PERIOD-INDEX PROBLEMS AND WILD RAMIFICATION

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ABSTRACT

This thesis consists of two main parts. The first part addresses period-index problems and symbol length problems of the p^∞-torsion part of Brauer groups of henselian discretely valued fields with residue fields of characteristic $p > 0$.

In Chapter 1, we provide an overview of the background, progress, and motivation behind period-index problems of Brauer groups and, more generally, Kato's groups. Chapter 2 recalls some properties of Brauer groups, especially the p-torsion part. In Chapter 3, we survey Kato's unit group filtration and Kato's Swan conductors, which are the main tools in this research area. We investigate the symbol length problems of certain groups related to absolute logarithmic differential forms over fields of characteristic $p > 0$. This symbol length problem plays an important role in the period-index problems of Kato's groups.

Chapter 4 presents a systematic investigation of period-index problems of the p-torsion part of Brauer groups of henselian discretely valued fields with residue fields of characteristic $p > 0$. We provide positive support for Chipchakov's conjecture on this topic. Assuming a conjecture on the symbol length, we offer a complete proof of Chipchakov's conjecture on the Brauer p-dimension of henselian discretely valued fields. We also generalize this idea to investigate the symbol length problem of higher Kato's groups, yielding results on the splitting dimension problems.

In Chapter 5, we use Kato's Swan conductor to investigate the period-index problem of the p-torsion part of Brauer groups of semiglobal fields. Semiglobal fields are intermediate entities between local fields and global fields. Using patching methods, we reduce the period-index problems to two types: period-index problems of henselian discretely valued fields and quotient fields of a complete local ring of Krull dimension 2. To study the second type, we employ a Gersten-type exact sequence of logarithmic de Rham cohomology with support, analogous to the Artin-Mumford ramification sequence. Both sequences are derived from the Bloch-Ogus spectral sequence. We compute the logarithmic de Rham cohomology with support and their connecting morphisms in this context. Using these computations, we obtain partial results on the period-index problem of semiglobal fields in characteristic $p > 0$.

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CHAPTER 1

INTRODUCTION

1.1 Motivation

Let k be a field. For any k-central simple algebra A, we denote by $\mathrm{per}(A)$ the order of its class in the Brauer group $\mathrm{Br}(k)$ (called the *period*) and by $\mathrm{ind}(A)$ its *index* which is the gcd of all the degrees of finite splitting fields. It is well-known that

$$\mathrm{per}(A) \mid \mathrm{ind}(A),$$

and these two integers have the same prime factors. Hence the period is bounded by the index and the index is bounded above by a power of the period. We use notion of the Brauer dimension to make this relationship precise. For a prime p, define the *Brauer dimension at p* as follows

$$\mathrm{Br.dim}_p(k) := \min_d \begin{cases} \mathrm{ind}(A) \mid \mathrm{per}(A)^d & \text{for any } A \in \mathrm{Br}(k)[p^n] \text{ and } n \in \mathbb{N}; \\ \infty & \text{otherwise.} \end{cases}$$

Then define the Brauer dimension of k to be

$$\mathrm{Br.dim}(k) = \sup_p \left\{ \mathrm{Br.dim}_p(k) \right\}.$$

In general, the Brauer dimension of a field can be finite or infinite. The period-index problem of a field is to investigate the Brauer dimension of the field.

An important class of fields for the period-index problem is that of C_m fields. For any positive integer m, we say a field k satisfies condition C_m if every homogeneous polynomial $f \in k[x_1, \cdots, x_n]$ of degree d with $d^m < n$ has a nontrivial zero in k^n [27]. Here are some properties of C_m fields:

1. If a field is C_m, then any finite extension is also C_m.
2. If a field is C_m, then any extension of transcendence degree n is C_{m+n}.
3. If a field k is C_m, then $k((t))$ the field of Laurent series is C_{m+1}.
4. The Brauer group of a C_1 field is 0. [37]

The examples of C_m fields include:

C_0 fields These are precisely the algebraically closed fields.

C_1 fields (quasi-algebraically closed fields)

- Finite fields
- The maximal unramified extension of a complete discretely valued field with a perfect residue field
- Complete discretely valued fields with algebraically closed residue fields.

C_m fields If V is a variety of dimension m over an algebraically closed field k, then the function field $k(V)$ is C_m.

Michael Artin [3] conjectured that $\text{Br.dim}(k) \leq 1$ for a C_2 field k. This conjecture has been proved in many cases by several authors. This conjecture has a natural extension to all C_m fields.

Conjecture 1.1.1

Let k be a C_m field. Then $\text{Br.dim}(k) \leq m - 1$.

It's important to note that the p-adic fields are not C_2. Guy Terjanian [42] identified the p-adic examples for all p that are not C_2. However, the Brauer dimension of p-adic fields (local fields) remains 1.

Many recent studies on the period-index problem have been inspired by this conjecture. The advances in tackling the period-index problems were reviewed [31] by the author. We briefly mention some of these here.

(i) For F a local or global field, $\text{Br.dim}(F) = 1$ (Albert-Brauer-Hasse-Noether [17]).

(ii) For a C_2 field F, $\text{Br.dim}_2(F) = \text{Br.dim}_3(F) \leq 1$ (Artin [3]).

(iii) For a finitely generated field F of transcendence degree 2 over an algebraically closed field, $\text{Br.dim}(F) = 1$ (de Jong [16], de Jong-Starr [40], and Lieblich [28]).

(iv) For a finitely generated field F of transcendence degree 1 over an l-adic field, $\text{Br.dim}_p(F) = 2$ for every prime $p \neq l$ (Saltman [36]).

(v) If F is a henselian discretely valued field with residue field k such that $\text{Br.dim}_p(l) \leq d$ for all finite extension l/k and all primes $p \neq \text{char}(k)$, then $\text{Br.dim}_p(F) \leq d + 1$ for all primes $p \neq \text{char}(k)$. (Harbater, Hartmann and Krashen [19]).

By looking at these recent works, we notice that the Brauer p-dimension of a field F is understood systematically when p is not equal to the residual characteristic of F. The main difficulty in the case that p coincides with the residual characteristic of F is caused by the *wild ramification* behavior as we explain now.

Let K be a henselian discretely valued field with the residue field F such that $\mathrm{char}(F) = p > 0$. Recall that every central simple algebra A over K is split by a finite separable extension of K with degree $\mathrm{ind}(A)$. We can understand the ramification behavior of a Brauer class through the ramification behavior of its separable splitting fields. A finite field extension L of K is called *tame* [43], if the residue field extension is separable and the ramification degree is invertible in the residue field F.

Let l be a prime different from p and $\omega \in \mathrm{Br}(K)[l]$. Then ω is split by a separable extension L of K with degree l^m for some $m \in \mathbb{N}$. Since $p \nmid l^m$, the ramification index and the residual degree of L/K are both prime to p. Hence, the splitting field L of ω is tame. When we work for the Brauer dimension away from p, we only need to deal with the tame extensions. When a Brauer class is split by a tame extension, we call it *tamely ramified*.

However, when we work with p-primary torsion Brauer classes over K, there are Brauer classes not split by any tame extensions.

Example 1.1.2

Let $K = \mathbb{F}_p((s))((t))$ be the field of iterated Laurent series over $\mathbb{F}_p$ in variables (s,t) with the complete discrete valuation given by the uniformizer t. The residue field of K is $F = \mathbb{F}_p((s))$.

Consider the K-central division algebra $[\frac{s}{t^p}, t) := \{\langle s,t\rangle \mid x^p - x = \frac{s}{t^p},\ y^p = t,\ y^{-1}xy = x+1\}$. It has the maximal order $B = \bar{O}_K\langle 1, tx, y, txy\rangle$. The residue division ring of B is, in fact, a purely inseparable extension of F given by $F[\bar{tx}]$. We can show there is no tame extension of K which splits $[\frac{s}{t^p}, t)$.

We will call these Brauer classes *wildly ramified*. Moreover, the wildly ramified Brauer classes are general members in the p-torsion part of Brauer groups, since the tamely ramified Brauer classes only form a subgroup of the p-torsion part of the Brauer group of F.

To investigate the Brauer p-dimension of a field K of residual characteristic $p > 0$, we need to interpret the p-torsion part of the Brauer group of K. There are two cases: equal characteristic case and mixed characteristic case. First, suppose that K is a field of characteristic $p > 0$. Then the p-torsion part of $\mathrm{Br}(K)$ is related to the logarithmic differential form $\Omega^1_{K,\log}$. In fact, we have the following

$$\mathrm{Br}(K)[p] \cong H^1_{\text{ét}}(K, \Omega^1_{K,\log}) \cong \Omega^1_K / \left(\mathcal{P}(\Omega^1_K) + d(K)\right),$$

where $\mathcal{P} : \Omega^1_K \to \Omega^1_K/d(K)$, $a\mathrm{dlog}(b) \mapsto (a^p - a)\mathrm{dlog}(b)$, and $d : K \to \Omega^1_K$ is the universal derivation.

Second, if K is a henselian discretely valued field of characteristic 0 containing a primitive p-th root of the unity, with the residue field F of characteristic $p > 0$, by the Bloch-Kato theorem [7], we have the following

$$\mathrm{Br}(K)[p] = H^2_{\text{ét}}(K, \mu_p) \cong H^2_{\text{ét}}(K, \mu_p^{\otimes 2}) \cong K^M_2(K)/p,$$

where μ_p is the group of p-th roots of the unity. This also holds for any field K of characteristic 0 containing the primitive p-th root of the unity, which is implied by the norm residue theorem proved by Voevodsky [45].

Now, in both cases, we can write a p-torsion Brauer classes as a sum of symbols. In the equal characteristic case, symbols are differential forms. In the mixed characteristic case, symbols are elements in the second Milnor K-group. Therefore, it leads to understand symbol algebras and associated symbol length problems. We will talk about them in Chapter 3.

To analyse the wild ramified Brauer classes over a henselian discretely valued field K, Kato defined an increasing filtration $\{M_i\}_{i\in\mathbb{N}}$ on $\mathrm{Br}(K)[p]$. Let $\alpha \in \mathrm{Br}(K)[p]$. Then we can define *Kato's Swan conductor* $\mathrm{sw}(\alpha)$ of α to be the minimal integer n such that $\alpha \in M_n$. Kato also described the consecutive quotients of this filtration. It is the fundamental tool to analyse the p-torsion part of Brauer groups of henselian discretely valued fields. We will also review them in Chapter 3.

1.2 Case: henselian discretely valued fields

Let us focus on henselian discretely valued fields first. Let K be a henselian discretely valued field with the residue field F of characteristic $p > 0$ and q be a prime number. If $q \neq p$, it is proved that $\mathrm{Br.dim}_q(K) \leq d + 1$ if $\mathrm{Br.dim}_q(F) \leq d$ for all finite extension E/F, by Harbater, Hartmann and Krashen [19]. We hope that similar results also hold when $q = p$. However, for any $n \geq 0$, there are examples of complete discretely valued field K with $\mathrm{Br.dim}_p(K) \geq n$ and $\mathrm{Br.dim}_p(F) = 0$ by Parimala and Suresh [2014].

In fact, there are bounds for the Brauer p-dimension of K in terms of the p-rank of F. If the p-rank of F is $n < \infty$, i.e. $[F : F^p] = p^n$, the Brauer p-dimension of F is no more than n [11, Corollary 3.4]. Moreover, Chipchakov proved that $\mathrm{Br.dim}_p(K) \geq n$ if $[F : F^p] = p^n$ and $\mathrm{Br.dim}_p(K)$ is infinite if and only if F/F^p is an infinite extension [13].

Conjecture 1.2.1 ([1])

Let K be a henselian discretely valued field with residue field F of characteristic $p > 0$. Assume that $[F : F^p] = p^n$. Then

$$n \leq \mathrm{Br.dim}_p(F) \leq n + 1.$$

When $n = 0$, the residue field F is perfect. Then there is no purely inseparable extension over F and no wildly ramified Brauer class over K. Therefore, the first nontrivial case of the conjecture is $n = 1$. Kato used the filtration and generalized Swan conductor to give the first result on the wildly ramified Brauer classes when F is complete. We state this elegant result in the following.

Theorem 1.2.2 (Proposition 4.2.1, [24, Section 4, Lemma 5])

Let K be a complete discretely valued field with the residue field F of characteristic $p > 0$. Suppose that $[F : F^p] = p$. Let $\omega \in \mathrm{Br}(K)[p]$ and $sw(\omega) > 0$. Then the division algebra D which represents ω is a degree p division algebra whose residue algebra is a purely inseparable field extension of degree p over F.

When the p-rank of the residue field is 1, it says that every wildly ramified Brauer class has equal period and index. It is clear for p-torsion Brauer classes. For higher p^∞-torsion classes, it

[1] [6, Conjecture 5.4] [13, Conjecture 1.1]

follows from the induction. Kato's proof can also be applied to the case of henselian discretely valued fields, since the results on the filtration and Kato's Swan conductor work in a similar way as ones in the complete discretely valued field case.

Next, we want to investigate the period-index bound when the p-rank of the residue field is greater than 1. We are going to state our main results in this direction. The proof generalizes Kato's ideas in the p-rank 1 case.

Theorem 1.2.3

Let K be a henselian discretely valued field with the residue field F of characteristic $p > 0$ and $[F : F^p] = p^n,\ n \in \mathbb{N}_{>0}$. Suppose that $\alpha \in \mathrm{Br}(K)[p]$ and $p \nmid \mathrm{sw}(\alpha) > 0$. Then $\mathrm{ind}(\alpha) \mid \mathrm{per}(\alpha)^n$.

Notice that we have a restriction on the Swan conductors of Brauer classes. To remove this restriction, however, we need estimates of the symbol length of two groups: Ω^1_F/Z^1_F and $K^M_2(F)/p$. The first group is the quotient of the absolute differential 1-forms modulo the closed differential 1-forms. The second group is the second Milnor K-group of F modulo by p. We propose the following conjecture on the symbol length of both groups.

Conjecture 1.2.4 (Theorem 3.5.3, Conjecture 3.5.7)

Let F be a field of characteristic $p > 0$ and $[F : F^p] = p^n, n \in \mathbb{N}$. Assume that F does not admit any finite extension of degree prime to p. Then both of the symbol length of the group Ω^1_F/Z^1_F and $K^M_2(F)/p$ are no more than $n - 1$.

For this conjecture, the known case is $(p, n) = (2, 2)$. While we discuss henselian discretely valued fields in both equal characteristic and mixed characteristic separately, these two groups appear differently in the two cases. In the equal characteristic case, we only need the symbol length result for the first group. However, in the mixed characteristic case, we need the symbol length results for both groups. These requirements follow from Kato's description of the consecutive quotients of the filtration of Brauer groups.

Proposition 1.2.5 ($(p, n) = (2, 2)$)

Let F be a field of characteristic $p = 2$ and $[F : F^p] = p^2$. Then

$$\mathrm{len}(\Omega^1_F / Z^1_F) = \mathrm{len}(K_2^M(F)/p) = 1.$$

The conjecture regarding the symbol length problem implies the conjecture about the Brauer p-dimensions.

Theorem 1.2.6 (Smaller Bounds for Wildly Ramified Brauer Classes)

Let K be a henselian discretely valued field with residue field F of characteristic $p > 0$ and $[F : F^p] = p^n, n \in \mathbb{N}$. Suppose that F does not admit any finite extension of degree prime to p. Let $\alpha \in \mathrm{Br}(K)[p]$ and $\mathrm{sw}(\alpha) > 0$. Then Conjecture 1.2.4 implies $\mathrm{ind}(\alpha) \mid \mathrm{per}(\alpha)^n$.

To approach the symbol length of Ω^1_F / Z^1_F, we will give several possible methods including the brutal force way and the Galois correspondence of purely inseparable extensions of height 1.

1.3 Case: semi-global fields

Next, we consider semi-global fields. A semi-global field is one-variable function field F over a complete discretely valued field K, i.e. the function field of a curve over K. Examples include $F = \mathbb{Q}_p(x)$, $F = k((t))(x)$, and any finite extension of these. These fields can be thought of as intermediate objects between global fields and local fields. A natural question to ask is what their Brauer p-dimensions are. Here is a list of known results regarding this question,

semi-global fields F	$\mathrm{Br.dim}_p(F)$
$\mathbb{F}_p((t))(x)$	2
$\mathbb{Q}_p(x)$	2
$\bar{\mathbb{F}}_p((t))(x)$	?
$\mathrm{Frac}(W(\bar{\mathbb{F}}_p))(x)$	?

where $\mathrm{Frac}(W(\bar{\mathbb{F}}_p))$ is the fraction field of the Witt ring of $\bar{\mathbb{F}}_p$. For the field $F = \bar{\mathbb{F}}_p((t))(x)$, as a C_2 field, we expect that $\mathrm{Br.dim}_p(F) = 1$. Indeed, we will demonstrate that there is a subgroup of p-torsion Brauer classes over F that satisfies this period-index bound.

Theorem 1.3.1 (Theorem 5.3.1)

Let X be a smooth projective curve over $k((t))$ where k is an algebraically closed fields of characteristic $p > 0$. Suppose that there is a model $\mathcal{X}$ over $k[[t]]$ with good reduction. Suppose that $\omega \in \mathrm{Br}(X)[p]$ satisfies $\mathrm{sw}_{\mathcal{X}}(\omega) < p$. Then $\mathrm{per}(\omega) = \mathrm{ind}(\omega)$.

We have $\mathrm{Br}(X) \hookrightarrow \mathrm{Br}(F)$ by the purity of Brauer groups, where F is the function field of X. We use Kato's Swan conductor to define a $\mathcal{X}$-Swan conductor for elements in $\mathrm{Br}(X)$ (Definition 5.3.4). The definition is based on the model $\mathcal{X}$ at the beginning. We do not know if the definition depends on the choice of the model with good reduction.

We use the patching methods to reduce the period-index problem of the semi-global field to two types of local period-index problems. The first type of period-index problem is addressed by considering period-index problems of complete discretely valued with residual p-rank 1. The second type of local period-index problem is analysed using a Gersten-type exact sequence (Theorem 5.1.4). We will discuss this Gersten-type exact sequence in detail in Chapter 5.

1.4 Period-index problems of higher Kato's groups

Brauer groups are special cases of Kato's groups. In the 1980s, Kato used differential forms to define groups $H^{n+1}(F, \mathbb{Z}/m(n))$ for a field F and a prime number m, even when m is not invertible in F. These groups generalize many arithmetical cohomological groups. For example,

- $H^1_{\text{ét}}(F, \mathbb{Z}/m(0)) \cong H^1_{\text{ét}}(F, \mathbb{Z}/m)$: the group classifying cyclic $\mathbb{Z}/m$-extensions of F.
- $H^2_{\text{ét}}(F, \mathbb{Z}/m(1)) \cong \mathrm{Br}(F)[m]$: the m-torsion subgroup of the Brauer group of F.

These higher cohomology groups have already been investigated from various perspectives. We could also discuss the period-index bounds for these groups.

Definition 1.4.1 ([21])

Let F be a field and p be a prime number. A field extension E/F is called a splitting field *for a class $\alpha \in H^i_{\text{ét}}(F, (\mathbb{Z}/p)(i-1))$, if the image of α_E of α under the natural map $H^i_{\text{ét}}(F, (\mathbb{Z}/p)(i-1)) \to H^i_{\text{ét}}(E, (\mathbb{Z}/p)(i-1))$ is trivial.*

The index *of a class $\alpha \in H^i_{\text{ét}}(F, (\mathbb{Z}/p)(i-1))$, denoted by $ind(\alpha)$, is the greatest common divisor of the degrees of splitting fields of α that are finite over F.*

It is clear that the definitions of period-index for higher Kato's groups are direct generalizations of those for torsion parts of Brauer groups.

Since Kato's filtration and Kato's Swan conductor can be defined for these higher cohomology groups, we also investigate the period-index bounds for these groups using a symbol length approach. More concretely, we prove that any wildly ramified element in $H^3_{\text{ét}}(K, (\mathbb{Z}/p)(2))$ is split by a purely inseparable extension of degree p, when K is a henselian discretely valued field with a residue field F of characteristic $p > 0$ and $[F : F^p] = p^2$.

Theorem 1.4.2 (Theorem 4.5.1, Theorem 4.5.2)

Let K be a henselian discretely valued field with the residue field F of characteristic $p > 0$. Suppose that $[F : F^p] = p^2$ and F does not admit any finite extension of degree prime to p. Let $\alpha \in H^3_{\text{ét}}(K, (\mathbb{Z}/p)(2))$ such that $sw(\alpha) > 0$. Then Conjecture 3.5.7 implies that $\alpha = \omega \wedge \dfrac{dc}{c}$ for some $\omega \in \Omega^1_K$ and $c \in K^\times$.

CHAPTER 2

REVIEW OF BRAUER GROUPS

2.1 Basic properties of Brauer groups

A central simple algebra A (CSA) over a field K is a finite-dimensional associative K-algebra A that is simple with center K.

Two central simple algebras A, A' are called *Morita equivalent* if there exist integers $r, s \in \mathbb{N}$ such that $A \otimes M_r(K) \simeq A' \otimes M_s(K)$ as K-algebras. By the Artin-Wedderburn theorem, a finite-dimensional simple algebra A is isomorphic to the matrix algebra $M_n(D)$ for a K-central division algebra D. Moreover, such a division algebra is uniquely determined by a central simple algebra.

The *Brauer group* of a field K is a torsion abelian group whose element are Morita equivalence classes of central simple algebras over K. The addition in the Brauer group is given by the tensor product of algebras.

As mentioned above, there is a unique central division algebra in each Brauer class. The *degree* $\deg(A)$ of a central simple algebra A is the integer n such that $\dim_K(A) = n^2$. Then we define the *index* $\mathrm{ind}(A)$ of a central simple algebra A to be the degree of the division algebra D associated to A by the Artin-Wedderburn theorem. In particular, note that the index is well-defined for a Brauer class. Also, for a Brauer class $[A]$ associated to a central simple algebra A, the *period* $\mathrm{per}(A)$ is its order in the Brauer group $\mathrm{Br}(K)$.

It is well-known that the period divides the index of a central simple algebra, and these two integers have the same prime factors. So the index divides a power of the period. The period-index problem asks if one can bound the index in terms of the power of the period. Here are the relevant definitions from the introduction.

