HEEGAARD FLOER D-INVARIANTS AND ITS APPLICATIONS

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ABSTRACT

This thesis studies Heegaard Floer d-invariants and its applications. The first result is applying d-invariants and linking form obstruction to prove an algebraically slice linear combination of L-space knots is not smoothly slice. The second result is the computation of d-invariants of splicing of circle bundles of higher genus surface.

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CHAPTER 1

INTRODUCTION

In the early 2000s, Ozsváth and Szabó introduced a collection of invariants for 3- and 4-manifolds called Heegaard Floer homology [OS04b][OS04a]. Given a based 3-manifold (Y, z) with a Spin^c structure $\mathfrak{s} \in \mathrm{Spin}^c(Y)$, Heegaard Floer homology associates $\mathbb{F}_2[U]$-modules with different flavors

$$HF^-(Y, z, \mathfrak{s}), \qquad HF^+(Y, z, \mathfrak{s}), \qquad HF^\infty(Y, z, \mathfrak{s}), \qquad \widehat{HF}(Y, z, \mathfrak{s}).$$

Heegaard Floer theory has been generalized in various senses and provides numerous invariants for the study of 3- and 4-dimensional manifolds, as well as contact and symplectic topology and knots. In this thesis, we will explore the Heegaard Floer d-invariants, which are invariants that encode the grading information of 3- and 4-dimensional manifolds and have various applications. For example, they have been used to reprove Donaldson's diagonalizable theorem in [OS03a, Theorem 9.1] and to study cosmetic surgery in [NW15].

We will talk about several computations of the d-invariants on certain manifolds and their applications.

1.1 Nonsliceness of algebraically slice knots

In [Rud76], Rudolph asks whether the set of algebraic knots is linearly independent in the knot concordance group C. An algebraic knot is, by definition, the connected link of an isolated singularity of a polynomial map $f : \mathbb{C}^2 \to \mathbb{C}$. It can also be defined as an iterated torus knot $T_{p_1,q_1;\cdots;p_n,q_n}$ with indices satisfying $p_i, q_i > 0$ and $q_{i+1} > p_i q_i p_{i+1}$.

A knot is algebraically slice if it is in the kernel of Levine's classifying homomorphism [Lev69]. Livingston and Melvin [LM83] observed that, for any knot K,

$$K_{p,q_1} \# T_{p,q_2} \# - K_{p,q_2} \# - T_{p,q_1}$$

is an algebraically slice knot. Here, $K_{p,q}$ is the (p, q)-cable of K.

When K is an algebraic knot, all the components in the above connected sum, up to mirror images, are algebraic knots provided the q_i's are large enough. Since the sliceness of a knot implies that it is algebraically slice, it is interesting to ask when the knot above is slice. In [KHL12], Hedden, Kirk and Livingston used Casson-Gordon invariants [CG86] to show that

Theorem 1.1 ([KHL12]). *For appropriately chosen integers q_i,*

$$T_{2,3;2,q_n}\#T_{2,q_1}\#-T_{2,3;2,q_1}\#-T_{2,q_n}$$

are not slice.

Using the metabelian Blanchfield pairing, introduced by Miller and Powell [MP18], this result is generalized by Conway, Kim, and Politarczyk [CKP23] to the following theorem.

Theorem 1.2 ([CKP23]). *Fix a prime power p. Let $\mathcal{S}_p$ be the set of iterated torus knots $T(p,q_1;p,q_2;\cdots;p,q_l)$, where the sequences $(q_1,q_2,\cdots,q_l)$ of positive integers that are coprime to p satisfy*

1) q_l is a prime;

2) for $i=1,\cdots,l-1$, the integer q_i is coprime to q_l when $l>1$;

The set $\mathcal{S}_p$ is linearly independent in the topological knot concordance group C^{top}.

In this thesis, we use the d-invariants obstruction from [HLR12] to show that for any L space knot K,

Theorem 1.3 ([Zha23a]). *$K_{2,k_1}\#T_{2,k_2}\#-K_{2,k_2}\#-T_{2,k_1}$ has infinite order in the knot concordance group C when k_1 and k_2 are any pair of distinct prime numbers greater than 3.*

As a corollary, we obtain a generalization of Theorem 1.1:

Corollary 1.3.1 ([Zha23a]). *$K_{2,k_1}\#T_{2,k_2}\#-K_{2,k_2}\#-T_{2,k_1}$ has infinite order in the knot concordance group C when K_{2,k_1} and K_{2,k_2} are algebraic knots.*

Note that all results about the nonsliceness of $K_{2,k_1}\#T_{2,k_2}\#-K_{2,k_2}\#-T_{2,k_1}$ have used Casson-Gordon invariants and our result is the first to use the Heegaard Floer homology. Moreover, the result of Conway, Kim and Politarczyk have the same p in the iterated cabling, whereas our results can cover cases when we use different p values in the iterated cabling.

1.2 d-invariants of splicing of circle bundles on higher genus surface

We compute the d-invariants of the splicing of circle bundles on higher genus surface in the pursuit of a Heegaard Floer proof of the 10/8 theorem. In 1982, Matusumoto conjectured that if M is a closed spin manifold, then $b_2(M) \geq (11/8)|\sigma(M)|$. This is known as the 11/8 conjecture. Here $b_2(M)$ represents the second Betti number and $\sigma(M)$ denotes the signature. Furuta proved the 10/8 theorem by studying the Pin(2) Seiberg-Witten theory [Fur01], which states that $b_2(M) \geq (10/8)|\sigma(M)| + 2$.

Using Heegaard-Floer homology, in collaboration with Hedden, we relate the proof of the 10/8 theorem to the computation of d-invariants of certain manifolds illustrated in Figure 1.1, which is the boundary of a 4 manifold with intersection form

$$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Figure 1.1 is the Kirby diagram of the manifolds H_{g_1,g_2}. Here, g_i denotes the genus of each surface and this graph denotes the splicing of two circle bundles with Euler number 0.

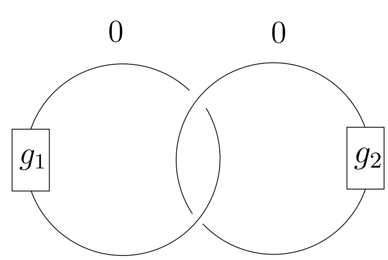


Figure 1.1 H_{g_1,g_2}

We compute d invariants using the link surgery formula from [MO10, Theorem 1.1].

Theorem 1.4 ([Zha23b]). $d_{\text{top}}(H_{g_1,g_2}) = |g_1 - g_2| - 2$*, when* $\min(g_1, g_2) \geq 1$.

More specifically, we decompose H_{g_1,g_2} into three components, using Zemke's bordered link surgery description along with the connected sum formula, to derive the link surgery complex for the framed link in Figure 1.1.

1.3 Outline

The thesis is organized as follows. In Chapter 2, we review Heegaard Floer homology, including the definition of d-invariants and the statement of link surgery formula. In Chapter 3, we give the proof of Theorem 1.3, which includes a discussion of the topology of branched covers and the computation of d-invariants. In Chapter 4, we relate the $10/8$ theorem to d-invariants, review Zemke's general surgery formula, and use it to compute the d-invariants. In Appendix **??**, we apply lattice homology to prove the special case of Theorem 1.3, where $K = T_{2,3}$.

CHAPTER 2

BACKGROUND

2.1 Preliminaries on Heegaard Floer homology and correction term

2.1.1 Heegaard Floer homology

Given a closed, oriented, based 3-manifold Y with a Spin^c structure $\mathfrak{s}$, Ozsváth and Szabó defined an invariant known as Heegaard Floer homology in [OS04b]. These invariants are $\mathbb{F}_2[U]$-modules, which come in different flavors, $\widehat{HF}$, HF^-, HF^∞ and HF^+.

In fact, these groups are related by functorially associated long exact sequences:

$$\cdots \to \widehat{HF}(Y,\mathfrak{s}) \xrightarrow{\hat{\iota}} HF^+(Y,\mathfrak{s}) \xrightarrow{U} HF^+(Y,\mathfrak{s}) \to \cdots$$

and

$$\cdots \to HF^-(Y,\mathfrak{s}) \xrightarrow{\iota} HF^\infty(Y,\mathfrak{s}) \xrightarrow{\pi} HF^+(Y,\mathfrak{s}) \to \cdots$$

Later in this thesis, we will use HF° to denote the Heegaard Floer homology when we do not specify the flavor. We will define HF° in Section 2.2, where we define the link Floer homology, and treat the Heegaard Floer homology of closed 3-manifolds as a special case of link Floer homology, such that there is only one basepoint on the Heegaard diagram.

When W is a smooth cobordism from a three-manifold Y_1 to Y_2, equipped with a Spin^c structure $\mathfrak{s}$ whose restrictions to the two boundary components are $\mathfrak{s}_1$ and $\mathfrak{s}_2$, respectively, then there are induced maps between the Heegaard Floer homology:

$$HF^\circ(Y_1,\mathfrak{s}_1) \xrightarrow{F^\circ_{W,\mathfrak{s}}} HF^\circ(Y_1,\mathfrak{s}_2).$$

Note that in the case where $\mathfrak{s}$ is a torsion Spin^c structure on Y, $HF^\circ(Y,\mathfrak{s})$ can be endowed with a relative $\mathbb{Z}$ grading. In Theorem 7.1 of [OS06], it has been shown that $HF^\circ(Y,\mathfrak{s})$ can be given an absolute $\mathbb{Q}$ grading $\widetilde{\mathrm{gr}}$ which lifts the relative $\mathbb{Z}$ grading. It is uniquely characterized by the following properties:

1) $\hat{\iota}$, ι and π above preserve the absolute grading

2) $\widehat{HF}(S^3)$ is supported in absolute grading zero

3) If W is a cobordism from Y_1 to Y_2, and $\xi \in HF^\infty(Y_1, \mathfrak{s}_1)$, then

$$\tilde{\mathrm{gr}}(F_{W,\mathfrak{s}}(\xi)) - \tilde{\mathrm{gr}}(\xi) = \tfrac{c_1(\mathfrak{s})^2 - 2\chi(W) - 3\sigma(W)}{4},$$

where $\mathfrak{s}_i = \mathfrak{s}|Y_i$ for $i = 1, 2$.

2.1.2 Correction terms

The definition of the Heegaard Floer correction terms depends on the structure of $HF^\infty(Y, \mathfrak{s})$. When $b_1(Y) = 0$, $HF^\infty(Y, \mathfrak{s}) = \mathbb{F}[U, U^{-1}]$ [OS04a, Theorem 10.1]. With the absolute grading, Ozsváth and Szabó defined a numerical invariant called the *correction term*, denoted by $d(Y, \mathfrak{s})$ [OS03a, Definition 4.1]:

Definition 2.1. $d(Y, \mathfrak{s}) = \min_{\alpha \neq 0 \in HF^+(Y,\mathfrak{s})}\{\tilde{\mathrm{gr}}(\alpha) | \alpha \in \mathrm{Im} U^k, \text{ for all } k \geq 0\}$.

The above definition is equivalent to the original definition provided in [OS03a], as both are the grading of the bottom element of the unique non-torsion tower. The terminology *correction term* reflects that $d(Y, \mathfrak{s})$ is the correction term in the formula [OS03a, Theorem 1.3], which relates the Euler characteristic of the reduced Heegaard Floer homology and the Casson invariant.

The d-invariants satisfy the following two properties,

1) (Additivity) $d(Y \# Y', \mathfrak{s} \# \mathfrak{s}') = d(Y, \mathfrak{s}) + d(Y', \mathfrak{s}')$: that is, d is additive under connected sums.

2) (Vanishing) Suppose $(Y, \mathfrak{s}) = \partial(W, \mathfrak{t})$, where W is a $\mathbb{Q}$-homology ball and $\mathfrak{t}$ is a Spin^c structure on W that restricts to $\mathfrak{s}$ on Y. Then $d(Y, \mathfrak{s}) = 0$.

When $b_1(Y) > 0$, $HF^\circ(Y, \mathfrak{s})$ is acted upon by the exterior algebra $\Lambda^*(H_1(Y; \mathbb{F})/\mathrm{Tors}$. This action is natural with respect to the cobordism map in the following sense: if elements $\gamma_i \in H_1(Y_i)/\mathrm{Tors}$ for $i = 1, 2$, are homologous in W, then

$$F_{W,\mathfrak{s}}(\gamma_1 \cdot \xi) = \gamma_2 \cdot F_{W,\mathfrak{s}}(\xi).$$

We say that $HF^\infty(Y)$ is *standard* if for each torsion Spin^c structure $\mathfrak{s}_0$,

$$HF^{\infty}(Y,\mathfrak{s}_0) \cong (\Lambda^b H^1(Y;F))\otimes_{\mathbb{F}}\mathbb{F}[U,U^{-1}]$$

as $\Lambda^b H_1(Y;F)\otimes_{\mathbb{F}}\mathbb{F}[U,U^{-1}]$-modules, where $b = b_1(Y)$. The $\Lambda^b H_1(Y;F)\otimes_{\mathbb{F}}\mathbb{F}[U,U^{-1}]$ is induced by the interior product between $H^1(Y)$ and $H_1(Y)$.

Osváth and Szabó [OS04a, Theorem 10.1] proved that $HF^{\infty}(Y)$ is *standard* when $b_1(Y) \leq 2$. When $b_1(Y) \geq 3$, it depends on the triple cup product structure by the results of Lidman [Lid10]. When $HF^{\infty}(Y)$ is *standard*, we can specify two generators using $H_1(Y)$ action: a "bottom-most" generator which is in the kernel of the action by $H_1(Y)$ and a "top-most" generator which is acted on non-trivially by any non-zero element in $\Lambda^b H_1(Y,F)$. Then the corresponding d-invariants are defined similarly to Definition 2.1, which uses the grading of the bottom elements of the image of these two towers under the map π.

Definition 2.2. *Let Y be a three-manifold with standard HF^{∞}, equipped with a torsion* Spin^c *structure $\mathfrak{s}$. $d_b(Y,\mathfrak{s})$ is the correction term that corresponds to the "bottom-most" generator, and $d_t(Y,\mathfrak{s})$ is the correction term corresponds to the "top-most" generator.*

2.2 Preliminaries on surgery formula

Given a three-manifold Y with a surgery description, we can compute $HF^{\circ}(Y)$ with the surgery formula. This formula requires the link Floer complex and the flip map. The Knot surgery formula was proved by Osváth and Szabó in [OS08] [OS10]. It has been generalized by Manolescu and Ozsváth to the case of surgery on an integral framed null-homologous link in [MO10]. Zemke provided a bordered interpretation of the link surgery formula in [Zem21a] and generalized it to a general surgery formula for arbitrary links in closed 3-manifolds in [Zem23].

2.2.1 Link Floer complex

In this section, we review the definition of generalized Heegaard Floer complexes for links following the convention in [MO10] and we treat the knot Floer complex as a special case of the link Floer complex, where the link has one connected component.

Definition 2.3. *A multi-pointed Heegaard diagram consists of $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$, where:*

- Σ *is a closed, oriented surface of genus g;*

- $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_{g+k-1}\}$ *is a collection of disjoint, simple closed curves on* Σ *which span a g-dimensional lattice of* $H_1(\Sigma; \mathbb{Z})$*, hence specify a handlebody* U_α*; the same goes for* $\boldsymbol{\beta} = \{\beta_1, \ldots, \beta_{g+k-1}\}$*, which specify a handlebody* U_β*;*

- $\mathbf{w} = \{w_1, \ldots, w_k\}$ *and* $\mathbf{z} = \{z_1, \ldots, z_m\}$ *(with* $k \geq m$*) are collections of points on* Σ *with the following property. Let* $\{A_i\}_{i=1}^k$ *be the connected components of* $\Sigma - \alpha_1 - \cdots - \alpha_{g+k-1}$ *and* $\{B_i\}_{i=1}^k$ *be the connected components of* $\Sigma - \beta_1 - \cdots - \beta_{g+k-1}$*. Then there is a permutation* σ *of* $\{1, \ldots, m\}$ *such that* $w_i \in A_i \cap B_i$ *for* $i = 1, \ldots, k$*, and* $z_i \in A_i \cap B_{\sigma(i)}$ *for* $i = 1, \ldots, m$*.*

A Heegaard diagram $\mathcal{H}$ describes a closed, connected, oriented 3-manifold $Y = U_\alpha \cup_\Sigma U_\beta$, and an oriented link $\vec{L} \subset Y$ (with $\ell \leq m$ components), obtained as follows. For $i = 1, \ldots, m$, we join w_i to z_i inside A_i by an arc which we then push by an isotopy into the handlebody U_α; then we join z_i to $w_{\sigma(i)}$ inside B_i by an arc which we then push into U_β. The union of these arcs (with the induced orientation) is the link $\vec{L}$. We then say that $\mathcal{H}$ is a multi-pointed Heegaard diagram representing $\vec{L} \subset Y$.

Remark 2.1. *In cases* $k = 1$ *and* $m = 0$*, the Heegaard diagram represents an empty link, which is simply a Heegaard diagram for the a pointed 3-manifold.*

We require that the Heegaard multi-diagram satisfies the following *admissibility* condition to ensure that there are only finitely many disks counting when defining the differential:

Definition 2.4. *Let* $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ *be a multi-pointed Heegaard diagram.*

(a) *A* region *in* $\mathcal{H}$ *is the closure of a connected component of* $\Sigma - (\alpha_1 \cup \cdots \cup \alpha_{g+k-1} \cup \beta_1 \cup \cdots \cup \beta_{g+k-1})$*;*

(b) *A* periodic domain *in* $\mathcal{H}$ *is a two-chain* ϕ *on* Σ *obtained as a linear combination of regions (with integer coefficients), such that the boundary of* ϕ *is a linear combination of* α *and* β *curves, and the local multiplicity of* ϕ *at every* $w_i \in \mathbf{w}$ *is zero.*

(c) *The diagram $\mathcal{H}$ is called* admissible *if every non-trivial periodic domain has some positive local multiplicities and some negative local multiplicities.*

From now on, we will assume that all the Heegaard diagrams in this paper are admissible. Moreover, we will use a more restrictive class of Heegaard diagrams.

Definition 2.5. *A Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ is called* link-minimal *if $m = \ell$; that is, each link component has only two basepoints.*

Definition 2.6. *A Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ for a nonempty link is called* minimally-pointed *if $k = m = \ell$; that is, each link component has only two basepoints, and there are no free basepoints.*

Definition 2.7. *A Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$ is called* basic *if it is minimally-pointed and, further, for each $i = 1, \ldots, \ell$, the basepoints w_i and z_i (which determine one of the link components) lie on each side of a beta curve β_i, and are not separated by any alpha curves.*

Remark 2.2. *Under the condition in Definition 2.7, β_i is a meridian for L_i.*

Given a link Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$, which describes an l-components link in an integral homology sphere $\vec{L} \subset Y$, the Heegaard diagram determines tori

$$\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_{g+k-1}, \ \mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_{g+k-1} \subset \mathrm{Sym}^{g+k-1}(\Sigma).$$

We define the generators of the *link Floer complex* $\mathbf{CFL}^-(\mathcal{H})$ as the intersection point $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. Each intersection point $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ is assigned a relative Spin^c structure on (Y, L), via a construction of non-vanishing vector fields, which we denote it by $\underline{\mathfrak{s}}(x)$.

For $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, we let $\pi_2(\mathbf{x}, \mathbf{y})$ be the set of homotopy classes of Whitney disks from $\mathbf{x}$ to $\mathbf{y}$ relative to $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$, as in [OS04b]. For each homotopy class of disks $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$, we denote by $n_{w_j}(\phi)$ and $n_{z_j}(\phi) \in \mathbb{Z}$ the multiplicity of w_j (resp. z_j) in the domain of ϕ. Furthermore, we let $\mu(\phi)$ be the Maslov index of ϕ.

Each generator $\mathbf{x}$ of the link Floer complex is bigraded by the Maslov grading $M(\mathbf{x}) \in \mathbb{Z}$ and an Alexander multi-grading, which takes value in the following set:

$$\mathbb{H}(L)_i = \frac{\mathrm{lk}(L_i, L - L_i)}{2} + \mathbb{Z} \subset \mathbb{Q}, \;\; \mathbb{H}(L) = \bigtimes_{i=1}^{\ell} \mathbb{H}(L)_i,$$

where lk denotes linking number. Let us also set

$$\overline{\mathbb{H}}(L)_i = \mathbb{H}(L)_i \cup \{-\infty, +\infty\}, \;\; \overline{\mathbb{H}}(L) = \bigtimes_{i=1}^{\ell} \overline{\mathbb{H}}(L)_i,$$

such that

$$A_i(\mathbf{x}) \in \mathbb{H}(L)_i, \; i \in \{1, \dots, \ell\}.$$

Let $\mathbb{W}_i$ and $\mathbb{Z}_i$ be the set of indices for the w's (resp. z's) belonging to the i^{th} component of the link. We then have

$$A_i(\mathbf{x}) - A_i(\mathbf{y}) = \sum_{j \in \mathbb{Z}_i} n_{z_j}(\phi) - \sum_{j \in \mathbb{W}_i} n_{w_j}(\phi),$$

where ϕ is any class in $\pi_2(\mathbf{x}, \mathbf{y})$.

Following [MO10], we define the completed *link Floer complex* $\mathbf{CFL}^-(\mathcal{H})$ as follows. We let $\mathbf{CFL}^-(\mathcal{H})$ be the free module over $\mathcal{R}(\mathcal{H}) = \mathbb{F}[[U_1, \dots, U_k]]$ generated by $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$, and equipped with the differential:

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \mu(\phi) = 1}} \#(\mathcal{M}(\phi)/\mathbb{R}) \cdot U_1^{n_{w_1}(\phi)} \cdots U_k^{n_{w_k}(\phi)} \mathbf{y}. \tag{2.1}$$

Here, $\mathcal{M}(\phi)$ is the moduli space of pseudo-holomorphic curves (solutions to Floer's equation) in the class ϕ, and $\mathbb{R}$ acts on $\mathcal{M}(\phi)$ by translations. Note that $\mathcal{M}(\phi)$ depends on the choice of a suitable path of almost complex structures on the symmetric product. We suppress the almost complex structures from notation for simplicity.

The Maslov grading M produces the homological grading on $\mathbf{CFL}^-(\mathcal{H})$, with each U_i decreasing M by two. Furthermore, each Alexander grading A_i defines a filtration on $\mathbf{CFL}^-(\mathcal{H})$, with U_i decreasing the filtration level A_i by one, and leaving A_j constant for $j \neq i$.

Remark 2.3. *We use the HF^+ version for the knot surgery formula later in this paper, and we define $HF^+(Y, K)$ as a special case of* $\mathbf{CFL}^-(\mathcal{H})$ *here.*

The Heegaard diagram $\mathcal{H}$ for (Y, K) is $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$, where we only have a pair of base points (w, z).

For each generator $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, the Alexander grading takes a value in $\mathbb{H}(K) = \mathbb{Z}$, which is given by evaluating $\underline{\mathfrak{s}}(x)$ on a Seifert Surface for K. For simplicity, we will also use $\underline{\mathfrak{s}}(x)$ to denote the $\mathbb{Z}$ we obtain from the evaluation.

We define $\mathbf{CFK}^\infty$ as the free module over $\mathcal{R}(\mathcal{H}) = \mathbb{F}[[U, U^{-1}]$ generated by $U^i x$, s.t. $x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, and equipped with the differential:

$$\partial \mathbf{x} = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(\mathbf{x}, \mathbf{y}) \\ \mu(\phi) = 1}} \#(\mathcal{M}(\phi)/\mathbb{R}) \cdot U^{n_w(\phi)} \mathbf{y}. \tag{2.2}$$

$\mathbf{CFK}^-$ is the subcomplex of $\mathbf{CFK}^\infty$ generated by $U^i x$, s.t. $i \leq 0$ and CFK^+ is defined as the quotient-complex $\mathbf{CFK}^\infty / \mathbf{CFK}^-$.

