THE ALGEBRAIC STRUCTURE OF TWISTED TOPOLOGICAL HOCHSCHILD HOMOLOGY

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ABSTRACT

Algebraic K-theory is an interesting invariant of rings and ring spectra which has connections to many mathematical fields including number theory, geometric topology, and algebraic geometry. While there is great interest in algebraic K-theory, it is difficult to compute. One successful approach is via trace methods. In this approach one utilizes trace maps from algebraic K-theory to more computable invariants which approximate algebraic K-theory. One of these trace maps is from algebraic K-theory to topological Hochschild homology (THH), which is an invariant of ring spectra. One of the main tools to compute THH is the Bökstedt spectral sequence, and the algebraic structure in this spectral sequence facilitates computations.

In recent years, several equivariant analogues of algebraic K-theory and THH have emerged. One such analogue is C_n-twisted THH, an invariant of ring C_n-spectra, which was defined by Angeltveit, Blumberg, Gerhardt, Hill, Lawson, and Mandell [ABG$^+$18]. To compute twisted THH there is an equivariant Bökstedt spectral sequence, constructed by Adamyk, Gerhardt, Hess, Klang, and Kong [AGH$^+$22].

This thesis explores the algebraic structures of twisted THH, and the equivariant Bökstedt spectral sequence. Classically, if A is a commutative ring spectrum, [EKMM97] and [MSV97] show that $\mathrm{THH}(A)$ is an A-Hopf algebra in the stable homotopy category. Angeltveit and Rognes extend this algebraic structure to the Bökstedt spectral sequence and prove that under some conditions, the Bökstedt spectral sequence is a spectral sequence of $H_*(A;\mathbb{F}_p)$-Hopf algebras for p prime [AR05]. In this thesis we show that for p prime and R a commutative ring C_p-spectrum, $\mathrm{THH}_{C_p}(R)$ is an R-algebra in the C_p-equivariant stable homotopy category. Further, for $p \geq 5$ prime and R a commutative ring C_p-spectrum, $\mathrm{THH}_{C_p}(R)$ is a non-counital R-bialgebra in the C_p-equivariant stable homotopy category. We also extend these results to the equivariant Bökstedt spectral sequence, proving that under appropriate flatness conditions it is a spectral sequence of non-counital bialgebras.

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CHAPTER 1

INTRODUCTION

Algebraic K-theory is an invariant of rings that has deep connections to many mathematical fields including number theory, algebraic geometry, and geometric topology. While algebraic K-theory is generally very difficult to compute, one successful approach is via trace methods.

Trace methods are tools that allow us to approximate algebraic K-theory. These approximations work by mapping from algebraic K-theory to more computable invariants. One example of such an approximation, the Dennis trace, relates algebraic K-theory to a classical invariant of rings, Hochschild homology (HH). For a ring A, the Dennis trace is a map:

$$K_*(A) \to HH_*(A).$$

For a closer approximation to algebraic K-theory, Bökstedt defined a topological analogue of HH called topological Hochschild homology (THH), which is an invariant of ring spectra [Bök85b]. There is a trace map, the topological Dennis trace, from algebraic K-theory to THH. Topological Hochschild homology has an S^1-action. Using this S^1-action one can define topological cyclic homology (TC) and the cyclotomic trace, which gives an even more accurate approximation to algebraic K-theory [BHM93]. Further, the topological Dennis trace factors through the cyclotomic trace:

$$K(R) \to \mathrm{TC}(R) \to \mathrm{THH}(R)$$

for a ring spectrum R.

One of the main tools we use to compute THH is the Bökstedt spectral sequence which relates HH to THH [Bök85b]. For k a field and A a ring this spectral sequence takes the form:

$$E^2_{*,*} = \mathrm{HH}_*(H_*(A;k)) \Rightarrow H_*(\mathrm{THH}(HA);k)$$

One way to facilitate spectral sequence calculations is to understand algebraic structures in the spectral sequence. Angeltveit and Rognes study the algebraic structure of the Bökstedt spectral sequence in [AR05]. Angeltveit and Rognes' results build off of results of [EKMM97] and

[MSV97]. Let us recall that a Hopf algebra can be thought of as both an algebra and a coalgebra with an antipode such that these structures are compatible.

Theorem 1.0.1 ([EKMM97, Corollary 3.4], [MSV97, Theorem I])**.** *For A a commutative ring spectrum,* $\mathrm{THH}(A)$ *is an A-Hopf algebra in the stable homotopy category.*

These authors prove this result by inducing the maps on THH from maps on the circle. This is possible because for A a commutative ring spectrum, $\mathrm{THH}(A) \cong A \otimes S^1$ [MSV97]. For example, the following fold map of spaces $S^1 \vee S^1 \to S^1$ induces the product map $\mathrm{THH}(A) \wedge_A \mathrm{THH}(A) \to \mathrm{THH}(A)$. Angeltveit and Rognes extend this result by using simplicial maps on the circle, allowing the algebraic structure to extend to the Bökstedt spectral sequence, proving the following result:

Theorem 1.0.2 ([AR05, Theorem 4.5])**.** *Let A be a commutative ring spectrum, and let p be prime. If each term of the Bökstedt spectral sequence, $E^r_{*,*}(A)$ for $r \geq 2$ is flat over $H_*(A;\mathbb{F}_p)$, then the Bökstedt spectral sequence is a spectral sequence of $H_*(A;\mathbb{F}_p)$-Hopf algebras.*

Angeltveit and Rognes then use this algebraic structure to facilitate many computations of THH [AR05].

In recent years, equivariant analogues of algebraic K-theory and topological Hochschild homology have emerged. Angeltveit, Blumberg, Gerhardt, Hill, Lawson, and Mandell construct a generalization of THH called C_n-twisted THH, for C_n a finite cyclic subgroup of S^1 [ABG+18]. This generalized theory is an invariant of C_n-equivariant ring spectra. Twisted THH is related to the equivariant algebraic K-theory of Merling [Mer17], and Malkiewich-Merling [MM19], as seen in [AGH+23] and [CGK].

To compute the equivariant homology of C_n-twisted THH, Adamyk, Gerhardt, Hess, Klang, and Kong construct an equivariant analogue to the Bökstedt spectral sequence.

Theorem 1.0.3 ([AGH+22, Theorem 4.2.7])**.** *Let $C_n = \langle \gamma \rangle$ be a finite subgroup of S^1. Let R be a ring C_n-spectrum and E a commutative ring C_n-spectrum such that γ acts trivially on E. If $\underline{E}_\star(R)$ is flat over $\underline{E}_\star$, then there is an equivariant Bökstedt spectral sequence*

$$E^2_{s,\star} = \underline{\mathrm{HH}}_s^{\underline{E}_\star, C_n}(\underline{E}_\star(R)) \Rightarrow \underline{E}_{s+\star}(i^*_{C_n} \mathrm{THH}_{C_n}(R)).$$

Here an object is underlined to indicate that it is a Mackey functor. To study equivariant homotopy theory, one needs equivariant analogues of familiar algebraic objects. Mackey functors arise naturally in equivariant homotopy theory as the equivariant analogue to abelian groups. Let G be a finite abelian group. For a G-equivariant spectrum R, the equivariant homotopy groups of R form a G-Mackey functor. The category of G-Mackey functors has a symmetric monoidal product called the box product, $\square$, allowing one to define an equivariant analogue to rings, called G-Green functors [Lew80].

In the spectral sequence above, $\underline{E}_\star$ denotes $\underline{\pi}_\star(E)$, the equivariant homotopy Mackey functors of the ring C_n-spectrum E. Also, $\underline{\mathrm{HH}}^{C_n}$ is Hochschild homology for Green functors, as defined by Blumberg, Gerhardt, Hill, and Lawson in [BGHL19]. While this equivariant Bökstedt spectral sequence opens the door for computations of twisted THH, as of yet, few computations appear in the literature.

Classically, many Bökstedt spectral sequence calculations are done with coefficients in a field, as this results in nicer behavior in the spectral sequence. Lewis defines G-Mackey fields to be commutative G-Green functors with no nontrivial ideals [Lew80]. If we use C_n-Mackey fields as the coefficients in the equivariant Bökstedt spectral sequence, the spectral sequence is easier to compute.

An important C_2-spectrum is $MU_{\mathbb{R}}$, the Real bordism spectrum. Hill, Hopkins, and Ravenel use $MU_{\mathbb{R}}$ in their solution of the Kervaire invariant one problem in [HHR16]. In this thesis, we compute the equivariant homology of $\mathrm{THH}_{C_2}(MU_{\mathbb{R}})$ with coefficients in the following C_2-Mackey field:

$$\underline{F}: \qquad \begin{tikzcd} \mathbb{F}_2 \arrow[d, bend right, "0"'] \\ 0 \arrow[u, bend right, "0"'] \end{tikzcd}$$

Theorem 1.0.4. *For $\underline{F}$ as above the $RO(C_2)$-graded equivariant homology of* $\mathrm{THH}_{C_2}(MU_{\mathbb{R}})$ *with coefficients in $\underline{F}$ is*

$$\underline{H}_\star(\mathrm{THH}_{C_2}(MU_{\mathbb{R}});\underline{F}) \cong H\underline{F}_\star[\beta_1,\beta_2,\ldots]\square_{H\underline{F}_\star}\Lambda_{H\underline{F}_\star}(z_1,z_2,\ldots)$$

as an $H\underline{F}_\star$-module. Here $|\beta_i| = i\rho$ and $|z_i| = 1 + i\rho$.

Classically, the algebraic structure in the Bökstedt spectral sequence has lead to computations of THH. In the current work, we study the algebraic structures of C_p-twisted THH and the equivariant Bökstedt spectral sequence. Angeltveit, Blumberg, Gerhardt, Hill, Lawson, and Mandell show in [ABG+18] that for a commutative C_n-ring spectrum R, $\mathrm{THH}_{C_n}(R)$ is $R \otimes_{C_n} S^1$. Using equivariant simplicial models of the circle, we demonstrate that C_p-twisted THH has the structure of an R-algebra.

Proposition 1.0.5. *For p prime and R a commutative C_p-ring spectrum,* $\mathrm{THH}_{C_p}(R)$ *is a commutative R-algebra in the C_p-equivariant stable homotopy category.*

For specific primes we can extend this algebraic structure to a bialgebra structure. Let us recall that, similarly to a Hopf algebra, a bialgebra is both an algebra and a coalgebra such that these structures are compatible. The key difference between the two algebraic structures is that a Hopf algebra has an antipode, and the definition of a bialgebra does not include an antipode.

Theorem 1.0.6. *Let R be a commutative ring C_p-spectrum and $p \geq 5$ prime. Then* $\mathrm{THH}_{C_p}(R)$ *is a non-counital, R-bialgebra in the C_p-equivariant stable homotopy category.*

Using the equivariant simplicial maps that provide these structures on C_p-twisted THH, we induce structures on the equivariant Bökstedt spectral sequence. Before we discuss these induced structures let us first recall the following related result.

Proposition 1.0.7 ([AGH+22, The 4.2.7])**.** *Let $C_n = \langle \gamma \rangle$ be a finite subgroup of S^1. Let R be a ring C_n-spectrum and E a commutative ring C_n-spectrum such that γ acts trivially on E. If R is a commutative ring C_n-spectrum, then the equivariant Bökstedt spectral sequence is a spectral sequence of $\underline{E}_\star$-algebras.*

In the current work we show this spectral sequence is a spectral sequence of $\underline{E}_\star(R)$-algebras.

Proposition 1.0.8. *For a prime p, let R and E be commutative ring C_p-spectra, such that the generator of C_p acts trivially on E and $\underline{E}_\star(R)$ is flat over $\underline{E}_\star$. The equivariant Bökstedt spectral sequence $E^r_{*,\star}$ is a spectral sequence of $\underline{E}_\star(R)$-algebras.*

Theorem 1.0.9. *For $p \geq 5$ prime, let R and E be commutative ring C_p-spectra, such that the generator of C_p acts trivially on E and $\underline{E}_\star(R)$ is flat over $\underline{E}_\star$. If each term of the equivariant Bökstedt spectral sequence $E^r_{*,\star}$ for $r \geq 2$ is flat over $\underline{E}_\star(R)$, then $E^r_{*,\star}$ is a spectral sequence of non-counital $\underline{E}_\star(R)$-bialgebras.*

1.1 Notation and conventions

Throughout this paper let G be a finite abelian group, and we are working with genuine orthogonal G-spectra indexed on a complete universe. We use $*$ to denote $\mathbb{Z}$-gradings, $\star$ to denote $RO(G)$-gradings, and $\bullet$ to denote simplicial gradings. Whenever discussing rotations, we mean counter clockwise rotations.

1.2 Organization

In Chapter 2 we recall the definitions and properties of Mackey functors, Green functors, and Mackey fields. We end the chapter by computing the $RO(C_p)$-graded homotopy groups of the Eilenberg-Mac Lane spectra of Mackey fields. We then recall the constructions of Hochschild homology (HH), topological Hochschild homology (THH), twisted HH, and twisted THH in Chapter 3. This chapter ends with a computation of the equivariant homology of twisted THH of the Real bordism spectrum.

In Chapter 4 we recall the classical story about the algebraic structure of THH and the Bökstedt spectral sequence. In Chapters 5 and 6 we study the algebraic structure of twisted THH and the equivariant Bökstedt spectral sequence respectively.

CHAPTER 2

MACKEY FUNCTORS

In this section we will recall the definition of a Mackey functor. Fix G a finite abelian group.

Definition 2.0.1. Let S, T and U be finite G-sets. A *span* from S to T is a diagram

$$S \longleftarrow U \longrightarrow T$$

where the maps are G-equivariant. An *isomorphism of spans* is a commutative diagram of finite G-sets

$$\begin{tikzcd} & U \arrow[dl] \arrow[dr] \arrow[dd, "\cong"] & \\ S & & T \\ & V \arrow[ul] \arrow[ur] & \end{tikzcd}$$

and the composition of spans is given by the pullback. Given two spans $S \leftarrow U_1 \rightarrow T$ and $S \leftarrow U_2 \rightarrow T$, there is a monoidal product via the disjoint union, $S \leftarrow U_1 \bigsqcup U_2 \rightarrow T$.

Definition 2.0.2. The *Burnside category* of G, denoted $\mathcal{A}_G$, has as objects finite G-sets. The morphism set $\mathcal{A}_G(S,T)$ is the group completion of the monoid of isomorphism classes of spans $S \leftarrow U \rightarrow T$.

Definition 2.0.3. A *G-Mackey functor* is an additive functor $\underline{M}\colon \mathcal{A}_G^{\mathrm{op}} \rightarrow \mathcal{A}b$ that sends disjoint unions to direct sums.

Recall that any finite G-set is isomorphic to $\bigsqcup_i G/H_i$ for $H_i \leq G$. Therefore one only needs to know $\underline{M}(G/H)$ for each $H \leq G$ to know $\underline{M}(S)$ for any finite G-set S.

Let Fin_G be the category of finite G-sets. A G-Mackey functor $\underline{M}$ is equivalent to a pair of additive functors

$$M_*, M^*\colon Fin_G \rightarrow \mathcal{A}b$$

which send disjoint unions to direct sums, and are covariant and contravariant, respectively, such that: for any $S \in Fin_G$, $M_*(S) = M^*(S)$, denoted $\underline{M}(S)$. Further, if the following diagram is a pullback in Fin_G:

$$\begin{tikzcd} W \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Z \arrow[r, "g"'] & Y \end{tikzcd}$$

$$\begin{tikzcd} \underline{M}(W) \arrow[r, "M_*(g')"] & \underline{M}(X) \\ \underline{M}(Z) \arrow[u, "M^*(f')"] \arrow[r, "M_*(g)"'] & \underline{M}(Y) \arrow[u, "M^*(f)"'] \end{tikzcd}$$

then the second diagram commutes.

Any sequence of subgroups $K \le H \le G$ induces a natural surjection $q_K^H \colon G/K \to G/H$. The homomorphism $M_*(q_K^H) \colon \underline{M}(G/K) \to \underline{M}(G/H)$ is called the *transfer map*, denoted tr_K^H or tr when K and H are clear from context. The homomorphism $M^*(q_K^H) \colon \underline{M}(G/H) \to \underline{M}(G/K)$ is called the *restriction map*, denoted res_K^H or res.

Definition 2.0.4. For a finite group G, the *Weyl group* with respect to $H \le G$ is defined as $WH := N_G(H)/H$ where $N_G(H)$ denotes the normalizer of H in G. Note that when G is abelian, $WH = G/H$.

The set of G-maps of G/H into itself is isomorphic to WH. Thus, the abelian group $\underline{M}(G/H)$ has a WH-action.

A *Lewis diagram* is a succinct way to describe a Mackey functor. For all $H \le G$, $\underline{M}(G/H)$ are written in the Lewis diagram. Also in the Lewis diagram are all the restriction maps res_K^H, and all the transfer maps tr_K^H for $K < H$ such that there are no subgroups K' such that $[K] < [K'] < [H]$. Here $[H]$ denotes the conjugacy class of H in G. In this paper we will leave off the Weyl actions in the Lewis diagrams. Consider a C_2-Mackey functor $\underline{M}$. Then a Lewis diagram for $\underline{M}$ has the following form:

$$\begin{tikzcd} \underline{M}(C_2/C_2) \arrow[d, bend right, "res"'] \\ \underline{M}(C_2/e) \arrow[u, bend right, "tr"'] \end{tikzcd}$$

Now that we have defined G-Mackey functors, we can now define maps between G-Mackey functors.

Definition 2.0.5. A *map between G-Mackey functors* $\phi\colon \underline{M} \to \underline{N}$ is a natural transformation between them. This map is defined by WH-equivariant group homomorphisms $\phi_H\colon \underline{M}(G/H) \to \underline{N}(G/H)$ for all $H \le G$. Further, these homomorphisms must respect the transfer and restriction maps. This can be visualized by the following diagram:

$$\begin{tikzcd} \underline{M}(G/H) \arrow[r, "\phi_H"] \arrow[d, bend right, "res"'] & \underline{N}(G/H) \arrow[d, bend right, "res"'] \\ \underline{M}(G/K) \arrow[r, "\phi_K"'] \arrow[u, bend right, "tr"'] & \underline{N}(G/K) \arrow[u, bend right, "tr"'] \end{tikzcd}$$

Let us consider some important examples of G-Mackey functors, starting with constant Mackey functors.

Example 2.0.6. Let L be an abelian group. The *constant G-Mackey functor* over L, denoted $\underline{L}$, is a Mackey functor where $\underline{L}(G/H) = L$ with a trivial WH-action. Each restriction map is the identity and each transfer map tr_K^H is multiplication by $|H/K|$.

Example 2.0.7. The constant C_2-Mackey functor $\underline{\mathbb{F}_2}$, is

$$\begin{tikzcd} \mathbb{F}_2 \arrow[d, bend right, "\mathrm{id}"'] \\ \mathbb{F}_2 \arrow[u, bend right, "0"'] \end{tikzcd}$$

where the Weyl actions are trivial.

Let us denote the set of maps in $\mathcal{A}_G$ from W to Y as $[W, Y]$. There is a natural family of Mackey functors defined in the following example.

Example 2.0.8 ([Lew80, Definition 1.1])**.** There is a *representable G-Mackey functor* $[-, S]$ for any $S \in \mathcal{A}_G$. Let us call this Mackey functor $\underline{P}_S$.

For the next example, we first need the following definition.

Definition 2.0.9. The *Burnside ring of G*, denoted $A(G)$, is the group completion of the monoid of isomorphism classes of finite G-sets under disjoint union. Multiplication in this ring is given by the Cartesian product of finite G-sets.

Example 2.0.10. The *Burnside Mackey functor for G*, denoted $\underline{A}_G$, or $\underline{A}$ when G is clear from context, is defined by $\underline{A}(G/H) = A(H)$ for all $H \leq G$. The transfer and restriction maps are given by induction and restriction maps on finite sets. More explicitly, for $K \leq H \leq G$, S a finite K-set, and T a finite H-set,

$$tr_K^H([S]) = [H \times_K S] \text{ and } res_K^H([T]) = [i_K^H(T)],$$

where $i_K^H\colon HSets \to KSets$ is the restriction functor. Additionally, $\underline{A}_G$ is $\underline{P}_{G/G}$.

Mackey functors are thought of as the equivariant analogue of abelian groups. While in classical homotopy theory many invariants take values in abelian groups, the equivariant analogues of those invariants take values in Mackey functors. The following is an important example that demonstrates this.

Example 2.0.11. Let X be a G-spectrum. For each $n \in \mathbb{Z}$, the equivariant homotopy groups of X assemble to a G-Mackey functor, denoted $\underline{\pi}_n(X)$, defined by

$$\underline{\pi}_n(X)(G/H) := \pi_n(X^H)$$

where X^H are the H fixed points of X.

This last example assembles into an integer graded G-Mackey functor. Before we define a graded G-Mackey functor let us first recall the notion of $RO(G)$-grading, where $RO(G)$ is the real representation ring of G. The following is an example of $RO(G)$ that will be used throughout this article.

Example 2.0.12. Any $\alpha \in RO(C_2)$ can be written as $n + m\sigma$ for σ the sign representation and $n, m \in \mathbb{Z}$.

Homotopy groups of G-spectra are naturally graded by $RO(G)$.

Definition 2.0.13. Let X be a G-spectrum. For all $\alpha = [\gamma] - [\beta] \in RO(G)$, the equivariant homotopy groups of X, denoted $\underline{\pi}_\alpha(X)$, are defined by

$$\underline{\pi}_\alpha(X)(G/H) = \pi_\alpha(X^H) = [S^\gamma \wedge G/H_+, S^\beta \wedge X]_G = [S^\gamma, S^\beta \wedge X]_H.$$

Let G be a finite abelian group, then for each subgroup $H \leq G$ and each $\alpha \in RO(G)$, there is a level wise G-actions on $\underline{\pi}_\alpha(E)$ is defined by

$$G \times \underline{\pi}_\alpha(E)(G/H) \to G/H \times \underline{\pi}_\alpha(E)(G/H) \to \underline{\pi}_\alpha(E)(G/H),$$

where the second map is the Weyl group action. These level-wise G-action assemble into a G-action on $\underline{\pi}_\alpha(E)$.

Remark 2.0.14. If G is a cyclic group, then any G-Mackey functor admits a G-action. Since G is cyclic, then every subgroup of G is normal so $WH = G/H$ is always a subgroup of G. So, for $\underline{M}$ a G-Mackey functor, $\underline{M}(G/H)$ is a G-module for all $H \leq G$ and all of the transfer and restriction maps are maps of G-modules. Therefore $\underline{M}$ admits a G-action.

Lewis and Mandell define $RO(G)$-graded G-Mackey functors in [LM06, Definition 2.2]

Definition 2.0.15. There are notions of $\mathbb{Z}$-graded and $RO(G)$-graded G-Mackey functors:

1. An *integer graded G-Mackey functor* is a collection of G-Mackey functors $\{\underline{M}_i\}_{i\in\mathbb{Z}}$, written as $\underline{M}_*$. A map of $\mathbb{Z}$-graded Mackey functors $\underline{M}_* \to \underline{N}_*$, is a collection of maps of Mackey functors $\{\underline{M}_i \to \underline{N}_i\}_{i\in\mathbb{Z}}$.

2. An *$RO(G)$-graded G-Mackey functor* is defined as a collection of G-Mackey functors $\{\underline{M}_\alpha\}_{\alpha\in RO(G)}$, written as $\underline{M}_\star$. A map of $RO(G)$-graded G-Mackey functors $\underline{M}_\star \to \underline{N}_\star$, is a collection of maps of G-Mackey functors $\{\underline{M}_\alpha \to \underline{N}_\alpha\}_{\alpha\in RO(G)}$.

Lewis defines an important example of a G-Mackey functor in [Lew80, Definition 5.5] called the J-Mackey functor. We will use these functors in our computations of equivariant homology and cohomology in Section 2.5. For our purposes, we will focus on the specific example of $G = C_p$, p prime.

Definition 2.0.16 ([Lew80, Definition 5.5])**.** Let $H \leq C_p$, p prime. The functor $J_{C_p/H}(V)$ for V a $\mathbb{Z}[WH]$-module is the following depending on H:

$$J_{C_p/e}(V): \quad \begin{tikzcd} V^{C_p} \arrow[d, bend right, "inc"'] \\ V \arrow[u, bend right, "tr"'] \end{tikzcd} \qquad\qquad J_{C_p/C_p}(V): \quad \begin{tikzcd} V \arrow[d, bend right, "0"'] \\ 0 \arrow[u, bend right, "0"'] \end{tikzcd}$$

where $tr(x) = \sum_{i=0}^{p-1} \gamma^i x$ for $x \in V$.

2.1 Box product and induction theories

The category of G-Mackey functors is an abelian category that has a symmetric monoidal product called the box product, denoted $\square$. The box product was first defined by Lewis in Section 1 of [Lew80].

Definition 2.1.1. Let $\underline{M}$ and $\underline{N}$ be G-Mackey functors. The *box product* $\underline{M} \,\square\, \underline{N}$ is given by a left Kan extension over the Cartesian product of finite G-sets

$$\begin{tikzcd}
\mathcal{A}_G \times \mathcal{A}_G \arrow[r, "\underline{M}\times\underline{N}"] \arrow[d, "-\times-"'] & \mathcal{A}b \times \mathcal{A}b \arrow[r, "-\otimes-"] & \mathcal{A}b \\
\mathcal{A}_G \arrow[urr, "\underline{M}\,\Box\,\underline{N}"'] & &
\end{tikzcd}$$

The unit for the box product is the Burnside Mackey functor $\underline{A}$, defined in Example 2.0.10.

Lewis demonstrates what the box product is for two C_p-Mackey functors in [Lew88].

Definition 2.1.2 ([Lew88])**.** Let p be prime and let us choose the generator γ for C_p. Let $\underline{M}$ and $\underline{N}$ be C_p-Mackey functors:

$$\underline{M}: \quad \begin{tikzcd} \underline{M}(C_p/C_p) \arrow[d, bend right, "res_{\underline{M}}"'] \\ \underline{M}(C_p/e) \arrow[u, bend right, "tr_{\underline{M}}"'] \end{tikzcd} \qquad \underline{N}: \quad \begin{tikzcd} \underline{N}(C_p/C_p) \arrow[d, bend right, "res_{\underline{N}}"'] \\ \underline{N}(C_p/e). \arrow[u, bend right, "tr_{\underline{N}}"'] \end{tikzcd}$$

We can inductively define $\underline{M}\,\Box\,\underline{N}$:

$$\begin{tikzcd} \Big(\underline{M}(C_p/C_p) \otimes \underline{N}(C_p/C_p) \oplus \big(\underline{M}(C_p/e) \otimes \underline{N}(C_p/e)\big)/C_p\Big)/_{FR} \arrow[d, bend right, "res"'] \\ \underline{M}(C_p/e) \otimes \underline{N}(C_p/e). \arrow[u, bend right, "tr"'] \end{tikzcd}$$

Let $x \in \underline{M}(C_p/e), y \in \underline{N}(C_p/e), a \in \underline{M}(C_p/C_p)$, and $b \in \underline{N}(C_p/C_p)$. The C_p-action on $\underline{M}(C_p/e) \otimes \underline{N}(C_p/e)$ is given by $\gamma(x \otimes y) = \gamma(x) \otimes \gamma(y)$. The quotient by the C_p-action, $\big(\underline{M}(C_p/e) \otimes \underline{N}(C_p/e)\big)/C_p$, is isomorphic to the image of the transfer, $Im(tr)$. The restriction map is defined by $res(a \otimes b) = res_{\underline{M}}(a) \otimes res_{\underline{N}}(b)$ and for any element $tr(z) \in Im(tr)$, $res(tr(z)) = z + \gamma z + \ldots + \gamma^{p-1} z$. The notation FR denotes the Frobenius reciprocity submodule which is generated by elements of the form:

$$(a \otimes tr_{\underline{N}}(y), 0) - (0, tr(res_{\underline{M}}(a) \otimes y)),$$

and

$$(tr_{\underline{M}}(x) \otimes b, 0) - (0, tr(x \otimes res_{\underline{N}}(b))).$$

Hill and Mazur extend this definition to $G = C_{p^n}$ in [HM19]. Since the subgroups of C_{p^n} are nested, one can inductively build the box product of two C_{p^n}-Mackey functors, say $\underline{M}$ and $\underline{N}$ by considering

$$(\underline{M} \square \underline{N})(C_{p^n}/C_{p^j}) = \big(\underline{M}(C_{p^n}/C_{p^j}) \otimes \underline{N}(C_{p^n}/C_{p^j}) \oplus \mathrm{Im}(tr_{C_{p^{j-1}}}^{C_{p^j}})\big)/_{FR}$$

where the restriction, transfers, and FR are defined as in the above definition.