Definition 2.1.1 (Brauer dimension [31])

- *Let K be a field. For a prime p, the* Brauer dimension at p, $\mathrm{Br.\,dim}_p(K)$*, is the smallest integer d such that for any $A \in \mathrm{Br}(K)[p^n]$, $\mathrm{ind}(A) \mid \mathrm{per}(A)^d$, and ∞ if no such number exists.*

- The Brauer dimension of K *is*

$$\text{Br.dim}(K) = \sup_{p} \left\{ \text{Br.dim}_p(K) \right\}.$$

The period-index problem asks if $\text{Br.dim}(K)$ is finite, and the local period-index problem asks if $\text{Br.dim}_p(K)$ is finite for an arbitrary prime p.

The Brauer group can also be defined in terms of Galois (étale) cohomology. We have

$$\text{Br}(K) = H^2_{\text{ét}}(K, \mathbb{G}_m),$$

where $\mathbb{G}_m$ denotes the sheaf of units in the structure sheaf.

In general, the Brauer group of a scheme is defined in terms of Azumaya algebras. An Azumaya algebra is a generalization of central simple algebras to R-algebras where R may not be a field. For a scheme X with structure sheaf $\mathcal{O}_X$, an Azumaya algebra on X is a coherent sheaf $\mathcal{A}$ of $\mathcal{O}_X$-algebras that is étale locally isomorphic to the sheaf of matrices over the structure sheaf. The Brauer group $\text{Br}(X)$ is an abelian group of equivalence classes of Azumaya algebras, with the addition given by the tensor product of algebras. Here two Azumaya algebras $\mathcal{A}, \mathcal{A}'$ are considered to be equivalent when $\text{M}_r(\mathcal{A}) \cong \text{M}_s(\mathcal{A}')$ as sheaves of $\mathcal{O}_X$-algebras for matrices of size $r \times r$ and $s \times s$ respectively.

As in the case of a field, we define the *cohomological Brauer group* of a quasi-compact scheme X to be the torsion subgroup of the étale cohomology group $H^2_{\text{ét}}(X, \mathbb{G}_m)$. The cohomology group $H^2_{\text{ét}}(X, \mathbb{G}_m)$ is torsion for a regular scheme X, but it may not be torsion in general.

We recall several well-known facts about Brauer groups in the following.

Theorem 2.1.2 (O. Gabber)
The Brauer group of a scheme X is equal to the cohomological Brauer group for any scheme with an ample line bundle.

For example, when X is quasi-projective over a field k, we have the coincidence of two Brauer groups.

Theorem 2.1.3 (Purity in codimension 1 [44])

For a Noetherian, integral, regular scheme X with function field K,

$$H^2_{\text{ét}}(X, \mathbb{G}_m) = \bigcap_{x \in X^{(1)}} H^2_{\text{ét}}(\mathcal{O}_{X,x}, \mathbb{G}_m) \text{ in } H^2_{\text{ét}}(K, \mathbb{G}_m).$$

2.2 Structure of p-primary part of Brauer groups

In this section, we assume all the fields have positive characteristic $p > 0$. We focus on the p-primary part of the Brauer groups. First, we recall the p-primary counterpart of the Merkurjev-Suslin theorem [1982]. The Merkurjev-Suslin theorem states that $\mathrm{Br}(K)[n]$ is generated by cyclic algebras of degree n when K contains a primitive n-th root of unity μ_n.

Firstly, we recall the Artin-Schreier-Witt theory of cyclic field extensions in positive characteristic:

Theorem 2.2.1 ([37])

Let k be a field of characteristic $p > 0$. Denote by $\mathcal{P} : W_r(k) \to W_r(k)$ the endmorphism of the length-r Witt ring that maps $(x_1, \cdots, x_r) \in W_r(k)$ to $(x_1^p, \cdots, x_r^p) - (x_1, \cdots, x_r)$. Then there exists a canonical isomorphism

$$W_r(k)/\mathcal{P}(W_r(k)) \cong H^1_{\text{ét}}(k, \mathbb{Z}/p^r).$$

Then we have the following theorem about the p^r-cyclic algebras (symbol algebras).

Proposition 2.2.2

Let K be a field of characteristic $p > 0$. For every $\omega \in Br(K)[p^r]$, we can write

$$\omega = \sum_i [a_i, b_i),$$

as a sum of p^r-symbol algebras where $a_i \in W_r(K)$ and $b_i \in K^\times$. The p^r-symbol algebra $[a_i, b_i)$ is defined by

$$[a_i, b_i) := \left\langle x, y \;\middle|\; \begin{array}{l} x \text{ is a primitive element of the Artin-Schreier-Witt extension defined by} \\ \mathcal{P}(x_1, \ldots, x_r) = a_i \text{ and with a generator } \sigma \text{ of the Galois group such that,} \\ y^{p^n} = b_i, \quad y^{-1}xy = \sigma(x). \end{array} \right\rangle.$$

The p^r-symbol algebra has index = period = p^r.

Example 2.2.3

Let $a, b \in K$ and consider the p-symbol algebra $[a, b)$. By definition,

$$[a, b) := \langle x, y \mid x^p - x = a, y^p = b, y^{-1}xy = x + 1\rangle.$$

This symbol algebra is the main object of our study, since we can reduce questions related to p^r-torsion Brauer classes to the p-symbol algebra by Theorem 2.2.7.

Next we relate the p-primary part of the Brauer group with the de Rham-Witt complex $W_r\Omega^1_K$ [22]. We can identify $\mathrm{Br}(K)[p^r]$ with the cokernel of

$$F - I : W_r\Omega^1_K \to W_r\Omega^1_K/dV^{r-1}(K), \tag{2.2.1}$$

where F is Frobenius morphism and I is the identity morphism.

Lemma 2.2.4 ([22])

Let K be a field of characteristic $p > 0$ and $r \in \mathbb{N}^+$.

$$\mathrm{Br}(K)[p^r] \cong W_r\Omega^1_K/\Big((F - I)W_r\Omega^1_K + dW_r(K)\Big). \tag{2.2.2}$$

Proof.

In fact, there exists an exact sequence of étale sheaves over the affine scheme $X = \mathrm{Spec}(K)$:

$$0 \longrightarrow W_r\Omega^1_{X,\log} \longrightarrow W_r\Omega^1_X \xrightarrow{F-I} W_r\Omega^1_X/dV^{r-1}O_X \longrightarrow 0\,,$$

which induces the cohomology group sequence

$$W_r\Omega^1_K \xrightarrow{F-I} W_r\Omega^1_K/dV^{r-1}K \xrightarrow{\delta_r} H^1_{\text{ét}}(K, W_r\Omega^1_{K,\log}) \longrightarrow 0\,,$$

since $H^1_{\text{ét}}(K, W_r\Omega^1_K) = 0$ by the quasi-coherence of $W_r\Omega^1_K$.

Also, there is another exact sequence of étale sheaves which relates the p^r-torsion part of the Brauer group with the logarithmic de Rham-Witt complex

$$0 \longrightarrow \mathbb{G}_m \xrightarrow{p^r} \mathbb{G}_m \longrightarrow W_r\Omega^1_{X,\log} \longrightarrow 0.$$

It induces the long exact cohomology sequence

$$0 \longrightarrow H^1_{\text{ét}}(K, W_r\Omega^1_{K,\log}) \longrightarrow H^2_{\text{ét}}(K, \mathbb{G}_m) \xrightarrow{p^r} H^2_{\text{ét}}(K, \mathbb{G}_m),$$

where $H^1_{\text{ét}}(K, \mathbb{G}_m) = 0$ by Hilbert's Theorem 90.

By using the relation $d = F^{r-1}dV^{r-1}$ [22, (2.18)], it is easy to see that δ_r induces an isomorphism between the cokernel of (2.2.1) and $\mathrm{Br}(K)[p^r]$. □

Now we use the structure of $W_r\Omega^1_K$ to describe p^r-torsion part of the Brauer group of K. We recall some facts about $W_r\Omega^1_K$ [2]. We use the notation $[a]_r := (a, 0, \cdots, 0) \in W_r(K)$.

Definition 2.2.5

$M^1_rK \subset W_r\Omega^1_K$ denotes the subgroup generated by the elements $[a]_rd[f]_r$ where $a \in K, f \in K^\times$.

Lemma 2.2.6 (Lemma 2.4, [2])

Let $M^1_rK \subset W_r\Omega^1_K$ denote the subgroup generated by multiplicative *elements $[a]_rd[f]_r$. Then we have*

$$W_r\Omega^1_K = \sum_{i=0}^{r-1} V^iM^1_{r-i}K + \sum_{i=0}^{r-1} dV^iK.$$

Moreover,

$$dW_r(K) = \sum_{i=0}^{r-1} dV^iK \subset W_r\Omega^1_K.$$

It follows that

$$\mathrm{Br}_{p^r}(K) \cong W_r\Omega^1_K \Big/ \Big((F - I)W_r\Omega^1_K + dW_r(K)\Big) \cong \sum_{i=0}^{r-1} \left[V^iM^1_{r-i}K\right]. \tag{2.2.3}$$

Then we relate the differential forms with symbol algebras by the following map

$$\begin{aligned} \delta_r :\ W_r\Omega^1_K/((F - I)W_r\Omega^1_K + dW_r(K)) &\longrightarrow \mathrm{Br}(K)[p^r] \\ a\ \mathrm{dlog}([b]_r) &\longmapsto [a, b), \end{aligned}$$

where $a \in W_r(K),\ b \in K^\times$, and $\mathrm{dlog}([b]_r) = [b]_r^{-1}d[b]_r$.

We denote the composite map $W_r\Omega^1_K \to \mathrm{Br}(K)[p^r] \to \mathrm{Br}(K)$ by δ_r as well. We have a commutative diagram

$$\begin{array}{ccc} W_{r-1}\Omega^1_K & \xrightarrow{V} & W_r\Omega^1_K \\ {\scriptstyle \delta_{r-1}}\downarrow & \swarrow {\scriptstyle \delta_r} & \\ \mathrm{Br}(K) & & \end{array}$$

Using the isomorphism (2.2.3), it is easy to give a direct proof of the following theorem.

Theorem 2.2.7 ([25])

For a field K of positive characteristic $p > 0$ and $m \in \mathbb{N}$, we have an exact sequence:

$$0 \longrightarrow \mathrm{Br}(K)[p^m] \xrightarrow{V} \mathrm{Br}(K)[p^{m+1}] \xrightarrow{R^1} \mathrm{Br}(K)[p] \longrightarrow 0, \tag{2.2.4}$$

where $R^1 : W_m\Omega^1_K \to \Omega^1_K$ sends $[a]_m \mathrm{dlog}([b]_m)$ to $a\, \mathrm{dlog}(b)$.

The Brauer dimension at p for a field of characteristic $p > 0$ is effectively controlled by the *rank* of the p-basis.

Definition 2.2.8 (p-basis and p-rank)

Let K be a field and $[K : K^p] = p^n$, $n \geq 0$. A p-basis of K is a subset $\{x_i\} \subset K$ such that the elements $x^E = \prod x_i^{e_i}, 0 \leq e_i < p$ form a basis of K over K^p, and the p-rank of K is the number of elements in the subset $\{x_i\}$. Hence the p-rank of K is n.

Proposition 2.2.9 ([11, Corollary 3.4])

Let K be a field with $[K : K^p] = p^n$. Then $\mathrm{Br.dim}_p(K) \leq n$.

Proof. For $r \in \mathbb{N}$ and a p-basis $\{a_i\}_{i=1}^n$ of K, by Theorem 2.2.7 and induction, every p^r-torsion Brauer class can be written as a sum of n symbol algebras $[c_i, a_i)$, where $c_i \in W_r(K)$ for $i \in \{1, \cdots, n\}$. Then the proposition follows from the following lemma. □

Lemma 2.2.10 ([1, Ch. VII, Lemma 13])

Let K be a field of characteristic $p > 0$. If A, B are two symbol algebras of degree p^m and p^n respectively, then $A \otimes B$ is Brauer equivalent to a symbol algebra of degree no more than p^{m+n}.

2.3 Brauer group of a complete discretely valued field

In this section, K denotes a complete discretely valued field with valuation ring O_K, residue field F and maximal ideal $m_K = (\pi)$. The valuation of K is denoted by v_K. Recall that a discrete valuation is a map $v_K: K \to \mathbb{Z} \cup \{\infty\}$ that satisfies:

(i) $v_K(a) = \infty$ if and only if $a = 0$;

(ii) $v_K(ab) = v_K(a) + v_K(b)$;

(iii) $v_K(a+b) \geq \min(v_K(a), v_K(b))$, with equality if $v_K(a) \neq v_K(b)$.

The valuation ring $O_K = v_K^{-1}(\mathbb{Z}_{\geq 0})$ is a complete local ring of Krull dimension 1. For a complete discretely valued field K, we can extend the complete valuation v_K to central simple division algebras over K and consider the residue division algebras. They are summarized in the following proposition.

Proposition 2.3.1 (Proposition 1.3.1, [5])

Let D be a central division K-algebra.

(i) The function $w : D \to \mathbb{Z} \cup \{\infty\}$ defined by $w(a) = v_K(\det(a))$ is a discrete valuation on D.

(ii) The set $B := \{a \in D \mid w(a) \geq 0\} = \{a \mid \det(a) \in O_K\}$ is the unique maximal O_K-order in D.

(iii) B is a local domain with maximal ideal $J := \{a \mid w(a) > 0\}$; the residue ring $\Delta = B/J$ is a division ring.

(iv) If π is an element of J such that $w(\pi)$ takes the minimal positive value, then $J = B\pi = \pi B$

Next we study the unique maximal order B in the above proposition. Let F' be the center of Δ. Then we have the integers d, e, e', f, n defined as follows:

$$d = w(\pi),\ J^e = m_K B,\ e' = [F' : F],\ f^2 = [\Delta : F'],\ n^2 = [D : K]. \tag{2.3.1}$$

Here n is the degree (index) of D, and also its degree.

Lemma 2.3.2 ([5, Lemma 1.3.7])

$ed = n$, and $ee'f^2 = n^2$.

Corollary 2.3.3

$[D : K] = 1$ *if and only if* $[\Delta : F] = 1$.

Proof. This is immediate from the above lemma. □

In the latter part, we are interested in the case that the residue field F is **quasi-algebraically closed**, i.e a C_1 field. Recall that a finite extension of a C_1 field is also C_1. Hence, the central division algebra Δ over F' will be isomorphic to F', since $\mathrm{Br}(F') = 0$. This implies $f = 1$.

Lemma 2.3.4

Suppose that the residue field F is C_1 and $[F : F^p] = p$. Then $e = e' = n$ and $d = 1$.

Proof.

We already have $f = 1$ and so it suffices to show $e' \leq n$ by Lemma 2.3.2. We will show that any field extension F' of F is simple. In this case, $F' = F[\alpha]$ and we choose $\beta \in B$ such that $\bar{\beta} = \alpha \in F'$. Then we have $e' \leq [K(\beta) : K] \leq n$, since $\mathrm{ind}(D)$ is n.

Now we prove that any finite field extension F' of F is simple. The field extension $F \subset F'$ can be written as a chain of field extensions $F \subset E \subset F'$ such that E is separable over F and F' is purely inseparable over E. It follows that $[E : E^p] = p$ by Lemma 2.3.5 below. Then the purely inseparable extension F'/E is simple. Set $F' = E[\alpha_1]$ and α_1 is algebraic over F. We can also denote $E = F[\alpha_2]$ by Theorem 2.3.6 below, since E/F is finite and separable. Finally, we get $F \subset F' = F[\alpha_1, \alpha_2]$ is simple by Theorem 2.3.6 again. □

Lemma 2.3.5 ([9, A.V.135, Corollary 3])

Let l/k be a finite or separable field extension of fields of characteristic p, and let n be the p-rank of k. Then the p-rank of l is also n.

Theorem 2.3.6 ([33, Theorem 5.1])

Let $l = k[\alpha_1, \cdots, \alpha_r]$ be a finite extension of k, and assume that $\alpha_2, \cdots, \alpha_r$ are separable over k (but not necessarily α_1). Then there exists a $\gamma \in E$ such that $l = k[\gamma]$.

CHAPTER 3

KATO'S GROUP AND SWAN CONDUCTOR

3.1 Kato's group

In the 1980s, Kato used differential forms to define groups $H^i_{\text{ét}}(k, (\mathbb{Z})/m(j))$ for a field k and any positive integer m, especially when m is not invertible in k. These groups generalize many well-known arithmetic cohomology groups. For example, we have $H^1_{\text{ét}}(k, \mathbb{Z}/m) \cong H^1_{\text{ét}}(k, \mathbb{Z}/m)$, the group classifying cyclic $\mathbb{Z}/m$-extensions of k with generators, and $H^2_{\text{ét}}(k, (\mathbb{Z}/m)(1)) \cong \text{Br}(k)[m]$, the m-torsion part of the Brauer group of k.

In fact, there is an explanation for Kato's groups: Voevodsky's étale motivic cohomology groups $H^i_{\text{ét}}(X, A(j))$ of a scheme X over a field k are defined for any abelian group A. They agree with Kato's groups when $X = \text{Spec}(k)$ and $A = \mathbb{Z}/m$ for any m.

It is especially of interest to investigate Kato's groups of a field k when k has *residual characteristic* $p > 0$.

Definition 3.1.1

We say a field k has residual characteristic $p > 0$ if it satisfies one of the following conditions:

(i) k is of characteristic $p > 0$;

(ii) k is a discretely valued field with a residue field of characteristic $p > 0$.

We will describe our approaches to Kato's groups in both cases.

3.1.1 Case: characteristic $p > 0$

Let us start with the definition of Kato's groups when k is of characteristic $p > 0$ and $m = p^r$, $r \in \mathbb{N}$. For $j \geq 0$, let $\Omega^j_k := \Omega^j_{k/\mathbb{Z}}$ be the group of absolute Kähler differential forms and $\Omega^j_{k,\log}$ be the subgroup of Ω^j_k generated by logarithmic differential $\dfrac{df_1}{f_1} \wedge \cdots \wedge \dfrac{df_j}{f_j}$ for $f_1, \ldots, f_j \in k^\times$. More generally, let $W_r\Omega^j_{k,\log}$ be the analogous group of logarithmic de Rham-Witt differentials [22]. Then we have the following

$$H^i_{\text{ét}}(k, (\mathbb{Z}/p^r)(j)) \cong H^{i-j}_{\text{ét}}(k, W_r\Omega^j_{\log}). \tag{3.1.1}$$

Since the étale p-cohomological dimension of k is at most 1 [17, Proposition 6.1.9],

$H^i_{ét}(k,(\mathbb{Z}/p^r)(j))$ is zero except when i is j or $j+1$. When $i=j$, Bloch, Gabber and Kato [7, Corollary 2.8] showed that

$$H^j_{ét}(k,(\mathbb{Z}/p^r)(j)) \cong H^0_{ét}(k,W_r\Omega^j_{\log}) \cong W_r\Omega^j_{k,\log} \cong K^M_j(k)/p^r, \tag{3.1.2}$$

where $K^M_j(k)$ is the Milnor K-group. When $i=j+1$, one way to describe these groups is in terms of Galois cohomology. First, we focus on the case $r=1$. Let k_s be a separable closure of k. Then

$$H^{j+1}_{ét}(k,(\mathbb{Z}/p)(j)) \cong H^1_{\mathrm{Gal}}(k,\Omega^j_{k_s,\log}). \tag{3.1.3}$$

To give a more precise description of the case $i=j+1$, we recall the original definition from Kato [26]. We define a group homomorphism $\mathcal{P}:\Omega^j_k \to \Omega^j_k/d\Omega^{j-1}_k$ by

$$\mathcal{P}\left(a\frac{db_1}{b_1}\wedge\cdots\wedge\frac{db_j}{b_j}\right) = (a^p-a)\frac{db_1}{b_1}\wedge\cdots\wedge\frac{db_j}{b_j}.$$

Then there is an exact sequence of groups

$$0 \longrightarrow H^0_{ét}(k,\Omega^1_{k,\log}) \longrightarrow \Omega^j_k \xrightarrow{\mathcal{P}} \Omega^j_k/d\Omega^{j-1}_k \longrightarrow H^1_{ét}(k,\Omega^j_{k,\log}) \longrightarrow 0.$$

Therefore, $H^1_{ét}(k,\Omega^j_{k,\log})$ is isomorphic to the cokernel of $\mathcal{P}$.

In conclusion, we have the following full description of $H^i_{ét}(k,(\mathbb{Z}/p)(j))$ for a field k of characteristic $p>0$:

$$H^i_{ét}(k,(\mathbb{Z}/p)(j)) \cong \begin{cases} \Omega^j_{k,\log} & \text{if } i=j \\ \Omega^j_k\Big/\left(\mathcal{P}(\Omega^j_k)+d\Omega^{j-1}_k\right) & \text{if } i=j+1 \\ 0 & \text{otherwise.} \end{cases} \tag{3.1.4}$$

Notice that these cohomology groups appear as subgroups or quotient groups of the group of the absolute Kähler differential forms. Hence, we can express an element in these groups as a sum of *symbols*. The symbols can be regarded as either equivalent classes of differential forms or elements in the Milnor K-groups by Bloch-Kato-Gabber [7].

3.1.2 Case: characteristic 0

Now let k be a field of characteristic 0 and p be a prime number. Moreover, assume k contains a primitive p-th root ζ of unity. This assumption assures that we can use symbols to investigate

Kato's groups. Recall the norm residue isomorphism theorem (Bloch-Kato conjecture), which is proved by Voevodsky [45].

Theorem 3.1.2 (Norm residue isomorphism theorem [45])

Let K be a field and p an integer invertible in K. Then

$$H^n(K, (\mathbb{Z}/p)(n)) \cong K_n^M(K)/p.$$

The norm residue isomorphism is firstly proved by Bloch and Kato [7] in the case of complete discretely valued fields. Then Murkerjev and Suslin [32] proved the case $n = 2$. Finally, Voevodsky [45] used the motivic cohomology to finish the general proof.

