechanism. The equation considered there does not have source terms. We will describe the crucial technique for shock formation in a simpler setting, and our equation involves a source term.

2.3.1 Main equation

The main equation (analogous to (2.1), (2.2), (2.3)) for this section is

$$\partial_t u + \partial_x (A(u)) = \tilde{F}(u)$$

where $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$ is the unknown, and $A : \mathbb{R}^2 \to \mathbb{R}^2$ and $\tilde{F} : \mathbb{R}^2 \to \mathbb{R}^2$ are functions with u as the input. We assume the equation satisfies the following assumption.

Assumption 1. *The main equation can be reduced to the evolution equations for Riemann invariants:*

$$(\partial_t + \lambda_1 \partial_x) v_1 = F_1$$

$$(\partial_t + \lambda_2 \partial_x) v_2 = F_2,$$

where $Span^{\mathbb{R}}\{u_1, u_2\} = Span^{\mathbb{R}}\{v_1, v_2\}$, and $\lambda_1 \neq \lambda_2$, F_1, F_2 are scalar functions of v_1 and v_2. When $F_1 = 0 = F_2$, v_1 and v_2 are the classical Riemann invariants.

Remark. *We provide a special case where the above assumption is satisfied. When the derivative dA can be decomposed to*

$$dA = PDP^{-1}$$

with D a diagonal matrix, and P^{-1} satisfying that $(P^{-1})_{11} = (P^{-1})_{21}$ are functions of u_1, and $(P^{-1})_{12}$, $(P^{-1})_{22}$ are constants, one can show that the assumption holds.

Since $\lambda_1 \neq \lambda_2$, we can use the characteristics to foliate the spacetime $\mathbb{R} \times \mathbb{R}$, assuming the solutions v_1, v_2 exist in a certain spacetime region, and therefore the functions λ_1, λ_2 of v_1, v_2 can be regarded as functions of (t,x). Regarding $\{0\}\times\mathbb{R}$ as the initial slice, we define the coordinates for characteristics as follows.

$$\{(t, X_i(t;z)) \mid t > 0\}$$

is the characteristic curve for $(\partial_t + \lambda_i \partial_x)$ passing through the point $(0,z)$, where $i = 1,2$ and $z \in \mathbb{R}$. In other words, X_i is x-coordinate of the characteristic curve, and we are using t as the parameter of the curve emanating from $(0,z)$.

The crucial property is as follows. Using the characteristic viewpoints, we actually have a *linear* control (not quadratic!) on the *total variation* of the Riemann invariants v_1, v_2. More specifically, on the one hand, the evolution equations for Riemann invariants implies

$$(\partial_t + \lambda_1\partial_x)(\partial_x v_1) = (\partial_x F_1) - (\partial_x \lambda_1)(\partial_x v_1).$$

On the other hand, the definition of X_1 gives an evolution equation for $\frac{\partial X_1}{\partial z}$

$$(\partial_t + \lambda_1\partial_x)\Big(\frac{\partial X_1}{\partial z}\Big) = \frac{d}{dt}\Big(\frac{\partial X_1}{\partial z}\Big) = \frac{\partial}{\partial z}\Big(\frac{dX_1}{dt}\Big) = \frac{\partial}{\partial z}\lambda_1 = (\partial_x\lambda_1)\Big(\frac{\partial X_1}{\partial z}\Big),$$

where $\frac{d}{dt} = (\partial_t + \lambda_1\partial_x)$ denotes the directional derivative along the characteristic. Combining these two, we see that we have an evolution for the integrand of the total variation.

$$(\partial_t + \lambda_1\partial_x)\Big((\partial_x v_1)\cdot\frac{\partial X_1}{\partial z}\Big) = (\partial_x F_1)\cdot\frac{\partial X_1}{\partial z}.$$

It is important that the $(\partial_x\lambda_1)$ terms cancel. The term $-(\partial_x\lambda_1)(\partial_x v_1)$ is expected to be the source for shock formation since it gives a term like $-(D_{v_1}\lambda_1)(\partial_x v_1)^2$ and the equation becomes a Riccati equation with finite blowup time if the sign is correct. What John found is that, despite this blowup tendancy, one can still get a control on the total variation, as we explain in the following important calculation. For $t > 0$,

$$\begin{aligned}
\int_{(\Gamma_t)} |\partial_x v_1| dx &= \int_{z_{LL}\ (\Gamma_t)}^{z_R} \Big|\partial_x v_1 \cdot \frac{\partial X_1}{\partial z}\Big| dz \\
&= \int_{z_L\ \ (\Gamma_0)}^{z_R} |\partial_x v_1| dz + \int_0^t \int_{z_{LL}\ (\Omega_t)}^{z_R} \Big|(\partial_x F_1)\cdot\frac{\partial X_1}{\partial z}\Big| dz dt
\end{aligned}$$

$$\leq \int_{z_L\ \ (\Gamma_0)}^{z_R} |\partial_x v_1| dz + \iint_{(\Omega_t)} |D_{v_1} F_1||\partial_x v_1| + |D_{v_2} F_1||\partial_x v_2| dx dt. \tag{2.15}$$

This means that, if we have control on the initial total variation of v_1, v_2, and if $|D_{v_1}F_1|$, $|D_{v_2}F_1|$ are pointwise uniformly bounded, we will have a closed feedback for the total variation of v_1 and v_2, and then we can do the Gronwall's inequality and run the bootstrap mechanism accordingly.

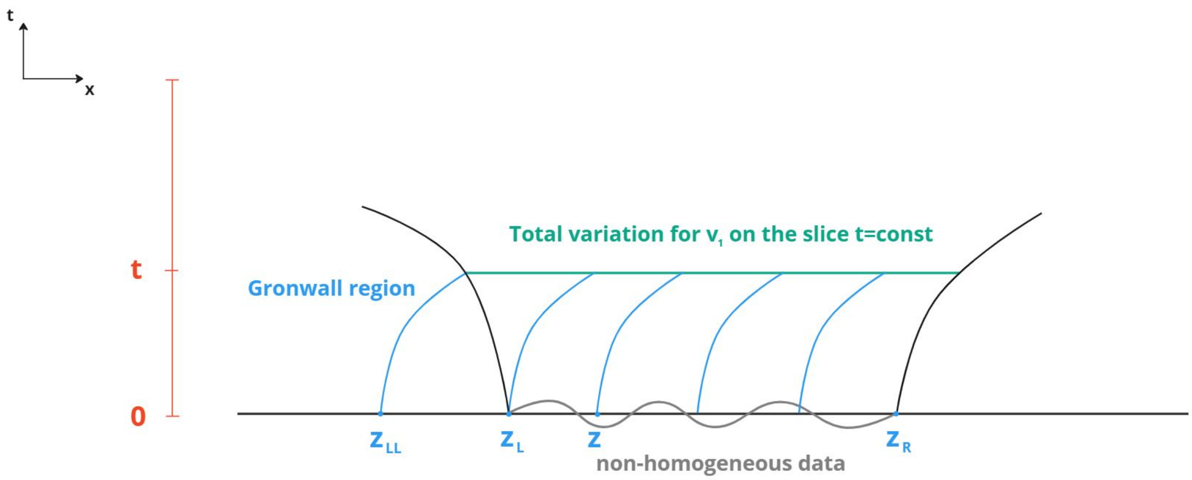

Figure 2.1 The strategy to control the total variation.

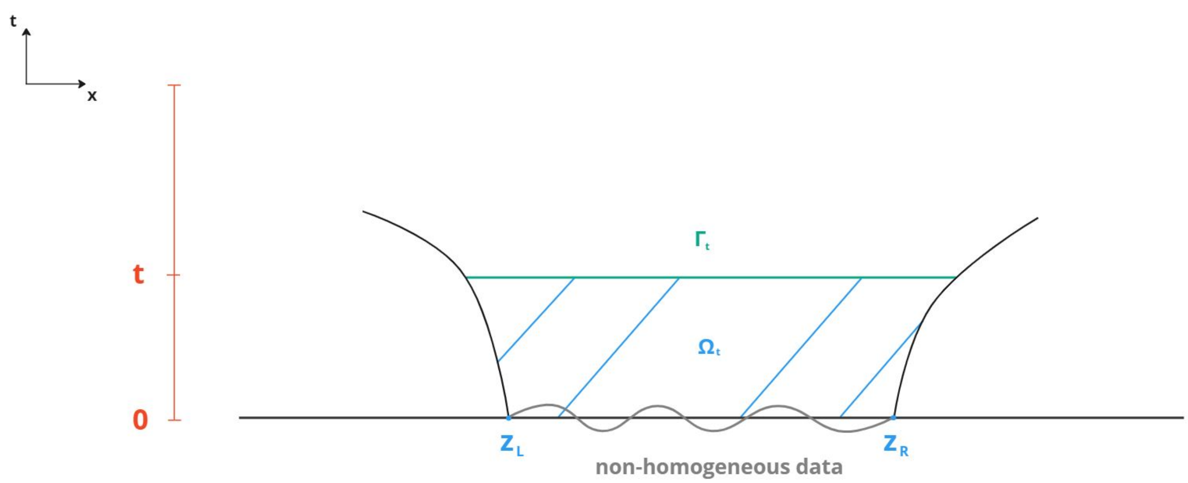

Figure 2.2 The picture for Γ_t, Ω_t.

Here we are using the notations (assuming $\lambda_1 < \lambda_2$)

$$z_L < z_R$$

$$\Gamma_t = \{(t,x) \mid X_2(t;z_L) \le x \le X_1(t;z_R)\}$$

$$\Omega_t = \{(\tau,x) \mid 0 \le \tau \le t, X_2(\tau;z_L) \le x \le X_1(\tau;z_R)\}.$$

We are also assuming the solutions exhibit homogeneous behavior outside the wave propagation cone so that we only have to focus on the region Ω_t.

Assumption 2. *The solution* (v_1, v_2)*, or equivalently* (u_1, u_2)*, satisfies*

$$(\partial_x v_1) = 0 = (\partial_x v_2)$$

on $(\mathbb{R}_+ \times \mathbb{R}) - \Omega_\infty$.

Proposition 2.3.1. *Assumption 1 and 2 imply the control of total variation (2.15).*

Remark. *If our equation allows homogeneous solutions, then Assumption 2 holds if the initial perturbation is 0 outside a compact subset* Γ_0*, according to the finite speed of propagation property.*

2.4 Piontwise blowup: Riccati equation

This section aims to establish Proposition 2.4.1. Continuing from the Assumption 1, we see a Riccati type structure for the evolution of $\partial_s v_1$:

$$(\partial_t + \lambda_1\partial_x)(\partial_x v_1) + (\partial_x\lambda_1)(\partial_x v_1) = (\partial_x F_1)$$

$$(\partial_t + \lambda_1\partial_x)(\partial_x v_1) + (D_{v_1}\lambda_1)(\partial_x v_1)^2 + (D_{v_2}\lambda_1)(\partial_x v_1)(\partial_x v_2) = (D_{v_1}F_1)(\partial_x v_1) + (D_{v_2}F_1)(\partial_x v_2).$$

Using the integral factor method for ordinary differential equations, we derive

$$(\partial_t + \lambda_1\partial_x)(e^f \cdot \partial_x v_1) = -e^{-f}(D_{v_1}\lambda_1)(e^f \cdot \partial_x v_1)^2 + e^f(D_{v_2}F_1)(\partial_x v_2), \tag{2.16}$$

where

$$f = \int_{0\ (X_1)}^{t} (D_{v_2}\lambda_1)(\partial_x v_2) - (D_{v_1}F_1)dt$$

is the integral over the curve $(t, X_1(t;z))$ and thus depends on the initial position $z \in \mathbb{R}$. Regarding the weighted derivative as a new variable $y = e^f \cdot \partial_x v_1$, one can see that there is a chance for y to blow up at finite time if the sign of $-(D_{v_1}\lambda_1)$ is correct, based on the finite blowup time property for a Riccati equation. This induces the following assumption.

Assumption 3. *Fix $z \in \mathbb{R}$ and let X_1 denote the characteristic for $(\partial_t + \lambda_1\partial_x)$ issuing from $(t, x) = (0, z)$. We assume there exist constants c, C, ϵ, which may depend on z, so that along X_1,*

- $(D_{v_1}\lambda_1) \geq c > 0$
- $c \leq e^f \leq C$
- $|\partial_s v_2| \leq \epsilon$
- $|D_{v_2}F_1| \leq C.$

Remark. *In Chapter 3, the lower bounds for $(D_{v_1}\lambda_1)$ and e^{-f} (the coefficient for $(e^f \cdot \partial_x v_1)^2$) are stronger. We also have to worry about whether the shock happens before the Big Bang blowup time there.*

Remark. *We only require $|\partial_s v_2| \leq \epsilon$ along X_1, not necessarily globally in space.*

Therefore, if Assumption 3 holds, the weighted derivative $e^f \cdot \partial_x v_1$, with sufficiently large initial data $\partial_x v_1(0, z)$, follows a Riccati equation (2.16) similar to

$$\frac{dy}{dt} \geq y^2,$$

which blows up at finite time. Since e^f has a positive lower bound, $\partial_s v_1$ also goes to infinity and thus concludes the shock formation.

Proposition 2.4.1. *Assumption 1, 2, 3 imply the shock formation within finite time.*

CHAPTER 3

INSTABILITY OF THE BIG BANG

3.1 Introduction

Starting with the homogeneous solutions to Einstein-Euler equations derived from Chapter 1, we prove the instability of the $r_* > 0$ Big Bang models under non-homogeneous, compactly supported perturbations. For notations, see 3.3.4.

Theorem 3.1.1. *Fix a background homogeneous fluid over $(r_*, r_0]$ having the Big Bang singularity at $r = r_* > 0$. The asymptote of $(\mathring{\rho}\mathring{\alpha})$ will determine $r_{mid} \in (r_*, r_0)$. We regard r_*, r_{mid}, r_0 as parameters (defined in Section 3.3.4). Fix $r_{cut} \in (r_*, r_{mid})$ which is not a parameter. There exist $LB = LB(r_{cut}, parameters)$, $\epsilon_{a,0} = \epsilon_{a,0}(LB, r_{cut}, parameters)$, $\epsilon_0 = \epsilon_0(LB, \epsilon_{a,0}, r_{cut}, parameters)$ so that for every compactly supported data, specified on $\{r = r_0\}$, satisfying*

- $E(r_0) = \int_{\Gamma_{r_0}} |\partial_s v_1| + |\partial_s v_2| ds < \epsilon_0$ *(small total variation)*
- $\int_{\Gamma_{r_0}} |\partial_s \ln \alpha| + |\partial_s \ln \beta| ds < \epsilon_{a,0}$ *(small metric total variation)*
- $|\partial_s v_2(r_0, s)| < \epsilon_{a,0}$ *for* $s \in [-1, 0]$ *(small opponent)*
- $\partial_s v_1(r_0, -\frac{1}{2}) \geq LB$,

$\partial_s v_1$ goes to infinity along the (λ_1) characteristic before $r = r_{cut}$ (meaning somewhere in $[r_{cut}, r_0]$). Moreover

$$\lim_{r_{cut} \to r_*^+} LB(r_{cut}) = 0.$$

We construct a sequence of initial data that satisfies the above constraint in Section 3.6. Section 3.3.4 includes all the notations.

During the time $r_{cut} \leq r \leq r_0$, it is guaranteed that the total variations $E(r)$, $\int_{\Gamma_r} |\partial_s \ln \alpha| + |\partial_s \ln \beta| ds$ remain small. Note that this control is valid only up to r_{cut}.

Proof. The proof is in Section 3.5.3. □

Remark. *The theorem says that, for a family of background solutions that satisfy certain condition involving C_2, we can find a sequence of initial data* ***that goes to background*** *in $W^{1,\infty}$ so that shock forms before $r = r_*$. In other words, this family of background solutions (modelling the Big Bang) are unstable.*

Remark. *We can of course arrange things so that $\epsilon_0 = \epsilon_{a,0}$. The reason we introduce another notation $\epsilon_{a,0}$ (where a for auxiliary) is to keep track of their contribution throughout the bootstrap framework.*

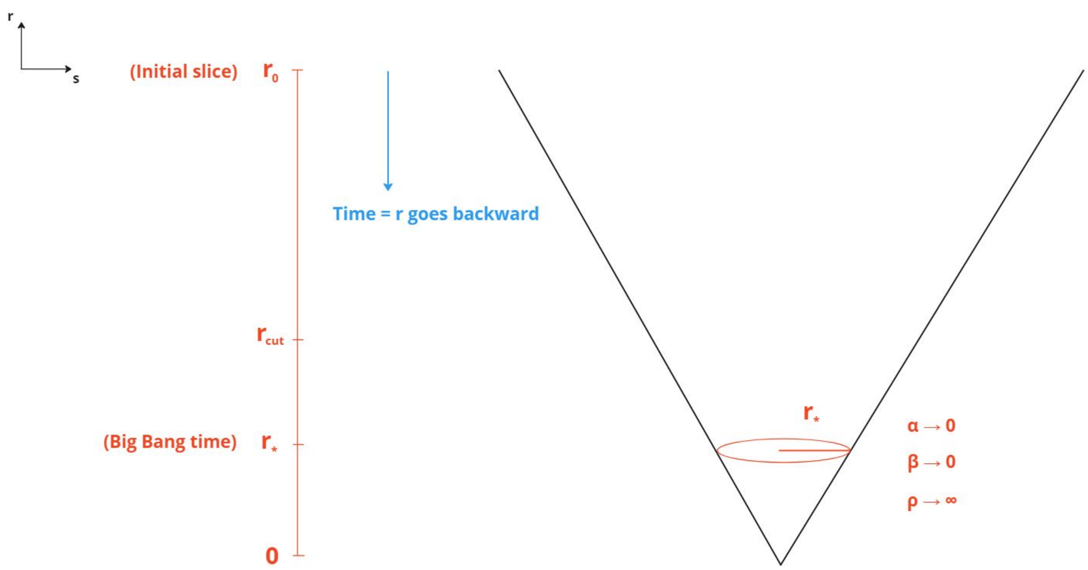

Figure 3.1 The picture for the big bang model considered in this work.

3.2 Main Equations

3.2.1 Geometry

As in [2], we consider warped product spacetimes defined as follows.

Definition 3.2.1. *A warped product spacetime is a spacetime $Q \times_r F$ with the metric*

$$\tilde{g} = g + r^2 h,$$

where Q is a simply-connected, 2-dimensional Lorentzian manifold with the metric g and F is an n-dimensional Riemannian manifold with the metric h. Here $r : Q \to (0, \infty)$ is a positive function on Q. We further assume that r serves as the time function for the spacetime satisfying

$$\langle dr, dr \rangle_g < 0.$$

Notice that in this paper, we assume all the dynamics for the warped product spacetime are exhibited in Q, and we regard F as a fixed fiber. Since Q is assumed to be simply-connected, we are able to assume the metric g has the following form

$$g = -\alpha dr^2 + \beta ds^2,$$

where α, β are positive functions on Q. The reason is as follows. By the simply-connectedness, we can construct the integral curves along ∇r. We define another function s on Q by setting these integral curves to be $s = const$. Thus, ∂_r is timelike since $\langle \nabla r, \nabla r \rangle_g = \langle dr, dr \rangle_g < 0$. ∂_s is spacelike since Q is Lorentzian. ∂_r is orthogonal to ∂_s since ∇r is orthogonal to $r = const$ slice.

3.2.2 Einstein-Euler equations

We consider the Einstein-Euler equations in this paper:

$$Ric^{[\tilde{g}]} = T - \frac{1}{n} tr(T)\tilde{g} + \frac{2}{n}\Lambda \tilde{g},$$

where $T = (p + \rho)\xi \otimes \xi + p\tilde{g}$ is the energy momentum tensor for a perfect fluid. Here p, ρ, ξ represent the pressure, density, and fluid velocity of the fluid respectively. We further impose the equation of state

$$p = \gamma \rho$$

for ultrarelativistic fluids. Here $0 < \sqrt{\gamma} < 1$ is the sound speed. Since the fluid veloctiy has unit length $\langle \xi, \xi \rangle = -1$, we use θ to parametrize $\xi = \xi^r \partial_r + \xi^s \partial_s$:

$$\sqrt{\alpha}\xi^r = \sqrt{1 + \theta^2}$$

$$\sqrt{\beta}\xi^s = \theta,$$

where $\theta \in \mathbb{R}$ is a scalar indicating how much the fluid velocity deviates from the time direction ∂_r. In summary, we have four unknowns in this paper: two metric components α, β and two fluid variables θ, ρ.

3.2.3 Reduced Einstein Field Equations

After expanding the definition of Ricci curvature, we get the following reduced Einstein field equations

$$\partial_r\left(\sqrt{\beta}\cdot r^n\rho^{\frac{1}{1+\gamma}}\cdot\sqrt{1+\theta^2}\right)+\partial_s\left(\sqrt{\alpha}\cdot r^n\rho^{\frac{1}{1+\gamma}}\cdot\theta\right)=0$$
$$\partial_r\left(\sqrt{\beta}\cdot\rho^{\frac{\gamma}{1+\gamma}}\cdot\theta\right)+\partial_s\left(\sqrt{\alpha}\cdot\rho^{\frac{\gamma}{1+\gamma}}\sqrt{1+\theta^2}\right)=0$$
$$(\partial_r\ln\alpha)=\left(2(1+\gamma)\theta^2+2\gamma\right)\cdot\frac{r}{n}(\rho\alpha)+\frac{n-1}{r}-\alpha\left(\frac{2\Lambda}{n}r-\frac{S^{[h]}}{nr}\right)$$
$$(\partial_r\ln\beta)=\left(2(1+\gamma)\theta^2+2\right)\cdot\frac{r}{n}(\rho\alpha)-\frac{n-1}{r}+\alpha\left(\frac{2\Lambda}{n}r-\frac{S^{[h]}}{nr}\right)$$

with the constraint

$$(\partial_s\ln\alpha)=-2(1+\gamma)\cdot\frac{r}{n}(\rho\alpha)\cdot\frac{\sqrt{\beta}}{\sqrt{\alpha}}\cdot\theta\sqrt{1+\theta^2}.$$

The main unknowns are θ, ρ, α, β (and become v_1, v_2, α, β later). The first two equations form a hyperbolic system and describe the evolution of the fluid variables θ and ρ. The third and the fourth equations describe the evolution of the metric components α and β. The last constraint equation is a condition for initial data and is compatible with the third equation. Once the initial data satisfies the constraint, the constraint holds forever.

Lemma 3.2.1. *The Einstein field equations are equivalent to the reduced Einstein field equations for smooth solutions.*

Proof. Assume the solution $(\theta, \rho, \alpha, \beta)$ satisfies the Einstein field equations. We have

1. Conservation of energy momentum tensor, $\nabla^a T_{ab} = 0$.

2. Einstein equation restricted on Q, $Ric^{[g]} - \frac{n}{r}\nabla^2 r = T_Q - \frac{1}{n}(tr_{\tilde{g}}T) + \frac{2}{n}\Lambda g$.

3. Einstein equation restricted on F, $Ric^{[h]} - \Big(r\Delta_g r + (n-1)\langle dr, dr\rangle_g\Big)h = T_F - \frac{1}{n}(tr_{\tilde{g}}T)(r^2 h) + \frac{2}{n}\Lambda(r^2 h)$.

Here T_Q, T_F denote the restriction of the energy momentum tensor T on the base Q and the fiber F, respectively. Expanding the conservation law gives the first two reduced Einstein field equations. Applying tr_g to the restrction on Q (simplified by using (2) on the $\partial_s \otimes \partial_s$ direction) and expanding the restriction on F give the third and the fourth reduced Einstein field equations. Applying the striction on Q to the $\partial_r \otimes \partial_s$ direction gives the constraint equation for $\ln\alpha$.

Conversely, assume the solution $(\theta, \rho, \alpha, \beta)$ satisfies the reduced Einstein field equations. Denote the tensor $Ric^{[\tilde{g}]} - T + \frac{1}{n}tr(T)\tilde{g} - \frac{2}{n}\Lambda\tilde{g}$ by $Einstein$. From the above argument, we know that the third and the fourth reduced Einstein equations imply $(Einstein)_F = 0$, where $(Einstein)_F$ is the Einstein tensor restricted on F. To prove $(Einstein)_Q = 0$, we observe that

- The constraint reduced Einstein equation implies $(Einstein)_Q(\partial_r, \partial_s) = 0$.
- The term $(Einstein)_Q(\partial_r, \partial_r)$ involves $(\partial_s^2\alpha)$ and $(\partial_r^2\beta)$. Therefore, we can start from the reduced Einstein equations for $(\partial_s \ln\alpha)$ and $(\partial_r \ln\beta)$, take one more derivative, and algebraically prove that $(Einstein)_Q(\partial_r, \partial_r) = 0$ holds. During the process, we also need the first two reduced Einstein equations.
- The way to prove $(Einstein)_Q(\partial_s, \partial_s) = 0$ is similar.

Indeed, since both the reduced Einstein equations (with a compatible constraint equation) and the Einstein equations are locally well-posed, they should be equivalent if one implies the other under enough regularity conditions. □

3.2.4 Riemann invariants

If we perform a matrix diagonalization on the hyperbolic system, the first two equations in the reduced Einstein field equations, we are able to get the following evolution equations for Riemann invariants v_1 and v_2 (refer to [15]). Notice that they are two evolution equations for two quantities

v_1 and v_2 along two different directions ($\partial_r + \lambda_1 \partial_s$ and $\partial_r + \lambda_2 \partial_s$ directions).

$$(\partial_r + \lambda_1 \partial_s)\Big(\underbrace{\frac{1}{2\sqrt{\gamma}} \ln\big(\sqrt{1+\theta^2}+\theta\big) + \frac{1}{2(1+\gamma)} \ln \rho + \frac{1}{2(1+\gamma)} \ln \alpha}_{v_1}\Big) = F_1$$

$$(\partial_r + \lambda_2 \partial_s)\Big(\underbrace{\frac{1}{2\sqrt{\gamma}} \ln\big(\sqrt{1+\theta^2}+\theta\big) - \frac{1}{2(1+\gamma)} \ln \rho - \frac{1}{2(1+\gamma)} \ln \alpha}_{v_2}\Big) = F_2.$$

Here the eigenvalues, or the speeds of propagation, are

$$\lambda_1 = \frac{\sqrt{\alpha}}{\sqrt{\beta}} \cdot \frac{(1-\gamma)\theta\sqrt{1+\theta^2} + \sqrt{\gamma}}{1+(1-\gamma)\theta^2}, \quad \lambda_2 = \frac{\sqrt{\alpha}}{\sqrt{\beta}} \cdot \frac{(1-\gamma)\theta\sqrt{1+\theta^2} - \sqrt{\gamma}}{1+(1-\gamma)\theta^2},$$

and the source terms are

$$\begin{aligned}
F_1 = &\Big(-\frac{1}{2} + \frac{\gamma}{1+\gamma}\Big) \cdot \frac{r}{n}(\rho\alpha) \\
&+ \Big(-\frac{n-1}{r} + \alpha\big(\frac{2}{n}\Lambda r - \frac{S^{[h]}}{nr}\big)\Big) \cdot \Big(\underbrace{-\frac{(1-\gamma)\theta\sqrt{1+\theta^2} + \sqrt{\gamma}}{4\sqrt{\gamma}\big(1+(1-\gamma)\theta^2\big)} - \frac{1}{2(1+\gamma)}}_{\text{bounded between} -\frac{1}{2(1+\gamma)} \pm \frac{1}{4\sqrt{\gamma}}}\Big) \\
&+ \frac{n}{2r} \cdot \frac{\sqrt{1+\theta^2}\big(\sqrt{\gamma}\theta - \sqrt{1+\theta^2}\big)}{1+(1-\gamma)\theta^2} \\
F_2 = &\Big(\frac{1}{2} - \frac{\gamma}{1+\gamma}\Big) \cdot \frac{r}{n}(\rho\alpha) \\
&+ \Big(-\frac{n-1}{r} + \alpha\big(\frac{2}{n}\Lambda r - \frac{S^{[h]}}{nr}\big)\Big) \cdot \Big(\underbrace{-\frac{(1-\gamma)\theta\sqrt{1+\theta^2} - \sqrt{\gamma}}{4\sqrt{\gamma}\big(1+(1-\gamma)\theta^2\big)} + \frac{1}{2(1+\gamma)}}_{\text{bounded between} \frac{1}{2(1+\gamma)} \pm \frac{1}{4\sqrt{\gamma}}}\Big) \\
&+ \frac{n}{2r} \cdot \frac{\sqrt{1+\theta^2}\big(\sqrt{\gamma}\theta + \sqrt{1+\theta^2}\big)}{1+(1-\gamma)\theta^2}.
\end{aligned}$$

For convenience, we record these Riemann invariants as a definition.

Definition 3.2.2. *We use*

$$v_1 = \frac{1}{2\sqrt{\gamma}} \ln\big(\sqrt{1+\theta^2}+\theta\big) + \frac{1}{2(1+\gamma)} \ln \rho + \frac{1}{2(1+\gamma)} \ln \alpha$$

$$v_2 = \frac{1}{2\sqrt{\gamma}} \ln\left(\sqrt{1+\theta^2} + \theta\right) - \frac{1}{2(1+\gamma)} \ln \rho - \frac{1}{2(1+\gamma)} \ln \alpha$$

to denote the Riemann invariants. They encode the information about the fluid variables θ and ρ. The $\ln \alpha$ *term is only for convenience and does not play a big role.*

Notice that for homogeneous fluids, $\theta = 0$, which implies $\lambda_1 = \frac{\sqrt{\alpha}}{\sqrt{\beta}} \cdot \sqrt{\gamma}$ and $\lambda_2 = -\frac{\sqrt{\alpha}}{\sqrt{\beta}} \cdot \sqrt{\gamma}$. The metric components α, β for the homogeneous fluids have the asymptotes (refer to Section 1.2)

$$\alpha \approx (r - r_*)^{\frac{2\gamma}{1-\gamma}}, \quad \beta \approx (r - r_*)^{\frac{2}{1-\gamma}}$$

as $r \longrightarrow r_*^+$. Therefore, since $0 < \gamma < 1$, the speeds λ_1, λ_2 go to infinity when r approaches r_* (Figure 3.2).

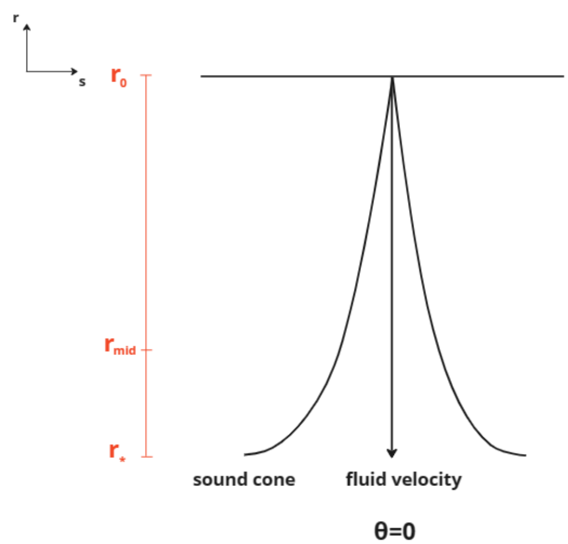

Figure 3.2 The speeds of propagation go to infinity close to r_* for homogeneous fluids.

3.3 Strategy

The fluid density ρ and the metric components α, β for the background homogeneous fluid exhibit singular behavior as $r \longrightarrow r_*^+$:

$$\alpha \approx (r - r_*)^{\frac{2\gamma}{1-\gamma}}, \quad \beta \approx (r - r_*)^{\frac{2}{1-\gamma}}, \quad \rho \approx (r - r_*)^{-\frac{1+\gamma}{1-\gamma}}.$$

Thus, after initial *nonhomogeneous* perturbation, we may expect these variables exhibit a similar, if not worse, singular behavior. The main difficulty is how to control quantities when the time r is close to r_*. The solution is that we avoid this issue by setting an r_{cut} with

$$r_* < r_{cut} < r_0.$$

The main philosophy is that, the equations may have unbounded coefficients in the time interval $(r_*, r_0]$, but for each *fix* $r_{cut} \in (r_*, r_0)$, the equations are expected to have large but *bounded* coefficients in $[r_{cut}, r_0]$. Our hope is to argue that the solution exists in $W^{1,1}$ during the time $[r_{cut}, r_0]$ but the derivative blows up pointwise before r_{cut} (meaning at some time in $[r_{cut}, r_0]$) for sufficiently large initial spatial derivative $\geq LB$, where LB is a lower bound depending on r_{cut}. We next claim that

$$\lim_{r_{cut} \to r_*} LB(r_{cut}) = 0$$

to establish the instability and conclude our main theorem. Note that this r_{cut} trick only makes sense when we aim to prove the *instability*: it is enough to construct a sequence of initial data that form shocks in $(r_*, r_0]$, where we label the sequence by r_{cut}, and the above $LB \to 0$ fact implies the instability for small initial non-homogeneous perturbation.

It turns out the above hope is true, and we actually use the total variation to claim the solution exists in $W^{1,1}$ during $[r_{cut}, r_0]$. Heuristically, we are trying to argue that $\|\partial_s v_1\|_{L^1(\{r=r\})} + \|\partial_s v_2\|_{L^1(\{r=r\})}$ remains small during $[r_{cut}, r_0]$, while $\partial_s v_1$ goes to infinity pointwise somewhere in $[r_{cut}, r_0]$. It is a basic fact in analysis that a function can go to infinity at one point while remaining small L^1 norm, and it is the main idea in John's work [15]. Our proof thus consists of two parts: total variation (for v_1, v_2) control and pointwise derivative ($\partial_s v_1$, $\partial_s v_2$) control. We describe the strategy for each part separately. Recall that our unknowns are v_1, v_2 (Riemann invariants, involving fluid variables) and α, β (metric components).

3.3.1 Total variation

Our total variation is defined as

$$E(r) = \int_{\Gamma_r} \big(|\partial_s v_1| + |\partial_s v_2|\big) ds$$

for $r_* < r \leq r_0$ (Figure 3.7). The idea is to use the trick in John's work [15] to argue that, if the initial total variation is small, it remains small up to $r = r_{cut}$. The idea of John's trick is to consider the evolution of the integrand of the total variation (Figure 3.3). It turns out that we can use Gronwall's inequality to argue that, if the initial perturbation has small total variation less than ϵ_0, it remains small in $[r_{cut}, r_0]$. Note that ϵ_0 depends on r_{cut}.

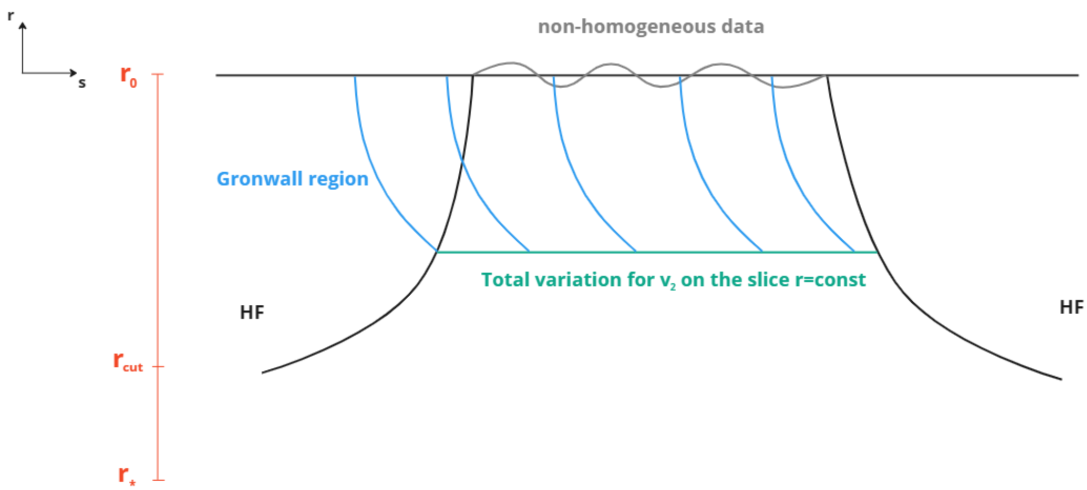

Figure 3.3 John's trick to control the total variation.

