TOPICS IN CLASSICAL AND QUANTUM KNOT INVARIANTS

By

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ABSTRACT

We continue the work done by Kalfagianni and Lee, where they gave two sided linear bounds for the crosscap number of alternating links in terms of certain coefficients of the Jones polynomial. In particular we find two-sided bounds for the crosscap number of the Conway sums of strongly alternating tangles in terms of certain coefficients of the Jones polynomial. Then we find families of links for which the crosscap number and these coefficients of the Jones polynomial grow independently. These families of links enable us to state the bounds for the crosscap number will not generalize to all links.

We also study the relationship of the span of the colored Jones polynomial to the crossing number of a family of knots cablings. In particular we use the degree of the colored Jones knot polynomials to show that the crossing number of a (p,q)-cable of an adequate knot with crossing number c is larger than $q^2\,c$. As an application we determine the crossing number of 2-cables of adequate knots.

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LIST OF ABBREVIATIONS

K	A knot or link
L	A link
S^1	A copy of the circle (one-sphere)
S^3	The 3-sphere.
$V(L)$	The Jones polynomial for a link L.
α_L, α'_L	The first and last coefficients of the Jones polynomial.
β_L, β'_L	The second and second to last coefficients of the Jones polynomial for L.
T_L	$\lvert\beta_L\rvert + \lvert\beta'_L\rvert$
$C(S)$	The crosscap number for a surface S.
$C(L)$	The crosscap number for a link L.
T_i	A tangle.
K_{iN}, K_{iD}	The link resulting from closing the tangle T_i via the numerator and denominator closures respectively.
k_L	The number of link components in L.
$c(K)$	The crossing number for a knot K.
$K_{p,q}$	The (p, q) cabling of K.
$N(K)$	The neighborhood of a knot K.
$\mathrm{wr}(K)$	The writhe of a knot K.
$K\#K^*$	The connected sum of knots K and K^*.
$D(L)$	A diagram for the link L.
$\mathbb{G}_A(L)$	The all A state graph for a link L.
$\mathbb{G}_B(L)$	The all B state graph for a link L.
$\mathbb{G}'_A(L)$	The reduced all A state graph for a link L.
$\mathbb{G}'_B(L)$	The reduced all B state graph for a link L.

e_A, e_B	The number of edges in the graphs $\mathbb{G}_A(L)$ and $\mathbb{G}_B(L)$ respectively.
e'_A, e'_B	The number of edges in the graphs $\mathbb{G}'_A(L)$ and $\mathbb{G}'_B(L)$ respectively.
Γ	The 4-valent graph which results from turning crossings of a knot to vertices.
$S(s)$	The Turaev surface for a Kauffman state s.
$\mathcal{S}(F)$	A linear skein for a surface F.
$\mathbb{SL}_2((C)$	The special linear group of degree 2 over the complex numbers.
TL_n	The Temperley-Lieb Algebra on n generators.
Σ	A Conway sphere.
$tw(D(L))$	The twist number for a link diagram.
$\ell_{\mathbf{in}}(D(L))$	The edges lost as we pass from $e_A + e_B$ to $e'_A + e'_B$.
$\ell_{\mathbf{ext}}(D(L))$	Edges we lose from identification when we take the Conway sum.
b_A, b_B	The bridges in a state graph.
$W_-(K)$	The negative Whitehead double of the link L with the blackboard framing.
$b(G)$	The first Betti number of the graph G.
$g_T(D)$	The Turaev genus for the knot diagram D.
$v_A(D)$	The number of vertices in $\mathbb{G}_A(L)$.
$v_B(D)$	The number of vertices in $\mathbb{G}_B(L)$.
$J_K(n)$	The n^{th} colored Jones polynomial of K.
$d_-[J_K(n)]$	The minimum degree of the colored Jones polynomial of K.
$d_+[J_K(n)]$	The maximum degree of the colored Jones polynomial of K.
d_{J_K}	The Jones diameter of K.

CHAPTER 1

INTRODUCTION

A *knot* K is a smooth (or piece-wise linear) embedding of S^1 into S^3 or $\mathbb{R}^3$. A *link* is the embedding of k disjoint copies of S^1 into S^3 or $\mathbb{R}^3$. In the case of link we call the individual images of the copies of S^1 the link components and say there are k link components, further we use either K or L as the standard notation for a link. We consider two links L_1 and L_2 to be equivalent if there exists an isotopy which takes us from L_1 to L_2. Telling if two knots are equivalent is a classical question in knot theory and one that turns out to be quite difficult.

When studying links we consider *link diagrams*. A link diagram is a projection of a link L onto the plane, $\mathbb{R}^2$, or sometimes the 2-sphere, such that we retain over/under information at the crossings. The link isotopy of link diagrams is done through a series of moves called the *Reidemeister moves* [32]. To highlight the difficulty of the question of whether two knot diagrams are equivalent, we note that it is a challenge to even recognize the *unknot*. The unknot is the trivial knot, in other words its simplest knot diagram is a copy of S^1. For the Jones polynomial, which we introduce in chapter two, it remains an open question whether the polynomial identifies the unknot [2].

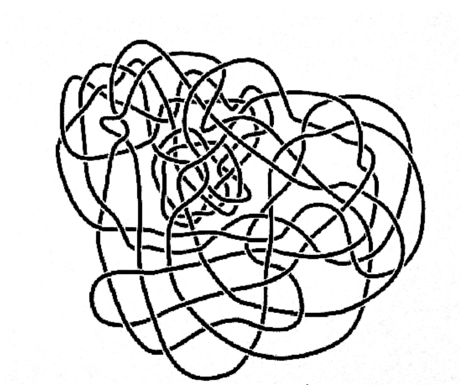

Figure 1.1 Haken's Gordian diagram of the unknot [12], which is difficult to recognize using isotopy.

Given that it is difficult to recognize links via isotopy we instead use a tool known as *knot invariants*. Knot invariants are other mathematical objects which we associate to links in such away that if two links are equivalent then their invariant must be equivalent. In particular if we define $f(L)$ to be a knot invariant for a link L, then given links L_1 and L_2 if $f(L_1) \neq f(L_2)$ then $L_1 \neq L_2$.

So, the study of knot theory lives largely in the realm of the study of knot invariants. In particular we study what knot invariants can tell us about knots and links, but also what knots and links can tell us about the category of mathematical objects different invariants come from. For this dissertation we will be using a family of link invariants known as quantum invariants to better understand the geometric invariants of knots and links. As we will see in the following chapters, geometric invariants of links are often challenging to compute for a link, while quantum invariants tend to be more algorithmic.

Another convenient property of certain quantum invariants (e.g. Jones polyonial or colored Jones polynomial) is that they behave nicely for a family of knots known as alternating knots and more generally adequate knots [32, 43, 25, 11]. Further, with these families of links we have been able to show that we can recover them from purely combinatorial approaches. For instance, the Jones polynomial can be recovered from the Bolobas-Riordan-Tutte polynomial for graphs [11].

The remainder of this introduction will be to give a layout of the dissertation, details of each chapter, and precise statements of the main results to be shown throughout.

1.1 Main Results and Layout of the Dissertation

This dissertation has been structured such that a knowledgeable reader can read the chapters in any order. But for those without the necessary background in knot theory, chapter 2 will provide a primer for concepts essential to the understanding of the results and their proofs.

1.1.1 Outline of Chapter 2

Chapter 2 will be used to provide the necessary background for the rest of the dissertation. The first section of this chapter will be introducing two important families of links, alternating and more generally adequate links. Further we recall graphs associated to link diagrams known as a Kauffman state graphs. Finally we end the section by introducing the Turaev surface and recall the information it holds about how close a link is to alternating. This section is important throughout our results as we regularly consider adequate links and reference the constructions stated here.

The next section of the background will be introducing classic link invariants, in other words those which stem from the geometry of our link. Our results in this dissertation have shared goal of being able to say more about these invariants. Then the last two sections of this chapter will give information about two quantum invariants, the Jones polynomial and its generalization the colored Jones polynomial. In the Jones polynomial section we will also show how state graphs can be used to calculate it, along with an interesting property between the Jones polynomial and crossing number for alternating links. In this chapter we will also give a construction of the colored Jones polynomial but one can still understand the results with the knowledge that the colored Jones polynomial is a sequence of Laurent polynomials.

1.1.2 Outline of Chapter 3

In this chapter we will present our results relating the crosscap number of knots and the coefficients of the Jones polynomial. We will start this chapter by defining the family of links we will be considering which are formed by tangle addition. Afterwards, we work towards the main result for the case where we have two tangles and then generalize to more than two tangles. Finally, we prove that there exist infinite families of knots for which the results do not generalize.

To start, let the Jones polynomial (see definition 2.3.1 for reference) for a link L be,

$$V(L) = \alpha_L t^n + \beta_L t^{n-1} + \cdots + \beta'_L t^{m+1} + \alpha'_L t^m, \qquad n, m \in \mathbb{Z},$$

where $\alpha_L, \alpha'_L \neq 0$. Then define $T_L = |\beta_L| + |\beta'_L|$. For chapter 3 this quantity will be the component of the Jones polynomial we will use throughout our results. The other important invariant is the crosscap number. The crosscap number is similar to the genus of a link but for spanning surfaces which are non-orientable. We will talk more about this in chapter 2 but a formula for the crosscap number for a surface S is as follows,

$$C(S) = 2 - \chi(S) - k,$$

where k is the number of boundary components for S and $\chi(S)$ the Euler characteristic of S. Then the crosscap number $C(L)$ for a link L is the minimum of $C(S)$ for all non-orientable spanning surfaces for L.

Quantum knot invariants have been used to better understand classical knot invariants and the geometry of knots and links for quite some time. In the 80's Kauffman [28] and Murasugi [37] showed that the span of the Jones polynomial, as seen in Theorem 2.3.8, realizes the crossing number for alternating links. More recent results from Futer, Kalfagianni, and Purcell showed how the coefficients of the colored Jones polynomial store information about the geometry of incompressible surfaces in the link complement and their strong relations to geometric structures and in particular hyperbolic geometry [17, 13, 16]. For example, specific coefficients of the colored Jones polynomial can coarsely define the volume of large families of hyperbolic links [15, 14]. In particular Dasbach and Lin [9] did this for all hyperbolic alternating links. Further, the Volume Conjecture [36] predicts that the asymptotics of the colored Jones polynomial can be used to calculate the volume of all hyperbolic links. Recent work by Kalfagianni and Lee [24] which this chapter references, continues this pattern of using quantum invariants to calculate the crosscap number of links.

For alternating links Kalfagianni and Lee showed that there exists linear bounds for the crosscap number with respect to certain coefficients of the Jones polynomial (see Theorem 1.1 in [24] for reference). Before their work the best known lower bound for all alternating links was $C(K) \geq 1$. On the other side Clark [7] observed that for any knot the crosscap number is bounded by

$$C(K) \leq 2g(K) + 1,$$

where $g(K)$ is the orientable genus of the knot. Kalfagianni and Lee's results gave an exact calculation of the crosscap number for 283 prime alternating knots on Knotinfo and improvements of 1472 prime alternating knots. Kindred [30] has also made progress in calculating the crosscap number, calculating it for alternating knots with less than 13 crossings. We hope our results in this chapter will pave the way for similar calculations for non-alternating knots which are less understood.

Motivation for such calculations also comes from the work that has been done concerning the orientable genus of links and the Alexander polynomial of a link and further the Heegard Floer homology of a link. Crowell [39] and Murasugi [38] have independently shown that the orientable genus of an alternating link is half the degree span of the Alexander polynomial of the knot. The Heegard Floer homology is now known to detect the genus of links [34]. There is also an algorithm using normal surface theory to calculate the orientable knot genus [4, 22]. The hope with the crosscap number is to find parallel results to Heegard Floer and orientable genus using quantum invariants. As of this dissertation the crosscap number outside of alternating knots and links is still poorly understood except for a few families of links [45].

We generalize the results of [24] to a family of links which are the Conway sum of strongly alternating tangles. We were able to do this using the alternating property of each individual tangle along with Lemma 3.3.4, plus a series of lemmas we prove throughout Chapter 3. These results give us bounds on the crosscap number of a link $C(L)$ for a family of links which are not necessarily alternating. The tangles we will be considering will be *non-splittable* which means that there does not exist a copy of $S^2 \subset S^3$ such that the components of L will be on opposite sides of S^2 with $S^2 \cap L = \emptyset$. The tangle diagrams will also be *twist reduced* which means that any copy of S^1 which intersects T in four points such that two are adjacent to one crossing two to another will bound a twist region, where a twist region is a maximal collection of bigons end to end. We will state the main result of this chapter here, refer to Definition 3.1.2 for strongly alternating, Definition 2.2.4 for crosscap number, Definition 3.1.3 and Figure 3.2 for a Conway sum, .

Theorem 1.1.1. *Let T_1 and T_2 be non-splittable, twist reduced, strongly alternating tangles whose Conway sum is a link L. Let $C(L)$ be the crosscap number of L and k_L be the number of components of L. Then, we have*

$$\left\lceil \frac{T_L}{6} \right\rceil - k_L \leq C(L) \leq 2T_L + k_L + 8,$$

where $T_L = |\beta_L| + |\beta'_L|$.

A key ingredient in the proof of Theorem 1.1.1, is Theorem 1.1.2 which we state below. Theorem 1.1.2 gives us bounds for $C(L)$ in terms of the crosscap numbers of the closures of the tangles which sum to L.

Theorem 1.1.2. *Let T_1 and T_2 be non-splittable, twist reduced, strongly alternating tangles, and* *let L be the link formed by the Conway sum of T_1 and T_2. Let K_{iN} and K_{iD} be the links formed by the numerator closure and denominator closure of T_i respectively, $i \in \{1, 2\}$. We have*

$$m - 2 \leq C(L) \leq m + 2$$

where $m = min\{C(K_{1N}) + C(K_{2N}), C(K_{1D}) + C(K_{2D})\}$.

Having Theorem 1.1.2 at hand we use a result of [24] and the additivity of twist numbers for strongly alternating tangles to find bounds for $C(L)$ in terms of the twist number of L. Then using these bounds and a generalization of Theorem 1.6 from [19] gives us Theorem 1.1.1.

After proving Theorem 1.1.1 for the Conway sum of two tangles we will generalize this to larger Conway sums. The proof for the case of larger Conway sums follows in the same vein as the proof for two tangles. Ultimately, we will end up with the following result:

Theorem 1.1.3. *Let $T_1, T_2, ..., T_l$ be non-splittable, twist reduced, strongly alternating tangles and let L be the link which results from taking the Conway Sum. Then let $C(L)$ be the crosscap number and k_L the number of link components in L, then*

$$\left\lceil \frac{T_L - 2l}{6} \right\rceil + 2 - k_L \leq C(L) \leq 2T_L + l + 6 + k_L.$$

After generalizing the results of [24] we consider the question of the extent to which Theorem 1.1.3 generalize to all knots. Then, we will introduce two infinite families of knots for which there do not exist linear bounds for $C(L)$ in terms of T_L for all links L in the given family. The first family we will look at is a family of torus knots. A *torus knot* is a (p, q)-curve on the torus such that p and q are relatively prime. Notice that when we project this curve into the plane we will have a knot diagram. For the family of torus knots we consider we will find that T_L will be at most two but $C(L)$ can be made arbitrarily large. This family will show that there does not exist a uniform upper bound for all links.

On the other hand we will consider a certain family of Whitehead doubles. A Whitehead double, $W(L)$ of a link is a satellite knot. See figure 3.11 for reference. For the Whitehead double of a knot it will always be true that $C(L) = 2$ which results directly from a result in [7]. But we will show that we can make T_L arbitrarily large for the family we look at which shows that a universal linear lower bound does not exist. To state this precisely we have the following result:

Theorem 1.1.4. *We have the following;*

(a) There exists a family of links for which $T_L \leq 2$, but $C(L)$ is arbitrarily large.

(b) There exists a family of links for which $C(L) \leq 3$, but T_L is arbitrarily large.

We point out that the families of link considered to prove Theorem 1.1.4 are both non-hyperbolic families. So, we are left with the following question.

Question 1.1.5. *Do there exist linear bounds for $C(L)$ in terms of T_L exist for all hyperbolic links?*

1.1.3 Outline of Chapter 4

In this chapter we will be using the colored Jones polynomial to get bounds for, and to calculate the crossing number for an infinite family of cable knots. Let K be a knot, then the crossing number for a knot, $c(K)$ is the minimal number of crossings for all knot diagrams representing K. The crossing number for knots has historically been intractable, especially for satellite knots. In particular we have the following question:

Question 1.1.6. *If we let C be the companion knot for a satellite knot K, is $c(K) \geq c(C)$? [29]*

Intuitively this seems to be an obvious conjecture as the diagram resulting from taking a cabling of C will have $c(K)$ equal to a multiple of $c(C)$ plus a constant depending on the pattern knot. Despite this the intractable nature of the crossing number has prevented a proof as of this dissertation.

Our work in this chapter will start with a stronger lower bound for cables of adequate knots. Then, using a result of [25] we give an exact calculation of the crossing number for an infinite family of knot cablings.

To state our results, for a knot K and co-prime integers p, q we define $K_{p,q}$ to be the (p, q) cabling of K. In other words $K_{p,q}$ is the curve on $\partial N(K)$ such that it wraps p times around the meridian and q times around the canonical longitude. See Figure 4.1 for reference. The writhe of any adequate diagram $D = D(K)$ for a knot K is an invariant of K. We will denote it by $\mathrm{wr}(K)$.

