

NONLINEAR ADIABATIC STABILITY FOR A GENERALIZED
REACTION-DIFFUSION SYSTEM

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ABSTRACT

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We examine a singularly perturbed, coupled, weakly damped, reaction-diffusion system in one space dimension. This system is examined in the semi-strong pulse interaction regime. We rigorously construct a slow manifold of N -pulse solutions. We identify neutral modes and uncouple them. We solve this reduced nonlinear N -dimensional system with a fixed point method, which generates an equilibrium solution for the reduced system. We turn the coupling back on and continue the slow manifold back to the original system. After analyzing the eigenvalue problem and using renormalization group methods, we show the approximate invariant manifold for the full system is adiabatically stable. We also derive an explicit formula for the pulse dynamics. This work is the first rigorous analysis of the weakly damped regime, in which the essential spectrum approaches the origin.

To my family. You are the best—.

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No road is long with good company. - old proverb

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Chapter 1

Introduction

The study of self-organizing pattern formation was first considered by Turing [31]. Systems of recent interest are the Gierer-Meinhardt model [18] and the Gray-Scott model [22]. These reaction-diffusion systems consist of activator components which drive pattern formation and inhibitor components which curtail the reaction. They model a variety of chemical reactions, including morphogenesis.

We study two classes of systems, the first encompassing a class of singularly perturbed reaction-diffusion equations

$$U_t = \epsilon^{-2}U_{xx} - \epsilon^\alpha \mu U + \epsilon^{-\beta} U^{\alpha 11} V^{\alpha 12}, \quad (1.1a)$$

$$V_t = V_{xx} - V + U^{\alpha 21} V^{\alpha 22}, \quad (1.1b)$$

with $\alpha_{11}, \alpha_{21} \geq 0$ and $\mu > 0$, $\alpha \geq 0$, $\beta \geq 0$, $\alpha_{12} \geq 2$ and $\alpha_{22} \geq 2$. We study this system for $0 < \epsilon \ll 1$, which introduces the novelty of an asymptotically small linear damping for the first component U .

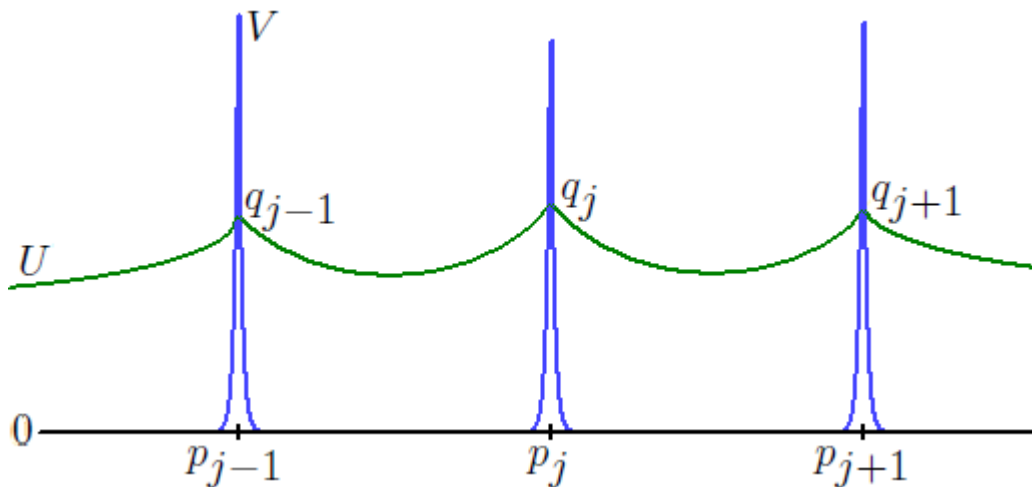


Figure 1.1: A typical quasi-steady solution structure for the coupled system (1.1) and (1.2). The V component is localized at the pulse positions p_j . The U component has an approximately constant value, q_j , on the narrow pulse intervals, and is slowly varying in between the pulses. For interpretation of the references to color in this and all other figures, the reader is referred to the electronic version of this dissertation.

The second class of systems we consider is referred to as an activator-inhibitor system, where the activator component, V , drives the reaction while the inhibitor component, U , curtails production of V . Many references in the literature, including [18], have modeled these systems with an equation of the form (1.1), with $\alpha_{21} < 0$. However, the singularity in the nonlinear term at $U = 0$ suggests unlimited production of the activator when the inhibitor is absent, which has no correspondence

to chemical reality. To be consistent with the chemical literature, see for example Chapter 26 of [1], we truncate the singularity and rewrite (1.1) as

$$U_t = \epsilon^{-2}U_{xx} - \epsilon^\alpha \mu U + \epsilon^{-\beta} U^{\alpha 11} V^{\alpha 12}, \quad (1.2a)$$

$$V_t = V_{xx} - V + \kappa(U)^{\alpha 21} V^{\alpha 22}, \quad (1.2b)$$

where the function κ is defined as

$$\kappa(s) = \begin{cases} s & s > 2\delta \\ \delta & 0 < s < \delta \end{cases}, \quad (1.3)$$

where κ is smooth for $s \in (\delta, 2\delta)$. For this modified system, the production rate of the activator V is bounded by $\frac{V^{\alpha 22}}{\delta^{\alpha 21}}$. For δ sufficiently small, we will show that the salient leading order dynamics of (1.1) are unaffected by the modification, while the artificial singularity at $U = 0$ has been removed. Combining both systems (and also the system that allows $\alpha_{11} < 0$), we introduce the system

$$U_t = \epsilon^{-2}U_{xx} - \epsilon^\alpha \mu U + \epsilon^{-\beta} \Gamma(U)^{\alpha 11} V^{\alpha 12}, \quad (1.4a)$$

$$V_t = V_{xx} - V + \Gamma(U)^{\alpha 21} V^{\alpha 22}, \quad (1.4b)$$

where Γ is defined as

$$\Gamma(U)^r = \begin{cases} U^r & r \geq 0 \\ \kappa(U)^r & r < 0 \end{cases}. \quad (1.5)$$

We write (1.4) in vector form $\vec{U} = (U, V)^T$:

$$\vec{U}_t = \mathcal{F}(\vec{U}). \quad (1.6)$$

For the system (1.4), we investigate the existence and the dynamics of localized N -pulse solutions in the second component V coupled via the long-range interaction of the delocalized component U . Within the framework of this singularly perturbed fast-slow system, this corresponds to the semi-strong pulse interaction regime, which is intermediate between the weak interaction regime and the strong interaction regime. In the weak interaction regime, both components U and V have sufficiently localized pulse structure, so that the mutual interaction of localized structures in each component is exponentially weak. Within the weak regime, there is no leading order influence of pulse location on the shape or the stability of the pulses. The weak interaction regime has been well-studied in reaction-diffusion systems, see [14], [15], [29], and [30]. In the context of (1.4), the strong interaction regime corresponds to the pulses of the V component being sufficiently close together that their point-wise interaction competes with other effects, leading to possibly self-replication, collision, or annihilation. There has been little theoretical investigation of the strong interac-

tion regime, which is typically investigated using numerical techniques.

The semi-strong pulse interaction regime is an intermediate regime between the weak interaction regime and the strong interaction regime. In a two-component semi-strong regime, one component is strongly localized and the other component is delocalized. The semi-strong regime typically arises in systems that are singularly perturbed, as a result of a dichotomy of scales in the diffusivity coefficients.

In the semi-strong interaction regime, the delocalized component acts as a mean-field which drives the motion and amplitudes of the localized pulses. In turn, the localized pulses couple to the delocalized components, in effect generating their own mean-field. This coupling between the localized and delocalized components has to be self-consistent. A key ingredient of this thesis is the construction of a self-consistent approximate invariant manifold for the system (1.4). Moreover, we rigorously reduce the full partial differential equation system (1.4) to this approximate invariant manifold, deriving leading order ordinary differential equations for the localized pulse dynamics.

The semi-strong regime has been studied both formally ([9], [10], [11], and [26]) and rigorously ([12] and [32]). These previous works have studied various systems including the Gierer-Meinhardt model [18] and the Gray-Scott model [22]. An N -pulse semi-strong interaction regime for the generalized Gierer-Meinhardt model is examined in [26]. Conditions for stability are established and ordinary differential

equations for the dynamics of quasi-equilibrium pulse solutions are determined. In [11], a general system that includes both the Gierer-Meinhardt and the Gray-Scott model is studied. The semi-strong two-pulse interaction is investigated and formal results for the asymptotic stability of their solution are determined, along with ordinary differential equations governing the dynamics of pulse positions.

Our work extends [12], where a 2-pulse semi-strong interaction regime is rigorously studied for the regularized Gierer-Meinhardt model, which corresponds to our system (1.4) with $\alpha = 0$, $\beta = 1$, $\alpha_{11} = 0$, $\alpha_{21} = -1$, and $\alpha_{12} = \alpha_{22} = 2$:

$$U_t = \epsilon^{-2}U_{xx} - \mu U + \epsilon^{-1}V^2 \quad (1.7a)$$

$$V_t = V_{xx} - V + \frac{V^2}{\kappa(U)}. \quad (1.7b)$$

The work in [12] rigorously determines an equation for pulse dynamics and establishes an asymptotic stability result for the reduced flow. In [32], the semi-strong regime is investigated for a 2, 3, or 4-pulse for a three-component coupled system with two inhibitors components and one activator component. Both of these two previous works use the renormalization group methodology to establish adiabatic stability of the quasi-invariant N -pulse manifold. This renormalization group methodology was developed in [20], and extended in [29]. The techniques in [29] have been applied to [23], [28], and [32] (for further details on renormalization group methodology see

[4], [5], [6], [7], [19], and [21]). In the context of [12], the renormalization group method is a means to obtain appropriate semigroup estimates on families of weakly time dependent linear operators.

The asymptotically weak linear damping term $\epsilon^\alpha \mu U$ present in (1.4) is a central and novel part of this thesis. For an appropriate linearization of the system, the asymptotic weakness of the linear damping term is manifested in the essential spectrum being asymptotically close to the origin. This thesis derives sufficient conditions under which the essential spectrum does not impinge upon the reduction of the full dynamics to a finite dimensional system. In particular, for the case $\alpha/2 + \beta = 1$, subject to the bound on the asymptotic decay rate,

$$0 \leq \alpha < 2/3, \tag{1.8}$$

we derive semigroup estimates which amount to a preservation of normal hyperbolicity despite the impinging essential spectrum. For larger α , our method breaks down as the flow on the manifold becomes comparable to the asymptotic decay rates; in other words, we lose normal hyperbolicity in the nonlinear semigroup. We believe these results are sharp, that is, a reduction to $\alpha > 2/3$ would require an extension of the approximate invariant manifold to include parts of the essential spectrum, which would be manifested as self-like structures (small in L^∞ , large in L^1) in the localized component V . However, we do not conduct analysis of the $\alpha > 2/3$ case in

this thesis.

This work contains the first rigorous construction of an N -pulse adiabatic manifold in the semi-strong regime. We identify the neutral modes of the system (1.4) and identify sufficient conditions under which they may be uncoupled at leading order to obtain a reduced formulation of the system. More specifically, we introduce the pulse positions $\vec{p}(t) \in \mathcal{K}_{l_0} \subset \mathbb{R}^N$ and the pulse amplitudes $\vec{q}(t) \in \mathbb{R}^N$, which evolve with respect to time. The set $\mathcal{K}_{l_0}$ is defined as

$$\mathcal{K}_{l_0} = \left\{ \vec{p} \mid |p_i - p_j| > l_0 |\ln \epsilon|, \ l_0 > 0, \ \forall i \neq j, \ 1 \leq i, j \leq N \right\}, \quad (1.9)$$

with l_0 sufficiently large so that the localized pulse overlap is $O(\epsilon^r)$ for $r \geq 2$. The N -pulse adiabatic solution is a function of the pulse positions $\vec{p} \in \mathbb{R}^N$, where the j -th localized pulse in the V component is centered at p_j for $j = 1, \dots, N$. The pulse positions are well-ordered, so that $p_i < p_j$ for $i < j$. The amplitude of the delocalized component U at each pulse position p_j is denoted by q_j , so $U(p_j) = q_j$ for $j = 1, \dots, N$ (see Figure 1.1).

For fixed pulse positions $\vec{p}$, we rigorously determine a self-consistent mean field for these N -pulse amplitudes $\vec{q} = \vec{q}(\vec{p})$ by a fixed point method. Using the implicit function theorem, we are able to generate the amplitudes $\vec{q}$ as local smooth functions of $\vec{p}$ and $\epsilon > 0$. For each $\vec{p} \in \mathcal{K}_{l_0}$ there exists at least one branch $\vec{q}(\vec{p})$, but often more than one. Also, there exists both a uniform lower and upper bound for at

least one branch $\vec{q}(\vec{p})$. For any branch with this uniform bound above and below, we can ignore the δ term in (1.3) (thus contained in the system (1.4)) so long as δ is chosen smaller than the uniform lower bound. We also formulate a non-bifurcation condition, that when met, will guarantee the local persistence of a particular branch $\vec{q}(\vec{p})$. Associated to each branch $\vec{q}(\vec{p})$, we construct an adiabatic N -pulse solution to (1.4) of the form

$$\Phi_{\vec{p}(t)}(x) = \begin{pmatrix} \Phi_1(x; \vec{p}(t), \vec{q}(t)) \\ \Phi_2(x; \vec{p}(t), \vec{q}(t)) \end{pmatrix}, \quad (1.10)$$

where Φ_1 corresponds to the U component and Φ_2 corresponds to the V component (see Section 2.1 and Section 2.2 for the explicit construction of $\Phi_{\vec{p}(t)}$). Under appropriate restrictions which we detail below, the N -pulse adiabatic solution $\Phi_{\vec{p}}$ serves as an adiabatic manifold with boundary for the system, generating a slow flow.

Linearizing the full system (1.4) about the adiabatic N -pulse solution generates the linearized operator $L_{\vec{p}(t)} = L(\vec{p}(t))$. The heart of the technical elements of this thesis is a detailed analysis of the linearized operator and the associated semigroup. A key step is the identification of a reduced linearization $\tilde{L}_{\vec{p}(t)} = \tilde{L}(\vec{p}(t))$. In particular, we show that there exists a $\nu > 0$ independent of ϵ such that $\sigma(\tilde{L}) \cap \{Re(\lambda) > -\nu\}$ can be decomposed into three parts. The first part is the essential spectrum, independent of pulse position $\vec{p}$. The essential spectrum consists of the set $(-\nu, -\epsilon^\alpha \mu]$, which lies within the left-half complex plane. Recall, the asymptotically small damp-

ing term results in an essential spectrum which is asymptotically close to the origin. The second part of the spectrum consists of N -point spectra, σ_0 , which corresponds to the translational modes of the localized pulses and whose eigenmodes lie in the tangent plane of the manifold at leading order. By an appropriate restriction on l_0 , which controls the localized pulse separation, we can restrict these eigenvalues to reside within $O(\epsilon^r)$ of the origin, for any $r \geq 2$ that we desire.

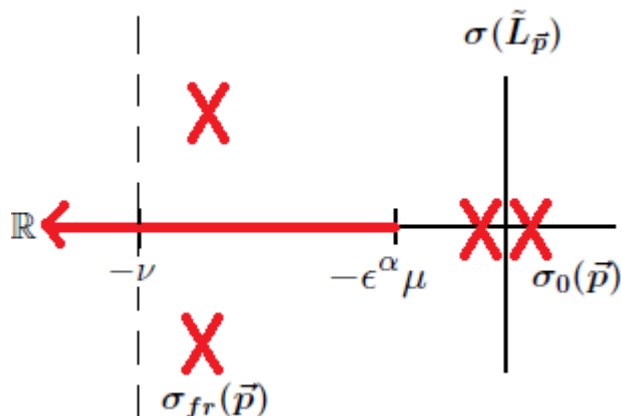


Figure 1.2: This figure illustrates the spectral decomposition of the reduced linearization for $N = 2$.

We show that the remainder of the spectrum, which we call the finite rank spectrum, σ_{fr} , can be characterized as solutions of N algebraic equations which we associate to the finite rank potentials in the reduced linearization. The finite rank spectrum evolves at leading order as the pulse positions evolve, and a key issue of this work is reducing the location of the spectral set σ_{fr} to an explicit set of algebraic equations. The following figure illustrates the evolution of the spectral set σ_{fr} for a 2-pulse solution to the regularized Gierer-Meinhardt system in [12] over a wide range

of values of the localized two-pulse positions $\vec{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$.

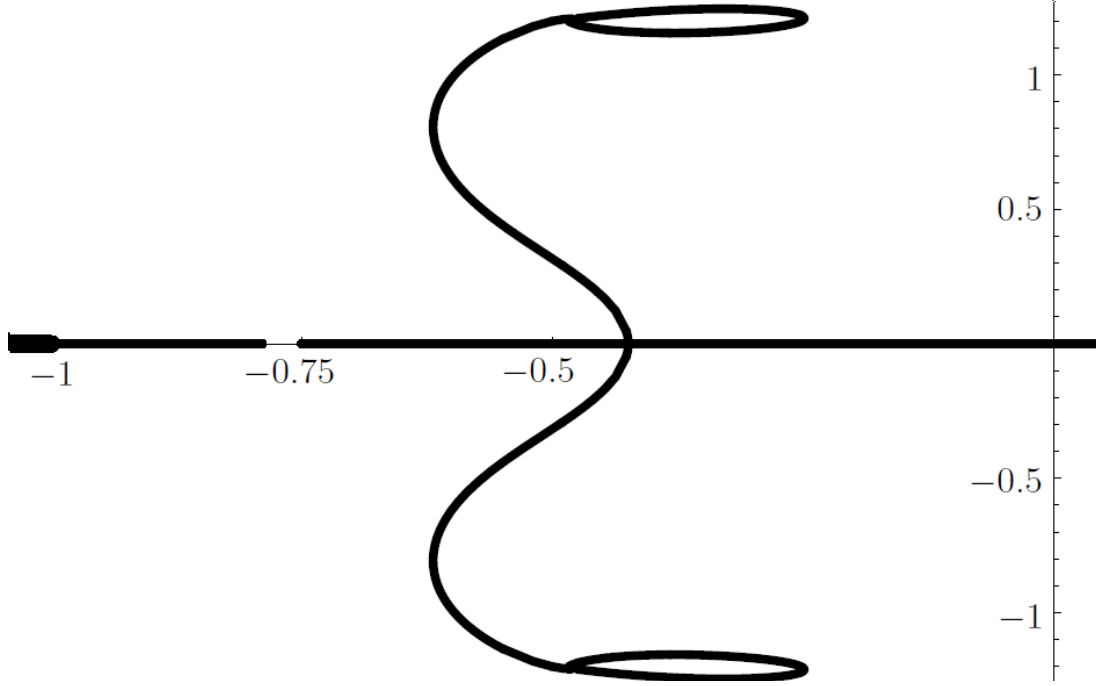


Figure 1.3: This is an illustration of the evolution of the finite rank spectrum for the 2-pulse regularized Gierer-Meinhardt system, for a variety of pulse positions. There are four finite rank eigenvalues. As the pulse separation approaches $+\infty$, the finite rank eigenvalues reside at the left-most point of the loops, corresponding to the weak regime. As the pulse separation decreases, the finite rank eigenvalues separate, one traversing the loop, and the other approaching the real axis, colliding with its complex conjugate, and splitting into a real pair with one approaching the origin and the other retreating towards the essential spectrum.

A novel feature of our analysis is the identification of a bifurcation parameter

$$\theta \equiv \alpha_{11} - \alpha_{12}\alpha_{21}/(\alpha_{22} - 1), \quad (1.11)$$

which balances the exponents of the nonlinear terms. We demonstrate the existence of the N -pulse manifold for $\theta \neq 1$, and numerically we observe a unique nontrivial N -pulse solution for $\theta < 1$, while for $\theta > 1$, we typically find multiple (up to 2^N) N -pulse solutions. Enumerating all of the branches, for each branch $\vec{q}^i = \vec{q}^i(\vec{p})$ of the N -pulse solution we define the adiabatic manifold of N -pulse solutions as

$$\mathfrak{M}^i = \left\{ \Phi \left(\cdot; \vec{p}, \vec{q}^i(\vec{p}) \right), \vec{p} \in \mathcal{K} \right\}. \quad (1.12)$$

where the set of admissible pulse positions $\mathcal{K} \subset \mathbb{R}^N$ is defined as

$$\mathcal{K} = \mathcal{K}^i \equiv \mathcal{K}_{l_0} \cap \mathcal{K}_{\nu}^i, \quad (1.13)$$

for $\mathcal{K}_{l_0}$ previously defined and $\mathcal{K}_{\nu}^i$ defined as

$$\mathcal{K}_{\nu}^i = \left\{ \vec{p} \mid \max \operatorname{Re} \left(\sigma_{fr} \left(\tilde{L}_{\vec{p}} \right) \right) < -\nu \right\}. \quad (1.14)$$

The set $\mathcal{K}_{\nu}^i$ imposes an explicit stability condition, which localizes the finite rank spectrum in the left-half complex plane. The stability condition not only rules out potential Hopf type bifurcation, in which N -pulses become unstable to oscillatory modes, but also saddle-node type bifurcation, in which a single N -pulse separates into two (or more) distinct N -pulse solutions. The stability condition is defined in

terms of eigenvalues of an explicit $N \times N$ matrix, in general the exact nature of the set $\mathcal{K}_{\mathcal{V}}^i$ depends sensitively upon the specific system studied. When $\vec{p} \in \mathcal{K}_{\mathcal{V}}^i$ the finite rank spectrum will never approach the origin, so there can be no splitting of an amplitude solution $\vec{q}^i$, so as a consequence, the non-bifurcation condition previously mentioned is enforced.

In order to discuss the possible pulse configurations contained in the set $\mathcal{K}$, we also introduce the set

$$\mathcal{K}_{weak} = \left\{ \vec{p} \in \mathbb{R}^n \mid \epsilon^{-1-\alpha/2} \ll p_{i+1} - p_i \right\}. \quad (1.15)$$

For our system (1.4), the set $\mathcal{K}_{weak}$ corresponds to the weak interaction regime, described earlier. There exists a unique non-degenerate N -pulse solution in the weak regime which consists of N well-separated copies of the 1-pulse. Since both localized and delocalized components are well-separated, the point spectrum in the weak regime consists of N exponentially close copies of the point spectrum of the 1-pulse. If there exists $\nu > 0$ such that the reduced linearization $\tilde{L}_1$ about this 1-pulse satisfies

$$\sigma(\tilde{L}_1) \setminus \{0\} \subset \{\lambda \mid \text{Re}(\lambda) < -\nu\}, \quad (1.16)$$

then $\mathcal{K}_{weak} \subset \mathcal{K}$. A second regime is the tight regime in which all the localized pulses, while still well-separated, are crowded into a region over which the delocalized

component is asymptotically constant. The set $\mathcal{K}_{tight}$ is defined as

$$\mathcal{K}_{tight} = \left\{ \vec{p} \in \mathbb{R}^n \mid l_0 |\ln \epsilon| < p_N - p_1 \ll \epsilon^{-1-\alpha/2} \right\}. \quad (1.17)$$

There can be multiple N -pulse branches in this regime, however all the branches are asymptotically close, that is

$$|\vec{q}^i(\vec{p}) - \vec{q}^j(\vec{p})| \ll 1 \quad (1.18)$$

for all branches $\vec{q}^i, \vec{q}^j$. Indeed, not only are the branches close, but the spectrum of the associated linearized operator is insensitive to the localized pulse positions $\vec{p} \in \mathcal{K}_{tight}$, so that either $\mathcal{K}_{tight} \subset \mathcal{K}_\nu^i$ for all branches of the tight regime, or $\mathcal{K}_{tight} \subset \left(\mathcal{K}_\nu^i\right)^c$ for all branches $\vec{q}$. We refer to the relative complement of the weak and tight regimes within $\mathcal{K}_{l_0}$ as the dynamic regime $\mathcal{K}_d$. For pulse positions $\vec{p} \in \mathcal{K}_d$, the spectrum changes by $O(1)$ amounts as $\vec{p}$ varies across $\mathcal{K}_d$.

Assumption 1.1. *We assume there exists a branch of N -pulse amplitude solutions $\vec{q}^i = \vec{q}^i(\vec{p})$ and a $d_0 > 0$, independent of ϵ , and a nonempty open set $\mathcal{K}_0^i \subset \mathcal{K}^i$, which consists of pulse configurations, $\vec{p}$, which are a minimum distance d_0 from the boundary ∂K^i of $\mathcal{K}^i$.*

Specifically, we define $\mathcal{K}_0^i$ such that

$$\mathcal{K}_0^i = \left\{ \vec{p} \in \mathcal{K}^i \mid d\left(\vec{p}, \partial K^i\right) \geq d_0 \right\}, \quad (1.19)$$

and introduce the adiabatic sub-manifold $\mathfrak{M}_0^i$ to be the graph of $\Phi_{\vec{p}}$ above $\mathcal{K}_0^i$. When i is fixed, we no longer notate it.

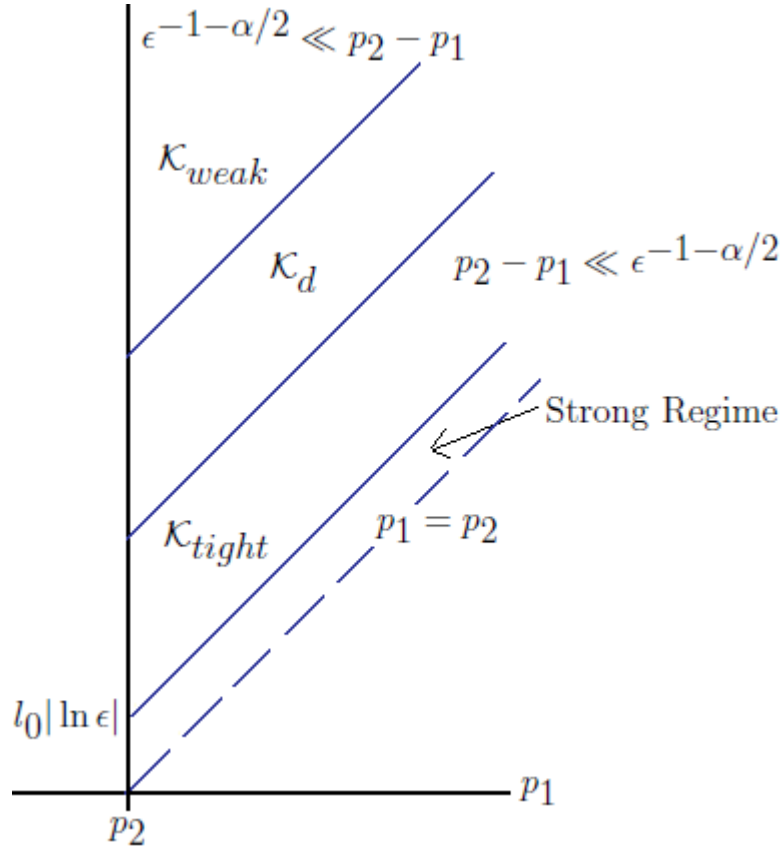


Figure 1.4: This figure illustrates, for $N = 2$ and $p_1 < p_2$, the minimum pulse separation $l_0 |\ln \epsilon|$ and also the pulse classes $\mathcal{K}_{tight}$, $\mathcal{K}_d$, and $\mathcal{K}_{weak}$.

To simplify our analysis, we impose the following restriction.

Simplification 1.1. *We assume that $\alpha \geq 0$ and $\beta \geq 0$ satisfy*

$$1 - \alpha/2 - \beta = 0. \tag{1.20}$$

This simplification assures that the delocalized component U (and Φ_1) is $O(1)$ in L^∞ , which simplifies analysis. Coupled with (1.8), this limits β to $2/3 < \beta \leq 1$.

The main result of this dissertation is an adiabatic stability result, which states that if the initial data to the system (1.4) begins sufficiently close to the adiabatic manifold $\mathfrak{M}_0$, then the full solution will decay to an asymptotically small layer of the adiabatic manifold $\mathfrak{M}$. There is also a time T_b , which is at least $O\left(\epsilon^{-(1+\omega)}\right)$, for any $0 < \omega < 1$, and perhaps $+\infty$, for which the full solution will remain inside this neighborhood of $\mathfrak{M}$, before $p(t)$ hits $\partial\mathcal{K}$, the boundary $\mathcal{K}$.

Once the full solution $\vec{U}$ has relaxed to the asymptotically small equilibrium layer of the adiabatic manifold $\mathfrak{M}$, we reduce the leading order pulse dynamics to an ordinary differential equation on the pulse positions $\vec{p}$. At leading order, the evolution of the pulse positions with respect to time depends on the pulse positions $\vec{p}$ and the amplitude branch $\vec{q} = \vec{q}(\vec{p})$ of the delocalized component at each pulse position.

Assumption 1.1 affords the existence of a branch of adiabatic N -pulse solutions determined by $\vec{q} = \vec{q}(\vec{p})$ over the domain $\mathcal{K} = \mathcal{K}_{l_0, \nu}$ for some $l_0, \nu > 0$ given. The assumption also provides for a $d_0 > 0$ such that $\mathcal{K}_0 \subset \mathcal{K}$ as defined in (1.19) is non-empty.

We introduce the spectral subset associated to the temporally decaying solutions of the semigroup generated by $\tilde{L}_{\vec{p}}$:

$$X_{\vec{p}} = \{\vec{U} \mid \|\vec{U}\|_X < \infty, \pi_{\vec{p}}\vec{U} = 0\}, \quad (1.21)$$

where $\pi_{\vec{p}}$ is the spectral projection associated to the N -point spectrum σ_0 near zero and the X -norm is defined in (1.40).

More specifically, we state the adiabatic stability result and a leading order pulse dynamics result in the following theorem.

Theorem 1.1. *Adiabatic stability and leading order pulse dynamics*

Let $\epsilon > 0$ be sufficiently small, while α and β satisfy Simplification 1.1 with $\alpha < 2/3$. Fix $\omega \in (0, 1)$, then the adiabatic manifold of N -pulse solutions (1.12) afforded by Assumption 1.1 is adiabatically stable up to $O\left(\epsilon^{(1+\omega)(1-\alpha/2)}\right)$. That is, there exist $M, M_0, T_b > 0$ such that for all initial data $\vec{U}_0$ of (1.4) which lie within $M_0\epsilon^\alpha |\ln \epsilon|^{-2}$ of $\mathfrak{M}_0$ in the X -norm (see (1.40)), the corresponding solutions of the system (1.4) can be uniquely decomposed as

$$\vec{U}(x, t) = \Phi_{\vec{p}(t)}(x) + W(x, t), \quad (1.22)$$

where $\Phi_{\vec{p}(t)}$ is an adiabatic N -pulse solution and the remainder $W \in X_{\vec{p}}$ satisfies

$$\|W(t)\|_X \leq M \left(e^{-\frac{\epsilon^\alpha}{2}\mu t} \|W_0\|_X + \epsilon^{(1+\omega)(1-\alpha/2)} \right), \quad (1.23)$$

for all $0 < t \leq T_b\epsilon^{-1-\omega}$. Moreover, during this time interval the pulse dynamics

reduce to

$$\frac{\partial \vec{p}}{\partial t} = \epsilon^{1+\alpha/2} Q^\theta \mathcal{A}(\vec{p}) \vec{q}^{-1} + O\left(\epsilon^2, \epsilon \|W(t)\|_X, \|W(t)\|_X^2\right), \quad (1.24)$$

where Q is the diagonal matrix of pulse amplitudes $\vec{q} = \vec{q}(\vec{p})$, the exponent is applied componentwise in $\vec{q}^{-1} = \left(q_1^{-1}, \dots, q_N^{-1}\right)^T$, and the antisymmetric matrix $\mathcal{A}(\vec{p})$ is defined in (6.244).

Elements of the Proof:

A key construction of this work is the reduced linearization. The construction of the reduced linearization allows us to characterize the point spectrum as it evolves under the pulse evolution. The exact linearization of F about $\Phi(\vec{p})$ for $\vec{p} \in \mathcal{K}$ takes the form

$$L_{\vec{p}} = \begin{pmatrix} L_{11}^e + \epsilon^{-\beta} \mathcal{V}_{11} & \epsilon^{-\beta} \mathcal{V}_{12} \\ \mathcal{V}_{21} & L_{22} \end{pmatrix}, \quad (1.25)$$

where $L_{11}^e = \epsilon^{-2} \partial_x^2 + \epsilon^\alpha \mu$ and $L_{22} = \partial_x^2 - 1 + \alpha_{22} \Phi_1^{\alpha_{21}} \Phi_2^{\alpha_{22}-1}$. Also $\mathcal{V}_{11}$ and $\mathcal{V}_{12}$ are potentials described in detail in Chapter 4. The point spectrum for this linearization is not easily characterized. To understand the exact linearization and the reduced linearization, it is first useful and informative to examine their reductions. The spectra of the diagonal system

$$\begin{pmatrix} L_{11}^e & 0 \\ 0 & L_{22} \end{pmatrix}, \quad (1.26)$$

is easy to characterize. The operator L_{11}^e produces only essential spectra, while $\sigma(L_{22})$ consists of N positive ground eigenvalues, N eigenvalues clustered near zero, and the remainder of the spectrum strictly bounded on the negative real axis. This linearization generates an unstable flow and coupling is needed to generate stability. The spectrum is unchanged if we consider the lower-triangular system

$$\begin{pmatrix} L_{11}^e & 0 \\ \mathcal{V}_{21} & L_{22} \end{pmatrix}, \quad (1.27)$$

for any potential $\mathcal{V}_{21}$. However, systems of the form

$$\tilde{L}_{\vec{p}} = \begin{pmatrix} L_{11}^e + \epsilon^{-\beta} J_{11} & \epsilon^{-\beta} J_{12} \\ \mathcal{V}_{21} & L_{22} \end{pmatrix}, \quad (1.28)$$

where J_{11} and J_{12} are *finite rank operators*, are sufficiently simple that their spectrum can be characterized, but are flexible enough to provide asymptotically accurate approximations of the full system (1.25). Indeed, since $(L_{11}^e)^{-1}$ is asymptotically small away from long-wavelength functions, we show that for an appropriate choice of finite-rank operator J_{1i} ($i = 1, 2$) the difference $(L_{11}^e)^{-1} \epsilon^{-\beta} (J_{1i} - \mathcal{V}_{1i})$ is small as a map on the weighted-windowed space $L_{1,\vec{p}}^1$ discussed below. This difference being small allows us to replace L with $\tilde{L}$ without impacting the leading order dynamics.

The salient element of the reduction of L to $\tilde{L}$ is to uniformly control the long-wavelength elements. This is achieved through the weighted $L^1_{1,\vec{p}}$ norm, which through a partition of unity $\{\chi_j\}_{j=1}^N$ centered about each pulse position, introduces locally weighted norms that control long-wavelength terms, *uniformly* for $\vec{p} \in \mathcal{K}$, in each χ_j window about the pulse at p_j . Specifically, we define the $L^1_{1,\vec{p}}$ norm as

$$\|f\|_{L^1_{1,\vec{p}}} = \sum_{j=1}^N \left\| \left(1 + |x - p_j|\right) \chi_j f \right\|_{L^1}. \quad (1.29)$$

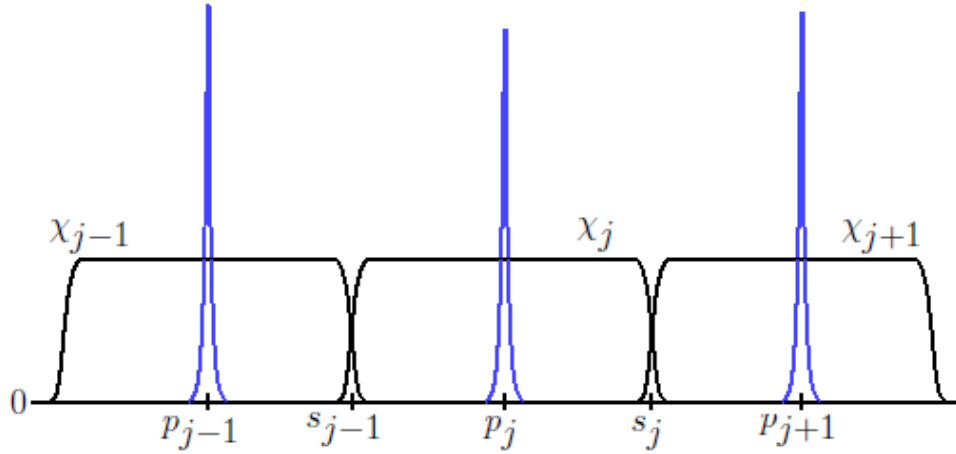


Figure 1.5: This illustrates the second component pulse paired with the partition of unity χ_j at three pulse locations.