3.3.2 Pointwise $\partial_s v$ behavior

In order to get the control on the pointwise behavior of $\partial_s v_1$, a natural way is to take a spatial derivative on $(\partial_r + \lambda_1 \partial_s) v_1 = F_1$. This yields

$$(\partial_r + \lambda_1 \partial_s)(\partial_s v_1) + (\partial_s \lambda_1)(\partial_s v_1) = (\partial_s F_1).$$

Note that $\lambda_1 = \lambda_1(\theta, \alpha, \beta) = \lambda_1(v_1, v_2, \alpha, \beta)$ and $F_1 = F_1(\theta, \rho, \alpha) = F_1(v_1, v_2, \alpha)$. It is worth noting that the source terms F_1 and F_2 do not depend on β. One can expect that $(\partial_s \lambda_1)$ will generate $(\partial_s v_1)$, $(\partial_s v_2)$, $(\partial_s \alpha)$, and $(\partial_s \beta)$. Thus, this can be regarded as a complicated Riccati equation since it involves $(\partial_s v_1)^2$ term (refer to Lemma 3.5.1). This Riccati structure is the main mechanism to generate shocks in John's work [15], but his equations differ from our system by the following features: his Riccati equation does not have linear terms, and his coefficents are all bounded. Fortunately, our goal is also different from his. John proved the shock formation for *all*

C^2-small data, while our goal is to prove shock formation for *some* sequence of initial data that converges to 0 in $W^{1,1}$.

Going back to our evolution for $(\partial_s v_1)$, there are several difficulties from $(\partial_s \lambda_1)$ (and also from $(\partial_s F_1)$):

- how to deal with linear $(\partial_s v_1)$ term,
- how to deal with linear $(\partial_s v_2)$ term, and
- how to deal with $(\partial_s \beta)$ term.

Our solution for the first issue is to use the integral factor method as in the ordinary differential equation context (refer to Lemma 3.5.1). The way to deal with $(\partial_s v_2)$ term is to use a bootstrap argument to claim it remains small *in a certain region* (refer to Lemma 3.5.9). The way to resolve the $(\partial_s \beta)$ issue is our main contribution (refer to Lemma 3.5.2), and we describe our method as follows.

Recall that α, β are our metric components. The reason why we only identify the issue for $(\partial_s \beta)$ but not for $(\partial_s \alpha)$ is because the feature of Einstein field equations under symmetry assumption naturally lacks an equation (constraint equation in our context) for *one* metric component. One can see this in our **reduced Einstein field equations**, where we have an equation for $(\partial_s \ln \alpha)$ but not for $(\partial_s \ln \beta)$. This is a difficulty in our argument since we only have control for quantities without derivative, so if there is no equation for $(\partial_s \ln \beta)$, there will be no control on $(\partial_s \ln \beta)$.

Our innovative way is to use the divergence structure of the hyperbolic system to control

$$\int_{(\lambda_1)} \lambda_1 (\partial_s \ln \beta) dr.$$

Note that this is the term that $(\partial_s \ln \beta)$ appears in the integral factor (refer to Lemma 3.5.1). Recall that our first equation in the **reduced Einstein field equations** is

$$\partial_r \Big(\underbrace{\sqrt{\beta} \cdot r^n \rho^{\frac{1}{1+\gamma}} \cdot \sqrt{1+\theta^2}}_{A} \Big) + \partial_s \Big(\underbrace{\sqrt{\alpha} \cdot r^n \rho^{\frac{1}{1+\gamma}} \cdot \theta}_{B} \Big) = 0.$$

If we denote the first parenthesis to be A and the second parenthesis to be B, we have the following calculation.

$$[\ln A - \ln A(r_0)]_{(\lambda_1)} = \int_{r_0\ (\lambda_1)}^{r} \frac{(\partial_r A) + \lambda_1(\partial_s A)}{A} dr = \int_{r_0\ (\lambda_1)}^{r} \frac{-(\partial_s B) + \lambda_1(\partial_s A)}{A} dr,$$

where $[\cdot]_{(\lambda_1)}$ and $\int_{r_0\,(\lambda_1)}^{r} dr$ denote the corresponding computation along the characteristic generated from $(\partial_r + \lambda_1\partial_s)$. Note that $(\partial_s A)$ involves the term we aim to estimate since A includes $\sqrt{\beta}$, while $(\partial_s B)$ *does not* include β. It turns out the integral on the above right hand side can be simplified to $\int_{(\lambda_1)} \lambda_1(\partial_s \ln\beta)$ plus the integral of a function of $(\partial_s \ln\alpha)$ and $(\partial_s v_2)$ (but no $(\partial_s v_1)$). We have an equation for $(\partial_s \ln\alpha)$, and we can actually integrate the $(\partial_s v_2)$ term. As a remark, one can alternatively apply John's trick to control $\int_{(\lambda_1)} |\partial_s v_2| dr$ although we did not choose this way. We can also estimate $[\ln A - \ln A(r_0)]_{(\lambda_1)}$ using the total variation and ∂_r integral curves. For more details, refer to Lemma 3.5.2.

After estimating the integral factor and other error terms, we have a precise behavior (up to a constant) of the integral factor. More specifically, we know how the integral factor depends on $r \in (r_*, r_{cut})$ (refer to Proposition 3.5.1). We thus aruge that $\lim_{r_{cut}\to r_*^+} LB(r_{cut}) = 0$ and prove our main theorem in Section 3.5.3.

3.3.3 Method of characteristics

Our proof is based on the method of characteristics. We define the notation for characteristics and relevant regions in this section.

Let $X_1(r;z)$ be the function so that $s = X_1(r;z)$ is the characteristic tangent to $(\partial_r + \lambda_1\partial_s)$ and starting from the point $(r,s) = (r_0, z)$. Similarly $X_2(r;z)$ is the function so that $s = X_2(r;z)$ is the characteristic tangent to $(\partial_r + \lambda_2\partial_s)$ and starting from the point $(r,s) = (r_0, z)$. Therefore, for example, the integral notation $\int_{r_0\,(\lambda_1)}^{r} f dr$ in the previous section means $\int_{r_0}^{r} f(r, X_1(r;z)) dr$. We may denote the characteristic $s = X_1(r;z)$ itself by X_1 or (λ_1) interchangeably.

Since we will estimate quantities along different characteristics, for a given point P in the spacetime Q, we define P_1 to be the point on the initial slice that is connected to P by the characteristic X_1. Similarly we define P_2 to be the point on the initial slice that is connected to P

by the characteristic X_2 (see Figure 3.4). We also define P_0 to be the point on the initial slice that is connected to P by the vertical straight line tangent to ∂_r (see Figure 3.5).

In addition to the curves capturing the evolution of the unknowns, we define another notation for total variation control. Given a point P in the spacetime Q, we define $\mathring{P}$ to be the point on the same time slice as P but lying in the background homogeneous fluid (HF) region (see Figure 3.6). For any quantity q, we denote $q(\mathring{P})$ by $\mathring{q}$.

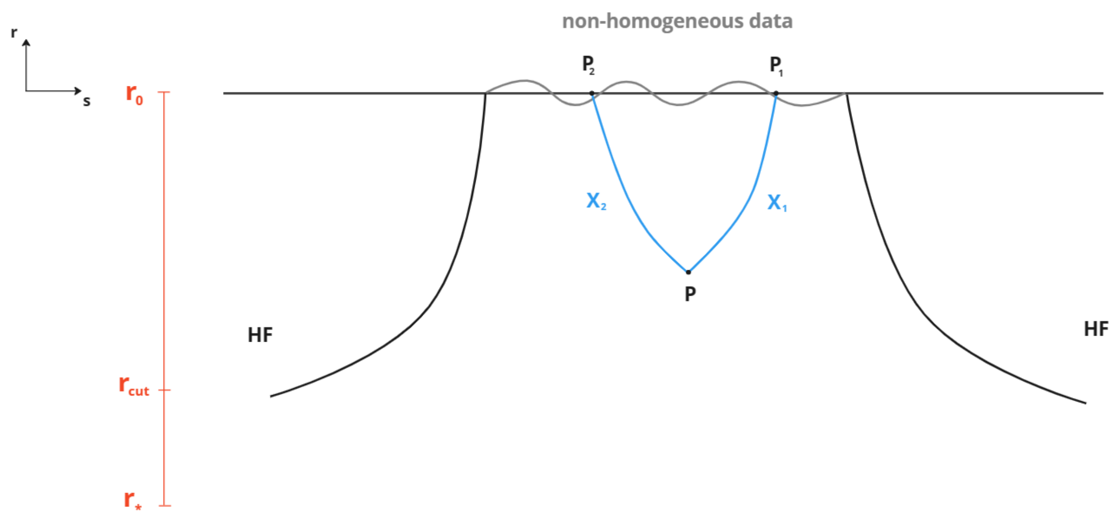

Figure 3.4 Characteristics along $(\partial_r + \lambda_1 \partial_s)$, $(\partial_r + \lambda_2 \partial_s)$.

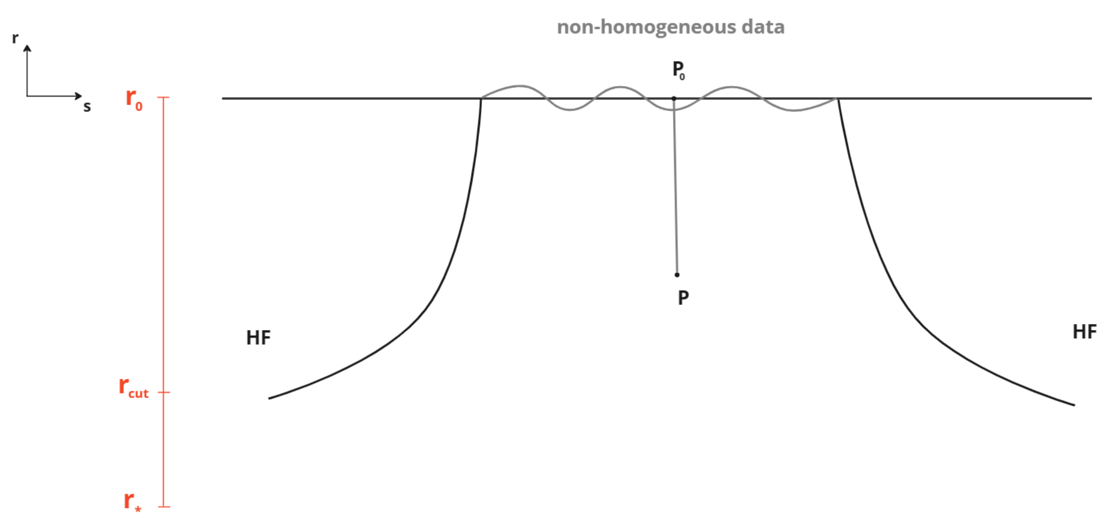

Figure 3.5 The grey vertical line is the integral curve for ∂_r.

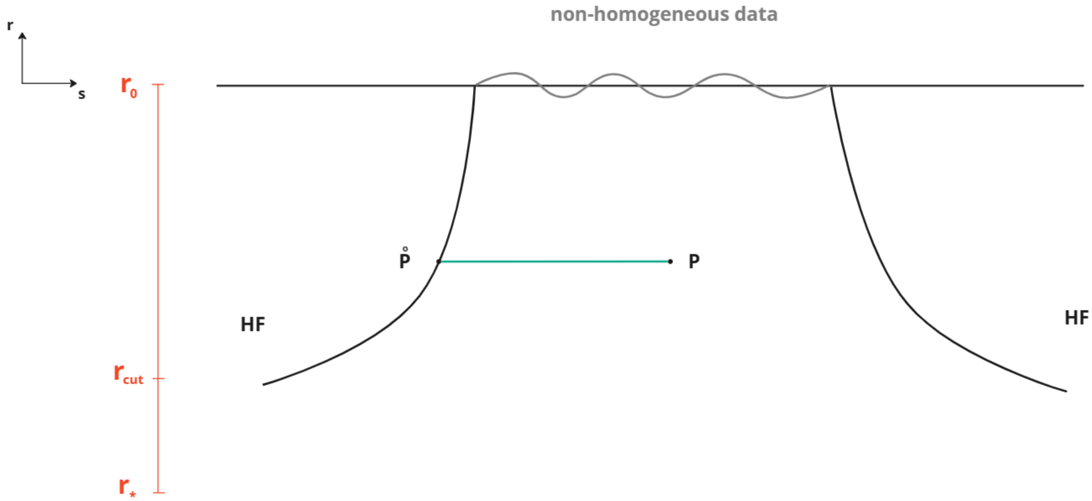

Figure 3.6 The green horizontal line denotes a constant time slice.

3.3.4 Notation

We give the definition and physical meaning of all the variables we will use in this paper.

- $(r, s) = (time, space)$ are independent variables.
    - r_0 is the initial time for the perturbation.
    - $r_* > 0$ is the blowup time for background homogeneous metric. See Figure 3.1.
    - $r_{mid} \in (r_*, r_0)$ is to capture the asymptote for $(\mathring{\rho}\mathring{\alpha})$. See Figure 3.10.
    - $r_{cut} \in (r_*, r_{mid})$ is the auxiliary time introduced in this paper.
- $(\theta, \rho, \alpha, \beta)$ are the main unknowns, which are functions of r and s.
    - (θ, ρ) are fluid variables, where θ is the angle between ξ and ∂_r (measuring how ξ deviates from ∂_r), and $\rho > 0$ is the density of the fluid.
    - (α, β) are metric components, where $\alpha > 0$ and $\beta > 0$.
    - On the background homogeneous fluid, these unknowns satisfy

$$\lim_{r \to r_*^+} \mathring{\alpha}(r) = 0, \quad \lim_{r \to r_*^+} \mathring{\beta}(r) = 0$$

$$\lim_{r\to r_*^+}\left(\frac{\mathring{\alpha}}{\mathring{\beta}}\right)=\infty$$

$$\mathring{\theta}=0 \quad \text{over } (r_*, r_0]$$

$$\lim_{r\to r_*^+}\mathring{\rho}(r)=\infty.$$

- $0<\sqrt{\gamma}<1$ is a parameter representing the sound speed.

- ξ is the fluid velocity satisfying $\langle\xi,\xi\rangle_g=-1$.

- $g=-\alpha dr^2+\beta ds^2$ is the metric for the 2-dimensional Lorentzian manifold Q.

- parameters = parameters(background profile, $r_{mid}, r_0, \gamma, n, \Lambda, S^{[h]}$) where background profile includes r_*, μ_*, τ_*, which are determined by a Big Bang background solution.

- We can also regard $(v_1, v_2, \alpha, \beta)$ as our main unknowns, where

$$v_1=\frac{1}{2\sqrt{\gamma}}\ln|\sqrt{1+\theta^2}+\theta|+\frac{1}{2(1+\gamma)}\ln(\rho\alpha),$$
$$v_2=\frac{1}{2\sqrt{\gamma}}\ln|\sqrt{1+\theta^2}+\theta|-\frac{1}{2(1+\gamma)}\ln(\rho\alpha)$$

  are Riemann invariatnts.

- For λ_1, λ_2, F_1, F_2, refer to Section 3.2.4.

- $s=X_1(r;z)$ denotes the characteristic starting from $(r,s)=(r_0,z)$ along the λ_1 direction. Similarly, $s=X_2(r;y)$ denotes the characteristic starting from $(r,s)=(r_0,y)$ along the λ_2 direction. See Figure 3.4 and Figure 3.9.

- ϵ, ϵ_a control the size of total variation over $[r_{cut}, r_0]$.

- ϵ_0, $\epsilon_{a,0}$ control the size of total variation on the initial slice.

- M generally means a constant that depends on r_{cut}. Usually $M=M(r_{cut}, parameters)$.

- C generally means a constant that does not depend on r_{cut}. Usually $C=C(parameters)$.

- $E(r)$ denotes the total variation on the $\{r = r\}$ time slice and is defined by

$$E(r) = \int_{\{r=r\}} |\partial_s v_1| + |\partial_s v_2| ds.$$

- For a P in the spacetime, define P_1, P_2 in Figure 3.4, P_0 in Figure 3.5, $\mathring{P}$ in Figure 3.6.

- $\mathring{q}$ denotes $q(\mathring{P})$.

- Γ_r and Ω_r are defined in Figure 3.7. z_L, z_{LL}, z_R, z_{RR} are defined in Figure 3.8. Ω_{r_{cut}, z_M} is defined in Lemma 3.5.9.

- δ_1, δ_2 depend on parameters, including r_* and r_{mid}

$$\delta_1 := \frac{1}{1+\gamma} - \frac{(1.1)(1-\gamma)}{2(1+\gamma)} \cdot \frac{r_{mid}}{r_*} > 0$$

$$\delta_2 := \frac{1}{1+\gamma} - \frac{(1-\gamma)}{(2.2)(1+\gamma)} > 0,$$

serving as exponents of the $\left(\frac{1}{r-r_*}\right)$ term in the integral factor. See Proposition 3.5.1.

3.4 Control of total variations

Proposition 3.4.1. *Fix $r_{cut} \in (r_*, r_0]$ and fix $0 < \epsilon < 1$. There exists an*

$$\epsilon_0 = \epsilon_0(r_{cut}, \epsilon, parameters)$$

so that the total variation always satisfies

$$E(r) = \int_{\Gamma_r} |\partial_s v_1| + |\partial_s v_2| ds \leq \epsilon \quad \text{for } r \in [r_{cut}, r_0]$$

as long as the initial data satisfies the ***Initial Data Assumption 1****. Moreover,*

$$|\theta| \leq 2\sqrt{\gamma}\epsilon$$

$$|\ln(\rho\alpha) - \ln(\mathring{\rho}\mathring{\alpha})| \leq (1+\gamma)\epsilon$$

$$\sup|\ln\alpha - \ln\mathring{\alpha}| + \sup|\ln\beta - \ln\mathring{\beta}| \leq \epsilon.$$

In particular, $|\theta|$, ρ, $|\ln\alpha|$, and $|\ln\beta|$ stay finite over the region $[r_{cut}, r_0]$.

We postpone the proof to **the end of Section 4**.

The reason we choose E to denote the total variation is that we are going to bound L^∞-norm of several quantities by this total variation E and run a bootstrap argument. This is similar to the role of energy in the wave equation context where we try to build L^∞-L^2 estimates. In the following lemma, we explain how we bound the L^∞-norm of our unknowns.

Lemma 3.4.1. *We estimate θ, $\sqrt{1+\theta^2}$, $\ln(\rho\alpha) - \ln(\mathring{\rho}\mathring{\alpha})$, and $\ln(\alpha) - \ln(\mathring{\alpha})$ by the total variation $E(r)$. For $\ln(\beta) - \ln(\mathring{\beta})$, we use $\sup E$ where $\sup$ denotes $\sup_{\Omega_r}$ and Ω_r is defined in the Figure 3.7. We use $\mathring{q}$ to dentoe $q(\mathring{P})$.*

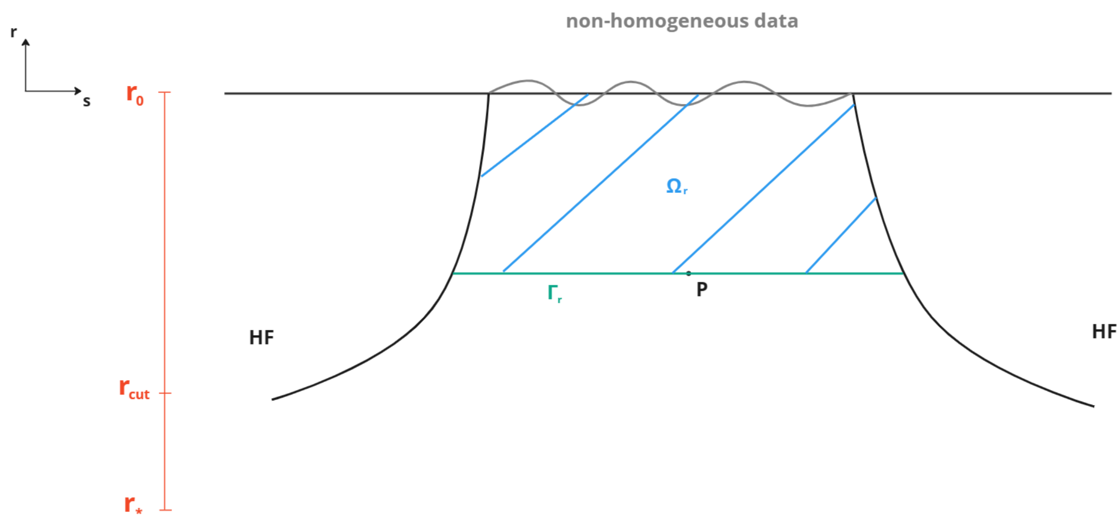

Figure 3.7 The picture for Γ_r and Ω_r.

Proof. For θ, we have (since $\theta(\mathring{P}) = 0$)

$$
\begin{aligned}
|\theta| &= |\theta(P) - \theta(\mathring{P})| \\
&\le \int_{\Gamma_r} |\partial_s \theta| ds \\
&\le \int_{\Gamma_r} \sqrt{\gamma}\sqrt{1+\theta^2}\big(|\partial_s v_1| + |\partial_s v_2|\big) ds
\end{aligned}
$$

$$\leq \sqrt{\gamma} \cdot \sup \sqrt{1+\theta^2} \int_{\Gamma_r} \left(|\partial_s v_1| + |\partial_s v_2|\right) ds$$

$$\leq \sqrt{\gamma} \cdot \sup \sqrt{1+\theta^2} \cdot E(r)$$

where Γ_r denotes $\{r = r\}$ slice inside of the sound cone as in Figure 3.7 and sup denotes $\sup_{\Omega_r}$. Here r is the time coordinate of P. Note that

$$\sup \sqrt{1+\theta^2} \leq 1 + \sup|\theta| \leq 1 + \sqrt{\gamma} \cdot \sup \sqrt{1+\theta^2} \cdot E.$$

For $\ln(\rho\alpha) - \ln(\mathring{\rho}\mathring{\alpha})$, we have

$$\begin{aligned} |\ln(\rho\alpha) - \ln(\mathring{\rho}\mathring{\alpha})| &= |\ln(\rho\alpha)(P) - \ln(\rho\alpha)(\mathring{P})| \\ &\leq \int_{\Gamma_r} |\partial_s \ln(\rho\alpha)| ds \\ &\leq (1+\gamma) \int_{\Gamma_r} \left(|\partial_s v_1| + |\partial_s v_2|\right) ds \\ &\leq (1+\gamma) E(r). \end{aligned}$$

For $\ln\alpha - \ln\mathring{\alpha}$, we have

$$\begin{aligned} &|\ln\alpha - \ln\mathring{\alpha}| = |\ln\alpha(P) - \ln\alpha(\mathring{P})| \\ &\quad \leq \int_{\Gamma_r} |\partial_s \ln\alpha| ds \\ &\quad = \int_{\Gamma_r} \frac{2(1+\gamma)}{n} \cdot r(\rho\alpha) \cdot \frac{\sqrt{\beta}}{\sqrt{\alpha}} \cdot |\theta| \sqrt{1+\theta^2} ds \\ &\quad \leq \frac{2(1+\gamma)}{n} \cdot r \sup(\rho\alpha) \cdot \sup \frac{\sqrt{\beta}}{\sqrt{\alpha}} \cdot \sup|\theta| \sup \sqrt{1+\theta^2} \cdot W(r) \\ &\quad \overset{\#}{\leq} \frac{2(1+\gamma)}{n} \cdot r \cdot (\mathring{\rho}\mathring{\alpha}) \cdot \frac{\sqrt{\mathring{\beta}}}{\sqrt{\mathring{\alpha}}} \cdot e^{(1+\gamma)E} \cdot e^{\ln\sqrt{\beta} - \ln\sqrt{\mathring{\beta}}} \cdot e^{\ln\sqrt{\mathring{\alpha}} - \ln\sqrt{\alpha}} \cdot \sup|\theta| \sup \sqrt{1+\theta^2} \cdot W(r) \\ &\quad \leq C_\alpha \cdot e^{(1+\gamma)E} \cdot e^{\ln\sqrt{\beta} - \ln\sqrt{\mathring{\beta}}} \cdot e^{\ln\sqrt{\mathring{\alpha}} - \ln\sqrt{\alpha}} \cdot \sup|\theta| \sup \sqrt{1+\theta^2} \cdot W(r) \end{aligned}$$

where $W(r) = \int_{\Gamma_r} 1 ds$ denotes the width for $\{r = r\}$, we use the above $\ln(\rho\alpha) - \ln(\mathring{\rho}\mathring{\alpha})$ estimate for $\overset{\#}{\leq}$, and

$$C_\alpha = C_\alpha(parameters).$$

Notice that the $\sup|\theta|$ on the right hand side is the key to make sure the right hand side is small. For $\ln\beta - \ln\mathring{\beta}$, we have

$$
\begin{aligned}
|\ln\beta - \ln\mathring{\beta}| &= |\ln\beta(P) - \ln\beta(\mathring{P})| \\
&\leq \int_{\Gamma_r} |\partial_s \ln\beta| ds \\
&\leq \int_{\Gamma_{r_0}} |\partial_s \ln\beta| ds + \iint_{\Omega_r} |\partial_r\partial_s \ln\beta| dr ds \\
&\leq \int_{\Gamma_{r_0}} |\partial_s \ln\beta| ds \\
&\quad + \iint_{\Omega_r} \frac{4(1+\gamma)}{n} \cdot |\theta||\partial_s\theta| \cdot r(\rho\alpha) + \Big(\frac{2(1+\gamma)}{n} \cdot \theta^2 + \frac{2}{n}\Big) \cdot r|\partial_s(\rho\alpha)| \\
&\quad\quad + \alpha(\partial_s \ln\alpha)\Big(\frac{2}{n}\Lambda r - \frac{S^{[h]}}{nr}\Big) dr ds \\
&\overset{\#}{\leq} \int_{\Gamma_{r_0}} |\partial_s \ln\beta| ds \\
&\quad + \frac{4(1+\gamma)}{n} \cdot (\sup|\theta|) \cdot (r_0 - r) \cdot \underbrace{\sqrt{\gamma} \sup\sqrt{1+\theta^2} \cdot \sup E}_{\int |\partial_s\theta| ds} \\
&\quad\quad \cdot \underbrace{\sup\Big(r(\mathring{\rho}\mathring{\alpha}) \cdot e^{\ln(\rho\alpha) - \ln(\mathring{\rho}\mathring{\alpha})}\Big)}_{r(\rho\alpha)} \\
&\quad + \Big(\frac{2(1+\gamma)}{n} \cdot \big(\sup\theta\big)^2 + \frac{2}{n}\Big) \cdot r_0(r_0 - r) \underbrace{(1+\gamma)\sup(\rho\alpha) \cdot \sup E}_{\int |\partial_s(\rho\alpha)| ds} \\
&\quad + \sup\alpha \cdot \underbrace{\frac{2(1+\gamma)}{n} \cdot r_0 \sup(\rho\alpha) \cdot \sup\frac{\sqrt{\beta}}{\sqrt{\alpha}} \cdot \sup|\theta| \sup\sqrt{1+\theta^2}}_{(\partial_s \ln\alpha)} \\
&\quad\quad \cdot \sup\Big(\frac{2}{n}\Lambda r - \frac{S^{[h]}}{nr}\Big) \cdot (r_0 - r) \sup W \\
&\leq \int_{\text{(initial slice)}} |\partial_s \ln\beta| ds \\
&\quad + M_\beta \cdot \sup|\theta| \sup\sqrt{1+\theta^2} \cdot \sup E \cdot e^{\sup|\ln(\rho\alpha) - \ln(\mathring{\rho}\mathring{\alpha})|}
\end{aligned}
$$

$$+ M_\beta\Big(1 + (\sup|\theta|)^2\Big) \cdot e^{\sup|\ln(\rho\alpha) - \ln(\mathring{\rho}\mathring{\alpha})|} \cdot \sup E$$

$$+ C_\beta \cdot e^{\sup|\ln\alpha - \ln\mathring{\alpha}|} \cdot e^{\sup|\ln(\rho\alpha) - \ln(\mathring{\rho}\mathring{\alpha})|} \cdot e^{\sup|\ln\sqrt{\beta} - \ln\sqrt{\mathring{\beta}}|} \cdot e^{\sup|\ln\sqrt{\mathring{\alpha}} - \ln\sqrt{\alpha}|}$$

$$\cdot \sup|\theta| \sup\sqrt{1+\theta^2} \cdot \sup W$$

where for $\overset{\#}{\leq}$ we rewrite $\partial_s(\rho\alpha)$ as $(\rho\alpha)(\partial_s \ln(\rho\alpha))$ and apply the above estimate to argue

$$\sup_{r \leq r' \leq r_0} \int_{\Gamma_{r'}} |\partial_s(\ln(\rho\alpha))| ds \leq (1+\gamma) \sup E.$$

In addition, we use $\sup W$ to denote $\sup_{r \leq r' \leq r_0} W(r')$. In the last inequality, we have that

$$M_\beta = M_\beta(r_{cut}, parameters)$$

goes to infinity at the rate $\frac{1}{r_{cut} - r_*}$ as $r_{cut} \to r_*$ and

$$C_\beta = C_\beta(parameters).$$

We also decompose $\sup(\rho\alpha)$ to $\sup(\mathring{\rho}\mathring{\alpha}) \cdot e^{\sup|\ln(\rho\alpha) - \ln(\mathring{\rho}\mathring{\alpha})|}$ and absorb $\sup(\mathring{\rho}\mathring{\alpha})$ to M_β or C_β constants.

□

Lemma 3.4.2. *We derive a more explicit form of $W(r)$. In particular,*

$$\sup W = \sup W(r_{cut}, parameters)$$

goes to infinity as $r_{cut} \to r_*$.

Proof. Let

$$z_L = \inf\{z \in \mathbb{R} \mid \text{initial perturbation}(r_0, z) \neq 0\}$$

$$z_R = \sup\{z \in \mathbb{R} \mid \text{initial perturbation}(r_0, z) \neq 0\}.$$

We have

$$W = X_2(r; z_R) - X_1(r; z_L)$$

$$= \int_{r_0}^{r} \lambda_2(r, X_2(r; z_R))dr - \int_{r_0}^{r} \lambda_1(r, X_1(r; z_L))dr$$

$$= \int_{r_0}^{r} \frac{\sqrt{\mathring{\alpha}}}{\sqrt{\mathring{\beta}}} \cdot \big(- \sqrt{\gamma}\big)dr - \int_{r_0}^{r} \frac{\sqrt{\mathring{\alpha}}}{\sqrt{\mathring{\beta}}} \cdot \big(\sqrt{\gamma}\big)dr$$

$$= 2\sqrt{\gamma} \int_{r}^{r_0} \frac{\sqrt{\mathring{\alpha}}}{\sqrt{\mathring{\beta}}} dr.$$

Therefore,

$$\sup W = \sup W(r_{cut}, parameters)$$

goes to infinity as $r_{cut} \longrightarrow r_*$.

□

We introduce the backbone of our total variation control: John's trick [15]. This trick is to use the evolution of the integrand of total variation and the Gronwall's inequality to get a control on the total variation. The main philosophy is that, although the evolution of $(\partial_s v_2)$ is a Riccati equation which does not give a good control, the evolution of $(\partial_s v_2) \cdot \frac{\partial X_2}{\partial z}$ (the integrand of the total variation) gives a linear equation and allows one to apply the Gronwall's inequality.

Lemma 3.4.3. *(John's trick) We have*

$$\int_{\Gamma_r} |\partial_s v_2| ds \leq \int_{\Gamma_{r_0}} |\partial_s v_2| ds + \iint_{\Omega_r} |D_{v_1} F_2||\partial_s v_1| + |D_{v_2} F_2||\partial_s v_2| + |D_\alpha F_2||\partial_s \alpha| dr ds.$$

Proof. Based on the evolution equations for $\partial_s v_1$ and $\partial_s v_2$ (using $\partial_s v_2$ as an example):

$$(\partial_r + \lambda_2 \partial_s) v_2 = F_2$$

$$(\partial_r + \lambda_2 \partial_s)(\partial_s v_2) = (\partial_s F_2) - (\partial_s \lambda_2)(\partial_s v_2)$$

and the evolution for the Jacobian

$$\frac{\partial}{\partial r}\Big(\frac{\partial X_2}{\partial z}\Big) = \frac{\partial}{\partial z}\Big(\frac{\partial X_2}{\partial r}\Big) = \frac{\partial}{\partial z}(\lambda_2) = (\partial_s \lambda_2)\Big(\frac{\partial X_2}{\partial z}\Big)$$

we find that the evolution equation for the integrand is

$$(\partial_r + \lambda_2 \partial_s)\Big(\partial_s v_2 \cdot \frac{\partial X_2}{\partial z}\Big) = (\partial_s F_2) \cdot \frac{\partial X_2}{\partial z}.$$

Note that the $(\partial_s \lambda_2)$ terms cancel, which indicates that there is no quadratic feedback if we consider the integrand of the total variation $(\partial_s v_2) \cdot \frac{\partial X_2}{\partial z}$. This is one of the key observations in John's paper and also one of the essential steps in this paper. Define $z_L = \inf\{z \in \mathbb{R} \mid \text{initial perturbation}(r_0, z) \neq 0\}$ and $z_R = \sup\{z \in \mathbb{R} \mid \text{initial perturbation}(r_0, z) \neq 0\}$ as before. For each $r \in [r_{cut}, r_0]$, define $z_{LL} \in \mathbb{R}$ on the initial slice so that $X_1(r, z_L) = X_2(r, z_{LL})$ and similarly $z_{RR} \in \mathbb{R}$ on the initial slice so that $X_2(r, z_R) = X_1(r, z_{RR})$ (see Figure 3.8). We have (recall Figure 3.7)

$$\begin{aligned}
\int_{\Gamma_r} |\partial_s v_2| ds &= \int_{Z_{LL}\ (\Gamma_r)}^{Z_R} |\partial_s v_2| \cdot \frac{\partial X_2}{\partial z} dz = \int_{Z_{LL}\ (\Gamma_r)}^{Z_R} \Big|\partial_s v_2 \cdot \frac{\partial X_2}{\partial z}\Big| dz \\
&\leq \int_{Z_{LL}\ (\Gamma_{r_0})}^{Z_R} \Big|\partial_s v_2 \cdot \frac{\partial X_2}{\partial z}\Big| dz + \int_{Z_{LL}}^{Z_R} \int_{r\ \ (\Omega_r)}^{r_0} \Big|\partial_s F_2 \cdot \frac{\partial X_2}{\partial z}\Big| dr dz \\
&= \int_{Z_L\ (\Gamma_{r_0})}^{Z_R} |\partial_s v_2| ds + \iint_{(\Omega_r)} |\partial_s F_2| dr ds \\
&\leq \int_{Z_L\ (\Gamma_{r_0})}^{Z_R} |\partial_s v_2| ds + \iint_{(\Omega_r)} |D_{v_1} F_2||\partial_s v_1| + |D_{v_2} F_2||\partial_s v_2| + |D_\alpha F_2||\partial_s \alpha| dr ds.
\end{aligned}$$

Notice that

$$\int_{Z_{LL}\ (\Gamma_{r_0})}^{Z_R} |\partial_s v_2| ds = \int_{Z_L\ (\Gamma_{r_0})}^{Z_R} |\partial_s v_2| ds$$

since $\partial_s v_2 = 0$ outside the interval $[z_L, z_R]$ on the initial slice. Thus, although the definition of z_{LL} depends on r, the term involving z_{LL} can be reduced to the one with z_L, and thus the result does not depend on r. The only dependence on r is through the spacetime integral over Ω_r.