Theorem 1.1.7. *For any adequate knot K with crossing number $c(K)$, and any coprime integers p, q, we have*

$$c(K_{p,q}) \geq q^2 \cdot c(K) + 1.$$

Theorem 1.1.7, combined with the results of [25], has significant applications in determining crossing numbers of prime satellite knots. In particular we have the following:

Corollary 1.1.8. *Let K be an adequate knot with crossing number $c(K)$ and writhe number* $\mathrm{wr}(K)$. *If $p = 2\,\mathrm{wr}(K) \pm 1$, then $K_{p,2}$ is non-adequate and $c(K_{p,2}) = 4c(K) + 1$.*

The proof of Corollary 1.1.8 shows that, when $p = q\,\mathrm{wr}(K) \pm 1$, we apply the $(p, 2)$-cabling operation to an adequate diagram $D = D(K)$ of K the resulting diagram is a minimum crossing diagram of the knot $c(K_{\pm 1,2})$. It should be compared with other results in the literature asserting that the crossing numbers of some important classes of knots are realized by a "special type" of knot diagrams. These classes include alternating and more generally adequate knots, torus knots and Montesinos knots [28, 37, 46] and untwisted Whitehead doubles of adequate knots of zero writhe number [25].

Theorem 1.1.7 allows to compute the crossing number of $(\pm 1, 2)$-cables of adequate knots that are equivalent to their mirror images (a.k.a. amphicheiral) since such knots are known have $\mathrm{wr}(K) = 0$. One way to obtain an amphicheiral adequate knot is to take the connected sum of an adequate knot with its mirror image (the mirror image of a knot is $K^* = r(K)$, where $r : S^3 \to S^3$ is an orientation reversing diffeomorphism.

Corollary 1.1.9. *For a knot K let K^* denote the mirror image of K and, for every $m > 0$, let $K^m := \#^m(K\#K^*)$ denote the connected sum of m-copies of $K\#K^*$. Suppose that K is adequate with crossing number $c(K)$. Then, the $(1, 2)$-cable $C_{1,2}(K_m)$ is non-adequate, and we have $c(K^m_{1,2}) = 4mc(K) + 1$.*

Note that out of the 2977 prime knots with up to 12 crossings, 1851 are listed as adequate on Knotinfo [34] and thus Corollary 1.1.9 applies to them. The method of the proof of Theorem 1.1.7 is similar to that used in [25] to determine the crossing number of untwisted Whitehead doubles of zero-writhe adequate knots. These Whitehead doubles and the cables $c(K_{\pm 1,2})$ of Theorem 1.1.7 are the only infinite non-adequate, non-composite satellite knots for which the crossing numbers have been determined.

Another result we show in this section relates to the open conjecture on the crossing number of connected sums [29]. $K = K_1 \# K_2$ is defined to be a *connected sum* of knots K_1 and K_2 if there exists a sphere such that $S \cap K$ is two points x, y and S bounds two 3-balls B_1 and B_2, such that if α is an embedded arc on S connecting x and y

$$(B_i \cap K) \cup \alpha = K_i.$$

Similar to satellite knots, the crossing number of connected sums is also not well understood. There remains an open conjecture that states $c(K_1 \# K_2) = c(K_1) + c(K_2)$, where # is the connected sum of knots. Similar to satellite knots this remains open as it is unclear if we can show that the resulting diagram for $K_1 \# K_2$ is a minimal crossing diagram for $K_1 \# K_2$.

Theorem 1.1.10. *Suppose that K is an adequate knot and let $K_1 = K_{p,2}$, where $p = q\,\mathrm{wr}(K) \pm 1$. Then for any adequate knot K_2, the connected sum $K_1 \# K_2$ is non-adequate and we have*

$$c(K_1 \# K_2) = c(K_1) + c(K_2).$$

CHAPTER 2

BACKGROUND

In this chapter we will cover the necessary background information needed throughout this document.

2.1 Alternating and Adequate Links

In this section we will define two families of knots which will come up regularly later in this paper. The first family of knots we will talk about is *alternating knots*.

Definition 2.1.1. A knot diagram D is called alternating if tracing along the diagram the crossings will alternate over/under. Then we call a knot K *alternating* if it has a knot diagram which is alternating.

Alternating knots generalize to alternating links by considering each of the link components independently enters crossings in an over/under pattern. Alternating knots are well understood, and therefore many tools exist to help us further study alternating knots and links.

A generalization of alternating knots and links which we will talk about here is a family known as *adequate knots*. All alternating knots and links are also adequate, which is why this is a natural family for us to consider next. As we will see throughout this document, adequate links have many useful properties with respect to both the geometry of a knot or link, and also with respect to the quantum invariants of a knot or link. It will also be an important hypothesis of many of our lemmas and theorems.

Definition 2.1.2. Let $D(L)$ be the diagram of a link L, we define the A *resolution* and B *resolution* of a crossing as shown in Figure 2.1. Then a *Kauffman state* is a choice of resolutions for each crossing in $D(L)$.

Definition 2.1.3. Next we define an *all A (resp. all B) state graph* $\mathbb{G}_A := \mathbb{G}_A(D(L))$ (resp. $\mathbb{G}_B(D(L))$) by adding edges where we performed resolutions and then contracting the simple closed curves to vertices. Let e_A (resp. e_B) be the number of edges in $\mathbb{G}_A(D(L))$ (resp. $\mathbb{G}_B(D(L))$). Further, if we identify all parallel edges (edges whose union bounds a disk) we find the *reduced state graph* $\mathbb{G}'_A := \mathbb{G}'_A(D(L))$ (resp. $\mathbb{G}'_B(D(L))$). Let e'_A (resp. e'_B) be the number of edges in

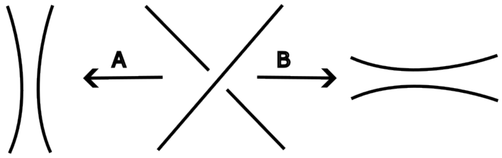

Figure 2.1 Here we see an A and B resolution for a crossing.

$\mathbb{G}'_A(D(L))$ (resp. $\mathbb{G}'_B(D(L))$). See Figure 2.2 for an example of this process.

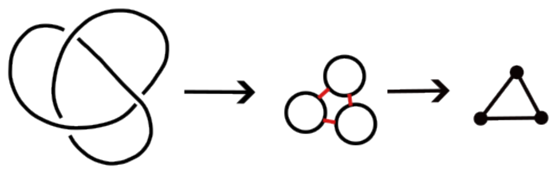

Figure 2.2 Left: the trefoil knot, Center: state circles with edges where the crossings were, Right: (reduced) state graph.

Now we are ready to define what it means for a link to be adequate.

Definition 2.1.4. We call a link diagram *A-adequate* (resp. *B-adequate*) if the A state (resp. B state) graph of the diagram has no one edge loops. A link diagram is called *adequate* if it is both A-adequate and B-adequate, and a link is adequate if it has a diagram which is adequate.

Another definition for adequacy has to do with the count of state circles after resolving the crossings.

Definition 2.1.5. We call a link diagram *A-adequate* (resp. *B-adequate*) if the all resolution A state (resp. B state) has more state circles than any state A' where all but one crossing is resolved with an A-resolution (resp. B resolution) with the one remaining crossing resolved via a B resolution (resp. A resolution). A link diagram is called *adequate* if it is both A-adequate and B-adequate, and a link is adequate if it has a diagram which is adequate.

It is not hard to see that these two definitions are equivalent. This arises from the fact that any one edge loop will represent a crossing which when flipped will result in a new state circle. The one edge loop will represent where we cut our state circle to get two state circles instead of one.

2.1.1 Turaev Surfaces

We can further develop the idea of a state graph to find a surface called the *Turaev surface* of a Kauffman state for a link diagram. We follow the construction of [47] and [8]. Let s be a choice of a state for a link diagram and $\hat{s}$ be the *dual state*. Where the dual state is defined to be the opposite choice of resolution at each of the crossings.

To build this surface we start by allowing $\Gamma \subset S^2$ to be the 4-valent graph arising from the projection of the link diagram onto S^2. In particular Γ will be the graph with vertices where our link diagram has crossings. We thicken S^2 to be $S^2 \times [-1, 1]$, with $\Gamma \subset S^2 \times \{0\}$. Outside of a neighborhood of the vertices of Γ our surface will intersect $\Gamma \times [-1, 1]$ inside this thickening. Then at each vertex of Γ we will add a saddle positioned so that at $S^2 \times \{1\}$ we will have the state circles of s and at $S^2 \times \{-1\}$ will be the state circles of $\hat{s}$, see Figure 2.3 for reference. Then we cap the state circles of both s and $\hat{s}$ with disks to create a closed surface $S(s)$. We call the resulting surface $S(s)$ the *Turaev surface* of s.

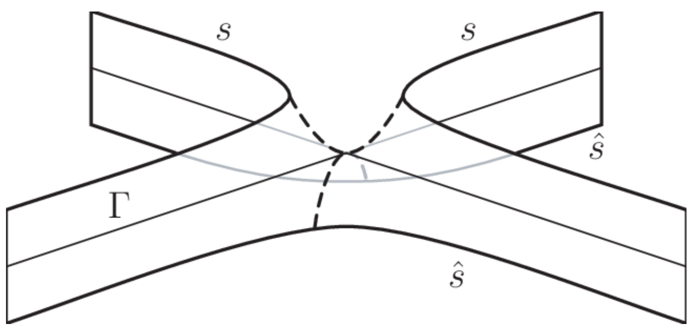

Figure 2.3 A saddle at one of the verices of Γ, to create the surface $S(s)$ we cap of the state circles shown by s and $\hat{s}$. We also see that away from the saddle the surface will include $\Gamma \times [-1, 1]$. [11]

Using the Turaev surface we are able to realize interesting properties of the link diagram. Before investigating these properties we introduce the following lemma from [11] for $G(s)$.

Lemma 2.1.6. *$S(s)$ is an unknotted surface which means that $S^3 \backslash S(s)$ is the disjoint union of two handlebodies.*

Now that we better understand the Turaev surface for a link we can define a link invariant known as the *Turaev genus* of a link.

Definition 2.1.7. Let s^* be the all A or all B state of a link diagram. If $g(S(s^*))$ is the genus of the surface $S(s^*)$, the *Turaev genus* $g_T(L)$ of a link L is the minimum for $g(S(s^*))$ over all diagrams for L.

The Turaev genus in a sense tells us how far our link is from alternating, in particular it gives the necessary genus for a surface on which are link diagram can be made alternating. This is shown through the following lemma and corollary [11].

Lemma 2.1.8. *When s is the all A-state or the all B-state, the link has an alternating projection to $S(s)$.*

Corollary 2.1.9. *A link has Turaev genus equal to zero if and only if the link is alternating.*

Corollary 2.1.9 follows directly from Lemma 2.1.8 since a link possessing an alternating projection on the sphere is the definition of alternating. We will use Turaev genus in our proofs later in this document, and give an explicit formula for the Turaev genus.

2.2 Classic Knot Invariants

In this section we will define the classic knot invariants which are relevant to this work. We will also give some historical results here which are not used in our results later in the paper. The first such invariant we will talk about is the *crossing number* of a knot.

Definition 2.2.1. The *crossing number* of a knot is the minimum number of crossings needed for a knot across all knot diagrams of that knot.

Given a knot K we will use $c(K)$ to represent the crossing number. If we let U be the unknot then $c(U) = 0$ and the unknot is the only knot with crossing number less than 3. The trefoil is the first nontrivial knot and has crossing number equal to three. Historically, given an arbitrary knot diagram the crossing number is quite intractable. As shown in Figure 1.1 we can make rather complex diagrams for the unknot using only isotopy.

Next we will define multiple knot invariants which arise from the spanning surface of a knot or link.

Definition 2.2.2. A *spanning surface* for a link L is a surface S with boundary such that $\partial(S) = L$.

Notice in Figure 2.4 that both of these spanning surfaces are non-orientable. Spanning surfaces

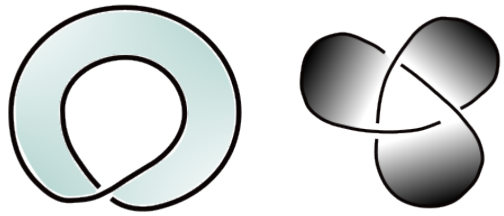

Figure 2.4 Left: A spanning surface for the unknot, Right: A spanning surface for the trefoil.

can also be orientable, for instance we can take the complement surface of the shading, for each of the knots in Figure 2.4. Both orientable and non-orientable surfaces give rise to link invariants. For orientable links we can find an invariant called the knot genus. Remember that for a surface S the genus is

$$g(S) = \frac{2 - \chi(S) - k}{2},$$

where $\chi(S)$ is the Euler characteristic of S and k the number of boundary components. Another definition is that the genus is the maximal number of non-intersecting closed curves we can cut S along without separating the surface. Now we are ready to define the genus of a knot.

Definition 2.2.3. The *knot genus* of a link $g(L)$ is the minimum genus across all orientable surfaces S which are spanning surfaces for the link.

We can define a similar invariant using the non-orientable spanning surfaces of a knot. To do so we must first define the *crosscap number* of a non-orientable surface. The crosscap number of a non-orientable surface S is

$$C(S) = 2 - \chi(S) - k$$

where $\chi(S)$ and k are defined as above. We can also view the crosscap number of a surface as the number of projected planes we need to construct our surface. Now we can define the crosscap number of a link.

Definition 2.2.4. The crosscap number of a link is the minimum value for $C(S)$ across all non-orientable surfaces which are bounded by the link.

2.3 Jones Polynomial

In this section we will define another knot and link invariant called the *Jones polynomial*. The Jones polynomial is a Laurent polynomial over $\mathbb{Z}$ and we will begin to define this invariant by showing the construction of the *Kauffman bracket polynomial*. We start with a knot or link diagram D and then define the Kauffman bracket polynomial to be $\langle D \rangle$. Then we get the following relationships between $\langle D \rangle$ and the Kauffman bracket of other diagrams. For rule 3 it relates the polynomial for a diagram with the two diagrams where a chosen crossing is resolved in both ways. Then rule 2 allows us to remove an unlinked trivial link component. Finally at the end we will use rule 1 to end up with a polynomial.

1. $\left\langle \bigcirc \right\rangle = 1$
2. $\left\langle L \cup \bigcirc \right\rangle = (-A^2 - A^{-2})\langle L \rangle$
3. $\left\langle \times \right\rangle = A \left\langle \asymp \right\rangle + A^{-1} \left\langle)(\right\rangle$.

It turns out that the Kauffman bracket polynomial is not a knot invariant under Reidemeister I moves as this will shift the polynomial by $A^{\pm 3}$ per twist. It will hold under the other two Reidemeister moves. We will account for this when we define the Jones polynomial using the Kauffman bracket polynomial.

Definition 2.3.1. Let $D(K)$ be a knot diagram of the knot K, and let $\langle D(K) \rangle$ be the Kauffman bracket polynomial for $D(K)$. Then we define $V(K)$, the *Jones polynomial* of K to be

$$V(K) = ((-A)^{-3w(D(K))} \langle D(K) \rangle)_{t^{1/2}=A^{-2}}.$$

Another definition for the Jones polynomial exists using Kauffman state graphs. To do this we will start by indexing the crossings for a given link diagram D, $1, 2, 3, \ldots, n$. Then we will define a function $s : \{1, 2, \ldots, n\} \to \{-1, 1\}$ where s maps to -1 if we resolve the crossing with a B-resolution and 1 if we resolve with an A-resolution. Notice that a choice of function s will give us a *Kauffman state*. Further, define a value $|sD|$ to be the number of state circles that arise in the Kauffman state defined by s. Then we can state the Kauffman bracket polynomial as follows:

Definition 2.3.2. If D is a link diagram with n crossings, the Kauffman bracket polynomial of D can be computed as below,

$$\langle D\rangle = \sum_s \left(A^{\sum_{i=1}^n s(i)}(-A^{-2} - A^2)^{|sD|-1}\right).$$

Where the first summation is over all 2^n possible functions s.

From here we get to the Jones polynomial by the same formula as in Definition 2.3.1. This relationship between the Jones polynomial and Kauffman states gives rise to many important results relating the crossing number of an adequate link to the Jones polynomial. For these results we will state them without proof but an interested reader can find them in Lickorish [32]. The proofs largely rely on Definition 2.3.2.

As mentioned when we defined the crossing number of a link, the crossing number is a historically intractable invariant to compute. However, the finding of Kauffman states and the Kauffman bracket polynomial resulted in a variety of useful discoveries concerning the crossing number for alternating links. We will start by letting s_A and s_B be the all A-state and all B-state respectively. Now we can formalize the alternative definition of adequacy we gave above.

Definition 2.3.3. We call a link diagram D A-adequate if $|s_A D| > |sD|$ for all states s where $\sum_{i=1}^n s(i) = n-2$. Similarly we call a link diagram B-adequate if $|s_B D| > |sD|$ for all states s where $\sum_{i=1}^n s(i) = n+2$, then if D is both A-adequate and B-adequate, D is adequate.

Then if we let $M\langle D\rangle$ be the maximum degree of the Kauffman bracket polynomial for D and $m\langle D\rangle$ be the minimum degree we get the following lemma.

