SOME PROBLEMS ON MANIFOLDS WITH LOWER BOUNDS ON RICCI CURVATURE

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A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

Mathematics-Doctor of Philosophy

2024

ABSTRACT

In this work, we delve into geometric analysis, particularly examining the interplay between lower bounds on Ricci curvature and specific functionals. Our exploration begins with an investigation into the implications of Yamabe invariants for asymptotically Poincaré-Einstein manifolds and their conformal boundaries under conditions of $Ric \geq -(n-1)g$. We establish a relationship wherein the type II Yamabe invariant of the conformal compactification of the manifold is bounded below by the Yamabe invariant of its conformal boundary. Additionally, we focus on compact manifolds with boundary where $Ric \geq 0$ and $II \geq 1$, obtaining partial results concerning Wang's conjecture.

ACKNOWLEDGEMENTS

This thesis represents not only my work at the keyboard but also a milestone in my life, made possible with the support of several individuals.

First and foremost, I would like to express my deepest gratitude to my advisor, Xiaodong Wang, whose expertise, understanding, and patience, added considerably to my graduate experience. I appreciate his vast knowledge and skill in differential geometry. Without his guidance, this thesis would not have been possible.

I would also like to thank my committee members, Kitagawa Jun, Thomas Parker, and Willie Wong, for their insightful comments and suggestions. Their feedback significantly improved the quality of this work.

I am deeply grateful to my family for their love, patience, and unwavering support.

I would also like to extend my heartfelt thanks to roommate Yuhan Jiang and friends from the group SMDJ8, whose camaraderie and support have been instrumental throughout this journey. Their understanding, laughter, and encouragement kept me grounded and motivated during the most challenging times. Your friendship means the world to me, and I am truly grateful to have each of you in my life.

This journey would not have been possible without all of you. Thank you.

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CHAPTER 1

YAMABE INVARIANTS FOR ASMPTOTICALLY POINCARÉ-EINSTEIN MANIFOLDS

Roughly speaking, a Poincaré-Einstein manifold is a non-compact manifold characterized by negative constant Ricci curvature and the admission of a conformal compactification. The investigation of Poincaré-Einstein manifolds is underpinned by a fundamental principle: the intricate interplay between the manifold's boundary and its interior. Given that we employ conformal transformations in defining Poincaré-Einstein manifolds, a natural inquiry arises concerning the existence of conformal invariants that exemplify this principle. Such inequality was introduced in [CLW17] with certain restrictions. Through collaborative efforts with X. Wang, we successfully eliminated these constraints, resulting in a comprehensive and unrestricted conclusion [WW21], [WW22]. This chapter will delve into the examination of these inequalities.

1.1 Asymptotically Poincaré-Einstein manifold

Poincaré-Einstein manifolds, which serve as the foundation for the AdS/CFT correspondence framework , have been the subject of extensive research over the past three decades, yielding significant advances in both mathematics and physics (see [Biq05], for instance).

The concept of the Poincaré-Einstein manifold emerges from an observation rooted in hyperbolic space $(\mathbb{H}^n, g_H)$. Utilizing the conformal ball model, this space can be effectively represented as $(\mathbb{B}^n, \frac{4}{(1-|x|^2)^2}dx^2)$, wherein dx^2 denotes the Euclidean metric. Through the application of the conformal factor $\frac{(1-|x|^2)^2}{4}$, $(\mathbb{H}^n, g_H)$ can be conformally compactified to the unit disk within Euclidean space. The boundary of this compactified space is commonly termed the "boundary at infinity" or the "conformal boundary." By summarizing this distinctive property in conjunction with the Ricci curvature equation $Ric_{g_H} = -(n-1)g_H$, we arrive at the comprehensive definition of Poincaré-Einstein manifolds:

Definition 1.1.1. *X is the interior of a compact manifold $\bar{X}$ with boundary M. (X, g_+) is called a $C^{3,\alpha}$ Poincaré-Einstein manifold if g_+ is a noncompact complete metric,*

$$Ric_{g_+} = -(n-1)g_+ \tag{1.1.1}$$

and $g = \rho^2 g_+$ *can be* $C^{3,\alpha}$ *extended to* $\bar{X}$ *by a boundary defining function* ρ*, i.e.*

$$\rho \in C^\infty(\bar{X}), \ \ \rho > 0 \ \ in \ \ X, \ \ \rho = 0 \ \ and \ \ d\rho \neq 0 \ \ on \ \ \partial X.$$

∂X*, together with the conformal class* $[\rho^2 g\big|_{\partial X}]$*, is called conformal infinity.*
If the Ricci curvature equation 1.1.1 *is replaced by* $Ric_{g_+} = -(n-1)g_+ + o(\rho^2)$*, we arrive at the definition for asymptotically Poincaré-Einstein manifolds.*

Apart from hyperbolic space $(\mathbb{B}^n, g_{\mathbb{H}})$, which serves as the prototype, Poincaré-Einstein manifolds also come in different ways.

Example 1.1.1. *Perturbation from* $(\mathbb{B}^n, g_{\mathbb{H}})$ *Let* h *be the standard round metric on* $\mathbb{S}^{n-1}$*. The work by J.Lee and C.Graham showed that if we perturb the metric on* $\mathbb{S}^{n-1}$ *slightly to* h'*, then there exists a corresponding* g' *satisfying (1.1.1) and* $(\mathbb{S}^{n-1}, h')$ *as its conformal boundary. [GL91]*

Example 1.1.2. *Let* (N^{n-1}, g_N) *be a compact manifold without boundary, and* $Ric_N = -(n-2)g_N$*, then*

$$(\mathbb{R} \times N, dt^2 + \cosh^2(t) g_N)$$

is a Poincaré-Einstein manifold with compactification $[0, 1] \times N$*.*

Note that the conformal boundary is $N \times \{\pm 1\}$. It has negative scalar curvature and is not connected. We will revisit this example later, as it serves to illustrate how the conformal boundary significantly influences the geometry of the entire manifold.

Given an asymptotically Poincaré-Einstein manifold, we want to study its geometry near conformal boundary. We start with the following result in [Lee94].

Theorem 1.1. *Let* (X, g_+) *be asymptotically Poincaré-Einstein manifold with* (M, h) *as its conformal boundary. For any* $h' \in [h]$*, there exists a boundary defining function* ρ *so that near conformal boundary* g *takes the form*

$$g = \frac{1}{\rho^2}(d\rho^2 \oplus h_\rho) \tag{1.1.2}$$

where $h_0 = h'$*. In particular,* $|d\rho|_{\rho^2 g_+} = 1$*.*

This is called Graham-Lee normal form. ρ is a distance function for $\bar{g}$, and its curvature can be computed using Riccati equation, Gauss-Codazzi equation and Codazzi-Mainardi equations. Pick local coordinates $\{x^i\}$ for $\partial X = M$, and $\{x_0 = \rho, x^i\}$ form local coordinates for X near conformal boundary. Apply (1.2.4), the traceless-Ricci curvature $E = Ric_{g_+} - \frac{R_{g_+}}{n} g_+$ are given in [BMW13] as

$$\begin{aligned}
2\rho E_{ij} &= -\rho h''_{ij} + \rho h^{pq} h'_{ip} h'_{jq} - \frac{\rho}{2} h^{pq} h'_{pq} h'_{ij} + (n-2) h'_{ij} + h^{pq} h'_{pq} h_{ij} + 2\rho Ric(h_\rho)_{ij} \\
E_{i0} &= \frac{1}{2} h^{pq} (\nabla_q h'_{ip} - \nabla_i h'_{pq}) \\
E_{00} &= -\frac{1}{2} h^{pq} h''_{pq} + \frac{1}{4} h^{pq} h^{kl} h'_{pk} h'_{ql} + \frac{1}{2\rho} h^{pq} h'_{pq}
\end{aligned} \tag{1.1.3}$$

where $'$ denotes $\frac{\partial}{\partial \rho}$. Set $\rho = 0$ in the first equation, and we get

$$(n-2)h' + tr_h(h')h = 0$$

This implies $h' = 0$, and therefore M is totally geodesic in $(\bar{X}, \bar{g})$. In particular M is umbilical in $(\bar{X}, \tilde{g})$ for any conformal compactification $\tilde{g}$ since the property of umbilicus is invariant under conformal change. Take derivative $\frac{\partial}{\partial \rho}$ k times to the first equation of (1.1.3), and we get

$$(n-1-k)\partial_\rho^k h + tr_h(\partial_\rho^k h)h = \partial_\rho^{k-1}(2\rho E)_{\rho=0} + (\text{terms containning} \quad \partial_\rho^l \quad \text{with} \quad l < k)$$

Now suppose $E \equiv 0$, i.e. (M, g_+) is Poincaré-Einstein. For $k < n-1$, the coefficients for $\partial_\rho^k h$ is non-zero. By induction, we could solve for $\partial_\rho^k h$ and thus get expansion for $\bar{g}$ near conformal boundary up to order $n-2$ if n is even and $n-1$ if n is odd. For example: i)$\partial_\rho^k h = 0$ for k odd and $k < n-1$; ii) if n is even, then $tr_h \partial_\rho^{n-1} h = 0$ and $\partial_\rho^{n-1} h$ is not determined. (See Proposition 2.7 in [Woo16], for example. The statement there is only for n odd, but the argument works also for even n's for orders below $n-1$). In particular, we can find the second order term

$$h'' = \begin{cases} -\frac{2}{n-3}\left(Ric\,(h) - \frac{R_h}{2(n-2)} h\right), & \text{if } n \geq 4; \\ -\frac{1}{2} h, & \text{if } n = 3. \end{cases} \tag{1.1.4}$$

1.2 Conformal Invariants

In this section I will introduce basic formulas under conformal change and then introduce Yamabe conformal invariants.

The Yamabe problem can be thought of as a continuation of uniformization theorem. For 2-dimensional spaces, all Riemannian surfaces are locally confomally Euclidean, and we have the uniformization theorem

Theorem 1.2. *Simply connected Riemann surface is biholomorphic to one of the following:*

- $\bar{\mathbb{C}} = \mathbb{S}^2$
- $\mathbb{C}$
- $\{z \in \mathbb{C} : |z| < 1\}$

As a result, all compact Riemannian surfaces admit a conformal metric of constant Gaussian curvature.

In dim>3, Weyl tensor is conformal invariant, thus obstruction for being locally conformally flat. But we could still ask the following: can we find a metric of constant scalar curvature within each conformal metric class. This is what Yamabe problem is about.

Given a Riemannian manifold (M^n, g) with $n \geq 3$ and local coordinates $\{x^i\}$. Under conformal change $\bar{g} = u^2 g$, the new Levi-Civita connection can be calculated by

$$\bar{\nabla}_X Y = \nabla_X Y + \frac{Xu}{u} Y + \frac{Yu}{u} X - \frac{g(X,Y)}{u} \nabla u \tag{1.2.1}$$

The conformal change of Hession can be computed as

$$\begin{aligned}
\bar{\nabla}^2 f(X,Y) &= \bar{g}\big(\bar{\nabla}_X \bar{\nabla} f, Y\big) \\
&\bar{g}\big(\bar{\nabla}_X \frac{1}{u^2} \nabla f, Y\big) \\
&= \bar{g}\big(-\frac{2Xu}{u^3} \nabla f + \frac{1}{u^2} \bar{\nabla}_X \nabla f, Y\big) \\
&= \bar{g}\Big(-\frac{2Xu}{u^3} \nabla f + \frac{1}{u^2} (\nabla_X \nabla f + \frac{Xu}{u} \nabla f + \frac{(\nabla f)u}{u} X - \frac{Xf}{u} \nabla u), Y\Big) \\
&= \nabla^2 f(X,Y) - \frac{1}{u}(Xu \cdot Yf + Xf \cdot Yu) + \frac{g(\nabla u, \nabla f)}{u} g(X,Y)
\end{aligned}$$

which is

$$\bar{\nabla}^2 f(X,Y) = \nabla^2 f - \frac{1}{u}(du \otimes df + df \otimes du) + \frac{g(\nabla u, \nabla f)}{u} g \tag{1.2.2}$$

Using the formula above, the Riemannian curvature can be computed as

$$\bar{R}_{ijkl} = u^2(R_{ijkl} - g_{ik}T_{jl} + g_{jl}T_{ik} - g_{il}T_{jk} - g_{jk}T_{il})$$

where

$$T_{ij} = \frac{\nabla_i \nabla_j u}{u} - 2\frac{\nabla_i u \nabla_j u}{u^2} + \frac{|du|^2}{2u^2} g_{ij}$$

Taking trace yields the formula for Ricci curvature and scalar curvature

$$\begin{aligned} \bar{R}_{ij} &= R_{ij} - (n-2)(\frac{\nabla_i \nabla_j u}{u} - 2\frac{\nabla_i u \nabla_j u}{u^2}) - (\frac{\Delta u}{u} + (n-3)\frac{|du|^2}{u^2})g_{ij} \\ \bar{R} &= \frac{1}{u^2}\Big(R - 2(n-1)\frac{\Delta u}{u} - (n-4)(n-1)\frac{|du|^2}{u^2}\Big) \end{aligned} \tag{1.2.3}$$

For scalar curvature we usually take the form $\bar{g} = u^{\frac{4}{n-2}} g$, and it takes the form

$$\bar{R} = u^{-\frac{n+2}{n-2}}(-\frac{4(n-1)}{n-2}\Delta u + Ru) \tag{1.2.4}$$

Remark 1.2.1. *For $n = 2$, we use the conformal change $\bar{g} = e^{2\phi} g$. The scalar curvature transforms by*

$$\bar{R} = e^{-2\phi}(-2\Delta u + R) \tag{1.2.5}$$

The operator $L_g(u) := -\frac{4(n-1)}{n-2}\Delta u + Ru$ is called conformal Laplacian. It has the following conformal invariance. Let $\bar{g} = u^{\frac{4}{n-2}} g$. Suppose there is a third conformal metric $g' = v^{\frac{4}{n-2}} g = (\frac{v}{u})^{\frac{4}{n-2}} \bar{g}$. Then by (1.2.4)

$$\begin{aligned} R' &= v^{-\frac{n+2}{n-2}} L_g(v) = (\frac{v}{u})^{-\frac{n+2}{n-2}} L_{\bar{g}}(\frac{v}{u}) \\ &\Rightarrow L_g(v) = u^{\frac{n+2}{n-2}} L_{\bar{g}}(\frac{v}{u}) \end{aligned} \tag{1.2.6}$$

Suppose X^n has a boundary Σ^{n-1} and let ν be the outer normal vector. Under conformal change $\bar{g} = u^{\frac{4}{n-2}} g$, the new normal vector becomes $\bar{\nu} = u^{-\frac{2}{n-2}} \nu$. Using (1.2.1), the second fundamental form II and mean curvature changes by

$$\begin{aligned}\bar{I}I(X,Y) &= u^{\frac{2}{n-2}}\left[II(X,Y)) + \frac{2}{(n-2)u}\frac{\partial u}{\partial \nu}g(X,Y)\right] \\ \bar{H} &= u^{-\frac{2}{n-2}}(H + \frac{2(n-1)}{(n-2)u}\frac{\partial u}{\partial \nu})\end{aligned} \tag{1.2.7}$$

Now we can define Yamabe invariants.

Definition 1.2.1. *Suppose (M^n, g) is a Riemannian manifold without boundary, the Yamabe invariant is defined to be*

$$Y(M,[g]) = \inf_{u\in H^1(M), u\neq 0} \frac{E_g(u)}{(\int_M u^{\frac{2n}{n-2}}dV)^{\frac{n-2}{n}}} \tag{1.2.8}$$

where

$$E_g(u) = \int_M \frac{4(n-1)}{n-2}|\nabla_M u|^2 + R_M u^2 dV$$

Remark 1.2.2. *Pick a sequence of functions which blows up locally and it can be shown that*

$$Y(M,[g]) \leq Y(\mathbb{S}^n, d\theta^2) \tag{1.2.9}$$

where $d\theta^2$ is the round metric in $\mathbb{S}^n$. See [Aub76].

If we write $g' = u^{\frac{4}{n-2}}g$ for $u > 0$, the integral in (1.2.8) can be rewritten as

$$\begin{aligned}\frac{\int_M \frac{4(n-1)}{n-2}|\nabla_g u|^2 + R_g u^2 \mathrm{dV}}{(\int_M u^{\frac{2n}{n-2}}\mathrm{dV})^{\frac{n-2}{n}}} &= \frac{\int_M u(-\frac{4(n-1)}{n-2}\Delta_g u + R_g u)\mathrm{dV}}{(\int_M u^{\frac{2n}{n-2}}\mathrm{dV})^{\frac{n-2}{n}}} \\ &= \frac{\int_M R_{g'}\mathrm{dV}_{g'}}{(\mathrm{Vol}(M,g'))^{\frac{n-2}{n}}}\end{aligned} \tag{1.2.10}$$

For general $u \in H^1(M)$, we can take $|u|$ and approximate it with positive functions in H^1. An equivalent definition for Yamabe invariant is thus derived

$$Y(M,[g]) = \inf_{g'\in[g]} \frac{\int_M R_{g'}\mathrm{dV}_{g'}}{(\mathrm{Vol}(M,g'))^{\frac{n-2}{n}}}$$

where $[g]$ represents the conformal class of g. Derived from this definition, it becomes evident that these two quantities remain invariant under conformal transformations, underscoring their pivotal role in the realm of conformal geometry research.

The Euler-Lagrangian equation for (1.2.8) is given by

$$L_g(u) = \lambda u^{\frac{n+2}{n-2}} \tag{1.2.11}$$

In conjunction with (1.2.4), the minimizer obtained from this equation provides a metric with constant scalar curvature. Consequently, the existence of the minimizer resolves the problem introduced at the beginning of this section. However, it is worth noting that in (1.2.8), we employ the $L^{\frac{2n}{n-2}}(X)$ norm in the denominator, and $\frac{2n}{n-2}$ represents the critical power for Sobolev embedding. While boundedness is assured, compactness is not guaranteed. To address this challenge, we employ the “lowering index" technique.

We define a new functional as

$$Y_q(M,[g]) = \inf_{u\in H^1(M), u\neq 0} \frac{E_g(u)}{\left(\int_M u^p \mathrm{dV}\right)^{\frac{2}{p}}} \tag{1.2.12}$$

where $p < \frac{2n}{n-2}$. These values of p are strictly below the critical Sobolev conjugate. Using the standard argument, the existence of minimizers u_p follows from the compactness of the inclusion $H^1(M) \subset L^p(M)$, and these u_p's satisfy the Euler-Lagrangian

$$L_g(u) = \lambda u^{p-1} \tag{1.2.13}$$

If we further impose the condition $|u|p = 1$, then $\lambda_p = Y_q(M,[g])$ in (1.2.12). Similar for (1.2.8). Trudinger[Tru68] and Aubin[Aub76] demonstrated that $\|u_p\|L^r$ is uniformly bounded for some $r > \frac{2n}{n-2}$ provided the inequality in (1.2.9) is strict. Consequently, u_p converges to a smooth solution u of (1.2.11), and u is a minimizer for (1.2.11). Thus, the primary challenge is reduced to establishing the strict inequality in (1.2.9), except for standard spheres. This problem was ultimately resolved by R. Schoen, who utilized the positive mass theorem to construct an appropriate test function. Combining all the elements above, we arrive at the following theorem:

Theorem 1.3. *Let (M,g) be a compact manifold without boundary. Then $Y(M,[g]) \leq Y(\mathbb{S}^n,[d\theta^2])$ with inequality iff round metric on $\mathbb{S}$. As a result, there exists a metric $g' \in [g]$ such that $R_{g'}$ is constant.*

For a comprehensive exploration of this problem, refer to [LP87] or Chapter 5 of [SY94].

For manifolds with boundary (M, Σ, g), we can ask the following two questions. Fix a conformal class $[g]$, can we find $g' \in [g]$ so that: I) $R_{g'} =$ constant, $H_{g'} = 0$; or II) $R_{g'} = 0$, $H_{g'} =$ constant. These two are called Type I and Type II Yamabe problem respectively. As in Yamabe problem, we can define the following two functional

Definition 1.2.2.

$$
\begin{aligned}
Y(X, M, [g]) &= \inf_{u \in H^1, u \neq 0} \frac{E(u)}{(\int_X u^{2n/(n-2)})^{(n-2)/n}} \qquad \textit{Type I} \\
Q(X, M, [g]) &= \inf_{u \in H^1, u \neq 0} \frac{E(u)}{(\int_M u^{2(n-1)/(n-2)})^{(n-2)/(n-1)}} \qquad \textit{Type II}
\end{aligned}
\tag{1.2.14}
$$

where

$$E(u) = \int_X \frac{4(n-1)}{n-2}|\nabla u|^2 + Ru^2 dV + 2\int_M Hu^2 dS$$

As before set $g' = u^{\frac{4}{n-2}} g$ for $u > 0$, then

$$
\begin{aligned}
&\int_X \frac{4(n-1)}{n-2}|\nabla u|^2 + Ru^2 \mathrm{dV} + 2\int_M Hu^2 \mathrm{dS} \\
=& \int_X u(-\frac{4(n-1)}{n-2}\Delta u + Ru)\mathrm{dV} + 2\int_M u^2 (H + \frac{2(n-1)}{u(n-2)}\frac{\partial u}{\partial \nu}\mathrm{dS} \\
=& \int_M R_{g'} \mathrm{dV}_{g'} + 2\int_\Sigma H_{g'} \mathrm{dS}_{g'}
\end{aligned}
$$

And (1.2.14) can be rewritten as

$$
\begin{aligned}
Y(M, \Sigma, [g]) &= \inf_{g' \in [g]} \frac{\int_M R_{g'} + 2\int_\Sigma H_{g'}}{\mathrm{Vol}(M, g')^{(n-2)/n}} \qquad \text{Type I} \\
Q(M, \Sigma, [g]) &= \inf_{g' \in [g]} \frac{\int_M R_{g'} + 2\int_\Sigma H_{g'}}{\mathrm{Area}(\Sigma, g'|_\Sigma)^{(n-2)/(n-1)}} \qquad \text{Type II}
\end{aligned}
$$

So these two minimum are conformal invariants. The corresponding Euler-Lagrangian equations are computed to be

$$
Type \quad I : \begin{cases} L_g(u) = \lambda u^{\frac{n+2}{n-2}} \\ \frac{\partial u}{\partial \nu} + \frac{n-2}{2(n-1)} Hu = 0 \end{cases}
\tag{1.2.15}
$$

$$
Type \quad II : \begin{cases} L_g(u) = 0 \\ \frac{\partial u}{\partial \nu} + \frac{n-2}{2(n-1)} Hu = \lambda u^{\frac{n}{n-2}} \end{cases}
\tag{1.2.16}
$$

So the minimizer of Type I and Type II Yamabe invariants solves the corresponding Yamabe problems respectively by (1.2.4) and (1.2.7). Again, by picking suitable test functions we have

$$\begin{aligned} Q(M,\Sigma,[g]) &\le Q(\mathbb{B}^n,\mathbb{S}^{n-1},[dx^2]) \\ Y(M,\Sigma,[g]) &\le \bar{Q}(\mathbb{S}^n_+,\mathbb{S}^{n-1},[ds^2]) \end{aligned} \tag{1.2.17}$$

And strict inequality implies the existence of minimizers by "lowering index" method. These problems are only partially solved. For Type I Yamabe problem, the strict inequality was verified in the following cases [Esc92b]

- $n = 3, 4, 5$;
- $n \ge 6$ and $\partial M = \Sigma$ is not umbilic.

For Type II, Escobar verified the following in [Esc92a]

- $n > 6$ and X has a nonumbilic boundary point;
- $n \ge 6$, with X locally flat and ∂X unbilic;
- $n = 4, 5$ and ∂X is umbilic;
- $n = 3$.