Hill and Mazur also show that the above definition also extends to an i-fold box product of C_{p^n}-Mackey functors in [HM19]. Say $\underline{M}_k$ is a C_{p^n}-Mackey functor for $1 \leq k \leq i$, the box product of these Mackey functors is defined by

$$\begin{gathered}(\underline{M}_1 \square \underline{M}_2 \square \dots \square \underline{M}_i)(C_{p^n}/C_{p^j}) = \\ \big(\underline{M}_1(C_{p^n}/C_{p^j}) \otimes \underline{M}_2(C_{p^n}/C_{p^j}) \otimes \dots \otimes \underline{M}_i(C_{p^n}/C_{p^j}) \oplus \mathrm{Im}(tr_{C_{p^{j-1}}}^{C_{p^j}})\big)/_{FR}.\end{gathered}$$

Here the Frobenius reciprocity submodule FR is generated by elements of the form

$$\begin{gathered}(m_1 \otimes m_2 \otimes \dots \otimes tr_K^{C_{p^j}}(b) \otimes \dots \otimes m_i, 0) - \\ (0, tr_K^{C_{p^j}}(res_K^{C_{p^j}}(m_1) \otimes res_K^{C_{p^j}}(m_2) \otimes \dots \otimes b \otimes \dots \otimes res_K^{C_{p^j}}(m_i)))\end{gathered}$$

where $b \in \underline{M}_k(C_{p^n}/K)$ for $1 \leq k \leq i$ and K any subgroup of C_{p^j}.

Proposition 2.1.3 ([Lew80, Lew88, Shu10]). *Let p be prime. For C_{p^n}-Mackey functors $\underline{M}$, $\underline{N}$ and $\underline{L}$, maps $\underline{M} \square \underline{N} \to \underline{L}$ are in natural bijective correspondence with collections of Weyl equivariant maps which satisfy certain conditions. Namely,*

$$f_j \colon \underline{M}(C_{p^n}/C_{p^j}) \otimes \underline{N}(C_{p^n}/C_{p^j}) \to \underline{L}(C_{p^n}/C_{p^j})$$

for all $0 \leq j \leq n$ such that the following compatibility conditions are satisfied:

1. $res_{C_{p^{j-1}}}^{C_{p^j}} \circ f_j = f_{j-1} \circ (res_{C_{p^{j-1}}}^{C_{p^j}} \otimes res_{C_{p^{j-1}}}^{C_{p^j}})$

2. $f_j \circ (tr^{C_{p^j}}_{C_{p^{j-1}}} \otimes id) = tr^{C_{p^j}}_{C_{p^{j-1}}} \circ f_{j-1} \circ (id \otimes res^{C_{p^j}}_{C_{p^{j-1}}})$

3. $f_j \circ (id \otimes tr^{C_{p^j}}_{C_{p^{j-1}}}) = tr^{C_{p^j}}_{C_{p^{j-1}}} \circ f_{j-1} \circ (res^{C_{p^j}}_{C_{p^{j-1}}} \otimes id)$

for all j.

There is also a box product on graded G-Mackey functors.

Definition 2.1.4 ([LM06, Definition 2.4])**.** Let $\underline{N}_\star$ and $\underline{M}_\star$ be $RO(G)$-graded G-Mackey functors. We define $\underline{N}_\star \square \underline{M}_\star$ as an $RO(G)$-graded G-Mackey functor such that for $\gamma \in RO(G)$

$$(\underline{N}_\star \square \underline{M}_\star)_\gamma = \bigoplus_{\gamma=\alpha+\beta} (\underline{N}_\alpha \square \underline{M}_\beta).$$

The definition is similar for $\mathbb{Z}$-graded Mackey functors.

The unit for the product on $RO(G)$-graded G-Mackey functors is $\underline{A}_\star$, which is $\underline{A}$ in degree 0 and $\underline{0}$ in all other degrees.

The following is an example of a Mackey functor which will prove to be useful shortly.

Example 2.1.5 ([Lew80, Definition 1.2])**.** Let $\underline{M}$ be a G-Mackey functor. For $S \in \mathcal{A}_G$ one can define $\underline{M}_S$ to be a G-Mackey functor where $\underline{M}_S(T) = \underline{M}(S \times T)$ for $T \in \mathcal{A}_G$.

Recall that $\underline{P}_S$ is the representable G-Mackey functor, $[-, S]$, as defined in Example 2.0.8. The following lemma shows us how we can use the definition of $\underline{P}_S$ to better understand $\underline{M}_S$ for any Mackey functor $\underline{M}$.

Proposition 2.1.6 ([Lew80, Lemma 1.6])**.** *Let $S \in \mathcal{A}_G$, and $\underline{M}$ be a G-Mackey functor. There are natural isomorphisms*

$$\underline{P}_S \square \underline{M} \cong \underline{M} \square \underline{P}_S \cong \underline{M}_S.$$

Next we will discuss H-characteristic and H-determined G-Mackey functors for $H \leq G$. These are nice classes of Mackey functors where, loosely, one can induce information about a Mackey functor $\underline{M}$ by knowing the value of $\underline{M}(G/H)$. A Mackey functor being H-determined is a stronger notion than a Mackey functor being H-characteristic. In [Lew80], Lewis refers to tools that help us to understand more about an H-characteristic or H-determined Mackey functor as induction theorems named after the classical induction theorems from representation theory. For more information see Section 4 of [Lew80].

We need a definition and some discussion before we can define an H-characteristic Mackey functor.

Definition 2.1.7 ([Lew80, Definition 5.1(a)]). A G-Mackey functor $\underline{M}$ is *H-bounded* if there is a subgroup H of G where $\underline{M}(G/K) = 0$ for $[K] < [H]$ and $\underline{M}(G/H) \neq 0$ if $\underline{M} \neq \underline{0}$.

Before the next definition, let us discuss how the map of G-sets $G/G \leftarrow G/H$ can induce the map $\underline{M} \to \underline{M}_{G/H}$. Recall that the morphism set of the Burnside category from S to T, $\mathcal{A}_G(S,T)$, is the group completion of the monoid of isomorphsim classes of spans $S \leftarrow U \to T$ for S, U, and T finite G-sets. The map of G-sets $G/G \leftarrow G/H$ generates an isomorphism class of spans in $\mathcal{A}_G(G/G, G/H)$, namely, $[G/G \leftarrow G/H \to G/H]$ where the map $G/H \to G/H$ is the identity map. This demonstrates how a map of G-sets $G/G \leftarrow G/H$ induces a map on a G-Mackey functor $\underline{M}$, $\underline{M} \to \underline{M}_{G/H}$.

Definition 2.1.8 ([Lew80, Definition 3.4]). Let $H \leq G$. A G-Mackey functor $\underline{M}$ is *H-characteristic* if the map $\underline{M} \to \underline{M}_{G/H}$, induced from the map of G-sets $G/G \leftarrow G/H$, is injective and $\underline{M}(G/K) = 0$ unless $[H] \leq [K]$.

Note that for $H \leq G$, a G-Mackey functor $\underline{M}$ being H-characteristic implies that $\underline{M}$ is H-bounded but the converse does not hold. In order to define an H-determined Mackey functor, we need the following definition.

Definition 2.1.9 ([Lew80, Definition 4.1]). We say a G-Mackey functor $\underline{M}$ satisfies *G/H-injective induction* if the diagram $\underline{M} \to \underline{M}_{G/H} \rightrightarrows \underline{M}_{G/H\times G/H}$ induced by the diagram $G/H \times G/H \rightrightarrows G/H \to G/G$ in Fin_G is an equalizer diagram. The two maps from $G/H \times G/H \to G/H$ are the projections onto the first and second components.

Lewis goes on to explain that to show a G-Mackey functor satisfies G/H-injective induction directly is difficult. Even so, this is a very useful form of induction as demonstrated by the following definition.

Definition 2.1.10 ([Lew80, Definition 5.1(b)]). A G-Mackey functor $\underline{M}$ is *H-determined* for $H \leq G$ if it is both H-bounded and satisfies G/H-injective induction.

Recall, that a G-Mackey functor being H-characteristic implies that it is H-bounded. In fact, if a G-Mackey functor is H-determined, then it is H-characteristic. Therefore the above definition is equivalent to the following.

Definition 2.1.11 ([Lew80]). A G-Mackey functor $\underline{M}$ is *H-determined* for $H \leq G$ if it is both H-characteristic and satisfies G/H-injective induction.

In fact Lewis shows that $J_{G/H}(V)$, two examples of which are seen in Definition 2.0.16, is H-determined for any $H \leq G$ and V a $\mathbb{Z}[WH]$-module in [Lew80, Lemma 5.6].

It is difficult to directly show that a G-Mackey functor satisfies G/H-injective induction. There are stronger forms of induction that are easier to verify.

Definition 2.1.12 ([Lew80, Definition 4.2]). We say a G-Mackey functor $\underline{M}$

1. is *G/H-projective* if the map $\underline{M}_{G/H} \to \underline{M}$ induced from the transfer map $G/G \leftarrow G/H$ is a split surjection.

2. is *G/H-injective* if the map $\underline{M} \to \underline{M}_{G/H}$ induced from the restriction map $G/H \to G/G$ is a split injection.

It turns out that if a G-Mackey functor is G/H-projective or G/H-injective then it satisfies G/H-injective induction. The following shows the relationship between G/H-injection and G/H-projection.

Proposition 2.1.13 ([Lew80, Proposition 4.4]). *For any G-Mackey functor $\underline{M}$, it is equivalent for $\underline{M}$ to be G/H-projective and G/H-injective.*

We can know a lot about a Mackey functor if we know it is S-projective, S a finite G-set, as demonstrated by the following induction theorem.

Proposition 2.1.14 ([Lew80, Proposition 4.4])**.** *Let $\underline{M}$ be a G-Mackey functor. The Mackey functor $\underline{M}$ is a direct summand of $\underline{M}_{G/H}$ if and only if $\underline{M}$ is G/H-projective.*

2.2 Green functors

Since Mackey functors are the equivariant analogues of abelian groups, one may ask how to define the equivariant analogue to rings. Now that we have a symmetric monoidal product, we can define an equivariant analogue to rings.

Definition 2.2.1 ([Lew80, Definition 2.1(a)])**.** A *G-Green functor* $\underline{R}$ is a Mackey functor with a unit map $\eta\colon \underline{A} \to \underline{R}$ and a multiplication map $\phi\colon \underline{R} \square \underline{R} \to \underline{R}$ such that the following diagrams commute:

$$\begin{tikzcd}
\underline{R} \square \underline{R} \square \underline{R} \arrow[r, "\phi \square 1"] \arrow[d, "1 \square \phi"'] & \underline{R} \square \underline{R} \arrow[d, "\phi"] \\
\underline{R} \square \underline{R} \arrow[r, "\phi"'] & \underline{R}
\end{tikzcd}$$

$$\begin{tikzcd}
\underline{A} \square \underline{R} \arrow[r, "\eta \square 1"] \arrow[dr, "\cong"'] & \underline{R} \square \underline{R} \arrow[d, "\phi"] & \underline{R} \square \underline{A} \arrow[l, "1 \square \eta"'] \arrow[dl, "\cong"] \\
& \underline{R} &
\end{tikzcd}$$

A G-Green functor is said to be commutative if the following diagram commutes

$$\begin{tikzcd}
\underline{R} \square \underline{R} \arrow[rr, "\tau"] \arrow[dr, "\phi"'] & & \underline{R} \square \underline{R} \arrow[dl, "\phi"] \\
& \underline{R} &
\end{tikzcd}$$

where τ swaps the two copies of $\underline{R}$.

To define graded Green functors which are commutative we must define the rotating isomorphism. The definition of the rotating isomorphism uses elements in the Burnside ring. Namely, for G a finite abelian group and $\alpha, \beta \in RO(G)$, the switch map $S^\alpha \wedge S^\beta \to S^\beta \wedge S^\alpha$ gives an element in the Burnside ring $A(G)$. Let us refer to this element as $s(\alpha, \beta)$. Further, for $\underline{N}_\star$, and $\underline{M}_\star$ $RO(G)$-graded Mackey functors, $s(\alpha, \beta)$ induces an automorphism $\underline{M}_\alpha \square \underline{N}_\beta \to \underline{M}_\alpha \square \underline{N}_\beta$. This automorphism, along with the symmetry isomorphism of abelian groups, gives an isomorphism $r_{\alpha,\beta} \colon \underline{M}_\alpha \square \underline{N}_\beta \to \underline{N}_\beta \square \underline{M}_\alpha$.

Definition 2.2.2. Let $\underline{M}_\star$ and $\underline{N}_\star$ be $RO(G)$-graded Mackey functors. The *rotating isomorphism*, noted $\tau \colon \underline{M}_\star \square \underline{N}_\star \to \underline{N}_\star \square \underline{M}_\star$ is defined on level $\gamma \in RO(G)$, by $r_{\alpha,\beta} \colon \underline{M}_\alpha \square \underline{N}_\beta \to \underline{N}_\beta \square \underline{M}_\alpha$, as defined above, for all $\alpha + \beta = \gamma$.

We are now ready to define an $\mathbb{Z}$ or $RO(G)$-graded G-Green functor.

Definition 2.2.3 ([LM06, Definition 3.1])**.** An *$RO(G)$-graded G-Green functor* $\underline{R}_\star$ is a collection of Mackey functors, $\{\underline{R}_\alpha\}_{\alpha \in RO(G)}$ with a unit map $\eta \colon \underline{A}_\star \to \underline{R}_\star$ and a multiplication map $\phi \colon \underline{R}_\star \square \underline{R}_\star \to \underline{R}_\star$ such that the following diagrams commute:

$$\begin{tikzcd}
\underline{R}_\star \square \underline{R}_\star \square \underline{R}_\star \arrow[r, "\phi \square 1"] \arrow[d, "1 \square \phi"'] & \underline{R}_\star \square \underline{R}_\star \arrow[d, "\phi"] \\
\underline{R}_\star \square \underline{R}_\star \arrow[r, "\phi"'] & \underline{R}_\star
\end{tikzcd}$$

$$\begin{tikzcd}
\underline{A}_\star \square \underline{R}_\star \arrow[r, "\eta \square 1"] \arrow[dr, "\cong"'] & \underline{R}_\star \square \underline{R}_\star \arrow[d, "\phi"] & \underline{R}_\star \square \underline{A}_\star \arrow[l, "1 \square \eta"'] \arrow[dl, "\cong"] \\
& \underline{R} &
\end{tikzcd}$$

An $RO(G)$-graded G-Green functor is said to be commutative if the following diagram commutes

$$\begin{tikzcd}
\underline{R}_\star \,\square\, \underline{R}_\star \arrow[rr, "\tau"] \arrow[dr, "\phi"'] & & \underline{R}_\star \,\square\, \underline{R}_\star \arrow[dl, "\phi"] \\
& \underline{R}_\star &
\end{tikzcd}$$

where τ is the rotating isomorphism defined in Definition 2.2.2.

This allows us to define Mackey modules, submodules, and ideals. Let us first consider the following definition.

Definition 2.2.4. A *subfunctor*, say $\underline{S}$, of a G-Mackey functor $\underline{M}$ is a functor from finite G-sets to abelian groups where $\underline{S}(G/H) \leq \underline{M}(G/H)$ for all $H \leq G$, where the transfer maps, restriction maps, and Weyl actions of $\underline{S}$ are induced from the corresponding maps and actions on $\underline{M}$.

Definition 2.2.5 ([Lew80, Definition 2.1(b)])**.** For $\underline{R}$ a G-Green functor, a *left* $\underline{R}$*-module* is a G-Mackey functor $\underline{M}$ with a module structure map $\xi\colon \underline{R} \,\square\, \underline{M} \to \underline{M}$ such that the following diagrams commute:

$$\begin{tikzcd}
\underline{R} \,\square\, \underline{R} \,\square\, \underline{M} \arrow[r, "\phi \,\square\, 1"] \arrow[d, "1 \,\square\, \xi"'] & \underline{R} \,\square\, \underline{M} \arrow[d, "\xi"] \\
\underline{R} \,\square\, \underline{M} \arrow[r, "\xi"'] & \underline{M}
\end{tikzcd}$$

$$\begin{tikzcd}
\underline{A} \,\square\, \underline{M} \arrow[r, "\eta \,\square\, 1"] \arrow[dr, "\cong"'] & \underline{R} \,\square\, \underline{M} \arrow[d, "\xi"] \\
& \underline{M}
\end{tikzcd}$$

where ϕ and η are the multiplication and unit map of $\underline{R}$ respectively. *Right* $\underline{R}$*-modules* and $\underline{R}$*-bimodules* are defined analogously. If $\underline{R}$ is commutative, then every left (right) $\underline{R}$-module is a $\underline{R}$-bimodule. An $\underline{R}$*-submodule* $\underline{N}$ of $\underline{M}$ is a subfunctor that is closed under the action of $\underline{R}$.

Lewis and Mandell define an $RO(G)$-graded module over an $RO(G)$-graded Green functor in [LM06].

Definition 2.2.6 ([LM06, Definition 3.2])**.** For $\underline{R}_\star$ an $RO(G)$-graded G-Green functor, a *left* $\underline{R}_\star$-*module* is an $RO(G)$-graded G-Mackey functor $\underline{M}_\star$ with a module structure map $\xi\colon \underline{R}_\star \square \underline{M}_\star \to \underline{M}_\star$ such that the following diagrams commute:

$$\begin{tikzcd}
\underline{R}_\star \square \underline{R}_\star \square \underline{M}_\star \arrow[r, "\phi \square 1"] \arrow[d, "1 \square \xi"'] & \underline{R}_\star \square \underline{M}_\star \arrow[d, "\xi"] \\
\underline{R}_\star \square \underline{M}_\star \arrow[r, "\xi"'] & \underline{M}_\star
\end{tikzcd}$$

$$\begin{tikzcd}
\underline{A}_\star \square \underline{M}_\star \arrow[r, "\eta \square 1"] \arrow[dr, "\cong"'] & \underline{R}_\star \square \underline{M}_\star \arrow[d, "\xi"] \\
 & \underline{M}_\star
\end{tikzcd}$$

where ϕ and η are the multiplication and unit map of $\underline{R}_\star$ respectively. *Right* $\underline{R}_\star$*-modules* and $\underline{R}_\star$*-bimodules* are defined analogously. If $\underline{R}_\star$ is commutative, then every left (right) $\underline{R}_\star$-module is an $\underline{R}_\star$-bimodule. An $RO(G)$-graded $\underline{R}_\star$*-submodule* $\underline{N}_\star$ of $\underline{M}_\star$ is a subfunctor on every level, meaning that $\underline{N}_\alpha$ is a subfunctor of $\underline{M}_\alpha$ for all $\alpha \in RO(G)$, where $\underline{N}_\star$ is closed under the action of $\underline{R}_\star$.

In the classical case, we have relative tensor products of abelian groups defined from a coequalizer diagram. The following is the equivariant analogue. Lewis originally defined the non-graded case in [Lew80] and Lewis and Mandell extended this to the $RO(G)$-graded case in [LM06, Definition 3.6(a)].

Definition 2.2.7 ([Lew80, LM06])**.** Let $\underline{L}_\star$ and $\underline{M}_\star$ be right and left $\underline{R}_\star$-modules, respectively, for $\underline{R}_\star$ an $RO(G)$-graded G-Green functor. Define $\underline{L}_\star \square_{\underline{R}_\star} \underline{M}_\star$ as the coequalizer in the category of $RO(G)$-graded Mackey functors

$$\underline{L}_\star \square \underline{R}_\star \square \underline{M}_\star \overset{\rho \square \mathrm{id}}{\underset{\mathrm{id} \square \lambda}{\rightrightarrows}} \underline{L}_\star \square \underline{M}_\star \to \underline{L}_\star \square_{\underline{R}_\star} \underline{M}_\star$$

for ρ and λ being the left and right module actions, respectively.

It is natural to now define the ideals of Green functors.

Definition 2.2.8 ([Lew80, Definition 2.1(c)])**.** Let $\underline{R}$ be a G-Green functor. A *left ideal* $\underline{I}$ of $\underline{R}$ is a submodule of $\underline{R}$ considered as a left module over itself. Analogously, one can define *right ideals* and *two sided ideals.*

The definition of an $RO(G)$-graded ideal can be derived from Lewis and Mandell's definition of an $RO(G)$-graded module.

Definition 2.2.9. Let $\underline{R}_\star$ be an $RO(G)$-graded G-Green functor. An *$RO(G)$-graded left ideal* $\underline{I}_\star$ of $\underline{R}_\star$ is an $RO(G)$-graded submodule of $\underline{R}_\star$ considered as a left module over itself. Analogously, one can define *$RO(G)$-graded right ideals* and *$RO(G)$-graded two sided ideals.*

Note that, as is true classically, if $\underline{I}_\star$ is an $RO(G)$-graded left ideal of $\underline{R}_\star$, then $\underline{I}_0$ must be a left ideal of $\underline{R}_0$. This is because by the definition of an $\underline{R}_\star$-subfunctor, there is an inclusion map $\underline{I}_0 \to \underline{R}_0$ and the module structure map on $\underline{R}_0 \square \underline{I}_0$ must land in $\underline{I}_0$.

We will need the notion of a flat $\underline{R}$-module later on in this paper. Lewis discusses this briefly after Proposition 2.4.

Definition 2.2.10 ([Lew80])**.** A left $\underline{R}$-module $\underline{M}$ is *flat* if the functor $- \square_{\underline{R}} \underline{M}$ from the category of right $\underline{R}$-modules to the category of G-Mackey functors is exact. The definitions of a flat right $\underline{R}$-module and flat $\underline{R}$-bimodule are defined analogously.

Lewis and Mandell extend this definition to $RO(G)$-gradings in [LM06, Theorem 4.5].

Definition 2.2.11. A left $\underline{R}_\star$-module $\underline{M}_\star$ is *flat* if the functor $- \square_{\underline{R}_\star} \underline{M}_\star$ from the category of right $\underline{R}_\star$-modules to the category of $RO(G)$-graded Mackey functors is exact. The definitions of a flat right $\underline{R}_\star$-module and flat $\underline{R}_\star$-bimodule are defined analogously.

Recall that for S a finite G-set and $\underline{M}$ a G-Mackey functor, $\underline{M}_S \cong \underline{M} \square \underline{P}_S$. Moreover, for $\underline{R}$ a G-Green functor and S a finite G-set, $\underline{R}_S$ is an $\underline{R}$-bimodule. Lewis discusses after Proposition 2.4 in [Lew80] that $- \square \underline{P}_S$, $\underline{R}_S \square_{\underline{R}} -$, and $- \square_{\underline{R}} \underline{R}_S$ are exact functors.

2.3 Equivariant spectra and fixed points

Let us recall some constructions for G-spectra, such as fixed points, and their relation to Mackey functors. Let us first define fixed points of a G-spectrum.

Definition 2.3.1. For G a finite group, $H \leq G$, and X a G-spectrum, the *H-fixed point spectrum* of X, X^H, is a $W_G H$-spectrum defined by:

$$X^H(V) := (X(V))^H$$

for V a G-representation that is fixed by H.

For E and D G-spectra, in general $(E \wedge D)^H$ is not equivalent to $E^H \wedge D^H$ for $H \leq G$. The geometric fixed points is another important notion of fixed points. We will need to work our way up to this definition.

Definition 2.3.2 ([LMSM86])**.** Let N be a normal subgroup of G. Denote $\mathcal{P}_N$ as the family of proper subgroups of N. Let $E\mathcal{P}_N$ denote the *classifying space* of $\mathcal{P}_N$ such that $E\mathcal{P}_N^H$ is empty and for any proper subgroup H, $E\mathcal{P}_N^H$ is weakly contractible.

We will always assume that $E\mathcal{P}_N$ is a G-CW complex and let $\tilde{E}\mathcal{P}_N$ be the mapping cone of $E\mathcal{P}_N \to *$.

Hill, Hopkins, and Ravenel discuss in more detail the construction of this classifying space in Section 2.5.2 of [HHR16].

Definition 2.3.3. Let X be a G-spectrum and N a normal subgroup of G. The *N-geometric fixed point functor* Φ^N, maps from the category of G-spectra to the category of G/N-spectra is defined by

$$\Phi^N(X) = (\tilde{E}\mathcal{P}_N \wedge X)^N.$$

For $H \trianglelefteq G$ and E and D G-spectra, we have

$$\Phi^H(E \wedge D) \simeq \Phi^H(E) \wedge \Phi^H(D).$$

Another important property of the geometric fixed point functor is that for a G-space X and $H \trianglelefteq G$ we have

$$\Phi^H(\Sigma^\infty X) \simeq \Sigma^\infty(X^H).$$

Proposition 2.3.4 ([LMSM86])**.** *For E a G-spectrum concentrated over $H \trianglelefteq G$ we have*

$$\Phi^H(E) \simeq E^H.$$

Let D and E be G-spectra. We let $[D, E]_G$ denote the homotopy classes of maps of G-spectra.

Proposition 2.3.5 ([LMSM86])**.** *Let E and D be G-spectra. If E is concentrated over H and $H \trianglelefteq G$, then*

$$[D, E]_G \cong [\Phi^H(D), (E)^H]_{WH}.$$

For a G-Mackey functor $\underline{M}$, there is an associated Eilenberg-Mac Lane G-spectrum, denoted $H\underline{M}$ (see, for example, [dS03, dSN09]). As is true classically, these Eilenberg-Mac Lane G-spectra are characterized by their $\mathbb{Z}$-graded homotopy groups:

$$\underline{\pi}_n(H\underline{M}) = \begin{cases} \underline{M} & n = 0 \\ \underline{0} & \text{else.} \end{cases}$$

We need the following definition to state another induction theorem which has to do with the Eilenberg-Mac Lane spectrum of a Mackey functor.

Definition 2.3.6. For $H \leq G$, a G-spectrum E is *concentrated over H* if $\underline{\pi}_\star^K(E) \neq \underline{0}$ if and only if K contains H up to conjugacy.

Proposition 2.3.7 ([Oru89, Remark 3.7])**.** *For any H-determined G-Mackey functor $\underline{M}$, the Eilenberg-Mac Lane spectrum $H\underline{M}$ is concentrated over H.*

The following is an interesting fact about H-determined Green functors, which we will use later in this paper. Oruç proves the following within the proof of Theorem 3.11 in [Oru89].

Proposition 2.3.8 ([Oru89])**.** *Let $\underline{R}$ be an H-determined, G/H-projective G-Green functor. Then if we consider $\Phi^H(H\underline{R})$ as non-equivariant, it is isomorphic to the non-equivariant Eilenberg-Mac Lane spectrum $H(\underline{R}(G/H))$.*

2.4 Mackey fields

Green functors are an equivariant analogue to rings, and we have recalled what it means for a Mackey functor to be an ideal of a Green functor, so now we can recall an equivariant analogue to fields.

Definition 2.4.1 ([Lew80, Definition 2.6(f)])**.** A *G-Mackey field $\underline{F}$* is a nonzero, commutative G-Green functor with no nontrivial proper ideals.

Let us define $RO(G)$-graded Mackey fields.

Definition 2.4.2. An *$RO(G)$-graded G-Mackey field* is a nonzero, commutative $RO(G)$-graded G-Green functor with no nontrivial $RO(G)$-graded ideals.

Before we recall some properties of Mackey fields, let us first cover some examples. One may guess that the constant Mackey functor over a field is always a Mackey field, but this is not always true.

Example 2.4.3. The constant C_2-Mackey functor over $\mathbb{F}_2$ is not a Mackey field. Recall that $\underline{\mathbb{F}}_2$ is the following

$$\begin{tikzcd} \mathbb{F}_2 \arrow[d, bend right, "\mathrm{id}"'] \\ \mathbb{F}_2 \arrow[u, bend right, "0"'] \end{tikzcd}$$

one can check that the following is an ideal of $\underline{\mathbb{F}}_2$

$$\begin{tikzcd} 0 \arrow[d, bend right, "0"'] \\ \mathbb{F}_2 \arrow[u, bend right, "0"'] \end{tikzcd}$$

which is not a trivial ideal, therefore $\underline{\mathbb{F}}_2$ is not a Mackey field.

It turns out that there are many cases where the constant Mackey functor over a field is in fact a Mackey field.

Example 2.4.4. The constant C_2-Mackey functor over $\mathbb{F}_p$ for p an odd prime is a Mackey field. Recall that $\underline{\mathbb{F}}_p$ is the following

$$\begin{tikzcd} \mathbb{F}_p \arrow[d, bend right, "\mathrm{id}"'] \\ \mathbb{F}_p \arrow[u, bend right, "2"'] \end{tikzcd}$$

where the transfer map is multiplication by two. One can check that this has no ideals. For example, the following Mackey functor, say $\underline{I}$, cannot be an ideal

$$\begin{tikzcd} 0 \arrow[d, bend right, "0"'] \\ \mathbb{F}_p \arrow[u, bend right, "0"'] \end{tikzcd}$$

as there is no inclusion map $\underline{I} \to \underline{\mathbb{F}}_p$ which respects the transfer and restriction maps. To see this consider the following

$$\begin{tikzcd} 0 \arrow[r, "0"] \arrow[d, bend right, "0"'] & \mathbb{F}_p \arrow[d, bend right, "\mathrm{id}"'] \\ \mathbb{F}_p \arrow[r, "\mathrm{id}"'] \arrow[u, bend right, "0"'] & \mathbb{F}_p \arrow[u, bend right, "2"'] \end{tikzcd}$$

this fails to be a map of Mackey functors as $\mathbb{F}_p \xrightarrow{inc} \mathbb{F}_p \xrightarrow{2} \mathbb{F}_p$ is not the same as $\mathbb{F}_p \xrightarrow{0} 0 \xrightarrow{0} \mathbb{F}_p$. One can do a similar argument for the other nontrivial candidate for an ideal of $\underline{\mathbb{F}}_p$

$$\begin{tikzcd} \mathbb{F}_p \arrow[d, bend right, "0"'] \\ 0. \arrow[u, bend right, "0"'] \end{tikzcd}$$

Therefore $\underline{\mathbb{F}}_p$ is a C_2-Mackey field.