Lemma 2.3.4. *Let D be a link diagram with n crossings. Then*

1. $M\langle D\rangle \leq n + 2|s_A D| - 2$, *with equality if D is A-adequate, and*

2. $m\langle D\rangle \geq -n - 2|s_B D| + 2$, *with equality if D is B-adequate.*

Combining the two inequalities in the previous lemma we get the following corollary,

Corollary 2.3.5. *If D is an adequate diagram, then*

$$M\langle D\rangle - m\langle D\rangle = 2n + 2|s_A D| + 2|s_B D| - 4.$$

Now we define a link diagram to be *connected* if there does not exist a simple closed curve such that the curve separates part of the link from the rest. Where a link diagram is considered *split* if it is possible to draw a simple closed curve γ such that $\gamma \cap D = \emptyset$ and γ separates D into two pieces. For connected diagrams, we can relate $|s_A D| + |s_B D|$ and the number of crossings a diagram has in the following way,

Lemma 2.3.6. *Let D be a connected n-crossing diagram.*

1. *If D is alternating, then* $|s_A D| + |s_B D| = n + 2$.
2. *If D is non-alternating and strongly prime, then* $|s_A D| + |s_B D| < n + 2$.

Where strongly prime is defined as follows:

Definition 2.3.7. We call a knot K *prime* if for any K_1 and K_2 such that $K = K_1 \# K_2$ one of K_1 or K_2 must be the unknot, then we call the knot *strongly prime* if the diagram of the unknot contains no crossings.

Then this can be used to prove the following conjecture by Tait [2], first shown by Kauffman, Thistlewaite, and Murasagi. First we define $B(V(L))$ to be the breadth of the Jones polynomial for a given link L. As a reminder, the breadth of a Laurent polynomial is the difference of the highest and lowest degrees of the polynomial.

Theorem 2.3.8. *Let D be a connected, n-crossing diagram of an oriented link L with Jones polynomial* $V(L)$*. Then,*

1. $B(V(L)) \leq n$;
2. *if D is alternating and reduced, then* $B(V(L)) = n$;
3. *if D is non-alternating and a prime diagram, then* $B(V(L)) < n$.

Where a *reduced knot* is a knot without a nugatory crossing or a crossing which can be removed by a half twist. It immediately follows that for an alternating link any diagram which realizes the crossing number must be reduced and alternating. Then we see that for alternating links the crossing number will be equal to $B(V(L))$. In [32] we can also see that an adequate diagram for a link also realizes the crossing number. As we will see later in the paper there are also results that relate the crossing number of adequate links to the colored Jones polynomial.

2.4 Colored Jones Polynomial

A generalization of the Jones polynomial which we will use during this dissertation is the colored Jones polynomial. The colored Jones polynomial is a sequence of Laurent polynomials J_n in $\mathbb{Z}[t, t^{-1}]$ indexed by $n \in \mathbb{N}$. The Jones polynomial will be the case where $n = 2$. Here we will define the colored Jones polynomial via the Kauffman bracket and the Jones-Wenzel idempotent of the Temperley-Lieb Algebra, a detailed description of which can be found in [32]. The colored Jones polynomial may also be defined via the representation theory of quantum $\mathbb{SL}_2(\mathbb{C})$ which we will not show here but details may be found in [41], [40], and [48].

2.4.1 The Temperley-Lieb Algebra

We begin by defining a *linear skein* $\mathcal{S}(F)$ over a surface F. $\mathcal{S}(F)$ is a vector space of formal linear sums over $\mathbb{C}$ of link diagrams in F quotiented by

1. $\left\langle D \cup \bigcirc \right\rangle = (-A^2 - A^{-2})\langle D \rangle$
2. $\left\langle \times \right\rangle = A \left\langle \asymp \right\rangle + A^{-1} \left\langle)(\right\rangle$

where A is an invertible variable.

Let $S(D, 2n)$ be the linear skein of the square representation of the disk, with n marked points along one side of the square and then n marked points along the opposite edge of the square. Given $d_1, d_2 \in S(D, 2n)$ we define a product $d_1 d_2$ as the juxtaposition of the two disks along the right side of d_1 and the left side of d_2 where we position the marked sides of the square to fall on the left and right.

Definition 2.4.1. Extending this product by linearity we arrive at a bilinear map

$$S(D, 2n) \times S(D, 2n) \to S(D, 2n)$$

which makes $S(D, 2n)$ into an algebra which we call the *Temperley-Lieb Algebra*, TL_n, over $\mathbb{C}$. TL_n has generators $1, e_1, ..., e_{n-1}$ shown below.

Figure 2.5 Generators of the Temperley-Lieb Algebra.

Note that in Figure 2.5 that on the left the n represents the n parallel strands in the generator 1, and on the right the $i-1$ and $n-(i+1)$ represent the number of parallel strands below and above the 'u' shaped strands.

There is a map from TL_n to $\mathcal{S}(\mathbb{R}^2)$ by projecting an element $d \in TL_n$ into $\mathbb{R}^2$ and connecting the marked points on the left side of d to the marked points on the right of d by n parallel strands. Notice that in the linear skein where $F = \mathbb{R}^2$, the image is reduced to an empty diagram with a coefficient a Laurent polynomial in A.

2.4.2 The Jones-Wenzl Idempotent

We now define the Jones-Wenzl idempotent, which is an element of the Temperley-Lieb algebra.

Definition 2.4.2. The *Jones-Wenzl idempotent* f_n is the unique element in TL_n which satisfies

(i.) $f_n e_i = 0 = e_i f_n$ for $1 \leq i \leq n-1$,

(ii.) $f_n - 1$ belongs to the algebra generated by $\{e_1, ..., e_{n-1}\}$,

(iii.) $f_n f_n = f_n$, and

Further Δ_n represents the polynomial that arises when we connect the n marked points on each side of f_n with parallel strands in the image of $\mathcal{S}(\mathbb{R}^2)$, which has the following form:

$$\Delta_n = \frac{(-1)^n (A^{2(n+1)} - A^{-2(n+1)})}{(A^2 - A^{-2})}.$$

We represent f_n by an empty box with n incoming parallel strands and n outgoing parallel strands as shown in Figure 2.6. Similarly, the closure Δ_n is represented by an empty box with n parallel incoming strands and n parallel outgoing strands, but in this case the n strands are closed as shown in Figure 2.6.

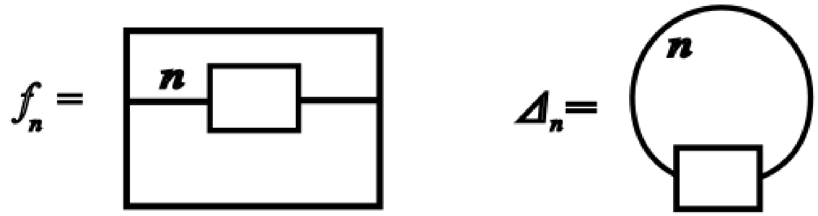

Figure 2.6 Left: The Jones Wenzl Idempotent, Right: its closure in $\mathcal{S}(\mathbb{R}^2)$.

The existence of f_n is proven in [32]. There is also a recursion formula for f_n which is important when working with idempotent so we will show it here diagrammatically. Notice that in Figure 2.7 the number above any line represents that many parallel strands, so for instance the leftmost diagram will have $n + 1$ parallel incoming and outgoing strands from the box. Also, the $\frac{\Delta_{n-1}}{\Delta_n}$ will be a rational function in A.

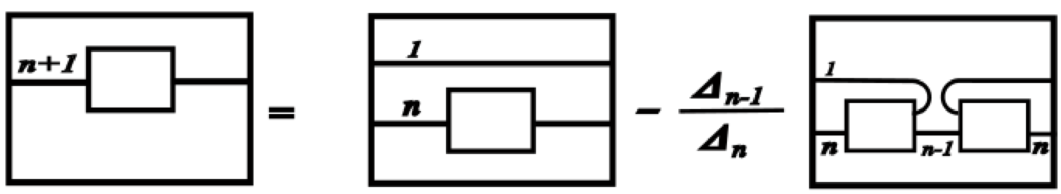

Figure 2.7 Recursion formula for the Jones Wenzl Idempotent.

Similar to the Kauffman bracket polynomial if we add a type-1 Reidemester move to the n-strands of f_n we will have equality if we multiply the skein element by a scalar in A [32].

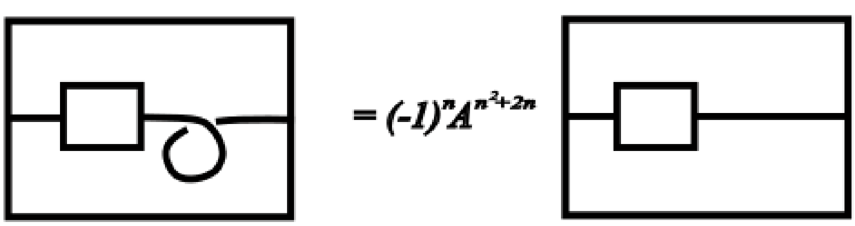

Figure 2.8 Effect of adding a twist to f_n.

2.4.3 A formula for the colored Jones polynomial

Before we give the formula for the colored Jones polynomial we define a family of polynomials defined by a recurrence relations called the *Chebyshev's polynomials of type 2* [31].

Definition 2.4.3. A *Chebyshev's polynomial of type 2* is a polynomial for an unoriented link diagram D defined by the following recurrence relation. Let $\mathcal{S}_0(D) = 1$ and $\mathcal{S}_1(D) = D$. Then for $n \geq 2$,

$$S_{n+1}(D) = DS_n(D) - S_{n-1}(D).$$

Now we can state a formula for the colored Jones polynomial. To start, if D is an unoriented link diagram of a link L with l link components, then D defines a multilinear map

$$\langle , ..., \rangle_D : \mathcal{S}(S^1 \times I) \times \cdots \times \mathcal{S}(S^1 \times I) \to \mathcal{S}(\mathbb{R}^2).$$

Where this is the Cartesian product of l copies of the linear skeins of the annulus $S^1 \times I$ into $\mathcal{S}(\mathbb{R}^2)$. We define this product by starting with an annulus corresponding to the blackboard framing of each component of the link, where the boundaries of the annulus run parallel to D. Figure 2.9 shows where the boundary of the annulus runs for a diagram D of the trefoil knot. For this setup

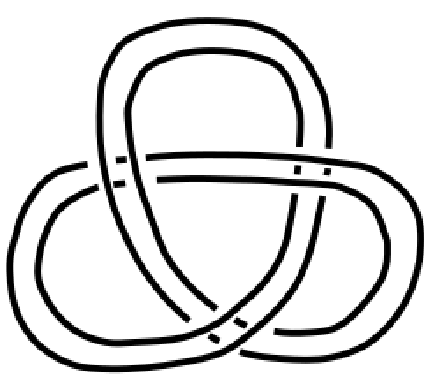

Figure 2.9 D^2 where D is the standard trefoil diagram.

we identify $S^1 \times I$ as the annulus, giving us a map from skein elements $S^1 \times I$ into the annulus corresponding to a link component of D.

Definition 2.4.4. The $n+1-kth$*-unreduced colored Jones polynomial* $J_K^{n+1}(q)$ is the polynomial obtained by the substitution $q^{1/2} = A^{-1/2}$ into the polynomial

$$\left((-1)^n A^{n^2+2n}\right)^{w(D)} \langle \Delta_n, ..., \Delta_n \rangle \in \mathcal{S}(R^2).$$

By induction and the recursion formula of f_n shown in Figure 2.7, we have that

$$\langle , \ldots, \Delta_n, \ldots, \rangle = \langle , \ldots, \alpha\Delta_n - \Delta_{n-1}, \ldots, \rangle$$

where α is the generator of $\mathcal{S}(S^1 \times I)$ consisting of a parallel curve encircling the annulus. Then if we extend by bilinearity we find a second formula for the colored Jones polynomial when substituting in $t^{1/2} = A^{-1/2}$ as below.

$$J_K(n+1, t) = \left(\left((-1)^n A^{n^2+2n} \right)^{w(D)} \langle \mathcal{S}_n(D) \rangle \right)_{t^{1/2}=A^{-1/2}}.$$

In chapter 4 we will be using the colored Jones polynomial to find bounds for the crossing number of a family of knot cablings and then calculate the crossing number for an infinite sub family of satellite knots. In particular, we will be using the span of the colored Jones polynomial, which is the difference between the maximum and minimum degrees of a polynomial, to calculate information about the crossing number for satellite knots. Garoufalidis [20] showed that the extremal degrees of the colored Jones polynomial are quasi-quadratic polynomials. As we will see in Chapter 4 this will be relevant for the proofs of multiple theorems and a key component of the statements of Proposition 4.1.3 and Lemma 4.1.4.

It has also been shown for certain families of knots that these extremal degrees of the colored Jones polynomial tell us the boundary slopes of knots. Garoufalidis [20] showed this for alternating knots, knots up to nine crossings, torus knots, and a family of 3-string pretzel knots. Futer, Kalfagianni, and Purcell [18] showed this for adequate knots, which is the family of knots we will use as the pattern knots for our satellite knots in Chapter 4. We mention this here, but we will not be talking about the slopes conjecture later in the dissertation, just using the extremal degrees.

CHAPTER 3

CROSSCAP NUMBER OF CONWAY SUMS

In this chapter we will explore the relationship between the crosscap number for links and certain coefficients of the Jones polynomial. The work and figures in this chapter come from a preprint by McConkey [35].

3.1 Tangles and the Upper Bound

We start with a couple of definitions.

Definition 3.1.1. A *tangle* is a graph in the plane contained within a box that intersects the box at the four corners with one-valent vertices, with all other vertices, contained inside the box, four-valent, and given over/under crossing data. We label the four, 1-valent vertices NW, NE, SE, SW, positioned according to Figure 3.1.

Definition 3.1.2. The *closure* of a tangle is the link which results when we connect the NW and NE points along the box and SW and SE points along the box as seen in the center panel of Figure 3.1, this is called the *numerator closure*. If we close as in the right hand panel of Figure 3.1 we call it the *denominator closure*. A tangle is *strongly alternating* if both closures are prime, reduced and alternating.

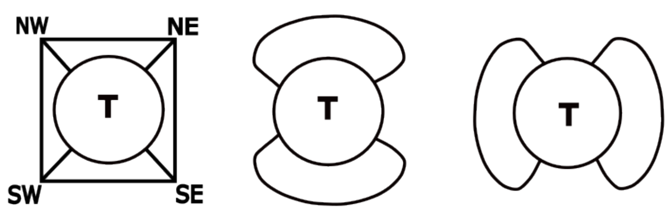

Figure 3.1 Left: A tangle inside a box with directional strands labelled, Center: numerator closure, Right: denominator closure.

Definition 3.1.3. A *Conway sphere* is a 2-sphere which intersects a knot or link transversely in four points. A *Conway sum* is a sum of tangles as shown in Figure 3.2. For our purposes, a Conway sphere Σ will be positioned such that it intersects a Conway sum at the four one valent vertices for one of the tangles in the sum. Notice if we let S be a spanning surface for our Conway sum, then

$S \cap \Sigma$ will contain two arcs and a possibly empty collection of simple closed curves.

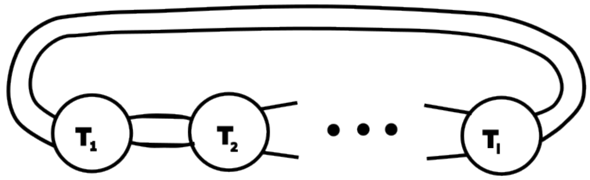

Figure 3.2 An example of a Conway sum of l tangles.

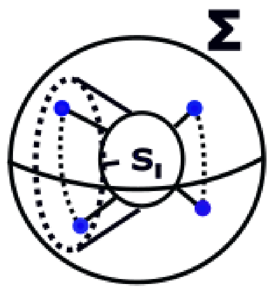

Figure 3.3 Here we see a tangle contained within a Conway sphere. The blue dots represent the intersection of T_i with Σ. Then the dotted lines are the intersections of S with Σ.

We are now ready to begin proving Theorem 1.1.2. We separate it into the upper and lower bounds, beginning with the upper bound:

Lemma 3.1.4. *Let T_1 and T_2 be a pair of non-splittable, strongly alternating tangles. Let L be the link formed by the Conway sum of T_1 and T_2. Let K_{iN} and K_{iD} be the numerator and denominator closures, respectively, of T_1 and T_2. If $C(L)$ is the crosscap number of L, then*

$$C(L) \leq min\{C(K_{1N}) + C(K_{2N}) + 2, C(K_{1D}) + C(K_{2D}) + 2\}.$$

Proof. Start with a pair of strongly alternating tangles, T_1 and T_2. Let K_{1N} and K_{2N} be the links acquired by the numerator closures of the tangles. Let S_1 and S_2 be non-orientable spanning surfaces which realize the crosscap numbers of K_{1N} and K_{2N} respectively. We find a spanning surface S of L by attaching S_1 and S_2 with a pair of bands bounded by the strands along which the Conway sum was taken. Notice that as we do not cut S_1 and S_2, S will also be non-orientable.

Now we study the relationship between $C(S)$ and the sum of $C(S_1)$ and $C(S_2)$. We remind the reader that $C(S) = 2 - \chi(S) - k$. The difference between S and the disjoint union of S_1 and S_2 is the two connecting bands used to construct S. So, $\chi(S) = \chi(S_1) + \chi(S_2) - 2$.

Next we compare the number of link components in L with the total in K_{1N} and K_{2N}. The gluing of the East strands of K_{1N} to the West strands of K_{2N} will reduce the number of components by 1 as we are connecting two disjoint links. The other attachment can increase or decrease the number of components by 1, or keep it the same. Then $k_L = k_{K_{1N}} + k_{K_{2N}} - \epsilon$ where $\epsilon = 0, 1,$ or 2.