A substantial amount of work has been dedicated to addressing these two problems; nevertheless, some cases still remain open. See, for instance, [Alm12], [Che09], [BC09], [Mar05], [Mar07], and others. Recall that Poincaré-Einstein manifolds have umbilical boundaries. Apply these results and direct arguments give us (see [CLW17])

Theorem 1.4. *Let X^n, g_+ be $C^{3,\alpha}$ Poincaré-Einstein manifold satisfying one of the following*

- $3 \le n \le 5$
- $n \ge 6$ *and X is spin*

Then there exists a conformal compactification $\bar{g} = \rho^2 g_+$ which is a minimizer for $Y(\bar{X}, M, [\bar{g}])$. Furthermore, $\bar{g}$ has constant scalar curvature and totally geodesic curvature.

Theorem 1.5. *Let X^n, g_+ be $C^{3,\alpha}$ Poincaré-Einstein manifold satisfying one of the following*

- $3 \leq n \leq 7$
- $n \geq 8$ *ad X is spin*
- $n \geq 8$ *and X is locally conformally flat*

Then there exists a conformal compactification $\bar{g} = \rho^2 g_+$ which is a minimizer for $Q(\bar{X}, M, [\bar{g}])$. Furthermore, $\bar{g}$ has vanishing scalar curvature and constant mean curvature.

Remark 1.2.3. *Some other approaches has been used to construct solutions to (1.2.16). Thus solutions will provide us with metric of zero scalar curvature and constant mean curvature, but they are not necessarily minimizers of Type II Yamabe invariant. See [Xu23].*

1.3 A Sharp Inequality

Having established the Yamabe invariants in the previous section, we will now formulate inequalities that establish a connection between the geometry of the boundary and the interior.

The work is initialized in [GH17]

Theorem 1.6. *Let (X, g_+) be a Poincaré-Einstein manifold satisfying one of the following*

$$a) \quad 3 \leq n \leq 5, \quad or \quad b) \quad X \quad is \quad spin.$$

Let $(\bar{X}, M, \bar{g})$ be its compactification and $\hat{g} = \bar{g}\big|_M$. Then

$$\frac{n}{n-2} Y(M, [\hat{g}]) \leq Y(\bar{X}, M, [\bar{g}]) I(\bar{X}, X, \bar{g})^2, \quad if \quad n \geq 4$$

$$12\pi\chi(M) \leq Y(\bar{X}, M, [\bar{g}]) I(\bar{X}, X, \bar{g})^2, \quad if \quad n = 3$$

where $I(\bar{X}, X, \bar{g}) = Vol(M, \hat{g})^{1/(n-1)} / Vol(\bar{X}, \bar{g})^{1/n}$. Moreover, if the equality holds, then $\bar{g}$ is Einstein and $\hat{g}$ has constant scalar curvature.

This inequality tells in a certain sense that for Poincaré-Einstein manifolds, the conformal geometry of the whole manifold can be controlled by the geometry of the conformal boundary. While the inequality represents a significant breakthrough in Poincaré-Einstein manifold research, it has limitations, notably that $I(\bar{X}, M, g)$ isn't conformally invariant. In [CLW17], X. Chen, M. Lai, and F. Wang introduced a new inequality (1.3.1) using $Q(\bar{X}, M, g)$ instead of $Y(\bar{X}, M, g)$.

Theorem 1.7. *Let (X, g_+) be a Poincaré-Einstein manifold with compactification $(\bar{X}, M, [\bar{g}])$. Suppose (X, g_+) satisfies one of the conditions in **Thm1.5** then*

$$\begin{aligned} Q(\bar{X}, M, [\bar{g}]) &\geq 2\sqrt{\frac{(n-1)}{n-2}Y(M, g\big|_M)}\ if\ n \geq 4 \\ Q(\bar{X}, M, [\bar{g}]) &\geq 4\sqrt{2\pi\chi(M)}\ if\ n = 3 \end{aligned} \tag{1.3.1}$$

Moreover, the equality holds iff (X, g_+) is isometric to hyperbolic space $(\mathbb{H}^n, g_{\mathbb{H}})$.

Sketch of proof By **Thm1.5**, the minimizer for $Q(\bar{X}, M, [\bar{g}])$ can be achieved. Say $\bar{g} = \rho^2 g_+$, without loss of generality. Use $Ric_{g_+} = -(n-1)g_+$ and (1.2.4) and traceless Ricci of $\bar{g}$ is given by

$$\bar{E} = -(n-2)\rho^{-1}\Big[\bar{\nabla}^2\rho - \frac{1}{n}(\Delta_{\bar{g}}\rho)\bar{g}\Big]$$

Integrating $\rho|\bar{E}|_{\bar{g}}\mathrm{dV}_{\bar{g}}$ by parts yields

$$\int_{\bar{X}} \rho|\bar{E}|_{\bar{g}}\mathrm{dV}_{\bar{g}} = \int_M \frac{1}{\rho}\big[\partial_\nu|\bar{\nabla}\rho|^2_{\bar{g}} + \frac{1}{\rho}(1 - |\bar{\nabla}\rho|^2_{\bar{g}})\partial_\nu\rho\big]\mathrm{dS}_{\hat{g}} \tag{1.3.2}$$

where ν is outer normal and $\hat{g} = \bar{g}|_M$. Since $\bar{g}$ has zero scalar curvature, by (1.2.4)

$$2\rho\Delta_{\bar{g}}\rho = n(|\bar{\nabla}\rho|^2_{\bar{g}} - 1)$$

By calculation in [Gra16], the equation above, together with (1.1.4) will give us local expansion for ρ near conformal boundary:

$$\partial_\nu\rho = 1, \quad \partial^2_\nu\rho = -\frac{1}{n-1}\bar{H}, \quad \partial^3_\nu\rho = \frac{1}{n-2}\hat{R} - \frac{1}{n-1}\bar{H}^2$$

where $\hat{R}$ is the scalar curvature for $\hat{g}$ and $\bar{H}$ is the mean curvature. Plug this into the integration above, we get

$$\frac{2}{(n-2)^2}\int_X \rho|\bar{E}|^2_{\bar{g}}\mathrm{dV}_{\bar{g}} = \int_M \Big(\frac{1}{n-2}\bar{H}^2 - \frac{1}{n-2}\hat{R}\Big)\mathrm{dS}_{\hat{g}}$$

(1.3.1) follows by noting that $\bar{H}$ is constant since $\bar{g}$ minimizes Type II Yamabe invariant. $\square$

This inequality tells in a certain sense that for Poincaré-Einstein manifolds, the conformal geometry of the whole manifold can be controlled by the geometry of the conformal boundary. However, their findings were constrained by two primary limitations. Firstly, their work rested upon the assumption that the minimizer of the second type Yamabe invariant could be realized. Furthermore, their approach was confined to Poincaré-Einstein manifolds, i.e. $Ric_{g_+} = -(n-1)g_+$. It's important to note that many of the properties associated with Poincaré-Einstein manifolds extend to asymptotically Poincaré-Einstein manifolds with $Ric_{g_+} \geq -(n-1)g_+$. The proof in [CLW17] highly depends on the vanishing of traceless Ricci curvature, so their method fails in general setting. In collaboration with X. Wang, we overcame these limitations, yielding the following result [WW21][WW22]. This inequality highlights the intricate relationship between the manifold's boundary and its interior, aligning with our guiding principle.

Theorem 1.8. (X, g_+) *asymptotically Poincaré-Einstein manifold with compactification* $(\bar{X}, M, \bar{g})$. *Suppose* $Ric_{g_+} \geq -(n-1)g_+$ *and the conformal infinity has nonnegative Yamabe invariant, then*

$$\begin{aligned} Q(\bar{X}, M, [\bar{g}]) &\geq 2\sqrt{\frac{(n-1)}{n-2} Y(M, \bar{g}\big|_M)} \; if \; n \geq 4 \\ Q(\bar{X}, M, \bar{g}) &\geq 4\sqrt{2\pi\chi(M)} \; if \; n = 3 \end{aligned} \tag{1.3.3}$$

Moreover, the equality holds iff (X, g_+) *is isometric to hyperbolic space* $(\mathbb{H}^n, g_{\mathbb{H}})$.

Proof. The proof consists of three parts. First, we will define modified Yamabe quotients and subsequently a quantity derived from it, playing a role analogous to $\rho|E|^2$ in [CLW17]. Next, we will analyze the asymptotic behavior of the function introduced in the initial step. Finally, we will prove a sequence of inequalities for each modified Yamabe quotients, the limit of which will yield the desired inequality. Finally we are going to prove rigidity, which in essence comes from [CLW17].

Let $\hat{g} = \bar{g}|_M$. Throughout the proof, operators and tensors with a + are defined with respect to g_+, those with a bar are defined with respect to $\bar{g}$, and those with a hat are defined with respect to $\hat{g}$.

Step 1

From **Cor1.1** in the next section, M is connected. By **Thm1.3** we can pick a $h \in [\bar{g}|_M]$ so that R_h is constant. Take Graham-Lee normal form (1.1.2). Lee [Lee94] constructed a function a positive smooth function ϕ on X s.t. $\Delta_+\phi = n\phi$ and near $\partial\overline{X}$

$$\phi = \rho^{-1} + \frac{R_h}{4(n-1)(n-2)}\rho + o\left(\rho^2\right)$$

He further proved that $|d\phi|_+^2 - \phi^2 \leq 0$ in the following way. Since we assume $Y(M,[h]) \geq 0$ and R_h is constant, $R_h \geq 0$. By a direct calculation, $|d\phi|^2 - \phi^2$ has a continuation extension to M and $|d\phi|_+^2 - \phi^2 \leq 0$ on M. By Bochner formula we have

$$\begin{aligned}
\Delta_+(|d\phi|_+^2 - \phi^2) &= 2g_+(\nabla_+\Delta_+\phi, \nabla_+\phi) + 2|\nabla_+^2\phi|_+^2 + 2Ric_+(\nabla_+\phi, \nabla_+\phi) - 2(\phi\Delta_+\phi + |\nabla_+\phi|_+^2) \\
&= 2n|d\phi|_+^2 + 2|\nabla_+^2\phi|_+^2 + 2Ric_+(\nabla_+\phi, \nabla_+\phi) - \frac{2}{n}|\Delta_+\phi|_+^2 - 2|\nabla_+\phi|_+^2 \\
&= 2(Ric_+(\nabla_+\phi, \nabla_+\phi) + (n-1)|d\phi|_+^2) + 2(|\nabla_+^2\phi|_+^2 - \frac{1}{n}|\Delta_+\phi|^2) \\
&\geq 0
\end{aligned}$$

As a result $|d\phi|_+^2 - \phi^2 \leq 0$ on X. Consider the metric $\widetilde{g} \coloneqq \phi^{-2}g_+$ on $\bar{X}$. Its scalar curvature is given by

$$\begin{aligned}
\widetilde{R} &= \phi^2\left(R_+ + 2(n-1)\phi^{-1}\Delta_+\phi - n(n-1)\phi^{-2}|d\phi|_+^2\right) \\
&\geq \phi^2(R_+ + n(n-1)) \geq 0
\end{aligned}$$

Moreover, by a direct calculation the boundary is totally geodesic. We consider the following modified energy functional

$$\tilde{E}(f) = E_{\overline{g}}(f) - \int_X (R_+ + n(n-1))\phi^2 f^2 dv_{\overline{g}}. \tag{1.3.4}$$

Note that $(R + n(n-1))\phi^2 \in C^{m-3,\alpha}\left(\overline{X}\right)$ under our assumptions. More explicitly, by (1.3.4)

$$\tilde{E}(f) = \int_X \left[\frac{4(n-1)}{n-2}|df|_{\overline{g}}^2 + \left(\widetilde{R} - (R + n(n-1))\phi^2\right)f^2\right] dv_{\overline{g}} \geq 0. \tag{1.3.5}$$

Since $R_+ + n(n-1) \geq 0$, we have

$$E_{\overline{g}}(f) \geq \widetilde{E}(f). \tag{1.3.6}$$

For $1 < q \le n/(n-2)$, consider

$$\tilde{\lambda}_q := \inf \frac{\widetilde{E}(f)}{\left(\int_M |f|^{q+1}\, d\sigma_{\overline{g}}\right)^{2/(q+1)}}. \tag{1.3.7}$$

Lemma 1.3.1. *Since $\tilde{E}(f) \ge 0$, $\lim_{q\nearrow n/(n-2)} \widetilde{\lambda}_q = \widetilde{\lambda}_{n/(n-2)}$.*

Proof of lemma

Pick a minimizing sequence u_i for $Q(X, M, [\bar{g}])$. For each u_i,

$$\lim_{q\nearrow\frac{n}{n-2}} \frac{\tilde{E}(u_i)}{(\int_M u_i^{q+1} \mathrm{dS}_{\hat{g}})^{\frac{2}{q+1}}} = \frac{\tilde{E}(u_i)}{(\int_M u_i^{\frac{2(n-1)}{n-2}} \mathrm{dS}_{\hat{g}})^{\frac{n-2}{n-1}}}$$

As a result $\limsup_{q\nearrow n/(n-2)} \widetilde{\lambda}_q \le \tilde{\lambda}_{n/(n-2)}$. Since $\tilde{E}(u) \ge 0$, by Hölder inequality

$$\frac{\tilde{E}(u)}{(\int_M u^{q+1} \mathrm{dS}_{\hat{g}})^{\frac{2}{q+1}}} \ge \frac{\tilde{E}(u)}{(\int_M u^{\frac{2(n-1)}{n-2}} \mathrm{dS}_{\hat{g}})^{\frac{n-2}{n-1}}} \mathrm{Area}(M, \hat{g})^{\frac{n-2}{(n-1)} - \frac{2}{q+1}}$$

As a result $\tilde{\lambda}_q \ge \tilde{\lambda}_{\frac{n}{n-2}} \mathrm{Area}(M, \hat{g})^{\frac{n-2}{(n-1)} - \frac{2}{q+1}}$. Take a limit, and we have

$$\liminf_{q\nearrow n/(n-2)} \widetilde{\lambda}_q \ge \tilde{\lambda}_{n/(n-2)}$$

□

Since $\tilde{E}(f) \ge 0$, it is easy to see that $\lim_{q\nearrow n/(n-2)} \widetilde{\lambda}_q = \widetilde{\lambda}_{n/(n-2)}$. Therefore, it suffices to prove the above theorem for $q < n/(n-2)$.

Since the trace operator $H^1\left(\overline{X}\right) \to L^{q+1}(\Sigma)$ is compact for $q < n/(n-2)$, by standard elliptic theory, the above infimum λ_q is achieved by a smooth, positive function f s.t.

$$\int_\Sigma f^{q+1} d\overline{\sigma} = 1 \tag{1.3.8}$$

and

$$\begin{cases} -\frac{4(n-1)}{n-2}\overline{\Delta} f + \overline{R} f = (R + n(n-1))\,\phi^2 f & \text{on } \overline{X}, \\ \frac{4(n-1)}{n-2}\frac{\partial f}{\partial \overline{\nu}} = \lambda_q f^q & \text{on } M. \end{cases} \tag{1.3.9}$$

By the conformal invariance of the conformal Laplacian, we have

$$\begin{aligned} L_g\left(f\phi^{-(n-2)/2}\right) &= \phi^{-(n+2)/2} L_{\overline{g}}(f) \\ &= (R + n(n-1))\, f\phi^{-(n-2)/2}. \end{aligned}$$

In other words, $u := f\phi^{-(n-2)/2}$ satisfies the following equation

$$-\Delta_{g_+}u = \frac{n(n-2)}{4}u. \tag{1.3.10}$$

Write $u = v^{-(n-2)/2}$. Then

$$\Delta_{g_+}v = \frac{n}{2}v^{-1}\left(|dv|_{g_+}^2 + v^2\right).$$

Equivalently $\Delta_{g_+}v - nv = \frac{n}{2}\Phi$ with $\Phi = v^{-1}\left(|dv|_{g_+}^2 - v^2\right)$.

Lemma 1.3.2. *We have*

$$\mathrm{div}_+\left(v^{-(n-2)}\nabla_+\Phi\right) = 2v^{-(n-2)}Q, \tag{1.3.11}$$

where

$$Q = \left|\nabla_+^2 v - \frac{\Delta_+ v}{n}g_+\right|_+^2 + Ric_+(\nabla_+ v, \nabla_+ v) + (n-1)|\nabla_+ v|_+^2 \geq 0.$$

All the computation is done with respect to g_+, but we drop the subscript to simplify the presentation.

Proof. As $v\Phi = |\nabla_+ v|^2 - v^2$, we have, by using the Bochner formula

$$\begin{aligned}\frac{1}{2}(v\Delta_+\phi + 2\langle\nabla_+ v, \nabla\phi\rangle_+ + \phi\Delta_+ v) &= \left|\nabla_+^2 v\right|_+^2 + \langle\nabla_+ v, \nabla_+\Delta_+ v\rangle_+ + Ric_+(\nabla_+ v, \nabla_+ v) - v\Delta_+ v - |\nabla_+ v|_+^2 \\ &= \frac{(\Delta_+ v)^2}{n} + \langle\nabla_+ v, \nabla_+\Delta_+ v\rangle_+ + v\Delta_+ v - n|\nabla_+ v|_+^2 + Q \\ &= \frac{\Delta_+ v}{n}(\Delta_+ v - nv) + \langle\nabla_+ v, \nabla_+(\Delta_+ v - nv)\rangle_+ + Q \\ &= \frac{1}{2}\Phi\Delta_+ v + \frac{n}{2}\langle\nabla_+ v, \nabla_+\Phi\rangle_+ + Q\end{aligned}$$

Thus,

$$\Delta\Phi_+ = (n-2)v^{-1}\langle\nabla_+ v, \nabla_+\Phi\rangle_+ + 2Q$$

or

$$\mathrm{div}_+\left(v^{-(n-2)}\nabla_+\Phi\right) = 2v^{-(n-2)}Q \geq 0.$$

□

$\mathrm{div}\left(v^{-(n-2)}\nabla\Phi\right)$ plays the role of traceless Ricci E in [CLW17].

Step 2

In this part we are going to figure out the asymptotical expansion for terms in div $\left(v^{-(n-2)}\nabla\Phi\right)$. We now consider the metric $g = u^{4/(n-2)}g_+$. Since $u = f\phi^{-(n-2)/2}$, we also have

$$g = f^{4/(n-2)}\phi^{-2}g_+ = f^{4/(n-2)}\widetilde{g}.$$

As $\partial\overline{X}$ is totally geodesic w.r.t. $\widetilde{g}$ and g is conformal to $\widetilde{g}$, we know that $\partial\overline{X}$ is umbilic w.r.t. g and its mean curvature, in view of the boundary condition of (1.3.9), is given by

$$H = \frac{\lambda_q}{2} f^{q-\frac{n}{n-2}}. \tag{1.3.12}$$

Set $\rho = u^{2/(n-2)} = v^{-1}$. By a direct calculation, the equation (1.3.10) becomes, using g as the background metric

$$2\rho\Delta\rho = n\left(|\nabla\rho|^2 - 1\right). \tag{1.3.13}$$

Let t be the geodesic distance to Σ w.r.t. g. We need the following lemma which is essentially contained in [CLW17].

Lemma 1.3.3. *Near $\Sigma = \partial\overline{X}$, we can write*

$$g = dt^2 + g_{ij}\,(t,x)\,dx_i dx_j,$$

where $\{x_1, \cdots, x_{n-1}\}$ are local coordinates on Σ. Then

$$\rho = t - \frac{H}{2\,(n-1)}t^2 + \frac{1}{6}\left(\frac{R^\Sigma}{n-2} - \frac{H^2}{n-1}\right)t^3 + o\left(t^3\right).$$

In particular,

$$\frac{\partial}{\partial\nu}\left[\rho^{-1}\left(|\nabla\rho|^2 - 1\right)\right]|_\Sigma = \frac{R^\Sigma}{n-2} - \frac{H^2}{n-1}.$$

Proof. For completeness, we present the proof showing that the Einstein condition is not required. In local coordinates

$$|\nabla\rho|^2 = \left(\frac{\partial\rho}{\partial t}\right)^2 + g^{ij}\frac{\partial\rho}{\partial x_i}\frac{\partial\rho}{\partial x_j},$$

$$\Delta\rho = \frac{\partial^2\rho}{\partial t^2} + \frac{\partial\log\sqrt{G}}{\partial t}\frac{\partial\rho}{\partial t} + \frac{1}{\sqrt{G}}\frac{\partial}{\partial x_i}\left(g^{ij}\sqrt{G}\frac{\partial\rho}{\partial x_j}\right).$$

Restricting (1.3.13) on Σ on which both ρ and r vanish with order 1 yields $\frac{\partial \rho}{\partial t}|_\Sigma = 1$.

Differentiating (1.3.13) in t yields

$$\frac{2}{n}\left(\frac{\partial \rho}{\partial t}\Delta\rho + \rho\frac{\partial}{\partial t}\Delta\rho\right) = 2\frac{\partial \rho}{\partial t}\frac{\partial^2 \rho}{\partial t^2} + 2g^{ij}\frac{\partial^2 \rho}{\partial x_i \partial t}\frac{\partial \rho}{\partial x_j} - g^{ik}g^{jl}\frac{\partial g_{kl}}{\partial t}\frac{\partial \rho}{\partial x_i}\frac{\partial \rho}{\partial x_j}. \tag{1.3.14}$$

Evaluating both sides on Σ yields

$$\frac{2}{n}\left(\frac{\partial^2 \rho}{\partial t^2} + \frac{\partial \log \sqrt{G}}{\partial t}\right)|_\Sigma = 2\frac{\partial^2 \rho}{\partial t^2}|_\Sigma.$$

Thus

$$\frac{\partial^2 \rho}{\partial t^2}|_\Sigma = \frac{1}{n-1}\frac{\partial \log \sqrt{G}}{\partial t}|_\Sigma = -\frac{H}{n-1}.$$

Differentiating the formula for $\Delta\rho$ we get

$$\begin{aligned}\frac{\partial}{\partial t}\Delta\rho|_\Sigma &= \left(\frac{\partial^3 \rho}{\partial t^3} + \frac{\partial^2 \log \sqrt{G}}{\partial t^2} + \frac{\partial \log \sqrt{G}}{\partial t}\frac{\partial^2 \rho}{\partial t^2}\right)|_\Sigma \\ &= \left(\frac{\partial^3 \rho}{\partial t^3} + \frac{\partial^2 \log \sqrt{G}}{\partial t^2} + \frac{H^2}{n-1}\right)|_\Sigma\end{aligned}$$

Differentiating (1.3.14) in r and evaluating on Σ, we obtain

$$\frac{2}{n}\left(\frac{\partial^2 \rho}{\partial t^2}\Delta\rho + 2\frac{\partial}{\partial t}\Delta\rho\right)|_\Sigma = 2\left(\frac{\partial^2 \rho}{\partial t^2}\right)^2|_\Sigma + 2\frac{\partial^3 \rho}{\partial t^3}|_\Sigma = \frac{2H^2}{(n-1)^2} + 2\frac{\partial^3 \rho}{\partial t^3}|_\Sigma.$$

Using the previous formulas, we arrive at

$$\frac{\partial^3 \rho}{\partial t^3}|_\Sigma = \frac{2}{n-2}\left(\frac{H^2}{n-1} + \frac{\partial^2 \log \sqrt{G}}{\partial t^2}|_\Sigma\right).$$

By a direct calculation, we also have

$$\frac{\partial^2 \log \sqrt{G}}{\partial t^2}|_\Sigma = -Ric\,(\nu,\nu) - \frac{H^2}{n-1}.$$

Therefore

$$\begin{aligned}\frac{\partial^3 \rho}{\partial t^3}|_\Sigma &= -\frac{2}{n-2}Ric\,(\nu,\nu) \\ &= \frac{R^\Sigma}{n-2} - \frac{H^2}{n-1},\end{aligned}$$

where we used the Gauss equation in the last step.