Recalling the decomposition $U = \Phi_{p(t)} + W$, we may rewrite the evolution equation (1.4) as an evolution for the remainder W and pulse positions $\vec{p} = \vec{p}(t)$ (analyzed in Chapter 6),

$$W_t + \frac{\partial \Phi}{\partial \vec{p}} \dot{\vec{p}} = R(\Phi) + \tilde{L}_{\vec{p}_0} W + \Delta L W + \mathcal{N}(W), \quad (1.30)$$

where the difference of the exact linearization and the reduced linearization is denoted $\Delta = \tilde{L} - L$ and where $W \in X_{\vec{p}}$, for the spectral subset $X_{\vec{p}}$ defined in (1.21). With the complimentary spectral projection $\tilde{\pi}_{\vec{p}_0}$, the W evolution is given by

$$W_t = -\tilde{\pi}_{\vec{p}_0} \left(\frac{\partial \Phi}{\partial \vec{p}} \dot{\vec{p}} \right) + \tilde{L}_{\vec{p}_0} W + \tilde{\pi}_{\vec{p}_0} (\Delta L W + \mathcal{N}(W)). \quad (1.31)$$

At an initial time t_n , the renormalization group process freezes $L = L_{\vec{p}_n}$, evolves the fast system for a finite time, and then uses a non-linear solve to update the slow components in a self-consistent way. If the secularity in $L_{\vec{p}} - L_{\vec{p}_n}$ can be controlled, then uniform estimates are obtained on a finite time interval, and the process may be iterated. We introduce the renormalization times $\{t_i\}_{i=1}^{\infty}$, where t_0 is the initial time for (1.31). We introduce the quantity,

$$T_1(t) = \sup_{t_i < s < t} e^{\frac{\epsilon}{2} \mu(s-t_i)} \|W(s)\|_X, \quad (1.32)$$

for $t_i < t < t_{i+1}$. We are able to reduce (1.31) to the following quadratic inequality,

$$T_1(t) \leq C |\ln \epsilon| \left(T_1(t_i) + \epsilon^{1-\alpha/2} + \epsilon^{-\alpha} T_1(t)^2 \right). \quad (1.33)$$

For the first iteration, $i = 0$, we recall the bound $T_1(t_0) \leq M_0 \epsilon^\alpha |\ln \epsilon|^2$ imposed by the initial proximity to $\mathfrak{M}_0$, and we rewrite the condition (1.33) as a quadratic inequality $g_0(r) \geq 0$ for $r = T_1$ where,

$$0 \leq g_0(r) = C|\ln \epsilon| \left(M_0 \epsilon^\alpha |\ln \epsilon|^2 + \epsilon^{1-\alpha/2} \right) - r + C|\ln \epsilon| \epsilon^{-\alpha} r^2. \quad (1.34)$$

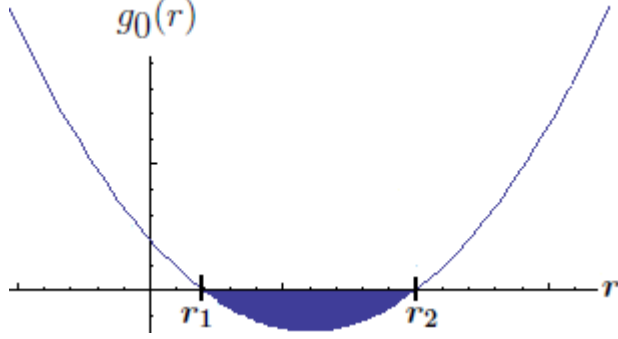


Figure 1.6: This figure generically illustrates the quadratic inequality (1.34) for $\alpha < 2/3$, so that $r_1 > 0$ and the remainder can be appropriately bounded. Our analysis reduces the size of the remainder to a quadratic inequality, so either the remainder starts smaller than r_1 and stays small or it begins larger the r_2 . The middle interval (r_1, r_2) is forbidden.

It follows from the quadratic formula that for $M_0 < 1/(4C)$ and $\alpha < 2/3$ there are two roots of $g_0 = 0$ for $\epsilon \ll 1$. Moreover, with this bound on α we can take the renormalization group time period $t_{i+1} - t_i$ sufficiently long to obtain decay of W . Successive iterations start with a smaller bound on $T_1(t_i)$ and yield tighter estimates on r_1 , until a limit is reached and subsequent evolution yields no further decay.

The renormalization group methodology yields the adiabatic stability result. The leading order pulse dynamics (evolution of $\vec{p}$) are obtained by projecting equation (1.30) onto the tangent plane of $\mathfrak{M}$. After sufficient decay of the remainder W , the dominant term is given by the projection of $F(\Phi(\vec{p}))$, which yields (1.24). The lower bound on the time, T_b , is obtained as an upper bound on the pulse dynamics, with

M_0 taken small enough that the remainder W decays to its adiabatic size before the pulses are within $d_0/2$ of $\partial\mathcal{K}$.

The following figure further illustrates the renormalization group process.

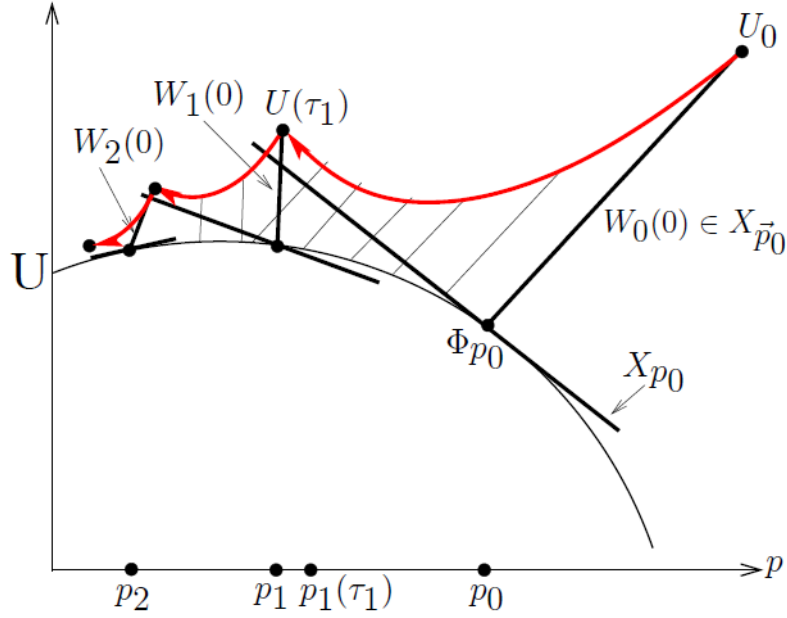


Figure 1.7: This figure illustrates the renormalization group technique. The initial condition is decomposed as $U_0 = \Phi_{p_0} + W_0(0)$, the linearization and associated spectral projections are frozen for a time interval sufficient to give decay, but not so long that the secular growth swamps the error. At the end of each renormalization interval, $U(\tau_1) = \Phi(\vec{p}(\tau_1)) + W(\tau_1)$ is reprojected into $U(\tau_1) = \Phi(\vec{p}_1) + W_1$, where $W_1 \in X_{\vec{p}_1}$. The process is iterated and the transient associated to the initial perturbation decays to the level of the accuracy of the approximate adiabatic solution.

1.1 Notation

We define the pulse positions

$$\vec{p} = (p_1, \dots, p_N)^T \in \mathbb{R}^N, \quad (1.35)$$

where N is the total number of pulses. We define the following norm:

$$\|f\|_{W_\xi^{1,1}} = \|\xi f\|_{L^1} + \|\partial_x f\|_{L^1}, \quad (1.36)$$

where ξ is a smooth, positive, compactly supported, mass one function, where

$$\xi_j = \xi(x - p_j). \quad (1.37)$$

The Sobolev-like norm $W_\xi^{1,1}$ controls L^∞ , since for any $x, y \in \mathbb{R}$,

$$|u(x)| \leq |u(y)| + \int_{-\infty}^{\infty} |u'(z)| dz, \quad (1.38)$$

which uses the fact that $u(x) - u(y) = \int_y^x u' dz$. Multiplying by the mass-one function $\xi(y)$, and integrating over all $y \in \mathbb{R}$, we have

$$|u(x)| \leq \|u'\|_{L^1} + \int_{-\infty}^{\infty} |\xi(y)u(y)| dy = \|u\|_{W_\xi^{1,1}}. \quad (1.39)$$

We define the following norm

$$\|F\|_X \equiv \|f_1\|_{W_\xi^{1,1}} + \|f_2\|_{H^1}, \quad (1.40)$$

for $F = (f_1, f_2)^T$. We define windowing a function f as

$$f = \sum_{j=1}^N f_j = \sum_{j=1}^N f \chi_j. \quad (1.41)$$

For $f \in [L^2(\mathbb{R})]^{N \times k}$ and $g \in [L^2(\mathbb{R})]^{N \times k}$, we define the tensor operator $f \otimes g$, acting on $h \in [L^2(\mathbb{R})]^k$ by

$$f \otimes g \cdot h = \left((h, g_1)_{L^2} f_1, \dots, (h, g_N)_{L^2} f_N \right)^T \in [L^2(\mathbb{R})]^{N \times k}, \quad (1.42)$$

where k denotes the number of components, so $k = 2$ for our system (1.4), and N is the number of pulses.

Chapter 2

Construction of an N -pulse Invariant Manifold.

In this chapter, we construct an N -pulse invariant manifold for a reduction of the system (1.1). We fix N -pulse positions at $\vec{p} = (p_1, \dots, p_N)^T$, and seek a manifold $\Phi = \Phi(x; \vec{p})$ as a graph above an N -dimensional set $\vec{p} \in \mathcal{K} \subset \mathbb{R}^N$. More specifically, we seek Φ which satisfies the invariance condition:

$$\tilde{\pi}_T(\vec{p}) \mathcal{F}(\Phi(\vec{p})) = 0, \tag{2.1}$$

where $\tilde{\pi}_T(\vec{p})$ is the projection complementary to the tangential plane of Φ , at $\Phi(\vec{p})$. The complementary tangential projection is written as $\tilde{\pi}_T \equiv I - \pi_T$, in terms of the

tangential projection:

$$\pi_T f \equiv \sum_{i=1}^N (f, B_i)_{L^2} B_i \in \mathbb{R}^2. \quad (2.2)$$

Here $B = (B_1, \dots, B_N)^T \in [L^2(\mathbb{R})]^{N \times 2}$ is a Gram-Schmidt Orthonormalization of $\left\{ \frac{\partial \Phi}{\partial p_i} \right\}_{i=1}^N$. We assume the family of vectors $\left\{ \frac{\partial \Phi}{\partial p_i} \right\}_{i=1}^N$ is linearly independent. So for each i, j :

$$(B_i, B_j)_{L^2} = \delta_{ij}, \quad (2.3)$$

where δ_{ij} is the Kronecker delta:

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}. \quad (2.4)$$

It is convenient to introduce the associated vector tangential projection:

$$\pi_T^N f \equiv B \otimes B \cdot f = \left((f, B_1)_{L^2} B_1, \dots, (f, B_N)_{L^2} B_N \right)^T \in \mathbb{R}^{N \times 2}, \quad (2.5)$$

where $B \in \mathbb{R}^{N \times 2}$ and $f \in \mathbb{R}^2$.

We show that if Φ satisfies (2.1) then its graph is invariant under the flow. For our general system (1.1), we decompose $\vec{U}$ as

$$\vec{U} = \Phi(\vec{p}) + W. \quad (2.6)$$

Using Proposition 6.1, which allows us to determine a base point where $\pi_T W = 0$, we choose W to be orthogonal to the tangent plane, so $\pi_T W = 0$ and $\tilde{\pi}_T W = W$. We linearize our system:

$$\nabla_{\vec{p}} \Phi \cdot \dot{\vec{p}} + W_t = \mathcal{F}(\Phi) + \mathcal{L}_\Phi W + \mathcal{N}(W). \quad (2.7)$$

Applying the projection and the complementary projection to (2.7), we have

$$\nabla_{\vec{p}} \Phi \cdot \dot{\vec{p}} + \pi_T W_t = \pi_T \mathcal{F}(\Phi) + \pi_T \mathcal{L}_\Phi W + \pi_T \mathcal{N}(W) \quad (2.8)$$

$$\tilde{\pi}_T W_t = \tilde{\pi}_T \mathcal{F}(\Phi) + \tilde{\pi}_T \mathcal{L}_\Phi \tilde{\pi}_T W + \tilde{\pi}_T \mathcal{N}(W), \quad (2.9)$$

where, by the construction of π_T ,

$$\pi_T \left(\nabla_{\vec{p}} \Phi \cdot \dot{\vec{p}} \right) = \nabla_{\vec{p}} \Phi \cdot \dot{\vec{p}}. \quad (2.10)$$

Indeed, we can represent each element of $\nabla_{\vec{p}} \Phi$ as

$$\frac{\partial \Phi}{\partial p_i} = \sum_{k=1}^N \alpha_k B_k, \quad (2.11)$$

so that

$$\pi_T \frac{\partial \Phi}{\partial p_i} = \sum_{j=1}^N \left(\sum_{k=1}^N \alpha_k B_k, B_j \right) B_j = \sum_{j=1}^N \alpha_j B_j = \frac{\partial \Phi}{\partial p_i}. \quad (2.12)$$

Moreover since $\pi_T W = 0$, it follows that

$$0 = \frac{\partial}{\partial t} (W, B_i)_{L^2} = (W_t, B_i)_{L^2} + \left(W, \frac{\partial B_i}{\partial t} \right)_{L^2}, \quad (2.13)$$

from which we conclude

$$\pi_T W_t = - \sum_{i=1}^N \left(W, \frac{\partial B_i}{\partial t} \right)_{L^2} B_i. \quad (2.14)$$

Assuming the invariance condition (2.1) holds, we apply it to (2.9), which yields the flow

$$W_t + \sum_{i=1}^N \left(W, \frac{\partial B_i}{\partial t} \right)_{L^2} B_i = \tilde{\pi}_T \mathcal{L}_\Phi \tilde{\pi}_T W + \tilde{\pi}_T \mathcal{N}(W). \quad (2.15)$$

The set $W = 0$ is invariant under this flow. On the $W = 0$ manifold, we reduce to the tangential flow

$$\nabla_{\vec{p}} \Phi \cdot \dot{\vec{p}} = \pi_T \mathcal{F}(\Phi). \quad (2.16)$$

Thus a smooth solution $\Phi_1(\vec{p})$ of (2.1) yields an invariant manifold of the flow to (1.1), which reduces the flow to the ODE (2.16).

Establishing a solution of (2.1) for our system (1.1) is beyond the scope of this

work. Instead we introduce the system:

$$\mathcal{F}(\vec{U}; \vec{p}, \delta) \equiv \begin{pmatrix} \mathcal{F}_1(U, V; \vec{p}, \delta) \\ \mathcal{F}_2(U, V; \vec{p}, \delta) \end{pmatrix}, \quad (2.17)$$

where

$$\mathcal{F}_1 \equiv \epsilon^{-2} U_{xx} - \epsilon^\alpha \mu U + \epsilon^{-\beta} \left((1 - \delta) \sum_{j=1}^N \chi_j U(p_j)^{\alpha 11} + \delta U^{\alpha 11} \right) V^{\alpha 12} \quad (2.18a)$$

$$\mathcal{F}_2 \equiv V_{xx} - V - (1 - \delta) \epsilon K \left(\frac{V}{\epsilon^2} \right) + \left((1 - \delta) \sum_{j=1}^N \chi_j U(p_j)^{\alpha 21} + \delta U^{\alpha 21} \right) V^{\alpha 22}, \quad (2.18b)$$

with

$$K(y) = \left\{ \sqrt{y} e^{-y}, \quad y \in [0, \infty) \right\}. \quad (2.19)$$

Here $\{\chi_j\}_{j=1}^N$ is a partition of unity where each χ_j is a C^∞ function that is 1 on $(s_{j-1} + 1, s_j - 1)$ with $s_j = \frac{pj+1-pj}{2}$, for $j = \{1, \dots, N-1\}$ while $s_0 = -\infty$ and $s_N = \infty$. On $(s_{j-1} - 1, s_{j-1} + 1]$ and $[s_j - 1, s_j + 1)$, χ_j decays smoothly to zero. The V component is localized about the N -pulse positions $\vec{p} = (p_1, \dots, p_N)^T$. The $K(V)$ term is added to remove any tail-tail interactions between these localized components.

We consider $\delta \in [0, 1]$, where $\delta = 1$ yields the original system (1.1), while $\delta = 0$

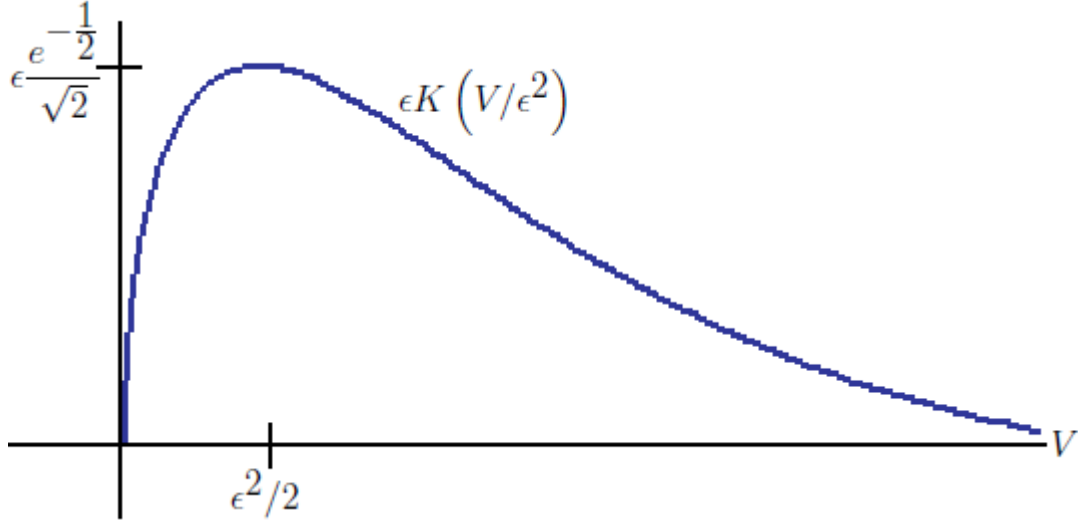


Figure 2.1: Notice the function $\epsilon K(V/\epsilon^2)$ exponentially decays for $V \gg \epsilon^2/2$.

reduces the inter-pulse coupling to a finite rank interaction. In particular

$$\mathcal{F}(\vec{U}; \vec{p}, \delta) = (1 - \delta)\mathcal{F}(\vec{U}; \vec{p}, 0) + \delta\mathcal{F}(\vec{U}), \quad (2.20)$$

where $\mathcal{F}(\vec{U})$ is the unperturbed system that does not depend explicitly on $\vec{p}$.

In the remainder of this chapter, we will construct a manifold $\Phi(\vec{p})$ which satisfies

$$\mathcal{F}(\Phi(\vec{p}); \vec{p}, \delta = 0) = 0. \quad (2.21)$$

For $\delta > 0$, we suggest how a contraction mapping argument could generate an invariant manifold with a slow normal velocity for $\delta = 1$. We leave to posterity the

verification of this condition. However, after this chapter, we rigorously study the quasi-invariance of the $\delta = 0$ manifold under the $\delta = 1$ flow.

2.1 Explicit form for Φ_2

We set $\delta = 0$ in (2.17), and fix the pulse positions $\vec{p} = (p_1, \dots, p_N)^T$. We look for $\Phi(x; \vec{p}, 0) = (\Phi_1, \Phi_2)^T$ which satisfies (2.21), that is,

$$0 = \epsilon^{-2} \partial_{xx} \Phi_1 - \epsilon^\alpha \mu \Phi_1 + \epsilon^{-\beta} \sum_{j=1}^N \chi_j \Phi_1(p_j)^{\alpha_{11}} \Phi_2^{\alpha_{12}} \quad (2.22a)$$

$$0 = \partial_{xx} \Phi_2 - \epsilon K(\Phi_2/\epsilon^2) - \Phi_2 + \sum_{j=1}^N \chi_j \Phi_1(p_j)^{\alpha_{21}} \Phi_2^{\alpha_{22}}. \quad (2.22b)$$

We introduce q_j :

$$q_j \equiv \Phi_1(p_j), \quad (2.23)$$

for $j = 1, \dots, N$. In this chapter, we solve the system

$$\vec{q} = \vec{q}(\vec{p}) = \Phi_1(\vec{p}). \quad (2.24)$$

We have the following theorem for the existence of this solution.

Theorem 2.1. *Self-consistent mean-field theorem*

Let $\delta = 0$ for the system (2.18), Φ be defined in (2.44), and $\vec{p} \in \mathcal{K}$. Then there

exists an $\epsilon_0 > 0$ such that for all ϵ satisfying $0 < \epsilon \leq \epsilon_0$, there exists a function $\mathcal{Q}(s, \epsilon) : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}^N$ which is smooth in s and ϵ such that

$$\vec{q} = \epsilon^{-\kappa} \mathcal{Q}(\epsilon^\eta \vec{p}, \epsilon), \quad (2.25)$$

where $\vec{q}$, so defined, solves (2.72), and $\Phi = \Phi(\vec{p}, \vec{q}(\vec{p}))$ solves (2.22). Here κ is defined as

$$\kappa \equiv \frac{1 - \alpha/2 - \beta}{\theta - 1} \quad (2.26)$$

for θ defined in (2.67), and η is defined as

$$\eta \equiv 1 + \alpha/2. \quad (2.27)$$

Moreover, in the limit as $\epsilon \rightarrow 0$, the rescaled variables $\tilde{q} = \epsilon^\kappa \vec{q}$ and $\tilde{p} = \epsilon^\eta \vec{p}$ solve

$$\tilde{q}(\tilde{p}) = \mathcal{M}(\tilde{p}) \tilde{q}^\theta, \quad (2.28)$$

where $\tilde{q}^\theta = (\tilde{q}_1^\theta, \dots, \tilde{q}_N^\theta)^T$ and $\mathcal{M}(\tilde{p})$ is defined in (2.82).

Furthermore, there exists constants $k, K > 0$ independent of ϵ and $\vec{p} \in \mathcal{K}$ such that the scaled pulse amplitudes $\tilde{q}$ are uniformly bounded above and below:

$$k < \|\tilde{q}\| < K. \quad (2.29)$$

Remark 2.1. *After this Chapter, we assume (1.20), which in turn implies that $\kappa = 0$. Thus after Chapter 2, the unscaled pulse amplitudes $\vec{q}$ are equivalent to the scaled pulse amplitudes $\tilde{q}$, so by (2.29), the unscaled pulse amplitudes $\vec{q}$ are bounded above and below.*

We construct our ansatz where we show (2.22b) has an exact solution of the form $\Phi_2(x) = \sum_{j=1}^N \phi_j(x)$, where each ϕ_j is compactly supported on $(s_{j-1} + 1, s_j - 1)$ and ϕ_j satisfies

$$0 = \phi_j'' - \epsilon K(\phi_j/\epsilon^2) - \phi_j + q_j^{\alpha_{21}} \phi_j^{\alpha_{22}}. \quad (2.30)$$

This equation has a first integral

$$\frac{\phi_j'^2}{2} = H(\phi_j) + \frac{\phi_j^2}{2} - \frac{q_j^{\alpha_{21}}}{\alpha_{22} + 1} \phi_j^{\alpha_{22} + 1}, \quad (2.31)$$

where $H(x) = \int_0^x \epsilon K(s/\epsilon^2) ds$. Isolating the left side, we define the homoclinic solution ϕ_j by the initial value problem

$$\left\{ \begin{array}{lcl} \phi_j' & = & \Omega(\phi_j) = \pm \sqrt{2H(\phi_j) + \phi_j^2 - \frac{2q_j^{\alpha_{21}}}{\alpha_{22} + 1} \phi_j^{\alpha_{22} + 1}} \\ \phi_j(p_j) & = & \phi_j^* \end{array} \right\}, \quad (2.32)$$

where $\Omega(\phi_j^*) = 0$.

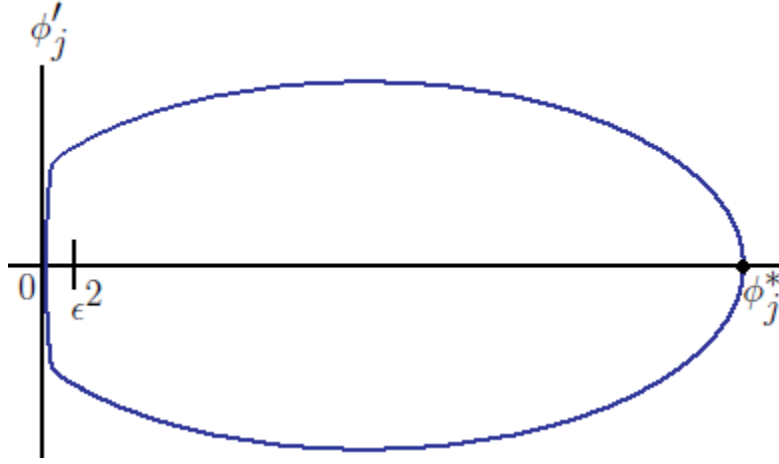


Figure 2.2: An illustration of the homoclinic orbit of ϕ_j' .

For $|\phi_j| \gg \epsilon^2$, we consider an asymptotic expansion of ϕ_j ,

$$\phi_j = \phi_j^0 + O(\epsilon). \quad (2.33)$$

Plugging this expansion into (2.30), at leading order we have

$$0 = \phi_j^{0''} - \phi_j^0 + q_j^{\alpha_{21}} \phi_j^{0\alpha_{22}}. \quad (2.34)$$

We can solve this exactly on $(s_{j-1} + 1, s_j - 1)$:

$$\phi_j^0(x) = \frac{\phi_0(x - p_j)}{q_j^{\alpha_{21}/(\alpha_{22}-1)}}, \quad (2.35)$$

where $\phi_0(x - p_j)$ is a pulse centered at $x = p_j$. We have that $\phi_j^0(x) \ll \epsilon^2$ well before x nears the boundaries of $(s_{j-1} + 1, s_j - 1)$ as long as the pulse separation is $p_j - p_{j-1} \gg O(|\ln \epsilon|)$. The set $\mathcal{K}_{l_0}$ where this condition is met is defined in (1.9).

Since ϕ_j is homoclinic to zero, we know that $\phi_j \rightarrow 0$ as $|x - p_j| \rightarrow \infty$. Once $\phi_j \ll \epsilon^2$, we can expand the exponential part of K about zero:

$$\epsilon K(\phi_j/\epsilon^2) = \sqrt{\phi_j} + O\left(\frac{\phi_j^{3/2}}{\epsilon^2}\right). \quad (2.36)$$

In this regime, (2.22b) will asymptotically scale to

$$0 = \phi_j'' - \sqrt{\phi_j}. \quad (2.37)$$

If we multiply by ϕ_j' and integrate, we have

$$\phi_j' = -\sqrt{\frac{4}{3}}\phi_j^{3/4}. \quad (2.38)$$

Separating variables, integrating, and simplifying, leads to

$$\phi_j(x) = 1/144 (x - c)^4. \quad (2.39)$$

The above is the tail behavior of ϕ_j , giving it compact support on the pulse interval

(s_{j-1}, s_j) . When $\phi_j(x) = O(\epsilon^2)$, $(x - c) = O(\epsilon^{1/2})$, where c can be determined to appropriately match function values. We define Φ_2 to be the sum of pulses that meets these asymptotic conditions:

$$\Phi_2 \equiv \sum_{j=1}^N \phi_j(x). \quad (2.40)$$

Then for any $\vec{p} \in \mathcal{K}_{l_0}$ and $\vec{q}(\vec{p}) \in \mathbb{R}^N$, Φ_2 is a steady state $\mathcal{F}(\Phi_2; \vec{p}, \delta = 0) = 0$.

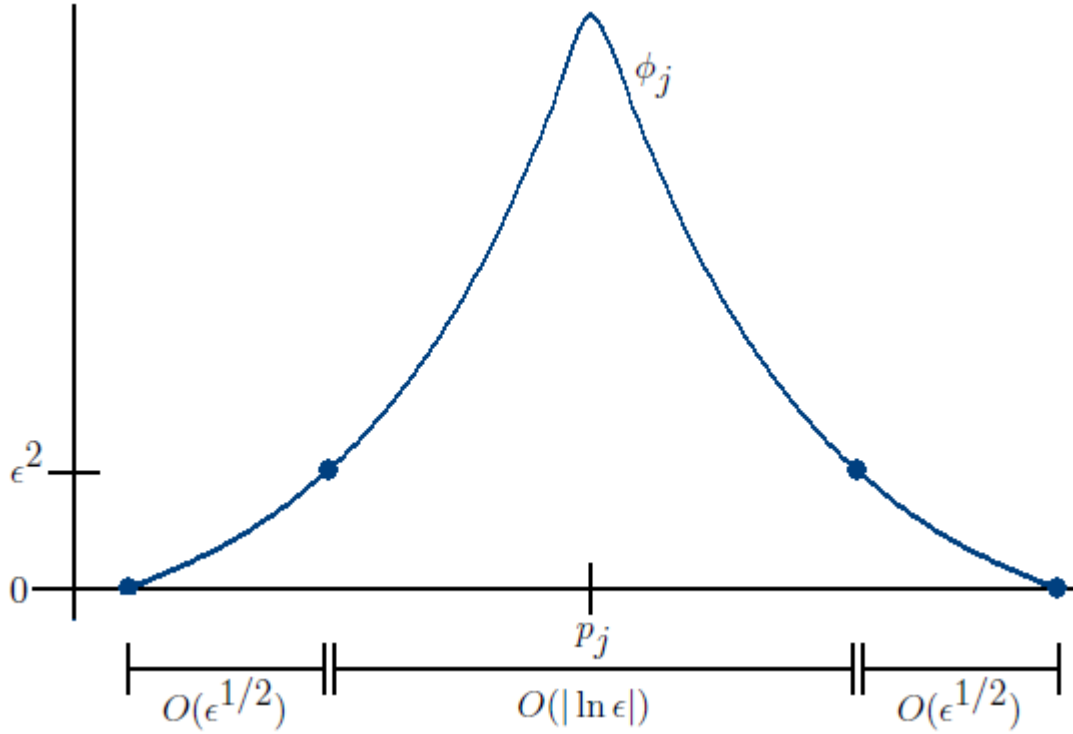


Figure 2.3: This is a cartoon of the pulse shape of Φ_2 at the pulse position p_j . Notice how it is localized.

2.2 Explicit form for Φ_1

To determine Φ_1 , we consider (2.22a) with our solution Φ_2 :

$$L_{11}^e \Phi_1 = \epsilon^{-\beta} \sum_{j=1}^N \chi_j q_j^{\alpha_{11}} \Phi_2^{\alpha_{12}} \quad (2.41)$$

$$= \epsilon^{-\beta} \sum_{j=1}^N q_j^{\alpha_{11}} \phi_j^{\alpha_{12}}(x), \quad (2.42)$$

where

$$L_{11}^e = -\epsilon^{-2} \partial_x^2 + \epsilon^\alpha \mu. \quad (2.43)$$

The inverse of L_{11}^e is denoted L_{11}^{-e} . We define Φ_1 as the solution to (2.42). In summary, we define Φ to be

$$\Phi(x, \vec{p}, \vec{q}) \equiv \begin{bmatrix} \Phi_1(x, \vec{p}, \vec{q}) \\ \Phi_2(x, \vec{p}, \vec{q}) \end{bmatrix} = \begin{bmatrix} \epsilon^{-\beta} L_{11}^{-e} \left(\sum_{j=1}^N q_j^{\alpha_{11}} \phi_j^{\alpha_{12}}(x) \right) \\ \sum_{j=1}^N \phi_j(x) \end{bmatrix}, \quad (2.44)$$

subject to (2.23).

2.3 Construction of the pulse amplitudes $\vec{q}(\vec{p})$

The condition $q_j = \Phi(p_j)$, requires $\vec{q}$ to satisfy the system

$$q_k = \epsilon^{-\beta} L_{11}^{-e} \left(\sum_{j=1}^N q_j^{\alpha_{11}} \phi_j^{\alpha_{12}(p_k)} \right), \quad (2.45)$$

for $k = 1, \dots, N$. The Green's function $G_\lambda(x)$ associated to $L_{11}^e + \lambda$ has the property that

$$(L_{11}^e + \lambda)^{-1} f = (G_\lambda * f)(x). \quad (2.46)$$

Using the Fourier transformation, G_λ is found to be

$$G_\lambda(x) \equiv \sqrt{\frac{\pi}{2}} \frac{\epsilon^2}{k_\lambda} e^{-k_\lambda |x|}, \quad (2.47)$$

where we introduce,

$$k_\lambda \equiv \epsilon \sqrt{\lambda + \epsilon^\alpha \mu}. \quad (2.48)$$

For $\lambda = 0$, we have

$$G_0(x) = \sqrt{\frac{\pi}{2\mu}} \epsilon^{1-\alpha/2} e^{-\epsilon^{1+\alpha/2} \sqrt{\mu} |x|}. \quad (2.49)$$

We proceed with an asymptotic reduction of (2.45). Applying (2.46) to (2.45) with $\lambda = 0$, we have

$$q_k = \epsilon^{-\beta} \left(G_0 * \sum_{j=1}^N q_j^{\alpha_{11}} \phi_j^{\alpha_{12}} \right) (p_k). \quad (2.50)$$

Writing this in vector form leads to:

$$\vec{q} = \epsilon^{1-\alpha/2-\beta} \mathcal{G} \vec{q}^{\alpha_{11}}, \quad (2.51)$$

with $\vec{q}^{\alpha_{11}} = (q_1^{\alpha_{11}}, \dots, q_N^{\alpha_{11}})^T$ and $\mathcal{G} \in \mathbb{R}^{N \times N}$, where

$$\mathcal{G}_{j,k} = \epsilon^{\alpha/2-1} \left(G_0 * \phi_j^{\alpha_{12}} \right) (p_k), \quad (2.52)$$

so $\mathcal{G}$ has no leading power in ϵ . We asymptotically expand $\mathcal{G}$ by substituting our asymptotic expansion (2.33) for ϕ_j . Here

$$\phi_j = \phi_j^0 + \epsilon \tilde{\phi}_j, \quad (2.53)$$

where ϕ_j has compact support and both ϕ_j^0 and $\tilde{\phi}_j$ decay exponentially. We define

$$\mathcal{G} = \mathcal{G}^0 + \epsilon \tilde{\mathcal{G}}, \quad (2.54)$$

where

$$\mathcal{G}_{j,k}^0 = \epsilon^{\alpha/2-1} \left(G_0 * \phi_j^{0\alpha_{12}} \right) (p_k), \quad (2.55)$$

and from a Taylor expansion and (2.35), we determine

$$\tilde{\mathcal{G}}_{j,k} = \alpha_{12} \epsilon^{\alpha/2-1} \left(G_0 * \left(\phi_j^0 \right)^{\alpha_{12}-1} \tilde{\phi}_j \right) (p_k) + O(\epsilon) \quad (2.56)$$

$$= \alpha_{12} \epsilon^{\alpha/2-1} q_j^{-\alpha_{21}(\alpha_{12}-1)/(\alpha_{22}-1)} \left(G_0 * \left(\phi_j^0 \right)^{\alpha_{12}-1} \tilde{\phi}_j \right) (p_k) + O(\epsilon). \quad (2.57)$$

The extra factors of q_j above are a result of replacing $\left(\phi_j^0 \right)^{\alpha_{12}-1}$ via (2.35). We write this matrix as:

$$\tilde{\mathcal{G}} = \tilde{\mathcal{G}}^{red} Q^{-\alpha_{21}(\alpha_{12}-1)/(\alpha_{22}-1)}, \quad (2.58)$$

with the $N \times N$ diagonal matrix $Q_{ij} = \left\{ q_i, \ i = j \middle| 0, \ i \neq j \right\}$ and

$$\tilde{\mathcal{G}}_{j,k}^{red} = \alpha_{12} \epsilon^{\alpha/2-1} \left(G_0 * \phi_0^{\alpha_{12}-1} \tilde{\phi}_j \right) (p_k). \quad (2.59)$$

Substituting $\mathcal{G} = \mathcal{G}^0$ into (2.51), we have, for any k ,

$$q_k = \epsilon^{-\beta} \left(G_0 * \sum_{j=1}^N q_j^{\alpha_{11}} \phi_j^{0\alpha_{12}} \right) (p_k) \quad (2.60)$$

$$= \sqrt{\frac{\pi}{2\mu}} \epsilon^{1-\alpha/2-\beta} \int e^{-\epsilon^{1+\alpha/2} \sqrt{\mu} |p_k - y|} \sum_{j=1}^N q_j^{\alpha_{11}} \phi_j^{0\alpha_{12}}(y) dy \quad (2.61)$$

$$= \sqrt{\frac{\pi}{2\mu}} \epsilon^{1-\alpha/2-\beta} \sum_{j=1}^N \int e^{-\epsilon^{1+\alpha/2} \sqrt{\mu} |p_k - y|} q_j^{\alpha_{11}} \phi_j^{0\alpha_{12}}(y) dy. \quad (2.62)$$

Next, we Taylor expand $e^{-\epsilon^{1+\alpha/2} \sqrt{\mu} |p_k - y|}$ about p_j :

$$e^{-\epsilon^{1+\frac{\alpha}{2}} \sqrt{\mu} |p_k - y|} = e^{-\epsilon^{1+\frac{\alpha}{2}} \sqrt{\mu} |p_k - p_j|} \quad (2.63)$$

$$- \epsilon^{1+\frac{\alpha}{2}} \sqrt{\mu} e^{-\epsilon^{1+\frac{\alpha}{2}} \sqrt{\mu} |p_k - p_j|} (y - p_j) + O(\epsilon^{2+\alpha}). \quad (2.64)$$

Substituting this into the integral in (2.62) and recalling (2.35), we have that:

$$\begin{aligned} \int e^{-\epsilon^{1+\alpha/2} \sqrt{\mu} |p_k - y|} q_j^{\alpha_{11}} \phi_j^{0\alpha_{12}}(y) dy &= \overline{\phi_0^{\alpha_{12}}} q_j^\theta e^{-\epsilon^{1+\alpha/2} \sqrt{\mu} |p_k - p_j|} \\ &\quad + h(\epsilon, \phi_0) q_j^\theta, \end{aligned} \quad (2.65)$$

where we define the mass $\bar{f} = \int_{-\infty}^{\infty} f(s) ds$. We also have

$$h(\epsilon, \phi_0) = O(\epsilon^{1+\alpha/2}), \quad (2.66)$$

since the decay from ϕ_j^0 dominates the polynomial terms from the Taylor expansion.