Now we substitute for $\chi(S)$ and k_L to find:

$$C(S) = 2 - \chi(S_1) - \chi(S_2) + 2 - k_{K_{1N}} - k_{K_{2N}} + \epsilon = C(K_{1N}) + C(K_{2N}) + \epsilon.$$

Hence:

$$C(L) \leq C(S) = C(K_{1N}) + C(K_{2N}) + 2$$

as $\epsilon = 2$, will give the weakest upper bound. By the same argument with the denominator closures of T_1 and T_2, we find that

$$C(L) \leq C(K_{1D}) + C(K_{2D}) + 2,$$

giving us the claim.

□

Here we remark that Lemma 3.1.4 will hold even if we take two general tangles. Notice in the proof that we do not use the fact that T_1 or T_2 are non-splittable or strongly alternating. As this will not hold true for the other statements we included these hypotheses for uniformity.

3.2 The Lower Bound

3.2.1 Technical Lemmas

Before we show the lower bound, we will need some more background, as well as some technical results. Lemma 3.2.1 below was discussed in the proof of Lemma 3.1.4.

Lemma 3.2.1. *Let L be the Conway sum of the tangles T_1 and T_2, and let K_1 and K_2 be closures of T_1 and T_2. If k_L is the number of link components for L, and k_1 and k_2 the number of link components for K_1 and K_2, respectively, then $k_L = k_1 + k_2 - \epsilon$ for $\epsilon = 0, 1, 2$.*

Definition 3.2.2. Let L be a link in S^3 and let $N(L)$ be a neighborhood of L. A spanning surface S of L in S^3 is defined to be *meridianally boundary compressible* if there exists a disk D embedded in $S^3 \backslash N(L)$, such that $\partial D = \alpha \cup \beta$ where $\alpha = D \cap \partial N(L)$ and $\beta = D \cap S$. Notice both α and β are arcs, β does not cut off a disk of S, $\partial D \cap \partial S$ cuts ∂S into two arcs ϕ_1, ϕ_2 and $\alpha \cup \phi_i$ is a meridian of the link for one of $i = 1, 2$ as shown in Figure 3.4. A spanning surface is said to be *meridianally boundary incompressible* if no such disk exists.

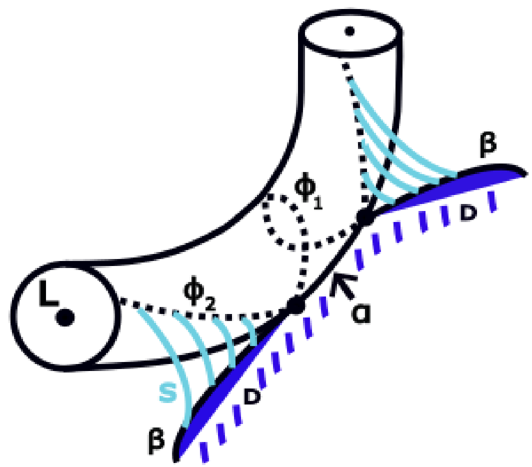

Figure 3.4 This figure shows a case where L is meridianally boundary compressible. As $\phi_1 \cup \alpha$ create a meridian of $N(L)$.

Definition 3.2.3. Given an alternating projection of a link L on S^2, we modify it so that in a neighborhood of each crossing, we have a ball whose equator lies on S^2 such that the over strand runs over the ball and the under goes underneath, see Figure 3.5 for reference. We call every such ball a *Menasco ball*, and we call such an embedding of L relative to S^2 a *Menasco projection* P with n crossings.

Definition 3.2.4. We say that a surface S intersects a Menasco ball B_i in a *crossing band* if $S \cap B_i$ consists of a disc bounded by the over and under strands on ∂B_i along with opposite arcs along the equator of B_i. We refer the reader to figures 28-30 in [3] for reference.

Let $F = S^2 / \bigcup_i B_i$, where the B_i are the Menasco balls for L. Given an incompressible (not necessarily meridianally) surface S spanning L, we can isotope S so that

Figure 3.5 Here we put a crossing into Menasco form, with the over strand running across the top of the ball and the lower strand running along the bottom.

i. $S \cap F$ is a collection of simple closed curves and arcs with endpoints on L or the equator of a Menasco ball.

ii. S is disjoint wherever possible from the interior of the B_i, including along $N(L)$. The only exception will be at crossing bands.

We say such a surface isotoped in this way is in *Menasco form*.

We will also need Lemma 5.1 from [3] by Adams and Kindred which we state here as Lemma 3.2.5. Lemma 3.2.5 is required to prove Lemma 3.2.6, which is proven as Corollary 5.2 in [3]. Lemma 3.2.6 is essential to our proof of the lower bound, as it guarantees the connectedness of any spanning surface of the numerator or denominator closures of tangles that we consider.

Lemma 3.2.5. *An incompressible and meridianally boundary incompressible surface S spanning an alternating link L can be isotoped relative to a given nontrivial Menasco projection P to obtain a crossing band.*

Lemma 3.2.6. *Any spanning surface for a non-splittable, alternating link is connected.*

Proof. This proof uses induction on the number of crossings a non-splittable, alternating link contains. As the unknot contains only one link component, any spanning surface for the unknot must be connected. Now consider a non-splittable alternating link L which has n crossings, and let S be a spanning surface for L then S is either incompressible and meridianally boundary incompressible or a finite sequence of compressions take S to an incompressible and meridianally boundary incompressible surface S'. Choose a reduced alternating diagram of L and put S' into

Menasco form relative to L. By Lemma 3.2.5, we can isotope S' such that S' contains a crossing band in at least one of the Menasco balls, M. Further, when we cut open the link along M, we find a spanning surface S'' for a non-splittable alternating link with fewer crossings L'. Part of the equator of M replaces the crossing strands and guarantees that S'' is a spanning surface of L'. Then, by induction, as S'' is a spanning surface for a link of $n-1$ crossings, it is connected. Regluing in the crossing band does not disconnect our surface, showing that S' and S are connected. □

Lemma 3.2.7. *Let Σ be a Conway sphere which intersects a Conway sum L of two strongly alternating tangles T_1 and T_2 in S^3 such that Σ separates T_1 and T_2. If we let S be a spanning surface of L and $S \cap \Sigma$ contains a simple closed curve γ such that γ does not separate the two arcs in $S \cap \Sigma$ on Σ, then there exists an isotopy on S which will eliminate γ.*

Proof. Assume there is only one closed curve γ contained in $S \cap \Sigma$. First we consider the case where cutting S along γ and gluing in disks along the resulting boundary components results in two disconnected closed components. Notice that this implies that S has a disconnected closed component, contradicting that S is a crosscap realizing surface.

Next assume that cutting along γ and gluing in disks does not result in a closed surface component. Then cutting S along Σ and gluing disks along the copies of γ will result in spanning surfaces for a closure of T_1 and a closure of T_2, both of which are connected by Lemma 3.2.6. Reversing this procedure everywhere but γ results in a new connected surface S' which also spans L. But as S' has two additional disks, $\chi(S') = \chi(S) + 2$, showing that $C(S') < C(S)$ contradicting that S is a crosscap realizing spanning surface.

The only remaining possibility is if cutting S along γ and gluing disks to the two resulting boundaries, results in a single closed surface component, U. Then U will separate S^3 into two disjoint spaces. As γ does not separate the two arcs on Σ, one side of U must not contain any part of S. But then we can move U to the opposite side of Σ and re-glue it to S along γ to find an isotopy of S for which γ is no longer in $\Sigma \cap S$.

In the case that we have multiple such closed curves along Σ we do the same as above starting with the innermost closed curve. The innermost closed curve in this case is the one that bounds an empty disk on Σ. Hence showing the claim. □

By Lemma 3.2.7, we can choose S such that the only simple closed curves in $\Sigma \cap S$ are those which bound two disks each containing an arc. Next we show that S can be chosen so that $S \cap \Sigma$ contains at most one such simple closed curve.

Lemma 3.2.8. *There exists a spanning surface S for L, where L is the Conway sum of two strongly alternating tangles, such that $C(S) = C(L)$ and $\Sigma \cap S$ contains at most one closed curve.*

Proof. By Lemma 3.2.7 we can assume that if $\Sigma \cap S$ contains closed curves $\gamma_1, ..., \gamma_n$, they each split Σ such that the two arcs lie on opposite disks. Assume we have $n > 1$ such closed curves in $\Sigma \cap S$, and let γ_1 and γ_2 be such that γ_1 bounds a disk on Σ such that no other γ_i are in the disk and γ_2 bounds a disk where the only closed curve in it is γ_1.

We now find a spanning surface S' such that $C(S') = C(L)$ and $S' \cap \Sigma$ contains $n - 2$ closed curves. We start with S and cut along γ_1 and γ_2 and then glue in annuli whose boundaries are a copy of γ_1 and a copy of γ_2. Then as the Euler characteristic of an annulus is 0, this cutting and gluing operation will result in $\chi(S) = \chi(S')$.

It remains to show that S' will be a connected surface. As in the proof of Lemma 3.2.7 we cut S along Σ and glue in disks along each γ_i except for γ_1 and γ_2 which we glue a pair of annuli. This results in spanning surfaces S_1 and S_2 for closures of T_1 and T_2 which by Lemma 3.2.6 are connected. If we reverse this procedure everywhere except γ_1 and γ_2 the result will be S' and as we only remove disks before regluing, S' will be connected. Hence, we have found a spanning surface S' for L such that $C(S') = C(L)$ and $S' \cap \Sigma$ contains two fewer closed curves. Hence, repeating for all such pairs of closed curves in $S \cap \Sigma$ we will find the claim. □

Lemma 3.2.9. *We can choose a surface S which spans a link L such that $C(S) = C(L)$ and cutting along Σ will not give us a closed surface component.*

Proof. This was shown in the proof of Lemma 3.2.7. □

3.2.2 Proof of the Lower Bound of Theorem 1.1.2

In this subsection we will prove the lower bound of Theorem 1.1.2, which will be restated as Lemma 3.2.10. We start by discussing what happens when we cut our link L along Σ. Assume that S is a non-orientable spanning surface for L with $C(S) = C(L)$. The two arcs on Σ will define how we close T_1 and T_2 after cutting. We let K_1 and K_2 be these closures. To see that K_1 and K_2 are the numerator or denominator closures consider a crossing which as a vertex in the tangle graph is adjacent to a 1-valent vertex. If the exterior regions for a tangle are the faces bounded by the box in the graph then one of the two exterior regions adjacent to the crossing must be included in S. This means the boundary of this region will result in the numerator or denominator closures.

We are now ready to prove the lower bound of Theorem 1.1.2.

Lemma 3.2.10. *Let T_1 and T_2 be non-splittable, strongly alternating tangles and L the link resulting from the Conway sum of T_1 and T_2. Also, let S be a spanning surface of L such that $C(L) = C(S)$. Then:*

$$C(K_1) + C(K_2) - 2 \leq C(L).$$

Proof. Let S be a non-orientable spanning surface for L such that $C(L) = C(S)$. By Lemma 3.2.8 S can be chosen such that the intersection of S with Σ contains at most one closed curve γ and that both disks γ bounds contain arcs. We cut S along Σ and if γ exists we glue a disk to each copy to get spanning surfaces S_1 and S_2 for K_1 and K_2 respectively. By Lemma 3.2.9 we know that S_1 and S_2 will not have closed components and by Lemma 3.2.6 S_1 and S_2 must be connected as K_1 and K_2 are alternating.

If k_1 and k_2 are the number of link components for K_1 and K_2 respectively, then by Lemma 3.2.1 $k_1 + k_2 - \epsilon = k_L$ where $\epsilon = 0, 1, 2$.

Next we consider how the Euler characteristics of S and the sum of the Euler characteristics of S_1 and S_2 will be related. We know that $\Sigma \cap S$ contains two arcs and at most one closed curve by Lemma 3.2.8. Then cutting the two arcs along Σ will increase the Euler characteristic by 2. Assume there are n closed curves, gluing disks along the two copies after cutting will further increase the Euler characteristic by $2n$. The final consideration we have to make is whether S_1 and S_2 are non-orientable, let $t = 0, 1, 2$ be the number of S_i which are orientable. Notice we will have to add t half twist bands to make sure all the S_i are non-orientable decreasing the Euler characteristic by t. Now we see that $\chi(S) = \chi(S_1) + \chi(S_2) - 2 - 2n + t$. Then:

$$C(S_1) + C(S_2) = 4 - \chi(S) - 2 - 2n + t - k_L - \epsilon = C(L) - 2n + t - \epsilon.$$

Simplifying we find

$$C(K_1) + C(K_2) + 2n - t + \epsilon \leq C(L).$$

This will be the weakest when $n = 0, t = 2$, and $\epsilon = 0$. Hence we find $C(K_1)+C(K_2)-2 \leq C(L)$. □

We restate Theorem 1.1.2:

Theorem 1.1.2. *Let T_1 and T_2 be non-splittable, twist reduced, strongly alternating tangles. Let L be the link formed by the Conway sum of T_1 and T_2. Let K_{iN} be the link formed by the numerator closure of T_i, $i \in \{1, 2\}$, similarly K_{iD} will be the link formed by the denominator closure. If we let $m = min\{C(K_{1N}) + C(K_{2N}), C(K_{1D}) + C(K_{2D})\}$ then,*

$$C(K_1) + C(K_2) - 2 \leq C(L) \leq m + 2.$$

Theorem 1.1.2 follows directly from Lemma 3.1.4 and Lemma 3.2.10. A similar result exists to Theorem 1.1.2 for the cross cap number of connected sums. In particular, Clark [7] showed with a strategy similar to our own, that if K_1 and K_2 are knots, then

$$C(K_1) + C(K_2) - 1 \leq C(K_1 \# K_2) \leq C(K_1) + C(K_2).$$

3.3 Crosscap Number, Twist Number, and the Jones Polynomial

3.3.1 Twist Number Bounds

Now we have a relationship between the crosscap numbers of the Conway sum of two tangles and the closures of the tangles which compose it. Unfortunately, the bounds depend upon the tangles and which closures we take. But we can use Theorem 1.1.2 to find bounds for $C(L)$ entirely dependent upon L. Before proceeding with the statements, we will need a definition.

Definition 3.3.1. The *twist number* of a link diagram or a tangle diagram is the number of twist regions a link diagram contains, where a *twist region* is a maximal collection of bigon regions contained end to end. We call a link diagram *twist-reduced* if any simple closed curve which meets the link diagram transversely at four points, with two points adjacent to one crossing and the other two another crossing, bounds a possibly empty collection of bigons arranged end to end between the two crossings.

We take a brief pause to mention that we can take the Conway sum of more than two tangles. In particular, for tangles $T_1, T_2, ..., T_n$, we can glue the eastern strands of T_i to the western strands of T_{i+1}. Then we glue the eastern strands of T_n to the western strands of T_1. See Figure 3.2 for an example.

Lemma 3.3.2. *Let $T_1, T_2, ..., T_n$ be strongly alternating tangle diagrams whose Conway sum is a link diagram $D(L)$. Then $tw(D(L)) = \sum_{i=1}^{n} tw(T_i)$, where $tw(D(L))$ is the twist number for $D(L)$ and $tw(T_i)$ the twist number for the tangle diagram T_i.*

Proof. First notice that taking the sum of tangles will not result in new twist regions. This is because, when taking a Conway sum, crossings that shared a twist region will still share a twist region and we introduce no new crossings. Therefore, $tw(D(L)) \leq \sum_{i=1}^{n} tw(T_i)$.

Now assume that $tw(D(L)) < \sum_{i=1}^{n} tw(T_i)$. Then for some i, a twist region in T_i and a twist region in T_{i+1} become one region in L. This implies there exists a simple closed curve γ which transversely intersects $D(L)$ twice in T_i and twice in T_{i+1}. If we think back to T_i lying in a unit square, then γ must intersect the north and south edges or the east and west edges of the square. In the first case, this shows that the denominator closure is not prime, and the second, the numerator closure is not prime. But as T_i is strongly alternating, this would be a contradiction, therefore $tw(D(L)) = \sum_{i=1}^{n} tw(T_i)$. □

Next we consider the relationship between the twist number of a tangle diagram and the twist numbers of diagrams of its closures.

Lemma 3.3.3. *Let T be a strongly alternating tangle diagram, and let $D(K)$ be the link diagram* *which comes from the numerator or denominator closure. Then:*

$$tw(T) - 2 \leq tw(D(K)) \leq tw(T).$$

Proof. The upperbound is true as we are not adding crossings when closing a tangle, and hence cannot create new twist regions.

The lower bound stems from the fact that, when we choose a closure for T we create two new potential bigons. If either region is a bigon, then it joins two twist regions. If both regions are bigons, the twist number is reduced by 2. In Figure 3.6 we see an example of a tangle where the lower bound is sharp for both closures. □

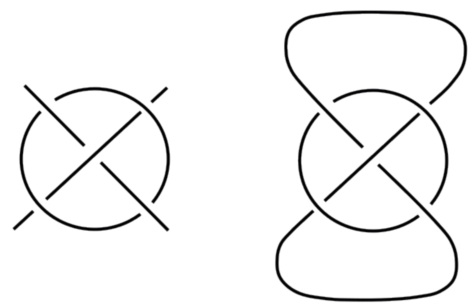

Figure 3.6 Left: A strongly alternating tangle with 5 twist regions. Right: The numerator closure with 3 twist regions. The denominator also results in 3 twist regions, showing the sharpness of the lower bound.

We will also need Theorem 3.8 from [24] which we state here. This Theorem allows us to relate the cross cap numbers of the closures of strongly alternating tangles to twist numbers of their diagrams.