Using the primitive p-th root ζ of the unity , we can identify $\mathbb{Z}/p = (\mathbb{Z}/p)(1) : 1 \mapsto \zeta$. Therefore, for any $i \in \mathbb{N}$, we have

$$H^i_{\text{ét}}(k, (\mathbb{Z}/p)(i-1)) = H^i_{\text{ét}}(k, (\mathbb{Z}/p)(i)) \cong K_i^M(k)/p. \tag{3.1.5}$$

Then we can describe the elements in Kato's groups by symbols from Milnor K-groups again.

3.2 Kato's Swan conductor

Let K be a complete discretely valued field with residue field F, and L be a finite Galois extension of K. Classically, the *Swan conductor* of a character of $\mathrm{Gal}(L/K)$ is defined in the case where the residue field of L is separable over F. Kato [26] provided a natural definition of the Swan conductor without requiring the residue field extension to be separable. More generally, he defined Swan conductors for elements in Kato's groups. The classical Swan conductor measures the wild ramification of the extension, while Kato's Swan conductor naturally extends this to measure the wild ramification of Brauer classes and other elements in higher Kato's groups.

As Kato's Swan conductors measure the wild ramification behaviors, we will concentrate on fields of residual characteristic $p > 0$ and Kato's groups with coefficient in $\mathbb{Z}/p$.

Notation 3.2.1 ([26])

Let K be a field. We define

$$H^q_p(K) := H^q_{\text{ét}}(K, (\mathbb{Z}/p)(q-1)).$$

When K is of characteristic $p > 0$, we have $(\mathbb{Z}/p)(q-1) \simeq \Omega^{q-1}_{\log}[-(q-1)]$ in $D^b(K_{\text{ét}})$. Then it follows that

$$H^q_p(K) = H^1_{\text{ét}}(K, \Omega^{q-1}_{K,\log}).$$

($\star$) In the rest of this chapter, we denote by K a henselian discretely valued field with the valuation v. Let O_K be the discrete valuation ring of K

$$O_K = \{x \in K \mid v(x) \geq 0\} \tag{3.2.1}$$

with the maximal ideal m, and let $F = O_K/m$ be the residue field.

Definition 3.2.2 (Unit group filtration)

Let $U_K = (O_K)^\times$ be the group of units in the ring O_K. For each $i \in \mathbb{N}$, consider the subgroup

$$U^i_K = \{x \in U_K \mid v(x-1) \geq i\} \text{ for } i \geq 1. \tag{3.2.2}$$

Then since $U_K \supset U^1_K \supset U^1_K \supset \cdots$, we have defined a decreasing filtration on U_K.

In the bounded derived category $D^b(K_{\text{ét}})$, we have an exact triangle

$$(\mathbb{Z}/p)(1) \longrightarrow \mathbb{G}_m \xrightarrow{p} \mathbb{G}_m \longrightarrow (\mathbb{Z}/p)(1)[1]\ . \tag{3.2.3}$$

Given $a \in K^\times = H^0(K, \mathbb{G}_m)$, we denote the image of a in $H^1_{\text{ét}}(K, (\mathbb{Z}/p)(1))$ by $\{a\}$.

Then we have the product maps:

$$H^q_n(K) \times (K^\times)^{\oplus r} \to H^{q+r}_n(K)$$

defined by $(\chi, a_1, \cdots, a_r) \mapsto \{\chi, a_1, \cdots, a_r\} := \chi \cup \{a_1\} \cup \cdots \cup \{a_r\}$.

Definition 3.2.3 (Kato's filtration [26, Proposition 6.3])

The increasing filtration $\{M^p_n\}_{n\geq 0}$ on $H^q_p(K)$ is defined by:

$$\chi \in M^p_n \iff \{\chi_L, 1 + \pi^{n+1} O_L\} = 0 \text{ in } H^{q+1}_p(L)$$

for any henselian discrete valuation field L over K such that $O_K \subset O_L$ and $m_L = O_L m_K$.

To see this filtration is well-defined, we refer to Kato's original paper [26, Proposition 1.8, Lemma 2.2]. We have $H^q_p(K) = \bigcup_{n \geq 0} M^p_n$. Now we are ready to define Kato's Swan conductors.

Definition 3.2.4 (Kato's Swan conductor [26, Definition 2.3])

Let $\chi \in H_p^q(K)$. *We define* Kato's Swan conductor $sw(\chi) \in \mathbb{N}$ *to be the minimum integer* $n \geq 0$ *such that* $\chi \in M_n^p$, *i.e.*

$$\mathrm{sw}_F(\chi) := \min\{n \in \mathbb{N} \mid \chi \in M_n^p\}.$$

As we mentioned earlier, the Kato's Swan conductor measures the wild ramifications of elements in Kato's groups. We usually consider the Kato's group $H_p^q(K)$ when the residue field F of K is of characteristic $p > 0$.

Notice that the above definition of Kato's filtration is independent of the characteristic of K. The following proposition tells that there is no wild ramification if we look at the Kato's groups with torsion away from the residual characteristic.

Proposition 3.2.5 ([26, Corollary 2.5])

Let K *be a henselian discretely valued field with the residue field* F *of characteristic* $p > 0$ *and* $l \neq p$ *be a prime. Then* $H_l^q(K) = M_0^l$ *for all* $q \in \mathbb{N}$.

When the torsion of Kato's group is understood from the context, we will simply denote the filtration by $\{M_n\}$. In the next two sections, we will describe the consecutive quotients of this filtration, based on the characteristic of K.

3.3 Equal characteristic case: char$(K) = p > 0$

Recall that K is a henselian discretely valued field with the discrete valuation v and the residue field F of characteristic $p > 0$. We assume $\mathrm{char}(K) = p > 0$ in this section. Then we have

$$H_p^q(K) = H^q_{\text{ét}}(K, (\mathbb{Z}/p)(q-1)) = H^1_{\text{ét}}(K, \Omega^{q-1}_{K,\log}) \cong \frac{\Omega_K^{q-1}}{(\mathrm{Fr} - I)\Omega_K^{q-1} + d\Omega_K^{q-2}},$$

where Fr is the Frobenius morphism. Kato generalized Brylinski's filtration [10] on Witt vectors to define an increasing filtration $\{M^j\}_{j \geq 0}$ on the p-primary Kato's groups. For our purpose, we only consider the p-torsion one $H_p^q(K)$. For $j \geq 0$, M^j is the subgroup of $H_p^q(K)$ generated by elements of the form

$$a\frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_{q-1}}{b_{q-1}}$$

with $a \in K$, $b_1, \ldots, b_{q-1} \in K^\times$, and $v(a) \geq -j$. It is clear that

$$0 \subset M^0 \subset M^1 \subset \cdots,$$

with $\bigcup_{j\geq 0} M^j = H_p^q(K)$. Kato proved that the two filtrations $\{M_j\}$ and $\{M^j\}$ coincide, that is, $M^j = M_j$ for each j [26, Theorem 3.2]. Therefore, we will use M_j in the following context for convenience.

Let $\pi \in \mathcal{O}_K$ be a uniformizer for v. For any $j > 0$, we define two homomorphisms depending on whether j is relatively prime to p or $p \mid j$. In each case, a simple computation shows that the homomorphim is well defined up to a choice of a uniformizer. First, consider the case when j is relatively prime to p. We define

$$\Omega_F^{q-1} \to M_j/M_{j-1}$$

by

$$\bar{a}\frac{d\bar{b}_1}{\bar{b}_1} \wedge \cdots \wedge \frac{d\bar{b}_{q-1}}{\bar{b}_{q-1}} \mapsto \frac{a}{\pi^j}\frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_{q-1}}{b_{q-1}} \pmod{M_{j-1}},$$

for $a \in \mathcal{O}_K$ and $b_1, \ldots, b_{q-1} \in \mathcal{O}_K^\times$.

Now we define the second homomorphism. Let Z_F^{q-1}, Z_F^{q-2} be the subgroup of closed forms in $\Omega_F^{q-1}, \Omega_F^{q-2}$ respectively. For $j > 0$ and $p \mid j$, define a homomorphism

$$\Omega_F^{q-1}/Z_F^{q-1} \oplus \Omega_F^{q-2}/Z_F^{q-2} \to M_j/M_{j-1}$$

as follows: On the first summand, it is defined as

$$\bar{a}\frac{d\bar{b}_1}{\bar{b}_1} \wedge \cdots \wedge \frac{d\bar{b}_{q-1}}{\bar{b}_{q-1}} \mapsto \frac{a}{\pi^j}\frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_{q-1}}{b_{q-1}} \pmod{M_{j-1}},$$

and for the second summand it is defined as

$$\bar{a}\frac{d\bar{b}_1}{\bar{b}_1} \wedge \cdots \wedge \frac{d\bar{b}_{q-2}}{\bar{b}_{q-2}} \mapsto \frac{a}{\pi^j}\frac{d\pi}{\pi} \wedge \frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_{q-2}}{b_{q-2}} \pmod{M_{j-1}},$$

where $a \in \mathcal{O}_K$ and $b_1, \ldots, b_{q-1} \in \mathcal{O}_K^\times$.

The homomorphisms are well defined (although they depend on the choice of uniformizer π). We recall Cartier's theorem in this context. It says that, for a field k of characteristic $p > 0$, $q \in \mathbb{N}$, the subgroups Z_k^q of closed forms in Ω_k^q is generated by the exact forms together with the forms of the form $a^p(db_1/b_1) \wedge \cdots \wedge (db_q/b_q)$ [23, Lemma 1.5.1].

To describe the subgroup M_0, we need to describe tame extensions of K [43]. We fix a discrete valuation v as above. An extension field of K is called *tame* with respect to v if it is a union of finite extensions of K for which the extension of residue fields is separable and the ramification degree is invertible in the residue field F. Let K_{tame} be the maximal tamely ramified extension of K (with respect to v) in a separable closure of K. Define the *tame* (or *tamely ramified*) subgroup of $H^q_{\text{ét}}(K, (\mathbb{Z}/p)(q-1))$ by

$$H^q_{\text{tame}}(K, (\mathbb{Z}/p)/p(q-1)) = \ker\Big(H^q_{\text{ét}}(K, (\mathbb{Z}/p)/p(q-1)) \to H^q_{\text{ét}}(K_{\text{tame}}, (\mathbb{Z}/p)/p(q-1))\Big).$$

There is residue homomorphism on the tamely ramified subgroup

$$\partial_v : H^q_{\text{tame}}(K, (\mathbb{Z}/p)(q-1)) \to H^{q-1}_{\text{ét}}(F, (\mathbb{Z}/p)(q-2)),$$

characterized by the property that

$$\partial_v(a\frac{d\pi}{\pi} \wedge \frac{db_1}{b_1} \wedge \cdots \wedge \frac{db_{q-2}}{b_{q-2}}) = \bar{a}\frac{d\bar{b}_1}{\bar{b}_1} \wedge \cdots \wedge \frac{d\bar{b}_{q-2}}{\bar{b}_{q-2}},$$

where $a \in O_K, b_1, \ldots, b_{q-2} \in O_K^\times$. Note that this description of elements of the tamely ramified subgroup follows from the theorem below. Then we define the *unramified* subgroup $H^q_{\text{nr}}(K, (\mathbb{Z}/p)(q-1))$ to be the kernel of the residue homomorphism ∂_v.

Theorem 3.3.1 (Equal characteristic case: $\text{char}(K) = p > 0$ [26, 43])

Let K be a henselian discretely valued field of characteristic $p > 0$ with the residue field F and q be a positive integer. Then the p-torsion Kato's group $H^q_p(K) = H^q_{\text{ét}}(K, (\mathbb{Z}/p)(q-1))$ has an increasing filtration $\{M_j\}_{j\geq 0}$ described as above, with isomorphisms (depending on the choice of a uniformizer)

$$M_j/M_{j-1} \cong \begin{cases} \Omega_F^{q-1} & \text{if } j > 0 \text{ and } p \nmid j, \\ \Omega_F^{q-1}/Z_F^{q-1} \oplus \Omega_F^{q-2}/Z_F^{q-2} & \text{if } j > 0 \text{ and } p \mid j. \end{cases}$$

Moreover, M_0 is the tame subgroup and there is a well-defined residue homomorphism on M_0, yielding an exact sequence

$$0 \longrightarrow H^q_{\mathrm{nr}}(K, (\mathbb{Z}/p)(q-1)) \longrightarrow H^q_{\mathrm{tame}}(K, (\mathbb{Z}/p)(q-1)) \xrightarrow{\partial_v} H^{q-1}_{\text{ét}}(F, (\mathbb{Z}/p)(q-2)) \longrightarrow 0,$$

where $H^q_{\mathrm{nr}}(K, (\mathbb{Z}/p)(q-1))$ is the unramified subgroup with respect to v. Finally, notice that $H^q_{\mathrm{nr}}(K, (\mathbb{Z}/p)(q-1)) \cong H^q_{\text{ét}}(F, (\mathbb{Z}/p)(q-1))$ by the henselian property of F.

3.4 Mixed characteristic case: char$(K) = 0$

Recall that K is a henselian discretely valued field with the discrete valuation v and the residue field F of characteristic $p > 0$. We assume $\mathrm{char}(K) = 0$ in this section. Furthermore, we will assume that K contains a primitive p-th root ζ of the unity. In general, when K does not contain a primitive p-th root of the unity, we can also describe the filtration $\{M_j\}_{j\geq 0}$ and their consecutive quotients [26, Proposition 4.1].

Let $e = v(p)$ and $N = ep(p-1)^{-1}$. These two numbers are integers. Notice that $v(\zeta - 1) = e(p-1)^{-1}$ and $p \mid N$. Using the primitive p-th root ζ of the unity, we can identify $\mathbb{Z}/p = (\mathbb{Z}/p)(1) : 1 \mapsto \zeta$ and $H^q_p(K) \cong H^q_{\text{ét}}(K, (\mathbb{Z}/p)(q))$. Then we can describe the elements in $H^q_p(K)$ by symbols from Milnor K-theory.

Theorem 3.4.1 (Bloch-Gabber-Kato Theorem [7])

Let F be a field of characteristic $p > 0$. For all integers $n \geq 0$, the differential symbol

$$\phi^n_F : K^M_n(F)/p \to H^n(F, (\mathbb{Z}/p)(n)) = \Omega^n_{F,\log}$$

is an isomorphism.

Kato uses the unit group filtration on O_K to define a decreasing filtration $\{M^j\}_{j\geq 0}$ on $H^q_p(K)$. For $j \geq 0$, M^j is the subgroup of $H^q_p(K)$ generated by elements of the form

$$\{a, b_1, \cdots, b_{q-1}\}$$

with $a \in U^j_K$ (Definition 3.2.2), $b_1, \ldots, b_q \in K^\times$. It is clear that

$$H^{q,q-1}(K) = M^0 \supset M^1 \supset \cdots \supset M^{ep(p-1)^{-1}} \supset M^{[ep(p-1)^{-1}+1]} = 0.$$

Notice that $M^n = 0$ for $n > ep(p-1)^{-1}$ by the henselian property of K. More precisely, when $n > ep(p-1)^{-1}$, $1 + \pi^n O_K \subset (1 + \pi^{n-e} O_K)^p$ by

$$(1 + \pi^{n-e} x)^p = 1 + p\pi^{n-e} x + \sum_{i=2}^{p-1} c_i x^i + \pi^{p(n-e)} x^p$$

with $v(c_i) > v(p\pi^{n-e}) = n$ and $p(n-e) > n$. Kato proved that the two filtrations $\{M_j\}$ and $\{M^{N-j}\}$ coincide, that is, $M_j = M^{N-j}$ for each j [26, Proposition 4.1]. Therefore, we will use M_j in the following context for convenience.

Let $\pi \in O_K$ be a uniformizer for v. For any $j > 0$, we define three homomorphisms depending on whether $p \nmid j$, $p \mid j < N$ and $j = N$. In each case, a simple computation shows that the homomorphim is well defined up to a choice of a uniformizer. First, consider the case when j is relatively prime to p. We define

$$\Omega_F^{q-1} \to M_j / M_{j-1}$$

by

$$\bar{a} \frac{d\bar{b}_1}{\bar{b}_1} \wedge \cdots \wedge \frac{d\bar{b}_{q-1}}{\bar{b}_{q-1}} \mapsto \{1 + \pi^{N-j} a, b_1, \cdots, b_{q-1}\} \pmod{M_{j-1}},$$

for $a \in O_K$ and $b_1, \ldots, b_{q-1} \in O_K^\times$.

Now we define the second homomorphism. Let Z_F^{q-1}, Z_F^{q-2} be the subgroup of closed forms in $\Omega_F^{q-1}, \Omega_F^{q-2}$ respectively. For $j > 0$ and $p \mid j$, define a homomorphism

$$\Omega_F^{q-1}/Z_F^{q-1} \oplus \Omega_F^{q-2}/Z_F^{q-2} \to M_j/M_{j-1}$$

as follows: On the first summand, it is defined as

$$\bar{a} \frac{d\bar{b}_1}{\bar{b}_1} \wedge \cdots \wedge \frac{d\bar{b}_{q-1}}{\bar{b}_{q-1}} \mapsto \{1 + \pi^{N-j} a, b_1, \cdots, b_{q-1}\} \pmod{M_{j-1}},$$

and for the second summand it is defined as

$$\bar{a} \frac{d\bar{b}_1}{\bar{b}_1} \wedge \cdots \wedge \frac{d\bar{b}_{q-2}}{\bar{b}_{q-2}} \mapsto \{1 + \pi^{N-j} a, b_1, \cdots, b_{q-2}, \pi\} \pmod{M_{j-1}},$$

where $a \in O_K$ and $b_1, \ldots, b_{q-1} \in O_K^\times$.

Finally, we define the third homomorphism. For $j = N$, define a homomorphism

$$K_q^M(F)/p \oplus K_{q-1}^M(F)/p \to M_N/M_{N-1}$$

as follows: On the first summand, it is defined as

$$\{\bar{a}_1, \cdots, \bar{a}_q\} \mapsto \{a_1, \cdots, a_q\},$$

and for the second summand it is defined as

$$\{\bar{a}_1, \cdots, \bar{a}_{q-1}\} \mapsto \{a_1, \cdots, a_{q-1}, \pi\}.$$

The homomorphisms are well defined (although they depend on the choice of uniformizer π).

There is residue homomorphism on the tamely ramified subgroup

$$\partial_v : H^q_{\text{tame}}(K, (\mathbb{Z}/p)(p-1)) \cong H^q_{\text{tame}}(K, (\mathbb{Z}/p)(p)) \to H^{q-1}_p(F),$$

characterized by the property that

$$\partial_v(\{1+\pi^N a, b_1, \cdots, b_{q-2}, \pi\}) = \bar{a}\frac{d\bar{b}_1}{\bar{b}_1} \wedge \cdots \wedge \frac{d\bar{b}_{q-2}}{\bar{b}_{q-2}},$$

where $a \in O_K, b_1, \ldots, b_{q-2} \in O_K^\times$.

Theorem 3.4.2 (char$(K) = 0$, mixed characteristic case)
Let K be a henselian discretely valued field of characteristic 0 with the valuation v and the residue field F of characteristic $p > 0$. Assume that K contains a primitive p-th root ζ of 1. Let $N = v(p)p(p-1)^{-1}$. Then $H^q_p(K) = H^q(K, (\mathbb{Z}/p)(q-1)) \cong H^q(K, (\mathbb{Z}/p)(q))$ has an increasing filtration $\{M_j\}_{j=0}^N$ as above, with isomorphisms (depending on the choice of a uniformizer)

$$M_j/M_{j-1} \cong \begin{cases} \Omega_F^{q-1} & \text{if } p \nmid j, \\ \Omega_F^{q-1}/Z_F^{q-1} \oplus \Omega_F^{q-2}/Z_F^{q-2} & \text{if } 0 < j < ep(p-1)^{-1} \text{ and } p \mid j, \\ K_q^M(F)/p \oplus K_{q-1}^M(F)/p & \text{if } j = N\ . \end{cases}$$

Moreover, M_0 is the tame subgroup and there is a well-defined residue homomorphism on M_0, yielding an exact sequence

$$0 \longrightarrow H^q_{\text{nr}}(K, (\mathbb{Z}/p)(q-1)) \longrightarrow H^q_{\text{tame}}(K, (\mathbb{Z}/p)(q-1)) \xrightarrow{\partial_v} H^{q-1}_{\text{ét}}(F, (\mathbb{Z}/p)(q-2)) \longrightarrow 0,$$

where $H^q_{\text{nr}}(K, (\mathbb{Z}/p)(q-1))$ is the unramified subgroup with respect to v. Finally, $H^q_{\text{nr}}(K, (\mathbb{Z}/p)(q-1)) \cong H^q_{\text{ét}}(F, (\mathbb{Z}/p)(q-1))$ by the henselian property of K.

3.5 Symbol length problem of groups $K_2^M(F)/p$ and Ω_F^1/Z_F^1

In this section, let F be a field of characteristic $p > 0$. we will investigate the symbol length problems groups $K_2^M(F)/p$ and Ω_F^1/Z_F^1.

3.5.1 Symbol length of $K_2^M(F)/p$

Definition 3.5.1 (Symbol length in $K_2^M(F)/p$)

Let k be a field. Let $\alpha \in K_2^M(F)/p$. The symbol length $\mathrm{len}(\alpha)$ *of α in $K_2^M(F)/p$ is defined to be the minimal integer m such that $\alpha = \{a_1, b_1\} + \cdots + \{a_m, b_m\}$ in $K_2^M(F)/p$.*

Then we define the symbol length of $K_2^M(F)/p$ by

$$\mathrm{len}(K_2^M(F)/p) := \sup_{\alpha}\{\mathrm{len}(\alpha)\}.$$

Recall that the p-rank of F is defined to be the integer $\log_p([F : F^p])$. We collect the known results when the p-rank of F is no more than 3.

Lemma 3.5.2 ([35, Lemma 1.3])

Let F be field of characteristic $p > 0$ and $[F : F^p] = p$. Then $K_2^M(F)/p = 0$.

Theorem 3.5.3 ([6, Theorem 3.4])

Let F be a field of characteristic $p > 0$ and $[F : F^p] = p^n, 2 \le n \le 3$. Assume that F does not admit any finite extension of degree prime to p. Then

$$\mathrm{len}(K_2^M(F)/p) \le \begin{cases} 1, & n = 2; \\ 3, & n = 3. \end{cases}$$

Notice that the assumption that F does not admit any finite extension of degree prime to p can be weakened to $F = F^{p-1} := \{x^{p-1} \mid x \in F\}$. In fact, the key lemma in the proof of Theorem 3.5.3 is the following.