Moreover, we can identify $U^i x$ with $[x, i, j]$, s.t $j = \underline{\mathfrak{s}}(x) + i$. Then $\mathbf{CFK}^-$ is identified with the subcomplex $[x, i, j]$, s.t. $i \leq 0$. From now on, we will use this version of the knot Floer complex when we talk about the knot surgery formula.

We also define the subcomplex that will be used in the surgery formula here. Given $\mathbf{s} = (s_1, \dots, s_\ell) \in \mathbb{H}(L)$, we define the *generalized Heegaard Floer complex*

$$\mathbf{A}^-(\mathcal{H}, \mathbf{s}) = \mathbf{A}^-(\mathcal{H}, s_1, \dots, s_\ell) = \mathbf{A}^-(\mathbb{T}_\alpha, \mathbb{T}_\beta, \mathbf{s})$$

to be the subcomplex of $\mathbf{CFL}^-(\mathcal{H})$ generated by elements $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ with $A_i(\mathbf{x}) \leq s_i$ for all $i = 1, \dots, \ell$.

Note that for the knot Floer complex, the generalized Floer complex $\mathbf{A}^-(\mathcal{H}, s)$ is the completion of the subcomplex $A_s^- = C\{\max(i, j - s) \leq 0\}$. The subcomplex used in the HF^+ version of the knot surgery formula is $A_s^+ = C\{\max(i, j - s) \geq 0\}$ and $B_s^+ = C\{i \geq 0\}$.

2.2.2 Knot and link surgery formula

In this section, we review the link surgery formula from [MO10]. We also describe the knot surgery formula for the plus flavor in Chapter 3 and Zemke's bordered surgery formula in Chapter 4.

We now describe the algebraic structure for the link surgery formula, which is called a *hypercube of chain complexes* in [MO10].

Define

$$\mathbb{E}_n = \{0, 1\}^n$$

as the set of vertices of the n-dimensional unit hypercube. If $\varepsilon, \varepsilon^{'} \in E_n$, we write $\varepsilon \leq \varepsilon^{'}$ if the inequality holds for each coordinate of ε and $\varepsilon^{'}$.

Definition 2.8. *An n-dimensional* hypercube of chain complexes *consists of a collection of $\mathbb{Z}$-graded vector spaces*

$$(C^{\varepsilon})_{\varepsilon\in\mathbb{E}_n}, \quad C^{\varepsilon} = \bigoplus_{*\in\mathbb{Z}} C^{\varepsilon}_*,$$

together with a collection of linear maps for each pair of indices $\varepsilon, \varepsilon^{'} \in E_n$ such that $\varepsilon \leq \varepsilon^{'}$

$$D_{\varepsilon,\varepsilon^{'}} : C^{\varepsilon} \to C^{\varepsilon^{'}},$$

The maps are required to satisfy the relations whenever ε and $\varepsilon^{''}$

$$\sum_{\varepsilon\leq\varepsilon^{'}\leq\varepsilon^{''}} D_{\varepsilon^{'},\varepsilon^{''}} \circ D_{\varepsilon,\varepsilon^{'}} = 0. \tag{2.3}$$

The input data for the link surgery complex is called a complete system of hyperboxes $\mathcal{H}$ for the link $\vec{L}$, which is defined in Section 8 of [MO10]. We will focus only on the *basic* system.

Let $\vec{L}$ be a n-component link and denote its components by $L_1, L_2, \cdots, L_n$. Fix a framing Λ for the link $\vec{L}$. For a component L_i of L, we let Λ_i be its induced framing, thought of as an element in $H_1(Y - L)$. The latter group can be identified with $\mathbb{Z}^n$ via the basis of oriented meridians for $\vec{L}$. Given a sublink $M \subseteq L$, we let $\Omega(M)$ be the set of all possible orientations on M. For $\vec{M} \in \Omega(M)$, we let $I_-(\vec{L}, \vec{M})$ denote the set of indices i such that the component L_i is in M and its orientation induced from $\vec{M}$ is opposite to the one induced from $\vec{L}$. Set

$$\Lambda_{\vec{L},\vec{M}} = \sum_{i\in I_-(\vec{L},\vec{M})} \Lambda_i \in H_1(Y - L) \cong \mathbb{Z}^n.$$

Let $Y_\Lambda(L)$ be the three-manifold obtained from Y by surgery on the framed link (L, Λ).

Given a basic Heegaard diagram $\mathcal{H}^L$ for L, the other diagrams appearing in a basic complete system $\mathcal{H}$ are the reductions

$$\mathcal{H}^{L-M} := r_M(\mathcal{H}^L),$$

obtained from $\mathcal{H}^L$ by deleting the z basepoints on the sublink M. Note that all the diagrams $\mathcal{H}^{L-M}$ are link-minimal. Let us denote the remaining components by $N := L - M$.

To the diagrams $\mathcal{H}^{L-M}$ we associate generalized Floer complexes $\mathfrak{A}^-(\mathcal{H}^{L-M}, \mathbf{s})$. These are modules over the ground ring

$$\mathcal{R} := \mathcal{R}(\mathcal{H}^L) = \mathbb{F}[[U_1, \ldots, U_n]].$$

There is a corresponding reduction on the Spin^c structures

$$\psi^M : \bar{\mathbb{H}}(L) \to \bar{\mathbb{H}}(N).$$

For each remaining component in $L_i \subset N$, we denote it by j_i. The map ψ^M is defined on each component by

$$\psi_i^M : \bar{\mathbb{H}}(L)_i \to \bar{\mathbb{H}}(N)_{j_i},\ s_i \to s_i - \frac{lk(L_i, M}{2}).$$

The surgery complex is the infinite direct product

$$C^-(\mathcal{H}, \Lambda) = \bigoplus_{M \subseteq L} \prod_{\mathbf{s} \in \mathbb{H}(L)} \mathfrak{A}^-(\mathcal{H}^{L-M}, \psi^M(\mathbf{s})). \tag{2.4}$$

To simplify the notation somewhat, we denote a typical term in the chain complex by

$$C_{\mathbf{s}}^{\varepsilon} = \mathfrak{A}^-(\mathcal{H}^{L-M}, \psi^M(\mathbf{s})), \tag{2.5}$$

where $\varepsilon = \varepsilon(M) = (\varepsilon_1, \cdots, \varepsilon_n) \in \{0, 1\}^n$ is such that $L_i \subseteq M$ if and only if $\varepsilon_i = 1$.

Furthermore, the differential on the complex $C^-(\mathcal{H}, \Lambda)$ is given by

$$D^-(\mathbf{s}, \mathbf{x}) = \sum_{N \subseteq L-M} \sum_{\vec{N} \in \Omega(N)} (\mathbf{s} + \Lambda_{\vec{L}, \vec{N}}, \Phi^{\vec{N}}_{\psi^M(\mathbf{s})}(\mathbf{x})),$$

for $\mathbf{s} \in \mathbb{H}(L)$ and $\mathbf{x} \in \mathfrak{A}^-(\mathcal{H}^{L-M}, \psi^M(\mathbf{s}))$. Note that in this formula, the maps

$$\Phi^{\vec{N}}_{\psi^M(\mathbf{s})} : \mathfrak{A}^-(\mathcal{H}^{L-M}, \psi^M(\mathbf{s})) \to \mathfrak{A}^-(\mathcal{H}^{L-M-N}, \psi^{M \cup N}(\mathbf{s}))$$

are constructed from polygon maps of the type considered in [MO10], depending on the choice of orientation. We omit their precise definition in this thesis and only give the definition for the knot case. Note that when $N = \emptyset$, the map $\Phi^{\vec{N}}_{\psi^M(\mathbf{s})}$ is just the usual differential on $\mathfrak{A}^-(\mathcal{H}^{L-M}, \psi^M(\mathbf{s}))$, counting holomorphic disks.

It has been shown in [MO10] that the above construction forms a hypercube of chain complexes and its homology is isomorphic to the Heegaard Floer homology of the surgery manifold $Y_\Lambda(L)$.

Theorem 2.1 ([MO10])**.** *Fix a complete system of hyperboxes $\mathcal{H}$ for an oriented, ℓ-component link $\vec{L}$ in an integral homology three-sphere Y, and fix a framing Λ of L. There is an isomorphism of homology groups:*

$$H_*(C^-(\mathcal{H}, \Lambda)) \cong \mathbf{HF}^-_*(Y_\Lambda(L)), \tag{2.6}$$

where $\mathbf{HF}^-$ is the completed version of Heegaard Floer homology over the power series ring $\mathbb{F}[[U]]$.

Remark 2.4. *In Theorem 2.1, the link surgery complex involves direct product, which is different from the knot surgery formula in [OS08], where a directed sum is used. Using directed sum only is only appropriate for computing the plus and hat flavor using an analogous surgery formula. For the minus flavor, the isomorphism does not hold (see the unknot computation in Example 2.2).*

To achieve the isomorphism for the minus flavor, we need to complete the direct sum, which gives us the direct product. In Section 4.2, there is a completion in the bordered surgery formula by the same reason and we will provide a more detailed discussion there.

We describe the knot surgery formula as an example of the link surgery formula.

Example 2.1. *When $\vec{L}$ has only one component $\vec{K}$, the hypercube is 1 dimensional, which forms a mapping cone. The mapping cone complex is a $\mathbb{F}[[U]]$-module*

$$C = \prod_{s\in\mathbb{Z}} C^0_{\mathbf{s}} \oplus \prod_{s\in\mathbb{Z}} C^1_s.$$

Here, C^0_s is the generalized Floer complex $\mathfrak{A}^-(\mathcal{H}^K, s)$, which is isomorphic to

$$A^-_s = C\{\max(i, j - s) \le 0\}$$

and each C^1_s is a copy of the complex

$$\mathfrak{A}^-(\mathcal{H}^\emptyset, 0) = \mathbf{CF}^-(\mathcal{H}^\emptyset),$$

whose homology is $\mathbf{HF}^-(Y)$. Let us denote it by B^-_s. For the m-surgery on K, the differentials on C consist of three parts. The first one is the self differential on A^-_s and B^-_s. The second is the differential which corresponds to the map that has the same orientation of $\vec{K}$

$$v^-_s = \Phi^K_s : A^-_s \longrightarrow B^-_s,$$

which is the inclusion. The third map corresponds to the opposite orientation of $\vec{K}$

$$h^-_s = \Phi^{-K}_s : A^-_s \longrightarrow B^-_{s+m},$$

which is the composition of the inclusion A^-_s to $C\{j \le s\}$ and the isomorphism between $C\{j \le s\}$ and B^-_s given by the flip map.

Thus, the complex C can be viewed as the mapping cone of the map

$$\prod_{s\in\mathbb{Z}} A^-_s \longrightarrow \prod_{s\in\mathbb{Z}} B^-_s, \quad (s, \mathbf{x}) \mapsto (s, v^-_s) + (s+m, h^-_s). \tag{2.7}$$

Example 2.2. *Using the above description, we can compute the $+1$ surgery on the unknot $\vec{U}$ in S^3. The knot Floer complex of U is generated by one element over $\mathbb{F}[[U]]$, hence we have*

$$A^-_s \cong B^-_s \cong \mathbb{F}[[U]],$$

we denote generators by a_s and b_s.

The flip map in this case is the identity, and the maps are

$$v^-_s = \begin{cases} 1 & \text{if } s \ge 0 \\ U^{-s} & \text{if } s \le 0, \end{cases} \qquad h^-_s = \begin{cases} U^s & \text{if } s \ge 0 \\ 1 & \text{if } s \le 0. \end{cases} \tag{2.8}$$

The homology of the complex C is then isomorphic to $\mathbb{F}[[U]]$, being freely generated by the element in the kernel

$$\sum_{s\in\mathbb{Z}} U^{|s|(|s|-1)/2} a_s.$$

The +1-surgery on the unknot is S^3, which has $HF^-(S^3) \cong \mathbb{F}[[U]]$. Hence, the above computation gives the right answer.

However, if we use the direct sum, the element $\sum_{s\in\mathbb{Z}} U^{|s|(|s|-1)/2} a_s$ which generates the homology no longer exists. Instead, we would have nontrivial cokernel in the new $C = \oplus C_s^1$. The cokernel would be generated as a $\mathbb{F}[[U]]$-module by classes $[b_i]$, $i \in \mathbb{Z}$, subjects to the relations:

$$[b_0] = [b_1] = U[b_{-1}] = U[b_2] = U^3[b_{-2}] = U^3[b_{-3}] = \cdots$$

Remark 2.5. *We also need another version of link Floer homology over the ring*

$$\mathbb{F}[\mathscr{U}_1, \cdots, \mathscr{U}_n, \mathscr{V}_1, \cdots, \mathscr{V}_n],$$

which is used in Zemke's bordered reinterpretation of link surgery formula. Let us denote it by $\mathcal{CFL}^-(\mathcal{H})$. $\mathcal{CFL}^-(\mathcal{H})$ is the free module over $\mathbb{F}[\mathscr{U}_1, \cdots, \mathscr{U}_n, \mathscr{V}_1, \cdots, \mathscr{V}_n]$ generated by $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$. The differential is defined similar to 2.1 except that we also count the **z** *basepoints, and we weight a holomorphic curve by*

$$\prod_{i=1}^{n} \mathscr{U}_i^{n_{w_i}(\phi)} \mathscr{V}_i^{n_{z_i}(\phi)}.$$

We have the following identification of subcomplexes:

Proposition 2.1 ([Zem21a])**.** *Suppose that $\mathcal{H}$ is a link minimal Heegaard diagram for a link L in S^3, which has no free basepoints. Let M be a sublink of L. Write $S_M \subset \mathbb{F}[\mathscr{U}_1, \ldots, \mathscr{U}_\ell, \mathscr{V}_1, \ldots, \mathscr{V}_\ell]$ for the multiplicatively closed subset generated by $\mathscr{V}_i$ for i such that $K_i \subset M$. Then there is an $\mathbb{F}[U_1, \ldots, U_\ell]$-equivariant chain isomorphism*

$$\bigoplus_{\mathbf{s}\in\mathbb{H}(L)} A^-(\mathcal{H}^{L-M}, \psi^M(\mathbf{s})) \cong S_M^{-1} \cdot \mathcal{CFL}(\mathcal{H}), \tag{2.9}$$

where we view U_i as acting by $\mathscr{U}_i\mathscr{V}_i$ on the right-hand side. Furthermore, if $\mathbf{s} \in \mathcal{H}(L)$, this isomorphism intertwines the summand $A^-(\mathcal{H}^{L-M}, \psi^M(\mathbf{s}))$ with the subspace of $S_M^{-1} \cdot \mathcal{CFL}(\mathcal{H})$ in Alexander multi-grading **s**.

We omit the proof of this proposition here. Note that the isomorphism in 2.9 is constructed by the map

$$A^-(\mathcal{H}^{L-M}, \psi^M(\mathbf{s})) \to S_M^{-1} \cdot \mathcal{CFL}(\mathcal{H})$$

via the formula

$$U_1^{i_1}\cdots U_\ell^{i_\ell}\cdot \mathbf{x} \mapsto \mathscr{U}_1^{i_1}\cdots \mathscr{U}_\ell^{i_\ell}\mathscr{V}_1^{s_1-A_1^L(\mathbf{x})+i_1}\cdots \mathscr{V}_\ell^{s_\ell-A_\ell^L(\mathbf{x})+i_\ell}\cdot \mathbf{x}.$$

We can also rephrase the link surgery formula in this setting. We only talk about the knot surgery formula here. For the general case, one may refer to Chapter.7 of [Zem21a].

Given a knot surgery complex $\mathbb{X}_\lambda(K) = \mathrm{Cone}(\prod_{s\in\mathbb{Z}} A_s^- \xrightarrow{v_s^-+h_s^-} \prod_{s\in\mathbb{Z}} B_s^-)$, $\prod_{s\in\mathbb{Z}} A_s^-$ *is identified with a completion of* $\mathcal{CFK}(K)$ *and* $\prod_{s\in\mathbb{Z}} B_s^-$ *is identified with a completion of* $\mathscr{V}^-\mathcal{CFK}(K)$ *by the above equivalence.*

Let $\mathbf{x}_1,\cdots,\mathbf{x}_2$ *be a free basis of* $\mathcal{CFK}(K)$ *over* $\mathbb{F}[\mathscr{U},\mathscr{V}]$. *Then the differential in the mapping cone is determined by the map on this basis together with the homomorphism of the coefficient ring:*

$$T:\mathbb{F}[\mathscr{U},\mathscr{V}]\to\mathbb{F}[\mathscr{U},\mathscr{V},\mathscr{V}^{-1}]$$

via the formula

$$T(\mathscr{U})=\mathscr{V}^{-1} \textit{ and } T(\mathscr{V})=\mathscr{U}\mathscr{V}^2,$$

and $I:\mathbb{F}[\mathscr{U},\mathscr{V}]\to\mathbb{F}[\mathscr{U},\mathscr{V},\mathscr{V}^{-1}]$, *which is the canonical inclusion.*

Using the above equivalence, we can compute the three summands of differentials on the generators in the mapping cone to this setting:

(i) The internal differential on each summand, let us denote it by ∂.

(ii) The map corresponds to the inclusion; let us denote it by v.

(iii) The map corresponds to the opposite orientation; let us denote it by h_λ.

Together with the coefficient ring homomorphism, given $\mathbf{a}\in\mathbb{F}[\mathscr{U},\mathscr{V}]$, *the differential of the mapping cone is given by:*

(i) $v(\mathbf{a}\cdot\mathbf{x}_i) = I(\mathbf{a})\cdot\mathbf{x}$

(ii) $h_\lambda(\mathbf{a}\cdot\mathbf{x}_i) = T(\mathbf{a})\cdot h_\lambda(\mathbf{x}_i)$

Using the above description, the complex of λ surgery on unknot $\mathbb{O}$ is the following.

$\mathcal{CFK}(\mathbb{O})$ *is generated by one generator. Hence the mapping cone is*

$$\mathbb{F}[[\mathscr{U},\mathscr{V}]] \xrightarrow{v+h_\lambda} \mathbb{F}[[\mathscr{U},\mathscr{V},\mathscr{V}^{-1}],$$

where

$$v(\mathscr{U}^i\mathscr{V}^j) = v(\mathscr{U}^i\mathscr{V}^j) \text{ and } h_\lambda(\mathscr{U}^i\mathscr{V}^j) = \mathscr{U}^j\mathscr{V}^{2j-i+\lambda}.$$

CHAPTER 3

NONSLICENESS OF ALGEBRAICALLY SLICE KNOTS

In this chapter, we give the proof of Theorem 1.3. In Section 3.1, we talk about the obstruction from the linking form and d-invariants. In Section 3.2, we describe the topology of 2-fold branched covers from different aspects. In Section 3.3, we compute the d-invariants using knot surgery formula and give the proof of Theorem 1.3. We also include the computation with lattice cohomology in the Appendix **??**.

3.1 Linking form and d-invariants obstruction

In this section, we review an obstruction for a knot to be slice from [HLR12]. Let K be a knot in S^3 and let $\Sigma(K)$ be the 2-fold branched cover of K. Suppose K is slice, which means K bounds a smooth disk in D^4. Then $\Sigma(K)$ bounds a $\mathbb{Z}/2\mathbb{Z}$-homology 4-ball W. Hence, to show the nonsliceness of K, it is enough to show the nonexistence of W.

The linking form on $\Sigma(K)$ provides an initial constraint on the pair $(W, \Sigma(K))$. To state it, recall that a subgroup $M \subset H_1(\Sigma(K))$ is called a $metabolizer$ if

- $|M|^2 = |T_1(Y)|$, where T_1 denotes the torsion subgroup of $H_1(Y)$, and
- The $\mathbb{Q}/\mathbb{Z}$-valued linking form on $H_1(Y)$ is identically zero on M.

If K is slice, then the $\mathbb{Z}/2\mathbb{Z}$-homology ball bounded by $\Sigma(K)$ gives rise, via the kernel of the inclusion induced map, $i_* : H_1(\Sigma(K)) \to H_1(W)$, to a metabolizer of $H_1(\Sigma(K))$[CG86]. This linking form can often be used to obstruct sliceness. Note, however, that when K is algebraically slice, this obstruction vanishes.

With Heegaard Floer homology, we get additional structures on the $metabolizer$. Recall that, for a rational homology 3-sphere Y, the Heegaard Floer homology of Y splits with respect to Spin^c structures over Y,

$$HF^+(Y) = \bigoplus_{t \in \text{Spin}^c(Y)} HF^+(Y, \mathfrak{s}),$$

and we can define the d-invariants $d(Y, \mathfrak{s})$ for each $\mathfrak{s} \in \text{Spin}^c(Y)$, $HF^+(Y, \mathfrak{t})$.

The obstruction is defined as a difference of correction terms.

Definition 3.1. *For Y a $\mathbb{Z}/2\mathbb{Z}$-homology sphere, define the relative d-invariants as $\bar{d}(Y, \mathfrak{s}) = d(Y, \mathfrak{s}) - d(Y, \mathfrak{s}_0)$, where $\mathfrak{s}_0$ is the unique spin structure on Y.*

When $H_1(Y; \mathbb{Z}/2\mathbb{Z}) = 0$, Poincare duality and the Chern class provide a bijection $\mathrm{Spin}^c \longleftrightarrow H_1(Y)$. Combining with the d-invariants property 2.1.2 above, we see that the d-invariants vanish on a metabolizer. One can package this using the following[HLR12],

Theorem 3.1. *Let P be a finite set of (distinct) odd primes. Suppose that W is a $\mathbb{Z}/2\mathbb{Z}$-homology 4-ball and $\partial W = \#_{p\in P} Y_p \# Y_1$, where*

(i) $p^k H_1(Y_p) = 0$ for each $p \in P$ and some $k \geq 0$.

(ii) Y_1 is a $\mathbb{Z}$-homology 3-sphere.

Then for each $p \in P$, there is a $metabolizer$ $M_p \subset H_1(Y_p)$ for which $\bar{d}(Y_p, \mathfrak{s}_{m_p}) = 0$ for all $m_p \in M_p$.

By using branched covers, the theorem yields the desired concordance obstruction.

Corollary 3.1.1. *Let $K = \#_{p\in P} K_p \# K_1$ be a connected sum of knots satisfying*
- *$p^k H_1(\Sigma(K_p)) = 0$ for each p in a set of primes, P, and some k,*
- *$H_1(\Sigma(K_1)) = 0$.*

Suppose K is slice.Then for each $p \in P$, there is a $metabolizer$ $M_p \subset H_1(\Sigma(K_p))$ for which $\bar{d}(\Sigma(K_p), \mathfrak{s}_{m_p}) = 0$ for all $m_p \in M_p$.

Note that Corollary 3.1.1 shows more; normally the linear combination $\sum K_p$ isn't concordant to any knot with $\det(K) = 1$.

3.2 Topology of the 2-fold branched covers

3.2.1 Surgery description by rational unknotting number one patterns

In this section, we use the algorithm from [DHMS22] to give a knot surgery description of $\Sigma_2(K_{2,p})$.

We first review the notion of *rational unknotting number one patterns*. For the definition of rational tangle and the bijection between the rational tangles in a fixed 3-ball B^3 and $\mathbb{Q} \cup \{\infty\}$, one can refer to section 2.1 in [DHMS22].

Definition 3.2. *Let $P \subseteq S^1 \times D^2$ be a pattern. We say that P has a rational unknotting number one if there exists a rational tangle T in P such that replacing T with another rational tangle $T^{'}$ gives a knot which is unknotted in the solid torus. We say that P has proper rational unknotting number one if $T^{'}$ can be taken to be a proper tangle replacement: that is, connecting the same two pairs of marked points as T.*

For a rational unknotting number one pattern P, we have

$$\Sigma_2(P(U)) \cong S^3_{p/q}(J)$$

for some strongly invertible knot J and surgery coefficient p/q. The claim is immediate from the Montesinos trick: since $P^{'}$ is an unknot, the branched double cover over $P^{'}$ is S^3. The 3-ball B^3 containing T' lifts to a solid torus in S^3, and replacing T' with T corresponds to doing surgery on the core of this solid torus. Moreover, we can explicitly produce J and the surgery coefficient p/q. Here, we will use $T_{2,k}$ to illustrate the procedure, which is given in Figure 3.1. For the general case, one can refer to [DHMS22].