Theorem 3.3.4. *Let $L \subset S^3$ be a link of k_L components with a prime, twist-reduced, alternating diagram $D(L)$. Suppose that $D(L)$ has $tw(D(L)) \geq 2$ twist regions. Let $C(L)$ denote the crosscap number of L. We have*

$$\left\lceil \frac{tw(D(L))}{3} \right\rceil + 2 - k_L \leq C(L) \leq tw(D(L)) + 2 - k_L$$

Furthermore, both bounds are sharp.

Now that we have that the twist number is additive for strongly alternating tangles by Lemma 3.3.2 and can relate $C(K_i)$ and $tw(T_i)$ by Lemma 3.3.3 and Theorem 3.3.4, we are ready to state Theorem 3.3.5

Theorem 3.3.5. *Let T_1 and T_2 be diagrams of non-splittable, strongly alternating, twist-reduced tangles whose Conway sum is a link diagram $D(L)$. Let $C(L)$ be the crosscap number of L, $tw(D(L))$ be the twist number of $D(L)$ and k_L be the number of link components in L. Then*

$$\left\lceil \frac{tw(D(L))}{3} \right\rceil - k_L \leq C(L) \leq tw(D(L)) + 4 - k_L.$$

Proof. We start with a lower bound in the proof of Lemma 3.2.10 with ambiguity on ϵ where $k_1 + k_2 - \epsilon = k_L$. Then $C(K_1) + C(K_2) - 2 + \epsilon \leq C(L)$. Let $D(K_1)$ and $D(K_2)$ be the diagrams of K_1 and K_2 that arise from cutting L as in Theorem 1.1.2. Notice that $tw(D(K_i)) \geq 2$ for $i = 1, 2$ as T_i is strongly alternating, and tangle diagrams with twist number 1 will have a non prime closure. Then by Lemma 3.3.4 we find for $i \in \{1, 2\}$ that $\left\lceil \frac{tw(D(K_i))}{3} \right\rceil + 2 - k_i \leq C(K_i)$ where K_i has k_i link components. So:

$$\left\lceil \frac{tw(D(K_1))}{3} \right\rceil + \left\lceil \frac{tw(D(K_2))}{3} \right\rceil + 2 + \epsilon - k_1 - k_2 \leq C(L).$$

By Lemma 3.3.3 $tw(T_i) - 2 \leq tw(D(K_i))$ and from Lemma 3.3.2 we know $tw(T_1) + tw(T_2) = tw(D(L))$, combining the two lemmas shows

$$\left\lceil \frac{tw(D(L))}{3} \right\rceil - 2 \leq \left\lceil \frac{tw(T_1) - 2}{3} \right\rceil + \left\lceil \frac{tw(T_2) - 2}{3} \right\rceil \tag{3.1}$$

$$\leq \left\lceil \frac{tw(D(K_1))}{3} \right\rceil + \left\lceil \frac{tw(D(K_2))}{3} \right\rceil. \tag{3.2}$$

Finally substituting in $k_1 + k_2 = k_l + \epsilon$, we find:

$$\left\lceil \frac{tw(D(L))}{3} \right\rceil - k_L \leq C(L).$$

Now we consider the upper bound. Similar to the lower bound we start with a step from Lemma 3.1.4, $C(L) \leq \min\{C(K_{1N}) + C(K_{2N}) + \epsilon, C(K_{1D}) + C(K_{2D}) + \epsilon\}$. By Lemma 3.3.3 we see that $tw(D(K_{iN})) \leq tw(T_i)$ and $tw(D(K_{iD})) \leq tw(T_i)$ and then by Theorem 3.3.4,

$$C(L) \leq tw(T_1) + tw(T_2) + 4 + \epsilon - k_1 - k_2.$$

Then substituting for $k_1 + k_2 = k_L + \epsilon$ and considering Lemma 3.3.2,

$$C(L) \leq tw(D(L)) + 4 - k_L.$$

Hence, showing the claim. $\square$

3.3.2 Jones Polynomial Bounds

From here we work to find bounds in terms of T_L, but first we have to generalize Theorem 1.6 in [19] which will allow us to relate the twist number of the diagram of a link L to T_L. Theorem 1.6 from [19] only considers knots but we need a similar result for links. We start by recalling some necessary vocabulary from Chapter 2.

Definition 3.3.6. Recall that if $D(L)$ is a diagram of the link L then the *A resolution* and *B resolution* for a crossing are as shown in Figure 2.1. Then a *Kauffman state* for $D(L)$ is a choice of resolution for all crossings in the diagram.

Next, we recall the construction of a *all A (resp. all B) state graph* $\mathbb{G}_A(D(L))$ (resp. $\mathbb{G}_B(D(L))$) by adding edges where we performed resolutions and then contracting the simple closed curves to vertices. Let e_A (resp. e_B) be the number of edges in $\mathbb{G}_A(D(L))$ (resp. $\mathbb{G}_B(D(L))$). Further, if we identify all parallel edges we find the *reduced state graph* $\mathbb{G}'_A(D(L))$ (resp. $\mathbb{G}'_B(D(L))$). Let e'_A (resp. e'_B) be the number of edges in $\mathbb{G}'_A(D(L))$ (resp. $\mathbb{G}'_B(D(L))$).

Now we restate the definition of link adequacy.

Definition 3.3.7. We call a link diagram *A-adequate* (resp. *B-adequate*) if the A state (resp. B state) graph of the diagram has no one edge loops. A link diagram is called *adequate* if it is both A-adequate and B-adequate, and a link is adequate if it has a diagram which is adequate.

Here we recall some terminology from [19].

Definition 3.3.8. Let $D(L)$ be the link diagram obtained by taking the Conway sum of strongly alternating tangles $T_1, ..., T_n$. Let $\ell_{\text{in}}(D(L))$ denote the loss of edges in $\mathbb{G}_A(D(L))$ and $\mathbb{G}_B(D(L))$ as we pass from $e_A + e_B$ to $e'_A + e'_B$ which come from equivalent crossings in the same tangle T_i. Then let $\ell_{\text{ext}}(D(L))$ be the number of edges we lose from identification when we take the Conway sum. It follows that $\ell_{\text{in}}(D(L)) + \ell_{\text{ext}}(D(L)) = e_A + e_B - e'_A - e'_B$.

For an alternating tangle diagram T, notice that the vertices of $\mathbb{G}_A(T)$ and $\mathbb{G}_B(T)$ are in 1-1 correspondence with the regions of T. Note that for the state graph of a tangle, if we consider the tangle lying within a disk, we have four exterior regions bounded by the disk. This means our state graphs have *interior vertices* those whose region lie entirely within the interior of the disk and two *exterior vertices* with corresponding region, with sides on the boundary of the disk.

For a tangle T_i, a *bridge* of $\mathbb{G}_A(T_i)$ (respectively $\mathbb{G}_B(T_i)$) is a subgraph consisting of an interior vertex v, and edges e', e'' which connect v to the exterior vertices v' and v''. We call the bridge *inadmissible* if the exterior vertices become identified in $\mathbb{G}_A(D(L))$ (respectively $\mathbb{G}_B(D(L))$), see Figure 3.7 for reference.

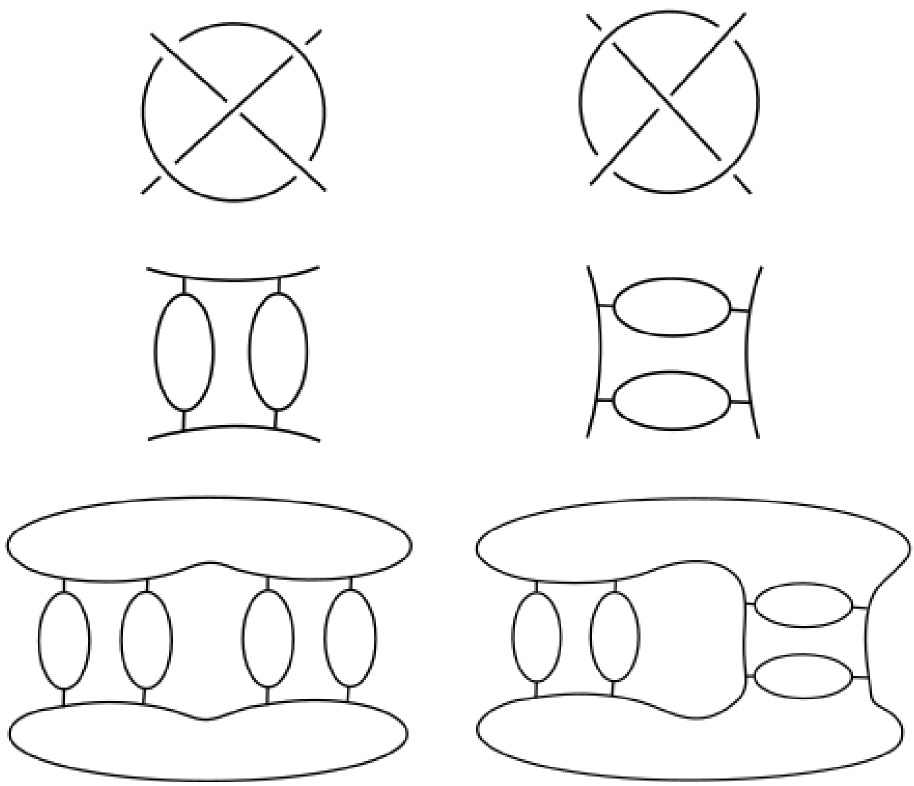

Figure 3.7 Top: Two strongly alternating tangles. Middle: The all A-resolution diagrams for the tangles. Bottom Left: The all A-state diagram for the Conway sum of two of the left tangles which has all admissible bridges. Bottom Right: The all A-state diagram for the Conway sum of one of both tangles, here the bridges are all inadmissible.

Now we find an upper bound for $\ell_{\text{ext}}(D(L))$ in terms of the twist number. This work will largely follow the proof of Lemma 5.4 in [19].

Lemma 3.3.9. *Let T_1 and T_2 be strongly alternating tangles whose Conway sum is a link diagram $D(L)$. Let k_L be number of link components in L. Then:*

$$\ell_{ext}(D(L)) \leq \frac{tw(D(L))}{2} + k_L + 4$$

Proof. For $T \in \{T_1, T_2\}$ let $b_A(T)$, $b_B(T)$ be the number of bridges in $\mathbb{G}_A(T)$ and $\mathbb{G}_B(T)$ respectively. Then the contribution of T to ℓ_{ext} will be at most $b_A(T) + b_B(T)$. Any other edge identification from moving to the reduced graph will still be counted by ℓ_{int}.

If b is a bridge there are two possibilities:

i. The edges e', e'' do not come from the resolutions of a single twist region.

ii. The edges e', e'' come from the resolutions of a single twist region.

Notice that for type ii bridges the two crossings which result in the edges are the only two in their respective twist region. Otherwise the two edges will not be adjacent to the exterior vertices or this would no longer constitute a twist region.

For type i bridges notice that when we pass from $\mathbb{G}_A(T)$ and $\mathbb{G}_B(T)$ to $\mathbb{G}'_A(T)$ and $\mathbb{G}'_B(T)$ the contributions to ℓ_{ext} is half the number of twist regions involved in such bridges. Unlike in [19] we can have more than one type ii bridge, as each additional type ii bridges creates a new link component.

Case 1: Suppose that $b_A(T) \geq 3$ or $b_B(T) \geq 3$. Without loss of generality let $b_A(T) \geq 3$, then $b_B(T) = 0$. If $b_B(T)$ were not zero then the B state bridge would cross the A state bridges, implying two internal vertices which is not a bridge.

There can be any number of type ii bridges, but we notice each bridge beyond the first will add a new link component. If we have only type ii bridges $b_A(T) \leq k_T$ where k_T is the number of tangle components. On the other hand if we only have type i bridges $b_A(T) \leq \frac{tw(T)}{2}$. Then for any mix of bridges we find that $b_A(T) + b_B(T) \leq \frac{tw(T)}{2} + k_T$.

Case 2: In this case we will consider $b_A(T) = b_B(T) = 2$. Then k must be at least 2, as the bridges in $\mathbb{G}_A(T)$ and $\mathbb{G}_B(T)$ will create a square resulting in a tangle second component. Also notice that as $\mathbb{G}_A(T)$ has two bridges there are at least two twist regions in T. Then $b_A(T)+b_B(T) \leq \frac{tw(T)}{2} + k_T + 1$.

Case 3: Either $b_A(T) \leq 2$ and $b_B(T) \leq 1$ or $b_A(T) \leq 1$ and $b_B(T) \leq 2$. Without loss of generality consider the first possibility. Then $b_A(T) + b_B(T)$ is at most three. If it's less than three we see that $k_T + 1 \geq 2$ so we need only consider when they sum to three. But as with the previous case we will have at least two twist regions as $\mathbb{G}_A(T)$ has two bridges. Thus, $b_A(T) + b_B(T) \leq \frac{tw(T)}{2} + k_T + 1$.

Then by Lemma 3.3.2 we know that the twist number is additive over Conway sums. By Lemma 3.2.1 $k_1 + k_2 \leq k_L + 2$. Then we find the following bound;

$$\ell_{\text{ext}} \leq \sum_{i=1}^{2} b_A(T_i) + b_B(T_i) \leq \frac{tw(D(L))}{2} + k_L + 4.$$

□

Now we have the tools necessary to prove the main lemma needed to find bounds for $C(L)$ in terms of T_L.

Lemma 3.3.10. *Let $T_1, ..., T_n$ be strongly alternating tangles whose Conway sum is a link diagram $D(L)$ for a link L. Then letting β_L and β'_L be the second and second-to-last coefficients of the Jones polynomial of L, $T_L = |\beta_L| + |\beta'_L|$, and k_L the number of link components, we have*

$$\frac{tw(D(L))}{2} - k_L - 2 \leq T_L \leq 2tw(D(L)).$$

Proof. We start by noting that we can mutate the link L in such a way that it either is alternating or the sum of T and T' where T is a positive strongly alternating tangle and T' is a negative strongly alternating tangle, without changing its Jones polynomial [32]. Where the positive and negative refer to whether the northwest strand originates from and overcrossing or an undercrossing. In the former case we have a stronger result by Dasbach and Lin [9] that $T_L = tw(L)$.

We will assume that L is not alternating. Then work by Lickorish and Thistlewaite [33] shows that $D(L)$ is adequate. Further, by propositions 1 and 5 of [33] we have $v_A + v_B = c$ where v_A is the number of vertices in $\mathbb{G}_A(D(L))$, v_B the same in $\mathbb{G}_B(D(L))$ and c the number of crossings in $D(L)$. Every edge we lose when passing from $\mathbb{G}_A(D(L))$ and $\mathbb{G}_B(D(L))$ to $\mathbb{G}'_A(D(L))$ and $\mathbb{G}'_B(D(L))$ comes from edges beyond the first in a twist region or a type (i) inadmissible bridge. By Lemma 5.2 in [19] we see that the number of edges lost from twist regions is $c - \mathrm{tw}(D(L))$. On the other hand type (i) inadmissible bridges for strongly alternating tangles will be precisely those which we lose from identification when taking the Conway sum, $\ell_{\mathrm{ext}}(D(L))$. Then the first part of the move from (3.4) to (3.5) below will come from the equality $e'_A + e'_B - e_A - e_B = -(c - tw(D(L)) + \ell_{ext})$. Work by Stoimenow shows that for an adequate link diagram, (see [9] for a proof)

$$T_L = e'_A + e'_B - v_A - v_B + 2 \tag{3.3}$$

$$= (e'_A + e'_B - e_A - e_B) + e_A + (e_B - v_A - v_B) + 2 \tag{3.4}$$

$$= -(c - tw(D(L)) + \ell_{ext}) + c + (c - v_A - v_B) + 2 \tag{3.5}$$

$$\geq tw(D(L)) - \ell_{ext} + 2 \tag{3.6}$$

$$\geq tw(D(L)) - \frac{tw(D(L))}{2} - k_L - 4 + 2 = \frac{tw(D(L))}{2} - k_L - 2. \tag{3.7}$$

The upper bound on T_L was shown by Futer, Kalfagianni and Purcell in [15]. □

Theorem 1.1.1. *Let T_1 and T_2 be non-splittable, twist reduced, strongly alternating tangles whose Conway sum is a link L. If $C(L)$ is the crosscap number of L, $T_L = |\beta_L| + |\beta'_L|$ and k_L is the number of link components in L we find that,*

$$\left\lceil \frac{T_L}{6} \right\rceil - k_L \leq C(L) \leq 2T_L + k_L + 8.$$

Proof. This follows immediately from Theorem 3.3.5 and Lemma 3.3.10. □

The following corollary is an immediate result from setting further restrictions on the twist numbers of our tangles.

Corollary 3.3.11. *Let T_1 and T_2 be twist reduced, non-splittable, strongly alternating tangles whose Conway sum is a link L. Assume that $tw(T_i) = tw(K_iN) = tw(K_iD)$. If $C(L)$ is the crosscap number of L, $T_L = |\beta_L| + |\beta'_L|$ and k_L is the number of link components in L, we have,*

$$\left\lceil \frac{T_L}{6} \right\rceil + 2 - k_L \leq C(L) \leq 2T_L + k_L + 8.$$

3.4 Generalizing to Larger Conway Sums of Tangles

Our goal in this section is to generalize Theorem 1.1.1 to Conway sums of more than two tangles. A *Conway sum* of more than 2 tangles is a closure where we connect diagrams of the tangles $T_1, T_2, ..., T_l$ linearly west to east shown in Figure 3.2. As with the case of the sum of two tangles, if we let L be our Conway sum and S a spanning surface, cutting S along a Conway sphere intersecting T_i will result in a spanning surface for either K_{iN} or K_{iD}, dictating the closure for the tangle. When we cut L, we position l Conway spheres such that Σ_i intersects L at the directional strands of T_i. We note that $S^3 \setminus \bigcup_i \Sigma_i$ will not be a sphere, but we are concerned with the surfaces within the interior of each Σ_i. $S/\cup_i \Sigma_i$ will be a collection of bands and tubes which we consider in the Euler characteristic change. This section will have similar results to the previous sections but with a factor for the number of tangles. We start with the following lemma which is a generalization of Lemma 3.2.1.