The second identity follows from a direct calculation. □

Step 3

In this part we will use lemma1.3.3 in (1.3.11) to get the main result. Integrating the identity (1.3.11) on $X_\varepsilon = \{t \geq \varepsilon\}$ yields

$$2\int_{X_\varepsilon} v^{-(n-2)} Q dv_{g_+} = \int_{\partial X_\varepsilon} v^{-(n-2)} \frac{\partial \Phi}{\partial \nu} d\sigma_{g_+}.$$

Since $g_+ = \rho^{-2} g$, we obtain by a direct calculation

$$\int_{\partial X_\varepsilon} v^{-(n-2)} \frac{\partial \Phi}{\partial \nu^+} d\sigma_{g_+} = \int_{\partial X_\varepsilon} \frac{\partial}{\partial \nu} \left[\rho^{-1}\left(|\nabla \rho|^2 - 1\right)\right] d\sigma_g.$$

Therefore

$$2\int_{X_\varepsilon} v^{-(n-2)} Q dv_{g_+} = \int_{\partial X_\varepsilon} \frac{\partial}{\partial \nu} \left[\rho^{-1}\left(|\nabla \rho|^2 - 1\right)\right] d\sigma_g.$$

Letting $\varepsilon \longrightarrow 0$, we obtain, in view of Lemma 1.3.3

$$2\int_X v^{-(n-2)} Q dv_{g_+} = \int_\Sigma \left(\frac{R^\Sigma}{n-2} - \frac{H^2}{n-1}\right) d\sigma_g \tag{1.3.15}$$

The rest of the argument is the same as in [WW21]. We present it for completeness. By (1.3.12) and the Holder inequality again

$$\begin{aligned}
\int_\Sigma H^2 d\sigma &= \left(\frac{\lambda_q}{2}\right)^2 \int_\Sigma f^{2\left(q - \frac{n}{n-2}\right)} f^{2(n-1)/(n-2)} d\overline{\sigma} \\
&= \left(\frac{\lambda_q}{2}\right)^2 \int_\Sigma f^{2\left(q - \frac{1}{n-2}\right)} d\overline{\sigma} \\
&\leq \left(\frac{\lambda_q}{2}\right)^2 \left(\int_\Sigma f^{q+1} d\overline{\sigma}\right)^{2\left(q - \frac{1}{n-2}\right)/(q+1)} V\left(\Sigma, \overline{g}\right)^{\left(\frac{n}{n-2} - q\right)/(q+1)} \\
&= \left(\frac{\lambda_q}{2}\right)^2 V\left(\Sigma, \overline{g}\right)^{\left(\frac{n}{n-2} - q\right)/(q+1)}.
\end{aligned}$$

Plugging the above inequality into (1.3.15), we obtain

$$2\int_X v^{-(n-2)} Q dv_{g_+} \leq \frac{\lambda_q^2}{4(n-1)} V\left(\Sigma, \overline{g}\right)^{\left(\frac{n}{n-2} - q\right)/(q+1)} - \frac{1}{n-2}\int_\Sigma R^\Sigma d\sigma. \tag{1.3.16}$$

When $n = 3$, this implies

$$\lambda_q^2 V\left(\Sigma, \overline{g}\right)^{(3-q)/(q+1)} \geq 32\pi \chi\left(\Sigma\right).$$

In the following, we assume $n > 3$. By (1.3.8) and the Hölder inequality

$$\begin{aligned} 1 &= \int_\Sigma f^{q+1} d\overline{\sigma} \\ &\le \left(\int_\Sigma f^{2(n-1)/(n-2)} d\overline{\sigma} \right)^{\frac{(q+1)(n-2)}{2(n-1)}} V\left(\Sigma, \overline{g}\right)^{\frac{n-q(n-2)}{2(n-1)}} \\ &= V\left(\Sigma, g\right)^{\frac{(q+1)(n-2)}{2(n-1)}} V\left(\Sigma, \overline{g}\right)^{\frac{n-q(n-2)}{2(n-1)}} \end{aligned}$$

Thus

$$V\left(\Sigma, \overline{g}\right)^{-\frac{n-q(n-2)}{(n-2)(q+1)}} \le V\left(\Sigma, g\right).$$

Plugging this inequality into (1.3.16) yields

$$\begin{aligned} &2 \int_X v^{-(n-2)} Q dv_{g_+} \\ \le &\frac{V\left(\Sigma, g\right)^{\frac{n-1}{n-3}}}{4\left(n-1\right)} \left[\widetilde{\lambda}_q^2 V\left(\Sigma, \overline{g}\right)^{\frac{2(n-q(n-2))}{(n-3)(q+1)}} - \frac{4\left(n-1\right)}{\left(n-2\right) V\left(\Sigma, g\right)^{\frac{n-1}{n-3}}} \int_\Sigma R^\Sigma d\sigma \right] \\ \le &\frac{V\left(\Sigma, g\right)^{\frac{n-1}{n-3}}}{4\left(n-1\right)} \left[\widetilde{\lambda}_q^2 V\left(\Sigma, \overline{g}\right)^{\frac{2(n-q(n-2))}{(n-3)(q+1)}} - \frac{4\left(n-1\right)}{\left(n-2\right)} Y\left(\Sigma, [\gamma]\right) \right]. \end{aligned}$$

Therefore

$$\widetilde{\lambda}_q^2 \ge \frac{4\left(n-1\right)}{\left(n-2\right)} Y\left(\Sigma\right) V\left(\Sigma, \overline{g}\right)^{-\frac{2(n-q(n-2))}{(n-3)(q+1)}}.$$

Finally let $q \nearrow \frac{n}{n-2}$ and we arrive at the desired inequality in Theorem 1.8.

Step 4

Suppose the equality in (1.3.3) holds for (X, g_+) as in **Thm1.8**. Let $(X, M, \hat{g})$ be its conformal boundary and $\hat{g} = \bar{g}|_M$. If $Q(\bar{X}, M, [\bar{g}]) = Q(\mathbb{B}^n, \mathbb{S}^{n-1}, dx^2)$, then $Y(M, [\hat{g}]) = Y(\mathbb{S}^{n-1}, [d\theta^2])$ from the equality. By **Thm1.3**, $(M, [\hat{g}])$ is the round metric on $\mathbb{S}^{n-1}$. Then **Thm1.10** implies that (X, g_+) is the standard hyperbolic space.

Now we suppose $Q(\bar{X}, M, [\bar{g}]) < Q(\mathbb{B}^n, \mathbb{S}^{n-1}, dx^2)$. In this case the minimizer for Type II Yamabe invariant can be realized, say $\bar{g} = \rho^2 g_+$. Note that we defined $\tilde{\lambda}_q$ and proved inequality for each $q < \frac{n}{n-2}$ in step 3. This is because we are not sure whether minimizer for $Q(\bar{X}, M, [\bar{g}])$ exists. Now since we have got minimizer, we can run the previous method directly and get

$$2 \int_X v^{2-n} Q \mathrm{dV}_{g_+} \le \frac{V(M, \hat{g})^{\frac{n-1}{n-3}}}{4(n-1)} \left[Q(\bar{X}, M, [\bar{g}]) 2 - \frac{4(n-1)}{n-2} Y(M, [\hat{g}]) \right]$$

Given the assumption that equality holds in (1.3.3), it follows that equality also holds in (1.3.6). This implies $R_+ = -n(n-1)$. Combining this with $Ric_+ \geq -(n-1)g_+$, we deduce $Ric_+ = -(n-1)g_+$. We now find ourselves in a situation analogous to that in [CLW17], and their approach is applicable here as well. For the sake of completeness, we provide a detailed proof.

We also get $Q \equiv 0$ and thus $\nabla_+^2 v = \frac{\Delta_+ v}{n} g_+$. Recall $g = \rho^2 g_+ = \frac{1}{v^2} g_+$. Compute $\nabla_+^2 \rho$

$$\begin{aligned}
\nabla_+^2 v(X,Y) &= g_+\Big((\nabla_+)_X \nabla_+ \frac{1}{\rho}, Y\Big) \\
&= -g_+\Big((\nabla_+)_X (\frac{1}{\rho^2} \nabla_+ \rho), Y\Big) \\
&= -\frac{1}{\rho^2} g_+\Big((\nabla_+)_X \nabla_+ \rho, Y\Big) + 2\frac{(X\rho)(Y\rho)}{\rho^3}
\end{aligned}$$

Taking trace and we get

$$\Delta_+ v = -\frac{1}{\rho^2}\Delta_+ \rho + 2\frac{|\nabla_+ \rho|_+}{\rho^3}$$

Substitute the above two equations into $\nabla_+^2 v = \frac{\Delta_+ v}{n} g_+$ and we have

$$\nabla_+^2 \rho = \frac{1}{n}\big(\Delta_+ \rho - \frac{|\nabla_+ \rho|_+^2}{\rho}\big) g_+ + \frac{2}{\rho} d\rho \otimes d\rho$$

We now aim to express the preceding equation in terms of $g = \rho^2 g_+$. Use (1.2.2), the three equation above give us

$$\nabla^2 \rho = \frac{\Delta \rho}{n} g \tag{1.3.17}$$

We obtain

$$\nabla_i \nabla_i \nabla_j \rho = \frac{1}{n} \nabla_j (\Delta \rho) \tag{1.3.18}$$

Use (1.2.3) and we get

$$\begin{aligned}
Ric_+ &= Ric - (n-2)\big(\rho \nabla^2 (\frac{1}{\rho}) - 2\rho^2 (d\frac{1}{\rho}) \otimes (d\frac{1}{\rho})\big) - \big(\rho \Delta \frac{1}{\rho} + (n-3)\rho^2 |d(\frac{1}{\rho})|^2\big) g \\
&= Ric + (n-2)\frac{1}{\rho}\nabla^2 \rho + \big(\frac{\Delta \rho}{\rho} + (n-1)\frac{|d\rho|^2}{\rho^2}\big) g \\
&= Ric + (n-1)\big(\frac{2\Delta \rho}{n\rho} - \frac{|d\rho|^2}{\rho^2}\big) g
\end{aligned} \tag{1.3.19}$$

We use (1.3.17) in the last equality. Recall that the scalar curvature of g is 0, (1.2.3) implies

$$R_+ = -n(n-1) = -n(n-1)|\nabla \rho|^2 + 2(n-1)\rho \Delta \rho$$

So we have $Ric \equiv 0$. So

$$\nabla_i \nabla_i \nabla_j \rho = \nabla_j \nabla_i \nabla_i \rho + Ric_{ij} \nabla_i \rho = \nabla_j (\Delta \rho) \tag{1.3.20}$$

Compare (1.3.18) and (1.3.20), we have $\nabla(\Delta\rho) = 0$, and therefore $\Delta\rho$ is constant. Use **lemma 1.3.3** and (1.3.13), we get the constant $\Delta\rho = -\frac{n}{n-1}H$ where H is the mean curvature for g. Apparently $H \neq 0$. If not, $\Delta\rho = 0$ and $\rho = 0$ on ∂X, which implies $\rho \equiv 0$, which is impossible. Set $w = -\frac{n-1}{nH}\rho$, then w satisfies

$$\begin{cases} \Delta w & = 1 \text{ in } X \\ w & = 0 \text{ on } M \\ \frac{\partial w}{\partial \nu} & = \frac{n-1}{nH} \text{ on } M \end{cases} \tag{1.3.21}$$

Integrate $(\Delta w)^2$

$$\begin{aligned} \frac{n-1}{n}\mathrm{Vol}(\bar{X}, g) &= \frac{n-1}{n}\int_X (\Delta w)^2 \mathrm{dV}_g \\ &= \int_X \left[(\Delta w)^2 - |\nabla w|^2\right] \mathrm{dV}_g \\ &= \int_M H \frac{\partial w}{\partial \nu} \mathrm{dS}_{\hat{g}} \\ &= (\frac{n-1}{n})^2 \int_M \frac{1}{H} \mathrm{dS}_{\hat{g}} \end{aligned}$$

where we used Reilly's formula in the third line. Therefore we arrive at

$$\int_{\partial X} \frac{n-1}{H} \mathrm{dS}_{\hat{g}} = n\mathrm{Vol}(\bar{X}, g)$$

Recall that $Ric = 0$. We conclude that (X, M, g) is isometric to Euclidean ball by [Mul87]. □

1.4 Geometry on Conformal Boundary Affects Geometry of the Interior

As discussed in the first section, a fundamental principle guiding the research on (asymptotically) Poincaré-Einstein manifolds is to comprehend the intersection between the geometry of (X, g_+) and the geometry of its conformal boundary. In this section, I will introduce preliminary results utilized in the preceding section and demonstrate how our **Thm1.8** exemplifies this principle. We start with a toplogy result.

Theorem 1.9. *Let (X, g_+) asymptotically Poincaré-Einstein manifold and $Ric \geq -(n-1)g_+$. If one connected component of its boundary has non-negative Yamabe invariant, then*

$$H_{n-1}(X;\mathbb{Z}) = 0.$$

In [WY99], E. Witten and S. Yau established the above theorem under the assumption that one boundary component has a positive Yamabe invariant. They introduced the brane action defined by

$$L(\Sigma) = \text{Area}(\Sigma) - nV(\Omega) \tag{1.4.1}$$

where $\Sigma = \partial\Omega$, and Ω is a domain in X. Given the conditions outlined in the theorem, and assuming a strictly positive Yamabe invariant, they demonstrated the following: 1) $L(\Sigma)$ admits a minimum through local calculations; 2) there exists a minimum in each nontrivial homology class if the boundary has a component of positive scalar curvature. Therefore $H_{n-1}(X;\mathbb{Z}) = 0$.

Later M.Cai and G. Galloway proved the zero Yamabe invaraint case using Riccati equation and Busemann functions. Let ρ be a boundary defining function and $\Sigma_\epsilon = \{\rho(x) = \epsilon\}$. They consider a new Busemann function given by

$$\begin{aligned} \beta_\epsilon(x) &= d(\Sigma_\epsilon, o) - d(x, \Sigma_\epsilon) \\ \beta(x) &= \lim_{\epsilon\to 0} \beta_\epsilon(x) \end{aligned} \tag{1.4.2}$$

Using the Riccati equation, they successfully demonstrated $\Delta\beta \geq n-1$, provided the conformal boundary has a zero Yamabe invariant. If X has more than one end, a carefully chosen ray can be constructed. Let b be the usual Busemann function associated with this ray, resulting in $\beta + b \leq 0$ with equality at an interior point. Since $Ric \geq -(n-1)g_+$, we have $\Delta b \geq -(n-1)$. Now, $\beta + b$ is a subharmonic function with an interior maximum point, implying $\beta + b \equiv 0$. This equality leads to the splitting $(X = \mathbb{R}\times\Sigma, g_+ = e^{2r} + h)$. At $r = -\infty$, we encounter a cusp, which contradicts the asymptotic Poincaré-Einstein condition.

By standard topology argument, we have

Corollary 1.1. *Under the same assumption as stated in the preceding theorem, it follows that ∂X is connected.*

By this corollary, manifolds in **Thm1.8** will have connected boundary and brings us no trouble.

Apart from topology, the conformal geometry of the boundary also affects the metric inside. We have the following rigidity result

Theorem 1.10. *Let (X, g_+) be an asymptotically Poincaré-Einstein manifolds with $Ric \geq -(n-1)g_+$. Suppose (X, g_+) has round sphere as its conformal boundary, then (X, g_+) is isometric to the hyperbolic space $(\mathbb{H}^n, g_{\mathbb{H}})$.*

The theorem was initially established by Q. Jie in [Qin03] for $n \leq 7$. In the context of hyperbolic spaces, we can consider the upper plane model. Q. Jie observed that if (X, g_+) has a round sphere as its conformal boundary, we can construct coordinate functions and utilize them to apply conformal transformations, resulting in an uncompact manifold with $\mathbb{R}^{n-1}$ as its boundary, akin to the upper plane model. Moreover, the scalar curvature is non-negative for the new metric. Consequently, we can glue two such manifolds along $\mathbb{R}^{n-1}$ to obtain an asymptotically flat manifold $(\tilde{X}, \tilde{g})$ with non-negative scalar curvature. Notably, its Arnowitt-Deser-Misner (ADM) mass $m_{ADM} = 0$. The positive mass theorem [SY79a][SY79b] then implies that $(\tilde{X}, \tilde{g})$ is the Euclidean space.
The general case was solved by S.Dutta and M.Javaheri in [DJ10], where they used a totally different method.

Thm1.8 serves as a compelling illustration of this principle. In the context of our theorem, as the conformal boundary becomes rounder and rounder, i.e., $Y(M, [g]) \nearrow Y(\mathbb{S}^{n-1}, d\theta^2)$, the second inequalities in both (1.2.17) and (1.3.3) compellingly lead to $Q(\bar{X}, M, [g]) \nearrow Q(\mathbb{B}^n, \mathbb{S}^{n-1}, dx^2)$, representing the compactification of $(\mathbb{H}^n, g_H)$. Therefore, our result can be interpreted as follows: as the conformal boundary approaches the standard sphere, the interior becomes increasingly "close" to the standard hyperbolic space.
In the context of **Thm1.5**, where the second inequality in (1.2.17) is strictly satisfied except for the case of $(\mathbb{B}^n, dx^2)$, the rigidity theorem can be derived from **Thm1.8**. The challenge lies in the fact that we still lack a complete solution to the type II Yamabe problem.

I'd like to mention another result by G.Li, Q.Jie and Y.Shi [LQS14]

Theorem 1.11. *For any $\epsilon > 0, \quad n \geq 4$, there exists $\delta > 0$ so that for any Poincaré-Einstein manifold (X, g_+), one gets*

$$|K_{g_+} + 1| \leq \epsilon$$

for all sectional curvature K, provided

$$Y(M, [\hat{g}]) \geq (1 - \delta) Y(\mathbb{S}^{n-1}, [ds^2]).$$

This theorem and **Thm1.8** complement each other.

1.5 Some Discussions on Compact Manifolds with Boundary

It is a natural question if the inequality in Theorem 1.8 holds for a compact Riemannian manifold (M^n, g) with $Ric \geq -(n-1)$ and $\Pi \geq 1$. We are motivated by the observation that some results for conformally compact manifolds follow from results for compact Riemannian manifolds by a limiting process. As an illustration, consider the following theorem by Lee.

Theorem 1.12. *(Lee [Lee94]) Let (X^n, g_+) be a conformally compact manifold whose conformal infinity has nonnegative Yamabe invariant. If $Ric\,(g_+) \geq -(n-1)\, g_+$ and (X^n, g_+) is asymptotically Poincare-Einstein, then the bottom of spectrum $\lambda_0\,(X^n, g_+) = (n-1)^2/4$.*

When the Yambabe invariant of the conformal infinity is positive, Lee's theorem follows from the following result for compact Riemannian manifolds.

Theorem 1.13. *Let (M^n, g) be a compact Riemannian manifold with $Ric \geq -(n-1)$. If along the boundary $\Sigma := \partial M$ we have the mean curvature $H \geq n - 1$, then the first Dirichlet eigenvalue*

$$\lambda_0\,(M) \geq \frac{(n-1)^2}{4}.$$

This theorem has a simple proof. Let r be the distance function to Σ. By standard method in Riemannian geometry, we have

$$\Delta r \leq -(n-1)$$

in the support sense. A direct calculation yields

$$\Delta e^{(n-1)r/2} \leq -\frac{(n-1)^2}{4} e^{(n-1)r/2}.$$

This implies $\lambda_0(M) \geq \frac{(n-1)^2}{4}$ (for technical details see [Wan02]).

We can deduce Lee's theorem from Theorem 1.13 when the conformal infinity has positive Yamabe invariant in the following way. As explained in Section 2, we pick a metric h on the conformal infinity with positive scalar curvature and then we have a good defining function r s.t. near the conformal infinity g_+ has a nice expansion (1.1.2). Then a simple calculation shows that the mean curvature of the boundary of $X_\varepsilon := \{r \geq \varepsilon\}$ satisfies

$$H = n - 1 + \frac{R_h}{2(n-2)}\varepsilon^2 + o\left(\varepsilon^2\right).$$

As $R_h > 0$, we have $H > n - 1$ if ε is small enough. By Theorem, $\lambda_0(X_\varepsilon) \geq \frac{(n-1)^2}{4}$. It follows that $\lambda_0(X) \geq \frac{(n-1)^2}{4}$. As the opposite inequality was known by [Maz88], we have $\lambda_0(X) = \frac{(n-1)^2}{4}$. When the conformal infinity has zero Yamabe invariant, the situation is more subtle. But by an idea in Cai-Galloway[CG99], a similar argument still works (cf. [Wan02]).

We now come back to Theorem 1.8. By the asymptotic expansion (1.1.2) the second fundamental form of ∂X_ε satisfies

$$\Pi_+ = (1 + O(\varepsilon))\, g_+,$$

i.e. all the principal curvatures are close to 1. This leads us to consider a compact Riemannian manifold (M^n, g) with $Ric \geq -(n-1)$ and $\Pi \geq 1$ on its boundary Σ and ask the question whether the inequality

$$Q(M, \Sigma, g) \geq 2\sqrt{\frac{(n-1)}{(n-2)}Y(\Sigma)} \text{ if } n \geq 4; \tag{1.5.1}$$

$$Q(M, \Sigma, g) \geq 4\sqrt{2\pi\chi(\Sigma)} \text{ if } n = 3$$

holds. The answer turns out to be no in general. To construct a counter example, we consider the hyperbolic space using the ball model $\mathbb{B}^n$ with the metric $g_{\mathbb{H}} = \frac{4}{\left(1-|x|^2\right)^2}dx^2$. For $0 < R < 1$, the Euclidean ball

$$\left\{x \in \mathbb{B}^n : |x|^2 = \sum_{i=1}^{n} x_i^2 \leq R\right\}$$

is a geodesic ball in $(\mathbb{B}^n, g_{\mathbb{H}})$ and the boundary has 2nd fundamental form $\Pi = \frac{1+R^2}{2R} I$. We now consider

$$M = \left\{ x \in \mathbb{B}^n : |x|^2 = \sum_{i=1}^{n-1} x_i^2 + k x_n^2 \leq R \right\},$$

where $k > 0$ is close to 1. Then $(M, g_{\mathbb{H}})$ is a compact hyperbolic manifold with boundary and on its boundary we have $\Pi \geq 1$ if k is sufficiently close to 1 by continuity. Since Σ with the induced metric is rotationally symmetric, it is conformally equivalent to the standard sphere $\mathbb{S}^{n-1}$. Thus, $Y(\Sigma) = Y\left(\mathbb{S}^{n-1}\right)$. But when $k \neq 1$, the boundary is not umbilic with respect to the Euclidean metric and hence not with respect to $g_{\mathbb{H}}$ either. By [Esc92a] and [Mar07], $Q(M, \Sigma, g_{\mathbb{H}}) < Q\left(\overline{\mathbb{B}^n}, \mathbb{S}^{n-1}\right)$. It follows that the inequality (1.5.1) is false.

Therefore, for a compact Riemannian manifold (M^n, g) with $Ric \geq -(n-1)$ and $\Pi \geq 1$ on its boundary Σ, it is more subtle to estimate its type II Yamabe invariant in terms of the boundary geometry. It is an interesting question and we do not have an explicit conjecture. Let us mention that in a similar setting, namely for a compact (M^n, g) with $Ric \geq 0$ and $\Pi \geq 1$ on its boundary Σ, there is a well-formulated conjecture [Wan19] on the type II Yamabe invariant in terms of the boundary area.

CHAPTER 2

LIOUVILLE TYPE THEOREMS ON MANIFOLDS WITH LOWER CURVATURE BOUND

One problem that lies in the center of geometric analysis is to understand how geometric conditions, such as curvature and fundamental forms, exert influence over the solutions of partial differential equations. In [Wan19], X.Wang proposed a conjecture that for manifolds with boundary, if the Ricci curvature is nonnegative and second fundament form is positive, then a series of elliptic PDEs doesn't admit non-constant solutions. Throughout this chapter, we will always assume that $\partial M = \Sigma$ is connected.