Lemma 3.5.4 ([25, Section 1, Lemma 3], [14, Lemma 3.2])

Let F be a field of characteristic $p > 0$ and E a purely inseparable extension of degree p of k. Assume $F = F^{p-1}$. Let $g : E \to F$ be a F-linear map. Then there exists a non-zero element $c \in E$ such that $g(c^i) = 0$ for $i = 1, \cdots, p-1$.

When $p = 2$, the condition $F = F^{p-1}$ is naturally satisfied. Then we have the following corollary.

Corollary 3.5.5

Let F be a field of characteristic $p = 2$ and $[F : F^p] = p^n, 2 \leq n \leq 3$. Then

$$\mathrm{len}(K_2^M(F)/p) = \begin{cases} 1, & n = 2; \\ 3, & n = 3. \end{cases}$$

3.5.2 Symbol length of Ω_F^1/Z_F^1

Definition 3.5.6 (Symbol length in Ω_F^1/Z_F^1)

Let F be a field of characteristic $p > 0$. Let $\alpha \in \Omega_F^1/Z_F^1$. The symbol length $\mathrm{len}(\alpha)$ *of α in Ω_F^1/Z_F^1 is defined to be the minimal integer m such that $\alpha = a_1 db_1 + \cdots + a_m db_m$ in Ω_F^1/Z_F^1.*

Then we define the symbol length of Ω_F^1/Z_F^1 by

$$\mathrm{len}(\Omega_F^1/Z_F^1) := \sup_{\alpha}\{\mathrm{len}(\alpha)\}.$$

The symbol length of Ω_F^1/Z_F^1 is clearly controlled by the p-rank of F. If the p-rank of F is 1, i.e. $[F : F^p] = p$, we have that $\Omega_F^1/Z_F^1 = 0$, since there is no nontrivial 2-form over F and every 1-form over F is closed. Meanwhile, if the p-rank of F is n, the symbol length of Ω_F^1/Z_F^1 is no more than n.

Following the observation for the case $[F : F^p] = p$, we make the following conjecture.

Conjecture 3.5.7

Let F be a field of characteristic $p > 0$ and $[F : F^p] = p^n$ for $n \in \mathbb{N}_{>0}$. Assume that F does not admit any finite extension of degree prime to p. Then $\mathrm{len}(\Omega_F^1/Z_F^1) \leq n - 1$.

The following proposition gives us the hint to make the conjecture.

Proposition 3.5.8

Let F be a field of characteristic $p > 0$ and $[F : F^p] = p^n, n \in \mathbb{N}_{>0}$. Suppose $\alpha \in \Omega_F^1/Z_F^1$. Then there exists a degree p^{n-1} inseparable field extension E/F such that $[\alpha_E] = 0$ in Ω_E^1/Z_E^1.

Proof. Since $[F : F^p] = p^n$, there exists a p-basis of F given by $\{x_1, \cdots, x_n\}$ for some $x_i \in F$. We have $\alpha = \sum_{i=1}^{n} f_i dx_i$ for some $f_i \in F$. Let $E = F[t_1, \cdots, t_{n-1}]/(t_1^p - x_1, \cdots, t_{n-1}^p - x_{n-1})$. It follows that $\alpha_E = f_n dx_n$, where $f_n = \sum_{j=0}^{p-1} g_j^p x_n^j$. For $j \neq p-1$, $g_j^p x_n^j dx_n = d(\frac{g_j^p x_n^{j+1}}{j+1}) \in Z_E^1$. When $j = p-1$, $g_{p-1}^p x^p \mathrm{dlog}(x) \in Z_E^1$ by Cartier's isomorphism. Hence, $[\alpha_E] = 0$ in Ω_E^1/Z_E^1. □

Besides the case $[F : F^p] = p$, we give evidence for Conjecture 3.5.7 in the case $p = 2$ and $n = 2$.

Lemma 3.5.9

Let F be a field of characteristic $p = 2$ and $[F : F^p] = p^2$. Then $\mathrm{len}(\Omega_F^1/Z_F^1) = 1$.

Proof.

Since $[F : F^p] = p^2$, there exist $s, t \in F$ such that the set $\{s^i t^j\}_{(i,j)}$ is a basis for F as an F^p-vector space.

Let $\alpha \in \Omega_F^1/Z_F^1$. Then for some $f, g \in F$, we get the following equalities modulo Z_F^1:

$$\begin{aligned}
\alpha &= f\mathrm{dlog}(s) + g\mathrm{dlog}(t) \\
&= \Big(\sum_{0\le i,j\le p-1} f_{ij}^p s^i t^j \mathrm{dlog}(s)\Big) + \Big(\sum_{0\le i,j\le p-1} g_{ij}^p s^i t^j \mathrm{dlog}(t)\Big) \\
&= \Big(f_{01}^2 t\mathrm{dlog}(s) + f_{11}^2 st\mathrm{dlog}(s)\Big) + \Big(g_{10}^2 s\mathrm{dlog}(t) + g_{11}^2 st\mathrm{dlog}(t)\Big) \\
&= f_{01}^2 t\mathrm{dlog}(s) + g_{10}^2 s\mathrm{dlog}(t) + \Big(f_{11}^2 - g_{11}^2\Big) st\mathrm{dlog}(s)
\end{aligned}$$

Now, suppose that $\alpha = adb \in \Omega_F^1/Z_F^1$. Then we have that

$$\begin{aligned}
adb =& \Big(\sum_{(0\le i,j\le p-1} a_{ij}^p s^i t^j\Big) d\Big(\sum_{0\le i,j\le p-1} b_{ij}^p s^i t^j\Big) \\
=& \Big(a_{01}^2 t + a_{10}^2 s + a_{11}^2 st\Big) d\Big(b_{01}^2 t + b_{10}^2 s + b_{11}^2 st\Big) \\
=& \Big(a_{11}^2 b_{10}^2 s^2 + a_{10}^2 b_{11}^2 s^2\Big) t\mathrm{dlog}(s) + \Big(a_{11}^2 b_{01}^2 t^2 + a_{01}^2 b_{11}^2 t^2\Big) s\mathrm{dlog}(t) + \Big(a_{10}^2 b_{01}^2 - a_{01}^2 b_{10}^2\Big) st\mathrm{dlog}(t).
\end{aligned}$$

Hence, it suffices to solve the following system of equations in the variables a_{ij}, b_{ij} for $0 \leq i, j \leq 1$:

$$\begin{aligned} f_{01}^2 &= a_{11}^2 b_{10}^2 s^2 + a_{10}^2 b_{11}^2 s^2 \\ g_{10}^2 &= a_{11}^2 b_{01}^2 t^2 + a_{01}^2 b_{11}^2 t^2 \\ f_{11}^2 - g_{11}^2 &= a_{10}^2 b_{01}^2 - a_{01}^2 b_{10}^2. \end{aligned}$$

Since F is of characteristic 2, it follows that

$$\begin{aligned} f_{01} &= a_{11} b_{10} s + a_{10} b_{11} s \\ g_{10} &= a_{11} b_{01} t + a_{01} b_{11} t \\ f_{11} + g_{11} &= a_{10} b_{01} + a_{01} b_{10}. \end{aligned}$$

Now, to solve this system of equations, we can write down a solution explicitly when $f_{01} \neq 0$. Let $a_{11} = 0$ and $b_{11} = 1$. Then we have that $a_{10} = \frac{f_{01}}{s}$ and $a_{01} = \frac{g_{10}}{t}$. Next, we take $b_{10} = 0$. It follows that $b_{01} = \frac{s(f_{11} + g_{11})}{f_{01}}$. The other case follows similarly.

Finally, we finish the proof in the case $p = 2$. More precisely, we have that

$$\alpha = (\frac{f_{01}^2}{s} + \frac{g_{10}^2}{t}) d(\frac{s^2(f_{11}^2 + g_{11}^2)}{f_{01}^2} t + st) \text{ in } \Omega_F^1 / Z_F^1. \tag{3.5.1}$$

Hence the symbol length is 1. □

For $(p, n) \neq (2, 2)$, we can also formulate the system of equations in a similar way. But the number of equations and variables increase exponentially as p and n increase. Hence, we will need a more nuanced approach in the general case.

We will provide a different approach to the symbol length problem of the group Ω_F^1 / Z_F^1 using the foliation and Galois theory of purely inseparable extensions in the appendix.

CHAPTER 4

PERIOD-INDEX PROBLEMS OF HENSELIAN DISCRETELY VALUED FIELDS

In this chapter, we show that it is sufficient to prove the Conjecture 1.2.1 for wildly ramified p-Brauer classes. Through out this chapter, let K be a henselian discretely valued field with the valuation v, valuation ring O_K and residue field F of characteristic $p > 0$. Suppose that $[F : F^p] = p^n, n \in \mathbb{N}$.

4.1 Reduction to the p-torsion part of Brauer group

Proposition 4.1.1 ([46, Proposition 2.1], [31, Proposition 6.1])

Suppose that a field K and all its finite extensions L, have the property that for all central simple A/L of period p satisfies $\operatorname{ind}(A) \leq p^m$. *Then, any A/K of period p^n satisfies* $\operatorname{ind}(A) \leq p^{mn}$.

Proposition 4.1.2 ([47, Proposition 5.3])

Suppose that K is a henselian discretely valued field with the residue field F of characteristic $p > 0$ and $[F : F^p] = p^n$. Let L be a finite extension of K. Then L is also a henselian discretely valued field with the residue field E and $[E : E^p] = p^n$.

Proof. We reduce to either case of a finite separable extension case or a purely inseparable simple extension case. When L/K is finite separable, the statement follows from Lemma 2.3.5 and [39, Remark 09E8]. When E/F is a purely inseparable simple extension, the statement follows from [39, Lemma 04GH] and lemmas 4.1.3, 4.1.4 below. □

Lemma 4.1.3 ([35, Lemma 3.1])

Let B be a regular local ring with field of fractions K, residue field κ and maximal ideal m. Let n be a natural number and $u \in B$ a unit such that $[\kappa(u^{\frac{1}{n}}) : \kappa] = n$. Then $B[u^{\frac{1}{n}}]$ is a regular local ring with residue field $\kappa(\bar{u}^{\frac{1}{n}})$.

Lemma 4.1.4 ([35, Lemma 3.2])

Let B be a regular local ring with field of fractions K, residue field κ and maximal ideal m. Let $\pi \in m$ be a regular prime and n a natural number. Then $B[\pi^{\frac{1}{n}}]$ is a regular local ring with residue field κ.

Combining these two propositions above, it suffices to verify Conjecture 1.2.1 for p-torsion Brauer classes. Moreover, we can assume that the residue field F does not admit any finite extension of degree prime to p by the lemma below.

Lemma 4.1.5 ([28])

Let K be a field and $\alpha \in \mathrm{Br}(K)$ a class annihilated by n. If L/K is a finite field extension of degree d and n is relatively prime to d, then $\mathrm{per}(\alpha) = \mathrm{per}(\alpha|_L)$ *and* $\mathrm{ind}(\alpha) = \mathrm{ind}(\alpha|_L)$.

Next we want to show that the tamely ramified classes satisfy the conjectured period-index bounds. The tamely ramified Brauer classes are exactly the elements in M_0 (Definition 3.2.3). By fixing a uniformizer π in K, we have the following split exact sequence

$$0 \longrightarrow \mathrm{Br}(K)[p] \longrightarrow \mathrm{Br}_{\mathrm{tame}}(K)[p] \xrightarrow{\partial} H^1_{\text{ét}}(F, \mathbb{Z}/p) \longrightarrow 0, \tag{4.1.1}$$

Since $[F : F^p] = p^n$, it follows that $\mathrm{Br.dim}_p(F) \leq n$ [11, Corollary 3.4]. So this takes care of one of the two components form the split sequence above. The elements arising from the H^1 term are split by degree-p extensions and so the conjectural bound follows in this case. So we get:

Lemma 4.1.6 (Tamely ramified p-torsion Brauer classes)

Let K be a henselian discretely valued field with the residue field F of characteristic $p > 0$. Assume that $[F : F^p] = p^n, n \in \mathbb{N}$. Let $\alpha \in \mathrm{Br}(K)[p]$ and $\mathrm{sw}(\alpha) = 0$. *Then* $\mathrm{ind}(\alpha) \mid \mathrm{per}(\alpha)^{n+1}$.

Notice that this lemma works for both equal characteristic case and mixed characteristic case.

4.2 Kato's results in the p-rank 1 case

In this section, we will recall Kato's results and the proof in the p-rank 1 case.

Proposition 4.2.1 ([24, Section 4, Lemma 5])

Let K be a complete field with a discrete valuation v and residue field F. Suppose that $\mathrm{char}(F) = p > 0$ *and* $[F : F^p] = p$. *Suppose that* $\omega \in \mathrm{Br}(K)[p]$ *and* $\omega \notin Br(K_{tame}/K)[p]$. *Then the division algebra D which represents ω is a degree p division algebra whose residue algebra is a purely inseparable field extension of degree p over F.*

Moreover, suppose that $\mathrm{char}(K) = p > 0$. *Let π be a uniformizer of K. In this case,* $D = [a, b)$

where $a \in K, b \in K^{\times}$, and it must have one of the following two forms:

$$(i)\ [\frac{f}{\pi^{pm}}, e\pi),\ \textit{where}\ f \in O_K,\ \bar{f} \notin F^p, m > 0, v(e) = 0.$$

$$(ii)\ [\frac{c}{\pi^{n}}, g),\ \textit{where}\ g \in O_K,\ \bar{g} \notin F^p, v(c) = 0\ \textit{and n is prime to}\ p.$$

In both case, D is decomposed by a totally ramified field extension of degree p and a field extension of degree p whose residue field is a purely inseparable extension.

We notice that Kato's proof can be generalized to the henselian case easily. Hence, we put the generalized result below with proof:

Theorem 4.2.2

Let K be a henselian field of characteristic $p > 0$ with a discrete valuation v and residue field F. Suppose that $[F : F^p] = p$. Suppose that $\omega \in \mathrm{Br}(K)[p]$ and $\omega \notin Br(K_{tame}/K)[p]$. Then the division algebra D which represents ω is a degree p divison algebra with inseparable residue field extension.

Moreover, let π be a uniformizer of K. Then $D = [a, b)$ for some $a \in K, b \in K^{\times}$ and it has one of the following two forms:

$$(i)\ [\frac{f}{\pi^{pm}}, e\pi),\ \textit{where}\ f \in O_K,\ \bar{f} \notin F^p, m > 0, v(e) = 0.$$

$$(ii)\ [\frac{c}{\pi^{n}}, g),\ \textit{where}\ g \in O_K,\ \bar{g} \notin F^p, v(c) = 0\ \textit{and n is prime to}\ p.$$

The following lemma plays the fundamental role in the proof. It explains how the Swan conductor of the class $a\mathrm{dlog}(1 + b)$ is affected by the valuations of a, b.

Lemma 4.2.3 (Kato [26])

Let $a, b \in K,\ i, j \in \mathbb{Z}$, and assume that $v_K(a) \geq -i,\ v_K(b) \geq j > 0$. Then we have

$$a\mathrm{dlog}(1 + b) \in M_{i-j}. \tag{4.2.1}$$

More precisely, if $a \neq 0$, we have

$$a\mathrm{dlog}(1 + b) + ab\mathrm{dlog}(a) \in M_{i-2j}. \tag{4.2.2}$$

Proof.

$$\begin{aligned}
a\mathrm{dlog}(1+b) &= \frac{a}{1+b}d(1+b)\\
&\equiv -(1+b)d(\frac{a}{1+b}) \bmod d(K)\\
&\equiv -bd(\frac{a}{1+b}) \bmod d(K)\\
&= -(ab)d(\frac{1}{1+b}) - (\frac{b}{1+b})da\\
&\equiv -\frac{b}{1+b}da \bmod M_{i-2j}\\
&\equiv -bda \bmod M_{i-2j}\\
&= -(ab)\mathrm{dlog}(a) \bmod M_{i-2j}.
\end{aligned}$$

□

Proof of Theorem 4.2.2.

The proof follows from the following two steps by induction on i:

(i) Hypotheses:

$$\begin{aligned}
&\omega \in \mathrm{Br}(K)[p],\\
&\omega \equiv [\frac{f}{\pi^{pm}}\mathrm{dlog}(\pi)] \bmod M_i,\\
&f \in O_K,\ \bar{f} \notin F^p,\ pm > i \geq 0,\\
&\pi \text{ is a uniformizer of } K.
\end{aligned}$$

Conclusion:

$$\begin{aligned}
&\text{There exist } f' \text{ and } \pi' \text{ such that}\\
&\omega \equiv [\frac{f'}{\pi^{pm}}\mathrm{dlog}(\pi')] \bmod M_{i-1}\\
&v(f'-f) \geq pm-i, \pi'/\pi \in U_K^{(pm-i)}.
\end{aligned}$$

(ii) Hypotheses:

$$\omega \in \mathrm{Br}(K)[p],$$
$$\omega \equiv [\frac{c}{\pi^n}\mathrm{dlog}(g)] \bmod M_i,$$
$$g \in O_K,\ \bar{g} \notin F^p, v(c) = 0, n \geq i \geq 0$$
$$\pi \text{ is a uniformizer of } K.$$

Conclusion:

There exist c' and g' such that

$$\omega \equiv [\frac{c'}{\pi^n}\mathrm{dlog}(g')] \bmod M_{i-1}$$
$$v(c'-c) > n-i, g'/g \in U_K^{(n-i)}.$$

We prove both of these simultaneously in two cases: $i > 0$ and $i = 0$.

(1) $\boldsymbol{i > 0}$: For (i), if $p \mid i$, the conclusion is clear since $M_i/M_{i-1} \cong F/F^p$ by fixing the uniformizer π. If $p \nmid i$, for any $g \in O_K$, by Lemma 4.2.3,

$$\begin{aligned}\frac{f}{\pi^{pm}}\mathrm{dlog}(1+h\pi^{pm-i}) &\equiv -\frac{fh\pi^{pm-i}}{\pi^{pm}}\mathrm{dlog}(\frac{f}{\pi^{pm}})\\ &= -\frac{fh}{\pi^i}\mathrm{dlog}(f) \bmod M_{pm-2(pm-i)}.\end{aligned}$$

Since $[k:k^p]=p$, we can find $h \in O_K$ such that $\omega - [\frac{f}{\pi^{pm}}\mathrm{dlog}(\pi(1+h\pi^{pm-i}))] \in M_{i-1}$. Hence the conclusion follows.

Then let us look at (ii). If $p \nmid i$, the conclusion follows since the p-rank of the residue field F is 1 and $M_i/M_{i-1} \cong \Omega^1_F$ by fixing the uniformizer π. If $p \mid i$, then for any $e \in O_K$, by Lemma 4.2.3 we have

$$\begin{aligned}\frac{c}{\pi^n}\mathrm{dlog}(1+e\pi^{n-i}) &\equiv -\frac{ce\pi^{n-i}}{\pi^n}\mathrm{dlog}(\frac{c}{\pi^n})\\ &= \underbrace{-\frac{ce}{\pi^i}\mathrm{dlog}(c)}_{Sw(*)<i} - \frac{nce}{\pi^i}\mathrm{dlog}(\pi)\\ &= -\frac{nce}{\pi^i}\mathrm{dlog}(\pi) \bmod M_{i-1}.\end{aligned}$$

The $(*)$ term has a Swan conductor smaller than i, since its residue corresponds to a term in $\Omega^1_F/Z^1_F \cong 0$. Now we can find $e \in O_K$ such that $\omega - \frac{c}{\pi^n}\mathrm{dlog}(g(1+e\pi^{n-i})) \in M_{i-1}$. Hence the conclusion follows in this case.

(2) $\boldsymbol{i = 0}$: First fix the uniformizer π. By Theorem 3.3.1, we have that

$$M_0 \cong \mathrm{Br}(F)[p] \oplus F/\mathcal{P}(F).$$

For both hypotheses, the proof proceeds in two steps: First, we modify the condition in each hypothesis so that the resulting symbol algebra is congruent to ω modulo $\mathrm{Br}(F)[p]$. In the second step, we finish the proof. We give details for (i) as an example.

Since $\omega - [\frac{f}{\pi^{pm}}\mathrm{dlog}(\pi)] \in M_0$, then there exists $f_1 \in O_K$ such that $\omega - [\frac{f + f_1\pi^{pm}}{\pi^{pm}}\mathrm{dlog}(\pi)] \in \mathrm{Br}(F)[p]$. Set $f' = f + f_1\pi^{pm}$. For any $h \in O_K$, by Lemma 4.2.3,

$$\begin{aligned}\frac{f'}{\pi^{pm}}\mathrm{dlog}(1 + h\pi^{pm}) &\equiv -f'h\mathrm{dlog}(\frac{f'}{\pi^{pm}})\\ &= -f'h\mathrm{dlog}(f') \text{ in } \mathrm{Br}(F)[p].\end{aligned}$$

We can find $h \in O_K$ such that $\omega - [\frac{f + f_1\pi^{pm}}{\pi^{pm}}\mathrm{dlog}(\pi(1 + h\pi^{pm}))] \in \mathrm{Br}(F)[p]$. Therefore the conclusion follows.

□

The following theorem was first proved in [12]. We give a different proof using ideas here.

Theorem 4.2.4 ([12, Theorem 2.3])

Let K be a henselian discretely valued field of characteristic $p > 0$ with the residue field F. Suppose that F is a local field. Then $\mathrm{Br.dim}(K) = 1$.

Proof. Since F is a local field of characteristic $p > 0$, we have that $k \cong \mathbb{F}_q((s))$, $q = p^n$. By Theorem 4.2.2, a wildly ramified Brauer class in $\mathrm{Br}(K)[p]$ is represented by a symbol algebra of degree p. So it suffices to show that a tamely ramified Brauer class in $\mathrm{Br}(K)$ has symbol length 1. Let $\omega \in \mathrm{Br}(K_{\mathrm{tame}}/K)[p]$. Then $\omega = [a, \pi) + [b, c)$ where a defines an unramified degree p Artin-Schreier extension of K and $[b, c) \in \mathrm{Br}(F)[p]$. By [37, Corollary 3, Page 194], $[b, c)$ is split by the degree p Artin-Schreier extension defined by a. Hence, $\alpha = [a, e)$ for some $e \in K^\times$.