(i) Let γ be a reference arc in B^3 which has one endpoint on each component of T', displayed in panel (2). When taking 2-fold branched cover of B^3 along T', we first cut B^3 at the disk bounds by each arc of T', which gives us a cylinder $D^2 \times I$ and γ is isotopic to $0 \times I$. Gluing two copies of cylinders gives a solid torus and the lift of γ in the solid torus is the core of this solid torus, i.e. J.

(ii) We also have a concrete description for the knot J. Let F_t be an isotopy of the solid torus moving $P^{'}$ into a local unknot in $S^1 \times D^2$. We then cut along the disk bounded by the unknot and glue two copies of the disk complement to get the 2-fold branched cover. We also keep

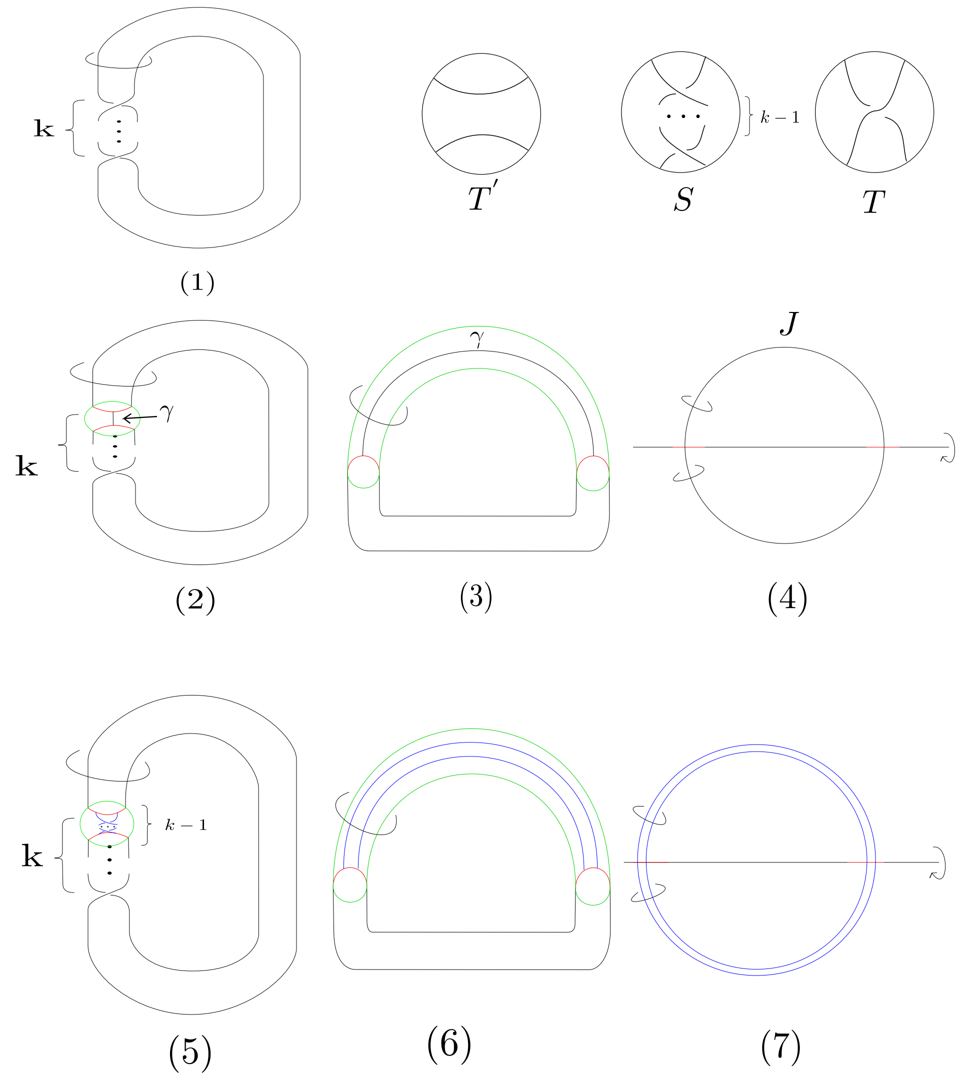


Figure 3.1 Montesinos trick

track of γ along F_t and lift it to the 2-fold branched cover. This gives the desired strongly invertible knot J, which in this case is just an unknot, displayed in panel (4).

(iii) To compute the surgery coefficient p/q, we must find the unique rational tangle S in B^3 which lifts to a pair of τ-equivariant Seifert framings of J, which can be done by running F_t backwards. We first find a τ-invariant Seifert framing of J in the 2-fold branched cover, then quotient it by τ and reverse the isotopy F_t to draw S in the original 3-ball B^3. Since J is a τ-equivariant unknot, we can pick a parallel copy of J to be Seifert framing. Quotienting this pair by τ gives us a pair of arcs. Keeping track of F_t backwards, it adds $k-1$ negative crossings to the pair of arcs. Hence, the rational tangle with $k-1$ negative crossings is the desired rational tangle in S in B^3. By the Montesinos trick, the surgery coefficient p/q is then precisely the rational number identified with the original tangle T relative to the choice of reference tangles $T_\infty = T^{'}$ and $T_0 = S$. In our example, the surgery coefficient is k. From the discussion above, we have

$$\Sigma_2(T_{2,k}) \cong S^3_k(U).$$

Let K be an oriented knot in S^3. We can now extend the discussion above to the branched cover of a cable knot $K_{2,k}$. Recall that $K_{2,k}$ can be constructed by taking the image of $T_{2,k}$ inside the gluing

$$S^3 \cong (S^3 - N(\mu)) \cup_{\partial N(\mu)} (S^3 - N(K))$$

formed by a boundary identification which maps a meridian μ of $T_{2,k}$ to a Seifert framing of K and a longitude of $T_{2,k}$ to a meridian of K.

Taking the 2-fold branched cover lifts $T_{2,k}$ to an unknot and the meridian μ to $\tilde{\mu} \cap \tau\tilde{\mu}$. Combining the discussion of the satellite operation above, we have

$$\Sigma_2(P(K)) \cong (S^3_k(J) - N(\tilde{\mu} - N(\tau\tilde{\mu})) \cup_{\partial N(\tilde{\mu})} (S^3 - N(K)) \cup_{\partial N(\tau\tilde{\mu})} (S^3 - N(K)),$$

which is illustrated in Figure 3.1. Since J is unknot, we have

$$\Sigma_2(P(K)) \cong S^3_k(K \# K^r)$$

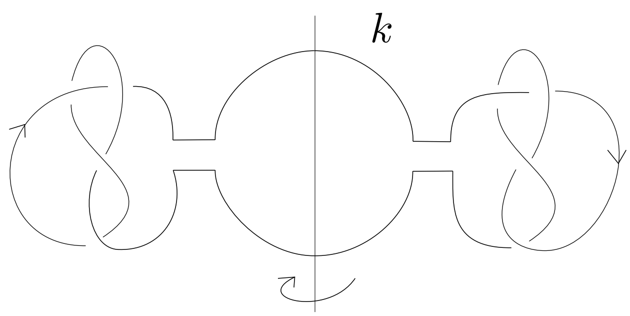


Figure 3.2 knot surgery description

Note that, since it is a k surgery on a knot in S^3, we have $|H_1(\Sigma_2(P(K))| = k$.

3.2.2 Topology of p-fold branched cover from complex polynomial

In this section, we study the topology of $\Sigma_p(K(p,q))$. Using the definition of torus knot with complex curve intersection, we give a link surgery description of the p-fold branched cover. Note that we take p-fold branched cover instead of 2-fold branched cover to make the manifold having nontrivial b_1, such that we have enough Spin^c structures to apply the d-invariants obstruction.

Using the same strategy as in Section 3.2, we first study the topology of $\Sigma_p(T_{p,q})$, with the description of the lifting of the meridian in $\Sigma_p(T_{p,q})$. Then we splice the knot complement to each lift to get the link surgery description of $\Sigma_p(K(p,q))$.

$T_{p,q}$ can be defined as the intersection of complex curves. Let

$$C_{p,q} = \{(z_2, z_3) \in C^2 \mid {z_2}^p + {z_3}^q = 0\}$$

and

$$S^2_\epsilon = \{(z_2, z_3) \in C^2 \mid |z_2|^2 + |z_3|^2 = \epsilon\}$$

be the two complex curves which intersect transversely. Define

$$T_{p,q} = C_{p,q} \pitchfork S^2_\epsilon.$$

The p-fold branched cover is the intersection of

$$C_{p,p,q} = \{(z_1, z_2, z_3) \in C^3 \mid {z_1}^p + {z_2}^p + {z_3}^q = 0\}$$

with

$$S^5_\epsilon = \{(z_1, z_2, z_3) \in C^3 \mid |z_1|^2 + |z_2|^2 + |z_3|^2 = \epsilon\},$$

which is denoted by $\Sigma(p,p,q)$ in [JN83, Chapter. 7]. Applying Theorem 7.2 of [JN83], $\Sigma(p,p,q)$ is a Seifert manifold $M(0;(1,r),p(q,s))$, where r,s are a pair of numbers such that $qr+ps=1$. $M(0;(1,r),p(q,s))$ can be represented by the plumbing diagram in Figure 3.3.

Note that, the Seifert manifold representation is only unique up to a set of operations ([JN83, Theorem 1.5]):

(i) Add or delete any Seifert pair $(\alpha,\beta) = (1,0)$

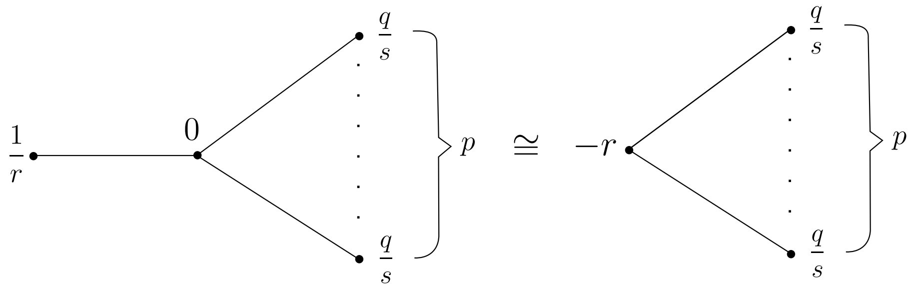


Figure 3.3 $M(0;(1,r),p(q,s))$

(ii) Replace any $(0,\pm 1)$ by $(0,\mp 1)$

(iii) Replace each (α_i,β_i) by $(\alpha_i,\beta_i+K_i\alpha_i)$ provided $\sum K_i = 0$

Applying continuous fraction to each $\frac{q}{s}$ vertex, we can represent it by the following plumbing diagram with only integer weights.

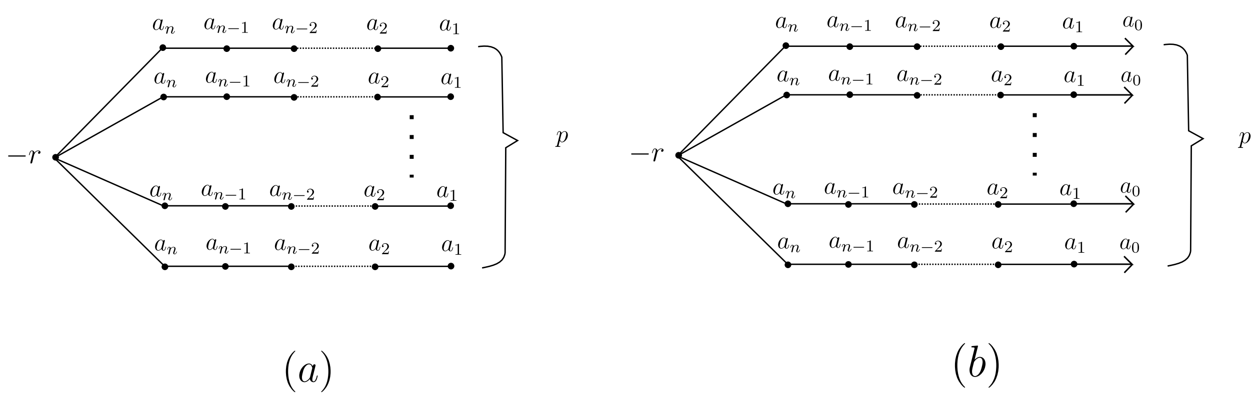


Figure 3.4 Plumbing diagram with integer weights

Embedding $T_{p,q}$ into the solid torus, one can check that the meridian of the solid torus is isotopic to $\{z_3 = 0\}$ in S^3. Hence, the lift of the meridian is also isotopic to $\{z_3 = 0\}$ in $M(0;(1,r),p(q,s))$, which corresponds to the core of p singular fibers with coefficient $\frac{q}{s}$. We use an arrow in Figure 3.4 to denote the singular fiber and a_0 to denote its framing. We can compute the framing of the core in the plumbing diagram in the following way.

Let us denote the meridian and longitude of the singular fiber complement by μ and λ and the meridian and longitude of the boundary of the singular fiber by μ' and λ'. The singular fiber is glued back with matrix $\begin{pmatrix} q & s \\ -p & r \end{pmatrix}$, which maps the $\lambda^{'}$ to $-p\mu + r\lambda$. Hence, to specify the framing in the plumbing diagram, we need to make sure the framing has $-p\mu + r\lambda$ as its image.

The meridian corresponds to the vertex with weight a_i by m_i. We have $m_1 = \mu$ and $m_0 = \mu^{'}$.

These following continuous fractions

$$\frac{q}{s} = a_n - \cfrac{1}{a_{n-1} - \cdots - \cfrac{1}{a_1}} \qquad \frac{q}{s'} = a_1 - \cfrac{1}{a_2 - \cdots - \cfrac{1}{a_n}}$$

satisfy $ss^{'} \equiv 1 \pmod q$. Hence $s^{'} = p + kq$, for some $k \in Z$. One can compute that, in the plumbing diagram, we have $m_0 = qm_n$ and $m_1 = \alpha m_n$, where

$$\frac{\alpha}{\beta} = a_2 - \cfrac{1}{a_3 - \cdots - \cfrac{1}{a_n}},$$

and we have $\frac{q}{s'} = a_1 - \frac{1}{\alpha/\beta} = \frac{a_1\alpha - \beta}{\alpha}$. In particular, we have $\alpha = s^{'} = p + kq$. Then for the framing a_0, we have the image of $\lambda^{'}$ is

$$-m_1 + a_0 m_0 = (-\alpha + a_0 q)m_n.$$

Hence, we should choose k as the framing.

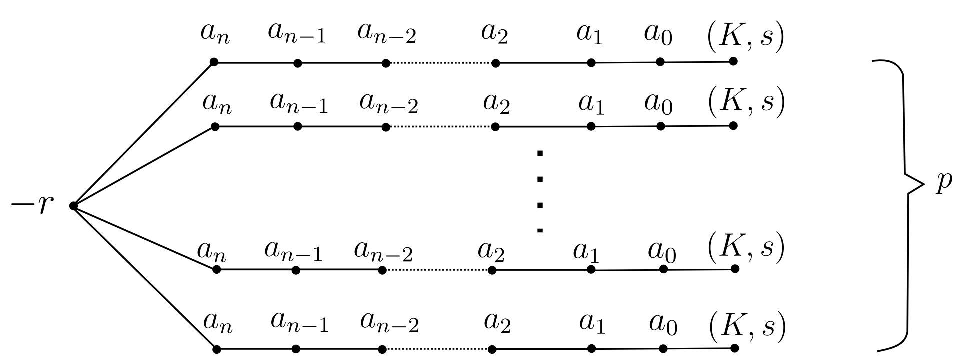


Figure 3.5 $\Sigma_p(K_{p,q})$

Splicing K to the core of singular fiber has the following description in Figure 3.5. We label the vertices with (K, s) indicate that it is the knot K with its Seifert framing.

Denote the meridian of each copy of K by m_i, we have

$$H_1(\Sigma_{p,q}(T_{p,q})) \cong \bigoplus_{p-1} Z/p,$$

which is generated by $m_1 - m_i, i = 1, \cdots, p-1$.

When $p = 2$, using the Seifert manifold description above and applying the operators in 3.2.2, we have the following two equivalent surgery descriptions of $\Sigma_2(T_{p,q})$ in Figure 3.6.

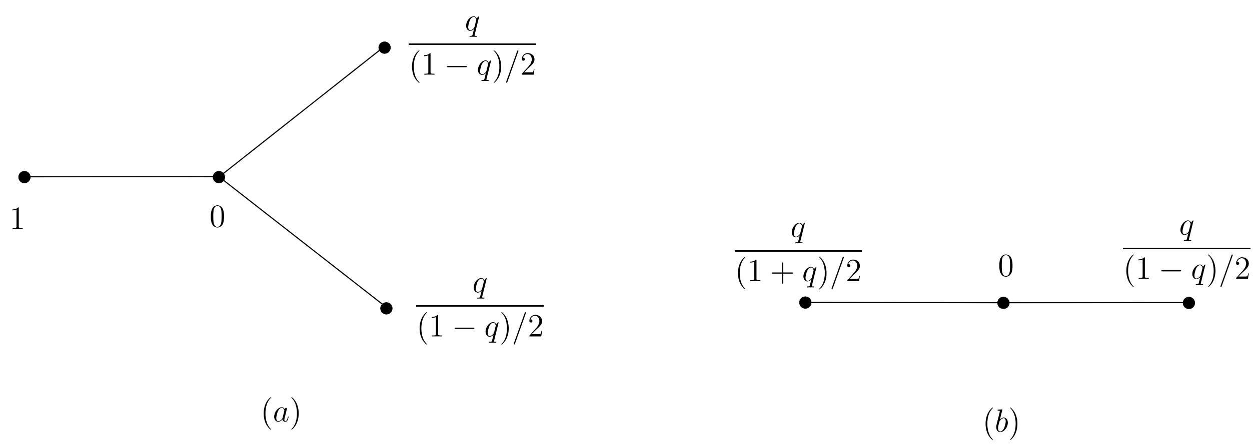


Figure 3.6 Seifert manifold of 2-fold branched cover

Using the description in Figure 3.6 (b), it is equivalent to gluing two solid torus via the matrix $\begin{pmatrix} q & 1 \\ -1 & 0 \end{pmatrix}$, which is equivalent to the 0 surgery on the unknot. The two cores of the singular fiber are homotopy to the meridians of the unknot with opposite orientations. Splicing the two meridians with the knot K, we get the same description as in section 3.2.1.

3.3 Computations with knot surgery formula

3.3.1 Knot Floer complex

We will give a description of $CFK^\infty(K\#K^r)$ in this subsection. For any knot K, we have a filtered chain homotopy equivalence

$$CFK^\infty(K\#K^r) \cong CFK^\infty(K) \otimes CFK^\infty(K^r).$$

from the connected sum formula [OS04b].

Since $CFK^\infty(K^r) \cong CFK^\infty(K)$, we have that $CFK^\infty(K\#K^r) \cong CFK^\infty(K) \otimes CFK^\infty(K)$.

When K is a L-space knot, the knot Floer complex is in a relatively simple form

$$CFK^\infty(K) \cong \mathrm{St}(K) \otimes \mathbb{Z}_2[U, U^{-1}].$$

Here, $\mathrm{St}(K)$ is the staircase complex associated to K. We have an example of $\mathrm{St}(T_{3,4})$ in Figure 3.7, where each dot represents a generator and the arrows represent differentials in the complex. The other complex in Figure 3.7 is the tensor complex, we omit some differentials induced from the second components for simplicity.

Following the notation from section 4.1 of [BL14], we can also denote this staircase complex by an array St(1,2,2,1). Each integer here denotes the length of the segments starting at the top left and moving to the bottom right in alternating right and downward steps. For a L-space knot K, the Alexander polynomial is in the form of $\Delta_K(t) = \Sigma_{i=0}^{2m}(-1)^i t^{n_i}$. We can get the staircase complex from the Alexander polynomial by $\mathrm{St}(K) = \mathrm{St}(n_{i+1} - n_i)$, where i runs from 0 to $2m - 1$.

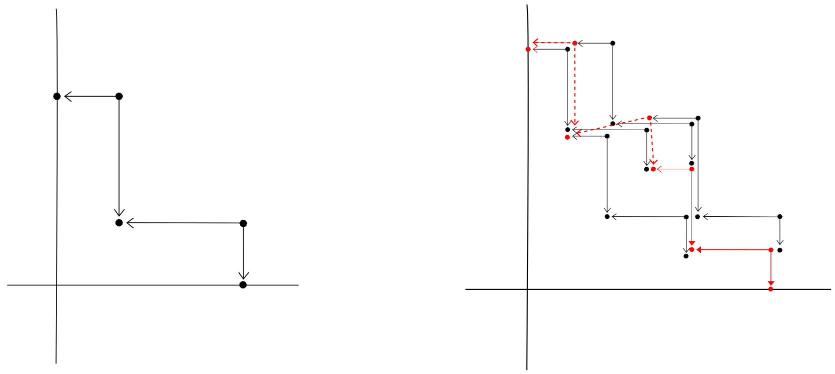
The absolute grading of the generator gives us a filtration on the staircase complex. The generator which does not have arrows pointing to other generators has grading 0 and we call these generators type **A**. Starting from top left, we denote these generators by a_1,a_2, . . . ,a_{m+1}. Similarly, we call the other generators which have nontrivial differentials type **B** and denote them by b_1,b_2, . . . , b_m.

In the tensor product $CFK^\infty(K) \otimes CFK^\infty(K)$, we have a subcomplex C', which is generated by the concatenation of $a_1 \otimes \mathrm{St}(K)$ and $\mathrm{St}(K) \otimes a_n$. We call the concatenation staircase the double of original staircase and denote it by $\mathrm{D}(\mathrm{St}(K))$. As an example, $\mathrm{D}(\mathrm{St}(T_{3,4}))$ is the red staircase in Figure 3.7.

Let $\tilde{C} := CFK^\infty(K\#K^r)/C'$ be the quotient complex.

Proposition 3.1. $H(\tilde{C}) \cong 0$.

Proof. We prove it by inductively quotienting the sub square complex from $\tilde{C}$. At each generator $b_i b_j$, we have the following square complex, which is demonstrated in 3.8, as a subcomplex of



$CFK^{\infty}(T_{3,4})$ $\quad$ $CFK^{\infty}(T_{3,4}\# -T_{3,4})$

Figure 3.7 knot Floer complex

$CFK^{\infty}(K\#K^r)$.

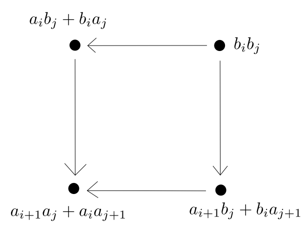
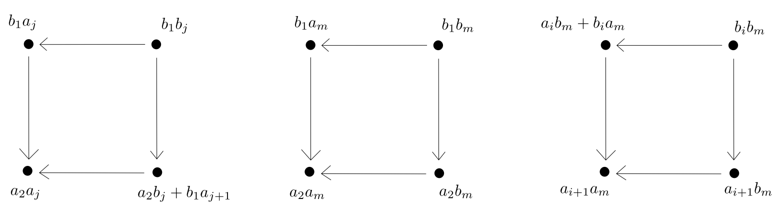


Figure 3.8 square complex

In $\tilde{C}$, when $i = 1$ or $j = m$, the square complexes become the ones in Figure 3.9.



square complex in $\tilde{C}$

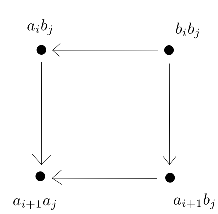


Figure 3.9 square complex in $\tilde{C}$

Quotienting these square complexes and performing a change of basis, the quotient complex we get is the same as deleting the square complex in the bottom of Figure 3.9 with $i = 1$ or $j = m$. Let us denote the new quotient complex by $\tilde{C}_1$.