Lemma 3.4.1. *Let L be the Conway sum of the tangles $T_1, T_2, ..., T_l$, and $K_1, K_2, ..., K_l$ are closures of the tangles. Then if k_L is the number of link components for L, and $k_1, k_2, ..., k_l$ the number of link components for each link respectively then $k_L = \sum_{i=1}^{l} k_i - l + \epsilon$ for $\epsilon = 0, 1, 2$.*

Theorem 3.4.2. *Let $T_1, T_2, ..., T_l$ be non-splittable, strongly alternating tangles, and let L be the Conway sum that results from the l tangles. If K_i is the closure of T_i resulting from cutting the crosscap realizing spanning surface for L for all $i \in \{1, 2, ..., l\}$ and K_{iN} is the numerator closure of T_i and K_{iD} the denominator closure, then we have:*

$$\sum_{i=1}^{l} C(K_i) - l \leq C(L) \leq min\{\sum_{i=1}^{l} C(K_{iN}) + l, \sum_{i=1}^{l} C(K_{iD}) + 2\}.$$

Proof. This proof will largely follow the work we did in Lemma 3.2.10 and Lemma 3.1.4. We will start by considering the upper bound.

First we consider the case where we have K_{iN} for all $i \in \{1, 2, ..., l\}$, and spanning surfaces S_i for each K_{iN} such that $C(S_i) = C(K_{iN})$. Unlike in Lemma 3.1.4 the NW and NE strands connect to different tangles and we will find the same for the SW and SE strands. Then the spanning surface resulting from the Conway sum will have northern and southern disks attached to each of the S_i by a band as seen in figure 3.8. Let this resulting surface be S.

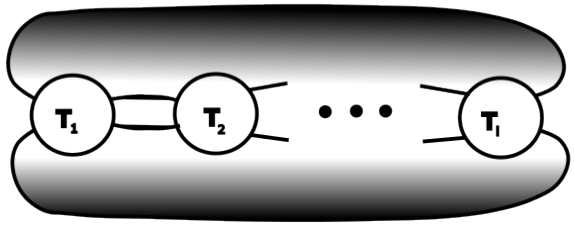

Figure 3.8 We see here that a surface would have to fill the shaded areas to connect the numerator closures of the tangles since the western and eastern boundaries of the S_i connect to separate tangles.

By this construction $\chi(S) = \sum_{i=1}^{l} \chi(S_i) - 2l + 2$ where the $2l$ comes from the bands connecting each surface to the disks, and the 2 from the disks themselves. Then by Lemma 3.4.1 we see that: $k_L = \sum_{i=1}^{l} k_{iN} - l + \epsilon$. Then

$$C(S) = 2 - (\sum_{i=1}^{l} \chi(S_i) - 2l + 2) - (\sum_{i=1}^{l} k_{iN} - l + \epsilon) = \sum_{i=1}^{l} C(K_{iN}) + l - \epsilon.$$

Then we see the weakest upperbound is when $\epsilon = 0$, so

$$C(L) \leq C(S) = \sum_{i=1}^{l} C(K_{iN}) + l.$$

Meanwhile the all denominator closure case will be similar to when $l = 2$. In particular $\chi(S) = \sum_{i=1}^{l} \chi(S_i) - l$ as we add bands to connect each of the S_i to their neighboring surfaces. By Lemma 3.4.1 and similar computations to the numerator closure case we find $C(L) \leq \sum_{i=1}^{l} C(K_{iD}) + 2$. We have a 2 instead of an l as we added half the number of bands in constructing S. Then we take the minimum of the denominator and numerator bounds to find an upperbound for $C(L)$.

Now we consider the lower-bound. Let S be a spanning surface for L such that $C(S) = C(L)$. Similar to Lemma 3.2.10 we will be considering the surfaces $S_1, S_2, ..., S_l$ that result from cutting along the Conway spheres Σ_i. By a similar argument to the one for Lemma 3.2.8, S can be chosen so that $\Sigma_i \cap S$ contains at most one closed curve for all i. Notice that if any of the steps in Lemma 3.2.8 were to disconnect the surface outside Σ_i, then for some other Σ_j, $i \neq j$, T_j would span a disconnected surface which is a contradiction to Lemma 3.2.6. Then when we cut along the Conway spheres we see that we are at most cutting along two arcs and a closed curve.

We know from Lemma 3.4.1 that $k_L = \sum_{i=1}^{l} k_i - l + \epsilon$ for $\epsilon = 0, 1, 2$. If we have closed curves along a Σ_i, when we cut we will have to add disks to both resulting boundary components which increases the Euler charactersitic by 2. For any surface resulting from cutting that is orientable we will have to add in a half twist band to make it non-orientable. Each such half twist band reduces the Euler characteristic by 1. Then $\sum_{i=1}^{l} \chi(S_i) = \chi(S) + t + c - b$ where $t = l$ or $t = 2l - 2$ depending on if the K_i are the denominator or numerator closures, c the number of closed curves on the Σ_i which can be as large as l and b the number of twist bands added to make the S_i non-orientable which also has maximum l. The value of t arises from the bands which sit in $S^3 \setminus \bigcup_i \Sigma_i$.

Now we see that

$$\sum_{i=1}^{l}(C(S_i)) = 2l - \sum_{i=1}^{l} \chi(S_i) - \sum_{i=1}^{l} k_i = 2l - \chi(S) - t - c + b - k_L - l + \epsilon.$$

Notice that the weakest upperbound for $\sum_{i=1}^{l}(C(S_i))$ is when t and c are minimal and b and ϵ are maximal. So this will be when we do not have any closed curves and each of the resulting S_i need half twist bands to make them non orientable. So:

$$\sum_{i=1}^{l}(C(S_i)) = 2l - \chi(S) - l + l - k_L - l + 2 = C(L) + l.$$

Then moving the l to the other side we see that $\sum_{i=1}^{l} C(K_i) - l \leq C(L)$, showing the claim.

□

As in section 3 we will now use Theorem 3.3.4 to find bounds for $C(L)$ in terms of $tw(D(L))$ where $tw(D(L))$ is the twist number for a diagram of L.

Theorem 3.4.3. *Let $T_1, T_2, ..., T_l$ be non-splittable, twist reduced, strongly alternating tangles and let $D(L)$ be the link diagram for the link L which results from taking the Conway Sum of the tangles. Let $tw(D(L))$ denote the twist number of $D(L)$ and $C(L)$ the crosscap number for L, then*

$$\left\lceil \frac{tw(D(L))}{3} \right\rceil + 2 - k_L \leq C(L) \leq tw(D(L)) + l + 2 - k_L.$$

Proof. We start with the bounds from Theorem 3.4.2. From here we use Lemma 3.3.4 and Lemma 3.3.3 to get bounds on $C(L)$ with respect to the twist numbers of diagrams of $T_1, ..., T_l$.

Combining these two statements we find

$$\sum_{i=1}^{l}\left\lceil\frac{tw(T_i)-2}{3}\right\rceil + l - \sum_{i=1}^{l} k_i \leq C(L) \leq \sum_{i=1}^{l} tw(T_i) + 2l + 2 - \sum_{i=1}^{l} k_i,$$

where k_i is the number of link components for each K_i.

We know that the twist number of strongly alternating tangles is additive over a Conway sum by Lemma 3.3.2. So the only detail left to consider is the relationship between k_L and $\sum_{i=1}^{l} k_i$. By Lemma 3.4.1 and a similar argument to that in Theorem 3.3.5 we find the claim:

$$\left\lceil\frac{tw(D(L))-2l}{3}\right\rceil + 2 - k_L \leq c(L) \leq tw(D(L)) + l + 2 - k_L.$$

□

The final piece of our puzzle is to find bounds in terms of T_L. To do this we use Lemma 3.3.10 and Theorem 3.4.3 and the result follows.

Theorem 1.1.3. *Let $T_1, T_2, ..., T_l$ be non-splittable, twist reduced, strongly alternating tangles and let L be the link which results from taking the Conway Sum. Then let $C(L)$ be the crosscap number and k_L the number of link components in L, then*

$$\left\lceil\frac{T_L-2l}{6}\right\rceil + 2 - k_L \leq C(L) \leq 2T_L + l + 6 + k_L.$$

With an additional constraint on our tangles we find the following corollary:

Corollary 3.4.4. *Let $T_1, T_2, ..., T_l$ be non-splittable, twist reduced, strongly alternating tangles such that $tw(T_i) = tw(D(K_{iN})) = tw(D(K_{iD}))$ for all $i \in \{1, ..., l\}$. Let L be the link which results from taking the Conway Sum, $C(L)$ the crosscap number, and k_L the number of link components in L, then*

$$\left\lceil\frac{T_L}{6}\right\rceil + 2 - k_L \leq C(L) \leq 2T_L + l + 6 + k_L.$$

3.5 Families where T_L and the Crosscap Number are Independent

We begin by recalling the following theorem from [24] which gives linear bounds for the crosscap number of an alternating link in terms of T_L, where $T_L = |\beta_L| + |\beta'_L|$ and β_L and β'_L are the second and second to last coefficients of the Jones polynomial of L respectively.

Theorem 3.5.1. *Let L be a non-split, prime alternating link with k-components and with crosscap number $C(L)$. Suppose that K is not a $(2, p)$ torus link. We have*

$$\left\lceil \frac{T_L}{3} \right\rceil + 2 - k \leq C(L) \leq T_L + 2 - k$$

where T_L is as above. Furthermore, both bounds are sharp.

In the previous sections we showed Theorem 3.5.1 generalizes to Conway sums of strongly alternating tangles. In this section we will show that Theorem 3.5.1 does not generalize to arbitrary knots.

Theorem 1.1.4. *We have the following;*

(a) There exists a family of links for which $T_L \leq 2$, but $C(L)$ is arbitrarily large.

(b) There exists a family of links for which $C(L) \leq 3$, but T_L is arbitrarily large.

3.5.1 Part (*a*) of theorem 1.1.4

In this section we will consider the following family of torus knots; $T(p, q)$, where $q = j$ and $p = 2 + 2jk$ for odd $j > 1$ and all natural numbers k. This family will allow us to prove part(a) of Theorem 1.1.4. We start with the following definition from Teragaito [45].

Definition 3.5.2. We define the value $N(p, q)$ from [45] for fractions $\frac{p}{q}$, where p and q are coprime, to begin write $\frac{p}{q}$ as a continued fraction,

$$\frac{p}{q} = [a_0, a_1, a_2, ..., a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}},$$

where the a_i are integers, $a_0 \geq 0$, $a_i > 0$ for $1 \leq i \leq n$, and $a_n > 1$. A continued fraction of this form is unique (cf [21]). Now we recursively define b_i as follows:

$$b_0 = a_0$$

$$b_i = \begin{cases} a_i & \text{if } b_{i-1} \neq a_{i-1} \text{ or if } \sum_{j=0}^{i-1} b_j \text{ is odd,} \\ 0 & \text{if } b_{i-1} = a_{i-1} \text{ and } \sum_{j=0}^{i-1} b_j \text{ is even.} \end{cases}$$

Then, $N(p,q) = \frac{1}{2}\sum_{i=1}^{n} b_i$. We say a torus knot K is even if the product of p and q is even and we say K is odd otherwise. Using these definitions we can state Theorem 1.1 from [45].

Theorem 3.5.3. *Let K be the non-trivial torus knot of type (p,q), where $p,q > 0$ and let F be a non-orientable spanning surface of K with $C(F) = C(K)$.*

(1) If K is even, then $C(K) = N(p,q)$ and the boundary slope of F is pq.

(2) If K is odd, then $C(K) = N(pq-1, p^2)$ (resp. $N(pq+1, p^2)$) and the boundary slope of F is $pq-1$ (resp. $pq+1$) if $xq \equiv -1$ (mod p) has an even (resp. odd) solution x satisfying $0 < x < p$.

We take advantage of (*1*) from Theorem 3.5.3 to both construct our family of torus link and prove Proposition 3.5.5 below. We also need the following lemma, which gives an explicit formula for the Jones polynomial of a torus knot originally given in Proposition 11.9 of [23], which allows us to calculate T_L for torus knots.

Lemma 3.5.4. *The Jones polynomial for a torus knot $T(p,q)$ is given by;*

$$V(T(p,q)) = t^{(p-1)(q-1)/2}\frac{1 - t^{p+1} - t^{q+1} + t^{p+q}}{1-t^2}.$$

Proposition 3.5.5. *Let $L = T(p,q)$ be the family of torus knots where $q > 1$ is odd and $p = 2+2qk$ for $k \in \mathbb{N}$, then $T_L \leq 2$ but $C(T(p,q))$ can be made arbitrarily large.*

Proof. Let $q > 1$ be odd, and $p = 2 + 2qk$ where k is a natural number. To show that for all such torus knots, $L = T(p,q)$, $C(L)$ does not have a universal upper bound with respect to T_L, we will show that as k goes to ∞, $C(L)$ also goes to ∞, but $T_L \leq 2$. We start by computing the crosscap number of $T(p,q)$ using Theorem 3.5.3.

First we notice that $\frac{p}{q} = \frac{2+2qk}{q} = 2k + \frac{1}{1+\frac{1}{2}}$. Then $A = [2k, 1, 2]$. Then by definition 3.5.2 $B = [2k, 0, 2]$. Finally, as pq is even,

$$C(L) = N(p,q) = \frac{2k+0+2}{2} = k+1.$$

Then as $k \to \infty$, $C(L)$ also goes to ∞.

Next by Lemma 3.5.4 we know that

$$\begin{aligned}
V(L) &= t^{(p-1)(q-1)/2} \frac{1 - t^{p+1} - t^{q+1} + t^{p+q}}{1 - t^2} \\
&= t^{((2+2qk)q-(2+2qk)-q+1)/2} \frac{1 - t^{2+2qk+1} - t^{q+1} + t^{2+2qk+q}}{1 - t^2} \\
&= t^{((2+2qk)q-(2+2qk)-q+1)/2} (-t^{2qk+q} - t^{2qk+q-2} - \\
&\qquad \cdots - t^{2+2qk+1} + t^{q-1} + \cdots + t^2 + 1).
\end{aligned}$$

The last step arises from taking the polynomial division. Therefore, given our choices of p and q we see that $T_L \leq 2$. □

Then Proposition 3.5.5 shows part (a) of Theorem 1.1.4.

3.5.2 Part (*b*) of theorem 1.1.4

In this section we will work to prove part (*b*) of Theorem 1.1.4 and the following theorem:

Theorem 3.5.6. *There does not exist a universal linear lower bound on $C(L)$ for all links, L, in terms of T_L.*

To prove this we will introduce a family of links for which $C(L)$ is uniformly bounded but T_L can be made arbitrarily large. These links will be constructed by using the Whitehead double defined here:

Definition 3.5.7. The *Whitehead double* of a knot L is the satellite of the unknot clasped inside of the torus. We call it a *positive* Whitehead double if the clasp is as in Figure 3.9 and a *negative* Whitehead double if not.

The particular family is defined in this next theorem:

Theorem 3.5.8. *Let $K_1, K_2, ..., K_n$ be alternating knots such that $\beta'_{K_i} \neq 0$. Then we let K be the connect sum of $K_1, K_2, ..., K_n$ such that K is alternating and $W_-(K)$ be the negative Whitehead double of K using the blackboard framing. Then $C(L) \leq 3$ and $|\beta'_L| \geq n$.*

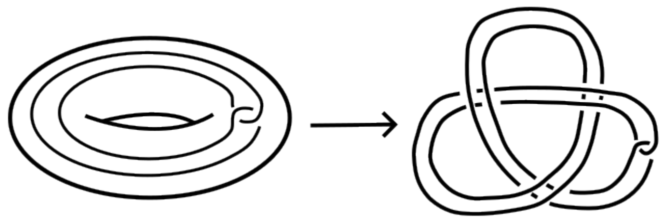

Figure 3.9 Here we see the unknot with a clasp contained inside the torus, then we see the resulting *positive* Whitehead double with the blackboard framing when map the torus to the trefoil.

Lemma 3.5.9. *If a link is B-adequate then the negative Whitehead double of the link using the blackboard framing is also B-adequate.*

A similar statement was proven in [6] as Proposition 7.1. They show it for the untwisted negative Whitehead double of a knot with non-negative writhe. The writhe of the knot introduces extra twists into the diagram of the untwisted Whitehead double, which can interfere with adequacy around the clasp.

Proof. We start by showing that the blackboard 2-cabling will be B-adequate. This is shown by Lickorish in [32] for n-cablings. Let D be a B-adequate diagram for our link and D^2 the 2 cabling. Notice in D^2 there will be four copies of each crossing in D. Then when we have the all B-resolution state we will end up with four parallel strands instead of two as we did in D. If we were to have a one edge loop, then two of the strands are part of the same state circle. But these state circles are copies of the state circles for D so this would contradict that D is adequate.