There are many other examples of Mackey fields. In fact any C_p-Mackey functor $\underline{M}$ where $\underline{M}(C_p/e) = 0$ and $\underline{M}(C_p/C_p) = F$ for F a field is a C_p-Mackey field. The following is another interesting example.

Example 2.4.5. The following is a C_2-Mackey field:

$$\begin{tikzcd} \mathbb{R} \arrow[d, bend right, "inc"'] \\ \mathbb{C} \arrow[u, bend right, "2\mathrm{Re}"'] \end{tikzcd}$$

where the C_2-action on $\mathbb{C}$ is complex conjugation, and the transfer map takes the real part of the complex number and multiplies it by two.

One may have noticed that for every example of a Mackey field $\underline{F}$, $\underline{F}(G/G)$ has been a field. This is not special to these examples, but is a feature of Mackey fields.

Proposition 2.4.6 ([Lew80, Proposition 3.9(f)])**.** *If $\underline{F}$ is a G-Mackey field, then $\underline{F}(G/G)$ is a field.*

Note that it is not necessarily true that $\underline{F}(G/H)$ is a field for $H \lneq G$. The following result will illuminate some of the interesting properties of Mackey fields.

Proposition 2.4.7 ([Lew80, Corollary 4.5])**.** *Let $\underline{R}$ be a G-Green functor, and $\underline{F}$ a G-Mackey field.*

1. *$\underline{R}$ is G/H-projective if and only if every module $\underline{M}$ over $\underline{R}$ is also G/H-projective.*

2. *If $\underline{F}(G/H) \neq 0$, then any module over $\underline{F}$ is G/H-projective.*

Since every Mackey field is nonzero, there must be some $H \leq G$ such that $\underline{F}(G/H) \neq 0$. Therefore, by the above proposition, $\underline{F}$ is G/H-projective for at least one $H \leq G$. Furthermore, since a Mackey functor being G/H-projective implies that it is G/H-injective and satisfies G/H-injective induction, then a Mackey field $\underline{F}$ satisfies all forms of G/H-induction for any $H \leq G$ such that $\underline{F}(G/H) \neq 0$.

Proposition 2.4.8 ([Lew80])**.** *Let $\underline{F}$ be a G-Mackey field. There exists some $H \leq G$ such that $\underline{F}$ is H-determined.*

Proof. We have already shown that for any $K' \leq G$ such that $\underline{F}(G/K') \neq 0$, $\underline{F}$ satisfies G/K'-injective induction, we must now show that there is some H such that $\underline{F}$ is H-bounded.

Since $\underline{F}$ cannot be the zero Mackey functor, there must be at least one subgroup of G, say H', such that $\underline{F}(G/H') \neq 0$. Choose a smallest, with respect to size, subgroup $H < H'$ such that $\underline{F}(G/H) \neq 0$. Then $\underline{F}(G/K) = 0$ for all $K < H$, therefore $\underline{F}$ is H-bounded. □

From this result, we can "sort" all G-Mackey fields by classes of subgroups $H \leq G$. Note that a Mackey functor can be H_1-characteristic and H_2-characteristic for $H_1 \neq H_2$. For example, a C_6-Mackey functor can be C_3-characteristic and C_2-characteristic.

We can also see that if $\underline{F}$ is H-determined, then by Proposition 2.4.7 every $\underline{F}$-module also satisfies all forms of G/H-induction.

Proposition 2.4.9 ([Lew80])**.** *Let $\underline{F}$ be an H-determined G-Mackey field. If $\underline{M}$ is an $\underline{F}$-module, then $\underline{M}$ is H'-determined for some $H' \not< H$.*

Proof. We have already shown that $\underline{M}$ satisfies G/H'-injective induction whenever $\underline{F}(G/H') \neq 0$. We must now show that $\underline{M}$ is H'-bounded for some such H'. If $\underline{M} = \underline{0}$, then we are done as $\underline{0}$ is K-bounded for any $K \leq G$.

If $\underline{M} \neq \underline{0}$, then we need to show that there exists some H' as above such that $\underline{M}(G/K) = 0$ for all $K < H'$, $\underline{M}(G/H') \neq 0$, and $H' \not< H$. Since $\underline{M} \neq \underline{0}$, there exists at least one minimal H' such that $\underline{M}(G/H') \neq 0$, where by minimal we mean that there exists no proper subgroup K of H' such that $\underline{M}(G/K) \neq 0$. Therefore, $\underline{M}$ is H'-bounded. We now need to show that $H' \not< H$.

By the assumption that $\underline{F}$ is H-determined, $\underline{F}(G/K)$ must be 0 for all $K \leq H$. Since $\underline{M}$ is an $\underline{F}$-module, the following diagram must commute

$$\begin{tikzcd} \underline{A} \,\Box\, \underline{M} \arrow[r, "\eta \,\Box\, 1"] \arrow[dr, "\cong"'] & \underline{F} \,\Box\, \underline{M} \arrow[d, "\xi"] \\ & \underline{M} \end{tikzcd}$$

for η the unit map for $\underline{F}$ and ξ the module structure map. If there exists a $K < H$ such that $\underline{M}(G/K) \neq 0$ then this diagram would not commute since one can show inductively that $(\underline{F} \,\Box\, \underline{M})(G/K) = 0$. Therefore H' is not a proper subgroup of H. $\square$

The following is a simple fact that will be useful in this paper.

Proposition 2.4.10. *For a G-Mackey field $\underline{F}$ we have that $\underline{F}(G/e) \neq 0$ if and only if $\underline{F}$ is e-determined.*

Proof. Let us assume that $\underline{F}(G/e) \neq 0$. Since $\underline{F}$ is a G-Mackey field, we know that it is H-determined for some $H \leq G$. This implies that $\underline{F}(G/K) = 0$ unless $H \leq K$. By our assumption $\underline{F}(G/e) \neq 0$, so $H \leq e$ and therefore $H = e$.

In the other direction, let us assume that $\underline{F}$ is e-determined. Then $\underline{F}$ is e-bounded, so $\underline{F}(G/e) \neq 0$. $\square$

The following follows from the previous proposition and the fact that any C_p-Mackey field must be either e-determined or C_p-determined.

Corollary 2.4.11. *Let p be prime. For a C_p-Mackey field $\underline{F}$, $\underline{F}$ is C_p-determined if and only if $\underline{F}(C_p/e) = 0$.*

2.5 Equivariant homotopy groups of Eilenberg-Mac Lane spectra

In this section, we will compute $\underline{\pi}_\star(H\underline{F})$ for any C_p-Mackey field $\underline{F}$ and p prime. Note that Ferland and Lewis computed these homotopy groups in [FL04]. That is, in Chapter 8 of [FL04] the authors compute $H\underline{M}_\star$ for $\underline{M}$ any C_p-Mackey functor. Considering $\underline{M}$ to be a C_p-Mackey field greatly simplifies this computation. Let us first recall the definition of equivariant homology and cohomology.

Define a_+ as the disjoint union of the discrete space a and a G-trivial point. For G-spectra D and E we regard $[D, E]_G$ as a G-Mackey functor by defining $[D, E]_G(a) = [\Sigma^\infty a_+ \wedge D, E]_G$ for a a finite G-set. Similarly for a G-space X, $[\Sigma^\infty X, E]_G(a) = [\Sigma^\infty a_+ \wedge \Sigma^\infty X, E]_G$ is a G-Mackey functor.

Definition 2.5.1 ([LMM81, LMSM86])**.** Assume E and D are G-spectra, and X a G-space. Then G-equivariant E*-cohomology* and E*-homology* of X and D are given by

$$\underline{E}^\star(X) = [\Sigma^\infty_+ X, S^\star \wedge E]_G, \qquad \underline{E}_\star(X) = [S^\star, \Sigma^\infty_+ X \wedge E]_G,$$
$$\tilde{\underline{E}}^\star(X) = [\Sigma^\infty X, S^\star \wedge E]_G, \qquad \tilde{\underline{E}}_\star(X) = [S^\star, \Sigma^\infty X \wedge E]_G,$$
$$\underline{E}^\star(D) = [D, S^\star \wedge E]_G, \quad \text{and} \quad \underline{E}_\star(D) = [S^\star, D \wedge E]_G.$$

Note that $\underline{\pi}_\star(E \wedge D) = \underline{E}_\star(D)$.

Oruç gives an explicit formula for the homology and cohomology of a G-spectrum with coefficients in G-Mackey functors under certain conditions in [Oru89]. In order to state this theorem, we need some observations and definitions.

Let V be a G-representation, and $\gamma \in G$, then there is a G-action on S^V, $\gamma\colon S^V \to S^V$. In fact, since fixed points have an action of the Weyl group, there is a WH action induced from γ, say $\underline{\gamma}\colon S^{V^H} \to S^{V^H}$.

Definition 2.5.2 ([Oru89, Definition 3.10(2)])**.** For any G-representation V, let $z_H(V)$ denote the $\mathbb{Z}[WH]$-module $H^{\dim(V^H)}(S^{V^H};\mathbb{Z}) \cong \mathbb{Z}$. The action of $\gamma \in WH$ on $z_H(V)$ is $\gamma x = (\deg \underline{\gamma})x$ for any $x \in z_H(V)$ and $\deg(\underline{\gamma})$ the non-equivariant degree. For any $\alpha = [V] - [W] \in RO(G)$. Let $z_H(\alpha) \cong z_H(V) \otimes z_H(W)$.

For $H \leq G$, if $|WH|$ is odd, then for every $\gamma \in WH$, $\deg(\underline{\gamma}) = 1$. Therefore, for any G-representation V, if $|WH|$ is odd, then $z_H(V) \cong \mathbb{Z}$ with a trivial WH-action. This paper focuses on $G = C_p$. In that case, $z_e(V) \cong \mathbb{Z}$ with the trivial C_p-action when p is an odd prime. We get something interesting when $p = 2$.

Example 2.5.3. Let $G = C_2$ and let γ be the nontrivial element of C_2. Let us compute $z_e(\ell + k\sigma)$ for $\ell, k \geq 0$. The map $\underline{\gamma}\colon S^{\ell+k\sigma} \to S^{\ell+k\sigma}$ has degree $(-1)^k$ since $\underline{\gamma}$ is flipping the $S^{\ell+k}$ sphere on k axes and fixing the other ℓ axes. Therefore, $\gamma x = (-1)^k x$ for $x \in z_e(\ell + k\sigma)$. Therefore, $z_e(\ell + k\sigma) \cong \mathbb{Z}$ has a trivial C_2-action when k is even and the C_2-action of multiplication by -1 when k is odd.

This example shows that the only way that $z_e(\ell + k\sigma)$ has a nontrivial C_2-action is if k is odd. We now have the tools to introduce Oruç's explicit computation of equivariant cohomology and homology of a G-spectrum with coefficients in certain Mackey functors. We have only defined J-Mackey functors for $G = C_p$, but Oruç's result is for a more general G. For the full definition of J-Mackey functors please refer to [Lew80, Definition 5.5].

Proposition 2.5.4 ([Oru89, Proposition 3.11])**.** *For $\underline{R}$ an H-determined, G/H-projective Green functor, $\underline{M}$ a module over $\underline{R}$, E a G-spectrum and $\alpha \in RO(G)$ we have*

$$\underline{H}_\alpha(E;\underline{M}) \cong J_{G/H}(H_{\dim(\alpha^H)}(\Phi^H(E);\underline{M}(G/H)) \otimes z_H(\alpha)), \text{ and}$$
$$\underline{H}^\alpha(E;\underline{M}) \cong J_{G/H}(H^{\dim(\alpha^H)}(\Phi^H(E);\underline{M}(G/H)) \otimes z_H(\alpha)).$$

Recall that any G-Mackey field is H-determined for some $H \leq G$. Let $\mathbb{S}_{C_p}$ and $\mathbb{S}$ be the units in the category of ring C_p-spectra, and the category of ring spectra, respectively. We can write $\underline{\pi}_\star(H\underline{F})$ as $\underline{H}_\star(\mathbb{S}_{C_p};\underline{F})$. The geometric fixed points of $\mathbb{S}_{C_p}$ are as follows, $\Phi^e(\mathbb{S}_{C_p}) = \mathbb{S}_{C_p}$, and $\Phi^{C_p}(\mathbb{S}_{C_p}) = \Phi^{C_p}(\Sigma^\infty_{C_p} S^0) = \Sigma^\infty((S^0)^{C_p}) = \Sigma^\infty S^0 = \mathbb{S}$.

For $\underline{F}$ an H-determined C_p-Mackey field, Proposition 2.5.4 shows that

$$\underline{\pi}_\alpha(H\underline{F}) \cong J_{C_p/H}(H_{\dim(\alpha^H)}(\Phi^H(\mathbb{S}_{C_p});\underline{F}(C_p/H)) \otimes z_H(\alpha)) \tag{2.5.1}$$

for any $\alpha \in RO(C_p)$. We can split the computation into two cases, when $\underline{F}$ is C_p-determined and when $\underline{F}$ is e-determined.

2.5.1 C_p-determined Mackey field coefficients

For this subsection, let $\underline{F}$ be a C_p-determined C_p-Mackey field for p prime. Then $\underline{F}(C_p/C_p)$ is a field, say k, and $\underline{F}(C_p/e) = 0$. There was a discussion around Example 2.5.3 which showed that $z_{C_p}(\alpha)$ is always congruent to $\mathbb{Z}$ with no group action. By Proposition 2.5.4 we have the following computation:

$$\begin{aligned}
\underline{\pi}_\alpha(H\underline{F}) &\cong J_{C_p/C_p}(H_{\dim(\alpha^{C_p})}(\mathbb{S};k) \otimes z_{C_p}(\alpha)) \\
&\cong J_{C_p/C_p}(H_{\dim(\alpha^{C_p})}(\mathbb{S};k)) \\
&\cong J_{C_p/C_p}(\pi_{\dim(\alpha^{C_p})}(Hk))
\end{aligned}$$

for any $\alpha \in RO(C_p)$. Note that this is only nonzero when $\dim(\alpha^{C_p}) = 0$, that is, when α is $k\sigma$ when $p = 2$ for $k \in \mathbb{Z}$. Using the J-Mackey functor computations in Definition 2.0.16 above we can simplify this computation to the following for p prime

$$\underline{\pi}_\alpha(H\underline{F}) \cong \begin{cases} \underline{F} & \deg(\alpha^{C_p}) = 0 \\ \underline{0} & \text{else.} \end{cases}$$

The $\mathbb{Z}$-graded homotopy groups of the Eilenberg-Mac Lane spectrum of a Mackey field gives a Mackey field since $\underline{\pi}_*(H\underline{F})$ is $\underline{F}$ in degree 0 and $\underline{0}$ in all other degrees. The question of whether the $RO(C_p)$-graded homotopy groups of the Eilenberg-Mac Lane spectrum of a Mackey field is a graded Mackey field is more complex since $\underline{\pi}_\star(H\underline{F})$ is nonzero in many degrees.

Proposition 2.5.5. *Let p be prime. For $\underline{F}$ a C_p-determined C_p-Mackey field, $H\underline{F}_\star$ is an $RO(C_p)$-graded Mackey field.*

Proof. Let $\underline{F}$ be such that $\underline{F}(C_p/C_p) = k$. By way of contradiction, say there exists a nontrivial, proper $RO(C_p)$-graded ideal $\underline{I}_\star$ of $H\underline{F}_\star$. The definition of a graded ideal says that there must be a module structure map $H\underline{F}_\alpha \,\square\, \underline{I}_\beta \to \underline{I}_{\alpha+\beta}$, and $\underline{I}_0$ must be an ideal of $H\underline{F}_0 = \underline{F}$. Therefore $\underline{I}_0$ is either $\underline{F}$ or $\underline{0}$. We will show that for either situation $\underline{I}_\star$ will be forced to be $\underline{0}_\star$ or $H\underline{F}_\star$, which will be a contradiction since $\underline{I}_\star$ is assumed to be a nontrivial, proper ideal.

If $\underline{I}_0 = \underline{F}$, choose $\alpha \in RO(C_p)$ such that $H\underline{F}_\alpha \neq \underline{0}$, so by above calculations since $H\underline{F}_\alpha \neq \underline{0}$, then $H\underline{F}_\alpha \cong \underline{F}$. Consider the module structure map $H\underline{F}_\alpha \square \underline{I}_0 \cong \underline{F} \square \underline{F} \to \underline{I}_\alpha$ and by Proposition 2.1.3 this map is determined by the module structure map: $k \otimes k \to \underline{I}_\alpha(C_p/C_p)$. As we know, when considering k as a module over itself, the module structure map is just the multiplication map, so $\underline{I}_\alpha(C_p/C_p)$ must be k or else that is not a multiplication map on k. Then we can see that $\underline{I}_\alpha$ must be $\underline{F}$. Thus, $\underline{I}_\alpha$ must be $\underline{F}$ for all α such that $H\underline{F}_\alpha \neq \underline{0}$ so $\underline{I}_\star \cong H\underline{F}_\star$.

If $\underline{I}_0 = \underline{0}$, choose $\alpha \in RO(C_p)$ such that $H\underline{F}_\alpha \neq \underline{0}$, so by above calculations since $H\underline{F}_\alpha \neq \underline{0}$, $H\underline{F}_\alpha \cong \underline{F}$. Consider the module structure map $H\underline{F}_\alpha \square \underline{I}_{-\alpha} \cong \underline{F} \square \underline{I}_{-\alpha} \to \underline{I}_0 = \underline{0}$ and by Proposition 2.1.3 this map is determined by the module structure map: $k \otimes \underline{I}_{-\alpha}(C_p/C_p) \to 0$. Recall that $\underline{I}_{-\alpha}$ is an ideal of $H\underline{F}_{-\alpha}$ which by the computation above is either $\underline{0}$ or $\underline{F}$. So $\underline{I}_{-\alpha}$ can either be $\underline{0}$ or $\underline{F}$. If $\underline{I}_{-\alpha} = \underline{F}$, then the multiplication map would be $k \otimes k \to 0$, but we assumed that k is a field so this is not a multiplication map. Therefore $\underline{I}_{-\alpha}$ must be $\underline{0}$. Note that if $\deg(\alpha^{C_p}) = \deg(-\alpha^{C_p})$ so by definition of $H\underline{F}_\star$, $H\underline{F}_\alpha = H\underline{F}_{-\alpha}$ for all $\alpha \in RO(C_p)$. Thus, $\underline{I}_\alpha$ must be $\underline{0}$ for all α such that $H\underline{F}_\alpha \neq \underline{0}$ so $\underline{I}_\star \cong \underline{0}_\star$.

Therefore, $H\underline{F}_\star$ is an $RO(C_p)$-graded Mackey field. $\square$

2.5.2 e-determined Mackey field coefficients

Let $\underline{F}$ be an e-determined C_p-Mackey field, for p prime. Then $\underline{F}(C_p/C_p)$ is a field and $\underline{F}(C_p/e) \neq 0$, say R. Example 2.5.3 shows that when $p = 2$, then $z_e(n+m\sigma)$ is congruent to $\mathbb{Z}$ with a nontrivial C_2-action when m is odd and congruent to $\mathbb{Z}$ with a trivial C_2-action when m is even. There was a discussion after Example 2.5.3 which shows that when p is an odd prime, $z_e(\alpha)$ is congruent to $\mathbb{Z}$ with a trivial C_p-action. By Proposition 2.5.4 we have the following computation:

$$\begin{aligned} \underline{\pi}_\alpha(H\underline{F}) &\cong J_{C_p/e}(H_{\dim(\alpha)}(\mathbb{S}_{C_p}; R) \otimes z_e(\alpha)) \\ &\cong J_{C_p/e}(\pi_{\dim(\alpha)}(HR) \otimes z_e(\alpha)) \end{aligned}$$

for any $\alpha \in RO(C_p)$. Note that $\pi_{\dim(\alpha)}(HR)$ is R when $\dim(\alpha) = 0$ and 0 else.

For the $p = 2$ case, using the J-Mackey functor computations in Definition 2.0.16 above we have the following:

$$\underline{\pi}_\alpha(H\underline{F}) \cong \begin{cases} \begin{tikzcd} R^{C_2} \arrow[d, bend right, "inc"'] \\ R \arrow[u, bend right, "1+\gamma"'] \end{tikzcd} & \alpha = k - k\sigma,\ k \text{ even} \\ \begin{tikzcd} (R \otimes \mathbb{Z})^{C_2} \arrow[d, bend right, "inc"'] \\ R \otimes \mathbb{Z} \arrow[u, bend right, "1+\gamma"'] \end{tikzcd} & \alpha = k - k\sigma,\ k \text{ odd} \\ \underline{0} & \text{else} \end{cases}$$

where the C_2-action is diagonal on $R \otimes \mathbb{Z}$ and the C_2-action on $\mathbb{Z}$ is multiplication by -1. If R is characteristic two, then $R \otimes \mathbb{Z} \cong R$ where the C_2-action is just the action on R and $(R \otimes \mathbb{Z})^{C_2} \cong R^{C_2}$. Since $H\underline{F}_0 \cong \underline{F}$, then every e-determined C_2-Mackey field can be written as the following:

$$\begin{tikzcd} R^{C_2} \arrow[d, bend right, "inc"'] \\ R \arrow[u, bend right, "1+\gamma"'] \end{tikzcd}$$

Note that this does not mean that every C_2-Mackey functor that has the above Lewis diagram is a C_2-Mackey field. For example $\underline{\mathbb{Z}}$ has this structure but is not a Mackey field as $\underline{\mathbb{Z}}(C_2/C_2) = \mathbb{Z}$ is not a field and $\underline{M}(G/G)$ must be a field if $\underline{M}$ is a G-Mackey field.

For the case when p is an odd prime, using the J-Mackey functor computations in Definition 2.0.16 above we have the following:

$$\underline{\pi}_\alpha(H\underline{F}) \cong \begin{cases} \begin{tikzcd} R^{C_p} \arrow[d, bend right, "inc"'] \\ R \arrow[u, bend right, "tr"'] \end{tikzcd} & \dim(\alpha) = 0 \\ \underline{0} & \text{else} \end{cases}$$

where the transfer map is the sum of γ^i for $0 \leq i \leq p-1$, γ the chosen generator of C_p. Since $H\underline{F}_0 \cong \underline{F}$, then every e-determined C_p-Mackey field for p an odd prime, can be written as the following:

$$\begin{tikzcd} R^{C_p} \arrow[d, bend right, "inc"'] \\ R. \arrow[u, bend right, "tr"'] \end{tikzcd}$$

Note that this does not mean that every C_p-Mackey functor for p an odd prime that has the above Lewis diagram is a C_p-Mackey field. For example $\underline{\mathbb{Z}}$ has this structure but is not a Mackey field as $\underline{\mathbb{Z}}(C_p/C_p) = \mathbb{Z}$ is not a field and $\underline{M}(G/G)$ must be a field if $\underline{M}$ is a G-Mackey field.

Further, Lewis discusses after Remark 7.4 in [Lew80] the following result.

Proposition 2.5.6 ([Lew80])**.** *If $\underline{F}$ is an H-determined G-Mackey field, then there must be an element $x \in \underline{F}(G/H)$ such that $tr_H^G(x) = 1 \in \underline{F}(G/G)$.*

Corollary 2.5.7. *Let p be prime. If $\underline{F}$ is an e-determined C_p-Mackey field, then the transfer map is nonzero.*

This result will greatly help in future proofs about e-determined C_p-Mackey fields.

As mentioned above, the question of whether the $RO(C_p)$-graded homotopy groups of the Eilenberg-Mac Lane spectrum of a Mackey field is a graded Mackey field is more complex than the $\mathbb{Z}$-graded homotopy groups since $\underline{\pi}_\star(H\underline{F})$ is nonzero in many degrees.

Proposition 2.5.8. *For $\underline{F}$ an e-determined C_p-Mackey field, and p an odd prime, $H\underline{F}_\star$ is an $RO(C_p)$-graded Mackey field.*

Proof. Let $\underline{F}$ be such that $\underline{F}(C_p/e) = R$, and $\underline{F}(C_p/C_p) = R^{C_p}$. By way of contradiction, say there exists a nontrivial, proper $RO(C_p)$-graded ideal $\underline{I}_\star$ of $H\underline{F}_\star$. The definition of a graded ideal says that there must be a module structure map $H\underline{F}_\alpha \,\Box\, \underline{I}_\beta \to \underline{I}_{\alpha+\beta}$, and $\underline{I}_0$ must be an ideal of $H\underline{F}_0 = \underline{F}$. Therefore $\underline{I}_0$ is either $\underline{F}$ or $\underline{0}$. We will show that for either situation $\underline{I}_\star$ will be forced to be $\underline{0}_\star$ or $H\underline{F}_\star$, which will be a contradiction since $\underline{I}_\star$ is assumed to be a nontrivial, proper ideal.

If $\underline{I}_0 = \underline{F}$, choose $\alpha \in RO(C_p)$ such that $H\underline{F}_\alpha \neq \underline{0}$, so by above calculations since $H\underline{F}_\alpha \neq \underline{0}$, then $H\underline{F}_\alpha \cong \underline{F}$. Consider the module structure map is $H\underline{F}_\alpha \Box \underline{I}_0 \cong \underline{F} \Box \underline{F} \to \underline{I}_\alpha$ and by Proposition 2.1.3 this map is determined by the module structure maps: $R \otimes R \to \underline{I}_\alpha(C_p/e)$ and $R^{C_p} \otimes R^{C_p} \to \underline{I}_\alpha(C_p/C_p)$. As we know, when considering a ring as a module over itself the module structure map is a multiplication map, so $\underline{I}_\alpha(C_p/e)$ must be R and $\underline{I}_\alpha(C_p/C_p)$ must be R^{C_p} or else the above mentioned maps are not multiplication maps on R and R^{C_p} respectively. Then we can see that $\underline{I}_\alpha$ must be $\underline{F}$. Thus, $\underline{I}_\alpha$ must be $\underline{F}$ for all α such that $H\underline{F}_\alpha \neq \underline{0}$ so $\underline{I}_\star \cong H\underline{F}_\star$.

If $\underline{I}_0 = \underline{0}$, choose $\alpha \in RO(C_p)$ such that $H\underline{F}_\alpha \neq \underline{0}$, so by above calculations since $H\underline{F}_\alpha \neq \underline{0}$, then $H\underline{F}_\alpha \cong \underline{F}$. Consider the module structure map $H\underline{F}_\alpha \Box \underline{I}_{-\alpha} \cong \underline{F} \Box \underline{I}_{-\alpha} \to \underline{I}_0 \cong \underline{0}$ and by Proposition 2.1.3 this map is determined by the module structure maps: $R \otimes \underline{I}_{-\alpha}(C_p/e) \to 0$, and $R^{C_p} \otimes \underline{I}_{-\alpha}(C_p/C_p) \to 0$. Since $\underline{I}_{-\alpha}$ is a submodule of $\underline{F}_{-\alpha}$, which by above computations is either $\underline{0}$ or $\underline{F}$, then $\underline{I}_{-\alpha}(C_p/e)$ is a submodule of R and $\underline{I}_{-\alpha}(C_p/C_p)$ is a submodule of R^{C_p}. The only submodule of R, say M, whose multiplication map is the zero map, is 0 itself so $\underline{I}_{-\alpha}(C_p/e)$ must be 0. Similarly we can show that $\underline{I}_{-\alpha}(C_p/C_p)$ must be 0. Then we can see that $\underline{I}_{-\alpha} \cong \underline{0}$. Note that if $\dim(\alpha) = \dim(-\alpha)$ so for p an odd prime, by definition of $H\underline{F}_\star$, $H\underline{F}_\alpha = H\underline{F}_{-\alpha}$ for all $\alpha \in RO(C_p)$. Thus, $\underline{I}_\alpha$ must be $\underline{0}$ for all α such that $H\underline{F}_\alpha \neq \underline{0}$ so $\underline{I}_\star \cong \underline{0}_\star$.

Therefore, $H\underline{F}_\star$ is an $RO(C_p)$-graded Mackey field. □

The proof is the same for the following proposition.

Proposition 2.5.9. *Let $\underline{F}$ be an e-determined C_2-Mackey field, where $\underline{F}(C_2/e) = R$. If R is such that $R \otimes \mathbb{Z} \cong R$ as $\mathbb{Z}[C_2]$-modules where $\mathbb{Z}$ has the C_2-action of multiplication by -1, then $H\underline{F}_\star$ is an $RO(C_2)$-graded Mackey field.*

Note that this does not mean that $H\underline{F}_\star$ is not an $RO(C_2)$-graded Mackey field for other e-determined C_2-Mackey fields $\underline{F}$.