2.1 Preparation

X. Wang has posed the following conjecture in [Wan19]:

Conjecture 2.1 (Wang's conjecture)**.** *Let $(M, \partial M = \Sigma, g)$ be a compact Riemannian manifold with boundary. Suppose Ric ≥ 0 on M, and II ≥ 1 on Σ where II is the second fundamental form, then the following PDE*

$$\begin{aligned} \Delta u &= 0 && on \quad M^n \\ \frac{\partial u}{\partial \nu} &= -\lambda u + u^q && on \quad \Sigma^{n-1} \end{aligned} \tag{2.1.1}$$

admits no non-constant positive solution provided $\lambda(q-1) \leq 1$ and $q \leq \frac{n}{n-2}$ unless (M, Σ, g) is isometric to $(\mathbb{B}^n, \mathbb{S}^{n-1}, dx^2)$, $q = \frac{n}{n-2}$ and u is given by

$$u_a = \Big[\frac{2}{n-2}\frac{1-|a|^2}{1+|a|^2|x|^2 - 2x\cdot a}\Big]^{\frac{n-2}{2}}$$

for some $a \in \mathbb{B}^n$.

The conjecture was proposed for the following reasons. Consider the following functional

$$\begin{aligned} E_{q,\lambda}(u) &= \frac{\int_M |\nabla u|^2 \mathrm{dV} + \lambda \int_\Sigma u^2 \mathrm{dS}}{(\int_\Sigma u^{q+1}\mathrm{dS})^{2/(q+1)}} \\ s(q,\lambda) &= \inf_{u\in H^1(M), u\neq 0} E(u) \end{aligned} \tag{2.1.2}$$

For convenience, we drop the index (q, λ) when it brings no confustion. By definition,

$$s(q, \lambda) \leq E(1) = \lambda|\Sigma|^{\frac{q-1}{1+q}} \tag{2.1.3}$$

Fix arbitrary u and take derivative in the direction of v, we have

$$\begin{aligned} \frac{\partial}{\partial t} E(u + tv)\Big|_{t=0} =& \frac{1}{(\int_\Sigma u^{q+1})^{\frac{4}{q+1}}} \Big[\Big(\int_M 2\langle \nabla u, \nabla v \rangle + 2\lambda \int_\Sigma uv \Big) \Big(\int_\Sigma u^{q+1} \Big)^{\frac{2}{q+1}} \\ & - 2\Big(\int_M (|\nabla u|^2 + \lambda u^2) \Big) \Big(\int_\Sigma u^{q+1} \Big)^{\frac{1-q}{q+1}} \int_M v u^q \Big] \\ =& \frac{2}{(\int_\Sigma u^{q+1})^{\frac{2}{q+1}}} \Big[\int_M v \Delta u + \int_\Sigma v \Big(\frac{\partial u}{\partial \nu} + u - E(u) \Big(\int_\Sigma u^{q+1} \Big)^{\frac{1-q}{1+q}} u^q \Big) \Big] \end{aligned} \tag{2.1.4}$$

Suppose u is a critical point of E, then $\frac{\partial}{\partial t} E(u + tv)\big|_{t=0} = 0$ for all smooth v, and therefore $\Delta u = 0$ in M and $\frac{\partial u}{\partial \nu} + u - E(u)(\int_\Sigma u^{q+1}) u^q = 0$. Since E is invariant under scaling, we could scale u to get rid of the coefficients before u^q, yielding (2.1.1). In summary, (2.1.1) arises as the Euler-Langrangian equation for (2.1.2).

If $q < \frac{n}{n-2}$, the trace embedding $H^1(M) \to L_{q+1}(\Sigma)$ is compact (see theorem 6.2 chapter 2 in [Nec11], for example), thereby enabling the attainment of the minimizer denoted as $u_{q,\lambda}$. Let us now consider a fixed value of q_0. As the parameter λ decreases, the weight of $\int_M |\nabla u|^2 \mathrm{dV}$ becomes increasingly prominent. To ensure that $u_{q_0,\lambda}$ attains minimizer, a concomitant decrease in $\int_M |u_{q_0,\lambda}|^2 \mathrm{dV}$ is expected. For example, we have the following lemma:

Lemma 2.1.1. *Let (M, Σ, g) be a compact Riemannian manifold with boundary. Suppose $s(q, \lambda)$ is achieved by constants for some $q \leq \frac{n}{n-2}$. Then for any $\mu < \lambda$, $s(q, \lambda)$ is only achieved by constants.*

Proof. For any fixed $u \in H^1(M)$ and $u \neq 0$, $E_{q,\mu}(u)$ is linear function in μ, and therefore concave. Since $s(q, \mu)$ is the infimum of concave functions, $s(q, \mu)$ is also a concave function in μ. Suppose $s(q, \lambda)$ is achieved by constant, we have $s(q, \lambda) = \lambda|\Sigma|^{\frac{q-1}{q+1}}$. We also have $s(q, 0) = 0$. By concavity,

$$s_{q,\mu} \geq \mu|\Sigma|^{\frac{q-1}{q+1}}$$

for $\mu < \lambda$. At the same time, we have $s_{q,\mu} \leq E_{q,\mu}(1) = \mu|\Sigma|^{\frac{1-q}{1+q}}$. So $s(q, \mu)$ is achieved by constants.

Suppose $s(q,\mu)$ is achieved by some non-constant u and $\mu < \lambda$, i.e. $E_{q,\mu}(u) = E_{q,\mu}(1) = s(q,\mu)$. Since u is non-constant, we must have $\int_M |\nabla u|^2 > 0$ and therefore $\frac{\int_\Sigma u^2}{(\int_\Sigma u^{(q+1)})^{2/(q+1)}} < |\Sigma|^{\frac{q-1}{q+1}}$. Use u as the test function for (q,λ), we obtain

$$\begin{aligned}E_{q,\lambda}(u) =& \frac{\int_M |\nabla u|^2 + \lambda \int_\Sigma u^2}{(\int_\Sigma u^{q+1})^{2/(q+1)}}\\ =& \frac{\int_M |\nabla u|^2 + \mu \int_\Sigma u^2}{(\int_\Sigma u^{q+1})^{2/(q+1)}} + (\lambda - \mu)\frac{\int_\Sigma u^2}{(\int_\Sigma u^{q+1})^{2/(q+1)}}\\ <& \mu|\Sigma|^{\frac{q-1}{q+1}} + (\lambda-\mu)|\Sigma|^{\frac{q-1}{q+1}} = s(q,\lambda)\end{aligned}$$

which is contradiction since we assumer $s(q,\lambda)$ is achieved by constant. □

Note that $u_{q,0}$ and $u_{1,\lambda}$ are both constants. Therefore, for values of $1 < q < \frac{n}{n-2}$, an intriguing possibility emerges: for each value of q, there might exist a threshold λ_q such that when $\lambda < \lambda_q$, the minimizer $u_{\lambda,q}$ will take constant values.

This phenomena was first found as Beckner inequality [Bec93]:

Theorem 2.1 (Beckner's inequality)**.** *For unit disk with Euclidean metric and $y \in H^1(\mathbb{B}^n)$, we have*

$$c_{n-1}^{\frac{q-1}{q+1}}\Big(\int_{\mathbb{S}^{n-1}} u^{q+1} dS\Big)^{\frac{2}{q+1}} \le \frac{1}{\lambda}\int_{\mathbb{B}^n} |\nabla u|^2 dV + \int_{\mathbb{S}^{n-1}} u^2 dS \tag{2.1.5}$$

provided that $1 \le q \le \frac{n}{n-2}$ and $\lambda(q-1) \le 1$, where $c_{n-1} = 2\pi^{(n-1)/2}/\Gamma\big((n-1)/2\big)$ is the volume of $n-1$ round sphere.

It follows from the inequality that the minimizers of $E_{\lambda,q}$ for unit ball are exclusively realized by constant functions. This intriguingly gives rise to the conjecture that (2.1.1) admits no non-constant solutions.

When $\lambda(q-1) > 1$, however, the $u \equiv 1$ is no longer minimizer for unit balls. The second variation of $E_{\lambda,q}$ at $u \equiv 1$ in the direction of v is

$$\begin{aligned}\frac{\partial^2}{\partial t^2} E_{\lambda,q}(1+tv)\Big|_{t=0} &= -\frac{2}{|\Sigma|^{2/(q+1)}}\Big[-\int_M v\Delta v + \int_\Sigma \big(\frac{\partial v}{\partial \nu} + \lambda v\big)v - \lambda q v^2\Big]\\ &= \frac{2}{|\Sigma|^{2/(q+1)}}\Big[\int_M |\nabla v|^2 - \lambda(q-1)\int_\Sigma v^2\Big]\end{aligned} \tag{2.1.6}$$

We can pick v to be the function associated to the first Steklov eigenvalue.

Definition 2.1.1. *The first Steklov eigenvalue is*

$$\lambda = \inf_{u \in H^1(M), u \neq 0} \frac{\int_M |\nabla u|^2}{\int_\Sigma u^2} \tag{2.1.7}$$

The corresponding Euler-Lagrangian equation, or the eigenfunction equation, is

$$\begin{aligned} \Delta u &= 0 \text{ in } M \\ \frac{\partial u}{\partial \nu} &= \lambda u \text{ on } \Sigma \end{aligned} \tag{2.1.8}$$

It's well known that the first Steklov eigenvalue for unit ball is 1 with n eigenfunctions given by coordinate functions. Pick v to be Steklov eigenfunction in (2.1.6), and we have

$$\frac{\partial^2}{\partial t^2} E_{\lambda,q}(1+tv)\Big|_{t=0} = \frac{2(1-\lambda(q-1))}{c_{n-1}^{2/(q+1)}} \int_{\mathbb{S}^{n-1}} v^2$$

As a consequence, the minimization of $E_{\lambda,q}$ by $u \equiv 1$ is unsuccessful if $\lambda(q-1) > 1$. However, the minimizer exist since the trace embedding is compact. This implies that the minimizer is a non-constant solution of (2.1.1). Therefore, the condition $\lambda(q-1) \leq 1$ is crucial and cannot be improved. These insights serve to clarify the conjecture for $\mathbb{B}^n$.

It's noteworthy to mention that the conjecture is fully resolved for unit balls as demonstrated in [GL23]. A natural progression from here is to delve into the intricate connection between geometric properties and the behavior of solutions of (2.1.1). This exploration is driven by the question of how geometric attributes influence the solutions of PDEs. A similar problem was resolved:

Theorem 2.2 (B.Véron and L.Véron [BV91])**.** *Let (M, Σ, g) be a compact Riemannian manifold with boundary.*

$$\begin{aligned} -\Delta u + \lambda u &= u^q \quad on \quad M^n \\ \frac{\partial u}{\partial \nu} &= 0 \quad on \quad \Sigma^{n-1} \end{aligned} \tag{2.1.9}$$

admits no non-constant solutions provided that $\lambda > 0$, $1 < q < \frac{n+2}{n-2}$ and $Ric \geq \frac{(n-1)(q-1)\lambda}{n} g$.

In the context of Wang's conjecture, there is famous Escobar conjecture.

Conjecture 2.2 (Escobar conjecture)**.** *Let* (M, Σ, g) *be a compact Riemannian manifold with boundary. Suppose* $Ric \geq 0$ *on* M*, and* $II \geq 1$ *on* Σ*. Then*

$$\int_M |\nabla u|^2 \geq \int_\Sigma u^2 \tag{2.1.10}$$

i.e. the first Steklov eigenvalue is no less than 1*.*

Under the condition of non-negative sectional curvature Escobar's conjecture was completely solved by C.Xia and C.Xiong in ([XX19]).
The insights gleaned from these findings, in conjunction with the outcomes concerning $\mathbb{B}^n$, culminate in Wang's conjecture.

If this conjecture holds true, it would lead to fascinating geometric implications. For instance, an intriguing consequence would be an upper bound on the area of Σ.

Conjecture 2.3. *Let* (M, Σ, g) *be as in conjecture 2.1. Then*

$$Area(\Sigma) \leq Area(\mathbb{S}^{n-1}) \tag{2.1.11}$$

Moreover, this inequality would only be realized by unit spheres as the boundary of unit disks.

. We can view this as an extension of the Bishop volume comparison theorem.
Consider the $E_{q,\frac{1}{q-1}}$. For $q < \frac{n}{n-2}$, $s(q, \frac{1}{q-1})$ can be achieved for some smooth function. If conjecture 2.1 holds true, then the only possible minimizer are constants, which implies

$$\frac{(q-1)\int_M |\nabla u|^2 + \int_\Sigma u^2}{(\int_\Sigma u^{(q+1)})^{2/(q+1)}} \geq |\Sigma|^{\frac{q-1}{q+1}}$$

for any $u \in H^1(M)$ and $u \neq 0$. Let $q \searrow$ in above inequality and we get

$$\frac{\frac{2}{n-2}\int_M |\nabla u|^2 + \int_\Sigma u^2}{(\int_\Sigma u^{2(n-1)/(n-2)})^{(n-2)/(n-1)}} \geq |\Sigma|^{\frac{1}{n-1}} \tag{2.1.12}$$

Recall the definition for type II Yamabe invariant in (1.2.14). Since $Ric \geq 0$ and $II \geq g|_\Sigma$, we get $Q(M, \Sigma, g) \geq \frac{4(n-1)}{n-2} s(\frac{n}{n-2}, \frac{n-2}{2})$. By the second inequality in (1.2.17), we finally arrive at (2.1.11).

The relationship between curvature and volume has a long and storied history in geometry. For manifolds without boundaries, the Bishop comparison theorem is a pivotal result. It asserts that a lower bound on the Ricci curvature results in an upper bound on the volume of geodesic balls. In the context of manifolds with boundaries, where we encounter second fundamental forms, a lower bound for this form could imply an upper bound for the distance from the boundary to the interior. When coupled with the Bishop comparison method, it naturally leads us to surmise such upper bounds on volume and area. However, tackling this problem is notably challenging. The conjecture we've presented offers a promising avenue to approach and potentially solve this intriguing problem.

A lot of work has been invested in exploring this conjecture; however, a significant portion of its components remain unsolved. As I see it, there are two primary challenges that contribute to the difficulty of addressing this conjecture.

Firstly, there exists a lack of comprehensive understanding regarding how Ricci curvature influences the solutions of (2.1.1). A notable advance in this direction was made in [GHW19], where it was demonstrated that by strengthening the Ricci curvature assumption to *sectional curvature* ≥ 0, a beneficial weight function emerges. This weight function proves advantageous during integration by parts, effectively nullifying bothersome boundary terms. This paves the way for the derivation of a weighted version of Reilly's formula, leading to partial results. Unfortunately, this method falters when the assumption is relaxed to non-negative Ricci curvature. In fact, no results exist in this setting.

I have obtained results in general Riemannian manifolds using different techniques. However, it's currently unclear how Ricci curvature, or even sectional curvature, influences the estimation in my approaches. I'm working to get over these difficulties to get a uniform estimate under curvature assumptions.

The second primary challenge emerges when $q \nearrow \frac{n}{n-2}$, particularly when the equality is reached. Notably, at $q = \frac{n}{n-2}$, the embedding $L_{\frac{2(n-1)}{n-2}}(\Sigma) \hookrightarrow H^1(M)$ becomes merely continuous without the compactness property, raising uncertainties about the existence of the minimizer. It's worth emphasizing that in this instance, (2.1.2) takes on a form reminiscent of the second-type Yamabe invariant defined in the preceding section. These intricacies contribute to the heightened complexity of addressing this conjecture.

Given this challenge, it might be worthwhile to concentrate on cases where q is close to 1. When we take the derivative with respect to p at $p = 1$, we obtain a log-Sobolev inequality. The verification of the log-Sobolev inequality would provide confidence in Wang's conjecture. I'm working in this direction and partial results are obtained. The results mentioned above will be presented in the following sections.

2.2 Pseudo Differential Operator

We start with an exploration of Dirichlet-to-Neumann operator. Using the property of Dirichlet-to-Neumann operator we can derive a non-existence theorem for (2.1.1) on general manifold with boundary without adding any restriction for curvature. In this section all the integration and Sobolev spaces will be with respect to Σ unless stated otherwise.

Definition 2.2.1.

$$\begin{gathered} DN : H^1(\Sigma) \longrightarrow L^2(\Sigma) \\ DN(f) = \frac{\partial u}{\partial \nu} \end{gathered} \tag{2.2.1}$$

where u is the harmonic extension for f, i.e.

$$\Delta u = 0 \quad \text{in } M$$

$$u|_\Sigma = f \quad \text{on } \Sigma$$

It's well known that DN is a first order elliptic pseudo differential operator. See [Tay96] chapter 1, for example. As a result

$$C_1 \|\nabla f\|_{L^2} \leq \|DN(f)\|_{L^2} \leq C_2 \|\nabla f\|_{L^2} \tag{2.2.2}$$

Define $\tilde{L}^2(\Sigma) \coloneqq \{f \in L^2(\Sigma) : \int_\Sigma f = 0\}$ and $\tilde{H}^1(\Sigma) = H^1(\Sigma) \cap A$. Using Poincaré lemma, we have $\|f\|_{l^2} \leq C\|\nabla f\|_{L^2}$ for $f \in \tilde{H}^1(\Sigma)$. Then (2.2.2) can be rewritten as

$$C_1 \|f\|_{H^1} \leq \|DN(f)\|_{L^2} \leq C_2 \|f\|_{H^1} \tag{2.2.3}$$

for $f \in \tilde{H}^1(\Sigma)$.

Suppose $f \in H^1(\Sigma)$ and $DN(f) = 0$. By (2.2.2) and employing the standard bootstrapping strategy, f must be a smooth function. Then, by the definition of DN and the maximal principle, the harmonic extension of f remains constant, implying f itself is constant. If we restrict to $\tilde{H}^1(\Sigma)$, then f must identically vanish. Consequently, DN is injective when restricted to $\tilde{H}^1(\Sigma)$.

Further more, DN is self-adjoint on $C^\infty(\Sigma)$. Let $f, g \in C^\infty(\Sigma)$ and u, v be their harmonic extension to M respectively. Then

$$\begin{aligned}\int_\Sigma DN(f)g &= \int_\Sigma \frac{\partial u}{\partial \nu} g \\ &= \int_M \langle \nabla u, \nabla v \rangle \\ &= \int_\Sigma \frac{\partial v}{\partial \nu} f = \int_\Sigma f DN(g)\end{aligned} \tag{2.2.4}$$

Let me introduce theorem 5.5 in chapter 3 of [LM90].

Theorem 2.3. *Let E be a hermitian vector bundle with connection over a compact Riemannian manifold, $\Gamma(E)$ the smooth sections for E. Suppose $P : \Gamma(E) \to \Gamma(E)$ is elliptic and self-adjoint, then there is an L^2-orthogonal direct sum decomposition:*

$$\Gamma(E) = ker(P) \oplus Im(P)$$

The statement is for hermitian bundles, but the argument works for real bundles with inner product structure as well. We already found the kernel of DN is given by constant $\mathbb{R}$. Given $g \in C^\infty(\Sigma)$, its L^2-orthogonal projection to $\mathbb{R}$ is $g - \frac{1}{\Sigma}\int_\Sigma g$, which lies in $\tilde{L}^2(\Sigma)$. By the theorem above, DN is surjective from $C^\infty(\Sigma) \cap \tilde{L}^2(\Sigma) \to C^\infty(\Sigma) \cap \tilde{L}^2(\Sigma)$, and therefore bijective.

Now assume $g \in \tilde{L}^2(\Sigma)$ which doesn't have to be smooth, then there exists a sequence $\{g_i\} \in C^\infty(\Sigma)$ so that $g_i \to g$ in $L^2(\Sigma)$. We can further assume that $\int_\Sigma g_i = 0$ by taking $g_i - \frac{1}{|\Sigma|}\int_\Sigma g_i$ instead. Then there exists f_i so that $DN(f_i) = g_i$. Use the first inequality in (2.2.3), $\{f_i\}$ is a Cauchy sequence and therefore converges to some $f \in H^1(\Sigma)$. It easy to see that $DN(f) = g$, and thus DN is actually surjective. Combine everything above, and we arrive at

Lemma 2.2.1. *When viewed as a map from $H^1(\Sigma)$ to $L^2(\Sigma)$, the image of DN is $\tilde{L}^2(\Sigma)$, and its kernel is $\mathbb{R}$. Furthermore, DN is self-adjoint when restricted to $C^\infty(\Sigma)$.*

For the Laplace equation, the existence of Green's function is a key tool for solving the equation. Given a compact manifold Σ without boundary, there exists a unique function $G(x, y)$ satisfying $\Delta_y G(x, y) = \delta_x(y)$. Consequently for any $f \in L^2(\Sigma)$ and $\int_\Sigma f = 0$, $u(x) \coloneqq \int_\Sigma G(x, y) f(y)dy$ solves $\Delta u = f$. In the context of Dirichlet-to-Neumann operator, a comparable kernel is anticipated. For $u \in C^\infty(\Sigma) \cap \tilde{L}^2$, define

$$T(u) = DN^{-1}(u - \frac{1}{|\Sigma|}\int_\Sigma u) \tag{2.2.5}$$

T is well-defined by lemma 2.2.1, and $T(u)$ is also $C^\infty(\Sigma)$, and thus in $\mathcal{D}(\Sigma)$. It defines a bilinear form $B(u, v) = \int_\Sigma T(u)v$. Obviously B satisfies the conditions in theorem 2.2.15 (explicit formulation will be given at the end of this section), and therefore there exists a kernel $K \in \mathcal{D}(\Sigma\times\Sigma)$ such that

$$B(u, v) = \int_\Sigma T(u)(x)v(x)dx = \int_{\Sigma\times\Sigma} u(x)v(y)K(x, y)dxdy$$

for any $u, v \in C^\infty(\Sigma)$. Since DN is a first order elliptic pseudo-differential operator, T is elliptic of order -1, and we have the following estimate for K from chapter 1, section 2 in [Tay96]

Lemma 2.2.2. *K is C^∞ off the diagonal in $\Sigma \times \Sigma$, and*

$$|K| \le Cd(x, y)^{2-n} \tag{2.2.6}$$

Σ has dimention $n - 1$, and therefore $\int_{\Sigma\times\Sigma} |u(x)v(y)K(x, y)|dxdy < \infty$. Therefore we could apply Fubini theorem

$$\int_\Sigma T(u)(x)v(x)dx = \int_\Sigma v(y)\big(\int_\Sigma u(x)K(x, y)dx\big)dy$$

Fix u and view v as the test function, we have

$$Tu(x) = \int_\Sigma K(x, y)u(y)dy$$

i.e.

$$u - \frac{1}{|\Sigma|}\int_\Sigma u = \int_\Sigma K(x,y)DN(u)(y)dy \tag{2.2.7}$$

Next we are going to use this expression to prove a non-existence theorem. For convenience, we scale u so that $\int_\Sigma u^{q+1} = 1$, and (2.1.1) becomes

$$\begin{aligned} \Delta u &= 0 && \text{on} \quad M^n \\ \frac{\partial u}{\partial \nu} &= -\lambda u + s(q,\lambda)u^q && \text{on} \quad \Sigma^{n-1} \\ \int_\Sigma u^{q+1} &= 1 \end{aligned} \tag{2.2.8}$$

Use (2.2.7) for (2.2.8),

$$\big(u - \frac{1}{\Sigma}\int_\Sigma u(x)\big) = \int_\Sigma K(x,y)(-\lambda u + su^q)(y)dy \tag{2.2.9}$$

By (2.2.6), $K(x,\cdot)$ is L^p for any $p < \frac{n-1}{n-2}$. Since Σ is compact, we can find a C independant of x such that

$$\|K(x,\cdot)\|_p \leq C, \forall x \in \Sigma$$

In the remaining of this section, C is a constant that depends on metric and q, but not λ. Let $p* = \frac{p}{p-1}$ be the conjugate of p, and we have $p* > n-1$. Then by Hölder inequality, the left hand side of (2.2.9) can be bounded by

$$\begin{aligned} LHS &\leq C\| - \lambda u + su^q\|_{p*} \\ &\leq C\lambda(\|u\|_{p*} + \|u^q\|_{p*}) \end{aligned}$$

(2.1.3) is used for the second inequality. Let $M = \sup u$, and $0 < t < 1$.

$$LHS \leq C\lambda M^t(\|u^{1-t}\|_{p*} + \|u^{q-t}\|_{p*}) \tag{2.2.10}$$

We aim to bound the right-hand side by $\int_\Sigma u^{q+1} = |\Sigma|$. By Hölder's inequality, this is achievable when $(q-t)p* \leq q+1$, which is equivalent to $q \leq \frac{tp*+1}{p*-1}$. Given that $q < \frac{n}{n-2}$, we can choose $0 < t < 1$ and $p* > n-1$ to meet the requirement.