Now we want to look at the negative Whitehead double of D using the blackboard framing. If we let the Whitehead double be $W_-(D)$ we will see that $G'(W_-(D))$ will be the same as $G'(D^2)$ but with an additional vertex and 2 new edges as we see in Figure 3.10. As resolving the clasp does not create a one edge loop we see that $W_-(D)$ is B-adequate. □

Here we remind the reader that $G'_B(D(L))$ for a link diagram $D(L)$ is the reduced all B-state graph. We continue with the following lemma:

Lemma 3.5.10. *If $W_-(D(L))$ is the negative Whitehead double of a B-adequate link diagram $D(L)$ using the blackboard framing, and $b(G)$ the first Betti number for a graph, then*

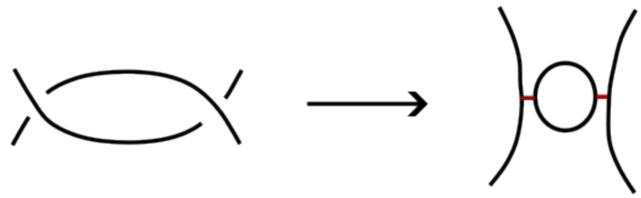

Figure 3.10 The result of the B-resolution on the clasp of a negative Whitehead double.

$b(\mathbb{G}'_B(W_-(D(L)))) = b(\mathbb{G}'_B(D(L))) + 1.$

Proof. Dasbach and Lin [10] showed in Lemma 2.5 that if D^2 is the two cabling of a B-adequate link diagram then $b(\mathbb{G}'_B(D^2)) = b(\mathbb{G}'_B(D(L)))$. For a graph G, $b(G) = e - v + 1$, where e is the number of edges and v the number of vertices. In the reduced graph when we take the two cabling every parallel copy of a state circle will also produce a new edge. Hence, the change in v and e will be the same between $\mathbb{G}'_B(D^2)$ and $\mathbb{G}'_B(D(L))$. Then when we move to $W_-(D(L))$ the clasp will add 2 edges and 1 vertex as we see in figure 3.10. Then we see that $b(\mathbb{G}'_B(W_-(D(L)))) = b(\mathbb{G}'_B(D^2)) + 1 = b(\mathbb{G}'_B(D(L))) + 1.$ □

The two previous lemmas allow us to see that the blackboard framing of the negative Whitehead double of an alternating link will be B-adequate. Also, we have a formula for the Betti number of the Whitehead double in relation to the first Betti number of the original link. The only remaining piece of the puzzle is to get from the first Betti number of the reduced B state graph to the second to last coefficient of the Jones polynomial. This comes from the following result proven by Stoimenow in Proposition 3.1 of [42].

Lemma 3.5.11. *If $D(L)$ is a B-adequate, connected diagram for a link, then in the representation of the Jones polynomial, $V(D(L))$, we have $\alpha'_{D(L)} = \pm 1$, $\alpha'_{D(L)}\beta'_{D(L)} \leq 0$, and*

$$|\beta'_{D(L)}| = e' - v' + 1 = b(\mathbb{G}'_B(D(L))),$$

where $\mathbb{G}'_B$ is the reduced all B state graph and e' and v' are the number of edges and vertices of the graph $\mathbb{G}'_B(D(L))$, respectively.

We now have the tools to prove Theorem 3.5.8. But first we show a more specific example of a family which satisfies Theorem 1.1.4 part (*b*).

Proposition 3.5.12. *Let $W_-(K_m)$ be the negative Whitehead double using the blackboard framing of the connect sum of m trefoils as in Figure 3.11. Then for all m, $C(W_-(K_m)) \leq 3$ and $T_{W_-(K_m)}$ grows with m. Therefore, $T_{W_-(K_m)}$ can be made arbitrarily large across the family of knots.*

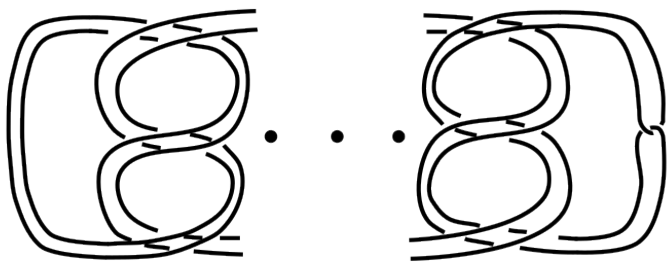

Figure 3.11 The negative Whitehead double of the connect sum of m trefoil knots.

Proof. The first part of the lemma is a direct result of [7] by Clark where he shows that $c(K) \leq 2g(K) + 1$ where $g(K)$ is the genus of the knot. For any Whitehead double we can find an oriented spanning surface with genus exactly one by taking the annulus with a double twisted band at the clasp. Then $C(W_-(K_m)) \leq 3$ as $g(W_-(K_m)) = 1$.

Now we will compute $\beta'_{W_-(K_m)}$ by finding $\mathbb{G}'_B(W_-(K_m))$. By Lemma 3.5.11 we only need to find the number of vertices and edges as $W_-(K_m)$ is B-adequate. By Lemma 3.5.11 and the graph $\mathbb{G}'_B(W_-(K_m))$ shown in Figure 3.12

$$|\beta'_{W_-(K_m)}| = e(\mathbb{G}'_B(W_-(K_m))) - v(\mathbb{G}'_B(W_-(K_m))) + 1 \tag{3.8}$$

$$= (5m + 3) - (4m + 3) + 1 = m. \tag{3.9}$$

Hence, showing that $T_{W_-(K_m)} \geq m$ for all k, proving the claim.

□

Here we will introduce a more general family of knots for which Theorem 1.1.4 part (*b*) holds true:

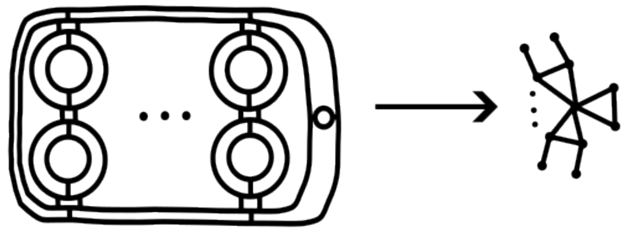

Figure 3.12 Left: The all B-state circle diagram. Right: The reduced state graph $\mathbb{G}'_B(L)$. Notice the disjoint dots are not nodes for the graph but represent that we have k copies of the subgraph on the left.

Theorem 3.5.8. *Let $K_1, K_2, ..., K_n$ be alternating knots such that $\beta'_{K_i} \neq 0$. Then let K be the connect sum of $K_1, K_2, ..., K_n$ such that K is alternating, and let $W_-(K)$ the negative Whitehead double of K using the blackboard framing. Then $C(W_-(K)) \leq 3$ and $|\beta'_{W_-(K)}| \geq n$.*

Proof. As in Lemma 3.5.12 for a Whitehead doubles such as $W_-(K)$, $C(W_-(K)) \leq 3$. Now we will work to compute $T_{W_-(K)}$. From [32] we know that the Jones polynomial for K will be the product of the Jones polynomials of the K_i. Then as all of the K_i are alternating, $\alpha'_{K_i} = \pm 1$ so $\beta'_K = \sum_{i=1}^n \pm\beta'_{K_i}$. From Lemma 3.5.11 we know that $\alpha'_{K_i}\beta'_{K_i} \leq 0$ which tells us that the signs of α'_{K_i} and β'_{K_i} do not match. If we let m be the number of the α'_i which are negative, then we see that $\beta'_K = \sum_{i=1}^n (-1)^{m\pm1}|\beta'_{K_i}|$. Hence, in our sum the signs match so $|\beta'_K| = \sum_{i=1}^n |\beta'_{K_i}|$. By our hypothesis $|\beta'_{K_i}| > 0$ for all i, hence $|\beta'_K| \geq n$.

By Lemma 3.5.9 we know that $W_-(K)$ will be B-adequate as K is alternating and therefore B-adequate. Then by Lemma 3.5.10 and Lemma 3.5.11 we see that $|\beta'_{W_-(K)}| = |\beta'_K| + 1$ and as $|\beta'_K|$ is at least as large as the number of knots in the connect sum so is $|\beta'_{W_-(K)}|$ and further $T_{W_-(K)}$. Then $T_{W_-(K)}$ will grow with n showing that it is unbounded across the family.

□

Lemma 3.5.12 and Theorem 3.5.8 both show that Theorem 1.1.4(b).

CHAPTER 4

CROSSING NUMBER OF CABLE KNOTS

In this chapter we will calculate the crossing number for an infinite family of cable knots. Similar to the previous chapter we will do this using a quantum invariant, in this case we will be using the colored Jones polynomial. The work and figures in this chapter come from a preprint by Kalfagianni and McConkey [26].

4.1 Jones Diameter

We will start this chapter by introducing some notation we will use throughout and introducing some tools we will need. We previously defined the colored Jones polynomial and in this chapter we will be using it to make calculations with respect to the crossing number of a family of cable knots. In particular we will be using a property of the colored Jones polynomial known as the Jones diameter which we define shortly. Afterwards we will introduce some results related to the Jones diameter which give us the tools we need for the main part of this chapter.

Here we state notation we will use in this chapter, some of which we have seen before. This is for the convenience of a reader whose interest lies in this chapter.

- $c(D)$ is the number of crossings of the knot diagram D.
- With an orientation on D, $c_+(D)$ and $c_-(D)$ are respectively the number of positive crossings and the number of negative crossings in the knot diagram D following the conventions of Figure 4.1.
- $v_A(D)$ is the number of state circles in the all-A state, and $v_B(D)$ is the number of state circles in the all-B state.
- The state graphs $\mathbb{G}_A(D)$ and $\mathbb{G}_B(D)$ have vertices the state circles of the all-A and all-B state respectively, and edge segments recording the original location of the crossing. In Figure 2.1, these segments are indicated in dashed line.
- The *writhe* of a knot diagram D, denoted by $\mathrm{wr}(D) := c_+(D) - c_-(D)$.

- The Turaev genus of D is defined by $2g_T(D) := 2 - v_A(D) - v_B(D) + c(D)$ [47, 11].

Figure 4.1 A positive crossing and a negative crossing.

Recall from definition 2.1.4 that a knot diagram $D = D(K)$ is *A-adequate* (resp. *B-adequate*) if $\mathbb{G}_A(D)$ (resp. $\mathbb{G}_B(D)$) has no one-edged loops. A knot is *adequate* if it admits a diagram $D = D(K)$ that is both A- and B-adequate [33, 32].

Notice that if $D = D(K)$ is an adequate diagram, then quantities $c(D)$, $c_\pm(D)$ [32, 28, 37, 46] as well as the Turaev genus $g_T(D)$ [1] are minimal over all diagrams representing K. As a result of this one can see that the writhe $\mathrm{wr}(D)$ for adequate diagrams of a given knot K is always constant. Thus, these are all invariants of K and we will denote them by $c(K)$, $c_\pm(K)$, $g_T(K)$, and $\mathrm{wr}(K)$ respectively.

Given a knot K let $J_K(n)$ denote its n-th colored Jones polynomial with variable in t. Let $d_+[J_K(n)]$ and $d_-[J_K(n)]$ denote the maximal and minimal degree of $J_K(n)$ in t and set

$$d[J_K(n)] := 4d_+[J_K(n)] - 4d_-[J_K(n)].$$

Lemma 4.1.1. *[32] Given a knot diagram $D = D(K)$, for all $n \in \mathbb{N}$, we have the following.*

(a) $d_+[J_K(n)] \leq \frac{c_+(D)}{2}n^2 + O(n)$ *and we have equality if D is B-adequate.*

(b) $d_-[J_K(n)] \geq -\frac{c_-(D)}{2}n^2 + O(n)$ *and we have equality if D is A-adequate.*

(c) $d[J_K(n)] \leq 2c(D)n^2 + (4 - 4g_T(D) - 2c(D))n + (4g_T(D) - 4)$*, and we have equality if D is adequate.*

The set of cluster points $\left\{n^{-2}d[J_K(n)]\right\}'_{n\in\mathbb{N}}$ is known to be finite and the point with the largest absolute value, denoted by dj_K, is called the *Jones diameter* of K.

The following, Theorem 4.1.2, gives us an essential relationship between the crossing number of K and its Jones diameter. The part of the theorem we will use for our main results is the non-adequate part as this allows us to directly calculate the crossing number for a family of cable knots.

Theorem 4.1.2. *[25]Let K be a knot with Jones diameter dj_K and crossing number $c(K)$. Then,*

$$dj_K \leq 2\,c(K),$$

with equality $dj_K = 2\,c(K)$ if and only if K is adequate.

In particular, if K is a non-adequate knot admitting a diagram $D = D(K)$ such that $dj_K = 2(c(D) - 1)$, then we have $c(D) = c(K)$.

The next couple of results we will state give formulae for the extreme degrees of the colored Jones polynomials for knots where the degrees $d_{\pm}[J_K(n)]$ are quadratic polynomials.

Proposition 4.1.3. *[27, 5] Suppose that K is a knot such that $d_+[J_K(n)] = a_2 n^2 + a_1 n + a_0$ and $d_-[J_K(n)] = a_2^* n^2 + a_1^* n + a_0^*$ are quadratic polynomials for all $n > 0$. Suppose, moreover, that $a_1 \leq 0$, $a_1^* \geq 0$ and that $\frac{p}{q} < 4a_2$ and $\frac{-p}{q} < 4a_2^*$.*

Then for n large enough,

$$4d_+[J_{K_{p,q}}(n)] = q^2 4a_2 n^2 + (q4a_1 + 2(q-1)(p - 4qa_2))n + A,$$

$$4d_-[J_{K_{p,q}}(n)] = -q^2 4a_2^* n^2 - (q4a_1^* + 2(q-1)(p - 4qa_2^*))n + A^*,$$

where $A, A^* \in \mathbb{Q}$ depend only on K and p, q.

Proof. The first equation is shown in [27] (see also [5]). To obtain the second equation we use the fact that, since $K^*_{-p,q} = (K_{p,q})^*$, we have $d_-[J_{K_{p,q}}(n)] = -d_+[J_{K^*_{-p,q}}(n)]$ and apply the first equation to $K^*_{-p,q}$.

$\square$

Here is a second result which handles the case not covered in Proposition 4.1.3.

Lemma 4.1.4. *[27, 5]Let the notation and setting be as in Proposition 4.1.3.*

If $\frac{p}{q} > 4a_2$, *then*

$$4d_+[J_{K_{p,q}}(n)] = pqn^2 + B,$$

where $B \in \mathbb{Q}$ *depends only on* K *and* p, q.

Similarly, if $\frac{-p}{q} > 4a_2^*$, *then*

$$4d_-[J_{K_{p,q}}(n)] = -pqn^2 + B^*,$$

where $B^* \in \mathbb{Q}$ *depends only on* K *and* p, q.

4.2 Lower bounds and admissible knots

We will say that a knot K is *admissible* if there is a diagram $D = D(K)$ such that we have

$$dj_K = 2\,(c(D) - 1).$$

Our interest in admissible knots comes from the fact that if K is admissible and non-adequate, then by Theorem 4.1.2, D is a minimal diagram (i.e. $c(D) = c(K)$).

Theorem 4.2.1. *Let* K *be an adequate knot and let* $c(K)$, $c_\pm(K)$ *and* $w(K)$ *be as above.*

(a) For any coprime integers p, q, *we have*

$$c(K_{p,q}) \geq q^2 \cdot c(K). \tag{4.1}$$

(b) The cable $K_{p,q}$ *is admissible if and only if* $q = 2$ *and* $p = q\,w(K) \pm 1$.

Proof. Since K is adequate, by Lemma 4.1.1,

$$4d_+[J_K(n)] - 4d_-[J_K(n)] = 2c(K)n^2 + (4 - 4g_T(K) - 2c(K))n + 4g_T(K) - 4, \tag{4.2}$$

for every $n \geq 0$.

We distinguish three cases.

Case 1. Suppose that $\frac{p}{q} < 2c_+(K)$ and $\frac{-p}{q} < 2c_-(K)$. Then, $d_+[J_K(n)]$ satisfies the hypothesis of Proposition 4.1.3 with $4a_2 = 2c_+(K) > 0$ and $d_-[J_K(n)] = -d_+[J_{K^*}(n)]$, where $d_+[J_{K^*}(n)]$ satisfies that hypothesis of Proposition 4.1.3 with $4a_2^* = 2c_+(K^*) = 2c_-(K)$. The requirement that

$a_1 \leq 0$ is satisfied since for adequate knots the linear terms of the degree of $J^*_K(n)$ are multiples of Euler characteristics of spanning surfaces of K. See [27, Lemmas 3.6, 3.7]. Now Proposition 4.1.3 implies that for sufficiently large n we have that $d_\pm[J_{K_{p,q}}(n)]$ is a quadratic polynomial and the Jones diameter of $K_{p,q}$ is $dj_K = q^2 c(K)$. Hence by Theorem 4.1.2 we get $c(K_{p,q}) \geq q^2 \cdot c(K)$ which proves part (a) of Theorem 1.1.7 in this case.