Suppose we have quotiented k times and got the quotient complex $\tilde{C}_k$. By the same argument above, we can quotient the sub square complex with $i = k + 1$ or $j = m - k$, which is the same as deleting the subcomplex in Figure 3.9 with $i = k+1$ or $j = m-k$ and get the new quotient complex $\tilde{C}_{k+1}$. Moreover, we have $\tilde{C}_m = 0$ since we have deleted all the generators.

Hence, we have $\tilde{C} \cong \tilde{C}_m \cong 0$. □

Combining Proposition 3.1 and the exact sequence for the pair $(CFK^\infty(K\#K^r), \mathrm{D}(\mathrm{St}(K)) \otimes \mathbb{Z}_2[U, U^{-1}])$, we have the following:

Proposition 3.2. $H(CFK^\infty(K\#K^r)) \cong H(\mathrm{D}(\mathrm{St}(K)) \otimes \mathbb{Z}_2[U, U^{-1}])$.

The middle generator $a_1 a_{n+1}$ of the double staircase is on the diagonal. This is easy to be shown since the staircase of any L-space knot is symmetric along the diagonal. The i-th filtration level of $a_1 a_{n+1}$ is the distance of $a_1 a_{n+1}$ to the j axis, which is equal to sum of length of horizontal arrows in the staircase: $\Sigma_{i \in 2Z+1} n_i$. Since $a_1 a_{n+1}$ is on the diagonal, we have

Proposition 3.3. *The bigrading of* $a_1 a_{n+1}$ *is* $(\Sigma_{i \in 2Z+1} n_i, \Sigma_{i \in 2Z+1} n_i)$.

3.3.2 Computation of d-invariants

We will use the argument from [NW15] to compute the d-invariants.

In their paper, they used the plus version of the integer surgery formula from [OS08]. Given a knot K in S^3, let $C = CFK^\infty(K)$ be the knot Floer complex associated to it. The plus integer surgery formula involves the following two subcomplex:

$$A_s^+ = C\{\max(i, j - s) \geq 0\} \qquad\qquad B^+ = C\{i \geq 0\}.$$

There are two canonical chain maps $v_s^+ : A_s^+ \to B^+$ and $h_s^+ : A_s^+ \to B^+$ as in [OS08]. We only need v_s^+ in this paper, which is the projection from A_s^+ onto $C\{i \geq 0\}$.

Let $\mathbb{A} = \otimes_{s \in \mathbb{Z}} A_s^+$ and $\mathbb{B} = \otimes_{s \in \mathbb{Z}} B_s^+$ and let $D_n^+ : \mathbb{A}^+ \to \mathbb{B}^+$ be the map

$$D_n^+(\{a_s\}_{s\in\mathbb{Z}}) = \{b_s\}_{s\in\mathbb{Z}},$$

where here

$$b_s = h_{s-k}^+(a_{s-n} + v_s^+(a_s).$$

Let $\mathbb{X}^+(k)$ denote the mapping cone of D_k^+.

Theorem 3.2 ([OS08])**.** *For any non-zero integer k, the homology of the mapping cone $\mathbb{X}_k^+$ of*

$$D_k^+ : \mathbb{A}^+ \to \mathbb{B}^+$$

is isomorphic to $HF^+(S_k^3(K))$.

In [NW15], Ni and Wu gave an efficient way to compute the d-invariants from the integer surgery formula. We first recall the notation from their paper.

Let

$$\mathfrak{A}_s^+ = H_*(A_s^+),\ \mathfrak{B}^+ = H_*(B^+).$$

Indeed, $B^+ = C\{i \geq 0\}$ is identified with $CF^+(S^3)$ and $\mathfrak{B}^+ \cong \mathcal{T}^+$. Here $\mathcal{T}^+ \cong \mathbb{Z}_2[U, U^{-1}]/\mathbb{Z}_2[U]$. Let

$$\mathfrak{v}_s^+, \mathfrak{h}_s^+ : \mathfrak{A}_s^+ \to \mathfrak{B}^+$$

be the map induced on homology.

Let $\mathfrak{A}_s^T = U^n\mathfrak{A}_s^+$ for $n \gg 0$, we have $\mathfrak{A}_s^T \cong \mathcal{T}^+$. Since each $\mathfrak{a}_s^+$ is a graded isomorphism at sufficiently high grading and is U-equivariant, $\mathfrak{a}_s^+ \mid \mathfrak{A}_s^T$ is modeled on multiplication by U^{V_s}. Note that the number V_s is an invariant. Also, by Proposition 3.2, we can use $\mathrm{D}(\mathrm{St}(K)) \otimes \mathbb{Z}_2[U, U^{-1}]$ to compute V_s. We have a useful property of V_s.

Proposition 3.4 ([NW15][Ras04])**.** $V_s \geq V_{s+1}$.

The formula given in [NW15] computes d-invariants of 3-manifold constructed from a rational surgery in S^3. In our case, we just need the formula in the integer surgery case.

Proposition 3.5 ([NW15]). *Suppose $k > 0$ and fix $0 \le i \le k-1$. Then*

$$d(S^3_k(K), i) = d(L(k,1), i) - 2\max\{V_i, V_{k-i}\}.$$

Combining this and the Proposition above, together with the symmetry of the d-invariants for lens space, we have

$$d(S^3_k, i) = d(S^3_k, k-i) = d(L(k,1), i) - 2V_i \text{ , when } 0 \le i \le (k-1)/2.$$

Lemma 3.3.1. *For a staircase* St $\subseteq C$ *and subcomplexes A^+_s and B^+, let us denote the restriction of* St *to the subcomplexes by $r(\mathrm{St})$. $H_*(r(\mathrm{St}))$ is nontrivial iff* St *is fully included in the subcomplex.*

Proof. Each staircase in the subcomplex is truncated by a horizontal line and a vertical line. Suppose it is not fully included in the subcomplex, since the staircase starts horizontally and ends vertically, each connected component of the remaining part is a staircase with an even number of generators. Hence, the homology will be trivial on these staircases. Below in Figure 3.10, we have $A^+_3(\mathrm{D}(\mathrm{St}(T(3,4))) \otimes \mathbb{Z}_2[U, U^{-1}])$ as an example. ◻

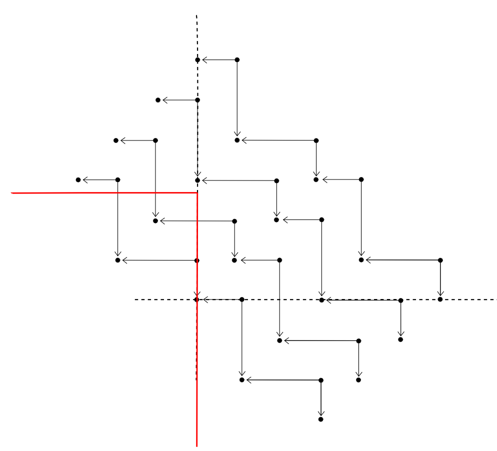


Figure 3.10 $A^+_3(\mathrm{D}(\mathrm{St}(T(3,4))) \otimes \mathbb{Z}_2[U, U^{-1}])$

Let us first study $\mathfrak{B}^+$. By Lemma 3.3.1, the bottom generator is represented by the staircase whose left top corner is on the j-axis, since it is the first staircase which is fully included in B^+. Let us denote it by St^B.

The first staircase included in $\mathfrak{A}_0^+$ is the one that has the middle generator at $(0,0)$, let us denote it by St^0. V_0 is the U-distance between these two staircases, i.e. the U power in $U^{V_0}St^B = St^0$. By Proposition 3.3, the bigrading of the middle term in St^B is $(\Sigma_{i\in 2Z+1}n_i, \Sigma_{i\in 2Z+1}n_i)$. Since the middle generator of St^0 is at $(0,0)$, we get $V_0 = \Sigma_{i\in 2Z+1}n_i$.

Let us denote the gap $V_s - V_0$ by $\bar{V}_s$. Note that when $s \geq 2\Sigma_{i\in 2Z+1}n_i$, $V_s = V_0$ and $\bar{V}_s = 0$. On the j axis, we denote the overlap with all the staircases by $O_{i=0}$ and the restriction of $O_{i=0}$ to $0 \leq j \leq s$ by $O_{i=0}^s$. We also denote the length of $O_{i=0}^s$ by $L_{i=0}^s$.

Proposition 3.6. *When* $0 \leq s \leq 2\Sigma_{i\in 2Z+1}n_i$, $\bar{V}_s = s - L_{i=0}^s$.

Proof. Let us look at the gap of $V_s - V_{s+1}$. Suppose $O_{i=0}^{s+1} \setminus O_{i=0}^s$ is nonempty, then the bottom most staircase of A_s^+ remains fully included in A_{s+1}^+. Hence, $V_s = V_{s+1}$ and $V_s - V_{s+1} = 0$. Suppose $O_{i=0}^{s+1} \setminus O_{i=0}^s$ is empty, then the bottom most staircase of A_{s+1}^+ is the one which is once above the bottom most staircase of A_s^+. Hence $V_s - V_{s+1} = 1$.

Sum all of these gaps up, we get the conclusion. □

Combining the propositions above, we have

Corollary 3.2.1. $\bar{d}(S_k^3(K,s) = \bar{d}(L(k,1),s) - s + L_{i=0}^s$, *when* $0 \leq s \leq (k-1)/2$.

3.3.3 Main theorem

In this subsection, we give the statement and proof of the main theorem.

Let us denote $\mathrm{Max}\{s | s - L_{i=0}^s = 0\}$ by $m(K)$. From the discussion of the staircase complex in Section 3.3.1, $m(K)$ is equal to the difference of degrees of the highest degree generator and the second top degree generator of $\Delta_K(t)$. For any polynomial $P(t) = \sum_{i=1}^n a_i t^{d_i}$ such that $d_i < d_{i+1}$, let $m(P(t)) = d_n - d_{n-1}$. Then $m(K) = m(\Delta_K(t))$.

It is been shown in [HW18] that, for any L space knot K,

$$\Delta_K(t) = t^g - t^{g-1} \cdots - t^{1-g} + t^{-g},$$

where g denotes the Seifert genus of K. Hence, we have $m(K) = 1$ for any L-space knot K.

Theorem 3.3. *For any L space knot K, $K_{2,k_1}\#-T_{2,k_1}\#-K_{2,k_2}\#T_{2,k_2}$ has infinite order in the smooth knot concordance group C when k_1 and k_2 are a pair of distinct prime numbers such that $k_1 > 3$ and $k_2 > 3$.*

Note that, since $m(K) = 1$, the assumption in Theorem 3.3 is the same as $k_i > 2m(K) + 1$.

In the special case, where K_{2,k_1} and K_{2,k_2} are algebraic knots, $k_i > 2pq > 3$, for some p, q which are co-prime. We have the following corollary:

Corollary 3.3.1. *$K_{2,k_1}\# - T_{2,k_1}\# - K_{2,k_2}\#T_{2,k_2}$ has infinite order in the smooth knot concordance group C when K_{2,k_1} and K_{2,k_2} are 2 distinct algebraic knots.*

Before proving the main theorem, let us study the linking form on the manifold $M = \Sigma_2(K_{2,k})\#-\Sigma_2(T_{2,k})$ first. From Section 3.2, we have $M \cong S^3_k(K\#K^r)\# - S^3_k(U)$, here U denotes the unknot. $H_1(M) \cong \mathbb{Z}/k\mathbb{Z} \oplus \mathbb{Z}/k\mathbb{Z}$, which is generated by the meridians of the surgery knots. Let us denote the meridian of $K\#K^r$ by α and the meridian of U by β. Then the linking form evaluating on these generators gives:

$$\lambda(\alpha, \alpha) \equiv 1/k,\ \lambda(\beta, \beta) \equiv -1/k,\ \lambda(\alpha, \beta) \equiv 0.\ (\text{mod } Z)$$

Let us denote the generators of $H_1(nM)$ by α_i and β_i, $1 \leq i \leq n$. Here, α_i are the meridians of each $K\#K^r$ and β_i are the meridians of each U. We also use M_i^1 and M_i^2 to denote the corresponding summands of $\Sigma_2(K_{2,k})$ and $\Sigma_2(T_{2,k})$. Via a change of basis, we can use α_i and $\alpha_i + \beta_i$ as the generators for $H_1(nM)$.

For a knot K satisfies the assumption in Theorem 3.3, we have

Lemma 3.3.2. *For any i, on the* Spin^c *structures correspond to the subgroup G_i generated by $\alpha_i + \beta_i$, there exists at least one* Spin^c *structure $\mathfrak{s}$, such that $\bar{d}(nM, \mathfrak{s}) \neq 0$.*

Proof. We prove it by contradiction. $G_i = \{l(\alpha_i + \beta_i) | l \in \mathbb{Z}/k\mathbb{Z}\}$. Suppose for each l, $\bar{d}(nM, l(\alpha_i + \beta_i)) = 0$. Then

$$\textstyle\sum_{l=1}^{k} \bar{d}(nM, l(\alpha_i + \beta_i)) = 0.$$

Using additivity of the relative d-invariants, we can rewrite it as

$$\sum_{l=1}^{k} \bar{d}(M_i^1, l\alpha_i) = \sum_{l=1}^{k} \bar{d}(M_i^2, l\beta_i).$$

By Corollary 3.2.1, $\bar{d}(M_i^1, l\alpha_i) = \bar{d}(M_i^2, l\beta_i) - l + L_{i=0}^{l}$, when $0 \leq l \leq (k-1)/2$.
Note that $-l + L_{i=0}^{l} \leq 0$ for any l. When $k > 2m(K) + 1$, $-(k-1)/2 + L_{i=0}^{(k-1)/2} < 0$ by the assumption. Hence, we have

$$\sum_{l=1}^{k} \bar{d}(M_i^1, l\alpha_i) < \sum_{l=1}^{k} \bar{d}(M_i^2, l\beta_i),$$

which contradicts the equation above. □

Let us denote the subgroup generated by $\alpha_i + \beta_i$ by $\tilde{G}$.

Lemma 3.3.3. *For any metabolizer G of $H_1(nM)$ with vanishing relative d-invariants, $G \cap \tilde{G} \neq \emptyset$.*

Proof. For any metabolizer G, we have $|G|^2 = k^{2n}$. Hence, $|G| = k^n$ and G is generated by n linearly independent elements $\{g_j = \sum_{i=1}^{n} a_{ij}\alpha_i + b_{ij}(\alpha_i + \beta_i) | j = 1, 2, \cdots, n\}$.

Suppose $G \cap \tilde{G} = \emptyset$, then $G \cong G/G \cap \tilde{G}$, which is generated by $g_j^{'} = \sum_{i=1}^{n} a_{ij}\alpha_i$. Since $|G| = k^n$, $\{g_j^{'}\}$ are linearly independent. Hence $\alpha_1 \in G/G \cap \tilde{G}$, which implies there exists $e = \alpha_1 + \sum_{i=1}^{n} c_i(\alpha_i + \beta_i) \in G$ for some $c_i \in \mathbb{Z}/k\mathbb{Z}$.

Suppose $c_1 \neq 0$, $e = (1 + c_1)\alpha_1 + c_1\beta_1 + \sum_{i=2} n(\alpha_i + \beta_i)$. Use the same argument from Lemma 3.3.2,

$$\sum_{l=1}^{k} \bar{d}(nM, le) \neq 0,$$

which contradicts the assumption.

Suppose $c_1 = 0$, $e = \alpha_1 + \sum_{i=2} n(\alpha_i + \beta_i)$.

$$\bar{d}(nM, le) = \bar{d}(\Sigma_2(K_{2,k}), l) + \sum_{i=2}^{n} \bar{d}(M_i, lc_i(\alpha_i + \beta i)) \not\equiv 0(\mathrm{mod} Z),$$

which also contradicts the assumption. Hence $G \cap \tilde{G} \neq \emptyset$. □

Proof of Theorem 3.3:

Proof. By Lemma 3.3.3, any metablizer G of $H_1(nM)$ contains an element $e \in \tilde{G}$. Then by Lemma 3.3.2 and additivity of relative d-invariants, there exists at least one Spin^c structures $\mathfrak{s}$, such that $\bar{d}(nM, \mathfrak{s}) \neq 0$. This shows the nonexistence of a metabolizer for which the relative d-invariants vanish. Then by Corollary 3.1.1, it proves the nonsliceness of $n(K_{2,k_1}\# - T_{2,k_1}\# - K_{2,k_2}\# T_{2,k_2})$. $\square$

Using the jump function for the Levine-Tristram signature of a knot, we can show the linearly independent of a set of knots.

For the knot K, let us use $r(K) = \{\theta_l\}$ to denote the set of numbers such that when evaluated at $\omega = e^{2\pi i \theta_l}$, the jump function is non-zero.

Corollary 3.3.2. *Let k_i be a set of distinct prime numbers such that, $k_i > 2m(K) + 1$ and $1/2k_i \notin r(K)$. Then the set of knots $\{T_{2,k_i}, K_{2,k_i}\}$ are linearly independent in the concordance group C.*

Proof. Consider a linear combination

$$J = \textstyle\sum_{i=1}^{N} n_i T_{2,k_i} + m_i K_{2,k_i}.$$

Suppose J is slice. Fix l, when we evaluate the jump function at $\omega = e^{2\pi i/2k_l}$. By the assumption, the only knots have non-zero jump function are T_{2,k_l} and K_{2,k_l}, both having jump equal to -1. Hence, $n_l = -m_l$.

$$J = \textstyle\sum_{i=1}^{N} m_i (K_{2,k_i} - T_{2,k_i}).$$

Using the same argument above, J is nonslice. $\square$

CHAPTER 4

D-INVARIANTS OF SPLICING OF CIRCLE BUNDLES OF HIGHER GENUS SURFACE

4.1 D=invariants obstruction to existence of certain intersection form

Given a smooth closed four-manifold X^4 with the intersection form $Q_X = mE_8 \oplus nH$. The d-invariants give a constraint on the pair (m, n) in the following way.

Consider a basis β for $H_2(X)$ and let $N(\Sigma_\beta) := N$ be a neighborhood of a collection of smooth, closed surfaces representing the elements in β. Note that, by adding tubes, we can arrange the surfaces to be embedded, with geometric intersection number equal to the algebraic intersection number, at the expense of increasing the genus. Let C be the complement of N in X, then it gives us a splitting of X as $X = N \cup_Y C$, which is illustrated in Figure 4.1. The intersection form Q_C of C is trivial.

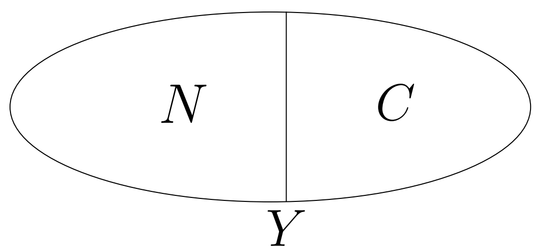


Figure 4.1 X

Given a four-manifold with boundary $(W, \partial W)$, the d-invariants of ∂W give constraints on the topology of W by the following Theorem from [OS03a].

Theorem 4.1 ([OS03a])**.** *Let $(W, \partial W)$ be a smooth four-manifold with boundary, such that W is negative semi-definite and the restriction map $H^1(W;\mathbb{Z}) \to H^1(Y;\mathbb{Z})$ is trivial. Suppose ∂W has standard HF^∞ and is equipped with a torsion* Spin^c *structure $\mathfrak{t}$, then we have the inequality:*

$$c_1(\mathfrak{s})^2 + b_2^-(W) \leq 4d_b(\partial W, \mathfrak{t}) + 2b_1(\partial W) \tag{4.1}$$

for all Spin^c *structures $\mathfrak{s}$ over W whose restriction to ∂W is $\mathfrak{t}$.*

When $H^1(W;\mathbb{Z}) \to H^1(Y;\mathbb{Z})$ is surjective, we have a similar inequality for $d_t(\partial W, \mathfrak{t})$:

$$c_1(\mathfrak{s})^2 + b_2^-(W) \leq 4d_t(\partial W, \mathfrak{t}) - 2b_1(\partial W). \tag{4.2}$$

We first apply Theorem 4.1 to obstruct mE_8 as the intersection form for a smooth, closed four-manifold as an example.

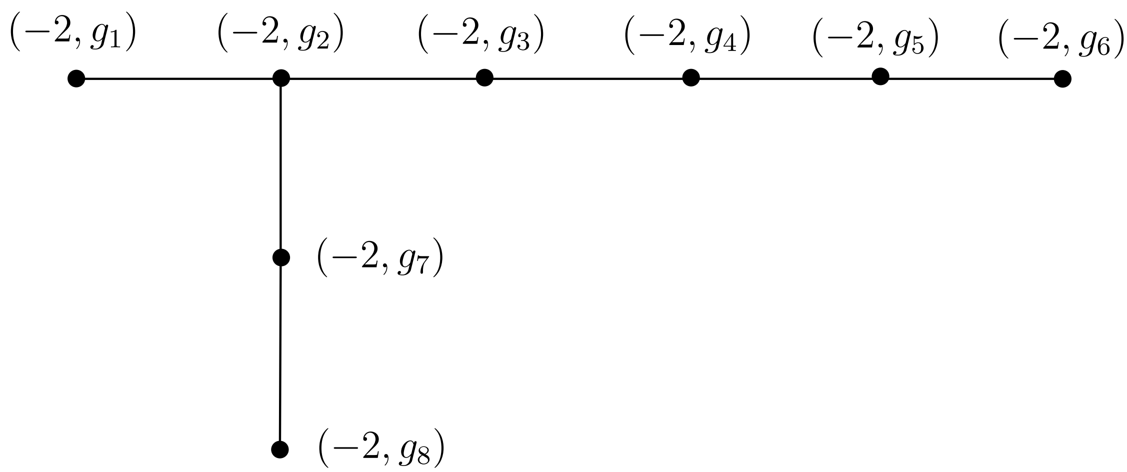


Figure 4.2 E_8

Example 4.1. *Figure 4.2 is a plumbing diagram for E_8, where each vertex represents a circle bundle over a genus g_i surface with Euler number -2. Denote the manifold corresponding to the E_8 plumbing by $(W(E_8), Y(E_8))$.*

Since $\det(E_8) = 1$, $\mathrm{H}_1(Y(E_8))$ *is generated by* $\mathrm{H}_1(\Sigma_{g_i})$. *Therefore, we have* $|\mathrm{H}_1(Y(E_8))| = 2\sum g_i$ *and* $H^1(W(E_8)) \to H^1(Y(E_8))$ *is surjective.*

Given a four-manifold X with $Q_X = mE_8$, we apply the splitting in Figure 4.1. Then by the discussion above, $H^1(N) \to H^1(Y)$ is surjective and $|\mathrm{H}_1(Y)| = 2\sum_{m,i} g_{m_i}$.

There is a unique torsion Spin^c *structure on Y, let us denote it by* $\mathfrak{t}$. *Since mE_8 is an even intersection form, $\vec{0}$ is a characteristic element. Let us use the corresponding* Spin^c *structure on $W(E_8)$. Then by the inequality 4.2, we have*

$$8m \leq 4d_t(Y, \mathfrak{t}) - 2(2\sum_{m,i} g_{m_i}) \tag{4.3}$$

On the other hand, since $H^1(C) = 0$, $H^1(C) \to H^1(-Y)$ is trivial. Taking the trivial Spin^c *structure on C and applying the inequality 4.1, we have:*

$$0 \leq 4d_b(-Y, \mathfrak{t}) + 2(2\sum_{m,i} g_{m_i}) \tag{4.4}$$

Combining the property that, $d_b(-Y, \mathfrak{t}) = -d_t(Y, \mathfrak{t})$, we have

$$\sum_{m,i} g_{m_i} \geq d_t(Y, \mathfrak{t}) \geq 2m + \sum_{m,i} g_{m_i},$$

which shows $m = 0$.