□

Proof of Proposition 3.4.1. We use a bootstrap argument.

Bootstrap Assumption.

$$E \leq \epsilon \quad \textit{over } [r, r_0]$$

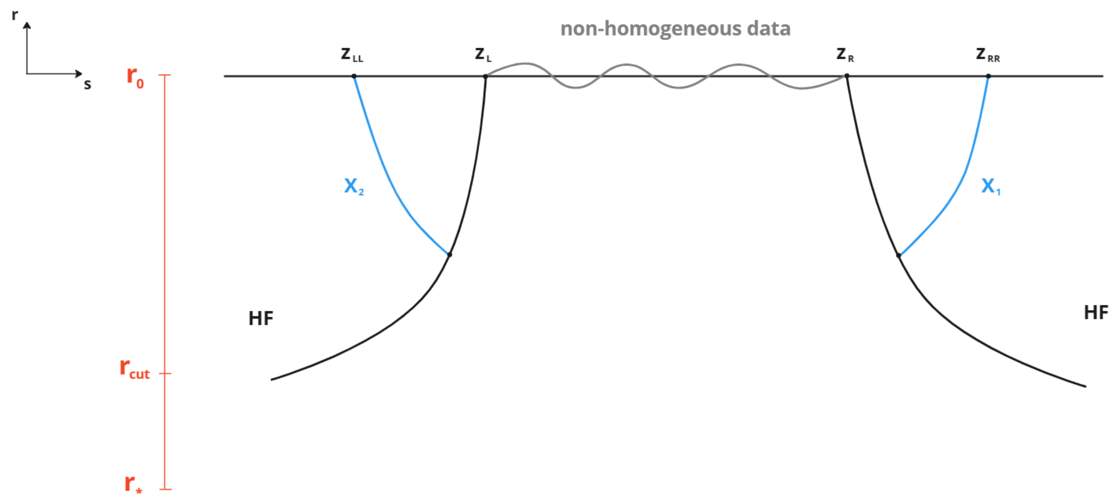

Figure 3.8 Definitions of z_L, z_{LL}, z_R, z_{RR}.

$$\sup \sqrt{1+\theta^2} \leq 2$$

$$\sup |\ln \alpha - \ln \mathring{\alpha}| + \sup |\ln \beta - \ln \mathring{\beta}| \leq \epsilon_a$$

$$\epsilon \leq \epsilon_a$$

with $r \in (r_*, r_0]$, $0 < \epsilon < 1$, $\sup = \sup_{\Omega_r}$.

The a in ϵ_a refers to auxiliary. The last assumption implies that we are trying to use the smallness of E to improve the estimate for $\sup |\ln \alpha - \ln \mathring{\alpha}|$ and $\sup |\ln \beta - \ln \mathring{\beta}|$.

Step 1. **Gronwall's inequality**

Looking at the inequality from Lemma 3.4.3

$$\int_{\Gamma_r} |\partial_s v_2| ds \leq \int_{\Gamma_{r_0}} |\partial_s v_2| ds + \iint_{\Omega_r} |D_{v_1} F_2||\partial_s v_1| + |D_{v_2} F_2||\partial_s v_2| + |D_\alpha F_2||\partial_s \alpha| dr ds,$$

we aim to estimate the right hand side in terms of E and apply the Gronwall's inequality. For $D_{v_1} F_2$, we have

$$D_{v_1} F_2 = \Big(\frac{1}{2} - \frac{\gamma}{1+\gamma}\Big) \cdot \frac{r}{n}(\rho\alpha) \cdot (1+\gamma)$$
$$+ \Big(- \frac{n-1}{r} + \alpha\big(\frac{2}{n}\Lambda r - \frac{S^{[h]}}{r}\big)\Big) \cdot D_\theta \bigg(- \frac{(1-\gamma)\theta\sqrt{1+\theta^2} - \sqrt{\gamma}}{4\sqrt{\gamma}(1+(1-\gamma)\theta^2)} \bigg) \cdot \sqrt{\gamma}\sqrt{1+\theta^2}$$

$$+\frac{n}{2r}\cdot D_\theta\left(\frac{\sqrt{1+\theta^2}\left(\sqrt{\gamma}\theta+\sqrt{1+\theta^2}\right)}{1+(1-\gamma)\theta^2}\right)\cdot\sqrt{\gamma}\sqrt{1+\theta^2}.$$

Therefore,

$$\begin{aligned}
\iint_{\Omega_r}|D_{v_1}F_2||\partial_s v_1|drds &\le \left(M\cdot e^{\sup|\ln(\rho\alpha)-\ln(\mathring{\rho}\mathring{\alpha})|}+C\cdot e^{\sup|\ln\alpha-\ln\mathring{\alpha}|}+C\right)\int_r^{r_0}E(r)dr\\
&\le \Bigg(M\cdot e^{(1+\gamma)\sup E}+C\cdot exp\Big(C_\alpha\cdot e^{(1+\gamma)\sup E}\cdot e^{\frac{1}{2}\epsilon_a}\cdot e^{\frac{1}{2}\epsilon_a}\cdot(\underbrace{\sqrt{\gamma}\cdot 2\cdot\sup E}_{\sup|\theta|})\\
&\qquad\cdot 2\cdot\sup W\Big)+C\Bigg)\cdot\int_r^{r_0}E(r)dr\\
&\le \left(M\cdot e^{(1+\gamma)\epsilon}+M+C\right)\int_r^{r_0}E(r)dr\\
&\le M_1\int_r^{r_0}E(r)dr.
\end{aligned}$$

In the first $\le$ we decompose $(\rho\alpha)$ to $(\mathring{\rho}\mathring{\alpha})\cdot e^{\ln(\rho\alpha)-\ln(\mathring{\rho}\mathring{\alpha})}$ and absorb $(\mathring{\rho}\mathring{\alpha})$ to M. In the second $\le$ we apply Lemma 3.4.1 and use the bootstrap assumption to estimate $\sqrt{1+\theta^2}$ by 2. Here we have

$$M=M(r_{cut},parameters),\quad M_1=M_1(r_{cut},parameters)$$

blow up when $r_{cut}\to r_*^+$ and

$$C=C(parameters)$$

remains bounded since $0<\gamma<1$. Similarly,

$$\iint_{\Omega_r}|D_{v_2}F_2||\partial_s v_2|drds\le M_1\int_r^{r_0}E(r)dr$$

For $|D_\alpha F_2||\partial_s\alpha|$, we have

$$\begin{aligned}
&\iint_{\Omega_r}|D_\alpha F_2||\partial_s\alpha|drds\\
&\le \iint_{\text{(blue region)}}\left|\frac{2}{n}\Lambda r-\frac{S^{[h]}}{nr}\right|\left|-\frac{(1-\gamma)\theta\sqrt{1+\theta^2}-\sqrt{\gamma}}{4\sqrt{\gamma}\left(1+(1-\gamma)\theta^2\right)}+\frac{1}{2(1+\gamma)}\right|\cdot\alpha|\partial_s\ln\alpha|drds\\
&\le C\cdot e^{\sup|\ln\alpha-\ln\mathring{\alpha}|}\Big(\frac{2(1+\gamma)}{n}\cdot r\cdot e^{\sup|\ln(\rho\alpha)-\ln(\mathring{\rho}\mathring{\alpha})|}\cdot e^{\sup|\ln\sqrt{\beta}-\ln\sqrt{\mathring{\beta}}|}\cdot e^{\sup|\ln\sqrt{\mathring{\alpha}}-\ln\sqrt{\alpha}|}
\end{aligned}$$

$$
\begin{aligned}
&\cdot \sup \sqrt{1+\theta^2}\Big) \cdot \iint_{\Omega_r} |\theta| dr ds \\
&\leq C \cdot exp\Big(e^{2\epsilon_a} \cdot e^{(1+\gamma)\epsilon} \cdot \sup W\Big) \cdot \int_r^{r_0} E(r) dr \\
&\leq M_2 \cdot \int_r^{r_0} E(r) dr.
\end{aligned}
$$

In the second $\leq$, we use the constraint equation from our **reduced Einstein field equations**. Notice that the terms $(\mathring{\rho}\mathring{\alpha}) \cdot \frac{\sqrt{\mathring{\beta}}}{\sqrt{\mathring{\alpha}}}$ can be absorbed to the constant C due to their asymptotes. In the third $\leq$, we incorporate the bootstrap assumption and replace $|\theta|$ by $E(r)$ up to a constant. Here

$$
M_2 = M_2(r_{cut}, parameters)
$$

goes to infinity as $r_{cut} \longrightarrow r_*$, and

$$
C = C(parameters)
$$

remains bounded. Together, we have

$$
E(r) \leq E(r_0) + 2M_1 \int_r^{r_0} E dr + 2M_2 \int_r^{r_0} E dr.
$$

By Gronwall's inequality, we have

$$
E \leq E(r_0) \cdot e^{(2M_1+2M_2)(r_0-r)} \leq E(r_0) \cdot e^{(2M_1+2M_2)(r_0-r_*)}.
$$

Step 2. **Improved estimate**

In order to close the bootstrap argument, we have to show that our bootstrap assumptions are improved. For the total variation E, we can choose the initial data (where M_3, M_4 are determined below)

$$
E(r_0) \leq \frac{\epsilon}{20 e^{(2M_1+2M_2)(r_0-r_*)}} \cdot \min\left\{\frac{1}{1+10M_3}, \frac{1}{1+10M_4}\right\},
$$

which implies the improved estimate

$$
E \leq \frac{\epsilon}{20} \cdot \min\left\{\frac{1}{1+10M_3}, \frac{1}{1+10M_4}\right\} \leq \frac{\epsilon}{20}.
$$

Since $0 < \epsilon < 1$ and $0 < \gamma < 1$, we have

$$\begin{aligned}
\sup \sqrt{1+\theta^2} &\leq 1 + |\theta| \\
&\leq 1 + \sqrt{\gamma} \cdot \sup \sqrt{1+\theta^2} \cdot E \\
&\leq 1 + \sqrt{\gamma} \cdot 2 \cdot \frac{\epsilon}{20} \\
&\leq 1 + \frac{\epsilon}{10} \leq \frac{3}{2}.
\end{aligned}$$

Finally, for $\sup |\ln \alpha - \ln \mathring{\alpha}|$ and $\sup |\ln \beta - \ln \mathring{\beta}|$, we have

$$\begin{aligned}
|\ln \alpha - \ln \mathring{\alpha}| &\leq C_\alpha \cdot e^{(1+\gamma)E} \cdot e^{\frac{1}{2}\epsilon_a} \cdot e^{\frac{1}{2}\epsilon_a} \cdot \left(\sup |\theta| \cdot \sup \sqrt{1+\theta^2} \right) \cdot \sup W \\
&\leq C_\alpha \cdot e^{(1+\gamma)} \cdot e \cdot \left(\sqrt{\gamma} \cdot 4 \cdot \sup E \right) \cdot \sup W \\
&\leq M_3 \cdot \sup E \\
&\leq \frac{\epsilon}{200} \leq \frac{\epsilon_a}{200}
\end{aligned}$$

with

$$M_3 = M_3(r_{cut}, parameters)$$

going to infinity as $r_{cut} \to r_*^+$, and

$$\begin{aligned}
|\ln \beta - \ln \mathring{\beta}| \leq &\int_{\Gamma_{r_0}} |\partial_s \ln \beta| ds \\
&+ M_\beta \cdot \Bigg(\underbrace{\sqrt{\gamma} \sup \sqrt{1+\theta^2} \sup E}_{\sup |\theta|} \Bigg) \cdot \sup \sqrt{1+\theta^2} \cdot \sup E \cdot e^{(1+\gamma)\epsilon} \\
&+ M_\beta \Big(1 + \sup |\theta|^2 \Big) \cdot e^{(1+\gamma)\epsilon} \cdot \sup E \\
&+ C_\beta \cdot e^{(1+\gamma)\epsilon} \cdot e^{2\epsilon_a} \cdot \Bigg(\underbrace{\sqrt{\gamma} \cdot \sup \sqrt{1+\theta^2} \cdot \sup E}_{\sup |\theta|} \Bigg) \cdot \sup \sqrt{1+\theta^2} \cdot \sup W \\
\leq &\int_{\Gamma_{r_0}} |\partial_s \ln \beta| ds + M_4 \cdot \sup E
\end{aligned}$$

$$\leq \int_{\Gamma_{r_0}} |\partial_s \ln \beta| ds + \frac{\epsilon}{200}$$

$$\leq \int_{\Gamma_{r_0}} |\partial_s \ln \beta| ds + \frac{\epsilon_a}{200}$$

$$\leq \frac{\epsilon_a}{200} + \frac{\epsilon_a}{200} = \frac{\epsilon_a}{100}$$

provided that the initial perturbation $\int_{\Gamma_{r_0}} |\partial_s \ln \beta| ds$ is sufficiently small. Here

$$M_4 = M_4(r_{cut}, parameters)$$

goes to infinity as $r_{cut} \longrightarrow r_*^+$.

Initial data Assumption 1.

$$E(r_0) \leq \frac{\epsilon}{20 e^{(2M_1+2M_2)(r_0-r_*)}} \cdot \min\left\{\frac{1}{1+10M_3}, \frac{1}{1+10M_4}\right\}$$

$$\int_{\text{(initial slice)}} |\partial_s \ln \beta| ds \leq \frac{\epsilon_a}{200}$$

where M_1 comes from $|D_{v_1} F_2||\partial_s v_1|$ in Step 1, M_2 comes from $|D_\alpha F_2||\partial_s \alpha|$ in Step 1, M_3 comes from $|\ln \alpha - \ln \mathring{\alpha}|$ in Step 2, and M_4 comes from $|\ln \beta - \ln \mathring{\beta}|$ in Step 2.

□

Remark. *From the computation, we see that*

$$M_3 = 4C_\alpha \cdot e^{2+\gamma} \cdot \sqrt{\gamma}\Big(\sup_{[r_{cut}, r_0]} W\Big)$$

$$M_4 = 4M_\beta \cdot \sqrt{\gamma} \cdot e^{(1+\gamma)} + M_\beta\Big(1 + 4\gamma\Big) \cdot e^{1+\gamma} + 4C_\beta \cdot \sqrt{\gamma} e^{3+\gamma} \cdot \Big(\sup_{[r_{cut}, r_0]} W\Big).$$

Note that $\sup_{[r_{cut}, r_0]} W$ will go to infinity as $r_{cut} \longrightarrow r_*^+$; this is the reason why we emphasize that M_3, M_4 depend on r_{cut}.

3.5 Riccati equation for derivatives

In this section, we derive the pointwise behavior of the derivative $(\partial_s v_1)$ when the total variation is small. This is reduced to deriving the pointwise behavior of the integral factor e^f as this involves the integral of derivatives along (λ_1) characteristic and is hard to control. The main difficulty is to control the integral of $(\partial_s \beta)$ term.

We begin with performing the integral factor method as in the ordinary differential equation context to absorb the linear terms of $(\partial_s v_1)$. The result is a Riccati equation for $(\partial_s v_1)$.

Lemma 3.5.1. *For $\partial_s v_1$, the derivative of the first Riemann invariant, we have*

$$(\partial_r + \lambda_1 \partial_s)(e^f \cdot \partial_s v_1) = -(e^{-f})(D_{v_1}\lambda_1)(e^f \cdot \partial_s v_1)^2 + e^f (D_{v_2}F_1)(\partial_s v_2) + e^f (D_\alpha F_1)(\partial_s \alpha)$$

where

$$-f = \int_{r_0 \ (\lambda_1)}^{r} (-D_{v_2}\lambda_1)(\partial_s v_2) - (D_\alpha \lambda_1)(\partial_s \alpha) - (D_\beta \lambda_1)(\partial_s \beta) + (D_{v_1}F_1) dr$$

with $dr < 0$.

Proof. Starting from the evolution equation for the first Riemann invariant

$$(\partial_r + \lambda_1 \partial_s) v_1 = F_1,$$

we take spatial derivative on both sides and get

$$(\partial_r + \lambda_1 \partial_s)(\partial_s v_1) = -(\partial_s \lambda_1)(\partial_s v_1) + (\partial_s F_1)$$

$$\begin{aligned}(\partial_r + \lambda_1 \partial_s)(\partial_s v_1) = {} & -(D_{v_1}\lambda_1)(\partial_s v_1)^2 \\ & + \Big(- (D_{v_2}\lambda_1)(\partial_s v_2) - (D_\alpha \lambda_1)(\partial_s \alpha) - (D_\beta \lambda_1)(\partial_s \beta) + (D_{v_1}F_1)\Big)(\partial_s v_1) \\ & + (D_{v_2}F_1)(\partial_s v_2) + (D_\alpha F_1)(\partial_s \alpha)\end{aligned}$$

$$(\partial_r + \lambda_1 \partial_s)(e^f \cdot \partial_s v_1) = -(e^{-f})(D_{v_1}\lambda_1)(e^f \cdot \partial_s v_1)^2 + e^f (D_{v_2}F_1)(\partial_s v_2) + e^f (D_\alpha F_1)(\partial_s \alpha).$$

□

Note that the quadratic term $(e^f \cdot \partial_s v_1)^2$ is the main driving force to generate a shock. We are hoping that the coefficient $D_{v_1}\lambda_1$ has a fixed sign and e^{-f} is nondegenerate. It turns out $D_{v_1}\lambda_1 > 0$ (refer to Lemma 3.5.7) and e^{-f} is nondegenerate (refer to Proposition 3.5.1) provided that the total variation is small. Regarding all the other terms as error terms, we prove our main theorem in Section 3.5.3.

Definition 3.5.1. *Our error term has different definitions in different contexts. For the terms inside of the integral factor e^{-f} or e^f, an error term is defined to be a constant term independent of r_{cut}. For example,*

$$C,\ M\epsilon$$

can be regarded as error terms. Although $M = M(r_{cut}, parameters)$ goes to infinity as $r_{cut} \to r_*^+$, ϵ can depend on r_{cut}, so $M\epsilon$ can be independent of r_{cut}. In the Riccati equation context, an error term is defined to be a small term that goes to 0 when $\epsilon \to 0$. For example,

$$M\epsilon$$

can be regarded as an error term. Notice that ϵ can depend on r_{cut}. Recall that ϵ is the notation for the upper bound of total variation.

It turns out that in the integral factor $-f$, only $(D_\beta\lambda_1)(\partial_s\beta)$ and $(D_{v_1}F_1)$ contribute non-error terms coming from the background homogeneous fluid.

3.5.1 Pointwise behavior of the integral factor $-f$

In this section, we try to analyze each term in the integral factor $-f$. We begin with the most difficult term.

Lemma 3.5.2. *We separate the background influence (with r_{cut}) and error terms (without r_{cut}) for the term*

$$\int_{r_0\ (\lambda_1)}^{r} -(D_\beta\lambda_1)(\partial_s\beta)dr$$

in the integral factor $-f$. It turns out the background influence is

$$-\frac{1-\gamma}{1+\gamma}\cdot\frac{1}{n}\int_r^{r_0} r(\mathring{\rho}\mathring{\alpha})dr+\frac{1}{1+\gamma}\ln(\mathring{\rho}\mathring{\alpha})-\frac{1}{1+\gamma}\ln(\mathring{\rho}(r_0)\mathring{\alpha}(r_0))$$

Proof. From the divergence structure of the first equation in the **reduced Einstein field equations**

$$\partial_r\Big(\underbrace{\sqrt{\beta}\cdot r^n\rho^{\frac{1}{1+\gamma}}\cdot\sqrt{1+\theta^2}}_{A}\Big)+\partial_s\Big(\underbrace{\sqrt{\alpha}\cdot r^n\rho^{\frac{1}{1+\gamma}}\cdot\theta}_{B}\Big)=0$$

and letting the first parenthesis be A, the second parenthesis be B, we have

$$\begin{aligned}
&[\ln(A)-\ln(A(r_0))]\Big|_{\lambda_1}\\
&=\int_{r_0\ (\lambda_1)}^{r}\frac{(\partial_r+\lambda_1\partial_s)A}{A}dr\\
&=\int_{r_0\ (\lambda_1)}^{r}\frac{-(\partial_s B)+\lambda_1(\partial_s A)}{A}dr\\
&=\int_{r_0\ (\lambda_1)}^{r}-\Big(\frac{1}{2}(\partial_s\ln\alpha)\cdot\frac{\sqrt{\alpha}}{\sqrt{\beta}}\cdot\frac{\theta}{\sqrt{1+\theta^2}}+\frac{1}{1+\gamma}(\partial_s\ln\rho)\cdot\frac{\sqrt{\alpha}}{\sqrt{\beta}}\cdot\frac{\theta}{\sqrt{1+\theta^2}}\\
&\qquad+(\partial_s\theta)\cdot\frac{\sqrt{\alpha}}{\sqrt{\beta}}\cdot\frac{1}{\sqrt{1+\theta^2}}\Big)\\
&\qquad+\lambda_1\cdot\Big(\frac{1}{2}(\partial_s\ln\beta)+\frac{1}{1+\gamma}(\partial_s\ln\rho)+(\partial_s\ln\sqrt{1+\theta^2})\Big)dr\\
&=\int_{r_0\ (\lambda_1)}^{r}\frac{1}{2}\cdot\lambda_1(\partial_s\ln\beta)-\frac{1}{2}\cdot(\partial_s\ln\alpha)\cdot\frac{\sqrt{\alpha}}{\sqrt{\beta}}\cdot\frac{\theta}{\sqrt{1+\theta^2}}\\
&\qquad+\Big(-\frac{1}{1+\gamma}\cdot\frac{\sqrt{\alpha}}{\sqrt{\beta}}\cdot\frac{\theta}{\sqrt{1+\theta^2}}+\frac{1}{1+\gamma}\cdot\lambda_1\Big)\Big(\underbrace{(1+\gamma)(\partial_s v_1-\partial_s v_2)-(\partial_s\ln\alpha)}_{(\partial_s\ln\rho)}\Big)\\
&\qquad+\Big(-\frac{\sqrt{\alpha}}{\sqrt{\beta}}\cdot\sqrt{\gamma}+\lambda_1\cdot\frac{\sqrt{\gamma}\theta}{\sqrt{1+\theta^2}}\Big)(\partial_s v_1+\partial_s v_2)dr\\
&=\int_{r_0\ (\lambda_1)}^{r}\frac{1}{2}\cdot\lambda_1(\partial_s\ln\beta)\\
&\qquad+\frac{\sqrt{\alpha}}{\sqrt{\beta}}\cdot\frac{1}{1+(1-\gamma)\theta^2}\cdot\Big(\Big(-\frac{1}{2}+\frac{\gamma}{1+\gamma}\Big)\cdot\frac{\theta}{\sqrt{1+\theta^2}}
\end{aligned}$$

$$- \frac{1-\gamma}{2} \cdot \frac{\theta^3}{\sqrt{1+\theta^2}} - \frac{1}{1+\gamma} \cdot \sqrt{\gamma}\Big)(\partial_s \ln \alpha)$$

$$- \frac{\sqrt{\alpha}}{\sqrt{\beta}} \cdot 2\sqrt{\gamma} \cdot \frac{1 - \frac{\sqrt{\gamma}\theta}{\sqrt{1+\theta^2}}}{1 + (1-\gamma)\theta^2} \cdot (\partial_s v_2) dr.$$

Notice that the first term $\frac{1}{2} \cdot \lambda_1(\partial_s \ln \beta)$ in the integrand is precisely the $-(D_\beta \lambda_1)(\partial_s \beta)$ term in $-f$. In order to derive the background influence, we have a closer look at $[\ln(A) - \ln(A(r_0))]\Big|_{\lambda_1}$. First we expand the definition of $\ln(A)$.

$$\ln(A) = \ln\Big(\frac{\sqrt{\beta}}{\alpha^{\frac{1}{1+\gamma}}} \cdot r^n (\rho\alpha)^{\frac{1}{1+\gamma}} \cdot \sqrt{1+\theta^2}\Big).$$

- Since the evolution equation we have for $\sqrt{\beta}$ is only along ∂_r direction, we estimate $\ln\Big(\frac{\sqrt{\beta}}{\alpha^{\frac{1}{1+\gamma}}}\Big)$ along this direction:

$$\Big[\ln\Big(\frac{\sqrt{\beta}}{\alpha^{\frac{1}{1+\gamma}}}\Big) - \ln\Big(\frac{\sqrt{\beta(r_0)}}{\alpha(r_0)^{\frac{1}{1+\gamma}}}\Big)\Big]\Big|_{\lambda_1}$$

$$= \ln\Big(\frac{\sqrt{\beta}}{\alpha^{\frac{1}{1+\gamma}}}\Big)(P) - \ln\Big(\frac{\sqrt{\beta}}{\alpha^{\frac{1}{1+\gamma}}}\Big)(P_0) + \ln\Big(\frac{\sqrt{\beta}}{\alpha^{\frac{1}{1+\gamma}}}\Big)(P_0) - \ln\Big(\frac{\sqrt{\beta}}{\alpha^{\frac{1}{1+\gamma}}}\Big)(P_1)$$

$$= \int_{r_0}^{r} \partial_r \ln\Big(\frac{\sqrt{\beta}}{\alpha^{\frac{1}{1+\gamma}}}\Big) dr + \Big(\underbrace{\ln\sqrt{\beta}(P_0) - \ln\sqrt{\beta}(P_1)}_{\text{2 initial perturbation for } \ln\sqrt{\beta}}\Big)$$

$$+ \Big(\underbrace{\ln(\alpha^{\frac{1}{1+\gamma}})(P_1) - \ln(\alpha^{\frac{1}{1+\gamma}})(P_0)}_{\text{2 initial perturbation for } \ln(\alpha^{\frac{1}{1+\gamma}})}\Big).$$

Note that

$$\partial_r \ln\Big(\frac{\sqrt{\beta}}{\alpha^{\frac{1}{1+\gamma}}}\Big) = \Big(\frac{1-\gamma}{1+\gamma} - (1-\gamma)\cdot\theta^2\Big) \cdot \frac{r}{n}(\rho\alpha) - \Big(\frac{1}{2} + \frac{1}{1+\gamma}\Big) \cdot \frac{n-1}{r}$$

$$+ \Big(\frac{1}{2} + \frac{1}{1+\gamma}\Big) \cdot \alpha\Big(\frac{2\Lambda}{n} r - \frac{S^{[h]}}{nr}\Big)$$

$$= \Big(\frac{1-\gamma}{1+\gamma} - (1-\gamma)\cdot\theta^2\Big) \cdot \frac{r}{n}(\mathring{\rho}\mathring{\alpha}) \cdot e^{\ln(\rho\alpha) - \ln(\mathring{\rho}\mathring{\alpha})}$$

$$- \Big(\frac{1}{2} + \frac{1}{1+\gamma}\Big) \cdot \frac{n-1}{r} + \Big(\frac{1}{2} + \frac{1}{1+\gamma}\Big) \cdot \alpha\Big(\frac{2\Lambda}{n} r - \frac{S^{[h]}}{nr}\Big).$$

Therefore, the contribution from the background to $\ln\left(\frac{\sqrt{\beta}}{\alpha^{\frac{1}{1+\gamma}}}\right)$ is

$$\frac{1-\gamma}{1+\gamma}\cdot\frac{1}{n}\int_{r_0}^{r} r(\mathring{\rho}\mathring{\alpha})dr = -\frac{1-\gamma}{1+\gamma}\cdot\frac{1}{n}\int_{r}^{r_0} r(\mathring{\rho}\mathring{\alpha})dr. \tag{3.1}$$

The remaining terms are bounded by

$$\int_r^{r_0} \frac{1-\gamma}{1+\gamma}\cdot\frac{r}{n}(\mathring{\rho}\mathring{\alpha})\cdot\left|e^{\ln(\rho\alpha)-\ln(\mathring{\rho}\mathring{\alpha})}-1\right| + (1-\gamma)\cdot\theta^2\cdot\frac{r}{n}(\mathring{\rho}\mathring{\alpha})\cdot e^{\ln(\rho\alpha)-\ln(\mathring{\rho}\mathring{\alpha})}$$
$$+2\cdot\frac{n-1}{r}+2\cdot\mathring{\alpha}\cdot e^{\ln\alpha-\ln\mathring{\alpha}}\left(\frac{2\Lambda}{n}r-\frac{S^{[h]}}{nr}\right)$$
$$\leq M\epsilon + M\epsilon^2 + C + C,$$

where we use Proposition 3.4.1 to estimate $|e^{\ln(\rho\alpha)-\ln(\mathring{\rho}\mathring{\alpha})}-1| \leq e|\ln(\rho\alpha)-\ln(\mathring{\rho}\mathring{\alpha})|$ and $\ln\alpha-\ln\mathring{\alpha}$, and we use the fact that $\mathring{\alpha}$ is bounded over the entire time interval $[r_*, r_0]$ (recall that $\lim_{r\to r_*}\mathring{\alpha}=0$). Here

$$M = M(r_{cut}, parameters)$$

goes to infinity as $r_{cut}\to r_*^+$ and

$$C = C(parameters)$$

remains bounded (recall that $r_* > 0$). Since ϵ is allowed to depend on r_{cut}, these terms can be regarded as error terms.

- Next, we consider the $\ln\left((\rho\alpha)^{\frac{1}{1+\gamma}}\right)$ term. Since we have control on the total variation of v_1 and v_2, we write this term as

$$\frac{1}{1+\gamma}\ln(\rho\alpha) = \frac{1}{1+\gamma}\ln(\mathring{\rho}\mathring{\alpha}) + \int_{\Gamma_r}(\partial_s v_1 - \partial_s v_2)ds.$$

Therefore, we have

$$\left[\frac{1}{1+\gamma}\ln(\rho\alpha)-\frac{1}{1+\gamma}\ln(\rho(r_0)\alpha(r_0))\right]\Big|_{\lambda_1} = \frac{1}{1+\gamma}\ln(\mathring{\rho}\mathring{\alpha})-\frac{1}{1+\gamma}\ln(\mathring{\rho}(r_0)\mathring{\alpha}(r_0)) \tag{3.2}$$
$$+\int_{\Gamma_r}(\partial_s v_1-\partial_s v_2)ds - \int_{\Gamma_r}(\partial_s v_1-\partial_s v_2)ds.$$

Notice that the last two integrals are bounded by 2ϵ (since $E \le \epsilon$ by Proposition 3.4.1) and hence can be taken as error terms.

- We consider the $\ln(r^n\sqrt{1+\theta^2})$ term. Since we know that $\sqrt{1+\theta^2} \le 2$ (by Proposition 3.4.1), we have

$$\left[\left|\ln\left(r^n\sqrt{1+\theta^2}\right) - \ln\left(r_0^n\sqrt{1+\theta_0^2}\right)\right|\right]_{\lambda_1} \le 2n|\ln r| + 2n|\ln r_0| \le C$$

where

$$C = C(parameters).$$

We still have to control the two error terms on the right hand side of the equation for $[\ln A - \ln A(r_0)]\Big|_{(\lambda_1)}$.

- The first error term is

$$\int_r^{r_0} \Big|\frac{\sqrt{\alpha}}{\sqrt{\beta}} \cdot \frac{1}{1+(1-\gamma)\theta^2} \cdot \Big(\Big(-\frac{1}{2} + \frac{\gamma}{1+\gamma} \cdot \frac{\theta}{\sqrt{1+\theta^2}} - \frac{1-\gamma}{2} \cdot \frac{\theta^3}{\sqrt{1+\theta^2}} - \frac{1}{1+\gamma} \cdot \sqrt{\gamma}\Big)(\partial_s \ln \alpha)\Big| dr \le M\epsilon$$

where ϵ comes from the θ in $(\partial_s \ln \alpha)$ equation (refer to **reduced Einstein field equations**) and M comes from $\frac{\sqrt{\alpha}}{\sqrt{\beta}}$. Here

$$M = M(r_{cut}, parameters)$$

goes to infinity when $r_{cut} \to r_*^+$.