Notice that a finite extension of a local field is still a local field. Hence, combining with Theorem 4.1.1, we get the desired conclusion. □

4.3 Equal characteristic case

In this section, we will prove the period-index result for p-torsion part of the Brauer group of a henselian discretely valued field of characteristic $p > 0$.

Let K be a henselian discretely valued field of characteristic $p > 0$ with the valuation v, valuation ring O_K and residue field F with $[F : F^p] = p^n, n \in \mathbb{N}$. Given a p-torsion Brauer class $\alpha \in \mathrm{Br}(K)[p]$, there are three cases: (i) $\mathrm{sw}(\alpha) = 0$, (ii) $p \nmid \mathrm{sw}(\alpha) > 0$ and (iii) $p \mid \mathrm{sw}(\alpha) > 0$. Recall Conjecture 3.5.7 mentioned in the previous chapter. We should point out that only Case (iii) is relevant to this conjecture. Now, Case (i) is already discussed in Lemma 4.1.6.

4.3.1 Case (ii): $p \nmid \mathbf{sw}(\alpha) > 0$

We will prove the following theorem in this subsection.

Theorem 4.3.1

Let F be a field of characteristic $p > 0$ and $[F : F^p] = p^n$, $n \in \mathbb{N}_{>0}$. Let K be a henselian discretely valued field of characteristic $p > 0$ with the residue field F. Suppose that $\alpha \in \mathrm{Br}(K)[p]$ and $p \nmid \mathrm{sw}(\alpha) > 0$. Then $\mathrm{ind}(\alpha) \mid \mathrm{per}(\alpha)^n$.

Proof.

Let $\{\bar{x}_1, \cdots, \bar{x}_n\}$ be a p-basis of F. Let $\{x_1, \cdots, x_n\}$ be the lifting of the p-basis in K and π be a uniformizer of K. Since $p \nmid \mathrm{sw}(\alpha) = k > 0$, we have that $\alpha \equiv \frac{a_1}{\pi^k}\mathrm{dlog}(x_1) + \cdots + \frac{a_n}{\pi^k}\mathrm{dlog}(x_n) \bmod M_{k-1}$, where either $a_i = 0$ or $v(a_i) = 0$, $a_i \neq 0$ for at least one i. The proof is based on the following downward induction on j:

Hypotheses:

$\alpha \in \mathrm{Br}(K)[p],\ 0 \le j < k,$

$\alpha \equiv [\frac{a_1}{\pi^k}\mathrm{dlog}(x_1) + \cdots + \frac{a_n}{\pi^k}\mathrm{dlog}(x_n)] \bmod M_j,$

either $\bar{a}_i = 0$ or $v(a_i) = 0$ for all $i \in \{1, \cdots, n\}$,

$\bar{a}_i \neq 0$ for at least one i,

$\{\bar{x}_1, \cdots, \bar{x}_n\}$is a p-basis of F, and π is a uniformizer of K.

Conclusion:

There exist $\{a'_i\}_i$, $\{x'_i\}_i$ and π' for $i \in \{1, \cdots, n\}$ such that

$\alpha \equiv [\frac{a'_1}{\pi'^k}\mathrm{dlog}(x'_1) + \cdots + \frac{a'_n}{\pi'^k}\mathrm{dlog}(x'_n)] \bmod M_{j-1},$

either $\bar{a}'_i = 0$ or $v(a'_i) = 0$ for all $i \in \{1, \cdots, n\}$,

$\bar{a}'_i \neq 0$ for at least one i,

$\{\bar{x}'_1, \cdots, \bar{x}'_n\}$is a p-basis of F, and π' is a uniformizer of K.

If $p \nmid j$, by fixing the uniformizer π, we have $M_j/M_{j-1} \cong \Omega^1_F$. Since $\{\bar{x}_1, \cdots, \bar{x}_n\}$ is a p-basis of F, the conclusion easily follows.

If $p \mid j > 0$, by fixing the uniformizer π, we have $M_j/M_{j-1} \cong \Omega^1_F/Z^1_F \oplus F/F^p$. Denote the projections from M_j/M_{j-1} to two direct components by P_1, P_2 respectively. WLOG, we assume that $\bar{a}_1 \neq 0$. For any $c \in O_K$, by Lemma 4.2.3,

$$\frac{a_1}{\pi^k}\mathrm{dlog}(1 + c\pi^{k-j}) \equiv -\frac{a_1 c \pi^{k-j}}{\pi^k}\mathrm{dlog}(\frac{a_1}{\pi^k}) \tag{4.3.1}$$

$$= -\frac{a_1 c}{\pi^j}\mathrm{dlog}(a_1) + \frac{k a_1 c}{\pi^j}\mathrm{dlog}(\pi) \bmod M_{k-2(k-j)}. \tag{4.3.2}$$

Since $\alpha - [\frac{a_1}{\pi^k}\mathrm{dlog}(x_1) + \cdots + \frac{a_n}{\pi^k}\mathrm{dlog}(x_n)] \in M_j$, we can choose c such that $\overline{k a_1 c} = P_2(\alpha - [\frac{a_1}{\pi^k}\mathrm{dlog}(x_1) + \cdots + \frac{a_n}{\pi^k}\mathrm{dlog}(x_n)])$. Then $\{x_1(1 + c\pi^{k-j}), x_2, \cdots, x_n\}$ gives a different lifting of the p-basis $\{\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n\}$. We use this new lifting to match the class from $\big(P_1(\alpha - [\frac{a_1}{\pi^k}\mathrm{dlog}(x_1(1 + c\pi^{k-j})) + \cdots + \frac{a_n}{\pi^k}\mathrm{dlog}(x_n)]), 0\big) \in \Omega^1_F/Z^1_F \oplus F/F^p$. The conclusion follows.

If $j = 0$, the proof is similar to the case ($p \mid j > 0$), since we treat the elements from Ω^1_F and Ω^1_F/Z^1_F in the same way. □

4.3.2 (iii) $p \mid \mathbf{sw}(\alpha) > 0$

Proposition 4.3.2

Let F be a field of characteristic $p > 0$ and $[F : F^p] = p^n$, $n \in \mathbb{N}_{>0}$. Let K be a henselian discretely valued field of characteristic $p > 0$ with the residue field F. Assume that Conjecture 3.5.7 is true *and F does not admit any finite extension of degree prime to p. Let $\alpha \in \mathrm{Br}(K)[p]$ and $p \mid \mathrm{sw}(\alpha) > 0$. Then $\mathrm{ind}(\alpha) \mid \mathrm{per}(\alpha)^n$.*

Proof.

The Conjecture 3.5.7 implies that the symbol length of the group Ω^1_F/Z^1_F is no more than $n-1$. Since $p \mid \mathrm{sw}(\alpha) = k > 0$, we have that $\alpha \equiv [\frac{a_1}{\pi^k}\mathrm{dlog}(x_1) + \cdots + \frac{a_n}{\pi^k}\mathrm{dlog}(x_{n-1}) + \frac{b}{\pi^k}\mathrm{dlog}(\pi)] \bmod M_{k-1}$. For $i \in \{1, \cdots, n-1\}$, either $\bar{a}_i = 0$ or $v(a_i) = 0$, while either $b = 0$ or $v(b) = 0$, $\bar{b} \notin F^p$. Not all $\bar{a}_i, b$ are zero and π is a uniformizer. $v(x_i) = 0$, $\{\bar{x}_1, \cdots, \bar{x}_{n-1}\}$ is a F^p-linearly independent set.

We shall discuss in cases $v(b) = 0$, $\bar{b} \notin F^p$ and $b = 0$ separately.

4.3.2.1 $v(b) = 0$, $\bar{b} \notin F^p$

The proof is based on the following induction on j:

Hypotheses:

$\alpha \in \mathrm{Br}(K)[p]$, $0 \leq j < k$,

$\alpha \equiv [\frac{a_1}{\pi^k}\mathrm{dlog}(x_1) + \cdots + \frac{a_{n-1}}{\pi^k}\mathrm{dlog}(x_{n-1}) + \frac{b}{\pi^k}\mathrm{dlog}(\pi)] \bmod M_j$,

either $\bar{a}_i = 0$ or $v(a_i) = 0$ for all $i \in \{1, \cdots, n\}$,

$v(b) = 0$, $\bar{b} \notin F^p$, and π is a uniformizer of K,

$v(x_i) = 0$, $\{\bar{x}_1, \cdots, \bar{x}_{n-1}\}$ is a F^p-linearly independent set.

Conclusion:

There exist $\{a_i'\}_i$, $\{x_i'\}_i$, b and π' for $i \in \{1, \cdots, n-1\}$ such that

$\alpha \equiv [\frac{a_1'}{\pi'^k}\mathrm{dlog}(x_1') + \cdots + \frac{a_{n-1}'}{\pi'^k}\mathrm{dlog}(x_{n-1}') + \frac{b'}{\pi'^k}\mathrm{dlog}(\pi')] \bmod M_{j-1}$,

either $\bar{a}_i' = 0$ or $v(a_i') = 0$ for all $i \in \{1, \cdots, n\}$,

$v(b') = 0$, $\bar{b}' \notin F^p$, and π' is a prime element of K,

$v(x_i') = 0$, $\{\bar{x}_1', \cdots, \bar{x}_{n-1}'\}$ is a F^p-linearly independent set.

Let $\alpha' = \alpha - [\frac{a_1}{\pi^k}\mathrm{dlog}(x_1) + \cdots + \frac{a_{n-1}}{\pi^k}\mathrm{dlog}(x_{n-1}) + \frac{b}{\pi^k}\mathrm{dlog}(\pi)]$. If $p \mid j > 0$, by fixing the uniformizer π, we have $M_j/M_{j-1} \cong \Omega_F^1/Z_F^1 \oplus F/F^p$. Since $\{\bar{x}_1, \cdots, \bar{x}_{n-1}\}$ is a F^p-linearly independent set and $[F : F^p] = p^n$, we can choose $x_n \in O_K$ such that $\{\bar{x}_1, \cdots, \bar{x}_n\}$ is a p-basis of F. Denote the projections from M_j/M_{j-1} to two direct components by P_1, P_2 respectively. Then $P_1(\alpha') = f_1\mathrm{dlog}(\bar{x}_1) + \cdots + f_n\mathrm{dlog}(\bar{x}_n)$. For any $c \in O_K$, by Lemma 4.2.3,

$$\frac{b}{\pi^k}\mathrm{dlog}(1 + c\pi^{k-j}) \equiv -\frac{bc}{\pi^j}\mathrm{dlog}(\frac{b}{\pi^k}) \tag{4.3.3}$$

$$= -\frac{bc}{\pi^j}\mathrm{dlog}(b) \bmod M_{k-2(k-j)}. \tag{4.3.4}$$

We can choose $c \in O_K$ such that $-\overline{bc}\mathrm{dlog}(\bar{b})$ coincides with f_n on $\mathrm{dlog}(\bar{x}_n)$ part. Let $g \in O_K$ be a lifting of $P_2(\alpha') \in k/k^p$. Then $P_1\big(\alpha - [\frac{a_1}{\pi^k}\mathrm{dlog}(x_1) + \cdots + \frac{a_{n-1}}{\pi^k}\mathrm{dlog}(x_{n-1}) + \frac{b + g\pi^{k-l}}{\pi^k}\mathrm{dlog}(\pi(1 + c\pi^{k-l}))]\big)$ is supported away from $\mathrm{dlog}(\bar{x}_n)$ and $P_2(\bullet) = 0$. Hence the conlusion follows.

If $j = 0$ or $p \nmid j > 0$, the proof is similar to the case $(p \mid j > 0)$, since we treat the elements from Ω_F^1, Ω_F^1/Z_F^1, $\mathrm{Br}(F)[p]$ in the same way. More precisely, we are using their liftings in Ω_F^1.

4.3.2.2 $b = 0$

In this case, the proof can be reduced to either the case $(p \nmid \mathrm{sw}(\alpha > 0)$ or the above case. Hence we finish the proof. □

4.4 Mixed characteristic case

In this section, we will prove the period-index result for p-torsion part of the Brauer group of a henselian discretely valued field of characteristic 0 with residual characteristic $p > 0$.

Let K be a henselian discretely valued field of characteristic 0 with the valuation v, valuation ring O_K and residue field F of characteristic $p > 0$. Let $[F : F^p] = p^n, n \in \mathbb{N}$. In the mixed characteristic case, we are not always able to express a Brauer class as a sum of symbol algebras, since K may not contain a primitive p-th root of unity. However, when addressing period-index problems, we can always reduce to the case where K contains a primitive p-th root of unity (as noted in Lemma 4.1.5).

In our setting, if the field K does not contain a primitive p-th root of unity, we can adjoin a primitive p-th root of unity ζ to the field K. The field extension $K(\zeta)/K$ is of degree $p - 1$. Hence it suffices to consider the period-index problem over $K(\zeta)$.

In the rest of this section, we assume that K contains a primitive p-th root ζ of the unity. Notice that $v(\zeta - 1) = \frac{v(p)}{p-1}$ and $p \mid N := \frac{pv(p)}{p-1}$. Given a p-torsion Brauer class $\alpha \in \mathrm{Br}(K)[p]$, there are four cases: (i) $\mathrm{sw}(\alpha) = 0$, (ii) $p \nmid \mathrm{sw}(\alpha) > 0$, (iii) $p \mid \mathrm{sw}(\alpha),\ 0 < \mathrm{sw}(\alpha) < N$ and (iv) $\mathrm{sw}(\alpha) = N$. We mentioned Conjecture 3.5.7 in the previous chapter. It is important to note that the case (iii) and (iv) are relevant to Conjecture 3.5.7. Additionally, the case (iv) requires the bound of symbol length of the group $K_2(F)/pK_2(F)$.

The case (i) has been addressed in Lemma 4.1.6. We will discuss the other three cases separately in the subsequent subsections.

4.4.1 (ii) $p \nmid \mathbf{sw}(\alpha) > 0$

We will prove the following theorem in this subsection.

Theorem 4.4.1

Let F be a field of characteristic $p > 0$ and $[F : F^p] = p^n$, $n \in \mathbb{N}_{>0}$. Let K be a henselian discretely valued field of characteristic 0 with the residue field F. Suppose that $\alpha \in \mathrm{Br}(K)[p]$ and $p \nmid \mathrm{sw}(\alpha) > 0$. Then $\mathrm{ind}(\alpha) \mid \mathrm{per}(\alpha)^n$.

Proof. Let $\{\bar{x}_1, \cdots, \bar{x}_n\}$ be a p-basis of F. Let $\{x_1, \cdots, x_n\}$ be a lifting of the p-basis and π be a uniformizer. Since $p \nmid \mathrm{sw}(\alpha) = k > 0$, we have that $\alpha \equiv [\{1 + \pi^{N-k}a_1, x_1\} + \cdots + \{1 + \pi^{N-k}a_n, x_n\}] \bmod M_{k-1}$, where either $a_i = 0$ or $v(a_i) = 0$, $a_i \neq 0$ for at least one i. The proof is based on the following induction on j:

Hypotheses:

$$\alpha \in \mathrm{Br}(K)[p],\ 0 \le j < k,$$

$$\alpha \equiv [\{1+\pi^{N-k}a_1, x_1\} + \cdots + \{1+\pi^{N-k}a_n, x_n\}] \bmod M_j,$$

either $\bar{a}_i = 0$ or $v(a_i) = 0$ for all $i \in \{1, \cdots, n\}$,

$\bar{a}_i \neq 0$ for at least one i,

$\{\bar{x}_1, \cdots, \bar{x}_n\}$is a p-basis of F, and π is a uniformizer of K.

Conclusion:

There exist $\{a'_i\}_i$, $\{x'_i\}_i$ and π' for $i \in \{1, \cdots, n\}$ such that

$$\alpha \equiv [\{1+\pi'^{N-k}a'_1, x'_1\} + \cdots + \{1+\pi'^{N-k}a'_n, x'_n\}] \bmod M_{j-1},$$

either $\bar{a}'_i = 0$ or $v(a'_i) = 0$ for all $i \in \{1, \cdots, n\}$,

$\bar{a}'_i \neq 0$ for at least one i,

$\{\bar{x}'_1, \cdots, \bar{x}'_n\}$is a p-basis of F, and π' is a uniformizer of K.

If $p \nmid j$, by fixing the uniformizer π, we have $M_j/M_{j-1} \cong \Omega^1_F$. Since $\{\bar{x}_1, \cdots, \bar{x}_n\}$ is a p-basis of F, the conclusion easily follows.

If $p \mid j > 0$, by fixing the uniformizer π, we have $M_j/M_{j-1} \cong \Omega^1_F/Z^1_F \oplus F/F^p$. Denote the projections from M_j/M_{j-1} to two direct components by P_1, P_2 respectively. WLOG, we assume that $\bar{a}_1 \neq 0$. For any $c \in O_K$, by Lemma 4.2.3,

$$\begin{aligned}\{1+\pi^{N-k}a_1, 1+c\pi^{k-j}\} &\equiv -\{1+\pi^{N-j}a_1c, -\pi^{N-k}a_1\} \\ &= -\{1+\pi^{N-j}a_1c, -a_1\} - \{1+\pi^{N-j}a_1c, \pi^{N-k}\} \bmod M_{k-2(k-j)} \\ &= -\{1+\pi^{N-j}a_1c, a_1\} + \{1-(N-k)\pi^{N-j}a_1c, \pi\} \bmod M_{j-1}\end{aligned}$$

Since $\alpha - [\{1+\pi^{N-k}a_1, x_1\} + \cdots + \{1+\pi^{N-k}a_n, x_n\}] \in M_j$, we can choose c such that $-\overline{(N-k)a_1c} = P_2(\alpha - [\{1+\pi^{N-k}a_1, x_1\} + \cdots + \{1+\pi^{N-k}a_n, x_n\}])$. Then $\{x_1(1+c\pi^{k-j}), x_2, \cdots, x_n\}$ gives a different lifting of the p-basis $\{\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n\}$. We use this new lifting to match the class from

$\big(P_1(\alpha - [\{1 + \pi^{N-k} a_1, x_1(1 + c\pi^{k-j})\} + \cdots + \{1 + \pi^{N-k} a_n, x_n\}]), 0\big) \in \Omega^1_F / Z^1_F \oplus F/F^p$. The conclusion follows.

If $j = 0$, the proof is similar to the case $(p \mid j > 0)$, since we treat the elements from Ω^1_F and Ω^1_F/Z^1_F in the same way. □

4.4.2 (iii) $p \mid \mathbf{sw}(\alpha),\ 0 < \mathbf{sw}(\alpha) < N$

We will prove the following theorem in this subsection.

Theorem 4.4.2

Let F be a field of characteristic $p > 0$ and $[F : F^p] = p^n,\ n \in \mathbb{N}_{>0}$. Let K be a henselian discretely valued field of characteristic 0 with the residue field F. Assume that Conjecture 3.5.7 is true *and F does not admit any finite extension of degree prime to p. Suppose that $\alpha \in Br(K)[p]$ and $p \mid sw(\alpha),\ 0 < sw(\alpha) < N$. Then $ind(\alpha) \mid per(\alpha)^n$.*

Proof.

The Conjecture 3.5.7 suggests that the symbol length of the group Ω^1_F/Z^1_F is no more than $n-1$. Since $p \mid \mathrm{sw}(\alpha) = k > 0$, we have that $\alpha \equiv [\{1 + \pi^{N-k} a_1, x_1\} + \cdots + \{1 + \pi^{N-k} a_{n-1}, x_{n-1}\} + \{1 + \pi^{N-k} b, \pi\}] \bmod M_{k-1}$. For $i \in \{1, \cdots, n-1\}$, either $a_i = 0$ or $v(a_i) = 0$, while either $b = 0$ or $v(b) = 0,\ \bar{b} \notin F^p$. Not all a_i, b are zero and π is a uniformizer. $v(x_i) = 0$, $\{\bar{x}_1, \cdots, \bar{x}_{n-1}\}$ is a F^p-linearly independent set.

We shall discuss in cases $v(b) = 0,\ \bar{b} \notin F^p$ and $b = 0$ separately.

4.4.2.1 $v(b) = 0,\ \bar{b} \notin F^p$

The proof is based on the following induction on j:

Hypotheses:

$\alpha \in \mathrm{Br}(K)[p],\ 0 \leq j < k,$

$\alpha \equiv [\{1+\pi^{N-k}a_1, x_1\} + \cdots + \{1+\pi^{N-k}a_{n-1}, x_{n-1}\} + \{1+\pi^{N-k}b, \pi\}] \bmod M_j,$

either $\bar{a}_i = 0$ or $v(a_i) = 0$ for all $i \in \{1, \cdots, n\}$,

$v(b) = 0,\ \bar{b} \notin F^p$, and π is a prime element of K,

$v(x_i) = 0$, $\{\bar{x}_1, \cdots, \bar{x}_{n-1}\}$ is a F^p-linearly independent set.

Conclusion:

There exist $\{a_i'\}_i$, $\{x_i'\}_i$, b and π' for $i \in \{1, \cdots, n-1\}$ such that

$\alpha \equiv [\{1+\pi'^{N-k}a_1', x_1'\} + \cdots + \{1+\pi'^{N-k}a_{n-1}', x_{n-1}'\} + \{1+\pi'^{N-k}b', \pi'\}] \bmod M_{j-1},$

either $\bar{a}_i' = 0$ or $v(a_i') = 0$ for all $i \in \{1, \cdots, n\}$,

$v(b') = 0,\ \bar{b}' \notin F^p$, and π' is a prime element of K,

$v(x_i') = 0$, $\{\bar{x}_1', \cdots, \bar{x}_{n-1}'\}$ is a F^p-linearly independent set.