For part (b), we recall that a diagram $D_{p,q}$ of $K_{p,q}$ is obtained as follows: Start with an adequate diagram $D = D(K)$ and take q parallel copies to obtain a diagram D^q. In other words, take the q-cabling of D following the blackboard framing. To obtain $D_{p,q}$ add *t-twists* to D^q, where $t := p - q\, wr(K)$ as follows: If $t < 0$ then a twist takes the leftmost string in D^q and slides it over the $q-1$ strings to the right; then we repeat the operation $|t|$-times. If $t > 0$ a twist takes the rightmost string in D^q and slides it over the $q-1$ strings to the left; then we repeat the operation $|t|$-times. Now

$$c(D_{p,q}) = q^2\, c(K) + |t|(q-1) = q^2\, c(K) + |p - q\, wr(K)|(q-1),$$

while $dj_K = 2q^2\, c(K)$. Now setting $2c(D_{p,q}) - 2 = dj_K$, we get $|p - q\, wr(K)|(q-1) = 1$ which gives that $q = 2$ and $p = q\, wr(K) \pm 1$. Similarly, if we set $p = q\, wr(K) \pm 1$ we find that $2c(D_{p,q}) - 2 = dj_K$ must also be true. Hence in this case both (a) and (b) hold.

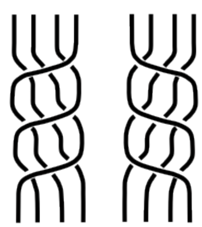

Figure 4.2 Left: 3 positive twists on four strands Right: 3 negative twists on four strands.

Case 2. Suppose that $\frac{p}{q} > 2c_+(K)$. Then by Lemma 4.1.4, $4d_+[J_{K_{p,q}}(n)] = pqn^2 + B$, where $B \in \mathbb{Q}$ depends only on K and p, q. Since $\frac{p}{q} > 2c_+(K)$, we get $pq > 2c_+(K)q^2$. On the other hand, since $\frac{-p}{q} < 0$, we clearly have $\frac{-p}{q} < 2c_-(K)$, and Proposition 4.1.3 applies to $d_+[J_{K^*_{-p,q}}(n)]$

for $4a_2^* = 2c_-(K)$. Then

$$4d_+[J_K(n)] - 4d_-[J_K(n)] = d_+[J_K(n)] + 4d_+[J_{K^*_{-p,q}}(n)] > q^2 \cdot c(K),$$

as desired. This finishes the proof for part (a) of the theorem. In this case, we don't get any admissible knots: first note that $p > 2qc_+(K) > q\,w(K)$. As in Case 1 we get a diagram $D_{p,q}$ of $K_{p,q}$ with

$$c(D_{p,q}) = q^2\, c(K) + (p - q\, w(K))(q - 1),$$

while $dj_K = 2q^2\, c_-(K) + p\, q$. Now setting $2c(D_{p,q}) - 2 = dj_K$, and after some straightforward algebra, we find that in order for $K_{p,q}$ to be admissible we must have

$$2(q^2 - q)\, c_-(K) + 2q\, c_+(K) + p\,(q - 2) - 2 = 0.$$

However, since $p, c(K) > 0$ and $q \geq 2$, above equation is never satisfied.

Case 3. Suppose that $\frac{-p}{q} > 2c_-(K) > 0$, in which case $\frac{p}{q} < 0 \leq 2c_-(K)$. This case is similar to Case 2 above.

□

Remark 4.2.2. In [43] inequality (4.1) is also verified, for some choices of p and q, using crossing number bounds obtained from the ordinary Jones polynomial in [44] and also from the 2-variable Kauffman polynomial. Theorem 1.1.7 shows that the colored Jones polynomial and the results of [25] provide better bounds for crossing numbers of satellite knots, allowing in particular exact computations for infinite families.

4.3 Non-adequacy results

To prove the stronger version of inequality (4.1), stated in Theorem 1.1.7, we need to know that the cables $K_{p,q}$ are not adequate. This is the main result in this section.

Theorem 4.3.1. *Let K be an adequate knot with crossing number $c(K) > 0$ and suppose that $\frac{p}{q} < 2c_+(K)$ and $\frac{-p}{q} < 2c_-(K)$. Then, the cable $K_{p,q}$ is non-adequate.*

To prove Theorem 4.3.1 we need the following lemma:

Lemma 4.3.2. *Let K be an adequate knot with crossing number $c(K) > 0$ and suppose that $\frac{p}{q} < 2c_+(K)$ and $\frac{-p}{q} < 2c_-(K)$. If $K_{p,q}$ is adequate, then $c(K_{p,q}) = q^2\, c(K)$.*

Proof. By our earlier discussion, for n large enough,

$$4d_+[J_{(K_{p,q}}(n)] - 4d_-[J_{K_{p,q}}(n)] = d_2 n^2 + d_1 n + d_0,$$

with $d_i \in \mathbb{Q}$. By Proposition 4.1.3, we compute $d_2 = q^2(4a_2 + 4a_2^*) = 2q^2 c(K)$. Now if $K_{p,q}$ is adequate, since $d_2 = 2c(K_{p,q})$, we must have $c(K_{p,q}) = q^2 c(K)$. □

We now give the proof of Theorem 4.3.1:

Proof. First, we let K, p, and q such that $t := p - q\, w(K) < 0$.

Recall that if K has an adequate diagram $D = D(K)$ with $c(D) = c_+(D) + c_-(D)$ crossings and the all-A (rep. all-B) resolution has $v_A = v_A(D)$ (resp. $v_B = v_B(D)$) state circles, then

$$4\, d_-[J_K(n)] = -2c_-(D)n^2 + 2(c(D) - v_A(D))n + 2v_A(D) - 2c_+(D), \tag{4.3}$$

$$4\, d_+[J_K(n)] = 2c_+(D)n^2 + 2(v_B(D) - c(D))n + 2c_-(D) - 2v_B(D). \tag{4.4}$$

Equation (4.3) holds for A-adequate diagrams $D = D(K)$. Thus in particular the quantities $c_-(D), v_A(D)$ are invariants of K (independent of the particular A-adequate diagram). Similarly, Equation (4.4) holds for B-adequate diagrams $D = D(K)$ and hence $c_+(D), v_B(D)$ are invariants of K. Recall also that $c(D) = c(K)$ since D is adequate.

Now we start with a knot K that has an adequate diagram D then $\mathrm{wr}(D) = \mathrm{wr}(K)$. Hence we have $c_+(D) = c_-(D) + \mathrm{wr}(K)$. Since D is B-adequate and $t < 0$, the cable $D_{p,q}$ is a B-adequate diagram of $K_{p,q}$ with $v_B(D_{p,q}) = q v_B(D)$ and $c_+(D_{p,q}) = q^2 c_+(D)$. See Figure 4.3. Furthermore, since as said above these quantities are invariants of $K_{p,q}$, they remain the same for all B-adequate diagrams of $K_{p,q}$.

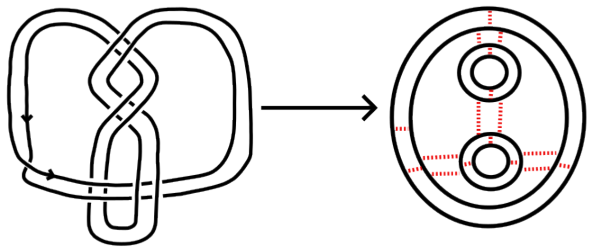

Figure 4.3 Left: the -1,2 cabling of the figure eight knot. Right: the B-state graph showing that the cabling is B-adequate.

Now assume, for a contradiction, that $K_{p,q}$ is adequate: Then, it has a diagram $\bar{D}$ that is both A and B-adequate. By above observation we must have $v_B(\bar{D}) = v_B(D_{p,q}) = qv_B(D)$ and $c_+(\bar{D}) = c_+(D_{p,q}) = q^2c_+(D)$.

By Lemma 4.3.2, $c(\bar{D}) = c(K_{p,q}) = q^2c(K)$.

Write

$$4\, d_+[J_{K_{p,q}}(n)] = xn^2 + yn + z,$$

for some $x, y, z \in \mathbb{Q}$.

For sufficiently large n we have two different expressions for x, y, z. On one hand, because $\bar{D}$ is adequate, we can use Equation (4.4) to determine x, y, z.

On the other hand, using $4\, d_+[J_{K^*_{-p,q}}(n)]$, x, y, z can be determined using Proposition 4.1.3 with a_2 and a_1 coming from Equation (4.4) applied to D.

We will use these two ways to find the quantity y. Applying Equation (4.4) to $\bar{D}$ we obtain

$$y = 2(v_B(\bar{D} - c(\bar{D}))) = 2qv_B(D) - 2q^2c(D) \tag{4.5}$$

On the other hand, using Proposition 4.1.3 with a_2 and a_1 coming from Equation (4.4) we have: $4a_2 = 2c_+(D) = c(D) + \mathrm{wr}(K)$. Also, we have $4a_1 = 2v_B(D) - 2c(D)$.

We obtain

$$y = q(4a_1) - 2q(q-1)(4a_2) + 2(q-1)p = 2qv_B(D) - 2q^2c(D) + 2(q-1)p - 2q(q-1)\mathrm{wr}(K). \tag{4.6}$$

It follows for the two expressions derived for y from Equations (4.5) and (4.6) to agree we must have

$$2q((q-1)2\mathrm{wr}(K)+p)-2p=0.$$

However this is impossible since $q>1$ and p,q are coprime. This contradiction shows that $K_{p,q}$ is non-adequate.

To deduce the result for $K_{p,q}$, with $t(K,p,q):=p-q\,w(K)>0$, let K^* denote the mirror image of K. Note that f $(K_{p,q})^*=K^*_{-p,q}$ and since being adequate is a property that is preserved under taking mirror images, it is enough to show that $K^*_{-p,q}$ is non-adequate. Since $t(K^*,-p,q):=-p-q\,w(K^*)=-t(K,p,q)<0$, the later result follows from the argument above. □

4.3.1 Proof of Theorem 1.1.7 and Corollary 1.1.8

By Theorem 4.2.1, we have

$$c(K_{p,q})\geq q^2c(K).$$

We need to show that this inequality is actually strict. Recall that by the proof of Theorem 4.2.1, if $\frac{p}{q}>2c_+(K)$ or $\frac{-p}{q}>2c_-(K)$, then the above inequality is strict so we need to only consider when $\frac{p}{q}<2c_+(K)$ and $\frac{-p}{q}<2c_-(K)$. By Theorem 4.3.1, $K_{p,q}$ is non-adequate. Hence by Theorem 4.1.2 again we have $2c(K_{p,q})\neq dj_K$ and the strict inequality follows. □

Next we discuss how to deduce Corollary 1.1.8:

Proof. If $q=2$ and $p=q\,w(K)\pm 1$, then by Theorem 4.2.1 $K_{p,q}$ is admissible. Thus by Theorem 4.1.2, the diagram $D_{p,2}$ constructed in the proof of Theorem 4.2.1 is minimal. That is $c(K_{p,2})=c(D_{p,2})=4\,c(K)+1$. □

4.4 Composite non-adequate knots

Here we give an application of Theorem 1.1.7 to the question on additivity of crossing numbers under the connected sum of knots. [29, Problems 1.67]. As already mentioned, for adequate knots the crossing number is additive under connected sum. The next result proves additivity for families of knots where one summand is adequate while the other is not.

Theorem 1.1.10. *Suppose that K is an adequate knot and let $K_1 := K_{p,2}$, where $p = 2\,w(K) \pm 1$. Then for any adequate knot K_2, the connected sum $K_1\#K_2$ is non-adequate and we have*

$$c(K_1\#K_2) = c(K_1) + c(K_2).$$

Before we proceed with the proof of the theorem we need some preparation. Given a knot K, such that for n large enough the degrees of the colored Jones polynomials of K are quadratic polynomials with rational coefficients, we will write

$$4\,d_+[J_K(n)] = x(K)n^2 + y(K)n + z(K) \text{ and } -4\,d_-[J_K(n)] = x^*(K)n^2 + y^*(K)n + z^*(K).$$

We also write

$$4d_+[J_K(n)] - 4d_+[J_K(n)] = d_2(K)n^2 + d_1(K)n + d_0(K).$$

Now let K_1, K_2 be as in the statement of Theorem 1.1.10. By assumption and Proposition 4.1.3, for n large enough the degrees of the colored Jones polynomials of both K_1 and K_2 are quadratic polynomials. For the proof we need the following well known lemma:

Lemma 4.4.1. *[32] For large enough n, the degrees $d_\pm[J_{K_1\#K_2}(n)]$ are polynomials, and we have the following.*

(a) $x(K_1\#K_2) = x(K_1) + x(K_2)$ *and* $x^*(K_1\#K_2) = x^*(K_1) + x^*(K_2)$.

(b) $y(K_1\#K_2) = y(K_1) + y(K_2) - 2$ *and* $y^*(K_1\#K_2) = y^*(K_1) + y^*(K_2) - 2$.

(c) $d_2(K_1\#K_2) = d_2(K_1) + d_2(K_2)$.

The second ingredient we need for the proof of Theorem 1.1.10 is the following lemma.

Lemma 4.4.2. *Let K be a non-trivial adequate knot, $p = 2\,w(K) \pm 1$ and let $K_1 := K_{p,2}$. Then for any adequate knot K_2, the connected sum $K_1\#K_2$ is non-adequate.*

Proof. The claim is proven by applying the arguments applied to $K_1 = K_{p,2}$ in the proofs of Lemma 4.3.2 and Theorem 4.3.1 to the knot $K_1\#K_2$ and using the fact that the degrees of the colored Jones polynomial are additive under connected sum.

First we claim that if $K_1 \# K_2$ were adequate then we would have

$$c(K_1 \# K_2) = 4c(K) + c(K_2) \tag{4.7}$$

Note that as $p = 2\,w(K) \pm 1$, we have $\frac{p}{2} < 2c_+(K)$ and $\frac{-p}{2} < 2c_-(K)$. Hence Proposition 4.1.3 applies to K_1. Now write

$$4d_+[J_{K_1\#K_2}(n)] - 4d_-[J_{K_1\#K_2}(n)] = d_2(K_1\#K_2)n^2 + d_1(K_1\#K_2)n + d_0(K_1\#K_2).$$

Since we assumed that $K_1\#K_2$ is adequate, we have $d_2(K_1\#K_2) = 2c(K_1\#K_2)$. On the other hand by Lemma 4.4.1, $d_2(K_1\#K_2) = d_2(K_1) + d_2(K_2) = 2 \cdot 4c(K) + 2c(K_2)$ which leads to (4.7).

Case 1. Suppose that $p - 2\,w(K) = -1 < 0$.

Start with $D = D(K)$ an adequate diagram and let $D_1 := D_{p,2}$ be constructed as in the proof of Theorem 4.2.1. Also let D_2 be an adequate diagram of K_2. As in the proof of Theorem 4.3.1 conclude that $D_1\#D_2$ is a B-adequate diagram for $K_1\#K_2$ and that the quantities $v_B(D_1\#D_2) = 2v_B(D) + v_B(D_2) - 1$ and $c_+(D_1\#D_2) = 4c_+(D) + c_+(D_2)$ are invariants of $K_1\#K_2$.

Let $\bar{D}$ be an adequate diagram. Then

$$v_B(\bar{D}) = v_B(D_1\#D_2) = 2v_B(D) + v_B(D_2) - 1 \text{ and } c_+(\bar{D}) = 4c_+(D) + c_+(D_2).$$

Next we will calculate the quantity $y(K_1\#K_2)$ of Lemma 4.4.1 in two ways: Firstly, since we assumed that $\bar{D}$ is an adequate diagram for $K_1\#K_2$, applying Equation (4.4), we get

$$y(K_1\#K_2) = 2(v_B(\bar{D}) - c(\bar{D})) = 2(2v_B(D) + v_B(D_2) - 1 - 4c(D) - c(D_2)).$$

Secondly, using by Proposition 4.1.3 we get $y(K_1) = 2(2v_B(D) - 4c(D) + p + 2\,\mathrm{wr}(K))$.

Then by Lemma 4.4.1,

$$y(K_1\#K_2) = y(K_1) + y(K_2) - 1 = 2(2v_B(D) - 4c(D) + p - 2\,\mathrm{wr}(K) + v_B(D_2) - c(D_2)) - 1.$$

Now note that in order for the two resulting expressions for $y(K_1\#K_2)$ to be equal we must have $2(p - 2\,\mathrm{wr}(K)) = 1$ which contradicts our assumption that $p - 2\,\mathrm{wr}(K) = -1$. We conclude that $K_1\#K_2$ is non-adequate.

Case 2. Assume now that $p - 2\,\mathrm{wr}(K) = 1$. Since $(K_{p,2})^* = K^*_{-p,2}$ and since being adequate is a property that is preserved under taking mirror images, it is enough to show that $K^*_{-p,2}\#K^*_2$ is non-adequate. Since $-p - 2\,\mathrm{wr}(K^*) = -(p - 2\mathrm{wr}(K)) = -1$, the later result follows from the argument above.

□

Now we give the proof of Theorem 1.1.10.

Proof. Note that if K is the unknot then so is $K_{p,2}$ and the result follows trivially. Suppose that K is a non-trivial knot. Then by Lemma 4.4.2 we obtain that $K_1\#K_2$ is non-adequate.

As discussed above $dj_{K_1} = 2(4c(K)) = 2(c(D_{\pm 1,2}) - 1)$. On the other hand, $dj_{K_2} = 2c(D_2) = 2c(K)$ where D_2 is an adequate diagram for K_2. Hence, by Lemma 4.4.1, $dj_{K_1\#K_2} = 2(c(D_1\#D_2) - 1)$, where $D_1 = D_{\pm 1,2}$. By Theorem 4.1.2,

$$c(K_1\#K_2) = c(D_1\#D_2) = c(D_1) + c(D_2) = c(K_1) + c(K_2),$$

where the last equality follows since, by Theorem 1.1.7, we have $c(K_1) = c(D_1) = c(D_{p,2})$. □

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