- The second error term is

$$\int_{r_0\ (\lambda_1)}^{r} -\frac{\sqrt{\alpha}}{\sqrt{\beta}} \cdot 2\sqrt{\gamma} \cdot \frac{1 - \frac{\sqrt{\gamma}\theta}{\sqrt{1+\theta^2}}}{1+(1-\gamma)\theta^2} \cdot (\partial_s v_2) dr.$$

One can apply John's trick to this term, but here we choose to use a more precise method to control this term. Since the integrand is similar to the evolution equation of θ along λ_1

direction, we argue that a major part of this is actually integrable along λ_1 direction. From the evolution of Riemann invariants (refer to Section 3.2.4), we have

$$(\partial_r + \lambda_1 \partial_s)\Big(\underbrace{\frac{1}{\sqrt{\gamma}} \ln\big(\sqrt{1+\theta^2}+\theta\big)}_{(v_1+v_2)}\Big) = (\lambda_1 - \lambda_2)(\partial_s v_2) + (F_1 + F_2)$$

$$\underbrace{\frac{\sqrt{\alpha}}{\sqrt{\beta}} \cdot \frac{2\sqrt{\gamma}}{1+(1-\gamma)\theta^2}}_{(\lambda_1-\lambda_2)} \cdot (\partial_s v_2) = (\partial_r + \lambda_1 \partial_s)\Big(\frac{1}{\sqrt{\gamma}} \ln\big(\sqrt{1+\theta^2}+\theta\big)\Big) - (F_1 + F_2)$$

$$\begin{aligned}
-\frac{\sqrt{\alpha}}{\sqrt{\beta}} \cdot 2\sqrt{\gamma} \cdot \frac{1-\frac{\sqrt{\gamma}\theta}{\sqrt{1+\theta^2}}}{1+(1-\gamma)\theta^2} \cdot (\partial_s v_2) &= -\Big(1 - \frac{\sqrt{\gamma}\theta}{\sqrt{1+\theta^2}}\Big) \cdot \frac{1}{\sqrt{\gamma}\sqrt{1+\theta^2}} (\partial_r + \lambda_1 \partial_s)\theta \\
&\quad + \Big(1 - \frac{\sqrt{\gamma}\theta}{\sqrt{1+\theta^2}}\Big)(F_1 + F_2) \\
&= (\partial_r + \lambda_1 \partial_s)\Big(\underbrace{-\frac{1}{\sqrt{\gamma}} \ln\big(\sqrt{1+\theta^2}+\theta\big) + \frac{1}{2}\ln(1+\theta^2)}_{Q(\theta)}\Big) \\
&\quad + \Big(1 - \frac{\sqrt{\gamma}\theta}{\sqrt{1+\theta^2}}\Big)(F_1 + F_2)
\end{aligned}$$

Integrating both sides along λ_1 direction, we have

$$\begin{aligned}
\Big|\int_{r_0\ (\lambda_1)}^{r} -\frac{\sqrt{\alpha}}{\sqrt{\beta}} \cdot 2\sqrt{\gamma} \cdot \frac{1-\frac{\sqrt{\gamma}\theta}{\sqrt{1+\theta^2}}}{1+(1-\gamma)\theta^2} \cdot (\partial_s v_2) dr\Big| &\le |Q(\theta(P))| + |Q(\theta(P_1))| \\
&\quad + \int_{r\ (\lambda_1)}^{r_0} |F_1 + F_2| dr \\
&\le C.
\end{aligned}$$

Note that $|\theta|$ remains small so the Q terms have no problem. There is no $(\rho\alpha)$ term in $F_1 + F_2$ so this term is also bounded. Here

$$C = C(parameters).$$

□

Lemma 3.5.3. *We separate the background influence and error terms for*

$$\int_{r_0\ (\lambda_1)}^{r} (D_{v_1} F_1) dr$$

in the integral factor $-f$*. It turns out the background influence is*

$$\Big(\frac{1+\gamma}{2} - \gamma\Big) \cdot \frac{1}{n} \int_{r}^{r_0} r(\mathring{\rho}\mathring{\alpha}) dr.$$

Proof. By the definition of F_1, we have

$$\begin{aligned} D_{v_1} F_1 &= (D_{(\rho\alpha)} F_1)(D_{v_1}(\rho\alpha)) + (D_\theta F_1)(D_{v_1}\theta) \\ &= \Big(-\frac{1}{2} + \frac{\gamma}{1+\gamma}\Big) \cdot \frac{r}{n}(\rho\alpha) \cdot (1+\gamma) + (D_\theta F_1) \cdot \sqrt{\gamma}\sqrt{1+\theta^2} \\ &= \Big(-\frac{1+\gamma}{2} + \gamma\Big) \cdot \frac{r}{n}(\mathring{\rho}\mathring{\alpha}) \cdot e^{\ln(\rho\alpha) - \ln(\mathring{\rho}\mathring{\alpha})} + (D_\theta F_1) \cdot \sqrt{\gamma}\sqrt{1+\theta^2}. \end{aligned}$$

Therefore, the contribution of the background to the $\int_{r_0\,(\lambda_1)}^{r} (D_{v_1} F_1) dr$ term is

$$\Big(\frac{1+\gamma}{2} - \gamma\Big) \cdot \frac{1}{n} \int_{r}^{r_0} r(\mathring{\rho}\mathring{\alpha}) dr. \tag{3.3}$$

Notice that the error terms are

$$\int_{r}^{r_0} \Big(\frac{1+\gamma}{2} - \gamma\Big) \cdot \frac{r}{n}(\mathring{\rho}\mathring{\alpha}) \cdot \Big|e^{\ln(\rho\alpha) - \ln(\mathring{\rho}\mathring{\alpha})} - 1\Big| + |D_\theta F_1| \cdot \sqrt{\gamma}\sqrt{1+\theta^2} dr$$

$$\leq M\epsilon + C,$$

where we use Proposition 3.4.1 to control $|e^{\ln(\rho\alpha) - \ln(\mathring{\rho}\mathring{\alpha})} - 1| \leq e|\ln(\rho\alpha) - \ln(\mathring{\rho}\mathring{\alpha})|$ and use the fact that $|D_\theta F_1| \cdot \sqrt{1+\theta^2}$ is bounded by a fraction of θ, of which the numerator and the denominator having the same degree. Here

$$M = M(r_{cut}, parameters)$$

goes to infinity as $r_{cut} \to r_*^+$ and

$$C = C(parameters).$$

□

Lemma 3.5.4. *We prove that the term*

$$\int_{r_0\ (\lambda_1)}^{r} -(D_{v_2}\lambda_1)(\partial_s v_2)dr$$

in the integral factor $-f$ can be regarded as an error term.

Proof. Recall that we have the evolution for Riemann invariants

$$(\partial_r + \lambda_1\partial_s)v_1 = F_1$$

$$(\partial_r + \lambda_2\partial_s)v_2 = F_2.$$

Adding these two equations, we get

$$(\partial_r + \lambda_1\partial_s)(v_1 + v_2) = (F_1 + F_2) + (\lambda_1 - \lambda_2)(\partial_s v_2)$$

$$(\lambda_1 - \lambda_2)(\partial_s v_2) = (\partial_r + \lambda_1\partial_s)(v_1 + v_2) - (F_1 + F_2)$$

$$\frac{\sqrt{\alpha}}{\sqrt{\beta}} \cdot \frac{2\sqrt{\gamma}}{1 + (1-\gamma)\theta^2} \cdot (\partial_s v_2) = (\partial_r + \lambda_1\partial_s)\Big(\frac{1}{\sqrt{\gamma}} \ln\big(\sqrt{1+\theta^2} + \theta\big)\Big) - (F_1 + F_2).$$

On the other hand, from the definition of λ_1 (refer to Section 3.2.4),

$$\begin{aligned} -(D_{v_2}\lambda_1)(\partial_s v_2) &= -(D_\theta\lambda_1) \cdot \sqrt{\gamma}\sqrt{1+\theta^2}(\partial_s v_2) \\ &= -\underbrace{\frac{\sqrt{\alpha}}{\sqrt{\beta}} \cdot \frac{1-\gamma}{\sqrt{1+\theta^2}\big(\sqrt{1+\theta^2} + \sqrt{\gamma}\theta\big)^2}}_{D_\theta\lambda_1} \cdot \sqrt{\gamma}\sqrt{1+\theta^2}(\partial_s v_2) \\ &= -\frac{\sqrt{\alpha}}{\sqrt{\beta}} \cdot \frac{\sqrt{\gamma}(1-\gamma)}{\big(\sqrt{1+\theta^2} + \sqrt{\gamma}\theta\big)^2} \cdot (\partial_s v_2). \end{aligned}$$

Putting these two equations together, we find that

$$\begin{aligned} -(D_{v_2}\lambda_1)(\partial_s v_2) = &-\frac{1-\gamma}{2} \cdot \frac{\sqrt{1+\theta^2} - \sqrt{\gamma}\theta}{\sqrt{1+\theta^2} + \sqrt{\gamma}\theta} \cdot (\partial_r + \lambda_1\partial_s)\Big(\frac{1}{\sqrt{\gamma}} \ln\big(\sqrt{1+\theta^2} + \theta\big)\Big) \\ &+ \frac{1-\gamma}{2} \cdot \frac{\sqrt{1+\theta^2} - \sqrt{\gamma}\theta}{\sqrt{1+\theta^2} + \sqrt{\gamma}\theta}(F_1 + F_2) \end{aligned}$$

$$
\begin{aligned}
&= -\frac{1-\gamma}{2\sqrt{\gamma}} \cdot \frac{\sqrt{1+\theta^2} - \sqrt{\gamma}\theta}{\sqrt{1+\theta^2} + \sqrt{\gamma}\theta} \cdot \frac{1}{\sqrt{1+\theta^2}} \cdot (\partial_r + \lambda_1 \partial_s)\theta \\
&\quad + \frac{1-\gamma}{2} \cdot \frac{\sqrt{1+\theta^2} - \sqrt{\gamma}\theta}{\sqrt{1+\theta^2} + \sqrt{\gamma}\theta}(F_1 + F_2) \\
&= -\frac{1-\gamma}{2\sqrt{\gamma}} \cdot (\partial_r + \lambda_1 \partial_s)\Big\{ -\frac{1-\sqrt{\gamma}}{1+\sqrt{\gamma}} \ln\big(1 - \frac{\theta}{\sqrt{1+\theta^2}+1}\big) \\
&\qquad + \frac{1+\sqrt{\gamma}}{1-\sqrt{\gamma}} \ln\big(1 + \frac{\theta}{\sqrt{1+\theta^2}+1}\big) \\
&\qquad - \frac{2\sqrt{\gamma}}{1-\gamma} \ln\big((\frac{\theta}{\sqrt{1+\theta^2}+1})^2 + 2\sqrt{\gamma}(\frac{\theta}{\sqrt{1+\theta^2}+1}) + 1\big)\Big\} \\
&\qquad + \frac{1-\gamma}{2} \cdot \frac{\sqrt{1+\theta^2} - \sqrt{\gamma}\theta}{\sqrt{1+\theta^2} + \sqrt{\gamma}\theta}(F_1 + F_2) \\
&= -\frac{1-\gamma}{2\sqrt{\gamma}} \cdot (\partial_r + \lambda_1 \partial_s) Q(\theta) + \frac{1-\gamma}{2} \cdot \frac{\sqrt{1+\theta^2} - \sqrt{\gamma}\theta}{\sqrt{1+\theta^2} + \sqrt{\gamma}\theta}(F_1 + F_2).
\end{aligned}
$$

where Q denotes the function inside of the big parenthesis. Since $Q(0) = 0$ and $|\theta|$ remains small, we conclude that

$$
\begin{aligned}
\Big|\int_{(\lambda_1)} -(D_{v_2}\lambda_1)(\partial_s v_2) dr\Big| &\le \Big|\frac{1-\gamma}{2\sqrt{\gamma}} \cdot Q(\theta(P))\Big| + \Big|\frac{1-\gamma}{2\sqrt{\gamma}} \cdot Q(\theta(P_1))\Big| \\
&\quad + (r_0 - r_*) \cdot \frac{1-\gamma}{2} \cdot \sup \frac{\sqrt{1+\theta^2} - \sqrt{\gamma}\theta}{\sqrt{1+\theta^2} + \sqrt{\gamma}\theta}(F_1 + F_2) \\
&\le C
\end{aligned}
$$

where we use the fact that $|F_1 + F_2|$ remains bounded since there is no $(\rho\alpha)$ term. Here

$$
C = C(parameters).
$$

□

Lemma 3.5.5. *We show that the $-(D_\alpha \lambda_1)(\partial_s \alpha)$ term in the integral factor $-f$ can be regarded as an erorr term (without r_{cut}).*

Proof. Notice that

$$
\begin{aligned}
-(D_\alpha \lambda_1)(\partial_s \alpha) &= -\frac{1}{2} \cdot \frac{\sqrt{\alpha}}{\sqrt{\beta}} \cdot \frac{(1-\gamma)\theta\sqrt{1+\theta^2}+\sqrt{\gamma}}{1+(1-\gamma)\theta^2} \cdot (\partial_s \ln \alpha) \\
&= \frac{1}{2} \cdot \frac{\sqrt{\alpha}}{\sqrt{\beta}} \cdot \frac{(1-\gamma)\theta\sqrt{1+\theta^2}+\sqrt{\gamma}}{1+(1-\gamma)\theta^2} \cdot \frac{2(1+\gamma)}{n} \cdot r(\rho\alpha) \cdot \frac{\sqrt{\beta}}{\sqrt{\alpha}} \cdot \theta\sqrt{1+\theta^2} \\
&= \frac{(1+\gamma)}{n} \cdot r(\rho\alpha) \cdot \frac{(1-\gamma)\theta\sqrt{1+\theta^2}+\sqrt{\gamma}}{1+(1-\gamma)\theta^2} \cdot \theta\sqrt{1+\theta^2} \\
&= \frac{(1+\gamma)}{n} \cdot r(\mathring{\rho}\mathring{\alpha}) \cdot e^{\ln(\rho\alpha)-\ln(\mathring{\rho}\mathring{\alpha})} \cdot \frac{(1-\gamma)\theta\sqrt{1+\theta^2}+\sqrt{\gamma}}{1+(1-\gamma)\theta^2} \cdot \theta\sqrt{1+\theta^2}.
\end{aligned}
$$

Therefore, we have

$$
\Big| \int_{r_0}^{r} -(D_\alpha \lambda_1)(\partial_s \alpha) dr \Big| \le \int_{r}^{r_0} \big| D_\alpha \lambda_1 \big| \big| \partial_s \alpha \big| dr \le M\epsilon,
$$

where M comes from $(\mathring{\rho}\mathring{\alpha})$ and ϵ comes from $|\theta|$. Here

$$
M = M(r_{cut}, parameters)
$$

goes to infinity as $r_{cut} \to r_*^+$.

□

For later convenience, we compute the asymptote for the background $(\mathring{\rho}\mathring{\alpha})$.

Lemma 3.5.6. *(Asymptote for $(\mathring{\rho}\mathring{\alpha})$.) There are constants*

$$
r_{mid} = r_{mid}(r_*, \gamma, \Lambda, S^{[h]}) > r_*, \quad C_{mid} = C_{mid}(r_0, r_{mid}, r_*, \gamma, \Lambda, S^{[h]})
$$

so that

$$
(\mathring{\rho}\mathring{\alpha})_L \le (\mathring{\rho}\mathring{\alpha}) \le (\mathring{\rho}\mathring{\alpha})_R, \quad r \in (r_*, r_0]
$$

where

$$
(\mathring{\rho}\mathring{\alpha})_L = \begin{cases} \frac{n}{(1-\gamma)(r-r_*)} \cdot \frac{1}{(1.1)r_*}, & r \in (r_*, r_{mid}] \\ \frac{1}{C_{mid}}, & r \in (r_{mid}, r_0] \end{cases}
$$

$$(\mathring{\rho}\mathring{\alpha})_R = \begin{cases} \frac{n}{(1-\gamma)(r-r_*)} \cdot \frac{(1.1)}{r_*}, & r \in (r_*, r_{mid}] \\ \\ C_{mid}, & r \in [r_{mid}, r_0] \end{cases}$$

with $\frac{r_{mid}}{r_*} < (1.1)$, $r_{mid} - r_* < 1$ *and* $r_0 > r_{mid}$.

Remark. *Note that* r_{mid} *does not depend on* r_0 *nor* r_{cut}.

Proof. We have

$$\mu = \mu_* + O((r - r_*)^{\frac{1+\gamma}{1-\gamma}})$$

$$\tau = \frac{1-\gamma}{n} \mu_*^{\frac{1+\gamma}{2\gamma}} \cdot r_*^{\frac{(n+1)(1-\gamma)}{2} - 1} (r - r_*) + O((r - r_*)^2).$$

Here the notation is

$$\mathring{\rho} = \frac{\mu^{\frac{1+\gamma}{2\gamma}}}{r^{\frac{(n+1)(1+\gamma)}{2}} \cdot \tau^{\frac{1+\gamma}{1-\gamma}}}$$

$$\mathring{\alpha} = r^{n-1} \cdot \tau^{\frac{2\gamma}{1-\gamma}}.$$

Therefore, we have

$$\begin{aligned} (\mathring{\rho}\mathring{\alpha}) &= \frac{\mu^{\frac{1+\gamma}{2\gamma}}}{\tau} \cdot r^{n-1-\frac{(n+1)(1+\gamma)}{2}} \\ &= \frac{\mu^{\frac{1+\gamma}{2\gamma}} \cdot r^{n-1-\frac{(n+1)(1+\gamma)}{2}}}{\frac{1-\gamma}{n} \mu_*^{\frac{1+\gamma}{2\gamma}} \cdot r_*^{\frac{(n+1)(1-\gamma)}{2}-1} (r - r_*) + O((r - r_*)^2)} \\ &= \frac{1}{r - r_*} \cdot \frac{n}{1-\gamma} \cdot \frac{\mu^{\frac{1+\gamma}{2\gamma}} \cdot r^{n-1-\frac{(n+1)(1+\gamma)}{2}}}{\mu_*^{\frac{1+\gamma}{2\gamma}} \cdot r_*^{\frac{(n+1)(1-\gamma)}{2}-1} + O(r - r_*)}. \end{aligned}$$

Since

$$\lim_{r \to r_*^+} \frac{\mu^{\frac{1+\gamma}{2\gamma}} \cdot r^{n-1-\frac{(n+1)(1+\gamma)}{2}}}{\mu_*^{\frac{1+\gamma}{2\gamma}} \cdot r_*^{\frac{(n+1)(1-\gamma)}{2}-1} + O(r - r_*)} = \frac{\mu_*^{\frac{1+\gamma}{2\gamma}} \cdot r_*^{n-1-\frac{(n+1)(1+\gamma)}{2}}}{\mu_*^{\frac{1+\gamma}{2\gamma}} \cdot r_*^{\frac{(n+1)(1-\gamma)}{2}-1}} = \frac{1}{r_*},$$

there is a time

$$r_{mid} = r_{mid}(r_*, \gamma, \Lambda, S^{[h]}) > r_*$$

so that

$$\frac{n}{(1-\gamma)(r-r_*)}\cdot\frac{1}{(1.1)r_*}\leq(\mathring{\rho}\mathring{\alpha})\leq\frac{n}{(1-\gamma)(r-r_*)}\cdot\frac{(1.1)}{r_*}$$

for $r_* < r \leq r_{mid}$ and $\frac{r_{mid}}{r_*} < (1.1)$. If $r_0 > r_{mid}$, then since $(\mathring{\rho}\mathring{\alpha})$ does not have a singularity over $[r_{mid}, r_0]$, there exists $C_{mid} = C_{mid}(r_0, r_{mid}, r_*, \gamma, \Lambda, S^{[h]})$ so that

$$\frac{1}{C_{mid}}\leq(\mathring{\rho}\mathring{\alpha})\leq C_{mid}$$

for $r_{mid} \leq r \leq r_0$. □

Proposition 3.5.1. *(Net influence on $-f$ from the background.) There exists a constnat*

$$C_3 = C_3(parameters)$$

so that

$$e^{-f}\geq\left(\frac{1}{r-r_*}\right)^{\delta_1}\cdot\frac{1}{C_3},\quad r\in[r_{cut},r_{mid}] \tag{3.4}$$

$$e^{-f}\geq\frac{1}{C_3},\quad r\in(r_{mid},r_0] \tag{3.5}$$

$$e^{f}\leq C_3,\quad r\in[r_{cut},r_0]. \tag{3.6}$$

In addition,

$$e^f(r_0)=1$$

from the definition of $-f$ (refer to Lemma 3.5.1).

Proof. Putting (3.1), (3.2), (3.3) together, we see that the net influence from the background solution is

$$-\frac{1-\gamma}{1+\gamma}\cdot\frac{1}{n}\int_r^{r_0}r(\mathring{\rho}\mathring{\alpha})dr+\frac{1}{1+\gamma}\big(\ln(\mathring{\rho}\mathring{\alpha})-\ln(\mathring{\rho}(r_0)\mathring{\alpha}(r_0))+\Big(\frac{1+\gamma}{2}-\gamma\Big)\cdot\frac{1}{n}\int_r^{r_0}r(\mathring{\rho}\mathring{\alpha})dr$$
$$=-\frac{(1-\gamma)^2}{2(1+\gamma)}\cdot\frac{1}{n}\int_r^{r_0}r(\mathring{\rho}\mathring{\alpha})dr+\frac{1}{1+\gamma}\big(\ln(\mathring{\rho}\mathring{\alpha})-\ln(\mathring{\rho}(r_0)\mathring{\alpha}(r_0))\big).$$

From Lemma 3.5.6, we have

$$(\mathring{\rho}\mathring{\alpha})_L \le (\mathring{\rho}\mathring{\alpha}) \le (\mathring{\rho}\mathring{\alpha})_R.$$

Therefore, we get the estimate for the background influence to $-f$

$$-\frac{(1-\gamma)^2}{2(1+\gamma)} \cdot \frac{1}{n}\int_r^{r_0} r(\mathring{\rho}\mathring{\alpha})_R dr + \frac{1}{1+\gamma}\big(\ln(\mathring{\rho}\mathring{\alpha})_L - \ln(\mathring{\rho}(r_0)\mathring{\alpha}(r_0))_R\big)$$

$$\le (\text{background influence to } -f)$$

$$\le -\frac{(1-\gamma)^2}{2(1+\gamma)} \cdot \frac{1}{n}\int_r^{r_0} r(\mathring{\rho}\mathring{\alpha})_L dr + \frac{1}{1+\gamma}\big(\ln(\mathring{\rho}\mathring{\alpha})_R - \ln(\mathring{\rho}(r_0)\mathring{\alpha}(r_0))_L\big).$$

Here we discuss two cases. For $r \in (r_*, r_{mid}]$,

$$-\frac{(1.1)(1-\gamma)}{2(1+\gamma)r_*}\int_r^{r_{mid}} \frac{r_{mid}}{r-r_*}dr - \frac{(1-\gamma)^2}{2n(1+\gamma)}\int_{r_{mid}}^{r_0}(r_0 C_{mid})dr$$
$$+\frac{1}{1+\gamma}\Big(\ln\Big(\frac{n}{(1.1)r_*(1-\gamma)(r-r_*)}\Big) - \ln\big(C_{mid}\big)\Big)$$

$$\le (\text{background influence to } -f)$$

$$\le -\frac{(1-\gamma)}{(2.2)r_*(1+\gamma)}\int_r^{r_{mid}} \frac{r_*}{r-r_*}dr - \frac{(1-\gamma)^2}{2n(1+\gamma)}\int_{r_{mid}}^{r_0}\frac{r_*}{C_{mid}}dr$$
$$+\frac{1}{1+\gamma}\Big(\ln\Big(\frac{(1.1)n}{r_*(1-\gamma)(r-r_*)}\Big) - \ln\Big(\frac{1}{C_{mid}}\Big)\Big),$$

where we use the definition of $(\mathring{\rho}\mathring{\alpha})_L$ and $(\mathring{\rho}\mathring{\alpha})_R$ in Lemma 3.5.6 and estimate r in the integrand. Applying exp to them, we have

$$\Big(\frac{1}{r-r_*}\Big)^{\frac{1}{1+\gamma}-\frac{(1.1)(1-\gamma)}{2(1+\gamma)}\cdot\frac{r_{mid}}{r_*}} \cdot \frac{1}{C} \le (\text{background influence to } e^{-f}) \le \Big(\frac{1}{r-r_*}\Big)^{\frac{1}{1+\gamma}-\frac{(1-\gamma)}{(2.2)(1+\gamma)}} \cdot C$$

for $r \in (r_*, r_{mid}]$. We know that

$$\delta_1 := \frac{1}{1+\gamma} - \frac{(1.1)(1-\gamma)}{2(1+\gamma)} \cdot \frac{r_{mid}}{r_*} > 0$$

$$\delta_2 := \frac{1}{1+\gamma} - \frac{(1-\gamma)}{(2.2)(1+\gamma)} > 0$$

since $\frac{r_{mid}}{r_*} < (1.1)$ by Lemma 3.5.6. Here

$$C = C(parameters).$$

By Lemma 3.5.2, Lemma 3.5.3, Lemma 3.5.4, Lemma 3.5.5, we have

$$\Big(\frac{1}{r-r_*}\Big)^{\delta_1}\cdot\frac{1}{C_3}\le e^{-f}\le\Big(\frac{1}{r-r_*}\Big)^{\delta_2}\cdot C_3,\quad r\in[r_{cut},r_{mid}]$$

for some

$$C_3=C_3(parameters).$$

We can rearrange C_3 so that

$$e^f\le C_3,\quad r\in(r_*,r_{mid}]$$

since C_3 can depend on r_{mid}. For $r\in(r_{mid},r_0]$, we have

$$-C\le(\text{background influence to }-f)\le C$$

and therefore

$$\frac{1}{C_3}\le e^{-f}\le C_3$$

after possibly making C_3 larger.

□

3.5.2 Pointwise behavior of the other terms

Lemma 3.5.7. *We compute the $(D_{v_1}\lambda_1)$ term.*

Proof.

$$(D_{v_1}\lambda_1)=(D_\theta\lambda_1)(D_{v_1}\theta)=\frac{\sqrt{\alpha}}{\sqrt{\beta}}\cdot\frac{1-\gamma}{\sqrt{1+\theta^2}\big(\sqrt{1+\theta^2}+\sqrt{\gamma}\theta\big)^2}\cdot\sqrt{\gamma}\sqrt{1+\theta^2}\ge C\Big(\frac{\sqrt{\mathring{\alpha}}}{\sqrt{\mathring{\beta}}}\Big)_L$$

since $|\theta|$, $|\ln\alpha-\ln\mathring{\alpha}|$, $|\ln\beta-\ln\mathring{\beta}|$ remain small. Furthermore, since $\frac{\sqrt{\mathring{\alpha}}}{\sqrt{\mathring{\beta}}}\approx\frac{1}{r-r_*}$, we have

$$\Big(\frac{\sqrt{\mathring{\alpha}}}{\sqrt{\mathring{\beta}}}\Big)_L=\begin{cases}\frac{C}{r-r_*}, & r\in(r_*,r_{mid}]\\[2ex] C, & r\in(r_{mid},r_0].\end{cases}$$

□

Lemma 3.5.8. *We show that $(D_\alpha F_1)(\partial_s \alpha)$ can be regarded as an error term, meaning its upper bound contains ϵ. Notice that here we are regarding F_1 as $F_1(v_1, v_2, \alpha)$.*

Proof. Notice that

$$\alpha(D_\alpha F_1) = \alpha\big(\frac{2}{n}\Lambda r - \frac{S^{[h]}}{nr}\big) \cdot \underbrace{\Big(-\frac{(1-\gamma)\theta\sqrt{1+\theta^2}+\sqrt{\gamma}}{4\sqrt{\gamma}\big(1+(1-\gamma)\theta^2\big)} - \frac{1}{2(1+\gamma)}\Big)}_{\text{bounded between} -\frac{1}{2(1+\gamma)} \pm \frac{1}{4\sqrt{\gamma}}}$$

$$(\partial_s \ln \alpha) = -2(1+\gamma) \cdot \frac{r}{n}(\rho\alpha) \cdot \frac{\sqrt{\beta}}{\sqrt{\alpha}} \cdot \theta\sqrt{1+\theta^2}.$$

Therefore, thanks to the θ inside of $(\partial_s \ln \alpha)$, we conclude that

$$|(D_\alpha F_1)(\partial_s \alpha)| \leq C\epsilon$$

where $C = C(parameters)$. □

Lemma 3.5.9. *We show that $|\partial_s v_2|$ is pointwise small over Ω_{r_{cut}, z_M}, as long as the initial data $\sup_{s\in[z_L, z_M]} |\partial_s v_2(r_0, s)|$ and ϵ (total variation upper bound) are sufficiently small. Here we apply John's trick again. Ω_{r_{cut}, z_M} is defined to be*

$$\Omega_{r_{cut}, z_M} = \{(r, X_1(r; z)) \mid r_{cut} \leq r \leq r_0,\ z_L \leq z \leq z_M\}.$$

See Figure 3.12.

Proof.

Step 1. The evolution equation for $(\partial_s v_2)$ is

$$\begin{aligned}(\partial_r + \lambda_2 \partial_s)(\partial_s v_2) &= -(\partial_s \lambda_2)(\partial_s v_2) + (\partial_s F_2)\\ &= -(D_{v_2}\lambda_2)(\partial_s v_2)^2 + \Big(-(D_{v_1}\lambda_2)(\partial_s v_1) - (D_\alpha \lambda_2)(\partial_s \alpha)\\ &\quad - (D_\beta \lambda_2)(\partial_s \beta) + (D_{v_2} F_2)\Big)(\partial_s v_2)\end{aligned}$$

$$+ (D_{v_1}F_2)(\partial_s v_1) + (D_\alpha F_2)(\partial_s \alpha).$$

Using the integral factor method, we get

$$(\partial_r + \lambda_2\partial_s)(e^{f_2} \cdot \partial_s v_2) = e^{f_2} \cdot (D_{v_1}F_2)(\partial_s v_1) + e^{f_2} \cdot (D_\alpha F_2)(\partial_s \alpha),$$

where

$$-f_2 = \int_{r_0\ (\lambda_2)}^{r} - (D_{v_2}\lambda_2)(\partial_s v_2) - (D_{v_1}\lambda_2)(\partial_s v_1) - (D_\alpha\lambda_2)(\partial_s\alpha) - (D_\beta\lambda_2)(\partial_s\beta) + (D_{v_2}F_2)dr.$$

Note that here we incoporate the quadratic term $(\partial_s v_2)^2$ into the integral factor. By the fundamental theorem of Calculus along (λ_2),

$$e^{f_2} \cdot (\partial_s v_2)(P) - (\partial_s v_2)(P_2) = \int_{r_0\ (\lambda_2)}^{r} e^{f_2} \cdot (D_{v_1}F_2)(\partial_s v_1) + e^{f_2} \cdot (D_\alpha F_2)(\partial_s\alpha)dr.$$

Step 2. (John's trick) Fix $z \in [z_{LL}, z_M)$. Let $y = y(r) \in [z_L, z_M]$ be the function so that

$$X_2(r; z) = X_1(r; y(r))$$

where $r \leq r_0$ satisfies $X_2(r, z) \leq X_1(r, z_M)$ so that $(r, X_2(r; z)) \in \Omega_{r_{cut}, z_M}$ (see Figure 3.12). Taking derivative with respect to r on both sides, we get

$$\begin{aligned} \lambda_2 &= \lambda_1 + \frac{\partial X_1}{\partial z} \cdot \frac{dy}{dr} \\ \frac{dr}{dy} &= \frac{1}{\lambda_2 - \lambda_1} \cdot \frac{\partial X_1}{\partial z} = -\frac{\sqrt{\beta}}{\sqrt{\alpha}} \cdot \frac{1 + (1-\gamma)\theta^2}{2\sqrt{\gamma}} \cdot \frac{\partial X_1}{\partial z}. \end{aligned}$$

This calculation aims to use (r, y) coordinate to foliate the spacetime region Ω_{r_{cut}, z_M}, as one can see the $drdy$ in the following computation (see Figure 3.9). Fix $P \in \Omega_{r_{cut}, z_M}$ and let $P_2 = (r_0, z)$. We have

$$\begin{aligned} &\int_{r\ (X_2(\cdot;z))}^{r_0} e^{f_2} \cdot |D_{v_1}F_2||\partial_s v_1| dr \\ &\qquad \leq \int_{z_M\ (X_2(\cdot;z))}^{z_L} e^{f_2} \cdot |D_{v_1}F_2||\partial_s v_1| \cdot \frac{dr}{dy} dy \end{aligned}$$

$$= \int_{z_L\ (X_2(\cdot;z))}^{z_M} e^{f_2} \cdot |D_{v_1} F_2| |\partial_s v_1| \cdot \frac{\sqrt{\beta}}{\sqrt{\alpha}} \cdot \frac{1 + (1-\gamma)\theta^2}{2\sqrt{\gamma}} \cdot \frac{\partial X_1}{\partial z} dy$$

$$\leq \sup_{\Omega_{r_{cut}, z_M}} \left(e^{f_2} \cdot \frac{\sqrt{\beta}}{\sqrt{\alpha}} |D_{v_1} F_2| \cdot \frac{1 + (1-\gamma)\theta^2}{2\sqrt{\gamma}} \right)$$

$$\int_{z_L\ (X_2(\cdot;z))}^{z_M} |\partial_s v_1| \cdot \frac{\partial X_1}{\partial z} dy$$

$$\leq \sup_{\Omega_{r_{cut}, z_M}} \left(e^{f_2} \cdot \frac{\sqrt{\beta}}{\sqrt{\alpha}} |D_{v_1} F_2| \cdot \frac{1 + (1-\gamma)\theta^2}{2\sqrt{\gamma}} \right)$$

$$\left(\int_{z_L\ (\Gamma_{r_0})}^{z_M} |\partial_s v_1| ds + \iint_{\Omega_{r_{cut}, z_M}} |\partial_s F_1| \cdot \frac{\partial X_1}{\partial z} dr dy \right)$$

$$= \sup_{\Omega_{r_{cut}, z_M}} \left(e^{f_2} \cdot \frac{\sqrt{\beta}}{\sqrt{\alpha}} |D_{v_1} F_2| \cdot \frac{1 + (1-\gamma)\theta^2}{2\sqrt{\gamma}} \right)$$

$$\left(\int_{z_L\ (\Gamma_{r_0})}^{z_M} |\partial_s v_1| ds + \iint_{\Omega_{r_{cut}, z_M}} |\partial_s F_1| dr ds \right).$$

Notice that we can make the initial data $\int_{z_L\ (\Gamma_{r_0})}^{z_R} |\partial_s v_1| ds$ small, and make

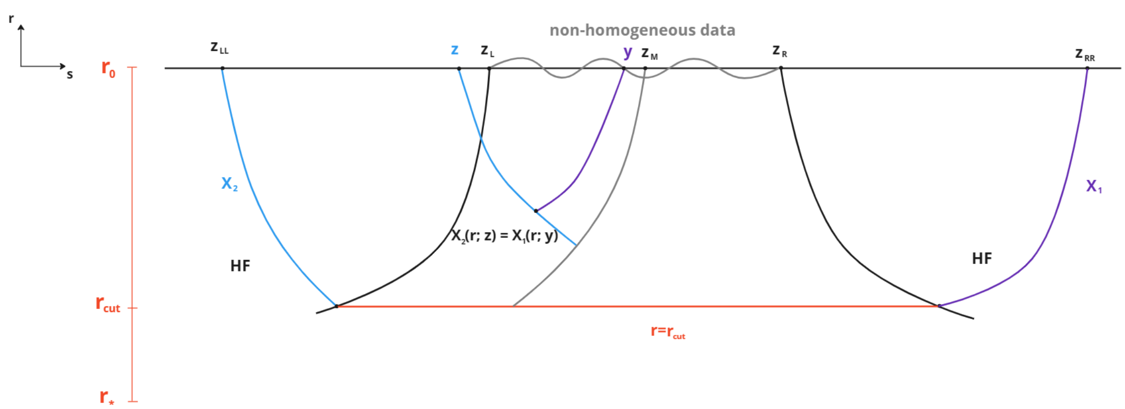

Figure 3.9 John's trick to foliate the spacetime by (r, y).

$$\iint_{\Omega_{r_{cut}, z_M}} |\partial_s F_1| dr ds \leq \iint_{\Omega_{r_{cut}, z_M}} |D_{v_1} F_1| |\partial_s v_1| + |D_{v_2} F_1| |\partial_s v_2| + |D_\alpha F_1| |\partial_s \alpha| dr ds$$

be bounded by $M\epsilon$ where $M = M(r_{cut}, parameters)$. Thus we conclude that

$$\int_{r \ (X_2(\cdot;z))}^{r_0} e^{f_2} \cdot |D_{v_1} F_2||\partial_s v_1| dr \leq \sup_{\Omega_{r_{cut}, z_M}} (e^{f_2}) \cdot M\epsilon.$$

Step 3. Note that

$$\int_{r \ (\lambda_2)}^{r_0} e^{f_2} \cdot |D_\alpha F_2||\partial_s \alpha| dr \leq \sup \left(e^{f_2} \cdot \alpha |D_\alpha F_2||\partial_s \ln \alpha| \right) \cdot (r_0 - r)$$

$$\leq \sup(e^{f_2}) \cdot C\epsilon$$

where $C = C(parameters)$ where ϵ comes from the $|\theta|$ inside of $(\partial_s \ln \alpha)$.

Step 4.

Bootstrap Assumption.

$$|\partial_s v_2| \leq \epsilon_a$$

with $0 < \epsilon_a < 1$.

With this assumption, we have

$$|f_2| \leq M\epsilon_a + M\epsilon + M\epsilon + M + M \leq 5M$$

where $M = M(r_{cut}, parameters)$. From the integral equation for $e^{f_2} \cdot (\partial_s v_2)$, we have

$$e^{f_2} \cdot |\partial_s v_2| \leq |(\partial_s v_2)(P_2)| + \int_{r_0 \ (\lambda_2)}^{r} e^{f_2} \cdot (D_{v_1} F_2)|\partial_s v_1| dr + \int_{r \ (\lambda_2)}^{r_0} e^{f_2} \cdot |D_\alpha F_2||\partial_s \alpha| dr$$

$$\leq |(\partial_s v_2)(P_2)| + e^{5M} \cdot M\epsilon + e^{5M} \cdot C\epsilon,$$

or

$$|\partial_s v_2(P)| \leq e^{5M} |(\partial_s v_2)(P_2)| + e^{10M} \cdot M\epsilon + e^{10M} \cdot C\epsilon.$$

Thus, if we choose the ϵ (total variation upper bound) small enough compared with ϵ_a and choose the initial data $\sup_{s \in [z_L, z_M]} |(\partial_s v_2)(r_0, s)|$ small enough, both depending on r_{cut}, we can make

$$e^{5M} |(\partial_s v_2)(P_2)| + e^{10M} \cdot M\epsilon + e^{10M} \cdot C\epsilon \leq \frac{1}{2}\epsilon_a.$$

Therefore, we are able to improve the estimate for $|\partial_s v_2|$ and therefore close the bootstrap argument.