For the left hand side, we might take x to be the maximal point for u. Again, by Hölder inequality and $\int_\Sigma u^{q+1} = |\Sigma|$, we have $\frac{1}{|\Sigma|}\int_\Sigma u \leq 1$. Together, we have

$$M - 1 \leq C\lambda M^t$$

Since $t < 1$, we arrive at the following

Lemma 2.2.3. *Let u be a solution of (2.2.8). Then we have*

$$\|u\|_\infty \leq C(M, n, q) \tag{2.2.11}$$

provided that $\lambda < 1$, where N is a constant that depends on n, q and $C(M, n, q)$ depends on the n, q and Riemannian manifold M.

Based on this L^∞ estimate, we can prove a non-existence theorem:

Theorem 2.4. *For each $1 < q < \frac{n}{n-2}$, there exists λ_q so that (2.2.8) only admits constant solutions for $\lambda \leq \lambda_0$. As a consequence, for these λ's*

$$s_{\lambda,q} = \lambda A(\Sigma)^{\frac{q-1}{q+1}}$$

We start with a lemma

Lemma 2.2.4. *Let u be a harmonic function on (M, Σ, g), then*

$$\int_M |\nabla u|^2 \leq \mu \int_\Sigma |\frac{\partial u}{\partial n}|^2 \tag{2.2.12}$$

where μ is the first Steklov eigenvalue.

Proof. Note that the inequality above is invariant under translation, so it suffices to prove the case $\int_M u = 0$.

$$\begin{aligned}\int_M |\nabla u|^2 &= \int_\Sigma u\frac{\partial u}{\partial n} \\ &\leq \frac{\epsilon}{2}\int_\Sigma u^2 + \frac{1}{2\epsilon}\int_\Sigma |\frac{\partial u}{\partial n}|^2 \\ &\leq \frac{\epsilon}{2\mu}\int_M |\nabla u|^2 + \frac{1}{2\epsilon}\int_\Sigma |\frac{\partial u}{\partial n}|^2\end{aligned}$$

Therefore

$$(2\epsilon - \frac{\epsilon^2}{\mu}) \int_M |\nabla u|^2 \leq \int_\Sigma |\frac{\partial u}{\partial n}|^2$$

The infimum of the quadratic on the left hand side is achieved for $\epsilon = \mu$ and, and the lemma follows. □

Proof of the theorem

$$\begin{aligned}
\mu \int_M |\nabla u|^2 &\leq \int_\Sigma |\frac{\partial u}{\partial n}|^2 \\
&= \int_\Sigma (su^q - \lambda u)\frac{\partial u}{\partial n} \\
&= \int_M (squ^{q-1} - 1)|\nabla u|^2 \\
&\leq \lambda \int_M (qu^{q-1} - 1)|\nabla u|^2 \\
&\leq C\lambda \int_M |\nabla u|^2
\end{aligned} \tag{2.2.13}$$

Therefore, if λ is small, 2.1.2 admit no non-constant minimizer. □

In Theorem 2.4, we investigated the solutions of (2.2.8), which arises as the minimizer for the functional. The same method can be applied to examine solutions for (2.1.1), which are not necessarily minimizers, but the trade-off is that $q < \frac{n-1}{n-2}$.

Theorem 2.5. *For each $1 < q < \frac{n-1}{n-2}$, there exists λ_q so that the equation only admits constant solutions for $\lambda \leq \lambda_0$.*

Proof. The method is similar, and I will only show the different parts. Recall that for equation (2.2.8) we have $\int_\Sigma u^{q+1} = |\Sigma|$ and we can control the right hand side of estimate (2.2.10). But for (2.1.1) we have to derive such an estimate. Integrate (2.1.1) by parts and we have

$$0 = \int_M \Delta u = \int_\Sigma -\lambda u + u^{q+1}$$

By Hölder inequality

$$\begin{aligned}
&\int u^q = \lambda \int u \leq \lambda (\int u^q)^{\frac{1}{q}} |\mathbb{S}^{n-1}|^{\frac{q-1}{q}} \\
\Rightarrow &\int u^q \leq \lambda^{\frac{q}{q-1}} |\mathbb{S}^{n-1}|
\end{aligned} \tag{2.2.14}$$

Now run the method for the previous theorem, the estimate (2.2.10) becomes

$$LHS \leq C(\lambda + 1)(\|u^{1-t}\|_{p*} + \|u^{q-t}\|_{p*})$$

where $0 < t < 1$ and $p* > n - 1$. We aim to bound the right-hand side by $\int_\Sigma u^q$, which requires $(q - t)p* \leq q$. This is feasible if $q \leq \frac{tp*}{p*-1}$, which implies $q < \frac{n-1}{n-2}$. $\square$

Remark 2.2.1. *In theorem 2.4, we initiate with an L^{q+1} bound, while in theorem 2.5, we can only derive an L^q estimate. This is the rationale behind assuming $q < \frac{n-1}{n-2}$ instead of $q < \frac{n}{n-2}$.*

To end this section, I will give explicit formulation of Schwartz kernel theorem. Let M be two compact Riemannian manifolds. We can define the following seminorms on $C^\infty(M)$ by

$$|u|_k \coloneqq \sup_{x \in M} \sum_{\alpha \leq k} |\nabla^\alpha u(x)|$$

These seminorms give topology to $C^\infty(M)$. A linear map T from $C^\infty(M)$ to $\mathbb{R}$ is continuous provided that there exists some C and k

$$T(u) \leq C|u|_k$$

for all $u \in C^\infty(M)$. Let $\mathcal{D}$ denote the space of distribution on M, i.e. all the continuous maps in the sense as above.

Suppose there is another Riammnian manifold N, and a map

$$T : C^\infty(M) \to \mathcal{D}(N)$$

Tu is a continuous operator on $C^\infty(N)$, and thus giving rise to a bilinear form B by the following:

$$B : C^\infty(M) \times C^\infty(N) \to \mathbb{R}$$

$$B(u, v) = \langle Tu, v \rangle, \quad u \in C^\infty(M),\ v \in C^\infty(N)$$

Finally, define $u \otimes v \in C^\infty(M \times N)$ by

$$u \otimes v(x, y) \coloneqq u(x)v(y),\ x \in M,\ y \in N$$

Given all these preparations, we have the Schwartz kernel theorem

Theorem 2.6 (Schwartz kernel theorem). *For any B as in above, there exists a distribution $K \in \mathcal{D}(M \times N)$ so that for $u \in C^\infty(M)$ and $v \in C^\infty(N)$ we have*

$$B(u, v) = \langle u \otimes v, K \rangle \tag{2.2.15}$$

2.3 Bootstrapping Strategy

The L^∞ estimate (2.2.11) can also be derived using standard bootstrapping strategy, and it's more straightforward. In this section (q, λ) will dropped for $s(q, \lambda)$ and $E_{q,\lambda}(u)$ for simplicity. C will be a constant that does not depend on λ or q and might change from line to line. All the integral and norms will be on the boundary Σ.

We start from $x_0 = q + 1$, and choose x_k inductively by

$$\frac{1}{x_{k+1}} = \frac{q}{x_k} - \frac{1}{n-1} \tag{2.3.1}$$

If $u \in L^{x_k}$ for $x_k \geq q + 1$, then $\frac{\partial u}{\partial n} = -\lambda u + s u^q \in L^{\frac{x_k}{q}}$. By Hölder inequality we have

$$\begin{aligned}
\|\frac{\partial u}{\partial n}\|_{\frac{x_k}{q}} &\leq \lambda \|u\|_{\frac{x_k}{q}} + s\|u^q\|_{\frac{x_k}{q}} \\
&\leq \lambda |\Sigma|^{\frac{q}{x_k} - \frac{1}{x_k}} \|u\|_{x_k} + s\|u\|_{x_k}^q \\
&\leq C\lambda(\|u\|_{x_k} + \|u\|_{x_k}^q)
\end{aligned}$$

We used (2.1.3) in the third line. Since $q \leq \frac{n}{n-2}$, x_k is increasing, and $x_k \geq q + 1$. Therefore, $\frac{q}{x_k} - \frac{1}{x_k}$ is bounded from both below and above, and that's why the constant C in the third line can be made independent of q. Since Dirichlet-to-Neumann operator is elliptic of order 1 (see chapter 1 of [Tay96], for example), we have

$$\begin{aligned}
\|u\|_{\frac{x_k}{q},1} &\leq C(\|\frac{\partial u}{\partial n}\|_{\frac{x_k}{q}} + \|u\|_{\frac{x_k}{q}}) \\
&\leq C(\lambda + 1)(\|u\|_{x_k} + \|u\|_{x_k}^q)
\end{aligned}$$

By Sobolev embedding theorem on the boundary and our choice of x_k, we hav

$$\|u\|_{x_{k+1}} \leq C\|u\|_{x_k/q,1} \leq C\big(\|u\|_{x_k} + \|u\|_{x_k}^q\big) \tag{2.3.2}$$

We used the assumption that $\lambda < 1$ in the second inequality. Note that the constant C might change for different k. But we are only taking finite bootstripe steps, so we can pick a universal constant C.

Lemma 2.3.1. *The sequence x_k will be negative in $K(n,q)$ steps, and $K(n,q)$ depends only on the dimension and an upper bound for q.*

This will imply an L^∞ bound by Sobolev inequality.

Proof of lemma:

Let $y_k = \frac{1}{x_k}$ and rewrite 2.3.1 as

$$y_{k+1} - y_k = (q-1)y_k - \frac{1}{n-1} \tag{2.3.3}$$

which implies $y_{k+1} < y_k$ if $y_k < \frac{1}{(q-1)(n-1)}$. By our assumption that $q < \frac{n}{n-2}$ we see that y_0 satisfy the inequality. So by induction $y_k < \frac{1}{(q-1)(n-1)}$ and $y_{k+1} < y_k$ $\forall k > 0$. We need to calculate how many steps it take so that $y_k < 0$. Again from (2.3.3) $y_{k+1} - y_k$ is decreasing, which means that y_k is decreasing faster and faster. So it takes at most

$$\frac{y_0}{y_0 - y_1} = \frac{\frac{1}{q+1}}{\frac{n-(n-2)q}{(q+1)(n-1)}} = \frac{n-1}{n-(n-2)q} \tag{2.3.4}$$

steps to make x_k negative. Let $K(n,q)$ be the least integer larger than $\frac{n-1}{n-(n-2)q}$, and this $K(n,q)$ is what we want in the lemma. $\square$

Use (2.3.2) and do induction, we have

$$\begin{aligned}
\|u\|_{x_{k+2}} &\le C(\|u\|_{x_{k+1}} + \|u\|^q_{x_{k+1}}) \\
&\le C\Big((\|u\|_{x_k} + \|u\|^q_{x_k}) + (\|u\|_{x_k} + \|u\|^q_{x_k})^q\Big) \\
&\le C\Big((\|u\|_{x_k} + \|u\|^q_{x_k}) + 2^q(\|u\|^q_{x_k} + \|u\|^{q^2}_{x_k})\Big) \\
&\le C(\|u\|_{x_k} + \|u\|^{q^2}_{x_k}) \\
&\dots \\
&\le C(\|u\|_{x_0} + \|u\|^{q^{k+2}}_{x_0})
\end{aligned} \tag{2.3.5}$$

where in the third line we used the inequality $(a+b)^q \le 2^q(a^q + b^q)$.

$$\|u\|_\infty \le C(M,n,q)\|u\|_{x_0} \tag{2.3.6}$$

Remark 2.3.1. *Note that if we q is bounded away from $\frac{n}{n-2}$, then from (2.3.4) we can have a uniform bound for K that doesn't depend on q. However, the constant C in (2.3.2) does depends on q and we fails to get a universal estimate. If we could find a universal constant, then (2.3.6) becomes*

$$\|u\|_\infty \leq C\lambda^N \|u\|_{x_0}$$

where both N and C are independent of q. And (2.2.13) is

$$\begin{aligned} \mu \int_M |\nabla u|^2 &\leq \lambda \int_M (qu^{q-1} - 1)|\nabla u|^2 \\ &\leq C\lambda(q(C\lambda)^{N(q-1)} - 1) \int_M |\nabla u|^2 \end{aligned}$$

We will be able to track how the critical λ_q changes with q, and it can easily seen that $\lambda_q \to \infty$ as $q \to 1$.

Remark 2.3.2. *One might inquire whether the bounds established in these two sections can be made universal when Ricci curvature and the second fundamental form are bounded below. Our interest lies in understanding how geometric conditions impact the solutions of PDEs. However, unlike the Laplacian operator, obtaining estimates for the Dirichlet-to-Neumann operator (DN) proves challenging.*

For instance, a comparison theorem for the heat kernel in terms of Ricci curvature is established in [CY81] and [LY86]. Under Ricci curvature restrictions, both lower and upper bounds for eigenvalues of the Laplacian operator can be derived. For further details, see Chapter 3 of [SY94] or [Li12]. However, these methods cannot be directly extended to the Dirichlet-to-Neumann operators. The lack of knowledge regarding how geometric conditions affect Dirichlet-to-Neumann operators poses a significant challenge in Conjecture 2.1.

2.4 Estimate of the Infimum

In the previous two sections we derived L^∞ estimate and then non-existence theorem for (2.2.8). Note that (2.2.8) comes from the Euler-Lagrangian equation of functional (2.1.2, so the non-existence theorem gives us $s(q, \lambda) = \lambda|\Sigma|^{\frac{1-q}{1+q}}$ for certain (q, λ)'s. If we closely examine the functional, we could get better estimate.

Let (M, Σ, g) be an arbitrary manifold with boundary. Throughout this section $\overline{\nabla}$ and ∇ denote gradient on M and Σ respectively. Integration without lower indices denotes integration on Σ. We begin with a lemma:

Lemma 2.4.1. *For $1 < q \leq q_0 < \frac{n+1}{n-1}$, and $\int f^{q+1} = |\Sigma|$, there exist a constant C that only depends on the metric and q_0 so that*

$$\int_\Sigma f^{2q}(\log f)^2 \leq C \int_\Sigma |\nabla f|^2 \tag{2.4.1}$$

Proof. We first show that $x^{2q}(\log x)^2 \leq C((x-1)^2 + |x-1|^{2q_1})$ where q_1 is chosen to lie in $(q_0, \frac{n+1}{n-1})$ and C only depends on q_0. This could be seen by looking into the following three cases.

$i)x \in (0, \dfrac{1}{2})$: $x^{2q}(\log x)^2$ is bounded above and $(x-1)^2$ is bounded below;

$ii)x \in [\dfrac{1}{2}, 2]$: x^{2q} is bounded above and $(\log x)^2$ is bounded by $(x-1)$

$iii)x \in (2, \infty)$: $x^{2q}(\log x)^2 \leq x^{2q_0}(\log x)^2$ and therefore uniformly bounded by $|x-1|^{2q_1}$.

Then we need to bound $\int (f-1)^2$ and $\int |f-1|^{2q_1}$ by $\int_\Sigma |\nabla f|^2$. We might apply Hölder inequality to "modify" the power for $\int |f-1|^{2q_1}$. Namely, use $\theta = \frac{1}{q_1}$ in Hölder inequality:

$$\int |f-1|^{2q_1} \leq \|f-1\|_{q_2}^{2q_1\theta} \|f-1\|_2^{2(1-\theta)q_1} \leq C\|f-1\|_{q_2}^2 \tag{2.4.2}$$

where $q_2 = \frac{2}{2-q_1} < \frac{2(n-1)}{n-3}$. This is where we use the assumption that $q_0 < \frac{n+1}{n-1}$. So q_2 is strictly below the Sobolev conjugate. The last inequality follows from $\int f^{q+1} = |\Sigma|$ and Hölder inequality again, and it's easy to see that C can be chosen independent of $q < q_1$. Now it suffices to show that $\|f-1\|_{q_2} \leq C\|\nabla f\|^2$ for f satisfying $\int f^{q+1} = A(\Sigma)$. This comes from a generalized Poincarè inequality. Let $\fint u := \frac{1}{|\Sigma|}\int u$ defined as the average of u over Σ, then

$$\|f - (\fint f^r)^{\frac{1}{r}}\|_{q_2} \leq C\|\nabla f\|_2 \tag{2.4.3}$$

for $r < q_0 + 1 < \frac{2(n-1)}{n-3}$ and C only depends on q_1 and q_2.(In our case and q_2 depends on q_0, so C only depends on q_0). The proof is by contradiction and modified from the standard proof. Suppose (2.4.3) is not true, and we can find a sequence r_i and f_i so that $\|\nabla f_i\|_2 \to 0$ and $\|f_i - (\fint f_i^{r_i})^{1/r_i}\|_{q_2} = 1$. By compactness, we might pick a subsequence so that $f_i - (\fint f_i^{r_i})^{1/r_i}$ converges in L^{q_2}. Since $\|\nabla f_i\|_2 \to 0$, the sequence $f_i - (\fint f_i^{r_i})^{1/r_i}$ converges in H^1 to a constant function. So we might write $f_i = a_i + h_i$ where a_i are constants and $h_i \to 0$ in H^1. As a result, $a_i - (\fint h_i^{r_i})^{1/r_i} \leq (\fint f_i^{r_i})^{1/r_i} \leq a_i + (\fint h_i^{r_i})^{1/r_i}$ by triangle inequality and it follows that $|f_i - (\fint f_i^{r_i})^{1/r_i}| \leq |h_i| + (\fint h_i^{r_i})^{1/r_i}$. This contradicts with $\|f_i - (\fint f_i^{r_i})^{1/r_i}\|_{q_2} = 1$. $\int |f-1|^2$ can be estimated in a similar way. □

In the lemma above, the base point is 1 and we measured distance from f to 1. That's why we have the three cases in the proof above. In order to apply this lemma we need a different normalization from (2.2.8).

$$\begin{aligned} \Delta u &= 0 && \text{on} \quad M^n \\ \frac{\partial u}{\partial \nu} &= -\lambda u + r(q,\lambda)u^q && \text{on} \quad \Sigma^{n-1} \\ \int_\Sigma u^{q+1} &= |\Sigma| \end{aligned} \tag{2.4.4}$$

If we assume u is a minimzer for $s(q,\lambda)$, multiply this equation by u and integrate by parts, it's easy to see that $s(q,\lambda) = r(q,\lambda)|\Sigma|^{\frac{q-1}{q+1}}$ and thus

$$\lambda \geq r(q,\lambda) \tag{2.4.5}$$

Theorem 2.7. *For $1 < q \leq q_0 < \frac{n+1}{n-1}$, there exists a constant C depending only on n, q_0 and the metric so that $s(q,\lambda) = \lambda|\Sigma|^{\frac{1-q}{1+q}}$ provided*

$$\lambda(q-1) < C \tag{2.4.6}$$

This method only works for $q < \frac{n+1}{n-1}$. The case $q < \frac{n}{n-2}$ will be dealt in the next section from a different viewpoint.

We'll need the following lemma:

Lemma 2.4.2 (Pohozaev ideneity). *Let* (M, Σ, g) *be a compact Riemannian manifold with boundary, and* $g' = g|_\Sigma$. *Suppose* u *is a smooth function and* X *is a smooth vector field. Then*

$$\int_M \langle \nabla_{\nabla u} X, \nabla u\rangle - \frac{1}{2}|\nabla u|^2 div_g X + (Xu)\Delta u = \int_\Sigma (\frac{\partial u}{\partial n}\langle X, \nabla u\rangle - \frac{1}{2}|\nabla u|^2\langle X, \vec{n}\rangle) \tag{2.4.7}$$

Proof. By direct calculation

$$\mathrm{div}(Xu)\nabla u = \langle \nabla_{\nabla u} X, \nabla u\rangle + \langle X, \nabla_{\nabla u}\nabla u\rangle + (Xu)\Delta u$$

$$\mathrm{div}(\frac{1}{2}|\nabla u|^2 X) = \frac{1}{2}|\nabla u|^2 \mathrm{div} X + \langle \nabla u, \nabla_X \nabla u\rangle = \frac{1}{2}|\nabla u|^2 \mathrm{div} X + \langle X, \nabla_{\nabla u}\nabla u\rangle$$

By taking the difference and applying integration by parts to the left-hand side, we obtain the desired equality. □

Proof. Throughout this proof C denotes some constants that only depends on q_0 and the metric g. And it might change from line to line. By Pohozaev identity (2.4.7) for harmonic function u and arbitrary smooth vector field X we have

$$\int_M (\langle \bar\nabla_{\bar\nabla u} X, \bar\nabla u\rangle - \frac{1}{2}|\bar\nabla u|^2 \mathrm{div}_g X) = \int_\Sigma (\frac{\partial u}{\partial n}\langle X, \nabla u\rangle - \frac{1}{2}|\bar\nabla u|^2\langle X, \vec{n}\rangle)$$

Fix a X satisfying $X\big|_\Sigma = \vec{n}$ in this equality and note that $\bar\nabla X$ is bounded by compactness,

$$\begin{aligned}\int_\Sigma |\nabla u|^2 &= \int_\Sigma (\frac{\partial u}{\partial n})^2 + \int_M (|\bar\nabla u|^2 \mathrm{div}_g X - 2\langle \bar\nabla_{\bar\nabla u} X, \bar\nabla u\rangle) \\ &\le \int_\Sigma (\frac{\partial u}{\partial n})^2 + C\int_M |\bar\nabla u|^2 \\ &\le C\int_\Sigma (\frac{\partial u}{\partial n})^2\end{aligned} \tag{2.4.8}$$

The last inequality follows by lemma 2.2.4. Adding $\frac{\lambda^2 - r^2}{\lambda}\int_M |\bar\nabla u^2| \ge 0$ (by (2.4.5)) to the right hand side and using (2.4.4), it becomes the following

$$\begin{aligned}\int |\nabla f|^2 &\le C(\int (\frac{\partial u}{\partial n})^2 + \frac{\lambda^2 - r^2}{\lambda}\int_M |\bar\nabla u^2|) \\ &= C[(r^2\int f^{2q} - 2\lambda r\int f^{q+1} + \lambda^2\int f^2) \\ &\quad + \frac{\lambda^2 - r^2}{\lambda}(r\int f^{q+1} - \lambda\int f^2)] \\ &= C[r^2(\int f^{2q} + f^2) + r(\frac{\lambda^2 - r^2}{\lambda} - 2\lambda)\int f^{q+1}]\end{aligned} \tag{2.4.9}$$

Consider the function $\phi(x) = a^{q+1+x} + a^{q+1-x}$ for $x \in [0, q-1]$ and $a > 0$. Taylor expansion implies that for some $\theta \in [0, q-1]$

$$\begin{aligned}\phi(q-1) = 2a^{q+1} + \frac{\phi''(\theta)}{2}(q-1)^2 &= 2a^{q+1} + \frac{(q-1)^2}{2}(\log a)^2 a^{q+1}(a^{\theta} + a^{-\theta}) \\ &\leq 2a^{q+1} + \frac{(q-1)^2}{2}(\log a)^2(a^{2q} + a^2)\end{aligned}$$

Using this estimate in 2.4.9,

$$\begin{aligned}\int |\nabla f|^2 &\leq C[\frac{(q-1)^2}{2} r^2 (\int (f^{2q} + f^2)\log f + r(\frac{-r^2}{\lambda} + 2r - \lambda)\int f^{q+1}] \\ &\leq \frac{C}{2}(q-1)^2 r^2 (\int f^{2q}(\log f)^2 + \int f^2(\log f)^2) \\ &\leq C(q-1)^2 r^2 \int f^{2q}(\log f)^2\end{aligned} \tag{2.4.10}$$

Using **Lemma 2.4.1** and (2.4.5), (2.4.10) becomes

$$\int |\nabla f|^2 \leq C(q-1)^2 \lambda^2 \int |\nabla f|^2$$

and f, and therefore u, will be constant if $(q-1)\lambda$ is small. □

Remark 2.4.1. *Note that (2.4.8) was obtained in (2.2.3) by directly utilizing the ellipticity of the Dirichlet-to-Neumann operator. However, it is challenging to discern how curvature conditions come into play in that method. (2.4.8) is more likely to be connected to geometry, and the problem is how to construct a nice vector field. This provides some information, but not precisely what we are seeking. This idea will come back later in a log-Sobolev inequality.*

One might hope to get a uniform bound for the constant in Dirichlet-to-Neumann operator, but such an estimate doesn't exist even under the condition of positive sectional curvature and $II \geq 1$ where II is the second fundamental form. Actually, consider the ellipse

$$\{(x, y) \big| x^2 + k^2 y^2 \leq 1\}$$

Under scaling of the metric $\bar{g} = k^{-2} g$, the second fundamental form can be arbitrarily large. At the same time both $|\bar{\nabla} f|^2 = k^2 |\nabla f|^2$ and $(\frac{\partial u}{\partial \bar{n}})^2 = k^2 (\frac{\partial u}{\partial n})^2$ are scaled by the same factor. So we might *forget about the restriction on the II. The normal vector and tangent vector are $\vec{n} = (x, k^2 y)$ and $\vec{v} = (k^2 y, x)$ respectively. For $u = x$ we calculate as follows*

$$\begin{aligned}\int (\frac{\partial u}{\partial n})^2 &= 4\int_0^1 ((1,0)\cdot\frac{\vec{n}}{|\vec{n}|})^2\sqrt{\frac{k^4y^2+x^2}{k^4y^2}}dx \\ &= 4\int_0^1 \frac{x^2}{k\sqrt{(k^2(1-x^2)+x^2)(1-x^2)}}dx \\ &\leq \frac{4}{k}\int_0^1 \frac{x}{\sqrt{1-x^2}} = \frac{4}{k}\end{aligned} \tag{2.4.11}$$

Also note that $\int |\nabla u|^2 + (\frac{\partial u}{\partial n})^2 = \int 1 \geq 4$. So $\int |\nabla u|^2$ can't be uniformly bounded by $\int (\frac{\partial u}{\partial n})^2$ under curvature and II restriction. Consider $u = y$, we see that $\int (\frac{\partial u}{\partial n})^2$ can't be uniformly bounded by $\int |\nabla u|^2$, either.