CHAPTER 3

CLASSICAL AND EQUIVARIANT HOCHSCHILD THEORIES

Throughout this paper, we will use equivariant analogues of Hochschild homology (HH) and topological Hochschild homology (THH); these equivariant analogues were defined in [BGHL19] and [ABG+18], respectively. In this section, we recall the definition of HH, a classical invariant of algebras, its topological analogue THH which was defined in [Bök85a], and discuss some tools used to compute them. We will then recall equivariant analogues of HH and THH, namely, Hochschild homology for Green functors [BGHL19, Definition 2.25] and twisted THH [ABG+18, Definition 8.2], which take as input equivariant rings and equivariant ring spectra, respectively.

3.1 Hochschild homology

For this section, let k be a commutative ring, let A be a k-algebra, and let all tensor products be over k. We will discuss two perspectives on Hochschild homology, one via the cyclic bar complex and the other using Tor-functors. We start by defining the cyclic bar complex.

Definition 3.1.1. Let k be a commutative ring. The *cyclic bar complex* for a k-algebra A, denoted $B^{cy}_\bullet(A)$, is a simplicial k-module such that $B^{cy}_n(A) = A^{\otimes n+1}$, where the face and degeneracy maps $d_i\colon A^{\otimes n+1} \to A^{\otimes n}$ and $s_i\colon A^{\otimes n+1} \to A^{\otimes n+2}$ are defined as follows:

$$d_i(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = \begin{cases} a_0 \otimes a_1 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n & 0 \leq i < n \\ a_n a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1} & i = n \end{cases}$$

$$s_i(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = a_0 \otimes a_1 \otimes \ldots \otimes a_i \otimes 1 \otimes \ldots \otimes a_n \qquad 0 \leq i \leq n.$$

To define Hochschild homology, we start by defining the *Hochschild complex*, $C_*(A)$. Define $C_n(A) = A^{\otimes n+1}$, such that the boundary map $b\colon A^{\otimes n+1} \to A^{\otimes n}$ is defined as $b = \sum_{i=0}^{n} (-1)^i d_i$. Using this, we can define $\mathrm{HH}(A)$.

Definition 3.1.2. Let k be a commutative ring, and A a k-algebra. The *Hochschild homology of A* is $\mathrm{HH}^k_n(A) := H_n(C_*(A))$.

By the Dold-Kan correspondence, there is an equivalent way to define Hochschild homology.

Definition 3.1.3. Let k be a commutative ring, and A a k-algebra. The *Hochschild homology of A* is $\mathrm{HH}_n^k(A) \coloneqq \pi_n(|B_\bullet^{cy}(A)|)$.

Interestingly, the cyclic bar complex has a cyclic operator on every level. In particular, there is a map $\tau\colon A^{\otimes n+1} \to A^{\otimes n+1}$ which rotates the last copy of A to the front for all $n \geq 0$. This map τ generates the C_{n+1}-action on the n^{th} level of the cyclic bar complex. This cyclic bar complex is a cyclic object and by Connes' theory of cyclic sets, the geometric realization of a cyclic object has an S^1-action [Con83].

Recall that $A - A$-bimodules agree with left $(A \otimes A^{\mathrm{op}})$-modules. We will write the enveloping algebra $A \otimes A^{\mathrm{op}}$ as A^e. We recall the following classical homological algebra result.

Proposition 3.1.4. *Let k be a commutative ring, and A a k-algebra. If A is flat as a module over k, then there is an isomorphism*

$$\mathrm{HH}_n^k(A) \cong \mathrm{Tor}_n^{A^e}(A, A).$$

3.2 Topological Hochschild homology

As one can see in the previous subsection, Hochschild homology is a purely algebraic object. Bökstedt developed a topological analogue to Hochschild homology, topological Hochschild homology (THH) [Bök85a]. Throughout this subsection, let R be a ring spectrum. We will start by defining the cyclic bar complex for ring spectra. Let $\tau\colon R^{\wedge n+1} \to R^{\wedge n+1}$ be the map that rotates the last copy of R to the front.

Definition 3.2.1. The *cyclic bar complex* for a ring spectrum R, denoted $B_\bullet^{cy}(R)$, is a simplicial spectrum such that $B_n^{cy}(R) = R^{\wedge n+1}$ where the face and degeneracy maps $d_i\colon R^{\wedge n+1} \to R^{\wedge n}$ and $s_i\colon R^{\wedge n+1} \to R^{\wedge n+2}$ are defined as follows:

$$d_i = \begin{cases} \mathrm{id}^i \wedge \phi \wedge \mathrm{id}^{n-i-1} & 0 \leq i < n \\ (\phi \wedge \mathrm{id}^{n-1}) \circ \tau & i = n \end{cases}$$

$$s_i = \mathrm{id}^{i+1} \wedge \eta \wedge \mathrm{id}^{n-i} \qquad 0 \leq i \leq n$$

where ϕ and η are the multiplication and unit maps of the ring spectrum R.

This is the topological analogue to the cyclic bar complex in algebra, as we replace k-algebras with ring spectra and the tensor with the smash product.

Definition 3.2.2 ([Bök85a]). The *topological Hochschild homology* of a ring spectrum R, $\mathrm{THH}(R)$, is the geometric realization of the cyclic bar complex, $|B^{cy}_\bullet(R)|$.

An advantage of Hochschild homology was that we could compute it using homological algebra. One of the main tools used to compute THH is the Bökstedt spectral sequence.

Theorem 3.2.3 ([Bök85b]). *Let R be a ring spectrum, and p prime. There is a* Bökstedt spectral sequence

$$E^2_{*,*} = \mathrm{HH}_*(H_*(R; \mathbb{F}_p)) \Rightarrow H_*(\mathrm{THH}(R); \mathbb{F}_p)$$

with differentials $d^r : E_{i,j} \to E_{i-r,j+r-1}$. This spectral sequence converges strongly.

This spectral sequence demonstrates a strong relationship between THH and its algebraic analogue. Bökstedt goes on to use this spectral sequence to compute $\mathrm{THH}(H\mathbb{F}_p)$ in [Bök85b] and many computations of THH have been made by other authors.

There are other useful perspectives on THH. Angeltveit, Blumberg, Gerhardt, Hill, Lawson, and Mandell show in [ABG$^+$18] that $\mathrm{THH}(R)$ can be written as an equivariant norm, $N_e^{S^1} R$. McClure, Schwänzl, and Vogt show in [MSV97] that for R a commutative ring spectrum $\mathrm{THH}(R) \cong |R \otimes S^1_\bullet|$.

3.3 Hochschild homology for Green functors

As discussed above, Green functors are an equivariant analogue to rings; therefore, it is natural to want an equivariant analogue of Hochschild homology to take an input of $\underline{R}$-algebras for $\underline{R}$ a

Green functor. In [BGHL19] the authors define an equivariant analogue to Hochschild homology, namely *Hochschild homology for Green functors*.

For this section, let $C_n = \langle \gamma \rangle$ be the cyclic group of order n where $\gamma = e^{2\pi i/n}$, let $\underline{R}$ be a commutative C_n-Green functor, let $\underline{M}$ be an $\underline{R}$-algebra, and all box products are over $\underline{R}$. Recall that for G a cyclic group, one can define a G-action on a G-Mackey functor Remark 2.0.14. Let $\alpha\colon \underline{M}^{\square m+1} \to \underline{M}^{\square m+1}$ be the map that rotates the last copy of $\underline{M}$ to the front and then acts on that $\underline{M}$ by γ.

We now recall the equivariant analogue of the cyclic bar construction defined in [BGHL19].

Definition 3.3.1 ([BGHL19, Definition 2.20])**.** Let $C_n = \langle \gamma \rangle$ and let $\underline{R}$ be a commutative C_n-Green functor. The *C_n-twisted cyclic bar complex* of $\underline{M}$ an $\underline{R}$-algebra, denoted $B_\bullet^{cy,C_n}(\underline{M})$, is a simplicial C_n-Mackey functor such that $B_m^{cy,C_n} = \underline{M}^{\square m+1}$, where the face and degeneracy maps $d_i\colon \underline{M}^{\square m+1} \to \underline{M}^{\square m}$ and $s_i\colon \underline{M}^{\square m+1} \to \underline{M}^{\square m+2}$ are defined as follows:

$$d_i = \begin{cases} \mathrm{id}^i \square \phi \square \mathrm{id}^{m-i-1} & 0 \le i < m \\ (\phi \square \mathrm{id}^{m-1}) \circ \alpha & i = m \end{cases}$$

$$s_i = \mathrm{id}^{i+1} \square \eta \square \mathrm{id}^{m-i} \qquad 0 \le i \le m$$

where ϕ and η are the multiplication and unit maps of $\underline{M}$.

There is an equivalence between the category of simplicial Mackey functors and the category of non-negatively graded dg Mackey functors by applying the Dold-Kan correspondence at each orbit. The homology of a simplicial Mackey functor is the homology of the associated normalized dg Mackey functor; details can be found in Section 4 of [BGHL19].

Definition 3.3.2 ([BGHL19, Definition 2.25])**.** Let $C_n = \langle \gamma \rangle$, let $\underline{R}$ be a commutative C_n-Green functor, and let $\underline{M}$ be an $\underline{R}$-algebra. The *Hochschild homology* of $\underline{M}$, is defined by

$$\underline{\mathrm{HH}}_i^{\underline{R},C_n}(\underline{M}) = H_i(B_\bullet^{cy,C_n}(\underline{M})).$$

Adamyk, Gerhardt, Hess, Klang, and Kong define Hochschild homology of graded Green functors in [AGH+22]. Let $\alpha\colon \underline{M}_\star^{\square m+1} \to \underline{M}_\star^{\square m+1}$ be the iteration of the rotating isomorphism (as defined in Definition 2.2.2) which moves the last copy of $\underline{M}_\star$ to the front, and then acts on that copy of $\underline{M}_\star$ by γ. Now let us define the C_n-twisted cyclic bar complex for $RO(C_n)$-graded Green functors.

Definition 3.3.3. [AGH+22, Definition 4.1.7] Let $C_n = \langle \gamma \rangle$, let $\underline{R}_\star$ be a commutative C_n-Green functor. The *C_n-twisted cyclic bar complex* of $\underline{M}_\star$ an $\underline{R}_\star$-algebra, denoted $B_\bullet^{C_n,cy}(\underline{M}_\star)$, is a simplicial $RO(C_n)$-graded Mackey functor such that $B_m^{cy,C_n} = \underline{M}_\star^{\square m+1}$, where the face and degeneracy maps $d_i \colon \underline{M}_\star^{\square m+1} \to \underline{M}_\star^{\square m}$ and $s_i \colon \underline{M}_\star^{\square m+1} \to \underline{M}_\star^{\square m+2}$ are defined as follows:

$$d_i = \begin{cases} \mathrm{id}^i \square \phi \square \mathrm{id}^{m-i-1} & 0 \le i < m \\ (\phi \square \mathrm{id}^{m-1}) \circ \alpha & i = m \end{cases}$$
$$s_i = \mathrm{id}^{i+1} \square \eta \square \mathrm{id}^{m-i} \qquad 0 \le i \le m$$

where ϕ and η are the multiplication and unit maps of $\underline{M}_\star$.

Definition 3.3.4. [AGH+22, Definition 4.1.8] Let $C_n = \langle \gamma \rangle$, let $\underline{R}_\star$ be a commutative $RO(C_n)$-graded Green functor, and let $\underline{M}_\star$ be an $\underline{R}_\star$-algebra. The *Hochschild homology for $RO(C_n)$-graded Green functors* of $\underline{M}_\star$, is defined by

$$\underline{\mathrm{HH}}_i^{\underline{R}_\star, C_n}(\underline{M}_\star) = H_i(B_\bullet^{cy,C_n}(\underline{M}_\star)).$$

Lewis and Mandell's paper [LM06] allows us to do homological algebra in the equivariant setting. As is true classically, there is a Tor functor perspective for Hochschild homology for Green functors. We recall the definition of Tor in this setting.

Definition 3.3.5 ([LM06])**.** Let $\underline{R}_\star$ be an $RO(G)$-graded Green functor. For $\underline{M}_\star$ and $\underline{N}_\star$ left and right $\underline{R}_\star$-modules respectively, $\underline{\mathrm{Tor}}_{s,\star}^{\underline{R}_\star}(\underline{N}_\star, \underline{M}_\star)$ is the s^{th} left derived functor of $\underline{N}_\star \square_{\underline{R}_\star} \underline{M}_\star$.

We will now define an $\underline{R}$-module which encodes this twisting information into the left module structure map. Let $\underline{R}$ be a C_n-Green functor, and $\underline{M}$ a left $\underline{R}$-module. Let us define ${}^\gamma\underline{M}$ as $\underline{M}$ with the left module map defined as

$$\begin{tikzcd} \underline{M} \,\square_{\underline{R}}\, \underline{M} \arrow[r, "\gamma\mu"] \arrow[d, "\gamma\square\mathrm{id}"'] & \underline{M} \\ \underline{M} \,\square_{\underline{R}}\, \underline{M} \arrow[ur, "\mu"'] & \end{tikzcd}$$

where μ is the left module map for $\underline{M}$ and $\gamma\colon \underline{M} \to \underline{M}$ acts on $\underline{M}$ by γ.

Proposition 3.3.6 ([AGH+22, Proposition 4.3.2])**.** *Let $C_n = \langle\gamma\rangle$ and $\underline{R}_\star$ be an $RO(C_n)$-graded commutative Green functor. If $\underline{M}_\star$ is an $\underline{R}_\star$-algebra and is flat as an $\underline{R}_\star$-module, there is a natural isomorphism*

$$\underline{\mathrm{HH}}_*^{\underline{R}_\star, C_n}(\underline{M}_\star) \cong \underline{\mathrm{Tor}}_{*,\star}^{\underline{M}_\star \square_{\underline{R}_\star} \underline{M}_\star^{op}}(\underline{M}_\star, {}^{\gamma}\underline{M}_\star).$$

3.4 Twisted topological Hochschild homology

For this section, let $C_n = \langle\gamma\rangle$ be the cyclic group of order n where $\gamma = e^{2\pi i/n}$, let R be a ring C_n-spectrum, and $\alpha\colon R^{\wedge m+1} \to R^{\wedge m+1}$ rotates the last copy of R to the front and acts on that copy of R by γ. In [ABG+18], Angeltveit, Blumberg, Gerhardt, Hill, Lawson, and Mandell define an equivariant analogue to THH which takes as input a ring C_n-spectra, namely C_n-*twisted* THH. The authors define C_n-twisted THH of a ring C_n-spectrum R to be the norm $N_{C_n}^{S^1} R$. The authors show that twisted THH can be defined using a twisted analogue of the cyclic bar complex.

Definition 3.4.1. Let $\gamma = e^{\frac{2\pi i}{n}}$ be the chosen generator of C_n. The C_n-*twisted cyclic bar complex* for a ring C_n-spectrum R, denoted $B_\bullet^{cy,C_n}(R)$, is a simplicial object such that $B_m^{cy,C_n} = R^{\wedge m+1}$, where the face and degeneracy maps, $d_i\colon R^{\wedge m+1} \to R^{\wedge m}$ and $s_i\colon R^{\wedge m+1} \to R^{\wedge m+2}$ are defined as follows:

$$d_i = \begin{cases} \mathrm{id}^i \wedge \phi \wedge \mathrm{id}^{m-i-1} & 0 \le i < m \\ (\phi \wedge \mathrm{id}^{m-1}) \circ \alpha & i = m \end{cases}$$

$$s_i = \mathrm{id}^{i+1} \wedge \eta \wedge \mathrm{id}^{m-i} \qquad 0 \le i \le m$$

where ϕ and η are the multiplication and unit maps of R.

In the following definition, we let $\mathcal{I}$ denote the change of universe functor.

Definition 3.4.2 ([ABG+18, Definition 8.2])**.** Let U be a complete S^1-universe, let $\widetilde{U} := i^*_{C_n} U$ be the pullback of the universe to C_n, and let R be a ring C_n-spectrum indexed on U. The *C_n-twisted topological Hochschild homology* of R is $\mathrm{THH}_{C_n}(R) = \mathcal{I}^U_{\mathbb{R}^\infty} |B^{cy,C_n}_\bullet(\mathcal{I}^{\mathbb{R}^\infty}_{\widetilde{U}} R)|$.

The work of Adamyk, Gerhardt, Hess, Klang, and Kong in [AGH+22, Theorem 4.2.7] shows that there is an equivariant analogue of the Bökstedt spectral sequence which demonstrates a relationship between Hochschild homology for Green functors and C_n-twisted THH. First, we must discuss the following facts.

For G an abelian group, and $\gamma \in G$, one can define a left γ-action on any genuine orthogonal G-spectrum, denoted $\ell_\gamma \colon X \to X$ (for more details see [Sch18, Section 3.1]). We will say that γ acts on a G-spectrum trivially if ℓ_γ is equivariantly homotopic to the identity map.

Theorem 3.4.3 ([AGH+22, Theorem 4.2.7])**.** *Let C_n be a finite subgroup of S^1 such that $C_n = \langle \gamma \rangle$. Let R be a ring C_n-spectrum and E a commutative ring C_n-spectrum such that γ acts trivially on E. If $\underline{E}_\star(R)$ is flat over $\underline{E}_\star$, then there is an equivariant Bökstedt spectral sequence*

$$E^2_{s,\star} = \underline{\mathrm{HH}}^{\underline{E}_\star, C_n}_s(\underline{E}_\star(R)) \Rightarrow \underline{E}_{s+\star}(i^*_{C_n} \mathrm{THH}_{C_n}(R))$$

where $d^r \colon E_{i,\alpha} \to E_{i-r,\alpha+r-1}$.

The category of orthogonal C_n-ring spectra and the category of unbased C_n-spaces are tensored over the category of unbased C_n-spaces. Let R be a commutative ring C_n-spectrum indexed over the trivial universe $\mathbb{R}^\infty$. Consider the functor $R \otimes_{C_n} (-)$ to be the coequalizer of the following diagram

$$R \otimes C_n \otimes (-) \underset{\ell \otimes \mathrm{id}}{\overset{\mathrm{id} \otimes r}{\rightrightarrows}} R \otimes (-)$$

where r is the C_n-action on $(-)$ and ℓ is the induced C_n-action on R.

The authors of [ABG+18] show that for U a complete S^1-universe and $\widetilde{U} = \iota^*_{C_n} U$, $\mathrm{THH}_{C_n}(R) \simeq \mathcal{I}^U_{\mathbb{R}^\infty}(R \otimes_{C_n} S^1)$. This definition will be heavily used in Chapter 5 in order to demonstrate the algebraic structure of twisted THH.

3.5 A computation of twisted THH

Very few computations of twisted THH have been done. In this section, we will compute $H\underline{F}_\star(\mathrm{THH}_{C_2}(MU_\mathbb{R}))$ for $\underline{F}$ the C_2-Mackey field such that $\underline{F}(C_2/C_2) = \mathbb{F}_2$ and $\underline{F}(C_2/e) = 0$, and $MU_\mathbb{R}$ the Real bordism spectrum. To do this computation, we will use the equivariant Bökstedt spectral sequence Theorem 3.4.3. Recall that to use this spectral sequence for this circumstance, we will need that γ, the non-trivial element of C_2, acts trivially on $H\underline{F}$, and that $H\underline{F}_\star(MU_\mathbb{R})$ is flat over $\underline{\pi}_\star(H\underline{F})$.

Recall that in Section 3.4 we discussed that one can define a C_p-spectrum to have a trivial γ-action, for $\gamma \in C_p$, if ℓ_γ is equivariantly homotopic to the identity map. Let X be a C_p-spectrum. If the Weyl action on $\underline{\pi}_\star(X)$ is trivial, then the generator γ of C_p induces the identity map on the $RO(C_p)$-graded homotopy groups of X. Furthermore, if the only element that induces the identity map in $X^\star X$ is the unit 1, then ℓ_γ must be equivariantly homotopic to the identity map.

If we consider X to be the Eilenberg-Mac Lane spectrum of the C_p-Mackey field $\underline{F}$ where $\underline{F}(C_p/C_p) = k$ and $\underline{F}(C_p/e) = 0$ then by the computations in Section 2.5 we know that whenever $\underline{\pi}_\alpha(H\underline{F}) \neq \underline{0}$ then $\underline{\pi}_\alpha(H\underline{F}) = \underline{F}$ which has a trivial Weyl action. Further, $\underline{\pi}_\alpha(H\underline{F}) \neq \underline{0}$ only when $\dim(\alpha^{C_p}) = 0$. Therefore $H\underline{F}_\star$ has a trivial Weyl action and the only degrees of $H\underline{F}^\star H\underline{F}$ we need to consider are the degrees α such that $\dim(\alpha^{C_p}) = 0$.

To compute $H\underline{F}^\star H\underline{F}$ we consider Proposition 2.3.8 which shows that we have the following isomorphism of non-equivariant ring spectra: $\Phi^{C_p}(H\underline{F}) \cong Hk$. Therefore, by Proposition 2.5.4 $H\underline{F}^\alpha H\underline{F}(C_p/C_p) \cong Hk^{\dim(\alpha^{C_p})}Hk$ and $H\underline{F}^\alpha H\underline{F}(C_p/e) = 0$.

Since we only need to consider $H\underline{F}^\alpha H\underline{F}$ for $\dim(\alpha^{C_p}) = 0$ then we only need to consider $Hk^0 Hk$. If we let $k = \mathbb{F}_2$, then the question reduces to which elements of $\mathbb{F}_2$ induce the identity on $H\mathbb{F}_2^0 H\mathbb{F}_2 \cong \mathbb{F}_2$. The only element of $\mathbb{F}_2$ which induces the identity on $\mathbb{F}_2$ is the unit 1. Therefore the action of γ on $H\underline{F}$ is trivial when $\underline{F}(C_2/C_2) = \mathbb{F}_2$, and $\underline{F}(C_2/e) = 0$.

We will now compute $H\underline{F}_\star(\mathrm{THH}_{C_2}(MU_\mathbb{R}))$ using the equivariant Bökstedt spectral sequence

$$E^2_{*,\star} = \underline{\mathrm{HH}}_*^{H\underline{F}_\star, C_2}(H\underline{F}_\star(MU_\mathbb{R})) \Rightarrow H\underline{F}_\star(\mathrm{THH}_{C_2}(MU_\mathbb{R}))$$

for $\underline{F}(C_2/C_2) = \mathbb{F}_2$ and $\underline{F}(C_2/e) = 0$. To aid our computations, we will recall what it means for a spectrum to be real oriented.

Consider $\mathbb{CP}^n$ and $\mathbb{CP}^\infty$ as pointed C_2-spaces under the action of complex conjugation, where the base point is $\mathbb{CP}^0$. Note that the C_2-fixed point spaces of $\mathbb{CP}^n$ and $\mathbb{CP}^\infty$ are $\mathbb{RP}^n$ and $\mathbb{RP}^\infty$ respectively.

Definition 3.5.1 ([Ara79])**.** Let E be a C_2-equivariant homotopy commutative ring spectrum. A *real orientation* of E is a class $x \in \widetilde{\underline{E}}^\rho(\mathbb{CP}^\infty)(C_2/C_2)$ whose restriction to

$$\widetilde{\underline{E}}^\rho(\mathbb{CP}^1)(C_2/C_2) \cong \underline{E}^0(pt)(C_2/C_2)$$

is the unit, where $\rho = 1 + \sigma$ is the regular representation. The spectrum E is *real oriented* if it has a real orientation.

The following corollary builds off of this work of Araki.

Corollary 3.5.2 ([HHR16, Corollary 5.18])**.** *If E is a real oriented spectrum, then there is a weak equivalence*

$$MU_{\mathbb{R}} \wedge E \simeq E \wedge \bigwedge_{i \geq 1} S^0[S^{i\rho}]$$

where $S^0[S^{i\rho}] = \bigvee_{j \geq 0} (S^{i\rho})^j$.

For $\underline{F}$ where $\underline{F}(C_2/C_2) = k$ and $\underline{F}(C_2/e) = 0$, let us consider for which k $H\underline{F}$ is real oriented.

Proposition 3.5.3. *Let $\underline{F}$ be the C_2-Mackey field where $\underline{F}(C_2/C_2) = k$ is a finite field and $\underline{F}(C_2/e) = 0$. Then, $H\underline{F}$ is real oriented if and only if k is characteristic 2.*

Proof. If $E := H\underline{F}$ is real oriented, then there exists an element $\bar{x} \in \widetilde{\underline{E}}^\rho(\mathbb{CP}^\infty)(C_2/C_2)$ that restricts to the unit of $\widetilde{\underline{E}}^\rho(\mathbb{CP}^1)(C_2/C_2)$. Using Definition 2.5.1, we have

$$\begin{aligned}\widetilde{\underline{E}}^\rho(\mathbb{CP}^\infty)(C_2/C_2) &\cong [\Sigma^\infty\mathbb{CP}^\infty, \Sigma^\rho E]_{C_2}(C_2/C_2)\\ &\cong [(C_2/C_2)_+ \wedge \Sigma^\infty\mathbb{CP}^\infty, \Sigma^\rho E]_{C_2}\\ &\cong [\Sigma^\infty\mathbb{CP}^\infty, \Sigma^\rho E]_{C_2}.\end{aligned}$$

By Proposition 2.3.7, E is concentrated on C_2, which means that $\underline{\pi}^H_\star(E) \neq 0$ if and only if H contains C_2 up to conjugation. Note that $\star - \rho$ ranges over all $RO(C_2)$-gradings. Therefore, an equivalent way to say that E is concentrated over C_2 is by saying that $\underline{\pi}^H_\star(S^\rho \wedge E) \cong \underline{\pi}^H_{\star-\rho}(E) \neq 0$ if and only if H contains C_2 up to conjugation. This implies that $S^\rho \wedge E$ is concentrated over C_2. Using Proposition 2.3.4, and Proposition 2.3.5 and the properties of geometric fixed points, we can continue our calculation in the following way:

$$\begin{aligned}
[\Sigma^\infty \mathbb{CP}^\infty, \Sigma^\rho E]_{C_2} &\cong [\Phi^{C_2}(\Sigma^\infty \mathbb{CP}^\infty), (S^\rho \wedge E)^{C_2}]_e \\
&\cong [\Sigma^\infty((\mathbb{CP}^\infty)^{C_2}), \Phi^{C_2}(S^\rho \wedge E)]_e \\
&\cong [\Sigma^\infty \mathbb{RP}^\infty, S^1 \wedge \Phi^{C_2}(E)]_e.
\end{aligned}$$

Since $[\Sigma^\infty \mathbb{RP}^\infty, S^1 \wedge \Phi^{C_2}(E)]_e$ is non-equivariant, we can use Proposition 2.3.8 to state

$$\begin{aligned}
[\Sigma^\infty \mathbb{RP}^\infty, S^1 \wedge \Phi^{C_2}(E)]_e &\cong [\Sigma^\infty \mathbb{RP}^\infty, S^1 \wedge Hk]_e \\
&\cong \widetilde{H}^1(\mathbb{RP}^\infty; k).
\end{aligned}$$

By a similar argument, we have that

$$\underline{\widetilde{E}}^\rho(\mathbb{CP}^1)(C_2/C_2) \cong \widetilde{H}^1(\mathbb{RP}^1; k).$$

We know that $H^1(\mathbb{RP}^1; k) = k$ and using the Universal Coefficient Theorem we have that $H^1(\mathbb{RP}^\infty; k) \cong \mathrm{Hom}(\mathbb{F}_2, k)$ is k when the characteristic of k is 2, and 0 else. Thus, if k is not characteristic two then there exists no $\bar{x} \in H^1(\mathbb{RP}^\infty; k)$ that maps to the unit in k, so $H\underline{F}$ would not be real oriented.

In the other direction, say k is characteristic two. We can use the cofibration sequence $\mathbb{CP}^1 \to \mathbb{CP}^\infty \to \mathbb{CP}^\infty/\mathbb{CP}^1$ to induce the following exact sequence:

$$\underline{\widetilde{E}}^\rho(\mathbb{CP}^\infty/\mathbb{CP}^1)(C_2/C_2) \to \underline{\widetilde{E}}^\rho(\mathbb{CP}^\infty)(C_2/C_2) \to \underline{\widetilde{E}}^\rho(\mathbb{CP}^1)(C_2/C_2).$$

By the computations above this gives an exact sequence:

$$\widetilde{H}^1((\mathbb{CP}^\infty/\mathbb{CP}^1)^{C_2}; k) \to \widetilde{H}^1(\mathbb{RP}^\infty; k) \to \widetilde{H}^1(\mathbb{RP}^1; k).$$

Since k is characteristic two, we get

$$\widetilde{H}^1((\mathbb{CP}^\infty/\mathbb{CP}^1)^{C_2};k) \to k \to k.$$

The C_2-action on the subset $\mathbb{CP}^1 \subset \mathbb{CP}^\infty$ is closed, therefore $(\mathbb{CP}^\infty/\mathbb{CP}^1)^{C_2} \simeq \mathbb{RP}^\infty/\mathbb{RP}^1$ which is connected. Since $\mathbb{RP}^\infty \backslash \mathbb{RP}^1$ is connected then $\mathrm{Ext}^1_{\mathbb{Z}}(H_0(\mathbb{RP}^\infty \backslash \mathbb{RP}^1;\mathbb{Z}),k) \cong \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z},k) \cong 0$. Further, since $\mathbb{RP}^\infty \backslash \mathbb{RP}^1$ has no 1-cells then $H_1(\mathbb{RP}^\infty \backslash \mathbb{RP}^1;\mathbb{Z}) \cong 0$. So the Universal Coefficient Theorem tells us that $\widetilde{H}^1((\mathbb{CP}^\infty/\mathbb{CP}^1)^{C_2};k) \cong 0$. Then the map $k \to k$ in this exact sequence is injective, which makes it an isomorphism since k is finite by assumption. So the identity element maps to the identity element, therefore $H\underline{F}$ is real oriented. □

Now that we know some examples of C_2-determined C_2-Mackey fields $\underline{F}$ which have an Eilenberg-Mac Lane spectrum that is real oriented, we can use Corollary 3.5.2 to obtain the following result.