We define

$$\theta \equiv \alpha_{11} - \alpha_{12}\alpha_{21}/(\alpha_{22} - 1). \quad (2.67)$$

Then with the previous reductions, at leading order

$$\vec{q} = \epsilon^{1-\alpha/2-\beta} \mathcal{M}(\vec{p}, \epsilon) |\vec{q}|^\theta, \quad (2.68)$$

where $|\vec{q}|^\theta \equiv (|q_1|^\theta, \dots, |q_N|^\theta)^T$. The matrix $\mathcal{M}$ is defined as

$$\mathcal{M}_{j,k}(\vec{p}, \epsilon) \equiv \overline{\phi_0^{\alpha_{12}}} \epsilon^{\alpha/2-1} G_{ij}^N = \sqrt{\frac{\pi}{2\mu}} \overline{\phi_0^{\alpha_{12}}} e^{-\epsilon^{1+\alpha/2} \sqrt{\mu} |p_k - p_j|}, \quad (2.69)$$

where $G_0^N = G_0^N(\vec{p})$ is defined as the two-point correlation matrix:

$$[G_0^N] \equiv G_{ij}^N = \sqrt{\frac{\pi}{2\mu}} \epsilon^{1-\alpha/2} e^{-\epsilon^{1+\alpha/2} \sqrt{\mu} |p_i - p_j|}. \quad (2.70)$$

We include the absolute value in (2.68) because we want to show the existence of a nontrivial $\vec{q}$ with every component positive. If we return to our exact equation (2.51), and substitute (2.54) for $\mathcal{G}$, we have

$$\vec{q} = \epsilon^{1-\alpha/2-\beta} \mathcal{M}(\vec{p}, \epsilon) |\vec{q}|^\theta + \epsilon^{2-\alpha/2-\beta} \tilde{\mathcal{G}} |\vec{q}|^{\alpha_{11}} + O(\epsilon^{1+\alpha/2}) |\vec{q}|^\theta. \quad (2.71)$$

To remove any $\vec{q}$ dependence in $\tilde{\mathcal{G}}$, we recall (2.58), and rewrite the above as

$$\vec{q}(\vec{p}) = \epsilon^{1-\alpha/2-\beta} \mathcal{M}(\vec{p}, \epsilon) |\vec{q}|^\theta + \epsilon^{2-\alpha/2-\beta} \tilde{\mathcal{G}}^{red} |\vec{q}|^{\theta + \frac{\alpha_{21}}{\alpha_{22}-1}} + O(\epsilon^{1+\alpha/2}) |\vec{q}|^\theta. \quad (2.72)$$

To eliminate the $\epsilon^{1-\alpha/2-\beta}$ from the leading term above, we rescale the amplitude variable as

$$\tilde{q} = \epsilon^\kappa \vec{q}. \quad (2.73)$$

We have

$$\kappa \equiv \frac{1 - \alpha/2 - \beta}{\theta - 1}, \quad (2.74)$$

where (2.72) becomes:

$$\tilde{q} \equiv \mathcal{M}(\vec{p}, \epsilon) |\tilde{q}|^\theta + \epsilon^\rho \tilde{\mathcal{G}}^{red}(\vec{p}, \epsilon) |\tilde{q}|^{\theta + \frac{\alpha_{21}}{\alpha_{22}-1}} + O(\epsilon^{\alpha+\beta}) |\tilde{q}|^\theta, \quad (2.75)$$

with

$$\rho = 1 + \frac{\alpha_{21}}{\alpha_{22} - 1} \left(\frac{1 - \alpha/2 - \beta}{1 - \theta} \right). \quad (2.76)$$

We need the condition that

$$\rho > 0, \quad (2.77)$$

to analyze the appropriate leading order problem. We rescale the pulse position variable to remove any ϵ dependence from $\mathcal{M}(\vec{p}, \epsilon)$ (and also from $\tilde{G}^{red}$):

$$\tilde{p} = \epsilon^\eta \vec{p}, \quad (2.78)$$

with

$$\eta = 1 + \alpha/2. \quad (2.79)$$

With this rescaling, (2.75) becomes

$$\tilde{q} = A(\tilde{p}, \tilde{q}, \epsilon), \quad (2.80)$$

where

$$A(\tilde{p}, \tilde{q}, \epsilon) \equiv \mathcal{M}(\tilde{p})|\tilde{q}|^\theta + \epsilon^\rho \tilde{\mathcal{G}}^{red}(\tilde{p})|\tilde{q}|^{\theta + \frac{\alpha 21}{\alpha 22 - 1}} + O(\epsilon^{\alpha + \beta})|\tilde{q}|^\theta, \quad (2.81)$$

and, in the rescaled variable, $\mathcal{M}(\tilde{p})$ has the componentwise form

$$\mathcal{M}_{j,k}(\tilde{p}) \equiv \sqrt{\frac{\pi}{2\mu}} \phi_0^{\alpha 12} e^{-\sqrt{\mu}|\tilde{p}_k - \tilde{p}_j|}. \quad (2.82)$$

First we examine the $\epsilon = 0$ case. The system (2.45), under this rescaling, reduces to

$$\tilde{q} = A(\tilde{p}, \tilde{q}, 0) = \mathcal{M}(\tilde{p})|\tilde{q}|^\theta. \quad (2.83)$$

If $\theta = 0$, we have the nontrivial positive solution where for each i :

$$\tilde{q}_i = \sum_{j=1}^N M_{i,j}. \quad (2.84)$$

We consider the case when $\theta \neq 0$ and $\theta \neq 1$. Examining (2.82) and (2.83), it is clear that

$$A_i(\tilde{p}, \tilde{q}, 0) = \mathcal{M}(\tilde{p})|\tilde{q}|^\theta \geq 0 \quad (2.85)$$

for all $\tilde{q} \in \mathbb{R}^N$ and each $i \in (1, \dots, N)$. Indeed, from (2.82) we see that every entry of $\mathcal{M}$ is positive and uniformly bounded. This also implies that there exists $0 < k < K$, which depend upon $\tilde{p}$, such that

$$k\|\tilde{q}\|^\theta \leq \|A(\tilde{p}, \tilde{q}, 0)\| \leq K\|\tilde{q}\|^\theta \quad (2.86)$$

for some vector norm $\|\cdot\|$. We define

$$B(\tilde{p}, \tilde{q}, 0) \equiv \frac{A(\tilde{p}, \tilde{q}, 0)}{\|\tilde{q}\|^\theta}, \quad (2.87)$$

and

$$S \equiv \{\tilde{q} : k \leq \|\tilde{q}\| \leq K, \tilde{q}_i \geq 0, i \in (1, \dots, N)\}, \quad (2.88)$$

where $\tilde{q} = (\tilde{q}_1, \dots, \tilde{q}_N)^T$. S is compact and B maps S into itself. S is a contractible

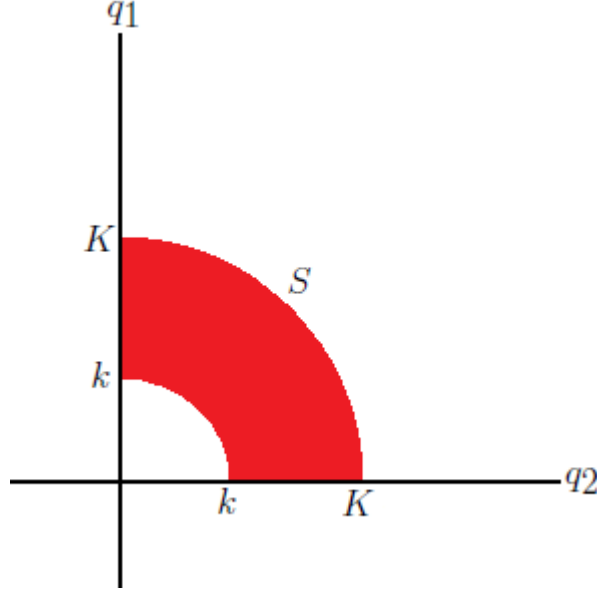


Figure 2.4: This is an illustration of S in two space dimensions.

manifold, where a contractible manifold is defined to be a manifold that can be continuously shrunk to any point inside itself. Moreover, B is continuous on S . Then by the Eilenberg-Montgomery fixed point theorem (which is a corollary to the Brouwer fixed point theorem), B has a fixed point $\tilde{q}_* \in S$.

We have that

$$A(\tilde{p}, \tilde{q}_*, 0) = \|\tilde{q}_*\|^\theta \tilde{q}_*, \quad (2.89)$$

and similarly for any constant s we have that

$$A(\tilde{p}, s\tilde{q}_*, 0) = s^\theta A(\tilde{p}, \tilde{q}_*, 0) = s^{\theta-1} \|\tilde{q}_*\|^\theta s\tilde{q}_*. \quad (2.90)$$

We choose

$$s \equiv \|\tilde{q}_*\|^{-\theta/(\theta-1)}, \quad (2.91)$$

so that A has the solution:

$$A(\tilde{p}, q_*, 0) = q_*, \quad (2.92)$$

where

$$q_* \equiv s\tilde{q}_*. \quad (2.93)$$

So there exists a nontrivial solution to (2.83) with every component positive. To show that solutions to (2.80) are locally unique for $\epsilon > 0$, we need the non-bifurcation condition (2.96) that gives generic local continuation. For $\vec{p} \in \mathcal{K}_\nu$, we have a locally unique solution to (2.80). Doing a continuation argument using the implicit function theorem, we define

$$\mathcal{B}(\tilde{p}, \tilde{q}, \epsilon) = \tilde{q} - A(\tilde{p}, \tilde{q}, \epsilon). \quad (2.94)$$

We have from (2.92) that $\mathcal{B}(\tilde{p}, q_*, 0) = 0$. Differentiating $\mathcal{B}(\tilde{p}, \tilde{q}, \epsilon)$ with respect to $\tilde{q}$,

evaluated at $(\tilde{q}, \epsilon) = (q_*, 0)$:

$$\nabla_{\tilde{q}} \mathcal{B}(\tilde{p}, q_*, 0) = I - \nabla_{\tilde{q}} A(\tilde{p}, q_*, 0) = I - \theta \mathcal{M}(\tilde{p}) |q_*(\tilde{p})|^{\theta-1}, \quad (2.95)$$

where $\mathcal{M}(\tilde{p})$ was defined at (2.82). Then the non-bifurcation condition is

$$\det \left(I - \theta \mathcal{M}(\tilde{p}) |q_*(\tilde{p})|^{\theta-1} \right) \neq 0. \quad (2.96)$$

The set $\mathcal{K}_\nu$ is the maximal open set of all pulse positions $\vec{p}$ beginning in the well-separated regime, and continuously extended into the semi-strong regime until values of $\vec{p}$ where (2.96) fails. By the implicit function theorem, for all $\vec{p} = \epsilon^{-\eta} \tilde{p} \in K_\nu$, there exists $\epsilon_0 > 0$ such that for $\epsilon_0 \geq \epsilon > 0$, we may extend:

$$\tilde{q}_* = \tilde{q}_*(\tilde{p}, \epsilon), \quad (2.97)$$

which solves (2.81) smoothly in $\tilde{p}$ and ϵ . Then in the original variables $\vec{p}$ and $\vec{q}$, (2.97) is equivalent to (2.25) in Theorem 2.1, listed again as

$$\vec{q} = \epsilon^{-\kappa} \mathcal{Q}(\epsilon^\eta \vec{p}, \epsilon), \quad (2.98)$$

where $\mathcal{Q}$ is equivalent to the continuous extension $\tilde{q}_*$ in (2.97).

2.4 Existence of an invariant manifold

We continue examining the case $\delta = 0$. From our positive solution q_* to (2.83), we construct $\tilde{\Phi}^*(\tilde{p})$. We have $\mathcal{F}(\tilde{\Phi}^*(\tilde{p}); \tilde{p}, \delta = 0) = 0$, so

$$\tilde{\pi}_T \mathcal{F}(\tilde{\Phi}^*(\tilde{p}); \tilde{p}, \delta = 0) = 0. \quad (2.99)$$

Thus we have an invariant manifold for the case $\delta = 0$. This case is even stronger in that we have a steady state solution, where the N -pulse positions are fixed.

For the case when $\delta \in (0, 1]$, we want to preserve invariance, so we need (2.1) to hold. By continuation, we will show a condition that will ensure

$$\tilde{\pi}_T(\Phi(\vec{p}, \delta), \vec{p}) (\mathcal{F}(\Phi(\vec{p}, \delta); \vec{p}, \delta)) = 0, \quad (2.100)$$

where $\tilde{\pi}_T(\Phi(\vec{p}, \delta), \vec{p})$ depends on $\Phi(\vec{p}, \delta)$ and $\vec{p}$. We differentiate the above with respect to δ :

$$\left(\nabla_{\Phi} \tilde{\pi}_T \frac{\partial \Phi}{\partial \delta} \right) (\mathcal{F}(\Phi(\vec{p}, \delta); \vec{p}, \delta)) + \tilde{\pi}_T(\Phi(\vec{p}, \delta), \vec{p}) \left(\mathcal{L}_{\vec{p}, \delta} \frac{\partial \Phi}{\partial \delta} + \frac{\partial \mathcal{F}}{\partial \delta}(\Phi(\vec{p}, \delta); \vec{p}, \delta) \right) = 0. \quad (2.101)$$

At $\delta = 0$, the first term is zero, so

$$\tilde{\pi}_T(\Phi(\vec{p}, 0), \vec{p}) \left(\mathcal{L}_{\vec{p}, 0} \frac{\partial \Phi(\vec{p}, 0)}{\partial \delta} \right) = -\tilde{\pi}_T(\Phi(\vec{p}, 0), \vec{p}) \left(\frac{\partial \mathcal{F}}{\partial \delta}(\Phi(\vec{p}, 0); \vec{p}, 0) \right). \quad (2.102)$$

This is equivalent to

$$\frac{\partial \Phi(\vec{p}, 0)}{\partial \delta} = - \left(\tilde{\pi}_T \mathcal{L}_{\vec{p}, 0} \tilde{\pi}_T \right)^{-1} \left(\frac{\partial \mathcal{F}}{\partial \delta} (\Phi(\vec{p}, 0); \vec{p}, 0) \right). \quad (2.103)$$

If (2.103) holds, we have the existence of an invariant manifold. A key step is to characterize the bounded invertibility of the conjugated operator $\tilde{\pi}_T \mathcal{L}_{\vec{p}, 0} \tilde{\pi}_T$.

Instead, we will continue our analysis of the case when $\delta = 1$ by using an approximate solution Φ . This will lead to a quasi-invariant manifold using a spectral projection defined after we analyze our eigenvalue problem.

2.5 Numerical Results

This section contains two numerical results. The first examines (2.22b) with the values $\Phi_1(p_j)^{\alpha_{21}} = 1$ and $\alpha_{22} = 2$. We further simplify the system by examining only one pulse position and replacing Φ_2 with Y . This choice of variables leads to:

$$Y_{xx} = \epsilon K(Y/\epsilon^2) + Y - Y^2. \quad (2.104)$$

Our boundary conditions are that $Y'(0) = 0$ and $Y(10) = 0$. We use the fact that $Y = \frac{3}{2}\text{sech}^2(Y/2)$ is a solution to this equation when the K term is not present to construct a solution to this equation with the K term present. Using the Matlab boundary value solver `bvp4c`, and solving for $x \in [0, 10]$, we have the following results in Figure 2.5 and Figure 2.6.

Our second numerical results demonstrates the behavior of the pulses and amplitudes in our system. Given pulse positions $\vec{p}$, we solve the nonlinear system (2.83) for the amplitudes $\vec{q}$. The system we are solving is $\vec{q} = \mathcal{M}(\vec{p})|\vec{q}|^\theta$, for $\mathcal{M}(\vec{p})$ defined in (2.69), and θ applied componentwise. Reducing the problem to three pulses, we take $\epsilon = 0.1$, $\alpha = 0$, and the coefficients of $\mathcal{M}$ to be 1. We solve this system beginning in the well-separated pulse regime, where $\mathcal{M}$ is almost diagonal, and continue the pulse positions closer together. As the Figure 2.7 illustrates, for $\theta = -1$, the pulse amplitudes converge as the pulse positions converge.

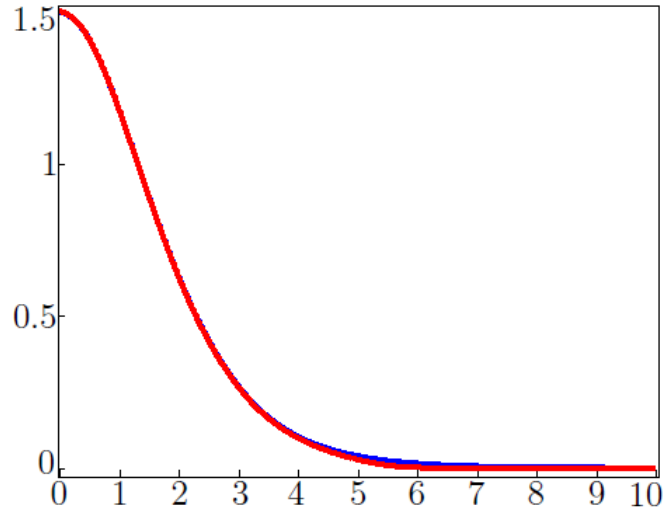


Figure 2.5: This figure is a graph of the solution to (2.104) without the K term in blue and the full solution to (2.104) in red. The solution to the full equation decays faster.

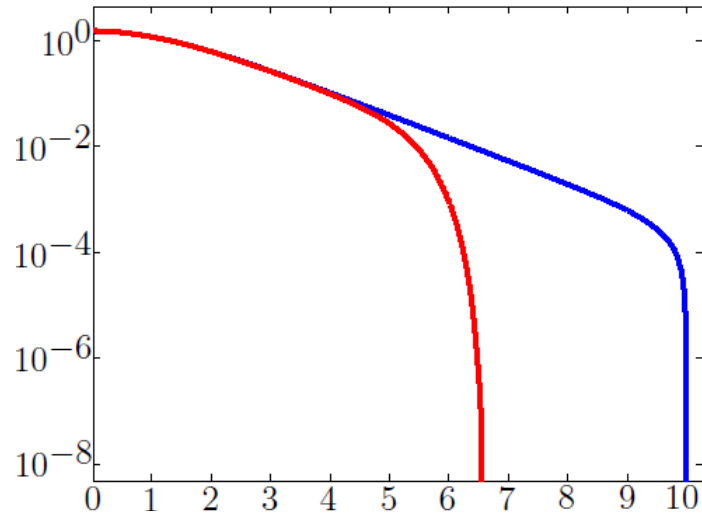


Figure 2.6: The fact that the solution to the full equation decays faster is more evident in this semi-log plot in the vertical coordinate. Again, the solution to (2.104) absent the K term is in blue and the solution to the full equation (2.104) is in red.

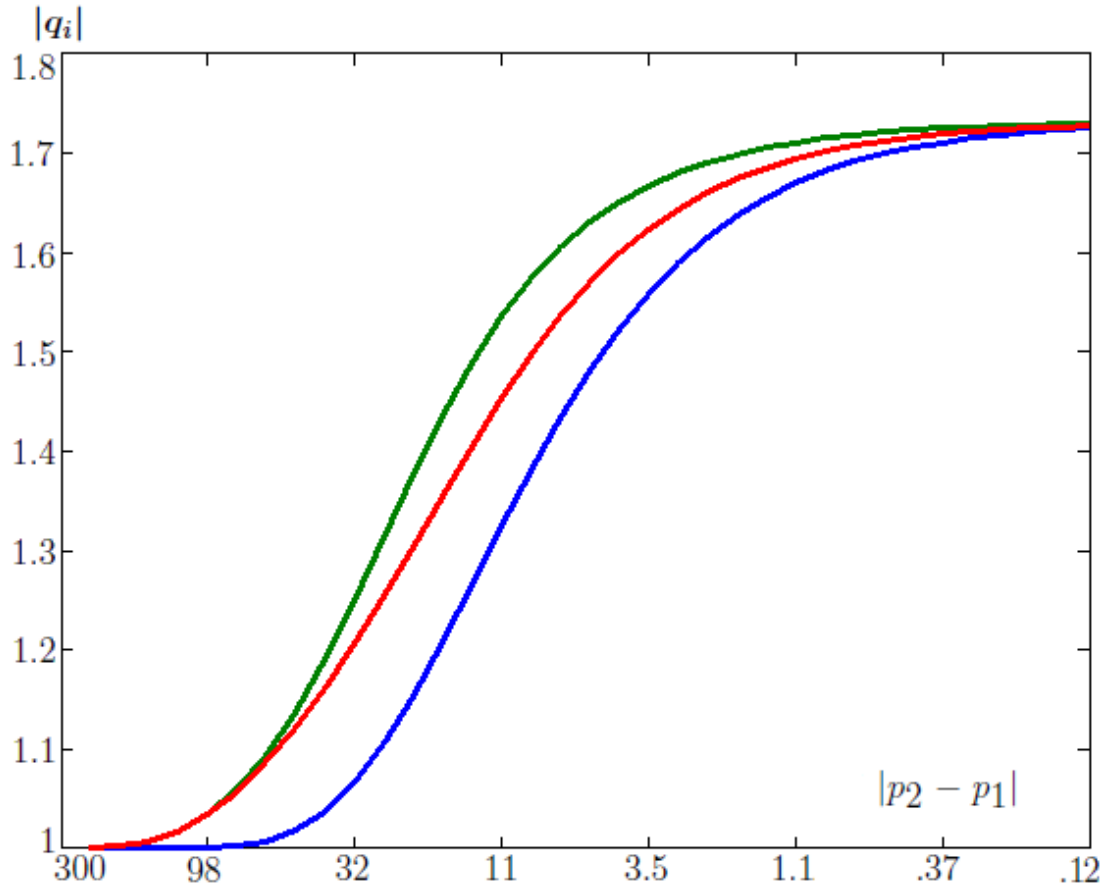


Figure 2.7: Each color is one component of the pulse amplitude, where pulse separation is varied which results in a change in amplitude. The horizontal axis is a log-scale of the separation between pulses, continued from a well-separated regime to a semi-strong regime. The values given are $|p_2 - p_1|$. The vertical axis is pulse amplitude.

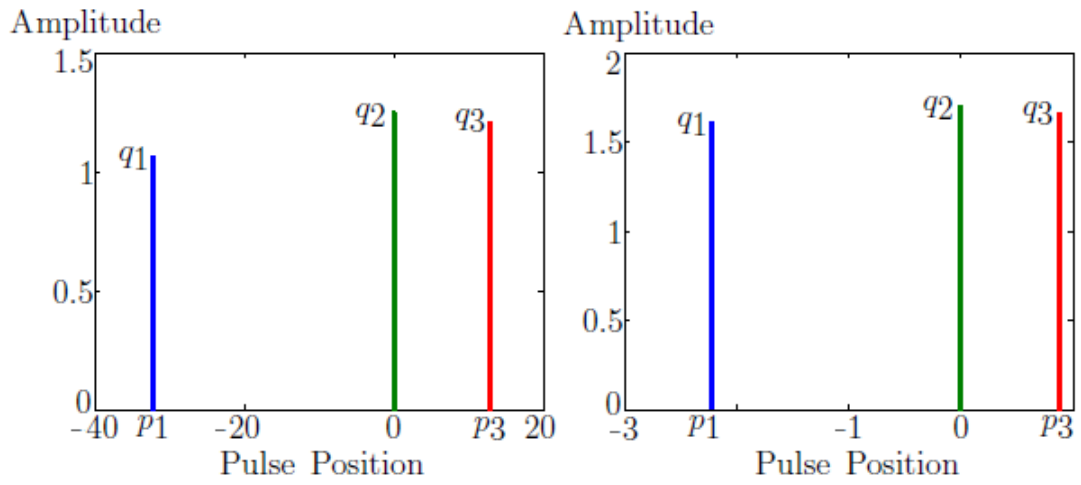


Figure 2.8: These two graphs illustrate the three pulses and their amplitudes for $|p_2 - p_1| = 33$ on the left and $|p_2 - p_1| = 1.6$ on the right, corresponding to the previous figure.

Chapter 3

Original System: Ansatz and Residual Estimates

We return to the system (1.1),

$$\partial_t U = \epsilon^{-2} U_{xx} - \epsilon^\alpha \mu U + \epsilon^{-\beta} U^{\alpha 11} V^{\alpha 12} \quad (3.1a)$$

$$\partial_t V = V_{xx} - V + U^{\alpha 21} V^{\alpha 22}. \quad (3.1b)$$

We show that the invariant manifold we constructed in Chapter 2 for the reduced system is a sufficiently accurate approximate invariant manifold for the original system to capture the leading order dynamics. We construct our ansatz as in the last chapter, where we solve (3.1b) for $V = \Phi_2$ at equilibrium, with $U = \Phi_1$ approxi-

mated by $q_j = \Phi_1(p_j)$ at each pulse position. We then solve (3.1a) at equilibrium for $\vec{q} = \Phi_1(\vec{p})$. This amounts to solving (2.22) with the $-\epsilon K(\Phi_2/\epsilon^2)$ term dropped.

With this slightly modified construction, we define

$$\Phi \equiv \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} \equiv \begin{bmatrix} \epsilon^{-\beta} L_{11}^{-e} \left(\sum_{j=1}^N q_j^{\alpha_{11}} \phi_j^{\alpha_{12}}(x) \right) \\ \sum_{j=1}^N \phi_j(x) \end{bmatrix}, \quad (3.2)$$

where ϕ_j solves

$$0 = \phi_j'' - \phi_j + q_j^{\alpha_{21}} \phi_j^{\alpha_{22}}. \quad (3.3)$$

It is also defined to be

$$\phi_j(x) \equiv \frac{\phi_0(x - p_j)}{q_j^{\alpha_{21}/(\alpha_{22}-1)}}, \quad (3.4)$$

where $\phi_0(x)$ solves

$$\phi_0'' - \phi_0 + \phi_0^{\alpha_{22}} = 0. \quad (3.5)$$

3.1 Linear estimates

The following lemma contains estimates used to develop resolvent estimates on the linearization, and thus to derive our semigroup estimates. The following estimates are used to obtain estimates on Φ_1 , which lead to residual estimates. Recall the definition of $L_{11}^e = -\epsilon^{-2} \partial_x^2 + \epsilon^\alpha \mu$ from (2.43), $k_\lambda \equiv \epsilon \sqrt{\lambda + \epsilon^\alpha \mu}$ defined in (2.48),

and that the χ_j 's form a partition of unity about the pulse positions $\vec{p}$. We have the following lemma.

Lemma 3.1. *There exists $C < \infty$ such that for any $f \in L^1(\mathbb{R})$ or $f \in W_\xi^{1,1}(\mathbb{R})$ and $\lambda \in \mathbb{C} \setminus (-\infty, -\epsilon^\alpha \mu)$, the following estimates hold:*

$$\|(L_{11}^e + \lambda)^{-1}f\|_{W_\xi^{1,1}} \leq C \frac{\epsilon^2}{\text{Re}(k_\lambda)} \|f\|_{L^1}, \quad (3.6)$$

$$\|(L_{11}^e + \lambda)^{-1}f\|_{W_\xi^{1,1}} \leq C \frac{\epsilon^2}{|k_\lambda| \text{Re}(k_\lambda)} \|f\|_{W_\xi^{1,1}}, \quad (3.7)$$

$$\|(L_{11}^e + \lambda)^{-1}f\|_{L^\infty} \leq C \frac{\epsilon^2}{|k_\lambda|} \|f\|_{L^1}, \quad (3.8)$$

$$\|\partial_x((L_{11}^e + \lambda)^{-1}f)\|_{L^\infty} \leq C \epsilon^2 \|f\|_{L^1}, \quad (3.9)$$

$$\|\partial_x((L_{11}^e + \lambda)^{-1}f)\|_{L^2} \leq C \frac{\epsilon^2}{\text{Re}(k_\lambda)^{1/2}} \|f\|_{L^1}, \quad (3.10)$$

$$\|\partial_x((L_{11}^e + \lambda)^{-1}f)\|_{L^1} \leq C \frac{\epsilon^2}{\text{Re}(k_\lambda)} \|f\|_{L^1}, \quad (3.11)$$

$$\|(L_{11}^e + \lambda)^{-1}f\|_{L^1} \leq C \frac{\epsilon^2}{|k_\lambda| \text{Re}(k_\lambda)} \|f\|_{L^1}, \quad (3.12)$$

$$\|(L_{11}^e + \lambda)^{-1}f\|_{L^2} \leq C \frac{\epsilon^2}{|k_\lambda| \text{Re}(k_\lambda)} \|f\|_{L^2}. \quad (3.13)$$

Moreover, for any $f \in L^1_{1,\vec{p}}(\mathbb{R})$, we have the following estimates

$$\|(L_{11}^e + \lambda)^{-1}f\|_{W_\xi^{1,1}} \leq C \frac{\epsilon^2}{\text{Re}(k_\lambda)} \left(|\otimes \vec{\chi} \cdot f| + |k_\lambda| \|f\|_{L^1_{1,\vec{p}}} \right), \quad (3.14)$$

$$\|(L_{11}^e + \lambda)^{-1}f\|_{L^\infty} \leq C \epsilon^2 \left(\frac{1}{|k_\lambda|} |\otimes \vec{\chi} \cdot f| + \|f\|_{L^1_{1,\vec{p}}} \right). \quad (3.15)$$

Proof: We define $g(x) \equiv (L_{11}^e + \lambda)^{-1}f = (G_\lambda * f)(x)$. Recall that $G_\lambda(x) = \sqrt{\frac{\pi}{2}} \frac{\epsilon^2}{k_\lambda} e^{-k_\lambda|x|}$. From the identity $g' = G'_\lambda * f$ and the L^p convolution estimates [25], we have

$$\|g'\|_{L^1} \leq \|G'_\lambda\|_{L^1} \|f\|_{L^1} \leq C \frac{\epsilon^2}{\text{Re}(k_\lambda)} \|f\|_{L^1}. \quad (3.16)$$

Similarly,

$$\|\xi g\|_{L^1} \leq \|\xi\|_{L^1} \|(G_\lambda * f)(x)\|_{L^\infty} \leq \|G_\lambda\|_{L^\infty} \|f\|_{L^1} \leq C \frac{\epsilon^2}{|k_\lambda|} \|f\|_{L^1}. \quad (3.17)$$

These two estimates establish (3.6). To prove (3.8) we observe that

$$\|g\|_{L^\infty} = \|(G_\lambda * f)(x)\|_{L^\infty} \leq \|G_\lambda\|_{L^\infty} \|f\|_{L^1} \leq C \frac{\epsilon^2}{|k_\lambda|} \|f\|_{L^1}. \quad (3.18)$$

We have (3.9), since

$$\|g'\|_{L^\infty} = \|(G'_\lambda * f)(x)\|_{L^\infty} \leq \|G'_\lambda\|_{L^\infty} \|f\|_{L^1} \leq C \epsilon^2 \|f\|_{L^1}. \quad (3.19)$$

In a similar manner we obtain (3.10), since

$$\|G'_\lambda\|_{L^2} = C \int \epsilon^4 e^{-2k_\lambda|x|} dx^{1/2} \leq C \frac{\epsilon^2}{Re(k_\lambda)^{1/2}}. \quad (3.20)$$

The next inequality (3.11) follows similarly from the estimate

$$\|G'_\lambda\|_{L^1} = C \int_{\mathbb{R}} \epsilon^2 e^{-k_\lambda|x|} dx \leq C \frac{\epsilon^2}{Re(k_\lambda)}. \quad (3.21)$$

For (3.12),

$$\|g\|_{L^1} = \|(G_\lambda * f)(x)\|_{L^1} \leq \|G_\lambda\|_{L^1} \|f\|_{L^1} \leq C \frac{\epsilon^2}{|k_\lambda| Re(k_\lambda)} \|f\|_{L^1}. \quad (3.22)$$

Similar to this, for (3.13),

$$\|g\|_{L^2} = \|(G_\lambda * f)(x)\|_{L^2} \leq \|G_\lambda\|_{L^1} \|f\|_{L^2} \leq C \frac{\epsilon^2}{|k_\lambda| Re(k_\lambda)} \|f\|_{L^2}. \quad (3.23)$$

Next we prove (3.7). For this case, we first observe that

$$\|\xi g\|_{L^1} \leq \|\xi\|_{L^1} \|(G_\lambda * f)(x)\|_{L^\infty} \quad (3.24)$$

$$\leq \|G_\lambda\|_{L^1} \|f\|_{L^\infty} \quad (3.25)$$

$$\leq C \frac{\epsilon^2}{|k_\lambda| Re(k_\lambda)} \|f\|_{W_\xi^{1,1}}. \quad (3.26)$$

We have our result if we combine the previous line with

$$\|g'\|_{L^1} = \|(G_\lambda * f')(x)\|_{L^1} \leq \|G_\lambda\|_{L^1} \|f'\|_{L^1} \leq C \frac{\epsilon^2}{|k_\lambda| \operatorname{Re}(k_\lambda)} \|f\|_{W_\xi^{1,1}}. \quad (3.27)$$

For the final two estimates, we decompose f as in (1.41), where $f = \sum_j f_j$ and $f_j = \chi_j f$, so that

$$g_j = G_\lambda * f_j \quad (3.28)$$

satisfies $g = \sum_j g_j$. Moreover using (1.29), we see that

$$\|f\|_{L_{1,\vec{p}}^1} = \sum_j \|f_j\|_{L_{1,j}^1}, \quad (3.29)$$

where $\|\cdot\|_{L_{1,j}^1} = \|(1 + |x - p_j|)\cdot\|_{L^1}$. We proceed by decomposing each f_j into a small mass and a massless part:

$$f_j = \bar{f}_j \xi_j + y'_j, \quad (3.30)$$

for $y_j \in L^1(\mathbb{R})$ and ξ_j defined in (1.37). Clearly for any f , $\|f\|_{L^1} \leq \|f\|_{L_{1,j}^1}$. Next

we examine

$$\|y_j\|_{L^1} = \int_{\mathbb{R}} \left[\partial_x(x - p_j) \right] |y_j| dx \quad (3.31)$$

$$\leq \int_{\mathbb{R}} |(x - p_j)y'_j| dx \quad (3.32)$$

$$= C \int_{\mathbb{R}} |(x - p_j) (f_j - \bar{f}_j \xi_j)| dx \quad (3.33)$$

$$\leq C \left(\|f_j\|_{L^1_{1,j}} + \|f\|_{L^1} \int_{\mathbb{R}} |(x - p_j) \xi_j| dx \right) \quad (3.34)$$

$$\leq C \|f_j\|_{L^1_{1,j}}. \quad (3.35)$$

We decompose $g_j = g_{j,1} + g_{j,0}$ where $g_{j,1} = \bar{f}_j G_\lambda * \xi_j$ and $g_{j,0} = G_\lambda * y'_j = G'_\lambda * y_j$.

Estimating $g_{j,1}$ using (3.6), we have

$$\|g_{j,1}\|_{W_\xi^{1,1}} = \bar{f}_j \|G_\lambda * \xi_j\|_{W_\xi^{1,1}} \leq C \frac{\epsilon^2}{\text{Re}(k_\lambda)} \bar{f}_j \|\xi_j\|_{L^1} \leq C \frac{\epsilon^2}{\text{Re}(k_\lambda)} \bar{f}_j. \quad (3.36)$$

The function G'_λ has a jump at $x = 0$. We deduce that

$$\partial_x g_{j,0} = 2G''_\lambda * y_j = \left[G''_\lambda \right] * y_j + \epsilon^2 y_j, \quad (3.37)$$

so that

$$G''_\lambda = \left\{ \begin{array}{ll} \left[G''_\lambda \right] & x \neq 0 \\ \epsilon^2 \delta_{x=0} & x = 0 \end{array} \right\}, \quad (3.38)$$

where the function $[G''_\lambda]$ is the point-wise second derivative of G_λ . Then

$$\|g'_{j,0}\|_{L^1} \leq \left(\| [G''_\lambda] \|_{L^1} + \epsilon^2 \right) \|y_j\|_{L^1} \leq C \frac{|k_\lambda|}{\text{Re}(k_\lambda)} \epsilon^2 \|f_j\|_{L^1_{1,j}}. \quad (3.39)$$

Using (3.9),

$$\|\xi g_{j,0}\|_{L^1} \leq \|\xi\|_{L^1} \|G'_\lambda * y_j\|_{L^\infty} \leq C \|G'_\lambda\|_{L^1} \|y_j\|_{L^1} \leq C \epsilon^2 \|f_j\|_{L^1_{1,j}}. \quad (3.40)$$

Summing over j , we have (3.14). The final inequality (3.15) follows using (3.9) and (3.8) respectively:

$$\|g_{j,0}\|_{L^\infty} = \|G'_\lambda * y_j\|_{L^\infty} \leq C \|G'_\lambda\|_{L^\infty} \|y_j\|_{L^1} \leq C \epsilon^2 \|f_j\|_{L^1_{1,j}} \quad (3.41)$$

$$\|\xi g_{j,0}\|_{L^\infty} = \bar{f}_j \|G_\lambda * \xi_j\|_{L^\infty} \leq C \bar{f}_j \|G_\lambda\|_{L^\infty} \|\xi_j\|_{L^1} \leq C \frac{\epsilon^2}{|k_\lambda|} \bar{f}_j \square \quad (3.42)$$

Recall from the introduction the assumption that $1 - \alpha/2 - \beta = 0$. We use this assumption and the previous lemma to prove the following bounds on $\Phi_1(\vec{p}, \delta = 1)$.