Let $\alpha' = \alpha - [\{1+\pi^{N-k}a_1, x_1\} + \cdots + \{1+\pi^{N-k}a_{n-1}, x_{n-1}\} + \{1+\pi^{N-k}b, \pi\}]$. If $p \mid j > 0$, by fixing the uniformizer π, we have $M_j/M_{j-1} \cong \Omega_F^1/Z_F^1 \oplus F/F^p$. Since $\{\bar{x}_1, \cdots, \bar{x}_{n-1}\}$ is a F^p-linearly independent set and $[F : F^p] = p^n$, we can choose $x_n \in O_K$ such that $\{\bar{x}_1, \cdots, \bar{x}_n\}$ is a p-basis of F. Denote the projections from M_j/M_{j-1} to two direct components by P_1, P_2 respectively. Then $P_1(\alpha') = f_1 \mathrm{dlog}(\bar{x}_1) + \cdots + f_n \mathrm{dlog}(\bar{x}_n)$. For any $c \in O_K$, by Lemma 4.2.3,

$$\{1+\pi^{N-k}b, 1+c\pi^{k-j}\} \equiv -\{1+\pi^{N-j}bc, -\pi^{N-k}b\} \tag{4.4.1}$$

$$= -\{1+\pi^{N-j}bc, -b\} \bmod M_{k-2(k-j)} \tag{4.4.2}$$

$$= \{1-\pi^{N-j}bc, b\} \bmod M_{j-1} \tag{4.4.3}$$

We can choose $c \in O_K$ such that $-\overline{bc}\mathrm{dlog}(\bar{b})$ coincides with f_n on $\mathrm{dlog}(\bar{x}_n)$ part. Let $g \in O_K$ be a lifting of $P_2(\alpha') \in F/F^p$. Then $P_1\big(\alpha - [\{1+\pi^{N-k}a_1, x_1\} + \cdots + \{1+\pi^{N-k}a_{n-1}, x_{n-1}\} + \{1+\pi^{N-k}(b+g\pi^{k-j}), \pi(1+c\pi^{k-j})\}]\big)$ is supported away from $\mathrm{dlog}(\bar{x}_n)$ and $P_2(\bullet) = 0$. Hence the conclusion follows.

If $j = 0$ or $p \nmid j > 0$, the proof is similar to the case ($p \mid j > 0$), since we treat the elements from Ω^1_F, Ω^1_F/Z^1_F, $\mathrm{Br}(F)[p]$ in the same way. More precisely, we are using their liftings in Ω^1_F.

4.4.2.2 $b = 0$

In this case, the proof can be reduced to either the case ($p \nmid \mathrm{sw}(\alpha > 0)$ or the above case. Hence we finish the proof. □

4.4.3 (iv) sw$(\alpha) = N$

We will prove the following theorem in this subsection.

Theorem 4.4.3

Let F be a field of characteristic $p > 0$ and $[F : F^p] = p^n$, $n \in \mathbb{N}_{>0}$. Let K be a henselian discretely valued field of characteristic 0 with the residue field F. Assume that Conjecture 3.5.7 is true *and F does not admit any finite extension of degree prime to p. Suppose that $\alpha \in Br(K)[p]$ and $sw(\alpha) = N$. Then $ind(\alpha) \mid per(\alpha)^n$.*

Proof.

The proof is similar to the case (iii). Fixing a uniformizer π, it follows that $M_N/M_{N-1} \cong K_2(F)/pK_2(F) \oplus K_1(F)/pK_1(F)$. Hence, at the starting point, we need both the symbol length bounds of $K_2(F)/pK_2(F)$ and Ω^1_F/Z^1_F. □

4.5 Symbol length problems of higher Kato's groups

In this section, we generalize the ideas from previous sections to investigate the symbol length problems of higher Kato's groups. Let K be a henselian discretely valued field of residual characteristic $p > 0$. We prove that any wildly ramified element in $H^3_{\text{ét}}(K, (\mathbb{Z}/p)(2))$ is split by a purely inseparable extension of degree p.

Theorem 4.5.1

Let K be a henselian discretely valued field of characteristic $p > 0$ with the residue field F. Suppose that $[F : F^p] = p^2$. Let $\alpha \in H^3_{\text{ét}}(K, (\mathbb{Z}/p)(2))$ such that $p \nmid sw(\alpha) > 0$. Then we have that $\alpha = \omega \wedge \dfrac{dc}{c}$ for some $\omega \in \Omega^1_K$ and $c \in K^\times$.

Proof. First, notice that $\Omega^2_F/Z^2_F = 0$, since $[F : F^p] = p^2$, which forces there is no non-trivial 3-form over F. By Theorem 3.3.1, we have that

$$M_j/M_{j-1} \cong \begin{cases} \Omega^2_F & \text{if } j > 0 \text{ and } p \nmid j, \\ \Omega^1_F/Z^1_F & \text{if } j > 0 \text{ and } p \mid j. \end{cases}$$

Let $\{\bar{x}_1, \bar{x}_2\}$ be a p-basis of F. Let $\{x_1, x_2\}$ be the lifting of the p-basis in K and let π be a uniformizer of K. Then $\{dx_1 \wedge dx_2\}$ gives a basis of Ω^2_F. Since $p \nmid \mathrm{sw}(\alpha) = k > 0$, we have that $\alpha \equiv \frac{a}{\pi^k}\mathrm{dlog}(x_1) \wedge \mathrm{dlog}(x_2) \bmod M_{k-1}$, where $v(a) = 0$. The proof is based on the following induction on j:

Hypotheses:

$$\alpha \in H^3_{\text{ét}}(K, (\mathbb{Z}/p)(2)),\ 0 \le j < k,$$
$$\alpha \equiv [\frac{a}{\pi^k}\mathrm{dlog}(x_1) \wedge \mathrm{dlog}(x_2) + \frac{b}{\pi^k}\mathrm{dlog}(x_2) \wedge \mathrm{dlog}(\pi)] \bmod M_j,$$
$$\bar{a} \neq 0,\ v(b) > 0,$$

$\{\bar{x}_1, \bar{x}_2\}$is a p-basis of F, and π is a uniformizer of K.

Conclusion:

There exist a', b' and x_2' such that

$$\alpha \equiv [\frac{a'}{\pi^k}\mathrm{dlog}(x_1) \wedge \mathrm{dlog}(x_2') + \frac{b'}{\pi^k}\mathrm{dlog}(x_2') \wedge \mathrm{dlog}(\pi)] \bmod M_{j-1},$$
$$\bar{a'} \neq 0,\ v(b') > 0,$$

$\{\bar{x}_1, \bar{x}_2'\}$is a p-basis of F, and π is a uniformizer of K.

Let $\alpha' = \alpha - [\frac{a}{\pi^k}\mathrm{dlog}(x_1) \wedge \mathrm{dlog}(x_2) + \frac{b}{\pi^k}\mathrm{dlog}(x_2) \wedge \mathrm{dlog}(\pi)]$. If $p \mid j > 0$, by fixing the uniformizer π, we have $M_j/M_{j-1} \cong \Omega^1_F/Z^1_F$. Since $\{\bar{x}_1, \bar{x}_2\}$ is a p-basis for F, we have $[\alpha'] \equiv f_1\mathrm{dlog}(\bar{x}_1) + f_2\mathrm{dlog}(\bar{x}_2)$ in M_j/M_{j-1}, where $f_1, f_2 \in F$. For any $c \in O_K$,

$$\frac{a}{\pi^k}\mathrm{dlog}(x_1) \wedge \mathrm{dlog}(1 + c\pi^{k-j}) \equiv -\frac{ac}{\pi^j}\mathrm{dlog}(x_1) \wedge \mathrm{dlog}(\frac{a}{\pi^k}) \tag{4.5.1}$$
$$= \frac{kac}{\pi^j}\mathrm{dlog}(x_1) \wedge \mathrm{dlog}(\pi) \bmod M_{k-2(k-j)}. \tag{4.5.2}$$

We can choose $c \in O_K$ such that $\overline{kac}$ coincides with f_1 and let $g \in O_K$ be a lifting of f_2. Then $\alpha - [\frac{a}{\pi^k}\mathrm{dlog}(x_1) \wedge \mathrm{dlog}(x_2(1+c\pi^{k-j})) + \frac{b+g\pi^{k-j}}{\pi^k}\mathrm{dlog}(x_2) \wedge \mathrm{dlog}(\pi)] \in M_{j-1}$. Moreover, denote $x_2' = x_2(1+c\pi^{k-j})$. It follows that $\alpha - [\frac{a}{\pi^k}\mathrm{dlog}(x_1) \wedge \mathrm{dlog}(x_2') + \frac{b+g\pi^{k-j}}{\pi^k}\mathrm{dlog}(x_2') \wedge \mathrm{dlog}(\pi)] \in M_{j-1}$. Hence the conclusion follows.

If $p \mid j > 0$, by fixing the uniformizer π, we have $M_j/M_{j-1} \cong \Omega^2_F$. Then the conclusion follows easily by using x_1, x_2 as the liftings of p-basis.

If $j = 0$, the proof is similar to the case $(p \mid j > 0)$, since $M_0 \cong H^2_{\text{ét}}(F, (\mathbb{Z}/p)(1)) \oplus H^3_{\text{ét}}(F, (\mathbb{Z}/p)(2))$. It is a combination of these two previous arguments. □

Theorem 4.5.2

Let K be a henselian discretely valued field of characteristic $p > 0$ with the residue field F. Suppose that $[F : F^p] = p^2$. Let $\alpha \in H^3_{\text{ét}}(K, (\mathbb{Z}/p)(2))$ such that $p \mid sw(\alpha) > 0$. Then Conjecture 3.5.7 implies that $\alpha = \omega \wedge \frac{dc}{c}$ for some $\omega \in \Omega^1_K$ and $c \in K^\times$.

Proof. First, notice that $\Omega^2_F/Z^2_F = 0$, since $[F : F^p] = p^2$, which forces there is no non-trivial 3-form over F. By Theorem 3.3.1, we have that

$$M_j/M_{j-1} \cong \begin{cases} \Omega^2_F & \text{if } j > 0 \text{ and } p \nmid j, \\ \Omega^1_F/Z^1_F & \text{if } j > 0 \text{ and } p \mid j. \end{cases}$$

Let $\{\bar{x}_1, \bar{x}_2\}$ be a p-basis of F. Let $\{x_1, x_2\}$ be the lifting of the p-basis in K and let π be a uniformizer of K. Then $\{dx_1 \wedge dx_2\}$ gives a basis of Ω^2_F. Since $p \mid \mathrm{sw}(\alpha) = k > 0$, Conjecture 3.5.7 implies that $\alpha \equiv \frac{a}{\pi^k}\mathrm{dlog}(b) \wedge \mathrm{dlog}(\pi) \bmod M_{k-1}$, where $v(a) = v(b) = 0$. Notice that $\{a, b\}$ defines a p-basis of F, since $\bar{a}\mathrm{dlog}(\bar{b})$ is non-trivial in Ω^1_F/Z^1_F. The proof is based on the following induction on j:

Hypotheses:

$$\alpha \in H^3_{\text{ét}}(K,(\mathbb{Z}/p)(2)),\ 0 \le j < k,$$

$$\alpha \equiv [\frac{c}{\pi^k}\text{dlog}(a) \wedge \text{dlog}(b) + \frac{a}{\pi^k}\text{dlog}(b) \wedge \text{dlog}(\pi)] \bmod M_j,$$

$$v(c) > 0,\ \bar{a}\text{dlog}(\bar{b}) \text{ is nonzero in } \Omega^1_F/Z^1_F,$$

$$\pi \text{ is a uniformizer of } K.$$

Conclusion:

There exist a', b' and c' such that

$$\alpha \equiv [\frac{c'}{\pi^k}\text{dlog}(a') \wedge \text{dlog}(b') + \frac{a'}{\pi^k}\text{dlog}(b') \wedge \text{dlog}(\pi)] \bmod M_{j-1},$$

$$v(c') > 0,\ \bar{a}'\text{dlog}(\bar{b}') \text{ is nonzero in } \Omega^1_F/Z^1_F,$$

$$\pi \text{ is a uniformizer of } K.$$

Let $\alpha' = \alpha - [\frac{c}{\pi^l}\text{dlog}(a) \wedge \text{dlog}(b) + \frac{a}{\pi^k}\text{dlog}(b) \wedge \text{dlog}(\pi)]$. If $p \mid j > 0$, by fixing the uniformizer π, we have $M_j/M_{j-1} \cong \Omega^1_F/Z^1_F$. Since $\{\bar{a}, \bar{b}\}$ is a p-basis for F, we have $[\alpha'] \equiv f_1\text{dlog}(\bar{a}_1) + f_2\text{dlog}(\bar{b}_2)$ in M_j/M_{j-1}, where $f_1, f_2 \in F$. For any $e \in O_K$,

$$\frac{a}{\pi^k}\text{dlog}((1+e\pi^{k-j})) \wedge \text{dlog}(\pi) \equiv -\frac{ae}{\pi^j}\text{dlog}(\frac{a}{\pi^k}) \wedge \text{dlog}(\pi) \tag{4.5.3}$$

$$= -\frac{ae}{\pi^j}\text{dlog}(a) \wedge \text{dlog}(\pi) \bmod M_{k-2(k-j)}. \tag{4.5.4}$$

We can choose $e \in O_K$ such that $-\overline{ae}$ coincides with f_1 and let $g \in O_K$ be a lifting of f_2. Then $\alpha - [\frac{c}{\pi^l}\text{dlog}(a) \wedge \text{dlog}(b) + \frac{a+g\pi^{k-j}}{\pi^k}\text{dlog}(b(1+e\pi^{k-j})) \wedge \text{dlog}(\pi)] \in M_{j-1}$. Moreover, denote $a' = a + g\pi^{k-j}$ and $b' = b(1+e\pi^{k-j})$. It follows that $\alpha - [\frac{c}{\pi^l}\text{dlog}(a') \wedge \text{dlog}(b') + \frac{a'}{\pi^k}\text{dlog}(b') \wedge \text{dlog}(\pi)] \in M_{j-1}$. Hence, the conclusion follows.

If $p \mid j > 0$, by fixing the uniformizer π, we have $M_j/M_{j-1} \cong \Omega^2_F$. Then the conclusion follows easily by using x_1, x_2 as the liftings of p-basis.

If $j = 0$, the proof is similar to the case ($p \mid j > 0$), since $M_0 \cong H^2_{\text{ét}}(F,(\mathbb{Z}/p)(1)) \oplus H^3_{\text{ét}}(F,(\mathbb{Z}/p)(2))$. It is a combination of these two previous arguments. □

CHAPTER 5

LOGARITHMIC DE RHAM COHOMOLOGY WITH SUPPORT

In this chapter, we will continue on our discussion of p-torsion part of Brauer groups of C_m fields. As we noted in the introduction,, there is one common class of C_m fields: the function fields of dimension $m-n$ algebraic varieties over C_n fields for $0 \leq n \leq m$. When studying the Brauer group of such a function field, an important concept is its behavior in codimension 1, since the Brauer group of a Noetherian, integral, regular scheme has purity in codimension 1 (Theorem 2.1.3). We restate this theorem here for clarity.

Theorem 5.0.1 (Purity in codimension 1 [44])

For a Noetherian, integral, regular scheme X with function field K,

$$H^2_{\text{ét}}(X, \mathbb{G}_m) = \bigcap_{x \in X^{(1)}} H^2_{\text{ét}}(O_{X,x}, \mathbb{G}_m) \textit{ in } H^2_{\text{ét}}(K, \mathbb{G}_m).$$

Recall that there is an injection from the Brauer group of the scheme X into the Brauer group of its function field K:

$$\mathrm{Br}(X) \hookrightarrow \mathrm{Br}(K).$$

Therefore, to understand the Brauer group of the function field K, it is essential to understand the cokernel of this homomorphism.

Artin and Mumford [4] provides significant insight into this area. When X is a surface, they showed that there exists an exact sequence that relates the Brauer group of X, the Brauer group of the function field K, and the ramification behavior of Brauer classes at both codimension 1 points and closed points (points of codimenison 2).

Theorem 5.0.2 ([41])

Let S be a smooth projective surface over an algebraically closed field k with $char(k) = p \geq 0$ and l be a prime number different from p. Suppose that $H^1_{ét}(S, \mathbb{Q}/\mathbb{Z}) = 0$. There is a canonical exact sequence

$$0 \longrightarrow Br_l(S) \longrightarrow Br_l(k(S)) \longrightarrow \bigoplus_{\textit{curves } C} H^1_{\text{ét}}(k(C), \mathbb{Z}/l) \longrightarrow \bigoplus_{\textit{closed points}} \mu_l^{-1} \longrightarrow \mu_l^{-1} \longrightarrow 0,$$

where μ_l denotes the l-th roots of unity.

In fact, this sequence could be derived from the Bloch-Ogus spectral sequence.

Theorem 5.0.3

Assume that X is smooth over a perfect field k of characteristic $p \geq 0$ and l be a prime number different from p. Then there is a spectral sequence

$$E_1^{pq} = \bigoplus_{x \in X^{(p)}} H^{q-p}(x, \mu_l^{\otimes n-p}) \Rightarrow H^{p+q}(X, \mu_l^{\otimes n}) \tag{5.0.1}$$

Here $X^{(p)}$ are the points of codimension p in X.

We should notice that the above discussion are restricted to the case when the torsion is prime to the base characteristic. We will give a systematic investigation of the case when the torsion is equal to the base characteristic.

5.1 Bloch-Ogus spectral sequence in positive characteristic

For the logarithmic de Rham cohomology, we also have the Bloch-Ogus spectral sequence.

Theorem 5.1.1 ([8, 15])

Let X be an equidimensional scheme over $\mathbb{F}_p$. Then we have the coniveau spectral sequence

$$E_1^{s,t} = \bigoplus_{x \in X^{(s)}} H_x^{s+t}(X, \Omega^i_{X,\log}) \Rightarrow E^{s+t} = H^{s+t}(X, \Omega^i_{X,\log})$$

converging to the logarithmic de Rham cohomology, where

$$H_x^m(X, \Omega^i_{X,\log}) := \varinjlim_{x \in U} H^m_{\bar{\{x\}} \cap U}(U, \Omega^i_{X,\log}) = H_x^m(X_x, \Omega^i_{X,\log})$$

and U runs through open neighborhoods of x in X (the last equality follows from the excision).

The complex of $E_1^{\bullet,q}$-terms

$$0 \longrightarrow \bigoplus_{x \in X^{(0)}} H_x^q(X, \Omega^i_{X,\log}) \longrightarrow \bigoplus_{x \in X^{(1)}} H_x^{q+1}(X, \Omega^i_{X,\log}) \longrightarrow \cdots$$

$$\longrightarrow \bigoplus_{x \in X^{(s)}} H_x^{q+s}(X, \Omega^i_{X,\log}) \longrightarrow \cdots$$

is usually called the Bloch-Ogus complex and denoted by $B^{q,i}(X)^\bullet$. It is a cohomological analogue of the Brown-Gersten-Quillen complex in algebraic K-theory.

We can also describe the E_2 page when considering k-schemes. Let k be a field of characteristic $p > 0$. For every integer m, we have the cohomology functor on k-schemes:

$$X \mapsto H^{m-i}_{\text{ét}}(X, \Omega^i_{X,\log}) = H^m_{\text{ét}}(X, \mathbb{Z}/p(i))$$

for a k-scheme X. For shorthand, we would write $H^m_p(X, i)$ in stead of the precise notation $H^{m-i}_{\text{ét}}(X, \Omega^i_{X,\log})$. The Zariski sheaf associated to the presheaf $U \mapsto H^m_p(U, i)$ is denoted by $\mathscr{H}^b_p(i)$.

For a smooth connected k-variety X, we define the *unramified cohomology group*

$$H^m_{\text{nr}}(X, \mathbb{Z}/p(i)) := H^0_{\text{Zar}}(X, \mathscr{H}^m_p(i)).$$

Then we have the following theorem which collects some well-known results.

Theorem 5.1.2

Let X be a smooth connected k-variety.

1. *We have the Bloch-Ogus spectral sequence*

$$E_2^{s,t} = H^s_{Zar}(X, \mathscr{H}^t_p(i)) \Rightarrow E^{s+t} = H^{s+t}_{\text{ét}}(X, \mathbb{Z}/p(i)) \tag{5.1.1}$$

with

$$E_2^{s,t} = 0 \textit{ if } b \notin \{i, i+1\}, \textit{ or if } a > b = i \tag{5.1.2}$$

and

$$E_2^{i,i} = H^i_{Zar}(X, \mathscr{H}^i_p(i)) \cong CH^i(X)/p. \tag{5.1.3}$$

2. *There are natural isomorphisms*

$$H^i(X, (\mathbb{Z}/p)(i)) \cong H^0_{Zar}(X, \mathscr{H}^i_p(i)) = H^i_{\text{nr}}(X, (\mathbb{Z}/p)(i)), \tag{5.1.4}$$

$$H^{2i+j}(X, (\mathbb{Z}/p)(i)) \cong H^{j+i-1}_{Zar}(X, \mathscr{H}^{i+1}_p(i)) \textit{ for } j \geq 1. \tag{5.1.5}$$

and

$$H^2_{\text{nr}}(X, (\mathbb{Z}/p)(1)) = H^0_{\text{Zar}}(X, \mathscr{H}^2_p(1)) \cong Br(X)[p]. \tag{5.1.6}$$

3. *For smooth proper connected k-varieties, the group $H^m_{\mathrm{nr}}(X, \mathbb{Z}/p(i))$ is a k-birational invariant.*

We want to finish up this section with the Gersten-type theorem which plays the most important role in the rest of this chapter. As an analogue of Gersten conjecture in algebraic K-theory, it is natural to expect that, if X is the spectrum of a regular local ring over $\mathbb{F}_p$, the Bloch-Ogus complex is acyclic in positive degree. In fact, we have the following:

Theorem 5.1.3 (Gersten-type theorem for Bloch-Ogus complex [38])
Let X be the spectrum of an equidimensional regular local ring over $\mathbb{F}_p$. Then we have

$$H^n(B^{q,i}(X)^\bullet) = \begin{cases} H^q(X, \Omega^i_{X,\log}) & (n = 0) \\ 0 & (n > 0). \end{cases}$$

It is proved in the case where X is a localization of a smooth scheme over a perfect field by Gros-Suwa [18].

This Gersten-type theorem provides us with the fundation to analyse the ramification behavior of a p-torsion Brauer class affine locally. We will mainly use the following version of Theorem 5.1.3.