Lemma 3.5.4. *Let $\underline{F}$ be a C_2-determined C_2-Mackey field. If $H\underline{F}$ is real oriented, then $H\underline{F}_\star(MU_\mathbb{R})$ is a free $H\underline{F}_\star$-module, that is,*

$$H\underline{F}_\star(MU_\mathbb{R}) \cong H\underline{F}_\star[b_1, b_2, \ldots]$$

where $\deg(b_i) = i\rho$.

Proof. Since $H\underline{F}$ is real oriented, we can use Corollary 3.5.2 to show that

$$MU_\mathbb{R} \wedge H\underline{F} \simeq H\underline{F} \wedge \bigwedge_{i\geq 1} S^0[S^{i\rho}]$$

which gives an isomorphism of $RO(C_2)$-graded Green functors

$$\underline{\pi}_\star(MU_\mathbb{R} \wedge H\underline{F}) \cong \underline{\pi}_\star(H\underline{F})[b_1, b_2, \ldots]$$

for $\deg(b_i) = i\rho$. □

There is a classical standard argument which is a result of Cartan and Eilenberg's Theorem X.6.1 in [CE99]. The argument is that for k a commutative ring, and A a commutative k-algebra that is flat as a module over k, then

$$\mathrm{Tor}_*^{A^e}(A,A) \cong A \otimes_k \mathrm{Tor}_*^A(k,k).$$

Using the homological algebra from [LM06] we can extend Cartan and Eilenberg's argument to the equivariant setting. That is, if $\underline{R}_\star$ is a commutative G-Green functor, and $\underline{M}_\star$ is a commutative $\underline{R}_\star$-algebra that is flat as a module over $\underline{R}_\star$, then

$$\underline{\mathrm{Tor}}_{*,\star}^{\underline{M}_\star \square_{\underline{R}_\star} \underline{M}_\star^{\mathrm{op}}}(\underline{M}_\star, \underline{M}_\star) \cong \underline{M}_\star \square_{\underline{R}_\star} \underline{\mathrm{Tor}}_{*,\star}^{\underline{M}_\star}(\underline{R}_\star, \underline{R}_\star).$$

We will use this in our calculations.

From the discussion in the beginning of this section and Proposition 3.5.3 we know that for the C_2-Mackey field $\underline{F}$ where $\underline{F}(C_2/C_2) = \mathbb{F}_2$, and $\underline{F}(C_2/e) = 0$, $H\underline{F}$ has a trivial C_2-action and is real oriented.

Theorem 3.5.5. *For $\underline{F}$ the C_2-Mackey field where $\underline{F}(C_2/C_2) = \mathbb{F}_2$, and $\underline{F}(C_2/e) = 0$,*

$$\underline{H}_\star(\mathrm{THH}_{C_2}(MU_\mathbb{R}); \underline{F}) \cong H\underline{F}_\star[b_1, b_2, \ldots]\square_{H\underline{F}_\star}\Lambda_{H\underline{F}_\star}(z_1, z_2, \ldots)$$

as an $H\underline{F}_\star$-module. Here $|b_i| = i\rho$ and $|z_i| = 1 + i\rho$.

Proof. Proposition 3.5.3 shows that $H\underline{F}$ has a trivial C_2-action. In order to use the Bökstedt spectral sequence, we need to show that $H\underline{F}_\star(MU_\mathbb{R})$ is flat over $H\underline{F}_\star$. The following isomorphism of $RO(C_2)$-graded Green functors is given by Lemma 3.5.4:

$$\underline{\pi}_\star(H\underline{F} \wedge MU_\mathbb{R}) \cong H\underline{F}_\star[b_1, b_2, \ldots]$$

where $\deg(b_i) = i\rho$. Therefore $H\underline{F}_\star(MU_\mathbb{R})$ is flat over $H\underline{F}_\star$. Since the appropriate conditions hold, we can use the equivariant Bökstedt spectral sequence

$$E^2_{s,\star} = \underline{\mathrm{HH}}_s^{H\underline{F}_\star, C_2}(\underline{H}_\star(MU_\mathbb{R}; \underline{F})) \Rightarrow \underline{H}_{s+\star}(i^*_{C_2}\,\mathrm{THH}_{C_2}(MU_\mathbb{R}); \underline{F}).$$

The E_2-term is

$$E^2_{s,\star} = \underline{\mathrm{HH}}_s^{H\underline{F}_\star, C_2}(H\underline{F}_\star[\beta_1, \beta_2, \ldots]) \cong H\underline{F}_\star[\beta_1, \beta_2, \ldots]\square_{H\underline{F}_\star}\Lambda_{H\underline{F}_\star}(z_1, z_2, \ldots)$$

where $\deg(b_i) = (0, i\rho)$ and $\deg(z_i) = (1, i\rho)$.

Recall that $d^r\colon E^r_{s,\alpha} \to E^r_{s-r,\alpha+r-1}$. Our spectrum $MU_\mathbb{R}$ is commutative, so by [AGH+22, Proposition 4.2.8] we can view this as a spectral sequence of $H\underline{F}_\star$-algebras. Consider the differential

d^2. We know that all the differentials are determined by what the differential does on the generators of the E^2 page, thus since the only generators are in the columns where $s = 0, 1$ then all of the differentials on the E^2-page are zero and the spectral sequence collapses. □

CHAPTER 4

HOPF STRUCTURE OF THE BÖKSTEDT SPECTRAL SEQUENCE

Throughout this section, let R be a commutative ring, and A a commutative ring spectrum.

Spectral sequences can have algebraic structures, and these structures can be very helpful when doing computations with said spectral sequences. More specifically, the algebraic structure of a spectral sequence can help one know more about the differentials of the spectral sequence. As mentioned in Section 3.2, the Bökstedt spectral sequence is one of the main tools we have to compute THH. In this section we will recall results of Angeltveit and Rognes in [AR05] which show that the Bökstedt spectral sequence has a Hopf algebra structure. These results we will recall extend the results of [EKMM97] and [MSV97] which demonstrate that for a commutative ring spectrum A, $\mathrm{THH}(A)$ is an A-Hopf algebra. In the future sections, namely Chapter 5 and Chapter 6, we will prove an equivariant analogue to these results for twisted THH and the equivariant Bökstedt spectral sequence, so this section is dedicated to recalling these classical results. We will start this section by recalling the algebraic definition of R-bialgebras and R-Hopf algebras.

Definition 4.0.1. Let R be a commutative ring. An R-*bialgebra* M is a unital, associative R-algebra as well as a counital, coassociative R-coalgebra such that the following diagrams commute:

$$
\begin{tikzcd}
M & & M \otimes M \arrow[ll, "\phi"'] \arrow[dl, "\varepsilon\otimes\varepsilon"] \\
& R \cong R \otimes R \arrow[from=ul, "\varepsilon"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& R \otimes R \cong R \arrow[dl, "\eta\otimes\eta"'] \arrow[dr, "\eta"] & \\
M \otimes M & & M \arrow[ll, "\psi"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
R \arrow[rr, "\mathrm{id}"] \arrow[dr, "\eta"'] & & R \\
& M \arrow[ur, "\varepsilon"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
M \otimes M \arrow[r, "\phi"] \arrow[d, "\psi\otimes\psi"'] & M \arrow[r, "\psi"] & M \otimes M \\
M \otimes M \otimes M \otimes M \arrow[rr, "\mathrm{id}\otimes\tau\otimes\mathrm{id}"'] & & M \otimes M \otimes M \otimes M \arrow[u, "\phi\otimes\phi"']
\end{tikzcd}
$$

where $\eta, \varepsilon, \phi, \psi$, and τ are the unit, counit, product, coproduct, and the map that swaps the two copies of M respectively.

Definition 4.0.2. Let R be a commutative ring. An R-*Hopf algebra* is an R-bialgebra with a map of R-modules $\chi\colon M \to M$, called the *antipode*, such that the following diagram commutes:

$$
\begin{tikzcd}
M \arrow[r, "\psi"] \arrow[d, "\psi"'] \arrow[dr, "\varepsilon"] & M \otimes M \arrow[dr, "\mathrm{id}\otimes\chi"] & \\
M \otimes M \arrow[dr, "\chi\otimes\mathrm{id}"'] & R \arrow[dr, "\eta"] & M \otimes M \arrow[d, "\phi"] \\
 & M \otimes M \arrow[r, "\phi"'] & M
\end{tikzcd}
$$

where $\eta, \varepsilon, \phi, \psi$, and χ are the unit, counit, product, coproduct, and antipode, respectively.

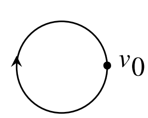
These algebraic definitions can be extended to ring spectra where tensor products are replaced with smash products, and rings are replaced with ring spectra. Further, one can extend these definitions to define spectral sequences of Hopf algebras. This will be discussed more in Chapter 6. In this section we will discuss the simplicial maps which can be used to prove that, for A a commutative ring spectrum, $\mathrm{THH}(A)$ is an A-Hopf algebra in the stable homotopy category. Then we will discuss how these maps induce the Hopf structure on the Bökstedt spectral sequence.

McClure, Schwänzel, and Vogt show in [MSV97] that for A a commutative ring spectrum, $\mathrm{THH}(A) \cong A \otimes S^1$. This result can be used to prove the following theorem by inducing the structure maps of $\mathrm{THH}(A)$ from maps on the circle.

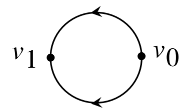
Theorem 4.0.3 ([EKMM97, Corollary 3.4], [MSV97, Theorem I])**.** *For A a commutative ring spectrum,* $\mathrm{THH}(A)$ *is an A-Hopf algebra in the stable homotopy category.*

Angeltveit and Rognes extend this result to the Bökstedt spectral sequence by considering simplicial maps on the circle [AR05]. We will first recall Angeltveit and Rognes' simplicial argument which proves that $\mathrm{THH}(A)$ is an A-Hopf algebra in the stable homotopy category.

Let us define three simplicial spaces $e, S^1_\bullet$ and $dS^1_\bullet$, where e is the point, $S^1_\bullet$ is the classical simplicial structure on the circle:



and $dS^1_\bullet$ is the following simplicial structure on the circle:



Note that $dS^1_\bullet = (\Delta^1_\bullet \sqcup \Delta^1_\bullet) \cup_{\partial\Delta^1_\bullet \sqcup \partial\Delta^1_\bullet} \partial\Delta^1_\bullet$.

Many of the maps needed to prove that $\mathrm{THH}(A)$ is an A-Hopf algebra are induced from the following simplicial maps

$$\begin{aligned}
&\eta\colon e \to S^1_\bullet \\
&\varepsilon\colon S^1_\bullet \to e \\
&\phi\colon S^1_\bullet \vee S^1_\bullet \to S^1_\bullet \\
&\psi\colon dS^1_\bullet \to S^1_\bullet \vee S^1_\bullet \\
&\chi\colon dS^1_\bullet \to dS^1_\bullet.
\end{aligned}$$

Here the map η includes the point into the basepoint of $S^1_\bullet$, the map ε crushes $S^1_\bullet$ to the point, the map ϕ folds the two copies of $S^1_\bullet$ together, the map ψ is the simplicial pinch map, and χ swaps the two 1-cells. Since tensoring with A preserves pushouts, $A \otimes (S^1_\bullet \vee S^1_\bullet)$ is isomorphic to the pushout of the simplicial span $A \otimes S^1_\bullet \leftarrow A \otimes e \to A \otimes S^1_\bullet$. By VII.1.6 in [EKMM97] we have the isomorphism $(|A \otimes S^1_\bullet|) \wedge_A (|A \otimes S^1_\bullet|) \cong \mathrm{THH}(A) \wedge_A \mathrm{THH}(A)$. So, we can identify $A \otimes (S^1_\bullet \vee S^1_\bullet)$ with $\mathrm{THH}(A) \wedge_A \mathrm{THH}(A)$. Let $\mathrm{dTHH}(A) := |A \otimes dS^1_\bullet|$. Lemma 3.8 in [AR05] says that the collapse map $\pi\colon dS^1_\bullet \to S^1_\bullet$, which collapses the second $\Delta^1_\bullet$, induces a weak equivalence $\pi\colon \mathrm{dTHH}(A) \to \mathrm{THH}(A)$. Therefore, the simplicial maps above induce the following maps of spectra

$$\begin{aligned}
&\eta\colon A \to \mathrm{THH}(A) \\
&\varepsilon\colon \mathrm{THH}(A) \to A \\
&\phi\colon \mathrm{THH}(A) \wedge_A \mathrm{THH}(A) \to \mathrm{THH}(A) \\
&\psi\colon \mathrm{THH}(A) \to \mathrm{THH}(A) \wedge_A \mathrm{THH}(A) \\
&\chi\colon \mathrm{THH}(A) \to \mathrm{THH}(A)
\end{aligned}$$

which are the unit, counit, product, coproduct and the antipode, respectively. Note that the coproduct map is the following $\psi\colon \mathrm{THH}(A) \xrightarrow{\simeq} \mathrm{dTHH}(A) \to \mathrm{THH}(A) \wedge_A \mathrm{THH}(A)$ and the antipode is the following $\chi\colon \text{THH}(A) \xrightarrow{\simeq} \text{dTHH}(A) \to \text{dTHH}(A) \xrightarrow{\simeq} \text{THH}(A)$ where the second map is induced by $\chi\colon dS^1_\bullet \to dS^1_\bullet$.

In order to show that $\text{THH}(A)$ is an A-Hopf algebra in the stable homotopy category, we need to show that a variety of diagrams commute up to homotopy. Note that we can reduce the problem to considering if diagrams of simplicial maps of circles commute up to simplicial homotopy. For example, we can reduce the problem of checking if the following diagram commutes:

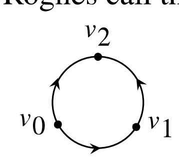
$$\begin{tikzcd}
\text{THH}(A) \wedge_A \text{THH}(A) \wedge_A \text{THH}(A) \arrow[r, "\text{id}\wedge\phi"] \arrow[d, "\phi\wedge\text{id}"'] & \text{THH}(A) \wedge_A \text{THH}(A) \arrow[d, "\phi"] \\
\text{THH}(A) \wedge_A \text{THH}(A) \arrow[r, "\phi"'] & \text{THH}(A)
\end{tikzcd}$$

to checking the commutativity of:

$$\begin{tikzcd}
S^1_\bullet \vee S^1_\bullet \vee S^1_\bullet \arrow[r, "\text{id}\vee\phi"] \arrow[d, "\phi\vee\text{id}"'] & S^1_\bullet \vee S^1_\bullet \arrow[d, "\phi"] \\
S^1_\bullet \vee S^1_\bullet \arrow[r, "\phi"'] & S^1_\bullet .
\end{tikzcd}$$

Recall that ϕ is the fold map, and so this diagram commutes since it does not matter what order the circles get folded together. Since this diagram of simplicial objects commutes, then the first diagram commutes. This demonstrates that the product map for $\text{THH}(A)$ is associative in the category of spectra.

In order to show that the coproduct map on $\text{THH}(A)$ is coassociative, we need a simplicial triple model of the circle. Angeltveit and Rognes call this $tS^1_\bullet$, which can be drawn as:



Angeltveit and Rognes show that $\text{tTHH}(A) := |A \otimes tS^1_\bullet|$ is weakly equivalent to $\text{THH}(A)$ within the proof of Theorem 3.9 in [AR05].

Let us now check if the coproduct $\psi\colon \text{THH}(A) \to \text{THH}(A) \wedge_A \text{THH}(A)$ is coassociative by checking that the following diagram commutes:

$$\begin{tikzcd}
tS^1_\bullet \arrow[r, "\psi_1"] \arrow[d, "\psi_2"'] & S^1_\bullet \vee dS^1_\bullet \arrow[d, "\mathrm{id}\vee\psi"] \\
dS^1_\bullet \vee S^1_\bullet \arrow[r, "\psi\vee\mathrm{id}"'] & S^1_\bullet \vee S^1_\bullet \vee S^1_\bullet
\end{tikzcd}$$

Here ψ_1 identifies v_0 and v_1 and ψ_2 identifies v_1 and v_2. This diagram commutes because either way we get the same wedge of three circles.

Since tTHH(A) and dTHH(A) are only weakly equivalent to THH(A), we need to be in the stable homotopy category for ψ to be coassociative.

The following are the remaining diagrams which need to commute, or to commute up to homotopy, in order to show that THH(A) is an A-Hopf algebra:

$$\begin{tikzcd}
dS^1_\bullet \vee dS^1_\bullet \arrow[r, "\phi'"] \arrow[d, "\psi\vee\psi"'] & dS^1_\bullet \arrow[r, "\psi"] & S^1_\bullet \vee S^1_\bullet \\
S^1_\bullet \vee S^1_\bullet \vee S^1_\bullet \vee S^1_\bullet \arrow[rr, "\mathrm{id}\vee\tau\vee\mathrm{id}"'] & & S^1_\bullet \vee S^1_\bullet \vee S^1_\bullet \vee S^1_\bullet \arrow[u, "\phi\vee\phi"']
\end{tikzcd}$$

$$\begin{tikzcd}
S^1_\bullet \vee S^1_\bullet \arrow[rr, "\phi"] \arrow[dr, "\varepsilon\vee\varepsilon"'] & & S^1_\bullet \arrow[dl, "\varepsilon"] \\
& e \vee e \cong e &
\end{tikzcd}$$

$$\begin{tikzcd}
& e \vee e \cong e \arrow[dl, "\eta\vee\eta"'] \arrow[dr, "\eta'"] & \\
S^1_\bullet \vee S^1_\bullet & & dS^1_\bullet \arrow[ll, "\psi"']
\end{tikzcd}$$

$$\begin{tikzcd}
e \arrow[dd, "\mathrm{id}"'] \arrow[dr, "\eta"] & \\
& S^1_\bullet \arrow[dl, "\varepsilon"] \\
e &
\end{tikzcd}$$

$$\begin{tikzcd}
tS^1_\bullet \arrow[r, "\psi_1"] \arrow[d, "\psi_2"'] \arrow[dr, "\varepsilon"] & S^1_\bullet \vee dS^1_\bullet \arrow[dr, "\mathrm{id}\vee\chi"] & \\
dS^1_\bullet \vee S^1_\bullet \arrow[dr, "\chi\vee\mathrm{id}"'] & e \arrow[dr, "\eta"] & S^1_\bullet \vee dS^1_\bullet \arrow[d, "\phi\circ(\mathrm{id}\vee\pi)"] \\
& dS^1_\bullet \vee S^1_\bullet \arrow[r, "\phi\circ(\pi\vee\mathrm{id})"'] & S^1_\bullet
\end{tikzcd}$$

where ϕ' is the simplicial fold map for $dS^1_\bullet$, τ swaps the two copies of $S^1_\bullet$, η' includes the point into the first point of $dS^1_\bullet$, and $\pi\colon dS^1_\bullet \to S^1_\bullet$ is the simplicial collapse map defined above. In fact, all of the above diagrams commute except the last diagram, which commutes up to homotopy. Angeltveit and Rognes discuss why the last diagram commutes up to homotopy in their proof of Theorem 3.9 in [AR05]. Thus, for A commutative ring spectrum, $\mathrm{THH}(A)$ is an A-Hopf algebra in the stable homotopy category.

Since Angeltveit and Rognes do this whole argument with simplicial circles then they are able to extend these structure maps to the Bökstedt spectral sequence in [AR05].

Theorem 4.0.4 ([AR05, Proposition 4.2])**.** *Let A be a commutative ring spectrum, and let p be prime. The Bökstedt spectral sequence $E^r_{*,*}(A)$ is a spectral sequence of commutative $H_*(A;\mathbb{F}_p)$-algebras.*

Further, they show that under some flatness conditions this spectral sequence has a coalgebraic structure as well.

Theorem 4.0.5 ([AR05, Theorem 4.5])**.** *Let A be a commutative ring spectrum, and let p be prime. If each term $E^r_{*,*}(A)$ for $r \geq 2$ is flat over $H_*(A;\mathbb{F}_p)$, then there is a coproduct*

$$\psi\colon E^r_{*,*}(A) \to E^r_{*,*}(A) \otimes_{H_*(A;\mathbb{F}_p)} E^r_{*,*}(A)$$

and $E^r_{*,*}(A)$ is a spectral sequence of $H_*(A;\mathbb{F}_p)$-Hopf algebras.

Indeed, Angeltveit and Rognes continue to do many computations of THH by leveraging this algebraic structure on the Bökstedt spectral sequence [AR05].

CHAPTER 5

ALGEBRAIC STRUCTURE ON TWISTED THH

In this section, we will explore the algebraic structure of C_p-twisted THH, p prime. Let U be a complete S^1-universe and let $\widetilde{U} := i^*_{C_p} U$. Let R be a commutative ring C_p-spectrum indexed over $\widetilde{U}$, let $\widetilde{R} := \mathcal{I}_{\widetilde{U}}^{\mathbb{R}^\infty} R$, let A be a commutative ring spectrum, and let the chosen generator of the group C_n be $\gamma := e^{2\pi i/n}$ unless otherwise specified. Recall that, by convention, our spectra are orthogonal spectra.

Recall the following simplicial model of S^1 from [Lod86], where $C_{n+1} = \{1, \gamma, \ldots, \gamma^n\}$ indicates the number of elements on each level,

$$
\begin{tikzcd}
\vdots \\
C_3 \arrow[d, shift right=8, "d_0"'] \arrow[d, "d_1"] \arrow[d, shift left=8, "d_2"] \\
C_2 \arrow[u, shift right=4, "s_0"] \arrow[u, shift left=4, "s_1"'] \arrow[d, shift right=4, "d_0"'] \arrow[d, shift left=4, "d_1"] \\
e \arrow[u, "s_0"]
\end{tikzcd}
$$

Let us call this model $S^1_\bullet$. The face and degeneracy maps are as follows:

$$
d_i(\gamma^j) = \begin{cases} \gamma^j & j \le i < n \\ \gamma^{j-1} & j > i \end{cases}
$$

$$
d_n(\gamma^j) = \begin{cases} \gamma^j & j < n \\ 1 & j = n \end{cases}
$$

$$
s_i(\gamma^j) = \begin{cases} \gamma^j & j \le i \\ \gamma^{j+1} & j > i. \end{cases}
$$

There are also maps $t\colon C_{n+1} \to C_{n+1}$ such that $t(\gamma^j)$ is γ^{j+1} for $j < n$ and 1 for $j = n$. It is notable that $d_n = d_0 \circ t\colon C_{n+1} \to C_{n+1}$ for all n.

Angeltveit and Rognes' classical argument, recalled in Chapter 4, requires additional models of the circle. For our equivariant proof, we will also need additional models and will construct these using the simplicial edgewise subdivision functor defined by Bökstedt, Hsiang, and Madsen in [BHM93]. The simplicial r-fold edgewise subdivision functor, $sd_r(-)$, is defined so that for a simplicial object $X_\bullet$,

$$sd_r(X_\bullet)_n = X_{(n+1)r-1}$$

with face and degeneracy maps $\bar{d}_i$ and $\bar{s}_i$ defined by

$$\bar{d}_i = d_i \circ d_{i+n+1} \circ \ldots \circ d_{i+(r-1)(n+1)}$$
$$\bar{s}_i = s_{i+(r-1)(n+2)} \circ \ldots \circ s_{i+(n+2)} \circ s_i$$

for d_i and s_i the face and degeneracy maps of the simplicial object $X_\bullet$.

Remark 5.0.1. Recall the simplicial relation that $d_i \circ d_j = d_{j-1} \circ d_i$ if $i < j$ and that in $S^1_\bullet$ $d_n = d_0 \circ t\colon C_{n+1} \to C_{n+1}$. It is also true that in $sd_r(S^1_\bullet)$, $\bar{d}_n = \bar{d}_0 \circ t$. To see this, consider that $\bar{d}_0 = d_0 \circ d_{n+1} \circ \ldots \circ d_{(r-1)n+r-1}$ and using the simplicial relation mentioned above, we can move d_0 to the front and get that $\bar{d}_0 = d_n \circ d_{2n+1} \circ \ldots \circ d_{(r-1)n+r-2} \circ d_0$. Consider $\bar{d}_0 \circ t = d_n \circ d_{2n+1} \circ \ldots \circ d_{(r-1)n+r-2} \circ d_0 \circ t = d_n \circ d_{2n+1} \circ \ldots \circ d_{(r-1)n+r-2} \circ d_{rn+r-1}$ which is $\bar{d}_n$.

Let us start by understanding the 2-fold edgewise simplicial subdivision of the circle, $sd_2(S^1_\bullet)$.

Example 5.0.2. Let us refer to $sd_2(S^1_\bullet)$ as $2S^1_\bullet$. By definition,

$$sd_2(S^1_\bullet)_n = S^1_{2n+1},$$
$$\bar{d}_i = d_i \circ d_{i+n+1}, \text{ and}$$
$$\bar{s}_i = s_{i+n+2} \circ s_i.$$

Therefore $2S^1_\bullet$ is

$$\begin{array}{c} \vdots \\ C_6 \\ \bar{d}_0 \downarrow \quad \bar{s}_0 \uparrow \quad \bar{d}_1 \downarrow \quad \bar{s}_1 \uparrow \quad \bar{d}_2 \downarrow \\ C_4 \\ \bar{d}_0 \downarrow \quad \bar{s}_0 \uparrow \quad \bar{d}_1 \downarrow \\ C_2 \end{array}$$

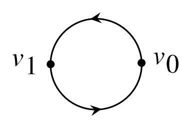
One can see that the only nondegenerate elements are $1, \gamma \in C_2$ and $\gamma, \gamma^3 \in C_4$, where the boundary of the 1-cell γ is defined by

$$\bar{d}_0(\gamma) = 1 \text{ and } \bar{d}_1(\gamma) = \gamma,$$

and the boundary of the 1-cell γ^3 is defined by

$$\bar{d}_0(\gamma^3) = \gamma \text{ and } \bar{d}_1(\gamma^3) = 1.$$

Therefore $2S^1_\bullet$ looks like:



where the C_2-action on $2S^1_\bullet$ is induced from applying the functor $sd_2(-)$ to $S^1_\bullet$. This action sends γ^i to $\gamma^{i+\frac{n}{2}}$, where $\gamma^n = 1$ in C_n. Therefore the C_2-action on $2S^1_\bullet$ is counter clockwise rotation by $180°$.

Remark 5.0.3. If we consider $2S^1_\bullet$ non-equivariantly, it is not the same as $dS^1_\bullet$ as defined in [AR05] and discussed earlier in Chapter 4. Non-equivariantly, $2S^1_\bullet$ is equivalent to $d'S^1_\bullet$ as defined in [AR05, Remark 3.6].

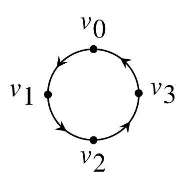
Example 5.0.4. We can similarly build $4S^1_\bullet := sd_4(S^1_\bullet)$. Note that this can be constructed by considering $sd_4(S^1_\bullet)$ or $sd_2(sd_2(S^1_\bullet))$. By definition,

$$sd_4(S^1_\bullet)_n = S^1_{4n+3},$$
$$\bar{d}_i = d_i \circ d_{i+n+1} \circ d_{i+2n+2} \circ d_{i+3n+3}, \text{ and}$$
$$\bar{s}_i = s_{i+3n+6} \circ s_{i+2n+4} \circ s_{i+n+2} \circ s_i.$$

Therefore $sd_4(S^1_\bullet)$ is

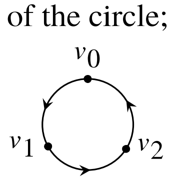
$$\begin{array}{c}
\vdots \\
C_{12} \\
\bar{d}_0 \downarrow \ \uparrow \bar{s}_0 \ \downarrow \bar{d}_1 \ \uparrow \bar{s}_1 \ \downarrow \bar{d}_2 \\
C_8 \\
\bar{d}_0 \downarrow \ \uparrow \bar{s}_0 \ \downarrow \bar{d}_1 \\
C_4
\end{array}$$

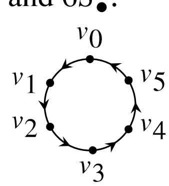
And $4S^1_\bullet$ looks like:



where the induced C_4-action is counter clockwise rotation by 90° and the induced C_2-action is counter clockwise rotation by 180°.

We can use this process to define $mS^1_\bullet$ for any positive integer m, which will have the C_m-action of counter clockwise rotation by $(\frac{360}{m})^\circ$. Notice that in order for $mS^1_\bullet$ to have a simplicial C_n-action of counter clockwise rotation by $(\frac{360}{n})^\circ$ then m must be a multiple of n. Consider two examples of C_3-equivariant simplicial models of the circle; $3S^1_\bullet$ and $6S^1_\bullet$:





where the induced C_3-action on both of these simplicial objects is counter clockwise rotation by $120°$.