Remark 2.4.2. *The idea of this proof comes from the paper by Ou and Lin [LO23]. We translate their work as follows: $\int_\Sigma |\nabla f|^2$ can be bounded by combination of $\int_\Sigma (\frac{\partial u}{\partial n})^2$ and $\int_M |\bar{\nabla} u|^2$. These two terms are "difference" of L^p norms by (2.2.8), and this "difference" can be bounded by $\int_\Sigma |\nabla f|^2$. In [LO23], this "difference" is measured by the ratio of L^p norm. We treated it differently by taking the subtraction.*

2.5 An ODE Approach

Consider $s_{\lambda,q} - A(\Sigma)^{\frac{q-1}{q+1}}\lambda \leq 0$. If this inequality is strict, $E_{q,\lambda}$ must have a non-constant minimizer. Using this minimizer, we are going to show that the strict negativity is preserved along some curve of λ, q which looks like (2.5.1) in the theorem below. But we know from the work of [GW20] that for unit ball, $q = \frac{n}{n-2}, \lambda = \frac{n-2}{2}$, $E_{q,\lambda}$ only admits constant minimizer, which gives some restriction on q, λ.

Lemma 2.5.1. *Let (M, Σ, g) be a Riemannian manifold with boundary so that $A(\Sigma) = 1$. Suppose $s(q_0, \lambda_0) - \lambda_0 \leq -\epsilon < 0$ for some (λ_0, q_0), then this inequality remains valid along the curve*

$$\frac{q+1}{q-1} = C(\lambda - \epsilon) \tag{2.5.1}$$

for $\lambda \leq \lambda_0$, where $C = \frac{q_0+1}{(q_0-1)(\lambda_0-\epsilon)}$ is chosen so that (λ_0, q_0) is on the curve.

Proof. Along the curve (2.5.1), $s(\lambda)$ and $q(\lambda)$ are functions of λ only. It suffices to show that for λ_1 satisfying $s(\lambda_1) = \lambda_1 - \epsilon$, $s(\lambda) - \lambda$ will be decreasing along the curve in the $-\lambda$ direction near λ_1. Note that $s(\lambda_1) = \lambda_1 - \epsilon$ implies the existence of a non-constant minimizer for E_{q_1,λ_1} satisfying (2.2.8), where $q_1 = q(\lambda_1)$. Denote it by u. Fix this u, and we want to show that $E_{q(\lambda),\lambda}(u)$ decreases fast enough in $-\lambda$ direction along the curve (2.5.1), and the theorem follows since $s(q(\lambda, \lambda) \leq E_{\lambda,q(\lambda)}(u)$ for all λ. Namely, we need to prove the following inequality:

$$\frac{\partial}{\partial(-\lambda)}(E_{\lambda,q(\lambda)}(u) - \lambda) < 0$$

One calculates that

$$\frac{\partial}{\partial q}(\int_\Sigma f^{q+1})^{\frac{2}{q+1}} = (\int_\Sigma f^{q+1})^{\frac{2}{q+1}}[-\frac{2}{(q+1)^2}\log\int_\Sigma f^{q+1} + \frac{2}{q+1}\frac{\int_\Sigma f^{q+1}\log f}{\int_\Sigma f^{q+1}}]$$

We assumed that $⨍_\Sigma f^{q_1+1} = A(\Sigma) = 1$, so

$$\frac{\partial}{\partial \lambda}E_{\lambda,q(\lambda)}(u)\big|_{\lambda_1} = \int_\Sigma f^2 - \frac{2s(\lambda_1)}{q_1+1}q'(\lambda_1)\int_\Sigma f^{q_1+1}\log f$$

So it suffices to show

$$\int_\Sigma f^2 - \frac{2s(\lambda_1)}{q+1}q'(\lambda_1)\int_\Sigma f^{q+1}\log f > 1 \tag{2.5.2}$$

By Hölder inequality and $A(\Sigma) = ⨍_\Sigma f^{q+1} = 1$, we have $\int_\Sigma f^{q+1+\epsilon} \geq 1$ for $\epsilon > 0$ and it follows that $\int_\Sigma f^{q+1}\log f \geq 0$. Also note that $\int_\Sigma f^x$ is a strict convex function in x, we have

$$\int_\Sigma f^2 + (q-1)\int_\Sigma f^{q+1}\log f > \int_\Sigma f^{q+1} = 1$$

This inequality is strict since u is not constant. So (2.5.2) holds provided

$$\frac{-2q'(\lambda)(\lambda-\epsilon)}{q+1} \geq q - 1$$

The solution for the equality is exactly (2.5.1), and we finish the proof. $\square$

Corollary 2.1. *For* $(\mathbb{B}^n, \mathbb{S}^{n-1}, g)$ *the standard metric,* $q \le \frac{n}{n-2}$, $s(q,\lambda)$ *is achieved only by constant functions for*

$$(q-1)(\lambda - \frac{n-2}{2(n-1)}) < \frac{n-2}{n-1} \tag{2.5.3}$$

Proof. We will first scale the metric so that the area of the boundary is 1, namely consider $(\mathbb{B}^n, \mathbb{S}^{n-1}, \overline{g} = k^2 g)$ where $k = A(\mathbb{S}^{n-1})^{\frac{1}{1-n}}$.The function $E_{q,\lambda}(u)$ changes as follows

$$\begin{aligned}\overline{E}_{q,\lambda}(u) &= \frac{\int_{\mathbb{B}^n} |\nabla_{\overline{g}} u|^2 \mathrm{dVol}_{\overline{g}} + \lambda \int_{\mathbb{S}^{n-1}} u^2 \mathrm{dS}_{\overline{g}}}{(\int_{\mathbb{S}^{n-1}} u^{q+1} \mathrm{dS}_{\overline{g}})^{\frac{2}{q+1}}} \\ &= k^{n-2-\frac{2(n-1)}{q+1}} \frac{\int_{\mathbb{B}^n} |\nabla_g u|^2 \mathrm{dVol}_g + k\lambda \int_{\mathbb{S}^{n-1}} u^2 \mathrm{dS}_g}{(\int_{\mathbb{S}^{n-1}} u^{q+1} \mathrm{dS}_g)^{\frac{2}{q+1}}} \\ &= k^{n-2-\frac{2(n-1)}{q+1}} Q_{k\lambda,q}(u)\end{aligned} \tag{2.5.4}$$

By the work of ([GW20]) for unit ball, (2.2.8) admit only constant solutions for $\lambda = \frac{n-2}{2}, q = \frac{n}{n-2}$. After the scaling (2.5.4), for $\bar{g} = k^2 g$ this critical point becomes $(q,\lambda) = (\frac{n}{n-2}, \frac{n-2}{2k})$. Suppose $\bar{s}_{q_0,\lambda_0} < \lambda_0$ for some (λ_0, q_0) satisfying

$$\lambda_1 < \frac{(n-2)(q+1)}{2k(n-1)(q-1)} \tag{2.5.5}$$

When ϵ is small enough, $\bar{s}_{q_0,\lambda_0} < \lambda_0 - \epsilon$. And by **Theorem 2.5.1** this inequality remains valid along (2.5.1) for $\lambda < \lambda_0$. In particular, we let $q = \frac{n}{n-2}$, then

$$\lambda = \frac{q+1}{C(q-1)} + \epsilon = (n-1)\frac{(q_0-1)(\lambda_0-\epsilon)}{q_0+1} + \epsilon \tag{2.5.6}$$

If (2.5.5) holds, we can make ϵ small so that $\lambda < \frac{n-2}{2k}$, which is contradiction since we must have $\bar{s}_{\frac{n-2}{2k},\frac{n}{n-2}} = \frac{n-2}{2k}$. Now transfer this result back to the unit ball and the proof is finished.

$\square$

Remark 2.5.1. *For standard balls Wang's conjecture been completely solved in [GL23], and the theorem above is also included.*

In the preceding sections, we demonstrated that for $q < \frac{n}{n-2}$, there exists a corresponding λ such that $s(q, \lambda)$ is only achieved by constants. Let's fix one such pair as (q_0, λ_0) and employ a similar argument to the one in the corollary above. This will yield a result similar to Theorem 2.7, but with $q_0 < \frac{n}{n-2}$ instead of $q_0 < \frac{n+1}{n-1}$. These two approaches are distinct and offer different perspectives on Wang's conjecture.

Remark 2.5.2. *The proof of the corollary does not rely on the specific structure of the manifold. The only instance where $(\mathbb{B}^n, \mathbb{S}^{n-1}, g)$ is involved is at the critical point $(\frac{n}{n-2}, \frac{n-2}{2})$. Therefore, our method is applicable to any manifold as long as one can compute such a critical point. The challenge lies in determining how to obtain such a point under curvature restrictions. When $q = \frac{n}{n-2}$, the problem is related to type II Yamabe problem. A breakthrough in Yamabe problem might help us find a critical (q, λ).*

2.6 Critial Power Case

If $q = \frac{n}{n-2}$, the trace operator is only continuous and fails to be compact, making the existence of minimizers more challenging. In this section, I will derive some existence results for the minimizer of $s\left(\frac{n}{n-2}, \lambda\right)$. It's difficult to determine whether these minimizers become constants for small λ's.

Lemma 2.6.1. *For any compact Riemannian manifold with boundary and $\lambda \geq 0$,*

$$\frac{4(n-1)}{n-2} s(\frac{n}{n-2}, \lambda) \leq Y(\mathbb{B}^n, \mathbb{S}^{n-1}, dx^2) \tag{2.6.1}$$

Proof. Fix a point $p \in \Sigma$. We can find a small neighborhood U of $p \in M$ so that $U = B^{n-1}(\delta) \times (0, \delta)$ for some small δ. Fix a cut-off function ϕ so that $\phi = 1$ in $B^{n-1}(\delta/2) \times (0, \delta/2)$ and vanishes outside U. Let $\{x_i\}_{i=1}^{n-1}$ be the coordinates for $B^{n-1}(\delta)$ and t coordinate for $(0, \delta)$. Define

$$v_\epsilon = (\frac{\epsilon}{(\epsilon + t)^2 + |y|^2})^{\frac{n-2}{2}} \tag{2.6.2}$$

Recall that we can use ϕv_ϵ as test functions to establish the type II Yamabe inequality (1.2.17). The differences between the functional of the type II Yamabe problem and $E_{\frac{n}{n-2},q}$ lie in the terms $\int_M Ru^2 dV$ and $\int_\Sigma \left(\frac{n-2}{2(n-1)}H - \lambda\right) u^2 dS$. Note that H and R remain bounded for a fixed metric. If we can demonstrate that $\int_M u^2 dV$ and $\int_\Sigma u^2 dV$ vanish as $\epsilon \to 0$ for $u = \phi v_\epsilon$, then the proof is complete. Let dV_E and dS_E be the volume form with respect to Euclidean space.

$$\begin{aligned}
\int_M (\phi v_\epsilon)^2 dV_g &\leq \int_U v_\epsilon^2 dV_g \\
&\leq C \int_U v_\epsilon^2 dV_E \\
&= C \int_0^\delta \int_{B^{n-1}(\delta)} \left(\frac{\epsilon}{(\epsilon+t)^2 + |y|^2}\right)^{n-2} dy dt \\
&= C \int_0^\delta \int_{B^{n-1}(\frac{\delta}{t+\epsilon})} \frac{\epsilon^{n-2}}{(t+\epsilon)^{n-3}} \frac{1}{(1+|z|^2)^{n-2}} dz dt \\
&\leq C \int_0^\delta \frac{\epsilon^{n-2}}{(t+\epsilon)^{n-3}} dt \\
&= C \int_0^{\frac{\delta}{\epsilon}} \frac{\epsilon^2}{(1+t)^{n-3}} dt \leq C\epsilon^2
\end{aligned}$$

We used change of variable in the third and fifth line. Similarly, we have

$$\begin{aligned}
\int_\Sigma (\phi v_\epsilon)^2 dS_g &\leq \int_\Sigma v_\epsilon^2 \\
&\leq C \int_{B^{n-1}(\delta)} v_\epsilon^2 dS_E \\
&= C \int_{B^{n-1}} \left(\frac{\epsilon}{\epsilon^2 + |y|^2}\right)^{n-2} dz \\
&= C \int_{B^{n-1}(\frac{\delta}{\epsilon})} \frac{\epsilon}{(1+|z|^2)^{(n-2)}} dz \leq C\epsilon
\end{aligned}$$

$$\begin{aligned}
\int_\Sigma (\phi v_\epsilon)^{\frac{2(n-1)}{n-2}} &\geq \int_{B^{n-1}(\delta/2)} v_\epsilon^{\frac{2(n-1)}{n-2}} dS_g \\
&\geq C \int_{B^{n-1}(\delta/2)} \left(\frac{\epsilon}{\epsilon^2 + |y|^2}\right)^{(n-1)} dS_E \\
&= C \int_{B^{n-1}(\delta/(2\epsilon))} \left(\frac{1}{1+|z|^2}\right)^{(n-1)} dz \geq C
\end{aligned} \tag{2.6.3}$$

Let $\epsilon \to 0$, and the three estimates prove the lemma. □

If $q < \frac{n}{n-2}$, such a bound doesn't exist.

Lemma 2.6.2. *$s(q,\lambda) \to \infty$ as $\lambda \to \infty$ for $q < \frac{n}{n-2}$.*

Proof. We prove by contradiction. Suppose not. Then there exists $\lambda_i \to$ and u_i such that $E_{q,\lambda_i}(u_i) < C$. Without loss of generality, we might assume $\|u_i\|_{L^{q+1}(\Sigma)} = 1$. Then we must have $\int_M |\nabla u_i|^2 < C$ and $\int_\Sigma u_i^2 \to 0$. By Alauglu theorem we can find a subsequence, still denoted by u_i, such that $u_i \rightharpoonup u_0$ and $\int_\Sigma u_0^2 = \lim \int_\Sigma u_i^2 = 0$. Since $q < \frac{n}{n-2}$, by compactness, we can pick a further subsequence so that $\|u_0\|_{L^{q+1}(\Sigma)} = \lim \|u_i\|_{L^{q+1}(\Sigma)} = 1$, which is a contradiction. □

According to the computations in Lemma 2.6.1, we have $v_\epsilon \to 0$ in $L^2(\Sigma)$, but they do not converge in $L^{\frac{2(n-1)}{n-2}}(\Sigma)$. This elucidates why the argument fails for $q = \frac{n}{n-2}$. In this critical case, where $s(\frac{n}{n-2},\lambda)$ is bounded in λ, the dynamics are quite different. The key observation is that if $s(\frac{n}{n-2},\lambda)$ stops increasing for large λ, it is likely minimized through a sequence of functions that blow up somewhere, with their $L^2(\Sigma)$ norms tending to zero. This phenomenon is akin to what is observed in (2.6.2). Consequently, $s(\frac{n}{n-2},\lambda)$ admits no minimizer in this scenario, not even constants. On the contrary, if $s(\frac{n}{n-2},\lambda)$ keeps increasing in λ, functions like (2.6.2) are ruled out as minimizing sequences. This exclusion opens up the possibility of obtaining a minimizer. These observations can be made concrete by the following theorem.

Theorem 2.8. *i):If there exists $\mu < \lambda$ such that $s(\frac{n}{n-2},\lambda) = s(\frac{n}{n-2},\mu)$, then $s(\frac{n}{n-2},\lambda)$ doesn't admit any minimizer.*
ii): If there exists $\lambda < \mu$ such that $s(\frac{n}{n-2},\lambda) < s(\frac{n}{n-2},\mu)$, then $s(\frac{n}{n-2},\lambda)$ admits a minimizer.

Proof. Part i):
Suppose $s(\frac{n}{n-2},\lambda)$ admit a minimizer u. Then

$$s(\frac{n}{n-2},\mu) \leq E_{\frac{n}{n-2},\mu}(u) < E_{\frac{n}{n-2},\lambda}(u) = s(\frac{n}{n-2},\lambda)$$

which contradicts our assumption.
Part ii):
Let u_i be a minimizing sequence for $s(\frac{n}{n-2},\lambda)$. We can scale u_i so that $\|u_i\|_{L^p} = 1$, where $p = \frac{2(n-1)}{n-2}$.

Then

$$\lim_{i\to\infty}\Big(\int_M |\nabla u_i|^2 + \lambda\int_\Sigma u_i^2\Big) = s(\frac{n}{n-2},\lambda) \tag{2.6.4}$$

Use these u_i as test-function for μ, and we have

$$\int_M |\nabla u_i|^2 + \mu\int_\Sigma u_i^2 \le s(\frac{n}{n-2},\mu) \tag{2.6.5}$$

By Alaoglu theorem and compactness we can get a subsequence so that

$$u_i \rightharpoonup u \text{ in } H^1(M),\ u_i \rightharpoonup u \text{ in } L^p(\Sigma),$$

$$u_i \to u \text{ in } L^2,\ u_i \to u \text{ a.e. in } M$$

Use these u_i as test-function for μ,

$$\int_M |\nabla u_i|^2 + \mu\int_\Sigma u_i^2 \ge s(\frac{n}{n-2},\mu) \tag{2.6.6}$$

Take the difference between the (2.6.4) and (2.6.5), and we get

$$(\mu-\lambda)\int_\Sigma u^2 = \lim(\mu-\lambda)\int_\Sigma u_i^2 \ge s(\frac{n}{n-2},\mu) - s(\frac{n}{n-2},\lambda) > 0$$

This is where we used our assumption. This rules out possibility that $u\equiv 0$, which happens in the proof of lemma 2.6.1.

Next we are going to show u minimizes $s(\frac{n}{n-2},\lambda)$. Since $u_i \rightharpoonup u$ in $L^p(\Sigma)$, we have $\|u\|_p \le 1$. Let $v_i = u_i - u$. By a result in [BL83],

$$1 = \lim\|u_i\|_p^p = \lim\|u+v_i\|_p^p = \|u\|_p^p + \lim\|v_i\|_p^p$$

Note that $u\ne 0$, so $\|u\|_p \le 1, \lim\|v_i\|_p \le 1$. Consequently

$$\begin{aligned}1 &\le \lim\|u\|_p^2 + \lim\|v_i\|_p^2\\ &\le \|u\|_p^2 + \lim\frac{1}{s(\frac{n}{n-2},\lambda)}\Big(\int_M|\nabla v_i|^2 + \lambda\int_\Sigma v_i^2\Big)\\ &= \|u\|_p^2 + \frac{1}{s(\frac{n}{n-2},\lambda)}\lim\int_M|\nabla v_i|^2\end{aligned}$$

v_i can be estimated using (2.6.4),

$$\int_M|\nabla u|^2 + \lambda\int_\Sigma u^2 + \lim\int_M|\nabla v_i|^2 = s(\frac{n}{n-2},\lambda)$$

where we used $v_i \rightharpoonup 0 \ \in H^1(M)$ and $v_i \to 0$ in $L^2(\Sigma)$. Combine the two equations above and we get

$$\frac{\int_M |\nabla u|^2 + \int_\Sigma u^2}{\|u\|_p^2} \leq s(\frac{n}{n-2}, \lambda)$$

So u is a minimizer. $\square$

Remark 2.6.1. *The proof of part ii) comes from [BN83] where the H.Brezis and L.Nirenberg proved similar results for a different equation*

$$-\Delta u = u^p + \lambda u \text{ on } M$$

$$u > 0 \text{ on } M$$

$$u = 0 \text{ on } \Sigma$$

Corollary 2.2. *For the unit disk, $s(\frac{n}{n-2}, \frac{n-2}{2})$ admits only constant minimizer for $\lambda < \frac{n-2}{2}$, and admit no minimizer for $\lambda > \frac{n}{n-2}$.*

Proof. It's well known that $s(\frac{n}{n-2}, \frac{n-2}{2}) = \frac{n-2}{2}|\Sigma|^{\frac{1}{n-1}}$. Then the result follows from the two theorems above and lemma 2.1.1. $\square$

For the critical power case, $s(\frac{n}{n-2}, \lambda)$ has a strong relationship with type II Yamabe problem. Use standard argument and we can get a similar existence theorem

Theorem 2.9. *If $\frac{4(n-1)}{n-2} s(\frac{n}{n-2}, \lambda) < Y(\mathbb{B}^n, \mathbb{S}^{n-1}, dx^2)$, then it admits a minimizer.*

Proof. The trick is again "lowering the index". For each $q < \frac{n}{n-2}$, 2.2.8 admits a solution u_q (it might be constant). If u_q's are uniforma bounded above, then the ellipticity of Dirichlet-to-Neumann operator implies a universal upper bound for u_q in C^k for any $k \in \mathbb{Z}_+$. Consequently s_q converges to a solution of (2.2.8) for $\frac{n}{n-2}$. So it suffices to show that there doen't exist such a L^∞ bound. Suppose on the contrary that there exits $q_k \to \frac{n}{n-2}$, u_k and $p_k \in \Sigma$ so that u_k minimizes $E_{q_k,\lambda}$ and $m_k := u_k(p_k) = \sup_{x \in M} u(x) \to \infty$. The idea is that we are going to show that by scaling u_k will "converge" locally around p to a solution of PDE in upper plane, and this contradicts the assumption $s(\frac{n}{n-2}, \lambda) < Y(\mathbb{B}^n, \mathbb{S}^{n-1}, dx^2)$. For convenience of readers, I will restate the equations for u_k

$$\begin{aligned} \Delta u_k &= 0 && \text{on} \quad M^n \\ \frac{\partial u_k}{\partial \nu} &= -\lambda u_k + s u_k^{q_k} && \text{on} \quad \Sigma^{n-1} \\ \int_\Sigma u_k^{q_k+1} &= 1 \end{aligned} \tag{2.6.7}$$

By compactness we might assume $p_k \to p \in \Sigma$. We might pick local coordinate upper ball $U_p(2\epsilon) \coloneqq \mathbb{B}^n(2\epsilon) \cap \{x_n \geq 0\}$ centered at p, where $\{x_i\}_{1 \leq i \leq n-1}$ is local normal coordinate for $p \in \Sigma$ and x_n is the coordinate in normal direction. Let $\delta_k = m_k^{1-q_k}$, and $v_k = \frac{1}{m_k} u_k(\delta_k x + p_k)$. Then v_k is defined in $U_p(\frac{\epsilon}{\delta_k})$ for large i's with radius $\frac{\epsilon}{\delta_k} \to \infty$. By (2.6.7), v_k locally satisfies

$$\begin{aligned} \frac{1}{b_k} \partial_j (a_k^{ij} \partial_i v_k) = 0 \text{ on } \quad & U_p(\frac{\epsilon}{\delta_k}) \\ -\frac{\partial v_k}{\partial x_n} + c_k v_k = s v_k^{q_k} \text{ on } \quad & U_p(\frac{\epsilon}{\delta_k}) \cap \{x_n = 0\} \end{aligned} \tag{2.6.8}$$

where

$$a_k^{ij}(x) = g^{ij}(\delta_k x + p_k) \to \delta_{ij}$$

$$b_k(x) = \sqrt{\det g(\delta_k x + p_k)} \to 1$$

$$c_k = \lambda m_k^{-q_k} \to 0$$

In the equation above we have an additional $-$ in front of $\frac{\partial v_k}{\partial x_n}$ because $\frac{\partial}{\partial x_n}$ is in the inner normal direction instead of outer normal direction.