In particular, the lemma assures that $\Phi_1 = O(1)$ in L^∞ .

Lemma 3.2. *There exists a constant $C < \infty$ such that*

$$\|\Phi_1\|_{L^\infty} \leq C \quad (3.43)$$

$$\|\partial_x \Phi_1\|_{L^\infty} \leq C\epsilon^{2-\beta} \quad (3.44)$$

$$\|\Phi_1\|_{L^1} \leq C\epsilon^{\beta-2} \quad (3.45)$$

$$\left\| \frac{\partial q_j}{\partial p_k} \right\|_{L^\infty} \leq C\epsilon^{2-\beta} \quad (3.46)$$

$$\|\partial_{p_k} \Phi_1\|_{L^1} \leq C \quad (3.47)$$

$$\|\partial_{p_k} \Phi_1\|_{L^\infty} \leq C\epsilon^{2-\beta}. \quad (3.48)$$

Proof: For the first inequality we use (3.8) and the assumption $1 - \alpha/2 - \beta = 0$:

$$\|\Phi_1\|_{L^\infty} = \left\| \epsilon^{-\beta} L_{11}^{-e} \left(\sum_{j=1}^N q_j^{\alpha_{11}} \phi_j^{\alpha_{12}} \right) \right\|_{L^\infty} \quad (3.49)$$

$$\leq C \left\| \sum_{j=1}^N q_j^{\alpha_{11}} \phi_j^{\alpha_{12}} \right\|_{L^1} \quad (3.50)$$

$$\leq C. \quad (3.51)$$

For (3.44) we have

$$\|\partial_x \Phi_1\|_{L^\infty} \leq C\epsilon^{-\beta} \left\| \partial_x \left(L_{11}^{-e} \left(\sum_{j=1}^N q_j^{\alpha_{11}} \phi_j^{\alpha_{12}} \right) \right) \right\|_{L^\infty} \quad (3.52)$$

$$\leq C\epsilon^{2-\beta} \left\| \sum_{j=1}^N q_j^{\alpha_{11}} \phi_j^{\alpha_{12}} \right\|_{L^1} \quad (3.53)$$

$$\leq C\epsilon^{2-\beta}, \quad (3.54)$$

where we used (3.9). For (3.45), using (3.12) we have

$$\|\Phi_1\|_{L^1} \leq C\epsilon^{-\beta} \left\| \left(L_{11}^{-e} \left(\sum_{j=1}^N q_j^{\alpha_{11}} \phi_j^{\alpha_{12}} \right) \right) \right\|_{L^1} \quad (3.55)$$

$$\leq C\epsilon^{\beta-2} \left\| \sum_{j=1}^N q_j^{\alpha_{11}} \phi_j^{\alpha_{12}} \right\|_{L^1} \quad (3.56)$$

$$\leq C\epsilon^{\beta-2}. \quad (3.57)$$

For (3.46), we examine q_j from the exact formulation (2.72) at leading order,

$$\left\| \frac{\partial q_j}{\partial p_k} \right\|_{L^\infty} \leq C \left\| \frac{\partial \left(\mathcal{M}(\vec{p}, \epsilon) |\vec{q}|^\theta \right)_j}{\partial p_k} \right\|_{L^\infty} \quad (3.58)$$

$$\leq C \left\| \frac{\partial \mathcal{M}(\vec{p}, \epsilon)}{\partial p_k} \right\|_{L^\infty} \quad (3.59)$$

$$\leq C\epsilon^{2-\beta}, \quad (3.60)$$

due to the $\epsilon^{\alpha/2+1} = \epsilon^{2-\beta}$ in the exponent of $\mathcal{M}(\vec{p}, \epsilon)$ defined in (2.69). For (3.47), we have that

$$\partial_{p_k} \Phi_1 = \epsilon^{-\beta} L_{11}^{-e} \partial_{p_k} \left(\sum_{j=1}^N q_j^{\alpha_{11}} \phi_j^{\alpha_{12}} \right) \quad (3.61)$$

$$= \epsilon^{-\beta} L_{11}^{-e} \left(-q_k^{\alpha_{11}} \partial_x \left(\phi_k^{\alpha_{12}} \right) - \sum_{j=1}^N \theta q_j^{\theta-1} \phi_0^{\alpha_{12}} \frac{\partial q_j}{\partial p_k} \right). \quad (3.62)$$

Then applying the L^1 norm to the above

$$\|\partial_{p_j} \Phi_1\|_{L^1} \leq \epsilon^{-\beta} C \|G_\lambda * \partial_x \left(\phi_k^{\alpha_{12}} \right)\|_{L^1} + \epsilon^{-\beta} C \|L_{11}^{-e} \sum_{j=1}^N \theta q_j^{\theta-1} \phi_0^{\alpha_{12}} \frac{\partial q_j}{\partial p_k}\|_{L^1}. \quad (3.63)$$

For the first term on the right,

$$\epsilon^{-\beta} C \|G_\lambda * \partial_x \left(\phi_k^{\alpha_{12}} \right)\|_{L^1} = \|\partial_x \left(G_\lambda * \left(\epsilon^{-\beta} q_k^{\alpha_{11}} \phi_k^{\alpha_{12}} \right) \right)\|_{L^1} \quad (3.64)$$

$$\leq C \epsilon^\beta \|\epsilon^{-\beta} q_k^{\alpha_{11}} \phi_k^{\alpha_{12}}\|_{L^1} \quad (3.65)$$

$$\leq C, \quad (3.66)$$

where we applied the estimate (3.11). For the second term on the right in (3.63),

applying the estimate (3.12) and (3.46) we have

$$\epsilon^{-\beta} C \|L_{11}^{-e} \sum_{j=1}^N \theta q_j^{\theta-1} \phi_0^{\alpha_{12}} \frac{\partial q_j}{\partial p_k}\|_{L^1} \leq \epsilon^{-\beta} C \sum_{j=1}^N \|\theta q_j^{\theta-1} \phi_0^{\alpha_{12}} \frac{\partial q_j}{\partial p_k}\|_{L^1} \quad (3.67)$$

$$\leq \epsilon^{\beta-2} C \sum_{j=1}^N \|\phi_0^{\alpha_{12}}\|_{L^1} \|\frac{\partial q_j}{\partial p_k}\|_{L^\infty} \quad (3.68)$$

$$\leq C. \quad (3.69)$$

We conclude that

$$\|\partial_{p_k} \Phi_1\|_{L^1} \leq C. \quad (3.70)$$

Proving (3.48) follows similarly where

$$\begin{aligned} \|\partial_{p_j} \Phi_1\|_{L^\infty} &\leq \epsilon^{-\beta} C \|G_\lambda * \partial_x (\phi_k^{\alpha_{12}})\|_{L^\infty} \\ &\quad + \epsilon^{-\beta} C \|L_{11}^{-e} \sum_{j=1}^N \theta q_j^{\theta-1} \phi_0^{\alpha_{12}} \frac{\partial q_j}{\partial p_k}\|_{L^\infty}. \end{aligned} \quad (3.71)$$

The first term on the right follows like above, using (3.9),

$$\epsilon^{-\beta} C \|G_\lambda * \partial_x (\phi_k^{\alpha_{12}})\|_{L^\infty} = \|\partial_x (G_\lambda * (\epsilon^{-\beta} q_k^{\alpha_{11}} \phi_k^{\alpha_{12}}))\|_{L^\infty} \quad (3.72)$$

$$\leq C \epsilon^2 \|\epsilon^{-\beta} q_k^{\alpha_{11}} \phi_k^{\alpha_{12}}\|_{L^1} \quad (3.73)$$

$$\leq C \epsilon^{2-\beta}. \quad (3.74)$$

For the second term on the right in (3.71), applying the estimate (3.8) and (3.46) we have

$$\epsilon^{-\beta} C \|L_{11}^{-e} \sum_{j=1}^N \theta q_j^{\theta-1} \phi_0^{\alpha_{12}} \frac{\partial q_j}{\partial p_k}\|_{L^\infty} \leq C \sum_{j=1}^N \|\theta q_j^{\theta-1} \phi_0^{\alpha_{12}} \frac{\partial q_j}{\partial p_k}\|_{L^1} \quad (3.75)$$

$$\leq C \sum_{j=1}^N \|\phi_0^{\alpha_{12}}\|_{L^1} \|\frac{\partial q_j}{\partial p_k}\|_{L^\infty} \quad (3.76)$$

$$\leq C \epsilon^{2-\beta}. \quad (3.77)$$

Then we conclude

$$\|\partial_{p_k} \Phi_1\|_{L^\infty} \leq C \epsilon^{2-\beta} \square \quad (3.78)$$

From the proof of (3.47), we conclude that

$$\frac{\partial \Phi_2}{\partial p_j} = -\frac{\partial \phi_j}{\partial x} + O\left(\epsilon^{2-\beta}\right), \quad (3.79)$$

in any L^p norm. The following corollary will later be used to determine the point spectrum.

Corollary 3.1. *There exists $C > 0$ such that for all $\lambda \in \mathbb{C} \setminus (-\infty, -\epsilon^\alpha \mu)$ and $\vec{p} \in \mathcal{K}$, the following holds*

$$\left| ((L_{11}^e + \lambda)^{-1} f, g)_{L^2} - (\otimes \vec{\chi} \cdot f)^T G_\lambda^N \otimes \vec{\chi} \cdot g \right| \leq C \epsilon^2 \|f\|_{L_{1,\vec{p}}^1} \|g\|_{L_{1,\vec{p}}^1}, \quad (3.80)$$

for all $f, g \in L^1_{1, \vec{p}}$, where $L^1_{1, \vec{p}}$ is defined in (1.29) and G^N_0 (for $\lambda = 0$) is the two-point correlation matrix defined in (2.70).

Proof: To show (3.80) we use the Taylor expansion

$$G_\lambda(y - x) = G_\lambda(p_i - p_j + (y - p_i) - (x - p_j)) \quad (3.81)$$

$$= G_\lambda(p_i - p_j) + G'_\lambda(s)((y - p_i) - (x - p_j)), \quad (3.82)$$

for some $s \in \mathbb{R}$. Windowing f and g as in (1.41) and substituting the above we have:

$$\left((L^e_{11} + \lambda)^{-1} f, g \right)_{L^2} = \sum_{i,j=1}^N \left((L^e_{11} + \lambda)^{-1} f_i, g_j \right)_{L^2} \quad (3.83)$$

$$= \sum_{i,j=1}^N \int \int G_\lambda(y - x) f_i(y) g_j(x) dy dx \quad (3.84)$$

$$\begin{aligned} &= \sum_{i,j=1}^N \int \int G_\lambda(p_i - p_j) f_i(y) g_j(x) dy dx \\ &+ \sum_{i,j=1}^N \int \int G'_\lambda(s)((y - p_i) - (x - p_j)) f_i(y) g_j(x) dy dx \end{aligned} \quad (3.85)$$

$$\begin{aligned} &= (\otimes \vec{\chi} \cdot f)^T G^N_\lambda \otimes \vec{\chi} \cdot g \\ &+ G'_\lambda(s) \sum_{i,j=1}^N \int \int ((y - p_i) - (x - p_j)) f_i(y) g_j(x) dy dx. \end{aligned} \quad (3.86)$$

We rearrange the above terms and apply the absolute value:

$$\left| ((L_{11}^e + \lambda)^{-1} f, g)_{L^2} - (\otimes \vec{\chi} \cdot f)^T G_\lambda^N \otimes \vec{\chi} \cdot g \right| = |\Upsilon|, \quad (3.87)$$

where

$$\Upsilon = G'_\lambda(s) \sum_{i,j=1}^N \int \int ((y - p_i) - (x - p_j)) f_i(y) g_j(x) dy dx. \quad (3.88)$$

Using the fact that $\|G'_\lambda\|_{L^\infty} \leq C\epsilon^2$, we estimate the last term of (3.87),

$$|\Upsilon| = \left| G'_\lambda(s) \sum_{i,j=1}^N \int \int ((y - p_i) - (x - p_j)) f_i(y) g_j(x) dy dx \right| \quad (3.89)$$

$$\leq C\epsilon^2 \sum_{i,j=1}^N \int \int (|y - p_i| + |x - p_j|) |f_i(y) g_j(x)| dy dx \quad (3.90)$$

$$\leq C\epsilon^2 \sum_{i,j=1}^N \int |y - p_i| f_i(y) dy \int |g_j(x)| dx + \int |x - p_j| g_j(x) dx \int |f_i(y)| dy \quad (3.91)$$

$$\leq C\epsilon^2 \|f\|_{L_{1,\vec{p}}^1} \|g\|_{L_{1,\vec{p}}^1} \square \quad (3.92)$$

3.2 Residual estimates

The residual is $R(\Phi) = \mathcal{F}(\Phi)$, which takes the form

$$R(\Phi) = \begin{pmatrix} R_1(\Phi) \\ R_2(\Phi) \end{pmatrix} = \begin{pmatrix} \epsilon^{-2} \partial_x^2 \Phi_1 - \epsilon^\alpha \mu \Phi_1 + \epsilon^{-\beta} \Phi_1^{\alpha 11} \Phi_2^{\alpha 12} \\ \partial_x^2 \Phi_2 - \Phi_2 + \Phi_1^{\alpha 21} \Phi_2^{\alpha 22} \end{pmatrix}. \quad (3.93)$$

We have the following properties for the residual:

Proposition 3.1. *Recall the definition of K_{l_0} in (1.9). Fix l_0 from this definition, then for all $\vec{p} \in K_{l_0}$, the residual has the following asymptotic formula*

$$\begin{pmatrix} R_1(\Phi) \\ R_2(\Phi) \end{pmatrix} = \begin{pmatrix} \epsilon^{-\beta} \sum_{j=1}^N (\Phi_1^{\alpha 11} - q_j^{\alpha 11}) \phi_j^{\alpha 12} + O(\epsilon^r) \\ \sum_{j=1}^N (\Phi_1^{\alpha 21} - q_j^{\alpha 21}) \phi_j^{\alpha 22} + O(\epsilon^r) \end{pmatrix}, \quad (3.94)$$

for $r = r(l_0) > 0$ large. Moreover, there exists $C > 0$, independent of ϵ and $\vec{p} \in \mathcal{K}_{l_0}$ such that for all $\vec{p} \in \mathcal{K}_{l_0}$ the following estimates hold,

$$\|R_1(\Phi)\|_{L^1} \leq C \epsilon^\alpha \quad (3.95)$$

$$\|R_2(\Phi)\|_{L^2} \leq C \epsilon^{2-\beta}. \quad (3.96)$$

Proof: We first examine $R_2(\Phi)$ in the L^2 norm. Adding and subtracting like

terms, we find

$$\begin{aligned}
\|R_2(\Phi)\|_{L^2} &\leq \left\| \sum_{j=1}^N \partial_x^2 \phi_j - \phi_j + q_j^{\alpha_{21}} \phi_j^{\alpha_{22}} \right\|_{L^2} \\
&+ \left\| \Phi_1^{\alpha_{21}} \left(\left(\sum_{j=1}^N \phi_j \right)^{\alpha_{22}} - \sum_{j=1}^N \phi_j^{\alpha_{22}} \right) \right\|_{L^2} \\
&+ \left\| \sum_{j=1}^N \left(\Phi_1^{\alpha_{21}} - q_j^{\alpha_{21}} \right) \phi_j^{\alpha_{22}} \right\|_{L^2}. \tag{3.97}
\end{aligned}$$

The first term above is zero by the definition of ϕ_j from (3.3). Next, using (3.43):

$$\left\| \Phi_1^{\alpha_{21}} \left(\left(\sum_{j=1}^N \phi_j \right)^{\alpha_{22}} - \sum_{j=1}^N \phi_j^{\alpha_{22}} \right) \right\|_{L^2} \leq C\epsilon^r \|\Phi_1^{\alpha_{21}}\|_{L^\infty} \tag{3.98}$$

$$\leq C\epsilon^r, \tag{3.99}$$

since

$$\left\| \left(\sum_{j=1}^N \phi_j \right)^{\alpha_{22}} - \sum_{j=1}^N \phi_j^{\alpha_{22}} \right\|_{L^2} \leq C\epsilon^r, \tag{3.100}$$

for $r \geq 2$, which follows from the fact that $\vec{p} \in \mathcal{K}_{l_0}$. In this space, the pulses are sufficiently separated so that the tail-tail interaction between ϕ_j and ϕ_k for $j \neq k$ is minimal. Finally, for the third term we Taylor expand $\Phi_1^{\alpha_{21}}$ under the sum at

$x = p_j$ for each j and use (3.44):

$$\left\| \sum_{j=1}^N (\Phi_1^{\alpha_{21}} - q_j^{\alpha_{21}}) \phi_j^{\alpha_{22}} \right\|_{L^2} \leq \left\| \sum_{j=1}^N \partial_x \left(\Phi_1^{\alpha_{21}} \right) (s_j) |x - p_j| \phi_j^{\alpha_{22}} \right\|_{L^2} \quad (3.101)$$

$$\leq \sum_{j=1}^N \left\| \partial_x (\Phi_1^{\alpha_{21}}) (s_j) \right\| \left\| |x - p_j| \phi_j^{\alpha_{22}} \right\|_{L^2} \quad (3.102)$$

$$\leq C \sum_{j=1}^N \left\| \partial_x (\Phi_1^{\alpha_{21}}) \right\|_{L^\infty} \quad (3.103)$$

$$\leq C \epsilon^{2-\beta}. \quad (3.104)$$

For the above, we used (3.43) and (3.44), where $s_j \in \mathbb{R}$ for each j , and the exponential decay in $\phi_j^{\alpha_{22}}$ dominates the linear growth of $|x - p_j|$. From (3.97), (3.99), and (3.104), we conclude that

$$\|R_2(\Phi)\|_{L^2} \leq C \epsilon^{2-\beta}, \quad (3.105)$$

which establishes (3.96). Next, we examine the L^1 norm of R_1 . From (3.94), we find

$$\begin{aligned} \|R_1(\Phi)\|_{L^1} \leq & \left\| -L_{11}^e \Phi_1 + \epsilon^{-\beta} \sum_{j=1}^N q_j^{\alpha_{11}} \phi_j^{\alpha_{12}} \right\|_{L^1} \\ & + \epsilon^{-\beta} \left\| \sum_{j=1}^N (\Phi_1^{\alpha_{11}} - q_j^{\alpha_{11}}) \phi_j^{\alpha_{12}} \right\|_{L^1} \\ & + \epsilon^{-\beta} \left\| \Phi_1^{\alpha_{11}} \left(\left(\sum_{j=1}^N \phi_j \right)^{\alpha_{12}} - \sum_{j=1}^N \phi_j^{\alpha_{12}} \right) \right\|_{L^1}. \end{aligned} \quad (3.106)$$

The first normed term above is zero by the definition of Φ_1 from (3.2). We estimate the second and the third terms as we did for Φ_2 . For the third term, we have that

$$\left\| \left(\sum_{j=1}^N \phi_j \right)^{\alpha_{12}} - \sum_{j=1}^N \phi_j^{\alpha_{12}} \right\|_{L^\infty} \leq C\epsilon^r, \quad (3.107)$$

for $\vec{p} \in \mathcal{K}_{l_0}$, so

$$\epsilon^{-\beta} \left\| \Phi_1^{\alpha_{11}} \left(\left(\sum_{j=1}^N \phi_j \right)^{\alpha_{12}} - \sum_{j=1}^N \phi_j^{\alpha_{12}} \right) \right\|_{L^1} \leq \epsilon^{r-\beta} \left\| \Phi_1^{\alpha_{11}} \right\|_{L^1} \quad (3.108)$$

$$\leq C\epsilon^{r-2}, \quad (3.109)$$

where we used (3.45). As before, we Taylor expand Φ_1 in the second term,

$$\left\| \sum_{j=1}^N (\Phi_1^{\alpha_{11}} - q_j^{\alpha_{11}}) \phi_j^{\alpha_{12}} \right\|_{L^1} \leq \left\| \sum_{j=1}^N \partial_x(\Phi_1^{\alpha_{11}})(s_j) |x - p_j| \phi_j^{\alpha_{12}} \right\|_{L^1} \quad (3.110)$$

$$\leq C \sum_{j=1}^N \left\| \partial_x(\Phi_1^{\alpha_{11}})(s_j) \right\| \left\| |x - p_j| \phi_j^{\alpha_{12}} \right\|_{L^1} \quad (3.111)$$

$$\leq C \sum_{j=1}^N \left\| \partial_x(\Phi_1^{\alpha_{11}}) \right\|_{L^\infty} \quad (3.112)$$

$$\leq C\epsilon^{2-\beta}, \quad (3.113)$$

with $s_j \in \mathbb{R}$ for each j and again we used (3.43) and (3.44). We deduce that

$$\epsilon^{-\beta} \left\| \sum_{j=1}^N (\Phi_1^{\alpha_{11}} - q_j^{\alpha_{11}}) \phi_j^{\alpha_{12}} \right\|_{L^1} \leq C \epsilon^{2-2\beta} = C \epsilon^\alpha. \quad (3.114)$$

Together (3.106), (3.109), and (3.113) yield

$$\|R_1(\Phi)\|_{L^1} \leq C \epsilon^\alpha. \quad (3.115)$$

which establishes (3.95). The asymptotic formula (3.94) for the residual follows by identifying the leading order terms $\square$

Chapter 4

Linearized Equation and Spectrum

We decompose solutions of (1.1) as

$$\begin{pmatrix} U \\ V \end{pmatrix} = \Phi_{\vec{p}} + W^*(x, t), \quad (4.1)$$

where $W^* = W + \vec{\Phi}_1$ and the pulse positions are functions of time $\vec{p} = \vec{p}(t)$. We are in a sense putting a correction term into our ansatz through the term W^* . We choose $\vec{\Phi}_1 = \left(\Phi_{1,1}, \Phi_{1,2}\right)^T$ such that

$$\tilde{L}_{\vec{p}_0} \vec{\Phi}_1 \equiv -\tilde{\pi}_{\vec{p}_0} R(\Phi), \quad (4.2)$$

so

$$\vec{\Phi}_1 \equiv -\tilde{L}_{\vec{p}_0}^{-1} \tilde{\pi}_{\vec{p}_0} R(\Phi). \quad (4.3)$$

$\tilde{L}$ is defined in (4.9) as the reduced linear operator frozen at the point $\vec{p}_0$, and $\tilde{\pi}$ is an orthogonal spectral projection defined in (5.13). $W = (W_1, W_2)^T$ is the remainder.

Inserting the decomposition (4.1) into the system (1.1) yields

$$W_t + \left(\frac{\partial \Phi}{\partial \vec{p}} + \frac{\partial \vec{\Phi}_1}{\partial \vec{p}} \right) \dot{\vec{p}} = R(\Phi) + L_{\vec{p}} \vec{\Phi}_1 + L_{\vec{p}} W + \mathcal{N}(\vec{\Phi}_1, W), \quad (4.4)$$

where the residual $R(\Phi)$ was defined in (3.93) and the linearized operator $L_{\vec{p}}$ is defined as

$$L_{\vec{p}} \equiv \begin{pmatrix} -L_{11}^e + \epsilon^{-\beta} \alpha_{11} \Phi_1^{\alpha_{11}-1} \Phi_2^{\alpha_{12}} & \epsilon^{-\beta} \alpha_{12} \Phi_1^{\alpha_{11}} \Phi_2^{\alpha_{12}-1} \\ \alpha_{21} \Phi_1^{\alpha_{21}-1} \Phi_2^{\alpha_{22}} & \partial_x^2 - 1 + \alpha_{22} \Phi_1^{\alpha_{21}} \Phi_2^{\alpha_{22}-1} \end{pmatrix}. \quad (4.5)$$

Up to constants, the nonlinearity is

$$\mathcal{N}(\vec{\Phi}_1, W) \equiv \begin{pmatrix} \mathcal{N}_1(\vec{\Phi}_1, W) \\ \mathcal{N}_2(\vec{\Phi}_1, W) \end{pmatrix}, \quad (4.6)$$

where

$$\begin{aligned}\mathcal{N}_1(\vec{\Phi}_1, W) = & \epsilon^{-\beta} \Phi_1^{\alpha_{11}-1} \Phi_2^{\alpha_{12}-1} W_1^* W_2^* + \Phi_1^{\alpha_{11}} \Phi_2^{\alpha_{12}-2} W_2^{*2} \\ & + \epsilon^{-\beta} \Phi_1^{\alpha_{11}-2} \Phi_2^{\alpha_{12}} W_1^{*2}\end{aligned}\quad (4.7)$$

$$\begin{aligned}\mathcal{N}_2(\vec{\Phi}_1, W) = & \Phi_1^{\alpha_{21}-1} \Phi_2^{\alpha_{22}-1} W_1^* W_2^* + \Phi_1^{\alpha_{21}} \Phi_2^{\alpha_{22}-2} W_2^{*2} \\ & + \Phi_1^{\alpha_{21}-2} \Phi_2^{\alpha_{22}} W_1^{*2},\end{aligned}\quad (4.8)$$

$$\text{with } \begin{pmatrix} W_1^* \\ W_2^* \end{pmatrix} = \begin{pmatrix} \Phi_{1,1} + W_1 \\ \Phi_{1,2} + W_2 \end{pmatrix}.$$

4.1 The reduced linearization

To simplify the study of the spectral problem we introduce the reduced linearization to be:

$$\tilde{L}_{\vec{p}} \equiv \begin{pmatrix} \tilde{L}_{11} & \epsilon^{-\beta} J_{12} \\ J_{21} & \tilde{L}_{22} \end{pmatrix} = \begin{pmatrix} -L_{11}^e & 0 \\ J_{21} & \tilde{L}_{22} \end{pmatrix} + \epsilon^{-\beta} \begin{pmatrix} J_{11} & J_{12} \\ 0 & 0 \end{pmatrix}, \quad (4.9)$$

where

$$\tilde{L}_{22} = \partial_x^2 - I + \alpha_{22} \sum_{j=1}^N \phi_0^{\alpha_{22}-1} (x - p_j). \quad (4.10)$$

We define the J_{21} component as:

$$J_{21} = \alpha_{21} \sum_{j=1}^N q_j^{\alpha_{21}-1} \phi_j^{\alpha_{22}}. \quad (4.11)$$

The potentials J_{11} and J_{12} are finite rank projections

$$J_{11} = \alpha_{11} \vec{\xi}^T \otimes (\Phi_2^{\alpha_{12}} Q^{\alpha_{11}-1} \vec{\chi}) \quad (4.12)$$

$$J_{12} = \alpha_{12} \vec{\xi}^T \otimes (\Phi_2^{\alpha_{12}-1} Q^{\alpha_{11}} \vec{\chi}), \quad (4.13)$$

where Q is the $N \times N$ diagonal matrix $Q_{jj} = q_j$ for each j . From Weyl's theorem on the essential spectra of compact perturbations of operators, we know that the essential spectrum of $L_{\vec{p}}$ and $\tilde{L}_{\vec{p}}$ coincide with that of L_{11}^e :

$$\sigma_{ess}(L_{\vec{p}}) = \sigma_{ess}(\tilde{L}_{\vec{p}}) = B = \left\{ -\epsilon^{-2} k^2 - \epsilon^\alpha \mu | k \in \mathbb{R} \right\}. \quad (4.14)$$

The difference $L_{\vec{p}} - \tilde{L}_{\vec{p}}$ is large, but it will enjoy the enhanced resolvent estimate (5.17), since the difference $L_{\vec{p}} - \tilde{L}_{\vec{p}}$ has no mass in each window χ_j of the partition of unity.

4.2 The point spectrum

Proposition 4.1. *The spectrum of $\tilde{L}$ can be broken into three parts: an essential part B , a part from the point spectrum of $\tilde{L}_{22}$, and a part controlled by the finite rank perturbations:*

$$\sigma(\tilde{L}_{\vec{p}}) \subset \left(B \cup \sigma_p(\tilde{L}_{22}) \cup \{ \lambda \mid \det(I + N_\lambda(\vec{p})) = 0 \} \right). \quad (4.15)$$

The $N \times N$ matrix N_λ is given by (4.25).

Proof: The following eigenvalue problem defines the point spectrum:

$$(\tilde{L}_{\vec{p}} - \lambda) \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (4.16)$$

This expands to

$$-(L_{11}^e + \lambda)\Psi_1 = -\epsilon^{-\beta}(J_{11}\Psi_1 + J_{12}\Psi_2) \quad (4.17)$$

$$(\tilde{L}_{22} - \lambda)\Psi_2 = -J_{21}\Psi_1. \quad (4.18)$$

Now if $\lambda \notin \sigma_p(\tilde{L}_{22}) \cup B$, then we can invert $\tilde{L}_{22} - \lambda$ in the second equation

$$\Psi_2 = -(\tilde{L}_{22} - \lambda)^{-1} J_{21} \Psi_1. \quad (4.19)$$

Substituting this into the first equation and inverting $L_{11}^e + \lambda$, we arrive at the following scalar problem:

$$\Psi_1 = \epsilon^{-\beta} (L_{11}^e + \lambda)^{-1} \left(J_{11} - J_{12} (\tilde{L}_{22} - \lambda)^{-1} J_{21} \right) \Psi_1. \quad (4.20)$$

Recalling (4.12)-(4.13) we regroup the right hand side into a single, finite rank operator,

$$\Psi_1 = \epsilon^{-\beta} (L_{11}^e + \lambda)^{-1} J_l^T \otimes J_r \cdot \Psi_1, \quad (4.21)$$

where the left and right components of the tensor product are

$$J_l = \vec{\xi} \quad (4.22)$$

$$J_r = (\alpha_{11} \Phi_2^{\alpha 12} Q^{\alpha 11} - \alpha_{12} \Phi_2^{\alpha 12} (\tilde{L}_{22} - \lambda)^{-1} J_{21} Q^{\alpha 11}) \vec{\chi}. \quad (4.23)$$

We project (4.21) with $\otimes J_r$

$$\otimes J_r \cdot \Psi_1 = \epsilon^{-\beta} \otimes J_r \cdot (L_{11}^e + \lambda)^{-1} J_l^T \otimes J_r \cdot \Psi_1, \quad (4.24)$$

and introduce the matrix

$$N_\lambda(\vec{p}) = -\epsilon^{-\beta} \otimes J_r \cdot (L_{11}^e + \lambda)^{-1} J_l^T, \quad (4.25)$$

so that the eigenvalue problem reduces to

$$(I + N_\lambda) \otimes J_r \cdot \Psi_1 = 0. \quad (4.26)$$

If $I + N_\lambda$ is invertible, then $\otimes J_r \cdot \Psi_1 = 0$ which from (4.21) implies $\Psi_1 = 0$, and from (4.21), we see that $\Psi_2 = 0$.

Conversely if $(I + N_\lambda)\vec{v} = 0$, then setting $\otimes J_r \cdot \Psi_1 = \vec{v}$ in (4.21) yields

$$\Psi_1 = \epsilon^{-\beta} (L_{11}^e + \lambda)^{-1} J_l^T \vec{v}. \quad (4.27)$$

Also Ψ_2 from (4.19) yields an eigenvector $\vec{\Psi} = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$ for the eigenvalue problem (4.16) for $\lambda \notin \sigma_p(\tilde{L}_{22}) \cup B$.

Hence $\lambda \in \mathbb{C} \setminus (B \cup \sigma_p(\tilde{L}_{22}))$ is an eigenvalue of $\tilde{L}$ if and only if $I + N_\lambda$ is invertible $\square$

Proposition 4.2. *Fix $l_0 > 0$. There exists $\nu > 0$ such that for all $\vec{p} \in \mathcal{K}_{l_0}$,*

$$\{Re(\lambda) > -\nu\} \cap \sigma_p(\tilde{L}_{\vec{p}}) = \sigma_0(\vec{p}) \cup \sigma_{fr}(\vec{p}), \quad (4.28)$$

where $\sigma_0(\vec{p})$ consists of N distinct $O(\epsilon^r)$ eigenvalues which are in $\sigma_p(\tilde{L}_{22})$. The set σ_{fr} is induced by the finite rank perturbations, and corresponds, up to multiplicity,

to the zeros of the N equations

$$\mathcal{R}(\lambda; \vec{p}) = \frac{1}{\alpha_{12}\alpha_{21}} \left(\alpha_{11} \overline{\phi_0^{\alpha_{12}}} - \frac{1}{\mu_j} \right), \quad (4.29)$$

for all j from 1 to N , where $\mathcal{R}$ is an explicitly known meromorphic function on $\mathbb{C} \setminus (-\infty, -1]$ given by (4.43). The μ_j 's are the N eigenvalues of the square matrix $\epsilon^{-\beta} Q^{\theta-1} G_{\lambda}^N(\vec{p})$. Moreover the eigenspace associated to σ_0 is contained, up to $O(\epsilon^r)$, within the space

$$\mathcal{V} = \text{span} \left\{ \begin{pmatrix} 0 \\ \phi'_1 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \phi'_N \end{pmatrix} \right\}. \quad (4.30)$$

Proof: We define the following reduced self-adjoint operator

$$\tilde{L}_{k,red} = \partial_x^2 - 1 + \alpha_{22} \phi_0^{\alpha_{22}-1} (x - p_k). \quad (4.31)$$

For the pulse separation l_0 sufficiently large, we can interpret $\tilde{L}_{22}$ as N spatially disjoint (windowed) operators. We analyze the point spectrum of $\tilde{L}_{k,red}$. We observe that $\phi'_0(x - p_k)$ is an eigenfunction of $\tilde{L}_{k,red}$ corresponding to the eigenvalue $\lambda_1 = 0$. This follows from the fact that ϕ_k solves (2.34). We can also apply the Sturm-Liouville Theory to this operator. This operator has real point spectrum, and we can order the eigenvalues. Since $\phi'_0(x - p_k)$ only has one zero, there is only one

positive eigenvalue, λ_0 , the groundstate whose eigenfunction ψ_0 has no zeros. This eigenvalue will also be $O(1)$, since (4.31) does not contain an ϵ . Similarly, all negative eigenvalues will be an $O(1)$ distance from $\lambda_1 = 0$, so there exists some $\nu > 0$ such that there are only two eigenvalues of $\tilde{L}_{k,red}$ where $\lambda_j > -\nu$, specifically for $j = 0$ or $j = 1$. The spectrum of each reduced operator satisfies

$$\sigma\left(\tilde{L}_{k,red}\right) \subset \{\lambda_1 = 0, \lambda_0\} \cup (-\infty, -\nu]. \quad (4.32)$$

Now we seek to determine our representation for σ_{f_r} . We simplify the inversion of $(\tilde{L}_{22} - \lambda)$ on J_{21} . We write J_{21} as N well-separated pulses, each localized about the pulse positions, so the inversion simplifies to

$$\begin{aligned} (\tilde{L}_{22} - \lambda)^{-1} J_{21} = & \alpha_{21} \sum_{k=1}^N q_k^{\alpha_{21}-1 - \frac{\alpha_{21}\alpha_{22}}{\alpha_{22}-1}} (\tilde{L}_{k,red} - \lambda)^{-1} \phi_0^{\alpha_{22}}(x - p_k) \\ & + O(\epsilon^r). \end{aligned} \quad (4.33)$$

We introduce

$$\Xi_0(x - p_k) \equiv (\tilde{L}_{k,red} - \lambda)^{-1} \phi_0^{\alpha_{22}}(x - p_k) \quad (4.34)$$

$$= (d_x^2 - 1 + \alpha_{22} \phi_0^{\alpha_{22}-1}(x - p_k) - \lambda)^{-1} \phi_0^{\alpha_{22}}(x - p_k), \quad (4.35)$$

so that

$$(\tilde{L}_{22} - \lambda)^{-1} J_{21} = \alpha_{21} \sum_{k=1}^N q_k^{\alpha_{21}-1-\alpha_{21}\alpha_{22}/(\alpha_{22}-1)} \Xi_0(x-p_k) + O(\epsilon^r). \quad (4.36)$$

Neglecting the near-neighbor interactions between each localized term, and using our definition of θ from (2.67), we may write J_r , defined in (4.23), as

$$J_r = (\alpha_{11}\Phi_2^{\alpha_{12}}Q^{\alpha_{11}-1} - \alpha_{12}\Phi_2^{\alpha_{12}-1}(\tilde{L}_{22} - \lambda)^{-1}J_{21}Q^{\alpha_{11}})\vec{\chi} \quad (4.37)$$

$$= \sum_{k=1}^N q_k^{\theta-1} \chi_k \left(\alpha_{11}\phi_0^{\alpha_{12}}(x-p_k) - \alpha_{12}\alpha_{21}\phi_0^{\alpha_{12}-1}(x-p_k)\Xi_0(x-p_k) \right). \quad (4.38)$$

The functions $\Xi_0(x-p_k)$ decay at an $O(1)$ rate depending on the distance of λ to $\sigma_{ess}(\tilde{L}_{22})$. Since $\vec{p} \in \mathcal{K}_{l_0}$, there exists a minimal pulse separation l_0 so that the products $\phi_0^{\alpha_{12}-1}(x-p_j)\Xi_0(x-p_k)$ and $\chi_j\Xi_0(x-p_k)$ are uniformly $O(\epsilon^r)$ for $r \geq 2$ when $j \neq k$. From (4.25) the (i, j) entry of the matrix N_λ is

$$N_{i,j} = -\epsilon^{-\beta} \left(J_{ri}, (L_{11}^e + \lambda)^{-1} J_{lj} \right)_{L^2} \quad (4.39)$$

$$\begin{aligned} &= -\epsilon^{-\beta} q_i^{\theta-1} \left(\alpha_{11}\phi_0^{\alpha_{12}}(x-p_i), (L_{11}^e + \lambda)^{-1} \xi_j \right)_{L^2} \\ &+ \epsilon^{-\beta} q_i^{\theta-1} \left(\alpha_{12}\alpha_{21}\phi_0^{\alpha_{12}-1}(x-p_i)\Xi_0(x-p_i), (L_{11}^e + \lambda)^{-1} \xi_j \right)_{L^2} \end{aligned} \quad (4.40)$$

Applying (3.80), we have

$$\begin{aligned}
N_{i,j} &= \epsilon^{-\beta} q_i^{\theta-1} \otimes \vec{\chi} \cdot \left(\alpha_{12} \alpha_{21} \phi_0^{\alpha_{12}-1} (x - p_i) \Xi_0(x - p_i) \right) G_{ij}^N \otimes \vec{\chi} \cdot \xi_j \\
&\quad - \epsilon^{-\beta} q_i^{\theta-1} \otimes \vec{\chi} \cdot \left(\alpha_{11} \phi_0^{\alpha_{12}} (x - p_i) \right) G_{ij}^N \otimes \vec{\chi} \cdot \xi_j + O(\epsilon^{2-\beta}) \quad (4.41)
\end{aligned}$$

$$= -\epsilon^{-\beta} q_i^{\theta-1} \left(\alpha_{11} \overline{\phi_0^{\alpha_{12}}} - \alpha_{12} \alpha_{21} \mathcal{R}(\lambda) \right) G_{ij}^N + O(\epsilon^{2-\beta}), \quad (4.42)$$

where we define

$$\mathcal{R}(\lambda) \equiv \left(\Xi_0, \phi_0^{\alpha_{12}-1} \right)_{L^2}. \quad (4.43)$$

We may represent N_λ as

$$N_\lambda = -\epsilon^{-\beta} \left(\alpha_{11} \overline{\phi_0^{\alpha_{12}}} - \alpha_{12} \alpha_{21} \mathcal{R}(\lambda) \right) Q^{\theta-1} G_\lambda^N + O(\epsilon^{2-\beta}). \quad (4.44)$$

The condition that $I + N_\lambda$ has a kernel is exactly (4.29).