In Section 3.4 we recalled the definition of twisted THH as defined by Angeltveit, Blumberg, Gerhardt, Hill, Lawson, and Mandell in [ABG+18]. We also discussed the different perspectives these authors gave us on twisted THH including, suppressing some change of universe notation, that for R a commutative ring C_n-spectrum $\mathrm{THH}_{C_n}(R) \simeq R \otimes_{C_n} S^1$. The following proposition demonstrates which simplicial model of the circle is suitable for this perspective.

Proposition 5.0.5. *Let R be a commutative ring C_p-spectrum indexed on the trivial universe $\mathbb{R}^\infty$, for p prime. Then $R \otimes_{C_p} pS^1_\bullet \cong B^{cy,C_p}_\bullet(R)$, the C_p-twisted cyclic bar construction.*

Proof. Let μ and η be the multiplication and unit maps of R respectively. To show that these simplicial objects are equivalent we will first show that every level is the same and then we will show that they have equivalent face and degeneracy maps.

The k-simplicies of $R \otimes_{C_p} pS^1_\bullet$ are defined by the following coequalizer diagram

$$R \otimes C_p \otimes pS^1_k \underset{\ell\otimes\mathrm{id}}{\overset{\mathrm{id}\otimes r}{\rightrightarrows}} R \otimes pS^1_k \longrightarrow R \otimes_{C_p} pS^1_k$$

where the map r is the C_p-action on pS^1_k and ℓ is the induced C_p-action on R.

Let $C_p = \langle\gamma\rangle$, and $pS^1_k = C_{pk+p} = \{1, x, \ldots, x^{pk+p-1}\}$. The induced C_p-action on the set of elements C_{pk+p} is defined by $\gamma x^i = x^j$ such that $j = i + k + 1(\mod pk + p)$. There is a C_{pk+p}-action on pS^1_k induced by $t\colon C_{pk+p} \to C_{pk+p}$ defined by $t(x^i) = x^j$ such that $j = i + 1(\mod pk + p)$.

As C_p-sets, $C_p \otimes pS^1_k = C_p \times C_{pk+p}$. Thus $R \otimes C_p \otimes pS^1_k$ can be written as $\bigwedge_{t=0}^{p-1}(\bigwedge_{s=0}^{pk+p-1} R_{x^s,\gamma^t})$. For ease of notation let us write $R_{s,t}$ instead of R_{x^s,γ^t}. Similarly, $R \otimes pS^1_k$ can be written as $\bigwedge_{s=0}^{pk+p-1} R_s$. With this notation $\mathrm{id} \otimes r\colon R_{s,t} \mapsto R_j$ such that $j = s + t(k+1)(\mod pk + p)$, and $\ell \otimes \mathrm{id}\colon R_{s,t} \mapsto {}_{\gamma^t}R_s$ where ${}_{\gamma^t}R$ indicates R which has been acted on by γ^t. By definition of the coequalizer, $R \otimes_{C_p} pS^1_k$ is the quotient space of $R \otimes pS^1_k$ where the quotient forces these two actions to agree. Recall that $\gamma^t x^s = x^j$ for $j = s + t(k+1)(\mod pk + p)$. Therefore $R \otimes_{C_p} pS^1_k = R^{\wedge k+1}$.

We will now show that the face and degeneracy maps from $R \otimes_{C_p} pS^1_k$ are equivalent to the face and degeneracy maps from $B^{cy,C_p}_k(R)$. We will start by considering the face and degeneracy maps of pS^1_k and induce the corresponding face and degeneracy maps of $R \otimes_{C_p} pS^1_k$.

Recall that $pS^1_k = sd_p(S^1_\bullet)_k$ with face and degeneracy maps $\bar{d}_i$ and $\bar{s}_i$ defined as follows:

$$\bar{d}_i = d_i \circ d_{i+k+1} \circ \ldots \circ d_{i+(p-1)(k+1)}$$
$$\bar{s}_i = s_{i+(p-1)(k+2)} \circ \ldots \circ s_{i+(k+2)} \circ s_i$$

where d_i and s_i are the face and degeneracy maps of $S^1_\bullet$, and $0 \leq i \leq k$.

Let us start by finding what the induced face maps are on $R \otimes_{C_p} pS^1_k$, say $\delta_i \colon R \otimes_{C_p} pS^1_k \to R \otimes_{C_p} pS^1_{k-1}$, for $0 \leq i \leq k$ and $k > 0$. The map $(\mathrm{id} \otimes \bar{d}_i) \colon R \otimes pS^1_k \to R \otimes pS^1_{k-1}$ applies the multiplication map to $R_i \wedge R_{i+1}$ as well as $R_{i+n(k+1)} \wedge R_{i+1+n(k+1)}$ for all $0 \leq n < p$. Therefore $\delta_i \colon R \otimes_{C_p} pS^1_k \to R \otimes_{C_p} pS^1_{k-1}$ is the map $\mathrm{id}^{\wedge i} \wedge \mu \wedge \mathrm{id}^{\wedge k-i-1}$ for $0 \leq i < k$.

Before figuring out what δ_k must be, recall that pS^1_k has a C_{pk+p}-action induced by the map t. Consider $(\mathrm{id} \otimes t) \colon R \otimes pS^1_k \to R \otimes pS^1_k$, this map rotates the last copy of R to the front. This map also rotates R_k into the position R_{k+1} was in, this is important as in the quotient $R \otimes_{C_p} pS^1_k$ we have that the following two are equivalent:

$$(\ell \otimes \mathrm{id})(R_{0,1}) = {}_\gamma R_0$$
$$(\mathrm{id} \otimes r)(R_{0,1}) = R_{k+1}$$

so the map that is induced on $R \otimes_{C_p} pS^1_k$ rotates the last copy of R to the front and acts on that copy of R by γ. Let us suggestively refer to this induced map as α_k.

Recall from Remark 5.0.1 that $\bar{d}_k = \bar{d}_0 \circ t$, so the last face map δ_k is induced from $\mathrm{id} \otimes \bar{d}_k = (\mathrm{id} \otimes \bar{d}_0) \circ (\mathrm{id} \otimes t) \colon R \otimes pS^1_k \to R \otimes pS^1_{k-1}$. The universal property of the coequalizer shows that the maps $(\mathrm{id} \otimes \bar{d}_0)$ and $(\mathrm{id} \otimes t)$ induce maps on $R \otimes_{C_p} pS^1_k$, namely δ_0 and α_k respectively. Further, by the uniqueness property, the map induced from their composition, $(\mathrm{id} \otimes \bar{d}_0) \circ (\mathrm{id} \otimes t)$ must be equivalent to the composition of the induced maps. Meaning, $\delta_k = \delta_0 \circ \alpha_k$.

We similarly induce the degeneracy maps of $R \otimes_{C_p} pS^1_k$, say $\sigma_i \colon R \otimes_{C_p} pS^1_k \to R \otimes_{C_p} pS^1_{k+1}$, for $0 \leq i \leq k$ and $k \geq 0$. By a similar argument as above we can show that these can be written as $\sigma_i = \mathrm{id}^{\wedge i+1} \wedge \eta \wedge \mathrm{id}^{\wedge k-i}$.

Recall that $B^{cy,C_p}_k(R) = R^{\wedge k+1}$ and the face and degeneracy maps from this level are the following:

$$d_i = \mathrm{id}^{\wedge i} \wedge \mu \wedge \mathrm{id}^{\wedge k-i-1}$$
$$d_k = d_0 \circ \alpha_k$$
$$s_j = \mathrm{id}^{\wedge j+1} \wedge \eta \wedge \mathrm{id}^{\wedge k-j}$$

for $0 \leq i < k$ and $0 \leq j \leq k$.

Therefore, $R \otimes_{C_p} pS^1_\bullet$ is isomorphic to $B^{cy,C_p}_\bullet(R)$. $\square$

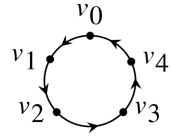
A result of this proposition is that for R a commutative ring C_p-spectrum indexed on C_p-universe $\widetilde{U}$, $|(\mathcal{I}^{\mathbb{R}^\infty}_{\widetilde{U}} R) \otimes_{C_p} pS^1_\bullet| \cong \mathrm{THH}_{C_p}(R)$. We can construct similar structures $(\mathcal{I}^{\mathbb{R}^\infty}_{\widetilde{U}} R) \otimes_{C_p} mpS^1_\bullet$, and we will refer to $|(\mathcal{I}^{\mathbb{R}^\infty}_{\widetilde{U}} R) \otimes_{C_p} mpS^1_\bullet|$ as $m\,\mathrm{THH}_{C_p}(R)$.

An equivariant analogue of Angeltveit and Rognes' result [AR05, Lemma 3.8] shows the following result.

Proposition 5.0.6. *Let U be a complete S^1-universe, and let $\widetilde{U} := i^*_{C_p} U$. Let R be a commutative ring C_p-spectrum indexed on the C_p-universe $\widetilde{U}$, for p prime. Then there is a C_p-weak equivalence*

$$\pi_m \colon m\,\mathrm{THH}_{C_p}(R) \to \mathrm{THH}_{C_p}(R).$$

It will also be important to consider simplicial objects that look the same non-equivariantly to $pS^1_\bullet$, but have different C_p-actions. For example, one can consider what looks like $5S^1_\bullet$



but with the C_5-action of counter clockwise rotation by 144°, 216°, or 288° instead of the usual counter clockwise rotation by 72°. Let us denote the C_p-simplicial space that resembles $pS^1_\bullet$ but has the C_p-action of counter clockwise rotation by $(\frac{n}{p}360)^\circ$ for $1 < n < p$ as $p_n S^1_\bullet$.

Proposition 5.0.7. *Let R be a commutative ring C_p-spectrum indexed over the trivial universe $\mathbb{R}^\infty$, p prime. For $1 < n < p$, the simplicial object $R \otimes_{C_p} p_n S^1_\bullet$ is isomorphic to the C_p-twisted cyclic bar construction $B^{cy,C_p}_\bullet(R)$.*

Proof. The proof is fairly similar to that of Proposition 5.0.5. Let μ and η be the multiplication and unit maps of R respectively. To show that these simplicial objects are equivalent we will first show that every level is the same and then we will show that they have equivalent face and degeneracy maps.

As $p_n S^1_\bullet$ resembles $pS^1_\bullet$, they have the same simplicial construction, but with different C_p-actions. Therefore $R \otimes_{C_p} p_n S^1_\bullet$ has the same number of copies of R as $B^{cy,C_p}_\bullet(R)$ on every level.

Now let us move on to show that the face and degeneracy maps are equivalent. Again, this argument will be similar to Proposition 5.0.5 and so we will start by discussing the C_{pk+p}-action on $R \otimes_{C_p} p_n S^1_\bullet$. Consider the coequalizer diagram

$$R \otimes C_p \otimes p_n S^1_k \underset{\ell \otimes \mathrm{id}}{\overset{\mathrm{id}\otimes r}{\rightrightarrows}} R \otimes p_n S^1_k \longrightarrow R \otimes_{C_p} p_n S^1_k$$

the C_p-action we consider here for ℓ is the same as it would be for the coequalizer diagram for $R \otimes_{C_p} pS^1_k$, while the C_p-action for r is different.

Let $C_p = \langle \gamma \rangle$, and $p_n S^1_k = C_{pk+p} = \{1, x, \dots, x^{pk+p-1}\}$. The induced C_p-action on the set of elements C_{pk+p} is defined by $\gamma x^i = x^j$ such that $j = i + n(k+1)(\mod pk+p)$. There is a C_{pk+p}-action on $p_n S^1_\bullet$ generated by $t\colon C_{pk+p} \to C_{pk+p}$ defined by $t(x^i) = x^j$ such that $j = i + 1($ mod $pk + p)$. This action is induced from $S^1_\bullet$. Note that unlike in Proposition 5.0.5, t^{k+1} does not generate the C_p-action on $p_n S^1_k$. Instead, $t^{n(k+1)}$ generates the C_p-action.

Let us use the notation ${}_{\gamma^t}\widetilde{R}$ to mean that $\widetilde{R}$ has been acted on by γ^t. The C_{pk+p}-action on $R \otimes_{C_p} p_n S^1_\bullet$ is the same as it was in the proof of Proposition 5.0.5, namely α_k:

$$(\alpha_k)(R_0 \wedge R_1 \wedge R_2 \wedge \dots \wedge R_k) = {}_\gamma R_k \wedge R_0 \wedge R_1 \wedge \dots \wedge R_{k-1}.$$

To show that the face and degeneracy maps of $R \otimes_{C_p} p_n S^1_\bullet$ are equivalent to those in $B^{cy,C_p}_\bullet(R)$ is the same as in Proposition 5.0.5. $\square$

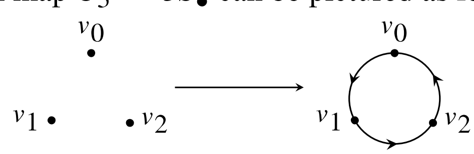
We are now ready to discuss the structure of twisted THH.

5.1 Algebraic structure

In this section, we will show that for R a commutative ring C_p-spectrum, $\mathrm{THH}_{C_p}(R)$ is a commutative R-algebra in the category of C_p-spectra for any prime p. The process for proving this is similar to the process that Angeltveit and Rognes use in [AR05] that we recalled in Chapter 4.

Recall that in [AR05] the simplicial map $\eta\colon e \to S^1_\bullet$ is the inclusion of the point, which induces the unit map $\eta\colon A \to \mathrm{THH}(A)$ by applying the functor $A \otimes (-)$. The equivariant analogue to this simplicial map is $\eta\colon C_p \to pS^1_\bullet$ which includes the p points into $pS^1_\bullet$. This induces the unit map $\eta\colon R \to \mathrm{THH}_{C_p}(R)$ by applying the functor $R \otimes_{C_p} (-)$. The intuition here is that we need C_p-equivariant analogues to the classical spaces used, so instead of a point we require the C_p-orbit of a point and a C_p-equivariant model of the circle that after applying the functor $R \otimes_{C_p} (-)$ gives us $\mathrm{THH}_{C_p}(R)$.

Example 5.1.1. The inclusion map $C_3 \to 3S^1_\bullet$ can be pictured as follows:

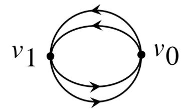


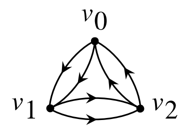
Let us consider the following pushout defined by the span of two copies of the simplicial map $\eta\colon C_p \to pS^1_\bullet$, for p prime:

$$\begin{tikzcd} C_p \arrow[r] \arrow[d] & pS^1_\bullet \arrow[d] \\ pS^1_\bullet \arrow[r] & X. \end{tikzcd}$$

Let us call this pushout $pS^1_\bullet \vee_{C_p} pS^1_\bullet := X$ and define the fold map $\phi\colon pS^1_\bullet \vee_{C_p} pS^1_\bullet \to pS^1_\bullet$ as folding the 1-cells together that share the same boundary. This chosen notation is meant to evoke that this is an equivariant analogue of the classical wedge.

Example 5.1.2. The spaces $2S^1_\bullet \vee_{C_2} 2S^1_\bullet$ and $3S^1_\bullet \vee_{C_3} 3S^1_\bullet$ can be depicted with the following diagrams, respectively:





where the C_2-action on the first diagram is counter clockwise rotation by 180°, and the C_3-action on the second diagram is counter clockwise rotation by 120°.

In order to show that this fold map induces the product map $\phi\colon \mathrm{THH}_{C_p}(R) \wedge_R \mathrm{THH}_{C_p}(R) \to \mathrm{THH}_{C_p}(R)$, we need to show $R \otimes_{C_p} (pS^1_\bullet \vee_{C_p} pS^1_\bullet)$ is congruent to $(R \otimes_{C_p} pS^1_\bullet) \wedge_R (R \otimes_{C_p} pS^1_\bullet)$ as simplicial C_p-spectra. This question reduces to if the functor $R \otimes_{C_p} (-)$ from C_p-spaces to C_p-spectra preserves pushouts.

Proposition 5.1.3. *Let R be a commutative ring C_p-spectrum and consider C_p as an unbased C_p-space, p prime. The functor $R \otimes_{C_p} (-)$ from the category of C_p-spaces to the category of commutative ring C_p-spectra preserves pushouts.*

Proof. Let $Z \leftarrow X \rightarrow Y$ define a pushout of C_p-spaces, say P. Since the pushout of the span $Z \leftarrow X \rightarrow Y$ is C_p-equivariant, and the functors $R \otimes C_p \otimes (-)$ and $R \otimes (-)$ preserve pushouts, then by definition of the left and right action maps which define the coequalizer $R \otimes_{C_p} (-)$ the top three-dimensional box of the following diagram commutes when considering only the left action maps, as well as when only considering the right action maps. Then by properties of coequalizers the arrows on the bottom two-dimensional square are induced in the following diagram:

$$
\begin{tikzcd}
R \otimes C_p \otimes X \arrow[rr] \arrow[dd, shift left] \arrow[dd, shift right] \arrow[dr] & & R \otimes C_p \otimes Y \arrow[dd, shift left] \arrow[dd, shift right] \arrow[dr] & \\
& R \otimes C_p \otimes Z \arrow[rr] \arrow[dd, shift left] \arrow[dd, shift right] & & R \otimes C_p \otimes P \arrow[dd, shift left] \arrow[dd, shift right] \\
R \otimes X \arrow[rr] \arrow[dr] \arrow[dd] & & R \otimes Y \arrow[dr] \arrow[dd] & \\
& R \otimes Z \arrow[rr] \arrow[dd] & & R \otimes P \arrow[dd] \\
R \otimes_{C_p} X \arrow[rr] \arrow[dr] & & R \otimes_{C_p} Y \arrow[dr] & \\
& R \otimes_{C_p} Z \arrow[rr] & & R \otimes_{C_p} P
\end{tikzcd}
$$

In order to show that the bottom square is a pushout diagram consider a commutative ring C_p-spectrum Q and two maps of ring C_p-spectra $q_1 \colon R \otimes_{C_p} Y \to Q$ and $q_2 \colon R \otimes_{C_p} Z \to Q$ such that the diagram commutes. We want to show that there is a unique map $h \colon R \otimes_{C_p} P \to Q$ that makes the diagram commute.

We can use q_1, q_2, and the universal property of pushouts to induce maps of ring C_p-spectra from both $R \otimes C_p \otimes P$ and $R \otimes P$ to Q which each respect both the left and right action maps. By the universal property of coequalizers this induces a unique map of ring C_p-spectra $h \colon R \otimes_{C_p} P \to Q$, and one can check that the diagram commutes. Note that in the diagram below, the squiggly lines are induced from the universal property of pushouts and the dashed line is induced by the universal property of coequalizers. Therefore $R \otimes_{C_p} (-)$ preserves pushouts.

$$
\begin{tikzcd}
R \otimes C_p \otimes X \arrow[rr] \arrow[dr] \arrow[dd, shift left] \arrow[dd, shift right] & & R \otimes C_p \otimes Y \arrow[dr] \arrow[dd, shift left] \arrow[dd, shift right] & & \\
& R \otimes C_p \otimes Z \arrow[rr] \arrow[dd, shift left] \arrow[dd, shift right] & & R \otimes C_p \otimes P \arrow[dd, shift left] \arrow[dd, shift right] \arrow[dddddr, rightsquigarrow] & \\
R \otimes X \arrow[rr] \arrow[dr] \arrow[dd] & & R \otimes Y \arrow[dr] \arrow[dd] & & \\
& R \otimes Z \arrow[rr] \arrow[dd] & & R \otimes P \arrow[dd] \arrow[dddr, rightsquigarrow] & \\
R \otimes_{C_p} X \arrow[rr] \arrow[dr] & & R \otimes_{C_p} Y \arrow[dr] \arrow[ddrr, bend left, "q_1"] & & \\
& R \otimes_{C_p} Z \arrow[rr] \arrow[drrr, bend right, "q_2"] & & R \otimes_{C_p} P \arrow[dr, dashed, "h"] & \\
& & & & Q
\end{tikzcd}
$$

$\square$

Now we can say that the simplicial maps:

$$
\begin{aligned}
\eta &\colon C_p \to pS^1_\bullet \\
\phi &\colon pS^1_\bullet \vee_{C_p} pS^1_\bullet \to pS^1_\bullet
\end{aligned}
$$

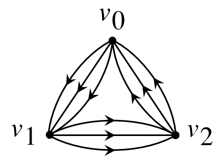
induce the following maps of commutative ring C_p-spectra:

$$
\begin{aligned}
\eta &\colon R \to \mathrm{THH}_{C_p}(R) \\
\phi &\colon \mathrm{THH}_{C_p}(R) \wedge_R \mathrm{THH}_{C_p}(R) \to \mathrm{THH}_{C_p}(R)
\end{aligned}
\tag{5.1.1}
$$

which are the unit and product maps respectively. We will use these maps to show that $\mathrm{THH}_{C_p}(R)$ is a commutative R-algebra.

In order to check associativity of this product map we will need to understand the pushout of the span $pS^1_\bullet \vee_{C_p} pS^1_\bullet \leftarrow C_p \to pS^1_\bullet$, which is $pS^1_\bullet \vee_{C_p} pS^1_\bullet \vee_{C_p} pS^1_\bullet$.

Example 5.1.4. Consider $3S^1_\bullet \vee_{C_3} 3S^1_\bullet \vee_{C_3} 3S^1_\bullet$ which can be pictured as:



Here $\phi \vee \mathrm{id}\colon 3S^1_\bullet \vee_{C_3} 3S^1_\bullet \vee_{C_3} 3S^1_\bullet \to 3S^1_\bullet \vee_{C_3} 3S^1_\bullet$ folds the outer copy of $3S^1_\bullet$ with the middle copy of $3S^1_\bullet$ and leaves the inner copy of $3S^1_\bullet$ alone. Similarly, $\mathrm{id} \vee \phi\colon 3S^1_\bullet \vee_{C_3} 3S^1_\bullet \vee_{C_3} 3S^1_\bullet \to 3S^1_\bullet \vee_{C_3} 3S^1_\bullet$ folds the inner copy of $3S^1_\bullet$ with the middle copy of $3S^1_\bullet$ and leaves the outer copy of $3S^1_\bullet$ alone.

Proposition 5.1.5. *Let p be prime. For a commutative ring C_p-spectrum R, $\mathrm{THH}_{C_p}(R)$ is a commutative R-algebra in the category of commutative ring C_p-spectra.*

Proof. We begin by checking associativity of the product map $\phi\colon \mathrm{THH}_{C_p}(R) \wedge_R \mathrm{THH}_{C_p}(R) \to \mathrm{THH}_{C_p}(R)$. For ease of notation, let $T := \mathrm{THH}_{C_p}(R)$. We need to verify that the following diagram commutes:

$$\begin{tikzcd}
T \wedge_R T \wedge_R T \arrow[r, "\mathrm{id}\wedge\phi"] \arrow[d, "\phi\wedge\mathrm{id}"'] & T \wedge_R T \arrow[d, "\phi"] \\
T \wedge_R T \arrow[r, "\phi"'] & T.
\end{tikzcd}$$

It is sufficient to show that the following diagram of C_p-simplicial spaces commutes

$$\begin{tikzcd}
pS^1_\bullet \vee_{C_p} pS^1_\bullet \vee_{C_p} pS^1_\bullet \arrow[r, "\mathrm{id}\vee\phi"] \arrow[d, "\phi\vee\mathrm{id}"'] & pS^1_\bullet \vee_{C_p} pS^1_\bullet \arrow[d, "\phi"] \\
pS^1_\bullet \vee_{C_p} pS^1_\bullet \arrow[r, "\phi"'] & pS^1_\bullet
\end{tikzcd}$$

where $pS^1_\bullet \vee_{C_p} pS^1_\bullet \vee_{C_p} pS^1_\bullet$ is the pushout of the span $pS^1_\bullet \vee_{C_p} pS^1_\bullet \leftarrow C_p \to pS^1_\bullet$. This is thought of as some equivariant analogue to the wedge of three circles, Example 5.1.4 shows this for $p = 3$. Note that $\mathrm{id} \vee \phi$ and $\phi \vee \mathrm{id}$ fold the inner two copies of $pS^1_\bullet$ together and the outer two copies of $pS^1_\bullet$ together respectively where ϕ folds the two copies of $pS^1_\bullet$ together. Therefore this diagram commutes.

To check unitality and commutativity of the product map, we need to show that the following diagrams commute:

$$\begin{tikzcd}
pS^1_\bullet \vee_{C_p} C_p \arrow[r, "\mathrm{id}\vee\eta"] \arrow[dr, "\simeq"'] & pS^1_\bullet \vee_{C_p} pS^1_\bullet \arrow[d, "\phi"] & C_p \vee_{C_p} pS^1_\bullet \arrow[l, "\eta\vee\mathrm{id}"'] \arrow[dl, "\simeq"] \\
& pS^1_\bullet &
\end{tikzcd}$$

$$\begin{tikzcd}
pS^1_\bullet \vee_{C_p} pS^1_\bullet \arrow[rr, "\tau"] \arrow[dr, "\phi"'] & & pS^1_\bullet \vee_{C_p} pS^1_\bullet \arrow[dl, "\phi"] \\
& pS^1_\bullet &
\end{tikzcd}$$

where $pS^1_\bullet \vee_{C_p} C_p \cong pS^1_\bullet$ is the pushout of the span $pS^1_\bullet \leftarrow C_p \rightarrow C_p$. The map $\mathrm{id} \vee \eta \colon pS^1_\bullet \vee_{C_p} C_p \rightarrow pS^1_\bullet \vee_{C_p} pS^1_\bullet$ is the identity on $pS^1_\bullet$ and includes C_p into the second copy of $pS^1_\bullet$, similarly $\eta \vee \mathrm{id} \colon C_p \vee_{C_p} pS^1_\bullet \rightarrow pS^1_\bullet \vee_{C_p} pS^1_\bullet$ is the identity on $pS^1_\bullet$ and includes C_p into the first copy of $pS^1_\bullet$. The map τ swaps the first and second copies of $pS^1_\bullet$.

For the unitality diagram, note that $\phi \colon pS^1_\bullet \vee_{C_p} pS^1_\bullet \rightarrow pS^1_\bullet$ is the fold map and the maps $\mathrm{id} \vee \eta$ and $\eta \vee \mathrm{id}$ both have an image of one copy of $pS^1_\bullet$, so this diagram commutes. For the commutativity diagram, the fold map has the same image no matter the position of the two copies of $pS^1_\bullet$.

Therefore, $\mathrm{THH}_{C_p}(R)$ is a commutative R-algebra. $\square$

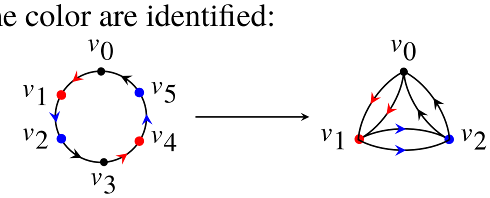
5.2 Coproduct Structure

In this section, we will show that for R a commutative ring C_p-spectrum, $\mathrm{THH}_{C_p}(R)$ is a non-counital R-coalgebra for $p \geq 5$ prime in the C_p-equivariant stable homotopy category. There will be remarks that explain why there is not the same coproduct map for $p = 2$ and why the coproduct for $p = 3$ is not coassociative.

If a counit map were to exist, it should be from $pS^1_\bullet$ to C_p, for p prime. There is no C_p-equivariant, simplicial way to map one connected component to C_p. Therefore, one cannot induce a counit map on $\mathrm{THH}_{C_p}(R)$ as we did for the unit map. Note that there could be a map $\mathrm{THH}_{C_p}(R) \to R$ that satisfies the properties of a counit, but for the purposes of this thesis, we would like to induce these maps from simplicial maps, so we will not explore these possible maps further in this paper.

Classically, the coproduct map on THH was induced from a pinch map on a double model of the circle. We will also use a pinch map to induce the coproduct structure on twisted THH. Consider, for $p \geq 3$, the pinch map on $2pS^1_\bullet$ which identifies opposite vertices. The case for $p = 2$ will be covered in Remark 5.2.2. The C_p-action on $2pS^1_\bullet$ sends v_0 to v_2, and the pinch map ψ' identifies v_i with v_j for $j = i + p(\mod 2p)$. The map ψ' has two copies of $p_2S^1_\bullet$ in its image. Recall that $p_nS^1_\bullet$ is the C_p-simplicial space that resembles $pS^1_\bullet$ but has the C_p-action of counter-clockwise rotation by $(\frac{n}{p}360)^\circ$ for $1 < n < p$.

Example 5.2.1. The following is a depiction of the pinch map for $p = 3$, $\psi' \colon 6S^1_\bullet \to 3_2S^1_\bullet \vee_{C_3} 3_2S^1_\bullet$ where the vertices of the same color are identified:



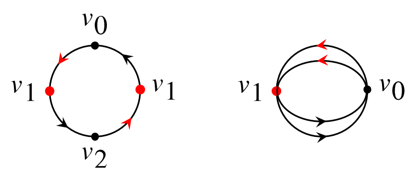
The colors of the arrows is to help keep track of where each 1-cell goes.