Theorem 5.1.4 (Gersten-type theorem [38])
Let X be the spectrum of a 2-equidimensional regular local ring over $\mathbb{F}_p$ with the unique closed point P and quotient field K. Then we have an exact sequence

$$0 \longrightarrow H^1_{\text{ét}}(X, \Omega^1_{X,log}) \longrightarrow H^1_{\text{ét}}(K, \Omega^1_{K,log}) \xrightarrow{\delta_1} \bigoplus_{x \in X^1} H^2_x(X, \Omega^1_{X,log}) \xrightarrow{\delta_2} H^3_P(X, \Omega^1_{X,log}) \longrightarrow 0.$$

5.2 Logarithmic de Rham cohomology of affine schemes with support

The goal of this section is to identify the morphisms δ_1, δ_2 and cohomology groups appeared in Theorem 5.1.4.

5.2.1 The morphism δ_1

Let $x \in X^{(1)}$. By the étale excision theorem [33], we have the following lemma.

Lemma 5.2.1

$$H^2_x(X, \Omega^1_{X,\log}) = H^2_x(X_x, \Omega^1_{\bullet,\log}) \cong H^2_x(O^h_{X,x}, \Omega^1_{\bullet,\log}).$$

Proof.

The first isomorphism follows from the étale excision theorem. The second isomorphism follows from [34, Corollary 1.28]. □

Furthermore, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^1(O^h_{X,x}, \Omega^1_{\bullet,\log}) & \longrightarrow & H^1(K^h, \Omega^1_{\bullet,\log}) & \xrightarrow{\delta_1} & H^2_x(O^h_{X,x}, \Omega^1_{\bullet,\log}) & \longrightarrow & 0 \\
 & & \downarrow \wr & & \downarrow \wr & & \vdots & & \\
0 & \longrightarrow & \mathrm{Br}(O^h_{X,x})[p] & \longrightarrow & \mathrm{Br}(K^h)[p] & \xrightarrow{\delta'_1} & \mathrm{Br}(K^h)[p] \big/ \mathrm{Br}(O^h_{X,x})[p] & \longrightarrow & 0
\end{array}$$

The first exact row comes from the long exact sequence in local cohomology in étale topology and the second exact row is the canonical exact sequence.

It follows that

$$H^2_x(X, \Omega^1_{X,\log}) \cong \mathrm{Br}(K^h)[p] \big/ \mathrm{Br}(O^h_{X,x})[p] \ , \tag{5.2.1}$$

and we can identify the morphism δ_1 with δ'_1, i.e. $\delta_1 = \delta'_1$.

5.2.2 The Morphism δ_2

Let $y \in X^{(1)}$ and $Y := \overline{\{y\}}$ be the closure of y in X.

Lemma 5.2.2 ([38])

Let X, Z be regular schemes over $\mathbb{F}_p$ and let $i : Z \hookrightarrow X$ be a regular closed immersion of codimension r. Then we have $\underline{H}^j_Z(X, O_X) = 0, \underline{H}^j_Z(X, W_m\Omega^1_X) = 0, \underline{H}^j_Z(X, W_m\Omega^i_X/dV^{m-1}\Omega^{i-1}_X) = 0$ for $j \neq r$.

Corollary 5.2.3

Let the notation be as above. Then we have $\underline{H}^j_Z(X, W_m\Omega^i_{X,\log}) = 0$ for $j \neq r, r+1$.

Also we have the following exact diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^1_Y(X, \Omega^1_{\bullet,\log}) & \longrightarrow & H^1_y(X - \{P\}, \Omega^1_{\bullet,\log}) & \xrightarrow{\delta_1^y} & H^2_P(X, \Omega^1_{\bullet,\log}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^1_Y(X, \Omega^1_{\bullet}) & \longrightarrow & H^1_y(X - \{P\}, \Omega^1_{\bullet}) & \xrightarrow{\delta_1^y} & H^2_P(X, \Omega^1_{\bullet}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle F-I} & & \downarrow{\scriptstyle F-I} & & \downarrow{\scriptstyle F-I} & & \\
0 & \longrightarrow & H^1_Y(X, \Omega^1_{\bullet}/d\mathcal{O}) & \longrightarrow & H^1_y(X - \{P\}, \Omega^1_{\bullet}/d\mathcal{O}) & \xrightarrow{\delta_1^y} & H^2_P(X, \Omega^1_{\bullet}/d\mathcal{O}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
H^2_P(X, \Omega^1_{\bullet,\log}) & \longrightarrow & H^2_Y(X, \Omega^1_{\bullet,\log}) & \longrightarrow & H^2_y(X - \{P\}, \Omega^1_{\bullet,\log}) & \xrightarrow{\delta_2^y} & H^3_P(X, \Omega^1_{\bullet,\log}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
\tag{5.2.2}
$$

Notice that we have $H^j_y(X, \bullet) = H^j_y(X - \{P\}, \bullet)$ by excision. In the above diagram, we are using cohomology groups instead of cohomology sheaves, since X is a strictly henselian local scheme.

In order to compute δ_2 and $H^3_P(X, \Omega^1_{X,\log})$, recall the following facts about the (étale) local cohomology.

Lemma 5.2.4 ([39, Lemma 0G74])

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $Z \subset X$ be a closed subset. Let K be an object of $D(\mathcal{O}_X)$ and denote K_{ab} its image in $D(\underline{\mathbb{Z}}_X)$. Then there is a canonical map $R\Gamma_Z(X, K) \to R\Gamma_Z(X, K_{ab})$ in $D(Ab)$.

Proposition 5.2.5 ([39, Lemma 0A46])

Let S be a scheme. Let $Z \subset S$ be a closed subscheme. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_S$-module and denote $\mathcal{F}^a$ the associated quasi-coherent sheaf on the small étale site of S. Then

$$H^q_Z(S_{Zar}, \mathcal{F}) = H^q_Z(S, \mathcal{F}^a).$$

Proposition 5.2.6

For any étale morphism $f : X \to Y$, $f^*\Omega^1_Y \to \Omega^1_X$ is an isomorphism of $\mathcal{O}_X$-modules.

By Proposition 5.2.6, we get $(\Omega^1_S)^a = \Omega^1_S$ on the small étale site of S. It is also known that $(\mathcal{O}_S)^a = \mathcal{O}_S$ (or G_a), where G_a is the additive group. Then by Proposition 5.2.5, the étale local cohomology groups of X agree with the Zariski local cohomology groups.

Now it suffices to calculate the étale local cohomology groups of $\Omega^1_X/d\mathcal{O}_X$. In fact, we have the following exact sequences on the small étale site of X

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{F} \mathcal{O}_X \longrightarrow d\mathcal{O}_X \longrightarrow 0, \tag{5.2.3}$$

$$0 \longrightarrow d\mathcal{O}_X \longrightarrow \Omega^1_X \longrightarrow \Omega^1_X/d\mathcal{O}_X \longrightarrow 0.$$

These sequences follow from the Cartier isomorphism [38, Corollary 2.5], since X is affine regular and F-finite. Passing to the cohomology sequences, we have the exact sequences

$$0 \longrightarrow H^1_Y(X,\mathcal{O}_X)/H^1_Y(X,\mathcal{O}_X)^p \xrightarrow{d} H^1_Y(X,\Omega^1_X) \longrightarrow H^1_Y(X,\Omega^1_X/d\mathcal{O}_X) \longrightarrow 0.$$

$$0 \longrightarrow H^2_P(X,\mathcal{O}_X)/H^2_P(X,\mathcal{O}_X)^p \xrightarrow{d} H^2_P(X,\Omega^1_X) \longrightarrow H^2_P(X,\Omega^1_X/d\mathcal{O}_X) \longrightarrow 0. \tag{5.2.4}$$

Using (5.2.4), it suffices to calculate the local cohomology groups of Ω^1_X and $\mathcal{O}_X$. They are computed by the Cech complex in the below.

Lemma 5.2.7 ([39, Lemma 0A6R])

Let A be a noetherian ring and let $I = (f_1, \cdots, f_r) \subset A$ be an ideal. Set $Z = V(I) \subset Spec(A)$. Then

$$R\Gamma_Z(A) \simeq (\ A \longrightarrow \prod_{i_0} A_{f_{i_0}} \longrightarrow \cdots \longrightarrow A_{f_1\cdots f_r}\ )$$

in $D(A)$. If M is an A-module, then we have

$$R\Gamma_Z(M) \simeq (\ M \longrightarrow \prod_{i_0} M_{f_{i_0}} \longrightarrow \cdots \longrightarrow M_{f_1\cdots f_r}\ )$$

in $D(A)$.

Recall that $X = \text{Spec } k\ [[\pi, t]]$, where k is an algebraically closed field. Let $R = k\ [[\pi, t]]$. Then π and t are regular primes of R. Denote by $V(\pi), V(t)$ the closures of codimension 1 points (π) and (t) respectively.

Then we have the following

$$
\begin{aligned}
H^1_{V(t)}(X, \Omega^1_X) &\cong \Omega^1_{R[\frac{1}{t}]} \Big/ \Omega^1_R \,, \\
H^1_{(t)}(X, \Omega^1_X) &\cong H^1_{(t)}(D(\pi), \Omega^1_{D(\pi)}) \cong \Omega^1_{R[\frac{1}{\pi t}]} \Big/ \Omega^1_{R[\frac{1}{\pi}]} \,, \\
H^2_P(X, \Omega^1_X) &\cong \Omega^1_{R[\frac{1}{\pi t}]} \Big/ (\Omega^1_{R[\frac{1}{\pi}]} + \Omega^1_{R[\frac{1}{t}]}) \\
H^1_{V(t)}(X, O_X) &\cong R[\frac{1}{t}] \,/R \,. \\
H^1_{(t)}(X, O^1_X) &\cong H^1_{(t)}(D(\pi), O^1_{D(\pi)}) \cong R[\frac{1}{\pi t}] \Big/ R[\frac{1}{\pi}] \,, \\
H^2_P(X, O^1_X) &\cong R[\frac{1}{\pi t}] \Big/ \left(R[\frac{1}{\pi}] + R[\frac{1}{t}] \right) .
\end{aligned}
$$

Combining with the exact sequence (5.2.4), it follows that

$$
\begin{aligned}
H^2_{V(t)}(X, \Omega^1_{X,\log}) &\cong \Omega^1_{R[\frac{1}{t}]} \Big/ \left(\Omega^1_R + (F - I)\Omega^1_{R[\frac{1}{t}]} + d(R[\frac{1}{t}]) \right) \\
&\cong \mathrm{Br}(R[\frac{1}{t}])[p] \,/\mathrm{Br}(R)[p] \,, \\
H^2_{(t)}(X, \Omega^1_{X,\log}) &= \Omega^1_{R[\frac{1}{\pi t}]} \Big/ \left(\Omega^1_{R[\frac{1}{\pi}]} + (F - I)\Omega^1_{R[\frac{1}{\pi t}]} + d(R[\frac{1}{\pi t}]) \right) \\
&\cong \mathrm{Br}(R[\frac{1}{\pi t}])[p] \Big/ \mathrm{Br}(R[\frac{1}{\pi}])[p] \,, \\
H^3_P(X, \Omega^1_{X,\log}) &= \Omega^1_{R[\frac{1}{\pi t}]} \Big/ \left(\Omega^1_{R[\frac{1}{\pi}]} + \Omega^1_{R[\frac{1}{t}]} + (F - I)\Omega^1_{R[\frac{1}{\pi t}]} + d(R[\frac{1}{\pi t}]) \right) \\
&\cong \mathrm{Br}(R[\frac{1}{\pi t}])[p] \Big/ \left(\mathrm{Br}(R[\frac{1}{\pi}])[p] + \mathrm{Br}(R[\frac{1}{t}])[p] \right) .
\end{aligned}
$$

Notice that we used the fact that the localization of a unique factorization domain (UFD) at a multiplicatively closed subset is also a UFD and the Picard group of a UFD is zero. Furthermore, from the last row of (5.2.2), we get the following identification

$$
\mathrm{Br}(R[\tfrac{1}{t}])[p] \,/\mathrm{Br}(R)[p] \longrightarrow \mathrm{Br}(R[\tfrac{1}{\pi t}])[p] \Big/ \mathrm{Br}(R[\tfrac{1}{\pi}])[p] \tag{5.2.5}
$$

$$
\xrightarrow{\delta_2^y} \mathrm{Br}(R[\tfrac{1}{\pi t}])[p] \Big/ \left(\mathrm{Br}(R[\tfrac{1}{\pi}])[p] + \mathrm{Br}(R[\tfrac{1}{t}])[p] \right) \longrightarrow 0.
$$

5.3 Period-index problems of semi-global fields in positive characteristic

In this section, we are going to prove the following theorem as an application of the logarithmic de Rham cohomology with support.

Theorem 5.3.1

Let X be a smooth projective curve over $k((t))$ where k is an algebraically closed fields of characteristic $p > 0$. Suppose that there is a model $\mathcal{X}$ over $k[[t]]$ with good reduction. Suppose that $\omega \in \mathrm{Br}(X)[p]$ satisfies $\mathrm{sw}_{\mathcal{X}}(\omega) < p$. Then $\mathrm{per}(\omega) = \mathrm{ind}(\omega)$.

Here the geometric Swan conductor $\mathrm{sw}_{\mathcal{X}}$ is defined based on the smooth model $\mathcal{X}$ of X. It is not clear if the definition is independent of the choice of the smooth model. We will review the notations and define the geometric Swan conductor (Definition 5.3.4) in the next subsection.

5.3.1 Notations

Let X be an algebraic curve over $K = k((t))$, where $k = \bar{k}$ is an algebraically closed field of characteristic $p > 0$. Denote by $F = K(X)$ the function field of X. Let $\mathcal{O}_K = k[[t]]$ be the complete discrete valuation ring with the field of fraction K and t the uniformizer. Denote by T the unique closed point of $\mathrm{Spec}(\mathcal{O}_K)$.

Definition 5.3.2

An integral model $\mathcal{X}$ of X is a 2-dimensional regular $\mathcal{O}_K$-scheme such that

(i) $p : \mathcal{X} \to \mathrm{Spec}(\mathcal{O}_K)$ is flat and proper;

(ii) There is an isomorphism of K-schemes $X \simeq \mathcal{X}_K$;

(iii) The reduced scheme $(Y = \mathcal{X} \times T)_{red}$ is a 1-dimensional (proper) schemes over T whose irreducible components are all regular and has normal crossings (i.e. $\mathcal{X}_T$ only has ordinary double points as singularities).

The existence of an integral model follows from the resolution of singularities of excellent 2-dimensional schemes ([29]), and the embedded resolution of the special fiber ([30]). If X admits a smooth integral model $\mathcal{X}$ over $\mathcal{O}_K$, we say that X has *good reduction* over $\mathcal{O}_K$. In this case, the special fiber $\mathcal{X}_T$ will have a single irreducible component that is a proper smooth curve over T.

$$\begin{array}{ccccc} X & \longrightarrow & \mathcal{X} & \longleftarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(K) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_K) & \longleftarrow & T = \mathrm{Spec}(k) \end{array}$$

For a closed point P of $\mathcal{X}$, let $\mathcal{O}_{\mathcal{X},P}$ denote the local ring at P, $\hat{\mathcal{O}}_{\mathcal{X},P}$ the completion of the regular local ring $\mathcal{O}_{\mathcal{X},P}$ at its maximal ideal and F_P the field of fractions of $\hat{\mathcal{O}}_{\mathcal{X},P}$. For an open subset U of an irreducible component of Y, let R_U be the ring consisting of elements in F which are regular on U. Then $\mathcal{O}_K \subset R_U$. Let $\hat{R}_U$ be the (t)-adic completion of R_U and F_U the field of fractions of $\hat{R}_U$.

Now suppose that the algebraic curve X has good reduction over $\mathcal{O}_K$. We have the following exact sequence by purity of the Brauer groups in codimension 1:

$$0 \longrightarrow \mathrm{Br}(\mathcal{X}) \longrightarrow \mathrm{Br}(K(X)) \xrightarrow{\oplus i_x} \bigoplus_{x \in X_0} \mathrm{Br}(\mathrm{Quot}(\hat{\mathcal{O}}_{X,x})).$$

For $\omega \in \mathrm{Br}(X)$, we have that $\omega \in \mathrm{Br}(\mathcal{O}_{X,x})$ for all $x \in X_0$.

Lemma 5.3.3

Let $f : X \to Y$ be a morphism of schemes. Let $y \in Y$ and $q : X' = X \times_Y \mathrm{Spec}\mathcal{O}_{Y,y} \to X$ be the projection morphism. Then $\mathcal{O}_{X',q^{-1}(x)} \cong \mathcal{O}_{X,x}$ for any $x \in X_y$.

Recall that every effective irreducible divisor $D \subset \mathcal{X}$ is either Y (D is vertical), or the closure of a closed point $x \in X_0$ of the generic fiber (D is horizontal). Using this lemma, it follows that $\omega \in \mathrm{Br}(\mathcal{O}_{\mathcal{X},x})$ for all $x \in X_0 \subset \mathcal{X}^{(1)}$. Hence we have that ω is ramified only along the vertical divisor Y. Hence we define Kato's Swan conductor of $\omega \in \mathrm{Br}(X)$ in the following way.

Definition 5.3.4 (Swan conductor for Brauer groups of curves)

Let k be an algebraically closed field of characteristic $p > 0$ and let X be an algebraic curve over $k((t))$. Suppose X has good reduction with the associated model $\mathcal{X} \to \mathrm{Spec}\ k[[t]]$. Denote by v_Y the valuation associated to the divisor Y and F the function field $k(X)$.

Then we define the $\mathcal{X}$-Swan conductor of $\omega \in \mathrm{Br}(X)[p]$ by

$$\mathrm{sw}_{\mathcal{X}}(\omega) = \mathrm{sw}_{F,v_Y}(\omega).$$

5.3.2 Sketch of the proof

In order to prove Theorem 5.3.1, we will use the patching method from [19], which reduces the global period-index problem to two types of local period-index problems. We continue to use the notations from last section.

Let η be a generic point of an irreducible component of Y and F_η the completion of F at the discrete valuation given by η. Let D be a central simple algebra over F. By [20, 5.8], there exists an irreducible open set U_η of Y containing η such that $\operatorname{ind}(D \otimes_F F_{U_\eta}) = \operatorname{ind}(D \otimes_F F_\eta)$.

Theorem 5.3.5 (Patching, [19, Theorem 5.1], [35, Page 228])

Let D be a central simple algebra over F of period p. Let S_0 be a finite set of closed points of $\mathcal{X}$ containing all the points of intersection of the components of Y and the support of the ramification divisor of D. Let S be a finite set of closed points of $\mathcal{X}$ containing S_0 and $Y \setminus (\cup U_\eta)$, where η varies over generic points of Y. Then

$$\operatorname{ind}(D) = \operatorname{lcm}\left\{\operatorname{ind}(D \otimes F_\zeta)\right\},$$

where ζ runs over S and irreducible components of $Y \setminus S$.

We apply this theorem in our situation. First, suppose $\zeta = U$ for some irreducible component U of $Y \setminus S$. Let η be the generic point of U. Then $U \subset U_\eta$. Since $F_{U_\eta} \subset F_U$, $\operatorname{ind}(D \otimes_F F_U) \mid \operatorname{ind}(D \otimes_F F_{U_\eta}) = \operatorname{ind}(D \otimes_F F_\eta)$. Since the residue field of the generic point of U is a function field of the curve over an algebraically closed field, by Theorem 4.2.2, we have $\operatorname{ind}(D \otimes F_\eta) | p$. Hence, $\operatorname{ind}(D \otimes_F F_U) \mid p$.

Next suppose $\zeta = P \in S$, where P is a closed point of $\mathcal{X}$. By the Cohen structure theorem for an equi-characteristic field [39, Tag 0C0S], we have

$$\hat{O}_{\mathcal{X},P} \cong k[[\pi, t]],$$

where π, t are local uniformizers at P. Notice that it is actually a k-algebra isomorphism, since the residue field k is naturally embedded into the complete local ring. In general, the Cohen's structure theorem only provides a ring isomorphism instead of a k-algebra isomorphism.

To analyze the period-index problem for the field $F_P = k((\pi, t))$, we will apply Theorem 5.1.4 to the 2-dimensional regular local ring $k[[\pi, t]]$. Notice that we have the condition $\operatorname{sw}_{\mathcal{X}}(\omega) < p$. We would relate the $\mathcal{X}$-Swan conductor to the local Kato's Swan conductor in the next subsection.

5.3.3 Local Swan Conductor

We use the notations from Theorem 5.1.4. There is a commutative diagram

$$\begin{array}{ccccc} \mathrm{Br}(F)[p] \simeq H^1(F, \Omega^1_{F,\log}) & \longrightarrow & H^1(\mathcal{O}_{\mathcal{X},P}, \Omega^1_{\bullet,\log}) & \longrightarrow & H^2_{(t)}\left(\mathcal{O}_{\mathcal{X},P}, \Omega^1_{\bullet,\log}\right), \\ & \searrow & \downarrow & & \downarrow \\ & & H^1(\hat{\mathcal{O}}_{\mathcal{X},P}, \Omega^1_{\bullet,\log}) & \longrightarrow & H^2_{(t)}\left(\hat{\mathcal{O}}_{\mathcal{X},P}, \Omega^1_{\bullet,\log}\right) \end{array}$$

where the horizontal row is part of the long exact sequence in local étale cohomology associated to the sheaf $\Omega^1_{\bullet,\log}$. By (5.2.1), we have the following isomorhisms

$$H^2_{(t)}(\mathcal{O}_{\mathcal{X},P}, \Omega^1_{\bullet,\log}) \cong \mathrm{Br}(\mathrm{Frac}((\mathcal{O}_{\mathcal{X},P})^h_{(t)}))[p]/\mathrm{Br}((\mathcal{O}_{\mathcal{X},P})^h_{(t)})[p], \tag{5.3.1}$$

$$H^2_{(t)}(\hat{\mathcal{O}}_{\mathcal{X},P}, \Omega^1_{\bullet,\log}) \cong \mathrm{Br}(\mathrm{Frac}((\hat{\mathcal{O}}_{\mathcal{X},P})^h_{(t)}))[p]/\mathrm{Br}((\hat{\mathcal{O}}_{\mathcal{X},P})^h_{(t)})[p]. \tag{5.3.2}$$

Notice that, for a prime ideal $\mathfrak{p}$ of a ring R, we denote by $R^h_{\mathfrak{p}}$ the henselization of the localization $R_{\mathfrak{p}}$ of the ring R at the prime ideal $\mathfrak{p}$.