Next we address the point spectrum of $\tilde{L}_{22}$. We treat this as a regular eigenvalue perturbation problem. The point spectrum of $\tilde{L}_{22}$ consists of clusters of N eigenvalues an $O(\epsilon^r)$ distance from $\lambda_1 = 0$ and λ_0 , and also negative point spectrum left of $-\nu$. We label the N eigenvalues near λ_0 as $\lambda_{0,k}$ for $k \in \{1, \dots, N\}$.

Claim 4.1. *For every k , $\lambda_{0,k}$ is not an eigenvalue for our full eigenvalue problem (4.17) and (4.18).*

Proof of claim: We have, up to $O(\epsilon^{2-\beta})$, that

$$\sum_{j=1}^N a_j \phi'_0(x - p_j) \in \ker(\tilde{L}_{22}) \quad (4.45)$$

$$\sum_{j=1}^N b_{j,k} \psi_0(x - p_j) \in \ker(\tilde{L}_{22} - \lambda_{0,k}). \quad (4.46)$$

We can apply Sturm-Liouville to order our eigenvalues. We arbitrarily let $\lambda_{0,1}$ be the ground state, so the corresponding eigenfunction has no zeros. Without loss of generality, we have that $b_{j,1} > 0$ for all j . For $\lambda_{0,2}$, without loss of generality $b_{j,2} > 0$ for every j except for $b_{N,2}$ which is negative, since the eigenfunction has exactly one zero. Due to the linear independence of each $\vec{b}_i = (b_{1,i}, \dots, b_{N,i})^T$ for $i = 1, \dots, N$, this argument follows so that if we arrange each $\vec{b}_i$ as a column of the matrix $\mathfrak{B}$, the resulting matrix is nonsingular,

$$\mathfrak{B} = (\vec{b}_1, \dots, \vec{b}_N) \in \mathbb{R}^N. \quad (4.47)$$

Consider the possibility that $\lambda = \lambda_{0,k}$ for every $k = 1, \dots, N$ is an eigenvalue for

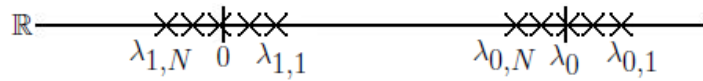


Figure 4.1: This illustrates the point spectrum of $\tilde{L}_{22}$ that is either positive or near zero. There are N eigenvalues within $O(\epsilon^r)$ of both $\lambda_1 = 0$ and λ_0 .

the eigenvalue problem (4.16). From (4.18) we have

$$(\tilde{L}_{22} - \lambda_{0,k})\Psi_2 = -J_{21}\Psi_1. \quad (4.48)$$

If we solve for Ψ_1 in (4.17) where J_{11} and J_{12} are finite rank, we find Ψ_1 is in the span of $(L_{11}^e + \lambda_{0,k})^{-1}\xi_i$ for $i = 1, \dots, N$. Thus, it is slowly varying in space. By the Fredholm Alternative, $-J_{21}\Psi_1$ must be orthogonal to everything contained in $\ker(\tilde{L}_{22} - \lambda_{0,k})$. Then

$$O(\epsilon^r) = \left\langle \sum_{j=1}^N b_{j,k} \psi_0(x - p_j), -J_{21}\Psi_1 \right\rangle, \quad (4.49)$$

for each k . This is equivalent to

$$\mathfrak{B}\vec{v} = O(\epsilon^r), \quad (4.50)$$

where for each j ,

$$v_j = \langle \psi_0(x - p_j), -J_{21}\Psi_1 \rangle. \quad (4.51)$$

$\mathfrak{B}$ is nonsingular, so for each j ,

$$O(\epsilon^T) = v_j = \langle \psi_0(x - p_j), -J_{21}\Psi_1 \rangle \quad (4.52)$$

$$= \langle \psi_0(x - p_j), -\alpha_{21} \sum_{k=1}^N \Psi_1 q_k^{\alpha_{21}-1} \phi_k^{\alpha_{22}} \rangle \quad (4.53)$$

$$= \langle \psi_0(x - p_j), -\alpha_{21} \Psi_1 q_j^{\alpha_{21}-1} \phi_j^{\alpha_{22}} \rangle \quad (4.54)$$

$$\approx -\alpha_{21} \Psi_1(p_j) q_j^{\alpha_{21}-1-\frac{\alpha_{21}\alpha_{22}}{\alpha_{22}-1}} \langle \psi_0(x - p_j), \phi_0^{\alpha_{22}}(x - p_j) \rangle \quad (4.55)$$

$$= O(1), \quad (4.56)$$

since Ψ_1 is slowly varying, q_j and $\Psi_1(p_j)$ cannot be zero, and $\psi_0(x - p_j)$ and $\phi_0^{\alpha_{22}}(x - p_j)$ have no zeros. Then we have a contradiction, so $\lambda_{0,k}$ cannot be an eigenvalue for (4.16) for any k $\square$

With the claim proven, we continue proving the proposition. On the other hand, when $N = 1$, $\begin{pmatrix} 0 \\ \phi'_0 \end{pmatrix}$ and $\lambda_1 = 0$ are an eigenfunction-eigenvalue pair for our system, since $\Psi_2 = \phi'_0$ satisfies (4.18) for $\Psi_1 = 0$. Also (4.17) is satisfied since

$J_{12}\phi'_0 = 0$. We show this below:

$$J_{12}\phi'_0 = \alpha_{12}\xi < (\phi_0^{\alpha_{12}-1} q^{\alpha_{11}} \chi), \phi'_0 > \quad (4.57)$$

$$= \alpha_{12}\xi q^{\alpha_{11}-\alpha_{21}(\alpha_{12}-1)/(\alpha_{22}-1)} < \phi_0^{\alpha_{12}-1}, \phi'_0(x - p_k) > \quad (4.58)$$

$$= 0. \quad (4.59)$$

For $N > 1$, the eigenvalue $\lambda_1 = 0$ breaks into N eigenvalues of size $O(\epsilon^T)$ for sufficiently large pulse separations. At leading order, the eigenspace associated to σ_0 is $\mathcal{V} = \ker(\tilde{L}_{22})$. This completes the proof of Proposition 4.2 $\square$

4.3 Finite rank spectrum

We have characterized the finite rank spectrum σ_{fr} in terms of the matrix $N_\lambda(\vec{p})$ defined in (4.25). The set σ_{fr} is the spectrum that moves as the pulse positions evolve. In order to control the evolution of this finite rank spectrum, we need $\vec{p} \in K_\nu$ to assure that σ_{fr} is bounded in the left-half complex plane away from the origin. From Proposition 4.1, σ_{fr} occurs only for λ such that $(I + N_\lambda)$ is singular. For all $\vec{p} \in \mathcal{K}_\nu$, we have $\sigma_{fr} \subset \mathcal{C}^*$, where $\mathcal{C}^*$ is appropriately contained in

$$\mathcal{C}^* \subset \mathcal{C}, \quad (4.60)$$

so that the matrix $(I + N_\lambda)$ is invertible in a neighborhood of the contour $\mathcal{C}$. Moreover as $|\lambda| \rightarrow \infty$ along $\mathcal{C}$, $|(I + N_\lambda)|^{-1} \rightarrow 0$. So by the continuity of $(I + N_\lambda)$, for all $\vec{p} \in \mathcal{K}_\nu$, there exists $C > 0$ such that we have the uniform bound,

$$|(I + N_\lambda)^{-1}| \leq C, \quad (4.61)$$

for all $\lambda \in \mathcal{C}$ and all λ to the left of $\mathcal{C}$.

4.4 Adjoint eigenfunction estimates

In this section we develop asymptotic expansions of the eigenfunctions $\{\Psi_k\}_{k=1}^N$ of $\tilde{L}$ that correspond to the algebraically small eigenvalues, and also the adjoint eigenfunctions $\{\Psi_k^\dagger\}_{k=1}^N$ that correspond to $\tilde{L}^\dagger$.

Lemma 4.1. *For $\vec{p} \in \mathcal{K}$, where $\mathcal{K}$ is defined in (1.13), the eigenspace corresponding to the algebraically small eigenvalues σ_0 is spanned by*

$$\Psi_k = \begin{pmatrix} 0 \\ \phi'_k \end{pmatrix} + O(\epsilon^r), \quad (4.62)$$

for $k \in \{1, \dots, N\}$. The space of adjoint eigenfunctions is spanned by the set $\{(\Psi_{1,k}^\dagger, \Psi_{2,k}^\dagger)^T\}_{k=1}^N$, given by (4.80) and (4.70) which satisfy the following esti-

mate:

$$\|\Psi_{1,k}^\dagger\|_{W_\xi^{1,1}} + \epsilon^\beta \|\Psi_{2,k}^\dagger - \phi'_k\|_{H^1} \leq C\epsilon^2, \quad (4.63)$$

for some $C > 0$ independent of ϵ and $\vec{p} \in \mathcal{K}$.

Proof: The previous proposition implies (4.62). The adjoint operator is given by:

$$\tilde{L}^\dagger = \begin{pmatrix} -L_{11}^e & J_{21} \\ 0 & \tilde{L}_{22} \end{pmatrix} + \epsilon^{-\beta} \begin{pmatrix} J_{11}^\dagger & 0 \\ J_{12}^\dagger & 0 \end{pmatrix}, \quad (4.64)$$

where

$$J_{11}^\dagger = \alpha_{11} \vec{\chi}^T \Phi_2^{\alpha_{12}} Q^{\alpha_{11}-1} \otimes \vec{\xi}, \quad (4.65)$$

and

$$J_{12}^\dagger = \alpha_{12} \vec{\chi}^T \Phi_2^{\alpha_{12}-1} Q^{\alpha_{11}} \otimes \vec{\xi}. \quad (4.66)$$

The eigenvalue problem for $\vec{\Psi}^\dagger = (\Psi_1^\dagger, \Psi_2^\dagger)^T$ is

$$(\tilde{L}^\dagger - \lambda) \begin{pmatrix} \Psi_1^\dagger \\ \Psi_2^\dagger \end{pmatrix} = 0. \quad (4.67)$$

Since we consider the small eigenvalues near zero we may neglect $\lambda = O(\epsilon^r)$. We

have the following two equations:

$$-L_{11}^e \Psi_1^\dagger = -J_{21} \Psi_2^\dagger - \epsilon^{-\beta} J_{11}^\dagger \Psi_1^\dagger \quad (4.68)$$

$$\tilde{L}_{22} \Psi_2^\dagger = -\epsilon^{-\beta} J_{12}^\dagger \Psi_1^\dagger. \quad (4.69)$$

Since $\phi'_k \in \ker \tilde{L}_{22}$, we form a basis $\{(\Psi_{1,k}^\dagger, \Psi_{2,k}^\dagger)^T\}_{k=1}^N$ of solutions to $\tilde{L}^\dagger \vec{\Psi}^\dagger$ where the second component is

$$\Psi_{2,k}^\dagger = \phi'_k - \epsilon^{-\beta} \tilde{L}_{22}^{-1} J_{12}^\dagger \Psi_{1,k}^\dagger. \quad (4.70)$$

Using the form of $\tilde{L}_{22}$ and that ϕ_k solves (3.3), we reduce to the following

$$\tilde{L}_{22}^{-1} \phi_k = \frac{x - p_k}{2} \phi'_k + \frac{1}{\alpha_{22} - 1} \phi_k + O(\epsilon^r), \quad (4.71)$$

which follows since $\tilde{L}_{22} \phi'_k = 0$ and $\vec{p} \in \mathcal{K}$. This is used to demonstrate the uniform boundedness of (4.100). Substituting (4.70) into (4.68), we have

$$\Psi_{1,k}^\dagger = L_{11}^{-e} [J_{21}(\phi'_k - \epsilon^{-\beta} \tilde{L}_{22}^{-1} J_{12}^\dagger \Psi_{1,k}^\dagger) + \epsilon^{-\beta} J_{11}^\dagger \Psi_{1,k}^\dagger] \quad (4.72)$$

$$= L_{11}^{-e} [J_{21} \phi'_k + \epsilon^{-\beta} J_r^{\dagger T} \otimes J_l^\dagger \cdot \Psi_{1,k}^\dagger], \quad (4.73)$$

where $J_l^\dagger = J_l = \vec{\xi}$ and from (4.23),

$$J_r^\dagger = \left(\alpha_{11} \Phi_2^{\alpha_{12}} Q^{\alpha_{11}-1} - \alpha_{12} J_{21} \tilde{L}_{22}^{-1} \Phi_2^{\alpha_{12}-1} Q^{\alpha_{11}} \right) \vec{\chi}. \quad (4.74)$$

Now we project (4.73) with $\otimes J_l^\dagger$, so

$$\otimes J_l^\dagger \cdot \Psi_{1,k}^\dagger = \otimes J_l^\dagger \cdot L_{11}^{-e} [J_{21} \phi'_k + \epsilon^{-\beta} J_r^{\dagger T} \otimes J_l^\dagger \cdot \Psi_{1,k}^\dagger]. \quad (4.75)$$

If we rearrange terms, we have

$$(I - \epsilon^{-\beta} \otimes J_l^\dagger \cdot L_{11}^{-e} J_r^{\dagger T}) \otimes J_l^\dagger \cdot \Psi_{1,k}^\dagger = \otimes J_l^\dagger \cdot L_{11}^{-e} J_{21} \phi'_k. \quad (4.76)$$

So

$$\otimes J_l^\dagger \cdot \Psi_{1,k}^\dagger = (I + N_\lambda^\dagger)^{-1} \otimes J_l^\dagger \cdot L_{11}^{-e} J_{21} \phi'_k, \quad (4.77)$$

where

$$N_\lambda^\dagger = -\epsilon^{-\beta} \otimes J_l^\dagger \cdot L_{11}^{-e} J_r^{\dagger T}. \quad (4.78)$$

Next, we plug (4.77) into (4.73) and factor to obtain

$$\Psi_{1,k}^\dagger = L_{11}^{-e} [J_{21} \phi'_k + \epsilon^{-\beta} J_r^{\dagger T} (I + N_\lambda^\dagger)^{-1} \otimes J_l^\dagger \cdot L_{11}^{-e} J_{21} \phi'_k] \quad (4.79)$$

$$= [I + \epsilon^{-\beta} L_{11}^{-e} J_r^{\dagger T} (I + N_\lambda^\dagger)^{-1} \otimes J_l^\dagger \cdot] L_{11}^{-e} J_{21} \phi'_k. \quad (4.80)$$

Using the estimate (3.14), we have

$$\|L_{11}^{-e} J_{21} \phi'_k\|_{W_\xi^{1,1}} \leq c \left(\epsilon^\beta |\otimes \vec{\chi} \cdot J_{21} \phi'_k| + \epsilon^2 \|J_{21} \phi'_k\|_{L_{1,\vec{p}}^1} \right), \quad (4.81)$$

where

$$J_{21} \phi'_k = \alpha_{21} q_k^{\alpha_{21}-1} \phi_k^{\alpha_{22}} \phi'_k + O(\epsilon^r). \quad (4.82)$$

Due to even-odd parity, this has algebraically small mass so

$$|\otimes \vec{\chi} \cdot J_{21} \phi'_k| = O(\epsilon^r). \quad (4.83)$$

In addition, $\|J_{21} \phi'_k\|_{L_{1,\vec{p}}^1} = O(1)$, since $J_{21} \phi'_k$ is exponentially decaying away from the pulse positions. Then

$$\|L_{11}^{-e} J_{21} \phi'_k\|_{W_\xi^{1,1}} \leq C \epsilon^2. \quad (4.84)$$

Also for (4.80), we use (3.6) and have

$$\|\epsilon^{-\beta} L_{11}^{-e} J_r^{\dagger T}\|_{W_\xi^{1,1}} \leq C \|J_r^{\dagger T}\|_{L^1} \quad (4.85)$$

$$\leq C, \quad (4.86)$$

since $J_r^\dagger$ is uniformly bounded in L^1 . Taking the $W_\xi^{1,1}$ norm of (4.80), we have

$$\|\Psi_{1,k}^\dagger\|_{W_\xi^{1,1}} = \|[I + \epsilon^{-\beta} L_{11}^{-e} J_r^{\dagger T} (I + N_\lambda^\dagger)^{-1} \otimes J_l^\dagger] L_{11}^{-e} J_{21} \phi'_k\|_{W_\xi^{1,1}} \quad (4.87)$$

$$\leq C\epsilon^2 + \|\epsilon^{-\beta} L_{11}^{-e} J_r^{\dagger T}\|_{W_\xi^{1,1}} \|(I + N_\lambda^\dagger)^{-1} \otimes J_l^\dagger \cdot L_{11}^{-e} J_{21} \phi'_k\|_{L^\infty} \quad (4.88)$$

$$\leq C\epsilon^2 + \|(I + N_\lambda^\dagger)^{-1}\|_{L^\infty} \left\| \int \xi L_{11}^{-e} J_{21} \phi'_k dx \right\| \quad (4.89)$$

$$\leq C\epsilon^2 (1 + \|(I + N_\lambda^\dagger)^{-1}\|_{L^\infty}), \quad (4.90)$$

since

$$\left| \int \xi_j L_{11}^{-e} J_{21} \phi'_k dx \right| \leq \|L_{11}^{-e} J_{21} \phi'_k\|_{W_\xi^{1,1}} \leq C\epsilon^2. \quad (4.91)$$

From (4.61), we similarly have that

$$\|(I + N_\lambda^\dagger)^{-1}\|_{L^\infty} \leq C, \quad (4.92)$$

and conclude

$$\|\Psi_{1,k}^\dagger\|_{W_\xi^{1,1}} \leq C\epsilon^2. \quad (4.93)$$

To achieve the second part of (4.63), we use (4.70) and have

$$\|\Psi_{2,k}^\dagger - \phi'_k\|_{H^1} = \|\epsilon^{-\beta} \tilde{L}_{22}^{-1} J_{12}^\dagger \Psi_{1,k}^\dagger\|_{H^1} \quad (4.94)$$

$$= \|\epsilon^{-\beta} \alpha_{12} \tilde{L}_{22}^{-1} \sum_i^N \chi_i \Phi_2^{\alpha_{12}-1} q_i^{\alpha_{11}} \langle \Psi_{1,k}^\dagger, \xi_i \rangle\|_{H^1} \quad (4.95)$$

$$= \|\epsilon^{-\beta} \alpha_{12} \sum_i^N \langle \Psi_{1,k}^\dagger, \xi_i \rangle \tilde{L}_{22}^{-1} \chi_i \Phi_2^{\alpha_{12}-1} q_i^{\alpha_{11}}\|_{H^1} \quad (4.96)$$

$$\leq C \epsilon^{-\beta} \sum_i^N \left| \langle \Psi_{1,k}^\dagger, \xi_i \rangle \right| \|\tilde{L}_{22}^{-1} \chi_i \Phi_2^{\alpha_{12}-1} q_i^{\alpha_{11}}\|_{H^1} \quad (4.97)$$

$$\leq C \epsilon^{-\beta} \|\Psi_{1,k}^\dagger\|_{W_\xi^{1,1}} \sum_i^N \|\tilde{L}_{22}^{-1} \chi_i \Phi_2^{\alpha_{12}-1}\|_{H^1} \quad (4.98)$$

$$\leq C \epsilon^{2-\beta}, \quad (4.99)$$

where as a consequence of (4.71) we have that

$$\tilde{L}_{22}^{-1} \chi_i \Phi_2^{\alpha_{12}-1} = \left(\frac{x - p_k}{2} \phi'_k + \frac{1}{\alpha_{22} - 1} \phi_k \right) \chi_i \Phi_2^{\alpha_{12}-1} + O(\epsilon^r), \quad (4.100)$$

which is uniformly bounded in H^1 $\square$

Chapter 5

Resolvent and Semigroup

Estimates

In this chapter we generate resolvent and semigroup estimates for our reduced operator $\tilde{L}_{\vec{p}}$. We fix a contour $\mathcal{C} \in \mathbb{C}$. We define $\mathcal{C}$ as

$$\mathcal{C} = \mathcal{C}_v \cup \mathcal{C}_{l-} \cup \mathcal{C}_{l+}, \quad (5.1)$$

where $\mathcal{C}_v = \left\{ -\frac{\epsilon\alpha\mu}{2} + is \mid s \in [-b, b] \right\}$, $\mathcal{C}_{l-} = \left\{ -ib + se^{-\frac{i5\pi}{6}} \mid s \in [-\infty, 0] \right\}$, and $\mathcal{C}_{l+} = \left\{ ib + se^{\frac{i5\pi}{6}} \mid s \in [-\infty, 0] \right\}$, for b positive, and independent of ϵ . We pick b sufficiently large so that $(\tilde{L}_{22} - \lambda)$, $(\tilde{L}_{22} - \lambda)$, and $I + N_\lambda$ are all invertible on $\mathcal{C}$.

The contour $\mathcal{C}$ is illustrated in Figure 5.1.

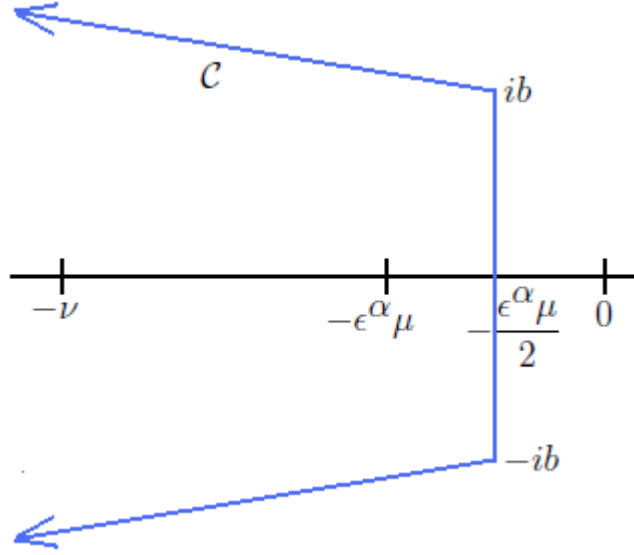


Figure 5.1: This is an illustration of our contour $\mathcal{C}$.

Given $F = (f_1, f_2)^T$ and $\lambda \in \mathcal{C}$, we examine the resolvent problem

$$(\tilde{L}_{\vec{p}} - \lambda I) \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}. \quad (5.2)$$

We invert the equation for g_2 and have

$$g_2 = (\tilde{L}_{22} - \lambda)^{-1} (f_2 - J_{21}g_1). \quad (5.3)$$

The equation for g_1 is

$$\left(L_{11}^e + \lambda - \epsilon^{-\beta} J_{11}\right) g_1 - \epsilon^{-\beta} J_{12} g_2 = -f_1. \quad (5.4)$$

If we substitute (5.3) into this and rearrange terms we obtain

$$(L_{11}^e + \lambda) g_1 - \epsilon^{-\beta} \left(J_{11} - J_{12}(\tilde{L}_{22} - \lambda)^{-1} J_{21}\right) g_1 = -KF, \quad (5.5)$$

where

$$KF = f_1 - \epsilon^{-\beta} J_{12}(\tilde{L}_{22} - \lambda)^{-1} f_2. \quad (5.6)$$

Recalling J_l and J_r from (4.22) and (4.23), we simplify (5.5) to

$$(L_{11}^e + \lambda) g_1 - \epsilon^{-\beta} J_l^T \otimes J_r \cdot g_1 = -KF. \quad (5.7)$$

If we invert the constant coefficient operator on the left and project with $\otimes J_r$, we have

$$\otimes J_r \cdot g_1 - \epsilon^{-\beta} \otimes J_r \cdot (L_{11}^e + \lambda)^{-1} J_l^T \otimes J_r \cdot g_1 = - \otimes J_r \cdot (L_{11}^e + \lambda)^{-1} KF. \quad (5.8)$$

Recalling the matrix N_λ from (4.25), we write the above as

$$(I + N_\lambda) \otimes J_r \cdot (L_{11}^e + \lambda)^{-1} J_l^T \otimes J_r \cdot g_1 = - \otimes J_r \cdot (L_{11}^e + \lambda)^{-1} K F, \quad (5.9)$$

and inverting we obtain an expression for the projection of g_1

$$\otimes J_r \cdot g_1 = -(I + N_\lambda)^{-1} \otimes J_r \cdot (L_{11}^e + \lambda)^{-1} K F. \quad (5.10)$$

If we substitute this into (5.7) and isolate g_1 , we establish the closed form expression

$$g_1 = (L_{11}^e + \lambda)^{-1} \left(\epsilon^{-\beta} J_l^T (I + N_\lambda)^{-1} \otimes J_r \cdot (L_{11}^e + \lambda)^{-1} - I \right) K F. \quad (5.11)$$

5.1 Spectral projections

The spectral projection associated to the N -point spectrum σ_0 near zero is defined by

$$\pi_{\vec{p}}^{\vec{U}} \equiv \sum_{j=1}^N \frac{(\vec{U}, \Psi_j^\dagger)}{(\Psi_j, \Psi_j^\dagger)} \Psi_j, \quad (5.12)$$

and the complementary projection is defined as

$$\tilde{\pi}_{\vec{p}}^{\vec{U}} \equiv I - \pi_{\vec{p}}^{\vec{U}}. \quad (5.13)$$

Recalling $\|\cdot\|_X$ defined in (1.40), we define the associated spectral subset that is associated to the temporally decaying solutions of the semigroup generated by $\tilde{L}_{\vec{p}}$:

$$X_{\vec{p}} \equiv \{\vec{U} \mid \|\vec{U}\|_X < \infty, \pi_{\vec{p}} \vec{U} = 0\}. \quad (5.14)$$

5.2 Resolvent estimates

Proposition 5.1. *For all λ on $\mathcal{C}$, $F \in X_{\vec{p}}$, and all $\vec{p} \in \mathcal{K}$, we have the following resolvent estimates for $\tilde{L}$,*

$$\|(\tilde{L} - \lambda)^{-1} F\|_X \leq C \frac{\epsilon^{2-\beta}}{\operatorname{Re}(k_\lambda)} \left(1 + \frac{\epsilon^{2-\beta}}{|k_\lambda|} |(I + N_\lambda)^{-1}| \right) \left(\epsilon^\beta \|f_1\|_{L_1} + \|f_2\|_{L_2} \right), \quad (5.15)$$

$$\begin{aligned} \|(\tilde{L} - \lambda)^{-1} F\|_X &\leq C \frac{\epsilon^2}{|k_\lambda| \operatorname{Re}(k_\lambda)} \left(1 + \frac{\epsilon^{2-\beta}}{\operatorname{Re}(k_\lambda)} |(I + N_\lambda)^{-1}| \right) \|f_1\|_{W_\xi^{1,1}} \\ &\quad + C \frac{\epsilon^{2-\beta}}{\operatorname{Re}(k_\lambda)} \left(1 + \frac{\epsilon^{2-\beta}}{\operatorname{Re}(k_\lambda)} |(I + N_\lambda)^{-1}| \right) \|f_2\|_{L_2}. \end{aligned} \quad (5.16)$$

If in addition the coarse-grained projection of f_1 is small, then we have the enhanced residual estimate

$$\begin{aligned} \|(\tilde{L} - \lambda)^{-1}F\|_X \leq & C \frac{\epsilon^{2-\beta}}{Re(k_\lambda)} |(I + N_\lambda)^{-1}| \frac{\epsilon^2}{|k_\lambda|} |\otimes \vec{\chi} \cdot f_1| + C \epsilon^2 \|f_1\|_{L^1_{1,\vec{p}}} \\ & + C \frac{\epsilon^{2-\beta}}{Re(k_\lambda)} \left(\epsilon^2 \|f_1\|_{L^1_{1,\vec{p}}} + \epsilon^\beta |\otimes \vec{\chi} \cdot f_1| + \|f_2\|_{L_2} \right). \end{aligned} \quad (5.17)$$

Proof: We have that

$$(\tilde{L} - \lambda)^{-1}F = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}. \quad (5.18)$$

We apply the $W_\xi^{1,1}$ norm to g_1 as represented in (5.11) and also use the estimate (3.6):

$$\begin{aligned} \|g_1\|_{W_\xi^{1,1}} \leq & C \frac{\epsilon^{2-\beta}}{Re(k_\lambda)} \|J_l^T (I + N_\lambda)^{-1} \otimes J_r \cdot (L_{11}^e + \lambda)^{-1} KF\|_{L^1} \\ & + C \frac{\epsilon^2}{Re(k_\lambda)} \|KF\|_{L^1}. \end{aligned} \quad (5.19)$$

From the definition of $J_l = \vec{\xi}$, and the fact that the L^1 norm of the components of $\vec{\xi}$ are each one, we have

$$\|J_l^T (I + N_\lambda)^{-1}\|_{L^1} \leq C |(I + N_\lambda)^{-1}|. \quad (5.20)$$

Contained within $\otimes J_r$ is $(\tilde{L}_{22} - \lambda)^{-1}$ which is uniformly invertible from L^2 to H^1 for $\lambda \in \mathcal{C}$ since $F \in X_{\vec{p}}$. So we have:

$$|\otimes J_r \cdot (L_{11}^e + \lambda)^{-1} KF| \leq \|J_r\|_{L^1} \|(L_{11}^e + \lambda)^{-1} KF\|_{L^\infty} \leq C \frac{\epsilon^2}{|k_\lambda|} \|KF\|_{L^1}, \quad (5.21)$$

where we used (3.8) and the fact that:

$$\|J_r\|_{L^1} = \|(\alpha_{11}\Phi_2^{\alpha_{12}}Q^{\alpha_{11}-1} - \alpha_{12}\Phi_2^{\alpha_{12}-1}(\tilde{L}_{22} - \lambda)^{-1}J_{21}Q^{\alpha_{11}})\vec{\chi}\|_{L^1} \quad (5.22)$$

$$\leq C\|\Phi_2^{\alpha_{12}}\vec{\chi}\|_{L^1} + C\|\Phi_2^{\alpha_{12}-1}(\tilde{L}_{22} - \lambda)^{-1}J_{21}\vec{\chi}\|_{L^1} \quad (5.23)$$

$$\leq C\left(1 + \|\Phi_2^{\alpha_{12}-1}\|_{L^2}\|(\tilde{L}_{22} - \lambda)^{-1}J_{21}\vec{\chi}\|_{L^2}\right) \quad (5.24)$$

$$\leq C\left(1 + \|J_{21}\vec{\chi}\|_{L^2}\right) \quad (5.25)$$

$$\leq C. \quad (5.26)$$

Applying (5.20) and (5.21) to (5.19), we have

$$\|g_1\|_{W_\xi^{1,1}} \leq C \frac{\epsilon^2}{Re(k_\lambda)} \left(\frac{\epsilon^{2-\beta}}{|k_\lambda|} |(I + N_\lambda)^{-1}| + 1 \right) \|KF\|_{L^1}. \quad (5.27)$$

Estimating the right hand side, we have that

$$\|KF\|_{L^1} \leq \|f_1\|_{L^1} + \epsilon^{-\beta} \|J_{12}(\tilde{L}_{22} - \lambda)^{-1}f_2\|_{L^1}. \quad (5.28)$$

Furthermore,

$$\|J_{12}(\tilde{L}_{22} - \lambda)^{-1}f_2\|_{L^1} \leq \|J_{12}\|_{L^2} \|(\tilde{L}_{22} - \lambda)^{-1}f_2\|_{L^2} \quad (5.29)$$

$$\leq C\|f_2\|_{L^2}. \quad (5.30)$$

Then we have:

$$\|g_1\|_{W_\xi^{1,1}} \leq C \frac{\epsilon^{2-\beta}}{Re(k_\lambda)} \left(\frac{\epsilon^{2-\beta}}{|k_\lambda|} |(I + N_\lambda)^{-1}| + 1 \right) (\epsilon^\beta \|f_1\|_{L^1} + \|f_2\|_{L^2}). \quad (5.31)$$

Next we take the H^1 norm of g_2 from (5.3):

$$\|g_2\|_{H^1} = \|(\tilde{L}_{22} - \lambda)^{-1}(f_2 - J_{21}g_1)\|_{H^1} \quad (5.32)$$

$$\leq C(\|f_2\|_{L^2} + \|J_{21}g_1\|_{L^2}) \quad (5.33)$$

$$\leq C(\|f_2\|_{L^2} + \|g_1\|_{W_\xi^{1,1}}). \quad (5.34)$$

Applying (5.31) to (5.32), and also combining these bounds, we have (5.15).

For (5.16), we again apply the $W_\xi^{1,1}$ norm to g_1 from (5.11) and then split the

estimate into two terms:

$$\begin{aligned} \|g_1\|_{W_\xi^{1,1}} &\leq \| (L_{11}^e + \lambda)^{-1} \epsilon^{-\beta} J_l^T (I + N_\lambda)^{-1} \otimes J_r \cdot (L_{11}^e + \lambda)^{-1} \|_{W_\xi^{1,1}} \\ &\quad + \| (L_{11}^e + \lambda)^{-1} KF \|_{W_\xi^{1,1}} \end{aligned} \quad (5.35)$$

$$\begin{aligned} &\leq C \frac{\epsilon^{2-\beta}}{Re(k_\lambda)} \| J_l^T (I + N_\lambda)^{-1} \otimes J_r \cdot (L_{11}^e + \lambda)^{-1} KF \|_{L^1} \\ &\quad + \| (L_{11}^e + \lambda)^{-1} KF \|_{W_\xi^{1,1}}, \end{aligned} \quad (5.36)$$

where we applied (3.6) to the first part. Addressing part of this term:

$$| \otimes J_r \cdot (L_{11}^e + \lambda)^{-1} KF | = | \langle J_r, (L_{11}^e + \lambda)^{-1} KF \rangle | \quad (5.37)$$

$$\leq \| J_r \|_{L^1} \| (L_{11}^e + \lambda)^{-1} KF \|_{L^\infty} \quad (5.38)$$

$$\leq C \| (L_{11}^e + \lambda)^{-1} KF \|_{W_\xi^{1,1}}. \quad (5.39)$$

Bounding J_l^T term as in (5.20), we have

$$\|g_1\|_{W_\xi^{1,1}} \leq C \left(1 + \frac{\epsilon^{2-\beta}}{Re(k_\lambda)} |(I + N_\lambda)^{-1}| \right) \| (L_{11}^e + \lambda)^{-1} KF \|_{W_\xi^{1,1}}. \quad (5.40)$$

Using (3.7) and (3.8) we obtain

$$\|(L_{11}^e + \lambda)^{-1}KF\|_{W_\xi^{1,1}} \leq C \left(\frac{\epsilon^2}{|k_\lambda|Re(k_\lambda)} \|f_1\|_{W_\xi^{1,1}} + \frac{\epsilon^{2-\beta}}{Re(k_\lambda)} \|f_2\|_{L_2} \right). \quad (5.41)$$

Combining these estimates, we have

$$\|g_1\|_{W_\xi^{1,1}} \leq C \frac{\epsilon^{2-\beta}}{Re(k_\lambda)} \left(1 + \frac{\epsilon^{2-\beta}}{Re(k_\lambda)} |(I + N_\lambda)^{-1}| \right) \left(\frac{\epsilon^\beta}{|k_\lambda|} \|f_1\|_{W_\xi^{1,1}} + \|f_2\|_{L_2} \right). \quad (5.42)$$

We bound $\|g_2\|_{H^1}$ as in (5.32) to obtain (5.16).