Note that $\frac{u_k}{m_k}$ has uniform L^∞ bound by its definition. Since they satisfy a similar equation

$$\Delta u_k = 0 \text{ on } M$$

$$\frac{\partial u_k}{\partial \nu} + c_k u_k = s u_k^{q_k} \text{ on} \Sigma$$

So u_k are uniform bounded in any $C_k(M)$ norm by ellipticity of Dirichlet-to-Neumann operator. v_k are defined in $U_p(\frac{\epsilon}{\delta_k})$ and $\partial_\alpha v_k(x) = \delta_k^{|\alpha|} \partial_\alpha u_k(\delta_k x + p_k)$ for $\delta_k \to 0$. So v_k are also uniformly bounded in any $C^k(U_p(R))$ for any fixed R. So we could pick a sub-sequence so that $v_i \to v$ so that

$$\begin{aligned} \Delta v = 0 \text{ on } \quad & \mathbb{H}_+^n \\ \frac{\partial v}{\partial \nu} = s v^{\frac{n}{n-2}} \text{ on } & \mathbb{R}^{n-1} \end{aligned} \tag{2.6.9}$$

where ν is the outer normal direction. (I failed to derive this convergence from (2.6.8) directly since Dirichlet-to-Neumann operator is global, while v_k is only defined locally. Also, the Schauder estimates can't be applied directly.)

Now if v has enough decay at infinity, we could multiply (2.6.9) by v on both sides and integrate to get

$$\int_{\mathbb{H}^n_+} |\nabla v|^2 = s \int_{\mathbb{R}^{n-1}} v^{\frac{2(n-1)}{n-2}} \tag{2.6.10}$$

Also, as a limit we probably have $\int_\Sigma v^{\frac{2(n-1)}{n-2}} \leq 1$. Note that $\mathbb{H}^n_+$ has vanishing mean curvature and scalar curvature. Use v as the test function for type II Yamabe problem and we get $Y(\mathbb{H}^n_+, \mathbb{R}^{n-1}, dx^2) \leq \frac{4(n-1)}{n-2}s$. However, it's well known that the upper half plane and unit disk are conformally equivalent through

$$\begin{aligned} F: \quad & \mathbb{B}^n \to \mathbb{H}^n \\ F(x_1, \cdots, x_n) =& \frac{1}{P}(2x_1, \cdots, 2x_{n-1}, 1 - x_1^2 \cdots - x_n^2) \\ F^* g_{\mathbb{H}^n} =& \frac{4}{P^2} g_{\mathbb{B}^n} := \phi^{\frac{4}{n-2}} g_{\mathbb{B}^n} \end{aligned} \tag{2.6.11}$$

where $P = x_1^2 + \cdots + x_{n-1}^2 + (1 - x_n)^2$. Use $v\phi$ as test function for $\mathbb{B}^n$, and we will get

$$Y(\mathbb{B}^n, \mathbb{S}^{n-1}, dx^2) = Y(\mathbb{H}^n_+, \mathbb{R}^{n-1}, dx^2) \leq \frac{4(n-1)}{n-2} s(\frac{n}{n-2}, \lambda)$$

which contradicts to assumption $s(\frac{n}{n-2}, \lambda) < Y(\mathbb{B}^n, \mathbb{S}^{n-1}, dx^2)$. Next we will fix the gaps in the argument above. Let $g' = g|_\Sigma$. We can compute in local coordinate

$$\begin{aligned} \int_{B_p^{n-1}(\frac{\epsilon}{\delta_k})} v_k^{q_k+1} \sqrt{\det(g')(p_k + \delta_k x)} dx &= \int_{B_{p_k}^{n-1}(\epsilon)} \frac{\sqrt{\det(g')(x)}}{\delta_k^{n-1} m_k^{q_k+1}} u_k^{q_k+1} dy \\ &\leq m_k^{(q-1)(n-1)-q-1} \int_\Sigma u_k^{q_k+1} dV_{g'} \\ &= m_k^{((n-2)q-n} \end{aligned}$$

We used change of variable $y = p_k + \delta_k x$ in the first line. Since $q < \frac{n}{n-2}$, $m_k^{(q-1)(n-1)-q-1} \leq 1$ for large k's. Since Similarly, we compute

$$\int_{U_p(\frac{\epsilon}{\delta_k})} a_k^{ij}(x)\partial_i v_k \partial_j v_k b_k(x)dx = \int_{U_p(\frac{\epsilon)}{\delta)k}} \frac{\delta_k^2}{m_k^2}\left(g^{ij}\partial_i u_k \partial_j u_k\right)(\delta_k x + p_k)dx$$
$$= m_k^{(n-2)q-n}\int_{U_{p_k}(\epsilon)} |\nabla u_k|^2 dV_g$$
$$\leq m_k^{(n-2)q-n}\|\nabla u_k\|_{L^2(M)} \leq C$$

Since $\sqrt{\det(g')(p_k+\delta_k x)} \longrightarrow 1$ and $a_k^{ij} \longrightarrow \delta_{ij}$, and v_k converges uniformly in any compact subset, by Fatou's lemma we arrive at

$$\begin{aligned} \int_{\mathbb{R}^{n-1}} v^{\frac{2(n-1)}{n-2}} \leq 1 \\ \int_{\mathbb{H}^n_+} |\nabla v|^2 \leq \infty \end{aligned} \tag{2.6.12}$$

Let $\eta(x)$ be a cut-off function in $\mathbb{H}^n_+$, and $v_R(x) = \eta(\frac{x}{R})v$. Then with the two bounds above we can verify that $v_R \longrightarrow v$ in $H^1(\mathbb{H}^n_+)$ and $L^{\frac{2(n-1)}{n-2}}(\mathbb{R}^{n-1})$ by showing

$$\int_{\mathbb{H}^n_+} |\nabla(v - v_R)|^2 \longrightarrow 0$$
$$\int_{\mathbb{R}^{n-1}} |v - v_R|^{\frac{2(n-1)}{n-2}} \longrightarrow 0$$

Multiply $\eta(\frac{x}{R})v$ to (2.6.9) and integrate by part, we get

$$\int_{\mathbb{H}^n_+} \langle \nabla v, \nabla v_R) \rangle = s\int_{\mathbb{R}^{n-1}} v^{\frac{n}{n-2}} v_R$$

Let $R \longrightarrow \infty$, and we get (2.6.10). $\square$

In view of the previous results, for a fixed Riemannian manifold with boundary we can ask the following questions:

i) does $\frac{4(n-1)}{n-2}s(\frac{n}{n-2},\lambda) \longrightarrow Y(\mathbb{B}^n, \mathbb{S}^{n-1}, dx^2)$ as $\lambda \longrightarrow \infty$

ii) does there exists a λ_0 so that $s(\frac{n}{n-2},\lambda_0) = s(\frac{n}{n-2},\lambda)$ for all $\lambda > \lambda_0$

iii) does there exists a λ_1 so that $s(\frac{n}{n-2},\lambda)$ admits only constant minimizer for all $\lambda < \lambda_1$

My guess is all these are correct, but I have not found a way to solve these.

CHAPTER 3

A LOG-SOBOLEV INEQUALITY

In this chapter, I will introduce the log-Sobolev inequality, which is closely related to Wang's conjecture. The validity of the log-Sobolev inequality provides a key insight into the confidence we can place in Wang's conjecture.

3.1 Motivation for Log-Sobolev inequality

Let $(M, \partial M = \Sigma, g)$ a Riemannian manifold with boundary. Wang's conjecture 2.1 relies on the boundedness of the trace operator: $H^1(M) \hookrightarrow L_{q+1}(\Sigma)$ for $q \le \frac{n}{n-2}$. The embedding is compact when the inequality is strict, but it weakens as q approaches $\frac{n}{n-2}$, ultimately losing compactness. Consequently, the conjecture becomes more challenging as q increases. For the critical power, the existence of the minimizer is uncertain due to the loss of compactness. Due to this, one might be interested in examining the behavior for small values of q. If Wang's conjecture is true, in its setting we have

$$\int_M |\nabla u|^2 + \lambda \int_\Sigma u^2 \ge |\Sigma|^{\frac{q-1}{q+1}} \lambda (\int_\Sigma u^{q+1})^{\frac{2}{q+1}}$$

for $\lambda(q-1) \le 1$ and $q \le \frac{n}{n-2}$. Let $\lambda = \frac{1}{q-1}$, and we get

$$\frac{q-1}{|\Sigma|} \int_M |\nabla u|^2 + \frac{1}{|\Sigma|} \int_\Sigma u^2 \ge (\frac{1}{|\Sigma|} \int_\Sigma u^{q+1})^{\frac{2}{q+1}}$$

Notice that the equality always holds for $q = 1$. Now for an arbitrary $u \in H^1(M)$ satisfying $\int_\Sigma u^2 = \Sigma$, take the limit $q \to 1$, and we arrive at

$$2 \int_M |\nabla u|^2 \ge \int_\Sigma u^2 \log(u^2) \tag{3.1.1}$$

Consider the functional

$$\begin{gathered} E(u) \coloneqq 2 \int_M |\nabla u|^2 - \int_\Sigma u^2 \log(u^2), \\ \text{where } u \in H^1(M), \ \int_\Sigma u^2 = |\Sigma| \end{gathered} \tag{3.1.2}$$

Then $E(u)$ is bounded below, which will be proved in the next section. Its Euler-Lagrangian equation is

$$\begin{aligned}\Delta u &= 0\\ \frac{\partial u}{\partial \nu} &= u\log u + \lambda u\end{aligned}$$

λ comes from the Lagrangian multiplier. Note that the second equation is not linear, and we could scale u to kill λu to get

$$\begin{aligned}\Delta u &= 0\\ \frac{\partial u}{\partial \nu} &= u\log u\end{aligned} \tag{3.1.3}$$

Just as Wang's conjecture, we can pose the following conjecture

Conjecture 3.1. *Let $(M, \partial M = \Sigma, g)$ be a compact Riemannian manifold with boundary. Suppose $Ric \geq 0$ on M, and $II \geq 1$ on Σ where II is the second fundamental form, then the following PDE*

$$\begin{aligned}\Delta u &= 0 \qquad on \quad M^n\\ \frac{\partial u}{\partial \nu} &= u\log u\end{aligned} \tag{3.1.4}$$

admits no solution other than $u \equiv 1$. Consequently,

$$2\int_M |\nabla u|^2 \geq \int_\Sigma u^2 \log(u^2) \tag{3.1.5}$$

for $u \in H^1(M)$ and $\int_\Sigma u^2 = |\Sigma|$.

3.2 Log-Sobolev Inequality in General Manifold

In this section, I will derive a log-Sobolev inequality for general Riemannian manifolds with boundary. Although a log-Sobolev inequality can be obtained using Theorem 2.4 and a similar argument as in the previous section, I will employ a different approach that provides additional information. These methods are modified from the work of [Rot81a], [Rot81b] and [Rot86], where O.Rothaus studied log-Sobolev inequality for manifolds without boundary.

For arbitrary $\rho > 0$, consider the functional

$$
\begin{aligned}
E_\rho(u) &:= \rho \int_M |\nabla u|^2 - \int_\Sigma u^2 \log(u^2), \\
\text{where } u \in H^1(M), &\int_\Sigma u^2 = |\Sigma| \qquad (3.2.1) \\
s(\rho) &= \inf_{u \in H^1(M)} E_\rho(u)
\end{aligned}
$$

Lemma 3.2.1. *$s_\rho > -\infty$ for any $\rho > 0$, and the infimum can be achieved.*

Proof. Throughout the proof C is a constant independent of and u and might change from line to line. Without loss of generality we might assume $u > 0$. Fix $0 < \epsilon < \frac{2}{n-2}$. Since we assume $\int_\Sigma u^2 dS = 1$ and log is a concave function, we mighe use Jensen's inequality for the measure $\frac{u^2}{|\Sigma|} dS$ and function u^ϵ, and we get

$$
\begin{aligned}
\frac{1}{|\Sigma|} \int_\Sigma u^2 \log u^2 dS &= \frac{2}{\epsilon} \int_\Sigma \log u^\epsilon \big(\frac{u^2}{|\epsilon|} dS\big) \\
&\leq \frac{2}{\epsilon} \log \big(\int_\Sigma \frac{1}{\Sigma} u^{2+\epsilon} dS \big)
\end{aligned}
$$

Use boundedness of trace operator,

$$
\begin{aligned}
\frac{2}{\epsilon} \log \big(\int_\Sigma \frac{1}{\Sigma} u^{2+\epsilon} dS \big) &\leq \frac{2}{\epsilon} \log \big(C \|u\|^{2+\epsilon}_{H^1(M)} \big) \\
&\leq \frac{2+\epsilon}{\epsilon} \log(\|u\|^2_{H^1(M)}) + C \qquad (3.2.2) \\
&\leq \rho \|u\|^2_{H^1(M)} + C
\end{aligned}
$$

In the last line we used that $\alpha x - \log x$ is bounded below in x for any fixed $\alpha > 0$.

The proof demonstrating the achievability of the infimum is standard. We can pick a minimizing sequence u_i.

$$
\begin{aligned}
s(\rho) + 1 &\geq E_\rho(u_i) \\
&= E_{\rho/2}(u_i) + \frac{\rho}{2} \int_M |\nabla u_i|^2 \geq s(\frac{\rho}{2}) + \frac{\rho}{2} \int_M |\nabla u_i|^2
\end{aligned}
$$

So u_i are bounded in $H^1(M)$. By Alauoglu theorem we can pick a subsequence that converges weakly to u in $H^1(M)$, and thus $\int_M |\nabla u|^2 \leq \lim \int_M |\nabla u_i|^2$. We only need to check that $\int_\Sigma u_i^2 \log u_i$ converges. Fix $\epsilon < \frac{1}{n-2}$, and we have $|(x^2 \log x)'| = |(2 \log x + 1)x| < C(1 + x^{1+\epsilon})$ for all

$x > 0$.Therefore

$$
\begin{aligned}
|\int_\Sigma u_i^2 \log u_i - \int_\Sigma u_j^2 \log u_j| &\le C \int_\Sigma |u_i - u_j| \max\{1 + u_i^{1+\epsilon}, 1 + u_j^{1+\epsilon}\} \\
&\le C \int_\Sigma |u_i - u_j| \big(1 + u_i^{1+\epsilon} + 1 + u_j^{1+\epsilon}\big) \\
&\le \|u_i - u_j\|_{L^2(\Sigma)} (1 + \|u_i\|_{L^{2+2\epsilon}} + \|u_j\|_{L^{2+2\epsilon}}) \\
&\le C(1 + \|u_i\|_{H^1(M)} + \|u_j\|_{H^1(M)}) \|u_i - u_j\|_{L^2(\Sigma)} \\
&\le C \|u_i - u_j\|_{L^2(\Sigma)}
\end{aligned}
$$

□

Now we can look at the Euler-Lagrangian equation for $s(\rho)$. By a similar computation as the previous section, we get

$$
\begin{aligned}
&\Delta u = 0 \ \text{ on } M \\
&\frac{\partial u}{\partial \nu} = \lambda u \log u \ \text{ in } \Sigma \quad \text{where } \lambda = \frac{2}{\rho}
\end{aligned}
\tag{3.2.3}
$$

For any $\rho > 0$, the function $s(\rho)$ is bounded and increasing with respect to ρ. Let u_ρ be its minimizer. It is expected that $\int_M |\nabla u_\rho|$ decreases to 0 as ρ approaches infinity. There might exist a critical ρ_0 such that $s(\rho_0)$ is achieved by $u \equiv 1$, enabling the establishment of a log-Sobolev inequality. See section 2 of [Rot81b]. Actually, we have the following stronger theorem.

Theorem 3.1. *There exists a λ_0 so that for $\lambda < \lambda_0$, (3.2.3) admits no solution other than $u \equiv 1$.*

Proof. The proof is similar to theorem 2.4. First we want to bound $\int_\Sigma u$. Integrate (3.2.3) by parts, we get $\int_\Sigma u \log u = 0$. Then

$$
\begin{aligned}
\int_\Sigma u &= \int_{\Sigma \cap \{u < e\}} u + \int_{\Sigma \cap \{u \ge e\}} u \\
&\le e|\Sigma| + \int_{\Sigma \cap \{u \ge e\}} u \log u \\
&= e|\Sigma| + \int_{\Sigma \cap \{u \le e\}} u \log u \le C_1
\end{aligned}
\tag{3.2.4}
$$

Let $K(x, y)$ be the Schwarz kernel for Dirichlet-to-Neumann operator. From section 3.2 we know that for $p < \frac{n-1}{n-2}$, $\|K(x, \cdot)\|_{L^p(\Sigma)} \leq C$ for all $x \in \Sigma$. Let $0 < t < 1$, $p* = \frac{p}{p-1} > 1$ and $M = \sup_{x \in \Sigma} u$

$$\begin{aligned}
|u(x) - \frac{1}{|\Sigma|} \int_{\Sigma} u| = \lambda| \int_{\Sigma} K(x, y) u(y) \log u(y) dy| \\
\leq C\lambda \|u \log u\|_{p*} \\
\leq C\lambda M^t \|u^{1-t} \log u\|_{p*}
\end{aligned}$$

Apparently, there exists a constant $C_2(p*, t)$ such that $x^{1-t} p * (\log x)^{p*} \leq C_2 + x$ for all $x > 0$ provided that $(1-t)p* < 1$, which is achievable. At maximal point, we have

$$\begin{aligned}
M - C_1 &\leq |u(x) - \frac{1}{|\Sigma|} \int_{\Sigma} u| \\
&\leq C\lambda M^t (\int_{\Sigma} C_2 + u) \leq C\lambda M^t
\end{aligned}$$

Therefore u is bounded provided λ is bounded above. Then use lemma 2.2.4, we have

$$\begin{aligned}
\int_M |\nabla u|^2 &\leq C \int_{\Sigma} |\frac{\partial u}{\partial \nu}|^2 \\
&= C\lambda \int_{\Sigma} u \log u \frac{\partial u}{\partial \nu} \\
&\leq C\lambda \int_{\Sigma} u \frac{\partial u}{\partial \nu} = C\lambda \int_M |\nabla u|^2
\end{aligned}$$

Note constant C is independant of λ as long as λ is bounded above. Therefore $\int_M |\nabla u|^2 = 0$ for small λ, thus $u \equiv 1$. □

3.3 Flow Method for Manifolds without Boundary

It's well known that on manifolds without boundary we can solve $u_t = \Delta u$ and u converges to the constant $\frac{1}{|\Sigma|} \int u_0$. If we run this flow and keep track of how $\int |\nabla u|^2$ decreases in t, hopefully we can get something. Actually, this idea works for Gaussian measure $d\mu = \frac{1}{(2\pi)^{n/2}} e^{-\frac{|x|^2}{2}} dx$ on $\mathbb{R}^n$. It's well known that $\int_{\mathbb{R}^n} d\mu = 1$, i.e. $d\mu$ is a probability on $\mathbb{R}^n$. Define

$$\mathbb{E}(f) := \int_{\mathbb{R}^n} f d\mu$$

$$\hat{\Delta} f := \Delta f - \langle x, \nabla f \rangle$$

$$u(t, x) = P_t f := \mathbb{E}_{\zeta} \big(f(e^{-t} x + \sqrt{1 - e^{-2t}} \zeta) \big)$$

Then u defined as above solves $u_t = \hat{\Delta} u$. Using this flow we can show that

Theorem 3.2. *If f is $C^1(\mathbb{R}^n)$, $\mathbb{E}(f) = 1$ and $\mathbb{E}(|\nabla f|)^2 \leq \infty$, then*

$$\mathbb{E}(f^2 \log f^2) \leq 2\mathbb{E}(|\nabla f|^2)$$

This method can be carried to Riemannian manifolds as follows

Theorem 3.3. *Let (M, g) be a compact Riemannian manifold without boundary. Suppose $Ric \geq (n-1)g$, then for all $f \in H^1(M)$, $f > 0$ and $\int_M f = |M|$, we have*

$$\frac{1}{2(n-1)} \int \frac{|\nabla f|^2}{f} \geq \int f \log f$$

If we pick f^2 in the inequality, we get

$$\frac{1}{n-1} \int |\nabla f|^2 \geq \int f^2 \log f$$

Proof. Let u be solutions of

$$\begin{aligned} u_t &= \Delta u \\ u(0, \cdot) &= f \end{aligned} \tag{3.3.1}$$

Then

$$\frac{\partial}{\partial t}(u \log u) = (\log u + 1)u_t = (\log u + 1)\Delta u$$
$$\frac{\partial}{\partial t} \int_M u \log u = -\int_M \frac{|\nabla u|^2}{u}$$

Note that $u \longrightarrow \frac{\int_M f}{|M|} = 1$ in $H^1(M)$. Integrate in time,

$$\int_0^\infty \int_M \frac{|\nabla u|^2}{u} dV dt = \int_M f \log f \tag{3.3.2}$$

Use Bochner's formula, we compute

$$\begin{aligned} \frac{\partial}{\partial t}|\nabla|^2 = 2\langle \nabla u_t, \nabla u \rangle &= 2\langle \nabla \Delta u, \nabla u \rangle \\ &= \Delta|\nabla u|^2 - 2|\nabla^2 u|^2 - 2Ric(\nabla u, \nabla u) \\ &\leq \Delta|\nabla u|^2 - 2|\nabla^2 u|^2 - 2(n-1)|\nabla u|^2 \end{aligned} \tag{3.3.3}$$

Let $v = \sqrt{u}$, then

$$4\frac{\partial}{\partial t}|\nabla v|^2 = \frac{\partial}{\partial t}\frac{|\nabla u^2|}{u} = \frac{1}{u}\frac{\partial |\nabla u|^2}{\partial t} - \frac{\Delta u|\nabla u|^2}{u^2}$$

$$\begin{aligned} 4\Delta|\nabla v|^2 &= \Delta\frac{|\nabla u|^2}{u} \\ &= \frac{\Delta|\nabla u|^2}{u} - 2\frac{\langle\nabla|\nabla u|^2, \nabla\rangle}{u^2} + |\nabla u|^2\Delta\frac{1}{u} \\ &= \frac{\Delta|\nabla u|^2}{u} - 2\frac{\langle\nabla|\nabla u|^2, \nabla\rangle}{u^2} + 2\frac{|\nabla u|^4}{u^3} - \frac{\Delta u|\nabla u|^2}{u^2} \end{aligned}$$

Take difference between the two equations above and use (3.3.3),

$$\begin{aligned} 4(\frac{\partial}{\partial t} - \Delta)|\nabla v|^2 &\leq -\frac{2(n-1)|\nabla u|^2}{u} - \frac{2|\nabla^2 u|^2}{u} + 2\frac{\langle\nabla|\nabla u|^2, \nabla\rangle}{u^2} - 2\frac{|\nabla u|^4}{u^3} \\ &= -\frac{2(n-1)|\nabla u|^2}{u} - \frac{2}{u}\sum_{1\leq i,j\leq n}\big(u_{ij}^2 - 2\frac{u_{ij}u_iu_j}{u} + \frac{u_i^2u_j^2}{u^2}\big) \\ &= -\frac{2(n-1)|\nabla u|^2}{u} - \frac{2}{u}\sum_{1\leq i,j\leq n}|u_{ij} - \frac{u_iu_j}{u}|^2 \\ &\leq -\frac{2(n-1)|\nabla u|^2}{u} \end{aligned} \tag{3.3.4}$$

Integrate this inequality in both space and time,

$$\begin{aligned} \int_0^\infty\int_M -\frac{2(n-1)|\nabla u|^2}{u} &\geq \int_0^\infty\int_M 4(\frac{\partial}{\partial t} - \Delta)|\nabla v|^2 dVdt \\ &= \int_0^\infty\int_M 4\frac{\partial}{\partial t}|\nabla v|^2 dVdt \\ &= \int_0^\infty \frac{\partial}{\partial t}\big(\int_M 4|\nabla v|^2 dV\big)dt \\ &= \lim_{t\to\infty}\int_M 4|\nabla v(t,\cdot)|^2 dV - \int_M 4|\nabla v(0,\cdot)|^2 dV \\ &= -\int_M \frac{|\nabla f|^2}{f}dV \end{aligned} \tag{3.3.5}$$

Combine (3.3.3) and (3.3.5), we get desired result. □

3.4 Sectional Curvature Results

Let $\rho(x) = d(x, \sigma)$ be the distance from the boundary. It's smooth away from the cut locus $\mathrm{Cut}(\Sigma)$, which is a closed set in the interior of M and is of measure zero. Consider $\psi := \rho^2 - \frac{\rho^2}{2}$. If (M, Σ, g) is assumed to have non-negative sectional curvature and $II \geq 1$, then by the Hessian comparison theorem (cf. [Kas82]),

$$-\nabla^2\phi \geq g$$

Furthermore, ψ has nice property near the boundary

$$\begin{aligned}\psi_\Sigma &= 0\\ \frac{\partial\psi}{\partial\nu} &= -1\end{aligned}$$

These prove advantageous when we use $\nabla\phi$ as the testing field in (2.4.7). But the problem is cut locus. To overcome this difficulty, in [XX19] the C.Xia and C.Xiong has the following construction.