Now, let us consider the following C_p-equivariant, simplicial pinch map for p an odd prime:

$$\psi' \colon 2pS^1_\bullet \to p_2S^1_\bullet \vee_{C_p} p_2S^1_\bullet$$

which identifies 0-cells that have the same C_2-orbit, that is, 0-cells opposite each other. Note that the C_2-action is induced on $2pS^1_\bullet$ from the functor $sd_2(-)$ as $2pS^1_\bullet = sd_2(pS^1_\bullet)$.

Remark 5.2.2. This pinch map which identifies opposite points is what makes this process not work for $p = 2$. Note that since $(\frac{2}{2}360)^\circ$ is just 360°, then if there were such a map, it would be from $4S^1_\bullet$ to a space that looks like $2S^1_\bullet \vee_{C_2} 2S^1_\bullet$ but with a trivial C_2-action, pictured here:



Let us refer to this space with trivial C_2-action as $X_\bullet$. In order for this to induce a coproduct map onto $\mathrm{THH}_{C_2}(R)$ then $|\widetilde{R} \otimes_{C_2} X_\bullet|$ would need to be C_2-weakly equivalent to $\mathrm{THH}_{C_2}(R) \wedge_R \mathrm{THH}_{C_2}(R)$, which is not true in general.

This pinch map, along with the S^1-homeomorphism π_2^{-1} defined in Proposition 5.0.6 gives the following map:

$$\psi := \psi' \circ \pi_2^{-1} \colon \ \mathrm{THH}_{C_p}(R) \to \mathrm{THH}_{C_p}(R) \wedge_R \mathrm{THH}_{C_p}(R) \tag{5.2.2}$$

which is the coproduct map.

Since the coproduct is not in general cocommutative in the classical case, one would (correctly) assume that the equivariant case will not be cocommutative in general.

We still would like to check if this coproduct is coassociative. Similarly to the classical case we need a “triple model" of our circle. In [AR05] this was $tS^1_\bullet$, as recalled in Chapter 4. Here we will consider $sd_3(pS^1_\bullet) = 3pS^1_\bullet$. The coproduct is not in general coassociative for $p = 3$, as explained in Remark 5.2.8.

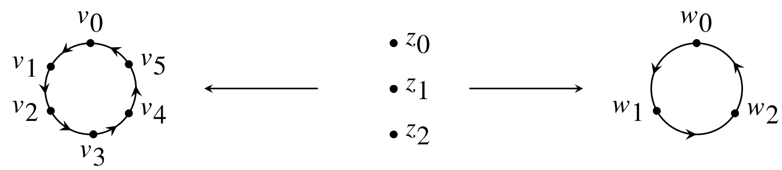
To check if ψ is coassociative, we will show later that it is sufficient to show that a diagram of the following form commutes

$$\begin{tikzcd}
3pS^1_\bullet \arrow[r, "\psi_1"] \arrow[d, "\psi_2"'] & X_\bullet \vee_{C_p} Y_\bullet \arrow[d, "\mathrm{id}\vee\psi'"] \\
Y_\bullet \vee_{C_p} X_\bullet \arrow[r, "\psi'\vee\mathrm{id}"'] & Y_\bullet \vee_{C_p} Y_\bullet \vee_{C_p} Y_\bullet.
\end{tikzcd}$$

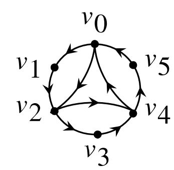
Where ψ_1 and ψ_2 are two different kinds of pinch maps and $X_\bullet$ is a simplicial space such that for R a commutative ring C_p-spectrum $|\widetilde{R} \otimes_{C_p} X_\bullet| \simeq 2\,\mathrm{THH}_{C_p}(R)$ and $Y_\bullet$ is such that $|\widetilde{R} \otimes_{C_p} Y_\bullet| \simeq \mathrm{THH}_{C_p}(R)$. These two pinch maps ψ_1 and ψ_2 on $3pS^1_\bullet$ for $p \geq 5$ are determined by identifying v_0 to v_p and v_{2p} respectively.

We can define $kpS^1_\bullet \vee_{C_p} mpS^1_\bullet$ as the pushout of the diagram $kpS^1_\bullet \leftarrow C_p \rightarrow mpS^1_\bullet$, where the left and right arrows both include C_p into a C_p-orbit of a 0-cell in the category of simplicial C_p-spaces. It does not matter which orbit of 0-cells. Consider the following example.

Example 5.2.3. Let us consider $6S^1_\bullet \vee_{C_3} 3S^1_\bullet$, this is the pushout of the following diagram:



where the left arrow is such that $z_0 \mapsto v_0$, $z_1 \mapsto v_2$, and $z_2 \mapsto v_4$ and the right arrow is such that $z_i \mapsto w_i$ for all i. The pushout of this diagram is pictured here:



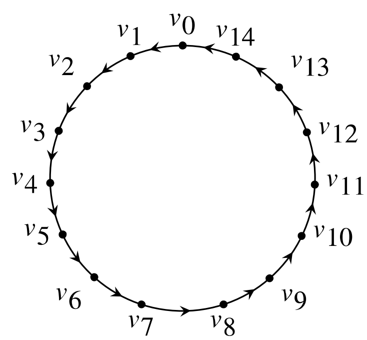
Note that the picture is the same even if z_0 maps to any v_i, as long as z_1 maps to v_j for $j = i + 2(\mod 6)$ and z_2 maps to v_k for $k = i + 4(\mod 6)$.

First, let us recall that $p_nS^1_\bullet$ is the C_p-simplicial space that resembles $pS^1_\bullet$ but has the C_p-action of counter clockwise rotation by $(\frac{n}{p}360)^\circ$ for $1 < n < p$. Recall the pinch map $\psi' \colon 2pS^1_\bullet \rightarrow pS^1_\bullet \vee_{C_p} pS^1_\bullet$ works by identifying opposite points. By abuse of notation let us denote the pinch map on $2p_nS^1_\bullet$ for $1 < n < p$ which identifies opposite points also as ψ'.

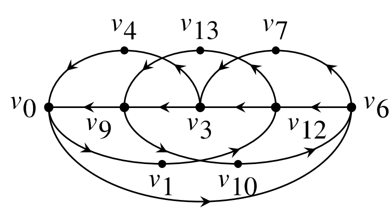
Proposition 5.2.4. *Let $p \geq 5$ be prime. Let ψ_1 and ψ_2 be the C_p-equivariant pinch maps on $3pS^1_\bullet$ such that ψ_1 is determined by identifying v_0 and v_p and ψ_2 is determined by identifying v_0 and v_{2p}. Then $\psi_1 \colon 3pS^1_\bullet \rightarrow p_3S^1_\bullet \vee_{C_p} 2p_{\frac{p+3}{2}}S^1_\bullet$, and $\psi_2 \colon 3pS^1_\bullet \rightarrow 2p_{\frac{p+3}{2}}S^1_\bullet \vee_{C_p} p_3S^1_\bullet$,* *such that* $(id \vee \psi') \circ \psi_1 \colon 3pS^1_\bullet \to p_3S^1_\bullet \vee_{C_p} p_3S^1_\bullet \vee_{C_p} p_3S^1_\bullet$ *and* $(\psi' \vee id) \circ \psi_2 \colon 3pS^1_\bullet \to p_3S^1_\bullet \vee_{C_p} p_3S^1_\bullet \vee_{C_p} p_3S^1_\bullet$.

The argument changes only slightly depending on if p is $1 \mod 3$ or $2 \mod 3$. In the hopes to build up some intuition, or to be used as an illustrative reference while reading the proof, we have included two examples. First, the easier of the two $p = 5$ is Example 5.2.5 and the example for $p = 7$ is very similar but may be helpful when reading the proof of the above proposition Example 5.2.6.

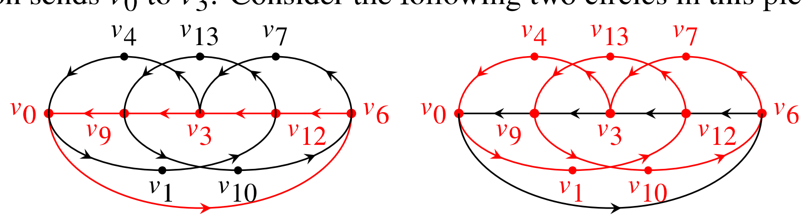
Example 5.2.5. For ψ_1 and ψ_2 as in Proposition 5.2.4, let us do a toy example for $p = 5$. First, let us picture $15S^1_\bullet$



Let us apply ψ_1 to $15S^1_\bullet$. By definition, ψ_1 identifies v_0 with v_5 and is C_5-equivariant, so there are many other vertices which will be identified, namely v_3 and v_8, v_6 and v_{11}, v_9 and v_{14}, v_{12} and v_2. This gives the following:

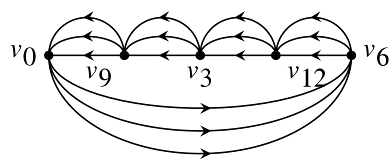


Here the C_5-action sends v_0 to v_3. Consider the following two circles in this picture:



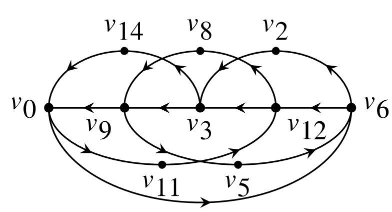
Recall that $p_nS^1_\bullet$ is the C_p-simplicial space that resembles $pS^1_\bullet$ but has the C_p-action of counter clockwise rotation by $(\frac{n}{p}360)^\circ$ for $1 < n < p$. The circle in red on the left is $5_m S^1_\bullet$ for some m and the circle in red on the right is $10_\ell S^1_\bullet$ for some ℓ. To figure out the C_5-actions on either circle we will consider how many rotations by $(\frac{1}{5}360)^\circ$ get us from v_0 to v_3. The circle colored in red on the left has the C_5-action of rotation by $(\frac{3}{5}360)^\circ$, meaning that one can trace three 1-cells in the positive orientation starting from v_0 to v_3, connecting through v_6, and v_{12}. The circle colored in red on the right has the C_5-action of rotation by $(\frac{4}{5}360)^\circ$, meaning that one can trace four pairs of 1-cells in the positive orientation starting from v_0 to v_3, connecting through v_{12}, v_9, and v_6. Therefore this whole space is $5_3 S^1_\bullet \vee_{C_5} 10_4 S^1_\bullet$.

The map $\mathrm{id} \vee \psi'$ applied to $5_3 S^1_\bullet \vee_{C_5} 10_4 S^1_\bullet$ identifies v_0 and v_{10}, v_9 and v_4, v_3 and v_{13}, v_{12} and v_7, and lastly v_6 and v_1. This gives the diagram:

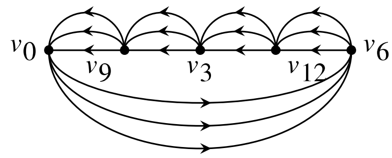


The C_5-action sends v_0 to v_3 so this is $5_3 S^1_\bullet \vee_{C_5} 5_3 S^1_\bullet \vee_{C_5} 5_3 S^1_\bullet$. Therefore for the case of $p = 5$, $(\mathrm{id} \vee \psi') \circ \psi_1$ is a map from $15 S^1_\bullet$ to $5_3 S^1_\bullet \vee_{C_5} 5_3 S^1_\bullet \vee_{C_5} 5_3 S^1_\bullet$.

Now let us apply ψ_2 to $15 S^1_\bullet$. By definition, ψ_2 identifies v_0 with v_{10} and is C_5-equivariant, so there are many other vertices which will be identified, namely v_3 and v_{13}, v_6 and v_1, v_9 and v_4, v_{12} and v_7. This gives the following:



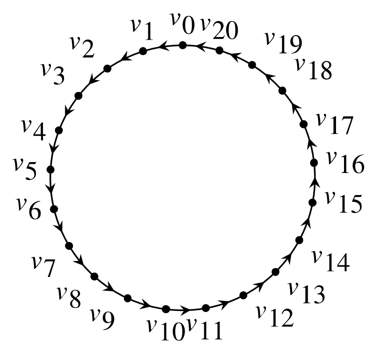
The C_5-action sends v_0 to v_3. Tracing through the two copies of the circle as we did above, one can see that this is $10_4 S^1_\bullet \vee_{C_5} 5_3 S^1_\bullet$. The map $\psi' \vee \mathrm{id}$ applied to $10_4 S^1_\bullet \vee_{C_5} 5_3 S^1_\bullet$ identifies v_0 and v_5, v_9 and v_{14}, v_3 and v_8, v_{12} and v_2, and lastly v_6 and v_{11}. This gives the diagram:



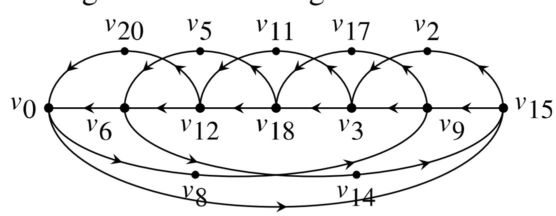
The C_5-action sends v_0 to v_3 so this is $5_3S^1_\bullet \vee_{C_5} 5_3S^1_\bullet \vee_{C_5} 5_3S^1_\bullet$. Therefore for the case of $p = 5$, $(\psi' \vee \mathrm{id}) \circ \psi_2$ is a map from $15S^1_\bullet$ to $5_3S^1_\bullet \vee_{C_5} 5_3S^1_\bullet \vee_{C_5} 5_3S^1_\bullet$.

We have now worked through the specific example of Proposition 5.2.4 for $p = 5$. For the case when p is $1 \mod 3$ the modular arithmetic is slightly different. For this reason, we have included the following example for $p = 7$. Note that these differences are minimal, but we wanted to include both illustrative examples for the reader.

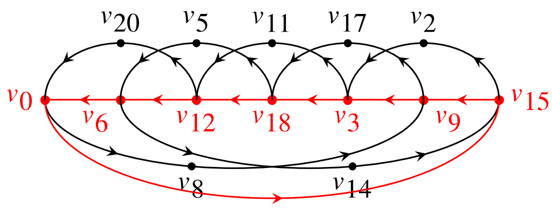
Example 5.2.6. For ψ_1 and ψ_2 as in Proposition 5.2.4, let us do a toy example for $p = 7$. First, let us picture $21S^1_\bullet$

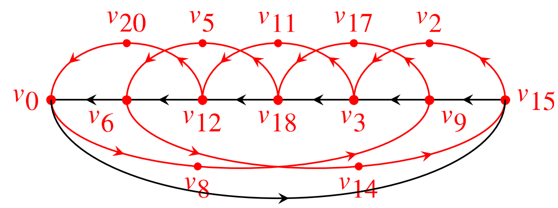


Let us apply ψ_1 to $21S^1_\bullet$. By definition, ψ_1 identifies v_0 with v_7 and is C_7-equivariant, so there are many other vertices which will be identified, namely v_3 and v_{10}, v_6 and v_{13}, v_9 and v_{16}, v_{12} and v_{19}, v_{15} and v_1, v_{18} and v_4. This gives the following:



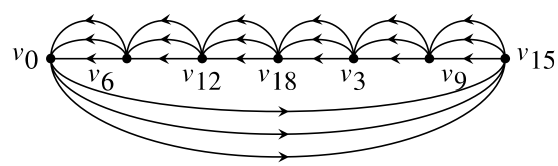
Here the C_7-action sends v_0 to v_3. Consider the following two circles in this picture:





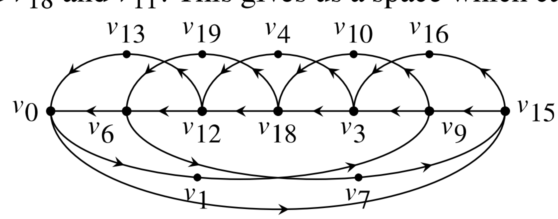
The circle in red on the left is $7_m S^1_\bullet$ for some m and the circle in red on the right is $14_\ell S^1_\bullet$ for some ℓ. To figure out the C_7-actions on either circle we will consider how many rotations by $(\frac{1}{7}360)^\circ$ get us from v_0 to v_3. The circle colored in red on the left has the C_7-action of rotation by $(\frac{3}{7}360)^\circ$, meaning that one can trace three 1-cells in the positive orientation starting from v_0 to v_3, connecting through v_{15}, and v_9. The circle colored in red on the right has the C_7-action of rotation by $(\frac{5}{7}360)^\circ$, meaning that one can trace five pairs of 1-cells in the positive orientation starting from v_0 to v_3, connecting through v_9, v_{18}, v_6 and v_{15}. Therefore this whole space is $7_3S^1_\bullet \vee_{C_7} 14_5S^1_\bullet$.

The map $\mathrm{id} \vee \psi'$ applied to $7_3S^1_\bullet \vee_{C_7} 14_5S^1_\bullet$ identifies v_0 and v_{14}, v_6 and v_{20}, v_{12} and v_5, v_{18} and v_{11}, v_3 and v_{17}, v_9 and v_2, and lastly, v_{15} and v_8. This gives the diagram:

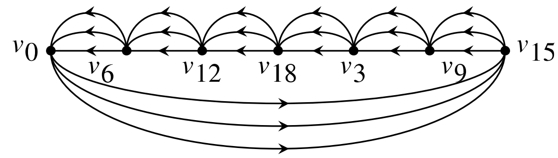


The C_7-action sends v_0 to v_3 so this is $7_3S^1_\bullet \vee_{C_7} 7_3S^1_\bullet \vee_{C_7} 7_3S^1_\bullet$. Therefore for the case of $p = 7$, $(\mathrm{id} \vee \psi') \circ \psi_1$ is a map from $21S^1_\bullet$ to $7_3S^1_\bullet \vee_{C_7} 7_3S^1_\bullet \vee_{C_7} 7_3S^1_\bullet$.

Now, let us apply ψ_2 to $21S^1_\bullet$. By definition, ψ_2 identifies v_0 and v_{14} and is C_7-equivariant, so there are many other vertices which will be identified, namely v_3 and v_{17}, v_6 and v_{20}, v_9 and v_2, v_{12} and v_5, v_{15} and v_8, and v_{18} and v_{11}. This gives us a space which can be considered as follows:



The C_7-action sends v_0 to v_3. Tracing through the two copies of the circle as we did above, one can find that this is $14_5S^1_\bullet \vee_{C_7} 7_3S^1_\bullet$. The map $\psi' \vee \mathrm{id}$ applied to $14_5S^1_\bullet \vee_{C_7} 7_3S^1_\bullet$ identifies v_0 and v_7, v_6 and v_{13}, v_{12} and v_{19}, v_{18} and v_4, v_3 and v_{10}, v_9 and v_{16}, and lastly, v_{15} and v_1. This gives the diagram:



The C_7-action sends v_0 to v_3 so this is $7_3S^1_\bullet \vee_{C_7} 7_3S^1_\bullet \vee_{C_7} 7_3S^1_\bullet$. Therefore for the case of $p = 7$, $(\psi' \vee \mathrm{id}) \circ \psi_2$ is a map from $21S^1_\bullet$ to $7_3S^1_\bullet \vee_{C_7} 7_3S^1_\bullet \vee_{C_7} 7_3S^1_\bullet$.

Now that we have given two examples, let us do the proof for a general $p \geq 5$.

Proof of Proposition 5.2.4. By our assumption, $p \geq 5$ is prime and the pinch maps ψ_1 and ψ_2 are determined by the fact that they are C_p-equivariant and identify v_0 to v_p and v_0 to v_{2p} respectively on $3pS^1_\bullet$. A big part of this proof is to show that ψ_1 sends $3pS^1_\bullet$ to $p_3S^1_\bullet \vee_{C_p} 2p_{\frac{p+3}{2}}S^1_\bullet$ and ψ_2 sends $3pS^1_\bullet$ to $2p_{\frac{p+3}{2}}S^1_\bullet \vee_{C_p} p_3S^1_\bullet$. In addition, we want to show that $(\mathrm{id} \vee \psi') \circ \psi_1$ and $(\psi' \vee \mathrm{id}) \circ \psi_2$ both send $3pS^1_\bullet$ to $p_3S^1_\bullet \vee_{C_p} p_3S^1_\bullet \vee_{C_p} p_3S^1_\bullet$. The proof strategy will be slightly different for ψ_1 and ψ_2, and whether $p \equiv 1 \pmod 3$ or $p \equiv 2 \pmod 3$, so there are four cases to consider. To prove what we want, we will take each of these cases through the following three steps.

Step 1: Show there is a copy of $p_3S^1_\bullet$ in the image of ψ_i.

Step 2: Show there is a copy of $2p_{\frac{p+3}{2}}S^1_\bullet$ in the image of ψ_i.

Step 3: Show that $(\mathrm{id} \vee \psi') \circ \psi_1$ or $(\psi' \vee \mathrm{id}) \circ \psi_2$ sends $3pS^1_\bullet$ to $p_3S^1_\bullet \vee_{C_p} p_3S^1_\bullet \vee_{C_p} p_3S^1_\bullet$.

In many of these arguments, we are counting 1-cells between vertices. Our convention will be that we are always "traveling" along 1-cells in the direction of their orientation. Note that this convention agrees with our definitions of $mpS^1_\bullet$ and $mp_nS^1_\bullet$. We will do one case, and the others are very similar with slightly different modular arithmetic.

Case 1) ψ_1 and $p \equiv 1 \pmod 3$.

Step 1: Since $p \equiv 1 \pmod 3$, then $p + 2 \equiv 0 \pmod 3$ and $2p + 1 \equiv 2 + 1 \equiv 0 \pmod 3$, so $p + 2 = 3k_1$ and $2p + 1 = 3k_2$ for some $k_1, k_2 \in \mathbb{Z}$. Since the C_p-action on $3pS^1_\bullet$ sends v_i to v_j for $j = i + 3 \pmod{3p}$ and ψ_1 is a C_p-equivariant map which identifies v_0 and v_p, then it must also identify v_{3k} and v_ℓ for $\ell \equiv 3k + p \pmod{3p}$. For example, ψ_1 identifies v_3 and v_{p+3},

v_{p+2} and v_{2p+2}, as well as v_{2p+1} and v_1 since we showed that $p+2$ and $2p+1$ can be written as multiples of 3. The vertex v_0 is connected by a 1-cell to v_1, which is identified to v_{2p+1}, which is connected by a 1-cell to v_{2p+2}, which is identified to v_{p+2}, which is connected by a 1-cell to v_{p+3}, which is identified to v_3. For the following picture, the dual labelling of the vertices indicates the identification of those vertices under ψ_1, also, we have organized the upper labels to be enumerated by numbers which are $0(\mod 3)$ and the bottom $1(\mod 3)$.

$$
\begin{array}{ccccccccc}
 & v_0 & & v_{2p+1} & & v_{p+2} & & v_3 & \\
\cdots & \bullet & \text{——} & \bullet & \text{——} & \bullet & \text{——} & \bullet & \cdots \\
 & v_p & & v_1 & & v_{2p+2} & & v_{p+3} &
\end{array}
$$

The following is an equation which tells us the enumeration of the top label of the vertex which is z 1-cells away from v_0: $z(1-p)(\mod 3p)$. This equation will help us to show that there is a copy of $p_3S^1_\bullet$ in the image of ψ_1.

Since $p \equiv 1(\mod 3)$ then there is a $k \in \mathbb{Z}$ such that $p = 3k+1$. There are two important values we need from the above equation, $z = p$ and $z = 3$: $p(1-p) = p-p^2 = p-p(3k+1) = -3pk \equiv 0(\mod 3p)$, and $3(1-p) = 3-3p \equiv 3(\mod 3p)$. The meaning behind these two computations are that there are p 1-cells on this simplicial space, and there are 3 1-cells between v_0 and v_3. Therefore $p_3S^1_\bullet$ is in the image of ψ_1.

Step 2: Since the map ψ_1 identifies v_{3k} and v_ℓ such that $\ell = 3k+p(\mod 3p)$ then ψ_1 identifies vertices whose enumeration is $0(\mod 3)$ to vertices whose enumeration is $1(\mod 3)$ since $3k+p \equiv 1(\mod 3)$. Since $p \equiv 1(\mod 3)$, then $p+1 \equiv 2(\mod 3)$. Therefore v_{p+1} is not identified to any other vertices under ψ_1, but v_0 is identified to v_p and v_{p+2} is identified to v_{2p+2}. This demonstrates that there is a copy of $2p_nS^1_\bullet$ in the image of ψ_1 as there are p copies of pairs of 1-cells like the pair of 1-cells between v_p and v_{p+2}. Now we need to show that $n = \frac{p+3}{2}$. The vertex v_0 is identified to v_p, which is connected by a pair of 1-cells to v_{p+2}, which is identified to v_{2p+2}, which is connected by a pair of 1-cells to v_{2p+4}. For the following picture, the dual labelling of the vertices indicates the identification of those vertices under ψ_1, also, we have organized the double labels so that the upper labels to be enumerated by numbers which are $0(\mod 3)$ and the bottom $1(\mod 3)$. The vertices which only have one label have an enumeration which is $2(\mod 3)$.

$$\begin{array}{ccccccccccc} & v_0 & & & & v_{p+2} & & & & v_{2p+4} & \\ \cdots & \bullet & \text{———} & \bullet & \text{———} & \bullet & \text{———} & \bullet & \text{———} & \bullet & \cdots \\ & v_p & & v_{p+1} & & v_{2p+2} & & v_{2p+3} & & v_4 & \end{array}$$

The following is an equation which tells us the enumeration of the top label of the vertex which is $2z$ 1-cells away from v_0: $z(p+2)(\mod 3p)$. This equation will help us to show that there is a copy of $2p_{\frac{p+3}{2}}S^1_\bullet$ in the image of ψ_1.

Since $p \equiv 1(\mod 3)$ then there is a $k \in \mathbb{Z}$ such that $p = 3k+1$. There are two important values we need from the above equation, $z = p$ and $z = \frac{p+3}{2}$, the first of these equations is: $p(p+2) = p^2 + 2p = p(3k+1) + 2p = 3pk + 3p \equiv 0(\mod 3p)$. The second of these equations is a bit more complicated. Since p is odd, and $p = 3k+1$ then k must be even, so let $k = 2\ell$ for some $\ell \in \mathbb{Z}$. We can now do the second equation:

$$\frac{p+3}{2}(p+2) = \frac{6\ell+4}{2}(p+2) = (3\ell+2)(p+2)$$
$$= 3p\ell + 2p + 6\ell + 4 = 3p\ell + 2p + (6\ell+1) + 3 =$$
$$3p(\ell+1) + 3 \equiv 3(\mod 3p).$$

The first of these equations shows that this simplicial space has $2p$ 1-cells, and the second equation shows us that this simplicial space has $\frac{p+3}{2}$ pairs of 1-cells between v_0 and v_3. Therefore $2p_{\frac{p+3}{2}}S^1_\bullet$ is in the image of ψ_1.

Step 3: Since $(\psi' \vee \mathrm{id})$ fixes the copy of $p_3S^1_\bullet$ then it suffices to show that $\psi' \colon 2p_{\frac{p+3}{2}}S^1_\bullet \to p_3S^1_\bullet \vee_{C_p} p_3S^1_\bullet$. By definition, ψ' identifies opposite vertices. The opposite vertex from v_0 will be p 1-cells away from v_0. Recall the following drawing of $2p_{\frac{p+3}{2}}S^1_\bullet$ from Step 2:

$$\begin{array}{ccccccccccc} & v_0 & & & & v_{p+2} & & & & v_{2p+4} & \\ \cdots & \bullet & \text{———} & \bullet & \text{———} & \bullet & \text{———} & \bullet & \text{———} & \bullet & \cdots \\ & v_p & & v_{p+1} & & v_{2p+2} & & v_{2p+3} & & v_4 & \end{array}$$

Recall from Step 2 that we can consider $p = 6\ell+1$ for some $\ell \in \mathbb{Z}$. Recall that the equation $z(p+2)(\mod 3p)$ tells us the enumeration of the top label of the vertex z pairs of 1-cells away from v_0. Therefore we would like to consider $(\frac{p+1}{2})(p+2) - 1(\mod 3p)$ to find out the enumeration of the vertex p 1-cells away from v_0. This computation is $(\frac{p+1}{2})(p+2)-1=(3\ell+1)(p+2)-1=3p\ell+p+6\ell+2-1=3p\ell+2p\equiv 2p(\mod 3p)$. Then p 1-cells away from v_0 is the vertex v_{2p}. Therefore ψ' will identify v_0 and v_{2p}.

By similar arguments as in the last two steps one can show that $\psi'\colon 2p_{\frac{p+3}{2}}S^1_\bullet \to p_3S^1_\bullet \vee_{C_p} p_3S^1_\bullet$.

The other cases are very similar, the only differences are that there are different formulas and the modular arithmetic looks different. $\square$

Now we can discuss the following proposition.