Given the intersection form $Q_X = mE_8 \oplus nH$, let us denote the submanifold corresponding to mE_8 by $\tilde{Y}_m$ and the submanifold corresponds to nH by Y_n. Let $g(\tilde{Y}_m)$ be the sum of the genera of all surfaces in $\tilde{Y}_m$ and $g(Y_n)$ be the sum of the genera of all surfaces in Y_n. By the similar argument as in Example 4.1, using the four-manifold C, we have

$$d_t(\tilde{Y}_m \# Y_n) = d_t(\tilde{Y}_m) + d_t(Y_n) \leq g_m + g_n.$$

Combining with the inequality 4.3, we have

$$2m + g_m + d_t(Y_n) \leq d_t(Y_m) + d_t(Y_n) \leq g_m + g_n,$$

hence

$$2m + d_t(Y_n) \leq g_n \tag{4.5}$$

Suppose we have the d-invariants $d_t(Y(H_{g_1,g_2})) \geq g_1 + g_2 - k$, then we can rewrite equation 4.5 as

$$\begin{aligned} 2m + g_n - nk &\leq g_n \\ 2m &\leq nk \end{aligned} \tag{4.6}$$

Note that, if $k = 2$, we recover the $10/8$ theorem.

However, we have

$$d_{\text{top}}(H_{g_1,g_2}) = |g_1 - g_2| - 2,$$

when $\min(g_1, g_2) \geq 1$.

We expect to improve the above argument to involutive Heegaard Floer setting and reprove the $10/8$ theorem for any genus. One could ask if we could improve the bound to $k = 3/2$ and prove the $11/8$ conjecture. However, by the properties of d-invariants, $d_t(Y(H_{g_1,g_2}) \in \mathbb{Z}$. Hence, the best we can do by this argument is the proof of $10/8$ theorem.

4.2 Zemke's general surgery formula

In this section, we review the bordered link surgery formula from [Zem21a] [Zem23]. We will use the version of link surgery formula from Remark 2.5.

4.2.1 Type-D module

In [Zem21a], Zemke reinterprets the link surgery formula in [MO10] as a type-D module. The type-D module is defined as following,

Definition 4.1. *If $\mathcal{A}$ is an associative algebra over $\mathbf{k}$, a right type-D module $N^{\mathcal{A}}$ is a right $\mathbf{k}$-module N, equipped with a $\mathbf{k}$-linear structure map*

$$\delta^1 : N \to N \otimes_{\mathbf{k}} \mathcal{A},$$

which satisfies

$$(\mathrm{id}_N \otimes \mu_2) \circ (\delta^1 \otimes \mathrm{id}_{\mathcal{A}}) \circ \delta^1 = 0. \tag{4.7}$$

Remark 4.1. *Given a type-D module $(N^{\mathcal{A}}, \delta^1)$, the associated pair $(N \otimes_{\mathbf{k}} \mathcal{A}, \delta^1 \otimes \mathrm{id}_{\mathcal{A}})$ is a chain complex. The type-D relation 4.7 is equivalent to $\delta^1 \otimes \mathrm{id}_{\mathcal{A}}$ being a differential on $N \otimes_{\mathbf{k}} \mathcal{A}$.*

We first talk about the type-D module for knots.

Definition 4.2. *The* knot surgery algebra $\mathcal{K}$ *is an algebra over the idempotent ring*

$$\mathbf{I} = \mathbf{I}_0 \oplus \mathbf{I}_1,$$

where each of $\mathbf{I}_i$ is rank 1 over $\mathbb{F}_2$. Let i_0 and i_1 be the generators of $\mathbf{I}_0$ and $\mathbf{I}_1$, respectively. We set

$$\mathbf{I}_0 \cdot \mathcal{K} \cdot \mathbf{I}_0 = \mathbb{F}[\mathcal{U}, \mathcal{V}] \quad \textit{and} \quad \mathbf{I}_1 \cdot \mathcal{K} \cdot \mathbf{I}_1 = \mathbb{F}[\mathcal{U}, \mathcal{V}, \mathcal{V}^{-1}].$$

Also

$$\mathbf{I}_0 \cdot \mathcal{K} \cdot \mathbf{I}_1 = 0.$$

There are two special algebra elements, $\sigma, \tau \in \mathbf{I}_1 \cdot \mathcal{K} \cdot \mathbf{I}_0$. These algebra elements satisfy the relation

$$\sigma \cdot f = I(f) \cdot \sigma \quad \textit{and} \quad \tau \cdot f = T(f) \cdot \tau,$$

for $f \in \mathbb{F}[\mathscr{U},\mathscr{V}] = \mathbf{I}_0 \cdot \mathcal{K} \cdot \mathbf{I}_0$. Here $T\colon \mathbb{F}[\mathscr{U},\mathscr{V}] \to \mathbb{F}[\mathscr{U},\mathscr{V},\mathscr{V}^{-1}]$ is the algebra homomorphism satisfying $T(\mathscr{U}) = \mathscr{V}^{-1}$ and $T(\mathscr{V}) = \mathscr{U}\mathscr{V}^2$. The map $I\colon \mathbb{F}[\mathscr{U},\mathscr{V}] \to \mathbb{F}[\mathscr{U},\mathscr{V},\mathscr{V}^{-1}]$ is the canonical inclusion.

In particular, $\mathbf{I}_1 \cdot \mathcal{K} \cdot \mathbf{I}_0$ is generated by two special algebra elements σ and τ, together with the left action of $\mathbb{F}[\mathscr{U},\mathscr{V},\mathscr{V}^{-1}]$, which satisfy the relations

$$\sigma\mathscr{U} = \mathscr{U}\sigma \quad \textit{and} \quad \sigma\mathscr{V} = \mathscr{V}\sigma$$

$$\tau\mathscr{U} = \mathscr{V}^{-1}\tau \quad \textit{and} \quad \tau\mathscr{V} = \mathscr{U}\mathscr{V}^2\tau.$$

Let $\mathbf{x}_1, \cdots, \mathbf{x}_2$ be a free basis of $\mathcal{CFK}(K)$ over $\mathbb{F}[\mathscr{U},\mathscr{V}]$. The type-$D$ module $\mathcal{X}_\lambda(K)^{\mathcal{K}}$ associated to $\mathbb{X}_\lambda(K)$ is defined as following:

(i) The underlying $\mathbf{I}$-module of $\mathcal{X}_\lambda(K)^{\mathcal{K}}$ is

$$\mathcal{X}_\lambda(K)^{\mathcal{K}} = \operatorname{Span}_{\mathbb{F}}(\mathbf{x}_1, \cdots, \mathbf{x}_n) \otimes_{\mathbb{F}} \mathbf{I}.$$

Over $\mathbb{F}$ each $\mathbf{x}_i$ contributes one generator in each idempotent. One can think of the generators in $\mathbf{I}_0$ correspond to $\prod_{s\in\mathbb{Z}} A_s^-$ and the generators in $\mathbf{I}_1$ correspond to $\prod_{s\in\mathbb{Z}} B_s^-$. We denote these generators by $\mathbf{x}_i^0$ and $\mathbf{x}_i^1$.

(ii) The map δ^1 on $\mathbb{X}_\lambda(K)$ contains three summands:

1) Internal δ^1 summands from the differential on $\mathcal{CFK}(K)$. If $\partial(\mathbf{x})$ contains a summand of $\mathbf{y} \cdot \mathscr{U}^n\mathscr{V}^m$, then $\delta^1(\mathbf{x}^\epsilon)$ contains a summand of $\mathbf{y}^\epsilon \otimes \mathscr{U}^n\mathscr{V}^m$, for $\epsilon \in \{0,1\}$.

2) The summand corresponds to the inclusion map v, such that $\delta^1(\mathbf{x}^0)$ contains a summand of the form $\mathbf{x}^1 \otimes \sigma$.

3) The summand corresponds to h_λ, such that if $\mathbf{y} \cdot \mathscr{U}^i\mathscr{V}^j$ is a summand of $h_\lambda(\mathbf{x})$ then we define $\delta^1(\mathbf{x}^\epsilon)$ to have a summand of the form

$$\mathbf{y}^1 \otimes \mathscr{U}^i\mathscr{V}^j\tau.$$

It is been proved in [Zem21a] that $\mathcal{X}_\lambda(K)^{\mathcal{K}}$ is a type-D module.

Example 4.2. *The type-D module associated to λ surgery on the unknot $\mathbb{O}$ is the following (compared to the knot surgery complex in Remark 2.5). The generators are We set*

$$\mathcal{D}_\lambda^{\mathcal{K}} \cdot \mathbf{I}_0 = \langle \mathbf{x}^0 \rangle \quad \text{and} \quad \mathcal{D}_\lambda^{\mathcal{K}} \cdot \mathbf{I}_1 = \langle \mathbf{x}^1 \rangle,$$

where $\langle \mathbf{x}^\varepsilon \rangle = \mathbb{F}$, spanned by $\mathbf{x}^\varepsilon$. The structure map is given by the formula

$$\delta^1(\mathbf{x}^0) = \mathbf{x}^1 \otimes (\sigma + \mathscr{V}^n \tau).$$

When the link has more than one component, we need to use the type-D modules over hypercubes. We begin with the 1-dimensional cube algebra $\mathcal{C}_1$. The algebra $\mathcal{C}_1$ is an algebra over the idempotent ring $\mathbf{I}_0 \oplus \mathbf{I}_1$, where $\mathbf{I}_\varepsilon \cong \mathbb{F}$. We set $\mathbf{I}_\varepsilon \cdot \mathcal{C}_1 \cdot \mathbf{I}_\varepsilon = \mathbf{I}_\varepsilon$, and we set

$$\mathbf{I}_1 \cdot \mathcal{C}_1 \cdot \mathbf{I}_0 = \langle \theta \rangle,$$

i.e. we set $\mathbf{I}_1 \cdot \mathcal{C}_1 \cdot \mathbf{I}_0 \cong \mathbb{F}$, generated by an element θ.

Next, we introduce the *cube algebra* $\mathcal{C}_n$. The definition is

$$\mathcal{C}_n = \mathcal{C}_1 \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathcal{C}_1 = \otimes_{\mathbb{F}}^n \mathcal{C}_1.$$

Note that this is an algebra over the idempotent ring $\mathbf{E}_n = \otimes_{\mathbb{F}}^n \mathbf{I}$. We define the type-$D$ module over hypercubes as the type-D module over $\mathcal{C}_n$.

Using the identification in Remark 4.7, we have the following proposition:

Proposition 4.1 ([Zem21a]). *The category of n-dimensional hypercubes of chain complexes is equivalent to the category of type-D modules over $\mathcal{C}_n$.*

Definition 4.3. *The link algebra $\mathcal{L}_\ell$ is defined as*

$$\mathcal{L}_\ell := \mathcal{K} \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathcal{K}.$$

We often write just $\mathcal{L}$, when ℓ is determined by context. We view $\mathcal{L}_\ell$ as being an algebra over the idempotent ring

$$\mathbf{E}_\ell := \mathbf{I} \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} \mathbf{I}.$$

The type-D module for links depends on a choice of auxiliary data, which is a *system of arcs* $\mathscr{A}$. Since the computation in this thesis is independent of the choice of the system of arcs, we omit the definition here. One can refer to Chapter.9 in [Zem21a] for more details.

Similar to the construction of the type-D module of the framed knot, the type-D module of framed link can be derived from the link surgery complex. Note that, There is an identification of $\mathcal{L}_\ell$-idempotents with points of the cube $\mathbb{E}_\ell$. If $M \subset L$, write $\varepsilon(M) \in \mathbb{E}_\ell$ for the coordinate such that $\varepsilon(M)_i = 0$ if $L_i \notin M$ and $\varepsilon(M)_i = 1$ if $L_i \in M$.

Let $\mathbf{x}_1, \dots, \mathbf{x}_n$ be a free basis of $\mathcal{CFL}(L)$ over $\mathbb{F}[\mathscr{U}_1, \dots, \mathscr{U}_\ell, \mathscr{V}_1, \dots, \mathscr{V}_\ell]$, then the corresponding type-D module is defined as the following:

(i) The generators of $\mathcal{X}_\Lambda(L, \mathscr{A})^{\mathcal{L}_\ell}$ are

$$\mathrm{Span}_{\mathbb{F}}(\mathbf{x}_1, \dots, \mathbf{x}_n) \otimes_{\mathbb{F}} \mathbf{E}_\ell$$

In particular, if $\mathbf{x}$ is a basis element of $\mathcal{CFL}(L)$, we have a generator $\mathbf{x}^\varepsilon$ for each $\varepsilon \in \mathbb{E}_\ell$. By Remark 2.5, we may view $\mathbf{x}^{\varepsilon(M)}$ as an element of the group $\prod_{\mathbf{s}\in\mathbb{H}(L)} \mathfrak{A}^-(\mathcal{H}^{L-M}, \psi^M(\mathbf{s}))$.

(ii) Suppose also that $\vec{N}$ is an oriented sublink of $L - M$, and that $\Phi^{\vec{N}}(\mathbf{x}^{\varepsilon(M)})$ has a summand of $\mathbf{y}^{\varepsilon(M\cup N)} \cdot f$, where we view f as an element of the 2ℓ-variable polynomial ring, localized at the variables $\mathscr{V}_i$ such that $L_i \subset N$. We may naturally view f as being an element of

$$\mathbf{E}_{\varepsilon(M\cup N)} \cdot \mathcal{L}_\ell \cdot \mathbf{E}_{\varepsilon(M\cup N)}.$$

There is an algebra element $t^{\vec{N}}_{\varepsilon(M),\varepsilon(M\cup N)} \in \mathbf{E}_{\varepsilon(M\cup N)} \cdot \mathcal{L}_\ell \cdot \mathbf{E}_{\varepsilon(M)}$ which is the tensor of σ_i for i such that $L_i \subset \vec{N}$ and L_i is oriented the same as L, τ_i for i such that $L_i \subset \vec{N}$ and L_i is oriented oppositely from L, and 1 for i such that $L_i \notin N$. With this notation, we declare $\delta^1(\mathbf{x}^\varepsilon)$ to have the summand

$$\mathbf{y}^{\varepsilon(M\cup N)} \otimes f \cdot t^{\vec{N}}_{\varepsilon(M),\varepsilon(M\cup N)}.$$

4.2.2 Type-A module of solid torus

One can also interpret the type-D module as a bordered invariant of the link complement. Similar to the construction in [LOT18], one can also associate a type-A structure to the link complement. The box tensor of the type-D structure of the link complement and the type-A structure of the solid torus recovers link surgery formula.

Definition 4.4. *Suppose $\mathcal{A}$ is an associative algebra. A* left A_∞-module ${}_{\mathcal{A}}M$ *over $\mathcal{A}$ is a left* $\mathbf{k}$*-module M equipped with* $\mathbf{k}$*-module map*

$$m_{j+1}\colon \mathcal{A}^{\otimes j} \otimes_{\mathbf{k}} M \to M$$

for each $j \geq 0$, such that if $a_n, \ldots, a_1 \in \mathcal{A}$ and $\mathbf{x} \in M$, then

$$\sum_{j=0}^{n} m_{n-j+1}(a_n, \ldots, a_{j+1}, m_{j+1}(a_j, \ldots, a_1, \mathbf{x})) + \sum_{k=1}^{n-1} m_n(a_n, \ldots, a_{k+1}a_k, \ldots, a_1, \mathbf{x}) = 0.$$

Given a type-D module $N^{\mathcal{A}}$ and a type-A module ${}_{\mathcal{A}}M$, the *box tensor product* of Lipshitz, Ozsváth and Thurston [LOT18] construct a chain complex.

The *box tensor product* $N^{\mathcal{A}} \boxtimes {}_{\mathcal{A}}M$ is the chain complex $N \otimes_{\mathbf{k}} M$, with differential $\partial^{\boxtimes}$ as follows. Denote that $\delta^j\colon N \to N \otimes \mathcal{A}^{\otimes j}$ is the map given inductively by $\delta^j = (\mathrm{id}_{\mathcal{A}^{\otimes j-1}} \otimes \delta^1) \circ \delta^{j-1}$ and $\delta^0 = \mathrm{id}$. Then the differential has the formula

$$\partial^{\boxtimes}(\mathbf{x} \otimes \mathbf{y}) = \sum_{j=0}^{\infty} (\mathrm{id}_N \otimes m_{j+1})(\delta^j(\mathbf{x}), \mathbf{y}).$$

The differential is usually depicted via the following diagram:

$$\partial^{\boxtimes}(\mathbf{x} \otimes \mathbf{y}) = \begin{tikzcd} \mathbf{x} \arrow[d] & \mathbf{y} \arrow[d] \\ \delta \arrow[d] \arrow[r, Rightarrow] & m \arrow[d] \\ {} & {} \end{tikzcd} \tag{4.8}$$

According to [LOT18], the map $\partial^{\boxtimes}$ is a differential whenever one of M or N satisfies a *boundedness* assumption.

Suppose that λ is an integer, the type-A module ${}_{\mathcal{K}}\mathcal{D}_\lambda$ for a solid torus is defined as the following:

Definition 4.5. *We set*

$$\mathbf{I}_0 \cdot \mathcal{D}_\lambda = \mathbb{F}[\mathscr{U}, \mathscr{V}] \quad \textit{and} \quad \mathbf{I}_1 \cdot \mathcal{D}_\lambda = \mathbb{F}[\mathscr{U}, \mathscr{V}, \mathscr{V}^{-1}].$$

We define the type-A structure map m_j on $\mathcal{D}_\lambda$ to be 0 unless $j = 2$. We define m_2 on $\mathcal{D}_\lambda$ as follows. If $f \in \mathbf{I}_i \cdot \mathcal{K} \cdot \mathbf{I}_i$ and $x \in \mathbf{I}_j \cdot \mathcal{D}_j$, then we define $m_2(f, x)$ to be $f \cdot x$ (ordinary multiplication of polynomials) if $i = j$ and to be 0 otherwise. If $x \in \mathbf{I}_0 \cdot \mathcal{D}_\lambda$, we define

$$m_2(\sigma, x) = I(x) \in \mathbb{F}[\mathscr{U}, \mathscr{V}, \mathscr{V}^{-1}] = \mathbf{I}_1 \cdot \mathcal{D}_\lambda.$$

Similarly, we define

$$m_2(\tau, x) = \mathscr{V}^\lambda \cdot T(x) \in \mathbb{F}[\mathscr{U}, \mathscr{V}, \mathscr{V}^{-1}],$$

where $\cdot$ denotes ordinary multiplication of polynomials.

Recall that in the definition of link surgery formula, it involves direct product instead of direct sum, which can be viewed as a completion with respect to the *cofinite topology*.

Definition 4.6. *If $(X_i)_{i \in I}$ is a family of groups, define the* cofinite subspace topology *on $\mathcal{X} = \oplus_{i \in I} X_i$ to be*

$$\mathcal{X}_{co(S)} ; = \bigoplus_{i \in I \setminus S} X_i$$

ranging over finite sets $S \subset I$. When $X_i \cong F$ for all i, we will refer to this topology as the cofinite basis topology.

It is not hard to see that the completion of the direct sum with this topology is direct product.

Similarly, we need to complete the type-A module ${}_{\mathcal{K}}\mathcal{D}_\lambda$. We view ${}_{\mathcal{K}}\mathcal{D}_\lambda$ as the direct sum of a copy of $\mathbb{F}$ over each Alexander and Maslov grading supported by the module, and complete it with respect to the cofinite basis topology. The completion is given by

$$\mathbf{I}_0 \cdot \boldsymbol{\mathcal{D}}_\lambda = \mathbb{F}[[\mathscr{U}, \mathscr{V}]], \quad \text{and} \quad \mathbf{I}_1 \cdot \boldsymbol{\mathcal{D}}_\lambda = \mathbb{F}[[\mathscr{U}, \mathscr{V}, \mathscr{V}^{-1}]].$$

It is not hard to see that the box tensor of type-D module of K with the 0-framed type-A module of the solid torus recovers the knot surgery formula:

$$\mathbb{X}_\lambda(K) \cong \mathcal{X}_\lambda(K)^{\mathcal{K}} \boxtimes {}_{\mathcal{K}}[\mathcal{D}_0]. \tag{4.9}$$

Note that, when taking the box tensor in this case, we use $X_i \otimes Y_j$ as the open subspace, where X_i and Y_j are open subspaces of $\mathcal{X}_\lambda(K)^{\mathcal{K}}$ and ${}_{\mathcal{K}}[\mathcal{D}_0]$ respectively. Withe this topology, the two completions coincide with each other.

Remark 4.2. *There is a similar construction for the link surgery formula, One can refer to Chapter.8 of [Zem21a] for more details.*

4.2.3 Type-DA bimodule and connected sum formula

We also need type-DA bimodules in the definition of the connected sum formula.

Definition 4.7. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are algebras over* $\mathbf{j}$ *and* $\mathbf{k}$*, respectively. A type-DA bimodule, denoted ${}_{\mathcal{A}}M^{\mathcal{B}}$, consists of a $(\mathbf{j},\mathbf{k})$-module M, equipped with $(\mathbf{j},\mathbf{k})$-linear structure morphisms*

$$\delta^1_{j+1} : \mathcal{A}^{\otimes j} \otimes_{\mathbf{j}} M \to M \otimes_{\mathbf{k}} \mathcal{B},$$

which satisfy the following structure relation:

$$\begin{aligned}
&\sum_{j=0}^{n} ((\mathrm{id}_M \otimes \mu_2) \circ (\delta^1_{n-j+1} \otimes \mathrm{id}_{\mathcal{B}}))(a_n, \dots, a_{j+1}, \delta^1_{j+1}(a_j, \dots, a_1, \mathbf{x})) \\
&+ \sum_{k=1}^{n-1} \delta^1_n(a_n, \dots, a_{k+1}a_k, \dots, a_1, \mathbf{x}) = 0.
\end{aligned}$$

The box tensor of a type-DA bimodule and type-D module gives a type-D module, and the differential is given by

$$\delta^{D\boxtimes AD} = \begin{tikzcd}
\mathbf{x} \arrow[d] & \mathbf{y} \arrow[dd] & \\
\delta \arrow[ddd] \arrow[dr, Rightarrow] & & \\
 & \delta^1 \arrow[d] \arrow[dr] & \\
 {} & {} & {}
\end{tikzcd}.$$

The bimodule used in the connected sum formula is the merge bimodule ${}_{\otimes^2\mathcal{K}}M_2{}^{\mathcal{K}}$.

(i) The underlying space is a $(\otimes^2\mathbf{I}, \mathbf{I})$ bimodule. As a vector space, M_2 is $\mathbf{I} = \mathbf{I}_0 + \mathbf{I}_1$. The left action of $\otimes^2\mathbf{I}$ is given by

$$(i_1 \otimes i_2) \cdot i = i_1 \cdot i_2 \cdot i.$$

The right $\mathbf{I}$ -action is the standard action.

(ii) There are two non-trivial structure maps on M_2, namely δ^1_2 and δ^1_{n+1}.

1) The map δ^1_2 is as follows. Suppose that $a_1 \otimes a_2$ is an elementary tensor in either

$$(\otimes^2 \mathbf{I}_0 \cdot \otimes^2 \mathcal{K} \cdot \otimes^2 \mathbf{I}_0) \quad \text{or} \quad (\otimes^2 \mathbf{I}_1 \cdot \otimes^2 \mathcal{K} \cdot \otimes^2 \mathbf{I}_1).$$

In this case, we set

$$\delta^1_2(a_1 \otimes a_2 \otimes i) = i \otimes a_1 a_2.$$

On any other elementary tensor, we set δ^1_2 to vanish.

2) We define δ^1_3 as follows. We set

$$\delta^1_3((1 \otimes \sigma) \otimes (\sigma \otimes 1) \otimes i_0) = i_1 \otimes \sigma, \quad \text{and} \quad \delta^1_3((1 \otimes \tau) \otimes (\tau \otimes 1) \otimes i_0) = i_1 \otimes \tau.$$

More generally, if a, b, c, d are monomials concentrated in single idempotents, then we set

$$\delta^1_3((a \otimes b\sigma) \otimes (c\sigma \otimes d) \otimes i_0) = i_0 \otimes abc\sigma d$$

and similarly for the τ terms. Note that we set

$$\delta^1_3((\sigma \otimes 1) \otimes (1 \otimes \sigma) \otimes i_0) = 0,$$

and similarly if τ replaces σ.