□

3.5.3 Proof of the Main Theorem

Proof of Theorem 3.1.1. Going back to the Riccati equation derived in Lemma 3.5.1

$$(\partial_r + \lambda_1 \partial_s)(e^f \cdot \partial_s v_1) = -(e^{-f})(D_{v_1}\lambda_1)(e^f \cdot \partial_s v_1)^2 + e^f(D_{v_2}F_1)(\partial_s v_2) + e^f(D_\alpha F_1)(\partial_s \alpha),$$

we have that, based on Proposition 3.5.1, Lemma 3.5.7, Lemma 3.5.8, and Lemma 3.5.9,

$$\begin{aligned} &-(\partial_r + \lambda_1 \partial_s)(e^f \cdot \partial_s v_1) \\ &\geq \begin{cases} \frac{1}{C_3(r-r_*)^{\delta_1}} \cdot \frac{1}{C(r-r_*)} \cdot (e^f \cdot \partial_s v_1)^2 - e^f(M\epsilon_a) - e^f(C\epsilon), & r \in [r_{cut}, r_{mid}] \\ \frac{1}{C_3} \cdot \frac{1}{C} \cdot (e^f \cdot \partial_s v_1)^2 - e^f(M\epsilon_a) - e^f(C\epsilon), & r \in (r_{mid}, r_0]. \end{cases} \end{aligned}$$

We apply the change of variables

- $y = (e^f \cdot \partial_s v_1)$ and
- $t = r_0 - r$, $-(\partial_r + \lambda_1 \partial_s) = \frac{d}{dt}$,

and let

$$a(t) = \begin{cases} \frac{1}{C_3(r-r_*)^{\delta_1}} \cdot \frac{1}{C(r-r_*)} \coloneqq \frac{c_0}{(r-r_*)^{1+\delta_1}}, & r \in (r_*, r_{mid}] \\ \frac{1}{C_3} \cdot \frac{1}{C} \coloneqq c_0, & r \in (r_{mid}, r_0] \end{cases}$$

for $0 \leq t \leq r_0 - r_{cut}$ (see Figure 3.10). The above inequality simplifies to

$$\frac{dy}{dt} \geq a(t) \cdot y^2 - e^f(M\epsilon_a) - e^f(C\epsilon).$$

Rearranging ϵ_a, ϵ to be even smaller so that

$$e^f(M\epsilon_a) + e^f(C\epsilon) \leq C_3(M\epsilon_a) + C_3(C\epsilon) \leq \frac{1}{2}a(t)y^2,$$

we have

$$\frac{dy}{dt} \geq \frac{1}{2} \cdot a(t) \cdot y^2.$$

Notice that the condition

$$C_3 M \epsilon_a + C_3 C \epsilon \leq \frac{1}{2} \cdot a(t) \cdot y^2$$

always holds since both $a(t)$ and y are increasing as t increases (since $r_{mid} - r_* < 1$). Thus we have

$$\int_{y_0}^{y} \frac{1}{y^2} dy \geq \int_0^{r_0 - r} \frac{1}{2} \cdot a(t) dt \geq \int_r^{r_{mid}} \frac{c_0}{2(r - r_*)^{1+\delta_1}} dr$$
$$\frac{1}{y_0} - \frac{1}{y} \geq \frac{c_0}{2\delta_1} \cdot \Big(\frac{1}{(r - r_*)^{\delta_1}} - \frac{1}{(r_{mid} - r_*)^{\delta_1}} \Big).$$

The above right hand side serving as a lower bound is a bit awkward since it is negative when $r \in (r_{mid}, r_0]$. One should however focus on $r \in [r_{cut}, r_{mid}]$. Therefore, if the initial data satisfies

$$\frac{1}{y_0} \leq \frac{c_0}{4\delta_1} \cdot \Big(\frac{1}{(r_{cut} - r_*)^{\delta_1}} - \frac{1}{(r_{mid} - r_*)^{\delta_1}} \Big),$$

y must go to infinity at some $r \in [r_{cut}, r_0]$. In other words, our lower bound

$$LB(r_{cut}, parameters)$$

for y_0 to generate a shock can be

$$\frac{1}{LB} = \frac{c_0}{4\delta_1} \cdot \Big(\frac{1}{(r_{cut} - r_*)^{\delta_1}} - \frac{1}{(r_{mid} - r_*)^{\delta_1}} \Big).$$

Note that

$$\lim_{r_{cut} \to r_*} LB = 0.$$

Since $e^f\big|_{r=r_0} = 1$, LB is a lower bound for both $y_0 = (e^f \cdot \partial_s v_1)\big|_{r=r_0}$ and $(\partial_s v_1)\big|_{r=r_0}$.

□

3.6 Initial data

3.6.1 Summary table for initial data

In this section, we use the following tables to summarize all the initial data assumption from previous chapters.

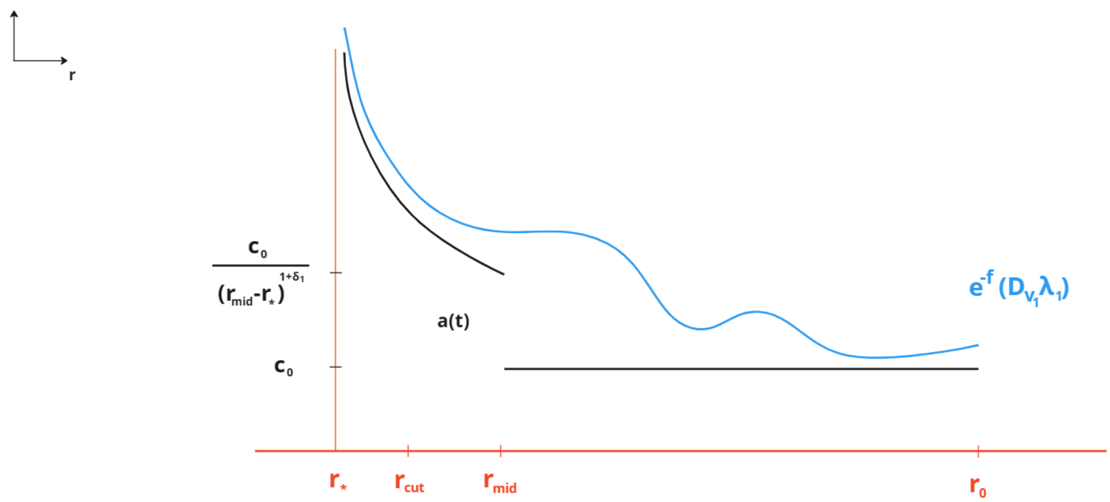

Figure 3.10 Lower bound $a(t)$.

Where	Target	Requirement
Proposition 3.4.1	$E \leq \epsilon$	$E\|_{r_0} \leq \frac{\epsilon}{20e^{(2M_1+2M_2)(r_0-r_*)}} \cdot \min\left\{\frac{1}{1+10M_3}, \frac{1}{1+10M_4}\right\}$
Assumption 1		$\int_{\text{(initial slice)}} \|\partial_s \ln\beta\| ds \leq \frac{\epsilon_a}{200}$
		$0 < \epsilon \leq \epsilon_a < 1$

Where	Main term	Error term
Lemma 3.5.2	$-\frac{1-\gamma}{1+\gamma} \cdot \frac{1}{n} \int_r^{r_0} r(\mathring{\rho}\mathring{\alpha})dr$	$M\epsilon + M\epsilon^2 + 2C$
		$\ln\sqrt{\beta}(P_0) - \ln\sqrt{\beta}(P_1)$
		$\ln(\alpha^{\frac{1}{1+\gamma}})(P_1) - \ln(\alpha^{\frac{1}{1+\gamma}})(P_0)$
	$\frac{1}{1+\gamma}\ln(\mathring{\rho}\mathring{\alpha}) - \frac{1}{1+\gamma}\ln(\mathring{\rho}\mathring{\alpha})(r_0)$	2ϵ
Lemma 3.5.3	$\left(\frac{1+\gamma}{2} - \gamma\right) \cdot \frac{1}{n}\int_r^{r_0} r(\mathring{\rho}\mathring{\alpha})dr$	$M\epsilon + C$

Where	Target	Requirement
Lemma 3.5.9	$\lvert\partial_s v_2\rvert$	$\lvert\partial_s v_2\rvert \leq e^{5M}\lvert(\partial_s v_2)(P_2)\rvert + e^{10M}\cdot M\epsilon + e^{10M}\cdot C\epsilon \leq \frac{1}{2}\epsilon_a$
Section 3.5.3		$C_0 M\epsilon_a + C_0 C\epsilon \leq \frac{c_0}{(r_0-r_*)^{1+\delta}}\cdot(y_0)^2$

Note that the smallness of $E(r_0)$, the total variation of v_1, v_2, will automatically imply the smallness of $\|(\partial_s \ln\alpha(r_0))\|_{L^1}$ by the θ in the constraint equation for $(\partial_s \ln\alpha)$ (refer to the **reduced Einstein field equations**) and the fact that the width $W(r_0) = z_R - z_L$ of the initial perturbation is finite. Therefore, as long as the $E(r_0)$ is small enough, the term $|\ln(\alpha^{\frac{1}{1+\gamma}})(P_1) - \ln(\alpha^{\frac{1}{1+\gamma}})(P_0)|$ will be small.

Therefore, in the later section, we try to pose the following conditions on the initial slice for $s \in [z_L, z_R]$:

- $v_1(r_0)$, $v_2(r_0)$ are perturbed but have a small total variation (small $E(r_0)$).
- $\partial_s v_1(r_0, s)$ has a large positive value for *some* $s \in (z_L, z_M)$.
- $\partial_s v_2(r_0, s)$ is small for *all* $s \in [z_L, z_M]$.
- $(\partial_s \alpha(r_0))$ is perturbed based on the constraint equation. In other words, $\|(\partial_s \ln\alpha(r_0))\|_{L^1}$ will be small since $E(r_0)$ is small.
- We do not perturb β for simplicity. In other words, $\beta(r_0, s) = \mathring{\beta}(r_0, s)$ for all $s \in [z_L, z_M]$.

3.6.2 Construction of initial data

In this section, we try to construct a sequence of initial data that satisfies the above conditions and works for shock formation. Firstly, we rewrite the constraint equation for $(\partial_s \ln\alpha)$ in terms of Riemann invariants v_1, v_2. The original constraint is

$$(\partial_s \ln\alpha) = -2(1+\gamma)\cdot\frac{r}{n}(\rho\alpha)\cdot\frac{\sqrt{\beta}}{\sqrt{\alpha}}\cdot\theta\sqrt{1+\theta^2}.$$

Based on the definition of v_1, v_2, we have that

$$\begin{aligned}\theta &= \frac{1}{2}\Big(e^{\sqrt{\gamma}(v_1+v_2)} - e^{-\sqrt{\gamma}(v_1+v_2)}\Big)\\ \sqrt{1+\theta^2} &= \frac{1}{2}\Big(e^{\sqrt{\gamma}(v_1+v_2)} + e^{-\sqrt{\gamma}(v_1+v_2)}\Big)\\ (\rho\alpha) &= e^{(1+\gamma)(v_1-v_2)}.\end{aligned}$$

Therefore, after replacement, we have

$$\begin{aligned}&(\partial_s \ln \alpha) =\\ &-2(1+\gamma)\cdot\frac{r}{n}\cdot e^{(1+\gamma)(v_1-v_2)}\cdot\frac{\sqrt{\beta}}{\sqrt{\alpha}}\cdot\frac{1}{2}\Big(e^{\sqrt{\gamma}(v_1+v_2)} - e^{-\sqrt{\gamma}(v_1+v_2)}\Big)\cdot\frac{1}{2}\Big(e^{\sqrt{\gamma}(v_1+v_2)} + e^{-\sqrt{\gamma}(v_1+v_2)}\Big),\end{aligned}$$

or equivalently,

$$\begin{aligned}&\big(\partial_s \sqrt{\alpha}\big) =\\ &-(1+\gamma)\cdot\frac{r}{n}\cdot e^{(1+\gamma)(v_1-v_2)}\cdot\sqrt{\beta}\cdot\frac{1}{4}\Big(e^{\sqrt{\gamma}(v_1+v_2)} - e^{-\sqrt{\gamma}(v_1+v_2)}\Big)\cdot\Big(e^{\sqrt{\gamma}(v_1+v_2)} + e^{-\sqrt{\gamma}(v_1+v_2)}\Big).\end{aligned}$$

This means that, if we impose the restriction of the perturbation of initial data that

- $v_1(r_0) + v_2(r_0)$ is an odd function
- $v_1(r_0) - v_2(r_0)$ is an even function
- β is an even function

with respect to s, we will get the right hand side of the above equation become an odd function and therefore it satisfies that

$$\int_{z_L}^{z_R} (\partial_s \sqrt{\alpha}) ds = 0,$$

which means we are allowed to solve for α along the initial slice.

In conclusion, our steps for choosing the initial data are as follows.

- Fix $r_{cut} \in (r_*, r_0)$.

- Fix $y_0 \geq LB$ that depends on r_{cut}.

- Choose sufficiently small ϵ, ϵ_a that may depend on r_{cut} and y_0 so that they satisfy all the requirements in the above tables. Then derive ϵ_0, $\epsilon_{a,0}$.

- Construct the initial data for v_1, v_2, β accordingly. For simplicity, we let $\beta(r_0) = \mathring{\beta}(r_0)$.

- Construct the initial data for α based on the above constraint equation.

As an example about how to construct the initial data for v_1, v_2 verifying the above restriction, we let $z_L = -1$, $z_M = 0$, $z_R = 1$, take a compactly supported smooth function $P : [-1, 0] \to \mathbb{R}$ of s (where P for Profile) so that

$$
\begin{aligned}
&\frac{d}{ds}P\Big|_{s=-\frac{1}{2}} = LB \\
&\left\|\frac{d}{ds}P\right\|_{L^\infty} \leq LB, \quad \int_{z_L}^{z_R}\left|\frac{d}{ds}P\right|ds \leq \epsilon_0, \quad \|P\|_{L^\infty} \leq \epsilon_0, \\
&\left|\left\{\frac{d}{ds}P \neq 0\right\}\right| \leq \frac{\epsilon_0}{LB}
\end{aligned}
$$

and construct functions $\chi^{(odd)}, \chi^{(even)}$ of s so that

$$
\chi^{(odd)}(s) = P(s) \quad \forall s \in [-1, 0], \quad \chi^{(odd)}(-s) = -\chi^{(odd)}(s) \quad \forall s \in [-1, 1]
$$

$$
\chi^{(even)}(s) = P(s) \quad \forall s \in [-1, 0], \quad \chi^{(even)}(-s) = \chi^{(even)}(s) \quad \forall s \in [-1, 1].
$$

The construction implies that, on the initial slice, $\chi^{(odd)}$ and $\chi^{(even)}$ have a large **positive** derivative at $s = -\frac{1}{2}$ while maintaining small total variations throughout the slice (see Figure 3.11). Next, let

$$
\begin{aligned}
v_1 &= \mathring{v}_1 + \frac{1}{2}\Big(\chi^{(odd)} + \chi^{(even)}\Big) \\
v_2 &= \mathring{v}_2 + \frac{1}{2}\Big(\chi^{(odd)} - \chi^{(even)}\Big) \\
\beta &= \mathring{\beta}.
\end{aligned}
$$

Notice that the definition of v_1 and v_2 implies that $\partial_s v_1\big|_{(r,s)=(r_0,-\frac{1}{2})} = \frac{d}{ds}P\big(-\frac{1}{2}\big) \geq LB$, which implies that

$$
y_0 = e^f \cdot \partial_s v_1\Big|_{(r,s)=(r_0,-\frac{1}{2})} \geq LB.
$$

In addition, for $\partial_s v_2$, we have

$$\partial_s v_2(r_0, s) = 0 \quad \forall s \in [-1, 0],$$

which verifies the requirement for boostrap to guarantee the smallness of $|\partial_s v_2|$ on the **left** of the initial slice (refer to Lemma 3.5.9). By finite speed of propagation property, we know that $|\partial_s v_2|$ remains pointwise small on the left of the interesting region, but we do not know what happens to $|\partial_s v_2|$ on the right part (see Figure 3.12).

Finally, observing that on the initial slice $v_1 + v_2 = \chi^{(odd)}$ is an odd function compactly supported in $s \in [-1, 1]$ and $v_1 - v_2 = (\mathring{v}_1 - \mathring{v}_2) + \chi^{(even)}$ is an even function which is equal to $(\mathring{v}_1 - \mathring{v}_2)$ at $s = -1, 1$, we see that it verifies the restriction of the initial perturbation stated above.

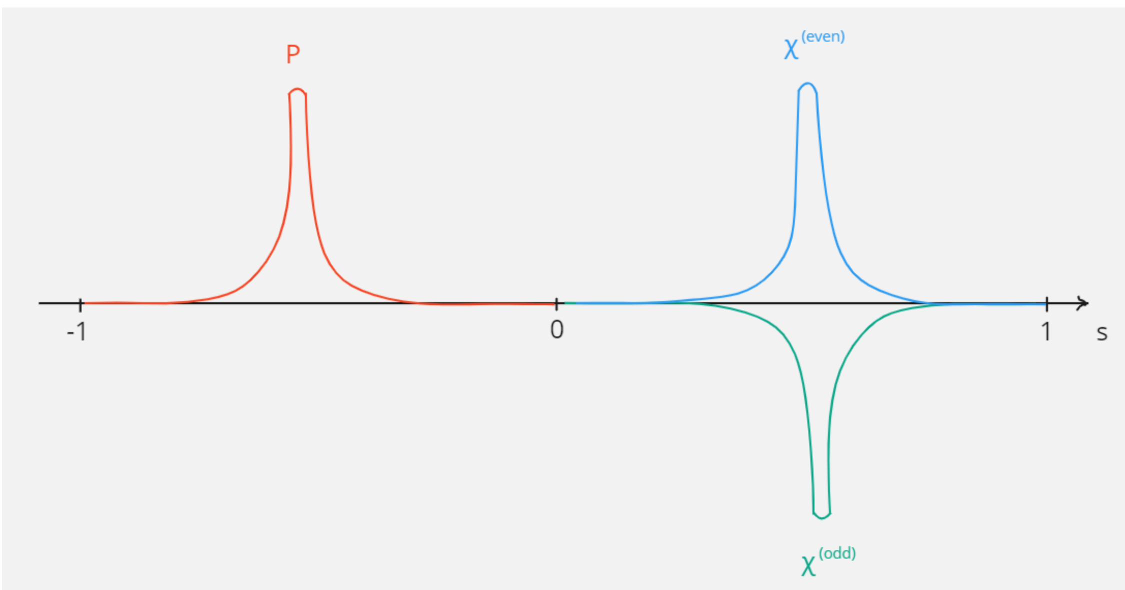

Figure 3.11 Construction of $\chi^{(odd)}$ and $\chi^{(even)}$.

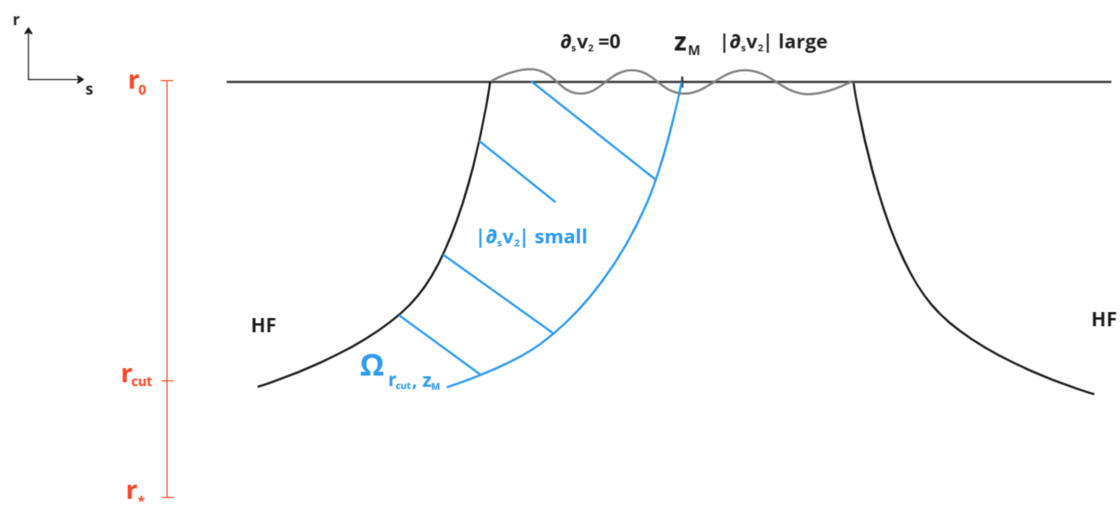

Figure 3.12 We apply Lemma 3.5.9 on the blue region Ω_{r_{cut},z_M}.

CHAPTER 4

STABILITY OF RELATIVISTIC FLUIDS ON FIXED BIG BANG SPACETIMES

This chapter aims to prove the stability of specially relativistic fluids on a fixed Big Bang spacetime, the metric of which derived from Section 1.2. In other words, we consider only Euler equations instead of Einstein-Euler equations.

4.1 The model problem and its homogeneous solutions

Our model problem is the relativistic Euler equations on a fixed warped product manifold $B\times_r F$, where B is a 1+1 Lorentzian manifold endowed with the metric

$$g = -\alpha dr^2 + \beta ds^2$$

and F is an n-dimensional Riemannian manifold representing the symmetry and regarded as a fiber. The fluid variables (θ, ρ) describing the underlying fluid are the primary variables of interest. In this paper, we analyze the following main equations

$$\partial_r\left(\sqrt{\beta}\cdot r^n \rho^{\frac{1}{1+\gamma}}\sqrt{1+\theta^2}\right) + \partial_s\left(\sqrt{\alpha}\cdot r^n \rho^{\frac{1}{1+\gamma}}\theta\right) = 0 \tag{4.1}$$

$$\partial_r\left(\sqrt{\beta}\cdot \rho^{\frac{\gamma}{1+\gamma}}\theta\right) + \partial_s\left(\sqrt{\alpha}\cdot \rho^{\frac{\gamma}{1+\gamma}}\sqrt{1+\theta^2}\right) = 0. \tag{4.2}$$

We proceed to establish the equivalence of Euler equations and our main equations. In the work by Wong and An [2], they computed the curvature of the warped product spacetime $B \times_r F$ and derived the Euler equations

$$div_g\left(r^n \rho^{\frac{1}{1+\gamma}}\xi\right) = 0$$

$$d\left(\rho^{\frac{\gamma}{1+\gamma}}\xi^\flat\right) = 0$$

in the fluid context, with the ultra-relativistic assumption $p = \gamma\rho$ as the equation of state. Here $0 < \sqrt{\gamma} < 1$ is a parameter representing sound speed, p is the fluid pressure, $\rho > 0$ is the fluid density, and ξ is a unit timelike vector field defined on B representing the fluid velocity and therefore satisfies the normalization condition $-\alpha(\xi^r)^2 + \beta(\xi^s)^2 = -1$. Since the metric components α, β are fixed, we parametrize ξ by the scalar unknown $\theta \in \mathbb{R}$ satisfying

$$\sqrt{\alpha}\xi^r = \sqrt{1+\theta^2}, \quad \sqrt{\beta}\xi^s = \theta.$$

We regard (θ, ρ) as fluid variables and they are the main unknown functions of (r, s) in this paper. Expanding the definition of divergence div_g and exterior derivative d, we have an equivalent system as our main equations.

Remark. *It is interesting noting that, the fixed fiber F does not play any role in our main equations. Its scalar curvature $S^{[h]}$ however is involved in Einstein-Euler equations as considered in Chapter 3.*

In a future work, we are able to classify all physical, spatially *homogeneous* solutions with a big bang singularity that solve the above Euler equations. It turns out in the fluid $(0 < \gamma < 1)$ and positive blowup time $(r_* > 0)$ case, we have the following asymptotic behavior for the metric components

Assumption 4. *We assume that the metric components α, β satisfy*

$$\alpha(r), \beta(r) > 0 \quad \forall r \in (r_*, r_0]$$

$$\lim_{r \to r_*^+} \alpha(r) = 0, \quad \lim_{r \to r_*^+} \beta(r) = 0$$

$$\lim_{r \to r_*^+} \partial_r \ln \beta(r) = \infty,$$

where $r_* > 0$ is the big bang blowup time. Geometrically, we are assuming ∂_s is a Killing vector field and the geometry has a certain asymptotic behavior when approcahing the big bang.

In order to better understand the Euler equations, we impose the spatial *homogenity* on the unknowns (θ, ρ) and analyze the equations. Assuming both θ and ρ are independent of s, the equations reduce to a coupled system of ordinary differential equations and read, after integraing with respect to r,

$$\sqrt{\beta} \cdot r^n \rho^{\frac{1}{1+\gamma}} \sqrt{1+\theta^2} = C_1$$

$$\sqrt{\beta} \cdot \rho^{\frac{\gamma}{1+\gamma}} \theta = C_2.$$

If θ is *nonzero* initially ($\theta(r_0) \neq 0$), we have $C_2 \neq 0$ and thus

$$\sqrt{1+\theta^2} = \frac{C_1}{\sqrt{\beta}} \cdot \frac{1}{r^n} \cdot \rho^{-\frac{1}{1+\gamma}}$$

$$\theta = \frac{C_2}{\sqrt{\beta}} \cdot \rho^{-\frac{\gamma}{1+\gamma}}.$$

Solving the inequality $\sqrt{1+\theta^2} \geq \theta$, we have

$$\frac{C_1}{C_2} \cdot \frac{1}{r^n} \geq \rho^{\frac{1-\gamma}{1+\gamma}}$$

and thus conclude that ρ remains bounded above for $r \in (r_*, r_0]$ since $r_* > 0$ and $0 < \gamma < 1$. This further implies that θ *blows up at least in a rate* $\frac{1}{\sqrt{\beta}}$. This rate plays an essential role in the remaining paper.

Remark. *These homogeneous solutions with* $\theta \neq 0$ *have no general relativisitc counterpart. In [28], it was observed that the spatially homogeneous Einstein-Euler system forces* ξ *to be parallel to* ∂_r *(or* $\theta = 0$*). This extra rigidity is a feature of Einstein's equations when large number of symmetries are present, and is what enables the full classification performed in a future paper.*

4.2 Dynamical Stability of homogeneous $\theta \neq 0$ solutions

In this section, we will establish the stability of homogeneous $\theta \neq 0$ solutions with *large enough* $\theta(r_0)$. Specifically, given any metric components α, β satisfying the Assumption 4 and given any $r_0 > r_*$, there exists a lower bound (depending on α, β, r_0) for $\theta(r_0)$ so that the homogeneous solutions with $\theta(r_0)$ greater than the lower bound are stable. In particular, shocks do not form before the big bang.

4.2.1 Evolution equations

We begin by transforming our main equations (4.1) and (4.2) to evolution equations. Our equations form a hyperbolic system on the $(1+1)$-dimensional Lorentzian manifold B; therefore, the method of characteristics is applicable. As indicated in [15], one derives the following equations after performing a standard diagonalization process

$$
\begin{aligned}
&(\partial_r + \lambda_1\partial_s)\underbrace{\Big(\frac{1}{2}\ln\big(\sqrt{1+\theta^2}+\theta\big) + \frac{1}{2\sqrt{\gamma}}\ln f\Big)}_{v_1} \\
&\qquad = \underbrace{\frac{\sqrt{\alpha}}{1+\frac{\sqrt{\gamma}\theta}{\sqrt{1+\theta^2}}}\Big\{\Big(-\frac{1}{4}\cdot\frac{\theta}{\sqrt{1+\theta^2}} - \frac{\sqrt{\gamma}}{4}\Big)(e_0\ln\beta) - \frac{n\sqrt{\gamma}}{2}(e_0\ln r)\Big\}}_{F_1} \qquad (4.3)
\end{aligned}
$$

$$
\begin{aligned}
&(\partial_r + \lambda_2\partial_s)\underbrace{\Big(\frac{1}{2}\ln\big(\sqrt{1+\theta^2}+\theta\big) - \frac{1}{2\sqrt{\gamma}}\ln f\Big)}_{v_2} \\
&\qquad = \underbrace{\frac{\sqrt{\alpha}}{1-\frac{\sqrt{\gamma}\theta}{\sqrt{1+\theta^2}}}\Big\{\Big(-\frac{1}{4}\cdot\frac{\theta}{\sqrt{1+\theta^2}} + \frac{\sqrt{\gamma}}{4}\Big)(e_0\ln\beta) + \frac{n\sqrt{\gamma}}{2}(e_0\ln r)\Big\}}_{F_2}. \qquad (4.4)
\end{aligned}
$$

Here $f = \rho^{\frac{\gamma}{1+\gamma}}$, $\lambda_1 = \frac{\sqrt{\alpha}}{\sqrt{\beta}} \cdot \frac{\sqrt{\gamma}\sqrt{1+\theta^2}+\theta}{\sqrt{1+\theta^2}+\sqrt{\gamma}\theta}$, $\lambda_2 = \frac{\sqrt{\alpha}}{\sqrt{\beta}} \cdot \frac{-\sqrt{\gamma}\sqrt{1+\theta^2}+\theta}{\sqrt{1+\theta^2}-\sqrt{\gamma}\theta}$. We use $e_0 = \frac{1}{\sqrt{\alpha}}\partial_r$ and $e_1 = \frac{1}{\sqrt{\beta}}\partial_s$ to denote an orthonormal frame on B. This system of equations describe the evolution of the two Riemann invariants v_1, v_2 in two different directions $(\partial_r + \lambda_1\partial_s)$, $(\partial_r + \lambda_2\partial_s)$. Since the coefficients in L_1, L_2 involve the unknown θ, one may ask whether this system satisfies the analogous genuinely nonlinear condition defined in [15]. The answer is *no*. Indeed, as $\theta \to \infty$, the derivative of the eigenvalues vanishes hence breaking the genuinely nonlinear requirement.

4.2.2 Strategy

Our strategy divides the time interval $(r_*, r_0]$ into two subintervals $(r_*, r_{mid}]$, $[r_{mid}, r_0]$ and performs two different tasks in each interval. r_{mid} is a parameter depending on the metric components α, β that will be chosen later but is expected to be close to r_*. In the first subinterval $(r_*, r_{mid}]$, we argue that the largeness of $\theta(r_{mid})$ enables a bootstrap argument and thus stabilizes the equation and prevents shock formation. In the second subinterval $[r_{mid}, r_0]$, we apply the standard Cauchy stability to find the appropriate lower bound for $\theta(r_0)$ to ensure the largeness of $\theta(r_{mid})$.

In order to execute our strategy, we have to estimate two quantities: the spatial derivative of Riemann invariants (as indicated in [15]), and θ (related to the heuristic above about how to break the genuinely nonlinear condition). For convenience, we denote the spatial derivative of Riemann invariants by $a = e_1\Big(\frac{1}{2}\ln\big(\sqrt{1+\theta^2}+\theta\big)+\frac{1}{2\sqrt{\gamma}}\ln f\Big)$ and $b = e_1\Big(\frac{1}{2}\ln\big(\sqrt{1+\theta^2}+\theta\big)-\frac{1}{2\sqrt{\gamma}}\ln f\Big)$.

4.2.3 Evolution equation for spatial derivatives a and b

We start by taking a spatial derivative e_1 on both Riemann invariant equations and get the evolution equations for the spatial derivatives a and b

$$\begin{aligned}-\partial_r\Big|_{X_1} a = &\frac{1-\gamma}{(1+\frac{\sqrt{\gamma}\theta}{\sqrt{1+\theta^2}})^2}\cdot\sqrt{\alpha}\cdot\frac{1}{1+\theta^2}(a+b)a+\frac{1}{4}\cdot\frac{1-\gamma}{(1+\frac{\sqrt{\gamma}\theta}{\sqrt{1+\theta^2}})^2}\cdot\frac{1}{1+\theta^2}(a+b)(\partial_r\ln\beta)\\ &-\frac{n\gamma}{2}\cdot\frac{1}{(1+\frac{\sqrt{\gamma}\theta}{\sqrt{1+\theta^2}})^2}\cdot\frac{1}{1+\theta^2}(a+b)(\partial_r\ln r)+\frac{1}{2}\cdot a(\partial_r\ln\beta)\end{aligned}$$

$$\begin{aligned}-\partial_r\Big|_{X_2} b = &\frac{1-\gamma}{(1-\frac{\sqrt{\gamma}\theta}{\sqrt{1+\theta^2}})^2}\cdot\sqrt{\alpha}\cdot\frac{1}{1+\theta^2}(a+b)b+\frac{1}{4}\cdot\frac{1-\gamma}{(1-\frac{\sqrt{\gamma}\theta}{\sqrt{1+\theta^2}})^2}\cdot\frac{1}{1+\theta^2}(a+b)(\partial_r\ln\beta)\\ &-\frac{n\gamma}{2}\cdot\frac{1}{(1-\frac{\sqrt{\gamma}\theta}{\sqrt{1+\theta^2}})^2}\cdot\frac{1}{1+\theta^2}(a+b)(\partial_r\ln r)+\frac{1}{2}\cdot b(\partial_r\ln\beta)\end{aligned}$$

where $-\partial_r\Big|_{X_1} = -(\partial_r+\lambda_1\partial_s)$ is the first characteristic direction and $-\partial_r\Big|_{X_2} = -(\partial_r+\lambda_2\partial_s)$ is the second characteristic direction. Observe that these two equations are coupled mainly through the term $\frac{1}{1+\theta^2}(a+b)$. If we assume the uniform bound for $\frac{1}{1+\theta^2}(|a|+|b|)$ and use the fact that the coefficients $\frac{1}{\left(1+\frac{\sqrt{\gamma}\theta}{\sqrt{1+\theta^2}}\right)^2}$ and $\frac{1}{\left(1-\frac{\sqrt{\gamma}\theta}{\sqrt{1+\theta^2}}\right)^2}$ are both in the range from $\frac{1}{\left(1+\sqrt{\gamma}\right)^2}$ to $\frac{1}{\left(1-\sqrt{\gamma}\right)^2}$, we can regard these two evolutions as linear ordinary differential equations along two characteristics and derive an upper bound for them *separately*. This is the idea we will apply in the first subinterval $(r_*, r_{mid}]$. A crucial ingredient of this idea is that we have to ensure the largeness of θ to make $\frac{1}{1+\theta^2}$ decay fast enough and close the bootstrap argument.

4.2.4 Evolution equation for θ

Based on the previous paragraph, we have to investigate how θ evolves in order to close the bootstrap argument. After adding equation (3), (4), (5), we get

$$-\partial_r\Big|_{X_1}\ln\left(\frac{\theta}{\sqrt{1+\theta^2}^{\gamma}}\right)=\frac{1-\gamma}{2}\cdot(\partial_r\ln\beta)-n\gamma\cdot(\partial_r\ln r)-2\sqrt{\gamma}\cdot\frac{\sqrt{1+\theta^2}}{\theta}\cdot\frac{1}{1+\theta^2}\cdot b\sqrt{\alpha}.$$

Notice that here we recover the homogeneous non-physical solution asymptote if $b = 0$:

$$\theta \approx \frac{1}{\sqrt{\beta}} \quad \text{as } r \to r_*^+$$

since r is assumed to be bounded and away from 0 (because $r_* > 0$). Therefore, if we can argue that the last term on the right hand side (the term involving $\frac{\sqrt{1+\theta^2}}{\theta} \cdot \sqrt{\alpha}$) can be regarded as an error term, we can ensure that the behavior of θ is similar to $\frac{1}{\sqrt{\beta}}$ at least when being close to the blowup time.