To obtain (5.17), we examine the case when the coarse-grained projection of f_1 is small. The bound on the f_2 component is the same as in (5.15), so without loss of generality we consider the case $F = (f_1, 0)^T$. Taking the $W_\xi^{1,1}$ norm of g_1 as represented in (5.11), we have

$$\|g_1\|_{W_\xi^{1,1}} = \|(L_{11}^e + \lambda)^{-1} \left(\epsilon^{-\beta} J_l^T (I + N_\lambda)^{-1} \otimes J_r \cdot (L_{11}^e + \lambda)^{-1} - I \right) f_1\|_{W_\xi^{1,1}} \quad (5.43)$$

$$\begin{aligned} &\leq C \epsilon^{-\beta} \|(L_{11}^e + \lambda)^{-1} J_l^T\|_{W_\xi^{1,1}} |(I + N_\lambda)^{-1}| \otimes J_r \cdot (L_{11}^e + \lambda)^{-1} f_1| \\ &+ C \|(L_{11}^e + \lambda)^{-1} f_1\|_{W_\xi^{1,1}}. \end{aligned} \quad (5.44)$$

Using (3.15) and the uniform L^1 bound on J_r we obtain

$$|\otimes J_r \cdot (L_{11}^e + \lambda)^{-1} f_1| \leq \|J_r\|_{L^1} \|(L_{11}^e + \lambda)^{-1} f_1\|_{L^\infty} \quad (5.45)$$

$$\leq C \left(\frac{\epsilon^2}{|k_\lambda|} |\otimes \vec{\chi} \cdot f_1| + \epsilon^2 \|f_1\|_{L_{1,\vec{p}}^1} \right). \quad (5.46)$$

From (3.6), we have the bound

$$\|(L_{11}^e + \lambda)^{-1} J_l^T\|_{W_\xi^{1,1}} \leq C \frac{\epsilon^2}{\text{Re}(k_\lambda)} \|J_l^T\|_{L^1} \leq C \frac{\epsilon^2}{\text{Re}(k_\lambda)}. \quad (5.47)$$

Finally, applying (3.14) to the remaining term,

$$\|(L_{11}^e + \lambda)^{-1} f_1\|_{W_\xi^{1,1}} \leq C \left(\frac{\epsilon^2}{\text{Re}(k_\lambda)} |\otimes \vec{\chi} \cdot f_1| + \epsilon^2 \frac{|k_\lambda|}{\text{Re}(k_\lambda)} \|f_1\|_{L_{1,\vec{p}}^1} \right). \quad (5.48)$$

Combining these estimates, we have (5.17) $\square$

We use the previous proposition to obtain the following estimate on $\vec{\Phi}_1$.

Lemma 5.1. *Fix the pulse separation $l_0 > 0$ sufficiently large, then there exists a constant $C > 0$ such that for all $\vec{p} \in \mathcal{K}$, we have following estimate:*

$$\|\vec{\Phi}_1\|_X \leq C \epsilon^{2-\beta}. \quad (5.49)$$

Proof: We apply the resolvent estimate (5.15) to the definition (4.3) of $\vec{\Phi}_1$ which yields

$$\|\vec{\Phi}_1\|_X = \|\tilde{L}_{\vec{p}_0}^{-1} \left(\tilde{\pi}_{\vec{p}_0} R(\Phi) \right)\|_X \quad (5.50)$$

$$\leq C \left(\epsilon^\beta \|[\tilde{\pi}_{\vec{p}_0} R(\Phi)]_1\|_{L_1} + \|[\tilde{\pi}_{\vec{p}_0} R(\Phi)]_2\|_{L_2} \right). \quad (5.51)$$

Using the residual estimate (3.95), we have

$$\|[\tilde{\pi}_{\vec{p}_0} R(\Phi)]_1\|_{L_1} \leq \|R_1(\Phi)\|_{L_1} + \|[\pi_{\vec{p}_0} R(\Phi)]_1\|_{L_1} \quad (5.52)$$

$$\leq C\epsilon^\alpha + \|[\pi_{\vec{p}_0} R(\Phi)]_1\|_{L_1}. \quad (5.53)$$

However, we have the estimate

$$\|[\pi_{\vec{p}_0} R(\Phi)]_1\|_{L_1} = \left\| \sum_{j=1}^N \frac{(R(\Phi), \Psi_j^\dagger)}{(\Psi_j, \Psi_j^\dagger)} \Psi_{1,j} \right\|_{L_1} \quad (5.54)$$

$$\begin{aligned} &\leq C \sum_{j=1}^N \|R_1(\Phi)\|_{L_1} \|\Psi_{1,j}^\dagger\|_{L^\infty} \|\Psi_{1,j}\|_{L_1} \\ &+ C \sum_{j=1}^N \|R_2(\Phi)\|_{L_2} \|\Psi_{2,j}^\dagger\|_{L_2} \|\Psi_{1,j}\|_{L_1} \end{aligned} \quad (5.55)$$

$$\leq C\epsilon^{2-\beta}, \quad (5.56)$$

which results from our previous residual estimates and adjoint eigenvector estimates.

Using the residual estimate (3.96), we have

$$\|[\tilde{\pi}_{\vec{p}_0} R(\Phi)]_2\|_{L_2} \leq \|R_2(\Phi)\|_{L_2} + \|[\pi_{\vec{p}_0} R(\Phi)]_2\|_{L_2} \quad (5.57)$$

$$\leq C\epsilon^{2-\beta}, \quad (5.58)$$

since as before

$$\|[\pi_{\vec{p}_0} R(\Phi)]_2\|_{L_2} = \left\| \sum_{j=1}^N \frac{(R(\Phi), \Psi_j^\dagger)}{(\Psi_j, \Psi_j^\dagger)} \Psi_{2,j} \right\|_{L_2} \quad (5.59)$$

$$\begin{aligned} &\leq C \sum_{j=1}^N \|R_1(\Phi)\|_{L_1} \|\Psi_{1,j}^\dagger\|_{L^\infty} \|\Psi_{2,j}\|_{L_2} \\ &+ C \sum_{j=1}^N \|R_2(\Phi)\|_{L_2} \|\Psi_{2,j}^\dagger\|_{L_2} \|\Psi_{2,j}\|_{L_2} \end{aligned} \quad (5.60)$$

$$\leq C\epsilon^{2-\beta}. \quad (5.61)$$

Since $\alpha + \beta = 2 - \beta$ from the assumption $1 - \alpha/2 - \beta = 0$, we have our result

$$\|\vec{\Phi}_1\|_X \leq C \left(\epsilon^\beta \|[\tilde{\pi}_{\vec{p}_0} R(\Phi)]_1\|_{L_1} + \|[\tilde{\pi}_{\vec{p}_0} R(\Phi)]_2\|_{L_2} \right) \leq C\epsilon^{2-\beta} \square \quad (5.62)$$

5.3 Semigroup estimates

For fixed $\vec{p} \in \mathcal{K}$, we see from classical results, e.g. [24], since $\tilde{L}$ is sectorial, we can generate its semigroup from the Laplace transform of its resolvent. With our contour $\mathcal{C}$, the semigroup S associated to $\tilde{L}$ is given by the contour integral

$$S(t)F = \frac{1}{2\pi i} \int_{\mathcal{C}} e^{\lambda t} (\lambda - \tilde{L})^{-1} F d\lambda, \quad (5.63)$$

where we assume that $F \in X_{\vec{p}}$. We have the following estimates on the semigroup.

Proposition 5.2. *For any $t_0 > 0$ there exists $C > 0$ such that for all $\vec{p} \in \mathcal{K}$, $F \in X_{\vec{p}}$, and $t \geq t_0$ the semigroup satisfies*

$$\|S(t)F\|_X \leq C e^{-\frac{\epsilon^\alpha}{2}\mu t} \left(\epsilon^\beta \|f_1\|_{L_1} + \|f_2\|_{L_2} \right) \quad (5.64)$$

$$\|S(t)F\|_X \leq C e^{-\frac{\epsilon^\alpha}{2}\mu t} \left(\ln(\epsilon^{-\alpha}) \|f_1\|_{W_\xi^{1,1}} + \|f_2\|_{L_2} \right) \leq C e^{-\frac{\epsilon^\alpha}{2}\mu t} \ln(\epsilon^{-\alpha}) \|F\|_X. \quad (5.65)$$

If in addition the coarse-grained projection of f_1 is small, then we have the improved estimate

$$\|S(t)F\|_X \leq C e^{-\frac{\epsilon^\alpha}{2}\mu t} \left(\epsilon^\beta |\otimes \vec{\chi} \cdot f_1| + \epsilon^2 \|f_1\|_{L_{1,\vec{p}}^1} + \|f_2\|_{L_2} \right). \quad (5.66)$$

Proof: Since $\vec{p} \in \mathcal{K}$ and from (4.61), we have that $(I + N_\lambda)^{-1}$ is uniformly bounded for all $\lambda \in \mathcal{C}$.

For each of the above semigroup estimates, we apply the appropriate resolvent estimate from Proposition 5.1, which leads to a class of integrals to bound. For (5.64) we find,

$$\|S(t)F\|_X = \left\| \frac{1}{2\pi i} \int_{\mathcal{C}} e^{\lambda t} (\lambda - \tilde{L})^{-1} F d\lambda \right\|_X \quad (5.67)$$

$$\leq C \int_{\mathcal{C}} \|(\lambda - \tilde{L})^{-1} F\|_X |e^{\lambda t}| d\lambda \quad (5.68)$$

$$\leq C \left(\epsilon^\beta \|f_1\|_{L_1} + \|f_2\|_{L_2} \right) \int_{\mathcal{C}} |e^{\lambda t}| \frac{\epsilon^{2-\beta}}{\operatorname{Re}(k_\lambda)} \left(1 + \frac{\epsilon^{2-\beta}}{|k_\lambda|} \right) d\lambda. \quad (5.69)$$

Similarly for (5.65) we see that

$$\|S(t)F\|_X \leq C \int_{\mathcal{C}} \|(\lambda - \tilde{L})^{-1} F\|_X |e^{\lambda t}| d\lambda \quad (5.70)$$

$$\begin{aligned} &\leq C \|f_1\|_{W_\xi^{1,1}} \int_{\mathcal{C}} |e^{\lambda t}| \frac{\epsilon^{2-\beta}}{\operatorname{Re}(k_\lambda)} \left(1 + \frac{\epsilon^{2-\beta}}{\operatorname{Re}(k_\lambda)} \right) \frac{\epsilon^\beta}{|k_\lambda|} d\lambda \\ &+ C \|f_2\|_{L_2} \int_{\mathcal{C}} |e^{\lambda t}| \frac{\epsilon^{2-\beta}}{\operatorname{Re}(k_\lambda)} \left(1 + \frac{\epsilon^{2-\beta}}{\operatorname{Re}(k_\lambda)} \right) d\lambda. \end{aligned} \quad (5.71)$$

While for (5.66) we have,

$$\|S(t)F\|_X \leq C \int_{\mathcal{C}} \|(\lambda - \tilde{L})^{-1}F\|_X |e^{\lambda t}| d\lambda \quad (5.72)$$

$$\begin{aligned} &\leq C \epsilon^2 \|f_1\|_{L_{1,\vec{p}}^1} \int_{\mathcal{C}} |e^{\lambda t}| \left(1 + \frac{\epsilon^{2-\beta}}{\operatorname{Re}(k_\lambda)}\right) d\lambda \\ &\quad + C \epsilon^\beta |\otimes \vec{\chi} \cdot f_1| \int_{\mathcal{C}} |e^{\lambda t}| \frac{\epsilon^{2-\beta}}{\operatorname{Re}(k_\lambda)} \left(1 + \frac{\epsilon^{2-\beta}}{|k_\lambda|}\right) d\lambda \\ &\quad + C \|f_2\|_{L_2} \int_{\mathcal{C}} |e^{\lambda t}| \frac{\epsilon^{2-\beta}}{\operatorname{Re}(k_\lambda)} d\lambda. \end{aligned} \quad (5.73)$$

The following claim estimates these integrals:

Claim 5.1. *Fix the contour $\mathcal{C}$ as in (5.1), then for all $\vec{p} \in \mathcal{K}$, there exists $C > 0$ such that*

$$\int_{\mathcal{C}} \frac{\epsilon^{2-\beta} |e^{\lambda t}|}{\operatorname{Re}(k_\lambda)} d\lambda \leq C \epsilon^{1-\beta} e^{-\frac{\epsilon^\alpha}{2} \mu t} \quad (5.74)$$

$$\int_{\mathcal{C}} \frac{\epsilon^{4-2\beta} |e^{\lambda t}|}{|k_\lambda| \operatorname{Re}(k_\lambda)} d\lambda \leq C \left(\epsilon^\alpha \ln(\epsilon^{-\alpha}) \right) e^{-\frac{\epsilon^\alpha}{2} \mu t} \quad (5.75)$$

$$\int_{\mathcal{C}} \frac{\epsilon^{4-2\beta} |e^{\lambda t}|}{\operatorname{Re}(k_\lambda)^2} d\lambda \leq C \left(\epsilon^\alpha \ln(\epsilon^{-\alpha}) \right) e^{-\frac{\epsilon^\alpha}{2} \mu t} \quad (5.76)$$

$$\int_{\mathcal{C}} \frac{\epsilon^2 |e^{\lambda t}|}{|k_\lambda| \operatorname{Re}(k_\lambda)} d\lambda \leq C \ln(\epsilon^{-\alpha}) e^{-\frac{\epsilon^\alpha}{2} \mu t} \quad (5.77)$$

$$\int_{\mathcal{C}} \frac{\epsilon^{4-\beta} |e^{\lambda t}|}{|k_\lambda| \operatorname{Re}(k_\lambda)^2} d\lambda \leq C \ln(\epsilon^{-\alpha}) e^{-\frac{\epsilon^\alpha}{2} \mu t}. \quad (5.78)$$

Proof of claim: The two angled parts of the contour $\mathcal{C}_{l-}$ and $\mathcal{C}_{l+}$ are straight-

forward to estimate, because the exponential decay is dominant as $Re(k_\lambda) \rightarrow -\infty$.

Our concern is along the vertical part of the contour $\mathcal{C}_v = \left\{ -\frac{\epsilon^\alpha \mu}{2} + is \mid s \in [-b, b] \right\}$.

On $\mathcal{C}_v$, we have

$$k_\lambda = \epsilon \sqrt{\epsilon^\alpha \mu + is} = \epsilon \sqrt[4]{\epsilon^{2\alpha} \mu^2 + s^2} e^{\frac{i \tan^{-1}(s \epsilon^{-\alpha} / \mu)}{2}}. \quad (5.79)$$

Using trigonometric properties,

$$|k_\lambda| = \epsilon \sqrt[4]{\epsilon^{2\alpha} \mu^2 + s^2} \quad (5.80)$$

$$Re(k_\lambda) = \epsilon \sqrt[4]{\epsilon^{2\alpha} \mu^2 + s^2} \cos \left(\frac{\tan^{-1}(s \epsilon^{-\alpha} / \mu)}{2} \right) \quad (5.81)$$

$$= \pm \epsilon \sqrt[4]{\epsilon^{2\alpha} \mu^2 + s^2} \sqrt{1/2 + \frac{\epsilon^\alpha \mu}{2 \sqrt{s^2 + \epsilon^{2\alpha} \mu^2}}}. \quad (5.82)$$

For $\mu = 0$ we have

$$|k_\lambda| = \epsilon \sqrt{|s|} \quad (5.83)$$

$$Re(k_\lambda) = \epsilon \sqrt{\frac{|s|}{2}}. \quad (5.84)$$

Applying this to (5.74) we have,

$$\int_{\mathcal{C}_v} \frac{\epsilon^{2-\beta} |e^{\lambda t}|}{\operatorname{Re}(k_\lambda)} d\lambda \quad (5.85)$$

$$\leq \epsilon^{1-\beta} e^{-\frac{\epsilon^\alpha}{2} \mu t} \left(- \int_{-b}^0 \sqrt{\frac{2}{|s|}} ds + \int_0^b \sqrt{\frac{2}{|s|}} ds \right) \quad (5.86)$$

$$\leq C e^{-\frac{\epsilon^\alpha}{2} \mu t} \sqrt{b} \epsilon^{1-\beta}. \quad (5.87)$$

For (5.75), we argue that

$$\int_{\mathcal{C}_v} \frac{\epsilon^{4-2\beta} |e^{\lambda t}|}{|k_\lambda| \operatorname{Re}(k_\lambda)} d\lambda = \int_{\mathcal{C}_v} \frac{\epsilon^{2+\alpha} |e^{\lambda t}|}{|k_\lambda| \operatorname{Re}(k_\lambda)} d\lambda \quad (5.88)$$

$$\leq C e^{-\frac{\epsilon^\alpha}{2} \mu t} \int_0^b \frac{\epsilon^\alpha}{\sqrt{\epsilon^{2\alpha} \mu^2 + s^2} \sqrt{1/2 + \frac{\epsilon^\alpha \mu}{2\sqrt{s^2 + \epsilon^{2\alpha} \mu^2}}}} ds \quad (5.89)$$

$$\leq C e^{-\frac{\epsilon^\alpha}{2} \mu t} \int_0^b \frac{\epsilon^\alpha}{\sqrt{\epsilon^{2\alpha} \mu^2 + s^2}} ds \quad (5.90)$$

$$= C e^{-\frac{\epsilon^\alpha}{2} \mu t} \int_0^b \frac{ds}{\sqrt{\mu^2 + \left(\frac{s}{\epsilon^\alpha}\right)^2}} \quad (5.91)$$

$$= C e^{-\frac{\epsilon^\alpha}{2} \mu t} \epsilon^\alpha \int_0^{b\epsilon^{-\alpha}} \frac{d\tilde{s}}{\sqrt{\mu^2 + \tilde{s}^2}}, \quad (5.92)$$

where we substituted $\tilde{s} = \frac{s}{\epsilon^\alpha}$. Then for the integral in the previous line, we have

$$\int_0^{b\epsilon^{-\alpha}} \frac{d\tilde{s}}{\sqrt{\mu^2 + \tilde{s}^2}} = \left[\sinh^{-1} \left(\frac{\tilde{s}}{\mu} \right) \right]_0^{b\epsilon^{-\alpha}} \quad (5.93)$$

$$= \sinh^{-1} \left(\frac{b\epsilon^{-\alpha}}{\mu} \right) \quad (5.94)$$

$$\leq 2 \ln \left(\frac{b\epsilon^{-\alpha}}{\mu} \right), \quad (5.95)$$

for ϵ sufficiently small

$$\epsilon^\alpha < \frac{.475b}{\mu}. \quad (5.96)$$

This gives us

$$\int_{\mathcal{C}_v} \frac{\epsilon^{4-2\beta} |e^{\lambda t}|}{|k_\lambda| \operatorname{Re}(k_\lambda)} d\lambda \leq C e^{-\frac{\epsilon^\alpha}{2} \mu t} \left(\epsilon^\alpha \ln(\epsilon^{-\alpha}) \right). \quad (5.97)$$

For (5.76), the bound follows as in (5.75), where

$$\int_{\mathcal{C}} \frac{\epsilon^{4-2\beta} |e^{\lambda t}|}{\operatorname{Re}(k_\lambda)^2} d\lambda \leq C e^{-\frac{\epsilon^\alpha}{2} \mu t} \left(\epsilon^\alpha \ln(\epsilon^{-\alpha}) \right). \quad (5.98)$$

We also have that (5.77) follows like (5.75), so

$$\int_{\mathcal{C}} \frac{\epsilon^2 |e^{\lambda t}|}{|k_\lambda| \operatorname{Re}(k_\lambda)} d\lambda \leq C e^{-\frac{\epsilon^\alpha}{2} \mu t} \ln(\epsilon^{-\alpha}). \quad (5.99)$$

The final integral to bound is (5.78). Again following the previous methods we have

that

$$\int_{\mathcal{C}} \frac{\epsilon^{4-\beta} |e^{\lambda t}|}{|k_{\lambda}| \operatorname{Re}(k_{\lambda})^2} d\lambda \leq C e^{-\frac{\epsilon^{\alpha}}{2} \mu t} \ln(\epsilon^{-\alpha}) \square \quad (5.100)$$

With the claim prove, we apply the estimates on these five integrals, which leads to all three semigroup estimates. Both (5.77) and (5.78) account for the $\ln(\epsilon^{-\alpha})$ in (5.65) $\square$

Chapter 6

Nonlinear Adiabatic Stability by Renormalization Group

In this chapter we prove the adiabatic stability results in (1.23) and derive the limiting pulse dynamics in (1.24). We assume at a time t_0 that our initial data $\vec{U}_0 = (U_0, V_0)^T$ satisfies

$$\|\vec{U}_0 - \Phi_{\vec{p}_*}\|_X \leq \delta, \tag{6.1}$$

for some $\delta > 0$ and $\vec{p}_* \in \mathcal{K}$. The following proposition allows us to choose a base point $\vec{p}_0$ about which we will develop a local coordinate system.

Proposition 6.1. *Fix δ sufficiently small. Given $\vec{p}_* \in \mathcal{K}$ for $\mathcal{K}$ defined in (1.13)*

and $\vec{U}_0$ satisfying the estimate $\|W_*\|_X \leq \delta$ for $W_* \equiv \vec{U}_0 - \Phi_{\vec{p}_*}$, there exists $M > 0$ and a smooth function $\mathcal{H} : X \rightarrow \mathcal{K}$ such that $\vec{p} = \vec{p}_* + \mathcal{H}(W_*)$ satisfies

$$W_0 \equiv \vec{U}_0 - \Phi_{\vec{p}} \in X_{\vec{p}}, \quad (6.2)$$

for $X_{\vec{p}}$ defined in (5.14). Moreover, if $W_* \in X_{\tilde{p}}$ for some $\tilde{p} \in \mathcal{K}$, then

$$|\vec{p} - \vec{p}_*| \leq M_1 \|W_*\|_X |\vec{p}_* - \tilde{p}|. \quad (6.3)$$

Proof: We may write $U_0 = \Phi_{\vec{p}_*} + W_*$ and $U_0 = \Phi_{\vec{p}} + W_0$, which implies that

$$W_0 = W_* + \Phi_{\vec{p}_*} - \Phi_{\vec{p}}. \quad (6.4)$$

The equation (6.2) requires that

$$0 = \pi_{\vec{p}} W_0 = \pi_{\vec{p}} (W_* + \Phi_{\vec{p}_*} - \Phi_{\vec{p}}), \quad (6.5)$$

which is equivalent to solving the system $\vec{\Lambda}(\vec{p}, W_*) = (\Lambda_1, \dots, \Lambda_N)^T = \vec{0}$ where

$$\Lambda_j(\vec{p}, W_*) = \langle W_* + \Phi_{\vec{p}_*} - \Phi_{\vec{p}}, \Psi_j^\dagger \rangle = 0. \quad (6.6)$$

It is clear that

$$\Lambda(\vec{p}_*, 0) = \vec{0}. \quad (6.7)$$

We examine the following gradient

$$\nabla_{\vec{p}} \Lambda|_{(\vec{p}=\vec{p}_*, W_*=0)} = - \left(\begin{array}{ccc} \langle \partial p_1 \Phi_{\vec{p}}, \Psi_1^\dagger \rangle & \cdots & \langle \partial p_N \Phi_{\vec{p}}, \Psi_1^\dagger \rangle \\ \vdots & \ddots & \vdots \\ \langle \partial p_1 \Phi_{\vec{p}}, \Psi_N^\dagger \rangle & \cdots & \langle \partial p_N \Phi_{\vec{p}}, \Psi_N^\dagger \rangle \end{array} \right)_{\vec{p}=\vec{p}_*}. \quad (6.8)$$

Using the asymptotic reductions (3.48) and (3.79),

$$\partial p_j \Phi_{\vec{p}} = \begin{pmatrix} 0 \\ \phi'_j \end{pmatrix} + O(\epsilon^{2-\beta}), \quad (6.9)$$

in L^∞ . Using (3.47), we have

$$\|\partial p_j \Phi_{1, \vec{p}}\|_{L^1} = O(1). \quad (6.10)$$

Also from (3.79),

$$\Psi_j^\dagger(\cdot, p_j) = \left(O(\epsilon^2), \phi'_j + O(\epsilon^{2-\beta}) \right)^T, \quad (6.11)$$

in L^∞ . Using these estimates, we reduce (6.8) to

$$\nabla_{\vec{p}} \Lambda|_{\vec{p}=\vec{p}_*, W_*=0} = -\|\phi'\|_{L^2}^2 Q^{-2\alpha_{21}/(\alpha_{22}-1)} + O(\epsilon^{2-\beta}), \quad (6.12)$$

where Q is the same diagonal matrix as before. From (2.29) in Theorem 2.1, we have that $\vec{q}$ is uniformly bounded from zero. Applying the implicit function theorem, we are guaranteed the existence of a smooth function $\mathcal{H} : X \rightarrow \mathcal{K}$ which provides the solution W_0 in a neighborhood about $(\vec{p}_*, 0)$.

Also if $W_* \in X_{\tilde{p}}$ for some $\tilde{p} \in \mathcal{K}$, then by definition

$$\langle W_*, \Psi_j^\dagger(\tilde{p}) \rangle = 0, \quad (6.13)$$

for all j from 1 to N . By the Mean Value Theorem, there exists $\vec{p}'$ such that

$$|\vec{p} - \vec{p}_*| |\nabla_{\vec{p}} \Lambda(\vec{p}')| = |\Lambda(\vec{p}, 0) - \Lambda(\vec{p}_*, 0)|, \quad (6.14)$$

where we have $|\nabla_{\vec{p}}\Lambda(\vec{p}')| = O(1)$. Then

$$|\vec{p} - \vec{p}_*| \leq M |\Lambda(\vec{p}_*, 0) - \Lambda(\vec{p}, 0)| \quad (6.15)$$

$$= M \left| \vec{0} - \begin{pmatrix} \vdots \\ \langle \Phi_{\vec{p}_*} - \Phi_{\vec{p}}, \Psi_j^\dagger(\vec{p}_*) \rangle \\ \vdots \end{pmatrix} \right| \quad (6.16)$$

$$= M \left| \begin{pmatrix} \vdots \\ \langle W_*, \Psi_j^\dagger(\vec{p}_*) - \Psi_j^\dagger(\tilde{p}) \rangle \\ \vdots \end{pmatrix} \right|, \quad (6.17)$$

since $\Phi_{\vec{p}_*} - \Phi_{\vec{p}} = W_0 - W_*$ and $\pi_{\vec{p}} W_0 = 0$. Using (6.13), the Hölder inequality, and the fact that the X -norm controls L^∞ , we have

$$|\vec{p} - \vec{p}_*| \leq M \sum_{j=1}^N \|\Psi_j^\dagger(\vec{p}_*) - \Psi_j^\dagger(\tilde{p})\|_{L^1} \|W_*\|_{L^\infty} \quad (6.18)$$

$$\leq M \sum_{j=1}^N \|\Psi_j^\dagger(\vec{p}_*) - \Psi_j^\dagger(\tilde{p})\|_{L^1} \|W_*\|_X. \quad (6.19)$$

To finish the proof, we use the Mean Value Theorem

$$\int |\Psi_j^\dagger(\vec{p}_*) - \Psi_j^\dagger(\tilde{p})| dx = \int |\vec{p}_* - \tilde{p}| |\nabla \Psi_j^\dagger(\vec{p}')| dx \quad (6.20)$$

$$\leq C |\vec{p}_* - \tilde{p}|. \quad (6.21)$$

Redefining our constant, we have (6.3) $\square$

6.1 Projected equations

We start the renormalization group procedure by freezing $\vec{p} = \vec{p}_0$ in $X_{\vec{p}_0}$, where $\vec{p}_0$ is the base point provided by Proposition 6.1. Then we rewrite (4.4), inserting $\Delta L \equiv L_{\vec{p}} - \tilde{L}_{\vec{p}_0}$, so that the evolution for the remainder W can be represented as

$$W_t + \left(\frac{\partial \Phi}{\partial \vec{p}} + \frac{\partial \vec{\Phi}_1}{\partial \vec{p}} \right) \dot{\vec{p}} = R(\Phi) + \tilde{L}_{\vec{p}_0} (\vec{\Phi}_1 + W) + \Delta L (\vec{\Phi}_1 + W) + \mathcal{N}(\vec{\Phi}_1, W) \quad (6.22)$$

$$W(x, 0) = W_0, \quad (6.23)$$

where $W \in X_{\vec{p}_0}$ and $\vec{p} = \vec{p}(t)$. To examine ΔL , we break it into secular and reductive parts, $\Delta L = \Delta_s L + \Delta_r L$, where

$$\Delta_s L \equiv L_{\vec{p}} - L_{\vec{p}_0} \quad (6.24)$$

$$\Delta_r L \equiv L_{\vec{p}_0} - \tilde{L}_{\vec{p}_0}. \quad (6.25)$$

We recognize from the definition of $\vec{\Phi}_1$ in (4.3) that $\pi_{\vec{p}_0} \tilde{L}_{\vec{p}_0} \vec{\Phi}_1 = 0$, since $\pi_{\vec{p}_0}$ and $\tilde{\pi}_{\vec{p}_0}$ are orthogonal projections. Similarly, we have

$$\pi_{\vec{p}_0} \frac{\partial \vec{\Phi}_1}{\partial \vec{p}} \dot{\vec{p}} = 0, \quad (6.26)$$

since

$$\pi_{\vec{p}_0} \frac{\partial \vec{\Phi}_1}{\partial p_j} \dot{p}_j = \frac{\left(\frac{\partial \vec{\Phi}_1}{\partial p_j} \dot{p}_j, \Psi_j^\dagger \right)}{\left(\Psi_j, \Psi_j^\dagger \right)} \Psi_j \quad (6.27)$$

$$= \frac{\left(\frac{\partial \vec{\Phi}_1}{\partial p_j}, \Psi_j^\dagger \right)}{\left(\Psi_j, \Psi_j^\dagger \right)} \Psi_j \dot{p}_j \quad (6.28)$$

$$= \pi_{\vec{p}_0} \frac{\partial \vec{\Phi}_1}{\partial p_j} \dot{p}_j \quad (6.29)$$

$$= \frac{\partial}{\partial p_j} \left(\pi_{\vec{p}_0} \vec{\Phi}_1 \right) \dot{p}_j \quad (6.30)$$

$$= 0 \quad (6.31)$$

since $\pi_{\vec{p}_0}$ and $\tilde{L}_{\vec{p}_0}$ commute, $\dot{p}_j$ is independent of x , and $\pi_{\vec{p}_0}$ is frozen, so the differential term can be removed. We impose the non-degeneracy condition $W \in X_{\vec{p}_0}$, which also implies $\pi_{\vec{p}_0} W_t = 0$, since $\pi_{\vec{p}_0}$ is independent of time. Since $W \in X_{\vec{p}_0}$, by definition we have

$$\pi_{\vec{p}_0} W = 0. \quad (6.32)$$

Also we have $\pi_{\vec{p}_0} \tilde{L}_{\vec{p}_0} W = \tilde{L}_{\vec{p}_0} \pi_{\vec{p}_0} W = 0$. Projecting both sides of (6.22) by $\pi_{\vec{p}_0}$, and applying the non-degeneracy condition, we have the following N equations:

$$\left(\frac{\partial \Phi}{\partial \vec{p}} \dot{\vec{p}}, \Psi_j^\dagger \right)_{L^2} = \left(R(\Phi) + \Delta L \left(\vec{\Phi}_1 + W \right) + \mathcal{N}(\vec{\Phi}_1, W), \Psi_j^\dagger \right)_{L^2}. \quad (6.33)$$

Using the L^∞ estimate on $\Psi_{j,1}^\dagger$ from (4.63), and (3.47), which assures that $\frac{\partial \Phi_1}{\partial p_j}$ is $O(1)$ in L^1 , then for the left side of (6.33) we have

$$\left(\frac{\partial \Phi}{\partial \vec{p}} \dot{\vec{p}}, \Psi_j^\dagger \right)_{L^2} = \left(-\|\phi'_0\|_{L^2}^2 Q^{-2\alpha_{21}/(\alpha_{22}-1)} + O(\epsilon^2) \right) \dot{\vec{p}}. \quad (6.34)$$

Then (6.33) is equivalent to

$$\left(-\|\phi'_0\|_{L^2}^2 Q^{\frac{-2\alpha_{21}}{\alpha_{22}-1}} + O(\epsilon^2) \right) \dot{\vec{p}} = \begin{pmatrix} (R(\Phi) + \Delta L \tilde{W} + \mathcal{N}(\vec{\Phi}_1, W), \Psi_1^\dagger)_{L^2} \\ \vdots \\ (R(\Phi) + \Delta L \tilde{W} + \mathcal{N}(\vec{\Phi}_1, W), \Psi_N^\dagger)_{L^2} \end{pmatrix}, \quad (6.35)$$

where $\tilde{W} = W + \vec{\Phi}_1$. Returning to (6.22), we apply the complimentary spectral projection. From our definition of $\vec{\Phi}_1$ in (4.3), we have

$$\tilde{\pi}_{\vec{p}_0} \left(R + \tilde{L}_{\vec{p}_0} \vec{\Phi}_1 \right) = \tilde{\pi}_{\vec{p}_0} \left(R - \tilde{\pi}_{\vec{p}_0} R \right) = \tilde{\pi}_{\vec{p}_0} \left(R - R + \pi_{\vec{p}_0} R \right) = 0. \quad (6.36)$$

So the evolution is given by

$$W_t = \tilde{R} + \tilde{L}_{\vec{p}_0} W + \tilde{\pi}_{\vec{p}_0} \left(\Delta L \left(W + \vec{\Phi}_1 \right) + \mathcal{N}(\vec{\Phi}_1, W) \right) \quad (6.37)$$

$$W(x, 0) = W_0, \quad (6.38)$$

where $\tilde{R} = -\tilde{\pi}_{\vec{p}_0} \left(\frac{\partial \Phi}{\partial \vec{p}} \dot{\vec{p}} \right) - \frac{\partial \vec{\Phi}_1}{\partial \vec{p}} \dot{\vec{p}}$ is the temporal component of the residual.

6.2 Decay of the remainder

In this section we establish uniform estimates on the decay of $\|W\|_X$ over the duration of each renormalization interval. We introduce the following two quantities:

$$T_1(t) = \sup_{t_0 < s < t} e^{\frac{\epsilon}{2} \mu(s-t_0)} \|W(s)\|_X \quad (6.39)$$

$$T_2(t) = \sup_{t_0 < s < t} |\vec{p}(s) - \vec{p}_0|. \quad (6.40)$$

Applying the variations of constants formula to (6.37), we have

$$W(x, t) = S(t-t_0)W_0 + \int_{t_0}^t S(t-s) \left(\tilde{R} + \tilde{\pi}_{\vec{p}_0} \left(\Delta L \left(\tilde{W} \right) + \mathcal{N}(\vec{\Phi}_1, W) \right) \right) ds, \quad (6.41)$$

where S is the semigroup generated by $\tilde{L}_{\vec{p}_0}$, and we recall the notation $\tilde{W} = W + \vec{\Phi}_1$.

In order to establish estimates on W in $\|\cdot\|_X$, we establish the following bounds.