Given a pair of framed links (Y_1, L_1, Λ_1) and (Y_2, L_2, Λ_2). Taking a pair of distinguished components K_1 and K_2, let $(Y_1 \# Y_2, L_1 \# L_2, \Lambda_1 + \Lambda_2)$ denote the connected sum of L_1 and L_2 at the distinguished components. Here, $\Lambda_1 + \Lambda_2$ is $\lambda_1 + \lambda_2$ at the connected sum of the distinguished components and remains the same for other components. We can compute the type-D module of the $(L_1 \# L_2, \Lambda_1 + \Lambda_2)$ by the connected sum formula:

Theorem 4.2 ([Zem21a])**.**

$$\mathcal{X}_{\Lambda_1+\Lambda_2}(Y_1 \# Y_2, L_1 \# L_2)^{\mathcal{L}_{\ell_1+\ell_2-1}} \simeq \left(\mathcal{X}_{\Lambda_1}(Y_1, L_1)^{\mathcal{L}_{\ell_1}} \otimes_{\mathbb{F}} \mathcal{X}_{\Lambda_2}(Y_2, L_2)^{\mathcal{L}_{\ell_2}} \right) \boxtimes {}_{\mathcal{K}^2} M^{\mathcal{K}}.$$

We can also gluing the solid torus to the distinguished component to get a one dimensional less hypercube, $\mathcal{X}_{\Lambda_1+\Lambda_2}(Y_1\#Y_2, L_1\#L_2)^{\mathcal{L}_{\ell_1+\ell_2-1}} \boxtimes_{\mathcal{K}} \mathcal{D}_0$:

$$\left(\mathcal{X}_{\Lambda_1}(Y_1, L_1)^{\mathcal{L}_{\ell_1}} \otimes_{\mathbb{F}} \mathcal{X}_{\Lambda_2}(Y_2, L_2)^{\mathcal{L}_{\ell_2}}\right) \boxtimes {}_{\mathcal{K}^2}M^{\mathcal{K}} \boxtimes_{\mathcal{K}} \mathcal{D}_0. \tag{4.10}$$

Thinking of $\boxtimes_{\mathcal{K}^2}M^{\mathcal{K}} \boxtimes_{\mathcal{K}} \mathcal{D}_0$ as a type-AA bimodule, which is denoted by ${}_{\mathcal{K}^2}[\mathbb{I}^{\supset}]$. Tensoring ${}_{\mathcal{K}^2}[\mathbb{I}^{\supset}]$ to a distinguished component has the effect as transforming the type-D module to the type-A module.

$$\begin{aligned}\mathcal{X}_{\Lambda_1+\Lambda_2}(Y_1\#Y_2, L_1\#L_2)^{\mathcal{L}_{\ell_1+\ell_2-1}} \boxtimes_{\mathcal{K}} \mathcal{D}_0 &= \left(\mathcal{X}_{\Lambda_1}(Y_1, L_1)^{\mathcal{L}_{\ell_1}} \otimes_{\mathbb{F}} \mathcal{X}_{\Lambda_2}(Y_2, L_2)^{\mathcal{L}_{\ell_2}}\right) \boxtimes {}_{\mathcal{K}^2}[\mathbb{I}^{\supset}] \\ &= \left(\mathcal{X}_{\Lambda_1}(Y_1, L_1)^{\mathcal{L}_{\ell_1}} \boxtimes_{\mathcal{K}} \mathcal{X}_{\Lambda_2}(Y_2, L_2)^{\mathcal{L}_{\ell_2-1}}\right)\end{aligned}$$

When both L_1 and L_2 have just one component, 4.10 is isomorphic to $\mathbb{X}_{\lambda_1+\lambda_2}(Y_1\#Y_2, K_1\#K_2)$. We can restate Theorem 4.2 as:

$$\mathbb{X}_{\lambda_1+\lambda_2}(Y_1\#Y_2, K_1\#K_2) \simeq \mathcal{X}_{\lambda_1}(Y_1, K_1)^{\mathcal{K}} \boxtimes {}_{\mathcal{K}}\mathcal{X}_{\lambda_2}(Y_2, K_2).$$

Gluing in the solid torus for the remaining link components, there is a similar statement of the pairing theorem for the link surgery formula. Let $C_{\Lambda_1}(L_1)$ and $C_{\Lambda_2}(L_2)$ be the link surgery hypercubes of Manolescu and Ozsváth. Write

$$C_{\Lambda_i}(L_i) \cong \mathrm{Cone}\left(\ C_0(L_i) \xrightarrow{F^{-K_i}+F^{K_i}} C_1(L_i)\ \right)$$

Here $C_\nu(L_i)$ consists of the complexes of all points of the cube $\mathbb{E}_{\ell_i}$ such that the coordinate for K_i is $\nu \in \{0,1\}$. Also, we are writing F^{K_i} (resp. F^{-K_i}) for the sum of the hypercube maps for all sublinks $\vec{N} \subset L_i$ which contain K_i (resp. $-K_i$).

Theorem 4.2 is equivalent to

Theorem 4.3. *The surgery hypercube $C_{Y_1\#Y_2,\Lambda_1+\Lambda_2}(L_1\#L_2)$ is homotopy equivalent to the $(\ell_1+\ell_2-1)$-dimensional hypercube*

$$\mathrm{Cone}\left(\ C_0(Y_1, L_1) \otimes C_0(Y_2, L_2) \xrightarrow{F^{K_1}\otimes F^{K_2}+F^{-K_1}\otimes F^{-K_2}} C_1(Y_1, L_1) \otimes C_1(Y_2, L_2)\ \right)$$

For each $\nu \in \{0,1\}$, the above complexes $C_\nu(L_1) \otimes C_\nu(L_2)$ are equipped with the tensor product differential $D_1^\nu \otimes \mathrm{id} + \mathrm{id} \otimes D_2^\nu$, where D_j^ν is the total differential of the hypercube $C_\nu(L_j)$ (i.e. the sum of the internal differentials as well as the hypercube maps).

4.3 Computation of d-invariants

In this section, we apply the connected sum formula to compute $d_{top}(H_{g_1,g_2})$. We decompose the link in Figure 1.1 into three summands in Figure 4.3, where we have two copies of the Borromean knot and a Hopf link in the middle.

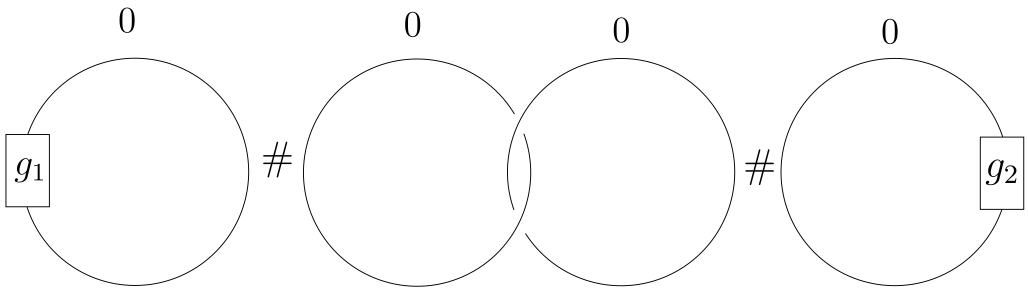


Figure 4.3 Connected sum decomposition of link surgery

4.3.1 Type-D module of Borromean knot

The knot surgery complex of Borromean knot with $\mathbb{Z}_2[U]$ coefficient is computed in [OS08], we rephrase it here with the $\mathbb{Z}_2[\mathscr{U},\mathscr{V}]$ coefficient.

Denote the genus g Borromean knot by B_g.

$$\mathcal{CFK}(B_g) \cong \Lambda^* H^1(\Sigma_g;\mathbb{Z}_2) \otimes \mathbb{Z}_2[\mathscr{U},\mathscr{V}],$$

such that for a generator $\mathbf{x} \in \Lambda^k H^1(\Sigma_g;\mathbb{Z}_2)$, $A(\mathbf{x}) = \mathrm{gr}(\mathbf{x}) = k - g$.

Note that $B_g \subset \#^{2g}(S^2 \times S^1)$. Under the identification $H_1(\Sigma;\mathbb{Z}_2) \cong H_1(\#^{2g}(S^2 \times S^1))$ and $HF^\infty(\#^{2g}(S^2 \times S^1) \cong \Lambda^* H^1(\Sigma_g;\mathbb{Z}_2) \otimes \mathbb{Z}_2[\mathscr{U},\mathscr{U}^{-1},\mathscr{V},\mathscr{V}^{-1}]$, the action of $\gamma \in H_1(\Sigma;\mathbb{Z}_2)$ is given by the formula

$$\gamma \cdot \mathbf{x} = \iota_\gamma \mathbf{x} \otimes \mathscr{V} + PD(\gamma) \wedge \mathbf{x} \otimes \mathscr{U}.$$

Let $\{\alpha_i^*, \beta_i^*\}_{i=1}^g$ be a symplectic basis of homology classes, there is an induced map

$$I : \Lambda^k H^1(\Sigma,\mathbb{Z}_2) \to \Lambda^k H^1(\Sigma,\mathbb{Z}_2),$$

which commutes with wedge product and satisfies $I(\alpha_i) = \beta_i$ and $I(\beta_i) = \alpha_i$. Together with the Hodge star operator

$$* : \Lambda^k H^1(\Sigma; \mathbb{Z}_2) \to \Lambda^{2g-k} H^1(\Sigma; \mathbb{Z}_2),$$

the flip map h is by the following formula. For $\mathbf{x} \in \Lambda^k H^1(\Sigma; \mathbb{Z}_2)$,

$$h : \mathbf{x} \to (*I\mathbf{x}) \otimes v^{2(k-g)}.$$

Note that, the knot surgery complex splits to the summands generated by the pairs $(x, h(x))$. The top generator of the H_1-action can be represented by $x_0 = \alpha_1 \wedge \beta_1 \wedge \cdots \wedge \alpha_g \wedge \beta_g$ or $x_1 = 1$. Hence, to compute $d_{top}(H_{g_1,g_2})$, we just need the type-D module for the pair $(\mathbf{x}_0, \mathbf{x}_1)$, and let us denote it by $B_g^{\mathcal{K}}$.

$B_g^{\mathcal{K}}$ is generated by $\mathrm{Span}(\mathbf{x}_0, \mathbf{x}_1) \otimes \mathbf{I}$. We denote the generator in the $\mathbf{I}_i$ idempotent by $\mathbf{x}_j^i$. The differential is given by

$$\delta^1(\mathbf{x}_0^0) = \mathbf{x}_0^1 \otimes \sigma + \mathbf{x}_1^1 \otimes \mathscr{V}^{2g}\tau,$$

$$\delta^1(\mathbf{x}_1^0) = \mathbf{x}_1^1 \otimes \sigma + \mathbf{x}_0^1 \otimes \mathscr{V}^{-2g}\tau.$$

4.3.2 Type-DA bimodule of Hopf link

We recall the type-DA bimodule for the negative Hopf link ${}_{\mathcal{K}_1}\mathcal{Z}^{\mathcal{K}_2}_{(\lambda_1,0)}$ from Chapter.17 of [Zem21a] in this section.

${}_{\mathcal{K}_1}\mathcal{Z}^{\mathcal{K}_2}_{(\lambda_1,0)}$ can be depicted by the following diagram:

$$\begin{tikzcd}
\mathcal{Z}_{0,0} \arrow[r, "A\text{-direction}"] \arrow[d, "D\text{-direction}"'] \arrow[dr] & \mathcal{Z}_{1,0} \arrow[d] \\
\mathcal{Z}_{0,1} \arrow[r] & \mathcal{Z}_{1,1}
\end{tikzcd}$$

The module is generated by two elements $\mathbf{a}$ and $\mathbf{d}$, such that the Alexander gradings $A = (A_1, A_2)$ and Maslov gradings $\mathrm{gr} = (\mathrm{gr}_{\mathbf{w}}, \mathrm{gr}_{\mathbf{z}})$ are

$$A(\mathbf{a}) = (0, 1) + (-\frac{1}{2}, -\frac{1}{2}), \qquad \mathrm{gr}(\mathbf{a}) = (0, 0),$$

$$A(\mathbf{d}) = (1, 0) + (-\frac{1}{2}, -\frac{1}{2}), \qquad \mathrm{gr}(\mathbf{d}) = (0, 0).$$

For each summand, we have

(i) $\mathcal{Z}_{0,0}$ is generated by $\langle \mathbf{a} \rangle[\mathscr{U}_1] \oplus \langle \mathbf{d} \rangle[\mathscr{V}]$ over $\mathbb{F}[\mathscr{U}_2, \mathscr{V}_2]$,

(ii) $\mathcal{Z}_{1,0}$ is generated by $\langle \mathbf{d} \rangle[\mathscr{V}, \mathscr{V}^{-1}]$ over $\mathbb{F}[\mathscr{U}_2, \mathscr{V}_2]$,

(iii) $\mathcal{Z}_{0,1}$ is generated by $\langle \mathbf{a} \rangle[\mathscr{U}_1] \oplus \langle \mathbf{d} \rangle[\mathscr{V}]$ over $\mathbb{F}[\mathscr{U}_2, \mathscr{V}_2, \mathscr{V}_2^{-1}]$,

(iv) $\mathcal{Z}_{1,1}$ is generated by $\langle \mathbf{d} \rangle[\mathscr{V}, \mathscr{V}^{-1}]$ over $\mathbb{F}[\mathscr{U}_2, \mathscr{V}_2, \mathscr{V}_2^{-1}]$.

The type-D structure maps are depicted as:

$$\delta_1^1 = \begin{tikzcd} \mathcal{Z}_{0,0} \arrow[d, "{}^{L_2}f_1^1 + {}^{-L_2}f_1^1"] & \mathcal{Z}_{1,0} \arrow[d, "{}^{L_2}g_1^1 + {}^{-L_2}g_1^1"] \\ \mathcal{Z}_{0,1} & \mathcal{Z}_{1,1} \end{tikzcd} \qquad \delta_2^1 = \begin{tikzcd} \mathcal{Z}_{0,0} \arrow[loop left, "m_2^1"] \arrow[r, "{}^{L_1}p_2^1 + {}^{-L_1}p_2^1"] & \mathcal{Z}_{1,0} \arrow[loop right, "m_2^1"] \\ \mathcal{Z}_{0,1} \arrow[loop left, "m_2^1"] \arrow[r, "{}^{L_1}q_2^1 + {}^{-L_1}q_2^1"] & \mathcal{Z}_{1,1} \arrow[loop right, "m_2^1"] \end{tikzcd}$$

$$\delta_3^1 = \begin{tikzcd} \mathcal{Z}_{0,0} \arrow[dr, "{}^{-H}\omega_3^1"] & \mathcal{Z}_{1,0} \\ \mathcal{Z}_{0,1} & \mathcal{Z}_{1,1} \end{tikzcd}$$

The type-D maps are computed in Chapter.17 of [Zem21a].

Proposition 4.2 ([Zem21a]). *Give the negative Hopf link framing $(\lambda_1, 0)$. The structure maps of the minimal model ${}_{\mathcal{K}_1}\mathcal{Z}^{\mathcal{K}_2}_{(\lambda_1,0)}$ are as follows:*

(i) For the summand $\mathcal{Z}_{0,0}$ and $\mathcal{Z}_{1,0}$, the m_2^1 is given by

a) $m_2^1(\mathscr{U}_1, \mathscr{U}_1^n \mathbf{a}) = \mathscr{U}_1^{n+1}\mathbf{a} \otimes 1.$

b) $m_2^1(\mathscr{V}_1, \mathscr{U}_1^n \mathbf{a}) = \begin{cases} \mathscr{U}_1^{n-1}\mathbf{a} \otimes \mathscr{U}_2\mathscr{V}_2 & \text{if } n > 0 \\ \mathbf{d} \otimes \mathscr{V}_2 & \text{if } n = 0. \end{cases}$

c) $m_2^1(\mathscr{V}_1, \mathscr{V}_1^m \mathbf{d}) = \mathscr{V}_1^{m+1}\mathbf{d} \otimes 1.$

d) $m_2^1(\mathscr{U}_1, \mathscr{V}_1^m \mathbf{d}) = \begin{cases} \mathscr{V}_1^{m-1}\mathbf{d} \otimes \mathscr{U}_2\mathscr{V}_2 & \text{if } m > 0 \\ \mathbf{a} \otimes \mathscr{U}_2 & \text{if } m = 0. \end{cases}$

For the summand $\mathcal{Z}_{0,1}$ *and* $\mathcal{Z}_{1,1}$*, the* m_2^1 *is given by*

$$m_2^1(\mathscr{U}_1^i\mathscr{V}_1^j, \mathscr{V}_1^n\mathbf{d}) = \mathscr{V}_1^{j+n-i}\mathbf{d}\otimes\mathscr{U}_2^i\mathscr{V}_2^i.$$

(ii) *The maps* ${}^{L_1}p_2^1$ *and* ${}^{L_1}q_2^1$ *are given by the same formulas as each other, as are* ${}^{-L_1}p_2^1$ *and* ${}^{-L_1}q_2^1$*. They are determined by the following formulas:*

a) ${}^{L_1}p_2^1(\sigma_1, \mathscr{U}_1^i\mathbf{a}) = \mathscr{V}_1^{-i-1}\mathbf{d}\otimes\mathscr{U}_2^i\mathscr{V}_2^{i+1}$ *and* ${}^{L_1}p_2^1(\sigma_1, \mathscr{V}_1^j\mathbf{d}) = \mathscr{V}_1^j\mathbf{d}\otimes 1$.

b) ${}^{-L_1}p_2^1(\tau_1, \mathscr{U}_1^i\mathbf{a}) = \mathscr{V}_1^{-i-1+\lambda_1}\mathbf{d}\otimes 1$ *and* ${}^{-L_1}p_2^1(\tau_1, \mathscr{V}_1^j\mathbf{d}) = \mathscr{V}_1^{j+\lambda_1}\mathbf{d}\otimes\mathscr{U}_2^{j+1}\mathscr{V}_2^j$.

c) ${}^{L_1}p_2^1(\tau_1, -) = 0$ *and* ${}^{-L_1}p_2^1(\sigma_1, -) = 0$.

(iii) *The maps for* $\pm L_2$ *are as follows:*

a) ${}^{L_2}f_1^1(\mathscr{U}_1^i\mathbf{a}) = \mathscr{U}_1^i\mathbf{a}\otimes\sigma_2$ *and* ${}^{L_2}f_1^1(\mathscr{V}_1^j\mathbf{d}) = \mathscr{V}_1^j\mathbf{d}\otimes\sigma_2$.

b) ${}^{-L_2}f_1^1(\mathscr{U}_1^i\mathbf{a}) = \mathscr{U}_1^{i+1}\mathbf{a}\otimes\tau_2$ *and*

$${}^{-L_2}f_1^1(\mathscr{V}_1^j\mathbf{d}) = \begin{cases} \mathbf{a}\otimes\mathscr{V}_2^{-1}\tau_2 & \text{if } j = 0 \\ \mathscr{V}_1^{j-1}\mathbf{d}\otimes\tau_2 & \text{if } j > 0. \end{cases}$$

c) ${}^{L_2}g_1^1(\mathscr{V}_1^i\mathbf{d}) = \mathscr{V}_1^i\mathbf{d}\otimes\sigma_2$.

d) ${}^{-L_2}g_1^1(\mathscr{V}_1^i\mathbf{d}) = \mathscr{V}_1^{i-1}\mathbf{d}\otimes\tau_2$.

(iv) *The map* ${}^{-H}\omega_3^1$ *is determined by the relations*

$$\begin{aligned} {}^{-H}\omega_3^1(\tau_1, \mathscr{V}_1^m, \mathscr{U}_1^i\mathbf{a}) &= \min(i+1, m)\mathscr{V}_1^{\lambda_1+m-i-2}\mathbf{d}\otimes\mathscr{U}_1^{m-1}\mathscr{V}_1^{m-1}\tau_2 \\ {}^{-H}\omega_3^1(\tau_1, \mathscr{U}_1^n, \mathscr{U}_1^i\mathbf{a}) &= 0 \\ {}^{-H}\omega_3^1(\tau_1, \mathscr{U}_1^n, \mathscr{V}_1^j\mathbf{d}) &= \min(n, j)\mathscr{V}_1^{j-n+\lambda_1-1}\mathbf{d}\otimes\mathscr{U}_1^{j-1}\mathscr{V}_2^{j-2}\tau_2 \\ {}^{-H}\omega_3^1(\tau_1, \mathscr{V}_1^m, \mathscr{V}_1^j\mathbf{d}) &= 0, \end{aligned}$$

and that ${}^{-H}\omega_3^1$ *vanishes if an algebra input is a multiple of* σ_1*. The map* ${}^{-H}\omega_3^1$ *also vanishes on pairs of algebra elements with other configurations of idempotents.*

Below is the summand ${}_{\mathcal{K}_1}\mathcal{Z}^{\mathcal{K}_2}_{(0,0)} \cdot \mathbf{I}_0$, where an arrow decorated with $a|b$ from $\mathbf{x}$ to $\mathbf{y}$ means that $\delta^1_2(a, \mathbf{x}) = \mathbf{y} \otimes b$ and we set $U_2 = \mathscr{U}_2\mathscr{V}_2$.

$$
\begin{tikzcd}
\cdots & \mathscr{U}_1^2\mathbf{a} \arrow[r, dashed, shift left, "\mathscr{V}_1|U_2"] \arrow[d, dashed, bend right, "\tau_1|1"'] \arrow[d, bend left, "\sigma_1|U_2^2\mathscr{V}_2"] & \mathscr{U}_2\mathbf{a} \arrow[l, shift left, "\mathscr{U}_1|1"] \arrow[r, dashed, shift left, "\mathscr{V}_1|U_2"] \arrow[d, dashed, bend right, "\tau_1|1"'] \arrow[d, bend left, "\sigma_1|U_2\mathscr{V}_2"] & \mathbf{a} \arrow[l, shift left, "\mathscr{U}_1|1"] \arrow[r, dashed, shift left, "\mathscr{V}_1|\mathscr{V}_2"] \arrow[d, dashed, bend right, "\tau_1|1"'] \arrow[d, bend left, "\sigma_1|\mathscr{V}_2"] & \mathbf{d} \arrow[l, shift left, "\mathscr{U}_1|\mathscr{U}_2"] \arrow[r, dashed, shift left, "\mathscr{V}_1|1"] \arrow[d, dashed, bend right, "\tau_1|\mathscr{U}_2"'] \arrow[d, bend left, "\sigma_1|1"] & \mathscr{V}_1\mathbf{d} \arrow[l, dashed, shift left, "\mathscr{U}_1|U_2"] \arrow[r, dashed, shift left, "\mathscr{V}_1|1"] \arrow[d, dashed, bend right, "\tau_1|\mathscr{U}_2U_2"'] \arrow[d, bend left, "\sigma_1|1"] & \mathscr{V}_1^2\mathbf{d} \arrow[l, dashed, shift left, "\mathscr{U}_1|U_2"] \arrow[d, dashed, bend right, "\tau_1|\mathscr{U}_2U_2^2"'] \arrow[d, bend left, "\sigma_1|1"] & \cdots \\
\cdots & \mathscr{V}_1^{-3}\mathbf{d} \arrow[r, dashed, shift left, "\mathscr{V}_1|1"] & \mathscr{V}_1^{-2}\mathbf{d} \arrow[l, dashed, shift left, "\mathscr{U}_1|U_2"] \arrow[r, dashed, shift left, "\mathscr{V}_1|1"] & \mathscr{V}_1^{-1}\mathbf{d} \arrow[l, dashed, shift left, "\mathscr{U}_1|U_2"] \arrow[r, dashed, shift left, "\mathscr{V}_1|1"] & \mathbf{d} \arrow[l, dashed, shift left, "\mathscr{U}_1|U_2"] \arrow[r, dashed, shift left, "\mathscr{V}_1|1"] & \mathscr{V}_1\mathbf{d} \arrow[l, dashed, shift left, "\mathscr{U}_1|U_2"] \arrow[r, dashed, shift left, "\mathscr{V}_1|1"] & \mathscr{V}_1^2\mathbf{d} \arrow[l, dashed, shift left, "\mathscr{U}_1|U_2"] & \cdots
\end{tikzcd}
$$

Note that, ${}_{\mathcal{K}_1}\mathcal{Z}^{\mathcal{K}_2}_{(0,0)} \cdot \mathbf{I}_1$ is the localization of ${}_{\mathcal{K}_1}\mathcal{Z}^{\mathcal{K}_2}_{(0,0)} \cdot \mathbf{I}_0$ at $\mathscr{V}_2$.