Theorem 3.4. *Suppose (M, Σ, g) has non-negative sectional curvature and $II \geq 1$. Fix a neighborhood C of $Cut(\Sigma)$ in the interior of M. Then for any $\epsilon > 0$, there exists a smooth non-negative function ψ_ϵ on M such that $\psi_\epsilon = \phi$ on $M \setminus C$ and*

$$-\nabla^2\psi_\epsilon \geq (1-\epsilon)g \tag{3.4.1}$$

In [GHW19], the authors use this function in Wang's conjecture and get the following

Theorem 3.5 (Q.Guo, F.Hang and X.Wang)**.** *Let (M, Σ, g) be as in Wang's conjecture. Then the only positive solutions to (2.1.1) is constant if $(q-1)\lambda \leq 1$ provided $2 \leq n \leq 8$ and $1 < q \leq \frac{4n}{5n-9}$. Consequently,*

$$\frac{q-1}{|\Sigma|}\int_M |\nabla u|^2 + \frac{1}{|\Sigma|}\int_\Sigma u^2 \geq (\frac{1}{|\Sigma|}\int_\Sigma u^{q+1})^{\frac{2}{q+1}} \tag{3.4.2}$$

for these (q, λ).

Their method also works for (3.1.3).

Theorem 3.6. *Let (M, Σ, g) be as in Wang's conjecture. Suppose $2 \leq n \leq 8$ Then the only positive solution to (3.1.3) is $u \equiv 1$.*

Proof. Let u be a solution to (3.1.3). Let a, b bw two constants that will be determined later. Set $u = v^{-a}$, then

$$\begin{aligned}\Delta v &= (a+1)\frac{|\nabla v|^2}{v}\\ \frac{\partial v}{\partial \nu} &= v\log v\end{aligned} \tag{3.4.3}$$

We have the following two lemmas from [GHW19]:

Lemma 3.4.1. *Suppose $\phi|_\Sigma = 0$ and $\frac{\partial \phi}{\partial \nu} = -1$, then for any smooth v and $b \in \mathbb{R}$*

$$\begin{aligned}
&\int_M (1-\frac{1}{n})(\Delta v)^2 v^b \phi + \frac{b}{2}\phi v^{b-2}|\nabla v|^2\big(3v\Delta v + (b-1)|\nabla v|^2\big) \\
&= \int_M v^b \nabla^2\phi(\nabla v, \nabla v) - |\nabla v|^2 v^b \Delta\phi - \frac{b}{2}|\nabla v|^2 v^{b-1}\langle \nabla v, \nabla \phi\rangle \\
&\quad + \big(|\nabla^2 v - \frac{\Delta v}{n} g|^2 + Ric(\nabla v, \nabla v)\big) v^b \phi - \int_\Sigma v^b |\nabla_\Sigma v|^2
\end{aligned} \tag{3.4.4}$$

Lemma 3.4.2. *The proof of the first lemma is similar to that of usual Reilly's formula, and the proof of the second one is based on Pohozaev identity (2.4.7). Under the same assumptions as in lemma 3.4.1, we have*

$$\begin{aligned}
&\int_M v^b \nabla^2\phi(\nabla v, \nabla v) + (v\Delta v + \frac{b}{2}|\nabla v|^2) v^{b-1}\langle \nabla v, \nabla\phi\rangle - \frac{1}{2} v^b |\nabla v|^2 \Delta\phi \\
&= \frac{1}{2}\int_\Sigma v^b (|\nabla_\Sigma v|^2 - (\frac{\partial v}{\partial \nu})^2)
\end{aligned} \tag{3.4.5}$$

Apply these two lemmas for v in (3.4.3), we get respectively

$$\begin{aligned}
Q :=& \quad \big(|\nabla^2 v - \frac{\Delta v}{n} g|^2 + Ric(\nabla v, \nabla v)\big) v^b \phi \\
=& \big((1-\frac{1}{n})(a+1)^2 + \frac{b(3a+b+2)}{2}\big)\int_M v^{b-2}|\nabla v|^4 \phi \\
&+ \int_M -v^b \nabla^2\phi(\nabla v, \nabla v) + |\nabla v|^2 v^b \Delta\phi + \frac{b}{2}|\nabla v|^2 v^{b-1}\langle\nabla v, \nabla\phi\rangle + \int_\Sigma v^b |\nabla_\Sigma v|^2
\end{aligned} \tag{3.4.6}$$

and

$$\begin{aligned}
&\int_M v^b \nabla^2\phi(\nabla v, \nabla v) + \big(a + 1 + \frac{b}{2}\big)\nabla v|^2 v^{b-1}\langle \nabla v, \nabla\phi\rangle - \frac{1}{2} v^b |\nabla v|^2 \Delta\phi \\
&= \frac{1}{2}\int_\Sigma v^b (|\nabla_\Sigma v|^2 - (\frac{\partial v}{\partial \nu})^2)
\end{aligned} \tag{3.4.7}$$

Combine these two equalities to eliminate terms involving $\langle \nabla v, \nabla \phi\rangle$, we get

$$\begin{aligned}
Q =& \big((1-\frac{1}{n})(a+1)^2 + \frac{b(3a+b+2)}{2}\big)\int_M v^{b-2}|\nabla v|^4\phi \\
&+ \int_M -\frac{a+1+b}{a+1+b/2} v^b \nabla^2\phi\langle\nabla v, \nabla v\rangle + \frac{a+1+3b/4}{a+1+b/2}|\nabla v|^2 v^b \Delta\phi \\
&+ \int_\Sigma -\frac{b/4}{a+1+b/2} v^b (\frac{\partial v}{\partial \nu})^2 + \frac{a+1+3b/4}{a+1+b/2} v^b |\nabla_\Sigma v|^2
\end{aligned}$$

Set $a + 1 + \frac{3b}{4} = 0$ to eliminate terms involving $\Delta\phi$ and $|\nabla_\Sigma v|^2$, and take ϕ to be ϕ_ϵ as in theorem 3.4, we get

$$\begin{aligned} Q_\epsilon \leq & \left(\frac{(5n-9-(n+9)a)(a+1)}{9n}\right) \int_M v^{b-2}|\nabla v|^4 \psi_\epsilon \\ & -(1-\epsilon) \int_C v^b |\nabla v|^2 + \int_{M\setminus C} v^b \nabla^2 \psi(\nabla v, \nabla v) + \int_\Sigma v^b (\frac{\partial v}{\partial \nu})^2 \end{aligned}$$

where

$$Q_\epsilon := \int_M \big(|\nabla^2 v - \frac{\Delta v}{n} g|^2 + Ric(\nabla v, \nabla v)\big) v^b \psi_\epsilon$$

Now let $\epsilon \to 0$ and then shrink C. Notice that $\Delta\phi \leq -g$ whenever its smooth. It yields

$$\begin{aligned} Q \leq & \left(\frac{(5n-9-(n+9)a)(a+1)}{9n}\right) \int_M v^{b-2}|\nabla v|^4 \psi_\epsilon \\ & - \int_M v^b |\nabla v|^2 + \int_\Sigma v^b (\frac{\partial v}{\partial \nu})^2 \end{aligned} \tag{3.4.8}$$

where

$$Q := \int_M \big(|\nabla^2 v - \frac{\Delta v}{n} g|^2 + Ric(\nabla v, \nabla v)\big) v^b \psi$$

Compute $\int_\Sigma v^b (\frac{\partial v}{\partial \nu})^2$ as follows

$$\begin{aligned} \int_\Sigma v^b (\frac{\partial v}{\partial \nu})^2 &= \lambda \int_\Sigma v^{b+1} \log v \frac{\partial v}{\partial \nu} \\ &= \lambda \int_M v^{b+1} \log v \Delta v + (b+1) v^b \log v |\nabla v|^2 + v^b |\nabla v|^2 \\ &= \lambda \int_M (a+b-2) v^b \log v |\nabla v|^2 + v^b |\nabla v|^2 \end{aligned}$$

Plug this equality in (3.3.5),

$$\begin{aligned} Q \leq & \left(\frac{(5n-9-(n+9)a)(a+1)}{9n}\right) \int_M v^{b-2}|\nabla v|^4 \psi_\epsilon \\ & + (\lambda - 1) \int_M v^b |\nabla v|^2 + \lambda(a+b-2) \int_M v^b \log v |\nabla v|^2 \end{aligned}$$

We want $a+b-2 = 0$ since we don't know the sign for $\int_M v^b \log v |\nabla v|^2$. Together with $a+1+\frac{3b}{4} = 0$, we get $a = 2,\ b = -4$. Additionally, we aim for $\frac{(5n-9-(n+9)a)(a+1)}{9n}$, which imposes the condition $n \leq 9$. This completes our theorem. □

Corollary 3.1. *Under the same assumptions,*

$$2 \int_M |\nabla u|^2 \geq \int_\Sigma u^2 \log(u^2)$$

for $u \in H^1(M)$ and $\int_\Sigma u^2 = |\Sigma|$

Remark 3.4.1. *If we take derivative with respect to q at $q = 1$ for (3.4.2), just as we did in section 4.1, we get the desired log-Sobolev inequality (3.1.1). But using their method, we also proves non-existence of non-constant solutions to (3.1.3), which is stronger.*

In [GHW19], the authors applied maximal principle for $n = 2$ and proved the following

Theorem 3.7 (Q.Guo, F.Hang and X.Wang)**.** *Let (M, Σ, g) be as in Wang's conjecture and $n = 2$. Then the only positive solutions to (2.1.1) is constant if $(q-1)\lambda \leq 1$ provided $q \geq 2$.*

This maximal principle also works for our case. But since it's fully covered by the previous result, I won't include it here.

3.5 Ricci Curvature Results

An obstacle in both Wang's Conjecture 2.1 and Conjecture 3.1 is the lack of a comprehensive understanding of how Ricci curvature affects the Dirichlet-to-Neumann operator. Although some partial results have been obtained under the assumption of sectional curvature ≥ 0, as discussed in the previous section and presented in [GHW19], no progress has been made under the condition $Ric \geq 0$. In this section, I will present a result in this direction.

Theorem 3.8. *(M^n, Σ, g) a Riemannian manifold with boundary. Suppose $n \leq 8$, $Ric \geq 0$ on M, $II \geq g_\Sigma$ and $Ric_\Sigma \geq (n-2)g_\Sigma$ on Σ, then there exists a λ_0 that only depends on the dimension so that*

$$\begin{aligned} \Delta u &= 0 \qquad\qquad on \quad M^n \\ \frac{\partial u}{\partial \nu} &= \lambda u \log u \quad on \quad \Sigma^{n-1} \\ \int_\Sigma u^2 dS &\leq A(\Sigma) \end{aligned} \tag{3.5.1}$$

admit no non-constant solution provided $\lambda \leq \lambda_0$

Proof. Let $u = v^{-\beta}$, then v satisfy the following

$$\begin{aligned} \Delta v &= (1+\beta)\frac{|\nabla v|^2}{v} \qquad \text{on} \quad M^n \\ \frac{\partial v}{\partial \nu} &= \lambda v \log v \qquad\quad \text{on} \quad \Sigma^{n-1} \end{aligned} \tag{3.5.2}$$

Define $E_{ij} = v_{ij} - \frac{\Delta v}{n}\delta_{ij}$ and $L_{ij} = \frac{v_i v_j}{v} - \frac{|\nabla v|^2}{nv}\delta_{ij}$. From the work of [LO23], we have

$$(v^a E_{ij} v_i)_j \geq v^a [E_{ij} + \frac{a + 2(\beta+1)\frac{n-1}{n}}{2} L_{ij}]^2 + c v^{a-2} |\nabla v|^4 \coloneqq Q \tag{3.5.3}$$

where

$$c = \frac{n-1}{n}(\beta+1)\frac{2+2\beta-n}{n} - \frac{n-1}{4n}[a + 2(\beta+1)\frac{n-1}{n}]^2 \tag{3.5.4}$$

In [LO23] Ou and Lin work on the unit ball. The calculation is essentially the same, and the only difference is that we used Bochner formula and finally get an inequality. Pick a frame $\{e\}_{1\leq\alpha\leq n-1}$ along the boundary, and let $e_n = \nu$ be the outer normal. Integrate by parts, and we have

$$\begin{aligned} \int_M \operatorname{div}\big(v^a E(\nabla v, \cdot)\big)\mathrm{dV} &= \int_\Sigma v^a E(\nabla v, \nu)\mathrm{dS} \\ &= \int_\Sigma v^a E(\nabla_\Sigma v, \nu)\mathrm{dS} + \int_\Sigma v^a E(v_n \nu, \nu)\mathrm{dS} = A + B \end{aligned} \tag{3.5.5}$$

We calculate A and B as follows.

$$\begin{aligned} A &= \int_\Sigma v^a \nabla^2 v(\nabla_\Sigma v, \nu)\mathrm{dS} \\ &= \int_\Sigma v^a \langle \nabla_\Sigma v, \nabla_\Sigma v_n \rangle - v^a \mathrm{II}(\nabla_\Sigma v, \nabla_\Sigma v)\mathrm{dS} \\ &\leq \int_\Sigma \lambda v^a \log v |\nabla_\Sigma v|^2 + (\lambda - 1) v^a |\nabla_\Sigma v|^2 \mathrm{dS} \end{aligned} \tag{3.5.6}$$

As for B, we have $\Delta = \Delta_\Sigma + (\frac{\partial}{\partial \nu})^2 + H\frac{\partial}{\partial \nu}$ on Σ. Using (3.5.2), B can be calculated as

$$
\begin{aligned}
B &= \int_\Sigma v^a v_n (v_{nn} - \frac{\Delta v}{n})\mathrm{dS} \\
&= \int_\Sigma v^a v_n \Big(\frac{(n-1)(1+\beta)}{n} \frac{|\nabla v|^2}{v} - \Delta_\Sigma v - H v_n \Big) \\
&= \int_\Sigma \frac{(n-1)(1+\beta)}{n} v^{a-1} v_n (|\nabla_\Sigma v|^2 + v_n^2) - H v^a v_n^2 - v^a v_n \Delta_\Sigma v \mathrm{dS} \\
&\leq \int_\Sigma \frac{(n-1)(1+\beta)}{n} v^{a-1} v_n (|\nabla_\Sigma v|^2 + v_n^2) - (n-1) v^a v_n^2 - v^a v_n \Delta_\Sigma v \mathrm{dS} \\
&= \int_\Sigma \lambda (a + 1 + \frac{(\beta+1)(n-1)}{n}) v^a \log v |\nabla_\Sigma v|^2 - (n-1)\lambda^2 v^{a+2} \log^2 v \\
&\quad + \lambda^3 \frac{(\beta+1)(n-1)}{n} v^{a+2} \log^3 v + \lambda v^a |\nabla_\Sigma v|^2 \mathrm{dS}
\end{aligned}
\tag{3.5.7}
$$

Combine (3.5.5), (3.5.6) and (3.5.7), we have

$$
\begin{aligned}
Q &\leq \int_\Sigma \lambda (a + 2 + \frac{(\beta+1)(n-1)}{n}) v^a \log v |\nabla_\Sigma v|^2 - (n-1)\lambda^2 v^{a+2} \log^2 v \\
&\quad + \lambda^3 \frac{(\beta+1)(n-1)}{n} v^{a+2} \log^3 v + (2\lambda - 1) v^a |\nabla_\Sigma v|^2 \mathrm{dS}
\end{aligned}
\tag{3.5.8}
$$

Set $x = \frac{(1+\beta)(n-1)}{n}$, and $a = -2 - x$ to kill the first term on the right hand side of the above inequality. It becomes

$$
Q \leq \int_\Sigma -(n-1)\lambda^2 v^{-x} \log^2 v + \lambda^3 x v^{-x} \log^3 v + (2\lambda - 1) v^{-x-2} |\nabla_\Sigma v|^2 \mathrm{dS} \tag{3.5.9}
$$

We further require $P \geq 0$ to make sure $Q \geq 0$. (3.5.4) becomes

$$
P(x) = -\frac{n^2 - 10n + 1}{4n(n-1)} x^2 - \frac{1}{n} x - \frac{n-1}{n} \tag{3.5.10}
$$

Next we estimate the middle term in (3.5.9) and lower the power for $\log^3 v$.

$$
\begin{aligned}
\lambda^3 x \int_\Sigma v^{-x} \log^3 v \mathrm{dS} &= \lambda^2 x \int_\Sigma v^{-x-1} \log^2 v v_n \mathrm{dS} \\
&= \lambda^2 x \int_M \Big((-x-1) \log^2 v + 2 \log v \Big) v^{-x-1} |\nabla_M v|^2 \mathrm{dV}
\end{aligned}
$$

We pick $x > 0$ so that $-x - 1 < 0$. Using $(-x-1)\log^2 v + 2\log v \leq c$ for constant $c > 0$. Note that a only depends on dimension, and therefore c. So the above equality becomes

$$
\begin{aligned}
\lambda^3 x \int_\Sigma v^{-x} \log^3 v \mathrm{dS} &\leq \lambda^2 cx \int_M v^{-x-2} |\nabla_M v|^2 \\
&= \frac{cx\lambda^3}{-x+\beta} \int_\Sigma v^{-x} \log v \\
&= \frac{c(n-1)x\lambda^3}{x-n-1} \int_\Sigma v^{-x} \log v \mathrm{dS}
\end{aligned} \tag{3.5.11}
$$

The second equality can be derived if we multiply (3.5.2) by v^{-x-1} and integrate by parts, which gives $\lambda \int_\Sigma v^{-x} \log v = (-x+\beta) \int_M v^{-x-2} |\nabla_M v|^2$. Let $w = v^{-x/2}$. Then (3.5.9) and (3.5.11) give us

$$
Q \leq \frac{4}{x^2} \int_\Sigma (2\lambda - 1) |\nabla_\Sigma w|^2 - (n-1)\lambda^2 w^2 \log^2 w - \frac{cx^2\lambda^3}{2(x-n-1)} w^2 \log w \mathrm{dS} \tag{3.5.12}
$$

Finally, we want to bound $\int_\Sigma w^2 \log w \mathrm{dS}$ by $\int_\Sigma |\nabla_\Sigma w|^2 \mathrm{dS}$ using theorem 3.3 since the coefficient for $w^2 \log w$ is positive and $Ric_\Sigma \geq g_\Sigma$ by our assumption, where Ric_Σ is Ricci curvature on Σ. Before that, let us figure out the sign for $\log \frac{\int_\Sigma w^2}{A(\Sigma)}$. $w^2 = u^{\frac{x}{\beta}}$, and $\frac{x}{\beta} < 2$ provided $x \geq \frac{2(n-1)}{n+1}$. By Hölder inequality for $\frac{\mathrm{dS}}{A(\Sigma)}$ and our assumption that $\int_\Sigma u^2 \leq A(\Sigma)$, $\frac{\int_\Sigma w^2}{A(\Sigma)} \leq 1$, and therefore the tail term in theorem 3.3 could be ignored. Now (3.5.12) is

$$
Q \leq \frac{4}{x^2} \int_\Sigma (2\lambda - 1 - \frac{c(n-1)x^2\lambda^3}{2(x-n-1)}) |\nabla_\Sigma w|^2 - (n-1)\lambda^2 w^2 \log^2 w \tag{3.5.13}
$$

where $\frac{2(n-1)}{n+1} \leq x < n-1$ and $P(x) \geq 0$. If we put $x = n-1$, $P(n-1) = \frac{(9-n)(n-1)^2}{4n}$, so admissible x could be found provided $2 \leq n \leq 8$. After picking such a x that only depends on dimension, c in (3.5.11) is also determined. If $\lambda > 0$ is small enough, we have from (3.5.13) that $0 \leq 0$ and therefore $u \equiv 1$. $\square$

Corollary 3.2. *Let (M, Σ, g) as in **theorem 3.8**, then*

$$
\int_M |\nabla u|^2 dV \geq \lambda_0 \int_\Sigma u^2 \log u dS \tag{3.5.14}
$$

for $\int_\Sigma u^2 dS = A(\Sigma)$.

Proof. For each $\lambda > 0$, we can show that

$$\int_M |\nabla u|^2 \mathrm{dV} - \lambda \int_\Sigma u^2 \log u \mathrm{dS} \tag{3.5.15}$$

is bounded below for $\int_\Sigma u^2 = A(\Sigma)$ and the infimum can be achieved. Let a_λ be the infimum and u be the minimizer. From the Euler-Lagrangian equation, we have

$$\begin{aligned} \Delta u &= 0 \quad \text{on } M \\ \frac{\partial u}{\partial \nu} &= \lambda u(\log u + a_\lambda) \quad \text{on } \Sigma \\ \int_\Sigma u^2 \mathrm{dS} &= A(\Sigma) \end{aligned}$$

We can scale to get rid of a_λ, namely take $v = e^{a_\lambda} u \leq u$. The equation for v is

$$\begin{aligned} \Delta v &= 0 \quad \text{on } M \\ \frac{\partial v}{\partial \nu} &= \lambda v \log v \quad \text{on } \Sigma \\ \int_\Sigma v^2 \mathrm{dS} &\leq \int_\Sigma u^2 \mathrm{dS} = A(\Sigma) \end{aligned}$$

The last inequality holds because as the infimum, $a_\lambda \leq 0$. For $\lambda \leq \lambda_0$, v is constant by theorem 3.8, and so is u. Therefore $a_\lambda = 0$ and the proof is done. □

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