Lemma 6.1. *Fix l_0 sufficiently large, then for all $\vec{p} \in \mathcal{K}$, there exists $C > 0$ such that*

$$\|[\Delta_s L \tilde{W}]_1\|_{L^1} \leq C \epsilon^{-\beta} T_2(t) \left(\|W\|_X + \epsilon^{2-\beta} \right) \quad (6.42)$$

$$\|[\Delta_s L \tilde{W}]_2\|_{L^2} \leq C T_2(t) \left(\|W\|_X + \epsilon^{2-\beta} \right) \quad (6.43)$$

$$|\otimes \vec{\chi} \cdot [\Delta_r L(W + \vec{\Phi}_1)]_1| \leq C \epsilon^{2-\beta} \left(\|W\|_X + \epsilon^{2-\beta} \right) \quad (6.44)$$

$$\|[\Delta L_r(W + \vec{\Phi}_1)]_1\|_{L^1_{1,\vec{p}}} \leq C \epsilon^{-\beta} \left(\|W\|_X + \epsilon^{2-\beta} \right) \quad (6.45)$$

$$\|[\Delta L_r(W + \vec{\Phi}_1)]_2\|_{L^2} \leq C \epsilon^{2-\beta} \left(\|W\|_X + \epsilon^{2-\beta} \right) \quad (6.46)$$

$$\|\mathcal{N}_1(\vec{\Phi}_1, W)\|_{L^2} \leq C \left(\|W\|_X^2 + \epsilon^{4-2\beta} \right) \quad (6.47)$$

$$\|\mathcal{N}_2(\vec{\Phi}_1, W)\|_{L^2} \leq C \left(\|W\|_X^2 + \epsilon^{4-2\beta} \right). \quad (6.48)$$

Proof: As before, we break ΔL into secular and reductive parts. We first examine the secular term $\Delta_s L$:

$$\begin{aligned} [\Delta_s L \tilde{W}]_1 &= \epsilon^{-\beta} \alpha_{11} \left(\Phi_{\vec{p},1}^{\alpha_{11}-1} \Phi_{\vec{p},2}^{\alpha_{12}} - \Phi_{\vec{p}_0,1}^{\alpha_{11}-1} \Phi_{\vec{p}_0,2}^{\alpha_{12}} \right) \tilde{W}_1 \\ &\quad + \epsilon^{-\beta} \alpha_{12} \left(\Phi_{\vec{p},1}^{\alpha_{11}} \Phi_{\vec{p},2}^{\alpha_{12}-1} - \Phi_{\vec{p}_0,1}^{\alpha_{11}} \Phi_{\vec{p}_0,2}^{\alpha_{12}-1} \right) \tilde{W}_2 \end{aligned} \quad (6.49)$$

$$\begin{aligned} [\Delta_s L \tilde{W}]_2 &= \alpha_{21} \left(\Phi_{\vec{p},1}^{\alpha_{21}-1} \Phi_{\vec{p},2}^{\alpha_{22}} - \Phi_{\vec{p}_0,1}^{\alpha_{21}-1} \Phi_{\vec{p}_0,2}^{\alpha_{22}} \right) \tilde{W}_1 \\ &\quad + \alpha_{22} \left(\Phi_{\vec{p},1}^{\alpha_{21}} \Phi_{\vec{p},2}^{\alpha_{22}-1} - \Phi_{\vec{p}_0,1}^{\alpha_{21}} \Phi_{\vec{p}_0,2}^{\alpha_{22}-1} \right) \tilde{W}_2. \end{aligned} \quad (6.50)$$

The Φ_2 terms are rapidly decaying away from each pulse position and Φ_1 is slowly varying. In particular, the Φ_2 terms are Lipschitz in $\vec{p}$ with an $O(1)$ constant while the slowly varying Φ_1 terms are Lipschitz in $\vec{p}$ with a small constant. However, Φ_1 always appears multiplied by Φ_2 , and the larger Lipschitz constant prevails. In particular, we estimate

$$\|\Phi_{\vec{p},2}^{\alpha_{22}} - \Phi_{\vec{p}_0,2}^{\alpha_{22}}\|_{H^1} + \|\Phi_{\vec{p},2}^{\alpha_{22}} - \Phi_{\vec{p}_0,2}^{\alpha_{22}}\|_{L^1} \leq C|\vec{p} - \vec{p}_0|. \quad (6.51)$$

Recalling the definition of T_2 from (6.40), we have

$$\begin{aligned} \|[\Delta_s L \tilde{W}]_1\|_{L^1} &\leq C\epsilon^{-\beta} \left(\|\Phi_{\vec{p},1}^{\alpha_{11}-1} \Phi_{\vec{p},2}^{\alpha_{12}} - \Phi_{\vec{p}_0,1}^{\alpha_{11}-1} \Phi_{\vec{p}_0,2}^{\alpha_{12}}\|_{L^1} \|\tilde{W}_1\|_{L^\infty} \right) \\ &\quad + C\epsilon^{-\beta} \left(\|(\Phi_{\vec{p},1}^{\alpha_{11}} \Phi_{\vec{p},2}^{\alpha_{12}-1} - \Phi_{\vec{p}_0,1}^{\alpha_{11}} \Phi_{\vec{p}_0,2}^{\alpha_{12}-1})\|_{L^2} \|\tilde{W}_2\|_{L^2} \right) \end{aligned} \quad (6.52)$$

$$\leq C\epsilon^{-\beta} T_2(t) \left(\|W\|_X + \epsilon^{2-\beta} \right), \quad (6.53)$$

where

$$\|\tilde{W}_1\|_{L^\infty} \leq \|W_1\|_{L^\infty} + \|\Phi_{1,1}\|_{L^\infty} \leq \|W\|_X + C\epsilon^{2-\beta}, \quad (6.54)$$

using the bound (5.49). Then we have established (6.42). The estimate (6.43) follows in the same manner, where

$$\|[\Delta_s L \tilde{W}]_2\|_{L^2} \leq C T_2(t) \left(\|W\|_X + \epsilon^{2-\beta} \right). \quad (6.55)$$

Next we examine the reductive term. The difference $L_{\vec{p}_0} - \tilde{L}_{\vec{p}_0}$ is large, but we are able to estimate this difference with the enhanced residual estimate (5.17). We choose the reduced linearization so it determines average values over each pulse region, so the differences of the two operators have little mass over each pulse region. The reductive term takes the form

$$[\Delta_r L \tilde{W}]_1 = \epsilon^{-\beta} \left(\alpha_{11} \Phi_{\vec{p}_0,1}^{\alpha_{11}-1} \Phi_{\vec{p}_0,2}^{\alpha_{12}} - J_{11}(\vec{p}_0) \right) \tilde{W}_1 \\ + \epsilon^{-\beta} \left(\alpha_{12} \Phi_{\vec{p}_0,1}^{\alpha_{11}} \Phi_{\vec{p}_0,2}^{\alpha_{12}-1} - J_{12}(\vec{p}_0) \right) \tilde{W}_2 \quad (6.56)$$

$$[\Delta_r L \tilde{W}]_2 = \left(\alpha_{21} \Phi_{\vec{p}_0,1}^{\alpha_{21}-1} \Phi_{\vec{p}_0,2}^{\alpha_{22}} - J_{21}(\vec{p}_0) \right) \tilde{W}_1 \\ + \alpha_{22} \left(\Phi_{\vec{p}_0,1}^{\alpha_{21}} \Phi_{\vec{p}_0,2}^{\alpha_{22}-1} - \sum_{j=1}^N \phi_0^{\alpha_{22}-1}(x - p_{j,0}) \right) \tilde{W}_2. \quad (6.57)$$

To show the estimate (6.44), we first define

$$\int \chi_j [\Delta_r L \tilde{W}]_1 dx = \mathfrak{L}(\tilde{W}_1) + \mathfrak{R}(\tilde{W}_2), \quad (6.58)$$

where

$$\mathfrak{L}(\tilde{W}_1) = \epsilon^{-\beta} \int \chi_j \left(\alpha_{11} \Phi_{\vec{p}_0,1}^{\alpha_{11}-1} \Phi_{\vec{p}_0,2}^{\alpha_{12}} - J_{11}(\vec{p}_0) \right) \tilde{W}_1 dx, \quad (6.59)$$

and

$$\mathfrak{R}(\tilde{W}_2) = \epsilon^{-\beta} \int \chi_j \left(\alpha_{12} \Phi_{\vec{p}_0,1}^{\alpha_{11}} \Phi_{\vec{p}_0,2}^{\alpha_{12}-1} - J_{12}(\vec{p}_0) \right) \tilde{W}_2 dx. \quad (6.60)$$

Next, we expand the potentials J_{11} and J_{12} underneath $\mathfrak{L}(\tilde{W}_1)$ and $\mathfrak{R}(\tilde{W}_2)$ respectively. For brevity, we only expand the $\mathfrak{R}(\tilde{W}_2)$ term.

$$\mathfrak{R}(\tilde{W}_2) = \epsilon^{-\beta} \alpha_{12} \int \chi_j \left(\Phi_{\vec{p}_0,1}^{\alpha_{11}} \Phi_{\vec{p}_0,2}^{\alpha_{12}-1} \tilde{W}_2 - \sum_j \int \chi_j \Phi_{\vec{p}_0,2}^{\alpha_{12}-1} q_j^{\alpha_{11}} \tilde{W}_2 dz \xi_j \right) dx \quad (6.61)$$

$$= \epsilon^{-\beta} \alpha_{12} \left(\int \chi_j \Phi_{\vec{p}_0,1}^{\alpha_{11}} \Phi_{\vec{p}_0,2}^{\alpha_{12}-1} \tilde{W}_2 dx - q_j^{\alpha_{11}} \int \chi_j \Phi_{\vec{p}_0,2}^{\alpha_{12}-1} \tilde{W}_2 dx \right), \quad (6.62)$$

since each ξ_j is mass one, and χ_j windows all ξ_k to zero except for $k = j$. Now we Taylor expand Φ_1 at each pulse location and use the fact that $\partial_x \Phi_1 = O(\epsilon^{2-\beta})$ in L^∞ near each pulse position. Again, for brevity, we continue with only the $\mathfrak{R}(\tilde{W}_2)$ term:

$$\begin{aligned} \mathfrak{R}(\tilde{W}_2) &= \epsilon^{-\beta} \alpha_{12} \int \chi_j \left(q_j^{\alpha_{11}} + \partial_x \left(\Phi_1^{\alpha_{11}} \right) (p') (p_{0,j} - p_j) \right) \Phi_{\vec{p}_0,2}^{\alpha_{12}-1} \tilde{W}_2 dx \\ &\quad - \epsilon^{-\beta} \alpha_{12} q_j^{\alpha_{11}} \int \chi_j \Phi_{\vec{p}_0,2}^{\alpha_{12}-1} \tilde{W}_2 dx \end{aligned} \quad (6.63)$$

$$\leq C \epsilon^{-\beta} \int \chi_j \partial_x \left(\Phi^{\alpha_{11}} \right) (p') (p_{0,j} - p_j) \Phi_{\vec{p}_0,2}^{\alpha_{12}-1} \tilde{W}_2 dx \quad (6.64)$$

$$\leq C \epsilon^{2-\beta} \|\chi_j \Phi_{\vec{p}_0,2}^{\alpha_{12}-1}\|_{L^1} \|\tilde{W}_2\|_{L^\infty} \quad (6.65)$$

$$\leq C \epsilon^{2-\beta} \left(\|W\|_X + \epsilon^{2-\beta} \right). \quad (6.66)$$

We have that

$$|\otimes \vec{\chi} \cdot [\Delta_r L(W + \vec{\Phi}_1)]_1| \leq C\epsilon^{2-\beta} \left(\|W\|_X + \epsilon^{2-\beta} \right). \quad (6.67)$$

Next we bound (6.45),

$$\begin{aligned} \|[\Delta L_r \tilde{W}]_1\|_{L^1_{1,\vec{p}}} &\leq C\epsilon^{-\beta} \left\| \left(\alpha_{11} \Phi_{\vec{p}_0,1}^{\alpha_{11}-1} \Phi_{\vec{p}_0,2}^{\alpha_{12}} - J_{11}(\vec{p}_0) \right) \tilde{W}_1 \right\|_{L^1_{1,\vec{p}}} \\ &\quad + C\epsilon^{-\beta} \left\| \left(\alpha_{12} \Phi_{\vec{p}_0,1}^{\alpha_{11}} \Phi_{\vec{p}_0,2}^{\alpha_{12}-1} - J_{12}(\vec{p}_0) \right) \tilde{W}_2 \right\|_{L^1_{1,\vec{p}}}. \end{aligned} \quad (6.68)$$

Since both terms follow similarly, for brevity we examine the first term:

$$C\epsilon^{-\beta} \left\| \left(\alpha_{11} \Phi_{\vec{p}_0,1}^{\alpha_{11}-1} \Phi_{\vec{p}_0,2}^{\alpha_{12}} - J_{11}(\vec{p}_0) \right) \tilde{W}_1 \right\|_{L^1_{1,\vec{p}}} \leq C\epsilon^{-\beta} \sum_{j=1}^N D_j(\tilde{W}_1), \quad (6.69)$$

where we notate

$$D_j(\tilde{W}_1) = \int \chi_j |(1 + |x - p_j|)| \left(\alpha_{11} \Phi_{\vec{p}_0,1}^{\alpha_{11}-1} \Phi_{\vec{p}_0,2}^{\alpha_{12}} - J_{11}(\vec{p}_0) \right) \tilde{W}_1 | dx. \quad (6.70)$$

Under the summation for any j :

$$\begin{aligned}
D_j(\tilde{W}_1) &\leq \int \chi_j |(1 + |x - p_j|)| \Phi_{\vec{p}_0,1}^{\alpha_{11}-1} \Phi_{\vec{p}_0,2}^{\alpha_{12}} \tilde{W}_1 | dx \\
&\quad + \int \chi_j |(1 + |x - p_j|)| \sum \int \chi_i \Phi_2^{\alpha_{12}} q_i^{\alpha_{11}-1} \tilde{W}_1 dz \xi_i dx \quad (6.71)
\end{aligned}$$

$$\begin{aligned}
&\leq C \|\chi_j (1 + |x - p_j|) \Phi_{\vec{p}_0,1}^{\alpha_{11}-1} \Phi_{\vec{p}_0,2}^{\alpha_{12}}\|_{L^1} \|\tilde{W}_1\|_{L^\infty} \\
&\quad + C \|\chi_j \Phi_2^{\alpha_{12}}\|_{L^1} \|\tilde{W}_1\|_{L^\infty} \int |\chi_j (1 + |x - p_j|) \xi_j| dx \quad (6.72)
\end{aligned}$$

$$\leq C \left(\|W\|_X + \epsilon^{2-\beta} \right), \quad (6.73)$$

where the linear growth term $(1 + |x - p_j|)$ is dominated by the exponentially decaying terms in each norm. Connecting the previous estimates we have:

$$\|[\Delta L_r(W + \vec{\Phi}_1)]_1\|_{L^1_{1,\vec{p}}} \leq C \epsilon^{-\beta} \left(\|W\|_X + \epsilon^{2-\beta} \right). \quad (6.74)$$

Finally, we bound (6.46)

$$\|[\Delta L_r \tilde{W}]_2\|_{L^2} \leq I(\tilde{W}_1) + J(\tilde{W}_2), \quad (6.75)$$

where we notate

$$I(\tilde{W}_1) = \|\alpha_{21} \left(\Phi_{\vec{p}_0,1}^{\alpha_{21}-1} \Phi_{\vec{p}_0,2}^{\alpha_{22}} - J_{21}(\vec{p}_0) \right) \tilde{W}_1\|_{L^2}, \quad (6.76)$$

and

$$J(\tilde{W}_2) = \|\alpha_{22} \left(\Phi_{\vec{p}_{0,1}}^{\alpha_{21}} \Phi_{\vec{p}_{0,2}}^{\alpha_{22}-1} - \sum_{j=1}^N \phi_0^{\alpha_{22}-1}(x - p_{j,0}) \right) \tilde{W}_2\|_{L^2}. \quad (6.77)$$

For the first term, we again Taylor expand Φ_1 underneath the summation:

$$I(\tilde{W}_1) = \left\| \left(\alpha_{21} \Phi_{\vec{p}_{0,1}}^{\alpha_{21}-1} \left(\sum_{j=1}^N \phi_j^{\alpha_{22}} + O(\epsilon^r) \right) - \alpha_{21} \sum_{j=1}^N q_j^{\alpha_{21}-1} \phi_j^{\alpha_{22}} \right) \tilde{W}_1 \right\|_{L^2} \quad (6.78)$$

$$\leq C \left\| \left(\sum_{j=1}^N \partial_x \left(\Phi_1^{\alpha_{21}-1} \right) (p'_j) (p_{0,j} - p_j) \phi_j^{\alpha_{22}} \right) \tilde{W}_1 \right\|_{L^2} \quad (6.79)$$

$$\leq C \|\partial_x \left(\Phi_1^{\alpha_{21}-1} \right)\|_{L^\infty} \left\| \sum_{j=1}^N (p_{0,j} - p_j) \phi_j^{\alpha_{22}} \tilde{W}_1 \right\|_{L^2} \quad (6.80)$$

$$\leq C \epsilon^{2-\beta} \|\tilde{W}_1\|_{L^\infty}. \quad (6.81)$$

Now we bound the other term again Taylor expanding Φ_1 about the pulse location:

$$J(\tilde{W}_2) = \left\| \left(\Phi_{\vec{p}_0,1}^{\alpha_{21}} \sum_{j=1}^N \phi_0^{\alpha_{22}-1} q_j^{-\alpha_{21}} - \sum_{j=1}^N \phi_0^{\alpha_{22}-1} \right) \tilde{W}_2 \right\|_{L^2} \quad (6.82)$$

$$= \left\| \left(\sum_{j=1}^N \phi_0^{\alpha_{22}-1} \frac{q_j^{\alpha_{21}} + O(\epsilon^{2-\beta})}{q_j^{\alpha_{21}}} - \sum_{j=1}^N \phi_0^{\alpha_{22}-1} \right) \tilde{W}_2 \right\|_{L^2} \quad (6.83)$$

$$\leq C \epsilon^{2-\beta} \left\| \sum_{j=1}^N \phi_0^{\alpha_{22}-1} \right\|_{L^2} \|\tilde{W}_2\|_{L^\infty} \quad (6.84)$$

$$\leq C \epsilon^{2-\beta} \left(\|W\|_X + \epsilon^{2-\beta} \right). \quad (6.85)$$

This gives us the following estimate:

$$\|[\Delta L_r(W + \vec{\Phi}_1)]_2\|_{L^2} \leq C \epsilon^{2-\beta} \left(\|W\|_X + \epsilon^{2-\beta} \right). \quad (6.86)$$

We estimate the first estimate on the nonlinearity in (6.47),

$$\begin{aligned}
\|\mathcal{N}_1(\vec{\Phi}_1, W)\|_{L^1} &\leq C\epsilon^{-\beta} \|\Phi_1^{\alpha_{11}-1} \Phi_2^{\alpha_{12}-1} \tilde{W}_1 \tilde{W}_2\|_{L^1} \\
&\quad + C\epsilon^{-\beta} \|\Phi_1^{\alpha_{11}} \Phi_2^{\alpha_{12}-2} \tilde{W}_2^2\|_{L^1} \\
&\quad + C\epsilon^{-\beta} \|\Phi_1^{\alpha_{11}-2} \Phi_2^{\alpha_{12}} \tilde{W}_1^2\|_{L^1} \tag{6.87}
\end{aligned}$$

$$\leq C\epsilon^{-\beta} \left(\|\tilde{W}_1\|_{L^\infty} \|\tilde{W}_2\|_{L^\infty} + \|\tilde{W}_2\|_{L^\infty}^2 + \|\tilde{W}_1\|_{L^\infty}^2 \right) \tag{6.88}$$

$$\leq C\epsilon^{2-\beta} \|\tilde{W}\|_X^2 \tag{6.89}$$

$$\leq C\epsilon^{2-\beta} \left(\|W\|_X^2 + \|\vec{\Phi}_1\|_X \|W\|_X + \|\vec{\Phi}_1\|_X^2 \right) \tag{6.90}$$

$$\leq C\epsilon^{2-\beta} \left(\|W\|_X^2 + \|\vec{\Phi}_1\|_X^2 \right) \tag{6.91}$$

$$\leq C\epsilon^{2-\beta} \left(\|W\|_X^2 + \epsilon^{4-2\beta} \right). \tag{6.92}$$

The next estimate follows in the same manner

$$\|\mathcal{N}_2(\vec{\Phi}_1, W)\|_{L^2} \leq C \left(\|W\|_X^2 + \epsilon^{4-2\beta} \right) \square \tag{6.93}$$

Then with these estimates, we have the following corollary.

Corollary 6.1. *Fix l_0 sufficiently large, then for all $\vec{p} \in \mathcal{K}$, there exists $C > 0$ such*

that

$$\|S(t-s) \left(\tilde{\pi}_{\vec{p}_0} \Delta_s L \tilde{W} \right) \|_X \leq C T_2(t) e^{-\frac{\epsilon^\alpha}{2} \mu(t-s)} \left(\|W\|_X + \epsilon^{2-\beta} \right) \quad (6.94)$$

$$\|S(t-s) \left(\tilde{\pi}_{\vec{p}_0} \Delta_r L \tilde{W} \right) \|_X \leq C e^{-\frac{\epsilon^\alpha}{2} \mu(t-s)} \epsilon^{2-\beta} \left(\|W\|_X + \epsilon^{2-\beta} \right) \quad (6.95)$$

$$\|S(t-s) \left(\tilde{\pi}_{\vec{p}_0} \mathcal{N}(\vec{\Phi}_1, W) \right) \|_X \leq C e^{-\frac{\epsilon^\alpha}{2} \mu(t-s)} \left(\|W\|_X^2 + \epsilon^{4-2\beta} \right). \quad (6.96)$$

Proof: The estimate (6.94) is achieved by using the semigroup estimate (5.64) and the estimates (6.42) and (6.43), where

$$\|S(t-s) \left(\tilde{\pi}_{\vec{p}_0} \Delta_s L \tilde{W} \right) \|_X \leq C e^{-\frac{\epsilon^\alpha}{2} \mu(t-s)} \left(\epsilon^\beta \|[\Delta_s L \tilde{W}]_1\|_{L^1} + \|[\Delta_s L \tilde{W}]_2\|_{L^2} \right) \quad (6.97)$$

$$\leq C e^{-\frac{\epsilon^\alpha}{2} \mu(t-s)} \left(\|W\|_X + \epsilon^{2-\beta} \right). \quad (6.98)$$

To achieve (6.95), we apply (5.66) and recall the estimates (6.44), (6.45), and (6.46).

This leads to

$$\begin{aligned}
\|S(t-s) \left(\tilde{\pi}_{\vec{p}_0} \Delta_r L \tilde{W} \right) \|_X &\leq C e^{-\frac{\epsilon^\alpha}{2} \mu t} \epsilon^\beta |\otimes \vec{\chi} \cdot [\Delta_r L(W + \vec{\Phi}_1)]_1| \\
&\quad + C e^{-\frac{\epsilon^\alpha}{2} \mu t} \epsilon^2 \|[\Delta_r L(W + \vec{\Phi}_1)]_1\|_{L^1_{1,\vec{p}}} \\
&\quad + C e^{-\frac{\epsilon^\alpha}{2} \mu t} \|[\Delta_r L(W + \vec{\Phi}_1)]_2\|_{L_2} \tag{6.99}
\end{aligned}$$

$$\leq C e^{-\frac{\epsilon^\alpha}{2} \mu t} \epsilon^{2-\beta} \left(\|W\|_X + \epsilon^{2-\beta} \right). \tag{6.100}$$

We apply (5.66) to (6.96), recalling the estimates (6.47) and (6.48),

$$\|S(t-s) \left(\tilde{\pi}_{\vec{p}_0} \mathcal{N}(\tilde{W}) \right) \|_X \leq C e^{-\frac{\epsilon^\alpha}{2} \mu(t-s)} \left(\epsilon^\beta \|\mathcal{N}_1(\tilde{W})\|_{L^1} + \|\mathcal{N}_2(\tilde{W})\|_{L^2} \right) \tag{6.101}$$

$$\leq C e^{-\frac{\epsilon^\alpha}{2} \mu(t-s)} \left(\|W\|_X^2 + \epsilon^{4-2\beta} \right) \square \tag{6.102}$$

We return to the previous projected equation in (6.35). The next lemma estimates terms on the right side of (6.35).

Lemma 6.2. *We have the following estimates for any j :*

$$\left(R_1, \Psi_{1,j}^\dagger\right)_{L^2} \leq C\epsilon^{4-2\beta} \quad (6.103)$$

$$\left([\Delta L(\vec{\Phi}_1 + W)]_1, \Psi_{1,j}^\dagger\right)_{L^2} \leq C\epsilon^{2-2\beta} \left(\|W\|_X + \epsilon^{2-\beta}\right) \quad (6.104)$$

$$\left(\mathcal{N}_1(\vec{\Phi}_1, W), \Psi_{1,j}^\dagger\right)_{L^2} \leq C\epsilon^{2-\beta} \left(\|W\|_X^2 + \epsilon^{4-2\beta}\right). \quad (6.105)$$

Proof: For (6.103), we use (3.8) and have

$$\left(R_1, \Psi_{1,j}^\dagger\right)_{L^2} \leq \|R_1\|_{L^1} \|\Psi_{1,j}^\dagger\|_{L^\infty} \leq C\epsilon^{4-2\beta}. \quad (6.106)$$

For (6.104), we write $\Delta L = \Delta_s L + \Delta_r L$, and estimate the secular terms first. We use the estimate (6.42),

$$\left([\Delta_s L \tilde{W}]_1, \Psi_{1,j}^\dagger\right)_{L^2} \leq \|[\Delta_s L \tilde{W}]_1\|_{L^1} \|\Psi_{1,j}^\dagger\|_{L^\infty} \quad (6.107)$$

$$\leq C\epsilon^{2-\beta} \left(\|W\|_X + \epsilon^{2-\beta}\right). \quad (6.108)$$

To complete the estimate of (6.104), we apply the estimate (6.45),

$$\left([\Delta_r L\tilde{W}]_1, \Psi_{1,j}^\dagger\right)_{L^2} \leq C \|[\Delta_r L\tilde{W}]_1\|_{L^1} \|\Psi_{1,j}^\dagger\|_{L^\infty} \quad (6.109)$$

$$\leq C \epsilon^{2-\beta} \|[\Delta_r L\tilde{W}]_1\|_{L^1_{1,\vec{p}}} \quad (6.110)$$

$$\leq C \epsilon^{2-2\beta} \left(\|W\|_X + \epsilon^{2-\beta}\right). \quad (6.111)$$

To estimate (6.105), we use the estimate (6.47),

$$\left(\mathcal{N}_1(\vec{\Phi}_1, W), \Psi_{1,j}^\dagger\right)_{L^2} \leq \|\mathcal{N}_1(\vec{\Phi}_1, W)\|_{L^1} \|\Psi_{1,j}^\dagger\|_{L^\infty} \quad (6.112)$$

$$\leq C \epsilon^{2-\beta} \left(\|W\|_X^2 + \epsilon^{4-2\beta}\right) \square \quad (6.113)$$

Applying the previous lemma and using (4.63) to expand $\Psi_{2,j}^\dagger$ in (6.33), we have the following equations for the evolution of the pulse position for each j :

$$\begin{aligned} \dot{p}_j = & - \frac{\left(R_2(\Phi) + [\Delta L(W + \vec{\Phi}_1)]_2 + \mathcal{N}_2(\vec{\Phi}_1, W), \phi'_j\right)_{L^2}}{q_j^{-2\alpha_{21}/(\alpha_{22}-1)} \|\phi'_0\|_{L^2}^2} \left(1 + O(\epsilon^2)\right) \\ & + O(\epsilon^{4-2\beta}, \epsilon^{2-2\beta} \|W\|_X, \epsilon^{2-\beta} \|W\|_X^2). \end{aligned} \quad (6.114)$$

Finally, we prove the following reduced residual estimate:

Lemma 6.3. *There exists $C > 0$ such that*

$$\|S(t-s)\tilde{\pi}_{\vec{p}_0}\tilde{R}\|_X \leq C\epsilon^2 e^{-\frac{\epsilon^\alpha}{2}\mu(t-s)}. \quad (6.115)$$

Proof: Using (5.64), we have

$$\begin{aligned} \|S(t-s)\tilde{\pi}_{\vec{p}_0}\tilde{R}\|_X &\leq C\epsilon^\beta e^{-\frac{\epsilon^\alpha}{2}\mu(t-s)} \|[\tilde{\pi}_{\vec{p}_0} \left(\frac{\partial\Phi}{\partial\vec{p}}\dot{\vec{p}} \right) + \frac{\partial\vec{\Phi}_1}{\partial\vec{p}}\dot{\vec{p}}]_1\|_{L^1} \\ &\quad + C e^{-\frac{\epsilon^\alpha}{2}\mu(t-s)} \|[\tilde{\pi}_{\vec{p}_0} \left(\frac{\partial\Phi}{\partial\vec{p}}\dot{\vec{p}} \right) + \frac{\partial\vec{\Phi}_1}{\partial\vec{p}}\dot{\vec{p}}]_2\|_{L^2}. \end{aligned} \quad (6.116)$$

We assume that $\dot{\vec{p}}$ is $O(\epsilon^{2-\beta})$ (we justify this later). First we examine the $\frac{\partial\vec{\Phi}_1}{\partial\vec{p}}$ terms, where by the definition of $\vec{\Phi}_1$,

$$\frac{\partial\vec{\Phi}_1}{\partial\vec{p}} = -\frac{\partial}{\partial\vec{p}}\tilde{L}_{\vec{p}_0}^{-1} \left(\pi_{\vec{p}_0} R(\Phi) \right) \quad (6.117)$$

$$= -\tilde{L}_{\vec{p}_0}^{-1} \left(\pi_{\vec{p}_0} \frac{\partial R(\Phi)}{\partial\vec{p}} \right), \quad (6.118)$$

where we can pull the differential through the linear operator and the projection because they are both frozen at $\vec{p} = \vec{p}_0$. Also, we have

$$\frac{\partial\vec{\Phi}_1}{\partial\vec{p}}\dot{\vec{p}} = \left(\sum_{j=1}^N \frac{\partial\vec{\Phi}_{1,1}}{\partial p_j} \dot{p}_j, \sum_{j=1}^N \frac{\partial\vec{\Phi}_{1,2}}{\partial p_j} \dot{p}_j \right)^T. \quad (6.119)$$

Then using our assumption on the size of $\dot{\vec{p}}$,

$$\epsilon^\beta \left(\left\| \left[\frac{\partial \vec{\Phi}_1}{\partial \vec{p}} \dot{\vec{p}} \right]_1 \right\|_{L^1} + \left\| \left[\frac{\partial \vec{\Phi}_1}{\partial \vec{p}} \dot{\vec{p}} \right]_2 \right\|_{L^2} \right) \leq \sum_{j=1}^N (\epsilon^\beta \left(\left\| \left[\frac{\partial \vec{\Phi}_1}{\partial p_j} \right]_1 \right\|_{L^1} + \left\| \left[\frac{\partial \vec{\Phi}_1}{\partial p_j} \right]_2 \right\|_{L^2} \right) \|\dot{p}_j\|_{L^\infty}) \quad (6.120)$$

$$\leq C\epsilon^2, \quad (6.121)$$

if for each j from 1 to N :

$$\left\| \left[\frac{\partial \vec{\Phi}_1}{\partial p_j} \right]_1 \right\|_{L^1} \leq C \quad (6.122)$$

$$\left\| \left[\frac{\partial \vec{\Phi}_1}{\partial p_j} \right]_2 \right\|_{L^2} \leq C\epsilon^\beta. \quad (6.123)$$

Examining the first of the two inequalities above, and using (3.12):

$$\left\| \left[\frac{\partial \vec{\Phi}_1}{\partial p_j} \right]_1 \right\|_{L^1} = \left\| \left[\tilde{L}_{\vec{p}_0}^{-1} \pi_{\vec{p}_0} \frac{\partial R(\Phi)}{\partial p_j} \right]_1 \right\|_{L^1} \leq C\epsilon^{-\alpha} \left\| \left[\pi_{\vec{p}_0} \frac{\partial R(\Phi)}{\partial p_j} \right]_1 \right\|_{L^1}, \quad (6.124)$$

where

$$\left\| \left[\pi_{\vec{p}0} \frac{\partial R(\Phi)}{\partial p_j} \right]_1 \right\|_{L^1} \leq C \left\| \sum_{i=1}^N \left(\left(\frac{\partial R_1(\Phi)}{\partial p_j}, \Psi_{i,1}^\dagger \right) + \left(\frac{\partial R_2(\Phi)}{\partial p_j}, \Psi_{i,2}^\dagger \right) \right) \Psi_{i,1} \right\|_{L^1} \quad (6.125)$$

$$\begin{aligned} &\leq C \sum_{i=1}^N \left\| \frac{\partial R_1(\Phi)}{\partial p_j} \right\|_{L^1} \|\Psi_{i,1}^\dagger\|_{L^\infty} \|\Psi_{i,1}\|_{L^1} \\ &+ C \sum_{i=1}^N \left\| \frac{\partial R_2(\Phi)}{\partial p_j} \right\|_{L^2} \|\Psi_{i,2}^\dagger\|_{L^2} \|\Psi_{i,1}\|_{L^1} \end{aligned} \quad (6.126)$$

$$\leq C\epsilon^2 \left\| \frac{\partial R_1(\Phi)}{\partial p_j} \right\|_{L^1} + \left\| \frac{\partial R_2(\Phi)}{\partial p_j} \right\|_{L^2}. \quad (6.127)$$

By expanding the definition of the residuals underneath the norm above, we have

$$\begin{aligned} \partial_{p_k} R_1(\Phi) &= \partial_{p_k} \left(-L_{11}^e \Phi_1 + \epsilon^{-\beta} \sum_{j=1}^N q_j^{\alpha_{11}} \phi_j^{\alpha_{12}} \right) \\ &+ \epsilon^{-\beta} \partial_{p_k} \left(\sum_{j=1}^N (\Phi_1^{\alpha_{11}} - q_j^{\alpha_{11}}) \phi_j^{\alpha_{12}} \right) \\ &+ \epsilon^{-\beta} \partial_{p_k} \left(\Phi_1^{\alpha_{11}} \left(\left(\sum_{j=1}^N \phi_j \right)^{\alpha_{12}} - \sum_{j=1}^N \phi_j^{\alpha_{12}} \right) \right). \end{aligned} \quad (6.128)$$

Similar to the proof of the residual estimates in Chapter 3, the first term is 0. Since everything in the third term is smooth and continuous, it follows as in the previous residual estimates that this term is $O(\epsilon^r)$ for $r \geq 2$. The middle term remains, where

as in the residual estimates, we Taylor expand about p_j (here $'$ denotes the derivative with respect to x):

$$(\Phi_1^{\alpha 11} - q_j^{\alpha 11})\phi_j^{\alpha 12} = (x - p_j) \left(\Phi_1^{\alpha 11} \right)' (p_j) \phi_j^{\alpha 12} + h.o.t.. \quad (6.129)$$

Then at leading order,

$$\partial_{p_k} \left(\sum_{j=1}^N (\Phi_1^{\alpha 11} - q_j^{\alpha 11}) \phi_j^{\alpha 12} \right) = \partial_{p_k} \left(\sum_{j=1}^N (x - p_j) \left(\Phi_1^{\alpha 11} \right)' (p_j) \phi_j^{\alpha 12} \right) \quad (6.130)$$

$$= \partial_{p_k} \left((x - p_k) \left(\Phi_1^{\alpha 11} \right)' (p_k) \phi_k^{\alpha 12} \right) \quad (6.131)$$

$$= - \left(\Phi_1^{\alpha 11} \right)' (p_k) \phi_k^{\alpha 12} + (x - p_k) \partial_{p_k} \left(\Phi_1^{\alpha 11} \right)' (p_k) \phi_k^{\alpha 12} \quad (6.132)$$

$$+ (x - p_k) \left(\Phi_1^{\alpha 11} \right)' (p_k) \partial_{p_k} \phi_k^{\alpha 12}. \quad (6.133)$$

By (3.44), we have

$$\| \left(\Phi_1^{\alpha 11} \right)' (p_k) \phi_k^{\alpha 12} \|_{L^1} \leq \| \left(\Phi_1^{\alpha 11} \right)' (p_k) \|_{L^\infty} \| \phi_k^{\alpha 12} \|_{L^1} \leq C \epsilon^{2-\beta}. \quad (6.134)$$

For the third term in (6.133), we know from (3.79) that in any L^p norm

$$\partial_{p_k} \phi_k = -\phi'_k + O(\epsilon^{2-\beta}). \quad (6.135)$$

Then

$$\|(\Phi_1^{\alpha_{11}})'(p_k)\|_{L^\infty} \|(x - p_k)(\Phi_1^{\alpha_{11}})'(p_k) \partial_{p_k} \phi_k^{\alpha_{12}}\|_{L^1} \leq C\epsilon^{2-\beta}, \quad (6.136)$$

since the exponential decay in $\partial_{p_k} \phi_k^{\alpha_{12}}$ dominates any linear growth. For the second term in (6.132), we use (3.48), so we have

$$\|(x - p_k) \partial_{p_k} (\Phi_1^{\alpha_{11}})'(p_k) \phi_k^{\alpha_{12}}\|_{L^1} \leq \|\partial_{p_k} (\Phi_1^{\alpha_{11}})'(p_k)\|_{L^\infty} \|(x - p_k) \phi_k^{\alpha_{12}}\|_{L^1} \quad (6.137)$$

$$\leq C\epsilon^{2-\beta}. \quad (6.138)$$

Combining these estimates in (6.128), we have

$$\|\partial_{p_k} R_1(\Phi)\|_{L^1} \leq C\epsilon^\alpha. \quad (6.139)$$

The estimate for the R_2 term follows similar to the above, where

$$\|\partial_{p_k} R_2(\Phi)\|_{L^1} \leq C\epsilon^{2-\beta}. \quad (6.140)$$

Combining these estimates, we have

$$\| \left[\pi_{\vec{p}0} \frac{\partial R(\Phi)}{\partial p_j} \right]_1 \|_{L^1} \leq C\epsilon^{2-\beta}. \quad (6.141)$$

Then combining these two estimates with (6.125) and (6.124), we have achieved a stronger condition than (6.122):

$$\| [\frac{\partial \vec{\Phi}_1}{\partial p_j}]_1 \|_{L^1} \leq C\epsilon^\beta. \quad (6.142)$$

Next we establish (6.123). Using (3.13):

$$\| [\frac{\partial \vec{\Phi}_1}{\partial p_j}]_2 \|_{L^2} = \| \left[\tilde{L}_{\vec{p}0}^{-1} \pi_{\vec{p}0} \frac{\partial R(\Phi)}{\partial p_j} \right]_2 \|_{L^2} \leq C\epsilon^{-\alpha} \| \left[\pi_{\vec{p}0} \frac{\partial R(\Phi)}{\partial p_j} \right]_2 \|_{L^2}. \quad (6.143)$$

Similar to the previous term

$$\begin{aligned} \| \left[\pi_{\vec{p}0} \frac{\partial R(\Phi)}{\partial p_j} \right]_2 \|_{L^2} &\leq C \sum_{i=1}^N \| \frac{\partial R_1(\Phi)}{\partial p_j} \|_{L^1} \| \Psi_{i,1}^\dagger \|_{L^\infty} \| \Psi_{i,2} \|_{L^2} \\ &\quad + C \sum_{i=1}^N \| \frac{\partial R_2(\Phi)}{\partial p_j} \|_{L^2} \| \Psi_{i,2}^\dagger \|_{L^2} \| \Psi_{i,2} \|_{L^2} \end{aligned} \quad (6.144)$$

$$\leq C\epsilon^2 \| \frac{\partial R_1(\Phi)}{\partial p_j} \|_{L^1} + \| \frac{\partial R_2(\Phi)}{\partial p_j} \|_{L^2}. \quad (6.145)$$