4.3.3 Link Floer complex

We first compute the type-D module of connected sum of genus g_1 Borromean knot and Hopf link, which is the type-D module of the dual knot of 0-surgery on the Borromean knot.

We compute $B_g^{\mathcal{K}} \boxtimes {}_{\mathcal{K}}\mathcal{Z}^{\mathcal{K}}_{0,0} \cdot \mathbf{I}_0$ first.

Let $\mathbf{x}_i^m := \mathbf{x}_i^0\mathscr{U}_1^m\mathbf{a}$, $\mathbf{y}_i^m := \mathbf{x}_i^0\mathscr{V}_1^m\mathbf{d}$, and $\mathbf{z}_i^m := \mathbf{x}_i^1\mathscr{V}_1^m\mathbf{d}$. Then $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are the generators of the box tensor.

The differentials are given by

$$
\begin{aligned}
\delta(\mathbf{x}_0^m) &= \mathbf{z}_0^{-m-1} \otimes \mathscr{U}_2^m\mathscr{V}_2^{m+1} + \mathbf{z}_1^{-m-1+2g} \\
\delta(\mathbf{x}_1^m) &= \mathbf{z}_1^{-m-1} \otimes \mathscr{U}_2^m\mathscr{V}_2^{m+1} + \mathbf{z}_0^{-m-1-2g} \\
\delta(\mathbf{y}_0^m) &= \mathbf{z}_0^m + \mathbf{z}_1^{m+2g} \otimes \mathscr{U}_2^{m+1}\mathscr{V}_2^m \\
\delta(\mathbf{y}_1^m) &= \mathbf{z}_1^m + \mathbf{z}_1^{m-2g} \otimes \mathscr{U}_2^{m+1}\mathscr{V}_2^m
\end{aligned}
$$

It can be depicted as following, where the coefficient are labeled on the arrow. The arrow without coefficient has coefficient 1.

$$
\begin{tikzcd}
\cdots & \mathbf{x}_0^{2g-1} \arrow[d, "\mathscr{U}^{2g-1}\mathscr{V}^{2g}"'] \arrow[dr, "\mathscr{U}_2"] & \mathbf{y}_1^0 \arrow[dl] \arrow[d] & \cdots & \mathbf{x}_0^g \arrow[d, "\mathscr{U}_2^g\mathscr{V}_2^{g+1}"'] \arrow[dr, "\mathscr{U}_2^g\mathscr{V}_2^{g-1}"] & \mathbf{y}_1^{g-1} \arrow[dl] \arrow[d] & \cdots & \mathbf{x}_0^0 \arrow[d, "\mathscr{V}_2"'] \arrow[dr, "\mathscr{U}_2^{2g}\mathscr{V}_2^{2g-1}"] & \mathbf{y}_1^{2g-1} \arrow[dl] \arrow[d] & \cdots \\
\cdots & \mathbf{z}_0^{-2g} & \mathbf{z}_1^0 & \cdots & \mathbf{z}_0^{-g-1} & \mathbf{z}_1^{g-1} & \cdots & \mathbf{z}_0^{-1} & \mathbf{z}_1^{2g-1} & \cdots
\end{tikzcd}
$$

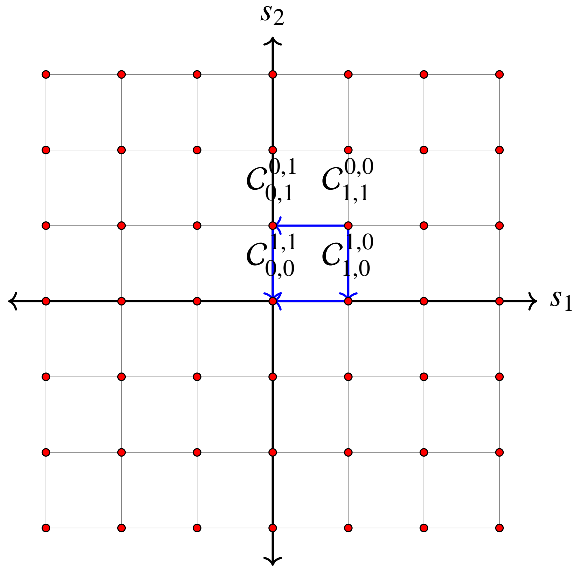


Figure 4.4 $C_{0,0}(B_{g_1}, B_{g_2})$ with a summand

$B_g^{\mathcal{K}} \boxtimes_{\mathcal{K}} \mathcal{Z}_{0,0}^{\mathcal{K}} \cdot \mathbf{I}_1$ is the localization of $B_g^{\mathcal{K}} \boxtimes_{\mathcal{K}} \mathcal{Z}_{0,0}^{\mathcal{K}} \cdot \mathbf{I}_1$ by $\mathscr{V}_{-1}$. Let us denote the corresponding generators by $\tilde{\mathbf{x}}_i^m$, $\tilde{\mathbf{y}}_i^m$, and $\tilde{\mathbf{z}}_i^m$.

Then the type-D structure map is given by

$$\delta^1(\mathbf{x}_i^m) = \tilde{\mathbf{x}}_i^m \otimes \sigma + \tilde{\mathbf{x}}_i^{m+1} \otimes \tau$$

$$\delta^1(\mathbf{y}_i^m) = \begin{cases} \tilde{\mathbf{y}}_i^m \otimes \sigma + \tilde{\mathbf{y}}_i^{m-1} \otimes \tau & \text{for } m > 0, \\ \tilde{\mathbf{y}}_i^0 \otimes \sigma + \tilde{\mathbf{x}}_i^0 \otimes \tau & \text{for } m = 0 \end{cases}$$

$$\delta^1(\mathbf{z}_i^m) = \tilde{\mathbf{z}}_i^m \otimes \sigma + \tilde{\mathbf{z}}_i^{m-1} \otimes \tau$$

We then apply the connected sum formula again and get the link surgery complex

$$C := C_{0,0}(B_{g_1}, B_{g_2}) = ((B_{g_1}^{\mathcal{K}} \boxtimes_{\mathcal{K}} \mathcal{Z}_{0,0}^{\mathcal{K}}) \otimes_{\mathbb{F}} B_{g_2}^{\mathcal{K}}) \boxtimes_{\mathcal{K}^2} M^{\mathcal{K}} \boxtimes_{\mathcal{K}} \mathcal{D}_0.$$

Let use $C_{s_1,s_2}^{\varepsilon_1,\varepsilon_2}$ to denote the summand of $C_{0,0}(B_{g_1}, B_{g_2})$, which has Alexander grading $(s_1, s_2) + (-\frac{1}{2}, -\frac{1}{2})$ and at $(\varepsilon_1, \varepsilon_2)$ vertex of the hypercube. We use Figure 4.4 to represent $C_{0,0}(B_{g_1}, B_{g_2})$, where each dot at (s_1, s_2) represent $\oplus_{\varepsilon_1,\varepsilon_2} C_{s_1,s_2}^{\varepsilon_1,\varepsilon_2}$. We also include a subcomplex in Figure 4.4.

We now describe the generators of $C^{\varepsilon_1,\varepsilon_2}$.

(i) $C^{0,0}$ is generated by $\mathbf{x}_{i,j}^m := \mathbf{x}_i^m \otimes \mathbf{x}_j^0$ and $\mathbf{y}_{i,j}^m := \mathbf{y}_i^m \otimes \mathbf{x}_j^0$.

(ii) $C^{1,0}$ is generated by $\mathbf{z}_{i,j}^{m} := \mathbf{z}_i^m \otimes \mathbf{x}_j^1$.

(iii) $C^{0,1}$ is generated by $\tilde{\mathbf{x}}_{i,j}^{m} := \tilde{\mathbf{x}}_i^m \otimes \mathbf{x}_j^0$ and $\tilde{\mathbf{y}}_{i,j}^{m} := \tilde{\mathbf{y}}_i^m \otimes \mathbf{x}_j^0$.

(iv) $C^{1,1}$ is generated by $\tilde{\mathbf{z}}_{i,j}^{m} := \tilde{\mathbf{z}}_i^m \otimes \mathbf{x}_j^1$.

The gradings at these generators are:

(i) $A(\mathbf{x}_{i,j}^m) = A(\tilde{\mathbf{x}}_{i,j}^m) = (A(\mathbf{x}_i), A(\mathbf{x}_j) + 1) + (-\frac{1}{2}, -\frac{1}{2})$ and $\mathrm{gr}(\mathbf{x}_{i,j}^m) = \mathrm{gr}(\tilde{\mathbf{x}}_{i,j}^m) = \mathrm{gr}(\mathbf{x}_i) + \mathrm{gr}(\mathbf{x}_j)$,

(ii) $A(\mathbf{y}_{i,j}^m) = A(\tilde{\mathbf{y}}_{i,j}^m) = (A(\mathbf{x}_i + 1), A(\mathbf{x}_j)) + (-\frac{1}{2}, -\frac{1}{2})$ and $\mathrm{gr}(\mathbf{y}_{i,j}^m) = \mathrm{gr}(\tilde{\mathbf{y}}_{i,j}^m) = \mathrm{gr}(\mathbf{x}_i) + \mathrm{gr}(\mathbf{x}_j)$,

(iii) $A(\mathbf{z}_{i,j}^m) = A(\tilde{\mathbf{z}}_{i,j}^m) = (A(\mathbf{x}_i + 1), A(\mathbf{x}_j)) + (-\frac{1}{2}, -\frac{1}{2})$ and $\mathrm{gr}(\mathbf{z}_{i,j}^m) = \mathrm{gr}(\tilde{\mathbf{z}}_{i,j}^m) = \mathrm{gr}(\mathbf{x}_i) + \mathrm{gr}(\mathbf{x}_j)$.

Recall that, for an element $\mathbf{e}$, the coefficient ring acts on the grading by the following:

$$A(\mathscr{U}_1^{k_1}\mathscr{V}_1^{l_1}\mathscr{U}_2^{k_2}\mathscr{V}_2^{l_2}\mathbf{e}) = A(\mathbf{e}) + (l_1 - k_1, l_2 - k_2),$$

$$\mathrm{gr}(\mathscr{U}_1^{k_1}\mathscr{V}_1^{l_1}\mathscr{U}_2^{k_2}\mathscr{V}_2^{l_2}\mathbf{e}) = \mathrm{gr}(\mathbf{e}) - 2k_1 - 2k_2.$$

We use $f^{\pm L_i}$ to denote the corresponding maps in the link surgery formula. We also define the flip map τ on the set $\{0, 1\}$, such that $\tau(0) = 1$ and $\tau(1) = 0$.

(i) $f^{L_1}(\mathbf{x}_{i,j}^m \otimes \mathscr{U}_2^k\mathscr{V}_2^l) = \mathbf{z}_{i,j}^{-m-1} \otimes \mathscr{U}_2^{k+m}\mathscr{V}_2^{l+m+1}$
$f^{L_1}(\mathbf{x}_{i,j}^m \otimes \mathscr{U}_2^k\mathscr{V}_2^l) = \mathbf{z}_{i,j}^{-m-1} \otimes \mathscr{U}_2^{k+m}\mathscr{V}_2^{l+m+1}$
$f^{L_1}(\mathbf{y}_{i,j}^m \otimes \mathscr{U}_2^k\mathscr{V}_2^l) = \mathbf{z}_{(i),j}^{m} \otimes \mathscr{U}_2^{k}\mathscr{V}_2^{l}$
$f^{L_1}(\mathbf{y}_{i,j}^m \otimes \mathscr{U}_2^k\mathscr{V}_2^l) = \mathbf{z}_{(i),j}^{m} \otimes \mathscr{U}_2^{k}\mathscr{V}_2^{l}$

(ii) $f^{-L_1}(\mathbf{x}_{i,j}^m \otimes \mathscr{U}_2^k\mathscr{V}_2^l) = \mathbf{z}_{\tau(i),j}^{A(\mathbf{x}_i)-A(\mathbf{x}_{\tau(i)})-m-1} \otimes \mathscr{U}_2^{k}\mathscr{V}_2^{l}$
$f^{-L_1}(\tilde{\mathbf{x}}_{i,j}^m \otimes \mathscr{U}_2^k\mathscr{V}_2^l) = \tilde{\mathbf{z}}_{\tau(i),j}^{A(\mathbf{x}_i)-A(\mathbf{x}_{\tau(i)})-m-1} \otimes \mathscr{U}_2^{k}\mathscr{V}_2^{l}$
$f^{-L_1}(\mathbf{y}_{i,j}^m \otimes \mathscr{U}_2^k\mathscr{V}_2^l) = \mathbf{z}_{\tau(i),j}^{A(\mathbf{x}_i)-A(\mathbf{x}_{\tau(i)})+m} \otimes \mathscr{U}_2^{k+m+1}\mathscr{V}_2^{l+m}$
$f^{-L_1}(\tilde{\mathbf{y}}_{i,j}^m \otimes \mathscr{U}_2^k\mathscr{V}_2^l) = \tilde{\mathbf{z}}_{\tau(i),j}^{A(\mathbf{x}_i)-A(\mathbf{x}_{\tau(i)})+m} \otimes \mathscr{U}_2^{k+m+1}\mathscr{V}_2^{l+m}$

(iii) $f^{L_2}(\mathbf{x}_{i,j}^m \otimes \mathscr{U}_2^k\mathscr{V}_2^l) = \tilde{\mathbf{x}}_{i,j}^{m} \otimes \mathscr{U}_2^{k}\mathscr{V}_2^{l}$
$f^{L_2}(\mathbf{y}_{i,j}^m \otimes \mathscr{U}_2^k\mathscr{V}_2^l) = \tilde{\mathbf{y}}_{i,j}^{m} \otimes \mathscr{U}_2^{k}\mathscr{V}_2^{l}$
$f^{L_2}(\mathbf{z}_{i,j}^m \otimes \mathscr{U}_2^k\mathscr{V}_2^l) = \tilde{\mathbf{z}}_{i,j}^{m} \otimes \mathscr{U}_2^{k}\mathscr{V}_2^{l}$

(iv) $f^{-L_2}(\mathbf{x}^m_{i,j} \otimes \mathscr{U}_2^k \mathscr{V}_2^l) = \tilde{\mathbf{x}}^{m+1}_{i,\tau(j)} \otimes \mathscr{U}_2^k \mathscr{V}_2^{l+A(x_j)-A(x_{\tau(j)})}$

$f^{-L_2}(\mathbf{y}^m_{i,j} \otimes \mathscr{U}_2^k \mathscr{V}_2^l) = \tilde{\mathbf{y}}^{m-1}_{i,\tau(j)} \otimes \mathscr{U}_2^k \mathscr{V}_2^{l+A(x_j)-A(x_{\tau(j)})}$, when $m > 0$

$f^{-L_2}(\mathbf{y}^0_{i,j} \otimes \mathscr{U}_2^k \mathscr{V}_2^l) = \tilde{\mathbf{x}}^{0}_{i,\tau(j)} \otimes \mathscr{U}_2^k \mathscr{V}_2^{l+A(x_j)-A(x_{\tau(j)})-1}$

$f^{-L_2}(\mathbf{z}^m_{i,j} \otimes \mathscr{U}_2^k \mathscr{V}_2^l) = \tilde{\mathbf{z}}^{m-1}_{i,\tau(j)} \otimes \mathscr{U}_2^k \mathscr{V}_2^{l+A(x_j)-A(x_{\tau(j)})}$

Based on the description above, one can check we have the following properties of the map in the link surgery complex.

Proposition 4.3. *(i) f^{L_1} induces isomorphism when $s_1 \geq 1 + g_1$,*

(ii) f^{L_2} induces isomorphism when $s_2 \geq 1 + g_2$,

(iii) f^{-L_1} induces isomorphism when $s_1 \leq -g_1$,

(iv) f^{-L_2} induces isomorphism when $s_2 \leq -g_2$.

4.3.4 Computation of d-invariants

We first quotient out the acyclic subcomplex in C to reduce it to a finitely generated module. To do it, we use the homological algebra argument in Chapter.4 of [MO10]. We will assign a specific filtration to the complex, and then prove that the *associated graded* complex is acyclic, which implies that the complex itself is acyclic.

Following the convention in [MO10], a *filtration* $\mathcal{F}$ on a $\mathcal{R}$-module $\mathcal{A}$ is a collection of $\mathcal{R}$-submodules $\{\mathcal{F}^i(\mathcal{A}) \mid i \in \mathbb{Z}\}$ of $\mathcal{A}$ such that $\mathcal{F}^i(\mathcal{A}) \subseteq \mathcal{F}^j(\mathcal{A})$ for all $i \leq j$. A filtration is called *bounded above* if $\mathcal{F}^i(\mathcal{A}) = \mathcal{A}$ for $i \gg 0$, and *bounded below* if $\mathcal{F}^i(\mathcal{A}) = 0$ for $i \ll 0$. A filtration is called *bounded* if it is both bounded above and bounded below.

If $\mathcal{A}$ is equipped with a differential ∂ that turns it into a chain complex, we say that the chain complex $(\mathcal{A}, \partial)$ is filtered by $\mathcal{F}$ if ∂ preserves each submodule $\mathcal{F}^i(\mathcal{A})$. The *associated graded* complex $\mathrm{gr}_{\mathcal{F}}\mathcal{A}$ is defined as

$$\mathrm{gr}_{\mathcal{F}}(\mathcal{A}) = \bigoplus_{i \in \mathbb{Z}} \big(\mathcal{F}^i(\mathcal{A})/\mathcal{F}^{i-1}(\mathcal{A})\big), \tag{4.11}$$

equipped with the differential induced from $\mathcal{F}$.

If $\mathcal{F}$ is a bounded filtration on a chain complex $(\mathcal{A}, \partial)$, a standard result from homological algebra says that if $\mathrm{gr}_{\mathcal{F}}(\mathcal{A})$ is acyclic, then $\mathcal{A}$ is acyclic as well.

A standard way to construct bounded filtrations is as follows. If $\mathcal{A}$ is freely generated over $\mathcal{R}$ by a collection of generators G, a bounded map $\mathcal{F} : G \to \mathbb{Z}$ defines a bounded filtration on $\mathcal{A}$ by letting $\mathcal{F}^i(\mathcal{A})$ be the submodule generated by the elements $g \in G$ with $\mathcal{F}(g) \leq i$.

Suppose now that we have a direct product of $\mathcal{R}$-modules

$$\mathcal{A} = \prod_{s \in S} \mathcal{A}_s,$$

indexed over a countable set S. Suppose further that each $\mathcal{A}_s$ is a free module over $\mathcal{R}$ with a set of generators G_s. Assume that $\mathcal{A}$ is equipped with a differential ∂.

In this situation, an assignment

$$\mathcal{F} : \bigcup_{s \in S} G_s \to \mathbb{Z}$$

specifies bounded filtrations on each $\mathcal{A}_s$. Together these produce a filtration on $\mathcal{A}$ given by

$$\mathcal{F}^i(\mathcal{A}) = \prod_{s \in S} \mathcal{F}^i(\mathcal{A}_s).$$

This filtration on $\mathcal{A}$ is generally neither bounded above nor bounded below. It is bounded above provided that there exists $i \gg 0$ such that $\mathcal{F}^i(\mathcal{A}_s) = \mathcal{A}_s$ for all s; that is, if $\mathcal{F}(g) \leq i$ for all $g \in G_s, s \in S$. We have the following:

Lemma 4.3.1. *Consider a module $\mathcal{A} = \prod_{s \in S} \mathcal{A}_s$, where each $\mathcal{A}_s$ is a freely generated over $\mathcal{R}$ by a set of generators G_s. Suppose $\mathcal{F} : \bigcup_{s \in S} G_s \to \mathbb{Z}$ defines a filtration on $\mathcal{A}$ that is bounded above. Further, suppose $\mathcal{A}$ is equipped with a differential ∂, and that the associated graded complex* $\mathrm{gr}_{\mathcal{F}}(\mathcal{A})$ *is acyclic. Then $\mathcal{A}$ itself is acyclic.*

Let $C_{(s_1,s_2) \leq (g_1,g_2)}$ denote the subcomplex $\prod_{s_1,s_2 \leq (g_1,g_2)} C^{\varepsilon_1,\varepsilon_2}_{s_1,s_2}$ and C_1 denote the quotient complex $C/C_{(s_1,s_2) \leq (g_1,g_2)}$.

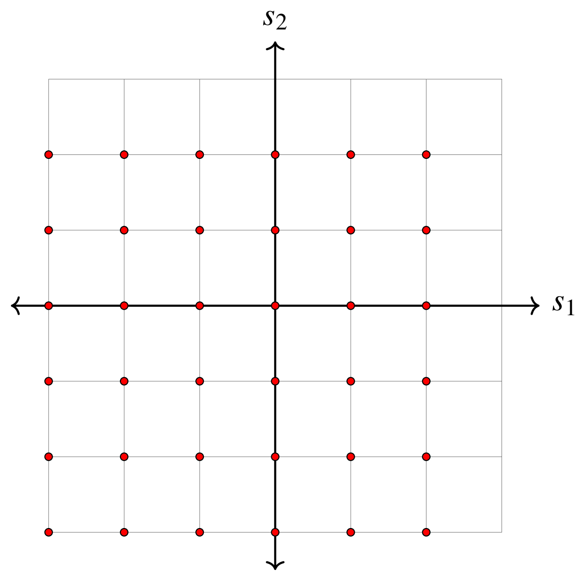


Figure 4.5 $C_{(s_1,s_2)\leq(2,2)}$, for $(g_1, g_2) = (2, 2)$ case

Lemma 4.3.2. $H_*(C_1) = 0$.

Proof. Given a generator $\mathbf{x}$ in $C^{\varepsilon_1,\varepsilon_2}_{s_1,s_2}$, define the filtration by

$$\mathcal{F}_{0,0}(\mathbf{x}) = -s_1 - s_2.$$

This filtration is bounded above on C_1. Hence, by Lemma 4.3.1, we just need to show the associated graded complex is acyclic. The associated graded complex are generated by $C^{\varepsilon_1,\varepsilon_2}_{s_1,s_2}$, such that either $s_1 > g_1$ or $s_2 > g_2$. The remaining differential are f^{L_1} and f^{L_2}. Then by Proposition 4.3, at least one of f^{L_1} or f^{L_2} induces isomorphism. Hence, the associated graded complex is acyclic. □

Hence, we have $H_*(C) \cong H_*(C_{(s_1,s_2)\leq(g_1,g_2)})$.

Let $C_{s_1\leq -g_1}$ denote the subcomplex of $C_{(s_1,s_2)\leq(g_1,g_2)}$ consist of $C^{\varepsilon_1,\varepsilon_2}_{s_1,s_2}$, such that $s_1 \leq -g_1$.

Lemma 4.3.3. $H_*(C_{s_1\leq -g_1}) = 0$.