Proposition 5.2.7. *Let R be a commutative ring C_p-spectrum, and $p \geq 5$ prime. Then* $\mathrm{THH}_{C_p}(R)$ *is a non-counital, coassociative R-coalgebra in the C_p-equivariant stable homotopy category.*

Proof. To prove coassociativity, we must show that the following diagram of C_p-spectra commutes, where $T := \mathrm{THH}_{C_p}(R)$

$$\begin{tikzcd} T \arrow[r, "\psi"] \arrow[d, "\psi"'] & T \wedge_R T \arrow[d, "\mathrm{id}\wedge\psi"] \\ T \wedge_R T \arrow[r, "\psi\wedge\mathrm{id}"'] & T \wedge_R T \wedge_R T. \end{tikzcd}$$

Since we are working in the homotopy category, it is sufficient to show that the following diagram of C_p-simplicial spaces commutes, where $3T := 3\,\mathrm{THH}_{C_p}(R)$ and $2T := 2\,\mathrm{THH}_{C_p}(R)$

$$\begin{tikzcd} 3T \arrow[r, "\psi_1"] \arrow[d, "\psi_2"'] & T \wedge_R 2T \arrow[d, "\mathrm{id}\wedge\psi"] \\ 2T \wedge_R T \arrow[r, "\psi\wedge\mathrm{id}"'] & T \wedge_R T \wedge_R T \end{tikzcd}$$

To prove this, it is sufficient to show that the following diagram of C_p-simplicial spaces commutes

$$\begin{tikzcd}
3pS^1_\bullet \arrow[r, "\psi_1"] \arrow[d, "\psi_2"'] & p_3S^1_\bullet \vee_{C_p} 2p_{\frac{p+3}{2}}S^1_\bullet \arrow[d, "\mathrm{id}\vee\psi'"] \\
2p_{\frac{p+3}{2}}S^1_\bullet \vee_{C_p} p_3S^1_\bullet \arrow[r, "\psi'\vee\mathrm{id}"'] & p_3S^1_\bullet \vee_{C_p} p_3S^1_\bullet \vee_{C_p} p_3S^1_\bullet
\end{tikzcd}$$

where ψ_1 is the C_p-equivariant pinch map determined by identifying the vertices v_0 and v_p and ψ_2 is the C_p-equivariant pinch map determined by identifying the vertices v_0 and v_{2p}. Proposition 5.2.4 discusses why this is the correct diagram to consider for these maps. The idea of the commutativity of this diagram is that ψ_1 is determined by identifying v_0 and v_p, and ψ' is determined by identifying v_0 and v_{2p} in $2p_{\frac{p+3}{2}}S^1_\bullet$, while the other direction goes in the reverse order first identifying v_0 and v_{2p} and then v_0 and v_p. The modular arithmetic works out such that whether one identifies v_0 with v_p or v_{2p} first or second it works out to be the same. This demonstrates that $(\mathrm{id}\vee\psi')\circ\psi_1 = (\psi'\vee\mathrm{id})\circ\psi_2$. This is shown for two specific examples in Example 5.2.5 and Example 5.2.6.

Therefore, $\mathrm{THH}_{C_p}(R)$ is a non-counital, coassociative R-coalgebra for $p \geq 5$. □

Note that we are working in the C_p-equivariant stable homotopy category because $3\,\mathrm{THH}_{C_p}(R)$ is only weakly equivalent to $\mathrm{THH}_{C_p}(R)$.

Now that we have thoroughly discussed the coproduct structure of $\mathrm{THH}_{C_p}(R)$ for $p \geq 5$, let us discuss the subtleties that arise when $p = 3$.

Remark 5.2.8. For the $p = 3$ case, one can not follow the same argument as in Proposition 5.2.7. Let us explore why this is. Consider $9S^1_\bullet$. The C_3-action on this space sends v_0 to v_3, and v_3 to v_6. If we identify v_0 with v_3 or v_6 as we defined ψ_1 and ψ_2, then we identify vertices with other vertices in their orbit. This would cause there to be fixed points in the image of these maps which we do not want for this particular approach. It is not clear how one could alter the definitions of ψ_1 and ψ_2 to make them work in this case. As is true with the counit, there may be a way to define a coassociative R-coproduct on $\mathrm{THH}_{C_3}(R)$, but we are focusing on inducing these algebraic structures from simplicial spaces, so we will not pursue this question further.

These algebraic and coalgebraic structures interact with each other to give the following theorem.

Theorem 5.2.9. *Let R be a commutative ring C_p-spectrum indexed on a complete C_p-universe, and let $p \geq 5$ be prime. Then $\mathrm{THH}_{C_p}(R)$ is a non-counital, R-bialgebra in the C_p-equivariant stable homotopy category.*

Proof. In Proposition 5.1.5 and Proposition 5.2.7 we showed that, under the assumed conditions, $\mathrm{THH}_{C_p}(R)$ is a non-counital R-algebra and R-coalgebra.

To show that these algebraic and coalgebraic structures are compatible to yield a bialgebraic structure, we need to show that the following diagram of C_p-spectra commutes, where $T := \mathrm{THH}_{C_p}(R)$

$$\begin{tikzcd}
T \wedge_R T \arrow[r, "\phi"] \arrow[d, "\psi\wedge\psi"'] & T \arrow[r, "\psi"] & T \wedge_R T \\
T \wedge_R T \wedge_R T \wedge_R T \arrow[rr, "\mathrm{id}\wedge\tau\wedge\mathrm{id}"'] & & T \wedge_R T \wedge_R T \wedge_R T \arrow[u, "\phi\wedge\phi"']
\end{tikzcd}$$

where τ swaps the two copies of T. Since we are working in the C_p-equivariant stable homotopy category it is sufficient to show that the following diagram of C_p-spectra commutes

$$\begin{tikzcd}
2T \wedge_R 2T \arrow[r, "\phi"] \arrow[d, "\psi'\wedge\psi'"'] & 2T \arrow[r, "\psi'"] & T \wedge_R T \\
T \wedge_R T \wedge_R T \wedge_R T \arrow[rr, "\mathrm{id}\wedge\tau\wedge\mathrm{id}"'] & & T \wedge_R T \wedge_R T \wedge_R T \arrow[u, "\phi\wedge\phi"']
\end{tikzcd}$$

To prove this, it is sufficient to show that the following diagram of C_p-simplicial spaces commutes:

$$\begin{tikzcd}
2pS^1_\bullet \vee_{C_p} 2pS^1_\bullet \arrow[r, "\phi"] \arrow[d, "\psi\vee\psi"'] & 2pS^1_\bullet \arrow[r, "\psi"] & pS^1_\bullet \vee_{C_p} pS^1_\bullet \\
pS^1_\bullet \vee_{C_p} pS^1_\bullet \vee_{C_p} pS^1_\bullet \vee_{C_p} pS^1_\bullet \arrow[rr, "\mathrm{id}\vee\tau\vee\mathrm{id}"'] & & pS^1_\bullet \vee_{C_p} pS^1_\bullet \vee_{C_p} pS^1_\bullet \vee_{C_p} pS^1_\bullet \arrow[u, "\phi\vee\phi"']
\end{tikzcd}$$

where ψ' and ϕ are the C_p-simplicial pinch and fold maps as defined above and τ swaps the two copies of $p_2S^1_\bullet$. To better understand the commutativity of this diagram, let us considered the following colored diagram where one of the starting circles is blue, the other starting circle is red, and a purple circle indicates that it is the image of a blue and red circle which were folded together:

$$\begin{tikzcd}
2pS^1_\bullet \vee_{C_p} 2pS^1_\bullet \arrow[r, "\phi"] \arrow[d, "\psi\vee\psi"'] & 2pS^1_\bullet \arrow[r, "\psi"] & pS^1_\bullet \vee_{C_p} pS^1_\bullet \\
pS^1_\bullet \vee_{C_p} pS^1_\bullet \vee_{C_p} pS^1_\bullet \vee_{C_p} pS^1_\bullet \arrow[rr, "\mathrm{id}\vee\tau\vee\mathrm{id}"'] & & pS^1_\bullet \vee_{C_p} pS^1_\bullet \vee_{C_p} pS^1_\bullet \vee_{C_p} pS^1_\bullet \arrow[u, "\phi\vee\phi"']
\end{tikzcd}$$

If we go along the top of the diagram, the two copies of $2pS^1_\bullet$ are folded together and then pinched into two copies of $pS^1_\bullet$. If we go along the bottom of the diagram, we first pinch both copies of $2pS^1_\bullet$. In order to fold the same 1-cells together that were folded together along the top of the diagram, we want to fold the first blue copy with the first red copy of $pS^1_\bullet$ and similarly with the last blue copy and last red copy. This is accomplished by the composition of maps $(\phi\vee\phi)\circ(\mathrm{id}\vee\tau\vee\mathrm{id})$.

To consider the next diagram, we will first set up some definitions. Say that the vertices of C_p are denoted $1, \gamma, \gamma^2$, and so on. Let us define $\eta'\colon C_p \to 2pS^1_\bullet$ to be the C_p-equivariant map determined by $\eta'\colon 1 \mapsto v_0$. Let us define $\eta \vee \eta\colon C_p \vee_{C_p} C_p \to p_2S^1_\bullet \vee_{C_p} p_2S^1_\bullet$ to be the C_p-equivariant map which includes each copy of C_p into the corresponding copy of $p_2S^1_\bullet$. Note that because we are "wedging over C_p" then this is really just including C_p into the vertices shared by the two copies of $p_2S^1_\bullet$. Let $\eta\vee\eta\colon 1 \mapsto v_0$.

To check that the unit and coproduct maps respect each other, we will show that the following diagram of C_p-simplicial spaces commutes

$$\begin{tikzcd}
& C_p \vee_{C_p} C_p \cong C_p \arrow[dl, "\eta\vee\eta"'] \arrow[dr, "\eta'"] & \\
p_2S^1_\bullet \vee_{C_p} p_2S^1_\bullet & & 2pS^1_\bullet \arrow[ll, "\psi'"]
\end{tikzcd}$$

By the definition of the maps we can see that this diagram commutes.

There are no other diagrams for us to check as we do not need to check any diagrams which include the counit. Therefore, $\mathrm{THH}_{C_p}(R)$ is a non-counital R-bialgebra in the C_p-equivariant stable homotopy category. $\square$

CHAPTER 6

ALGEBRAIC STRUCTURE ON THE EQUIVARIANT BÖKSTEDT SPECTRAL SEQUENCE

In this section, we will discuss the algebraic structure of the equivariant Bökstedt spectral sequence that is induced from the algebraic structure on twisted THH. Let us recall the equivariant Bökstedt spectral sequence.

Theorem 6.0.1 ([AGH+22, Theorem 4.2.7])**.** *Let $C_n = \langle \gamma \rangle$ be a finite subgroup of S^1. Let R be a ring C_n-spectrum and E a commutative ring C_n-spectrum such that γ acts trivially on E. If $\underline{E}_\star(R)$ is flat over $\underline{E}_\star$, then there is an equivariant Bökstedt spectral sequence*

$$E^2_{s,\star} = \underline{\mathrm{HH}}_s^{\underline{E}_\star, C_n}(\underline{E}_\star(R)) \Rightarrow \underline{E}_{s+\star}(i^*_{C_n} \mathrm{THH}_{C_n}(R))$$

where $d^r : E^r_{s,\alpha} \to E^r_{s-r,\alpha+r-1}$.

These authors also show that this equivariant Bökstedt spectral sequence has some algebraic structure.

Proposition 6.0.2 ([AGH+22, Proposition 4.2.8])**.** *Let E and R be as in Theorem 6.0.1. If R is a commutative ring C_n-spectrum, then the equivariant Bökstedt spectral sequence is a spectral sequence of $RO(C_n)$-graded algebras over $\underline{E}_\star$.*

In Chapter 5 it was shown that $\mathrm{THH}_{C_p}(R)$ is a non-counital R-bialgebra for $p \geq 5$ and a commutative R-algebra for $p = 2, 3$. The structure maps which we used to demonstrate these structures on $\mathrm{THH}_{C_p}(R)$, namely (5.1.1) and (5.2.2), are all induced from simplicial maps. Since they are induced from simplicial maps, they respect the skeletal filtration on $\mathrm{THH}_{C_p}(R)$. We will use this fact to induce the structure maps on the equivariant Bökstedt spectral sequence which will be used to demonstrate its algebraic structure.

6.1 Algebraic structure

Let us recall an equivariant analogue to a differential bigraded algebra.

Definition 6.1.1. Let p be prime, $\underline{R}$ a commutative C_p-Green functor, and $\underline{M}$ a commutative $\underline{R}$-algebra that is flat over $\underline{R}$. A *differential* $(\mathbb{Z}, RO(C_p))$*-graded* $\underline{M}$*-algebra* is a collection of $(\mathbb{Z}, RO(C_p))$-graded $\underline{M}$-bimodules and a differential, $(E_{*,\star}, d)$, with the following maps of $\underline{M}$-bimodules:

$$d\colon \bigoplus_{s+\alpha=\beta} E_{s,\alpha} \to \bigoplus_{r+\gamma=\beta-1} E_{r,\gamma},$$

$$\mu\colon E_{s,\alpha} \,\square_{\underline{M}}\, E_{r,\beta} \to E_{s+r,\alpha+\beta}, \text{ and}$$

$$\eta\colon \underline{M} \to E_{*,\star}.$$

The first map defines the differential, the second map is the multiplication map, and the third map is the unit map. These maps must make all the usual associativity and unitality diagrams commute. The differential d must be compatible with the product map in the sense that it satisfies the Leibniz rule:

$$d \circ \mu = \mu \circ (d \,\square_{\underline{M}}\, \mathrm{id} + (-1)^{s+\dim(\alpha^{C_p})} \mathrm{id} \,\square_{\underline{M}}\, d).$$

Now, let us recall an equivariant analogue to a spectral sequence of algebras. Note that the flatness assumption in the following definition is so that the equivariant Künneth spectral sequence, as defined in [LM06, Theorem 1.3], collapses to give the Künneth isomorphism.

Definition 6.1.2. Let p be prime, $\underline{R}$ a commutative C_p-Green functor, and $\underline{M}$ a commutative $\underline{R}$-algebra which is flat over $\underline{R}$. A C_p-equivariant *spectral sequence of* $\underline{M}$*-algebras* is a collection of differential $(\mathbb{Z}, RO(C_p))$-graded $\underline{M}$-algebras $\{E^r_{*,\star}, d^r\}$, with multiplication maps ϕ_r such that ϕ_{r+1} is the composite

$$\begin{aligned}\phi_{r+1}\colon E^{r+1}_{*,\star} \,\square_{\underline{M}}\, E^{r+1}_{*,\star} &\xrightarrow{\cong} H_*(E^r_{*,\star}) \,\square_{\underline{M}}\, H_*(E^r_{*,\star}) \xrightarrow{p} \\ H_*(E^r_{*,\star} \,\square_{\underline{M}}\, E^r_{*,\star}) &\xrightarrow{H_*(\phi_r)} H_*(E^r_{*,\star}) \xrightarrow{\cong} E^{r+1}_{*,\star},\end{aligned}$$

where the homomorphism p is induced from the homology cross product map.

Using this definition and the simplicial maps from earlier sections, we can prove the following proposition.

Proposition 6.1.3. *Let p be prime, and let R and E be commutative ring C_p-spectra, such that the generator of C_p acts trivially on E, and $\underline{E}_\star(R)$ is flat over $\underline{E}_\star$. The equivariant Bökstedt spectral sequence $E^r_{*,\star}$ is a spectral sequence of commutative $\underline{E}_\star(R)$-algebras.*

Proof. Unless otherwise specified, every box product is over $\underline{E}_\star$. Let $'E^r_{*,\star}$ be the spectral sequence associated to the skeleton filtration on $R \otimes_{C_p} (pS^1_\bullet \vee_{C_p} pS^1_\bullet) \cong \mathrm{THH}_{C_p}(R) \wedge_R \mathrm{THH}_{C_p}(R)$. We know that $\mathrm{THH}_{C_p}(R)$ is a commutative R-algebra by Proposition 5.1.5, and all the relevant structure maps (5.1.1) are induced from C_p-equivariant simplicial maps, so they respect the skeleton filtration on $\mathrm{THH}_{C_p}(R)$. As a result, the "fold" maps $\phi\colon {'E^r_{*,\star}} \to E^r_{*,\star}$ respect the differentials of the Bökstedt spectral sequence.

One can consider $\mathrm{THH}_{C_p}(R)_\bullet$, the simplicial C_p-spectrum $R \otimes_{C_p} pS^1_\bullet \cong B^{cy,C_p}_\bullet(R)$. Since $\mathrm{THH}_{C_p}(R)_s = R^{\wedge s+1}$, and by our assumption that $\underline{E}_\star(R)$ is flat over $\underline{E}_\star$ then we have the following isomorphism:

$$\underline{E}_\star(R^{\wedge(s+1)} \wedge_R R^{\wedge(s+1)}) \cong \underline{E}_\star(R)^{\Box s+1} \Box_{\underline{E}_\star(R)} \underline{E}_\star(R)^{\Box s+1}.$$

By definition, the left hand side is $'E^1_{s,\star}$ and the right hand side is $E^1_{s,\star} \Box_{\underline{E}_\star(R)} E^1_{s,\star}$.

The following map is induced from the homology cross product map

$$H_*(E^r_{*,\star}) \Box_{\underline{E}_\star(R)} H_*(E^r_{*,\star}) \to H_*(E^r_{*,\star} \Box_{\underline{E}_\star(R)} E^r_{*,\star}).$$

Therefore we have a map

$$E^2_{*,\star} \Box_{\underline{E}_\star(R)} E^2_{*,\star} \cong H_*(E^1_{*,\star}) \Box_{\underline{E}_\star(R)} H_*(E^1_{*,\star}) \to H_*(E^1_{*,\star} \Box_{\underline{E}_\star(R)} E^1_{*,\star}) \cong H_*('E^1_{*,\star}) \cong {'E^2_{*,\star}}$$

which, by induction, induces the following maps

$$E^r_{*,\star} \Box_{\underline{E}_\star(R)} E^r_{*,\star} \to {'E^r_{*,\star}}$$

for all $r \geq 2$.

Then we can define the composite map of spectral sequences

$$\phi_r\colon E^r_{*,\star} \;\square_{\underline{E}_\star(R)}\; E^r_{*,\star} \longrightarrow {}'E^r_{*,\star} \xrightarrow{\phi} E^r_{*,\star}$$

for any $r \geq 1$. The maps ϕ_r respect differentials for all r since ϕ and the product maps respect the differentials. Since all of the necessary maps respect the differential, the commutative diagrams in Proposition 5.1.5 induce the necessary commutative diagrams for the equivariant Bökstedt spectral sequence. Therefore $E^r_{*,\star}$ is a spectral sequence of commutative $\underline{E}_\star(R)$-algebras. $\square$

6.2 Coalgebraic structure

Let us recall an equivariant analogue to a differential bigraded coalgebra.

Definition 6.2.1. Let p be prime, $\underline{R}$ a commutative C_p-Green functor, and $\underline{M}$ a commutative $\underline{R}$-algebra that is flat over $\underline{R}$. A *differential* $(\mathbb{Z}, RO(C_p))$*-graded* $\underline{M}$*-coalgebra* is a collection of $(\mathbb{Z}, RO(C_p))$-graded $\underline{M}$-bimodules and a differential $(E_{*,\star}, d)$ with the following maps of $\underline{M}$-bimodules:

$$d\colon \bigoplus_{s+\alpha=\beta} E_{s,\alpha} \to \bigoplus_{r+\gamma=\beta-1} E_{r,\gamma},$$

$$\Delta\colon E_{s,\alpha} \to \bigoplus_{\substack{u+w=s\\ \beta+\gamma=\alpha}} E_{u,\beta} \,\square_{\underline{M}}\, E_{w,\gamma}, \text{ and}$$

$$\varepsilon\colon E_{*,\star} \to \underline{M}.$$

The first map defines the differential, the second map is the comultiplication map, and the third map is the counit map. These maps must make all the usual coassociativity and counitality diagrams commute. The differential d must be compatible with the coproduct map in the sense that it satisfies the coLeibniz rule:

$$\Delta \circ d = (d \,\square_{\underline{M}}\, \mathrm{id} + (-1)^{s+\dim(\alpha^{C_p})} \mathrm{id} \,\square_{\underline{M}}\, d) \circ \Delta.$$

Now we can recall an equivariant analogue to a spectral sequence of coalgebras.

Definition 6.2.2. Let p be prime, $\underline{R}$ a commutative C_p-Green functor, $\underline{M}$ a commutative $\underline{R}$-algebra that is flat over $\underline{R}$, and $(E^r_{*,\star}, d)$ be differential $(\mathbb{Z}, RO(C_p))$-graded $\underline{M}$-coalgebras such that $E^r_{*,\star}$ is flat as an $\underline{R}$-module for $r \geq 2$. A C_p-equivariant *spectral sequence of $\underline{M}$-coalgebras* is a collection of differential $(\mathbb{Z}, RO(C_p))$-graded $\underline{M}$-coalgebras $\{E^r_{*,*}, d^r\}$, with comultiplication maps ψ_r such that ψ_{r+1} is the composite

$$\psi_{r+1} \colon E^{r+1}_{*,\star} \xrightarrow{\cong} H_*(E^r_{*,\star}) \xrightarrow{H_*(\psi_r)} H_*(E^r_{*,\star} \square_{\underline{R}} E^r_{*,\star}) \xrightarrow{p^{-1}}$$
$$H_*(E^r_{*,\star}) \square_{\underline{R}} H_*(E^r_{*,\star}) \xrightarrow{\cong} E^{r+1}_{*,\star} \square_{\underline{R}} E^{r+1}_{*,\star},$$

where the homomorphism p is the Künneth isomorphism.

This definition allows us to discuss the coalgebraic structures of the equivariant Bökstedt spectral sequence.

Proposition 6.2.3. *For $p \geq 5$ prime, let R and E be commutative ring C_p-spectra, such that the generator of C_p acts trivially on E, and $\underline{E}_\star(R)$ is flat over $\underline{E}_\star$. If each term of the equivariant Bökstedt spectral sequence $E^r_{*,\star}$ for $r \geq 2$ is flat over $\underline{E}_\star(R)$, then there is a coproduct $\psi_r \colon E^r_{*,\star} \to E^r_{*,\star} \square_{\underline{E}_\star(R)} E^r_{*,\star}$, and $E^r_{*,\star}$ is a spectral sequence of non-counital $\underline{E}_\star(R)$-coalgebras.*

Proof. Unless otherwise specified, every box product is over $\underline{E}_\star$. Let $2E^r_{*,\star}$ be the spectral sequence associated to the skeleton filtration on $R \otimes_{C_p} 2pS^1_\bullet$, let ${}_2E^r_{*,\star}$ be the spectral sequence associated to the skeleton filtration on $R \otimes_{C_p} p_2S^1_\bullet$, let ${}'E^r_{*,\star}$ be the spectral sequence associated to the skeleton filtration on $R \otimes_{C_p} (pS^1_\bullet \vee_{C_p} pS^1_\bullet)$, and let ${}'_2E^r_{*,\star}$ be the spectral sequence associated to the skeleton filtration on $R \otimes_{C_p} (p_2S^1_\bullet \vee p_2S^1_\bullet)$.

We know that $\mathrm{THH}_{C_p}(R)$ is a non-counital R-coalgebra in the stable homotopy category by Proposition 5.2.7, and the coproduct map (5.2.2) is induced from the C_p-equivariant simplicial pinch map, so this map respects the skeleton filtration on $\mathrm{THH}_{C_p}(R)$, so $\psi \colon 2E^r_{*,\star} \to {}'_2E^r_{*,\star}$, which is induced from the pinch map, respects the differentials of the Bökstedt spectral sequence. Consider the following sequence of maps which are defined below:

$$E^r_{*,\star} \xleftarrow{\pi_2} 2E^r_{*,\star} \xrightarrow{\psi} {}'_2E^r_{*,\star} \xrightarrow{\xi} {}'E^r_{*,\star} \xleftarrow{pr} E^r_{*,\star} \square_{\underline{E}_\star(R)} E^r_{*,\star}.$$

Here π_2 is the algebraic analogue of the weak equivalence defined in Proposition 5.0.6 and is an isomorphism for $r \geq 2$. The map ξ is an algebraic analogue of the isomorphism defined in Proposition 5.0.7 and is an isomorphism for $r \geq 1$. The last map p_r takes some work to define and is described below.

One can consider $\mathrm{THH}_{C_p}(R)_\bullet$, the simplicial C_p-spectrum $R \otimes_{C_p} pS^1_\bullet \cong B^{cy,C_p}_\bullet(R)$. In [LM06, Theorem 1.3] Lewis and Mandell define an equivariant analogue to the Künneth spectral sequence. Our flatness assumption gives that the equivariant Künneth spectral sequence collapses for $\underline{E}_\star(\mathrm{THH}_{C_p}(R)_s \wedge_R \mathrm{THH}_{C_p}(R)_s))$, therefore it is isomorphic to $\underline{E}_\star(\mathrm{THH}_{C_p}(R)_s) \,\Box_{\underline{E}_\star(R)}\, \underline{E}_\star(\mathrm{THH}_{C_p}(R)_s)$. Since $\mathrm{THH}_{C_p}(R)_s = R^{\wedge s+1}$ then this isomorphism can be written as:

$$\underline{E}_\star(R^{\wedge(s+1)} \wedge_R R^{\wedge(s+1)}) \cong \underline{E}_\star(R)^{\Box s+1} \,\Box_{\underline{E}_\star(R)}\, \underline{E}_\star(R)^{\Box s+1}.$$

The left hand side is $'E^1_{s,\star}$ and the right hand side is $E^1_{s,\star} \,\Box_{\underline{E}_\star(R)}\, E^1_{s,\star}$. For a fixed s, one can define a shuffle map $[E^1_{*,\star} \,\Box_{\underline{E}_\star(R)}\, E^1_{*,\star}]_{s,\star} \to E^1_{s,\star} \,\Box_{\underline{E}_\star(R)}\, E^1_{s,\star}$. The Eilenberg-Zilber theorem can be applied to any bisimplicial object in an abelian category [Wei94, Theorem 8.5.1]. Using the Eilenberg-Zilber theorem, we can show that the map $p_1\colon E^1_{*,\star} \,\Box_{\underline{E}_\star(R)}\, E^1_{*,\star} \to {'E}^1_{*,\star}$ is an isomorphism on homology. By assumption, $E^2_{*,\star}$ is flat over $\underline{E}_\star(R)$ therefore the equivariant Künneth spectral sequence collapses to the Künneth isomorphism. By the Künneth isomorphism, and the Eilenberg-Zilber theorem we have the following isomorphism:

$$p_2\colon E^2_{*,\star} \,\Box_{\underline{E}_\star(R)}\, E^2_{*,\star} \cong H_*(E^1_{*,\star} \,\Box_{\underline{E}_\star(R)}\, E^1_{*,\star}) \xrightarrow{\cong} H_*({'E}^1_{*,\star}) \cong {'E}^2_{*,\star}.$$

Choose $r \geq 2$, using an inductive argument, assume that $p_r\colon E^r_{*,\star} \,\Box_{\underline{E}_\star(R)}\, E^r_{*,\star} \to {'E}^r_{*,\star}$ is an isomorphism. By assumption $E^{r+1}_{*,\star}$ is flat over $\underline{E}_\star(R)$ so we can use the equivariant Künneth spectral sequence to get the following isomorphism:

$$p_{r+1}\colon E^{r+1}_{*,\star} \,\Box_{\underline{E}_\star(R)}\, E^{r+1}_{*,\star} \cong H_*(E^r_{*,\star} \,\Box_{\underline{E}_\star(R)}\, E^r_{*,\star}) \xrightarrow{\cong} H_*({'E}^r_{*,\star}) \cong {'E}^{r+1}_{*,\star}.$$

We can now define the coproduct ψ_r on $E^r_{*,\star}$ as $p_r^{-1} \circ \xi \circ \psi \circ \pi_2^{-1}$ for $r \geq 2$. These isomorphisms and ψ respect the skeletal filtrations they are respectively defined on, therefore ψ_r respects the skeletal filtration it is defined on for all r. Since ψ_r respects the skeletal filtration, it also respects the differentials for all r. Since all of the necessary maps respect the differential, the commutative diagrams in Proposition 5.2.7 induce the necessary commutative diagrams for the equivariant Bökstedt spectral sequence. Therefore $E^r_{*,\star}$ is a spectral sequence of non-counital $\underline{E}_\star(R)$-coalgebras. $\square$

In fact, combining the results and arguments from Proposition 6.1.3 and Proposition 6.2.3 we have the following result.

Theorem 6.2.4. *For $p \geq 5$ prime, let R and E be commutative ring C_p-spectra, such that the generator of C_p acts trivially on E and $\underline{E}_\star(R)$ is flat over $\underline{E}_\star$. If each term of the equivariant Bökstedt spectral sequence $E^r_{*,\star}$ for $r \geq 2$ is flat over $\underline{E}_\star(R)$, then $E^r_{*,\star}$ is a spectral sequence of non-counital $\underline{E}_\star(R)$-bialgebras.*

Proof. Since all necessary maps respect the differential, the commutative diagrams in Theorem 5.2.9 induce the necessary commutative diagrams for the equivariant Bökstedt spectral sequence. Therefore, $E^r_{*,\star}$ is a non-counital $\underline{E}_\star(R)$-bialgebra spectral sequence. $\square$

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