The using (6.139) and (6.140), we have (6.123):

$$\|[\frac{\partial \vec{\Phi}_1}{\partial p_j}]_2\|_{L^2} \leq C\epsilon^\beta. \quad (6.146)$$

Now we examine the $\frac{\partial \Phi}{\partial \vec{p}}$ terms:

$$\|[\tilde{\pi}_{\vec{p}_0} \frac{\partial \Phi}{\partial \vec{p}} \dot{\vec{p}}]_1\|_{L^1} = \|\frac{\partial \Phi_1}{\partial \vec{p}} \dot{\vec{p}} - [\pi_{\vec{p}_0} \frac{\partial \Phi}{\partial \vec{p}} \dot{\vec{p}}]_1\|_{L^1} \quad (6.147)$$

$$\begin{aligned} &\leq \|\frac{\partial \Phi_1}{\partial \vec{p}} \dot{\vec{p}}\|_{L^1} + C \sum_{j=1}^N \|\frac{\partial \Phi_1}{\partial \vec{p}} \dot{\vec{p}}\|_{L^1} \|\Psi_{1,j}^\dagger\|_{L^\infty} \|\Psi_{1,j}\|_{L^1} \\ &+ C \sum_{j=1}^N \|\frac{\partial \Phi_2}{\partial \vec{p}} \dot{\vec{p}}\|_{L^2} \|\Psi_{2,j}^\dagger\|_{L^2} \|\Psi_{1,j}\|_{L^1} \end{aligned} \quad (6.148)$$

$$\leq C(1 + \epsilon^2) \|\frac{\partial \Phi_1}{\partial \vec{p}} \dot{\vec{p}}\|_{L^1} + C \|\frac{\partial \Phi_2}{\partial \vec{p}} \dot{\vec{p}}\|_{L^2} \quad (6.149)$$

$$\leq C \sum_{j=1}^N \left(\|\frac{\partial \Phi_1}{\partial p_j} \dot{p}_j\|_{L^1} + \|\frac{\partial \Phi_2}{\partial p_j} \dot{p}_j\|_{L^2} \right) \quad (6.150)$$

$$\leq C \sum_{j=1}^N \left(\|\frac{\partial \Phi_1}{\partial p_j}\|_{L^1} + \|\frac{\partial \Phi_2}{\partial p_j}\|_{L^2} \right) \|\dot{p}_j\|_{L^\infty} \quad (6.151)$$

$$\leq C\epsilon^{2-\beta}, \quad (6.152)$$

using (3.47), (3.79), the form of the small eigenfunctions in (4.62), and the estimates (4.63) on the adjoint eigenfunctions. From (3.79) and (4.62), we have in any L^p

norm,

$$\frac{\partial \Phi_2}{\partial p_j} = -\Psi_{2,j} + O(\epsilon^{2-\beta}). \quad (6.153)$$

Using this, we have

$$[\tilde{\pi}_{\vec{p}0} \frac{\partial \Phi}{\partial \vec{p}} \dot{\vec{p}}]_2 = \frac{\partial \Phi_2}{\partial \vec{p}} \dot{\vec{p}} - [\pi_{\vec{p}0} \frac{\partial \Phi}{\partial \vec{p}} \dot{\vec{p}}]_2 \quad (6.154)$$

$$= - \sum_{j=1}^N \Psi_{2,j} \dot{p}_j - \sum_{j=1}^N \frac{\left(\frac{\partial \Phi_1}{\partial \vec{p}} \dot{\vec{p}}, \Psi_{1,j}^\dagger \right) + \left(\frac{\partial \Phi_2}{\partial \vec{p}} \dot{\vec{p}}, \Psi_{2,j}^\dagger \right)}{\left(\Psi_{2,j}, \Psi_{2,j}^\dagger \right)} \Psi_{2,j} \quad (6.155)$$

$$= - \sum_{j=1}^N \Psi_{2,j} \dot{p}_j + \frac{\left(\frac{\partial \Phi_1}{\partial \vec{p}} \dot{\vec{p}}, \Psi_{1,j}^\dagger \right)}{\left(\Psi_{2,j}, \Psi_{2,j}^\dagger \right)} \Psi_{2,j} \\ + \sum_{j=1}^N \frac{\left(\Psi_{2,j} - O(\epsilon^{2-\beta}), \Psi_{2,j}^\dagger \right) \dot{p}_j}{\left(\Psi_{2,j}, \Psi_{2,j}^\dagger \right)} \Psi_{2,j} \quad (6.156)$$

$$= - \sum_{j=1}^N \frac{\left(\frac{\partial \Phi_1}{\partial \vec{p}} \dot{\vec{p}}, \Psi_{1,j}^\dagger \right)}{\left(\Psi_{2,j}, \Psi_{2,j}^\dagger \right)} \Psi_{2,j} + O(\epsilon^{2-\beta}) \frac{\left(1, \Psi_{2,j}^\dagger \right) \dot{p}_j}{\left(\Psi_{2,j}, \Psi_{2,j}^\dagger \right)} \Psi_{2,j}. \quad (6.157)$$

Then once again in the L^2 norm

$$\begin{aligned} \|[\tilde{\pi}_{\vec{p}} \frac{\partial \Phi}{\partial \vec{p}} \dot{\vec{p}}]_2\|_{L^2} &\leq \left\| \sum_{j=1}^N \frac{\left(\frac{\partial \Phi_1}{\partial \vec{p}} \dot{\vec{p}}, \Psi_{1,j}^\dagger \right)}{\left(\Psi_{2,j}, \Psi_{2,j}^\dagger \right)} \Psi_{2,j} \right\|_{L^2} \\ &\quad + \epsilon^{2-\beta} \left\| \frac{\left(1, \Psi_{2,j}^\dagger \right) \dot{p}_j}{\left(\Psi_{2,j}, \Psi_{2,j}^\dagger \right)} \Psi_{2,j} \right\|_{L^2} \end{aligned} \quad (6.158)$$

$$\begin{aligned} &\leq C \sum_{j=1}^N \left\| \frac{\partial \Phi_1}{\partial \vec{p}} \dot{\vec{p}} \right\|_{L^1} \|\Psi_{1,j}^\dagger\|_{L^\infty} \|\Psi_{2,j}\|_{L^2} \\ &\quad + \epsilon^{2-\beta} \|\Psi_{2,j}^\dagger\|_{L^1} \|\dot{p}_j\|_{L^\infty} \|\Psi_{2,j}\|_{L^2} \end{aligned} \quad (6.159)$$

$$\leq C \epsilon^{4-\beta}. \quad (6.160)$$

If we combine these results, we establish this lemma $\square$

To develop estimates on $\|W\|_X$, we take the X -norm of (6.41) and use the semigroup estimate (5.65) on the initial term. We obtain

$$\begin{aligned} \|W(t)\|_X &\leq C e^{-\frac{\epsilon^\alpha}{2} \mu(t-t_0)} \ln(\epsilon^{-\alpha}) \|W(t_0)\|_X + C \int_{t_0}^t e^{-\frac{\epsilon^\alpha}{2} \mu(t-s)} \|W\|_X^2 ds \\ &\quad + C \int_{t_0}^t e^{-\frac{\epsilon^\alpha}{2} \mu(t-s)} \left[\epsilon^2 + \left(T_2(t) + \epsilon^{2-\beta} \right) \left(\epsilon^{2-\beta} + \|W\|_X \right) \right] ds, \end{aligned} \quad (6.161)$$

where T_2 is defined in (6.40). We evaluate the previous line at $t = t'$, multiply by $e^{\frac{\epsilon^\alpha}{2} \mu(t'-t_0)}$, and take the sup over $t' \in (t_0, t)$. Recalling the definition of T_1 in

(6.39), we find

$$\begin{aligned}
T_1(t) \leq & C \left(|\ln(\epsilon^{-\alpha})| T_1(t_0) + \int_{t_0}^t \epsilon^{2-\beta} (T_2(t) + \epsilon^\beta) e^{\frac{\epsilon^\alpha}{2} \mu(s-t_0)} ds \right) \\
& + C \left(\int_{t_0}^t (T_2(t) + \epsilon^{2-\beta}) T_1(t) + T_1(t)^2 e^{\frac{\epsilon^\alpha}{2} \mu(t_0-s)} ds \right) \quad (6.162)
\end{aligned}$$

$$\begin{aligned}
\leq & C \left(|\ln(\epsilon^{-\alpha})| T_1(t_0) + \epsilon^{2-\beta-\alpha} (T_2(t) + \epsilon^\beta) e^{\frac{\epsilon^\alpha}{2} \mu(\Delta t)} \right) \\
& + C \left((T_2(t) + \epsilon^{2-\beta}) \Delta t T_1(t) + \epsilon^{-\alpha} T_1(t)^2 \right), \quad (6.163)
\end{aligned}$$

where $\Delta t = t - t_0$. The following lemma allows us to bound T_2 in terms of T_1 , Δt , and ϵ .

Lemma 6.4.

$$T_2(t) \leq C_1 (\epsilon^{2-\beta} \Delta t + \epsilon^2 + \epsilon^{-\alpha} T_1^2). \quad (6.164)$$

Proof: Applying the Mean Value Theorem, to the definition (6.40) of T_2 , we find

$$T_2(t) \leq \int_{t_0}^{t_0+\Delta t} |\dot{p}(s)| ds. \quad (6.165)$$

Recalling (6.114), the residual estimate from (3.96), the estimates on $[\Delta L(W +$

$\vec{\Phi}_1)]_2$ in (6.43) and (6.46), and the estimate on $\mathcal{N}_2(\vec{\Phi}_1, W)$ in (6.48), we have

$$\begin{aligned} T_2(t) &\leq C \int_{t_0}^{t_0+\Delta t} \|R_2(\Phi)\|_{L^2} + T_1(t)^2 e^{\epsilon^\alpha \mu(t_0-s)} ds \\ &\quad + C \int_{t_0}^{t_0+\Delta t} \left(T_2(t) + \epsilon^{2-\beta} \right) \left(e^{\frac{\epsilon^\alpha}{2} \mu(t_0-s)} T_1(t) + \epsilon^{2-\beta} \right) ds \end{aligned} \quad (6.166)$$

$$\leq C \left(\epsilon^{2-\beta} \Delta t + \left(T_2(t) + \epsilon^{2-\beta} \right) \left(\epsilon^{-\alpha} T_1(t) + \epsilon^{2-\beta} \Delta t \right) + \epsilon^{-\alpha} T_1(t)^2 \right). \quad (6.167)$$

To control T_2 , we impose the following constraint on T_1

$$T_1(t) \leq \frac{\epsilon^\alpha}{2C}, \quad (6.168)$$

where the constant C , from the right side of (6.167), is independent of $\vec{p} \in \mathcal{K}$ and ϵ , but may depend upon l_0 , the minimal pulse separation. We will show that the set

$$\left\{ t \mid T_1(t) \leq \frac{\epsilon^\alpha}{2C} \right\}, \quad (6.169)$$

is a forward invariant set under the flow.

Using this constraint (6.168), the $\epsilon^{-\alpha} T_1(t) T_2(t)$ term may be subtracted from

both sides, and multiplying by 2, we have

$$T_2(t) \leq C \left(\epsilon^{2-\beta} \Delta t + \epsilon^{2-\beta} \Delta t T_2(t) + \epsilon^\beta T_1(t) + \epsilon^{4-2\beta} \Delta t + \epsilon^{-\alpha} T_1(t)^2 \right). \quad (6.170)$$

Similarly, for Δt satisfying the constraint

$$\Delta t \leq \frac{\epsilon^{\beta-2}}{2C}, \quad (6.171)$$

we can remove the $\epsilon^{2-\beta} \Delta t T_2(t)$ term and eliminate T_2 from the right side of (6.170).

Using Young's inequality on the $\epsilon^\beta T_1(t)$ term, that is

$$\epsilon^\beta T_1(t) = \epsilon^{\beta+\alpha/2} \epsilon^{-\alpha/2} T_1(t) = \epsilon \epsilon^{-\alpha/2} T_1(t) \leq C \left(\epsilon^2 + \epsilon^{-\alpha} T_1(t)^2 \right), \quad (6.172)$$

we arrive at the statement of the lemma

$$T_2(t) \leq C_1 \left(\epsilon^{2-\beta} \Delta t + \epsilon^2 + \epsilon^{-\alpha} T_1(t)^2 \right) \square \quad (6.173)$$

Applying Lemma 6.4 to (6.163), we obtain

$$\begin{aligned} T_1(t) \leq & C \left(|\ln(\epsilon^{-\alpha})| T_1(t_0) + \epsilon^\beta (\epsilon^{2-\beta} \Delta t + \epsilon^2 + \epsilon^{-\alpha} T_1(t)^2 + \epsilon^\beta) e^{\frac{\epsilon^\alpha}{2} \mu(\Delta t)} \right) \\ & + C \left((\epsilon^{-\alpha} T_1(t)^2 + \epsilon^{2-\beta} (1 + \Delta t)) \Delta t T_1(t) + \epsilon^{-\alpha} T_1(t)^2 \right). \end{aligned} \quad (6.174)$$

To control T_1 , we fix $0 < \omega < 1$ and impose an additional, more exigent, constraint on Δt

$$\Delta t \leq \frac{2}{\mu} \epsilon^{-\alpha} \ln \left(\epsilon^{-\omega\beta} \right), \quad (6.175)$$

so that, in particular

$$e^{\frac{\epsilon^\alpha}{2} \mu(\Delta t)} \leq \epsilon^{-\omega\beta}, \quad (6.176)$$

and from (6.168),

$$\Delta t T_1 = O(|\ln \epsilon|). \quad (6.177)$$

Imposing this condition, and keeping only leading order terms in (6.174) we obtain

$$T_1(t) \leq C|\ln \epsilon| \left(T_1(t_0) + \epsilon^\beta + \epsilon^{-\alpha} T_1(t)^2 \right). \quad (6.178)$$

The corresponding quadratic equation in r ,

$$g(r) = C|\ln \epsilon| \left(T_1(t_0) + \epsilon^\beta \right) - r + C|\ln \epsilon| \epsilon^{-\alpha} r^2, \quad (6.179)$$

has two positive roots $0 < r_1 < r_2$ if ϵ is sufficiently small, if

$$\beta - \alpha > 0, \quad (6.180)$$

and if

$$\epsilon^{-\alpha} T_1(t_0) |\ln \epsilon|^2 \quad (6.181)$$

is sufficiently small. In light of the assumption that $1 - \alpha/2 - \beta = 0$ from (1.20), then (6.180) implies

$$\alpha < 2/3. \quad (6.182)$$

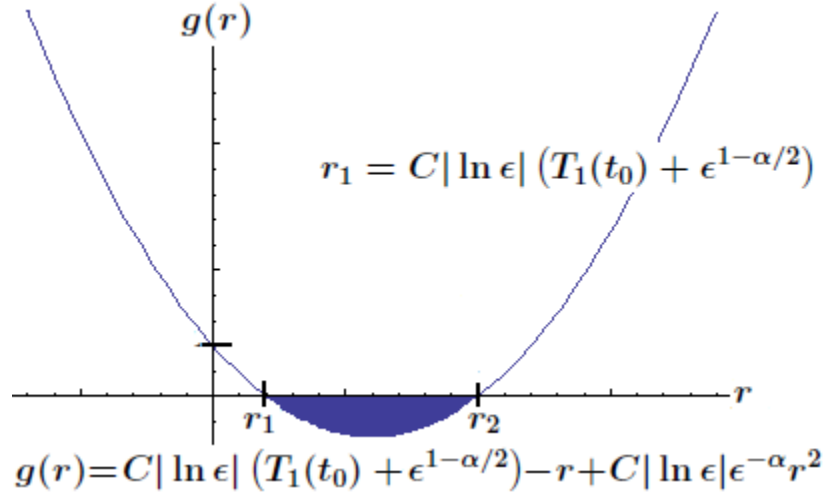


Figure 6.1: This illustrates the quadratic function $g(r)$, where either the remainder starts smaller than r_1 and stays small or it begins larger the r_2 . The middle interval (r_1, r_2) is forbidden.

The constraint (6.181) becomes

$$T_1(0) \leq \frac{C_0 \epsilon^\alpha}{|\ln(\epsilon)|^2}, \quad (6.183)$$

for C_0 sufficiently small, but independent of ϵ and $\vec{p} \in \mathcal{K}$. Under these conditions, the smaller root at leading order is

$$r_1 = C|\ln \epsilon| \left(T_1(t_0) + \epsilon^\beta \right), \quad (6.184)$$

and the region (r_1, r_2) is excluded for $T_1(t)$. In particular if

$$T_1(t_0) \leq r_1, \quad (6.185)$$

then

$$T_1(t) \leq r_1, \quad (6.186)$$

for Δt satisfying (6.175). In particular

$$T_1(t) \leq C|\ln \epsilon| \left(T_1(t_0) + \epsilon^\beta \right), \quad (6.187)$$

which in terms of W , for Δt exactly satisfying (6.175), becomes

$$\|W(t)\|_X \leq C\epsilon^{\omega\beta} \left(\|W(t_0)\|_X + \epsilon^\beta \right), \quad (6.188)$$

for any $0 < \omega < 1$ and any $t \in (t_0, t_0 + \Delta t)$, where the $|\ln \epsilon|$ term is absorbed in the $\epsilon^{\omega\beta}$ term by taking ω slightly smaller.

6.3 The renormalization group iteration

We can now iterate the estimates above to an equilibrium, much as in the application of renormalization group methods to statistical mechanics. We break our evolution equation into a series of initial value problems. We fix $\omega < 1$ and define the renormalization times as $t_n = t_{n-1} + \Delta t$, where Δt is given in (6.175). On the disjoint intervals $I_n = [t_{n-1}, t_n]$, we have initial data $W(t_n) \in X_{\vec{p}_n}$, with $T_{1,n}$ and $T_{2,n}$ corresponding to (6.39) and (6.40). The renormalization group map $\mathfrak{G}$ takes the initial data $W_{n-1} = W(t_{n-1})$ for the initial value problem on I_{n-1} and pulse positions $\vec{p}_{n-1} = \vec{p}(t_{n-1})$ and returns the initial data $W_n = W(t_n)$ and pulse positions $\vec{p}_n = \vec{p}(t_n)$ for the next initial value problem:

$$\mathfrak{G} \begin{pmatrix} W_{n-1} \\ \vec{p}(t_{n-1}) \end{pmatrix} = \begin{pmatrix} W_n \\ \vec{p}(t_n) \end{pmatrix}. \quad (6.189)$$

This map includes both the evolution under the flow, and a reprojection under Proposition 6.1. The initial data and the new base point $\vec{p}_n$ are obtained from $W(t_n^-)$, the right end point of the evolution of W over I_{n-1} . This process is illustrated in Figure 1.1.

To bound the renormalization group map, we must control the secular jump under

the projection. From Lemma 6.1,

$$p_n = p(t_n^-) + \mathcal{H} \left(W(t_n^-) \right). \quad (6.190)$$

Since $W(t_n^-) \in X_{p_{n-1}}$, we apply the estimate (6.3), which becomes

$$|\vec{p}_n - \vec{p}(t_n^-)| \leq M_1 \|W(t_n^-)\|_X T_2(\Delta t) \quad (6.191)$$

$$\leq M_1 \|W(t_n^-)\|_X \left(\epsilon^\beta + \|W(t_n^-)\|_X \right) \|W(t_n^-)\|_X, \quad (6.192)$$

where we applied estimates on T_2 from (6.164) and Δt from (6.175). We decompose the solution at time t_n as

$$\vec{U}(t_n) = \Phi_{\vec{p}(t_n^-)} + W(t_n^-) = \Phi_{\vec{p}(t_n)} + W(t_n). \quad (6.193)$$

We can bound the jump of W at each renormalization as

$$\|W(t_n) - W(t_n^-)\|_X = \|\Phi_{\vec{p}(t_n^-)} - \Phi_{\vec{p}(t_n)}\|_X \quad (6.194)$$

$$\leq C |\vec{p}(t_n^-) - \vec{p}(t_n)| \quad (6.195)$$

$$\leq M_1 \left(\epsilon^\beta + \epsilon^{-\alpha} T_{1,n-1}^2 \right) \|W(t_n^-)\|_X \quad (6.196)$$

$$\leq M_1 \left(\epsilon^\beta + \|W(t_n^-)\|_X \right) \|W(t_n^-)\|_X, \quad (6.197)$$

since Φ is X -Lipschitz in $\vec{p}$. Since $\epsilon^{-\alpha} T_1^2 \ll 1$, the renormalization step is asymptotically negligible and we recover the estimate

$$\|W_n\|_X \leq M\epsilon^{\omega\beta} \left(\|W_{n-1}\|_X + \epsilon^\beta \right). \quad (6.198)$$

To control $\|W\|_X$ on the long, renormalization group time scale, we may introduce

$$\eta_{n+1} = M\epsilon^{\omega\beta} \left(\eta_n + \epsilon^\beta \right) \quad (6.199)$$

$$\eta_0 = \|W_0\|_X, \quad (6.200)$$

so that

$$\|W_n\|_X \leq \eta_n. \quad (6.201)$$

It is easy to verify that

$$\eta_n \rightarrow \frac{M}{1 - \epsilon^{\omega\beta} M} \epsilon^{(1+\omega)\beta}, \quad (6.202)$$

as $n \rightarrow \infty$ for η_0 small enough. The overall evolution for W may be written as

$$\|W\|_X \leq M \left(e^{-\frac{\epsilon^\alpha}{2} \mu t} \|W_0\|_X + \epsilon^{(1+\omega)\beta} \right), \quad (6.203)$$

for any $0 < \omega < 1$.

6.4 Long-time asymptotics

After the residual has relaxed into its $O\left(\epsilon^{(1+\omega)\beta}\right)$ equilibrium layer about the manifold Φ , the pulse dynamics in (6.114) reduce to

$$\dot{p}_j = -\frac{\left(R_2(\Phi) + [\Delta L(W + \vec{\Phi}_1)]_2 + \mathcal{N}_2(\vec{\Phi}_1, W), \phi'_j\right)_{L^2}}{q_j^{-2\alpha_{21}/(\alpha_{22}-1)} \|\phi'_0\|_{L^2}^2} + O\left(\epsilon^2\right). \quad (6.204)$$

Using (6.175), (6.183), and (6.187) to estimate (6.164), we have

$$T_2(t) \leq C_1 \left(\epsilon^{2-\beta} \Delta t + \epsilon^2 + \epsilon^{-\alpha} T_1(t)^2 \right) \quad (6.205)$$

$$\leq C \left(\epsilon^\beta + \epsilon^2 + \epsilon^{-\alpha} \left(|\ln \epsilon| \left(T_1(t_0) + \epsilon^\beta \right) \right)^2 \right) \quad (6.206)$$

$$\leq C \epsilon^\beta, \quad (6.207)$$

if

$$T_1(t_0) \leq \frac{\epsilon^{\frac{1}{2} + \frac{\alpha}{4}}}{|\ln(\epsilon)|}. \quad (6.208)$$

The above it is more exigent then (6.183), so it is the constraint on $\|W_0\|_X$ in Theorem 1.1. We use (6.43), (6.46), and (6.207) to bound the following:

$$\|[\Delta L \tilde{W}]_2\|_{L^2} \leq C \left(\epsilon^{2-\beta} + T_2(t) \right) \left(\|W\|_X + \epsilon^{2-\beta} \right) \quad (6.209)$$

$$\leq C \left(\epsilon^2 + \epsilon^\beta \|W\|_X \right) \quad (6.210)$$

$$\leq C \epsilon^2. \quad (6.211)$$

Also for the nonlinearity, using (6.48) we have

$$\|\mathcal{N}_2(\tilde{W})\|_{L^2} \leq C \left(\|W\|_X^2 + \epsilon^{4-2\beta} \right) \leq C \epsilon^{4-2\beta}. \quad (6.212)$$

So the evolution equation reduces (6.204) to

$$\dot{p}_j = - \frac{\left(R_2(\Phi), \phi'_j \right)_{L^2}}{q_j^{-2\alpha_{21}/(\alpha_{22}-1)} \|\phi'_0\|_{L^2}^2} + O\left(\epsilon^2\right). \quad (6.213)$$

Taylor expanding, we can reduce $\left(R_2(\Phi), \phi'_j\right)_{L^2}$ to

$$\left(R_2(\Phi), \phi'_j\right)_{L^2} = \left(\sum_{i=1}^N \left(\Phi_1^{\alpha_{21}} - q_i^{\alpha_{21}}\right) \phi_i^{\alpha_{22}}, \phi'_j\right)_{L^2} \quad (6.214)$$

$$= \left(\left(\Phi_1^{\alpha_{21}}\right)'(p_j)(x - p_j)\phi_j^{\alpha_{22}}, \phi'_j\right)_{L^2} + O\left(\epsilon^2\right) \quad (6.215)$$

$$= \alpha_{21} \Phi_1^{\alpha_{21}-1}(p_j) \Phi'_1(p_j) \left((x - p_j)\phi_j^{\alpha_{22}}, \phi'_j\right)_{L^2} + O\left(\epsilon^2\right) \quad (6.216)$$

$$= \alpha_{21} q_j^{\alpha_{21}-1} \Phi'_1(p_j) \left((x - p_j)\phi_j^{\alpha_{22}}, \phi'_j\right)_{L^2} + O\left(\epsilon^2\right). \quad (6.217)$$

Continuing, we integrate by parts on the above inner product where

$$\left(R_2(\Phi), \phi'_j\right)_{L^2} = \alpha_{21} q_j^{\alpha_{21}-1} \Phi'_1(p_j) \left((x - p_j)\phi_j^{\alpha_{22}}, \phi'_j\right)_{L^2} + O\left(\epsilon^2\right) \quad (6.218)$$

$$= -\alpha_{21} q_j^{\alpha_{21}-1} \Phi'_1(p_j) \left(1, \frac{\phi_j^{\alpha_{22}+1}}{\alpha_{22}+1}\right)_{L^2} + O\left(\epsilon^2\right) \quad (6.219)$$

$$= -\frac{\alpha_{21}}{\alpha_{22}+1} q_j^{\alpha_{21}-1} \Phi'_1(p_j) \overline{\phi_j^{\alpha_{22}+1}} + O\left(\epsilon^2\right). \quad (6.220)$$

Substituting the definition $\phi_j(x) = \frac{\phi_0(x - p_j)}{q_j^{\alpha_{21}/(\alpha_{22}-1)}}$ from (3.4), the above expands to

$$\left(R_2(\Phi), \phi'_j\right)_{L^2} = \frac{-\alpha_{21}}{\alpha_{22}+1} q_j^{\alpha_{21}-1} \Phi'_1(p_j) \overline{\phi_j^{\alpha_{22}+1}} + O(\epsilon^2) \quad (6.221)$$

$$= -\frac{\Phi'_1(p_j)^{\alpha_{21}}}{\alpha_{22}+1} \frac{q_j^{\alpha_{21}-1}}{q_j^{\alpha_{21}(\alpha_{22}+1)/(\alpha_{22}-1)}} \overline{\phi_0^{\alpha_{22}+1}} + O(\epsilon^2) \quad (6.222)$$

$$= -\frac{\Phi'_1(p_j)^{\alpha_{21}}}{\alpha_{22}+1} q_j^{\alpha_{21}-1-\frac{\alpha_{21}(\alpha_{22}+1)}{\alpha_{22}-1}} \|\phi_0\|_{L^{\alpha_{22}+1}}^{\alpha_{22}+1} + O(\epsilon^2). \quad (6.223)$$

Now returning to (6.213), and substituting the above, we have

$$\dot{p}_j = -\frac{\left(R_2(\Phi), \phi'_j\right)_{L^2}}{q_j^{-2\alpha_{21}/(\alpha_{22}-1)} \|\phi'_0\|_{L^2}^2} + O(\epsilon^2) \quad (6.224)$$

$$= \frac{\Phi'_1(p_j)^{\alpha_{21}}}{\alpha_{22}+1} \frac{\|\phi_0\|_{L^{\alpha_{22}+1}}^{\alpha_{22}+1}}{\|\phi'_0\|_{L^2}^2} q_j^{\alpha_{21}-1-\frac{\alpha_{21}(\alpha_{22}+1)}{\alpha_{22}-1} + \frac{2\alpha_{21}}{\alpha_{22}-1}} + O(\epsilon^2) \quad (6.225)$$

$$= \frac{\alpha_{21}}{\alpha_{22}+1} \frac{\|\phi_0\|_{L^{\alpha_{22}+1}}^{\alpha_{22}+1}}{\|\phi'_0\|_{L^2}^2} \frac{1}{q_j} \Phi'_1(p_j) + O(\epsilon^2), \quad (6.226)$$

where

$$\alpha_{21} - 1 - \frac{\alpha_{21}(\alpha_{22} + 1)}{\alpha_{22} - 1} + 2\alpha_{21}/(\alpha_{22} - 1) = \alpha_{21} - 1 + \frac{-\alpha_{21}\alpha_{22} - \alpha_{21} + 2\alpha_{21}}{\alpha_{22} - 1} \quad (6.227)$$

$$= \alpha_{21} - 1 + \frac{-\alpha_{21}(\alpha_{22} - 1)}{\alpha_{22} - 1} \quad (6.228)$$

$$= -1. \quad (6.229)$$

Then we can express the dynamics of the pulse position as

$$\dot{p}_j = \frac{\alpha_{21}}{\alpha_{22} + 1} \frac{\|\phi_0\|_{L^{\alpha_{22}+1}}^{\alpha_{22}+1}}{\|\phi'_0\|_{L^2}^2} \frac{1}{q_j} \Phi'_1(p_j) + O(\epsilon^2). \quad (6.230)$$

We know from (3.44), that $\Phi'_1(p_j)$ is $O(\epsilon^{2-\beta})$ in L^∞ , so the above will generate leading order dynamics. We want to determine a representation for $\Phi'_1(p_j)$. We substitute the definition of $\phi_j(x)$ from (3.4) into the definition of Φ_1 in (3.2), where

$$\Phi_1 = \epsilon^{-\beta} L_{11}^{-e} \left(\sum_{j=1}^N q_j^{\alpha_{11}} \phi_j^{\alpha_{12}}(x) \right) \quad (6.231)$$

$$= \epsilon^{-\beta} L_{11}^{-e} \left(\sum_{j=1}^N q_j^{\alpha_{11} - \alpha_{21}\alpha_{12}/(\alpha_{22}-1)} \phi_0^{\alpha_{12}}(x - p_j) \right) \quad (6.232)$$

Differentiating the above, we have

$$\Phi'_1 = \epsilon^{-\beta} \alpha_{12} L_{11}^{-e} \left(\sum_{j=1}^N q_j^\theta \phi_0^{\alpha_{12}-1} (x - p_j) \phi'_0(x - p_j) \right), \quad (6.233)$$

for θ defined in (2.67). Then substituting the above, we expand term $\Phi'_1(p_j)$ in (6.230) as

$$\Phi'_1(p_j) = \left(\Phi'_1(x), \delta p_j \right) \quad (6.234)$$

$$= \epsilon^{-\beta} \alpha_{12} \left(\sum_{k=1}^N q_k^\theta \phi_0^{\alpha_{12}-1} \phi'_0, L_{11}^{-e} \delta p_j \right) \quad (6.235)$$

$$= \epsilon^{-\beta} \alpha_{12} \left(\sum_{k=1}^N q_k^\theta \phi_0^{\alpha_{12}-1} \phi'_0, G_0(x - p_j) \right) \quad (6.236)$$

$$= \alpha_{12} \left(\sum_{k=1}^N q_k^\theta \phi_0^{\alpha_{12}-1} \phi'_0, \sqrt{\frac{\pi}{2\mu}} e^{-\epsilon^{1+\alpha/2} \sqrt{\mu}|x-p_j|} \right), \quad (6.237)$$

where $G_0(x) = \sqrt{\frac{\pi}{2\mu}} \epsilon^{1-\alpha/2} e^{-\epsilon^{1+\alpha/2} \sqrt{\mu}|x|}$, the Green's function defined in (2.49).

We also used the fact that $1 - \alpha/2 - \beta = 0$. Then continuing the above and integrating

by parts, we have

$$\Phi_1'(p_j) = \alpha_{12} \left(\sum_{k=1}^N q_k^\theta \phi_0^{\alpha_{12}-1} \phi_0', \sqrt{\frac{\pi}{2\mu}} e^{-\epsilon^{1+\alpha/2} \sqrt{\mu}|x-p_j|} \right) \quad (6.238)$$

$$= \sqrt{\frac{\pi}{2\mu}} \left(\sum_{k=1}^N q_k^\theta \left(\phi_0^{\alpha_{12}}(x-p_k) \right)', e^{-\epsilon^{1+\alpha/2} \sqrt{\mu}|x-p_j|} \right) \quad (6.239)$$

$$= \sqrt{\frac{\pi}{2}} \epsilon^{1+\alpha/2} \sum_{k=1}^N q_k^\theta \left(\phi_0^{\alpha_{12}}(x-p_k), \text{sign}(x-p_j) e^{-\epsilon^{1+\alpha/2} \sqrt{\mu}|x-p_j|} \right) \quad (6.240)$$

$$= \sqrt{\frac{\pi}{2}} \epsilon^{2-\beta} \sum_{k=1}^N q_k^\theta \left(\phi_0^{\alpha_{12}}(x-p_k), \text{sign}(p_k-p_j) e^{-\epsilon^{1+\alpha/2} \sqrt{\mu}|p_k-p_j|} \right) + O\left(\epsilon^{4-2\beta}\right) \quad (6.241)$$

$$= \sqrt{\frac{\pi}{2}} \epsilon^{2-\beta} \phi_0^{\alpha_{12}} \sum_{k=1}^N q_k^\theta \left(\text{sign}(p_k-p_j) e^{-\epsilon^{1+\alpha/2} \sqrt{\mu}|p_k-p_j|} \right) + O\left(\epsilon^{4-2\beta}\right). \quad (6.242)$$

Combining this with (6.230) in vector form yields the following representation of the pulse dynamics:

$$\dot{\vec{p}} = \epsilon^{2-\beta} Q^\theta \mathcal{A}(\vec{p}) \vec{q}^{-1} + O\left(\epsilon^2\right), \quad (6.243)$$

where Q is the diagonal matrix of the amplitudes $\vec{q}$ and the antisymmetric matrix $\mathcal{A}(\vec{p})$ is defined componentwise as

$$\mathcal{A}_{kj} = \begin{cases} \sqrt{\frac{\pi}{2}} \frac{\alpha_{21}}{\alpha_{22}+1} \frac{\|\phi_0\|_{L^2}^{\alpha_{22}+1}}{\|\phi'_0\|_{L^2}^2} \phi_0^{\alpha_{12}} e^{-\epsilon^{1+\alpha/2} \sqrt{\mu} |p_k - p_j|} & k > j \\ 0 & k = j \\ -\sqrt{\frac{\pi}{2}} \frac{\alpha_{21}}{\alpha_{22}+1} \frac{\|\phi_0\|_{L^2}^{\alpha_{22}+1}}{\|\phi'_0\|_{L^2}^2} \phi_0^{\alpha_{12}} e^{-\epsilon^{1+\alpha/2} \sqrt{\mu} |p_k - p_j|} & k < j \end{cases}. \quad (6.244)$$

Then the pulse position dynamics are a function of the amplitudes $\vec{q}$, where the pulse amplitudes are a function of the pulse positions $\vec{p}$.

Chapter 7

Future Work

In the future, we would like to investigate a more general system and prove similar results. The following list is three natural extensions to the research of this dissertation.

- Extend the adiabatic stability results for very weak damping, that is for $\alpha \geq \frac{2}{3}$.
- Extend the adiabatic stability results for a more general nonlinearity and include multiple activator components.
- Extend to two space dimensions to investigate the existence of radial spot solutions and study the interactions of N -spot patterns.

Extending to the case of very weak damping, $\alpha \geq \frac{2}{3}$, requires handling the spectrum differently. Our previous methods fail because of a loss of normal hyperbolicity:

the stable eigenspace becomes too close to the center eigenspace. A methodology to handle this differently would be to split the essential spectrum into two parts: $(-\infty, -\rho)$ and $[-\rho, -\epsilon^\alpha \mu]$ for $\rho > 0$, where the smaller interval is called the nose of the essential spectrum. One would need to introduce two different spectral projections, developed as appropriate contour-deformation limits of the resolvent. The semigroup estimates would also be established as a limit in which the contour of integration relaxes onto the nose of the essential spectrum. A key difficulty in this analysis would be projecting the essential spectrum off of the nose and assuring that the spectral projection faithfully captures any embedded eigenvalues, uniformly as $\epsilon \rightarrow 0^+$.

The second extension would examine a more general nonlinearity. With a general nonlinearity, the problem could encompass many equations including the regularized Gierer-Meinhardt equation and the Gray-Scott equation. The goal would be to unify previous research and new results under one inclusive work. This would offer a general system that could be used for many specific reaction-diffusion equations. One could also pursue the problem in the case of multiple activator components to achieve similar results.

The third extension would be to prove the existence of radial spot solutions for a general equation and study their stability properties. One could also examine the interaction of multiple spot patterns. Here U and V are functions of two space

dimensions. Previous results have formally been extended to the two-dimensional spatial spot problems, including [8], [16], [17], [27], [33], [35], [36], and [37]. Many of these works include the Gierer-Meinhardt or the Gray-Scott model. In the future, we hope to rigorously extend similar results to the two-dimensional setting.

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