4.2.5 Bootstrap argument for $(r_*, r_{mid}]$

In this section, we present our bootstrap assumption and try to incorporate this assumption into the evolution equations we previously have derived. Our bootstrap assumption is

$$\frac{1}{1+\theta^2}\Big(|a| + |b|\Big) \leq M. \tag{4.5}$$

We begin with deriving a uniform bound for the spatial derivatives of Riemann invariants $|a|$ and $|b|$.

Lemma 4.2.1. *(Estimate for $|a|$ and $|b|$) Assume the bootstrap assumption 4.5 holds with the constant M. Then we have*

$$|a|+|b| \leq e^{\frac{1-\gamma}{(1-\sqrt{\gamma})^2}\cdot M \int_{r_*}^{r_{mid}} \sqrt{\alpha} dr} \cdot \frac{\sqrt{\beta(r_{mid})}}{\sqrt{\beta}}$$

$$\cdot \left(\sup_{r=r_{mid}} |a| + \sup_{r=r_{mid}} |b| + \frac{1}{2} \cdot \frac{1-\gamma}{\left(1-\sqrt{\gamma}\right)^2} \cdot M \ln\left(\frac{\beta(r_{mid})}{\beta}\right) + n\gamma \cdot \frac{1}{\left(1-\sqrt{\gamma}\right)^2} \cdot M \ln\left(\frac{r_{mid}}{r}\right)\right)$$

for any $r \in (r_*, r_{mid}]$, where r_{mid} is a parameter that will be chosen in the next Lemma.

Proof. As mentioned in Section 3.3, we try to incorporate the bootstrap assumption 4.5 to the evolution equations for a and b and try to integrate the equations along the characteristics. Since the computation for a and b are mostly the same, we only perform the calculation for a here. From the evolution equation of a, we have

$$\Big| -\partial_r\big|_{X_1} a\Big| \leq \frac{1-\gamma}{\left(1-\sqrt{\gamma}\right)^2} \cdot \sqrt{\alpha} \cdot M|a| + \frac{1}{4} \cdot \frac{1-\gamma}{\left(1-\sqrt{\gamma}\right)^2} \cdot M(\partial_r \ln \beta)$$

$$+\frac{n\gamma}{2}\cdot\frac{1}{\left(1-\sqrt{\gamma}\right)^2}\cdot M(\partial_r \ln r)+\frac{1}{2}\cdot|a|(\partial_r \ln\beta).$$

We integrate the inequality along X_1 direction, where X_1 is the characteristic generated by the vector field $\partial_r+\lambda_1\partial_s$

$$|a|\leq\sup_{r=r_{mid}}|a|+\frac{1}{4}\cdot\frac{1-\gamma}{\left(1-\sqrt{\gamma}\right)^2}\cdot M\ln\left(\frac{\beta_0}{\beta}\right)+\frac{n\gamma}{2}\cdot\frac{1}{\left(1-\sqrt{\gamma}\right)^2}\cdot M\ln\left(\frac{r_{mid}}{r}\right)$$
$$+\int_0^t\left(\frac{1-\gamma}{\left(1-\sqrt{\gamma}\right)^2}\cdot\sqrt{\alpha}\cdot M+\frac{1}{2}(\partial_r\ln\beta)\right)|a(r_{mid}-t,X_1(t))|dt.$$

where $t=r_{mid}-r$. By Gronwall's inequality, we have the following estimate for $|a|$

$$|a|\leq e^{\frac{1-\gamma}{(1-\sqrt{\gamma})^2}\cdot M\int_{r_*}^{r_{mid}}\sqrt{\alpha}dr}\cdot\frac{\sqrt{\beta(r_{mid})}}{\sqrt{\beta}}\cdot$$
$$\left(\sup_{r=r_{mid}}|a|+\frac{1}{4}\cdot\frac{1-\gamma}{\left(1-\sqrt{\gamma}\right)^2}\cdot M\ln\left(\frac{\beta(r_{mid})}{\beta}\right)+\frac{n\gamma}{2}\cdot\frac{1}{\left(1-\sqrt{\gamma}\right)^2}\cdot M\ln\left(\frac{r_{mid}}{r}\right)\right).$$

A similar process along the other characteristic X_2 generated by the vector field $\partial_r+\lambda_2\partial_s$ gives the analogous estimate for b and therefore the result.

□

Lemma 4.2.2. *(Estimate for θ.) Assume the bootstrap assumption 4.5 holds with M. There exists $r_{mid}>r_*$ so that θ is increasing over $(r_*,r_{mid}]$ (meaning $-\partial_r\big|_{X_1}\theta>0$) and satisfies*

$$\theta^{1-\gamma}\geq\sqrt{2}^{\gamma}\cdot\frac{\theta(r_{mid})}{\sqrt{1+\theta(r_{mid})^2}^{\gamma}}\cdot\left(\frac{\beta(r_{mid})}{\beta}\right)^{\frac{1-\gamma}{2}}\cdot\left(\frac{r_*}{r_{mid}}\right)^{n\gamma}\cdot e^{-2\sqrt{\gamma}RM\int_{r_*}^{r_{mid}}\sqrt{\alpha}dr}.$$

as long as $\theta(r_{mid})\geq 1$. Here $\frac{\beta(r_{mid})}{\beta}$ ranges in $[1,\infty)$ and $R=\left(\sup_{\theta\in[1,\infty)}\frac{\sqrt{1+\theta^2}}{\theta}\right)$.

Proof. Similarly to the previous lemma, we replace those terms in Section 3.4 involving $\frac{1}{1+\theta^2}\cdot b$ by M based on the bootstrap assumption 4.5

$$-\partial_r\Big|_{X_1}\ln\left(\frac{\theta}{\sqrt{1+\theta^2}^{\gamma}}\right)\geq\frac{1-\gamma}{2}\cdot(\partial_r\ln\beta)-n\gamma\cdot(\partial_r\ln r)-2\sqrt{\gamma}\cdot\left(\sup_{\theta\in[1,\infty)}\frac{\sqrt{1+\theta^2}}{\theta}\right)\cdot M\sqrt{\alpha}$$

provided that θ is always greater than 1. In order to preserve this $\theta\geq 1$ condition, *we are restricted to a region that is close to r_*.* Notice that since α,β satisfy the Assumption 4 (meaning

$\lim_{r\to r_*^+}(\partial_r \ln\beta) = \infty$, $\sqrt{\alpha}$ is bounded) and $r_* > 0$ (meaning $(\partial_r \ln r)$ is bounded), there exists an r_{mid} so that restricted to the time interval $(r_*, r_{mid}]$, θ *keeps increasing*, assuming θ begins with a value greater than 1. In addition, we will make sure that r_{mid} is close enough to r_* so that $\frac{\beta(r_{mid})}{\beta}$ *ranges in* $[1,\infty)$. This range will be used in the subsequent proposition. Returning to the inequality, we have

$$-\partial_r\Big|_{X_1} \ln\left(\frac{\theta}{\sqrt{1+\theta^2}^{\gamma}}\right) \geq \frac{1-\gamma}{2}\cdot(\partial_r \ln\beta) - n\gamma\cdot(\partial_r \ln r) - 2\sqrt{\gamma}\cdot R\cdot M\sqrt{\alpha}.$$

Integrating this inequality along X_1, we arrive at

$$\frac{\theta}{\sqrt{1+\theta^2}^{\gamma}} \geq \frac{\theta(r_{mid})}{\sqrt{1+\theta(r_{mid})^2}^{\gamma}}\cdot\left(\frac{\beta(r_{mid})}{\beta}\right)^{\frac{1-\gamma}{2}}\cdot\left(\frac{r_*}{r_{mid}}\right)^{n\gamma}\cdot e^{-2\sqrt{\gamma}RM\int_{r_*}^{r_{mid}}\sqrt{\alpha}dr}.$$

In order to isolate the desired quantity θ, we make use of the fact that $\theta \geq 1$ and get a lower bound for θ:

$$\theta^{1-\gamma} \geq \sqrt{2}^{\gamma}\cdot\frac{\theta(r_{mid})}{\sqrt{1+\theta(r_{mid})^2}^{\gamma}}\cdot\left(\frac{\beta(r_{mid})}{\beta}\right)^{\frac{1-\gamma}{2}}\cdot\left(\frac{r_*}{r_{mid}}\right)^{n\gamma}\cdot e^{-2\sqrt{\gamma}RM\int_{r_*}^{r_{mid}}\sqrt{\alpha}dr}.$$

□

With all the ingredients in this section, we now present the crucial argument in this paper.

Proposition 4.2.1. *Fix a pair of metric components (α,β) satisfying Assumption 4 with positive blowup time $r_* > 0$, and fix any bootstrap constant $M > 0$. Then there exist*

$$r_{mid} = r_{mid}(M,\gamma) > r_*, \quad \theta_{mid}^{LB} = \theta_{mid}^{LB}(M,\gamma,r_*,r_{mid}) \geq 1$$

(where LB is for Lower Bound) so that as long as the initial data of the solution to the main equations (4.1), (4.2) satisfy

- $\inf_{r=r_{mid}}\theta \geq \theta_{mid}^{LB}$
- $\sup_{r=r_{mid}}|a| + \sup_{r=r_{mid}}|b| \leq M$

we have that

$$\frac{1}{1+\theta^2}\Big(|a|+|b|\Big) \leq M$$

always holds. In particular, when $r \in (r_*, r_{mid}]$, since θ remains finite, shock will not form.

Proof. Assume the bootstrap condition (4.5) holds with constant M. By Lemma 4.2.1 and Lemma 4.2.2, we have

$$\frac{1}{1+\theta^2}\Big(|a|+|b|\Big) \leq e^{\frac{1-\gamma}{(1-\sqrt{\gamma})^2}\cdot M\int_{r_*}^{r_{mid}}\sqrt{\alpha}dt}\cdot\sqrt{x}\cdot$$

$$\frac{\Big(\sup_{r=r_{mid}}(|a|+|b|)+\frac{1}{2}\cdot\frac{1-\gamma}{(1-\sqrt{\gamma})^2}\cdot M\ln(x)+n\gamma\cdot\frac{1}{(1-\sqrt{\gamma})^2}\cdot M\ln\left(\frac{r_{mid}}{r}\right)\Big)}{1+2^{\frac{\gamma}{1-\gamma}}\cdot\left(\frac{\theta_{mid}^{LB}}{\sqrt{1+(\theta_{mid}^{LB})^2}^{\gamma}}\right)^{\frac{2}{1-\gamma}}\cdot x\cdot\left(\frac{r_*}{r_{mid}}\right)^{\frac{2n\gamma}{1-\gamma}}\cdot e^{-4\sqrt{\gamma}RM\int_{r_*}^{r_{mid}}\sqrt{\alpha}dr}}.$$

where $x=\frac{\beta(r_{mid})}{\beta}$ ranges in $[1,\infty)$ from Lemma 4.2.2. Since

$$\frac{\sqrt{x}+\sqrt{x}\ln(x)}{x}$$

is a bounded function for $x\in[1,\infty)$, we know that when θ_{mid}^{LB} is large enough, the term $\frac{\theta_{mid}^{LB}}{\sqrt{1+(\theta_{mid}^{LB})^2}^{\gamma}}$ in the denominator is large, so the right hand side of the above inequality will be *strictly less* than M, which closes the bootstrap argument. The only remaining unproven thing is that $\theta<\infty$ for $r\in(r_*,r_{mid}]$. This can be done using the evolution equation for θ from Section 3.4

$$-\partial_r\ln\left(\frac{\theta}{\sqrt{1+\theta^2}^{\gamma}}\right)=\frac{1-\gamma}{2}\cdot(\partial_r\ln\beta)-n\gamma\cdot(\partial_r\ln r)-2\sqrt{\gamma}\cdot\frac{\sqrt{1+\theta^2}}{\theta}\cdot\frac{1}{1+\theta^2}\cdot b\sqrt{\alpha}.$$

Since the bootstrap assumption 4.5 holds, we have an upper bound

$$-\partial_r\ln\left(\frac{\theta}{\sqrt{1+\theta^2}^{\gamma}}\right)\leq\frac{1-\gamma}{2}\cdot(\partial_r\ln\beta)-n\gamma\cdot(\partial_r\ln r)+2\sqrt{\gamma}RM\sqrt{\alpha}$$

with R defined in Lemma 4.2.2. Since the right hand side is finite (but not bounded) for $r\in(r_*,r_{mid}]$, θ remains finite, and therefore

$$|a|+|b|\leq M(1+\theta^2)$$

implies the boundedness of the spatial derivatives. □

4.2.6 Bootstrap argument for $[r_{mid},r_0]$

In this section, we apply the standard Cauchy stability for hyperbolic system in the time interval $[r_{mid},r_0]$. Specifically, we have

Proposition 4.2.2. *Given any $r_0 > r_{mid} > r_*$, any $\theta_{mid}^{LB} \geq 1$, any homogeneous background solution $(\theta_{homo}, \rho_{homo})$ satisfying the Euler equations (4.1), (4.2), with $\theta_{homo}(r_{mid}) > \theta_{mid}^{LB}$, and any width $W_0 > 0$ of initial perturbation defined by*

$$W_0 = |\{\theta(r_0) \neq \theta_{homo}(r_0)\}|,$$

there exists $\epsilon_0 > 0$ so that as long as

$$\|a(r_0)\|_{L^\infty} + \|b(r_0)\|_{L^\infty} < \epsilon_0,$$

we have $\|a(r_{mid})\|_{L^\infty}$, $\|b(r_{mid})\|_{L^\infty}$ are finite and

$$\theta(r_{mid}) > \theta_{mid}^{LB}.$$

Proof. We will use a bootstrap argument here. Our bootstrap assumption is

$$\|a(r)\|_{L^\infty} + \|b(r)\|_{L^\infty} < \epsilon$$

$$\|\theta(r)\|_{L^\infty} \leq \|\theta_{homo}(r)\|_{L^\infty} + 1$$

for $r \in [r_{mid}, r_0]$ and for some $0 < \epsilon < 1$ that will be chosen later. Using the evolution equations for a, b derived in Section 3.3, we have

$$\Big| -\partial_r\big|_{X_1} a\Big| \leq \frac{1-\gamma}{\big(1+\frac{\sqrt{\gamma}\theta}{\sqrt{1+\theta^2}}\big)^2} \cdot \sqrt{\alpha} \cdot \frac{1}{1+\theta^2} \cdot (2\epsilon) \cdot |a| + \frac{1}{4} \cdot \frac{1-\gamma}{\big(1+\frac{\sqrt{\gamma}\theta}{\sqrt{1+\theta^2}}\big)^2} \cdot \frac{1}{1+\theta^2}(|a|+|b|)(\partial_r \ln\beta)$$

$$+ \frac{n\gamma}{2} \cdot \frac{1}{\big(1+\frac{\sqrt{\gamma}\theta}{\sqrt{1+\theta^2}}\big)^2} \cdot \frac{1}{1+\theta^2}(|a|+|b|)(\partial_r \ln r) + \frac{1}{2} \cdot |a|(\partial_r \ln\beta)$$

$$\leq C_1 \cdot (2\epsilon) \cdot \|a(r)\|_{L^\infty} + C_1 \cdot (\|a(r)\|_{L^\infty} + \|b(r)\|_{L^\infty})$$

where $C_1 = C_1(\sup_{r\in[r_{mid},r_0]} \sqrt{\alpha}, \sup_{r\in[r_{mid},r_0]}(\partial_r \ln\beta), \sup_{r\in[r_{mid},r_0]}(\partial_r \ln r))$. Integrating this inequality along X_1, we have

$$|a| \leq \|a(r_0)\|_{L^\infty} + 3C_1 \int_r^{r_0} \Big(\|a(r)\|_{L^\infty} + \|b(r)\|_{L^\infty} \Big) dr,$$

and therefore

$$\|a(r)\|_{L^\infty} \le \|a(r_0)\|_{L^\infty} + 3C_1 \int_r^{r_0} \Big(\|a(r)\|_{L^\infty} + \|b(r)\|_{L^\infty} \Big) dr.$$

We can derive the analogous inequality for b, which together with the above inequality leads to

$$\Big(\|a(r)\|_{L^\infty} + \|b(r)\|_{L^\infty} \Big) \le \Big(\|a(r_0)\|_{L^\infty} + \|b(r_0)\|_{L^\infty} \Big) + 6C_1 \int_r^{r_0} \Big(\|a(r)\|_{L^\infty} + \|b(r)\|_{L^\infty} \Big) dr.$$

According to the Gronwall's inequality, we arrive at an L^∞ control of the derivatives

$$\begin{aligned}\Big(\|a(r)\|_{L^\infty} + \|b(r)\|_{L^\infty} \Big) &\le \Big(\|a(r_0)\|_{L^\infty} + \|b(r_0)\|_{L^\infty} \Big) e^{6C_1(r_0 - r_{mid})} \\ &\le \epsilon_0 \cdot e^{6C_1(r_0 - r_{mid})}.\end{aligned}$$

This means that if we choose the upper bound ϵ_0 of the initial perturbation to be sufficiently small depending on ϵ and C_1, we can ensure $(\|a(r)\|_{L^\infty} + \|b(r)\|_{L^\infty}) < \frac{1}{2}\epsilon$, an improved estimate for a and b. For θ, we have

$$\begin{aligned}|\theta(r) - \theta_{homo}(r)| &\le \int_{\{r=r\}} |\partial_s \theta| ds \\ &= \int_{\{r=r\}} \sqrt{1+\theta^2} \Big| (\partial_s v_1) + (\partial_s v_2) \Big| ds \\ &\le \int_{\{r=r\}} \sqrt{1+\theta^2} \big(|\partial_s v_1| + |\partial_s v_2| \big) ds \\ &\le \sqrt{1 + \big(\theta_{homo}(r) + 1\big)^2} \cdot \sqrt{\beta(r)} \int_{\{r=r\}} (|a| + |b|) ds \\ &\le \sqrt{1 + \big(\theta_{homo}(r) + 1\big)^2} \cdot \sqrt{\beta(r)} \epsilon \cdot W(r)\end{aligned}$$

where $W(r)$ is the width of $\{a \ne 0\} \cup \{b \ne 0\}$ and is uniformly bounded during $r \in [r_{mid}, r_0]$ by finite speed of propagation (due to the uniform bounds of eigenvalues $|\lambda_1|, |\lambda_2| \le \frac{\sqrt{\alpha}}{\sqrt{\beta}}$). We proceed to ensure the largeness of $\theta(r_{mid})$. If we choose ϵ so that

$$\epsilon = \min \Big\{ 1, \frac{1}{2} \cdot \frac{1}{\sup_{r \in [r_{mid}, r_0]} \sqrt{1 + \big(\theta_{homo}(r) + 1\big)^2} \cdot \sqrt{\beta(r)} \cdot W(r)} \Big\},$$

we can improve the estimate for θ and thus close the bootstrap argument. In order to ensure the largeness of θ, we do the same computation

$$|\theta(r_{mid}) - \theta_{homo}(r_{mid})| \leq \sqrt{1 + \left(\theta_{homo}(r_{mid}) + 1\right)^2} \cdot \sqrt{\beta(r_{mid})}\epsilon \cdot W(r_{mid})$$

but this time we choose ϵ to be even smaller so that

$$|\theta(r_{mid}) - \theta_{homo}(r_{mid})| \leq \frac{1}{2}\left(\theta_{homo}(r_{mid}) - \theta^{LB}_{mid}\right),$$

whic implies that $\theta(r_{mid}) > \theta^{LB}_{mid}$. □

Lemma 4.2.3. *The solution (θ, ρ) satisfying the conditions both in Proposition 4.2.1 and Proposition 4.2.2 exists in $W^{1,\infty}$.*

Proof. By the finite speed of propagation, we have

$$\begin{aligned}|\ln\rho(r) - \ln\rho_{homo}(r)| &\leq \int_{\{r=r\}} \frac{1+\gamma}{\sqrt{\gamma}}|\partial_s(v_1 - v_2)|ds \\ &\leq \frac{1+\gamma}{\sqrt{\gamma}} \cdot \sqrt{\beta}\Big(\|a\|_{L^\infty} + \|b\|_{L^\infty} \Big) \cdot W(r),\end{aligned}$$

which is finite (but not bounded) in both $r \in (r_*, r_{mid}]$ and $r \in [r_{mid}, r_0]$ cases. θ, $|a|$, $|b|$ are also bounded as shown in the previous propositions in both cases. □

4.2.7 Main Theorem

Our main theorem in this paper is

Theorem 4.2.1. *Fix a pair of metric components (α, β) satisfying Assumption 4 with positive blowup time $r_* > 0$ and $r_0 > r_*$, and fix any constant $M > 0$. Then there exists θ^{LB}_0 so that the homogeneous solutions with $\theta_{homo}(r_0) > \theta^{LB}_0$ are stable. More specifically, there exists $\epsilon_0 > 0$ so that as long as*

$$\|a(r_0)\|_{L^\infty} + \|b(r_0)\|_{L^\infty} < \epsilon_0,$$

shock will not form and the solution exists in $W^{1,\infty}$ before the blowup time $r = r_*$.

Proof. By Proposition 4.2.1, there exist r_{mid} and θ_{mid}^{LB} so that the homogeneous solutions with $\theta_{homo}(r_{mid}) > \theta_{mid}^{LB}$ are stable. Since the homogeneous solutions satisfy a system of ordinary differential equations, there exists a corresponding θ_0^{LB} so that the homogeneous solutions with $\theta_{homo}(r_0) > \theta_0^{LB}$ implies $\theta_{homo}(r_{mid}) > \theta_{mid}^{LB}$, and by Proposition 4.2.2 (for $[r_{mid}, r_0]$) and Proposition 4.2.1 (for $(r_*, r_{mid}]$), they are stable. The $W^{1,\infty}$ claim is from Lemma 4.2.3. □

Remark. *This theorem states that the homogeneous $\theta \neq 0$ solutions are dynamically stable as long as the angle θ between the fluid velocity and time direction $-\partial_r$ is sufficiently large. Intuitively, as long as the homogeneous fluid drives away from time direction far enough, the blowup rate of θ beats the mechanism for shock formation, thus preventing the occurrence of shock. Notice that the largeness of θ is essential in our proof when we improve the bootstrap estimate in $r \in (r_*, r_{mid}]$ region. We introduced the parameter r_{mid} to ensure the monotonicity of the fluid variable θ and the metric component β in the $(r_*, r_{mid}]$ region, relying on the asymptotic behavior described in Assumption 4.*

CHAPTER 5

STABILITY OF MEMBRANE EQUATIONS

This chapter aims to prove the global existence of membrane equations for sufficiently small initial data. This is a separate work from previous chapters.

Membrane equation is a historically interesting problem. In Euclidean space, it describes a membrane minimizing the area with a given boundary, while in Lorentzian spacetime, it represents the world sheet of an extended object without external force (see [12]). This paper aims to prove the global existence of the Lorentzian-type membrane equation

$$\partial_i \left(\frac{m^{ij}\partial_j u}{\sqrt{1 + m^{ab}\partial_a u \partial_b u}} \right) = 0 \tag{5.1}$$

for sufficiently small initial data, where $(x^0, x^1, x^2, x^3) = (t, x^1, x^2, \theta)$ represents the coordinate for $\mathbb{R}^{1,2} \times \mathbb{T}^1$ and m^{ij} is the component of the Minkowski metric (including $\mathbb{T}^1$ as a periodic space variable)

$$m = -dt^2 + d(x^1)^2 + d(x^2)^2 + d\theta^2.$$

Previously in Lindblad's work [20], he proved the global existence for small initial data on classical Minkowski spacetime $\mathbb{R}^{1,n}$, where he used the vector field method, proposed by Klainerman [18], to achieve the global existence of membrane equation with space dimension greater than 1 for compactly supported initial data. We record the main ideas in his proof and address the difference between his strategy and ours.

The vector field method is to use the appropriately chosen weighted vector fields to gain the decay of derivatives. The general strategy of this method is

1. applying prior inequalities involving
   a) the energy of weighted vector fields (acting on the unknown u),
   b) the pointwise bound of the derivatives of u, and
   c) the nonhomogeneous term from the differential equation being considered, and

2. running the bootstrap argument to argue the boundedness of the energy, and thus the boundedness of the derivatives of u.

According to the well-known criterion for the global existence of quasilinear wave equation, it is sufficient to have the boundedness of $|\partial u| + |\partial^2 u|$ to ensure the global existence (where ∂ may be $\partial_t, \partial_1, \partial_2$, or ∂_θ in our case). To close the bootstrap argument, one has to bound the nonhomogeneous term by energy with an appropriate decay so that the integrand becomes integrable. In Lindblad's argument, he uses three different inequalities: two energy estimates and one L^∞-L^1 estimate. He also makes use of the null structure of the Lorentzian membrane equation to close the bootstrap argument.

In order to apply the estimates mentioned above, Lindblad commutes the membrane equation with Λ^I and derive an equation for $\Box\Lambda^I u$, in which the right hand side consists of terms falling into three catogories: terms that are multilinear in u, of divergence form, and of null form. Here Λ^I may be the composition of

$$
\begin{aligned}
&\partial_t, \partial_{x^1}, ..., \partial_{x^n} \\
&S = t\partial_t + \sum_{i=1}^{n} x^i \partial_i \\
&L^i = t\partial_i + x^i \partial_t, \quad 1 \le i \le n \\
&\Omega_{ij} = x^i \partial_j - x^j \partial_i, \quad 1 \le i, j \le n.
\end{aligned}
$$

Among these vector fields, the dilation field S is an obstacle for generalizing the strategy to our case $\mathbb{R}^{1,2} \times \mathbb{T}^1$ since there is no naturally analogous dilation field on $\mathbb{T}^1$. In order to resolve this issue, we observe where Lindblad uses this dilation field. The L^∞-L^1 estimate

$$
\begin{aligned}
|w(t,x)| \le & C(1+t+|x|)^{-(n-1)/2} \\
& \left(\sum_{|I| \le n-1} \int_0^t \left\| \Lambda^I F(s,.)/(1+s+|.|)^{(n-1)/2} \right\|_{L^1} ds + C(f,g)\epsilon \right),
\end{aligned}
$$

requires the dilation field to work. Another place involving S is when he takes advantage of the null structure of the membrane equation, the estimate

$$|Q(\phi,\psi)| \leq C(1+t+|x|)^{-1}(|\partial\phi||\Lambda\psi| + |\Lambda\phi||\partial\psi|)$$

also requires the dilation field to be one of the vector fields Λ. Based on these two observations, we can not directly apply the same argument to $\mathbb{R}^{1,2}\times\mathbb{T}^1$ case. Instead, we only use

$$\partial_t, \partial_\theta, L^i = t\partial_i + x^i\partial_t, \quad 1 \leq i \leq 2$$

to form a Lie algebra and apply the vector field method. Notice that Ω_{ij} can be expressed in terms of L^i when $t \neq 0$. In addition, we foliate the spacetime region lying inside the future light cone ($t > |x|$) with the hyperbolic curves ($t^2 - |x|^2 = \text{const}$). This was introduced by LeFloch and Ma in [19] where they established the global well-posedness of nonlinear wave equations and Klein-Gordon equations on $\mathbb{R}^{1+3}$. This foliation helps us capture the decay along the $t - x = \text{const}$ direction and enables us to close the bootstrap argument.

Another related work is Ifrim and Stingo's paper (see [14]). In their work, they proved the global existence for a coupled Klein-Gordon equation on $\mathbb{R}^{1,2}$ with small data. In their paper, they dyadically decomposed the spacetime region $\{(t,x)\big||t-|x|| \lesssim t+|x|, t>0\}$ and used the constant-time-slice energy to bound the weighted spacetime energy. The relation between their equation and our membrane equation is that, by expanding our solution in Fourier series with respect to θ: $u = \sum_{n=-\infty}^{\infty} u_n e^{in\theta}$ and plugging this into our main equation (5.1) (see Lemma 5.1.2)

$$\big(1 + m^{ab}\partial_a u\partial_b u\big)m^{ij}\partial_i\partial_j u = m^{ik}m^{jl}\partial_k u\partial_l u\partial_i\partial_j u,$$

we would arrive at

$$\begin{aligned}
&\Box_{t,x}u_n - n^2 u_n + \sum_{n_1+n_2+n_3=n}\left(-\partial_t u_{n_1}\partial_t u_{n_2} + \sum_{j=1}^{2}\partial_j u_{n_1}\partial_j u_{n_2} - (n_1 n_2)u_{n_1}u_{n_2}\right)\\
&\quad\times(\Box_{t,x}u_{n_3} - (n_3)^2 u_{n_3})\\
&= \sum_{n_1+n_2+n_3=n}\Bigg(\partial_t u_{n_1}\partial_t u_{n_2}\partial_t^2 u_{n_3} - 2\sum_{j=1}^{2}\partial_t u_{n_1}\partial_j u_{n_2}\partial_t\partial_j u_{n_3} + \sum_{i,j=1}^{2}\partial_i u_{n_1}\partial_j u_{n_2}\partial_i\partial_j u_{n_3}
\end{aligned}$$

$$+ 2\partial_t u_{n_1}(n_2)u_{n_2}(n_3)\partial_t u_{n_3} - 2\sum_{j=1}^{2}\partial_j u(n_2)u_{n_2}(n_3)\partial_i u_{n_3}$$
$$+ (n_1 n_2 n_3^2)u_{n_1}u_{n_2}u_{n_3}\Bigg)$$

This can be regarded as a strongly coupled system of Klein-Gordan and wave equations on $\mathbb{R}^{1,2}$.

In this paper, we are going to show the global existence of (5.1), which is recorded in Theorem 5.5.2. In section 2, we begin with the setup for the geometry of the spacetime and computation of geometric quantities that are involved with the energy. In section 3, we prove the global Sobolev inequality, which plays an essential role in this paper to derive the desired decay. This proof is parallel to the proof in [28]. Section 4 works for all the derivative estimates we need later, using the global Sobolev inequality and the exploit of the null structure in the Lorentzian membrane equation. We treat $\partial_t^2 u$ specially because our energy only involves $\partial_t u$ but no $\partial_t^2 u$. Section 5 establishes the comparability of the two versions of energy introduced in Section 2. Section 6 shows the energy estimate using the divergence theorem and the bootstrap mechanism.

5.1 Geometry and Energy

We will use the coordinate (t, x^1, x^2, θ) to represent the points in our spacetime throughout this paper. In order to adapt the hyperboloidal foliation method, we parametrize the spacetime region lying inside the future null cone, $\left\{(t, x^1, x^2, \theta)\big||x| < t\right\}$, with $(\tau, \rho, \phi, \theta)$:

$$t = \tau\cosh(\rho)$$
$$x^1 = \tau\sinh(\rho)\cos\phi$$
$$x^2 = \tau\sinh(\rho)\sin\phi,$$

where $\tau \in [2, \infty), \rho \in [0, \infty), \phi \in [0, 2\pi), \theta \in [0, 2\pi)$. The following Lorentz boost vector fields would be used in this paper.

$$L^1 := t\partial_1 + x^1\partial_t = (\cos\phi)\partial_\rho - (\coth\rho\sin\phi)\partial_\phi$$
$$L^2 := t\partial_2 + x^2\partial_t = (\sin\phi)\partial_\rho + (\coth\rho\cos\phi)\partial_\phi.$$

Notice that the latter expression only works for $\rho > 0$, but L^1, L^2 are well-defined even when $\rho = 0$. We introduce a basic computation for the inverse matrix.

Lemma 5.1.1. *Let $g_{ij} = m_{ij} + \partial_i u \partial_j u$, then*

$$g^{ij} = m^{ij} - \frac{1}{1 + m^{ab}\partial_a u \partial_b u}\left(m^{ik}m^{jl}\partial_k u \partial_l u\right).$$

Proof. Let M, G be the matrix representations of m_{ij}, g_{ij} respectively, and let v be the column vector representing $\partial_i u$. Then we have $G = M + vv^T$ and, by the Sherman-Morrison formula,

$$G^{-1} = M - \frac{Mvv^T M}{1 + v^T M v},$$

which gives g^{ij}. □

This metric g plays a role since our main equation can be rewritten using g with a much simpler structure than (5.1).

Lemma 5.1.2. *The equation* (5.1) *is equivalent to*

$$\Box_g u = 0,$$

where $g_{ij} = m_{ij} + \partial_i u \partial_j u$.

Proof. From the definition of $\Box_g$, we have

$$\Box_g u = g^{ij}\partial_i \partial_j u - g^{ij}\Gamma^k_{ij}\partial_k u.$$

To calculate the second term on the right hand side, we calculate Γ^k_{ij} first.

$$\begin{aligned}
\Gamma^k_{ij} &= \frac{1}{2}g^{kl}(\partial_i g_{lj} + \partial_j g_{il} - \partial_l g_{ij}) \\
&= \frac{1}{2}g^{kl}(\partial_i(\partial_l u \partial_j u) + \partial_j(\partial_i u \partial_l u) - \partial_l(\partial_i u \partial_j u)) \\
&= g^{kl}\partial_l u(\partial_i \partial_j u),
\end{aligned}$$

which implies

$$\Box_g u = g^{ij}(\partial_i \partial_j u)(1 - g^{kl}\partial_k u \partial_l u).$$

Observe that

$$g^{ij}\partial_i\partial_j u = \left(m^{ij} - \frac{1}{1+m^{ab}\partial_a u\partial_b u}\left(m^{ik}m^{jl}\partial_k u\partial_l u\right)\right)\partial_i\partial_j u$$
$$= m^{ij}(\partial_i\partial_j u) - \frac{1}{1+m^{ab}\partial_a u\partial_b u}m^{ik}m^{jl}\partial_k u\partial_l u(\partial_i\partial_j u)$$

is equivalent to (5.1) and

$$g^{ij}\partial_i u\partial_j u = m^{ij}\partial_i u\partial_j u - \frac{1}{1+m^{ab}\partial_a u\partial_b u}(m^{ik}m^{jl}\partial_k u\partial_l\partial_i u\partial_j u)$$
$$= \sigma - \frac{1}{1+\sigma}\sigma^2$$
$$= \frac{\sigma}{1+\sigma}$$

is never 1, where $\sigma = m^{ij}\partial_i u\partial_j u$. □

Remark. *Lemma 5.1.2 implies that u could be regarded as a wave map. See [27] for more details.*

Using the Minkowski metric m_{ij} and the dynamic metric g_{ij}, we can define the corresponding tensors

$$Q^i_j[u;m] = m^{ik}\partial_k u\partial_j u - \frac{1}{2}\sigma[u;m]\delta^i_j$$
$$Q^i_j[u;g] = g^{ik}\partial_k u\partial_j u - \frac{1}{2}\sigma[u;g]\delta^i_j,$$

and currents

$${}^{(X)}J[u;m] = Q^i_j[u;m]X^j\partial_i$$
$${}^{(X)}J[u;g] = Q^i_j[u;g]X^j\partial_i,$$

where

$$\sigma[u;m] = m^{ij}\partial_i u\partial_j u$$
$$\sigma[u;g] = g^{ij}\partial_i u\partial_j u.$$

To apply the divergence theorem, we note that the divergence of the current ${}^{(X)}J$ is given by

$$div_g\left({}^{(X)}J\right) = \Box_g u(\nabla_j u)X^j + \nabla^i u \nabla_j u \nabla_i X^j - \frac{1}{2}\nabla^k u \nabla_k u \nabla_i X^i. \tag{5.2}$$

If we plug ∂_t into X, the above expression simplifies to

$$div_g\left({}^{(\partial_t)}J\right) = \Box_g u(\nabla_t u) + (\nabla^i u \nabla_j u)\Gamma^j_{it} - \frac{1}{2}(\nabla^k u \nabla_k u)\Gamma^i_{it}.$$

Since are going to apply the divergence theorem on

$$\{(t, x^1, x^2, \theta) | \tau_0 \leq \tau \leq \tau_1\},$$

which is bounded by $\Sigma_\tau = \{(t, x^1, x^2, \theta) | \sqrt{t^2 - |x|^2} = \tau\}$ with τ equals τ_0 and τ_1 respectively, we define our energy to be

$$\begin{aligned}
\mathcal{E}_\tau[u;m]^2 &= 2\int_{\Sigma_\tau} \langle {}^{(\partial_t)}J[u;m], \partial_\tau \rangle_m dS_m \\
\mathcal{E}_\tau[u;g]^2 &= 2\int_{\Sigma_\tau} \langle {}^{(\partial_t)}J[u;g], \vec{n} \rangle_g dS_g \\
\mathcal{E}^{\leq s}_\tau[u;m] &= \sum_{|\gamma_1|+|\gamma_2|\leq s} \mathcal{E}_\tau[L^{\gamma_1}\partial_\theta^{\gamma_2} u; m] \\
\mathcal{E}^{\leq s}_\tau[u;g] &= \sum_{|\gamma_1|+|\gamma_2|\leq s} \mathcal{E}_\tau[L^{\gamma_1}\partial_\theta^{\gamma_2} u; g]
\end{aligned}$$

where $\vec{n}$ is the future-directed normal vector of Γ_τ with respect to g. In section 5, we are going to establish the comparability of the energies with respect to m and with respect to g. In other words, we will be able to ensure the smallness of derivative of u and thus smallness of $g_{ij} - m_{ij}$.