Using the diagram above, we can give a result which relates the $\mathcal{X}$-Swan conductor to the local cohomology groups as in (5.3.1) and (5.3.2).

Proposition 5.3.6

Let X be an algebraic curve over $k((t))$ with a smooth integral model $\mathcal{X} \to$ Spec $k[[t]]$ and $\omega \in \mathrm{Br}(X)[p]$. Then

$$\mathrm{sw}_{Frac((\hat{\mathcal{O}}_{\mathcal{X},P})^h_{(t)})}(\omega) = \mathrm{sw}_{Frac((\mathcal{O}_{\mathcal{X},P})^h_{(t)})}(\omega) = \mathrm{sw}_{\mathcal{X}}(\omega).$$

Proof of Proposition 5.3.6.

The second equality follows from Definition 5.3.4. For the first one, by Lemma 5.3.7, we can take $K = \mathrm{Frac}((\mathcal{O}_{\mathcal{X},P})^h_{(t)})$ and $L = \mathrm{Frac}((\hat{\mathcal{O}}_{\mathcal{X},P})^h_{(t)})$. Then it suffices to show that the residue field extension is separable, since t is the uniformizer in both fields. The residue field extension is given by $k(\pi) \to k((\pi))$, which is a completion morphism. The separability is given by Lemma 5.3.8. □

Lemma 5.3.7 ([26, Lemma 6.2, Page 119])

Let $K \subset L$ be two henselian discretely valued fields such that $\mathcal{O}_K \subset \mathcal{O}_L$ and $m_L = \mathcal{O}_L m_K$. Assume *that the residue field of L is separable over the residue field of K. Then, for any* $\omega \in Br(K)[p]$*, we have*

$$\mathrm{sw}_K(\omega) = \mathrm{sw}_L(\omega).$$

Lemma 5.3.8

Let F *be a discretely valued field with* $[F : F^p] = p$ *and* $\hat{F}$ *be its completion. Then the completion morphism* $F \to \hat{F}$ *is separable.*

Proof.

We prove it by contradiction. Suppose that there exists an algebraic extension E/F inside $\hat{F}$ which is not separable. Then we can decompose E/F as a chain of field extensions $E/L/F$ where L is separable over F and E/L is purely inseparable. Moreover, let π be a uniformizer of F. Then we have that π is still a uniformizer in L, and it gives a p-basis of L/L^p. Since E/L is purely inseparable, there exists $a \in E \subset \hat{F}$ such that $a^p \in L \setminus L^p$. Let $b = a^p$. It follows that $b = \sum_{i=0}^{p-1} f_i^p \pi^i$ in L (also in $\hat{F}$) such that $f_i \neq 0$ for some $i > 0$. However, notice that π is a uniformizer of $\hat{F}$. Therefore, it implies that b is not a p-power in $\hat{F}$, which is a contradiction. Hence the conclusion follows. □

5.3.4 The end of the proof

Theorem 5.3.9

Let $R = k[[\pi, t]]$*,* $X = \mathrm{Spec}(R)$*,* $K = \mathrm{Frac}(R)$ *and* $\omega \in \mathrm{Br}(K)[p]$ *which ramifies only along* (t) *with* $\mathrm{sw}_{K,(t)}(\omega) = m < p$*. Then* $\omega = [*, \pi)$*.*

Proof. By Theorem 5.1.4, we have the exact sequence

$$0 \longrightarrow H^1(X, \Omega^1_{X,\log}) \longrightarrow H^1(K, \Omega^1_{K,\log}) \xrightarrow{\delta_1} \bigoplus_{x \in X^1} H^2_x(X, \Omega^1_{X,\log}) \xrightarrow{\delta_2} H^3_P(X, \Omega^1_{X,\log}) \longrightarrow 0.$$

Since X is affine and regular, $H^1(X_{\text{ét}}, \Omega^1_{X,\log}) \simeq \mathrm{Br}(R)[p] \simeq \mathrm{Br}(k)[p] = 0$. So the last exact sequence reduces to

$$0 \longrightarrow H^1(K, \Omega^1_{K,\log}) \xrightarrow{\delta_1} \bigoplus_{x \in X^1} H^2_x(X, \Omega^1_{X,\log}) \xrightarrow{\delta_2} H^3_P(X, \Omega^1_{X,\log}) \longrightarrow 0.$$

Our goal is to find a symbol algebra that represents ω in the Brauer group of K. Since $\mathrm{sw}_{K,(t)}(\omega) = m < p$, by Theorem 4.2.2, we have that

$$\omega = [\frac{f_m}{t^m}, \pi) + [\frac{f_{m-1}}{t^{m-1}}, \pi) + \cdots + [f_0, t) \text{ in } \mathrm{Br}(K)[p],$$

where $f_i \in \pi \cdot k[[\pi]]$ for all i. The choices of f_i follow from the identification in (5.2.5). Since that π is a regular element, we have a consistent way to lift the elements from $k((\pi))$.

Since $f_0 \in k[[\pi]]$ and k is algebraically closed, by Hensel's lemma, we have $[f_0, t) \simeq 0$. Hence, $\omega = [\frac{f_m}{t^m} + \cdots + \frac{f_1}{t}, \pi)$. □

Now we are ready to finish the proof of Theorem 5.3.1 based on Theorem 5.3.5.

Proof of Theorem 5.3.1.

First, suppose $\zeta = U$ for some irreducible component U of $Y \setminus S$. Let η be the generic point of U. Then $U \subset U_\eta$. Since $F_{U_\eta} \subset F_U$, $\mathrm{ind}(D \otimes_F F_U) \mid \mathrm{ind}(D \otimes_F F_{U_\eta}) = \mathrm{ind}(D \otimes_F F_\eta)$. Since the residue field of the generic point of U is a function field of the curve over an algebraically closed field, by Theorem 4.2.2, we have $\mathrm{ind}(D \otimes F_\eta) | p$. Hence, $\mathrm{ind}(D \otimes_F F_U) \mid p$.

Second, suppose $\zeta = P \in S$, where P is a closed point of $\mathcal{X}$. Combining Proposition 5.3.6 and Theorem 5.3.9, we have $\mathrm{ind}(D \otimes F_\zeta) | p$.

Finally, by Theorem 5.3.5, we have that $\mathrm{per}(\omega) = \mathrm{ind}(\omega)$. □

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APPENDIX A

SYMBOL LENGTH AND FOLIATION THEORY

Let F be a field of characteristic $p > 0$ and $[F : F^p] = p^n, n \in \mathbb{N}_{>0}$. We can approach the symbol length problem of Ω^1_F / Z^1_F using the Galois theory of purely inseparable extensions.

Let L/K be a field extension of characteristic $p > 0$. The vector space $\mathrm{Der}_K(L)$ of K-derivations $D : L \to L$ is closed under forming commutators $[D, D']$ and p-fold compositions $D^{[p]}$ in the associative ring $\mathrm{End}_K(L)$. We can view $\mathrm{Der}_K(L)$ as a *Lie algebra* over K, endowed the map $D \mapsto D^{[p]}$ as an additional structure. This phenomenon only happens in characteristic $p > 0$. We call them *restricted Lie algebras*.

A restricted Lie algebra (p-Lie algebra) over K is a Lie algebra $\mathfrak{g}$ over K, together with a map $\mathfrak{g} \to \mathfrak{g}$, $x \mapsto x^{[p]}$ called the p-map, subject to the following three axioms:

(R 1) We have $\mathrm{ad}_{x^{[p]}} = (\mathrm{ad}_x)^p$ for all vectors $x \in \mathfrak{g}$.

(R 2) Moreover $(\lambda \cdot x)^{[p]} = \lambda^p \cdot x^{[p]}$ for all vectors $x \in \mathfrak{g}$ and scalars $\lambda \in K$.

(R 3) The formula $(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{r=1}^{p-1} s_r(x, y)$ holds for all $x, y \in \mathfrak{g}$.

Here the summands $s_r(x, y)$ are universal expressions defined by

$$s_r(t_0, t_1) = -\frac{1}{r} \sum_u (\mathrm{ad}_{t_{u(1)}} \circ \cdots \circ (\mathrm{ad}_{t_{u(p-1)}}(t_1),$$

where $\mathrm{ad}_a(x) = [a, x]$ denotes the *adjoint representation*, and the index runs over all maps $u : \{1, \cdots, p-1\} \to \{0, 1\}$ taking the value zero exactly r times.

Restricted Lie algebras were introduced and studied by Jacobson. It appears in the Galois theory of purely inseparable extensions. We will recall it below. We say L/K has exponent 1 if $x^p \in K$ for all $x \in L$.

Theorem A.0.1 (Jacobson)

Let L/K be a finite purely inseparable field extension of exponent one. There is an inclusion-reversing bijection between L/K-restricted Lie algebra $Der_E(L) \subset Der_K(L)$ and intermediate field extension $K \subset E \subset L$.

Recall that there is an isomorphism of F-vector spaces $\mathrm{Hom}_F(\Omega^1_F, F) \cong^{\phi} \mathrm{Der}_{F^p}(F)$, where

$f \in \mathrm{Hom}_F(\Omega^1_F, F) \mapsto \phi(f)(a) = f(da)$. Notice that Z^1_F has dimension $(p^n + n - 1)$ as a F^p-vector space.

Let $\alpha \in \Omega^1_F/Z^1_F$ and $\beta \in Z^1_F$. Consider the F-subspace of $\mathrm{Hom}_F(\Omega^1_F, F)$ defined by $V(\alpha+\beta) = \{f \in \mathrm{Hom}_F(\Omega^1_F, F) \mid f(\alpha+\beta) = 0\}$. Denote the image of $V(\alpha+\beta)$ in $\mathrm{Der}_{F^p}(F)$ also by $V(\alpha+\beta)$. Then the existence of a restricted F-subspace of $V(\alpha+\beta)$ would be equivalent to the symbol length conjecture 3.5.7. Let us investigate the special case $(p, n) = (2, 2)$ using this approach.

Let $\omega \in \mathrm{Der}_{F^p}(F)$ and $\{\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\}$ be a F-basis of $\mathrm{Der}_{F^p}(F)$ given by a p-basis $\{s, t\}$ of F. Then we have that $\omega = f\frac{\partial}{\partial s} + g\frac{\partial}{\partial t}$ for $f, g \in F$.

Since $\dim_F \mathrm{Der}_{F^p}(F) = 2$, we want to find out the conditions on f and g such that $\omega^{[p]} = k\omega$ for $k \in F$. If either of f or g is 0, it is obvious $\omega^{[p]} = 0$. Hence we assume that $f, g \neq 0$. Since $p = 2$,

$$\omega^{[2]}(s) = \omega^{[2]}(ds) = \omega(f) \tag{A.0.1}$$

$$= \omega(df) = \omega(f_s ds + f_t dt) \tag{A.0.2}$$

$$= f_s f + f_t g. \tag{A.0.3}$$

Similarly, we have that $\omega^{[2]}(t) = g_s f + g_t g$. Hence, $\omega^{[2]} = k\omega$ is equivalent to the existence of a solution to the following equation

$$(f_s f + f_t g)g = (g_s f + g_t g)f. \tag{A.0.4}$$

Since we are considering the F-vector space $\{l \cdot \omega \mid l \in F\}$, we can further assume that $g = 1$. Then the equation reduces to

$$f_s f + f_t = 0. \tag{A.0.5}$$

Now we take the expansion of f over F^p. Let $f = f_{00}^2 + f_{01}^2 t + f_{10}^2 s + f_{11}^2 st$. Then $f_s = f_{10}^2 + f_{11}^2 t$ and $f_t = f_{01}^2 + f_{11}^2 s$. It follows that

$$(f_{01}^2 + f_{00}^2 f_{10}^2 + f_{11}^2 f_{01}^2 t^2) + (f_{11}^2 + f_{10}^4 + f_{11}^4 t^2)s + (f_{11}^2 f_{00}^2 + f_{10}^2 f_{01}^2)t = 0. \tag{A.0.6}$$

It is equivalent to the following system of equations

$$\begin{cases} f_{01} + f_{00}f_{10} + f_{11}f_{01}t &= 0 \\ f_{11} + f_{10}^2 + f_{11}^2 t &= 0 \\ f_{11}f_{00} + f_{10}f_{01} &= 0. \end{cases} \tag{A.0.7}$$

If $f_{11} = 0$, it implies $f_{10} = f_{01} = 0$. Hence we get the first kind of solutions $f = f_{00}^2$. When $f_{11} \neq 0$, we notice that the determinant of the first and the third equations respect to variables f_{00} and f_{01} is

$$\begin{vmatrix} f_{10} & 1 + f_{11}t \\ f_{11} & f_{10} \end{vmatrix} = f_{11} + f_{10}^2 + f_{11}^2 t. \tag{A.0.8}$$

Therefore, the second equation of (A.0.7) is more *independent* comparing with others. Moreover, since $f_{11} \neq 0$, we can divide both sides of the second equation by f_{11}^2

$$(\frac{f_{10}}{f_{11}})^2 + \frac{1}{f_{11}} + t = 0. \tag{A.0.9}$$

Let $m = \dfrac{f_{10}}{f_{11}} \in F$. Then we have $f_{11} = \dfrac{1}{m^2 + t}$ and $f_{10} = \dfrac{m}{m^2 + t}$. Hence $f_{00} = mn$, $f_{01} = n$ for $n \in F$, and $f = m^2n^2 + n^2t + \dfrac{m^2 s}{(m^2+t)^2} + \dfrac{st}{(m^2+t)^2}$.

Summarizing the above discussions, we get the following theorem which classifies all the proper restricted F-subspaces of $\mathrm{Der}_{F^p}(F)$.

Theorem A.0.2 (Classification of proper restricted subspaces of $\mathrm{Der}_{F^p}(F)$)
Let F be a field of characteristic $p = 2$ and $[F : F^p] = p^2$. Let $\{s, t\}$ be a p-basis of F and $\omega = f\frac{\partial}{\partial s} + g\frac{\partial}{\partial t} \in Der_{F^p}(F) \cong Hom_F(\Omega_F^1, F)$. Then the F-subspace of $Der_{F^p}(F)$ generated by ω is restricted if and only if $[f : g] \in \mathbb{P}_F^1$ takes values in the following cases:

(i) $[1 : 0]$, $[0 : 1]$;

(ii) $[k^2 : 1]$ *for* $k \in F^\times$;

(iii) $[m^2n^2 + n^2t + \dfrac{m^2 s}{(m^2+t)^2} + \dfrac{st}{(m^2+t)^2} : 1]$ *for* $m, n \in F$.

Now we can turn back to the symbol length problem of the group Ω_F^1/Z_F^1. Keep the assumptions in Theorem A.0.2. Let $\alpha \in \Omega_F^1/Z_F^1$. Then $\alpha = m^2 t \mathrm{dlog}(s) + n^2 s \mathrm{dlog}(t) + l^2 st \mathrm{dlog}(s)$, where

$f, g, h \in F$. Let $\beta \in Z_F^1$. Then $\beta = a^2\mathrm{dlog}(s) + b^2\mathrm{dlog}(t) + d(c) = a^2\mathrm{dlog}(s) + b^2\mathrm{dlog}(t) + d(c_{01}^2 t + c_{10}^2 s + c_{11}^2 st)$, where $a, b, c = c_{00}^2 + c_{01}^2 t + c_{10}^2 s + c_{11}^2 st \in F$.

$$\alpha + \beta = (\frac{m^2 t}{s} + l^2 t + \frac{a^2}{s} + c_{10}^2 + c_{11}^2 t)ds + (\frac{n^2 s}{t} + b^2 + c_{01}^2 + c_{11}^2 s)dt. \tag{A.0.10}$$

Let $\omega \in V(\alpha + \beta)$, i.e. $\omega(\alpha + \beta) = 0$. Suppose that $\omega = f\frac{\partial}{\partial s} + g\frac{\partial}{\partial t}$. It follows that

$$\omega(\alpha + \beta) = (\frac{m^2 t}{s} + l^2 t + \frac{a^2}{s} + c_{10}^2 + c_{11}^2 t)f + (\frac{n^2 s}{t} + b^2 + c_{01}^2 + c_{11}^2 s)g = 0 \tag{A.0.11}$$

Hence

$$f = \frac{\frac{n^2 s}{t} + b^2 + c_{01}^2 + c_{11}^2 s}{\frac{m^2 t}{s} + l^2 t + \frac{a^2}{s} + c_{10}^2 + c_{11}^2 t}.$$

The computation here will get complicated.

From (3.5.1), we find that α is split by the purely inseparable extension defined by $y^2 = \frac{s^2 l^2}{m^2} t + st$. The restricted F-subspace corresponding to the 1-form $d(\frac{s^2 l^2}{m^2} t + st) = [tds + (\frac{s^2 l^2}{m^2} + s)dt]$ is $(\frac{s^2 l^2}{m^2 t} + \frac{s}{t})\frac{\partial}{\partial s} + \frac{\partial}{\partial t}$.

APPENDIX B

RAMIFICATION OF CENTRAL DIVISION ALGEBRAS (p-RANK 1 CASE)

In this appendix, let K be a complete discretely valued field with the valuation v and the residue field F of characteristic $p > 0$, where $[F : F^p] = p$. We want to use the structure of p-torsion part of $\mathrm{Br}(K)$ to understand the ramification behavior of p^n-torsion part of $\mathrm{Br}(K)$. We mainly consider two cases: (1) $\mathrm{Br.dim}_p(E) = 0$ for all finite extension E/F, and (2) F is a local field.

B.1 Br.dim$_p(E) = 0$ for all finite extension E/F

In this case, $\mathrm{Br}(K)[p]$ has symbol length 1. Hence, every central division algebra of period p over K is cyclic and has ramification index p and a degree p residue field extension. Now for a period p^n central division algebra A over K, it has degree p^n by Theorem 4.2.2. Hence we can assume $[A : K] = p^{2n}$. This gives $e = e' = p^n$ (Notation 2.3.1). The residue division algebra of A is commutative and hence a field. It has degree p^n over K. We want to describe this residue field using the structure of $\mathrm{Br}(K)[p]$.

Next we explain how to read the information related to the residue field extension. Consider $p^{n-1}[A]$, where $[A]$ indicates the class of A as above in $\mathrm{Br}(K)$. This class has period p. Hence, there exists a field extension L_1/K of degree p with the degree p residue field extension E_1/F.

Then we consider $[A]_{L_1}$, the image of $[A]$ in $\mathrm{Br}(L_1)$. The field E_1 is a complete discretely valued field with the residue field E_1 which is a degree p field extension of F. Now E_1/F is either an Artin-Schreier extension or a purely inseparable extension. By Proposition 4.1.2 and the assumptions on F, $\mathrm{Br}(E_1)[p] = 0$ and $[E_1 : E_1^p] = p$. The period of $[A]_{L_1}$ is p^{n-1}. Otherwise, the index of $[A]_{L_1}$ is less than p^{n-1}. Then a splitting field of $[A]_{L_1}$ would have the degree over K less than $p^n (= p \cdot p^{n-1})$, which is a contradiction.

Hence, we can repeat the argument above to get a composition of field extensions of degree p, $K \subset L_1 \subset \cdots \subset L_n$, such that the composition of residue field extensions, $F \subset E_1 \subset \cdots \subset E_n$, consists of either Artin-Schreier extension or purely inseparable extension of degree p. L_n is a splitting field of A with degree p^n.

This gives the following theorem:

Theorem B.1.1

Suppose that the field F satisfies $[F : F^p] = p$ and $Br.dim_p(E) = 0$ for all finite extensions E/F. Let K be a complete discretely valued field with the residue field F. Then the degree of a central division algebra of period p^n over K is p^n. Moreover, it admits a splitting field of degree p^n such that the residue field extension is of degree p^n which is a composition of either Artin-Schreier extension of degree p or purely inseparable extension of degree p. The ramification index is also p^n.

In fact, we have an easy way to determine the separable degree and the inseparable degree of the residue field extension.

Corollary B.1.2

We continue with the same assumptions as in Theorem B.1.1. Let A be a central division algebra over K of period p^n. Denote the order of $[A]_{tame}$ in $Br(K_{\text{tame}})$ by $p^m(\leq p^n)$, where K_{tame} is the maximal tame extension of $k((\pi))$. Then the residue field B of A has degree p^n over k with separable degree $[B : F]_s = p^{n-m}$ and inseparable degree $[B : F]_i = p^m$.

Proof.

Consider the class $p^m[A]$. It is split by a tame extension of degree p^{n-m} over K. More precisely, this tame extension has ramification index 1 and residual degree p^{n-m}. Hence, the proof reduces to the case $m = n$. This case just follows from Theorem B.1.1. □

B.1.1 F a local field

In general, if $\text{Br.dim}_p(E) > 0$ for a finite extension E/F, the situation is more complicated, since there exist nontrivial division algebras over the residue field E. However, when F is a local field, we have the following theorem similar to Theorem B.1.1.

Theorem B.1.3

Suppose that F is a local field, i.e. $F \cong \mathbb{F}_q((t)), q = p^n$. Then the degree of a central division algebra of period p^n over K is p^n. Moreover, it admits a splitting field of degree p^n such that the *residue field extension is of degree p^n and it is a composition of either Artin-Schreier extension of degree p or purely inseparable extension of degree p. The ramification index is also p^n.*

Proof.

The proof is similar to the proof of Theorem B.1.1. The local field condition on F is used for Theorem 4.2.4. It follows that a tamely ramified Brauer class in $\mathrm{Br}(K)[p]$ is split by a tame Artin-Schreier extension of degree p over K. □

Similarly, we have the following corollary.

Corollary B.1.4

Suppose that F is a local field, i.e. $F \cong \mathbb{F}_q((t)), q = p^n$. Let A be a central division algebra over K of period p^n. Denote the order of $[A]_{tame}$ in $Br(K_{\text{tame}})$ by $p^m(\le p^n)$, where K_{tame} is the maximal tame extension of K. Then the residue field B of A has degree p^n over F with separable degree $[B : F]_s = p^{n-m}$ and inseparable degree $[B : F]_i = p^m$.

Remark B.1.5

In fact, the condition on F can be replaced by F is p-quasilocal and almost perfect using [12, Theorem 2.3]. The key ingredient of the proof is the fact that $Br.dim_p(E) = 1$ for all fintie extension E/F.