Proof. Using the filtration $\mathcal{F}_{0,1} = s_1 + s_2 + \varepsilon_1$, which is bounded above, the associated graded complex are

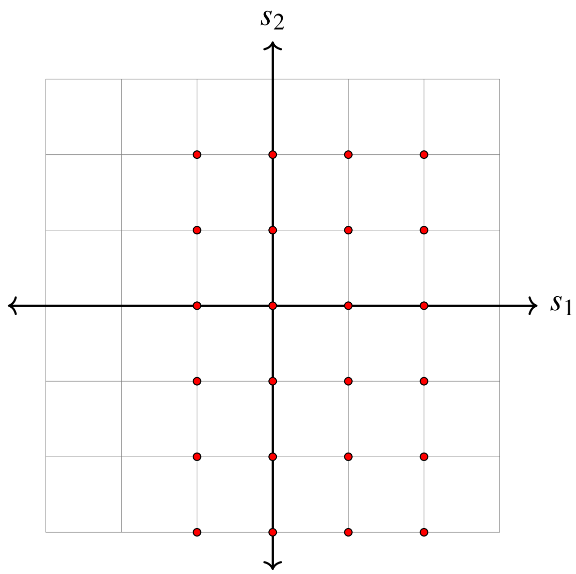


Figure 4.6 C_2, for $(g_1, g_2) = (2, 2)$ case

$$\begin{tikzcd} C^{0,1}_{s_1,s_2} \arrow[d, "f^{-L_1}"] & C^{0,0}_{s_1,s_2} \arrow[l] \arrow[d, "f^{-L_1}"] \\ C^{1,1}_{s_1,s_2-1} & C^{1,0}_{s_1,s_2-1} \arrow[l] \end{tikzcd},$$

when $s_2 + \varepsilon_1 \leq g_2$ and

$$C^{1,1}_{s_1,g_2} \longleftarrow C^{1,0}_{s_1,g_2} .$$

Since by Proposition 4.3, f^{-L_2} induces isomorphism when $s_1 \leq -g_1$, the associated graded complex with $s_2 + \varepsilon_1 \leq g_2$ is acyclic. For $s_2 + \varepsilon_1 = g_2$, f^{L_2} induces an isomorphism, hence it is also acyclic. $\square$

Let $C_2 := C_1/C_{s_1 \leq -g_1}$, then we have $H_*(C_2) \cong H_*(C)$.

Similarly, we can quotient out the subcomplex $C^{\varepsilon_1,\varepsilon_2}_{s_1,s_2}$, such that $s_1 + \varepsilon_2 > 1 - g_1$ and $s_2 \leq 1 - g_2$. Let us denote the quotient complex by C_3.

Note that, f^{-L_1} also induces an isomorphism on $\begin{tikzcd} C^{0,0}_{1-g_1,s_2} \arrow[d, "f^{-L_1}"] \\ C^{1,0}_{1-g_1,s_2-1} \end{tikzcd}$, when $s_2 \leq -g_2$. Let us quotient

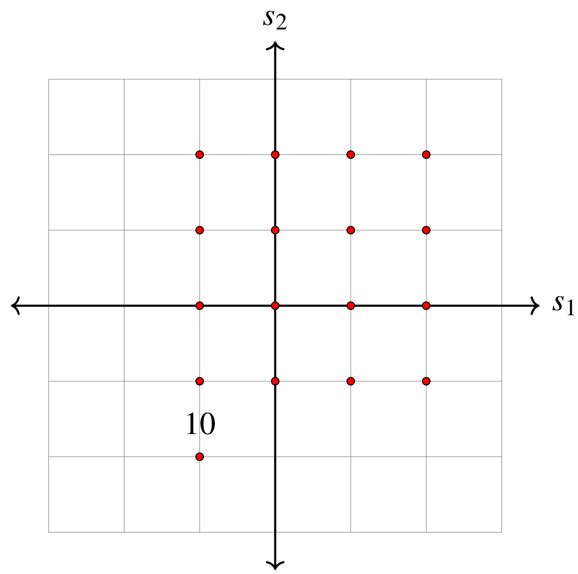


Figure 4.7 C_4, for $(g_1, g_2) = (2, 2)$ case

it and denote the quotient complex by C_4. Based on the discussion above, we have

Proposition 4.4. $H_*(C_4) \cong H_*(C)$.

By localizing $\mathscr{U}_2$, it is not hard to see that the towers can be represented by $C_{1-g_1,-g_2} + \bigoplus_{(s_1,s_2)\in S}(f^{-L_1} + f^{-L_2})(C^{0,0}_{s_1,s_2})$, where S is a subset of lattice in $[1-g_1, \cdots, g_1] \times [1-g_2, \cdots, g_2]$. Hence, to compute the d-invariants, we need to find the top grading of these different representations of the towers.

Given a framed link (L, Λ), let $W_\Lambda(L)$ be the corresponding cobordism 4-manifold and $\mathfrak{z}_{\mathbf{s}}$ the Spin^c structure on $W_\Lambda(L)$ corresponds to the Alexander grading $\mathbf{s}$. The absolute grading of the elements in the link surgery formula are given by the following formula:

Theorem .4 ([Zem23])**.** *Suppose $L \subset S^3$ is a link with integer framing Λ and $\mathfrak{s} \in \mathrm{Spin}^c(S^3_\Lambda(L))$ is torsion. Then the isomorphism $\boldsymbol{CF}^-(S^3_\Lambda(L), \mathfrak{s}) \simeq \mathbb{X}_\Lambda(L, \mathfrak{s})$ is absolutely graded if on each $\mathbb{X}^\varepsilon_\Lambda(L, \mathbf{s}) \subset \mathbb{X}_\Lambda(L)$ we use the Maslov grading*

$$\tilde{\mathrm{gr}} := \mathrm{gr}_{\mathbf{w}} + \frac{c_1(\mathfrak{z}_{\mathbf{s}})^2 - 2\chi(W_\Lambda(L)) - 3\sigma(W_\Lambda(L))}{4} + |L| - |\varepsilon|$$

where $\mathrm{gr}_{\mathbf{w}}$ is the internal Maslov grading from $\mathcal{CFL}(L)$.

In our case, for the Alexander grading $\mathbf{s} = (s_1, s_2)$, $c_1(\mathfrak{z}_{\mathbf{s}})^2 = -8s_1s_2$. Since the representations are in either $C^{0,1}$ or $C^{1,0}$. Given an element $\mathbf{e}_{s_1,s_2} \in C^{\varepsilon_1,\varepsilon_2}_{s_1,s_2}$,

$$\tilde{\mathrm{gr}} = \mathrm{gr}_{\mathbf{w}} - 2s_1s_2 - 2.$$

Without loss of generality, we assume that $g_1 \geq g_2$. Given an element $\mathbf{e} = \Sigma_{s_1,s_2}\mathbf{e}_{s_1,s_2}$, such that $\mathbf{e}$ represents a top generator of the towers. Suppose $e_{s_1,s_2} \in C^{\varepsilon_1,\varepsilon_2}_{s_1,s_2}$, let $\tilde{\mathrm{gr}}(\mathbf{e}_{s_1,s_2})$ denotes the grading of the top generator of $C^{\varepsilon_1,\varepsilon_2}_{s_1,s_2}$. It is not hard to see that

$$\tilde{\mathrm{gr}}(\mathbf{e}) = \min_{s_1,s_2}\tilde{\mathrm{gr}}(\mathbf{e}_{s_1,s_2}).$$

We claim that for any $\mathbf{e}$ that represents the top generator of the towers,

$$\max(\tilde{\mathrm{gr}}(\mathbf{e})) = g_1 - g_2 - 2.$$

Proof. Let us first check $\mathbf{e} \in C^{1,0}_{1-g_1,-g_2}$. $C^{1,0}_{1-g_1,-g_2}$ is generated by 4 elements, $\mathbf{z}^{-2g_1}_{0,0} \otimes \mathscr{U}^{2g_2}_2$, $\mathbf{z}^{-2g_1}_{0,1}$, $\mathbf{z}^{0}_{1,0} \otimes \mathscr{U}^{2g_2}_2$, and $\mathbf{z}^{0}_{1,1}$.

Hence $\max(\tilde{\mathrm{gr}}(\mathbf{e})) = \tilde{\mathrm{gr}}(\mathbf{z}^{-2g_1}_{0,1}) = g_1 - g_2 - 2(g_1-1)g_2 - 2 \leq g_1 - g_2 - 2$.

Given a representation $\mathbf{e}$ other than $C^{1,0}_{1-g_1,-g_2}$. Note that, $\mathbf{e}$ cannot be contained in the subcomplex $\bigoplus_{(s_1,s_2)} C^{\varepsilon_1,\varepsilon_2}_{s_1,s_2}$, such that $s_1s_2 > 0$ for all $(s_1, s_2) \in S$. Hence, we have

$$\max(\tilde{\mathrm{gr}}(\mathbf{e})) = \max_S(\min_{(s_1,s_2)\in S}\tilde{\mathrm{gr}}(\mathbf{e}_{s_1,s_2})) \leq \max(\mathrm{gr}_{\mathbf{w}}(\mathbf{e}_{s_1,s_2})) - 2,$$

where $s_1s_2 \geq 0$.

Let $\mathbf{e}\{i, j\}$ denotes the elements which tensors with $\mathbf{x}_i$ and $\mathbf{x}_j$. Since the representations contain $(f^{-L_1} + f^{-L_2})(C^{0,0}_{s_1,s_2})$, both $\mathbf{x}_{i,j}$ and $\mathbf{x}_{\tau(i),\tau(j)}$ are contained in $\mathbf{e}$, for a given (i, j). Hence, $\max(\mathrm{gr}_{\mathbf{w}}(\mathbf{e}_{s_1,s_2})) = g_1 - g_2$.

Hence, to prove the claim, we only need to find an element $\mathbf{e}$ with $\tilde{\mathrm{gr}}(\mathbf{e}) = g_1 - g_2 - 2$.

Let $\mathbf{e}$ be the representations on the slope $s_1+s_2 = 1$, such that, when s_1 is even, we take elements in $C^{1,0}_{s_1,s_2}$, and when when s_1 is odd, we take elements in $C^{0,1}_{s_1,s_2}$.

When s_1 is even, the top generators of $C^{1,0}_{s_1,s_2}$ are

$$\mathbf{z}^{-g_1-1+2k}_{0,1} \otimes \mathscr{V}^{g_2+1-2k}_2,$$

where $k \in [\lfloor \frac{-g_2}{2} \rfloor, \lfloor \frac{g_2-1}{2} \rfloor]$.

When s_1 is odd, the top generators of $C^{0,1}_{s_1,s_2}$ are

$$\tilde{\mathbf{y}}_{1,0}^{g_1+2k} \otimes \mathscr{V}_2^{-g_2-2k},$$

where $k \in [\lfloor \frac{1-g_2}{2} \rfloor, \lfloor \frac{g_2}{2} \rfloor]$.

We have the minimal grading of these generators as $g_1 - g_2$. Hence, $\tilde{\mathrm{gr}}(\mathbf{e}) = g_1 - g_2 - 2$.

$\square$

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APPENDIX A

D INVARIANTS COMPUTATION VIA LATTICE COHOMOLOGY

When K is an algebraic knot, $\Sigma_2(K_{2,q})$ is a graph manifold. In this case, we can compute the d-invariants using lattice cohomology. In this appendix, we give the computation of the d-invariants of $\Sigma_2(T_{2,3;2,q})$.

Plumbing diagram of $\Sigma_2(T_{2,3;2,q})$

In this section, we give a plumbing diagram description for $\Sigma_2(T_{2,3;2,q})$, for $q \geq 7$, using the algorithm from section 3.2.2, which is illustrated in Figure A.1. Let $q = 2l + 1$, we have the plumbing diagram for $\Sigma_2(T_{2,q})$ in Figure A.1 (a) with the framing -1 for the singular fiber. $T_{2,3}$ has the plumbing description as in Figure A.1 (b), where the arrow denotes $T_{2,3}$ with Seifert framing -6. Splicing them together, we obtain the plumbing diagram in Figure A.1 (c). Applying Kirby calculation to it, we get the simple plumbing diagram in Figure A.1 (d).

Lattice cohomology

In [OS03b], Ozsváth and Szabó gave an algorithm to compute Heegaard Floer homology of a plumbed three-manifold obtained from a plumbing tree which is negative definite and with at most two *bad vertice* (which is a vertex such that the weight exceeds minus the valence). This algorithm is later formalized by Némethi in [Ném05] and [Ném08], where he defined *lattice cohomology* for the plumbing diagram $\mathbb{HF}(Y(G))$ and proved it is isomorphic to the Heegaard Floer homology for *almost rational* graphs. This isomorphism is generalized to all the pluming diagram by Zemke in [Zem21b]. Since the plumbing diagram for $\Sigma_2(T_{2,3;2,q})$ has two bad vertices, we will use the d-invariants formula in [OS03b] to compute.

Given a plumbing diagram G, let us use v to denote its vertices. The plumbing diagram gives rise to a four-manifold with boundary $X(G)$. Let $Y(G)$ be the boundary of $X(G)$.

$H_2(X;\mathbb{Z})$ is spanned by the spheres which correspond to each vertex v. Taking Poincaré dual to each sphere, we can also view v as a basis for $H^2(X, Y;\mathbb{Z})$. We also have the following exact sequence:

$$0 \to H^2(X, Y; \mathbb{Z}) \to H^2(X; \mathbb{Z}) \to H^1(Y; \mathbb{Z}) \to 0 \tag{A.1}$$

Note that we identify $H^2(X; \mathbb{Z})$ with a subset of $H^2(X, Y; \mathbb{Q})$. The set of characteristic elements is defined by

$$\text{Char} = \text{Char}(G) := \{k \in H^2(X; \mathbb{Z}) | k(v) + (v, v) \in 2\mathbb{Z}\}$$

for every v. The first Chern class gives an identification of the set of Spin^c structures of X with $\text{Char}(G)$. Using the exact sequence A.1, the set of Spin^c structures over Y is identified with the set of $H^2(X, Y; \mathbb{Z})$-orbits in $\text{Char}(G)$. Let us denote the characteristic classes corresponding to the Spin^c $\mathfrak{s}$ over Y by $\text{Char}_{\mathfrak{s}}(G)$. We have the following formula to compute d-invariants:

Proposition A.1 ([OS03b]). *Let G be a negative-definite graph with at most two bad points, and fix a* Spin^c *structure $\mathfrak{s}$ over Y. Then,*

$$d(Y(G), \mathfrak{s}) = \max_{\{k \in \text{Char}_{\mathfrak{s}}(G)\}} \frac{k^2 + |G|}{4}.$$

Computation

To compute the d-invariants in the above expression, we need to represent the Spin^c structure first. Let us denote the plumbing diagram of two-fold branched cover of $T_{2,3;2,p}$ by M_p. From section 3.2, we have

$$H_1(Y(M_p)) = \mathbf{Z}/p\mathbf{Z},$$

which has an one-to-one correspondence with $\text{Spin}^c(M_p)$.

The intersect form A_p of M_p is

$$\begin{bmatrix} -3 & & 1 & & & & \\ & -2 & 1 & & & & \\ 1 & 1 & -2 & & & & \\ & & & \overbrace{\ddots}^{2l-6} & & & \\ & & & & -2 & 1 & 1 \\ & & & & 1 & -2 & \\ & & & & 1 & & -3 \end{bmatrix}$$

where we have $(2l-6)$'s -2 vertices in the middle linked as Hopf link.

Denote the vector $(-1, 0, 0, \overbrace{0, \cdots, 0}^{2l-6}, 0, 0, -1)^T$ by w_p. Then for any characteristic vector k, it can be represented by

$$A_p k = w_p + 2\mathbf{Z}.$$

Since p is a prime number, by the discussion from section A, we can use the following set to represent $\mathrm{Spin}^c(Y(M_p))$:

$$k_i = A_p{}^{-1}(w_p + iI),$$

here $i \in \{0, 1, \cdots, p\}$ and $I = (1, 0, 0, \cdots)^T$.

Using $[k]$ to denote the $H^2(X, Y; \mathbb{Z})$-orbit of k, hence they represent the same Spin^c structure on Y. For simplicity, let us use χ_k to denote the function $\chi_k : L \to \mathbb{Z}$:

$$\chi_k(x) := -(k(x) + (x, x))/2,$$

and $m_k = \min_{x \in L} \chi_k(x)$. For any $k' \in [k]$, we can find a $x \in L$, s.t. $k' = k + 2x$. $(k')^2 = (k + 2x, k + 2x) = (k, k) - 8\chi_k(x)$. Hence,

$$k^2 - 8\min \chi_k = \max_{k' \in [k]} (k')^2.$$

Now we get a new expression for the d-invariants:

$$d(Y(M),[k]) = -2m_k + \tfrac{k^2+|G|}{4}.$$

Then the relative d-invariants $\bar{d}(M,[k_i]) = -2(m_{k_i} - m_{k_0}) + \frac{{k_i}^2 - {k_0}^2}{4}$.

We give an explicit computation of the relative d-invariants here.

$$\begin{aligned} {k_i}^2 &= (w_p + iI){A_p}^{-1}(w_p + iI) \\ k_i^2 - {k_0}^2 &= 2iI^T {A_p}^{-1} w_p + i^2 I^T {A_p}^{-1} \end{aligned}$$

For the first term, we need to solve the equation $A_p v_p = w_p$, then $iI^T_T {A_p}^{-1} w_p$ is the first element of v_p. After solving the equation, we get $v_p = (1,1,2,\overbrace{2,\cdots,2}^{2l-6},2,1,1)$ and $iI^T {A_p}^{-1} w_p = 1$.

The second term $I^T {A_p}^{-1} I = {A_p}^{-1}{}_{11}$ and it is easy to compute, which is the (1,1) minor of A_p. One can check that $I^T {A_p}^{-1} I = -\frac{p-4}{p}$. Then we have:

$$\bar{d}(M,[k_i]) = -2(m_{k_i} - m_{k_0}) - \tfrac{p-4}{4p} i^2 + \tfrac{i}{2}.$$

Let $x = (a_1, a_2, \cdots, a_r)$, $2m_{k_i} = \max_x (k_i(x) + (x,x))$,

$$\begin{aligned} (k_i(x) + (x,x)) &= (i-1)a_1 - a_n - 3{a_1}^2 - 2{a_2}^2 - \cdots - 2{a_{n-1}}^2 - 3{a_n}^2 + 2a_1 a_3 + \cdots + 2a_{n-2}a_n \\ &= -{a_1}^2 + (l-1)a_1 - 2(a_2 - \frac{1}{2}a_3)^2 - \frac{1}{2}(a_3 - 2a_1)^2 - (a_2 - a_3)^2 - \cdots \\ &- (a_{n-2} - a_{n-1})^2 - 2(a_{n-1} - \frac{1}{2}a_{n-2})^2 - \frac{1}{2}(a_{n-2} - 2a_n)^2 - {a_n}^2 - a_n \end{aligned} \tag{A.2}$$

Hence, $2m_{k_i} = -\max_{a_1}((i-1)a_1 - a_1)$. In particular, $2m_{k_0}$ achieves at the vertex $\tilde{x}$ with $a_1 = 0$ and

$$2m_{k_i} - 2m_{k_0} \leq -2\chi_{k_0}(\tilde{x}) - 2\chi_{k_i}(\tilde{x}) = -ia_1 = 0. \tag{A.3}$$

Linearly independence

Using the computation above, we can prove the linearly independence as in Chapter 3.

Since $d(L(p,1),j) = -\frac{1}{4} + \frac{(2j-p)^2}{4p}$, we have

$$\bar{d}(L(p,1),j) = \tfrac{j^2}{p} - j.$$

Suppose we have i, j, s.t. $\bar{d}(Y(M), [k_i]) + \bar{d}(L(p, 1), j) = 0$, then

$$\frac{p-4}{4p}i^2 - \frac{i}{2} + \frac{j^2-jp}{p} \equiv 0(mod\mathbf{Z}),$$
$$\frac{p-4}{4}i^2 - \frac{pi}{2} + j^2 \equiv 0(modp).$$

Hence, i must be an even number, suppose $i = 2n$, $n \in (0, 1, \cdots, \frac{p-1}{2})$, then we have

$$j^2 - 4n^2 \equiv 0(modp),$$

$$(j - 2n)(j + 2n) \equiv 0(modp),$$

which means, we have $j = 2n$ or $p - 2n$. Together, we have p possible solutions. However, when we put it back to the equation, we have

$$\frac{p-4}{4p}i^2 - \frac{i}{2} + \frac{j^2-jp}{p} = n(n-3),$$
$$2(m_{k_i} - m_{k_0}) = n(3-n).$$

When $0 < n < 3$, $n(3-n) > 0$, but by equation A.2, $2(m_{k_i} - m_{k_0}) \leq 0$. Hence, we have less than p solutions, which obstructs the existence of the metabolizer.

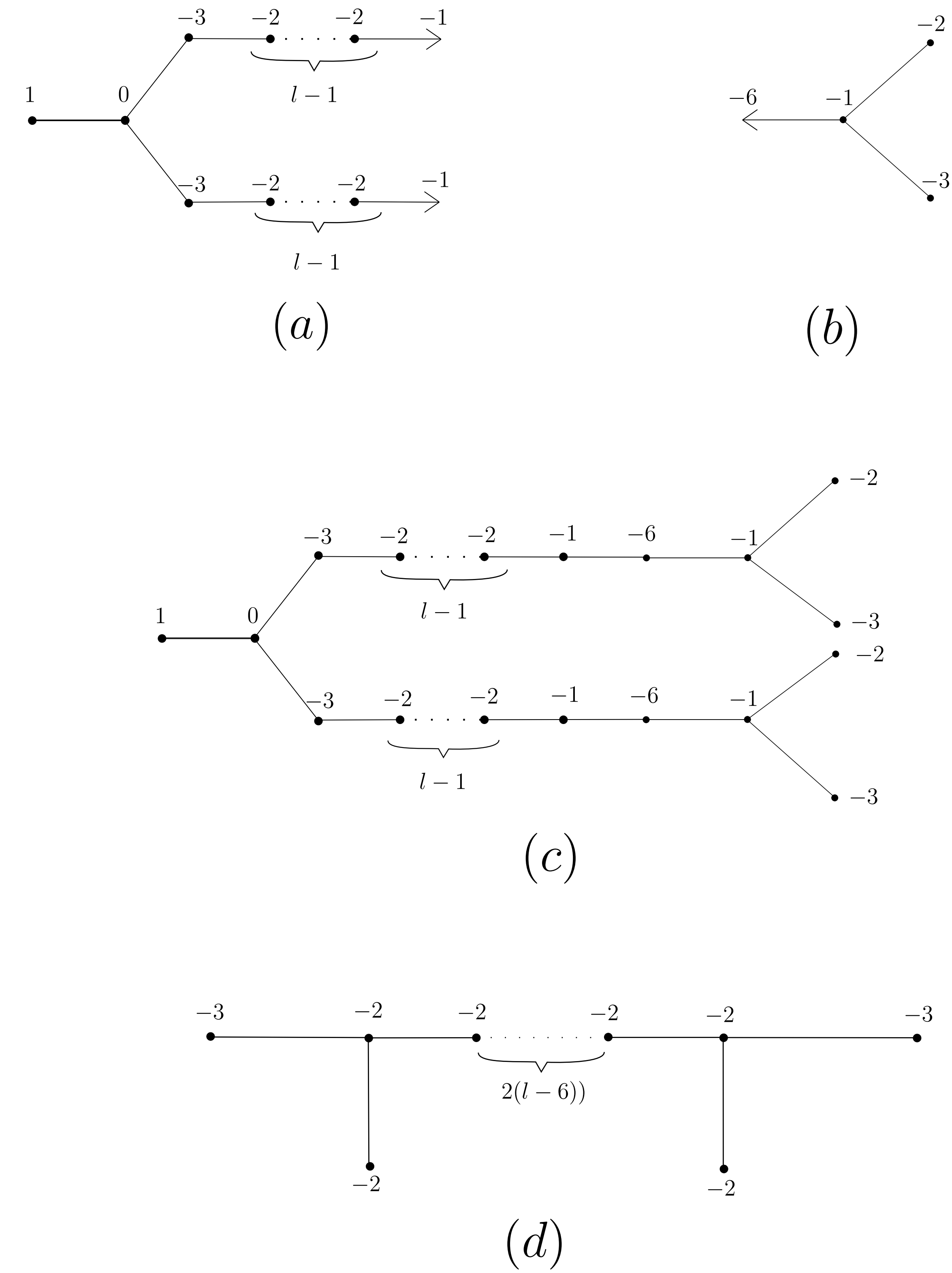

Figure A.1 Plumbing diagram