Observe that we can complete the square for the integrand in $\mathcal{E}_\tau[u;m]$:

$$\begin{aligned}
&\langle {}^{(\partial_t)}J[u;m], \partial_\tau \rangle \\
&= \langle Q^i_j (\partial_t)^j \partial_i, \cosh(\rho)\partial_t + \sinh(\rho)\cos\phi \partial_{x^1} + \sinh(\rho)\sin\phi \partial_{x^2} \rangle \\
&= \left(-\nabla^t u \nabla_t u + \frac{1}{2}\sigma\right)\cosh(\rho) + (\nabla^1 u \nabla_t u)\sinh(\rho)\cos\phi + (\nabla^2 u \nabla_t u)\sinh(\rho)\sin\phi
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\left((\nabla_t u)^2 + (\nabla_1 u)^2 + (\nabla_2 u)^2 + (\nabla_\theta u)^2\right)\cosh(\rho) \\
&\quad + (\nabla_1 u \nabla_t u)\sinh(\rho)\cos\phi + (\nabla_2 u \nabla_t u)\sinh(\rho)\sin\phi \\
&= \frac{1}{2}\left(\sqrt{\cosh(\rho)}\nabla_1 u + \frac{\sinh(\rho)}{\sqrt{\cosh(\rho)}}\cos\phi\nabla_t u\right)^2 + \frac{1}{2}\left(\sqrt{\cosh(\rho)}\nabla_2 u + \frac{\sinh(\rho)}{\sqrt{\cosh(\rho)}}\sin\phi\nabla_t u\right)^2 \\
&\quad + \frac{1}{2}\frac{1}{\cosh(\rho)}(\nabla_t u)^2 + \frac{1}{2}\cosh(\rho)(\nabla_\theta u)^2 \\
&= \frac{1}{2}\frac{1}{\tau^2\cosh(\rho)}|L^1 u|^2 + \frac{1}{2}\frac{1}{\tau^2\cosh(\rho)}|L^2 u|^2 + \frac{1}{2}\frac{1}{\cosh(\rho)}|\partial_t u|^2 + \frac{1}{2}\cosh(\rho)|\partial_\theta u|^2,
\end{aligned}
$$

where the ∇ above denotes the connection with respect to the Minkowski metric m_{ij}.

5.2 Sobolev inequality

The following theorem is mostly following the analogous one in [28].

Theorem 5.2.1. *(L^∞-L^2 estimate.) Let $l \in \mathbb{R}$. Then*

$$
|f(\tau,\rho,\phi,\theta)|^2 \lesssim \tau^{-2}(\cosh\rho)^{-1-l}\sum_{|\gamma_1|+|\gamma_2|\le 2}\int_{\Sigma_\tau}(\cosh\rho)^l|L^{\gamma_1}\partial_\theta^{\gamma_2} f|^2 dvol_{\tilde{\Sigma}_\tau}
$$

Proof. We discuss two cases separately.

Case 1: $\rho < \frac{5}{3}$. In this case, we are going to apply the standard Sobolev inequality with the metric h_1 (on Σ_τ), where

$$
\begin{aligned}
h_1 &= d\rho^2 + \sinh(\rho)^2 d\phi^2 + d\theta^2 \\
h_\tau &= \tau^2\left(d\rho^2 + \sinh(\rho)^2 d\phi^2\right) + d\theta^2.
\end{aligned}
$$

By the Sobolev inequality, we have

$$
\begin{aligned}
|f(\tau,\rho,\phi,\theta)|^2 &\lesssim \sum_{|\gamma|\le 2}\int_{\Sigma_\tau\cap\{\rho<2\}}|\nabla^\gamma f|^2_{h_1} dvol_{h_1} \\
&\lesssim \sum_{|\gamma_1|+|\gamma_2|\le 2}\int_{\Sigma_\tau\cap\{\rho<2\}}|L^{\gamma_1}\partial_\theta^{\gamma_2} f|^2 dvol_{h_1} \\
&= \tau^{-2}\sum_{|\gamma_1|+|\gamma_2|\le 2}\int_{\Sigma_\tau\cap\{\rho<2\}}|L^{\gamma_1}\partial_\theta^{\gamma_2} f|^2 dvol_{h_\tau} \\
&\approx \tau^{-2}\cosh(\rho)^{-1-l}\sum_{|\gamma_1|+|\gamma_2|\le 2}\int_{\Sigma_\tau\cap\{\rho<2\}}\cosh(\rho)^l|L^{\gamma_1}\partial_\theta^{\gamma_2} f|^2 dvol_{h_\tau},
\end{aligned}
$$

where the reasons are as follows. The second $\lesssim$ is because

$$
\begin{aligned}
|\nabla f|_{h_1}^2 &= (\partial_\rho f)^2 + \frac{1}{\sinh(\rho)^2}(\partial_\phi f)^2 + (\partial_\theta f)^2 \\
&\leq |L^1 f|^2 + |L^2 f|^2 + |\partial_\theta f|^2
\end{aligned}
$$

and

$$
\begin{aligned}
|\nabla\nabla f|_{h_1}^2 &= (h_1)^{ik}(h_1)^{jl}\nabla\nabla f(\partial_i, \partial_j)\nabla\nabla f(\partial_k, \partial_l) \\
&\lesssim \sum_{|\gamma_1|+|\gamma_2|\leq 2} |L^{\gamma_1}\partial_\theta^{\gamma_2} f|^2.
\end{aligned}
$$

The $\approx$ is because we are focusing on a compact region (of (ρ, ϕ, θ)).

<u>Case 2</u>: $\rho > \frac{4}{3}$. In this case, we are going to use the metric h_0 instead, where

$$
h_0 = d\rho^2 + d\phi^2 + d\theta^2.
$$

By the Sobolev inequality,

$$
\begin{aligned}
&|f(\tau, \rho, \phi, \theta)\cosh(\rho)^{l/2}\sinh(\rho)^{1/2}|^2 \\
&\quad\lesssim \sum_{|\gamma|\leq 2}\int_{\Sigma_\tau\cap\{\rho>1\}} \left|\nabla^\gamma\left(f\cosh(\rho)^{l/2}\sinh(\rho)^{1/2}\right)\right|_{h_0}^2 dvol_{h_0} \\
&\quad\approx \sum_{|\gamma|\leq 2}\int_{\Sigma_\tau\cap\{\rho>1\}} \cosh(\rho)^l\,|\nabla^\gamma f|_{h_0}^2\,dvol_{h_1} \\
&\quad\lesssim \sum_{|\gamma_1|+|\gamma_2|\leq 2}\int_{\Sigma_\tau\cap\{\rho>1\}} \cosh(\rho)^l\left|L^{\gamma_1}\partial_\theta^{\gamma_2} f\right| dvol_{h_1} \\
&\quad= \tau^{-2}\sum_{|\gamma_1|+|\gamma_2|\leq 2}\int_{\Sigma_\tau\cap\{\rho>1\}} \cosh(\rho)^l\left|L^{\gamma_1}\partial_\theta^{\gamma_2} f\right|^2 dvol_{h_\tau},
\end{aligned}
$$

where the reasons are as follows. The $\approx$ is because we exclude an open neighborhood of $\rho = 0$, and thus $\sinh(\rho)$ and $\cosh(\rho)$ are comparable. The second $\lesssim$ is because

$$
\begin{aligned}
|\nabla f|_{h_0}^2 &= (\partial_\rho f)^2 + (\partial_\phi f)^2 + (\partial_\theta f)^2 \\
&\leq |L^1 f|^2 + |L^2 f|^2 + |\partial_\theta f|^2
\end{aligned}
$$

and

$$
\begin{aligned}
|\nabla\nabla f|^2_{h_0} &= (h_0)^{ik}(h_0)^{jl}(\partial_i\partial_j f)(\partial_k\partial_l f) \\
&\leq \sum_{|\gamma_1|+|\gamma_2|\leq 2} |L^{\gamma_1}\partial_\theta^{\gamma_2} f|^2.
\end{aligned}
$$

□

5.3 Estimate for derivatives

Lemma 5.3.1. *We calculate the following three terms in this lemma.*

$$
\begin{aligned}
&m^{ab}\partial_a u\partial_b u = A - 1 \\
&m^{ij}\partial_i\partial_j u = -\frac{1}{\cosh(\rho)^2}\partial_t\partial_t u + B \\
&m^{ik}m^{jl}\partial_k u\partial_l u\partial_i\partial_j u.
\end{aligned}
$$

Proof. Using the identity $\partial_i = \frac{1}{t}L^i - \frac{x^i}{t}\partial_t$ for $i = 1, 2$, we have

$$
\begin{aligned}
m^{ab}\partial_a u\partial_b u &= -(\partial_t u)^2 + \sum_{i=1}^{2}\left(\frac{1}{t}L^i u - \frac{x^i}{t}\partial_t u\right)^2 + (\partial_\theta u)^2 \\
&= -\frac{1}{\cosh(\rho)^2}(\partial_t u)^2 + \sum_{i=1}^{2}\frac{1}{t^2}|L^i u|^2 - 2\sum_{i=1}^{2}\frac{1}{t}\frac{x^i}{t}(L^i u\partial_t u) + (\partial_\theta u)^2 \\
&:= A - 1,
\end{aligned}
$$

$$
\begin{aligned}
m^{ij}\partial_i\partial_j u &= -\partial_t\partial_t u + \sum_{i=1}^{2}\left(\frac{1}{t}L^i - \frac{x^i}{t}\partial_t\right)\left(\frac{1}{t}L^i u - \frac{x^i}{t}\partial_t u\right) + \partial_\theta\partial_\theta u \\
&= -\partial_t\partial_t u + \sum_{i=1}^{2}\Big(\frac{1}{t^2}L^i L^i u - \frac{1}{t}\frac{x^i}{t}L^i\partial_t u - \frac{x^i}{t}\frac{1}{t}\partial_t L^i u + \frac{x^i}{t}\frac{x^i}{t}\partial_t\partial_t u \\
&\qquad - \frac{1}{t}\frac{x^i}{t^2}L^i u - \frac{1}{t}(1 - \frac{x^i x^i}{t^2})\partial_t u + \frac{x^i}{t}\frac{1}{t^2}L^i u - \frac{x^i}{t}\frac{x^i}{t^2}\partial_t u\Big) + \partial_\theta\partial_\theta u \\
&= -\frac{1}{\cosh(\rho)^2}\partial_t\partial_t u + \sum_{i=1}^{2}\left(\frac{1}{t^2}L^i L^i u - \frac{1}{t}\frac{x^i}{t}L^i\partial_t u - \frac{x^i}{t}\frac{1}{t}\partial_t L^i u - \frac{1}{t}\partial_t u\right) \\
&\quad + \partial_\theta\partial_\theta u \\
&:= -\frac{1}{\cosh(\rho)^2}\partial_t\partial_t u + B
\end{aligned}
$$

and

$$
\begin{aligned}
&m^{ik}m^{jl}\partial_k u \partial_l u \partial_i \partial_j u = \\
&\quad \partial_t u \partial_t u (\partial_t \partial_t u) - 2\sum_{i=1}^{2} \partial_t u \left(\frac{1}{t}L^i u - \frac{x^i}{t}\partial_t u\right)\left(\frac{1}{t}L^i \partial_t u - \frac{x^i}{t}\partial_t \partial_t u\right) \\
&\quad + \sum_{i,j=1}^{2} \left(\frac{1}{t}L^i u - \frac{x^i}{t}\partial_t u\right)\left(\frac{1}{t}L^j u - \frac{x^j}{t}\partial_t u\right)\left(\left(\frac{1}{t}L^i - \frac{x^i}{t}\partial_t\right)\left(\frac{1}{t}L^j u - \frac{x^j}{t}\partial_t u\right)\right) \\
&\quad - 2\partial_t u \partial_\theta u \partial_t \partial_\theta u + 2\sum_{i=1}^{2}\left(\frac{1}{t}L^i u - \frac{x^i}{t}\partial_t u\right)\partial_\theta u \left(\frac{1}{t}L^i \partial_\theta u - \frac{x^i}{t}\partial_t \partial_\theta u\right) \\
&\quad + \partial_\theta u \partial_\theta u \partial_\theta \partial_\theta u \\
&= \left(\frac{1}{\cosh(\rho)^4}\partial_t u \partial_t u + 2\sum_{i=1}^{2}\frac{1}{t}\frac{x^i}{t}\partial_t u L^i u \right. \\
&\quad \left. + \sum_{i,j=1}^{2}\frac{1}{t^2}\frac{x^i}{t}\frac{x^j}{t}L^i u L^j u - 2\sum_{i,j=1}^{2}\frac{1}{t}\frac{x^i}{t}\frac{(x^j)^2}{t^2}L^i u \partial_t u\right)(\partial_t \partial_t u) \\
&\quad - 2\sum_{i=1}^{2}\frac{1}{t^2}\partial_t u L^i u (L^i \partial_t u) + 2\sum_{i=1}^{2}\frac{x^i}{t}\frac{1}{t}\partial_t u \partial_t u (L^i \partial_t u) \\
&\quad + \sum_{i,j=1}^{2}\left(\frac{1}{t}L^i u - \frac{x^i}{t}\partial_t u\right)\left(\frac{1}{t}L^j u - \frac{x^j}{t}\partial_t u\right) \\
&\qquad \left(\frac{1}{t^2}(L^i L^j u) - \frac{1}{t}\frac{x^j}{t}(L^i \partial_t u) - \frac{x^i}{t}\frac{1}{t}(\partial_t L^j u)\right. \\
&\qquad \left. - \frac{1}{t}\frac{x^i}{t^2}L^j u - \frac{1}{t}(\delta_{ij} - \frac{x^i x^j}{t^2})\partial_t u + \frac{x^i}{t}\frac{1}{t^2}L^j u - \frac{x^i}{t}\frac{x^j}{t^2}\partial_t u\right) \\
&\quad - 2\partial_t u \partial_\theta u (\partial_t \partial_\theta u) + 2\sum_{i=1}^{2}\frac{1}{t^2}L^i u \partial_\theta u (L^i \partial_\theta u) - 2\sum_{i=1}^{2}\frac{1}{t}\frac{x^i}{t}L^i u \partial_\theta u (\partial_t \partial_\theta u) \\
&\quad - 2\sum_{i=1}^{2}\frac{1}{t}\frac{x^i}{t}\partial_t u \partial_\theta u (L^i \partial_\theta u) + \frac{2|x|^2}{t^2}\partial_t u \partial_\theta u (\partial_t \partial_\theta u) + \partial_\theta u \partial_\theta u (\partial_\theta \partial_\theta u) \\
&:= \left(\frac{1}{\cosh(\rho)^4}|\partial_t u|^2 + \frac{2}{\cosh(\rho)^2}\sum_{i=1}^{2}\frac{1}{t}\frac{x^i}{t}\partial_t u L^i u + \sum_{i,j=1}^{2}\frac{1}{t^2}\frac{x^i}{t}\frac{x^j}{t}L^i u L^j u\right)(\partial_t^2 u) \\
&\quad + C,
\end{aligned}
$$

where

$$A = A(t, x, \partial_t u, Lu, \partial_\theta u)$$

$$B = B(t, x, \partial_t u, Lu, LLu, L\partial_t u)$$

$$C = C(t, x, \partial_t u, Lu, LLu, L\partial_t u, L\partial_\theta u, \partial_t\partial_\theta u, \partial_\theta\partial_\theta u).$$

□

Remark. *By the notation above, we could simplify the equation* $g^{ij}\partial_i\partial_j u = 0$*, or*

$$m^{ij}\partial_i\partial_j u = \frac{1}{1 + m^{ab}\partial_a u\partial_b u}(m^{ik}m^{jl}\partial_k u\partial_l u\partial_i\partial_j u),$$

to the following identity

$$\begin{aligned} &A\left(-\frac{1}{\cosh(\rho)^2}\partial_t^2 u + B\right) \\ &= (\partial_t^2 u)\left(\frac{1}{\cosh(\rho)^4}|\partial_t u|^2 + \frac{2}{\cosh(\rho)^2}\sum_{i=1}^{2}\frac{1}{t}\frac{x^i}{t}\partial_t u L^i u + \sum_{i,j=1}^{2}\frac{1}{t^2}\frac{x^i}{t}\frac{x^j}{t}L^i u L^j u\right) + C. \end{aligned}$$

The point is that, since our energy does not involve second derivative with respect to time, we need the main equation to help control $\partial_t^2 u$*. Therefore, we solve for* $\partial_t^2 u$*:*

$$\partial_t^2 u = \cosh(\rho)^2 \frac{AB - C}{1 + \sum_{i=1}^{2}\frac{1}{t^2}|L^i u|^2 + |\partial_\theta u|^2 + \cosh(\rho)^2\sum_{i,j=1}^{2}\frac{1}{t^2}\frac{x^i}{t}\frac{x^j}{t}L^i u L^j u}. \tag{5.3}$$

We begin with estimating pointwise upper bound for derivative with at most one ∂_t.

Lemma 5.3.2. *We have*

$$|L^{\gamma_1}\partial_\theta^{\gamma_2}\partial_t u| \lesssim \frac{1}{\tau}\mathcal{E}_\tau^{\leq s}[u; m]$$

$$|L^{\gamma_1}\partial_\theta^{\gamma_2}L^i u| \lesssim \mathcal{E}_\tau^{\leq s}[u; m]$$

$$|L^{\gamma_1}\partial_\theta^{\gamma_2}\partial_\theta u| \lesssim \frac{1}{\tau\cosh(\rho)}\mathcal{E}_\tau^{\leq s}[u; m]$$

for $|\gamma_1| + |\gamma_2| + 3 \leq s + 1$ *and* $i = 1, 2$.

Proof. Using the global Sobolev inequality Theorem 5.2.1, we have

$$\begin{aligned}
|L^{\gamma_1}\partial_\theta^{\gamma_2}\partial_t u|^2 &\lesssim \frac{1}{\tau^2}\sum_{|\gamma_1'|+|\gamma_2'|\le 2}\int_{\Sigma_\tau}\frac{1}{\cosh(\rho)}|L^{\gamma_1'}\partial_\theta^{\gamma_2'}L^{\gamma_1}\partial_\theta^{\gamma_2}\partial_t u|^2 dvol\\
&\lesssim \frac{1}{\tau^2}\sum_{|\gamma_1'|+|\gamma_2'|\le s}\int_{\Sigma_\tau}\left(\frac{1}{\cosh(\rho)}|\partial_t L^{\gamma_1'}\partial_\theta^{\gamma_2'}u|^2+\sum_{i=1}^{2}\frac{1}{\cosh(\rho)}|\partial_i L^{\gamma_1'}\partial_\theta^{\gamma_2'}u|^2\right)\\
&\le \frac{1}{\tau^2}\mathcal{E}_\tau^{\le s}[u;m]^2,
\end{aligned}$$

where we use $[L^i,\partial_t]=-\partial_i$, $[L^i,\partial_j]=-\delta_{ij}\partial_t$ and $\partial_i=\frac{1}{t}L^i-\frac{x^i}{t}\partial_t$,

$$\begin{aligned}
|L^{\gamma_1}\partial_\theta^{\gamma_2}L^i u|^2 &\lesssim \frac{1}{\tau^2}\sum_{|\gamma_1'|+|\gamma_2'|\le 2}\int_{\Sigma_\tau}\frac{1}{\cosh(\rho)}|L^{\gamma_1'}\partial_\theta^{\gamma_2'}L^{\gamma_1}\partial_\theta^{\gamma_2}L^i u|^2 dvol\\
&\le \sum_{j=1}^{2}\sum_{|\gamma_1'|+|\gamma_2'|\le s}\int_{\Sigma_\tau}\frac{1}{\tau^2\cosh(\rho)}|L^j L^{\gamma_1'}\partial_\theta^{\gamma_2'}u|^2 dvol\\
&\lesssim \mathcal{E}_\tau^{\le s}[u;m]^2,
\end{aligned}$$

and

$$\begin{aligned}
|L^{\gamma_1}\partial_\theta^{\gamma_2}\partial_\theta u|^2 &\lesssim \frac{1}{\tau^2\cosh(\rho)^2}\sum_{|\gamma_1'|+|\gamma_2'|\le 2}\int_{\Sigma_\tau}\cosh(\rho)|L^{\gamma_1'}\partial_\theta^{\gamma_2'}L^{\gamma_1}\partial_\theta^{\gamma_2}\partial_\theta u|^2 dvol\\
&= \frac{1}{\tau^2\cosh(\rho)^2}\sum_{|\gamma_1'|+|\gamma_2'|\le s}\int_{\Sigma_\tau}\cosh(\rho)|\partial_\theta L^{\gamma_1'}\partial_\theta^{\gamma_2'}u|^2 dvol\\
&\le \frac{1}{\tau^2\cosh(\rho)^2}\mathcal{E}_\tau^{\le s}[u;m]^2.
\end{aligned}$$

□

Now we proceed to estimating the pointwise upper bound for second derivative with respect to time. This requires the following bootstrap assumption, which we will assume from now on.

Bootstrap Assumption. *Our bootstrap assumption is*

$$\mathcal{E}_\tau^{\le s}[u;m]\le\epsilon \tag{5.4}$$

for $\tau\in[\tau_1,\tau_2]$, *where* $2\le\tau_1<\tau_2$ *are arbitrary, and* $0<\epsilon\le 1$ *and* s *will be chosen later.*

Lemma 5.3.3. *If $\Box_g u = 0$ and u satisfies the bootstrap assumption 5.4, then*

$$|L^{\gamma_1}\partial_t L^{\gamma_2}\partial_t L^{\gamma_3}\partial_\theta^{\gamma_4} u| \lesssim \frac{\cosh(\rho)}{\tau}\mathcal{E}_\tau^{\leq s}[u;m]$$

for $|\gamma| + 4 \leq s + 1$ and ϵ sufficiently small, where $|\gamma| = |\gamma_1| + |\gamma_2| + |\gamma_3| + |\gamma_4|$.

Proof. From *Remark 2*, we have a pointwise estimate for $\partial_t^2 u$:

$$\begin{aligned}
|\partial_t^2 u| &= \cosh(\rho)^2 \frac{|AB - C|}{1 + \sum_{i=1}^2 \frac{1}{t^2}|L^i u|^2 + |\partial_\theta u|^2 + \cosh(\rho)^2 \sum_{i,j=1}^2 \frac{1}{t^2}\frac{x^i}{t}\frac{x^j}{t} L^i u L^j u} \\
&\lesssim \cosh(\rho)^2 \frac{|AB - C|}{1 - \frac{4}{\tau^2}\epsilon^2} \\
&\lesssim \cosh(\rho)^2 |AB - C|
\end{aligned}$$

when ϵ is sufficiently small, where the first $\lesssim$ is because of the bootstrap assumption 5.4. Using Lemma 5.3.2, we find that

$$AB - C \lesssim \frac{1}{\tau\cosh(\rho)}\mathcal{E}_\tau^{\leq s}[u;m],$$

which gives the result when $|\gamma| = 0$. To deal with the case $|\gamma| > 0$, we observe that

$$\begin{aligned}
|L^{\gamma_1}\partial_\theta^{\gamma_2}\partial_t^2 u| \lesssim \cosh(\rho)^2 &\sum_{|\gamma'|+|\gamma''| \leq |\gamma|} |L^{\gamma_1'}\partial_\theta^{\gamma_2'}(AB - C)| \\
&\left| L^{\gamma_1''}\partial_\theta^{\gamma_2''}\left(\frac{1}{1 + \sum_{i=1}^2 \frac{1}{t^2}|L^i u|^2 + |\partial_\theta u|^2 + \cosh(\rho)^2 \sum_{i,j=1}^2 \frac{1}{t^2}\frac{x^i}{t}\frac{x^j}{t} L^i u L^j u}\right)\right|.
\end{aligned}$$

Since $|\gamma| + 4 \leq s + 1$ and $AB - C$ involves at most the second order derivatives, we get the pointwise estimates

$$\begin{aligned}
&|L^{\gamma_1'}\partial_\theta^{\gamma_2'}(AB - C)| \lesssim \frac{1}{\tau\cosh(\rho)}\mathcal{E}_\tau^{\leq s}[u;m], \\
&\left| L^{\gamma_1''}\partial_\theta^{\gamma_2''}\left(\frac{1}{1 + \sum_{i=1}^2 \frac{1}{t^2}|L^i u|^2 + |\partial_\theta u|^2 + \cosh(\rho)^2 \sum_{i,j=1}^2 \frac{1}{t^2}\frac{x^i}{t}\frac{x^j}{t} L^i u L^j u}\right)\right| \\
&\qquad \lesssim \frac{1}{\tau^2}\mathcal{E}_\tau^{\leq s}[u;m]
\end{aligned}$$

when $|\gamma''| > 0$. This implies that

$$|L^{\gamma_1}\partial_\theta^{\gamma_2}\partial_t^2 u| \lesssim \frac{\cosh(\rho)}{\tau}\mathcal{E}_\tau^{\leq s}[u;m].$$

To do induction on $|\gamma|$, we observe that

$$L^{\gamma_1}\partial_t L^{\gamma_2}\partial_t L^{\gamma_3}\partial_\theta^{\gamma_4}u = L^{\gamma_1}L^{\gamma_2}L^{\gamma_3}\partial_\theta^{\gamma_4}\partial_t^2 u + L^{\gamma_1}\partial_t L^{\gamma_2}[\partial_t, L^{\gamma_3}]\partial_\theta^{\gamma_4}u$$
$$+ L^{\gamma_1}[\partial_t, L^{\gamma_2}L^{\gamma_3}]\partial_\theta^{\gamma_4}\partial_t u.$$

Since $[\partial_t, L^{\gamma_3}]$ and $[\partial_t, L^{\gamma_2}L^{\gamma_3}]$ are linear combinations of ∂_t and $\partial_i = \frac{1}{t}L^i - \frac{x^i}{t}\partial_t$, and they decrease the order of the derivatives, the result follows by induction hypothesis with the aid of Lemma 5.3.2. □

Remark. *The operator L^i preserves the decays $\frac{1}{t}$ and $\frac{x^i}{t}$, which means that*

$$L^i\left(\frac{1}{t}f\right) \le \frac{1}{t}(|L^i f| + |f|)$$

and

$$L^i\left(\frac{x^j}{t}f\right) \lesssim |L^i f| + |f|.$$

It also preserves $\cosh(\rho)^2$ by the following way:

$$L^i\left(\cosh(\rho)^2\right) \lesssim \cosh(\rho)^2.$$

Proposition 5.3.1. *Let $s \ge 5$. If $\Box_g u = 0$ and u satisfies the bootstrap assumption 5.4, then*

$$\int_{\Sigma_\tau} |\left([\Box_g, L^{\alpha_1}\partial_\theta^{\alpha_2}]u\right)\left(\partial_t L^{\alpha_1}\partial_\theta^{\alpha_2}u\right)| dvol \lesssim \frac{1}{\tau^2}\mathcal{E}_\tau^{\le s}[u;m]^4$$

for any $|\alpha| = |\alpha_1| + |\alpha_2| \le s$, where the implicit constant depends only on s.

Proof. We decompose the bracket first.

$$[L^{\alpha_1}\partial_\theta^{\alpha_2}, \Box_g]u = [L^{\alpha_1}\partial_\theta^{\alpha_2}, m^{ij}\partial_i\partial_j]u + [L^{\alpha_1}\partial_\theta^{\alpha_2}, (g^{ij} - m^{ij})\partial_i\partial_j]u.$$

The first term on the right hand side vanishes since

$$[L^k, m^{ij}\partial_i\partial_j] = [t\partial_k + x^k\partial_t, -\partial_t^2 + \partial_1^2 + \partial_2^2 + \partial_\theta^2] = 0.$$

The second term on the right hand side is

$$L^{\alpha_1}\partial_\theta^{\alpha_2}\left(\frac{-1}{1 + m^{ab}\partial_a u\partial_b u}m^{ik}m^{jl}\partial_k u\partial_l u\partial_i\partial_j u\right)$$

$$-\left(\frac{-1}{1+m^{ab}\partial_a u\partial_b u}\right) m^{ik}m^{jl}\partial_k u\partial_l u\partial_i\partial_j(L^{\alpha_1}\partial_\theta^{\alpha_2}u)$$

The first term is of the form

$$L^{\gamma_1}\partial_\theta^{\gamma_2}\left(\frac{-1}{1+m^{ab}\partial_a u\partial_b u}\right) L^{\gamma_3}\partial_\theta^{\gamma_4}(m^{ik}m^{jl}\partial_k u\partial_l u\partial_i\partial_j u).$$

By Lemma 5.3.1, the last part becomes

$$L^{\gamma_3}\partial_\theta^{\gamma_4}\left(\Big(\frac{1}{\cosh(\rho)^4}|\partial_t u|^2+\frac{2}{\cosh(\rho)^2}\sum_{i=1}^{2}\frac{1}{t}\frac{x^i}{t}\partial_t u L^i u+\sum_{i,j=1}^{2}\frac{1}{t^2}\frac{x^i}{t}\frac{x^j}{t}L^i u L^j u\Big)(\partial_t^2 u)+C\right).$$

If the $\partial_t^2 u$ absorbs the highest order of derivatives, the integral (neglecting the C term for the moment) would be bounded by

$$\begin{aligned}
&\int_{\Sigma_\tau}\frac{1}{\tau^2\cosh(\rho)^2}\mathcal{E}_\tau^{\leq s}[u;m]^2\left|L^{\gamma_3'}\partial_\theta^{\gamma_4'}\partial_t^2 u\right||\partial_t L^{\alpha_1}\partial_\theta^{\alpha_2}u|dS\\
&\quad\leq\frac{1}{\tau^2}\mathcal{E}_\tau^{\leq s}[u;m]^2\sqrt{\int_{\Sigma_\tau}\frac{1}{\cosh(\rho)^3}|L^{\gamma_3'}\partial_\theta^{\gamma_4'}\partial_t^2 u|^2dS}\sqrt{\int_{\Sigma_\tau}\frac{1}{\cosh(\rho)}|\partial_t L^{\alpha_1}\partial_\theta^{\alpha_2}u|^2dS}\\
&\quad\lesssim\frac{1}{\tau^2}\mathcal{E}_\tau^{\leq s}[u;m]^4,
\end{aligned}$$

where we use (5.3) to replace $\partial_t^2 u$ and Lemma 5.3.2 at the last step.

If the $\partial_t^2 u$ does not involve the highest order of derivatives, the integral (again, neglecting the C term), according to Lemma 5.3.3, is bounded by

$$\begin{aligned}
&\int_{\Sigma_\tau}\frac{1}{\tau\cosh(\rho)^2}\left(|L^{\gamma_3'}\partial_\theta^{\gamma_4'}\partial_t u|+\sum_{i=1}^{2}\frac{1}{\tau}|L^{\gamma_3'}\partial_\theta^{\gamma_4'}L^i u|\right)\frac{\cosh(\rho)}{\tau}\mathcal{E}_\tau^{\leq s}[u;m]^2|\partial_t L^{\alpha_1}\partial_\theta^{\alpha_2}u|dS\\
&\quad=\frac{1}{\tau^2}\mathcal{E}_\tau^{\leq s}[u;m]^2\int_{\Sigma_\tau}\frac{1}{\cosh(\rho)}\left(|L^{\gamma_3'}\partial_\theta^{\gamma_4'}\partial_t u|+\sum_{i=1}^{2}\frac{1}{\tau}|L^{\gamma_3'}\partial_\theta^{\gamma_4'}L^i u|\right)|\partial_t L^{\alpha_1}\partial_\theta^{\alpha_2}u|dS\\
&\quad\leq\frac{1}{\tau^2}\mathcal{E}_\tau^{\leq s}[u;m]^2\sqrt{\int_{\Sigma_\tau}\frac{3}{\cosh(\rho)}\left(|L^{\gamma_3'}\partial_\theta^{\gamma_4'}\partial_t u|^2+\sum_{i=1}^{2}\frac{1}{\tau^2}|L^{\gamma_3'}\partial_\theta^{\gamma_4'}L^i u|^2\right)dS}\\
&\qquad\times\sqrt{\int_{\Sigma_\tau}\frac{1}{\cosh(\rho)}|\partial_t L^{\alpha_1}\partial_\theta^{\alpha_2}u|^2dS}\\
&\quad\lesssim\frac{1}{\tau^2}\mathcal{E}_\tau^{\leq s}[u;m]^4.
\end{aligned}$$

Considering the C term, the integral is bounded by

$$\int_{\Sigma_\tau} \frac{1}{\tau^2\cosh(\rho)} \mathcal{E}_\tau^{\le s}[u;m]^2 \left(|L^{\gamma_3'}\partial_\theta^{\gamma_4'}\partial_t u| + \sum_{i=1}^{2} \frac{1}{\tau} |L^{\gamma_3'}\partial_\theta^{\gamma_4'} L^i u| \right) |\partial_t L^{\alpha_1}\partial_\theta^{\alpha_2} u| dS$$
$$+ \int_{\Sigma_\tau} \frac{1}{\tau^2} \mathcal{E}_\tau^{\le s}[u;m]^2 \left(\sqrt{\cosh(\rho)} |L^{\gamma_3'}\partial_\theta^{\gamma_4'}\partial_\theta u| \right) \left(\frac{1}{\sqrt{\cosh(\rho)}} |\partial_t L^{\alpha_1}\partial_\theta^{\alpha_2} u| \right) dS$$
$$\lesssim \frac{1}{\tau^2} \mathcal{E}_\tau^{\le s}[u;m]^4,$$

where we use the Hölder inequality again as above. □

Remark. *In the above proof, the term*

$$L^{\gamma_1}\partial_\theta^{\gamma_2} \left(\frac{-1}{1 + m^{ab}\partial_a u \partial_b u} \right)$$

is bounded by a constant depending only on s when ϵ is sufficiently small.

Proposition 5.3.2. *Let $s \ge 5$. If $\Box_g u = 0$ and u satisfies the bootstrap assumption 5.4, then*

$$\int_{\Sigma_\tau} |\nabla^i v \nabla_j v \Gamma_{it}^j| dvol \lesssim \frac{1}{\tau^2} \mathcal{E}_\tau^{\le s}[u;m]^4$$

and

$$\int_{\Sigma_\tau} |\nabla^k v \nabla_k v \Gamma_{it}^i| dvol \lesssim \frac{1}{\tau^2} \mathcal{E}_\tau^{\le s}[u;m]^4,$$

where $v = L^{\alpha_1}\partial_\theta^{\alpha_2} u$, $0 \le |\alpha| \le s$, and ∇ is the connection with respect to g.

Proof.

$$\begin{aligned} \nabla^i v \nabla_j v \Gamma_{it}^j &= g^{ik}\partial_k v \partial_j v \left(\frac{1}{2} g^{jl} (\partial_i g_{lt} + \partial_t g_{il} - \partial_l g_{it}) \right) \\ &= \frac{1}{2} g^{ik} g^{jl} \partial_k v \partial_j v \big(\partial_i(\partial_l u \partial_t u) + \partial_t(\partial_i u \partial_l u) - \partial_l(\partial_i u \partial_t u) \big) \\ &= g^{ik} g^{jl} \partial_k v \partial_j v (\partial_i \partial_t u) \partial_l u \end{aligned}$$

Using Lemma 5.1.1, we could decompose the above expression into four terms:

$$m^{ik} m^{jl} \partial_k v \partial_j v (\partial_i \partial_t u) \partial_l u$$

$$\left(-\frac{1}{1+m^{ab}\partial_a u\partial_b u}m^{ip}m^{kq}\partial_p u\partial_q u\right)m^{jl}\partial_k v\partial_j v(\partial_i\partial_t u)\partial_l u$$

$$m^{ik}\left(-\frac{1}{1+m^{ab}\partial_a u\partial_b u}m^{jr}m^{ls}\partial_r u\partial_s u\right)\partial_k v\partial_j v(\partial_i\partial_t u)\partial_l u$$

$$\left(\frac{1}{1+m^{ab}\partial_a u\partial_b u}m^{ip}m^{kq}\partial_p u\partial_q u\right)\left(\frac{1}{1+m^{ab}\partial_a u\partial_b u}m^{jr}m^{ls}\partial_r u\partial_s u\right)\partial_k v\partial_j v(\partial_i\partial_t u)\partial_l u.$$

The last three terms have the desired estimate from the following naive estimates:

$$|\partial u| \lesssim \frac{1}{\tau}\mathcal{E}_\tau^{\le s}[u;m],$$

where ∂u denotes $\partial_t u, \partial_1 u, \partial_2 u$, or $\partial_\theta u$.

For the first term, we have the following expansion:

$$\begin{aligned}
&m^{ik}m^{jl}\partial_k v\partial_l v(\partial_i\partial_t u)\partial_j u = \\
&\quad(\partial_t v)^2(\partial_t^2 u)\partial_t u - \sum_{j=1}^{2}\partial_t v\left(\frac{1}{t}L^j v - \frac{x^j}{t}\partial_t v\right)(\partial_t^2 u)\left(\frac{1}{t}L^j u - \frac{x^j}{t}\partial_t u\right) \\
&\quad - \sum_{i=1}^{2}\left(\frac{1}{t}L^i v - \frac{x^i}{t}\partial_t v\right)\partial_t v\left(\frac{1}{t}L^i(\partial_t u) - \frac{x^i}{t}\partial_t^2 u\right)\partial_t u \\
&\quad + \sum_{i,j=1}^{2}\left(\frac{1}{t}L^i v - \frac{x^i}{t}\partial_t v\right)\left(\frac{1}{t}L^j v - \frac{x^j}{t}\partial_t v\right)\left(\frac{1}{t}L^i(\partial_t u) - \frac{x^i}{t}\partial_t^2 u\right)\left(\frac{1}{t}L^j u - \frac{x^j}{t}\partial_t u\right) \\
&\quad - \partial_t v\partial_\theta v(\partial_t^2 u)\partial_\theta u - \partial_\theta v\partial_t v(\partial_\theta\partial_t u)\partial_t u \\
&\quad + \sum_{i=1}^{2}\left(\frac{1}{t}L^i v - \frac{x^i}{t}\partial_t v\right)\partial_\theta v\left(\frac{1}{t}L^i(\partial_t u) - \frac{x^i}{t}\partial_t^2 u\right)\partial_\theta u \\
&\quad + \sum_{j=1}^{2}\partial_\theta v\left(\frac{1}{t}L^j v - \frac{x^j}{t}\partial_t v\right)(\partial_\theta\partial_t u)\left(\frac{1}{t}L^j u - \frac{x^j}{t}\partial_t u\right) + \partial_\theta v\partial_\theta v(\partial_\theta\partial_t u)\partial_\theta u \\
&= (\partial_t^2 u)\Bigg(\frac{1}{\cosh(\rho)^4}(\partial_t v)^2\partial_t u \\
&\quad + \sum_{j=1}^{2}\left(-\frac{1}{t^2}\partial_t v L^j v L^j u + \frac{1}{t}\frac{x^j}{t}\frac{1}{\cosh(\rho)^2}\partial_t v L^j v\partial_t u + \frac{1}{t}\frac{x^j}{t}\frac{1}{\cosh(\rho)^2}(\partial_t v)^2 L^j u\right) \\
&\quad + \sum_{i=1}^{2}\frac{1}{t}\frac{x^i}{t}\frac{1}{\cosh(\rho)^2}L^i v\partial_t v\partial_t u - \sum_{i,j=1}^{2}\frac{1}{t^3}\frac{x^i}{t}L^i v L^j v L^j u + \sum_{i,j=1}^{2}\frac{1}{t^2}\frac{x^i}{t}\frac{x^j}{t}L^i v L^j v\partial_t u \\
&\quad + \sum_{i,j=1}^{2}\frac{1}{t^2}\frac{x^i}{t}\frac{x^j}{t}L^i v\partial_t v L^j u + \sum_{i,j=1}^{2}\frac{1}{t^2}\frac{x^i}{t}\frac{x^i}{t}\partial_t v L^j v L^j u
\end{aligned}$$

$$
\begin{aligned}
&- \partial_t v \partial_\theta v \partial_\theta u - \sum_{i=1}^{2} \frac{1}{t}\frac{x^i}{t} L^i v \partial_\theta v \partial_\theta u + \sum_{i=1}^{2} \frac{x^i}{t}\frac{x^i}{t} \partial_t v \partial_\theta v \partial_\theta u\Bigg) \\
&- \sum_{i=1}^{2} \frac{1}{t^2} L^i v \partial_t v (L^i \partial_t u)\partial_t u + \sum_{i=1}^{2} \frac{1}{t}\frac{x^i}{t} (\partial_t v)^2 (L^i \partial_t u)\partial_t u \\
&+ \sum_{i,j=1}^{2} \left(\frac{1}{t}L^i v - \frac{x^i}{t}\partial_t v\right)\left(\frac{1}{t}L^j v - \frac{x^j}{t}\partial_t v\right)\left(\frac{1}{t}(L^i \partial_t u)\right)\left(\frac{1}{t}L^j u - \frac{x^j}{t}\partial_t u\right) \\
&- \partial_\theta v \partial_t v (\partial_\theta \partial_t u)\partial_t u + \sum_{i=1}^{2} \left(\frac{1}{t}L^i v - \frac{x^i}{t}\partial_t v\right)\partial_\theta v \left(\frac{1}{t}L^i(\partial_t u)\right)\partial_\theta u \\
&+ \sum_{j=1}^{2} \partial_\theta v \left(\frac{1}{t}L^j v - \frac{x^j}{t}\partial_t v\right)(\partial_\theta \partial_t u)\left(\frac{1}{t}L^j u - \frac{x^j}{t}\partial_t u\right) + \partial_\theta v \partial_\theta v (\partial_\theta \partial_t u)\partial_\theta u,
\end{aligned}
$$

where we combine the terms in the parenthesis after $\partial_t^2 u$ due to its extraordinary decay. Using Lemma 5.3.2, Lemma 5.3.3, and the Hölder inequality, we have proved the first inequality.

Similarly,

$$
\begin{aligned}
\nabla^k v \nabla_k v \Gamma^i_{it} &= g^{kl}\partial_l v \partial_k v \left(\frac{1}{2} g^{ij}(\partial_i g_{jt} + \partial_t g_{ij} - \partial_j g_{it})\right) \\
&= \frac{1}{2} g^{ij} g^{kl} \partial_l v \partial_k v \big(\partial_i(\partial_j u \partial_t u) + \partial_t(\partial_i u \partial_j u) - \partial_j(\partial_i u \partial_t u)\big) \\
&= g^{ij} g^{kl} \partial_l v \partial_k v (\partial_i \partial_t u)\partial_j u.
\end{aligned}
$$

It could be decomposed into the following terms:

$$
\begin{aligned}
& m^{ij} m^{kl} \partial_l v \partial_k v (\partial_i \partial_t u)\partial_j u \\
& \left(-\frac{1}{1 + m^{ab}\partial_a u \partial_b u} m^{ip} m^{jq} \partial_p u \partial_q u\right) m^{kl} \partial_l v \partial_k v (\partial_i \partial_t u)\partial_j u \\
& m^{ij}\left(-\frac{1}{1 + m^{ab}\partial_a u \partial_b u} m^{kr} m^{ls} \partial_r u \partial_s u\right) \partial_l v \partial_k v (\partial_i \partial_t u)\partial_j u \\
& \left(\frac{1}{1 + m^{ab}\partial_a u \partial_b u} m^{ip} m^{jq} \partial_p u \partial_q u\right)\left(\frac{1}{1 + m^{ab}\partial_a u \partial_b u} m^{kr} m^{ls} \partial_r u \partial_s u\right) \partial_l v \partial_k v (\partial_i \partial_t u)\partial_j u.
\end{aligned}
$$

As above, the last three terms have the desired decay. To deal with the first term, observe that

$$
m^{kl}\partial_k v \partial_l v = -(\partial_t v)^2 + \sum_{i=1}^{2} \left(\frac{1}{t}L^i v - \frac{x^i}{t}\partial_t v\right)\left(\frac{1}{t}L^i v - \frac{x^i}{t}\partial_t v\right) + (\partial_\theta v)^2
$$

$$= -\frac{1}{\cosh(\rho)^2}(\partial_t v)^2 + \frac{1}{t^2}(|L^1 v|^2 + |L^2 v|^2) - 2\sum_{i=1}^{2}\frac{1}{t}\frac{x^i}{t}L^i v\partial_t v + (\partial_\theta v)^2$$

and

$$\begin{aligned} m^{ij}(\partial_i\partial_t u)\partial_j u &= -(\partial_t^2 u)\partial_t u + \sum_{i=1}^{2}\left(\frac{1}{t}L^i(\partial_t u) - \frac{x^i}{t}\partial_t^2 u\right)\left(\frac{1}{t}L^i u - \frac{x^i}{t}\partial_t u\right) \\ &\quad + (\partial_\theta\partial_t u)\partial_\theta u \\ &= -\frac{1}{\cosh(\rho)^2}(\partial_t^2 u)\partial_t u + \sum_{i=1}^{2}\frac{1}{t^2}(L^i\partial_t u)L^i u - \sum_{i=1}^{2}\frac{1}{t}\frac{x^i}{t}(L^i\partial_t u)\partial_t u \\ &\quad - \sum_{i=1}^{2}\frac{1}{t}\frac{x^i}{t}(\partial_t^2 u)L^i u + (\partial_\theta\partial_t u)\partial_\theta u. \end{aligned}$$

Using Lemma 5.3.2, Lemma 5.3.3, and the Hölder inequality, we obtain the desired decay. □

5.4 Energy Comparability

In this section, we aim to show the comparability of the two types of energy. The main reference is [1].

Proposition 5.4.1. *If u satisfies the bootstrap assumption 5.4, then $\mathcal{E}_\tau^{\le s}[u;m]$ and $\mathcal{E}_\tau^{\le s}[u;g]$ are comparable for $2 \le \tau_1 \le \tau \le \tau_2$.*

From now on, we would always assume the bootstrap assumption 5.4. In order to prove this result, we need some geometric computations. In this section, we denote the matrix associated to the metric g_{ij} by G, the matrix associated to the metric m_{ij} by M, and the vector $[v_0\partial_t + v_1\partial_1 + v_2\partial_2 + v_3\partial_\theta]$ by

$$\begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

We begin by deriving the explicit expression for normal vectors on Σ_τ with respect to m and g.

Lemma 5.4.1. *For each surface Σ_τ, the normal vectors with respect to m and g are*

$$[\vec{n}_m] = M^{-1}w$$

$$[\vec{n}_g] = \frac{1}{\sqrt{-w^T G^{-1} w}} G^{-1} w,$$

where

$$w = \begin{bmatrix} -\cosh(\rho) \\ \sinh(\rho)\cos\phi \\ \sinh(\rho)\sin\phi \\ 0 \end{bmatrix}.$$

Remark. $w^T M^{-1} w = -1$.

Proof. From

$$[\partial_\rho] = \begin{bmatrix} \tau\sinh(\rho) \\ \tau\cosh(\rho)\cos\phi \\ \tau\cosh(\rho)\sin\phi \\ 0 \end{bmatrix}$$

and

$$[\partial_\phi] = \begin{bmatrix} 0 \\ -\tau\sinh(\rho)\sin\phi \\ \tau\sinh(\rho)\cos\phi \\ 0 \end{bmatrix},$$

we could check that $[\partial_\rho]^T G[\vec{n}_g] = 0$ and $[\partial_\phi]^T G[\vec{n}_g] = 0$. Furthermore, $[\vec{n}_g]^T G[\vec{n}_g] = -1$ implies that $[\vec{n}_g]$ is the desired (past-pointing) normal vector. □

Note that the bootstrap assumption 5.4 ensures the negativity of $w^T G^{-1} w$. Intuitively, when the ϵ in the bootstrap assumption 5.4 is sufficiently small, $w^T G^{-1} w$ will be close to $w^T M^{-1} w = -1$, as proved in the following lemma. From the definition of ${}^{(X)}J$, we have

$$\left[{}^{(\partial_t)}J[v;g]\right] = \begin{bmatrix} \nabla^t v \nabla_t v - \frac{1}{2}(\nabla^k v \nabla_k v) \\ \nabla^1 v \nabla_t v \\ \nabla^2 v \nabla_t v \\ \nabla^\theta v \nabla_t v \end{bmatrix}.$$

With the aid of Lemma 5.4.1, we are able to compute the product:

$$\begin{aligned}\langle {}^{(\partial_t)}J[v;g], \vec{n}_g\rangle_g &= \left[{}^{(\partial_t)}J[v;g]\right]^T G\left(\frac{1}{\sqrt{-w^T G^{-1} w}} G^{-1} w\right) \\ &= \frac{1}{\sqrt{-w^T G^{-1} w}}\Bigg(\frac{1}{2}(\nabla^t v \nabla_t v - \nabla^1 v \nabla_1 v - \nabla^2 v \nabla_2 v - \nabla^\theta v \nabla_\theta v)(-\cosh(\rho)) \\ &\qquad + (\nabla^1 v \nabla_t v)(\sinh(\rho)\cos\phi) + (\nabla^2 v \nabla_t v)(\sinh(\rho)\sin\phi)\Bigg),\end{aligned}$$

where the connection ∇ is with respect to g.

Lemma 5.4.2. *Let $v = L^{\alpha_1}\partial_\theta^{\alpha_2} u$, where $|\alpha| \le s$. Then*

$$\langle {}^{(\partial_t)}J[v;g], \vec{n}_g\rangle_g$$

and

$$\langle {}^{(\partial_t)}J[v;m], \vec{n}_m\rangle_m$$

are comparable provided that the ϵ in the bootstrap assumption 5.4 is sufficiently small.

Proof. We are going to show that

1. $w^T G^{-1} w$ and $w^T M^{-1} w$ are comparable.

2. The two versions, with respect to g and m, of

$$\begin{aligned}&\frac{1}{2}(\nabla^t v \nabla_t v - \nabla^1 v \nabla_1 v - \nabla^2 v \nabla_2 v - \nabla^\theta v \nabla_\theta v)(-\cosh(\rho)) \\ &\qquad + (\nabla^1 v \nabla_t v)(\sinh(\rho)\cos\phi) + (\nabla^2 v \nabla_t v)(\sinh(\rho)\sin\phi)\end{aligned}$$

   are comparable.

For the first claim, observe that

$$\begin{aligned}\left|w^T G^{-1} w - w^T M^{-1} w\right| &= \left|-\frac{1}{1 + m^{ab}\partial_a u \partial_b u} m^{ik} m^{jl}(\partial_k u \partial_l u) w_i w_j\right| \\ &\lesssim \frac{1}{\tau^2}\mathcal{E}_\tau^{\le s}[u;m]^2(\cosh(\rho)^2) \\ &\le \mathcal{E}_\tau^{\le s}[u;m]^2.\end{aligned}$$

Using the fact that $w^T M^{-1} w = -1$, we establish the first claim. For the second claim, by using Lemma 5.1.1, it is sufficient to show that the following terms (approximately the difference of the two versions) are relatively small compared to the m version:

$$\Big(\frac{1}{2}(m^{tk}m^{pl}\partial_k u\partial_l u\partial_p v\partial_t v - m^{1k}m^{pl}\partial_k u\partial_l u\partial_p v\partial_1 v - m^{2k}m^{pl}\partial_k u\partial_l u\partial_p v\partial_2 v)$$
$$\times(-\cosh(\rho))\Big)$$
$$+ (m^{1k}m^{pl}\partial_k u\partial_l u\partial_p v\partial_t v)(\sinh(\rho)\cos\phi)$$
$$+ (m^{2k}m^{pl}\partial_k u\partial_l u\partial_p v\partial_t v)(\sinh(\rho)\sin\phi),$$

which is bounded by

$$\frac{\cosh(\rho)}{\tau^2}\mathcal{E}_\tau^{\leq s}[u;m]^2(|\partial_t v|^2 + |\partial_1 v|^2 + |\partial_2 v|^2 + |\partial_\theta v|^2)$$

up to a constant. On the other hand, the m version is

$$\frac{1}{2}(-|\partial_t v|^2 - |\partial_1 v|^2 - |\partial_2 v|^2 - |\partial_\theta v|^2)(-\cosh(\rho))$$
$$+ (\partial_1 v\partial_t v)(\sinh(\rho)\cos\phi) + (\partial_2 v\partial_t v)(\sinh(\rho)\sin\phi)$$
$$\geq \frac{1}{2}\cosh(\rho)(|\partial_t v|^2 + |\partial_1 v|^2 + |\partial_2 v|^2 + |\partial_\theta v|^2)$$
$$-\frac{1}{2}\sinh(\rho)\Big(|\partial_t v|^2(\cos^2\phi) + |\partial_1 v|^2 + |\partial_t v|^2(\sin^2\phi) + |\partial_2 v|^2\Big)$$
$$\geq \frac{1}{4}\frac{1}{\cosh(\rho)}(|\partial_t v|^2 + |\partial_1 v|^2 + |\partial_2 v|^2 + |\partial_\theta v|^2).$$

Therefore, as long as $\mathcal{E}_\tau^{\leq s}[u;m]$ is small enough, the second claim holds. □

Remark. *In the above proof, we use a rough estimate*

$$|\partial u| \lesssim \frac{1}{\tau}\mathcal{E}_\tau^{\leq s}[u;m],$$

where ∂ may be $\partial_t, \partial_1, \partial_2$, or ∂_θ. This is a rough version of Lemma 5.3.2.

Lemma 5.4.3. *The two versions of volume form,*

$$dvol_{(\Sigma_\tau;g)}$$

and

$$dvol_{(\Sigma_\tau;m)},$$

are comparable provided that the ϵ *in the bootstrap assumption 5.4 is sufficiently small.*

Proof. We will denote the column vector

$$\begin{bmatrix} \partial_t u \\ \partial_1 u \\ \partial_2 u \\ \partial_\theta u \end{bmatrix}$$

by v, as in the proof of Lemma 5.1.1. Observe that

$$\begin{aligned} \det G &= \det\left((I + vv^T M)M\right) \\ &= (1 + v^T M v)\det(M) \end{aligned}$$

and

$$|v^T M v| = |m^{ab}\partial_a u \partial_b u| \lesssim \frac{1}{\tau^2}\mathcal{E}_\tau^{\leq s}[u;m],$$

we have the desired result provided that $\mathcal{E}_\tau^{\leq s}[u;m]$ is sufficiently small. □

Proof of Proposition 5.1. It follows from Lemma 5.4.2 and Lemma 5.4.3. □

5.5 Global Existence

We now prove the energy estimate that is essential to our paper. We argue that the integrand in the divergence theorem can be estimated by an integrable function of τ over $[2,\infty)$, and thus prove the claim.

Theorem 5.5.1. *We have the energy inequality*

$$\max_{\tau_1\leq\tau\leq\tau_2} \mathcal{E}_\tau^{\leq s}[u;g] \lesssim \left(\mathcal{E}_{\tau_1}^{\leq s}[u;g] + \max_{\tau_1\leq\tau\leq\tau_2} \mathcal{E}_\tau^{\leq s}[u;g]^3\right)$$

for any $\tau_2 \geq \tau_1 \geq 2$.

Proof. Let $v = L^{\alpha_1}\partial_\theta^{\alpha_2} u$ with $|\alpha| \leq s$. We have

$$\begin{aligned}
&\frac{1}{2}\mathcal{E}_{\tau_2'}[v;g]^2 - \frac{1}{2}\mathcal{E}_{\tau_1}[v;g]^2 \\
&\quad = \int_{\Sigma_{\tau_2'}} \langle {}^{(\partial_t)}J, \vec{n}_g\rangle_g dS_g - \int_{\Sigma_{\tau_1}} \langle {}^{(\partial_t)}J, \vec{n}_g\rangle_g dS_g \\
&\quad = \int_{\tau_1}^{\tau_2'} \int_{\Sigma_\tau} div_g({}^{(\partial_t)}J) \frac{1}{\sqrt{-\langle \nabla\tau, \nabla\tau\rangle_g}} dS_g d\tau \\
&\quad \lesssim \int_{\tau_1}^{\tau_2'} \int_{\Sigma_\tau} \left| \Box_g v(\nabla_t v) + (\nabla^i v \nabla_j v)\Gamma_{it}^j - \frac{1}{2}(\nabla^k v \nabla_k v)\Gamma_{it}^i \right| dS_g d\tau \\
&\quad \lesssim \int_{\tau_1}^{\tau_2'} \frac{1}{\tau^2}\mathcal{E}_\tau^{\leq s}[u;m]^4 d\tau \\
&\quad \leq \frac{1}{2}\max_{\tau_1 \leq \tau \leq \tau_2} \mathcal{E}_\tau^{\leq s}[u;m]^4 \\
&\quad \lesssim \frac{1}{2}\max_{\tau_1 \leq \tau \leq \tau_2} \mathcal{E}_\tau^{\leq s}[u;g]^4
\end{aligned}$$

for every $\tau_1 \leq \tau_2' \leq \tau_2$. The first $\lesssim$ follows from the fact that

$$\begin{aligned}
\langle \nabla\tau, \nabla\tau\rangle_g &= g^{ij}\partial_i\tau\partial_j\tau \\
&= m^{ij}\partial_i\tau\partial_j\tau - \left(\frac{1}{1 + m^{ab}\partial_a u \partial_b u}\right) m^{ik}m^{jl}\partial_k u \partial_l u \partial_i \tau \partial_j \tau \\
&= -1 - \left(\frac{1}{1 + m^{ab}\partial_a u \partial_b u}\right)\left((\partial_t u)^2\left(\cosh(\rho) - \frac{\sinh(\rho)^2}{\cosh(\rho)}\right)^2\right. \\
&\qquad \left. + \sum_{j=1}^{2}(\partial_t u)(L^j u)\left(\frac{2x^j}{\tau^2} - \frac{|x|^2}{t^2}\frac{2x^j}{\tau^2}\right) + \sum_{i,j=1}^{2}\frac{1}{t^2}\frac{x^i}{\tau}\frac{x^j}{\tau}L^i u L^j u\right) \\
&\approx -1
\end{aligned}$$

provided that ϵ is small enough, the second $\lesssim$ follows from **Proposition 4.1** and **Proposition 4.2**, and the last $\lesssim$ follows from **Proposition 5.1**. This implies that

$$\max_{\tau_1 \leq \tau \leq \tau_2} \mathcal{E}_\tau^{\leq s}[u;g]^2 \lesssim \left(\mathcal{E}_{\tau_1}^{\leq s}[u;g]^2 + \max_{\tau_1 \leq \tau \leq \tau_2}\mathcal{E}_\tau^{\leq s}[u;g]^4\right),$$

which gives the desired result. ◻

It is clear that in the above proof, the implicit constants for $\lesssim$ do not depend on τ_2 thanks to the integrability of $\frac{1}{\tau^2}$. Therefore, we have the following corollary. There exists a constant $C_2 > 0$ so that the energy inequality

$$\max_{2\le\tau\le\tau_2} \mathcal{E}_\tau^{\le s}[u;m] \le C_s \left(\mathcal{E}_2^{\le s}[u;m] + \max_{2\le\tau\le\tau_2} \mathcal{E}_\tau^{\le s}[u;m]^3 \right)$$

holds for every $\tau_2 \ge 2$.

Theorem 5.5.2. *There exists an* $\epsilon_0 > 0$ *so that the equation* (5.1)

$$\partial_i \left(\frac{m^{ij}\partial_j u}{\sqrt{1 + m^{kl}\partial_k u \partial_l u}} \right) = 0$$

has a global solution in $\mathbb{R}^{1,2} \times \mathbb{T}^1$ *provided that*

$$\mathcal{E}_2^{\le s}[u;m] \le \epsilon_0.$$

Proof. It is sufficient to show that there exists an ϵ_0 so that

$$\mathcal{E}_\tau^{\le s}[u;m] \le 2C_s\epsilon_0$$

for $\tau \ge 2$ since if this is true, Lemma 5.3.2 and Lemma 5.3.3 imply that

$$|\partial\partial u| + |\partial u| \lesssim \mathcal{E}_\tau^{\le s}[u;m] \le 2C_s\epsilon_0,$$

where ∂ may be $\partial_t, \partial_1, \partial_2$ or ∂_θ, and therefore the solution can be continued according to the standard local well-posedness results. To show the boundedness of $\mathcal{E}_\tau^{\le s}[u;m]$, it is sufficient to show that

$$\max_{2\le\tau\le\tau_2} \mathcal{E}_\tau^{\le s}[u;m] \le 4C_s\epsilon_0$$

implies

$$\max_{2\le\tau\le\tau_2} \mathcal{E}_\tau^{\le s}[u;m] \le 2C_s\epsilon_0.$$

If the ϵ_0 is chosen to be small enough so that $\epsilon = 4C_s\epsilon_0$ satisfies the small requirement in all the previous lemmas and propositions, we have (by Corollary 5.5)

$$\begin{aligned} \max_{\tau_1\le\tau\le\tau_2} \mathcal{E}_\tau^{\le s}[u;m] &\le C_s(\epsilon_0 + (4C_s\epsilon_0)^3) \\ &\le C_s(\epsilon_0 + \epsilon_0) \end{aligned}$$

provided that

$$\epsilon_0^2 \le \frac{1}{4^3 C_s^3}.$$

Therefore we close the boostrap argument. □

CHAPTER 6

CONCLUSIONS

I will summarize my progress and compare our results with others in this chapter. My work is restricted to warped product spacetimes.

In Section 1.2, we consider the **homogeneous** Einstein-Fluid equations with the equation of state $p = \gamma\rho$ where $\sqrt{\gamma}$ is the sound speed within 3 ranges : $\gamma = 0$ (dust case), $0 < \gamma < 1$ (fluid case), and $\gamma = 1$ (stiff-fluid case). In each case, we classify all the physical solutions that have a Big Bang singularity and derive the asymptotes of the unknowns close to the singularity $r = r_*$. In the fluid case, r_* may be positive ($r_* > 0$) or zero ($r_* = 0$, such as the Friedmann-Lemaître-Robertson-Walker spacetime).

In Chapter 3, we investigate the stability of those $r_* > 0$ homogeneous solutions under **nonhomogeneous**, compactly supported perturbations on the initial slice $\{r = r_0\}$. We proved that there exists a sequence of initial perturbations that goes to 0 in $W^{1,\infty}$ so that each perturbation generates a shock before the Big Bang $r = r_*$. In other words, these $r_* > 0$ homogeneous solutions are unstable.

In Chapter 4, we consider Euler's equations in a special relativity setting. That is, we do not consider the full Einstein-Fluid equations; instead, we only focus on half of the system, dropping the feedback from fluid variables to the metric. We assume the metric is a fixed function of time, and consider the dynamic evolution of fluid variables. Surprisingly, these $r_* > 0$ models are stable under this special relativity setting. This means that the mechanism for generating shocks not only relies on the structure of the fluid equations (specially relativistic fluids), but also involves the evolution of metric components (Einstein-Fluid equations).

In Chapter 5, we investigate the stability of the membrane equation, an equation for the vanishing mean curvature. We consider the space $\mathbb{R}^{1,2} \times \mathbb{T}^1$ involving a compact factor $\mathbb{T}^1$ and apply the standard vector field method with the modification for $\mathbb{T}^1$. It turns out the extra compact factor does not hurt the integrability of the coefficients and thus the energy remains small throughout the time, estabilishing the global existence.

Regarding the stability of the Big Bang, in [24], Rodnianski and Speck proved the stability of Friedmann-Lemaître-Robertson-Walker spacetimes with certain topology, governed by Einstein-scalar field equations. In [11], Fournodavlos, Rodnianski, and Speck proved the stability of Kasner solutions, governed by Einstein-vaccum or Einstein-scalar field equations. In both papers, they do not consider Einstein-Fluie equations.

For the shock formation trend, Riemann introduced the concept of Riemann invariants in [23]. He considered an isentropic fluid with the plane symmetry and used the Riemann invariants to prove that shocks can form from smooth initial data. In [15], John proposed a more general condition, genuinely nonlinear condition, as a sufficient condition for shocks to form for a one-dimensional hyperbolic system without source terms. He used the total variations of the unknowns to control the solution and used the Riccati structure to prove the existence of shocks, which is introduced in our Chapter 2. The first clear picture about specailly relativistic fluids was established in [6] by Christodoulou. He provided sharp sufficient conditions to generate shocks for 3-dimensional relativistic fluids and geometric information of the boundary of maximally extended classical solutions. In [22], Rendall and Stahl proved the existence of shocks for a large class of solutions governed by Einstein-Fluid equations under plane symmetry.

BIBLIOGRAPHY

[1] Leonardo Abbrescia, Willie Wai-Yeung Wong, *Global nearly-plane-symmetric solutions to the membrane equation*, Submitted (2019), arXiv:1903.03553v2

[2] Xinliang An, Willie Wai Yeung Wong, *Warped product space-times*, Classical and Quantum Gravity, Volume 35, Number 2 (2018).

[3] Simon Brendle, *Hypersurfaces in Minkowski Space with vanishing mean curvature*, Comm. Pure Appl. Math. 55 (2002), no. 10, 1249–1279. MR1912097

[4] Yvonne Choquet-Bruhat, *General relativity and Einstein's equations*, Oxford University Press (2009).

[5] Demetrios CHristodoulou, *The problem of a self-gravitating scalar field*, Commun. Math. Phys. 105, 337–361 (1986).

[6] Demetrios Christodoulou, *The formation of shocks in 3-Dimensional fluids*, EMS Monographs in Mathematics (2007).

[7] Demetrios Christodoulou, Andre Lisibach, *Shock development in spherical symmetry*, Annals of PDE 2 (2015), 1–246.

[8] Demetrios Christodoulou, Andre Lisibach, *The Euler equations of compressible fluid flow*, American Mathematical Society Vol. 44, No. 4 (2007), 581–602.

[9] Mihalis Dafermos, *Spherically symmetric spacetimes with a trapped surface*, Classical Quant. Grav. 22.11 (2005), 2221–2232.

[10] Lawrence C. Evans, *Partial differential equations*, 2nd edition, American Mathematical Society, Vol. 19 (2010).

[11] Grigorios Fournodavlos, Igor Rodnianski, Jared Speck, *Stable Big Bang formation for Einstein's equations: The complete sub-critical regime*, Journal of the American Mathematical Society 36 (2023), 827–916.

[12] Jens Hoppe, *Relativistic membranes*, J. Phys. A 46 (2013), no. 2. MR3005897

[13] Lars Hörmander, *Lectures on Nonlinear Hyperbolic Differential Equations*, Springer Verlag (1997).

[14] Mihaela Ifrim, Annalaura Stingo, *Almost global well-posedness for quasilinear strongly coupled wave-Klein-Gordon systems in two space dimensions*, Submitted (2019).

[15] Fritz John, *Formation of singularities in one-dimensional nonlinear wave propagation*, Communications on pure and applied mathematics (1974), Volume 27, 377—405.

[16] Fritz John, *Blow-up for quasi-linear wave equations in three space dimensions*, Communications on pure and applied mathematics (1981), Volume 34, 29–51.

[17] Tosio Kato, *The Cauchy problem for quasi-linear symmetric hyperbolic systems*, Archive for Rational Mechanics and Analysis (1975), Volume 58, 181–205.

[18] Klainerman Sergiu, *The null condition and global existence to nonlinear wave equations*, Lectures in Appl. Math., 23, Amer. Math. Soc. 293–326(1986). MR0837683

[19] Philippe G. LeFloch, Yue Ma *The hyperboloidal foliation method*,

[20] Hans Lindblad, *A remark on Global existence for small initial data of the minimal surface equation in Minkowskian space time*. Proc. Amer. Math. Soc. 132(2004), no. 4, 1095-1102. MR2045426 (2005a:35203)

[21] Eric Poisson, Clifford M. Will *Gravity*, Cambridge University Press (2014).

[22] Alan D. Rendall, Fredrik Stahl *Shock waves in plane symmetric spacetimes*, Comm. Partial Differential Equations 33, no. 10-–12 (2008).

[23] Bernhard Riemann *The propagation of planar air waves of finite amplitude*, Abhandlungen der Gesellschaft der Wissenschaften zu Göttingen, Mathematischphysikalische Klasse 8, 43 (1858-59).

[24] Igor Rodnianski, Jared Speck *Stable Big Bang formation in near-FLRW solutions to the Einstein-scalar field and Einstein-stiff fluid systems*, Selecta Math. (N.S.) 24 (2018), no. 5, 4293—4459.

[25] Christopher D. Sogge, *Lectures on Non-Linear Wave Equations*, International Press of Boston, Incorporated; 2nd Revised edition (2013).

[26] Eleuterio F. Toro, *Riemann Solvers and Numerical Methods for Fluid Dynamics*, Springer Berlin, Heidelberg XXIV 724 (2010).

[27] Willie Wai-Yeung Wong, *Singularities of axially symmetric time-like minimal submanifolds in Minkowski space*, Sumitted (2016), arXiv:1607.06846.

[28] Willie Wai-Yeung Wong, *Small data global existence and decay for two dimensional wave maps*, Sumitted (2017), arXiv:1712.07684.